UNIVERSITY OF CAPE TOWN

DEPARTMENT OF MATHEMATICS

The Strict Topology;

Theory, Generalizations and Applications.

by

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CONTENTS.

INTRODUCTION.

The strict topology β was first defined on the space of bounded complex-valued continuous functions $C_b(X)$, on a locally compact Hausdorff space X, by Buck [2]. It was found to have many applications in Approximation Theory, spectral synthesis, spaces of bounded holomorphic functions and multipliers of Banach Algebras. For references to these applications see Conway's paper [4].

The norm topology does not seem the appropriate topology to place on $C_b(X)$ if one wishes to study $C_b(X)$ as a topological vector space, particularly if one wishes to relate properties in $C_b(X)$ with those of X (see Chapter 5). If $C_b(X)$ is given the norm topology, then the dual is a space of measures on the Stone-Čech compactification βX. It would be preferable to have a topology on $C_b(X)$ so that the dual is a space of measures on X and which coincides with the norm topology when X is compact. The strict topology satisfies these conditions.

In Chapter 1 we give the basic definitions and properties of the strict topology and we also find its topological dual space. Most of this work has been done by Buck [2], but certain aspects of his proofs have been enlarged upon and modified. We have included a brief discussion of the compact-open topology to serve as a comparison to the strict topology. Finally we discuss the duality between $C_b(X)$ and its dual using a functional analytic approach rather than the measure-theoretic approach adopted by Conway [4].

Wiweger [25] has defined the concept of a mixed topology on a bi-topological space. In Chapter 2 we examine a generalization of the mixed topology introduced by Cooper [6]. Using two different methods, we show that the strict topology is an example of such a mixed topology.

The theory of weighted spaces has been expounded by Summers in [21] and [22]. In Chapter 3 we clarify certain statements in Summers' work and show that $C_b(X)$ with the strict topology is a weighted space. We also give a new method for proving the completeness of $(C_b(X),\beta)$.

Chapter 4 considers a method of generalizing the strict topology and certain applications to approximation theory are given. In Chapter 5 we have included a discussion of the relations between the topological properties of $(C_b(X),\beta)$ and those of X. The final part of Chapter 5 considers necessary and sufficient conditions for $C_b(X)$ to be the topological dual of a Banach space satisfying certain conditions. We have also included a brief discussion of the particular case when X is taken to be the positive integers with the discrete topology.

Theorems 1.0.7, 1.0.8, 1.1.1, 1.1.3, 3.0.4, 3.1.1, 5.0.9, the necessity of 5.1.2 and 5.1.7 are original. New proofs have been given for Theorems 1.0.6 (ii) and (iii), 1.1.2, the necessity of 1.1.4, 2.0.6, 3.1.2, 3.1.3, 4.0.8, 5.0.4, 5.0.5 and 5.0.6. In the remaining work some details of known proofs have been supplied and the exposition clarified.

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PROPERTIES OF THE STRICT TOPOLOGY.

1.0.1. We shall begin by describing the space which will be used in this dissertation.

Unless otherwise specified X will denote a locally compact, Hausdorff topological space.

$C(X)$: the vector space of all continuous real or complex-valued functions on X.

$C_b(X)$: the vector space of bounded continuous real or complex-valued funtions on X.

$C_o(X)$: the vector space of continuous functions which vanish at infinity. That is, $\varphi \in C_o(X)$ if given any $\varepsilon > 0$, there is a compact set $K \subset X$ such that $|\varphi(x)| < \varepsilon$ for all $x \notin K$.

$C_K(X)$: the vector space of continuous functions with compact support. The support of a function $f \in C(X)$ is the closure in X of the set $\{x \in X: f(x) \neq 0\}$ and will be denoted by supp f.

$L(X)$: the space of lower semi-continuous functions on X. Recall that a real valued funtion f on X is lower semi-continuous if $\{x \in X: f(x) \leq a\}$ is closed for each real number a. A function f is upper semi-continuous if $-f$ is lower semi-continuous.

$M(X)$: the space of bounded Radon measures defined on X. For a definition, see [1] page 50.

$C^+(X)$: the space consisting of the non-negative functions in $C(X)$. That is

$$C^+(X) = \{f \in C(X): f(x) \geq 0 \text{ for all } x \in X\}.$$

$C_b^+(X)$, $C_o^+(X)$ and $L^+(X)$ are defined similarly.

We give $C_b(X)$ the following locally convex structures:-

σ: the Banach space topology defined by the norm $\|f\| = \sup\{|f(x)| : x \in X\}$, called the norm topology.

κ: the topology defined by the family of seminorms $\|f\|_K = p_K(f) = \sup\{|f(x)| : x \in K\}$, where K runs through the compact subsets of X and called the compact-open topology.

β: the topology defined by the family of seminorms $\|f\|_\varphi = \sup\{|f(x)\varphi(x)| : x \in X\}$, where $\varphi \in C_o(X)$ and called the strict topology.

Note (a): The strict topology on the space $C_b(X)$, for X locally compact was introduced by Buck [2].

(b): If $V_\varphi = \{f \in C_b(X) : \|f\varphi\| \leqslant 1\}$ where $\varphi \in C_o(X)$ then the sets $\{V_\varphi\}$ form a neighbourhood basis at zero for the strict topology.

(c): The compact open topology is generated by the seminorms $\|f\|_\varphi = \sup\{|f(x)\varphi(x)| : x \in X\}$ where $\varphi \in C_K(X)$.

(d): Each $\varphi \in C_o(X)$ has σ-compact support. For each $n \in N$, define

$$K_n = \{x \in X : |\varphi(x)| \geqslant \frac{1}{n}\}.$$

Then K_n is compact and $\operatorname{supp}\varphi = \bigcup_{n=1}^{\infty} K_n$ is σ-compact. It is tempting to conjecture that the strict topology is generated by the family $\{p_K(f) : K \subset X, K \text{ is } \sigma\text{-compact}\}$, of seminorms. This is false as the counter-example below shows. Let $\mathbb{R}$ denote the real numbers with its usual topology. Since R itself is σ-compact, as $R = \bigcup_{n=1}^{\infty}[-n,n]$, the topology generated by the above family of seminorms is the norm topology on $C_b(R)$. We shall see in Theorem 1.0.7., that if X is not compact then $\beta \neq \sigma$. Note also that the fact that $\mathrm{supp}\varphi$ is σ-compact need not imply that $\varphi \in C_o(X)$. For example, consider the constant function $1(x) = 1$ for all $x \in R$.

1.0.2. Lemma. Suppose X is a locally compact topological space. If $\{x_n\}$ is a discrete sequence of points in X and $\{c_n\}$ a null sequence of positive real numbers, than there is a $\varphi \in C_o(X)$ such that $\varphi(x_n) = c_n$. Furthermore, if $f \in C(X)$ such that $\varphi f \in C_o(X)$ for every $\varphi \in C_o(X)$, then $f \in C_b(X)$. (By a discrete sequence, we mean a sequence with no points of accumulation, or equivalently, the given topology coincides with the discrete topology on the sequence.)

Proof: Since X is locally compact, we may choose a sequence $\{K_n\}$ of compact sets with $x_n \in \mathrm{int}K_n$ and $K_m \cap K_n = \varphi$ for all $m \neq n$. For each n, choose $\varphi_n \in C(X)$ satisfying

$$\begin{array}{ll} 0 \leqslant \varphi_n(x) \leqslant 1 & \text{for all } x \in X, \\ \varphi_n(x_n) = 1 & \text{and} \\ \varphi_n(x) = 0 & \text{if } x \notin K_n. \end{array}$$

Let $\varphi(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x)$. Then $\varphi(x_n) = c_n$. Also $\varphi \in C_o(X)$, since if $\varepsilon > 0$ we can choose n such that $c_m < \varepsilon$ for all $m > n$. Then if $x \notin \bigcup_{i=1}^{n} K_i$ either $x \in K_m$ for $m > n$ or $x \notin \bigcup_{n=1}^{\infty} K_n$.

If $x \in K_m$, then $\varphi(x) \leq c_m < \varepsilon$. If $x \notin \bigcup_{n=1}^{\infty} K_n$, then $\varphi(x) = 0 < \varepsilon$ as required.

Suppose $f \in C(X)$ as in the statement of the lemma and suppose that f is not bounded on X. Then for each $n \in \mathbb{N}$, choose $x_n \in X$ such that $|f(x_n)| \geq n$. Since f is continuous the sequence $\{x_n\}$ may be chosen to be discrete. By above, choose $\varphi \in C_o(X)$ so that

$$\varphi(x_n) = \frac{1}{f(x_n)} \qquad \text{for each } n.$$

So $\varphi f(x_n) = 1$ for $n = 1,2,\ldots$ Thus $\varphi f \notin C_o(X)$ which is a contradiction. Suppose $\varphi f \in C_o(X)$ and let $\varepsilon > 0$. Then there exists a compact set K such that $|\varphi f(x)| < \varepsilon$ for $x \notin K$. Hence $\{x_n\} \subset K$ and since K is compact this sequence has a point of accumulation.

Remarks (a): Before giving some basic properties of the strict topology on $C_b(X)$ we will state a few well known results relating to the compact-open and norm topologies.

(b): In the norm topology, $C_b(X)$ is a Banach space (complete normed space), $C_K(X)$ is not dense in $C_b(X)$ but its closure is the complete space $C_o(X)$ with the norm topology.

(c): In the compact-open topology, $C_b(X)$ is a locally convex linear space. It is metrizable only if X is σ-compact (see below). $C_K(X)$ is κ-dense in $C_b(X)$. Also $(C_b(X),\kappa)$ is not complete, the completion being $(C(X),\kappa)$ when X is a k-space (see definition 4.0.3.).

(d): Recall that a topological space X is σ-compact if $X = \bigcup_{n=1}^{\infty} K_n$ where each K_n is compact. X is regularly σ-compact if X is σ-compact and the compact sets $\{K_n\}$ may be chosen so that

$$\phi \neq \operatorname{int}K_1 \subset K_1 \subset \operatorname{int}K_2 \subset K_2 \ \ldots$$

1.0.3. <u>Lemma.</u> Every σ-compact, locally compact topological space X is regularly σ-compact.

<u>Proof</u>: Let X be as above. Then $X = \bigcup_{n=1}^{\infty} K_n$ where each K_n is compact.

Define $K_1' = K_1$ and for each $x \in K_1'$ choose an open neighbourhood V_x of x such that $\overline{V}_x$ is compact. This is possible since X is locally compact. Then $\{V_x : x \in K_1'\}$ is an open cover of K_1' so there is a finite subcover $\{V_{x_1}, \ldots, V_{x_n}\}$. Let

$$K_2' = \bigcup_{i=1}^{n} \overline{V}_{x_i} \supset \operatorname{int}K_2' \supset \bigcup_{i=1}^{n} V_{x_i} \supset K_1'.$$

Now $K_2' \cup K_2$ is compact, so proceeding as before we find K_3' compact such that

$$K_3' \supset \operatorname{int}K_3' \supset K_2' \cup K_2 \supset K_2'.$$

Continue in this way to obtain a sequence $\{K_n'\}$ of compact subsets of X such that $\bigcup_{n=1}^{\infty} K_n' = X$ and

$$K_1' \subset \operatorname{int}K_2' \subset K_2' \subset \operatorname{int}K_3' \subset \ldots$$

Hence X is regularly σ-compact as required.

1.0.4. **Lemma.** Suppose X is regularly σ-compact where $X = \bigcup_{n=1}^{\infty} K_n$ and $K_1 \subset \mathrm{int}K_2 \subset \ldots$ If K is any compact subset of X, then $K \subset K_n$ for some n.

Proof: Suppose that the conclusion is false. Choose $x_1 \in K$ such that $x_1 \notin K_1$. Let n_2 be the smallest integer such that $x_1 \in K_{n_2}$. Continue in this way to obtain a sequence $\{x_{n_r}\} \subset K$ such that $x_r \notin K_{n_r}$, $x_r \in K_{n_{r+1}}$. Clearly

$$X = \bigcup_{r=1}^{\infty} K_{n_r} \quad \text{and} \quad K_{n_r} \subset \mathrm{int}K_{n_{r+1}} \quad \text{for all } r.$$

Relabel the K_{n_r} by replacing K_{n_r} with K_r. Since K is compact, the sequence $\{x_r\}$ has a point of accumulation x_o say, where $x_o \in K$. Since $x_o \in X = \bigcup_{r=1}^{\infty} K_r$, there is an m such that $x_o \in K_m \subset \mathrm{int}K_{m+1}$. Now if $r > m$ then $x_r \notin K_r$ and $x_r \in K_{r+1}$. But $K_m \subset \mathrm{int}K_{m+1} \subset \ldots \subset \mathrm{int}K_{r-1} \subset K_r$, so $x_r \notin \mathrm{int}K_{m+1}$ for all $r > m$. That is, $x_o \notin \overline{\{x_r\}}_{r>m}$ since $x_o \in \mathrm{int}K_{m+1}$ which is open.

So x_o is not a point of accumulation of $\{x_n\}$ which gives the desired contradiction.

1.0.5. **Theorem.** Suppose X is a σ-compact, locally compact topological space. Then $(C(X),\kappa)$ is metrizable.

Proof: The compact-open topology is generated by the seminorms $p_K(f) = \sup\{|f(x)| : x \in K\}$ where K runs through the compact subsets of X. By Lemma 1.0.3. we may let $X = \bigcup_{r=1}^{\infty} K_r$ where $K_1 \subset \mathrm{int}K_2 \subset K_2 \subset \ldots$ We shall show that the countable family of seminorms $\{p_{K_n}\}$ generates a local base at zero for κ. It is sufficient to show that for each p_K there is a p_{K_n} such that $p_K \leqslant p_{K_n}$.

Let K be any compact set, then by Lemma 1.0.4. there is an integer n such that $K \subset K_n$. Clearly $p_K \leqslant p_{K_n}$. Also $(C(X),\kappa)$ is Hausdorff, since if $f \neq 0$, $f \in C(X)$ we may choose $x_o \in X$ such that $f(x_o) = r > 0$. Choose ε so that $0 < \varepsilon < r$. Since f is continuous and X is locally compact, there is an open neighbourhood U of x_o such that $f(x) > \varepsilon$ for all $x \in U$ and $\overline{U}$ is compact.

Consider the seminorm $p_{\overline{U}}(f) = \sup\{|f(x)| : x \in \overline{U}\}$. Then $V = \{g \in C(X) : p_{\overline{U}}(g) \leqslant \varepsilon\}$ is a basic κ-neighbourhood of zero and $f \notin V$.

The result now follows from a standard theorem on the metrizability of linear topological spaces (see [15], page 163).

In Chapter 5 we shall discuss the metrisability of $(C_b(X),\beta)$. We shall now list and prove some of the basic and important facts concerning the strict topology which will be used repeatedly throughout this work.

1.0.6. Theorem. Let X be a locally compact Hausdorff topological space. Then

(i) $\kappa \leqslant \beta \leqslant \sigma$,

(ii) the β-bounded and norm bounded sets of $C_b(X)$ are the same,

(iii) the topology β coincides with κ on the norm bounded sets,

(iv) a sequence $\{f_n\}$ in $C_b(X)$ is β-convergent if and only if it is norm bounded and κ-convergent,

(v) the subspace $C_K(X)$ is β-dense in $C_b(X)$,

(vi) $(C_b(X),\beta)$ is a complete locally convex topological vector space.

Proof: (i) Since $\|f\|_\varphi = \|f\varphi\| \leqslant \|f\|\|\varphi\|$ it follows that $\beta \leqslant \sigma$. Conversely, if K is a compact subset of X, choose $\varphi \in C_o(X)$ such that $\varphi(x) = 1$ for all $x \in K$. Then $\|f\|_K \leqslant \|f\|_\varphi$ so that $\kappa \leqslant \beta$.

(ii) Since $\beta \leqslant \sigma$, the σ-bounded sets are β-bounded. Now suppose A is β-bounded but not σ-bounded. Let $\lambda_1 > 0$, then there exists $f_1 \in A$ and $x_1 \in X$ such that $|f_1(x_1)| > \lambda_1$. Since f_1 is continuous there is an open neighbourhood N_{x_1} of x_1 such that $\overline{N}_{x_1}$ (the closure of N_{x_1}) is compact and $|f_1(x)| > \lambda_1$ for all x in N_{x_1}. Since $\kappa = \beta$, A is κ-bounded and so there is a $\lambda_2 > 0$ such that $|f(x)| \leqslant \lambda_2$ for all $x \in \overline{N}_{x_1}$ and for all $f \in A$. Clearly $\lambda_2 > \lambda_1$. Now A is not σ-bounded, so there is an $f_2 \in A$ and $x_2 \in X$ such that $|f_2(x_2)| > \lambda_2$. Note that $x_2 \notin \overline{N}_{x_1}$. Since f_2 is continuous there is an open neighbourhood N_{x_2} of x_2 such that $\overline{N}_{x_2}$ is compact, $N_{x_2} \cap N_{x_1} = \phi$ and $|f_2(x)| > \lambda_2$ for all $x \in N_{x_2}$.

Noting that $\overline{N}_{x_1} \cup \overline{N}_{x_2}$ is compact we may proceed in this way to obtain a discrete sequence $\{x_n\}$ in X and a sequence $\{f_n\}$ in A such that $|f_n(x_n)| > \lambda_n$, where $\{\lambda_n\}$ is an unbounded increasing sequence of positive real numbers. By Lemma 1.0.2. we may choose $\varphi \in C_o(X)$ such that $\varphi(x_n) = \lambda_n^{-\frac{1}{2}}$ for each n. Then

$$p_\varphi(f_n) = \|f_n\varphi\| \geqslant |f_n(x_n)\varphi(x_n)| \geqslant \lambda_n^{\frac{1}{2}}.$$

This contradicts the fact that A is β-bounded.

(iii) Let B be a norm bounded subset of $C_b(X)$. It is sufficient to show that κ coincides with β on the absolutely convex hull of B. So we may suppose that B is absolutely convex. Since B is σ-bounded, there is an $M > 0$ such that $\|f\| \leqslant M$ for all $f \in B$. Suppose $V_\varphi \cap B$ is any $\beta|_B$-neighbourhood of zero, where $V_\varphi = \{f \in C_b(X): \|f\varphi\| \leqslant 1\}$ and $\varphi \in C_o(X)$. Choose a compact set $K \subset X$ such that $|\varphi(x)| \leqslant \frac{1}{M}$ for all $x \notin K$.

Let $V_K = \{f \in C_b(X): \|f\|_K \leqslant \frac{1}{\|\varphi\|}\}$. Then V_K is a κ-neighbourhood of zero. We show that $V_K \cap B \subset V_\varphi \cap B$. If $f \in V_K \cap B$ then if $x \notin K$, $|f(x)\varphi(x)| = |f(x)||\varphi(x)| \leqslant M.\frac{1}{M} = 1$ and if $x \in K$, $|f(x)\varphi(x)| = |f(x)||\varphi(x)| \leqslant \frac{1}{\|\varphi\|}.\|\varphi\| = 1$. Therefore $|f(x)\varphi(x)| \leqslant 1$ for all $x \in X$ and thus $f \in V_\varphi \cap B$. So $V_\varphi \cap B$ is a $\kappa|_B$-neighbourhood of zero. Since $\kappa \leqslant \beta$, every $\kappa|_B$-neighbourhood of zero is a $\beta|_B$-neighbourhood of zero. Thus $\kappa|_B$ and $\beta|_B$ have the same neighbourhoods of zero.

Let $f \in B$. Any $\beta|_B$-neighbourhood of f in B is of the form $(f + V)\cap B$, where V is a β-neighbourhood of zero in $C_b(X)$. Since $\beta|_B$ and $\kappa|_B$ have the same neighbourhoods of zero, there is an absolutely convex κ-neighbourhood U in $C_b(X)$ such that $U \cap B \subset \frac{1}{2}V$. We show that $(f + U)\cap B \subset (f + V)\cap B$. Let $g \in (f + U)\cap B$, then $g = f + u$ for some $u \in U$ and $f + u \in B$. Thus $g - f \in U$, $u \in B - B$ and so $g - f \in B - B$. Since B is absolutely convex, $B - B \subset 2B$ and since U is balanced, $U \subset 2U$. Thus $g - f \in 2(U\cap B) \subset V$. So $g \in f + V$ and since $g \in B$ it follows that $g \in (f + V)\cap B$ as required. Thus $(f + V)\cap B$

is a $\kappa|_B$ neighbourhood of f in B and so $\beta|_B \leq \kappa|_B$. Since $\kappa \leq \beta$, the reverse inequality is true and so the result follows.

(iv) Suppose $\{f_n\}$ is a β-convergent sequence to $f \in C_b(X)$. Now $\{f_n\}$ is β-bounded, hence norm bounded. Since $\kappa \leq \beta$, $\{f_n\}$ is κ-convergent to f. Conversely, if $\{f_n\}$ is norm bounded and $f_n \xrightarrow{\kappa} f$, then $f_n \xrightarrow{\beta} f$ since $\kappa = \beta$ on the norm bounded sets.

(v) Let $f \in C_b(X)$. We shall find a net in $C_K(X)$ which is β-convergent to f. Suppose K is a compact set in X. Choose $\psi_K \in C_K(X)$ such that

$$\psi_K(x) = 1 \qquad \text{if } x \in K \text{ and}$$
$$0 \leq \psi_K(x) \leq 1 \qquad \text{for all } x \in X.$$

Let $f_K = \psi_K f$. Since $\psi_K \in C_K(X)$ and f is bounded it follows that $f_K \in C_K(X)$. Partial order the compact subsets of X by inclusion. Then $\{f_K\}$ with K compact is a net in $C_K(X)$. Now $\{f_K\}$ is norm bounded since

$$\|f_K\| = \|\psi_K f\| \leq \|\psi_K\|\|f\| = \|f\| \quad \text{for all compact } K.$$

Also f_K is κ-convergent to f, since if $K_1 \subset K$ and K_1 is compact then $\|f_K - f\|_{K_1} = 0$. The result now follows from part (iv) above.

(vi) We will show in Chapter 2 and again in Chapter 3 how a proof of this result may be obtained. (See Theorems 2.1.2. and 3.1.3.)

We shall now investigate under what conditions the topologies κ, β and σ coincide.

1.0.7. Theorem. (i) $\beta = \sigma$ if and only if X is compact.

(ii) If $\kappa = \beta$ then X is pseudocompact.

Proof: (i) From Theorem 1.0.6. (v) and since $C_K(X) \subset C_o(X) \subset C_b(X)$ it follows that $C_o(X)$ is β-dense in $C_b(X)$. Since $\beta = \sigma$, we have $C_o(X)$ is σ-dense in $C_b(X)$. But $C_o(X)$ is σ-complete so that $C_o(X) = C_b(X)$. Hence X is compact. Since if X were not compact then the constant function $1(x) = 1$, for all $x \in X$, is in $C_b(X)$ but not in $C_o(X)$. Conversely, if X is compact, then $\sigma = \kappa$. Since $\kappa \leqslant \beta \leqslant \sigma$ we have $\beta = \sigma$.

(ii) Suppose $\kappa = \beta$. Now $C_K(X)$ is κ-dense in $C(X)$ and $C_K(X)$ is β-dense in $C_b(X)$. But $C_b(X)$ is β-complete, hence $C_b(X) = C(X)$ and X is pseudocompact.

Note: We shall give an example where $\kappa = \beta < \sigma$. (See Theorem 5.0.9.)

1.0.8. Theorem. If X is a locally compact topological space then $\kappa = \beta$ if and only if $C_K(X) = C_o(X)$.

Proof: Suppose $\kappa = \beta$ and let $\varphi \in C_o(X)$. We must show that φ has compact support. Now, $V_\varphi = \{f \in C_b(X) : ||f\varphi|| \leqslant 1\}$ is a β-neighbourhood and hence a κ-neighbourhood of zero. So there is a compact set $K \subset X$ and an $\varepsilon > 0$ such that $V_{K,\varepsilon} \subset V_\varphi$ where

$V_{K,\varepsilon} = \{f \in C_b(X) : \|f\|_K \leq \varepsilon\}$ is a basic κ-neighbourhood of zero.

So, if $\|f\|_K \leq \varepsilon$ then $\|f\varphi\| \leq 1$ ...(1).

We claim that $\operatorname{supp}\varphi \subset K$, since if not, we can choose $x_0 \notin K$ such that $\varphi(x_0) = r \neq 0$. For each $n \in \mathbb{N}$, choose $f_n \in C_b(X)$ such that $f_n(x) = 0$ for all $x \in K$ and $f_n(x_0) = n$. Choose n_0 so that $n_0|r| > 1$. Then $\|f_{n_0}\|_K \leq \varepsilon$, but $|f_n\varphi(x_0)| = n_0|r| > 1$ and hence $\|f_n\varphi\| > 1$. This contradicts (1) and so $\operatorname{supp}\varphi$ is compact.

The converse follows from the definitions of κ and β.

We shall now give a result which suggests that the strict topology is a more natural topology on $C_b(X)$ than the norm topology whenever X is locally compact. We shall need the following lemma.

1.0.9. Lemma. Each measure μ in $M(X)$ has a σ-compact support.

Proof: It is known that the uniform dual of $C_0(X)$ is $M(X)$. Since $C_K(X)$ is σ-dense in $C_0(X)$ it follows that the uniform dual of $C_K(X)$ is also $M(X)$. Let μ be a positive measure on X and define a linear functional L on $C_K(X)$ by

$$L(f) = \int_X f d\mu.$$

Since μ is bounded, L is σ-continuous and there exists a number $M = \|L\|$ such that $|L(f)| \leq M\|f\|$ for all $f \in C_K(X)$. Choose $f_n \in C_K(X)$ with $\|f_n\| = 1$ and $\lim L(f_n) = M$. Let $K_n = \operatorname{supp} f_n$ and let $S_\mu = \bigcup_{n=1}^{\infty} K_n$. Since $f \in C_K(X)$, each K_n is compact and thus S_μ is σ-compact. We show that $\operatorname{supp}\mu \subset S_\mu$. Let $g \in C_K(X)$ be any function which is zero on S_μ. We can assume that $\|g\| < \frac{1}{2}$. Then, $\|f_n + g\| = \|f_n - g\| = 1$ for each n, so that

$$|L(f_n \pm g)| = |L(f_n) \pm L(g)| \leqslant M.$$

Letting n tend to infinity and noting that $\lim L(f_n) = M$, we see that $L(g) = 0$. Hence the behaviour of a function f off S_μ does not affect the value of L(f) so that

$$L(f) = \int_{S_\mu} f d\mu \qquad \text{for all } f \in C_K(X).$$

1.1.0. __Theorem.__ If X is a locally compact Hausdorff topological space, then the strict dual of $C_b(X)$ is M(X).

__Proof:__ Let L be any β-continuous linear functional on $C_b(X)$. Since $\beta \leqslant \sigma$, L is σ-continuous on the subspace $C_K(X)$. Thus, by Lemma 1.0.9, there is a $\mu \in M(X)$ such that

$$L(f) = \int_{S_\mu} f d\mu \qquad \text{for all } f \in C_K(X) \quad \ldots(1)$$

Since $C_K(X)$ is β-dense in $C_b(X)$ it is sufficient to show that (1) defines a β-continuous linear functional on $C_K(X)$. Let $S_\mu = \bigcup_{n=1}^{\infty} K_n$, $a_n = \mu(K_n - K_{n-1})$ and put $K_0 = \varphi$. Then $a_n \geqslant 0$ and $\sum_{n=1}^{\infty} a_n = M$. Choose a sequence $\{b_n\}$ of non-negative real numbers monotone decreasing to zero and satisfying

$$\sum_{n=1}^{\infty} \frac{a_n}{b_n} < 2M.$$

Let $c_n = b_n - b_{n+1}$ so that $b_n = c_n + c_{n+1} + \ldots$ Construct $\psi_n \in C_K^+(X)$ with $\|\psi_n\| = 1$, $\psi_n(x) = 1$ for $x \in K_n$ and $\psi_n(x) = 0$ for $x \notin K_{n+1}$.

Let $\varphi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$. Then $\varphi \in C_0(X)$: if $\varepsilon > 0$, choose an integer n so that $b_n < \varepsilon$. Then if $x \notin K_n$, $\psi_r(x) = 0$ for $r=1,2,3,\ldots,n-1$ so that $\varphi(x) \leq c_n + c_{n+1} + \ldots = b_n < \varepsilon$. Also φ is zero except on S_μ and

$$\varphi(x) \geq b_n \qquad \text{for } x \in K_n - K_{n-1}.$$

The function $\frac{1}{\varphi}$ is continuous on S_μ and if $L_n = K_n - K_{n-1}$ then

$$\int_{S_\mu} \frac{1}{\varphi} d\mu = \sum_{n=1}^{\infty} \int_{L_n} \frac{1}{\varphi} d\mu \leq \sum_{n=1}^{\infty} \frac{a_n}{b_n} < 2M.$$

Let $f \in C_K(X)$ with $\|f\|_\varphi < \frac{\varepsilon}{2M}$. Then

$$\left| \int_X f d\mu \right| \leq \frac{\varepsilon}{2M} \int_{S_\mu} \frac{1}{\varphi} d\mu < \varepsilon$$

and hence (1) defines a β-continuous linear functional on $C_K(X)$ as required.

Notes: (a) We can now define a pairing functional between $C_b(X)$ and $M(X)$ by

$$\langle f, \mu \rangle = \int f d\mu.$$

(b) One advantage of the strict topology over that of the norm topology is that the space $M(X)$ is relatively easy to work with. The uniform dual of $C_b(X)$ is $M(\beta X)$ where βX is the Stone-Čech compactification of X and is somewhat unwieldy.

(c) The norm topology on $M(X)$ is the usual variation norm; namely the strong topology when regarded as the dual normed space to $(C_0(X), \sigma)$. The strong topology on $M(X)$, regarded as the dual of $(C_b(X), \beta)$, is the norm topology since the β-bounded and norm bounded sets of $C_b(X)$ are the same.

(d) Buck raised the following question: "Is it true that the strict topology coincides with the Mackey topology, the strongest locally convex topology on $C_b(X)$ having $M(X)$ as its duel?"

As yet no characterization of those spaces X for which the answer is affirmative is available, but Conway [4] has shown that $(C_b(X),\beta)$ is a Mackey space whenever X is paracompact.

A linear topological space E is a Mackey space if and only if every weak* compact convex circled subset of the dual E' is equicontinuous. It is therefore necessary to characterize the β-equicontinuous subsets of $M(X)$. But an absolutely convex set $H \subset M(X)$ is β-equicontinuous if and only if

$$H^o = \{f \in C_b(X) : |\langle f,\mu\rangle| \leq 1 \text{ for all } \mu \in H\}$$

is a β-neighbourhood of zero in $C_b(X)$. This is so if and only if there is a $\varphi \in C_o(X)$ such that $V_\varphi \subset H^o$. That is, $H \subset V_\varphi^o \subset M(X)$ where

$$V_\varphi^o = \{\mu \in M(X) : |\langle f,\mu\rangle| \leq 1 \text{ for all } f \in V_\varphi\}.$$

So it becomes necessary to characterize the sets V_φ^o where $\varphi \in C_o(X)$.

Before proceeding we indicate briefly how to extend the domain of $\mu \in M(X)$ from $C_b(X)$ to $L(X)$. For details see [12], Chapter 3, section 11. If $\mu \in M(X)$ and $g \in L^+$, let

$D_g = \{h \in C_b^+ : h \leq g\}$ where $h \leqslant g$ means $h(x) \leqslant g(x)$ for all $x \in X$. Then $g = \sup\{h : h \in D_g\}$. We can then define

$$\int g d\mu = \sup\{\int h d\mu : h \in D_g\}.$$

An analogous result holds for upper semi-continuous functions.

1.1.1. Lemma. Let E be a closed subset of X. The following statements are equivalent:

(a) $\mu(K) = 0$ if $K \subset E$ and K is compact.

(b) $\langle f,\mu\rangle = 0$ for all $f \in C_b(X)$ such that $\mathrm{supp} f \subset E$.

Proof: (a) implies (b).

Since $\mu(E) = \sup\{\mu(K): K \text{ compact}, K \subset E\}$ it follows that $\mu(E) = 0$. Let $f \in C_b(X)$ satisfy $\mathrm{supp} f \subset E$. Then

$$\int f d\mu = \int_E f d\mu = 0. \quad \text{So } \langle f,\mu\rangle = 0 \text{ as required.}$$

(b) implies (a).

Let K be compact, $K \subset E$. Then χ_K, the characteristic function of K is upper semi-continuous:

If $a > 1$, then $\{x: \chi_K(x) \geq a\} = \varphi$;
If $0 < a \leqslant 1$, then $\{x: \chi_K(x) \geq a\} = K$ and
if $a \leqslant 0$, then $\{x: \chi_K(x) \geq a\} = X$.

Since X is Hausdorff, φ, K and X are closed. Consider

$$U_{\chi_K} = \{f \in C_b(X) : f \geq \chi_K \text{ and supp } f \subset E\}.$$

Then $\chi_K = \inf\{f \in C_b(X) : f \in U_{\chi_K}\}$. By hypothesis, $\langle f,\mu\rangle = 0$ for all $f \in U_{\chi_K}$, so

$\langle \chi_K,\mu\rangle = 0$ and hence $\mu(K) = 0$ as required.

Note: This result shows the relationship between $\mu \in M(X)$ regarded as a Radon measure on X and as a bounded linear functional on $C_b(X)$.

1.1.2. Theorem. If $\varphi \in C_o(X)$ and $V_\varphi = \{f \in C_b(X): \|f\varphi\| \leqslant 1\}$, then $V_\varphi^o = \{\mu \in M(X): \mu$ vanishes off $N(\varphi)$ and $\|\frac{\mu}{\varphi}\| \leqslant 1\}$ where $N(\varphi) = \{x \in X: \varphi(x) \neq 0\}$ and $<f, \frac{\mu}{\varphi}> = <\frac{f}{\varphi}, \mu>$. Note that μ vanishes off $N(\varphi)$ if $\mu(K) = 0$ for every compact set K such that $K \cap N(\varphi) = \varphi$.

Proof: Let $\mu \in V_\varphi^o$, that is $\|f\varphi\| \leqslant 1$ implies $|<f,\mu>| \leqslant 1$. We will show that μ vanishes off $N(\varphi)$. By the previous lemma, it is enough to show that $\mu(f) = <f,\mu> = 0$ for all $f \in C_b(X)$ satisfying supp$f \subset E$. We may assume that $\varphi \geqslant 0$. Let $g \in C_b(X)$ such that supp$g \subset E$. Then $g\varphi(x) = 0$ for all $x \in X$ and so $ng\varphi(x) = 0$ for all $x \in X$ and $n \in \mathbb{N}$. Hence $\|ng\varphi\| \leqslant 1$ for all $n \in \mathbb{N}$ and since $\mu \in V_\varphi^o$ we have $|<ng,\mu>| \leqslant 1$ for all $n \in \mathbb{N}$. Hence $|<g,\mu>| = 0$ as required.

Let $f \in V_\varphi$. We may suppose that $f \geqslant 0$. We shall show that $\frac{f}{\varphi}$ is integrable with respect to μ and $|<\frac{f}{\varphi},\mu>| \leqslant 1$. Let

$$h(x) = \frac{f(x)}{\varphi(x)} \qquad \text{if } \varphi(x) \neq 0 \text{ and}$$

$$h(x) = 0 \qquad \text{if } \varphi(x) = 0.$$

Then h is lower semi-continuous. If $x \in X$ and $x_\alpha \to x$, $x_\alpha < x$ it is sufficient to show that $\lim \inf h(x_\alpha) \geqslant h(x)$. If $\varphi(x) \neq 0$, then h is continuous at x and so $\lim \inf h(x_\alpha) = h(x)$. If $\varphi(x) = 0$, then $h(x) = 0$ by definition. But $h(x_\alpha) \geqslant 0$ for all x_α and so $\lim \inf h(x_\alpha) \geqslant h(x)$.

Let $g \in L(X)$ such that $\|g\varphi\| \leqslant 1$. Then if $D_g = \{h \in C^+(X): h \leqslant g\}$ we have $\|h\varphi\| \leqslant 1$ for all $h \in D_g$. Since $\mu \in V_\varphi^o$ we have $|<h,\mu>| \leqslant 1$ for all $h \in D_g$.

But $<g,\mu> = \lim\{<h,\mu>: h \epsilon D_g\}$, hence $|<g,\mu>| \leqslant 1$. So we have shown that if $\|g\varphi\| \leqslant 1$ and $g \in L(X)$ then $|<g,\mu>| \leqslant 1$.

Let $g = h$ be defined as above. Then if $\|f\| \leqslant 1$ we have $|<\frac{f}{\varphi},\mu>| \leqslant 1$ so that $\|\frac{\mu}{\varphi}\| \leqslant 1$ and h is integrable with respect to μ.

Conversely, if μ vanishes off $N(\varphi)$ and $\|\frac{\mu}{\varphi}\| \leqslant 1$ then $\|f\varphi\| \leqslant 1$ implies $|<f,\mu>| = |<\frac{f\varphi}{\varphi},\mu>| \leqslant \|f\varphi\|\|\frac{\mu}{\varphi}\| \leqslant 1$ as required.

Note: (i) This theorem is due initially to Glicksberg [10] and the proof was later "simplified" by Conway [4]. Note that $\frac{\mu}{\varphi}(f) = <\frac{f}{\varphi},\mu>$, but $\mu \in M(X)$ is initially defined only for continuous functions and it is clear that $\frac{f}{\varphi}$ need not be continuous. It is, however, lower semi-continuous and hence the necessity to extend the domain of each $\mu \in M(X)$ from $C_b(X)$ to the larger space $L(X)$. Both the proofs of Glicksberg and Conway tend to gloss over this aspect of the proof.

(ii) The functions of the form $\frac{1}{\varphi}$ are not very manageable and Conway has given an alternate characterization of the β-equicontinuous sets. (See Theorem 1.1.4.) We shall prove the necessity using a functional analytic approach, but before proceeding we shall require the following lemma.

1.1.3. Lemma. Suppose X is a locally compact Hausdorff topological space and H is a subset of M(X). The following statements are equivalent:-

(i) For all $\varepsilon > 0$, there is a compact set $K \subset X$ such that $|\mu|(X \sim K) < \varepsilon$ for all $\mu \in H$.

(ii) For all $\varepsilon > 0$, there is a compact set $K \subset X$ such that, if $\|f\| \leqslant 1$ and $\text{supp}f \cap K = \varphi$ then $|\langle f,\mu\rangle| \leqslant \varepsilon$ for all $\mu \in H$.

Proof: (i) implies (ii).

Let $\varepsilon > 0$. There exists a compact set $K \subset X$ such that $|\mu|(X \sim K) < \varepsilon$. Suppose that $\|f\| \leqslant 1$ and $\text{supp}f \cap K = \varphi$. Then $\text{supp}f \subset X \sim K$. Hence $|\mu|(\text{supp}f) \leqslant \varepsilon$ for all $\mu \in H$. If $\mu \in H$, then

$$\begin{aligned} |\langle f,\mu\rangle| &= \left|\int f d\mu\right| \\ &= \left|\int_{\text{suppf}} f d\mu\right| \\ &\leqslant \left|\int \chi_{\text{suppf}} d\mu\right| && \text{since } \|f\| \leqslant 1 \\ &\leqslant |\mu|(\text{suppf}) \leqslant \varepsilon && \text{as required.} \end{aligned}$$

(ii) implies (i).

Let $\varepsilon > 0$. By hypothesis there is a compact set $K \subset X$ such that if $\|f\| \leqslant 1$ and $\text{supp}f \cap K = \varphi$ then $|\langle f,\mu\rangle| \leqslant \frac{\varepsilon}{2}$, for all $\mu \in H$. Suppose that $f \geqslant 0$ and let $g \in C_b(X)$ such that $|g| \leqslant f$ (pointwise), $\text{supp}f \cap K = \varphi$ and $\|f\| \leqslant 1$. Then $\text{supp}g \cap K = \varphi$ so that $|\langle g,\mu\rangle| \leqslant \frac{\varepsilon}{2}$. Hence $\sup\{|\langle g,\mu\rangle| : |g| \leqslant f, g \in C_b(X)\} \leqslant \frac{\varepsilon}{2}$. It follows from [1] page 55, that $\langle f,|\mu|\rangle = |\mu|(f) \leqslant \frac{\varepsilon}{2}$.

Now suppose that f is not necessarily positive and satisfies $\|f\| \leqslant 1$ and $\text{supp}f \cap K = \varphi$. Decompose f as follows: $f = f^+ - f^-$ where

$$f^+ = \frac{|f| + f}{2} \quad \text{and} \quad f^- = \frac{|f| - f}{2}.$$

Then $\mathrm{supp}f^+ \cap K = \varphi$, $\mathrm{supp}f^- \cap K = \varphi$, $f^+ \geqslant 0$, $f^- \geqslant 0$, $\|f^+\| \leqslant 1$ and $\|f^-\| \leqslant 1$. By above $|\mu|(f^+) \leqslant \frac{\varepsilon}{2}$ and $|\mu|(f^-) \leqslant \frac{\varepsilon}{2}$. Similarly we can write $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$.

$$\text{Now } ||\mu|(f)| = ||\mu|(f^+ - f^-)| \leqslant |\mu|(f^+) + |\mu|(f^-)$$
$$\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It follows from (ii) that for all $\varepsilon > 0$, there is a compact set $K \subset X$ such that if $\|f\| \leqslant 1$ and $\mathrm{supp}f \cap K = \varphi$ then $||\mu|(f)| \leqslant \varepsilon$. Let L be any compact set satisfying $L \subset X \sim K$. If $U_{\chi_L} = \{f \in C_b(X): f \geqslant \chi_L, \|f\| \leqslant 1, \mathrm{supp}f \cap K = \varphi\}$ then $\chi_L = \inf\{f: f \in U_{\chi_L}\}$. By hypothesis $||\mu|(f)| \leqslant \varepsilon$ for all $f \in U_{\chi_L}$, hence $||\mu|(\chi_L)| \leqslant \varepsilon$ and so $|\mu|(L) \leqslant \varepsilon$ as required.

1.1.4. Theorem. A set $H \subset M(X)$ is β-equicontinuous if and only if

(i) H is uniformly bounded and

(ii) For every $\varepsilon > 0$, there is a compact set $K \subset X$ such that $|\mu|(X \sim K) \leqslant \varepsilon$ for all $\mu \in H$.

Proof: Necessity.

Suppose $H \subset M(X)$ is β-equicontinuous, then there is a $\varphi \in C_0(X)$ such that $H \subset V_\varphi^0$. By Theorem 1.1.2. $\|\frac{\mu}{\varphi}\| \leqslant 1$ for all $\mu \in H$ and hence $\|\mu\| \leqslant \|\varphi\|$ for all $\mu \in H$ and so H is uniformly bounded. Also H^0 is a β-neighbourhood of zero in $C_b(X)$. Since $\kappa = \beta$ on the norm bounded sets, we have $H^0 \cap rB$ is a $\kappa|_{rB}$-neighbourhood of zero in rB for every $r > 0$, where B is the unit ball in $C_b(X)$. Hence for any $r > 0$, there is a compact set $K \subset X$ such that if $\|f\| \leqslant r$ and $p_K(f) \leqslant 1$ then $|\langle f, \mu\rangle| \leqslant 1$ for all $\mu \in H$. Call this statement "(p)".

Let $\varepsilon > 0$ be given and put $r = \frac{1}{\varepsilon}$. By (p) there is a compact set $K \subset X$ such that if $\|f\| \leqslant \frac{1}{\varepsilon}$ and $p_K(f) \leqslant 1$ then $|<f,\mu>| \leqslant 1$. Suppose that $\|f\| \leqslant 1$ and $\operatorname{supp} f \cap K = \varphi$, then $p_K(\frac{1}{\varepsilon}.f) = 0$ and $\|\frac{f}{\varepsilon}\| \leqslant \frac{1}{\varepsilon} = r$. So by statement (p), $|<\frac{f}{\varepsilon},\mu>| \leqslant 1$ and hence $|<f,\mu>| \leqslant \varepsilon$ for all $\mu \in H$. This shows that (ii) of the previous lemma holds, hence so does (i) and the result follows.

Sufficiency.

Suppose H satisfies the conditions of the theorem. Since H is uniformly bounded, we may suppose that $\|\mu\| \leqslant 1$ for all $\mu \in H$. By induction we can obtain a sequence $\{K_n\}$ of compact subsets of X such that $K_n \subset \operatorname{int} K_{n+1}$ and $|\mu|(X \sim K_n) \leqslant (\frac{1}{4})^n$ for all $\mu \in H$. For each integer $n \geqslant 1$, let $\varphi_n \in C_K(X)$ be such that $\varphi_n(x) = 1$ for $x \in K_n$, $0 \leqslant \varphi_n \leqslant 1$ and $\varphi_n(x) = 0$ if $x \notin K_{n+1}$.

Put $\varphi(x) = 2\sum_{n=1}^{\infty}(\frac{1}{2})^n\varphi_n(x)$, then $\varphi \in C_o(X)$ since the sequence $\{(\frac{1}{2})^n\}$ converges to zero. Also $N(\varphi) = \bigcup_{n=1}^{\infty} K_n$. We shall show that $H \subset V_\varphi^o$. If K is a compact subset of X and $K \cap N(\varphi) = \varphi$ then $K \cap K_n = \varphi$ for all $n \geqslant 1$. Thus $|\mu|(K) \leqslant (\frac{1}{4})^n$ for all n and so $|\mu|(K) = 0$. Hence each $\mu \in H$ vanishes off $N(\varphi)$. If $x \in K_n \sim K_{n-1}$ for $n \geqslant 2$, then $\varphi_1(x) = \ldots = \varphi_{n-2}(x) = 0$ and $\varphi_k(x) = 1$ for $k \geqslant n$.

Therefore

$$\varphi(x) = 2\sum_{i=n-1}^{\infty}(\tfrac{1}{2})^i\varphi_i(x) \geqslant 2\sum_{i=n}^{\infty}(\tfrac{1}{2})^i = 2(\tfrac{1}{2})^{n-1}.$$

If $\mu \in H$ and $n \geqslant 2$, then

$$\int_{K_n \sim K_{n-1}} (\frac{1}{\varphi})d|\mu| \leqslant (\tfrac{1}{2})2^{n-1}(\tfrac{1}{4})^{n-1} \leqslant (\tfrac{1}{2})^n.$$

and so

$$\int\left(\frac{1}{\varphi}\right)d|\mu| = \int_{K_1}\left(\frac{1}{\varphi}\right)d|\mu| + \sum_{n=2}^{\infty}\int\left(\frac{1}{\varphi}\right)d|\mu|$$
$$\leq \left(\tfrac{1}{2}\right)\|\mu\| + \sum_{n=2}^{\infty}\left(\tfrac{1}{2}\right)^n \leq 1.$$

Thus by Theorem 1.1.2, $H \subset V_\varphi^o$ and H is therefore β-equicontinuous.

Note: (i) It is thought that a proof of the sufficiency of Theorem 1.1.4. can be obtained using functional analytic methods rather than the measure theoretic approach adopted by Conway. The author has however not succeeded in achieving this.

(ii) We shall show in Chapter 2, that, with the aid of this theorem, the strict topology is a mixed topology.

(iii) Conway utilizes this characterization of the β-equicontinuous subsets of M(X) to show that whenever X is paracompact, the strict topology is a Mackey topology. The details may be found in [4].

We shall show in Chapter 5, that when X is taken to be the positive integers with the discrete topology, a simple application of theorem 1.1.4. yields the result that β is the Mackey topology on $C_b(\mathbb{N})$. We are also able to give an example of a dense subspace of a Mackey space which is not itself a Mackey space.

THE MIXED TOPOLOGY AND APPLICATIONS

We now present the mixed topology defined on a linear space E with two locally convex Hausdorff topologies. This was initially introduced by Wiweger and extensively studied in [25] and [26] and is closely related to the earlier work of Alexiewicz, Orlicz and Semadeni on two-norm spaces.

We will also show that the strict topology, introduced in the previous chapter is a mixed topology - a result obtained independently by Webb [24] and Cooper [6]. We will point out how typically known properties of the strict topology may be deduced from Wiweger's work.

Most of Wiweger's results required that one of the topologies τ on E be normable. We shall define the mixed topology with the less restrictive requirement that (E,τ) be a (DF)-space. This is the natural generalization of Wiweger's definition and was first introduced by Cooper [6].

2.0.1. Definition. A (DF)-space is a locally convex space (E,τ) which possesses a fundamental sequence (see notes below) $\{B_n\}$ of bounded sets and has the property that if $\{U_n\}$ is a sequence of closed, absolutely convex neighbourhoods of zero such that $U = \bigcap_{n=1}^{\infty} U_n$ absorbs bounded sets of E, then U is also a neighbourhood of zero.

Notes: (i) A sequence $\{B_n\}$ of bounded sets is fundamental if, when B is any bounded subset of E there is an n such that $B \subset B_n$. Observe that if E has a fundamental sequence $\{B_n\}$ of bounded sets then there is a fundamental sequence $\{B_n'\}$ of bounded sets such that each B_n' is absolutely convex and $B_n' + B_n' \subset B_{n+1}'$ for each $n \geqslant 1$. Let B_1' be the absolutely convex hull of B_1. Then $B_1' + B_1'$ is bounded and so there is an integer n_1 such that $B_1' + B_1' \subset B_{n_1}$. Let B_2' be the absolutely convex hull of $\bigcup_{i=1}^{n_1} B_i$. Then B_2' is bounded and $B_1' + B_1' \subset B_{n_1} \subset B_2'$. Proceeding in this way we obtain the sequence.

(ii) By considering polars of the sets U_n in E' we note that the definition of a (DF)-space is equivalent to (a) E has a fundamental sequence of bounded sets and (b) every strongly bounded countable union of equicontinuous subsets of E' is equicontinuous.

(iii) It is known that the class of (DF)-spaces contains all the normed spaces and the strong duals of metrisable locally convex spaces, both of which are of frequent occurrence in analysis. Also, a linear map L from a (DF)-space E into a locally convex space F is continuous if and only if $L|_{B_n}$ is continuous for each n. For a detailed study of these spaces refer to Grothendieck [11].

2.0.2. Definition. Let E be a vector space with two locally convex topologies τ^* and τ satisfying:

(i) $\tau^* \leqslant \tau$,

(ii) (E,τ) is a (DF)-space with a fundamental sequence $\{B_n\}$ of absolutely convex bounded sets such that $B_n + B_n \subset B_{n+1}$ for all n, and

(iii) each B_n is τ^*-closed.

If $\{U_n^*\}$ is a sequence of absolutely convex τ^*-neighbourhoods of zero, let

$$\gamma(U_n^*) = \bigcup_{n=1}^{\infty}(U_1^*\cap B_1 + \dots + U_n^*\cap B_n).$$

Then the set of all such $\gamma(U_n^*)$ forms a basis of neighbourhoods of zero for a locally convex topology $\gamma[\tau^*,\tau]$ on E and will be called the mixed topology generated by τ^* and τ. When no confusion can arise we will denote the mixed topology simply by γ.

Note: It follows from 2.0.1, note (i) that part (ii) of the above definition may be replaced by: (E,τ) is a (DF)-space.

The following propositions give some of the main properties of the mixed topology. The proofs are taken from Cooper [6], page 587.

2.0.3. **Proposition.** Let E, τ^*, τ and γ be defined as in Definition 2.0.2. Then

(i) For any τ-bounded set B, $\tau^*|_B = \gamma|_B$;

(ii) A collection H of linear maps from E into a locally convex space F is γ-equicontinuous if and only if $H|_{B_n}$ is τ^*-equicontinuous for each n. In particular, a linear map L from E into F is γ-continuous if and only if $L|_{B_n}$ is τ^*-continuous for each n;

(iii) The mixed topology γ is the finest locally convex topology on E coinciding with τ^* on the τ-bounded sets.

Proof: (i) Suppose B is a τ-bounded subset of E. Choose a positive integer r such that $B - B \subset B_r$. Let $B \cap (x_o + \gamma(U_n^*))$ be a $\gamma|_B$-neighbourhood of $x_o \in B$. Put $U^* = U_r^*$. Then $U^* \cap (B-B) \subset U_r^* \cap B_r \subset \gamma(U_n^*)$ so that $(x_o + U^*) \cap B \subset (x_o + \gamma(U_n^*)) \cap B$. Hence $(x_o + \gamma(U_n^*)) \cap B$ is a $\tau^*|_B$-neighbourhood of $x_o \in B$. The converse follows since $\tau^* \leq \gamma$ (see Proposition 2.0.4.).

(ii) If H is γ-equicontinuous on E, then H is γ-equicontinuous on the τ-bounded sets. Since $\tau^* \leq \gamma$, H is τ^*-equicontinuous on the τ-bounded sets.

Conversely, if $H|_{B_n}$ is τ^*-equicontinuous for each n, let W be an absolutely convex neighbourhood of zero in F and put

$$W_n = (\tfrac{1}{2})^n W.$$

Then we can find τ^*-neighbourhoods $\{U_n^*\}$ so that $h(U_n^* \cap B_n) \subset W_n$ for each $h \in H$. So for any n and $h \in H$ we have

$$h(U_1^* \cap B_1 + \ldots U_n^* \cap B_n) \subset W_1 + \ldots + W_n \subset W.$$

Hence $h(\gamma(U_n^*)) \subset W$ for each $h \in H$ so that H is γ-equicontinuous.

(iii) By (i), $\gamma = \tau^*$ on τ-bounded sets. Suppose γ' is a locally convex topology with $\gamma' = \tau^*$ on τ-bounded sets. Consider the identity map $i:(E,\gamma) \to (E,\gamma')$. By (ii), i is continuous and so γ is finer than γ'.

Note: Because of this last result, we could have defined the mixed topology generated by τ^* and τ as being the finest locally convex topology agreeing with τ^* on the τ-bounded sets.

2.0.4. Proposition. Given E, τ^*, τ and γ as before, then

(i) $\tau^* \leq \gamma \leq \tau$;

(ii) γ is independent of the choice of $\{B_n\}$;

(iii) a sequence $\{x_n\}$ in E is γ-convergent to zero if and only if $\{x_n\}$ is τ-bounded and $\{x_n\}$ is τ^*-convergent to zero;

(iv) γ and τ have the same bounded sets;

(v) a set $K \subset E$ is γ-compact if and only if it is τ-bounded and τ^*-compact.

Proof: (i) Consider the identity map $i:(E,\gamma) \to (E,\tau^*)$. By Proposition 2.0.3. (i), i is τ^*-continuous on τ-bounded sets and so by Proposition 2.0.3. (ii), i is γ-continuous. Thus $\tau^* \leq \gamma$. Now consider the identity map $i:(E,\tau) \to (E,\gamma)$. By Proposition 2.0.3. (i) and the fact that $\tau^* \leq \tau$, it follows that i is τ-continuous on τ-bounded sets. By Note 2.0.1. (iii) it follows that i is continuous and hence $\gamma \leq \tau$.

(ii) Let $\{B_n\}$ and $\{C_n\}$ be appropriate sequences of bounded sets in E and let γ_1 and γ_2 be the corresponding mixed topologies. Consider the identity map $i:(E,\gamma_1) \to (E,\gamma_2)$. By Proposition 2.0.3. (i), $\gamma_1|_B = \gamma_2|_B = \tau^*|_B$ for any bounded set B and so by Proposition 2.0.3. (ii) it follows that i is continuous. Hence $\gamma_2 \leq \gamma_1$. Similarly $\gamma_1 \leq \gamma_2$ and the result follows.

(iii) Suppose $\{x_n\}$ is γ-convergent to zero. Since $\tau^* \leq \gamma$, $\{x_n\}$ is τ^*-convergent to zero. We must show that $\{x_n\}$ is τ-bounded. If this were false, we could find a subsequence $\{x_{k_n}\}$ so that $x_{k_n} \notin B_n$. Since B_n is τ^*-closed we can choose a τ^*-neighbourhood U_n^* so that

$$x_{k_n} \notin B_n + 2U_n^*.$$

We can also suppose that $U_n^* + U_n^* \subset U_{n-1}^*$ for $n > 1$. Then for each $n > 1$,

$$\begin{aligned}\gamma(U) &= \bigcup_{k=1}^{\infty}(U_1^* \cap B_1 + \dots + U_k^* \cap B_k)\\ &\subset \bigcup_{p=1}^{\infty}(B_1 + \dots + B_{n-1} + U_n^* + U_{n+1}^* + \dots + U_{n+p}^*)\\ &\subset B_n + 2U_n^*.\end{aligned}$$

(Note that $B_i + B_i \subset B_{i+1}$ for each i, so that $B_1 + \dots + B_{n-1} \subset B_n$.) Hence $x_{k_n} \notin \gamma(U)$ for each n, which contradicts the fact that $\{x_n\}$ is γ-convergent to zero.

Conversely, if $\{x_n\}$ is τ-bounded and τ^*-convergent to zero, then by Proposition 2.0.3. (i), $\{x_n\}$ is γ-convergent to zero.

(iv) Since $\gamma \leqslant \tau$, every τ-bounded set is γ-bounded. Suppose that B is a γ-bounded set. Let $\{x_n\}$ be a sequence in B and let $\{\lambda_n\}$ be a null sequence of positive scalars. Then $\{\sqrt{\lambda_n}x_n\}$ is a γ-null sequence (since $\sqrt{\lambda_n} \to 0$) and so is τ-bounded. Hence $\lambda_n x_n \to 0$ in τ and so B is τ-bounded.

(v) Suppose A is γ-compact. Then A is γ-bounded and so by (iv), A is τ-bounded. By Proposition 2.0.3. (i), $\tau^* = \gamma$ on A and so A is τ^*-compact. Conversely, if A is τ-bounded and τ^*-compact, then by Proposition 2.0.3. (i), $\tau^* = \gamma$ on A and so A is γ-compact.

<u>Note</u>: The reader will notice the strong similarities between the preceeding propositions and Theorem 1.0.6, concerning the strict topology. This resemblance was one of the main reasons which motivated the inquiry to discover whether the strict topology was in fact a mixed topology. We now proceed to carry out this investigation.

2.0.5. Consider the space $C_b(X)$, where X is a locally compact Hausdorff topological space, with the locally convex structures κ and σ defined as in Chapter 1. By Theorem 1.0.6. (i), we have $\kappa \leqslant \sigma$. Now $(C_b(X), \sigma)$ is a normed space and hence a (DF)-space.

Let $B_n = \{f \in C_b(X) : \|f\| \leqslant 2^n\}$ for each positive integer n. Each B_n is an absolutely convex and σ-bounded neighbourhood of zero and hence is a fundamental sequence of σ-bounded sets which also satisfies the property that

$$B_n + B_n \subset B_{n+1} \qquad \text{for each } n.$$

To show that each B_n is κ-closed, it is sufficient to show that the unit ball $B = \{f \in C_b(X) : \|f\| \leqslant 1\}$ is closed. Let $f \in C_b(X)$ be a κ point of accumulation of B. Suppose that $\{f_\alpha\}$, $\alpha \in A$, where A is some index set, is a net in B which is κ-convergent to f. Since $\{x\} \subset X$ is compact for each $x \in X$ it follows that the net $\{f_\alpha\}$ converges pointwise to f. Since $f_\alpha \in B$, we have $|f_\alpha(x)| \leqslant 1$ for each $\alpha \in A$ and for each $x \in X$. It follows that $|f(x)| \leqslant 1$ for each $x \in X$ and hence $\|f\| \leqslant 1$. So $f \in B$ as required.

We can now define the mixed topology $\gamma[\kappa, \sigma]$ on $C_b(X)$ using Definition 2.0.2.

2.0.6. Theorem. Let X be a locally compact Hausdorff topological space. Then the strict topology on $C_b(X)$ is the mixed topology $\gamma[\kappa,\sigma]$.

Proof: By Theorem 2.0.3. (iii), γ is the finest locally convex topology on $C_b(X)$ which coincides with κ on the σ-bounded sets. But by Theorem 1.0.6, β is a locally convex topology on $C_b(X)$ which coincides with κ on the σ-bounded sets. Hence $\beta \leqslant \gamma$.

To obtain the reverse inclusion, let U be a closed absolutely convex γ-neighbourhood of zero in $C_b(X)$. Before proceeding, note that it follows easily from Proposition 2.0.3. (ii) that $(C_b(X),\gamma)' = M(X) = (C_b(X),\beta)'$. Thus $U^o \subset M(X)$ is γ-equicontinuous. Hence U^o is weak*($\sigma(M(X),C_b(X))$) compact and so U^o is strongly ($\beta(M(X),C_b(X))$) bounded. Thus (i) of Theorem 1.1.4. is satisfied.

Now $U = U^{oo}$ is a γ-neighbourhood of zero in $C_b(X)$. Since $\gamma = \kappa$ on the norm bounded sets it follows that $U \cap rB$ is a $\kappa|_{rB}$-neighbourhood of zero in rB for every $r > 0$, where B is the unit ball in $C_b(X)$. Following the same line of reasoning used in the proof of the necessity of Theorem 1.1.4, we see that (ii) of Theorem 1.1.4. is satisfied. Thus by Theorem 1.1.4, U^o is β-equicontinuous and thus U is a β-neighbourhood of zero in $C_b(X)$. Hence $\gamma \leqslant \beta$ and the proof is complete

Note: The above proof requires the aid of Theorem 1.1.4. We shall show now that it is possible to prove this result without the aid of Conway's equicontinuity criterion. Proceeding in this direction, we shall require the following result of Wiweger [25], the proof of which is long and technical and will be omitted.

2.0.7. Proposition. Let E be a linear topological space with locally convex structures τ^* and τ defined as in Definition 2.0.2. Suppose that the following conditions hold:

(i) The topology τ is a normed topology defined by the norm $\|.\|$ which satisfies the condition that if $x \in E$, then $\|x\| = \sup\{p_\alpha(x): \alpha\in\Lambda\}$ where $\{p_\alpha: \alpha\in\Lambda\}$ is a family of defining seminorms for τ^*.

(ii) If $\{p_n: n=1,2,\ldots\}$ is a subsequence of $\{p_\alpha: \alpha\in\Lambda\}$, and if $x \in E$ and $\varepsilon > 0$, then for every positive integer n, there are vectors y, z of E such that $x = y + z$, $p_i(z) = 0$ for $i = 1,2,\ldots,n$ and $\|y\| \leq \max\{p_i(x): i=1,2,\ldots,n\} + \varepsilon$. Then the sets of the form

$$\bigcap_{i=1}^{\infty}\{x\in E: p_i(x)\leq\alpha_i\}$$

form a base of neighbourhoods of zero for γ as the $\{p_i\}$ ranges over the countable subsets of Λ and the $\{\alpha_i\}$ ranges over all sequences of positive real numbers such that $\alpha_i \to \infty$ as $i \to \infty$.

2.0.8. Theorem. Let X be a locally compact Hausdorff topological space. If $E = C_b(X)$, $\tau^* = \kappa$ and $\tau = \sigma$, then κ and σ satisfy the conditions of the previous theorem. It follows that the mixed topology $\gamma[\kappa,\sigma]$ has a base of neighbourhoods of the form

$$\bigcap_{i=1}^{\infty}\{f\in C_b(X):\ p_{K_i}(f)\leq\alpha_i\}$$

where $\{K_i\}$ is a sequence of compact subsets of X and $0 < \alpha_i \to \infty$ as $i \to \infty$.

Proof: Now $p_K(f) \leq \|f\|$ for each compact subset $K \subset X$. Hence $\sup\{p_K(f)\colon K\subset X,\ K \text{ compact}\} \leq \|f\|$. Since each singleton set $\{x\} \subset X$ is compact, it follows that $\sup\{p_K(f)\colon K\subset X,\ K \text{ compact}\} = \|f\|$. Hence (i) of Proposition 2.0.7. holds. To show that (ii) holds, let $\{K_n\}$ be a sequence of compact subsets of X and let $f \in C_b(X)$ and $\varepsilon > 0$. Suppose that n is a positive integer and put $K = \bigcup_{i=1}^{n} K_i$. Then K is compact and hence f is uniformly continuous on K. Thus since X is locally compact, we can find a compact neighbourhood K' of K (i.e. $K \subset \operatorname{int} K'$) such that $p_{K'}(f) \leq p_K(f) + \varepsilon$. Choose $\varphi \in C_K(X)$ such that $\|\varphi\| \leq 1$, $\varphi(x) = 1$ if $x \in K$ and $\varphi(x) = 0$ if $x \notin K'$. Now let $g = \varphi f$, $h = (1-\varphi)f$ so that $f = g + h$. If $x \in K$ then $h(x) = 0$ so that $p_{K_i}(h) = 0$ for $i = 1,2,\ldots,n$. Also

$$\begin{aligned}
\|g\| &= p_{K'}(g) && \text{since } \operatorname{supp} g \subset \operatorname{supp}\varphi \subset K',\\
&\leq p_{K'}(f) && \text{since } |g(x)| \leq |f(x)| \text{ for all } x \in X,\\
&\leq p_K(f) + \varepsilon && \text{by construction}\\
&\leq \max\{p_{K_i}(f)\colon i=1,2,\ldots,n\} + \varepsilon.
\end{aligned}$$

Hence (ii) of Proposition 2.0.7. holds and the proof is complete.

Note: The above theorem was originally proved by Wiweger in [25].

We now include a lemma dealing with the construction of a certain function $\varphi \in C_o(X)$ which Cooper took for granted but whose existence requires a non-trivial proof. The details of the following lemma were supplied by my supervisor Dr. J.H.Webb.

2.0.9. Lemma. Let X be a locally compact Hausdorff topological space and let $\{K_n\}$ be a sequence of compact subsets of X such that

$$\phi \neq \text{int}K_1 \subset K_1 \subset \text{int}K_2 \subset K_2 \ldots \qquad \text{and let}$$

$\{c_n\}$ be a sequence of positive real numbers monotone decreasing to zero. Then there exists a non-negative function $\varphi \in C_o(X)$ such that $x \in K_n$ if and only if $\varphi(x) \geq c_n$ for each $n = 1,2,\ldots$

Proof: Choose a real number c_o such that $c_o > c_1$. Let $x_o \in \text{int}K_1$. Then $\{x_o\}$ and $K_1 \sim \text{int}K_1$ are disjoint closed subsets of K_1. Construct φ_1 continuous on K_1 such that

$$\begin{aligned} \varphi_1(x_o) &= c_o && \text{and} \\ \varphi_1(x) &= c_1 && \text{for } x \in K_1 \sim \text{int}K_1 \text{ and} \\ c_1 \leq \varphi_1(x) &\leq c_o && \text{for all } x \in K. \end{aligned}$$

Now $K_1 \sim \text{int}K_1$ and $K_2 \sim \text{int}K_2$ are disjoint closed subsets of K_2. Construct φ_2 continuous on K_2 such that

$$\begin{aligned} \varphi_2(x) &= c_1 && \text{on } K_1 \sim \text{int}K_1, \\ \varphi_2(x) &= c_2 && \text{on } K_2 \sim \text{int}K_2 \text{ and} \\ c_2 \leq \varphi_2(x) &\leq c_1 && \text{for all } x \in K_2. \end{aligned}$$

Continue in this way to get φ_n continuous on K_n. Define

$$\varphi(x) = \sup\{\varphi_n(x) : x \in K_n\} \qquad \text{if } x \in K \text{ and}$$
$$\varphi(x) = 0 \qquad \text{if } x \notin K.$$

We shall show that φ is continuous on X. Let $K = \bigcup_{n=1}^{\infty} K_n$ and let $x_o \in K$. Then there is a unique m such that $x_o \in K_m \sim K_{m-1}$. By definition of φ, $\varphi(x_o) = \varphi_m(x_o)$.

Now $K_m = (K_m \sim \text{int}K_m) \cup (\text{int}K_m \sim K_{m-1}) \cup K_{m-1}$. Suppose $x_o \in \text{int}K_m \sim K_{m-1}$. Then $\text{int}K_m \sim K_{m-1}$ is an open subset of $K_m \sim K_{m-1}$ and $\varphi = \varphi_m$ on this set. Since φ_m is continuous at x_o it follows that φ is continuous at x_o.

Suppose $x_o \in K_m \sim \text{int}K_m$, then $\varphi_m(x_o) = \varphi_{m+1}(x_o)$. Given $\varepsilon > 0$, choose a neighbourhood U of x_o such that

$$|\varphi_m(x) - \varphi_m(x_o)| < \varepsilon \qquad \text{for all } x \in U \cap K_m \text{ and}$$
$$|\varphi_{m+1}(x) - \varphi_{m+1}(x_o)| < \varepsilon \qquad \text{for all } x \in U \cap K_{m+1} \text{ and}$$

$U \subset \text{int}K_{m+1}$ and $U \cap K_{m-1} = \phi$. Then on U, $\varphi(x)$ is either $\varphi_m(x)$ or $\varphi_{m+1}(x)$. In either case $|\varphi(x) - \varphi(x_o)| < \varepsilon$, so φ is continuous at x_o.

Suppose $x_o \notin K$, then $\varphi(x_o) = 0$ by definition. Given $\varepsilon > 0$, choose c_n so that $c_n < \varepsilon$. Now $x_o \notin K_n$ so there is a neighbourhood U of x_o such that $U \cap K_n = \phi$. But

$$U = (U \cap K) \cup (U \sim K).$$
On $U \cap K$, $\varphi(x) < c_n < \varepsilon$.
On $U \sim K$, $\varphi(x) = 0 < \varepsilon$.

So $|\varphi(x)| < \varepsilon$ for all $x \in U$, hence φ is continuous at x_o. Now $\varphi \in C_o(X)$. If $\varepsilon > 0$, choose $c_n < \varepsilon$ so that if $x \notin K_n$ then $0 \leqslant \varphi(x) \leqslant c_n < \varepsilon$. Thus $|\varphi(x)| < \varepsilon$ for all $x \notin K_n$ and φ satisfies the conditions of the theorem.

2.1.0. **Theorem.** If X is a locally compact Hausdorff topological space, then the strict topology on $C_b(X)$ is the mixed topology $\gamma[\kappa,\sigma]$.

Proof: We present an alternative way of showing that $\beta \leq \gamma$. Consider the identity map $i:(C_b(X),\gamma) \to (C_b(X),\beta)$. We shall show that i is κ-continuous on $B_n = \{f \in C_b(X): \|f\| \leq 2^n\}$. Let $V_\varphi = \{f \in C_b(X): \|f\varphi\| \leq \varepsilon\}$ where $\varphi \in C_o(X)$, be a β-neighbourhood of zero and $\varphi \neq 0$. Since $\varphi \in C_o(X)$, there is a compact set $K \subset X$ such that $|\varphi(x)| < \varepsilon(\frac{1}{2})^n$ if $x \notin K$. Then

$$U = \{f \in C_b(X): p_K(f) \leq \frac{\varepsilon}{\|\varphi\|}\}$$

is a κ-neighbourhood of zero. We claim that $B_n \cap U \subset V_\varphi$. If $f \in B_n \cap U$, then $\|f\| \leq 2^n$ and $|f(x)| \leq \frac{\varepsilon}{\|\varphi\|}$ for $x \in K$.

$$\text{If } x \in K \text{ then } |f(x)\varphi(x)| = |\varphi(x)||f(x)| \leq \frac{\|\varphi\|\varepsilon}{\|\varphi\|} = \varepsilon.$$

$$\text{If } x \notin K \text{ then } |f(x)\varphi(x)| = |f(x)||\varphi(x)| \leq 2^n\varepsilon(\tfrac{1}{2})^n = \varepsilon.$$

So $\|f\varphi\| \leq \varepsilon$ and so $f \in V_\varphi$. It follows from Proposition 2.0.3. (ii) that i is continuous and so $\beta \leq \gamma$.

For the reverse inclusion, suppose that U is a γ-neighbourhood of zero. By Theorem 2.0.8. we may suppose that

$$U = \bigcap_{i=1}^{\infty}\{f \in C_b(X): p_{K_i}(f) < a_i\}.$$

Since X is locally compact, the compact sets can be chosen so that $K_1 \subset \operatorname{int}K_2 \subset K_2 \subset \ldots$ (see Lemma 1.0.3.) and we may suppose that the sequence $\{a_i\}$ is monotone increasing. Let $c_i = \frac{1}{a_i}$, then by the previous lemma we obtain a non-negative function

$\varphi \in C_o(X)$ such that $x \in K_n$ if and only if $\varphi(x) \geqslant c_n$. Then $V_\varphi \subset U$. For if $f \in V_\varphi$ then $|f(x)\varphi(x)| \leqslant 1$ for all $x \in X$. If $x \in K_n$ then

$$|f(x)| = \left|\frac{f(x)\varphi(x)}{\varphi(x)}\right| = \frac{|f(x)\varphi(x)|}{|\varphi(x)|}.$$

Hence

$$|f(x)| \leqslant \frac{1}{|\varphi(x)|} \leqslant \frac{1}{c_n} = a_n.$$

So $p_{K_n}(f) \leqslant a_n$ for each n. So $f \in U$ and U is a β-neighbourhood of zero as required.

Notes: (i) The compact-open and norm topologies may be defined on $C_b(X)$ without the requirement that X be locally compact. In fact we can define them when X is an arbitrary topological space. In 2.0.5. no use was made of the local compactness property so that κ and σ are such that $\gamma[\kappa,\sigma]$ can be defined. This enables us to generalize the strict topology to the case where X is an arbitrary topological space. Using Propositions 2.0.3. and 2.0.4. concerning the mixed topology, it is clear that the known properties of the strict topology on $C_b(X)$, for X locally compact, extend to arbitrary X.

(ii) Suppose that X is completely regular and that β is defined on $C_b(X)$ by means of $C_o(X)$ in the normal way. Then if $\kappa \leqslant \beta$ we shall show that X is locally compact. Let $x \in X$. Then $\{x\}$ is compact, so since $\kappa \leqslant \beta$, there is a $\varphi \in C_o(X)$ such that $\varphi(x) \neq 0$. The set $\{y: |\varphi(y)| \geqslant \frac{1}{2}\varphi(x)\}$ is a compact neighbourhood of x. Thus X is locally compact.

2.1.1. Definition. Let X be an arbitrary topological space. The strict topology on $C_b(X)$ is taken to be the mixed topology $\gamma[\kappa,\sigma]$.

We shall now discuss completeness properties of $C(X)$ and $C_b(X)$ with the topologies κ and β respectively.

2.1.2. Definition. A topological space X is a k-space if it satisfies the following condition: a set $A \subset X$ is closed if $A \cap K$ is closed for every compact set $K \subset X$.

2.1.3. Lemma. Let X be a topological space. Let G denote the family of all complex-valued functions on X and let F denote the family of all complex-valued functions on X which are continuous on each compact set $K \subset X$. Then

(i) a net $\{f_\alpha : \alpha \in A\}$ in G is κ-convergent to $g \in G$ if and only if it is κ-Cauchy in G and pointwise convergent to g.

(ii) G is κ-complete.

(iii) F is κ-complete.

Proof: (i) The necessity is obvious. For the sufficiency suppose that $\{f_\alpha : \alpha \in A\}$ is a κ-Cauchy net in G and $f_\alpha(x) \to g(x)$ for all $x \in X$. Let K be any compact subset of X and let $\varepsilon > 0$. Then there is an $\alpha \in A$ such that

$$m,n \geqslant \alpha \Rightarrow |f_m(x) - f_n(x)| \leqslant \varepsilon \qquad \text{for all } x \in K.$$

Since $f_n(x) \to g(x)$ for each $x \in X$ we have $m \geqslant \alpha$ implying

$|f_m(x) - g(x)| \leq \varepsilon$ for all $x \in K$. Thus $f_m \xrightarrow{\kappa} g$.

(ii) Let $\{f_\alpha : \alpha \in A\}$ be a κ-Cauchy net in G. If $x \in X$, then $\{f_\alpha(x)\}$ is a Cauchy sequence of complex numbers. Since the complex numbers, with the usual topology, is complete, there is a complex number $g(x)$ such that $f_\alpha(x) \to g(x)$. Then $g \in G$ and by (i), $f_\alpha \xrightarrow{\kappa} g$.

(iii) Since G is complete and $F \subset G$, it is sufficient to show that F is κ-closed in G. Let f be a κ-point of accumulation of F in G and suppose that $\{f_\alpha : \alpha \in A\}$ is a net in F which is κ-convergent to f. Let K be any subset of X. Then $\{f_\alpha|_K\}$ converges uniformly to $f|_K$. Now each f_α is continuous on K and since $C(K)$ is norm complete, it follows that f is continuous on K. Thus $f \in F$ as required.

2.1.4. <u>Theorem.</u> Let X be a k-space. Then $C(X)$ is κ-complete.

<u>Proof:</u> We shall show that $F = C(X)$ where F is defined as in the previous lemma. Clearly $C(X) \subset F$. For the reverse inclusion, suppose that $f \in F$. Then f is continuous on each compact subset of X. Let B be a closed subset of the complex numbers. Then $K \cap f^{-1}(B)$ is closed for every compact subset $K \subset X$. Since X is a k-space, it follows that $f^{-1}(B)$ is closed and so $f \in C(X)$.

So by (iii) of Lemma 2.1.3. it follows that $C(X)$ is κ-complete

We shall now prove an analogous result for $(C_b(X),\beta)$. We shall require the following proposition due to Raikov [17] concerning completeness in locally convex spaces.

2.1.5. Proposition. Let E be a locally convex space and $\{F_n\}$ a sequence of absolutely convex subsets of E satisfying

(i) $F_n + F_n \subset F_{n+1}$ for each n;

(ii) an absolutely convex subset V of E is a neighbourhood of zero in E if and only if $V \cap F_n$ is a neighbourhood of zero in F_n for each n and

(iii) each F_n is a complete subset of E.

Then E is complete.

2.1.6. Corollary. If X is a k-space, then $(C_b(X),\beta)$ is complete.

Proof: The sequence $\{B_n\}$, where $B_n = \{f \in C_b(X) : \|f\| \leq 2^n\}$ is a sequence of absolutely convex subsets of E satisfying $B_n + B_n \subset B_{n+1}$ for each n. So (i) of Proposition 2.1.5. holds.

By Proposition 2.0.3, β is the finest locally convex topology on $C_b(X)$ which coincides with κ on the norm bounded sets, so that (ii) of Proposition 2.1.5. is satisfied.

It follows from 2.0.5. that B_n is κ-closed in C(X). But C(X) is κ-complete (Theorem 2.1.4.) so that B_n is κ-complete. Since B_n is norm-bounded it follows that B_n is β-complete and the result now follows from Proposition 2.1.5.

WEIGHTED SPACES.

This chapter will be concerned with weighted spaces of continuous functions. These spaces are well known in connection with approximation theory (see for example [16]) and provide a general setting for the study of continuous function spaces encountered in analysis.

We will show that the space $(C_b(X),\beta)$, discussed in the previous chapters, is an example of a weighted space. The work of this chapter is centered around that of Summers [21] and [22]. We shall also give an alternative proof of a theorem by Summers using a general linear topological result.

Throughout this chapter X will be a completely regular T_1 topological space unless otherwise specified.

Notation: Let $B(X)$ denote the space of all complex-valued bounded functions on X.

$B_o(X)$ will denote the space of all complex-valued functions which vanish at infinity and $N(X)$ will denote those functions f on X with the property that $|f|$ is upper semi-continuous on X. Upper semi-continuous will be abbreviated as u.s.c.

Let $N(f) = \{x\in X: f(x)\neq 0\}$ and suppf will be the closure of $N(f)$ in X. Clearly

$$C_b(X) = C(X)\cap B(X) \qquad \text{and}$$

$$C_o(X) = C(X)\cap B_o(X).$$

3.0.1. Definition. A Nachbin family V on X is a set of non-negative u.s.c. functions on X satisfying the condition that if $u, v \in V$ and $\lambda \geqslant 0$, then there is a $w \in V$ such that

$$\lambda u, \lambda v \leqslant w.$$

$u \leqslant v$ means $u(x) \leqslant v(x)$ for each $x \in X$. Each $v \in V$ will be termed a weight.

3.0.2. Definition. Suppose V is a Nachbin family on X, then the corresponding weighted space is

$$CV_o(X) = \{f \in C(X): fv \in B_o(X) \text{ for every } v \in V\},$$

endowed with the weighted topology $w(V)$ generated by the set of seminorms $\{p_v : v \in V\}$ where $p_v(f) = \|fv\|$ for each $f \in CV_o(X)$.

Note: (i) It is easy to check that $CV_o(X)$ is a locally convex space with a base of closed, absolutely convex neighbourhoods of zero given by the sets

$$V_v = \{f \in CV_o(X): \|fv\| \leqslant 1\}$$

as v runs through V.

(ii) The above definition, given by Summers [21], differs from that given by Nachbin [16] page 62, and thus invites speculation as to whether the weighted function spaces indeed coincide. The definitions differ only in that Nachbin uses a different set of weights. We shall call the set used by Nachbin a weighted family.

3.0.3. Definition. A weighted family V on X is a set of non-negative u.s.c. functions on X satisfying the condition that if u,v $\in$ V, then there is a $\lambda > 0$ and a $w \in V$ such that $\lambda u, \lambda v \leq w$.

It is clear that every Nachbin family is a weighted family, but the converse is not true as is seen by the following example:-

Let $X = \{x\}$ be a one-point set. If $u(x) = 1$, $r(x) = 2$ and $w(x) = 3$ then $V = \{u,r,w\}$ is a weighted family but not a Nachbin family. Nevertheless we have the following theorem.

3.0.4. Theorem. Given any weighted family $\tilde{V}$ and the corresponding space $C\tilde{V}_o(X)$ with the topology $w(\tilde{V})$, then there is a Nachbin family V such that $CV_o(X) = C\tilde{V}_o(X)$ and $w(V) = w(\tilde{V})$.

Proof: Given $\tilde{V}$ define $V = \{\lambda\tilde{v}: \lambda \geq 0, \tilde{v} \in \tilde{V}\}$. Then V is a Nachbin family on X since suppose $\alpha\tilde{u}, \alpha\tilde{v} \in V$ where $\tilde{u},\tilde{v} \in \tilde{V}$ and let $\lambda > 0$. Since $\tilde{V}$ is a weighted family, there is a $\mu > 0$ and $\tilde{w} \in \tilde{V}$ such that $\mu\tilde{u},\mu\tilde{v} \leq \tilde{w}$. Let $\gamma = \max\{\alpha,\beta\}$ and put $w = \lambda\gamma\frac{\tilde{w}}{\mu}$. Then $w \in V$ and

$$\lambda\alpha\tilde{u} \leq \lambda\gamma\tilde{u} \leq \lambda\gamma\frac{\tilde{w}}{\mu} = w.$$

Also

$$\lambda\beta\tilde{v} \leq \lambda\gamma\tilde{v} \leq \lambda\gamma\frac{\tilde{w}}{\mu} = w \qquad \text{as required.}$$

It is clear that $CV_o(X) \subset C\tilde{V}_o(X)$. Conversely, let $f \in C\tilde{V}_o(X)$ then $f\tilde{v} \in B_o(X)$ for all $\tilde{v} \in \tilde{V}$. Let $v \in V$, then $v = \alpha\tilde{v}$ for some $\alpha > 0$, $\tilde{v} \in \tilde{V}$. Then $fv = \alpha f\tilde{v} \in B_o(X)$, so that $f \in CV_o(X)$. Since $\tilde{V} \subset V$, we have $w(\tilde{V}) \leq w(V)$. Consider a basic $w(V)$ neighbourhood:

$$V_v = \{f \in CV_o(X) : \|fv\| \leq 1\}.$$

Now $v = \lambda\tilde{v}$ for some $\lambda > 0$ and $\tilde{v} \in \tilde{V}$, so

$$\begin{aligned} V_v &= \{f \in CV_o(X) : \|f\tilde{v}\| \leq \tfrac{1}{\lambda}\} \\ &= \tfrac{1}{\lambda}\{f \in CV_o(X) : \|f\tilde{v}\| \leq 1\} \end{aligned}$$

which is a basic $w(\tilde{V})$ neighbourhood and so $w(V) = w(\tilde{V})$.

Note: This result justifies Summers' use of the term "Nachbin family".

3.0.5. Definition. If U and V are two Nachbin families on X, then $U \leq V$ if for each $u \in U$, there is a $v \in V$ such that $u \leq v$. This is a partial order on the class of all Nachbin families on X. We shall say that $U \approx V$ if $U \leq V$ and $V \leq U$.

3.0.6. Theorem. If U and V are Nachbin families on X with $U \leq V$, then

(i) $CV_o(X) \subset CU_o(X)$ and

(ii) $w(U)|_{CV_o(X)} \leq w(V)$.

Proof: (i) Suppose $f \in CV_o(X)$ and let $u \in U$. Since $U \leq V$, there is a $v \in V$ such that $u \leq v$. As $f \in CV_o(X)$ we have $fv \in B_o(X)$ and hence $fu \in B_o(X)$.

Since u was arbitrary, $f \in CU_o(X)$.

(ii) It is sufficient to show that if $u \in U$ then there is a $v \in V$ such that

$$V_v \subset V_u \cap CV_o(X).$$

Since $U \leqslant V$, there is a $v \in V$ such that $u \leqslant v$. Let $f \in V_v$, then $\|fv\| \leqslant 1$ and $f \in CV_o(X)$. Since $0 < u \leqslant v$, $\|fu\| \leqslant 1$ and so $f \in V_u$ as required.

An immediate consequence of this theorem is the following result

3.0.7. Corollary. If U and V are Nachbin families on X with $U \approx V$, then $CU_o(X) = CV_o(X)$ and $w(U) = w(V)$.

3.0.8. Examples of Weighted spaces.

Let X be a locally compact Hausdorff space where χ_A denotes the characteristic function of a subset $A \subset X$. Let $V = \{\lambda\chi_K : \lambda \geqslant 0, K \subset X, K \text{ compact}\}$. V is a Nachbin family on X since $\lambda\chi_K$ is u.s.c. and if $\lambda_1\chi_{K_1}, \lambda_2\chi_{K_2} \in V$ and $\lambda \geqslant 0$ then let $\mu = \max\{\lambda\lambda_1, \lambda\lambda_2\}$. Then $\lambda\lambda_1\chi_{K_1}, \lambda\lambda_2\chi_{K_2} \leqslant \mu\chi_{K_1 \cup K_2} \in V$.

We show that $CV_o(X) = C(X)$. Now $CV_o(X) \subset C(X)$ by definition. Conversely, if $f \in C(X)$ we must show that $fv \in B_o(X)$ for all $v \in V$. Consider $\lambda\chi_K \in V$, then $f(\lambda\chi_K) = \lambda f\chi_K$. Now $\lambda f\chi_K(x) = \lambda f(x)$ if $x \in K$ and is zero otherwise. But λf is a continuous function on the compact set K and is therefore bounded. So $\lambda f\chi_K \in B_o(X)$ as required. Furthermore, $w(V)$ is the compact-open topology κ, since $w(V)$ is defined by the seminorms

$$p_v(f) = \|fv\| \qquad \text{where } v \in V, \text{ so}$$

$$p_K(f) = \|f\chi_K\| = \sup\{|f(x)| : x \in K\} \qquad \text{where}$$

K runs through the compact subsets of X. This is precisely the way in which the compact-open topology is defined.

We show now that $V \approx C_K^+(X)$. Given $\lambda\chi_K \in V$, then since X is locally compact we can find a compact set K' of X such that $K \subset \operatorname{int} K'$ and a function $\varphi \in C_K^+(X)$ such that

$$\begin{aligned} \varphi(x) &= \lambda && \text{if } x \in K \text{ and} \\ \varphi(x) &= 0 && \text{if } x \notin K'. \end{aligned}$$

Clearly $\lambda\chi_K \leqslant \varphi$ so that $V \leqslant C_K^+(X)$. Conversely, given $\varphi \in C_K^+(X)$ then $\operatorname{supp}\varphi = K$ is compact. Letting $\lambda = \|\varphi\|$ we have $\varphi \leqslant \lambda\chi_K$ so that $C_K^+(X) \leqslant V$. Hence $V \approx C_K^+(X)$.

The following theorem therefore follows:-

3.0.9. Theorem. Let X be a locally compact Hausdorff topological space. Then $(C(X),\kappa)$ is a weighted space and the compact-open topology is generated by the seminorms

$$p_\varphi(f) = \|f\varphi\|$$

as φ runs through $C_K^+(X)$. It is of interest to compare this with the definition of the strict topology in Chapter 1.

3.1.0. Theorem. If X is a locally compact Hausdorff topological space and if $V = C_o^+(X)$, then $(CV_o(X), w(V)) = (C_b(X),\beta)$.

Proof: Suppose $f \in CV_o(X)$, then

$$f\varphi \in B_o(X) \quad \text{for all } \varphi \in C_o^+(X).$$

But since both f and φ are continuous we have $f\varphi \in C_o(X)$ for all $\varphi \in C_o^+(X)$. By Lemma 1.0.2. it follows that $f \in C_b(X)$. Conversely, if $f \in C_b(X)$ and $\varphi \in C_o^+(X)$ then $f\varphi \in C_o(X) \subset$

$B_o(X)$ so that $f \in CV_o(X)$. So

$$CV_o(X) = C_b(X).$$

The weighted topology $w(V)$ is generated by the seminorms $p_\varphi(f) = \|f\varphi\|$ as φ runs through $C_o^+(X)$. This is, by definition, the strict topology.

Corollary 3.1.2. is a result due to Summers [21]. We shall prove it using the following general result on linear topological spaces.

3.1.1. Theorem. Suppose (E,τ^*) and (F,τ) are Hausdorff linear topological spaces with $F \subset E$, $\tau^*|_F \leq \tau$ and τ has a base of neighbourhoods which are τ^*-closed in E.

Then if (E,τ^*) is complete, (F,τ) is complete.

Proof: Let $\{x_\alpha\}$ be a τ-Cauchy net in F. Since $\tau^*|_F \leq \tau$, $\{x_\alpha\}$ is a τ^*-Cauchy net in E because for every τ-neighbourhood V of zero in F, there exists an α_o such that $\alpha,\beta \geq \alpha_o$ implies $x_\alpha - x_\beta \in V$.

Hence for every $\tau^*|_F$-neighbourhood $W \cap F$ say, where W is a τ^*-neighbourhood of zero in E,

$$x_\alpha - x_\beta \in W \cap F \qquad \text{for } \alpha,\beta \geq \alpha_1 \text{ for some } \alpha_1.$$

Hence

$$x_\alpha - x_\beta \in W \qquad \text{for } \alpha\ \beta \geq \alpha_1 \text{ so}$$

$\{x_\alpha\}$ is a τ^*-Cauchy net in E. But (E,τ^*) is complete, so there is an $x_o \in E$ such that $x_\alpha \xrightarrow{\tau^*} x_o$.

Let V be a τ^*-closed τ-neighbourhood of zero in F. Since $\{x_\alpha\}$ is a τ-Cauchy net there is an α_0 such that

$$\alpha, \beta \geq \alpha_0 \quad \text{implies } x_\alpha - x_\beta \in V.$$

Hence

$$\alpha, \beta \geq \alpha_0 \quad \text{implies } x_\alpha \in x_\beta + V.$$

Thus

$$\beta \geq \alpha_0 \quad \text{implies } x_0 \in x_\beta + V \text{ since } x_\beta + V \text{ is}$$

τ^*-closed. Hence $x_0 \in F$ since $x_\beta + V \subset F$. Also $\beta \geq \alpha_0$ implies $x_0 - x_\beta \in V$ and since such V's form a base of τ-neighbourhoods of zero in F it follows that $x_\beta \xrightarrow{\tau} x_0 \in F$.

So F is τ-complete.

3.1.2. <u>Corollary.</u> Let U be a Nachbin family on X satisfying

(i) if $x \in X$, there is $u \in U$ with $u(x) > 0$.
(ii) $CU_0(X)$ is complete.

If V is a Nachbin family on X with $U \leq V$ then $CV_0(X)$ is complete.

<u>Proof</u>: Let $F = CV_0(X)$, $E = CU_0(X)$, $\tau = w(V)$ and $\tau^* = w(U)$. It follows readily from the definitions that $F \subset E$ and $\tau^*|_F \leq \tau$. Also (i) guarantees that τ^* is Hausdorff and since $U \leq V$, τ is Hausdorff.

We now show that τ has a base of neighbourhoods which are τ^*-closed in E. (F, τ) has a base of absolutely convex neighbourhoods of zero given by the sets

$$V_v = \{f \in CV_0(X) : \|fv\| \leq 1, \ v \in V\}.$$

We show that each V_v is τ^*-closed in E. Let $\{f_\alpha\} \subset V_v$ be a net in V_v such that

$$f_\alpha \overset{*}{\underset{\tau}{\to}} f \in CU_o(X).$$

We shall show that $f \in V_v$. Now $\{f_\alpha u\}$ converges uniformly to fu for every $u \in U$.

If $x \in X$, then by (i) there is $u \in U$ such that $u(x) > 0$. So $f_\alpha(x) \to f(x)$ for all $x \in X$. Let $v \in V$, then $f_\alpha v(x) \to fv(x)$ for all $x \in X$. Since $f_\alpha \in V_v$,

$$|f_\alpha v(x)| \leqslant 1 \quad \text{for all } \alpha \text{ and for all } x \in X$$

hence

$$|fv(x)| \leqslant 1 \quad \text{for all } x \in X$$

so

$$\|fv\| \leqslant 1.$$

This is true for all $v \in V$, hence $f \in V_v$ as required. It follows from the previous theorem that $CV_o(X)$ is complete.

3.1.3. __Theorem.__ If X is a locally compact Hausdorff topological space, then $(C_b(X),\beta)$ is complete.

__Proof:__ Let $U = \{\lambda\chi_K : \lambda \geqslant 0, K \subset X, K \text{ compact}\}$ as in 3.0.8. Then

$$(CU_o(X), w(U)) = (C(X), \kappa)$$

If $x \in X$, let K be a compact neighbourhood of x, then $\chi_K(x) > 0$, so (i) of Corollary 3.1.2. holds. Also, it is known that $(C(X),\kappa)$ is complete whenever X is locally compact (in fact whenever X is a k-space) so that (ii) of Corollary

3.1.2. holds. Let $V = C_0^+(X)$ so that by Theorem 3.1.0.

$$(CV_0(X), w(V)) = (C_b(X), \beta).$$

Now $U \leqslant V$. (To see this, follow the method employed in 3.0.8) It follows from the previous corollary that $(C_b(X), \beta)$ is complete.

APPLICATIONS OF THE STRICT TOPOLOGY TO APPROXIMATION THEORY.

4.0.1. We have, up to know, considered the strict topology on $C_b(X)$ for X locally compact Hausdorff. In Chapter 2 we generalized the strict topology to the situation where X was an arbitrary topological space. This involved the use of the mixed topology whose basic sets, which are described in terms of infinite intersections, are rather clumsy to work with. In this chapter we will generalize the strict topology in a more natural way and then we will proceed to consider applications to approximation theory.

Recall that when X was a locally compact Hausdorff topological space, the strict topology was defined by means of a set of seminorms determined by the elements of $C_o(X)$. (See Chapter 1.) By Lemma 2.0.9. we were also guaranteed that $C_o(X)$ was non-trivial. Suppose X is the space Q of rational numbers with the usual topology. We shall show that Q is not locally compact and that $C_o(Q) = \{0\}$. It follows that the strict topology on $C_b(Q)$ is the indiscrete topology.

To overcome this difficulty we will follow a method introduced by Giles [8]. It consists in letting $B_o(X)$, the space of bounded functions (not necessarily continuous) which vanish at infinity, play the role normally allocated to $C_o(X)$.

We show now that Q is not locally compact and that $C_o(Q) = \{0\}$. Let $V = U \cap Q$ be a neighbourhood of $x \in Q$, where U is a neighbourhood of x in $\mathbb{R}$. Let $y \in U$ be an irrational number. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a sequence in V which converges to y. Thus V is not sequentially compact. Since $\mathbb{Q}$ is metrisable, it follows that V is not compact and hence $\mathbb{Q}$ is not locally compact. Now assume that $f \neq 0$ and $f \in C_o(\mathbb{Q})$. Choose $x_o \in \mathbb{Q}$ such that $f(x_o) \neq 0$. Let $\varepsilon = \frac{1}{2}|f(x_o)|$, then there is a compact set $K \subset \mathbb{Q}$ such that

$$x \notin K \Rightarrow |f(x)| < \varepsilon.$$

Then $x_o \in \{x: f(x) > \varepsilon\} \subset K$. Since f is continuous, $\{x: f(x) > \varepsilon\}$ is open in $\mathbb{Q}$ and so K is a compact neighbourhood of x_o. This gives the desired contradiction.

4.0.2. Definition. Let X be any topological space, then the "generalized strict topology" on $C_b(X)$ is defined by the family of seminorms

$$p_\psi(f) = \|f\psi\| = \sup\{|f(x)\psi(x)| : x \in X\}$$

where $\psi \in B_o(X)$.

The next theorem justifies this definition.

4.0.3. Theorem. If X is a locally compact Hausdorff topological space, then the generalized strict topology on $C_b(X)$ coincides with the strict topology β.

Proof: Since $C_o(X) \subset B_o(X)$, the generalized strict topology on $C_b(X)$ is finer than the strict topology.

Conversely, let $U_\psi = \{f \in C_b(X) : \|f\psi\| \leq 1\}$ be a generalized strict neighbourhood of zero where $\psi \in B_o(X)$. We shall show that there is a $\varphi \in C_o(X)$ such that $U_\psi \supset U_\varphi$. Since ψ is bounded, we may suppose that $\|\psi\| \leq 1$. For each positive integer n, choose a compact set K_n such that $|\psi(x)| < 2^{-n}$ for all $x \notin K_n$. The sequence $\{K_n\}$ may also be chosen so that

$$K_1 \subset K_2 \subset K_3 \subset \dots.$$

Since X is locally compact we may choose $\varphi_n \in C_o^+(X)$ with $\varphi_n(x) = 2^{-n}$ if $x \in K_n$ and $\|\varphi_n\| = 2^{-n}$. Let $\varphi(x) = \sum_{n=1}^{\infty} \varphi_n(x)$ for each $x \in X$. Since the series $\sum_{n=1}^{\infty} 2^{-n}$ converges, it follows that $\varphi \in C_o(X)$. Also $U_\varphi \subset U_\psi$. Let $f \in U_\varphi$. Then $\|f\varphi\| \leq 1$. Let $x \in X$, then either $x \notin \bigcup_{n=1}^{\infty} K_n$ or $x \in K_{n+1} \sim K_n$ for some n.

In the first case $\psi(x) = 0$ so that $|f(x)\psi(x)| = 0$. In the second case $|\psi(x)| < 2^{-n}$ but

$$\begin{aligned} \varphi(x) &\geq 2^{-(n+1)} + 2^{-(n+2)} + \dots \\ &= 2^{-n}. \end{aligned}$$

Hence $|f(x)\psi(x)| < 2^{-n}|f(x)| \leq |f(x)||\varphi(x)|$. It follows that $f \in U_\psi$ as required.

We are now justified in denoting the generalized strict topology by β. It follows from Theorem 2.1.4. that if X is a k-space then C(X) with the compact-open topology is complete. We shall show that a similar result holds for $C_b(X)$ with the generalized strict topology and that the essential properties of the strict topology carry over to the general situation.

4.0.4. Theorem. Let X be any topological space. Then

(i) β and σ have the same bounded sets,

(ii) β and κ coincide on the norm bounded sets,

and (iii) if X is a k-space, then $(C_b(X),\beta)$ is complete.

Proof: The proofs of (i) and (ii) are identical to those of their counterparts in Theorem 1.0.6. where $C_o(X)$ is replaced by $B_o(X)$.

(iii) Let $\{f_\alpha\}$ be β-Cauchy net in $C_b(X)$. Since $\kappa \leq \beta$, $\{f_\alpha\}$ is a κ-Cauchy net in $C_b(X)$ and since X is a k-space, $f_\alpha \xrightarrow{\kappa} f$ where $f \in C(X)$. Let $\psi \in B_o(X)$, then $\{\psi f_\alpha\}$ is a σ-Cauchy net in $B(X)$. But $B(X)$ is norm complete so $\psi f_\alpha \xrightarrow{\sigma} g$ where $g \in B(X)$. Hence $\psi f_\alpha(x) \rightarrow g(x)$ for each $x \in X$, but also $\psi f_\alpha(x) \rightarrow \psi f(x)$ for each $x \in X$. So $g = \psi f$ and hence $\psi f \in B(X)$ for each $\psi \in B_o(X)$. It follows from Lemma 1.0.2. that $f \in C_b(X)$ and for each $\psi \in B_o(X)$, $\psi f_\alpha \xrightarrow{\sigma} \psi f$. Thus $f_\alpha \xrightarrow{\beta} f$ as required.

Notes: 1. Gulick and Sentilles have recently independently investigated the strict topology for $C_b(X)$ where X is completely regular, but their papers are still to be published. A reference to this fact may be found in Giles [8].

2. We shall now show how the generalized strict topology may be successfully applied to approximation theorems where conditions on the space X are significantly relaxed. In what follows $C_b(X)$ will be the space of all bounded real-valued continuous functions. The

classical Stone-Weiestrass theorem states that if X is compact, then a uniformly closed subalgebra of $C_b(X)$ which separates points and which, for each $x \in X$, contains a function g with $g(x) \neq 0$, is $C_b(X)$ itself.

4.0.5. <u>Theorem.</u> Suppose X is locally compact and U is a β-closed subalgebra of $C_b(X)$. If U separates points of X and contains a function vanishing nowhere, then $U = C_b(X)$.

<u>Notes</u>: 1. The proof may be found in [2]. This theorem is however unsatisfactory because of its insistence on the condition that U contain a function vanishing nowhere. This condition is certainly not necessary because $C_K(X)$ contains no non-vanishing function if X is locally compact but not compact. But yet $C_K(X)$ is a β-dense subalgebra of $C_b(X)$. (See Theorem 1.0.6. (v).)

2. Glicksberg [10] removed this condition by making use of Bishop's generalized Stone-Weierstrass theorem. His proof however is far from being elementary and we shall give a proof due to Todd [23] which follows simply from the following modified result of Buck's [2].

4.0.6. <u>Theorem.</u> Let X be a locally compact, σ-compact topological space. If U is a β-closed subalgebra of $C_b(X)$ which separates points of X and which for each $x \in X$, contains a function g with $g(x) \neq 0$, then $U = C_b(X)$.

<u>Proof</u>: By Theorem 4.0.5. it is sufficient to find a function

in U which vanishes nowhere on X. Since X is σ-compact, $X = \bigcup_{n=1}^{\infty} K_n$ where K_n is compact in X for each positive integer n. By the standard Stone-Weierstrass theorem, $U|_{K_n}$ is uniformly dense in $C(K_n)$. Hence there is an $f_n \in U$ such that $f_n(x) \geqslant 1$ for all $x \in K_n$. Let $c_n = \|f_n\|$ and define

$$g_n = \left(\frac{f_n}{nc_n}\right)^2.$$

Since U is an algebra, it follows that $g_n \in U$ and that $g_n(x) > 0$ for all $x \in K_n$ and non-negative on X. Also $\|g_n\| \leqslant \frac{1}{n^2}$ so that $\sum_{i=1}^{n} g_i$ converges uniformly and hence strictly to a function $g \in U$. Since $X = \bigcup_{n=1}^{\infty} K_n$ and since g_n is strictly positive on K_n, it follows that g is strictly positive on X.

We now drop the condition that X be σ-compact.

4.0.7. <u>Theorem.</u> Let X be a locally compact Hausdorff space. Let U be a strictly closed subalgebra of $C_b(X)$ which separates points of X and which for each $x \in X$, contains a function g with $g(x) \neq 0$, then $U = C_b(X)$.

<u>Proof</u>: Let $f \in C_b(X)$, $\varphi \in C_o(X)$ and $\varepsilon > 0$ be given. Let $K_n = \{x\colon |\varphi(x)| \geqslant \frac{1}{n}\}$. Then $K_n \subset \operatorname{int} K_{n+1}$ and so $K = \bigcup_{n=1}^{\infty} K_n$ is regularly σ-compact and locally compact under the relative topology induced by X. So applying Theorem 4.0.6. it follows that $U|_K$ is β-dense in $C_b(K)$. Hence there is a function $g \in U$ such that $|(f-g)\varphi(x)| < \varepsilon$ for all $x \in K$. Since φ is zero outside K, we have $|(f-g)\varphi(x)| < \varepsilon$ for all $x \in X$ and so $\|(f-g)\varphi\| < \varepsilon$. So f is in the β-closure of U and since U is β-closed, $f \in U$. Thus $U = C_b(X)$ as required.

Note: Theorem 4.0.7. was proved by Todd in [23].

Using the generalized strict topology β on $C_b(X)$ we shall improve on the previous theorem by dropping the requirement that X be locally compact. This improvement (Theorem 4.0.9.) is due to Giles [8] and requires Lemma 4.0.8. which Giles claims follows from Gelfand's representation theorem for commutative C^*-algebras. This reference seems rather strange since we shall prove the Lemma using the classical Stone-Weierstrass theorem.

4.0.8. Lemma. Let X be any topological space and let U be a β-closed subalgebra of $C_b(X)$. If $\varphi: \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $\varphi(0) = 0$, then $f \in U$ implies $\varphi \circ f \in U$ where $\varphi \circ f(x) = \varphi(f(x))$.

Proof: Suppose $f \in U$. Since f is bounded, there is a real number $a = \|f\|$ such that $f(X) \subset [-a,a]$. Since $[-a,a]$ is compact we have, by the classical Stone-Weiestrass theorem, a sequence of polynomials $\{\varphi_n\}$ which converges uniformly on $[-a,a]$ to φ and satisfying $\varphi_n(0) = 0$ for each n.. Thus $\varphi_n \circ f$ converges uniformly on X to $\varphi \circ f$. Now $\varphi_n \circ f$ is the sum of powers of f and since U is an algebra, it follows that $\varphi_n \circ f \in U$ for each n. But U is β-closed and hence σ-closed in $C_b(X)$ so that $\varphi \circ f \in U$ as required.

4.0.9. Theorem. Let X be any topological space. If U is a β-closed subalgebra of $C_b(X)$ which separates points of X and which for each $x \in X$, contains a function g with $g(x) \neq 0$, then $U = C_b(X)$.

Proof: Let $f \in C_b(X)$. We shall show that f is in the β-closure of U and hence in U. Let $\varepsilon > 0$ and $\psi \in B_o(X)$. We must find a $g \in U$ so that $\|(f-g)\psi\| < \varepsilon$.

Choose $M > \max\{\|f\|, \|\psi\|\}$. Given $\varepsilon' > 0$, choose K compact, $K \subset X$ such that $|\psi(x)| < \varepsilon'$ if $x \notin K$. By the classical Stone-Weierstrass theorem we have $U|_K$ is σ-dense in $C(K)$ and so there is an $h \in U$ with $\|(f-h)|_K\| < \varepsilon'$. Suppose $\varepsilon' < M$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} \varphi(\lambda) &= \lambda && \text{if } |\lambda| \leqslant 2M, \\ \varphi(\lambda) &= 2M && \text{if } \lambda > 2M \text{ and} \\ \varphi(\lambda) &= -2M && \text{if } \lambda < -2M. \end{aligned}$$

Now φ is continuous and so by Lemma 4.0.8. it follows that $\varphi \circ h \in U$. Let $g = \varphi \circ h$ then clearly $\|g\| \leqslant 2M$.

If $x \in K$, then

$$\begin{aligned} |(f(x)-g(x))\psi(x)| &= |f(x)-g(x)|\,|\psi(x)| \\ &< \varepsilon'\|\psi\| < M\varepsilon'. \end{aligned}$$

If $x \notin K$, then

$$\begin{aligned} |(f(x)-g(x))\psi(x)| &= |f(x)-g(x)|\,|\psi(x)| \\ &\leqslant (\|f\| + \|g\|)\varepsilon' \\ &< (M+2M)\varepsilon' \\ &= 3M\varepsilon'. \end{aligned}$$

So by choosing $\varepsilon' < \frac{\varepsilon}{3M}$ we get $\|(f-g)\psi\| < \varepsilon$ as required.

RELATIONSHIPS BETWEEN X AND $C_b(X)$ AND OTHER RESULTS.

This chapter will be concerned with relationships between the topological properties of $(C_b(X),\beta)$ and the topological properties of the underlying space X. One such result on the compact-open topology has already been given in Chapter 1 (see Theorem 1.0.5.). We will then briefly discuss the particular case when $X = \mathbb{N}$, the positive integers with the discrete topology. Finally we will consider necessary and sufficient conditions for $C_b(X)$ to be a topological dual of a Banach space. This treatment was motivated by the work of Rubel and Shields[18].

5.0.1. Theorem. Let X be a locally compact Hausdorff topological space. Then X is discrete if and only if $(C_b(X),\beta)$ is semi-reflexive.

Proof: Suppose X is discrete. Let $F \in M(X)'$. We must show that there is an $f \in C_b(X)$ such that $F(\mu) = \langle f,\mu\rangle = \int f d\mu$ for all $\mu \in M(X)$. Define $f(x) = F(\delta_x)$ where δ_x is the point measure at $x \in X$ and is defined as follows: $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ if $x \notin A$, where A is any subset of X. Now $\delta_x \in M(X)$. If $\{f_\alpha\}$ is a net in $C_b(X)$ such that $f_\alpha \xrightarrow{\beta} f$ where $f \in C_b(X)$, we must show that $\int f_\alpha d\delta_x \to \int f d\delta_x$. Now $\int f_\alpha d\delta_x = f_\alpha(x)$ and $\int f d\delta_x = f(x)$. Since β-convergence implies pointwise convergence it follows that $\delta_x \in M(X)$. Since F is a bounded linear functional it follows that f is bounded and so since X is discrete, $f \in C_b(X)$. Let $\mu \in M(X)$. Now each $\mu \in M(X)$ has σ-compact support (Lemma 1.0.9.). Since X is discrete, the σ-compact sets are precisely the countable sets. Hence there is a sequence $\{s_n\}$ in X such that $\mu(X) = \sum_{i=1}^{\infty}\mu(\{s_i\})$. Let $a_i = \mu(\{s_i\})$ and as μ is bounded, $\sum_{i=1}^{\infty} a_i < \infty$. Now $\mu = \mu^+ - \mu^-$ where μ^+ and μ^- are positive measures on X. Since $|\mu| = \mu^+ + \mu^-$, it follows that $|\mu| \in M(X)$ and hence $\sum_{i=1}^{\infty}|a_i| < \infty$. Thus $\{a_i\} \in l^1$ and so $\mu = \sum_{i=1}^{\infty} a_i\delta_{s_i}$. Now F is continuous, therefore $F(\mu) = \sum_{i=1}^{\infty} a_i F(\delta_{s_i}) = \sum_{i=1}^{\infty} a_i f(s_i) = \langle f,\mu\rangle$ as required.

Conversely, suppose $(C_b(X),\beta)$ is semireflexive. Let $x \in X$. We shall show that $\{x\}$ is open in X. Let $F(\mu) = \mu(\{x\})$ for all $\mu \in M(X)$. Then $F \in M(X)'$ and by hypothesis there is an $f \in C_b(X)$ such that $F(\mu) = \langle f,\mu\rangle$ for all $\mu \in M(X)$. So $\mu(\{x\}) = \int f d\mu$ for all $\mu \in M(X)$. Thus f is the characteristic function of $\{x\}$ and since f is continuous it follows that $\{x\}$ is open in X.

Note: In the above theorem the proof of the necessity was taken from [3] and the sufficiency from [4].

5.0.2. Theorem. Let X be a locally compact, σ-compact Hausdorff topological space. Then X is metrisable if and only if $(C_b(X),\beta)$ is separable.

Proof: Suppose X is metrisable. Then since X is σ-compact, X satisfies the second axiom of countability (i.e. the topology on X has a countable base $\{B_n\}$ of open sets). For each B_n, $X \sim B_n$ is closed, so since X is metrisable we can find a function $f_n \in C_b(X)$ such that f_n vanishes only on $X \sim B_n$. For example we may define f_n by $f_n(x) = d(x,X\sim B_n)$, where d is the metric on X. Let U be the countable subalgebra of $C_b(X)$ generated by $\{f_n: n\in\mathbb{N}\}$ over the rationals. Since X is Hausdorff, U separates points of X. If $x \in X$, choose B_n so that $x \in B_n$. Then $f_n(x) \neq 0$ and so by applying Theorem 4.0.7, it follows that U is a countable β-dense subset of $C_b(X)$ and so $(C_b(X),\beta)$ is separable.

Conversely, suppose that $(C_b(X),\beta)$ is separable and let S be a countable β-dense subset of $C_b(X)$. Let U_s be the countable subalgebra of $C_b(X)$ generated by S over the rationals. Since X is locally compact and σ-compact, we can find a $\varphi \in C_o(X)$ which vanishes nowhere on X. (See Lemma 2.0.9.) Let $\varphi U_s = \{\varphi f: f\in U_s\}$. Since $\varphi \in C_o(X)$ and f is bounded, $\varphi f \in C_o(X)$ so that $\varphi U_s \subset C_o(X)$. But U_s is β-dense in $C_b(X)$ so φU_s is uniformly dense in $C_o(X)$. Let B be the unit ball of φU_s and for each $f \in B$, define $E_f: X \to [-1,1]$ by $E_f(x) = f(x)$. This defines a map E of X into the product $\pi\{[-1,1]: f\in B\}$ by mapping a point x of X into the member of the product whose f-th coordinate is f(x). The mapping E is continuous and is a homeomorphism of X onto E(X) if B distinguishes points and closed sets. (See [13], page 116.) That this is so follows because U_s is dense in $C_b(X)$ hence separates points and closed sets. Thus so does φU_s, since φ vanishes nowhere, and therefore B distinguishes points and closed sets. It follows that X is metrisable.

Note: Recall that a barrel in a locally convex Hausdorff linear topological space (E,τ) is a closed absolutely convex absorbent subset of E. A subset A of E is bornivorous if A absorbs each bounded subset of E. (Suppose B is bounded, then there exists a scalar λ such that $B \subset \lambda A$.)

5.0.3. <u>Definition.</u> (a) A space (E,τ) is barrelled (quasi-barrelled) if every barrel (bornivorous barrel) is a neighbourhood of zero in E.

(b) A space (E,τ) is bornological if any absolutely convex set which absorbs all the bounded subsets of E is a neighbourhood of zero in E.

<u>Notes</u>: (i) Given a locally convex Hausdorff linear topological space (E,τ), let τ^b be the finest topology on E having the same bounded sets as τ. Then (E,τ) is bornological if and only if $\tau = \tau^b$. Suppose (E,τ) is bornological. If U is an absolutely convex τ^b-neighbourhood of zero, then U absorbs all the bounded subsets of E and so U is a τ-neighbourhood of zero. Thus $\tau = \tau^b$. Conversely, if $\tau = \tau^b$, then let U be an absolutely convex set which absorbs all the bounded subsets of E. By definition of τ^b, it follows that U is a neighbourhood of zero in E. Thus (E,τ) is bornological.

(ii) Every metrisable locally convex space (E,τ) is bornological (see [15], page 380).

(iii) Every normed space is quasi-barrelled. Every sequentially complete quasi-barrelled space is barrelled. Every metrisable locally convex space is quasi-barrelled (see [15], pages 368 and 369).

5.0.4. Theorem. Let X be any topological space. Then X is compact if and only if $(C_b(X),\beta)$ is quasi-barrelled.

Proof: If X is compact, then by Theorem 1.0.7.(i) it follows that $\beta = \sigma$. Hence $(C_b(X),\beta)$ is barrelled and thus quasi-barrelled.

Conversely, suppose that $(C_b(X),\beta)$ is quasi-barrelled. The unit ball $B = \{f \in C_b(X): \|f\| \leq 1\}$ is absolutely convex. It is also κ-closed (see 2.0.5.) and since $\kappa \leq \beta$, it is β-closed. Now B is a σ-neighbourhood of zero in $C_b(X)$ and hence B absorbs all the σ-bounded sets. But the σ-bounded sets are the same as the β-bounded sets (see Theorem 4.0.4.(i)) so that B is a bornivorous barrel in $(C_b(X),\beta)$. Hence B is a β-neighbourhood of zero and so $\beta = \sigma$. Thus by Theorem 1.0.7.(i), X is compact.

Note: If X is compact, then $(C_b(X),\beta)$ is clearly metrisable and bornological. Furthermore, every metrisable space is bornological and every bornological space is quasi-barrelled, so we have the two following corollaries.

5.0.5. Corollary. Let X be any topological space. Then X is compact if and only if $(C_b(X),\beta)$ is bornological.

5.0.6. Corollary. Let X be any topological space. Then X is compact if and only if $(C_b(X),\beta)$ is metrisable.

We will now consider $C_b(X)$ where X is the space of positive integers $\mathbb{N}$ with the discrete topology. Note that, in this case, $C_b(X) = l^\infty$, $M(X) = l^1$ and $C_o(X) = c_o$. Recall that a Hausdorff topological space is paracompact if each open cover has an open locally finite refinement. It is clear that $\mathbb{N}$ is paracompact since if U is an open cover of $\mathbb{N}$, let V consist of all single-point sets. V is an open locally finite refinement of U.

Conway [4] has shown that $(C_b(X),\beta)$ is a Mackey space provided that X is paracompact. Recall that if (E,τ) is a locally convex linear topological space and E' is its topological dual, then the Mackey topology $\mu(E,E')$ on E is the finest locally convex topology on E having E' as its dual space. It follows from Conway's result that (l^∞,β) is a Mackey space, but it will be instructive to include a direct proof of this fact.

Note that Theorem 1.1.2. adapted to this situation reads as follows:- A set $H \subset l^1$ is β-equicontinuous if and only if (i) H is uniformly bounded and (ii) for every $\varepsilon > 0$, there is an integer N such that $\sum_{i=N+1}^{\infty} |a_i| \leq \varepsilon$ for every $\xi = \{x_i\}_{i=1}^{\infty}$ in H.

5.0.7. Theorem. The space (l^∞,β) is a Mackey space.

Proof: Recall that the Mackey topology $\mu(l^\infty,l^1)$ has a base consisting of polars of the weak*-relatively compact subsets of l^1. Now l^1 is a perfect space (see [15], page 406) and so if H is a weak*-relatively compact subset of l^1, then by [15], page 415, H is weak*-relatively sequentially compact. (This is a result of Schur, see for example [7], page 296.) Again by [15], page 415, it follows that H is norm relatively compact. We may suppose that H is absolutely convex and closed, so that $H^{oo} = H$. We must show that H^{o} is a β-neighbourhood of zero in l^{∞}. In other words, it is sufficient to show that $H^{oo} = H$ is β-equicontinuous. Let $\varepsilon > 0$. Since H is norm relatively compact, there are $\xi_1, \xi_2, \dots, \xi_n$ in H so that

$$H \subset \bigcup_{k=1}^{n} \{\xi : \|\xi - \xi_k\| < \tfrac{\varepsilon}{2}\}$$

For each $k = 1, \dots, n$ let $\xi_k = \{x_i^{(k)}\}_{i=1}^{\infty}$.

Since $\xi_k \in l^1$, choose N_k such that

$$\sum_{i=N_k+1}^{\infty} |x_i^{(k)}| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, \dots, N_k\}$. Then

$$\sum_{i=N+1}^{\infty} |x_i^{(k)}| < \frac{\varepsilon}{2} \qquad \text{for } 1 \leq k \leq n.$$

If $\xi = \{x_i\}_{i=1}^{\infty} \in H$, then $\|\xi - \xi_k\| < \frac{\varepsilon}{2}$ for some k.

Since $|x_i| \leq |x_i - x_i^{(k)}| + |x_i^{(k)}|$ for each i, we have

$$\sum_{i=N+1}^{\infty} |x_i| \leq \sum_{i=N+1}^{\infty} |x_i - x_i^{(k)}| + \sum_{i=N+1}^{\infty} |x_i^{(k)}|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It follows from the note preceeding the theorem that H is β-equicontinuous and so (l^{∞}, β) is a Mackey space.

Notes: (i) It is known that the completion of a Mackey space is a Mackey space. That the converse is false is shown by the following counter-example. By Theorem 1.0.6.(v), c_o is β-dense in l^∞. Since (l^∞,β) is complete (Theorem 1.0.6.(vi)) it follows that the completion of (c_o,β) is (l^∞,β). So the dual of (c_o,β) is l^1, but it is known that the dual of (c_o,σ) is l^1. Since the norm topology is properly stronger than β, it follows that (c_o,β) is not a Mackey space. The above argument will hold whenever X is paracompact.

(ii) It follows from the theorems in this chapter that (l^∞,β) is semireflexive and separable but is neither barrelled, bornological nor metrisable.

(iii) For more details about the space $(l^\infty(S),\beta)$ for S discrete, see [3] and [5].

5.0.8. We shall now give an example of a topological space where $\kappa = \beta < \sigma$. From Theorem 1.0.7. we must restrict our attention to a pseudocompact, non-compact space X. Such a space is provided by the space $X = W$, where W is the set of all ordinals less than ω_1, the first uncountable ordinal, with the usual interval topology. This space has been extensively studied and we shall content ourselves with pointing out a few facts about it. For details, see [9]. Let $W^* = W \cup \{\omega_1\}$ and let $W(\alpha)$ be the set of all ordinals less than α. W is pseudocompact, but not compact and W^* is the one-point compactification of W. Every function $f \in C(W)$ is constant on a tail $W \sim W(\alpha)$.

5.0.9. Theorem. If W is defined as above and κ, β and σ are defined in the usual way on C(W), then $\kappa = \beta < \sigma$.

Proof: Since W is not compact, it follows from Theorem 1.0.7.(i) that $\beta < \sigma$. Using Theorem 1.0.8, it is sufficient to show that $C_K(W) = C_o(W)$. Clearly $C_K(W) \subset C_o(W)$. For the reverse inclusion, let $f \in C_o(W)$. Extend f to a function $f^\beta \in C(W^*)$ by defining $f^\beta(\omega_1)$ to be the final constant value of f. Since f vanishes at infinity, $f^\beta(\omega_1) = 0$. Since f^β is constant on a tail, it follows that there is an open set U in W^* such that $f^\beta(\omega_1) = 0$ for all $\omega \in U$ and $\omega_1 \in U$. $W^* \sim U$ is thus a closed subset of the compact space W^* and is therefore compact. But since $\omega_1 \in U$, $W^* \sim U \subset W$ and so $W^* \sim U$ is a compact subset of W on which f vanishes. Hence $f \in C_K(W)$ and the theorem is proved.

5.1.0. Definition. Let (E,τ) be a linear topological space and E' its topological dual. The weak*-topology $\sigma(E',E)$ on E' is generated by polars of the finite subsets of E. On E', τ° will denote the topology of uniform convergence on the precompact subsets of E.

Note: If (E,τ) is a Banach space then τ° is the topology of uniform convergence on the compact subsets of E and is the finest topology on E' coinciding with $\sigma(E',E)$ on the equicontinuous subsets of E'. (See [14], page 212.)

5.1.1. Suppose X is the open disc $D = \{z \in C : |z| < 1\}$ in the complex plane with the usual topology. Then $H^\infty(D)$ will denote the subspace of $C_b(D)$ consisting of the bounded analytic functions on D. This space has been extensively studied by Rubel and Shields[18]. As in Chapter 1, the strict topology β on $H^\infty(D)$ is defined by means of the pseudo-norms on $H^\infty(D)$ generated by the points of $C_o(D)$. Rubel and Shields [18] have found that this particular subspace $H^\infty(D)$ is the dual of a separable Banach space (E,τ) such that $\beta = \tau^\circ$. As a natural extension of these ideas we may ask: "Under what conditions is the space $C_b(X)$ the dual of a Banach space (E,τ) satisfying $\beta = \tau^\circ$?" The ramifications of this question motivates the ensuing discussion. We will also show that, if X is not compact, the subspace $C_o(X)$ of $C_b(X)$ can never be the dual of a Banach space.

5.1.2. Theorem. The unit ball of $C_b(X)$ is κ-compact if and only if $C_b(X)$ is the dual of a Banach space (E,τ) with $\beta = \tau^\circ$.

Proof: For a proof of the necessity, see [20]. Conversely, the unit sphere

$$C = \{f\in E' : |\langle\mu,f\rangle|\leq 1,\ \mu\in E,\ \|\mu\|\leq 1\}$$

is $\sigma(E',E)$-compact by Alaoglu's Theorem (see [14], page 155). Now C is equicontinuous since it is the polar of the unit ball in E and so

$$\tau^\circ|_C = \sigma(E',E)|_C.$$

Thus C is τ°-compact and by hypothesis C is β-compact. So since $\kappa \leq \beta$, it follows that C is κ-compact.

5.1.3. Definition. Let E be a vector space with real scalars. If $x,y \in E$ then by (x,y) we shall mean the open interval $\{\alpha x + (1-\alpha)y : 0<\alpha<1\}$. Suppose that M is a subset of E. A point $z \in M$ is called an extreme point of M if it belongs to no open interval $(x,y) \subset M$.

We shall require the following well known Krein-Milman theorem.

5.1.4. Theorem. Let A be a convex compact subset of a locally convex real linear topological space E. Then A has an extreme point and A is the closed convex extension of the set of all its extreme points.

5.1.5. Lemma. Let X be a locally compact Hausdorff topological space. Then any extreme point f of the unit ball

$$B = \{f \in C_b(X) : \|f\| \leqslant 1\}$$

in $C_b(X)$ satisfies $|f(x)| = 1$ for all $x \in X$. On any component $P \subset X$, $f|_P$ is either identically one or identically minus one.

Proof: Let f be an extreme point of the unit ball B and suppose $|f(x_o)| < 1$ for some $x_o \in X$. Since f is continuous, there is a compact neighbourhood U of x such that $|f(x)| < 1$ for all $x \in U$. Since U is compact, there is a positive real number r such that $|f(x)| \leqslant r < 1$ for all $x \in U$. Choose $g \in C_K^+(X)$ with $\text{supp} g \subset U$, $g \neq 0$, $\|g\| \leqslant 1-r$. Then $|f(x) + g(x)| < 1$ for all $x \in U$. Then on suppg we have

$f(x) - g(x) < f(x) < f(x) + g(x)$ so that f is an interior point of $(f-g, f+g)$. This contradicts the fact that f is an extreme point and so $|f(x)| = 1$ for all $x \in X$.

The last part of the lemma follows since P is open and f is continuous on P.

5.1.6. Theorem. If X is a locally compact but not compact topological space, then $C_0(X)$ is not the dual of any Banach space.

Proof: Suppose that $C_0(X) = (E, \tau)'$ where (E, τ) is a Banach space. Then the unit ball $B = \{f \in C_0(X) : \|f\| \leq 1\}$ in $C_0(X)$ is weak*-compact and convex but has no extreme points. Given $f \in C_0(X)$ and $0 < \varepsilon < 1$, there is a compact set K such that $|f(x)| < \varepsilon < 1$ for all $x \notin K$. By Lemma 5.1.5, f is not an extreme point and so by the Krein-Milman theorem we have $B = \{\varphi\}$. This gives us the desired contradiction.

5.1.7. Theorem. Let X be a locally compact Hausdorff topological space. If $C_b(X)$ is the dual of a Banach space then X is totally disconnected.

Proof: Let P be any component of X and let f be an extreme point of the unit ball B of $C_b(X)$. By Lemma 5.1.5, $f|_P$ is either identically one or identically minus one. Hence the closed convex extension of the set of extreme points produces only constant functions when restricted to P. But $C_b(X)$ separates points of X and so P contains only one point. Thus X is totally disconnected.

BIBLIOGRAPHY.

[1] BOURBAKI, N. "Integration", Chapters I - IV, Actual. Scient. et Ind., No. 1175, Paris, 1952.

[2] BUCK, R.C. "Bounded continuous functions on a locally compact space", Michigan Math. J. 5, (1958), 95 - 104.

[3] COLLINS, H.S. "On the space $l^\infty(S)$, with the strict topology", Math. Zeit. 106, (1968), 361 - 373.

[4] CONWAY, J.B. "The strict topology and compactness in the space of measures II", Trans. Amer. Math. Soc., 126, (1967), 474 - 486.

[5] CONWAY, J.B. "Subspaces of $(C(S),\beta)$, the space (l^∞,β), and (H^∞,β)", Bull. Amer. Math. Soc., 72, (1966), 79 - 81.

[6] COOPER, J.B. "The strict topology and spaces with mixed topologies", Proc. Amer. Math. Soc, 30, No3, (1971), 583 - 592.

[7] DUNFORD, N. and SCHWARTZ, J.T. "Linear Operators", New York, Interscience Vol. I, 1958.

[8] GILES, R. "A generalization of the strict topology", Trans. Amer. Math. Soc., 161, (1971), 467 - 474.

[9] GILMAN, L. and JERISON, M. "Rings of continuous functions", Princeton, N.J. Van Nostrand, 1960.

[10] GLICKSBERG, I. "Bishop's generalized Stone-Weierstrass theorem for the strict topology", Proc. Amer. Math. Soc., 14, (1963), 329 - 333.

[11] GROTHENDIECK, A. "Sur les éspaces (F) et (DF)", Summa Brasil. Math. 3, (1954), 57 - 122.

[12] HEWITT, E, and ROSS, K.A. "Abstract harmonic analysis", Berlin, Springer, Vol. I, 1963.

[13] KELLEY, J.L. "General Topology", New York, Van Nostrand, 1955.

[14] KELLEY, J.L., NAMIOKA, I. and co-authors. "Linear topological spaces", New York, Van Nostrand, 1963.

[15] KOTHE, G. "Topological vector spaces I", Springer-Verlag Berlin . Heidelberg . New York, 1969.

[16] NACHBIN, L. "Elements of approximation theory", Princeton N.J. Van Nostrand, 1967.

[17] RAIKOV, D.A. "A criterion of completeness in locally convex spaces", Uspehi Mat Nauk 14, (1959), No 1 (85), 223 - 229.

[18] RUBEL, L. and SHIELDS, A. "The space of bounded analytic functions on a region", Ann. Inst. Fourier, Grenoble 16, 1 (1966), 235 - 277.

[19] SCHAEFER, H.H. "Topological vector spaces", New York, MacMillan, 1966.

[20] SHAPIRO, J.H. "The bounded weak star topology and the general strict topology", J. Func. Anal. 8, (1971), 275 - 286.

[21] SUMMERS, W.H. "A representation theorem for bi-equicontinuous completed tensor products of weighted spaces", Trans. Amer. Math. Soc 146, (1969), 121 - 131.

[22] SUMMERS, W.H. "Dual spaces of weighted spaces", Trans. Amer. Math. Soc. 151, (1970), 323 - 333.

[23] TODD, C. "Stone-Weierstrass theorems for the strict topology", Proc. Amer. Math. Soc. 16 (1965), 654 - 659.

[24] WEBB, J.H. "The strict topology is mixed", S.A. Math. Soc. (1969).

[25] WIWEGER, A. "Linear spaces with mixed topology", Studia Math. 20, (1961), 47 - 68.

[26] WIWEGER, A. "A topologisation of Saks spaces", Bull. de l'academie Polonaise des Sciences III, Vol. V, No 8, (1957).