

The copyright of this thesis vests in the author. No quotation from it or information derived from it is to be published without full acknowledgement of the source. The thesis is to be used for private study or non-commercial research purposes only.

Published by the University of Cape Town (UCT) in terms of the non-exclusive license granted to UCT by the author.

An Investigation of Short Rate Models and the Pricing of Contingent Claims in a South African Setting

Chris Jones

29 November, 2010



Presented for the Degree of Masters of Science in Mathematics of Finance in the Department of Statistical Science, University of Cape Town

Supervisor: Professor Ronald Becker

Plagiarism Declaration

1. I know that plagiarism is wrong. Plagiarism is to use another's work and pretend that it is one's own.
2. Each significant contribution to, and quotation in this dissertation from the work of other people has been attributed, and has been cited and referenced.
3. This dissertation is my own work.
4. I have not allowed, and will not allow, anyone to copy my work with the intention of passing it off as his or her own work.

Abstract

This dissertation investigates the dynamics of interest rates through the modelling of the short rate – the spot interest rate that applies for an infinitesimally short period of time. By modelling such a rate via a diffusion process, one is able to characterize the entire yield curve and price plain vanilla options. The aim is to investigate which of the more popular short rate models is best suited for pricing such options, which are actively traded in the market. Thus one can then use such models to price more exotic options, as such options are typically less frequently traded in the market.

Although much literature exists with regards to the theoretical side of short rate models, discussion about the practical implementation of such models is limited, particularly from a South African perspective. This paper intends to not only cover the theory behind short rate models, but also to describe the unique South African market practicalities that exist, providing a framework to pricing plain vanilla options using the more popular short rate models, before justifying which models appear most appropriate to use in the South African environment.

I would like to thank Professor Becker for all his support and guidance throughout this thesis. His comments and suggestions proved most helpful.

Contents

1	Introduction	9
1.1	Dissertation Outline	10
2	Introduction to Bond Markets	11
2.1	Bank Account	11
2.2	Pure Discount Bond	11
2.3	Spot Rates	11
2.3.1	Arbitrage	12
2.4	Forward Rate	12
2.5	Short Rate	12
2.6	LIBOR and JIBAR	12
2.7	Options on Bonds	12
2.8	Interest Rate Caps	12
2.9	Interest Rate Floors	13
2.10	Interest Rate Swaps	14
2.11	Interest Rate Swaptions	14
3	No-Arbitrage Pricing and Numeraire Change	16
3.1	Basic Setup	16
3.2	Equivalent Martingale Measure	18
3.3	Change of Numeraire	19
3.4	Girsanov's Theorem and Ito's Lemma	19
3.5	Put-Call Parity	20
3.5.1	Bond Options	20
3.5.2	Caps and Floors	21
3.6	Pricing of Caps and Floors	22
3.7	Pricing of Swaps	22
3.8	Pricing of Swaptions	23
3.8.1	One Factor Short Rate Models	24
3.8.2	Multi-Factor Short Rate Models	25

4 Pricing of Options using Black's Model	26
4.1 Options on Bonds	26
4.2 Caps and Floors	26
4.3 Swaptions	27
4.4 Consistency in Pricing Formulas	28
4.5 Suitability of Black's Model for Pricing Interest Rate Options	28
5 Market Practicalities	30
5.1 Short Rate	30
5.2 Pure Discount Bonds	30
5.2.1 Available Data	31
5.2.2 Bootstrap Method	31
5.3 Caps and Floors	32
5.4 Individual Caplets and Floorlets	33
6 Affine Term Structure Models	35
6.1 Bond Options	36
6.1.1 Application to Bond Options	38
7 Vasicek Model	40
7.1 Pure Discount Bond	41
7.2 Cap and Floors	42
8 Ho-Lee Model	46
8.1 Pure Discount Bond	46
8.2 Short Rate Dynamics Revisited	48
8.3 Cap and Floors	49
9 Cox-Ingersoll-Ross Model	52
9.1 Pure Discount Bond	54
9.2 Cap and Floors	55
10 Black-Karasinski Model	58
10.1 Simplifying the Short Rate Equation	59
10.2 Moments of the Short Rate Revisited	60
10.3 Pure Discount Bond	61
10.4 Cap and Floors	64
11 Two Factor Vasicek Model	66
11.1 Pure Discount Bond	69
11.2 Cap and Floors	74
12 Two-Factor Cox-Ingersoll-Ross Model	81
12.1 Pure Discount Bond	82
12.2 Cap and Floors	83

13 Calibrating Short Rate Models	85
13.1 Parameterization	85
13.2 Data	87
13.3 Forms of Short Rate Models Compared	87
14 Comparison of Models	89
14.1 Testing Non-Nested, Non-Linear Regression Models	90
15 Empirical Results - Fit to Yield Curve	93
15.1 Fit to Pure Discount Bond Prices and the Yield Curve	93
15.1.1 Goodness-of-Fit	95
15.1.2 Parameter Estimates under each short rate Model	97
15.1.3 Example of Fit to the Yield Curve	102
15.1.4 Davidson and MacKinnon's P-Tests	102
15.2 Fit to Cap and Floor Prices	104
15.2.1 Goodness-of-Fit	104
15.3 Fit to Cap and Floor Prices after Parameters Recalibrated	105
16 Conclusion	108

List of Figures

10.1 Black-Karasinski: Branching of Non-Standard Trinomial Tree	61
10.2 Black-Karasinski: Tree Notation	62
15.1 Calibration Errors of the Short Rate Models	94
15.2 Average Absolute Value of Error Term	96
15.3 Standard Deviation of the Error Term	96
15.4 Ho-Lee Parameter Values	97
15.5 Black-Karasinski Parameter Values	98
15.6 CIR One Factor Parameter Values	98
15.7 Vasicek One Factor Parameter Values	99
15.8 CIR Two Factor (Process X) Parameter Values	100
15.9 CIR Two Factor (Process Y) Parameter Values	100
15.10 Vasicek Two Factor (Short Rate) Parameter Values	101
15.11 Vasicek Two Factor (Short Term Mean Reversion Level) Pa- parameter Values	101
15.12 Yield Curve on 18 October 2007	103
15.13 Standard Deviation of the Error Term for Cap Prices	105
15.14 Standard Deviation of the Error Term for Cap Prices	107

List of Tables

15.1 Goodness-of-Fit Statistics	95
15.2 Results of Pairwise P-Tests	104
15.3 Goodness-of-Fit Statistics of Cap Prices	105
15.4 Goodness-of-Fit Statistics of Cap Prices	106

University of Cape Town

Chapter 1

Introduction

The earliest attempt to model interest rates was published by Vasicek (1977), whereby the short rate was used as the factor driving the entire yield curve. This paper led to the development of various alternative models (e.g., Cox-Ingersoll-Ross, Hull-White, Ho-Lee, Brennan and Schwartz) each attempting to best explain the underlying interest rate process.

Part of the appeal of short rate models is the simplicity of the models and the ease in solving these models numerically. The fact that “principal component analysis shows that 80% – 90% of the price variation in the bond market can be explained by a single factor,” adds credence to the use of short rate models, a fact highlighted by Ouwehand (2008). However, detractors emphasize the facts that the short rate does not exist in reality and the yield curve derived through such models is typically constrained in how it can evolve as reasons why such models should not be used. Nevertheless, an understanding of short rate models and the principles underlying these models is essential in providing one with a firm foundation for the modelling of interest rates.

This paper builds on the initial groundwork laid by Svoboda (2002) in her paper “An investigation of various interest rate models and their calibration in the South African market.” Svoboda’s paper introduces the more popular short rate models, as well as explains how one can calibrate such models to the yield curve in the South African market.

From the perspective of a South African reader, this paper will extend Svoboda’s work by: i. explaining the market practicalities that exist in the South African interest rate market; ii. investigating further short rate models such as the two factor forms of the Vasicek and Cox-Ingersoll-Ross models; iii. showing how one can price caps and swaptions when the short rate is assumed to follow certain dynamics; iv. and examining which short

rate models are most effective at pricing plain vanilla interest rate options over an extended period of time, when comparing the model-implied price to the traded market price.

1.1 Dissertation Outline

Chapter 2 introduces the basic instruments which one encounters in the bond market before Chapter 3 lays down the foundations for the arbitrage-free pricing of contingent claims and provides approaches to price various interest rate options in an arbitrage-free world. Chapter 4 investigates the use of Black's model to price interest rate options (market convention is to use this model when pricing such options). Chapter 5 provides the reader with a basic understanding of the market practicalities that one will encounter in the South Africa interest rate market whilst Chapter 6 introduces a family of short rate models, termed "Affine Term Structure Models."

Chapters 7 to 12 introduce some of the more widely used short rate models, showing one: i. how to derive the short rate from the diffusion process; ii. price a pure discount bond when the short rate follows certain dynamics; iii. and justify the various formulae for pricing caps, assuming the short rate follows certain dynamics.

Chapter 13 explains how one can calibrate a short rate model to the yield curve, thus obtaining values for the various parameters. Chapter 14 builds on these results by providing a framework to compare various models, each of which purports to explain the same dependent variable. Chapter 15 compares the fit of the short rate models covered in the paper to pure discount bond prices, using goodness-of-fit tests Davidson and MacKinnon's C Test. The chapter then goes on to investigate the ability of each of the short rate models covered to generate prices for at-the-money caps which are close to those observed in the South African interest rate market. One would be sceptical of using a short rate model to price exotic options if the prices generated by that model for plain vanilla options were far from consistent with those actually observed in the market. Chapter 16 highlights the key results from the paper, and provides recommendations as to which short rate models appear most appropriate for pricing contingent claims in the South African market.

Chapter 2

Introduction to Bond Markets

This chapter will provide an introduction to the more common instruments found in the bond market and terms used in relation to the bond market.

2.1 Bank Account

B_t represents the value at time t of R1 invested in the bank account at time zero. The bank account is assumed to evolve under the following differential equation

$$dB_t = r_t B_t dt,$$

where r_t is the instantaneous interest rate at time t , known as the short rate. Thus the value of the bank account at time t is

$$B_t = \exp\left(\int_0^t r_s ds\right).$$

2.2 Pure Discount Bond

$P(t, T)$ denotes the value at time t of a pure discount bond, a contract whereby it is agreed that the issuer will pay the holder R1 at time T , with no intermediary cashflows between time t and time T . For the purpose of this thesis, these bonds are assumed to have zero default risk.

2.3 Spot Rates

The continuously-compounded spot rate at time t for maturity T , denoted $R(t, T)$, is the constant rate which an investment at time t of $P(t, T)$ would need to earn in order to yield R1 at time T . i.e.,

$$R(t, T) := \frac{-\ln(P(t, T))}{T - t}.$$

2.3.1 Arbitrage

In an arbitrage-free world, the spot rate can never be negative. If this were to occur, then a pure discount bond would trade at a value greater than its par value, resulting in an arbitrage opportunity.

2.4 Forward Rate

The continuously-compounded forward rate at time t , denoted $F(t, T, S)$, is the constant rate which applies between times T and S , where $t \leq T \leq S$. i.e.,

$$F(t, T, S) := \frac{1}{S - T} \ln \frac{P(t, T)}{P(t, S)}.$$

2.5 Short Rate

The short rate r_t is defined as the instantaneous interest rate at time t applicable for next momentary period dt . Thus

$$r_t = \lim_{T \rightarrow t} R(t, T) = R(t, t).$$

2.6 LIBOR and JIBAR

LIBOR, the London Inter Bank Offer Rate, is defined by Jorion (2007) as a “benchmark cost of borrowing for highly rated (AA) credits.” Hull (2006) defines it as “the rate at which a bank is willing to lend to other banks.” This rate is published daily for a range of borrowing periods. JIBAR, or Johannesburg Inter Bank Agreed Rate, is the South African equivalent of LIBOR.

2.7 Options on Bonds

A European call option on a pure discount bond provides the holder with the right to purchase this bond at an agreed point in the future for an agreed price. A European put option on a pure discount bond provides the holder with the right to sell the bond at an agreed point in the future for an agreed price.

2.8 Interest Rate Caps

Hull (2006) describes an interest rate cap as “an option that provides a payoff when a specified interest rate is above a certain level. The interest rate is a floating rate that is reset periodically.” Thus, such an instrument

may be used as insurance against the possibility that the underlying interest rate rises above a certain level.

Example. *You wish to protect against the risk that LIBOR rates starting in three and six months time rises above 6%. This level above which the interest rate must rise in order for you to receive a payoff is known as the cap rate or strike rate, denoted r_K . In this example the interest rate is reset every three months (i.e., in three and six months time) thus the period in-between resets is three-months and is known as the tenor, denoted τ_i . If the underlying interest rate is above the cap rate on the reset date then the cap payment for that period is the product of the the face of the instrument (termed the ‘principal amount’), the tenor and the difference between the underlying rate and cap rate. i.e.,*

$$N\tau_i[\max(r_i - r_K, 0)] \quad \text{where } N \text{ is the Principal amount.}$$

Suppose you entered into a nine-month cap on three-month LIBOR with a cap rate of 6% and a tenor of three months and the underlying interest rate on the reset date in 3 and 6 months time is 8% and 6% respectively. Then your payments would be $N\frac{1}{4}(8\% - 6\%) = 0.5\%N$ and $R0$. (In order to simplify matters, day count issues have been ignored; however such issues will be introduced at a later stage).

A cap can be viewed as a portfolio of a series of options on the underlying interest rate, with each option known as a caplet. Thus when valuing a cap, one can sum the discounted values of each of the caplets. i.e.,

$$\begin{aligned} \text{Cap} &= \sum_{i=1}^n P(t, T_i) N\tau_i[\max(r_i - r_K, 0)] \\ &:= \sum_{i=1}^n P(t, T_i) \text{Cpl}_i. \end{aligned}$$

Hull (2006) notes that caps are usually defined so that the initial LIBOR rate, even if it is greater than the cap rate, does not lead to a payoff on the first reset date.

2.9 Interest Rate Floors

An interest rate floor, as defined by Hull (2006), is “an option which provides a payoff when an interest rate is below a certain level.” Thus a floor can be used as insurance against the possibility that the interest rate falls below a certain level. The floor will provide the following payoff at each reset date:

$$N\tau_i[\max(r_K - r_i, 0)].$$

Similar to a cap, a floor can be viewed as a portfolio of a series of options on the underlying interest rate with each option known as a floorlet. One can also value the floor by simply summing the discounted values of each of the floorlets. i.e.,

$$\begin{aligned}\text{Floor} &= \sum_{i=1}^n P(t, T_i) N \tau_i [\max(r_K - r_i, 0)] \\ &:= \sum_{i=1}^n P(t, T_i) \text{Flr}_i.\end{aligned}$$

As is the case with caps, floors are usually defined so that the initial LIBOR rate, even if it is less than the floor rate, does not lead to a payoff on the first reset date.

2.10 Interest Rate Swaps

Hull (2006) defines an interest rate swap as an agreement between two parties to exchange cash flows in the future based on the future value of an interest rate. The agreement defines the dates when the cash flows are to be paid and the way in which these cash flows are to be calculated.

The most common interest rate swap is a ‘plain vanilla’ interest rate swap where one party agrees to pay cash flows based on a fixed rate and receive cash flows based on a floating interest rate, termed ‘Pay-Fixed Swap’. ‘Pay-Floating Swap’ is the party on the opposite side which pays cash flows based on a floating interest rate and receives cash flows based on the fixed rate. These payments are both based on the same notional principal amount. This notional principal is not exchanged. In reality the fixed and floating payments are netted-off against one another thus only one of the parties makes a payment at each predetermined date.

In South Africa, the fixed and floating legs are typically exchanged quarterly. This is not necessarily the case in other jurisdictions, as noted by Brigo and Mercurio (2006), where typically the fixed leg involves semi-annual or annual payments, and the floating leg involves quarterly or semi-annual payments.

2.11 Interest Rate Swaptions

Interest rate swaptions are options on interest rate swaps whereby, as Hull (2006) explains, “they give the holder the right to enter into a certain interest rate swap at a certain time in the future.” Swaptions can be used to provide insurance against interest rates moving beyond a certain level, whilst allowing the holder to benefit from favourable interest rate movements.

A payer swaption is an option giving the holder the right to enter into a pay-fixed swap whilst a receiver swaption is an option giving the holder the right to enter into a pay-floating swap.

Chapter 3

No-Arbitrage Pricing and Numeraire Change

The purpose of the first half of this chapter is to lay the foundations for the arbitrage-free pricing of contingent claims, and is based around the books of Pelsser (2004) and Brigo and Mercurio (2006) and the lecture notes of Ouwehand (2008). The second half of the chapter will focus on the relationship between options and the pricing of such options in a world where arbitrage does not exist. This chapter will highlight the key concepts required, however for more a more formal and rigorous handling of this subject, one is referred to Musiela and Rutkowski (2007) or Hunt and Kennedy (2000).

3.1 Basic Setup

Throughout this paper we consider a market where continuous trading occurs, no transaction costs are incurred, markets are sufficiently liquid for every security, short sales are allowed and there is perfect divisibility of assets. The trading interval is limited to a finite period $[0, T]$. The market model is the tuple

$$M = (\Omega, F, \mathbb{P}, (F_t)_{t \geq 0}, (S_t^0, \dots, S_t^N)_{t \geq 0}),$$

where (Ω, F, \mathbb{P}) is a probability space with Ω denoting the sample space with elements $\omega \in \Omega$, F denoting a σ -algebra on Ω , and \mathbb{P} denoting a probability measure on (Ω, F) . The uncertainty is resolved over $[0, T]$ according to a filtration (F_t) satisfying the usual conditions. $S_t = (S_t^0, \dots, S_t^N)$ is an $(N + 1)$ -dimensional adapted cadlag semi-martingale.

We assume that Ω comes with a K -dimensional Brownian motion $W_t = (W_t^1, \dots, W_t^K)$ which generates the filtration (F_t) . Furthermore, we assume that there exist assets which are traded in the market, called ‘marketed assets’ and that the prices, S_t , of these marketed assets can be modelled via

Itô processes

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t,$$

where $\mu(t, S_t)$, $\sigma(t, S_t)$ are deterministic.

Under these conditions, the marketed asset price process is (strong) markov.

The asset indexed by 0 is a bank account. Its price evolves under the following process

$$dS_t^0 = r_t S_t^0 dt,$$

where $S_0^0 = 1$ and r_t is the short rate at time t . For the remainder of this paper, $S_t^0 = B_t$.

Definition 1. A trading strategy or portfolio is a predictable process $\phi_t = (\phi_t^0, \dots, \phi_t^N)$ which is integrable with respect to the semi-martingale S_t where ϕ_t^n denotes the holdings in asset S_t^n at time t . The value of the portfolio at time t is

$$V_t(\phi) = \phi_t \cdot S_t = \sum_{n=0}^N \phi_t^n S_t^n.$$

One requires the trading strategies to be self-financing:

$$dV_t = d(\phi_t \cdot S_t) = \phi_t \cdot dS_t + S_t \cdot d\phi_t + d[S, \phi]_t = \phi_t \cdot dS_t,$$

thus

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u \cdot dS_u = V_0(\phi) + G_t(\phi),$$

where the gains process $G_t(\phi)$ is defined as $\int_0^t \phi_u \cdot dS_u$. Hence the value of the portfolio at time t equals the initial portfolio value plus the gain or loss over the period. No additional funds are added or removed.

Definition 2. A trading strategy ϕ is self-financing if and only if $G_t(\phi) = \int_0^t \phi_u \cdot dS_u$ i.e., if and only if $d(\phi_t \cdot S_t) = \phi_t \cdot dS_t$.

A European contingent claim, C , is a derivative which, at some future time T has a payoff which is a known Borel-measurable function of asset prices at time T . i.e.,

$$C_T = f(S_T).$$

Thus C_T is an F_T -measurable random variable.

Definition 3. A European contingent claim, C , is said to be attainable if and only if there exists a self-financing strategy ϕ_t such that $C_T = V_T(\phi)$. where T is the exercise date of the contingent claim. Then ϕ is called a replicating portfolio for C .

Definition 4. *A financial market is complete if and only if every contingent claim is attainable.*

If the market is arbitrage-free, and since it is assumed that trading costs are not incurred, then the value of a replicating portfolio at time t gives a unique value for the contingent claim, C . Hence one can value the contingent claim by valuing the replicating portfolio, a process called ‘pricing by arbitrage’, see e.g., Pelsser (2004). If the conditions are met under which a market is arbitrage-free and complete, then all contingent claims can be priced by arbitrage.

3.2 Equivalent Martingale Measure

Any price process, N_t , which has strictly positive prices for all $t \in [0, T]$ is termed a ‘numeraire.’ Ouwehand (2008) describes a numeraire as “a unit into which other assets are translated. Thus, if S_t is the price of S in money, then $\hat{S}_t = \frac{S_t}{N_t}$ is the price of S in units of N .” Typically the bank account $B_t = S_t^0$ is chosen as the numeraire. In such a case, $\bar{S}_t = \frac{S_t}{B_t}$ is the discounted value of S at time t .

Definition 5. *Suppose N is a numeraire. A measure \mathbb{Q} on (Ω, F) is an Equivalent Martingale Measure (EMM) for numeraire N if and only if*

1. \mathbb{Q} is equivalent to \mathbb{P} . i.e., both measures have the same null-sets,
2. $\hat{S}_t = \frac{S_t}{N_t}$ is a \mathbb{Q} -martingale.

An Equivalent Martingale Measure associated with the bank account is called a risk-neutral measure.

Theorem 1. *If an EMM \mathbb{Q} exists for some numeraire N , then there are no arbitrage opportunities. (Ouwehand, 2008)*

Theorem 2 (Risk-Neutral Valuation). *Suppose that X is an attainable contingent claim, and that \mathbb{Q} is an EMM for numeraire N . Then,*

$$\hat{X}_t = \mathbb{E}_{\mathbb{Q}}[\hat{X}_T | F_t],$$

i.e.,

$$X_t = N_t \mathbb{E}_{\mathbb{Q}} \left[\frac{X_T}{N_T} \middle| F_t \right].$$

(Ouwehand, 2008)

3.3 Change of Numeraire

Geman et al (1995) noted that the EMM \mathbb{Q} is not necessarily the most convenient measure to use when pricing a contingent claim. In such instances, one is better suited to use a change of numeraire. Jamshidian (1989) illustrates this point, using a change of numeraire when the short rate is stochastic in order to price a bond option.

It is critical to understand the impact of a change in numeraire on a self-financing portfolio and a contingent claim which is attainable:

Proposition 1. *A self-financing portfolio remains self-financing under a change of numeraire. (Ouweland, 2008)*

Corollary 1. *If a contingent claim is attainable in a given numeraire, it is also attainable in any other numeraire, and the replicating portfolio is the same. (Ouweland, 2008)*

This section ends with a proposition which is an extension of the Risk-Neutral Valuation theorem and provides one with a fundamental tool for the pricing of contingent claims.

Proposition 2. *Assume that there exists a numeraire N and a probability measure \mathbb{Q}^N , equivalent to the initial \mathbb{Q}_0 , such that the price of any traded asset X (without intermediate payments) relative to N is a martingale under \mathbb{Q}^N . i.e.,*

$$\frac{X_t}{N_t} = \mathbb{E}_{\mathbb{Q}^N} \left[\frac{X_T}{N_T} \middle| \mathcal{F}_t \right].$$

Let U be an arbitrary numeraire. Then there exists a probability measure \mathbb{Q}^U , equivalent to the initial \mathbb{Q}_0 , such that the price of any attainable claim $\hat{Y} = \frac{Y}{U}$ is a martingale under \mathbb{Q}^U

$$\frac{Y_t}{U_t} = \mathbb{E}_{\mathbb{Q}^U} \left[\frac{Y_T}{U_T} \middle| \mathcal{F}_t \right].$$

Moreover the Radon-Nikodym derivative defining the measure \mathbb{Q}^U is given by

$$\frac{d\mathbb{Q}^U}{d\mathbb{Q}^N} = \frac{U_T N_0}{U_0 N_T}.$$

(Brigo and Mercurio, 2006)

3.4 Girsanov's Theorem and Ito's Lemma

Girsanov's theorem provides one with an understanding of the impact of a change in numeraire on the stochastic differential equation of an asset, particularly how the drift of an asset will change due to a change in numeraire.

Theorem 3 (Girsanov's Theorem). *For any stochastic process λ_t such that*

$$\int_0^t \lambda_s^2 ds < \infty,$$

with probability one, consider the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} = \zeta_t$ given by

$$\zeta_t = \exp\left(\int_0^t \lambda_s dW_s - \frac{1}{2} \int_0^t \lambda_s^2 ds\right),$$

where W is a Brownian motion under the measure \mathbb{P} . Under the measure \mathbb{Q} the process

$$\hat{W}_t = W_t - \int_0^t \lambda_s ds,$$

is also a Brownian motion. (Pelsser, 2004)

A key point to note from this result is that a change in numeraire only results in a change in the drift of the underlying asset, not in a change in the volatility.

Another key result from stochastic calculus is Ito's Lemma.

Lemma 1 (Ito's Lemma). *Suppose that X is a one-dimensional stochastic process given by the differential equation*

$$dX_t = \mu(t, \omega)dt + \sigma(t, \omega)dW_t,$$

and a function $f(t, X)$ of the process X exists, then provided that f is sufficiently differentiable

$$df = \left(\frac{\partial f(t, X)}{\partial t} + \mu(t, \omega) \frac{\partial f(t, X)}{\partial X} + \frac{1}{2} \sigma(t, \omega)^2 \frac{\partial^2 f(t, X)}{\partial X^2} \right) dt + \sigma(t, \omega) \frac{\partial f(t, X)}{\partial X} dW.$$

(Pelsser, 2004)

3.5 Put-Call Parity

Put-Call parity describes the relationship between options and the underlying assets where the characteristics of the various options are identical.

3.5.1 Bond Options

Consider the following two portfolios:

1. Long one Call Option and K units of a pure discount bond expiring at T . i.e., $C_t + KP(t, T)$.

2. Long one Put Option and one pure discount bond expiring at S . i.e., $P_t + P(t, S)$.

Note: The terms of the options are identical, with a strike price of K and the underlying a pure discount bond expiring at S – i.e., $P(t, S)$. The pure discount bonds $P(t, T)$ and $P(t, S)$ have face values of 1.

At time T , if the pure discount bond expiring at time S is greater than the strike price, K , portfolio 1 will receive $P(T, S) - K + K = P(T, S)$ whilst portfolio 2 will receive $0 + P(T, S) = P(T, S)$. If the pure discount bond expiring at time S is less than the strike price, K , portfolio 1 will receive K whilst portfolio 2 will receive $P(T, S) + K - P(T, S) = K$. Since the cash flows of the two portfolios are equivalent at time T , the two portfolios must be equivalent in value at any point in time. i.e.,

$$C_t + KP(t, T) = P_t + P(t, S).$$

3.5.2 Caps and Floors

Consider the following two portfolios:

1. Long one Cap.
2. Long one Floor and one Pay-Fixed Swap.

The cap and floor are assumed to have the same term, tenor, strike rate and underlying interest rate (LIBOR). The fixed rate of the swap is assumed to be the same as the strike rate whilst the floating rate of the swap is assumed to be the same as the underlying interest rate of the cap and floor. The payment dates of the swap are assumed to be equivalent to those of the cap and floor.

For any period, if LIBOR is greater than the strike rate, portfolio 1 will receive $\text{LIBOR} - r_K$ whilst portfolio 2 will receive $0 + \text{LIBOR} - r_K$. If LIBOR is less than the strike rate, portfolio 1 will not receive any payment, whilst portfolio 2 will receive $r_K - \text{LIBOR}$ and will pay $r_K - \text{LIBOR}$ (i.e., the nett payment is zero). Since the payment dates of the two portfolios are equivalent, and the cash flows of each of the portfolios are equivalent at any payment date, assuming no-arbitrage, the two portfolios must be equivalent in value at any point in time. i.e.,

$$\text{Cap}_t = \text{Floor}_t + \text{Pay-Fixed Swap}_t.$$

Note that a ‘plain vanilla’ interest rate swap with the same term and periods between reset dates as a cap or floor will involve one more payment date

than the cap or floor, occurring at the first reset date. Thus the above Pay-Fixed Swap is not a ‘plain vanilla’ interest rate swap, but instead can be valued at time 0 as

$$\text{‘Plain Vanilla’ Pay-Fixed Swap} = 1 + e^{-R_1\tau_1} + R\alpha_1 e^{-R_1\tau_1},$$

where

- τ_1 is the period 0 to t_1 , the first reset date,
- R_1 is the continuously compounded rate applicable over the period 0 to t_1 ,
- α_1 is the length of the period from 0 to t_1 , and thus equal to τ_1 .

3.6 Pricing of Caps and Floors

The price of a cap can be shown to be equivalent to a series of European put options on pure discount bonds. This equivalence proves extremely useful in the pricing of caps under short rate models.

The payoff of a caplet at T_n is $[L(T_{n-1}, T_n) - K]^+ \tau_n$ which is equivalent to $[L(T_{n-1}, T_n) - K]^+ \tau_n P(T_{n-1}, T_n)$ at T_{n-1} . Thus the caplet payoff at T_{n-1} is:

$$\begin{aligned} &= [(1 + L(T_{n-1}, T_n)\tau_n) - (1 + K\tau_n)]^+ P(T_{n-1}, T_n) \\ &= (1 + K\tau_n) \left(\frac{[1 + L(T_{n-1}, T_n)\tau_n] P(T_{n-1}, T_n)}{(1 + K\tau_n)} - P(T_{n-1}, T_n) \right)^+ \\ &= (1 + K\tau_n) \left(\frac{1}{(1 + K\tau_n)} - P(T_{n-1}, T_n) \right)^+. \end{aligned}$$

Thus, a caplet with payoff at T_n is equivalent to $(1 + K\tau_n)$ many puts on $P(t, T_n)$ with Strike $\frac{1}{(1 + K\tau_n)}$ and maturity T_{n-1} .

Similarly a floorlet with payoff $[K - L(T_{n-1}, T_n)]^+ \tau_n$ at T_n is equivalent to $(1 + K\tau_n)$ many calls on $P(t, T_n)$ with Strike $\frac{1}{(1 + K\tau_n)}$ and maturity T_{n-1} .

Finally, the value of a cap or floor can be found by summing up the prices of the respective caplets or floorlets.

3.7 Pricing of Swaps

A ‘plain-vanilla’ swap can be valued by separately valuing the series of cash-flows based on the fixed rate and those based on the floating rate. One of two approaches may be used to value the swap.

The first approach follows the market practicality in that the notional principals are not exchanged. As West (2008) explains, the value of the cash flows based on the fixed rate, R , where the swap consists of n payments, is relatively easy to determine. i.e.,

$$V_{\text{fix}} = R \sum_{i=1}^n \alpha_i e^{-R_i \tau_i},$$

where

R_i is the continuously compounded rate for the period zero to t_i ,

τ_i is the period zero to t_i ,

α_i is the length of the i th three-month period, from period t_{i-1} to t_i .

The value of the cash flows based on the floating interest rate is slightly more difficult to determine. West's approach in his 2008 paper is to manipulate the cashflows into an equivalent form in such a way that the value of this equivalent form is far easier to determine. Hence, the value of the floating-leg of the swap at its initiation is shown to be

$$V_{\text{float}} = 1 - e^{-R_n \tau_n}.$$

Since the fixed rate is set such that these values of the two series of cashflows equate one another at inception

$$R = \frac{1 - e^{-R_n \tau_n}}{\sum_{i=1}^n \alpha_i e^{-R_i \tau_i}}. \quad (3.1)$$

The second approach assumes that the principals are exchanged at maturity of the swap. Thus, the series of fixed rate cashflows can be valued as a fixed rate bond with coupon size R . i.e.,

$$V_{\text{fix}} = R \sum_{i=1}^n \alpha_i e^{-R_i \tau_i} + e^{-R_n \tau_n}.$$

The series of floating rate cashflows can be valued as a floating rate bond with coupon based on the floating rate. The value of such a bond at its inception (or immediately after a coupon has been paid) is equal to the principal amount of the swap. i.e.,

$$V_{\text{float}} = 1.$$

3.8 Pricing of Swaptions

When valuing a swaption it is convenient to use the latter approach in the previous section. Thus, one can view a swaption as an option to exchange

a fixed rate bond for the principal amount underlying the swap. Hence, a payer swaption can be regarded as a put option on a coupon paying bond with the strike price equal to the principal amount. A receiver swaption can be viewed as a call option on a coupon bearing bond with the strike price equal to the principal amount.

3.8.1 One Factor Short Rate Models

If the short rate is modelled using a one factor model then, as Hull (2006) explains, one is able to express the price of a coupon paying bond as the sum of European options on pure discount bonds. This is a generalization of Jamshidian's 1989 decomposition. i.e.,

$$C^{K,\tau}(t, r) = \sum_i Y_i C^{K_i, \tau, T_i}(t, r), \quad (3.2)$$

where

- $C^{K,\tau}(t, r)$ is the price at time t of a call option on a coupon-paying bond, B , whereby the call option expires at time τ and has strike K and the bond expires at time T_N ,
- Y_i is the size of the cashflow at time T_i ,
- $C^{K_i, \tau, T_i}(t, r)$ is the price at time t of a call option on a pure discount bond, $P(t, T_i)$, whereby the call option expires at time τ and has strike K_i and the pure discount bond expires at time T_i .

Hull (2006) describes the process in order to value the option on the coupon paying bond using the above formula:

1. Calculate r^* , the value of the short rate such that the value of the coupon paying bond is equal to the strike price of the option at the option expiration date, τ . Under one factor short rate models, the value of a pure discount bond is a decreasing function of the short rate, thus r^* is unique.
2. Calculate the price of each of the pure discount bonds at the option expiration date, τ , where the short rate is r^* .
3. Set the strike price, K_i , of each of the options on the pure discount bonds equal to the value of the pure discount bonds derived in the previous step.
4. Value each of the options on the pure discount bonds at time t , given the actual short rate, r .
5. Use formula 3.2 to derive the value at time t of the option on the coupon paying bond.

3.8.2 Multi-Factor Short Rate Models

Jamshidian's (1989) decomposition for coupon paying bonds, and hence swaptions, is not applicable for multi-factor models as the value of the pure discount bond is not necessarily a decreasing function of the short rate. Thus, one needs to use an alternative approach to value swaptions under such short rate models, such as Monte Carlo simulation.

Chapter 4

Pricing of Options using Black's Model

Market convention is to price bond options, interest rate caps and floors and swaptions using the Black model. Black (1976) was the first to show how one can value European futures options by extending the Black-Scholes model, which was first published in 1973.

4.1 Options on Bonds

If one assumes that the underlying bond price $P(t, T)$ at the maturity of the option, time T , is distributed lognormally, that F_B is the forward bond price with volatility σ_B , and the strike price of the option is K then the call option on the bond at time 0 can be priced using Black's formula. i.e.,

$$C_0 = P(0, T)[F_B N(d_1) - K N(d_2)].$$

Similarly, the formula to value a put option on the bond at time 0 is

$$P_0 = P(0, T)[K(1 - N(d_2)) - F_B(1 - N(d_1))],$$

where

$$d_1 = \frac{\ln \frac{F_B}{K} + \frac{1}{2} \sigma_B^2 T}{\sigma_B \sqrt{T}},$$
$$d_2 = d_1 - \sigma_B \sqrt{T}.$$

4.2 Caps and Floors

If one assumes that the future underlying interest rate r_i is distributed lognormally with volatility σ_i . Then the interest rate option can be priced using Black's formula. i.e.,

$$\text{Cpl}_i(0) = N\tau_i P(0, t_{i+1})[f_i(0; t_i, t_{i+1})N(d_1) - r_K N(d_2)],$$

and

$$\text{Flr}_i(0) = N\tau_i P(0, t_{i+1}) [r_K(1 - N(d_2)) - f_i(0; t_i, t_{i+1})(1 - N(d_1))],$$

where

$$d_1 = \frac{\ln \frac{f_i(0; t_i, t_{i+1})}{r_K} + \frac{1}{2}\sigma_i^2 t_i}{\sigma_i \sqrt{t_i}},$$

$$d_2 = d_1 - \sigma_i \sqrt{t_i},$$

$f_i(t; t_i, t_{i+1})$ is the forward rate for the period t_i to t_{i+1} at time t .

4.3 Swaptions

Hull (2006) shows that if one assumes that the underlying swap rate at the maturity of the option s_T is distributed lognormally with volatility σ , then one is able to price the swaption using Black's formula. In order to derive the price, consider a payer swaption where the holder has the right to pay the fixed rate s_K and receive JIBAR on a swap that consists of m payments per year for n years and starts in T years with notional N (i.e., the fixed payment is $\frac{s_K N}{m}$).

At time T , if the holder of the option decides to exercise the option, then the holder can immediately hedge away interest rate risk by entering into an equal but opposite position by purchasing a par-floating swap at the swap rate s_T . Obviously the holder will only exercise this option if the option is of value. i.e., if s_T is greater than s_K . Otherwise the holder of the option will simply let the option lapse.

Thus, the holder will receive the following payoff m times per year for n years

$$\frac{N}{m} \max(s_T - s_K, 0).$$

Since the swap rate s_T is lognormally distributed with volatility σ and assuming that the payments occur at time T_i where $i = 1, 2, 3, \dots, nm$, then the value of the payment at time zero for the cashflow at time T_i is

$$\frac{N}{m} P(0, T_{i+1}) [s_0 N(d_1) - s_K N(d_2)],$$

where

$$d_1 = \frac{\ln \frac{s_0}{s_K} + \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}},$$

$$d_2 = d_1 - \sigma \sqrt{T}.$$

Thus, the value of the payer swaption at time 0 is

$$\begin{aligned} PS(0) &= \sum_{i=1}^{nm} \frac{N}{m} P(0, T_{i+1}) [s_0 N(d_1) - s_K N(d_2)] \\ &= \frac{NA}{m} [s_0 N(d_1) - s_K N(d_2)], \end{aligned}$$

where

$$A = \sum_{i=1}^{nm} P(0, T_{i+1}).$$

Similarly, one can find the value of the receiver swaption at time 0 which is

$$RS(0) = \frac{NA}{m} [s_K(1 - N(d_1)) - s_0(1 - N(d_2))].$$

4.4 Consistency in Pricing Formulas

Black's formula has been used to value an option on a bond, a cap and a swaption. On each occasion a different variable has been assumed to be lognormally distributed – the option on a bond assumes that the future bond price is lognormally distributed, a cap assumes that the future interest rate is lognormally distributed whilst the swaption assumes that the future swap rate is lognormally distributed. Although each of these formulas are consistent by themselves, Hull (2006) points out that these formulas are not consistent with one another as only one of the underlying variables can be assumed to be lognormally distributed.

One can price the caplet or floorlet directly using Black's formula if one assumes that the interest rate underlying each of the caplets or floorlets is lognormally distributed. Alternatively, since a caplet (floorlet) can be viewed as a number of puts (calls) on a pure discount bond – as was shown in the previous section – one can price the pure discount bond and hence derive the price of the caplet or floorlet. Such an approach assumes that the bond price is lognormally distributed. Since these two assumptions are not consistent with one another, the values derived under the two different approaches will differ with West (2009) noting that the difference between these two answers “typically differs at the 5th decimal place”.

4.5 Suitability of Black's Model for Pricing Interest Rate Options

Black's model relies on two approximations, which Ouwehand (2008) points out are particularly questionable when pricing interest rate options. The

4.5. SUITABILITY OF BLACK'S MODEL FOR PRICING INTEREST RATE OPTIONS 29

key assumption of Black's model is that a market variable, X , is distributed lognormally under the risk-neutral measure.

Throughout this section, we will consider a call option with maturity T and strike K . Thus, the value of such an option at time zero is

$$C_0 = \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_0^T r_t dt} (X_T - K)^+ \right].$$

The first approximation is that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_0^T r_t dt} (X_T - K)^+ \right] &= \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_0^T r_t dt} \right] \mathbb{E}_{\mathbb{Q}} [(X_T - K)^+] \\ &= P(0, T) \mathbb{E}_{\mathbb{Q}} [(X_T - K)^+]. \end{aligned}$$

Such an approximation is only true if the interest rate is independent of the market variable. This is obviously not true for interest rate options, where the market variable is dependent on the interest rate.

The second approximation is that the forward price of X is a \mathbb{Q} -martingale. i.e.,

$$F_T = \mathbb{E}_{\mathbb{Q}}(X_T) = F_0(\text{the forward price of } X \text{ at time } 0).$$

Under the risk-neutral measure, the expected value of X_T is the futures price. Since the interest rate is stochastic, this is not equivalent to the forward price.

Ouwehand (2008) goes on to show that these two approximations are valid for interest rate instruments such as bond options, caps, floors and swaptions, as long as certain assumptions are made about the distribution of various interest rate instruments under the appropriate measures. Thus Black's model can be used to price interest rate options such as bond options, caps and floors, and swaptions.

Chapter 5

Market Practicalities

This section aims to provide the reader with an understanding of how to apply the theory and concepts discussed in this paper thereby providing a linkage between the theoretical and practical worlds.

5.1 Short Rate

The short rate is defined as the instantaneous interest rate at that point in time however such a rate is purely a theoretical concept. In South Africa the shortest rate that is available is the overnight rate. However, as West (2008) notes and West (2009) discusses in detail, using such a rate as a proxy for the short rate is probably not advisable. Cuchiero (2007) agrees with West, reasoning that such a rate typically is highly volatile and has a “low correlation with other yields.” She states that the one-month or three-month spot rates are better proxies, with one of the reasons for this being the liquidity of these rates. Chapman et al (1998) shows that, from a US context, using a three-month rate as a proxy for the short rate does not introduce economically significant biases for one factor affine short rate models.

Thus, the one-month JIBAR will be used as a proxy for the short rate whilst the three-month JIBAR will be used as a proxy for the short-term mean-reversion level for the two factor Vasicek model.

5.2 Pure Discount Bonds

In order to find the value of a pure discount bond or the interest rate over a specific period, one requires a yield curve. Since the pure discount bond market in South Africa is relatively small, with approximately R60bn in outstanding debt according to ASSA (2010), one cannot derive these yields solely from these traded pure discount bonds. Thus, one is required to

bootstrap the yield curve, a technique which uses available data and certain rules to derive the yield curve.

5.2.1 Available Data

One is faced with the choice of using swaps or government bonds in order to derive the yield curve through bootstrapping. Although government bonds initially appear the obvious choice, and are the instrument used in other countries such as the US, this is not the case in South Africa.

The disadvantages of using government bonds in South Africa is the lack of liquidity of all but a few issues resulting in a lack of data points and a need for subjective liquidity adjustments. The lack of liquidity is exacerbated by Basel I's recognition for capital adequacy. Under Basel I, SA government bonds with a maturity of less than three years were fully recognised, hence banks had a preference for short-term bonds resulting in very little trade in bonds as soon as their maturities decreased below three years. Although Basel II does not distinguish between short- and long-term bonds for capital recognition, major South African banks still tend to prefer short-dated bonds, as highlighted by ASSA (2010).

The disadvantage of using swaps is that the credit risk is also priced into the yield (although with plain-vanilla swaps, this credit risk is minimal since the notional is not actually swapped). However, according to West (2009), the sophistication and liquidity of the swap market in South Africa means that swap rates are typically used nowadays when constructing the risk-free yield curve. PWC's 2008 Long-Term Insurance survey highlighted the fact that, in South Africa, swap rates are preferred but that there is no definitive answer, with 50% of insurers surveyed using swap rates, 33% bonds and 17% undecided. Thus, swap rate data will be used.

Two points to note: Firstly, the use of swaps or bonds is mutually exclusive. This is particularly relevant when bootstrapping the real yield curve where few instruments trade. The combined used of such instruments would almost certainly result in one obtaining a yield curve which presents arbitrage opportunities. Secondly, the use of the term 'swap' in this sections incorporates the overnight rate, one-month and three-month JIBAR rates, FRA rates out to two years and swap rates with maturities ranging from two years to 30 years.

5.2.2 Bootstrap Method

In order to 'fill-in' the missing data points, one requires an interpolation method. A wide variety of interpolation methods exist. However the method

used in this model is the ‘Monotone Convex Interpolation Method’, as derived by Hagan and West (2006) and Hagan and West (2008). This method has been specifically designed with interest rate interpolation at hand, hence it is the only method (according to Hagan and West (2006)) that displays all of the following characteristics:

- Produces forward rates which are always positive (Negative forward rates for a nominal yield curve would result in arbitrage).
- The interpolation method is local hence a change to a rate at one point in the curve will not affect the entire curve’s shape.
- Produces continuous forward rates.
- Produces stable forward rates thus a change to a rate at one point in the curve does not result in a significant change in the forward curve.
- The hedges of financial instruments are local thus when using this bootstrap approach for risk management, delta risk is assigned to instruments close to the given term of the risky cashflow to be hedged.

West (2008) describes the approach which one should take in order to bootstrap the yield curve once one has chosen a bootstrap method using the available data, including how one can overcome the problem of holes in the term structure.

5.3 Caps and Floors

Certain market practicalities exist with regard to caps and floors, some of which are unique to the South African market. A generalized interest rate option – n -year option with tenor period of m -months - is used throughout this section to better illustrate the points.

1. The exact reset dates will occur $m, 2m, 3m, \dots, 12n$ months from today and will follow the ‘Modified Following Rule.’

Modified Following Rule. *In order to determine the exact date n months from today one needs to apply the following two rules, termed the ‘Modified Following Rule’, and as described by West (2008):*

- (a) *The day has to be in the month which is exactly n -months from the current month,*
- (b) *The day should be the first business day on or after the date with the same day number as the current day. However, if this contradicts the previous rule, the day is the last business day of the month n -months from the current month.*

2. The options are settled in arrears. e.g., The option dependent on the interest rate between t_i and t_{i+1} will only be settled at time t_{i+1} .
3. The options usually do not include protection against the initial LIBOR rate. i.e., The underlying interest rate of the first option will cover months m to $2m$ (time t_1 to t_2) and a cap (floor) will consist of $\frac{12n}{m} - 1$ caplets (floorlets).
4. The options usually quoted are At-The-Money options and are quoted as a single volatility value.
5. The strike rate of an At-The-Money option is the rate which equates the value of the cap and the floor. In order to find this rate, one can either try to equate these two values or use put-call parity:

$$\text{Cap} = \text{Floor} + \text{Pay-Fixed Swap}.$$

Since this strike rate equates the value of the cap with the value of the floor, one is ultimately solving for the rate which makes the swap's value equal to zero, where the swap's characteristics (reset dates, underlying interest rate, etc) exactly match those of the cap and the floor (i.e., the swap is not a typical 'plain vanilla' interest rate swap). Thus, the formula to find the strike rate (fixed-rate) needs to be slightly adapted from that shown in 3.1.

Thus, if one wishes to determine the strike rate, R_K , for an At-The-Money cap with a tenor of three-months and a term of n -quarters (i.e., $3n$ months) then, at time zero

$$R_K = \frac{e^{-R_1\tau_1} - e^{-R_n\tau_n}}{\sum_{i=2}^n \alpha_i e^{-R_i\tau_i}},$$

where

- R_i is the continuously compounded rate for the period zero to t_i ,
- τ_i is the period zero to t_i ,
- α_i is the length of the i th three-month period, from period t_{i-1} to t_i .

5.4 Individual Caplets and Floorlets

Individual caplets and floorlets have slightly different market practicalities as compared to caps and floors. A generalized form of such an option is a nxm option, which has the following features

1. The start date of the options is n -months from the current date whilst the end-date is $(n-m)$ months from the start date. Both of these

dates follow the ‘Modified Following Rule’. This point is especially important as $(n-m)$ months from the start date which is m -months from the current date is not necessarily equal to n -months from the current date, due to the ‘Modified Following Rule.’

2. The option is settled in advance. e.g., The option dependent on the interest rate between t_i and t_{i+1} will be settled at time t_i , as soon as the applicable interest rate over the life of the option is known.
3. Since this option is settled in advance, the payment is discounted. The payoff is thus:

$$\text{Cpl} = \frac{N\tau_i \max(r_i - r_K, 0)}{1 + r_i \tau_i},$$

and

$$\text{Flr} = \frac{N\tau_i \max(r_K - r_i, 0)}{1 + r_i \tau_i}.$$

4. The options usually quoted are At-The-Money options and are quoted as a volatility value.
5. The strike rate for an At-The-Money option is simply the forward rate.

Chapter 6

Affine Term Structure Models

A short rate model is said to possess affine term structure (ATS) if the price of a pure discount bond is given by $P(t, T) = e^{A(t, T) - B(t, T)r_t}$ where $A(t, T)$ and $B(t, T)$ are deterministic functions. Not all short rate models are affine term structure models - the Black-Karasinski model is one such example which is not affine. However the fact that an ATS model guarantees an explicit formula for the price of a pure discount bond is one of the reasons why this class of short rate models has gained popularity.

Assume a short rate model with the following risk neutral dynamics:

$$dr_t = \mu(t, r)dt + \sigma(t, r)dW_t.$$

This class of short rate models requires both the drift and the volatility squared to be affine functions of the short rate. i.e.,:

$$\begin{aligned}\mu(t, r) &= \alpha(t)r + \beta(t), \\ \sigma^2(t, r) &= \gamma(t)r + \delta(t),\end{aligned}$$

where α , β , γ and δ are deterministic functions of t .

One is able to obtain a system of differential equations, allowing one to solve for $A(t, T)$ and $B(t, T)$. i.e.,:

$$\frac{dP(t, T)}{P(t, T)} = (A_t - rB_t - B\mu(t, r) + \frac{1}{2}\sigma(t, r)^2B^2)dt - \sigma(t, r)BdW_t,$$

where A_t represents $A(t, T)$ differentiated with respect to t , and B_t represents $B(t, T)$ differentiated with respect to t .

But under the risk neutral dynamics, the drift of $P(t, T)$ is r thus

$$\begin{aligned} A_t - rB_t - B\mu(t, r) + \frac{1}{2}\sigma^2(t, r)B^2 &= r \\ A_t - rB_t - B\alpha(t)r - B\beta(t) + \frac{1}{2}\gamma(t)rB^2 + \frac{1}{2}\delta(t)B^2 &= r \\ A_t - B\beta(t) + \frac{1}{2}\delta(t)B^2 &= r(1 + B_t + B\alpha(t) - \frac{1}{2}\gamma(t)B^2). \end{aligned} \tag{6.1}$$

The lefthand side of equation 6.1 is independent of r whereas the right hand side of the equation contains r . This can only occur if both sides are identical to zero, allowing one to find the coupled system of differential equations:

$$\begin{cases} A_t(t, T) = B(t, T)\beta(t) - \frac{1}{2}\delta(t)B(t, T)^2 \\ A(T, T) = 0 \end{cases}$$

$$\begin{cases} B_t(t, T) = -\alpha(t)B(t, T) + \frac{1}{2}\gamma(t)B(t, T)^2 - 1 \\ B(T, T) = 0 \end{cases}$$

When attempting to solve this series of equations, one should solve for $B(t, T)$ first as this equation does not contain A . The solution for $B(t, T)$ can then be plugged into the equation for $A(t, T)$ which is solved by integrating both sides between t and T .

6.1 Bond Options

Theorem 6.1.1. *If $\hat{S}_t = \frac{S_t}{P(t, T)}$ is an Itô process of the form $\frac{d\hat{S}_t}{\hat{S}_t} = \mu(t)dt + \sigma(t)dW_t$, and if $\sigma(t)$ is deterministic, then the value of a European call C with maturity T and strike K on underlying S is given by*

$$C_0 = S_0N(d_1) - KP(0, T)N(d_2),$$

where

$$\sigma_{av} = \sqrt{\frac{1}{T} \int_0^T \sigma(t)^2 dt},$$

$$d_1 = \frac{\ln \frac{S_0}{KP(0, T)} + \frac{1}{2}\sigma_{av}^2 T}{\sigma_{av}\sqrt{T}}, \quad d_2 = d_1 - \sigma_{av}\sqrt{T}.$$

Put-call parity can be used to derive the price of the corresponding European put option

$$P_0 = KP(0, T)(1 - N(d_2)) - S_0(1 - N(d_1)),$$

with d_1 , d_2 and σ_{av} defined as above.

Proof.

$$\begin{aligned}
\frac{d\hat{S}_t}{\hat{S}_t} &= \mu(t)dt + \sigma(t)dW_t, \\
\therefore d\ln(\hat{S}_t) &= \frac{1}{\hat{S}_t}d\hat{S} - \frac{1}{2(\hat{S}_t)^2}\sigma^2(t)(\hat{S}_t)^2dt \\
&= \mu(t)dt + \sigma(t)dW_t - \frac{1}{2}\sigma^2(t)dt \\
&= \left(\mu(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)dW_t, \\
\therefore \ln(\hat{S}_t) &= \ln(\hat{S}_0) + \int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)dW_s, \\
\therefore \hat{S}_t &= \hat{S}_0 \exp \left[\int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)dW_s \right].
\end{aligned}$$

However, given the EMM \mathbb{Q}^T , \hat{S}_t is a martingale under the \mathbb{Q}^T measure. Thus

$$\hat{S}_t = \hat{S}_0 \exp \left[\int_0^t -\frac{1}{2}\sigma^2(s)ds + \int_0^t \sigma(s)d\hat{W}_s \right].$$

Thus $\ln(\hat{S}_t) = X$ is normally distributed with the following mean and variance

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^T} \left[\ln(\hat{S}_t) \right] &= \ln(\hat{S}_0) - \int_0^t \frac{1}{2}\sigma^2(s)ds := M, \\
\text{Var}_{\mathbb{Q}^T} \left[\ln(\hat{S}_t) \right] &= \int_0^t \sigma^2(s)ds := V^2.
\end{aligned}$$

Hence one can calculate the expected payoff at time T

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^T} \left[\max(e^X - K, 0) \right] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{(x-M)^2}{2V^2}} \max(e^x - K, 0) dz \\
&= \int_{\ln K}^{\infty} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{(x-M)^2}{2V^2}} (e^x - K) dz \\
&= \int_{\frac{\ln K - M}{V}}^{\infty} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{1}{2}x^2} (K - e^{M+Vx}) dx \\
&= \int_{\frac{\ln K - M}{V}}^{\infty} \frac{1}{\sqrt{2\pi V^2}} e^{M+Vx - \frac{1}{2}x^2} dx \\
&\quad - \int_{\frac{\ln K - M}{V}}^{\infty} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{1}{2}x^2} K dx \\
&= e^{M + \frac{1}{2}V^2} \int_{\frac{\ln K - M}{V}}^{\infty} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{1}{2}V^2 + Vx - \frac{1}{2}x^2} dx \\
&\quad - K \int_{\frac{\ln K - M}{V}}^{\infty} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{1}{2}x^2} dx
\end{aligned}$$

$$\begin{aligned}
&= e^{M+\frac{1}{2}V^2} \int_{\frac{\ln K-M}{V}}^{\infty} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{1}{2}(x-V)^2} dx \\
&\quad - K \int_{\frac{\ln K-M}{V}}^{\infty} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{1}{2}x^2} dx \\
&= e^{M+\frac{1}{2}V^2} \int_{\frac{\ln K-M-V^2}{V}}^{\infty} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{x^2}{2}} dx \\
&\quad - K \int_{\frac{\ln K-M}{V}}^{\infty} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{1}{2}x^2} dx \\
&= e^{M+\frac{1}{2}V^2} N\left(\frac{M - \ln K + V^2}{V}\right) - KN\left(\frac{M - \ln K}{V}\right) \\
&= \hat{S}_0 N\left(\frac{\ln \frac{\hat{S}_0}{K} + \frac{1}{2} \int_0^t \sigma^2(t)}{\int_0^t \sigma(t)}\right) - KN\left(\frac{\ln \frac{\hat{S}_0}{K} - \frac{1}{2} \int_0^t \sigma^2(t)}{\int_0^t \sigma(t)}\right).
\end{aligned}$$

Thus the expected value of the Call at time zero is

$$= S_0 N(d_1) - KP(0, T)N(d_2),$$

where

$$d_1 = \left(\frac{\ln \frac{S_0}{KP(0, T)} + \frac{1}{2} \int_0^t \sigma^2(t)}{\int_0^t \sigma(t)} \right), \quad d_2 = d_1 - \int_0^t \sigma^2(t).$$

□

6.1.1 Application to Bond Options

Consider a European call P on a pure discount bond $P(t, S)$ with strike K and maturity T where $S > T$. For ATS models, with risk-neutral dynamics $dr_t = \mu(t, r)dt + \sigma(t, r)dW_t$, it has been shown that the bond prices have the following dynamics under the risk-neutral measure:

$$\frac{dP(t, T)}{P(t, T)} = rdt - \sigma(t, r)B(t, T)dW_t.$$

Let $\hat{P}_t = \frac{P(t, S)}{P(t, T)}$. Thus

$$\begin{aligned}
\frac{d\hat{P}_t}{\hat{P}_t} &= \frac{dP(t, S)}{P(t, S)} - \frac{dP(t, T)}{P(t, T)} + \left(\frac{dP(t, T)}{P(t, T)} \right)^2 - \left(\frac{dP(t, S)}{P(t, S)} \right) \left(\frac{dP(t, T)}{P(t, T)} \right) \\
&= (\sigma^2(t, r)B(t, T)[B(t, T) - B(t, S)]dt + \sigma(t, r)(B(t, T) - B(t, S))dW_t.
\end{aligned}$$

Thus

$$P_0 = KP(0, T)(1 - N(d_2)) - P(0, S)(1 - N(d_1)),$$

where

$$\sigma_{av} = \sqrt{\frac{1}{T} \int_0^T \sigma^2(t, r) (B(t, T) - B(t, S))^2 dt},$$
$$d_1 = \frac{\ln \frac{S_0}{KP(0, T)} + \frac{1}{2} \sigma_{av}^2 T}{\sigma_{av} \sqrt{T}}, \quad d_2 = d_1 - \sigma_{av} \sqrt{T}.$$

University of Cape Town

Chapter 7

Vasicek Model

Vasicek (1977) models the instantaneous spot rate as a Ornstein-Uhlenbeck process:

$$dr_t = k(\phi - r_t)dt + \sigma dW_t, \quad (7.1)$$

where k, ϕ, σ, r_0 are positive constants.

Integrating equation 7.1, one can solve for r_t :

$$\begin{aligned} (dr_t)e^{kt} &= k\phi e^{kt}dt - r_t k e^{kt}dt + \sigma e^{kt}dW_t, \\ d(r_t e^{kt}) &= k\phi e^{kt}dt + \sigma e^{kt}dW_t, \\ r_t e^{kt} &= r_s e^{ks} + k\phi \int_s^t e^{k\tau} d\tau + \sigma \int_s^t e^{k\tau} dW_\tau, \\ r_t e^{kt} &= r_s e^{ks} + \phi(e^{kt} - e^{ks}) + \sigma \int_s^t e^{k\tau} dW_\tau, \\ r_t &= r_s e^{-k(t-s)} + \phi(1 - e^{-k(t-s)}) + \sigma \int_s^t e^{-k(t-\tau)} dW_\tau. \end{aligned} \quad (7.2)$$

Thus r_t , conditional on F_s , is distributed normally with the following moments:

$$\begin{aligned} \mathbb{E}(r_t|F_s) &= r_s e^{-k(t-s)} + \phi(1 - e^{-k(t-s)}), \\ \text{Var}(r_t|F_s) &= \mathbb{E} \left[(r_t - \mathbb{E}(r_t|F_s))^2 \right] \\ &= \mathbb{E} \left[\sigma^2 \int_s^t e^{-2k(t-\tau)} d\tau \right] \\ &= \frac{\sigma^2}{2k} [1 - e^{-2k(t-s)}]. \end{aligned}$$

The long-term mean and variance can be found by taking the limit as t tends towards infinity:

$$\begin{aligned}\lim_{t \rightarrow \infty} \mathbb{E}(r_t | F_s) &= \phi, \\ \lim_{t \rightarrow \infty} \text{Var}(r_t | F_s) &= \frac{\sigma^2}{2k}.\end{aligned}$$

One can thus clearly see several key attributes of the Vasicek model:

1. The short rate experiences mean reversion, and over the long-term will tend towards ϕ .
2. There is no certainty that the short rate will always be positive. In reality, nominal interest rates can never be negative, however the Vasicek model can not ensure that such a situation does not occur - a major drawback of the model.

7.1 Pure Discount Bond

$dr_t = (k\phi - kr_t)dt + \sigma dW_t$ thus, in terms of the parameters for the Generalized Affine Term Structure Model, $\alpha = -k, \beta = k\phi, \gamma = 0$ and $\delta = \sigma^2$, with all parameters constant.

The system of differential equations that must be solved is therefore:

$$\begin{cases} A_t &= k\phi B - \frac{1}{2}\sigma^2 B^2 \\ A(T, T) &= 0 \end{cases}$$

$$\begin{cases} B_t &= kB - 1 \\ B(T, T) &= 0 \end{cases}$$

$B(t, T)$ is a Riccati equation. However, since the coefficient of B^2 is 0, the Riccati equation simplifies to become a linear differential equation, which can be solved as follows:

$$\begin{aligned}\Rightarrow \frac{dB}{kB-1} &= dt, \\ \Rightarrow \frac{1}{k} \ln(kB-1) &= t + C, \\ \Rightarrow kB-1 &= C^* e^{kt}, \\ \Rightarrow B(t, T) &= \frac{1}{k} + C^{**} e^{kt},\end{aligned}$$

but $B(T, T)$ must meet the boundary condition. i.e.,

$$\begin{aligned}
 B(T, T) &= 0, \\
 \therefore \frac{1}{k} + C^{**} e^{kT} &= 0, \\
 \therefore C^{**} &= -\frac{1}{k} e^{-kT}, \\
 \therefore B(t, T) &= \frac{1}{k} (1 - e^{-k(T-t)}). \tag{7.3}
 \end{aligned}$$

Substituting $B(t, T)$ into A_t , one can solve for $A(t, T)$:

$$\begin{aligned}
 \Rightarrow A_t &= \phi(1 - e^{-k(T-t)}) - \frac{\sigma^2}{2k^2} (1 - e^{-k(T-t)})^2, \\
 \Rightarrow A(t, T) &= A(T, T) - \int_t^T A_s ds \\
 &= - \int_t^T [\phi(1 - e^{-k(T-s)}) - \frac{\sigma^2}{2k^2} (1 - e^{-k(T-s)})^2] ds \\
 &= -\phi(T-t) + \frac{\phi}{k} (1 - e^{-k(T-t)}) + \frac{\sigma^2}{2k^2} (T-t) - \frac{\sigma^2}{k^3} (1 - e^{-k(T-t)}) + \\
 &\quad \frac{\sigma^2}{4k^3} (1 - e^{-2k(T-t)}) \\
 &= -\phi(T-t) + \phi B + \frac{\sigma^2}{2k^2} (T-t) + \frac{\sigma^2}{4k^3} (-4 + 4e^{-k(T-t)} + 1 - e^{-2k(T-t)}) \\
 &= -\phi(T-t) + \phi B + \frac{\sigma^2}{2k^2} (T-t) + \frac{\sigma^2}{4k} [-\frac{1}{k^2} (1 - 2e^{-k(T-t)} + e^{-2k(T-t)} + \\
 &\quad 2(1 - e^{-k(T-t)})] \\
 &= -\phi(T-t) + \phi B + \frac{\sigma^2}{2k^2} (T-t) - \frac{\sigma^2 B^2}{4k} - \frac{B\sigma^2}{2k^2} \\
 &= -\frac{\sigma^2 B^2}{4k} + \frac{(B - (T-t))(k^2 \phi - \frac{\sigma^2}{2})}{k^2}. \tag{7.4}
 \end{aligned}$$

Thus, the price of a pure discount bond, $P(t, T) = e^{A(t, T) - B(t, T)r_t}$, has been found for the Vasicek model, with the values for $A(t, T)$ and $B(t, T)$ represented by 7.4 and 7.3 respectively.

Since the Vasicek model consists of only three parameters and these parameters are time independent, when fitting this model to the observed term structure, the model is over-determined and can only fit exactly to three bonds.

7.2 Cap and Floors

In order to price a caplet, one must first find the value of a European put option on a pure discount bond and then use the relationship between the

put option and the caplet.

Since σ is constant and $B(t, T) = \frac{1}{k}(1 - e^{-k(T-t)})$,

$$\begin{aligned}\sigma_{av}^2 T &= \int_0^T \frac{\sigma^2}{k^2} (e^{-kS} - e^{-kT})^2 e^{2kt} dt \\ &= \frac{\sigma^2}{2k^3} (e^{-kS} - e^{-kT})^2 (e^{2kT} - 1),\end{aligned}$$

And the price of the put option is

$$P_0 = KP(0, T)(1 - N(d_2)) - P(0, S)(1 - N(d_1)),$$

where

$$d_1 = \frac{\ln \frac{P(0, S)}{KP(0, T)} + \frac{1}{2}\sigma_{av}^2 T}{\sigma_{av}\sqrt{T}}, \quad d_2 = d_1 - \sigma_{av}\sqrt{T}.$$

Example 1. *If the short rate is assumed to possess the following risk-neutral dynamics*

$$dr_t = k(\phi - r_t)dt + \sigma dW_t$$

with $k = 0.1$, $\phi = 0.05$ and $\sigma = 0.1$ and a caplet has the payoff $0.25(L - 0.0475)^+$ at expiry in 1 year, where L is the three-month spot LIBOR rate in nine months' time, r_0 is 5% and $P(0, T)$ is a pure discount bond with face value 1 then

$$\begin{aligned}K &= \frac{1}{(1 + K_{Caplet}\tau)} \\ &= \frac{1}{(1 + 0.0475(0.25))} \\ &= 0.9882643607, \\ \sigma_{av} &= \sqrt{\frac{\frac{\sigma^2}{2k^3} (e^{-kS} - e^{-kT})^2 (e^{2kT} - 1)}{T}} \\ &= \sqrt{\frac{\frac{0.1^2}{2(0.1)^3} (e^{-0.1(1)} - e^{-0.1(0.75)})^2 (e^{2(0.1)(0.75)} - 1)}{0.75}} \\ &= 0.0237925029, \\ B(0, T) &= \frac{1}{k}(1 - e^{-kT}) \\ &= \frac{1}{0.1}(1 - e^{-0.1(0.75)}) \\ &= 0.7225651367\end{aligned}$$

(7.5)

$$\begin{aligned}
A(0, T) &= -\frac{\sigma^2 B^2}{4k} + \frac{(B - T)(k^2 \phi - \frac{\sigma^2}{2})}{k^2} \\
&= -\frac{0.1^2(0.7225651367)^2}{4(0.1)} + \frac{(0.7225651367 - 0.75)(0.1^2(0.05) - \frac{0.1^2}{2})}{0.1^2} \\
&= -0.0007068209
\end{aligned}$$

$$\begin{aligned}
P(0, T) &= e^{A(0, T) - B(0, T)r_0} \\
&= e^{-0.0007068209 - 0.7225651367(0.05)} \\
&= 0.9638350801
\end{aligned}$$

$$\begin{aligned}
B(0, S) &= \frac{1}{k}(1 - e^{-kS}) \\
&= \frac{1}{0.1}(1 - e^{-0.1(1)}) \\
&= 0.9516258196
\end{aligned}$$

$$\begin{aligned}
A(0, S) &= -\frac{\sigma^2 B^2}{4k} + \frac{(B - S)(k^2 \phi - \frac{\sigma^2}{2})}{k^2} \\
&= -\frac{0.1^2(0.7225651367)^2}{4(0.1)} + \frac{(0.7225651367 - 1)(0.1^2(0.05) - \frac{0.1^2}{2})}{0.1^2} \\
&= -0.0008714114
\end{aligned}$$

$$\begin{aligned}
P(0, S) &= e^{A(0, S) - B(0, S)r_0} \\
&= e^{-0.0008714114 - 0.9516258196(0.05)} \\
&= 0.9527023988
\end{aligned}$$

$$\begin{aligned}
d_1 &= \frac{\ln \frac{P(0, S)}{KP(0, T)} + \frac{1}{2}\sigma_{av}^2 T}{\sigma_{av}\sqrt{T}} \\
&= 0.0193983885
\end{aligned}$$

$$\begin{aligned}
d_2 &= d_1 - \sigma_{av}\sqrt{T} \\
&= -0.0012065235
\end{aligned}$$

$$\begin{aligned}
P_0 &= KP(0, T)(1 - N(d_2)) - P(0, S)(1 - N(d_1)) \\
&= 0.9882643607(0.9638350801)(1 - N(-0.0012065235)) \\
&\quad - 0.9527023988(1 - N(0.0193983885)) \\
&= 0.0077415580,
\end{aligned}$$

and thus the value of the caplet is

$$\begin{aligned} Cpl &= (1 + K_{Caplet}\tau)P_0 \\ &= (1 + 0.0475(0.25))0.0077415580 \\ &= 0.0078334890. \end{aligned}$$

Chapter 8

Ho-Lee Model

Ho and Lee (1986) proposed the first no-arbitrage model of the term structure. The model was presented in the form of a binomial tree of bond prices and consisted of two parameters - the standard deviation of the short rate and the market price of risk of the short rate. It has since been shown that the continuous time limit of the Ho-Lee model is:

$$dr_t = \phi(t)dt + \sigma dW_t, \quad (8.1)$$

where σ, r_0 are positive constants and $\phi(t)$ is a deterministic function.

Integrating equation 8.1, one can solve for r_t :

$$r_t = r_0 + \int_0^t \phi(s)ds + \sigma W_t.$$

8.1 Pure Discount Bond

$dr_t = \phi(t)dt + \sigma dW_t$ thus, in terms of the parameters for the Generalized Affine Term Structure Model, $\alpha = 0, \beta = \phi(t), \gamma = 0$ and $\delta = \sigma^2$, with σ constant and $\phi(t)$ deterministic.

The system of differential equations that must be solved is therefore:

$$\begin{cases} A_t & = \phi(t)B - \frac{1}{2}\sigma^2 B^2 \\ A(T, T) & = 0 \end{cases}$$

$$\begin{cases} B_t & = -1 \\ B(T, T) & = 0 \end{cases}$$

One can easily solve for $B(t, T)$:

$$\begin{aligned}
 &\Rightarrow B_t + 1 = 0, \\
 &\Rightarrow B + t + C = 0, \\
 \text{but } &B(T, T) = -(C + T) = 0, \\
 &\therefore C = -T, \\
 &\therefore B(t, T) = T - t.
 \end{aligned} \tag{8.2}$$

Substituting $B(t, T)$ into A_t , one can solve for $A(t, T)$:

$$\begin{aligned}
 \Rightarrow A_t &= \phi(t)(T - t) - \frac{\sigma^2(T - t)^2}{2}, \\
 \Rightarrow A(t, T) &= A(T, T) - \int_t^T A_s ds \\
 &= - \int_t^T [\phi(s)(T - s)] ds + \int_t^T \left[\frac{\sigma^2(T - s)^2}{2} \right] ds \\
 &= - \int_t^T [\phi(s)(T - s)] ds + \frac{\sigma^2(T - t)^3}{6}.
 \end{aligned} \tag{8.3}$$

Thus, the price of a pure discount bond, $P(t, T) = e^{A(t, T) - B(t, T)r_t}$, has been found for the Ho-Lee model, with the values for $A(t, T)$ and $B(t, T)$ represented by 8.3 and 8.2 respectively.

Since $\phi(t)$ is a function of time and not a constant, one is able to choose $\phi(t)$ such that the model fits the initial term structure.

$$\begin{aligned}
 -\ln P(0, T) &= -A(0, T) + B(0, T)r_0 \\
 &= \int_0^T [\phi(s)(T - s)] ds - \frac{\sigma^2(T - t)^3}{6} + Tr_0, \\
 \text{but} \\
 f(t, T) &= -\frac{\partial P(t, T)}{\partial T}, \\
 \therefore f(0, T) &= \phi(T)(T - T) + \int_0^T \phi(s) ds - \frac{\sigma^2 T^2}{2} + r_0 \\
 &= \int_0^T \phi(s) ds - \frac{\sigma^2 T^2}{2} + r_0, \\
 \Rightarrow f_T(0, T) &= \phi(T) - \sigma^2 T, \\
 \Rightarrow \phi(T) &= f_T(0, T) + \sigma^2 T,
 \end{aligned}$$

i.e.,:

$$\phi(t) = f_T(0, t) + \sigma^2 t, \quad \text{where} \quad F_T(0, t) = \left. \frac{\partial f(0, T)}{\partial T} \right|_{T=t}, \quad (8.4)$$

and $f^*(0, t)$ is the instantaneous forward price for a maturity t observed at time zero.

Substituting 8.4 into 8.3, one can see that $A(t, T)$ is a function of: the instantaneous forward rate as seen at time zero; observed bond prices at time zero; volatility of the short rate; current time t ; and time to maturity. i.e.,:

$$\begin{aligned} A(t, T) &= - \int_t^T [\phi(s)(T-s)] ds + \frac{\sigma^2(T-t)^3}{6} \\ &= - \int_t^T [f_T^*(0, s)(T-s)] ds - \int_t^T [\sigma^2 s(T-s)] ds + \frac{\sigma^2(T-t)^3}{6} \\ &= - [f^*(0, s)(T-s)] \Big|_t^T - \int_t^T f^*(0, s) ds - \frac{\sigma^2 t(T-t)^2}{2} \\ &= f^*(0, t)(T-t) - \int_t^T \left[-\frac{\partial P^*(0, s)}{\partial T} ds \right] - \frac{\sigma^2 t(T-t)^2}{2} \\ &= f^*(0, t)(T-t) + \ln P^*(0, T) - \ln P^*(0, t) - \frac{\sigma^2 t(T-t)^2}{2}, \end{aligned}$$

where $P^*(0, t)$ is the pure discount bond price with maturity t observed at time zero.

Thus

$$P(t, T) = \frac{P(0, T)}{P(0, t)} e^{f^*(0, t)(T-t) - \frac{\sigma^2 t(T-t)^2}{2} - (T-t)r_t}.$$

8.2 Short Rate Dynamics Revisited

$$\begin{aligned} r_t &= r_0 + \int_0^t \phi(s) ds + \sigma W_t \\ &= r_0 + f^*(0, t) + \frac{\sigma^2 t^2}{2} - r_0 + \sigma W_t \\ &= f^*(0, t) + \frac{\sigma^2 t^2}{2} + \sigma W_t. \end{aligned}$$

Thus r_t , conditional on F_s , is distributed normally with the following moments:

$$\begin{aligned}\mathbb{E}(r_t|F_s) &= f^*(0, t) + \frac{\sigma^2 t^2}{2}, \\ \text{Var}(r_t|F_s) &= \sigma^2 t.\end{aligned}$$

The long-term mean and variance can be found by taking the limit as t tends towards infinity:

$$\begin{aligned}\lim_{t \rightarrow \infty} \mathbb{E}(r_t|F_s) &\rightarrow \infty, \\ \lim_{t \rightarrow \infty} \text{Var}(r_t|F_s) &\rightarrow \infty.\end{aligned}$$

One can thus clearly see several key attributes of the Ho-Lee model:

1. The short rate does not experience mean reversion, and over the long-term will tend towards ∞ (under the risk-neutral measure). Such a feature is highly unrealistic.
2. There is no certainty that the short rate will always be positive. In reality, nominal interest rates can never be negative, however the Ho-Lee model can not ensure that such a situation does not occur. This is a major drawback of the model.

8.3 Cap and Floors

In order to price a caplet, one must first find the value of a European put option on a pure discount bond and then use the relationship between the put option and the caplet.

Since σ is constant and $B(t, T) = T - t$,

$$\begin{aligned}\sigma_{av}^2 T &= \int_0^T \sigma^2 (T - S)^2 dt \\ &= \sigma^2 (T - S)^2 T.\end{aligned}$$

And the price of the put option is

$$P_0 = KP(0, T)(1 - N(d_2)) - P(0, S)(1 - N(d_1)),$$

where

$$d_1 = \frac{\ln \frac{P(0, S)}{KP(0, T)} + \frac{1}{2} \sigma^2 (S - T)^2 T}{\sigma (S - T) \sqrt{T}}, \quad d_2 = d_1 - \sigma (S - T) \sqrt{T}.$$

Example 1. *If the short rate is assumed to possess the following risk-neutral dynamics*

$$dr_t = \phi(t)dt + \sigma dW_t,$$

with $\phi = 0.01$ and $\sigma = 0.1$ and a caplet has the payoff $0.25(L - 0.0475)^+$ at expiry in 1 year, where L is the three-month spot LIBOR rate in nine months' time, r_0 is 5% and $P(0, T)$ is a pure discount bond with face value 1 then

$$\begin{aligned} K &= \frac{1}{(1 + K_{\text{Caplet}}\tau)} \\ &= \frac{1}{(1 + 0.0475(0.25))} \\ &= 0.9882643607, \\ \sigma_{av} &= \sigma(S - T) \\ &= 0.1(0.25) \\ &= 0.025, \\ B(0, T) &= T \\ &= 0.75 \\ A(0, T) &= -\frac{\phi T^2}{2} + \frac{\sigma^2 T^3}{6} \\ &= -\frac{0.01(0.75)^2}{2} + \frac{0.1^2(0.75)^3}{6} \\ &= -0.002109375 \\ P(0, T) &= e^{A(0, T) - B(0, T)r_0} \\ &= 0.9611648208 \\ B(0, S) &= S \\ &= 1 \\ A(0, S) &= -\frac{\phi S^2}{2} + \frac{\sigma^2 S^3}{6} \\ &= -\frac{0.01(1)^2}{2} + \frac{0.1^2(1)^3}{6} \\ &= -0.0033333333 \end{aligned} \tag{8.5}$$

$$\begin{aligned}P(0, S) &= e^{A(0, S) - B(0, S)r_0} \\ &= 0.9480639385\end{aligned}$$

$$\begin{aligned}P_0 &= KP(0, T)(1 - N(d_2)) - P(0, S)(1 - N(d_1)) \\ &= 0.9882643607(0.9611648208)(1 - N(-0.0994561073)) \\ &\quad - 0.9480639385(1 - N(-0.0778054722)) \\ &= 0.0108437605,\end{aligned}$$

and thus the value of the caplet is

$$\begin{aligned}Cpl &= (1 + K_{Caplet}\tau)P_0 \\ &= (1 + 0.0475(0.25))0.0108437605 \\ &= 0.0109725302.\end{aligned}$$

University of Cape Town

Chapter 9

Cox-Ingersoll-Ross Model

One of the major disadvantages with the Vasicek model is that the model allows the short rate to be negative. Cox, Ingersoll and Ross proposed an alternative model in 1985 whereby the short rate is always positive:

$$dr_t = k(\phi - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad (9.1)$$

where k, ϕ, σ, r_0 are positive constants.

The standard deviation of the change in the short rate over an instantaneous period is proportional to the square root of the short rate. Thus, as the short rate decreases towards zero, its standard deviation decreases.

Integrating equation 9.1, one can solve for r_t :

$$\begin{aligned} (dr_t)e^{kt} &= k\phi e^{kt}dt - r_t k e^{kt}dt + e^{kt}\sigma\sqrt{r_t}dW_t, \\ d(r_t e^{kt}) &= k\phi e^{kt} + e^{kt}\sigma\sqrt{r_t}dW_t, \\ r_t e^{kt} &= r_s e^{ks} + k\phi \int_s^t e^{k\tau}d\tau + \sigma \int_s^t e^{k\tau}\sqrt{r_\tau}dW_\tau, \\ r_t e^{kt} &= r_s e^{ks} + \phi(e^{kt} - e^{ks}) + \sigma \int_s^t e^{k\tau}\sqrt{r_\tau}dW_\tau, \\ r_t &= r_s e^{-k(t-s)} + \phi(1 - e^{-k(t-s)}) + \sigma \int_s^t e^{-k(t-\tau)}\sqrt{r_\tau}dW_\tau. \end{aligned} \quad (9.2)$$

Although it is straightforward to compute the expectation of the short rate, the variance requires some manipulation:

$$\mathbb{E}(r_t|F_s) = r_s e^{-k(t-s)} + \phi(1 - e^{-k(t-s)}),$$

Let $X_t = e^{kt}r_t$ and substitute this into 9.2:

$$\begin{aligned}
dX_t &= k\phi e^{kt} + e^{kt}\sigma\sqrt{r_t}dW_t \\
&= k\phi e^{kt} + e^{\frac{kt}{2}}\sigma\sqrt{X_t}dW_t, \\
d(X_t^2) &= 2X_t dX_t + (dX_t)^2 \\
&= 2X_t k\phi e^{kt} + 2X_t^{\frac{3}{2}}e^{\frac{kt}{2}}\sigma dW_t + e^{kt}\sigma^2 X_t dt, \\
\therefore X_t^2 &= X_s^2 + (2k\phi + \sigma^2) \int_s^t e^{k\tau} X_\tau d\tau + 2\sigma \int_s^t e^{\frac{k\tau}{2}} X_\tau^{\frac{3}{2}} dW_\tau, \\
\therefore \mathbb{E}(X_t^2|F_s) &= X_s^2 + (2k\phi + \sigma^2) \int_s^t e^{k\tau} \mathbb{E}(X_\tau) d\tau \\
&= r_s^2 e^{2ks} + (2k\phi + \sigma^2) \int_s^t e^{k\tau} (r_s e^{ks} + \phi(e^{k\tau} - e^{ks})) d\tau \\
&= r_s^2 e^{2ks} + \frac{(2k\phi + \sigma^2)}{k} (r_s - \phi)(e^{k(t+s)} - e^{2ks}) + \frac{(2k\phi + \sigma^2)}{2k} \phi(e^{2kt} - e^{2ks}), \\
\therefore \mathbb{E}(r_t^2|F_s) &= r_s^2 e^{-2k(t-s)} + \frac{(2k\phi + \sigma^2)}{k} (r_s - \phi)(e^{-k(t-s)} - e^{-2k(t-s)}) \\
&\quad + \frac{(2k\phi + \sigma^2)}{2k} \phi(1 - e^{-2k(t-s)}), \\
\therefore \text{Var}(r_t|F_s) &= r_s^2 e^{-2k(t-s)} + \frac{(2k\phi + \sigma^2)}{k} (r_s - \phi)(e^{-k(t-s)} - e^{-2k(t-s)}) \\
&\quad + \frac{(2k\phi + \sigma^2)}{2k} \phi(1 - e^{-2k(t-s)}) - (r_s e^{-k(t-s)} + \phi(1 - e^{-k(t-s)}))^2 \\
&= r_s^2 e^{-2k(t-s)} + \frac{(2k\phi + \sigma^2)}{k} (r_s - \phi)(e^{-k(t-s)} - e^{-2k(t-s)}) \\
&\quad + \frac{(2k\phi + \sigma^2)}{2k} \phi(1 - e^{-2k(t-s)}) - r_s^2 e^{-2k(t-s)} \\
&\quad - 2r_s \phi(e^{-k(t-s)} - e^{-2k(t-s)}) - \phi^2(1 - e^{-2k(t-s)})^2 \\
&= \frac{r_s \sigma^2}{k} (e^{-k(t-s)} - e^{-2k(t-s)}) + \frac{\phi \sigma^2}{2k} (1 - 2e^{-k(t-s)} + e^{-2k(t-s)}).
\end{aligned}$$

The long-term mean and variance can be found by taking the limit as t tends towards infinity:

$$\begin{aligned}
\lim_{t \rightarrow \infty} \mathbb{E}(r_t|F_s) &= \phi, \\
\lim_{t \rightarrow \infty} \text{Var}(r_t|F_s) &= \frac{\phi \sigma^2}{2k}.
\end{aligned}$$

Similar to the Vaiscek model, the short rate experiences mean reversion, and over the long-term will tend towards ϕ . The Cox-Ingersoll-Ross model however does not follow the normal distribution. Svoboda (2002) shows r_t , conditional on F_s , instead follows a non-central chi-squared distribution. Svoboda (2002) also shows that, provided the initial short rate is positive

and $2k\phi \geq \sigma^2$, then the short rate is assured of always remaining positive - a highly preferable feature of any short rate model.

9.1 Pure Discount Bond

$dr_t = (k\phi - kr_t)dt + \sigma\sqrt{r_t}dW_t$ thus, in terms of the parameters for the Generalized Affine Term Structure Model, $\alpha = -k, \beta = k\phi, \gamma = \sigma^2$ and $\delta = 0$, with all parameters constant.

The system of differential equations that must be solved is therefore:

$$\begin{cases} A_t & = k\phi B \\ A(T, T) & = 0 \end{cases}$$

$$\begin{cases} B_t & = kB + \frac{1}{2}\sigma^2 B^2 - 1 \\ B(T, T) & = 0 \end{cases}$$

Ouwehand (2008) shows that, in order to solve the Riccati equation for $B(t, T)$, one can try a solution of the form:

$$B(t, T) = \frac{X(t)}{cX(t) + d}$$

$$\therefore B_t = \frac{X_t}{cX(t) + d} - \frac{cX(t)X_t}{(cX + d)^2}$$

Substituting back into the equation for $B(t, T)$:

$$\frac{X_t}{cX(t) + d} - \frac{cX(t)X_t}{(cX + d)^2} = \frac{kX(t)}{cX(t) + d} + \frac{1}{2}\sigma^2 \left(\frac{X(t)}{cX(t) + d} \right)^2 - 1$$

$$\therefore X_t(cX(t) + d) - cX(t)X_t = kX(t)(cX(t) + d) + \frac{1}{2}\sigma^2 X^2(t) - (cX(t) + d)^2$$

$$\therefore 0 = -dX_t + X^2(t)(kc + \frac{1}{2}\sigma^2 - c^2) + X(t)(kd - 2cd) - d^2.$$

If one chooses c such that $kc + \frac{1}{2}\sigma^2 - c^2 = 0$ (i.e., $c = \frac{k + \sqrt{k^2 + 2\sigma^2}}{2}$ or $\frac{k - \sqrt{k^2 + 2\sigma^2}}{2}$) and sets $\kappa = 2c - k = \sqrt{k^2 + 2\sigma^2}$ then the differential equation reduces to the following:

$$\frac{X_t}{\kappa X(t) + d} = -1,$$

$$\Rightarrow \frac{1}{\kappa} \ln(\kappa X(t) + d) = -t + C,$$

$$\Rightarrow \kappa X(t) + d = C^* e^{-\kappa t},$$

$$\Rightarrow X(t) = \frac{-d}{\kappa} + C^{**} e^{-\kappa t},$$

but $B(T, T) = 0$, allowing one to solve for C^{**}

$$\therefore X(T) = \frac{-d}{\kappa} + C^{**} e^{-\kappa T} = 0, \quad (9.3)$$

$$\therefore C^{**} = \frac{d}{\kappa} e^{\kappa T}. \quad (9.4)$$

Substituting 9.4 back into 9.3 allows one to solve for $X(t)$ and thus $B(t, T)$

$$\begin{aligned} \therefore X(t) &= \frac{d}{\kappa} \left(e^{\kappa(T-t)} - 1 \right), \\ \therefore B(t, T) &= \frac{\frac{d}{\kappa} (e^{\kappa(T-t)} - 1)}{\frac{cd}{\kappa} (e^{\kappa(T-t)} - 1) + d} \\ &= \frac{e^{\kappa(T-t)} - 1}{c(e^{\kappa(T-t)} - 1) + \kappa} \\ &= \frac{e^{\kappa(T-t)} - 1}{\frac{1}{2}(\kappa + k)(e^{\kappa(T-t)} - 1) + \kappa} \\ &= \frac{2(e^{\kappa(T-t)} - 1)}{(\kappa + k)(e^{\kappa(T-t)} - 1) + 2\kappa}. \end{aligned} \quad (9.5)$$

Substituting $B(t, T)$ into A_t , one can solve for $A(t, T)$:

$$\begin{aligned} \Rightarrow A_t &= k\phi B, \\ \Rightarrow A(t, T) &= A(T, T) - \int_t^T A_s ds \\ &= - \int_t^T k\phi B(s, T) ds. \end{aligned}$$

Svoboda (2002) solves for $A(t, T)$ to find:

$$A(t, T) = \frac{2k\phi}{\sigma^2} \ln \left[\frac{2\kappa e^{\frac{1}{2}(k+\kappa)(T-t)}}{2\kappa + (k + \kappa)(e^{\kappa(T-t)} - 1)} \right]. \quad (9.6)$$

Thus, the price of a pure discount bond, $P(t, T) = e^{A(t, T) - B(t, T)r_t}$, has been found for the Cox-Ingersoll-Ross model, with the values for $A(t, T)$ and $B(t, T)$ represented by 9.6 and 9.5 respectively.

9.2 Cap and Floors

In order to price a caplet, one must first find the value of a European put option on a pure discount bond and then use the relationship between the put option and the caplet. Since the standard deviation of the change in the short rate is not independent of the short rate, Theorem 6.1.1 can not be used to find the value of this option.

Cox, Ingersoll and Ross (1985) show that the value of a call option on a pure discount bond is:

$$C_t = P(t, S)\chi^2\left(2\bar{r}[\rho + \psi + B(T, S)]; \frac{4k\phi}{\sigma^2}; \frac{2\rho^2 r_t e^{\kappa(T-t)}}{\rho + \psi + B(T, S)}\right) - KP(t, T)\chi^2\left(2\bar{r}[\rho + \psi]; \frac{4k\phi}{\sigma^2}; \frac{2\rho^2 r_t e^{\kappa(T-t)}}{\rho + \psi}\right),$$

where

$$\begin{aligned}\bar{r} &= \frac{\ln\left(\frac{A(T, S)}{K}\right)}{B(T, S)}, \\ \psi &= \frac{k + \kappa}{\sigma^2}, \\ \rho &= \frac{2\kappa}{\sigma^2(e^{\kappa(T-t)} - 1)},\end{aligned}$$

and $\chi^2(\cdot; v; \lambda)$ is the non-central chi-squared distribution with v degrees of freedom and non-central parameter λ .

The value of the corresponding put option can be found through Put-Call Parity

Example 1. *If the short rate is assumed to possess the following risk-neutral dynamics*

$$dr_t = k(\phi - r_t)dt + \sigma\sqrt{r_t}dW_t,$$

with $k = 0.1$, $\phi = 0.05$ and $\sigma = 0.1$ and a caplet has the payoff $0.25(L - 0.0475)^+$ at expiry in 1 year, where L is the three-month spot LIBOR rate in nine months' time, r_0 is 5% and $P(0, T)$ is a pure discount bond with face value 1 then

$$C_0 = P(0, S)\chi^2\left(2\bar{r}[\rho + \psi + B(T, S)]; \frac{4k\phi}{\sigma^2}; \frac{2\rho^2 r_0 e^{\kappa(T)}}{\rho + \psi + B(T, S)}\right) - KP(0, T)\chi^2\left(2\bar{r}[\rho + \psi]; \frac{4k\phi}{\sigma^2}; \frac{2\rho^2 r_0 e^{\kappa(T)}}{\rho + \psi}\right)$$

$$\begin{aligned}
&= (0.9513028793)\chi^2 \left(2(0.0471901756)[249.7210531646 + 27.3205080757 + 0.2468754833]; \right. \\
&\quad \left. \frac{4(0.1)(0.05)}{0.1^2}; \frac{2(249.7210531646)^2(0.05)e^{0.1732050808(0.75)}}{249.7210531646 + 27.3205080757 + 0.2468754833} \right) \\
&\quad - 0.9882643607(0.9632264061)\chi^2 \left(2(0.0471901756)[249.7210531646 + 27.3205080757]; \right. \\
&\quad \left. \frac{4(0.1)(0.05)}{0.1^2}; \frac{2(249.7210531646)^2(0.05)e^{0.1732050808(0.75)}}{249.7210531646 + 27.3205080757} \right) \\
&= 0.0014204286,
\end{aligned}$$

$$\begin{aligned}
P_0 &= C_0 - P(0, S) + KP(0, T) \\
&= 0.0014204286 - 0.9513028793 + 0.9882643607(0.9632264061) \\
&= 0.0020398777,
\end{aligned}$$

and thus the value of the caplet is

$$\begin{aligned}
Cpl &= (1 + K_{Caplet}\tau)P_0 \\
&= (1 + 0.0475(0.25))0.0020398777 \\
&= 0.0020641012.
\end{aligned}$$

Chapter 10

Black-Karasinski Model

Black and Karasinski (1991) proposed a short rate model that guaranteed positive interest rates by assuming that the instantaneous change in the short rate is the exponential of an Ornstein-Uhlenbeck process. i.e.,:

$$d \ln(r_t) = [\phi(t) - a(t) \ln(r_t)]dt + \sigma(t)dW_t,$$

where r_0 is a positive constant and $\phi(t), a(t), \sigma(t)$ are deterministic functions. For the remainder of this chapter, we will assume that $a(t) = a, \sigma(t) = \sigma$ (i.e., these two parameters are constant). Thus

$$d \ln(r_t) = [\phi(t) - a \ln(r_t)]dt + \sigma dW_t. \quad (10.1)$$

Integrating equation 10.1, one can solve for r_t :

$$\begin{aligned} d(\ln(r_t))e^{at} &= \phi(t)e^{at}dt - \ln(r_t)ae^{at}dt + \sigma e^{at}dW_t, \\ d(\ln(r_t)e^{at}) &= \phi(t)e^{at} + \sigma e^{at}dW_t, \\ \ln(r_t)e^{at} &= \ln(r_s)e^{as} + \int_s^t \phi(\tau)e^{a\tau}d\tau + \sigma \int_s^t e^{a\tau}dW_\tau, \\ \ln(r_t) &= \ln(r_s)e^{-a(t-s)} + \int_s^t \phi(\tau)e^{-a(t-\tau)}d\tau + \sigma \int_s^t e^{-a(t-\tau)}dW_\tau, \\ r_t &= e^{\ln(r_s)e^{-a(t-s)} + \int_s^t \phi(\tau)e^{-a(t-\tau)}d\tau + \sigma \int_s^t e^{-a(t-\tau)}dW_\tau}. \end{aligned} \quad (10.2)$$

Let $\ln(r_t) = X$. Thus X , conditional on F_s , is distributed normally with the following moments:

$$\begin{aligned} \mathbb{E}(X|F_s) &= \ln(r_s)e^{-a(t-s)} + \int_s^t \phi(\tau)e^{-a(t-\tau)}d\tau := M, \\ \text{Var}(X|F_s) &= \frac{\sigma^2}{2a} [1 - e^{-2a(t-s)}] := V^2. \end{aligned}$$

If $Y = e^X$, the mean and variance of Y can be found as follows:

$$\begin{aligned}
\mathbb{E}(Y) &= \mathbb{E}(e^X) \\
&= \int_{-\infty}^{\infty} e^x \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{(x-M)^2}{2V^2}} dx \\
&= \int_{-\infty}^{\infty} e^{M+z} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{z^2}{2V^2}} dz \\
&= e^M \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{z^2 - 2zV^2 + V^2}{2V^2}} dz \\
&= e^M \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{(z-V^2)^2}{2V^2} + \frac{V^2}{2}} dz \\
&= e^{M + \frac{V^2}{2}}, \\
\mathbb{E}(Y^2) &= \mathbb{E}(e^{2X}) \\
&= \int_{-\infty}^{\infty} e^{2x} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{(x-M)^2}{2V^2}} dx \\
&= \int_{-\infty}^{\infty} e^{2(M+z)} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{z^2}{2V^2}} dz \\
&= e^{2M} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{4zV^2 - z^2}{2V^2}} dz \\
&= e^{2M} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{(z-2V^2)^2}{2V^2} + 2V^2} dz \\
&= e^{2M + 2V^2}, \\
\text{Var}(Y) &= \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 \\
&= e^{2M + 2V^2} (e^{V^2} - 1).
\end{aligned}$$

Since $r_t = Y$, r_t , conditional on F_s , is distributed lognormally with the following moments:

$$\begin{aligned}
\mathbb{E}(r_t | F_s) &= e^{\ln(r_s)e^{-a(t-s)} + \int_s^t \phi(\tau)e^{-a(t-\tau)} d\tau + \frac{\sigma^2}{4a} [1 - e^{-2a(t-s)}]}, \\
\text{Var}(r_t | F_s) &= e^{2\ln(r_s)e^{-a(t-s)} + 2\int_s^t \phi(\tau)e^{-a(t-\tau)} d\tau + \frac{\sigma^2}{2a} [1 - e^{-2a(t-s)}]} \left(e^{\frac{\sigma^2}{2a} [1 - e^{-2a(t-s)}]} - 1 \right).
\end{aligned}$$

10.1 Simplifying the Short Rate Equation

One is able to break-down equation 10.2 into a deterministic function and a stochastic equation. Define r_t^* and $\alpha(t)$ as follows:

$$r_t^* = \ln r_t - \alpha(t), \quad (10.3)$$

$$\alpha(t) = \ln(r_0)e^{-at} + \int_0^t \phi(\tau)e^{-a(t-\tau)} d\tau. \quad (10.4)$$

Thus r_t^* can be found as follows:

$$\begin{aligned}
r_t^* &= \ln(r_t) - \alpha(t) \\
&= \ln(r_s)e^{-a(t-s)} + \int_s^t \phi(\tau)e^{-a(t-\tau)}d\tau + \sigma \int_s^t e^{-a(t-\tau)}dW_\tau - \alpha(t) \\
&= \alpha(s)e^{-a(t-s)} + r_s^*e^{-a(t-s)} + \int_s^t \phi(\tau)e^{-a(t-\tau)}d\tau + \sigma \int_s^t e^{-a(t-\tau)}dW_\tau - \alpha(t) \\
&= \ln(r_0)e^{-at} + \int_0^s \phi(\tau)e^{-a(t-\tau)}d\tau + \int_s^t \phi(\tau)e^{-a(t-\tau)}d\tau - \alpha(t) \\
&\quad + r_s^*e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-\tau)}dW_\tau \\
&= \ln(r_0)e^{-at} + \int_0^t \phi(\tau)e^{-a(t-\tau)}d\tau + r_s^*e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-\tau)}dW_\tau - \alpha(t) \\
&= r_s^*e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-\tau)}dW_\tau. \tag{10.5}
\end{aligned}$$

One can easily to show that $dr_t = -ar_t^*dt + \sigma dW_t$ is a solution to equation 10.5. i.e.,

$$\begin{aligned}
d(r_t^*)e^{at} &= -ae^{at}r_t^* + \sigma e^{at}dW_t \\
d(r_t^*e^{at}) &= \sigma e^{at}dW_t \\
r_t^*e^{at} &= r_s^*e^{as} + \sigma \int_s^t e^{a\tau}dW_\tau \\
r_t^* &= r_s^*e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-\tau)}dW_\tau, \tag{10.6}
\end{aligned}$$

Thus

$$\ln r_t = r_t^* + \alpha(t),$$

where

$$\begin{aligned}
\alpha(t) &= \ln(r_0)e^{-at} + \int_0^t \phi(\tau)e^{-a(t-\tau)}d\tau, \\
dr_t &= -ar_t^*dt + \sigma dW_t.
\end{aligned}$$

10.2 Moments of the Short Rate Revisited

$$\begin{aligned}
\mathbb{E}(r_t|F_s) &= e^{\ln(r_s)e^{-a(t-s)} + \int_s^t \phi(\tau)e^{-a(t-\tau)}d\tau + \frac{\sigma^2}{4a}[1-e^{-2a(t-s)}]}, \\
\therefore \lim_{t \rightarrow \infty} \mathbb{E}(r_t|F_s) &= \exp\left(\lim_{t \rightarrow \infty} \alpha(t) + \frac{\sigma^2}{4a}\right).
\end{aligned}$$

10.3 Pure Discount Bond

In contrast to the short rate models previously reviewed, the Black-Karasinski model does not provide an analytically tractable solution for the bond price. Thus, one is required to use methods such as constructing a tree or Monte-Carlo simulation in order to price pure discount bonds.

A tree is a discrete-time representation of the stochastic process whereby one assumes that the interest rate for each discrete time-period has the same dynamics as that of the short rate. As Hull (2006) points out, one usually constructs a trinomial tree, as opposed to a binomial tree, due to the “extra degree of freedom.”

When constructing a typical trinomial tree, the interest rate is allowed to either increase, decrease or stay constant over each discrete time-period (Figure 10.1a). However, as discussed previously, interest rates have been shown to experience mean-reversion. Thus, Hull et al (1993) and Hull et al (1994a) proposed the use of a non-standard branching tree. Under such a method, if the interest rate is exceptionally low then the rate can either stay constant, increase or increase significantly (Figure 10.1b). Similarly, if the interest rate is significantly high then the rate can either stay constant, decrease or decrease significantly (Figure 10.1c). i.e., The non-standard branching approach incorporates a form of mean-reversion. For the remainder of this chapter, we will use a non-standard branching tree.

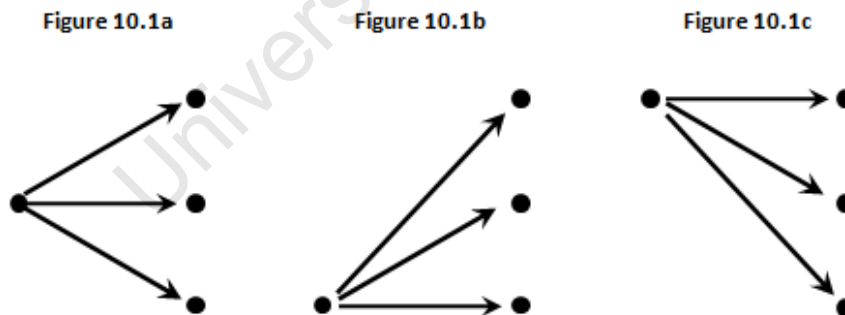


Figure 10.1: Non-Standard Branching Trinomial Tree

When constructing the tree, one needs to fix the time horizon T and set the discrete time period $\Delta t_i = t_{i+1} - t_i$. Note that this time period is time dependent thus it need not be constant – an important feature when confronted with the practicality of day counts.

In terms of notation, i is used to denote the time period (i.e., $i = 0, 1, 2, \dots, N$) whilst j informs the height of the node.

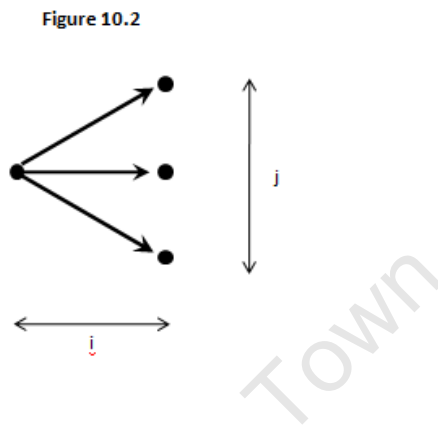


Figure 10.2: Black-Karasinski: Tree Notation

As explained by Brigo and Mercurio (2006) and Hull (2006), the construction of the trinomial tree consists of a two-step process. Firstly, one needs to construct a tree for a process r^* . The risk-neutral dynamics of the continuous version of this process are:

$$dr_t^* = -ar_t^*dt + \sigma(t)dW_t,$$

where

$$r_t^* = \ln r_t - \alpha(t),$$

where $r_0^* = 0$. The second step involves displacing r_t^* such that pure discount bonds priced through the trinomial tree exactly match the market prices of these bonds. $\alpha(t)$, as defined in 10.4, is used to displace the pure discount bond prices, with r_t determined using equation 10.3. i.e.,

$$r_t = e^{\alpha(t)+r_t^*}.$$

Under the first step, one requires both the expected change in r_t^* and variance of the change in r_t^* :

$$\begin{aligned} \mathbb{E}(r_{i+1}^* | r_i^* = r_{i,j}^*) &= r_{i,j}^* e^{-a\Delta t_i} =: M_{i,j}, \\ \text{Var}(r_{i+1}^* | r_i^* = r_{i,j}^*) &= \frac{\sigma^2}{2a} (1 - e^{-2a\Delta t_i}) =: V_{i,j}^2. \end{aligned}$$

Δr_i^* is the space between interest rates at a discrete time period i with $r_{i,j}^* = j\Delta r_i^*$. Brigo and Mercurio (2006) recommends one set $\Delta r_i^* = V_{i-1}\sqrt{3}$

with Hull (2006) noting that such a choice proves to be good in terms of error minimization.

At each discrete time period, $r_{i,j}^*$ can proceed to $r_{i+1,k+1}^*$, $r_{i+1,k}^*$ or $r_{i+1,k-1}^*$ with probabilities p_u , p_m and p_d respectively. Since the middle node is $r_{i+1,k}^*$, one chooses k such that this node is as close as possible to $M_{i,j}$, as defined in 10.6. Thus, k is set as

$$k = \text{round} \left(\frac{M_{i,j}}{\Delta r_{i+1}^*} \right).$$

The probabilities for the possible paths at each point in time are chosen to match the expected change in r_t^* and variance of the change in r_t^* over the next time period Δt_i . Since these probabilities must sum to unity, one is able to solve the three simultaneous equations. One should note that, due to the probabilities being chosen according to the values of $M_{i,j}$ and $V_{i,j}$, these probabilities will be dependent on the form of non-standard branching.

$$\begin{cases} p_u = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} + \frac{\eta_{j,k}}{2\sqrt{3}V_i} \\ p_m = \frac{2}{3} - \frac{\eta_{j,k}^2}{3V_i^2} \\ p_d = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} - \frac{\eta_{j,k}}{2\sqrt{3}V_i} \end{cases}$$

where

$$\eta_{j,k} = M_{i,j} - x_{i+1,k}.$$

As Brigo and Mercurio (2006) point out, these probabilities are guaranteed to be non-negative.

Thus, one has a fully determined trinomial tree for r_t^* , including the minimum and maximum values for the height of the nodes $j_{min}(i)$ and $j_{max}(i)$ respectively.

The second step of the process involves shifting the nodes of the tree at each discrete time period so that the theoretical pure discount bond prices match the observed pure discount bond prices, through the use of $\alpha(t)$.

Let $Q_{i,j}$ denote the present value of the option which pays R1 if, and only if, the node (i, j) is reached and zero otherwise. Let $P^*(0, t)$ be the price observed in the market of a pure discount bond which pays R1 at time t . $q(h, j)$ is defined as the probability of moving from node (i, h) to node

$(i + 1, j)$.

$$\begin{aligned} P^*(0, t_1) &= e^{-r_{0,j}\Delta t_0} \\ &= e^{-e^{(\alpha(0)+r_0^*)\Delta t_0}} \\ &= e^{-e^{(\alpha(0))\Delta t_0}}, \\ \therefore \alpha(0) &= \ln\left(\frac{-\ln(P^*(0, t_1))}{\Delta t_0}\right). \end{aligned}$$

A recursive relationship between $Q_{i+1,j}$ and $\alpha(i)$ follows. One can use the following generalized formula to solve for $Q_{i,j}$:

$$Q_{i+1,j} = \sum_h Q_{i,h} q(h, j) e^{(-e^{(\alpha(i)+h\Delta r_i^*)\Delta t_i})}.$$

And then use the derived values for $Q_{i+1,j}$ to solve for $\alpha(i)$ where $\alpha(i)$ is the value which sets the following equation equal to zero:

$$p^*(0, t_{i+1}) - \sum_{j=j_{min}(i)}^{j_{max}(i)} Q_{i,j} e^{(-e^{(\alpha(i)+j\Delta r_i^*)\Delta t_i})}.$$

One can use the Newton-Raphson method to solve for such a value. In such an instance, one requires the derivative of the function. i.e.,

$$\sum_{j=j_{min}(i)}^{j_{max}(i)} Q_{i,j} e^{(-e^{(\alpha(i)+j\Delta r_i^*)\Delta t_i})} e^{(\alpha(i)+j\Delta r_i^*)\Delta t_i}.$$

Once one has solved for $\alpha(i)$ for all discrete time periods, one can use the values derived in step one for $r_{i,j}^*$ to solve for $r_{i,j}$ through the use of the discrete form of equation 10.5:

$$r_{i,j} = e^{\alpha(i)+r_{i,j}^*}.$$

10.4 Cap and Floors

In order to price a caplet, one should construct the trinomial tree as described previously. From the tree one can determine the payoff for each path and multiply this payoff by the probability of taking that path. The value of the caplet is then the sum across all possible pathways of the present value of this number.

Example 1. *If the short rate is assumed to possess the following risk-neutral dynamics*

$$d \ln(r_t) = [\phi(t) - a(t) \ln(r_t)]dt + \sigma(t)dW_t,$$

with $a = 0.1$, $\sigma = 0.1$ and $\alpha(t) = -2.995732 \forall t$ where $\alpha(t)$ is defined as:

$$\alpha(t) = \ln(r_0)e^{-at} + \int_0^t \phi(\tau)e^{-a(t-\tau)} d\tau,$$

and a caplet has the payoff $0.25(L - 0.06)^+$ at expiry in 1 year, where L is the three-month spot LIBOR rate in nine months' time, r_0 is 5%, $\Delta t_i =$ one month and $P(0, T)$ is a pure discount bond with face value 1 then

$$Cpl = 0.0169557505.$$

University of Cape Town

Chapter 11

Two Factor Vasicek Model

An extension of the Vasicek model is the two factor Vasicek model, where the second stochastic factor allows the mean-reversion level for the short rate to follow a mean-reverting stochastic process, with the aim for this model of better fitting empirical data, and more realistically modelling the dynamics of short rates.

Hibbert, Mowbray and Turnbull (2001) discuss the fact that this model can also be viewed as a special case of the two factor Hull-White model, first described in their 1994 paper. The risk-neutral dynamics of the short rate under this model are:

$$dr_{1,t} = k_1(r_{2,t} - r_{1,t})dt + \sigma_1 dW_{1,t}, \quad (11.1)$$

$$dr_{2,t} = k_2(\phi - r_{2,t})dt + \sigma_2 dW_{2,t}, \quad (11.2)$$

where r_1 is the short rate at time t , r_2 is the mean-reversion level of the short rate at time t , $k_1, k_2, \phi, \sigma_1, \sigma_2, r_{1,0}, r_{2,0}$ are positive constants and $W_{1,t}$ and $W_{2,t}$ are Brownian motions which are independent of one another.

Integrating equation 11.2, one can solve for $r_{2,t}$, as shown in chapter 7, Hence

$$r_{2,t} = r_{2,s}e^{-k_2(t-s)} + \phi(1 - e^{-k_2(t-s)}) + \sigma_2 \int_s^t e^{-k_2(t-\tau)} dW_{2,\tau}. \quad (11.3)$$

In order to solve for the short rate, one needs to substitute equation 11.3 in equation 11.1 and integrate the equation:

$$\begin{aligned} dr_{1,t} &= k_1(r_{2,t} - r_{1,t})dt + \sigma_1 dW_{1,t} \\ &= k_1[r_{2,s}e^{-k_2(t-s)} + \phi(1 - e^{-k_2(t-s)}) + \sigma_2 \int_s^t e^{-k_2(t-\tau)} dW_{2,\tau} - r_{1,t}]dt + \sigma_1 dW_{1,t} \\ &= k_1 r_{2,s} e^{-k_2(t-s)} dt + k_1 \phi (1 - e^{-k_2(t-s)}) dt + k_1 \sigma_2 \left(\int_s^t e^{-k_2(t-\tau)} dW_{2,\tau} \right) dt \\ &\quad - k_1 r_{1,t} dt + \sigma_1 dW_{1,t}, \end{aligned}$$

$$\begin{aligned}
\Rightarrow e^{k_1 t} dr_{1,t} &= k_1 r_{2,s} e^{-k_2(t-s)+k_1 t} dt + k_1 \phi e^{k_1 t} - k_1 \phi e^{-k_2(t-s)+k_1 t} dt \\
&\quad + k_1 \sigma_2 e^{k_1 t} \left(\int_s^t e^{-k_2(t-\tau)} dW_{2,\tau} \right) dt - k_1 e^{k_1 t} r_{1,t} dt + \sigma_1 e^{k_1 t} dW_{1,t}, \\
\Rightarrow d(e^{k_1 t} r_{1,t}) &= k_1 r_{2,s} e^{-k_2(t-s)+k_1 t} dt + k_1 \phi e^{k_1 t} dt - k_1 \phi e^{-k_2(t-s)+k_1 t} dt \\
&\quad + k_1 \sigma_2 e^{k_1 t} \left(\int_s^t e^{-k_2(t-\tau)} dW_{2,\tau} \right) dt + \sigma_1 e^{k_1 t} dW_{1,t}, \\
\Rightarrow e^{k_1 t} r_{1,t} &= e^{k_1 s} r_{1,s} + k_1 r_{2,s} e^{k_2 s} \int_s^t e^{-(k_2-k_1)\tau} d\tau + k_1 \phi \int_s^t e^{k_1 \tau} d\tau - k_1 \phi e^{k_2 s} \int_s^t e^{-(k_2-k_1)\tau} d\tau \\
&\quad + k_1 \sigma_2 \int_s^t e^{-(k_2-k_1)\tau} \left(\int_s^\tau e^{k_2 m} dW_{2,m} \right) d\tau + \sigma_1 \int_s^t e^{k_1 \tau} dW_{1,\tau}, \\
\Rightarrow r_{1,t} &= r_{1,s} e^{-k_1(t-s)} + \frac{k_1 r_{2,s} e^{k_2 s - k_1 t}}{k_1 - k_2} \left(e^{-(k_2-k_1)t} - e^{-(k_2-k_1)s} \right) + \phi(1 - e^{-k_1(t-s)}) \\
&\quad - \frac{k_1 \phi e^{k_2 s - k_1 t}}{k_1 - k_2} \left(e^{-(k_2-k_1)t} - e^{-(k_2-k_1)s} \right) \\
&\quad + k_1 \sigma_2 e^{-k_1 t} \int_s^t e^{-(k_2-k_1)\tau} \left(\int_s^\tau e^{k_2 m} dW_{2,m} \right) d\tau + \sigma_1 \int_s^t e^{-k_1(t-\tau)} dW_{1,\tau} \\
&= r_{1,s} e^{-k_1(t-s)} + \frac{k_1}{k_1 - k_2} (r_{2,s} - \phi) \left(e^{-k_2(t-s)} - e^{-k_1(t-s)} \right) + \phi(1 - e^{-k_1(t-s)}) \\
&\quad + k_1 \sigma_2 e^{-k_1 t} \int_{m=s}^{m=t} e^{k_2 m} \left(\int_{\tau=m}^{\tau=t} e^{-(k_2-k_1)\tau} d\tau \right) dW_{2,m} + \sigma_1 \int_s^t e^{-k_1(t-\tau)} dW_{1,\tau} \\
&= r_{1,s} e^{-k_1(t-s)} + \frac{k_1}{k_1 - k_2} (r_{2,s} - \phi) \left(e^{-k_2(t-s)} - e^{-k_1(t-s)} \right) + \phi(1 - e^{-k_1(t-s)}) \\
&\quad + k_1 \sigma_2 e^{-k_1 t} \int_s^t e^{k_2 m} \left(\frac{e^{-(k_2-k_1)t} - e^{-(k_2-k_1)m}}{k_1 - k_2} \right) dW_{2,m} + \sigma_1 \int_s^t e^{-k_1(t-\tau)} dW_{1,\tau} \\
&= r_{1,s} e^{-k_1(t-s)} + \frac{k_1}{k_1 - k_2} (r_{2,s} - \phi) \left(e^{-k_2(t-s)} - e^{-k_1(t-s)} \right) + \phi(1 - e^{-k_1(t-s)}) \\
&\quad + k_1 \sigma_2 \frac{e^{-k_2 t}}{k_1 - k_2} \int_s^t e^{k_2 m} dW_{2,m} - k_1 \sigma_2 \frac{e^{-k_1 t}}{k_1 - k_2} \int_s^t e^{k_1 m} dW_{2,m} \\
&\quad + \sigma_1 \int_s^t e^{-k_1(t-\tau)} dW_{1,\tau}. \tag{11.4}
\end{aligned}$$

Thus the short-term mean-reversion level, $r_{2,t}$, conditional on F_s , is distributed normally with the following moments:

$$\begin{aligned}
\mathbb{E}(r_{2,t}|F_s) &= r_{2,s} e^{-k_2(t-s)} + \phi(1 - e^{-k_2(t-s)}), \\
\text{Var}(r_{2,t}|F_s) &= \frac{\sigma_2^2}{2k_2} [1 - e^{-2k_2(t-s)}].
\end{aligned}$$

The short rate, $r_{1,t}$, conditional on F_s , is distributed normally with the following moments:

$$\begin{aligned}\mathbb{E}(r_{1,t}|F_s) &= r_{1,s}e^{-k_1(t-s)} + \frac{k_1}{k_1 - k_2} (r_{2,s} - \phi) \left(e^{-k_2(t-s)} - e^{-k_1(t-s)} \right) + \phi(1 - e^{-k_1(t-s)}), \\ \text{Var}(r_{1,t}|F_s) &= \mathbb{E}[(r_{1,t}|F_s) - \mathbb{E}(r_{1,t}|F_s)]^2 \\ &= \frac{(k_1\sigma_2)^2}{(k_1 - k_2)^2} \left(e^{-2k_2t} \int_s^t e^{2k_2m} dm - 2e^{-(k_1+k_2)t} \int_s^t e^{(k_1+k_2)m} dm \right. \\ &\quad \left. + e^{-2k_1t} \int_s^t e^{2k_1m} dm \right) + \sigma_1^2 \int_s^t e^{-2k_1(t-\tau)} dW_\tau \\ &= \frac{(k_1\sigma_2)^2}{(k_1 - k_2)^2} \left(\frac{(1 - e^{-2k_2(t-s)})}{2k_2} - \frac{2}{(k_1 + k_2)} \left(1 - e^{-(k_1+k_2)(t-s)} \right) \right. \\ &\quad \left. + \frac{(1 - e^{-2k_1(t-s)})}{2k_1} \right) + \frac{\sigma_1^2}{2k_1} \left(1 - e^{-2k_1(t-s)} \right).\end{aligned}$$

The long-term mean and variance of both the short rate and the short-term mean reversion level can be found by taking the limit as t tends towards infinity:

Short Rate:

$$\begin{aligned}\lim_{t \rightarrow \infty} \mathbb{E}(r_{1,t}|F_s) &= \phi, \\ \lim_{t \rightarrow \infty} \text{Var}(r_{1,t}|F_s) &= \frac{(k_1\sigma_2)^2}{(k_1 - k_2)^2} \left(\frac{1}{2k_2} - \frac{2}{(k_1 + k_2)} + \frac{1}{2k_1} \right) + \frac{\sigma_1^2}{2k_1}.\end{aligned}$$

Short-Term Mean-Reversion Level:

$$\begin{aligned}\lim_{t \rightarrow \infty} \mathbb{E}(r_{2,t}|F_s) &= \phi, \\ \lim_{t \rightarrow \infty} \text{Var}(r_{2,t}|F_s) &= \frac{\sigma_2^2}{2k}.\end{aligned}$$

The two factor Vasicek model displays several key attributes which the one factor Vasicek model displays:

1. The short rate experiences mean reversion, and over the long-term will tend towards ϕ .
2. There is no certainty that the short rate will always be positive. In reality, nominal interest rates can never be negative however the Vasicek model can not ensure that such a situation does not occur. This is a major drawback of the model.

11.1 Pure Discount Bond

$$P(t, T) = \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T r_{1,u} du} \middle| F_t \right)$$

Thus, in order to price a pure discount bond under the two factor Vasicek model, one needs to evaluate the integral of the short rate. Brigo and Mercurio (2006) show how one can compute this integral for a similar two factor model. Computing the pure discount bond price for the Vasicek model is more complicated, however the methodology will be based on Brigo and Mercurio's approach.

As Brigo and Mercurio (2006) explain, stochastic integration by parts allows one to manipulate the integral of the short rate into a more manageable equation. i.e.,

$$\int_t^T r_{1,u} du = Tr_{1,T} - tr_{1,t} - \int_t^T u dr_{1,u} = \int_t^T (T-u) dr_{1,u} + (T-t)r_{1,t}. \quad (11.5)$$

Substituting equation 11.1 in equation 11.5:

$$\int_t^T (T-u) dr_{1,u} + (T-t)r_{1,t} = \int_t^T (T-u)k_1(r_{2,u} - r_{1,u})du + \int_t^T (T-u)\sigma_1 dW_{1,u} + (T-t)r_{1,t}. \quad (11.6)$$

One can then substitute equations 11.3 and 11.4 in equation 11.6 thus

$$\begin{aligned} \int_t^T r_{1,u} du &= \int_t^T (T-u)k_1 r_{2,t} e^{-k_2(u-t)} du \\ &+ \int_t^T (T-u)k_1 \phi (1 - e^{-k_2(u-t)}) du \\ &+ \int_t^T (T-u)k_1 \sigma_2 \left(\int_t^u e^{-k_2(u-s)} dW_{2,s} \right) du \\ &- \int_t^T (T-u)k_1 r_{1,t} e^{-k_1(u-t)} du \\ &- \int_t^T (T-u)k_1 \frac{k_1}{(k_1 - k_2)} (r_{2,t} - \phi) \left(e^{-k_2(u-t)} - e^{-k_1(u-t)} \right) du \end{aligned}$$

$$\begin{aligned}
& - \int_t^T (T-u)k_1\phi(1 - e^{-k_1(u-t)})du \\
& - \int_t^T (T-u)k_1^2\sigma_2 \frac{e^{-k_2u}}{(k_1 - k_2)} \left(\int_t^u e^{k_2m} dW_{2,m} \right) du \\
& + \int_t^T (T-u)k_1^2\sigma_2 \frac{e^{-k_1u}}{(k_1 - k_2)} \left(\int_t^u e^{k_1m} dW_{2,m} \right) du \\
& - \int_t^T (T-u)k_1\sigma_1 \left(\int_t^u e^{-k_1(u-s)} dW_{1,s} \right) du \\
& + \int_t^T (T-u)\sigma_1 dW_{1,u} \\
& + (T-t)r_{1,t}.
\end{aligned} \tag{11.7}$$

Equation 11.7 can be solved by separating the various components of the equation, and using integration by parts or interchanging the integrals to solve each of these components. i.e.,

$$\begin{aligned}
\int_t^T (T-u)k_1r_{2,t}e^{-k_2(u-t)} du &= r_{2,t}k_1 \frac{(T-t)}{k_2} - \int_t^T \frac{e^{-k_2(u-t)}}{k_2} du \\
&= r_{2,t}k_1 \frac{(T-t)}{k_2} + \frac{(e^{-k_2(T-t)} - 1)}{k_2^2} r_{2,t}k_1,
\end{aligned} \tag{11.8}$$

$$- \int_t^T (T-u)k_1r_{1,t}e^{-k_1(u-t)} du = -r_{1,t}(T-t) - \frac{(e^{-k_1(T-t)} - 1)}{k_1} r_{1,t}, \tag{11.9}$$

$$\begin{aligned}
\int_t^T (T-u)k_1\phi(1 - e^{-k_2(u-t)})du &= \int_t^T (T-u)k_1\phi du - \int_t^T (T-u)k_1\phi e^{-k_2(u-t)} du \\
&= k_1\phi \left(\frac{T^2}{2} - Tt + \frac{t^2}{2} \right) - k_1\phi \frac{(T-t)}{k_2} \\
&\quad - \frac{(e^{-k_2(T-t)} - 1)}{k_2^2} k_1\phi,
\end{aligned} \tag{11.10}$$

$$\begin{aligned}
- \int_t^T (T-u)k_1\phi(1 - e^{-k_1(u-t)})du &= -k_1\phi \left(\frac{T^2}{2} - Tt + \frac{t^2}{2} \right) + \phi(T-t) \\
&\quad + \frac{(e^{-k_1(T-t)} - 1)}{k_1} \phi,
\end{aligned} \tag{11.11}$$

$$\begin{aligned}
& - \int_t^T (T-u)k_1 \frac{k_1}{(k_1 - k_2)} (r_{2,t} - \phi) \\
& \quad \times \left(e^{-k_2(u-t)} - e^{-k_1(u-t)} \right) du = \frac{-k_1^2}{(k_1 - k_2)} (r_{2,t} - \phi) \left(\int_t^T (T-u)e^{-k_2(u-t)} du \right. \\
& \quad \quad \left. - \int_t^T (T-u)e^{-k_1(u-t)} du \right) \\
& = \frac{-k_1^2}{(k_1 - k_2)} (r_{2,t} - \phi) \left(\frac{(T-t)}{k_2} + \frac{(e^{-k_2(T-t)} - 1)}{k_2^2} \right. \\
& \quad \quad \left. - \frac{(T-t)}{k_1} - \frac{(e^{-k_1(T-t)} - 1)}{k_1^2} \right), \quad (11.12)
\end{aligned}$$

$$\begin{aligned}
\int_t^T (T-u)k_1\sigma_2 \int_t^u e^{-k_2(u-s)} dW_{2,s} & = k_1\sigma_2 \int_{u=t}^{u=T} (T-u)e^{-k_2u} \left(\int_{s=t}^{s=u} e^{k_2s} dW_{2,s} \right) du \\
& = k_1\sigma_2 \int_{s=t}^{s=T} \left(\int_{u=s}^{u=T} (T-u)e^{-k_2u} du \right) e^{k_2s} dW_{2,s} \\
& = k_1\sigma_2 \int_t^T \left(\frac{(T-s)e^{-k_2s}}{k_2} + \frac{e^{-k_2T} - e^{-k_2s}}{k_2^2} \right) e^{k_2s} dW_{2,s} \\
& = k_1\sigma_2 \int_t^T \left(\frac{(T-s)}{k_2} + \frac{e^{-k_2(T-s)} - 1}{k_2^2} \right) dW_{2,s}, \quad (11.13)
\end{aligned}$$

$$\begin{aligned}
- \int_t^T (T-u)k_1\sigma_1 \int_t^u e^{-k_1(u-s)} dW_{1,s} & = -\sigma_1 \int_t^T \left((T-s) + \frac{e^{-k_1(T-s)} - 1}{k_1} \right) dW_{1,s}, \\
& \quad (11.14)
\end{aligned}$$

$$\begin{aligned}
- \int_t^T (T-u)k_1^2\sigma_2 \frac{e^{-k_2u}}{(k_1 - k_2)} \left(\int_t^u e^{k_2m} dW_{2,m} \right) du & = -\frac{k_1^2\sigma_2}{(k_1 - k_2)} \int_{u=t}^{u=T} (T-u)e^{-k_2u} \\
& \quad \times \left(\int_{m=t}^{m=u} e^{k_2m} dW_{2,m} \right) du \\
& = -\frac{k_1^2\sigma_2}{(k_1 - k_2)} \int_t^T \left(\frac{(T-s)}{k_2} \right. \\
& \quad \quad \left. + \frac{e^{-k_2(T-s)} - 1}{k_2^2} \right) dW_{2,s}, \quad (11.15)
\end{aligned}$$

$$\begin{aligned}
\int_t^T (T-u)k_1^2\sigma_2\frac{e^{-k_1u}}{(k_1-k_2)}\left(\int_t^ue^{k_1m}dW_{2,m}\right)du &= \frac{k_1^2\sigma_2}{(k_1-k_2)}\int_{u=t}^{u=T}(T-u)e^{-k_1u} \\
&\quad \times\left(\int_{m=t}^{m=u}e^{k_1m}dW_{2,m}\right)du \\
&= \frac{\sigma_2}{(k_1-k_2)}\int_t^T\left(k_1(T-s)\right. \\
&\quad \left.+e^{-k_1(T-s)}-1\right)dW_{2,s}.
\end{aligned} \tag{11.16}$$

Thus one can substitute equations 11.8 to 11.16 into equation 11.7:

$$\begin{aligned}
\int_t^T r_{1,u}du &= -r_{1,t}\frac{(e^{-k_1(T-t)}-1)}{k_1}-\frac{r_{2,t}k_1}{k_1-k_2}\left(T-t+\frac{e^{-k_2(T-t)}-1}{k_2}\right) \\
&\quad +\frac{r_{2,t}k_1}{k_1-k_2}\left(T-t+\frac{e^{-k_1(T-t)}-1}{k_1}\right) \\
&\quad -k_1\phi\left(1-\frac{k_1}{k_1-k_2}\right)\left(\frac{T-t}{k_2}+\frac{e^{-k_2(T-t)}-1}{k_2^2}\right) \\
&\quad +\phi(T-t)\left(1-\frac{k_1}{k_1-k_2}\right) \\
&\quad +\phi\left(1-\frac{k_1}{k_1-k_2}\right)\left(\frac{e^{-k_1(T-t)}-1}{k_1^2}\right) \\
&\quad -\frac{k_1\sigma_2}{k_1-k_2}\int_t^T\left((T-s)+\frac{e^{-k_2(T-s)}-1}{k_2}\right)dW_{2,s} \\
&\quad +\frac{\sigma_2}{(k_1-k_2)}\int_t^T\left(k_1(T-s)+e^{-k_1(T-s)}-1\right)dW_{2,s} \\
&\quad -\sigma_1\int_t^T\left(\frac{e^{-k_1(T-s)}-1}{k_1}\right)dW_{1,s}
\end{aligned}$$

$$\begin{aligned}
&= -r_{1,t} \frac{(e^{-k_1(T-t)} - 1)}{k_1} - r_{2,t} \frac{k_1}{k_1 - k_2} \left(\frac{e^{-k_2(T-t)} - 1}{k_2} - \frac{e^{-k_1(T-t)} - 1}{k_1} \right) \\
&\quad + \phi \left(\frac{e^{-k_1(T-t)} - 1}{k_1} \right) + \phi(T-t) \\
&\quad + \frac{k_1 \phi}{k_1 - k_2} \left(\frac{e^{-k_2(T-t)} - 1}{k_2} - \frac{e^{-k_1(T-t)} - 1}{k_1} \right) \\
&\quad - \frac{k_1 \sigma_2}{k_2(k_1 - k_2)} \int_t^T (e^{-k_2(T-s)} - 1) dW_{2,s} \\
&\quad + \frac{\sigma_2}{(k_1 - k_2)} \int_t^T (e^{-k_1(T-s)} - 1) dW_{2,s} \\
&\quad - \frac{\sigma_1}{k_1} \int_t^T (e^{-k_1(T-s)} - 1) dW_{1,s}.
\end{aligned}$$

In order to evaluate the price of a pure discount bond, one is required to evaluate the first two moments of the integral of the short rate. i.e.,

$$\begin{aligned}
M(t, T) &= \mathbb{E} \left(\int_t^T r_{1,u} du \middle| F_t \right) = -r_{1,t} \frac{(e^{-k_1(T-t)} - 1)}{k_1} - r_{2,t} \frac{k_1}{k_1 - k_2} \left(\frac{e^{-k_2(T-t)} - 1}{k_2} - \frac{e^{-k_1(T-t)} - 1}{k_1} \right) \\
&\quad + \phi \left(\frac{e^{-k_1(T-t)} - 1}{k_1} \right) + \phi(T-t) \\
&\quad + \frac{k_1 \phi}{k_1 - k_2} \left(\frac{e^{-k_2(T-t)} - 1}{k_2} - \frac{e^{-k_1(T-t)} - 1}{k_1} \right), \\
V(t, T) &= \text{Var} \left(\int_t^T r_{1,u} du \middle| F_t \right) = \frac{(k_1 \sigma_2)^2}{k_2^2 (k_1 - k_2)^2} \left(\frac{1 - e^{-2k_2(T-t)}}{2k_2} - \frac{2(1 - e^{-k_2(T-t)})}{k_2} + T - t \right) \\
&\quad + \left(\frac{\sigma_2^2}{(k_1 - k_2)^2} + \frac{\sigma_1^2}{k_1^2} \right) \left(\frac{1 - e^{-2k_1(T-t)}}{2k_1} - \frac{2(1 - e^{-k_1(T-t)})}{k_1} \right. \\
&\quad \left. + T - t \right) - \frac{2k_1 \sigma_2^2}{k_2(k_1 - k_2)^2} \left(\frac{1 - e^{-(k_1+k_2)(T-t)}}{k_1 + k_2} - \frac{1 - e^{-k_1(T-t)}}{k_1} \right. \\
&\quad \left. - \frac{1 - e^{-k_2(T-t)}}{k_2} + T - t \right).
\end{aligned}$$

Where the variance has been calculated using the following facts

$$\begin{aligned}
\text{Var} \left(\int_t^T (e^{-k_2(T-s)} - 1) dW_{2,s} \middle| F_t \right) &= \int_t^T (e^{-2k_2(T-s)} - 2e^{-k_2(T-s)} + 1) ds \\
&= \left(\frac{1 - e^{-2k_2(T-t)}}{2k_2} - \frac{2(1 - e^{-k_2(T-t)})}{k_2} + T - t \right),
\end{aligned}$$

$$\begin{aligned}
& \text{Covar} \left[\left(\int_t^T \left(e^{-k_2(T-s)} - 1 \right) dW_{2,s} \right) \left(\int_t^T \left(e^{-k_1(T-s)} - 1 \right) dW_{2,s} \right) \middle| F_t \right] \\
&= \int_t^T \left(e^{-(k_1+k_2)(T-s)} - e^{-k_1(T-s)} - e^{-k_2(T-s)} + 1 \right) ds \\
&= \left(\frac{1 - e^{-(k_1+k_2)(T-t)}}{k_1 + k_2} - \frac{(1 - e^{-k_1(T-t)})}{k_1} - \frac{(1 - e^{-k_2(T-t)})}{k_2} + T - t \right).
\end{aligned}$$

Since the integral of the short rate is normally distributed with mean $M(t, T)$ and variance $V(t, T)$:

$$P(t, T) = \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T r_{1,u} du} \middle| F_t \right) = e^{-M(t, T) + \frac{V(t, T)}{2}} = e^{A(T-t) - r_{1,t} B_1(T-t) - r_{2,t} B_2(T-t)},$$

where

$$\begin{aligned}
B_1(T-t) &= \frac{1 - e^{-k_1(T-t)}}{k_1}, \\
B_2(T-t) &= \frac{k_1}{k_1 - k_2} \left(\frac{1 - e^{-k_2(T-t)}}{k_2} - \frac{1 - e^{-k_1(T-t)}}{k_1} \right), \\
A(T-t) &= (B_1(T-t) - (T-t)) \left(\phi - \frac{\sigma_1^2}{2k_1^2} \right) + B_2(T-t) \phi - \frac{\sigma_1^2 B_1(T-t)^2}{4k_1} \\
&\quad + \frac{\sigma_2^2}{2} \left(\frac{T-t}{k_2^2} - \frac{2(B_1(T-t) + B_2(T-t))}{k_2^2} + \frac{1 - e^{-2k_1(T-t)}}{(k_1 - k_2)^2 2k_1} \right. \\
&\quad \left. - \frac{2k_1(1 - e^{-(k_1+k_2)(T-t)})}{k_2(k_1 - k_2)^2(k_1 + k_2)} + \frac{k_1^2(1 - e^{-2k_2(T-t)})}{k_2^2(k_1 - k_2)^2 2k_2} \right).
\end{aligned}$$

11.2 Cap and Floors

In order to price a caplet, one must first find the value of a European put option on a pure discount bond and then use the relationship between the put option and the caplet. The price at time t of this European put option with maturity T and strike K written on a pure discount bond with payoff $R1$ at maturity S is:

$$P_t = \mathbb{E} \left[e^{-\int_t^T r_s ds} (K - P(T, S))^+ \middle| F_t \right].$$

Computing this expectation requires knowledge about the joint distribution of the contingent claim $X_T = (K - P(T, S))^+$ and the bank account process $B_T = e^{\int_t^T r_s ds}$. As Jamshidian (1989) explains, in order to compute this expectation, one needs to change the probability measure. Thus, the numeraire is changed to $P(t, T)$ with \mathbb{Q}^T the corresponding Equivalent

Martingale Measure.

$$\begin{aligned}\frac{P_t}{B_0} &= \mathbb{E} \left[\frac{(K - P(T, S))^+}{B_T} \middle| F_t \right], \\ \Rightarrow \frac{P_t}{P(t, T)} &= \mathbb{E}_{\mathbb{Q}^T} \left[\frac{(K - P(T, S))^+}{P(T, T)} \middle| F_t \right], \\ \Rightarrow P_t &= P(t, T) \mathbb{E}_{\mathbb{Q}^T} [(K - P(T, S))^+ | F_t].\end{aligned}$$

In order to understand the dynamics of the short rate under the EMM and how the market price of risk has changed under the change in numeraire, one needs to compute the Radon-Nikodym process and then use Girsanov's theorem. i.e.,

$$\begin{aligned}\frac{d\mathbb{Q}^T}{d\mathbb{Q}} &= \frac{B_0 P(T, T)}{B_T P(0, T)} \\ &= \frac{e^{-\int_0^T r_s ds}}{P(0, T)} \\ &= e^{-\int_0^T r_s ds + M(0, T) - \frac{1}{2} V(0, T)} \\ &= \exp \left(\frac{k_1 \sigma_2}{k_2 (k_1 - k_2)} \int_0^T (e^{-k_2(T-s)} - 1) dW_{2,s} - \frac{\sigma_2}{(k_1 - k_2)} \int_0^T (e^{-k_1(T-s)} - 1) dW_{2,s} \right. \\ &\quad \left. + \frac{\sigma_1}{k_1} \int_0^T (e^{-k_1(T-s)} - 1) dW_{1,s} - \frac{1}{2} V(0, T) \right).\end{aligned}$$

The Girsanov theorem implies that the two processes $W_{1,t}^T$ and $W_{2,t}^T$ defined as

$$\begin{aligned}W_{1,t}^T &= W_{1,t} - \frac{\sigma_1}{k_1} (e^{-k_1(T-t)} - 1) dt, \\ W_{2,t}^T &= W_{2,t} - \frac{k_1 \sigma_2}{k_2 (k_1 - k_2)} (e^{-k_2(T-t)} - 1) dt + \frac{\sigma_2}{(k_1 - k_2)} (e^{-k_1(T-t)} - 1) dt,\end{aligned}$$

are two independent Brownian motion under the \mathbb{Q}^T measure. The dynamics of the short rate process $r_{1,t}$ and the short-term mean-reversion level $r_{2,t}$ under \mathbb{Q}^T are thus

$$dr_{1,t} = k_1 (r_{2,t} - r_{1,t}) dt + \frac{\sigma_1}{k_1} (e^{-k_1(T-t)} - 1) dt + \sigma_1 dW_{1,t}^T, \quad (11.17)$$

$$dr_{2,t} = k_2 (\phi - r_{2,t}) dt + \frac{k_1 \sigma_2}{k_2 (k_1 - k_2)} (e^{-k_2(T-t)} - 1) dt - \frac{\sigma_2}{(k_1 - k_2)} (e^{-k_1(T-t)} - 1) dt + \sigma_2 dW_{2,t}^T. \quad (11.18)$$

Integrating equation 11.18, one can solve for $r_{2,t}$ and then input this solution into equation 11.17. Integrating this formula, one can thus solve for

$r_{1,t}$. Thus, the short-term mean-reversion level, $r_{2,t}$, conditional on F_s , is distributed normally with the following variance:

$$\text{Var}_{\mathbb{Q}^T}(r_{2,t}|F_s) = \frac{\sigma_2^2}{2k_2}[1 - e^{-2k_2(t-s)}].$$

The short rate, $r_{1,t}$, conditional on F_s , is distributed normally with the following variance:

$$\begin{aligned} \text{Var}_{\mathbb{Q}^T}(r_{1,t}|F_s) &= \frac{(k_1\sigma_2)^2}{(k_1 - k_2)^2} \left(\frac{(1 - e^{-2k_2(t-s)})}{2k_2} - \frac{2}{(k_1 + k_2)} \left(1 - e^{-(k_1+k_2)(t-s)} \right) \right. \\ &\quad \left. + \frac{(1 - e^{-2k_1(t-s)})}{2k_1} \right) + \frac{\sigma_1^2}{2k_1} \left(1 - e^{-2k_1(t-s)} \right). \end{aligned}$$

The covariance between the short rate and the short-term mean-reversion level is

$$\begin{aligned} \text{Covar}_{\mathbb{Q}^T}(r_{1,t}, r_{2,t}|F_s) &= \frac{k_1\sigma_2^2 e^{-2k_2t}}{k_1 - k_2} \int_s^t e^{2k_2m} dW_{2,m} - \frac{k_1\sigma_2^2 e^{-(k_1+k_2)t}}{k_1 - k_2} \int_s^t e^{(k_1+k_2)m} dW_{2,m} \\ &= \frac{k_1\sigma_2^2}{k_1 - k_2} \left(\frac{1 - e^{-2k_2(t-s)}}{2k_2} - \frac{1 - e^{-(k_1+k_2)(t-s)}}{k_1 + k_2} \right). \end{aligned} \tag{11.19}$$

Let $P(T, S) = e^z$,

$$z = A(S - T) - r_{1,T}B_1(S - T) - r_{2,T}B_2(S - T).$$

Thus z is distributed normally with mean M_p and variance V_p^2 .

$$\begin{aligned} M_p &= A(S - T) - B_1(S - T)\mathbb{E}_{\mathbb{Q}^T}(r_{1,T}) - B_2(S - T)\mathbb{E}_{\mathbb{Q}^T}(r_{2,T}), \\ V_p^2 &= (B_1(S - T))^2\text{Var}_{\mathbb{Q}^T}(r_{1,T}) + (B_2(S - T))^2\text{Var}_{\mathbb{Q}^T}(r_{2,T}) \\ &\quad + 2(B_1(S - T))(B_2(S - T))\text{Covar}_{\mathbb{Q}^T}(r_{1,T}, r_{2,T}). \end{aligned}$$

Thus one is able to evaluate the price of a European put option on a pure discount bond:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^T} [(K - e^z)^+ | F_t] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(V_p)^2}} e^{-\frac{(z-M_p)^2}{2V_p^2}} (K - e^z)^+ dz \\
&= \int_{-\infty}^{\ln K} \frac{1}{\sqrt{2\pi(V_p)^2}} e^{-\frac{(z-M_p)^2}{2V_p^2}} (K - e^z) dz \\
&= \int_{-\infty}^{\frac{\ln K - M_p}{V_p}} \frac{1}{\sqrt{2\pi(V_p)^2}} e^{-\frac{1}{2}x^2} (K - e^{M_p + V_p x}) dx \\
&= \int_{-\infty}^{\frac{\ln K - M_p}{V_p}} \frac{1}{\sqrt{2\pi(V_p)^2}} e^{-\frac{1}{2}x^2} K dx \\
&\quad - \int_{-\infty}^{\frac{\ln K - M_p}{V_p}} \frac{1}{\sqrt{2\pi(V_p)^2}} e^{M_p + V_p x - \frac{1}{2}x^2} dx \\
&= K \int_{-\infty}^{\frac{\ln K - M_p}{V_p}} \frac{1}{\sqrt{2\pi(V_p)^2}} e^{-\frac{1}{2}x^2} dx \\
&\quad - e^{M_p + \frac{1}{2}V_p^2} \int_{-\infty}^{\frac{\ln K - M_p}{V_p}} \frac{1}{\sqrt{2\pi(V_p)^2}} e^{-\frac{1}{2}V_p^2 + V_p x - \frac{1}{2}x^2} dx \\
&= K \int_{-\infty}^{\frac{\ln K - M_p}{V_p}} \frac{1}{\sqrt{2\pi(V_p)^2}} e^{-\frac{1}{2}x^2} dx \\
&\quad - e^{M_p + \frac{1}{2}V_p^2} \int_{-\infty}^{\frac{\ln K - M_p}{V_p}} \frac{1}{\sqrt{2\pi(V_p)^2}} e^{-\frac{1}{2}(x - V_p)^2} dx \\
&= K \int_{-\infty}^{\frac{\ln K - M_p}{V_p}} \frac{1}{\sqrt{2\pi(V_p)^2}} e^{-\frac{1}{2}x^2} dx \\
&\quad - e^{M_p + \frac{1}{2}V_p^2} \int_{-\infty}^{\frac{\ln K - M_p - V_p^2}{V_p}} \frac{1}{\sqrt{2\pi(V_p)^2}} e^{-\frac{x^2}{2}} dx \\
&= K \left(1 - N \left(\frac{M_p - \ln K}{V_p} \right) \right) - e^{M_p + \frac{1}{2}V_p^2} \left(1 - N \left(\frac{M_p - \ln K + V_p^2}{V_p} \right) \right).
\end{aligned}$$

Under the EMM \mathbb{Q}^T

$$\frac{X_t}{P(t, T)} = \mathbb{E}_{\mathbb{Q}^T} \left[\frac{X_T}{P(T, T)} | F_t \right].$$

Thus substituting $P(\tau, S)$ for X_τ

$$\begin{aligned}
\frac{P(t, S)}{P(t, T)} &= \mathbb{E}_{\mathbb{Q}^T} \left[\frac{P(T, S)}{P(T, T)} | F_t \right] \\
&= \mathbb{E}_{\mathbb{Q}^T} [P(T, S) | F_t].
\end{aligned}$$

But previously it was shown that $\ln P(T, S)$ is distributed normally with mean M_p and variance V_p^2 , thus

$$\frac{P(t, S)}{P(t, T)} = \mathbb{E}_{\mathbb{Q}^T} [P(T, S) | F_t] = e^{M_p + \frac{1}{2} V_p^2}.$$

Thus

$$\begin{aligned} M_p &= \ln \left(\frac{P(t, S)}{P(t, T)} \right) - \frac{1}{2} V_p^2, \\ \therefore \frac{M_p - \ln K}{V_p} &= \frac{\ln \left(\frac{P(t, S)}{K P(t, T)} \right) - \frac{1}{2} V_p^2}{V_p}, \\ \text{and } \frac{M_p - \ln K + V_p^2}{V_p} &= \frac{\ln \left(\frac{P(t, S)}{K P(t, T)} \right) + \frac{1}{2} V_p^2}{V_p}. \end{aligned}$$

Hence the price of a European put option on a pure discount bond is

$$\begin{aligned} P_t &= P(t, T) \left[K (1 - N(d_2)) - e^{M_p + \frac{1}{2} V_p^2} (1 - N(d_1)) \right] \\ &= K P(t, T) (1 - N(d_2)) - P(t, S) (1 - N(d_1)), \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln \left(\frac{P(t, S)}{K P(t, T)} \right) + \frac{1}{2} V_p^2}{V_p}, \\ d_2 &= d_1 - V_p, \\ V_p^2 &= (B_1(S - T))^2 \text{Var}_{\mathbb{Q}^T}(r_{1, T}) + (B_2(S - T))^2 \text{Var}_{\mathbb{Q}^T}(r_{2, T}) \\ &\quad + 2(B_1(S - T))(B_2(S - T)) \text{Covar}_{\mathbb{Q}^T}(r_{1, T}, r_{2, T}). \end{aligned}$$

The value of the corresponding call option can be found through Put-Call Parity.

Example 1. If the short rate, $r_{1,t}$, and the short-term mean-reversion level, $r_{2,t}$, are assumed to possess the following risk-neutral dynamics

$$\begin{aligned} dr_{1,t} &= k_1(r_{2,t} - r_{1,t})dt + \sigma_1 dW_{1,t}, \\ dr_{2,t} &= k_2(\phi - r_{2,t})dt + \sigma_2 dW_{2,t}, \end{aligned}$$

with $k_1 = 0.1$, $\phi = 0.05$, $\sigma_1 = 0.1$, $k_2 = 0.05$ and $\sigma_2 = 0.05$ and a caplet has the payoff $0.25(L - 0.0475)^+$ at expiry in 1 year, where L is the three-month spot LIBOR rate in nine months' time, $r_{1,0}$ is 5%, $r_{2,0} = 4.5\%$ and $P(0, T)$

is a pure discount bond with face value 1 then

$$\begin{aligned} B_1(T) &= \frac{1 - e^{-k_1 T}}{k_1} \\ &= \frac{1 - e^{-0.1(0.75)}}{0.1} \\ &= 0.7225651367 \end{aligned}$$

$$\begin{aligned} B_2(T) &= \frac{k_1}{k_1 - k_2} \left(\frac{1 - e^{-k_2 T}}{k_2} - \frac{1 - e^{-k_1 T}}{k_1} \right) \\ &= \frac{0.1}{0.1 - 0.05} \left(\frac{1 - e^{-0.05(0.75)}}{0.05} - \frac{1 - e^{-0.1(0.75)}}{0.1} \right) \\ &= 0.0270930177 \end{aligned}$$

$$A(T) = 0.0006479693$$

$$\begin{aligned} P(0, T) &= e^{A(T) - B_1(T)r_{1,0} - B_2(T)r_{2,0}} \\ &= 0.9639657893 \end{aligned}$$

$$\begin{aligned} B_1(S) &= \frac{1 - e^{-k_1 S}}{k_1} \\ &= \frac{1 - e^{-0.1(1)}}{0.1} \\ &= 0.9516258196 \end{aligned}$$

$$\begin{aligned} B_2(S) &= \frac{k_1}{k_1 - k_2} \left(\frac{1 - e^{-k_2 S}}{k_2} - \frac{1 - e^{-k_1 S}}{k_1} \right) \\ &= \frac{0.1}{0.1 - 0.05} \left(\frac{1 - e^{-0.05(1)}}{0.05} - \frac{1 - e^{-0.1(1)}}{0.1} \right) \\ &= 0.04757138069 \end{aligned}$$

$$A(S) = 0.0015077329$$

$$\begin{aligned} P(0, S) &= e^{A(S) - B_1(S)r_{1,0} - B_2(S)r_{2,0}} \\ &= 0.9529295808 \end{aligned}$$

$$\begin{aligned} P_0 &= KP(0, T)(1 - N(d_2)) - P(0, S)(1 - N(d_1)) \\ &= 0.9882643607(0.9639657893)(1 - N(0.0037748811)) \\ &\quad - 0.9529295808(1 - N(0.0243874337)) \\ &= 0.0076973762, \end{aligned}$$

and thus the value of the caplet is

$$\begin{aligned} Cpl &= (1 + K_{Caplet}\tau)P_0 \\ &= (1 + 0.0475(0.25))0.0076973762 \\ &= 0.0077887825. \end{aligned}$$

University of Cape Town

Chapter 12

Two-Factor Cox-Ingersoll-Ross Model

An extension of the Cox-Ingersoll-Ross model is the two factor model whereby the short rate is the sum of two independent processes, with each process modelled using a one factor CIR model. i.e.,

$$\begin{aligned}r_t &= x_t + y_t, \\dx_t &= k_x(\phi_x - x_t)dt + \sigma_x\sqrt{x_t}dW_{t,x}, \\dy_t &= k_y(\phi_y - y_t)dt + \sigma_y\sqrt{y_t}dW_{t,y},\end{aligned}$$

where $k_x, \phi_x, \sigma_x, x_0, k_y, \phi_y, \sigma_y, y_0$ are positive constants.

Since these two processes are independent of one another, one can easily solve for the short rate as well as the expectation and variance of the short rate, based on the results of the one factor CIR model in Chapter 9. i.e.,

$$\begin{aligned}x_t &= x_s e^{-k_x(t-s)} + \phi_x(1 - e^{-k_x(t-s)}) + \sigma_x \int_s^t e^{-k_x(t-\tau)} \sqrt{x_\tau} dW_{\tau,x}, \\y_t &= y_s e^{-k_y(t-s)} + \phi_y(1 - e^{-k_y(t-s)}) + \sigma_y \int_s^t e^{-k_y(t-\tau)} \sqrt{y_\tau} dW_{\tau,y}, \\ \therefore r_t &= x_s e^{-k_x(t-s)} + y_s e^{-k_y(t-s)} + \phi_x(1 - e^{-k_x(t-s)}) + \phi_y(1 - e^{-k_y(t-s)}) \\ &\quad + \sigma_x \int_s^t e^{-k_x(t-\tau)} \sqrt{x_\tau} dW_{\tau,x} + \sigma_y \int_s^t e^{-k_y(t-\tau)} \sqrt{y_\tau} dW_{\tau,y}.\end{aligned}$$

The expectation and variance of the short rate are thus:

$$\begin{aligned}\mathbb{E}(r_t|F_s) &= x_s e^{-k_x(t-s)} + y_s e^{-k_y(t-s)} + \phi_x(1 - e^{-k_x(t-s)}) + \phi_y(1 - e^{-k_y(t-s)}), \\ \text{Var}(r_t|F_s) &= \frac{x_s \sigma_x^2}{k_x} (e^{-k_x(t-s)} - e^{-2k_x(t-s)}) + \frac{y_s \sigma_y^2}{k_y} (e^{-k_y(t-s)} - e^{-2k_y(t-s)}) \\ &\quad + \frac{\phi_x \sigma_x^2}{2k_x} (1 - 2e^{-k_x(t-s)} + e^{-2k_x(t-s)}) + \frac{\phi_y \sigma_y^2}{2k_y} (1 - 2e^{-k_y(t-s)} + e^{-2k_y(t-s)}).\end{aligned}$$

The long-term mean and variance can be found by taking the limit as t tends towards infinity:

$$\begin{aligned}\lim_{t \rightarrow \infty} \mathbb{E}(r_t | F_s) &= \phi_x + \phi_y, \\ \lim_{t \rightarrow \infty} \text{Var}(r_t | F_s) &= \frac{\phi_x \sigma_x^2}{2k_x} + \frac{\phi_y \sigma_y^2}{2k_y}.\end{aligned}$$

As is the case with the one factor model, the two factor CIR model exhibits mean-reversion. The second preferable feature of the short rate under these dynamics is that the short rate is guaranteed to remain positive provided that the initial processes are both positive and $2k_x \phi_x \geq \sigma_x^2$ and $2k_y \phi_y \geq \sigma_y^2$.

Note that the independence of the two processes are key in the analytical tractability of the short rate. Brigo and Mercurio (2006) discuss the fact that such a property limits the shape of the curve of the absolute volatilities of the instantaneous forward rates, thus such a model may be deemed inappropriate under certain market conditions when calibrating to market data.

Allowing for correlation between these two processes (more specifically, between the Brownian motions of these two processes) would no longer limit the shape of the curve of the absolute volatilities of the instantaneous forward rates to such a degree however, as the joint distribution of these two processes is not known, the loss in tractability of the short rate far outweighs any potential benefits.

12.1 Pure Discount Bond

The price of a pure discount bond under the two factor Cox-Ingersoll-Ross model can be decomposed into two separate expectations, due to the independence of the two processes. These expectations have already been calculated in Chapter 9. i.e.,

$$\begin{aligned}P(t, T) &= \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T r_u du} \middle| F_t \right) \\ &= \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T (x_u + y_u) du} \middle| F_t \right) \\ &= \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T x_u du} \middle| F_t \right) \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T y_u du} \middle| F_t \right) \\ &= \left(e^{A_x(t, T) - B_x(t, T)x_t} \right) \left(e^{A_y(t, T) - B_y(t, T)y_t} \right) \\ &= e^{A_x(t, T) + A_y(t, T) - B_x(t, T)x_t - B_y(t, T)y_t},\end{aligned}$$

where, for $i = x, y$

$$\begin{aligned} B_i(t, T) &= \frac{2(e^{\kappa_i(T-t)} - 1)}{(\kappa_i + k_i)(e^{\kappa_i(T-t)} - 1) + 2\kappa_i}, \\ A_i(t, T) &= \frac{2k_i\phi_i}{\sigma_i^2} \ln \left[\frac{2\kappa_i e^{\frac{1}{2}(k_i + \kappa_i)(T-t)}}{2\kappa_i + (k_i + \kappa_i)(e^{\kappa_i(T-t)} - 1)} \right], \\ \kappa_i &= \sqrt{k_i^2 + 2\sigma_i^2}. \end{aligned}$$

12.2 Cap and Floors

In order to price a caplet, one must first find the value of a European put option on a pure discount bond and then use the relationship between the put option and the caplet. As discussed in Jamshidian (1989) and shown in the previous chapter, one can use a change in measure to the Equivalent Martingale Measure \mathbb{Q}^T in order to simplify the calculation of the expectation. i.e.,

$$\begin{aligned} P_t &= \mathbb{E} \left[e^{-\int_t^T r_s ds} (K - P(T, S))^+ | F_t \right] \\ &= P(t, T) \mathbb{E}_{\mathbb{Q}^T} [(K - P(T, S))^+ | F_t]. \end{aligned}$$

In order to compute this expectation, one requires knowledge of the dynamics of the two processes under \mathbb{Q}^T . The dynamics for x_t are shown below, as derived in Brigo and Mercurio (2006). Those for y_t can be similarly displayed.

$$dx_t = (k_x \phi_x - [k_x + B_x(t, T)\sigma_x^2]x_t)dt + \sigma_x \sqrt{x_t} dW_{t,x}^T,$$

where

$$dW_{t,x}^T = dW_{t,x} + B_x(t, T)\sigma_x \sqrt{x_t} dt,$$

with a non-central chi-squared distribution function given by

$$p_{x_t|x_s}^T(z) = q_x(t, s) p_{\chi^2(\nu_x, \delta_x(t,s))}(q_x(t, s)z),$$

where

$$\begin{aligned} \nu_x &= \frac{4k_x\phi_x}{\sigma_x^2}, \\ \psi_x &= \frac{k_x + \kappa_x}{\sigma_x^2}, \\ q_x(t, s) &= 2[\rho_x(t-s) + \psi_x + B_x(t, T)], \\ \delta_x(t, s) &= \frac{4\rho_x(t-s)^2 x_s e^{\kappa_x(t-s)}}{q_x(t, s)}, \\ \rho_x(t-s) &= \frac{2\kappa_x}{\sigma_x^2(e^{\kappa_x(t-s)} - 1)}. \end{aligned}$$

Thus

$$\begin{aligned} P_t &= P(t, T) \mathbb{E}_{\mathbb{Q}^T} [(K - P(T, S))^+ | F_t] \\ &= P(t, T) \int_0^\infty \int_0^\infty (K - P(T, S))^+ p_{x_t|x_s}^T(z_1) p_{y_t|y_s}^T(z_2) dz_1 dz_2. \end{aligned}$$

Chen and Scott (1992) reveal how one can reduce the multivariate integrals to a univariate integral, allowing one to calculate various options prices including that of a European put option. Using their approach, the price of a European call option is found to be:

$$\begin{aligned} C_t &= P(t, T) \int_0^\infty \int_0^\infty (P(T, S) - K)^+ p_{x_t|x_s}^T(z_1) p_{y_t|y_s}^T(z_2) dz_1 dz_2 \\ &= P(t, S) \chi^2(L_x^*, L_y^*; \nu_x, \nu_y, \delta_x(T, t), \delta_y(T, t)) \\ &\quad - P(t, T) K \chi^2(L_x, L_y; \nu_x, \nu_y, \delta_x^*(T, t), \delta_y^*(T, t)), \end{aligned}$$

where

$$\chi^2(L_x, L_y; \nu_x, \nu_y, \delta_x^*, \delta_y^*) = \int_0^{L_y} F^* \left(L_x - \frac{L_x z_y}{L_y}; \nu_x, \delta_x^* \right) f^*(z_y; \nu_y, \delta_y^*) dz_y,$$

where F^* and f^* are the standard non-central chi-squared probability distribution function and density function respectively, and

$$\begin{aligned} C^* &= \ln \frac{A_x(T, S) A_y(T, S)}{K}, \\ y_x^* &= \frac{C^*}{B_x(T, S)}, \\ L_x^* &= 2[\rho_x(T-t) + \psi_x + B_x(T, S)] y_x^*, \\ L_x &= 2[\rho_x(T-t) + \psi_x] y_x^*, \\ \delta_x(T, t)^* &= \frac{2\rho_x(T-t)^2 x_s e^{\kappa_x(T-t)}}{\rho_x(T-t) + \psi_x}. \end{aligned}$$

Example 1. If the short rate is assumed to possess the following risk-neutral dynamics

$$\begin{aligned} r_t &= x_t + y_t, \\ dx_t &= k_x(\phi_x - x_t)dt + \sigma_x \sqrt{x_t} dW_{t,x}, \\ dy_t &= k_y(\phi_y - y_t)dt + \sigma_y \sqrt{y_t} dW_{t,y}, \end{aligned}$$

with $k_1 = 0.2$, $\phi_1 = 0.05$, $\sigma_1 = 0.15$, $k_2 = 0.005$, $\phi_2 = 0.03$ and $\sigma_2 = 0.075$ and a caplet has the payoff $0.25(L - 0.0475)^+$ at expiry in 1 year, where L is the three-month spot LIBOR rate in nine months' time, x_0 is 2%, y_0 is 3% and $P(0, T)$ is a pure discount bond with face value 1 then

$$P_0 = 0.0027495499,$$

and thus the value of the caplet is

$$C_{pl} = 0.0027822008.$$

Chapter 13

Calibrating Short Rate Models

The previous chapters in this thesis have focused on the theory behind some of the more common short rate models, including how these models can be used to value contingent claims such as caps, floors or swaptions. This chapter aims to provide the reader with an understanding of how one calibrates a short rate model. This will provide the foundation for the following three chapters which review the results of empirical comparisons of the short rate models discussed in this thesis.

13.1 Parameterization

In order to test the ability of a short rate model to price contingent claims, one needs to calibrate that model to market prices. This process consists of fitting values of instruments or indicators determined through the short rate model to those observed in the market. For example, one could determine parameter values by fitting the short rate model to cap and floor prices or to the yield curve. The chosen parameters are those which enable the theoretical values to fit as closely as possible to the observed values in the market, where ‘closeness’ is a measure defined by the user.

Examples of measures for ‘closeness’ include:

- Squared difference between theoretical and observed values

$$\sum_{i=1}^n (V_i^* - V_i(\beta))^2.$$

- Absolute value of the relative difference between theoretical and ob-

served values

$$\sum_{i=1}^n \frac{|V_i^* - V_i(\beta)|}{V^*},$$

where

V_i^* = is the observed value of instrument i ,

$V_i(\beta)$ = is the theoretical value of instrument i given the vector of parameters, β .

One of the major disadvantages of the first approach is that it is biased by the size of the value of the instrument. i.e., the closeness measure is far more sensitive to pricing errors when the value of the instrument or indicator is large. Thus, if one were using this measure when fitting the model to prices of pure discount bonds, the fit would not necessarily be as close for long-term maturities as for values for shorter-term maturities.

The ability of the theoretical values derived by the short rate model to fit values observed in the market depends on the number of parameters in the model versus the number of primary instruments to which the model is fitted. If there are more instruments or indicators than parameters, the model is over-determined and thus it is highly unlikely that the theoretical prices will match the observed prices exactly for all instruments or indicators.

Models such as the Ho-Lee model and Black-Karasinski overcome this problem through the use of parameters which are deterministic functions as opposed to constants. This allows these models to fit the values of a series of instruments or indicators exactly and allow one to fit the model to another series of instruments or indicators using the remaining parameters. The disadvantage of such models is that the deterministic parameters are harder to justify intuitively.

Example. *The risk-neutral dynamics of the short rate under the Ho-Lee model are:*

$$dr_t = \phi(t)dt + \sigma dW_t.$$

Since $\phi(t)$ is a deterministic function, one can fit the Ho-Lee model exactly to the yield curve and then choose a value for σ which fits the model as close as possible to caps prices.

The approach taken in this thesis is to fit each of the short rate models to the yield curve by comparing the absolute value of the relative difference in the theoretical and observed values of pure discount bonds. i.e., the parameters chosen are those which minimized the following equation:

$$\sum_{i=1}^n \frac{|P_i^*(t, T) - P_i(t, T; \beta)|}{P^*(t, T)},$$

where

$P_i^*(t, T)$ = is the observed value at time t of pure discount bond Bond i with maturity T ,
 $P_i(t, T; \beta)$ = is the theoretical value at time t of pure discount bond i with maturity T
 given the vector of parameters, β .

Since this equation is dependant on multiple variables, an algorithm to minimize multivariate functions is required. One such algorithm is Nelder-Mead which has gained popularity due to its powerfulness. Rouah et al (2007) note that this algorithm is “easy to implement, and it converges very quickly regardless of which starting values are used.” Thus Nelder-Mead will be used in this thesis for calibrating the short rate models, with all programming completed in Visual Basic.

13.2 Data

Two different sources were used to obtain the data:

- Interest rate data was obtained from I-Net Bridge. This data consists of one-month and three-month JIBAR; 3x6, 6x9, 9x12, 12x15, 15x18 and 18x21 FRAs; and three-year, four-year, five-year, six-year, seven-year, eight-year, nine-year, ten-year, twelve-year and fifteen-year Swaps.
- Cap and floor data was obtained from Rand Merchant Bank. This data consists of the mid-volatility of bid and offer quotes for At-the-Money options of caps with maturities ranging from one-year to ten-years with three-month tenors.

480 data points were considered for the period 27 March 2006 to 27 February 2008. As discussed in Chapter 5, the ‘Monotone Convex Interpolation Method’ has been used to complete the yield curve.

I am extremely grateful to Petrus Bosman at Prescient Securities for providing the cap and floor data.

13.3 Forms of Short Rate Models Compared

The short rate dynamics for two of the models discussed in this thesis consist of deterministic functions for at least one of the parameters in that model. This allows these models to fit the yield curve exactly, however it does not allow for an effective comparison across all of the models of their ability to fit the yield curve. Thus, special cases of these two short rate models will be considered whereby the deterministic function is instead constrained to be a constant. Thus the dynamics of each of the models considered are:

- Vasicek One Factor Model

$$dr_t = k(\phi - r_t)dt + \sigma dW_t,$$

- Ho-Lee Model

$$dr_t = \phi dt + \sigma dW_t,$$

- Cox-Ingersoll-Ross One Factor Model

$$dr_t = k(\phi - r_t)dt + \sigma\sqrt{r_t}dW_t,$$

- Black-Karasinski Model

$$d\ln(r_t) = [\phi - a\ln(r_t)]dt + \sigma dW_t,$$

- Vasicek Two Factor Model

$$\begin{aligned} dr_{1,t} &= k_1(r_{2,t} - r_{1,t})dt + \sigma_1 dW_{1,t}, \\ dr_{2,t} &= k_2(\phi - r_{2,t})dt + \sigma_2 dW_{2,t}, \end{aligned}$$

- Cox-Ingersoll-Ross Two Factor Model

$$\begin{aligned} r_t &= x_t + y_t, \\ dx_t &= k_x(\phi_x - x_t)dt + \sigma_x\sqrt{x_t}dW_{t,x}, \\ dy_t &= k_y(\phi_y - y_t)dt + \sigma_y\sqrt{y_t}dW_{t,y}. \end{aligned}$$

Chapter 14

Comparison of Models

This chapter will provide the foundation for the comparison of the empirical ability of each of the short rate models to fit the yield curve, by introducing a variety of approaches to conduct this comparison. Pelsser (2004) points out that one can use goodness-of-fit criteria such as R^2 , adjusted- R^2 or the standard error of the regression; or one can use a more formal testing procedure to compare each short rate model against one another.

Goodness-of-fit criteria are easy to calculate and are typically found to be fairly intuitive statistics. e.g., R^2 is the proportion of variability in the dependent variable that can be explained by variability in the independent variables. However, a critical disadvantage with using such measures to compare models is that goodness-of-fit criteria “do not take into consideration the losses associated with choosing an incorrect model,” as highlighted by Judge et al (1982), and discussed by Pelsser (2004). e.g., When comparing two models which are both unsuitable, goodness-of-fit criteria will not inform one that neither model is appropriate for the task at hand.

Davidson and MacKinnon (1981) (1993) and Greene (2003) provide alternative approaches to testing different economic theories or models which “purport to explain the same dependent variable.” A group of these approaches which will be used for this thesis are the tests for non-nested, non-linear regression models. Non-nested describes the fact that the models being compared are not special cases of one another. Non-linear describes the fact that the dependent variable - the yield of a pure discount bond with maturity T - is not a linear combination of the independent variables - the short rate and time to maturity of the pure discount bond. (Note: Parameter constraints on the two factor CIR model prevent the one factor CIR model from being described as a special case of the two factor model).

14.1 Testing Non-Nested, Non-Linear Regression Models

No consensus approach exists for the testing of non-nested, non-linear regression models. Fisher and McAleer (1981), Mizon and Richard (1986) and Greene (2003) all discuss some of the possible approaches which one can use. Pelsser (2004), when comparing the ability of short rate models to fit cap and floor prices, prefers the approach discussed by Davidson and MacKinnon (1981) (1993). This thesis will adopt the same approach.

Under Davidson and MacKinnon's (1981) (1993) approach, one considers a model whose validity one wishes to test. i.e.,

$$H_0 : y_i = f(X_i, \beta) + \epsilon_{0,i},$$

where

- y_i is the i th observation of the dependent variable,
- X_i is a vector of the i th observations of the explanatory variables,
- β is a vector of parameters to be estimated,
- $\epsilon_{0,i}$ is the error term of the i th observation of the model and is assumed to be normally and independently distributed (NID) with a mean of 0 and a variance of σ_0^2 .

One is faced with an alternative model which also purports to explain the same dependent variable. i.e.

$$H_1 : y_i = g(Z_i, \gamma) + \epsilon_{1,i},$$

where

- y_i is the i th observation of the dependent variable,
- Z_i is a vector of the i th observations of the explanatory variables,
- γ is a vector of parameters to be estimated,
- $\epsilon_{1,i}$ is the error term of the i th observation of the model and is assumed to be normally and independently distributed (NID) with a mean of 0 and a variance of σ_0^2 .

Since these two models are non-nested, the truth of H_0 implies that the alternative hypothesis, H_1 , is false, and vice versa.

In order to perform the tests on H_0 and H_1 , one is required to embed the

two competing models into an artificial compound model through the use of an artificial parameter α . i.e.,

$$H_C : y_i = (1 - \alpha)f(X_i, \beta) + \alpha g(Z_i, \gamma) + \epsilon_i, \quad (14.1)$$

If H_0 is true, then the true value of α is 0. In the case that α is significantly different from 0 and 1, the artificial compound model can be used as a specification test, providing “an indication in which direction one needs to search for a better model,” a significant advantage as highlighted by Pelsser (2004).

One of the problems with the artificial compound model in 14.1 is that, in most cases, the model can not be estimated as some of the parameters will not be separately identifiable, a point highlighted in both Davidson and MacKinnon (1993) and Pelsser (2004). Davidson and MacKinnon (1981) suggest a possible solution to this problem whereby one replaces the parameter vector of H_1 with estimates of these parameters, assuming H_1 is true. Thus one can replace γ with an estimate $\hat{\gamma}$, derived through non-linear least squares, and thus obtain the model

$$H_C : y_i = (1 - \alpha)f(X_i, \beta) + \alpha \hat{g} + \epsilon_i, \quad (14.2)$$

where $\hat{g} = g(Z_i, \hat{\gamma})$.

Two possible approaches exist to test the null hypothesis:

- If H_0 is linear then one can perform the J-Test whereby α and β are estimated jointly, with the t-statistic used to test the null hypothesis that $\alpha = 0$.
- If H_0 is non-linear, then one can conduct the P-Test. By using the Taylor expansion of $f_i \approx \hat{f}_i + \hat{F}_i(\beta - \hat{\beta})$, where \hat{F}_i is the matrix of partial derivatives of f with respect to β evaluated at $\hat{\beta}$, the non-linear regression 14.2 can be linearized to obtain the Gauss-Newton regression

$$H_C : y_i - \hat{f}_i = \alpha(\hat{g}_i - \hat{f}_i) + \hat{F}_i b + \epsilon_i, \quad (14.3)$$

where one can use the t-statistic for $\alpha = 0$ from this regression.

Davidson and MacKinnon (1993) point out that the J-Test and the P-Test are “asymptotically equivalent under H_0 .” Thus, if one of these tests is “asymptotically valid, both of them must be” although they may “yield different results in small samples.”

The t-statistics from the J-Test and P-Test are based on the assumption that H_0 is true. Thus, one cannot use these equations to test H_1 and instead is required to reverse the roles of H_0 and H_1 before repeating the test. Thus, four possible outcomes exist:

- Accept both H_0 and H_1 ,
- Accept H_0 and reject H_1 ,
- Reject H_0 and accept H_1 ,
- Reject both H_0 and H_1 .

University of Cape Town

Chapter 15

Empirical Results - Fit to Yield Curve

Six short rate models, in the form discussed in Chapter 13.3, were calibrated and tested on a daily basis across the entire dataset of 480 observation dates in order to understand the empirical ability of each model to fit the yield curve. This section reviews the results of these tests.

15.1 Fit to Pure Discount Bond Prices and the Yield Curve

The calibration of these models involved minimizing the sum across 60 maturity dates of the absolute value of the relative difference in pure discount bond prices, as elaborated on in Chapter 13. Figure 15.1 shows this minimum value for each model at each data point. One should note that certain constraints were implemented during this calibration process - e.g., the volatility was constrained to only positive numbers; $2k\phi$ was constrained to being greater than σ^2 in the Cox-Ingersoll-Ross one factor model in order to ensure that the short rate remains positive.

Figure 15.1 clearly illustrates that the Ho-Lee model is the poorest fit of the six models whilst the Cox-Ingersoll-Ross one factor model struggles to fit the observed values towards the beginning of the dataset (May 2006 - October 2006). The Vasicek two factor model generally appears to be the best fitting model across the entire data set.

Although this value which has been minimized provides the reader with a firm understanding of the ability of each model to fit the data, it is intuitively more appealing to view the model fit in terms of the annualised

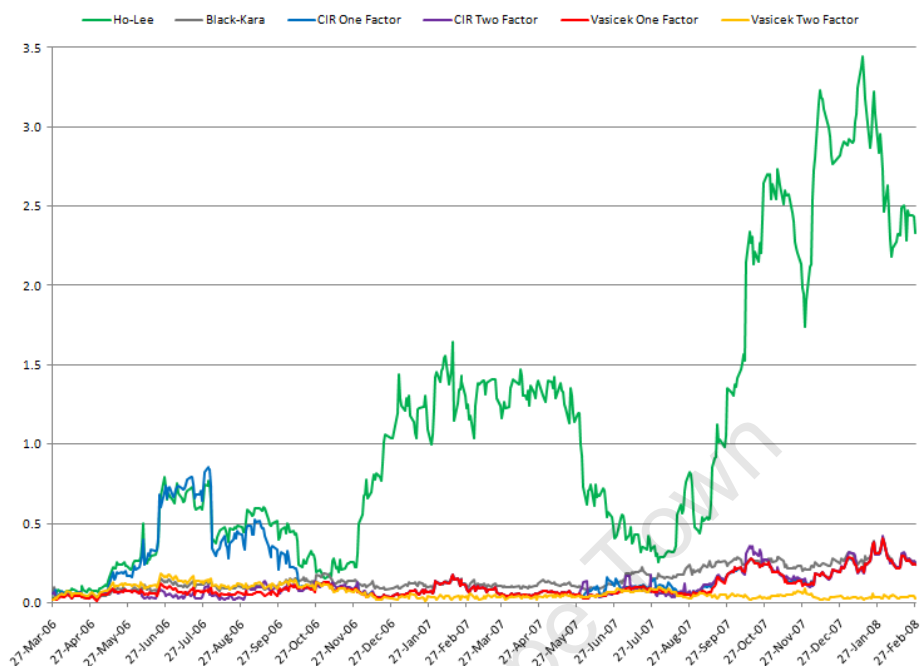


Figure 15.1: Sum of absolute value of relative difference between theoretical and observed PDB values

yield of pure discount bonds. i.e.,

$$y_{i,\tau} = -\ln \frac{P(t, T; \beta)}{\tau},$$

where

$y_{i,\tau}$ is the model implied annualized yield of pure discount bond with maturity τ at observation i ,

$P(t, T; \beta)$ is the theoretical value at time i of a pure discount bond with maturity T given the vector of parameters, β ,

τ is the term to maturity of the bond, $T-i$.

One can compare this model implied annualized yield to the annualized yield observed in the market. i.e.,

$$y_{i,\tau}^* = y_{i,\tau} + \epsilon_i,$$

where

$y_{i,\tau}^*$ is the observed annualized yield of pure discount bond with maturity τ at observation i ,

ϵ is the error term at observation i which is assumed to be distributed normally with a mean of 0 and a standard deviation of σ_ϵ .

15.1.1 Goodness-of-Fit

Two statistics which provide the reader with an indication of the goodness-of-fit of the annualized yields of each of the short rate models are the average absolute value of the error term across the different maturities at each observation date and the standard deviation of the error term, σ_ϵ . Table 15.1 contains summary statistics for these two goodness-of-fit indicators, including the minimum, median, maximum, average and quartiles. Note: ‘Q1’ and ‘Q3’ denote the first and third quartile respectively. Figure 15.2 shows the average of the first indicator across the different maturities, for each datapoint and for all six models whilst Figure 15.3 shows the values for the second indicator across these observation dates.

		Mean	St Dev	Min	Q1	Median	Q3	Max
Ho Lee	$ \epsilon $	0.286%	0.200%	0.012%	0.135%	0.245%	0.330%	0.871%
	σ_ϵ	0.263%	0.144%	0.022%	0.164%	0.241%	0.297%	0.650%
Black-Kara	$ \epsilon $	0.158%	0.070%	0.014%	0.117%	0.146%	0.204%	0.354%
	σ_ϵ	0.422%	0.204%	0.031%	0.292%	0.394%	0.531%	0.989%
CIR One Factor	$ \epsilon $	0.075%	0.050%	0.008%	0.041%	0.063%	0.080%	0.259%
	σ_ϵ	0.102%	0.049%	0.014%	0.068%	0.089%	0.127%	0.251%
CIR Two Factor	$ \epsilon $	0.062%	0.043%	0.007%	0.038%	0.051%	0.074%	0.297%
	σ_ϵ	0.078%	0.035%	0.012%	0.055%	0.074%	0.100%	0.206%
Vasicek One Factor	$ \epsilon $	0.047%	0.020%	0.006%	0.033%	0.043%	0.063%	0.099%
	σ_ϵ	0.077%	0.030%	0.013%	0.058%	0.076%	0.096%	0.163%
Vasicek Two Factor	$ \epsilon $	0.042%	0.024%	0.007%	0.023%	0.033%	0.060%	0.109%
	σ_ϵ	0.077%	0.044%	0.011%	0.042%	0.065%	0.107%	0.205%

Table 15.1: Summary statistics of annualized yields computed over the 480 observation dates

As shown previously in Figure 15.1, the Ho-Lee model has a significantly poorer fit of the yield curve as compared to the other five models. This view is supported when one views the two goodness-of-fit parameters for the Ho-Lee model. The Black-Karasinski model also has a poor fit in terms of the standard deviation of the error term, averaging 0.422%.

Over the initial period (March 2006 - November 2006), the Vasicek one factor model and Cox-Ingersoll-Ross two factor model appear to fit the data the best, although the latter model has a poor fit during June 2006. For the remaining observation dates the best fitting model is the Vasicek two factor model. As expected, the two factor Vasicek model fits the data better than the one factor model, as is the case for the Cox-Ingersoll-Ross models. Interestingly, however, is the fact that the maximum value of the average absolute error term over the 480 observation dates is larger for both of the two factor models as compared to their respective one factor models. Generally, the Vasicek models appear to fit the data better than the Cox-Ingersoll-Ross models.

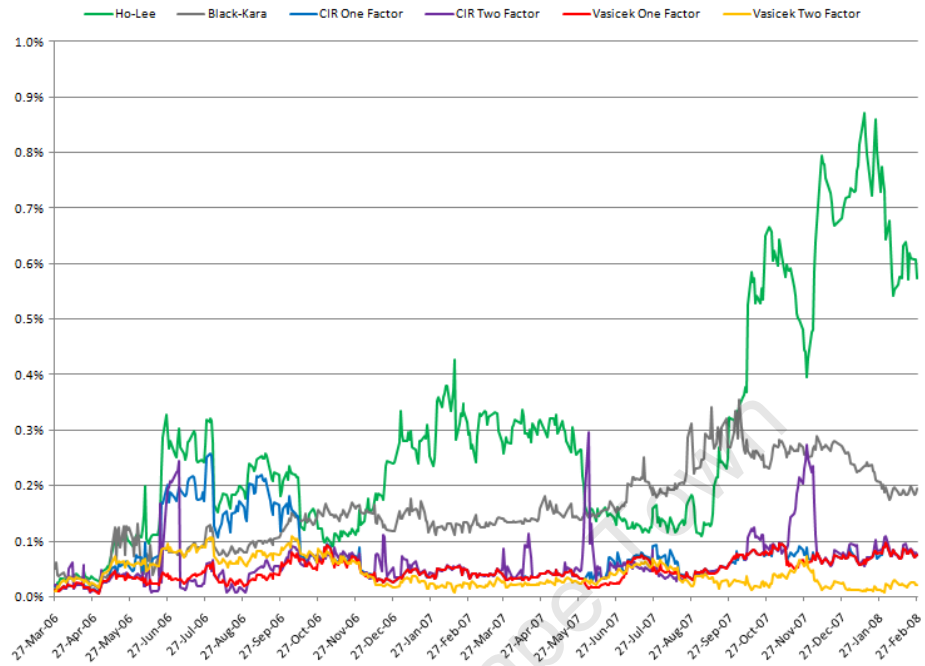


Figure 15.2: Average Absolute Value of Error Term

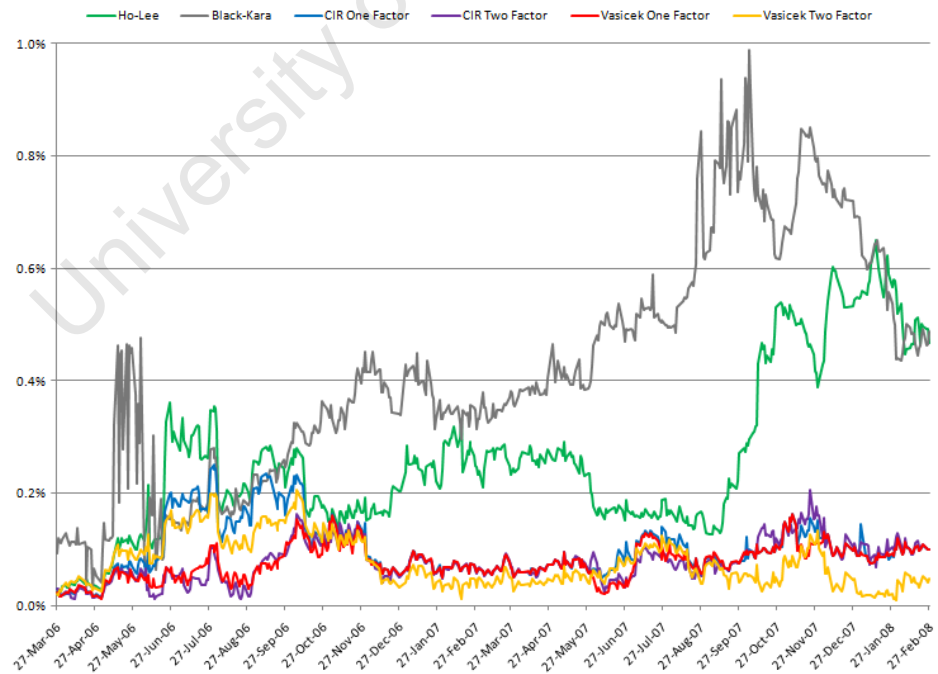


Figure 15.3: Standard Deviation of the Error Term

15.1. FIT TO PURE DISCOUNT BOND PRICES AND THE YIELD CURVE97



Figure 15.4: Ho-Lee Parameter Values

Based on the first goodness-of-fit measure - the average absolute value of the error term - the Vasicek two factor model fits the observed values best. Based on the second goodness-of-fit measure - the standard deviation of the error term - the Cox-Ingersoll-Ross two factor model and the two Vasicek models all fit the data relatively well.

15.1.2 Parameter Estimates under each short rate Model

The individual parameter estimates for the six short rate models are shown in Figure 15.4 to Figure 15.11.

The Ho-Lee model has been shown to fit the observed yield curve poorly. One possible reason for this is that the risk-neutral dynamics of the model do not allow for mean reversion of the short rate. Wu and Zhang (1996) found interest rates to be mean-reverting when testing across a range of OECD countries, whilst Pelsser (2004) notes that “on the basis of economic theory, there are compelling arguments for the mean-reversion of interest rates.” Figure 15.4 shows the drift parameter to consistently remain close to zero, thus one is essentially trying to fit the model to the observed prices using only one parameter, σ .

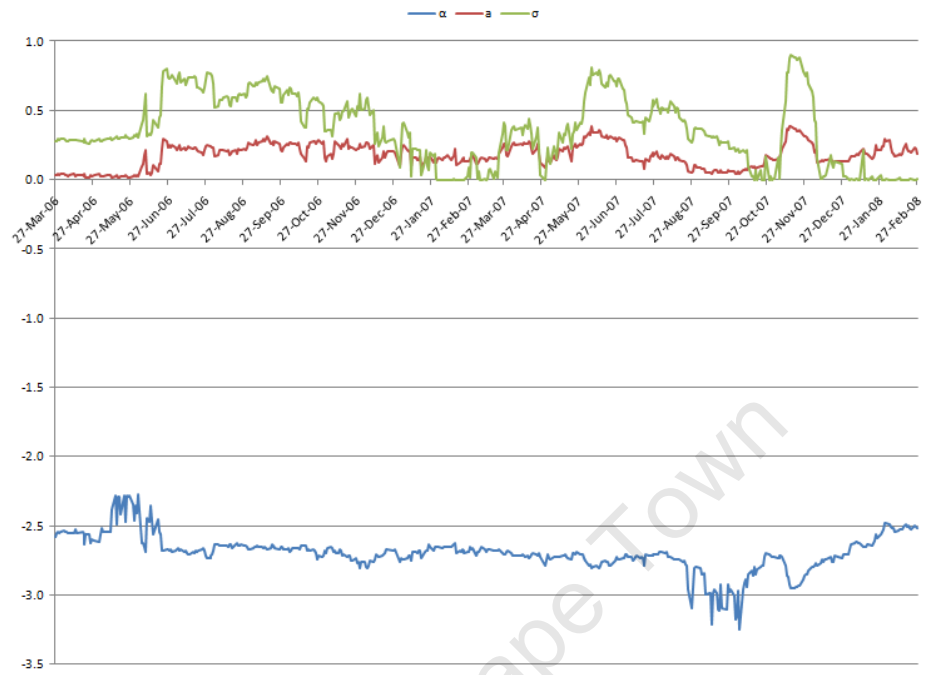


Figure 15.5: Black-Karasinski Parameter Values

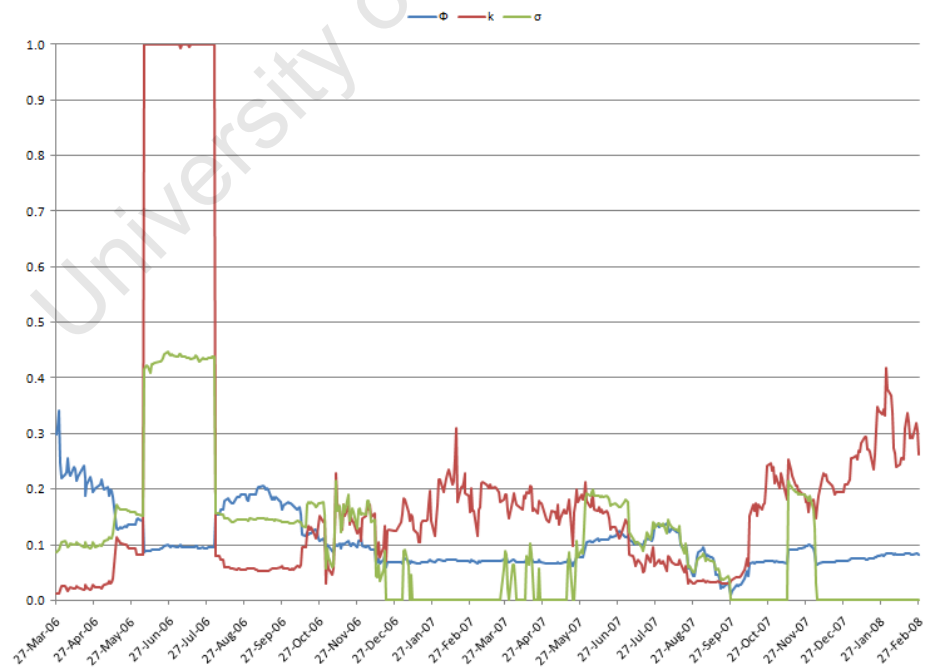


Figure 15.6: CIR One Factor Parameter Values

15.1. FIT TO PURE DISCOUNT BOND PRICES AND THE YIELD CURVE 99

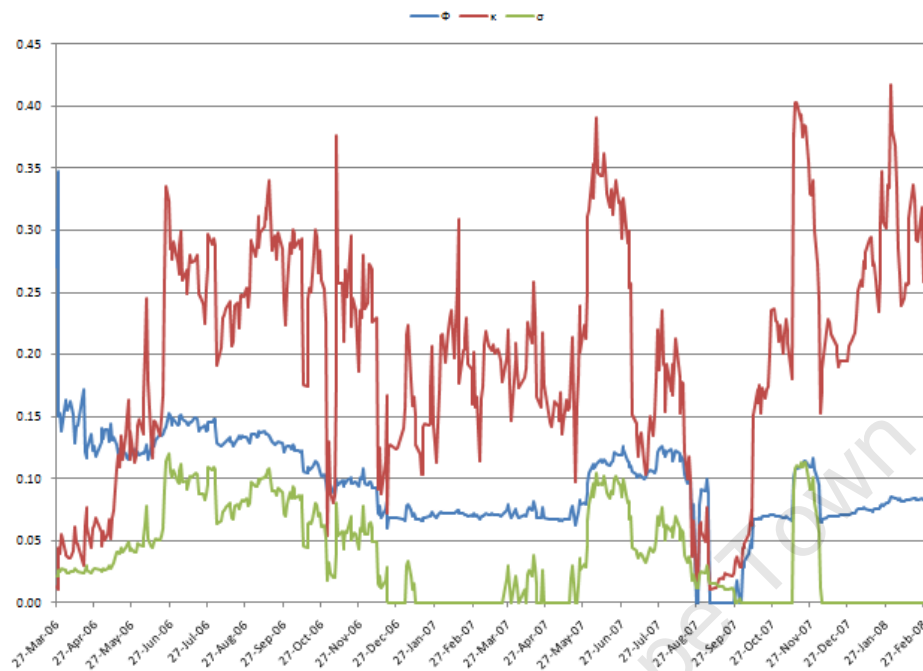


Figure 15.7: Vasicek One Factor Parameter Values

The parameter values in the Cox-Ingersoll-Ross one factor model, as shown in Figure 15.6, appear difficult to justify intuitively. The volatility of the model is $\sigma\sqrt{r_t}$. Since σ is close to zero for extended periods over the set of observations, the volatility of the model over these periods will be extremely close to zero, resulting in the Brownian motion having minimal impact on the value of the short rate. The Vasicek one factor model experiences the same issue with regards to the volatility of the model, with σ close to zero for extended periods over the set of observations, as shown in Figure 15.7.

The Cox-Ingersoll-Ross two factor model is appealing theoretically due to the fact that one can model short-term fluctuations in the short rate whilst still allowing for extended periods of either high or low interest rates. Under the model, one of the independent processes, X , is expected to have high values for the speed of reversion and volatility parameters whilst the other process, Y , would have low values for these two parameters. Figure 15.8 and Figure 15.9 show the speed of reversion to be consistent with this theory however the volatility to be inconsistent with this theory. Since the volatility of the processes are $\sigma_X\sqrt{X_t}$ and $\sigma_Y\sqrt{Y_t}$ respectively and σ_X and σ_Y are close to zero for extended periods of time, the volatility for each of the processes is low.

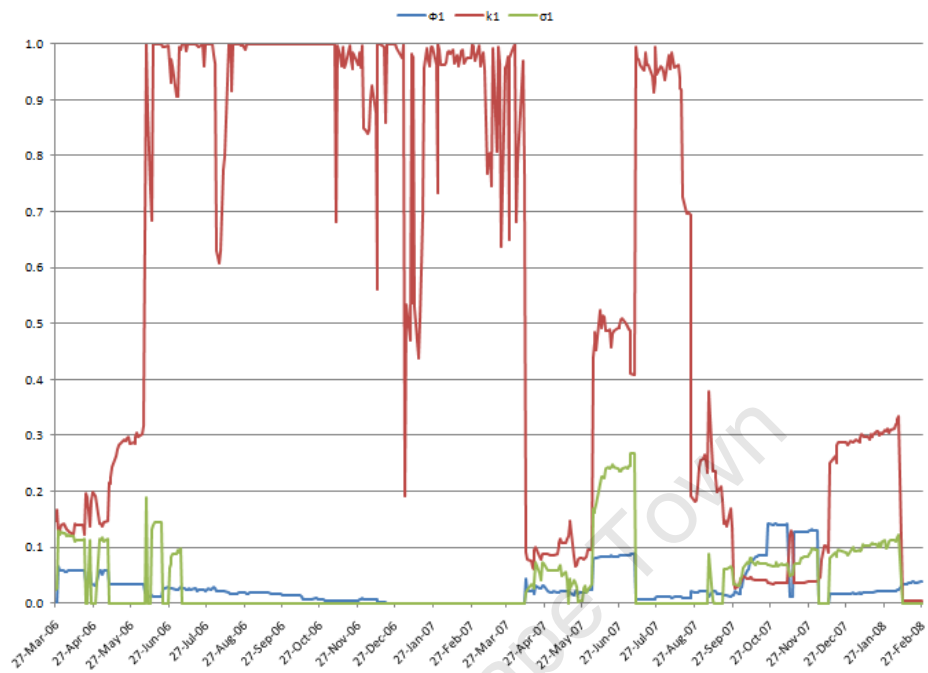


Figure 15.8: CIR Two Factor (Process X) Parameter Values

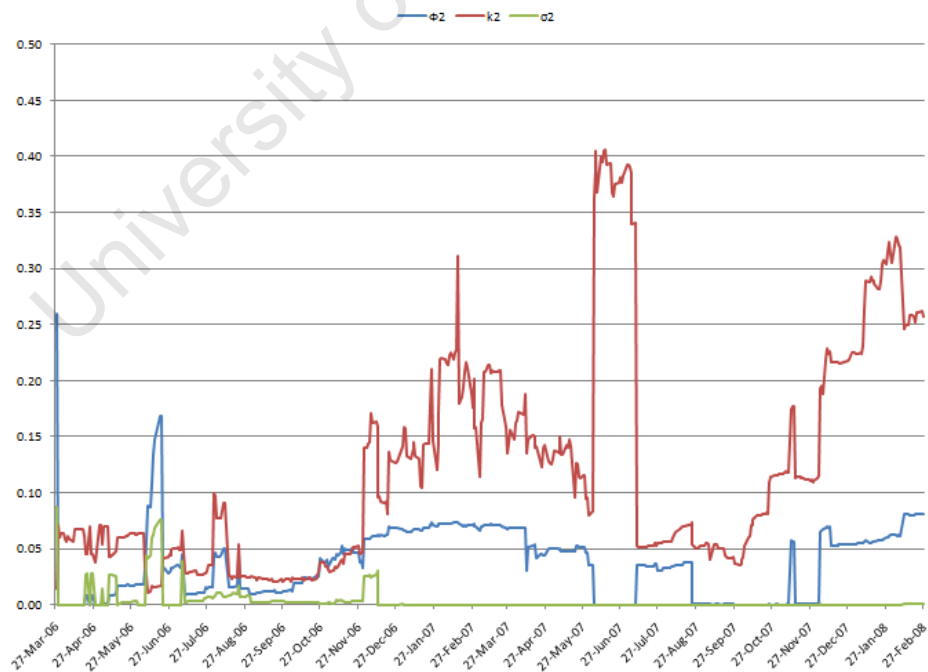


Figure 15.9: CIR Two Factor (Process Y) Parameter Values

15.1. FIT TO PURE DISCOUNT BOND PRICES AND THE YIELD CURVE 101



Figure 15.10: Vasicek Two Factor (Short Rate) Parameter Values

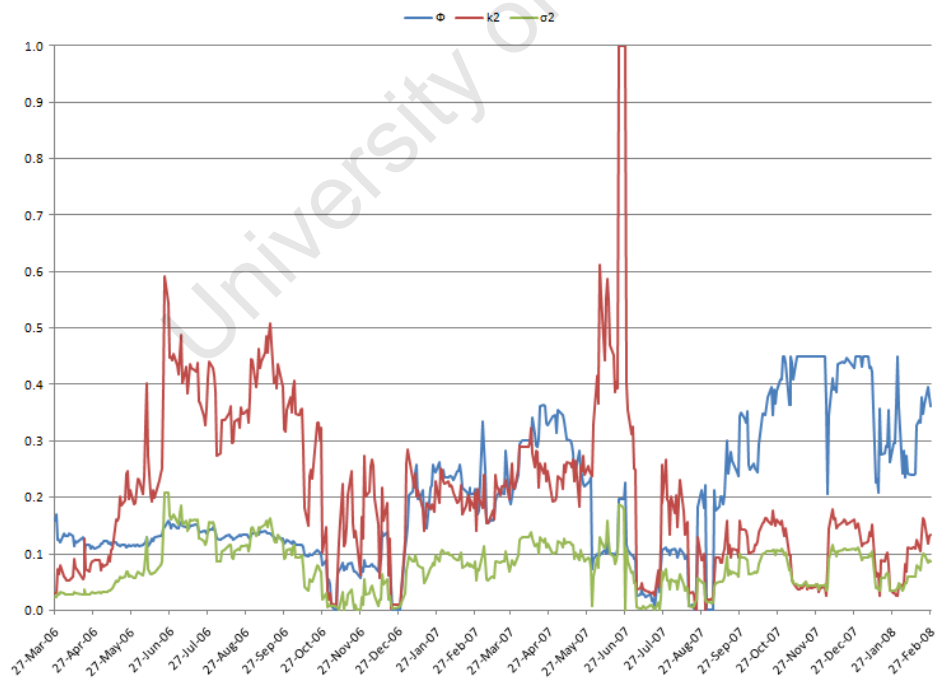


Figure 15.11: Vasicek Two Factor (Short Rate Mean Reversion Level) Parameter Values

Under the Vasicek two factor model, the short rate reverts to a short-term mean-reversion level which in term reverts to a long-term mean-reversion level. This feature is appealing as it allows for short term fluctuations in the short rate whilst still allowing for extended periods of either high or low interest rates. Figure 15.10 reveals that over extended periods of time the short rate has a high speed of reversion towards the short-term mean-reversion level with a low volatility parameter for this process. Over the remaining periods, the speed of reversion is far lower, with higher volatility parameters resulting in the short rate reverting to the short-term mean-reversion levels at a far slower rate. Figure 15.11 shows that the short-term mean-reversion process typically reverts slowly to the long-term risk-neutral mean-reversion level although this level appears highly volatile over the middle and end observation dates and appears excessively high over the end observation dates.

15.1.3 Example of Fit to the Yield Curve

In order to provide the reader with an indication of the fit of the six models, Figure 15.12 shows the observed yield curve on 18 October 2007, as well as the yield curve derived from the six Short Rate models. At this date the yield curve is inverted, forming a convex shape. i.e., the yield of debt instruments of a longer-term maturity have a lower yield than that of short-term instruments of the same quality. A characteristic of this curve differentiating it from a typical inverted yield curve is that the yield initially increases slightly with the term before decreasing and forming the inverted shape. This characteristic results in the short rate models struggling to fit the observed prices.

It is immediately apparent that both the Ho-Lee model and Black-Karasinski models are unable to produce an inverted convex form. The Ho-Lee model's resultant concave shape is a poor fit of the observed prices whilst the Black-Karasinski model fits the observed prices extremely poorly at the short term of the curve, as it attempts to fit a humped curve to the observed prices. The Cox-Ingersoll-Ross one factor and two factor models, as well as the Vasicek one factor model are unable to fit the observed prices as their curvature is too shallow. The Vasicek two factor model closely matches the observed prices, except at the short-end of the curve where the Vasicek model is unable to reproduce the increasing then decreasing characteristic of the observed yield curve.

15.1.4 Davidson and MacKinnon's P-Tests

Although the goodness-of-fit measures provide one insight into which models should be preferred, it provides little indication of the actual suitability of

15.1. FIT TO PURE DISCOUNT BOND PRICES AND THE YIELD CURVE 103

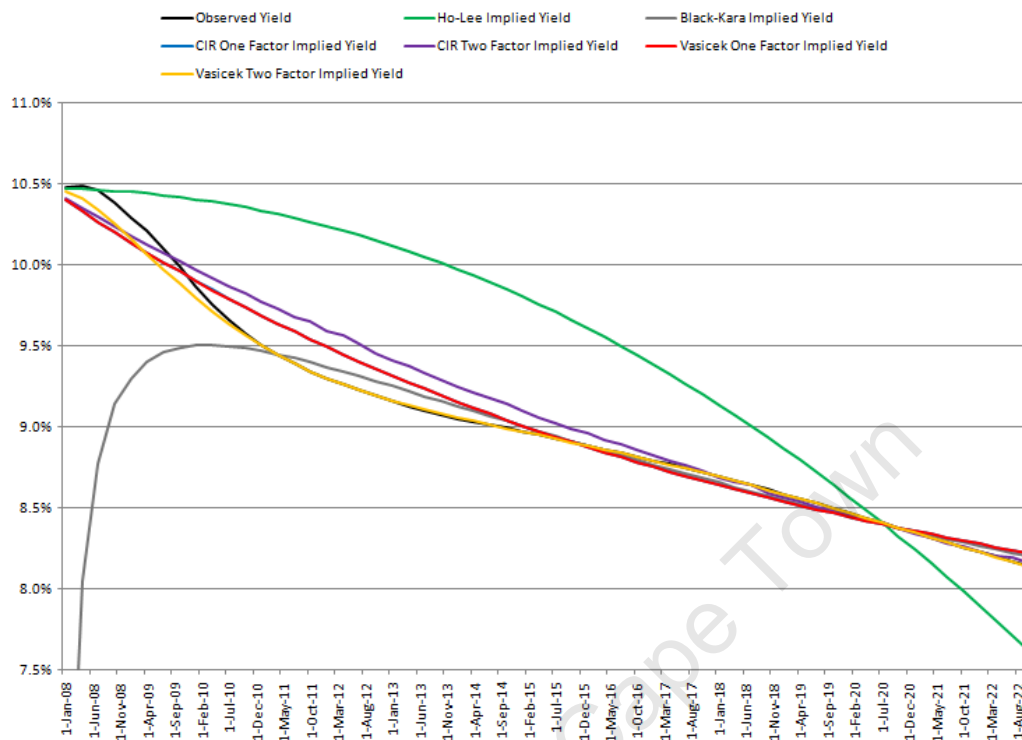


Figure 15.12: Yield Curve on 18 October 2007

each model. Davidson and MacKinnon’s P-Test, as discussed in Chapter 14, allows one to more formally test the ability of the various short rate models to fit the observed prices. Table 15.2 provides the results of the various pairwise P-Tests which have been performed, summarizing the number of times each of the four possible outcomes occurred across the 480 observation dates for each pairwise test.

None of the six models are overwhelmingly accepted as being able to suitably fit the observed yield curve. Over the 15 pairwise tests, both models being tested were rejected over a minimum of 58% of the 480 observation dates. The Vasicek two factor models appears more suitable than the rest of the short rate models as under each pairwise test, the Vasicek model always outperms the alternative model. The Cox-Ingersoll-Ross one factor model and Vasicek one factor model appear to have a similar suitability in fitting the data, with both models being accepted under the pairwise test over 17.9% of the observation dates.

Acc. Vas 2F & Vas 1F	Acc. Vas 2F / Rej. Vas 1F	Rej. Vas 2F / Acc. Vas 1F	Rej. Vas 2F & Vas 1F	Acc. Vas 1F & HL	Acc. Vas 1F / Rej. HL	Rej. Vas 1F / Acc. HL	Rej. Vas 1F & HL
0.2%	4.6%	1.3%	94.0%	0.0%	0.6%	14.4%	85.0%
Acc. Vas 2F & CIR 2F	Acc. Vas 2F / Rej. CIR 2F	Rej. Vas 2F / Acc. CIR 2F	Rej. Vas 2F & CIR 2F	Acc. CIR 2F & CIR 1F	Acc. CIR 2F / Rej. CIR 1F	Rej. CIR 2F / Acc. CIR 1F	Rej. CIR 2F & CIR 1F
2.5%	11.3%	11.0%	75.2%	5.0%	20.0%	14.6%	60.4%
Acc. Vas 2F & CIR 1F	Acc. Vas 2F / Rej. CIR 1F	Rej. Vas 2F / Acc. CIR 1F	Rej. Vas 2F & CIR 1F	Acc. CIR 2F & BK	Acc. CIR 2F / Rej. BK	Rej. CIR 2F / Acc. BK	Rej. CIR 2F & BK
0.0%	9.2%	2.7%	88.1%	0.2%	7.9%	1.5%	90.4%
Acc. Vas 2F & BK	Acc. Vas 2F / Rej. BK	Rej. Vas 2F / Acc. BK	Rej. Vas 2F & BK	Acc. CIR 2F & HL	Acc. CIR 2F / Rej. HL	Rej. CIR 2F / Acc. HL	Rej. CIR 2F & HL
2.1%	24.0%	1.3%	72.7%	1.5%	19.2%	12.5%	66.9%
Acc. Vas 2F & HL	Acc. Vas 2F / Rej. HL	Rej. Vas 2F / Acc. HL	Rej. Vas 2F & HL	Acc. CIR 1F & BK	Acc. CIR 1F / Rej. BK	Rej. CIR 1F / Acc. BK	Rej. CIR 1F & BK
2.7%	11.3%	2.3%	83.8%	0.0%	12.7%	2.7%	84.6%
Acc. Vas 1F & CIR 2F	Acc. Vas 1F / Rej. CIR 2F	Rej. Vas 1F / Acc. CIR 2F	Rej. Vas 1F & CIR 2F	Acc. CIR 1F & HL	Acc. CIR 1F / Rej. HL	Rej. CIR 1F / Acc. HL	Rej. CIR 1F & HL
3.8%	17.1%	21.0%	58.1%	9.4%	18.1%	5.6%	66.9%
Acc. Vas 1F & CIR 1F	Acc. Vas 1F / Rej. CIR 1F	Rej. Vas 1F / Acc. CIR 1F	Rej. Vas 1F & CIR 1F	Acc. BK & HL	Acc. BK / Rej. HL	Rej. BK / Acc. HL	Rej. BK & HL
17.9%	7.1%	6.0%	69.0%	0.0%	7.9%	21.0%	71.0%
Acc. Vas 1F & BK	Acc. Vas 1F / Rej. BK	Rej. Vas 1F / Acc. BK	Rej. Vas 1F & BK				
0.0%	9.8%	1.7%	88.5%				

Table 15.2: Results of Pairwise P-Tests

15.2 Fit to Cap and Floor Prices

The parameters determined when calibrating the short rate models to the observed values of pure discount bonds, as described in Chapter 15.1, were used to investigate the ability of these short rate models to price contingent claims such as caps and floors. The rationale behind using the same parameter values, even though such values were unlikely to be those which would minimize the difference between the observed and theoretical cap and floor prices, was to investigate whether a short rate model which fitted the yield curve relatively well would be able to provide realistic values when pricing contingent claims.

15.2.1 Goodness-of-Fit

Table 15.3 contains summary statistics for the two goodness-of-fit indicators which have previously been used. Figure 15.13 shows the second of these indicators - the standard error of the regression - across the different maturities, for each datapoint and for all six models.

One can clearly see that these short rate models struggle to provide values for at-the-money caps and floors which are close to those observed in the market. The best fitting model is the Ho-Lee model - a model which

15.3. FIT TO CAP AND FLOOR PRICES AFTER PARAMETERS RECALIBRATED 105

		Mean	St Dev	Min	Q1	Median	Q3	Max
Ho Lee	Ave $ \epsilon $	1.190%	0.601%	0.545%	0.781%	0.926%	1.513%	3.205%
	σ_ϵ	0.833%	0.601%	0.145%	0.637%	0.782%	0.993%	1.523%
Black-Kara	Ave $ \epsilon $	3.549%	2.014%	0.191%	2.225%	2.779%	5.229%	7.993%
	σ_ϵ	1.803%	2.014%	0.110%	1.234%	1.626%	2.583%	3.646%
CIR One Factor	Ave $ \epsilon $	4.093%	1.725%	0.047%	3.037%	3.701%	5.230%	7.570%
	σ_ϵ	2.251%	1.725%	0.047%	1.876%	2.253%	2.928%	3.556%
Vasicek One Factor	Ave $ \epsilon $	5.412%	3.955%	0.023%	2.296%	3.756%	8.752%	14.597%
	σ_ϵ	2.936%	3.955%	0.023%	1.382%	2.227%	4.650%	7.527%
Vasicek Two Factor	Ave $ \epsilon $	12.983%	8.622%	0.219%	5.402%	11.709%	20.526%	28.915%
	σ_ϵ	7.181%	8.622%	0.094%	3.269%	6.891%	11.065%	15.560%

Table 15.3: Summary Statistics of Fit to Cap Prices Computed over the 480 Observation Dates

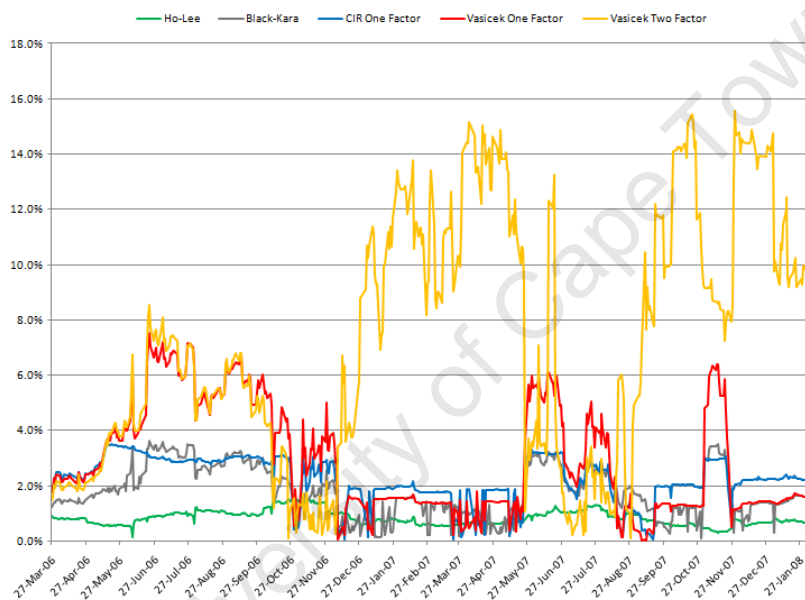


Figure 15.13: Standard Deviation of the Error Term for Cap Prices

had a fairly poor fit to the observed annualized yields and pure discount bond. The Vasicek two factor model was one of the better fitting models to the observed annualized yields and pure discount bond prices however, when using these same parameters, this model is an extremely poor fit to the observed cap and floor prices.

15.3 Fit to Cap and Floor Prices after Parameters Recalibrated

Since the short rates models calibrated to the yield curve do not provide realistic values when pricing caps and floors, the short rate models have

been recalibrated. On this occasion, the models were calibrated so as to minimize the pricing difference between the theoretical and observed value of an at-the-money cap with term ten years and tenor three months. Table 15.4 contains summary statistics for the two goodness-of-fit indicators which have previously been used, as well as for the average error. This last measure will allow one to see whether the short rate model is typically overpricing or underpricing the cap. If this measure is negative, then the actual price is smaller than the theoretical price hence the model is overpricing the instrument. Figure 15.14 shows the second of these indicators - the standard error of the regression - across the different maturities, for each datapoint and for all six models.

One can clearly see that the Cox-Ingersoll-Ross fits the observed prices the

		Mean	St Dev	Min	Q1	Median	Q3	Max
Ho Lee	Ave ϵ	-0.252%	0.277%	-1.518%	-0.437%	-0.300%	-0.114%	0.377%
	Ave $ \epsilon $	0.381%	0.155%	0.220%	0.282%	0.364%	0.448%	1.518%
	σ_ϵ	0.306%	0.042%	0.191%	0.274%	0.300%	0.330%	0.438%
Black-Kara	Ave ϵ	0.156%	0.218%	-0.303%	-0.060%	0.098%	0.382%	0.554%
	Ave $ \epsilon $	0.219%	0.165%	0.028%	0.085%	0.122%	0.398%	0.554%
	σ_ϵ	0.145%	0.095%	0.032%	0.071%	0.088%	0.234%	0.427%
CIR One Factor	Ave ϵ	-0.059%	0.171%	-0.465%	-0.189%	-0.030%	0.004%	0.608%
	Ave $ \epsilon $	0.137%	0.122%	0.018%	0.051%	0.096%	0.201%	0.608%
	σ_ϵ	0.086%	0.059%	0.021%	0.056%	0.079%	0.098%	0.386%
Vasicek One Factor	Ave ϵ	-0.283%	0.230%	-0.516%	-0.394%	-0.360%	-0.280%	0.600%
	Ave $ \epsilon $	0.359%	0.065%	0.231%	0.295%	0.367%	0.397%	0.600%
	σ_ϵ	0.202%	0.039%	0.133%	0.173%	0.200%	0.226%	0.351%
Vasicek Two Factor	Ave ϵ	-0.055%	0.143%	-0.258%	-0.142%	-0.108%	-0.017%	0.448%
	Ave $ \epsilon $	0.151%	0.076%	0.056%	0.102%	0.140%	0.170%	0.448%
	σ_ϵ	0.120%	0.037%	0.061%	0.104%	0.119%	0.135%	0.264%

Table 15.4: Summary Statistics of Fit to Cap Prices Computed over 100 Observation Dates

closest over the 100 observation dates. The Vasicek two factor model and Black-Karasinski model also appear to provide suitable prices for the caps. Interestingly, the Black-Karasinski model is the only model which does not consistently overprice the caps. The reason for this is the underlying distributions of the models as there is a significant difference in skewness between the normal and chi-squared distributions and the lognormal distribution.

15.3. FIT TO CAP AND FLOOR PRICES AFTER PARAMETERS RECALIBRATED107

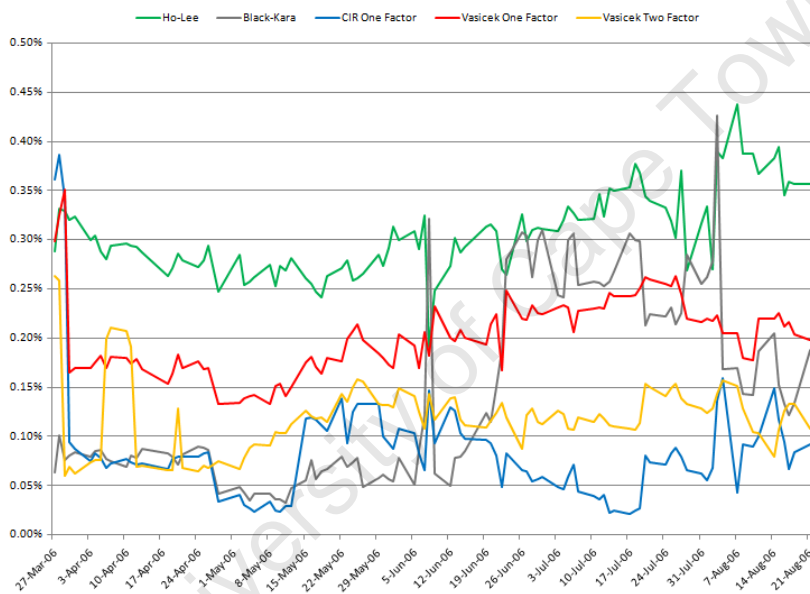


Figure 15.14: Standard Deviation of the Error Term for Cap Prices

Chapter 16

Conclusion

The appeal of short rate models is their level of simplicity, as they allow one to determine the entire yield curve through only one factor. However, this level of simplicity comes at a cost. Without the introduction of a time-dependent parameter, short rate models are unable to fit exactly to the observed values.

The aim of this thesis was to investigate the ability of short rate models to fit the yield curve and hence, to price contingent claims such as caps and floors. The initial hypothesis was that the parameter values determined through the calibration process of fitting the short rate models to the yield curve would provide fairly accurate values when used to price contingent claims. Unfortunately this proved not to be the case. All six of the short rate models struggled to closely fit the cap prices. The model which had performed the best in terms of fitting to the observed yield curve, had the poorest fit to the cap prices.

Once new parameter values had been determined after recalibrating the short rate models to the cap and floor prices, the fit of the six short rate models was significantly closer to the observed option prices. This inconsistency in parameter values makes the short rate models less appealing as one cannot simply calibrate to the yield curve when seeking to price an exotic interest rate option. Instead one needs to find an instrument that is frequently traded and whose price process is believed to be similar to that of the exotic option, in order to effectively calibrate the short rate models.

In terms of fitting to the observed yield curve, the Vasicek two factor model was the most appealing model in terms of the goodness-of-fit measures, although the Cox-Ingersoll-Ross two factor model and the Vasicek one factor model both performed well under one of the goodness-of-fit measures. The drawback of goodness-of-fit measures is that they do not provide one with an

indication when none of the models are appropriate, instead one should use a non-nested regression regression model test, such as Davidson and MacKinnon's P-Test. When conducting pairwise tests using the P-Test, none of the models provided overwhelming evidence that they could fit closely to the observed values.

Once the short rate models had been recalibrated, this time with the models fitted to the cap prices, the fit of all of the models to the cap prices was significantly improved. The Cox-Ingersoll-Ross one factor model provided the best fit to the data, according to the goodness-of-fit criteria, however the Vasicek two factor model and Black-Karasinski model also both fitted the observed data suitably. The Black-Karasinski model was the only model which did not consistently overprice the cap values.

Bibliography

- [1] ASSA (Actuarial Society of South Africa) (2010) *Subject F205 Investment Applications*, Johannesburg: ASSA
- [2] F. Black (1976) *The Pricing of Commodity Contracts*, Journal of Financial Economics: 3, 167-179
- [3] F. Black, P. Karasinski(1991) *Bond and Option Pricing when Short Rates are Lognormal*, Financial Analysts Journal: 47, 52-59
- [4] M. Brigo, F. Mercurio (2006) *Interest Rate Models - Theory and Practice (With Smaile, Inflation and Credit)*, Second Edition. New York: Springer
- [5] A. Cairns (2004) *Interest Rate Models: An Introduction*, United Kingdom: Princeton University Press
- [6] A. Cairns (2004) *Interest Rate Models*, Encyclopedia of Actuarial Science
<http://www.ma.hw.ac.uk/~andrewc/papers/ajgc33.pdf>
- [7] D. Chapman, J. Long, N. Pearson (1998) *Using Proxies for the Short Rate: When are Three Months Like an Instant*, The Review of Financial Studies: 12 [4], 763-806
- [8] R. Chen, L. Scott (1992) *Pricing Interest Rate Options in a Two-Factor Cox-Ingersoll-Ross Model of the Term Structure*, The Review of Financial Studies: 5[4], 613-636
- [9] J. Cox, J. Ingersoll, S. Ross (1985) *A Theory of the Term Structure of Interest Rates*, Econometrica: 53, 385-407
- [10] C. Cuchiero (2005) *Affine Interest Rate Models - Theory and Practice*
<http://www.vif.ac.at/cuchiero/thesis.pdf>
- [11] R. Davidson, J. MacKinnon (1981) *Several Tests for Model Specification in the Presence of Alternative Hypotheses*, Econometrica: 49 [3], 781 - 793

- [12] R. Davidson, J. MacKinnon (1993) *Estimation and Inference in Econometrics*, New York: Oxford University Press
- [13] G. Fisher, M. McAleer (1981) *Alternative Procedures and Associated Tests of Significance for Non-Nested Hypotheses*, Journal of Econometrics: 16, 103 - 119
- [14] H. Geman, N. El Karoui, J. Rochet (1995) *Changes of Numeriare, Changes of Probability Measure and Pricing of Options*, Journal of Applied Probability: 32, 443-458
- [15] W. Greene (2003) *Econometric Analysis*, Fifth Edition. New Jersey: Prentice Hall
- [16] J. Hibbert, P. Mowbray, C. Turnbull (2001) *A Stochastic Asset Model and Calibration for Long-Term Financial Planning Purposes*, Technical Report: Barrie and Hibbert Ltd
- [17] T. Ho, S. Lee (1986) *Term Structure Movements and the Pricing of Interest Rate Contingent Claims*, The Journal of Finance: 41, 1011-1029
- [18] J. Hull (2006) *Options, Futures and other Derivatives*, Sixth Edition. Upper Saddle River, New Jersey: Pearson Prentice-Hall
- [19] J. Hull *Technical Note No 14*: Options, Futures and other Derivatives*, <http://rotman.utoronto.ca/~hull/TechnicalNotes/TechnicalNote14.pdf>
- [20] J. Hull *Technical Note No 15*: Options, Futures and other Derivatives*, <http://rotman.utoronto.ca/~hull/TechnicalNotes/TechnicalNote15.pdf>
- [21] J. Hull, A. White (1993) *One-Factor Interest Rate Models and the Valuation of Interest Rate Derivative Securities*, Journal of Finance and Quantitative Analysis: 28, 235-254
- [22] J. Hull, A. White (1994a) *Branching Out*, Risk: 7, 34-37
- [23] J. Hull, A. White (1994b) *Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models*, The Journal of Derivatives: 2, 37-47
- [24] P. Hagan, G. West (2006) *Interpolation Methods for Curve Construction*, Applied Mathematical Finance: 13[2], 89-129
- [25] P. Hagan, G. West (2008) *Methods for Constructing a Yield Curve*, WILMOTT Magazine: May, 70-81

- [26] P. Hunt, J. Kennedy (2000) *Financial Derivatives in Theory and Practice*, Chichester: Wiley
- [27] F. Jamshidian (1989) *An Exact Bond Option Formula*, The Journal of Finance: 44, 205-209
- [28] P. Jorion (2007) *Financial Risk Manager Handbook*, Fourth Edition. New Jersey: Wiley Finance
- [29] G. Judge, C. Hill, W. Griffiths, H. Lutkepohl, T. Lee (1982) *Introduction to the Theory and Practice of Econometrics* New York: John Wiley and Sons
- [30] G. Mizon, J-F. Richard (1986) *The Encompassing Principle and its Application to Testing Non-Nested Hypotheses*, *Econometrica*: 54, 657-678
- [31] M. Musiela, M. Rutkowski (2007) *Martingale Methods in Financial Modelling*, Second Edition Berlin: Springer
- [32] P. Ouwehand (2008) *Interest Rate Modelling*, Lecture Notes
- [33] A. Pelsser (2004) *Efficient Methods for Valuing Interest Rate Derivatives*, Second Edition. Great Britain: Springer
- [34] PWC (2008) *PGN 110 ESG and Calibration Survey*
http://pwc.com/en_ZA/za/assets/pdf/pwc-110-insurance-survey-08.pdf
- [35] D. Rouah, G. Vainberg (2007) *Option Pricing Models and Volatility using Excel-VBA* New Jersey: Wiley Finance
- [36] S. Svoboda (2002) *An Investigation of Various Interest Rate Models and their Calibration in the South African Market*
<http://www.finmod.co.za/research.html>
- [37] A. Uboldi *Modelling Interest Rates for Public Debt Management*
<http://www.mat.uniroma1.it/ricerca/dottorato/TESI/ARCHIVIO/ulbodiadamo.pdf>
- [38] O. Vasicek (1977) *An Equilibrium Characterization of the Term Structure*, *Journal of Financial Economics*: 5, 177-188
- [39] G. West (2008) *South African Financial Markets*
<http://www.finmod.co.za/safm.pdf>
- [40] G. West (2009) *Interest Rate Derivatives: Lecture Notes*
<http://www.finmod.co.za/ird.pdf>

- [41] Y. Wu, H. Zhang (1996) *Mean Reversion in Interest Rates: New Evidence from a Panel of OECD Countries*, *Journal of Money, Credit and Banking*: 28[4.1]

University of Cape Town