A FACTORIAL STUDY OF ARITHMETICAL ABILITY

by

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A C K N O W L E D G M E N T S

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CHAPTER 1.

SURVEY OF PREVIOUS INVESTIGATIONS

The problem of human abilities and of the organization or structure of the mind has from time immemorial been a subject of speculation and study for the layman as well as the psychologist. The layman's notions as to what abilities exist, and what each ability includes or excludes, are extremely loose. Teachers and parents generally accept the existance of specific abilities. Such remarks as: "John has a good head for figures"; "Mary has a poor memory", etc., are heard every day. A specefic mathematical ability with which alone this thesis is concerned, is generally accepted by laymen.

In the history of Mathematics, too, there are indications that arithmetical ability is distinct from general intellectual ability. Binet[1], Scripture[2], Mitchell[3] and others have given biographical accounts of mathematical prodigies who had little or no formal education and gave evidence of their unusual ability at a very early age, although in other respects they were distinctly stupid. Superior intellects,too, performed feats of calculation in early childhood, that to

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1. Binet,A.: Grands Calculateurs, 1894
2. Scripture, E.W.: Arithmetical Prodigies,Amer.J.Psych. 1891
3. Mitchell, F.D. Mathematical Prodigies, Amer. J. Psych.1907

the adults about them, seemed to savour of the miraculous. Here then we have evidence of what appears to be a specific native ability for number.

Since the development of mental tests and correlational methods in the present century numerous studies have been made, which shed light on this problem. Burt[1] in a pioneer study in 1917 showed that achievement in various school subjects could be accounted for by (a) a general factor — general educational ability, (b) a number of group-factors among which there is an arithmetical factor and (c) specific factors in each test.

Spearman[2] who originated factor methods, showed that diverse tests of mental ability usually gave inter-correlations which could be wholly accounted for by a single general factor plus specific factors. He claimed that "of 'special abilities' sufficiently broad to admit of measurement..... there have been but the scantiest indications ..... Among the exceptional cases where specific correlations and group factors do become of appreciable magnitude, the four most important have been in respect of what may be called the logical, the mechanical, the psychological and the arithmetical abilities".

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1. Burt,C., The Distribution and Relations of Educational Abilities, L.C.C. Report, 1917, pp. 51 - 54.
2. Spearman, C., The Abilities of Man, 1927, pp.241-42.

Using the results of Collar, Rogers and others and applying what he calls 'the sole valid criterion' namely the tetrad equation, Spearman further claimed that (a) " the inter- correlations of arithmetical abilities are traceable to a single factor over and above *g* , and (b) " there appears no real basis for the common opinion which would take arithmetic and geometry to furnish one single special ability. Their union as mathematics seems rather to be one of practical convenience".

These findings which were confirmed by Alexander[1] in 1935, indicate that on the cognitive side achievement in arithmetic can be accounted for by two factors, namely a general factor and a specific factor common to all the arithmetical processes.

The opposite of this view is that held by Thorndike[2] and other American psychologists who conceived the mind to consist of a large number of highly particularised and independent abilities. To take but one example of Thorndike's analysis: " there are at least seven minor functions involved in two-place column addition each of which is psychologically distinct and requires distinct educational treatment".

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1. Alexander, W.P. , Intelligence: Concrete and Abstract, Brit. J. Psych. , Mon. Sup., 19, 1935, p. 86.

2. Thorndike, E.L., The Psychology of Arithmetic, p.52.

In a study of scholastic achievement at different levels of maturity Kelley[1] in 1928 found , amongst others, a specific arithmetical factor.

Cunningham and Price[2] in a nationwide survey of arithmetical ability in Australia showed by methods of partial correlation that " we may regard the ability to work problems in arithmetic and the ability to compute accurately as being almost, though not quite distinct from one another".

In 1938 Thurstone[3] using multiple factor methods developed by himself, found in a battery of 56 tests generally supposed to be saturated with "g", a number of independent primary factors and no common factor. One of the clearest of these primaries was number ability characteristically defined by tests of addition and multiplication. The test of arithmetical reasoning was found to be saturated mainly with a reasoning factor and an inductive factor together with other minor factors.

Subsequent studies by Wright[4] and Woodrow[5] confirmed Thurstone's main findings.

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1. Kelley,T.L., Crossroads in the Mind of Man, p.104.
2. Cunningham, K.S., and Price, W.T., The Standardisation of an Australian Arithmetic Test.
3. Thurstone, L.L., Primary Mental Abilities, 1938, pp.79-91
4. Wright, Ruth,E., A Factorial Analysis of the Original Stanford-Binet Scale, Psychometrika,4,3,1939.

In 1941 the writer[6] showed in a master's thesis that at least three and probably four factors, not including a general common factor, are necessary to explain the correlations between tests of arithmetical ability, thus confirming Thurstone's finding in this respect. The factors were (a) a reasoning factor functioning in such tests as arithmetical problems and number series, (b) a number factor functioning in the simplest arithmetical processes such as dot-counting and addition, (c) a second number factor functioning in the more involved mechanical processes such as subtraction, multiplication, division and fractions, and (d) an attention-span factor functioning mainly in problems and fractions.

The results of these multiple-factor studies appeared to be in agreement with the hypothesis that achievement in arithmetic is due to a number of factors functioning independently of one another. The present study was therefore undertaken with the object of finding these factors and determining their exact nature.

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5. Woodrow, H., The Common Factors in Fifty-two Mental Tests. Psychometrika, 4, 2, 1939.
6. Olckers, P.J., "'n Faktorontleding van 'n Battery Reken-Toetse", on file at the library of the University of S.A.

CHAPTER 2.

PRELIMINARY INVESTIGATION.

In order to test the hypothesis that the inter-correlations between tests of arithmetical ability can only be accounted for by a set of independent factors, and to obtain an approximate delineation of the field a battery of tests in Arithmetic as well as tests of general intelligence was assembled. It consisted of the following:

THE TEST BATTERY

1. Addition: A graded test varying in difficulty from a single column of two digits to five columns of five digits each. Time: 4 minutes; Scoring formula: 1 mark for each column.
2. Subtraction: A graded test varying in difficulty from a single column to five columns. Time: 4 minutes; Scoring formula: 1 mark for each answer.
3. Multiplication: A graded test varying in difficulty from one digit by one digit to five digits by two digits. Time: 4 minutes; Scoring formula: 1 mark for each figure in the multiplier.
4. Division: A graded test varying in difficulty from one digit by one digit two five digits by two digits. Time: 4 minutes; Scoring formula: 1 mark for each digit in the divisor. (These tests were devised and standardized by F.T.Milne.[1]

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1. Milne,F.T.,The Use of Scholastic Tests in S.A. Schools, National Bureau of Educ.and Soc. Res.,Pretoria,pp.29-43.

5. Roman Numerals: This test consisted of 25 numbers written in Roman numerals, which had to be converted into the ordinary notation. The key was given and explained and the test proper was preceded by a fore-exercise. Time: 4 minutes; Scoring formula: 1 mark for each item correct.

6. Dot Counting: This test consisted of 20 rows of dots grouped in different ways, which had to be counted. Time: 3 minutes; Scoring formula: 1 mark for each row counted correctly.

7. Repetition of Digits: This test consisted of fifteen items ranging from 4 digits to nine digits presented orally. Scoring formula: 1 mark for each item.

8. Fractions: This was a mixed test of the four rules applied to ordinary and decimal fractions, 24 items in all. Time: 7 minutes; Scoring formula: 1 mark for each item.

9. Problems: This test was standardised by Milne and consisted of 28 items. Time: 10 minutes; Scoring formula: 1 mark for each item.

(The following tests were taken from the S.A. Group Test of Intelligence)

10. Classification: In this test the subjects were asked to underline the one word in the row which did not belong to the same group as the others, e.g.

water coffee bread tea milk 20 items in all.

Time: 2½ minutes; Score: Number correct.

11. Analogies: This was a multiple choice test in which the subjects were asked to complete analogies like the following: clothes man; feathers .....coat hat bird tail 30 items in all. Time: 4 min. Score: Number correct.

12. Number Series: This test was similar to well-known tests of this type, e.g.

2 4 6 (8) (10) 20 items in all.

Time: 8½min. Score: number of items correct.

13. Letter Test: In this test the subjects had to respond to a series of questions by letters selected from a series given at the top of the page, e.g. Write down the second letter in the row. 20 items in all. Time: 8 minutes. Score: Number correct.

14. Same-Opposite: Here 40 pairs of words were given, which the subjects had to mark as similar or opposite. Time: 4 min. Score: Number correct minus number wrong.

15. Figure Test: Here the subjects had to answer 20 questions in connection with composite geometrical figures. Time: 8 minutes; Score: number correct.

THE SUBJECTS

The subjects were 112 standard VII pupils of the Parow High School. There were 46 girls and 66 boys. The ages varied from 12 years 9 months to 16 years 8 months with an average of 14 years 4 months (S.D. 3.27 months) ?

The subjects were selected on the basis of school standard attained rather than on the basis of age to rule out as far as possible the influence of instruction in arithmetic. If the subjects had all been taken from the 14 year group it would have been possible to have pupils ranging, as far as scholastic attainment is concerned, from Std.V to Std IX with the result that some of the correlations especially those between the arithmetic tests, were bound to be spurious.

The I.Q.s of the group ranged from 82 to 132 with an average of 105.98 (S.D. 9.24). The group was therefore fairly homogeneous with respect to both scholastic attainment and intelligence.

THE APPLICATION OF THE TESTS

The tests were given in three sessions lasting from 30 to 40 minutes each. Each test was preceded by an explanation on the board and a fore-exercise. Between each pair of tests there was a break for handing out and taking in papers, during which time the subjects were allowed to relax and talk freely. All the subjects co-operated well and were keen to take the tests as it relieved them from ordinary routine work.

THE RESULTS

The distributions, means and standard deviations of the raw scores are given in table 1.

TABLE 1

Distribution of raw scores

Addition		Subtraction		Multiplication		Division		Roman Numerals	
Score	f	Score	f	Score	f	Score	f	Score	f
19	2	23	1	16	1	19	1	0	1
20	2	24	1	17	1	20	2	1	1
21	5	25	2	18	3	21	0	2	1
22	2	26	5	19	1	22	2	3	3
23	1	27	6	20	5	23	5	4	4
24	2	28	9	21	8	24	12	5	2
25	11	29	9	22	6	25	11	6	2
26	9	30	13	23	11	26	13	7	2
27	9	31	22	24	8	27	22	8	15
28	19	32	17	25	9	28	17	9	17
29	14	33	11	26	8	29	15	10	15
30	10	34	6	27	13	30	6	11	17
31	10	35	5	28	10	31	3	12	8
32	8	36	2	29	8	32	1	13	8
33	5	37	2	30	8	33	1	14	5
34	1	38	1	31	4	34	1	15	5
35	1			32	5			16	1
36	1			33	1			17	1
				34	0				
				35	1				
				36	0				
				37	1				
M.	27.9		30.8		25.7		26.8		9.8
S.D.	3.5		2.8		4.0		2.8		3.1

Table 1 (continued)

Dot Counting		Repetition of digits		Fractions		Problems		Classification	
Score	f	Score	f	Score	f	Score	f	Score	f
7	3	0	2	2	2	6	1	5	1
8	0	1	2	3	1	7	2	6	1
9	1	2	3	4	3	8	3	7	3
10	3	3	14	5	7	9	7	8	6
11	8	4	15	6	5	10	6	9	6
12	9	5	21	7	6	11	11	10	10
13	15	6	18	8	7	12	12	11	19
14	10	7	19	9	10	13	15	12	23
15	22	8	8	10	16	14	18	13	14
16	20	9	4	11	16	15	15	14	12
17	7	10	4	12	13	16	8	15	6
18	9	11	2	13	11	17	5	16	4
19	2			14	8	18	4	17	6
20	3			15	3	19	4	18	0
				16	2	20	3	19	1
				17	1	21	0		
				18	0	22	0		
				19	1	23	1		
M.	14.4		5.5		10.1		13.6		12.0
S.D.	2.7		2.2		3.9		3.2		2.6

Table 1 (continued)

Analogies		Number Series		Letter Test		Same Opposite		Figures	
Score	f	Score	f	Score	f	Score	f	Score	f
10	6	0	2	1	11	20	13	3	2
11	1	1	0	2	1	21	1	4	1
12	2	2	1	3	0	22	2	5	0
13	5	3	3	4	1	23	1	6	2
14	8	4	2	5	2	24	4	7	12
15	2	5	2	6	4	25	2	8	8
16	3	6	9	7	4	26	8	9	9
17	4	7	8	8	7	27	1	10	14
18	9	8	12	9	12	28	9	11	14
19	10	9	11	10	12	29	1	12	16
20	5	10	14	11	14	30	15	13	13
21	9	11	12	12	11	31	9	14	8
22	5	12	7	13	12	32	11	15	4
23	6	13	14	14	10	33	2	16	4
24	13	14	3	15	7	34	15	17	2
25	6	15	6	16	4	35	4	18	2
26	9	16	3			36	11	19	1
27	4	17	1			37	3		
28	4	18	1			38	3		
29	1					39	2		
						40	2		
M.	19.9		9.8		9.9		30.1		11.0
S.D.	5.0		3.5		4.1		5.5		3.1

The factorial methods assume that each test performance can be expressed in first approximation as a linear function of the several factors, and hence that each of the primary abilities is distributed normally in the experimental population. From this it follows that the distributions of the scores in the tests should be normal. The most complete procedure would therefore be to normalize each of the distributions of raw scores before computing the correlation coefficients. An inspection of table 1, however, shows that the distributions are unimodal, as was to be expected, and fairly normal. It was therefore deemed unnecessary to normalise any of the raw distributions.

THE INTER-CORRELATIONS

The inter-correlations between the fifteen tests were calculated by the Bravais-Pearson formula to four places of decimals. To save space the results multiplied by 100, are given correct to two figures in table 2.

Of the 105 correlation coefficients all are positive excepting seven which are small. The correlations are generally low, which is probably due to the homogeneity of the group. Kelley[1] contends that " to secure a reliability coefficient of .40 from a group composed of children in a

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1. Kelley, T.L., The Reliability of Test Scores, J. Educ. Res., May 1921, p.374.

TABLE 2

THE CORRELATION MATRIX

Test	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	..														
2	49														
3	47	57													
4	49	59	55												
5	09	08	-04	05											
6	34	32	39	24	-04										
7	11	03	07	24	09	07									
8	14	40	29	39	34	00	20								
9	18	27	12	36	35	02	32	47							
10	21	20	13	11	28	-04	18	28	20						
11	05	12	08	15	41	07	26	27	24	28					
12	27	21	25	17	20	11	05	47	51	36	28				
13	13	04	14	16	22	10	20	28	17	51	35	25			
14	02	-02	-02	06	22	-20	15	31	43	31	28	11	20		
15	15	22	18	31	19	-12	04	42	35	36	18	37	32	18	..

single grade is probably indicative of greater, not less, reliability than to secure a reliability coefficient of .90 from a group composed of children from the second to the twelfth grades.

The average correlation coefficient is .2262 (± .0605) which is more than 3½ times its P.E. It may therefore be assumed that there is a positive relationship between the inter-correlated variables.

THE FACTORIAL ANALYSIS

The correlation matrix was factored to five factors by Thurstone's[1] centroid method with guessed communalities. In the absence of a more reliable criterion the residuals were compared with the P.E. of the average correlation coefficient to determine whether further factors should be removed. After five factors had been removed the average residual fell to .0360 which is well below .0605, the P.E of the average correlation coefficient, thereby indicating that further factors would be unreliable. Only 20 of the 105 residuals were above this value, the highest being .1189. It was therefore decided to retain five factors only, as sufficient to explain the inter-correlations between the 15 tests. The results are recorded in table 3.

1(a) Thurstone, L.L., The Vectors of Mind, pp. 232 - 250

(b) Thomson, G.H., The Factorial Analysis of Human Ability, pp. 30 - 41.

(c) Guilford, J.P., Psychometric Methods, pp. 457 - 513.

TABLE 5

THE CENTROID MATRIX

	I	II	III	IV	V	h_j^2
Test 1	.4758	-.4263	-.1526	-.1058	-.0872	.4441
2	.5563	-.5172	.1199	.1129	.0965	.5183
3	.5041	-.5505	-.0824	.0569	.0903	.5794
4	.5917	-.3941	.2095	.1279	.1324	.5832
5	.3611	.3246	.1173	-.1222	-.0896	.2725
6	.1089	-.4175	-.0917	-.1278	-.2167	.2817
7	.3016	.0088	.1760	-.2200	-.0200	.1786
8	.6523	.1560	.1694	.2749	.0895	.5753
9	.6242	.2330	.2979	.1400	-.2289	.6047
10	.5418	.2587	-.3093	-.1828	.2437	.5490
11	.4687	.2690	.0655	-.3014	-.0781	.3533
12	.5642	.1321	-.2090	.2573	-.2739	.5207
13	.4776	.2528	-.3108	-.2456	.1935	.4767
14	.3062	.4011	.1970	-.0269	.1234	.3094
15	.4897	.1757	-.0803	.3042	.2112	.4143

THE ROTATION OF THE AXES

Factorists using Thurstone's method of analysis, are generally agreed that the first factors which emerge from the 'centroid' process need not have psychological significance, and that in an attempt to interpret these factors psychologically the reference axes have to be rotated to a

significant position.[1]

Burt[2] who prefers a group-factor method, contends that the interpretation is best made from the centroid matrix by accepting a general factor and bipolar difference - factors. Column I of table 3 would thus represent a common factor, column II a bipolar factor indicating a dichotomous classification of the fifteen tests into two main categories--verbal and non-verbal, column III is a doubly bipolar factor, etc. A closer inspection of table 3, however, would show that this supposed ~~common~~ general factor plays an equally important part in practically all the tests in the battery with the exception of no.3 6 and 7, a fact which is difficult to explain on the basis of 'relational perception' as Burt would have it. The further classification of the tests into verbal, non-verbal, numerical or visual types is very superficial and hardly illuminating. To be of value an explanation should as far as possible be causal and not merely descriptive.

To solve this problem Thurstone introduced the principle of what he calls 'simple structure', which demands (a) positive loadings throughout, and (b) a maximum

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1.(a) Thurstone, L.L., The Vectors of Mind, pp.150 -197.
(b) Thomson, G.H., op. cit. pp. 242 - 256.

2. Burt, C., The Factors of the Mind, pp.310 - 319.

number of zero loadings in each column, on the assumption that a single factor must represent a single ability which acts positively unless it is absent from a performance and further that it is almost certain that no one test will involve all the factors or abilities. Two objections may be raised against this procedure, namely (a) that it excludes the possibility of a general factor and (b) that it is too mechanical and hence is bound to lead to anomalous results.

A third procedure has been developed by Reyburn and Taylor[1], which to the writer seems the only truly scientific method. It starts from the point of view that " common factors, to be of value to us , must not be relative to the batteries of tests into which they enter and by which they are measured,.....and that all that the analysis of a battery can yield is a hypothesis or the confirmation of a hypothesis" Consequently using all the knowledge that we already have about hte tests in the battery a hypothesis is constructed concerning a common factor and a single axis located, which is provisionally accepted only if the factor loadings measured on it are reasonable and intelligible for all the variables in the battery. The whole centroid matrix

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1. Reyburn,H.A., and Taylor,J.G., On the Interpretation of Common Factors: A Criticism and a Statement, Psychometrika, March 1943, pp. 53 - 64.

is then rotated in such a way that all the other axes are at right angles to one another and to the first. Thereafter the process is repeated dropping the first column. Finally the factors are considered together and adjusted where their intelligibility and coherence with other analyses can be improved.

Applying this method in the present investigation, the first axis was passed through test 7, Repetition of digits, because with this test there is a definite break in the correlation matrix as an inspection of table 2 will show. Further this type of test had been found by Spearman[1] to be a disturbing factor in his test for a general factor. It was thought that this axis would show up some sort of immediate memory factor which seemed to enter into many of the tests. The second axis was passed through tests 5, Roman Numerals, 9, Problems, 12, Number Series, and 14, Same-Opposite, on the assumption that these tests involved a common factor of an inductive-deductive nature, such a factor having been found by Thurstone[2] and others. The third axis was passed through tests 1, 3 and 6 for the number factor which had been found by different investigators. The remaining columns were then plotted against one another

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1. Spearman,C., The Abilities of Man, p.288.
2. Thurstone, L.L., and T.G., Factorial Studies of Intelligence, Univ.of Chicago Press,1941,p.6.

and it was found that an axis through tests 2, 4, 8 and 15 would minimise the negative loadings. The fifth column was then adjusted with the second and the third with the first by slight rotations. In this manner the transformation matrix given in table 4 was obtained. The resulting factor matrix is given in table 5.

TABLE 4

THE TRANSFORMATION MATRIX

	I	II	III	IV	V
I	.5014	.2907	.6640	.3289	.3589
II	.4434	.3876	-.5366	-.4004	.4521
III	.5623	.1170	-2802	.3237	-.6975
IV	-.4743	.6956	-.2155	.4920	.0493
V	-.1033	-.5172	-.3820	.6197	.4375

TABLE 5

THE FACTOR MATRIX

		I	II	III	IV	V
1	Addition	.0152	-.0696	.6375	.1792	.0177
2	Subtraction	.0728	.0070	.5544	.5405	-.0840
3	Multiplication	-.0553	-.0836	.6065	.4435	.0217
4	Division	.1927	.0643	.4675	.5652	-.0597
5	Roman Numerals	.4397	.2059	.0993	-.0889	.1421
6	Dot-Counting	-.0962	-.0945	.4855	.0025	-.1618
7	Repetition of Digits	.3918	.0000	.1583	.0000	.0000
8	Fractions	.3753	.4178	.2152	.4009	.2295
9	Problems	.4938	.5224	.2632	.1355	.0159
10	Classification	.3243	-.0317	.2539	.0356	.6139
11	Analogies	.5260	.0789	.2432	-.1291	.1857
12	Number Series	.0736	.5114	.4115	.0218	.2895
13	Letter Test	.3044	-.0783	.2582	-.0377	.5564
14	Same-Opposite	.4677	.1850	-.1084	.0671	.2004
15	Figures	.1558	.3034	.1072	.3453	.4088

THE INTERPRETATION OF THE FACTORS

FACTOR I: The tests with factor-loadings of .30 or higher in column I of table 5 are

(11) Analogies ..... .5260

(9) Problems ..... .4938

(14) Same-Opposite..... .4677

(	(15) Roman Numerals	.4397
	(7) Repetition of Digits.....	.3918
	(8) Fractions	.3753
	(10) Classification.....	.3243
	(13) Letter Test.....	.3044

From an inspection of these tests and introspection by testees it was clear that concentration and retentive power played an important role in the successful solution of every one of these tests. As was expected test 7, Repetition of Digits, showed a considerable loading on this factor. It was therefore tentatively identified as a retention factor.

FACTOR II: The tests with significant loadings in the second column are

	(9) Problems	.5224
	(12) Number Series	.5114
	(8) Fractions	.4178
	(15) Figures.....	.3034
	(5) Roman Numerals.....	.2059

The ability to select essentials and to see relationships seemed to play an important part in all these tests. It was therefore tentatively identified as a reasoning factor of an inductive-deductive character. The high loading of Number Series on this factor seemed to stress its inductive character. The projections of all the tests on this axis seemed to be in agreement with this hypothesis with

the exception of the classification and analogies tests. It was difficult to see why these tests which also involved relational thinking should have negligible loadings on this factor. These tests, however, appeared to differ from the problems and number series tests in this respect that they did not require selection and the application of general principles to the same extent.

FACTOR III. In the third column the tests with significant loadings are

(1)	Addition	.6375
(3)	Multiplication	.6065
(2)	Subtraction	.5544
(6)	Dot-Counting	.4855
(4)	Division	.4675
(12)	Number Series	.4115
(9)	Problems	.2632
(13)	Letter Test	.2582
(10)	Classification	.2539
(11)	Analogies	.2432

From the above table it was clear that the simplest numerical tests had the highest saturations of this factor, and that the more involved the processes become the lower the loadings are. Speed of association seemed to play an important role in all these tests. It was therefore tentatively described as speed in performing simple well-practised tasks of a continuous nature. The loadings of all

the tests in the battery were in agreement with this hypothesis. The only one that was questioned was Number Series with a loading of .4115. Introspection, however, showed that the first items of this test were so simple that the answer followed immediately upon reading the question. Furthermore such series as 2 4 6 8 10, etc are fairly well practised.

Another significant fact about this factor, which was revealed by this analysis, is that it enters to a lesser extent into some non-numerical tests in which speed seems to play an important role.

FACTOR IV: The tests with significant loadings in the fourth column are

(4)	Division.....	.5652
(2)	Subtraction.....	.5405
(3)	Multiplication.....	.4435
(8)	Fractions.....	.4009
(15)	Figures	.3453

Four of these tests dealt with number and the fifth with geometrical figures where the relative size and position of the separate parts were of importance. These tests seemed to differ from the rest by the greater amount of manipulation involved in them. This factor was therefore tentatively described as facility in manipulating symbols according to well-known rules.

FACTOR V: In the fifth column the tests with significant loadings are

(10) Classification	.6139
(13) Letter Test	.5564
(15) Figures	.4086
(12) Number Series	.2895
(14) Same-Opposite	.2004
(8) Fractions	.2295

As this was the residual factor and it was difficult to decide what these tests had in common, no attempt was made to define this factor. The classification and letter tests suggested a verbal factor, whereas the figures test pointed to a space factor.

DISCUSSION

This analysis yielded several interesting results. Firstly no general factor was established, although there was nothing in the method of rotating the axis to prevent this. The arithmetical tests as well as the tests in the S.A. Group Test, which were supposed to be saturated with a general common factor, divided their variance among in - dependent factors, thus confirming Thurstone's finding in this respect.

Secondly no common factor was found to function in all the tests of arithmetical ability. On the contrary four major factors were involved, namely, a retention factor, a reasoning factor, a speed factor and a manipulation factor.

Thirdly none of these factors appeared to be confined to arithmetic alone. The number factor, e.g., which was characteristically defined by the tests of addition and multiplication, entered to a lesser extent into some of the purely verbal tests. Consequently there appears to be no grounds for the popular acceptance of a specific arithmetical ability. Number tests appear to constitute but good examples of a more basic ability.

The hypotheses created by these findings, however, required further testing in a different battery and with a different experimental population before they could be accepted as being in any way objective, for as Thomson[1] has pointed out, it is " very doubtful indeed whether any factors and any factor loadings have absolute meaning. They appear to be entirely dependent upon the population in which they are measured, and for their definition there would be required not only a given set of tests and a given technical procedure in analysis, but also a given population of persons".

In addition there were several points in connection with the exact nature of the common factors, which had to be cleared up. Most of the tests used in this battery were too complex to make a clear interpretation of the factors possible. It was therefore necessary to devise simple tests to measure the different abilities involved in arithmetic.

1. Thomson, G.H., Op. cit., p.194.

CHAPTER 3

THE SECOND TEST EXPERIMENT.

In order to test the hypotheses which were constructed on the results of the preliminary investigation, a new test battery was drawn up, consisting of the following tests:

THE TEST BATTERY:

1. Four Rules A. This test consisted of alternate rows of two-digit addition, subtraction, multiplication and division sums with whole numbers, involving not more than one step each, e.g.

Add $\begin{array}{ccc} 4 & 5 & 7 \\ 3 & 0 & 8 \\ \hline \end{array}$ ..... Multiply $\begin{array}{ccc} 4 & 5 & 0 \\ 3 & 6 & 8 \\ \hline \end{array}$

Subtract $\begin{array}{ccc} 11 & 9 & 5 \\ 6 & 3 & 0 \\ \hline \end{array}$ ..... Divide $4\overline{)12}$ $8\overline{)24}$ $7\overline{)42}$ .....

Time: 4 minutes. Score: number correct.

It was thought that this test would define the nature of the speed factor involved in computation.

2. Simple Opposites: This test consisted of 80 simple words of which the subjects were required to indicate the opposite by writing the first letter only. Time: 2½ min. Score: number correct.

It was thought that this type of word test would call for the same sort of ability as the simple number tests.

3. Alphabet: This test consisted of 80 pairs of letters, e.g., AB MN IJ , etc. The subjects were instructed to write down the letter following upon each pair in the alphabet. Time: 2½ min. Score: number correct.

This test was designed to determine to what extent associative power and practice with the material involved entered into number ability.

4. Four Rules B: This test consisted of alternate pairs of addition, subtraction, multiplication and division sums, involving four steps each. The subjects were instructed to do the numbers in order and not to pick out one type. Time: 6 minutes. Score: number correct.

5. Cancellation of 9's: This test consisted of rows of figures and the subjects were asked to cancel the 9's. Time: 3 minutes. Score: number cancelled.

This test was included in the battery as a speed test.

6. Four Rules C. This test consisted of items similar to test 4, involving column addition, subtraction, long multiplication and division. Time: 8 minutes. Score: 2 marks per item. Tests 4 and 6 were designed to check the hypothesis that different factors are involved in computation depending upon the complexity of the calculation and the amount of manipulation involved.

7. Fractions: This test consisted of 20 items comprising addition, subtraction , multiplication and division of

ordinary fractions. Time: 4 min. Score: Number correct.

8. Change: This test consisted of 20 items like the following:

	Price	Coins presented	Change
1.	3d.	sixpence	(3d)

Time: 2 minutes. Score: number correct.

9. Cancellation: This test consisted of rows of figures. The subjects were asked to cancel the 0's immediately proceded by a three, unless the 0 is followed by a seven, in which case the 7 had to be cancelled. Sufficient practice in order to become acquainted with the rule, was given before the test proper. Time: 4 min. Score: number cancelled.

It was thought that this test would give an indication of the correct nature of the retention factor identified in the preliminary investigation.

10. Problems: This test consisted of 20 graded items covering the main principles involved in arithmetic.
Time: 5 minutes. Score: number correct.

11. Number Series: This test was similar to well-known tests of this type in which the subjects were asked to complete 25 series by writing down two numbers each time.
Time: 6 minutes. Score: number correct.

12. Definitions: This test consisted of 30 definitions of terms used in arithmetic. The subjects were asked to select the first letter of the word defined from a group of five letters given in brackets, e.g.

1. The unit with which petrol is measured (E S P G T)

Time: 3 minutes. Score: number correct.

13. Repetition of Digits: This test consisted of five groups of six numbers each, ranging from four digits to eight digits. The digits were read out at the rate of about one per second. The subjects were instructed to hold up their pencils and to write only on the command which was given after each number had been read out.
Score: number of items written down correctly.

THE SUBJECTS:

The subjects were 100 Junior Certificate pupils of the Parow, Bellville and Hassan High Schools. There were 45 girls and 55 boys. All took a course in Commercial Arithmetic. Like in the preliminary investigation the subjects were selected on the basis of school standard reached in order to make the group as homogeneous as possible with regard to instruction in arithmetic.

ADMINISTRATION OF TESTS AND RESULTS.

The tests were administered in the same way as in the preliminary investigation in two sections on different days. The subjects co-operated well and worked hard throughout the testing time.

The distributions, means and standard deviations of the raw scores are given in table 6.

TABLE 6

DISTRIBUTION OF RAW SCORES

Four Rules A		Opposites		Alphabet		Four Rules B	
Score	f	Score	f	Score	f	Score	f
40–46	5	30–35	1	15–21	4	9– 12	3
47–53	9	36–40	6	22–28	5	13–16	5
54–60	10	41–45	9	29–35	13	17–20	14
61–67	12	46–50	22	36–42	12	21–24	18
68–74	20	51–55	20	43–49	17	25–28	27
75–81	18	56–60	20	50–56	12	29–32	19
82–88	14	61–65	15	57–63	14	33–36	10
89–95	7	66–70	4	64–70	10	37–40	2
96–102	2	71–75	5	71–77	9	41–44	1
103–109	2			78–84	4	45–48	1
110–116	1						
M.	71.72		53.60		49.19		25.74
S.D.	15.12		8.55		16.17		6.52

Cancellation A		Four Rules C		Fractions		Change	
Score	f	Score	f	Score	f	Score	f
16–18	2	10–11	5	2– 3	6	3– 4	2
19–21	4	12–13	6	4– 5	11	5– 6	4
22–24	8	14–15	10	6– 7	14	7– 8	10
25–27	6	16–17	14	8– 9	19	9–10	20
28–30	15	18–19	14	10–11	13	11–12	33
31–33	14	20–21	26	12–13	20	13–14	23
34–36	16	22–23	9	14–15	12	15–16	8
37–39	14	24–25	6	16–17	5		
40–42	9	26–27	6				
43–45	9	28–29	1				
46–49	3	30–31	3				
M.	30.65		19.24		9.60		11.08
S.D.	7.20		4.58		3.82		2.66

TABLE 6 (continued)

Cancell.D.		Problems		Number S.		Definitions		Repetition of Digits	
Score	f	Score	f	Score	f	Score	f	Score	f
20-25	1	2- 3	3	1- 2	13	6- 7	3	10-11	5
26-31	5	4- 5	7	3- 4	7	8- 9	11	12-13	9
32-37	3	6- 7	20	5- 6	9	10-11	15	14-15	17
38-43	7	8-9	25	7- 8	11	12-13	24	16-17	14
44-49	18	10-11	26	9-10	23	14-15	20	18-19	10
50-55	20	12-13	7	11-12	17	16-17	14	20-21	21
56-61	16	14-15	10	13-14	13	18-19	11	22-23	9
62-67	14	16-17	1	15-16	5	20-21	2	24-25	8
68-73	12	18-19	1	17-18	2			26-27	6
74-79	4							28-29	1
M.	64.3		11.2		8.78		13.36		18.54
S.D.	11.76		3.14		4.24		3.32		4.5

An inspection of this table shows that the distributions of the raw scores in all the tests are fairly normal. None of the distributions deviate sufficiently from the normal to justify adjustment. It was therefore assumed that the distributions are of the correct form for factor-analysis.

THE INTER-CORRELATIONS

The inter-correlations between the thirteen tests were calculated by the product-moment method to four decimal figures. To save space the results are given in table 7 correct to two figures and with the decimal point preceding the entries omitted.

TABLE 7

THE CORRELATION MATRIX

Test	1	2	3	4	5	6	7	8	9	10	11	12	13
1													
2	34												
3	30	36											
4	67	21	31										
5	49	40	39	30									
6	58	07	18	64	08								
7	45	05	-01	42	08	51							
8	27	17	-06	38	02	31	44						
9	05	12	11	15	46	08	09	11					
10	05	06	08	13	-01	07	33	50	02				
11	20	19	-00	28	05	29	53	56	24	49			
12	27	25	03	18	16	17	42	21	08	37	21		
13	-05	12	15	10	03	14	07	10	00	16	16	16	

Of the 78 correlation coefficients all are positive excepting 5 which are small. As was expected the coefficients are again low, the average being .2214 ($\pm$ .0642), which is approximately 3½ times the P.E. It may therefore be assumed that there is a positive relationship between the inter-correlated variables.

THE FACTORIAL ANALYSIS.

The correlation matrix was factored to four factors

by the centroid method with guessed communalities. After the fourth factor had been removed the average residual fell to .047 which was well below the P.E. of the average correlation coefficient, thereby indicating that further factors would not be reliable. The results are given in table 8.

TABLE 8

THE CENTROID MATRIX

	I	II	III	IV
1 Four Rules A	.676	-.060	.533	-.167
2 Opposites	.437	.424	.061	-.062
3 Alphabet	.343	.356	.296	.185
4 Four Rules B	.703	-.168	.395	.062
5 Cancellation A	.461	.520	.172	-.276
6 Four Rules C	.596	-.349	.327	.116
7 Fractions	.608	-.369	-.120	-.012
8 Change	.550	-.383	-.223	-.136
9 Cancellation B	.307	.305	-.137	-.253
10 Problems	.427	-.162	-.504	.167
11 Number Series	.593	-.271	-.400	-.206
12 Definitions	.457	.039	-.251	-.060
13 Rep.of Digits	.210	.082	-.109	.309

THE ROTATION OF THE AXES

Following the principles set out on page 18 above, the axes were rotated to a meaningful position. As a first

attempt axes were passed through tests 1, 5, and 11 in succession in order to locate the number, speed and reasoning factors respectively found in the preliminary investigation. Contrary to expectation it was then found that tests 1 and 4 had almost zero loadings on the speed factor. The reason for this was that as the tests of computation were positively correlated with the cancellation test, the number plane passing through them had absorbed their potential projections on the speed factor located by an axis through the cancellation test. As the preliminary investigation had shown that tests of computation are complex, involving at least two factors, it was assumed that as far as this battery was concerned the cancellation test measured a single ability. Consequently axes were passed through tests 13, 5 and 1 in succession in order to locate the retention , speed and manipulation factors, respectively, found in the preliminary investigation. This left the reasoning factor as a residual.

The transformation matrix is given in table 9 and the factorial matrix in table 10.

TABLE 9

THE TRANSFORMATION MATRIX

	I	II	III	IV
I	.528	.544	.500	.418
II	.206	.658	-.624	-.370
III	-.274	.258	.599	-.706
IV	.777	-.453	.037	-.436

TABLE 10

THE FACTORIAL MATRIX.

	I	II	III	IV	h_j^2	* Reliability
1 Four Rules A	.069	.541	.689	.000	.775	.822
2 Opposites	.253	.561	-.012	.010	.379	.912
3 Alphabet	.317	.413	.134	-.278	.367	.988
4 Four Rules B	.277	.346	.695	.050	.683	.931
5 CancellationA	.089	.763	.000	.000	.590	.907
6 Four Rules C	.243	.126	.716	.097	.598	.831
7 Fractions	.269	.062	.462	.481	.522	.830
8 Change	.167	.051	.375	.588	.517	.819
9 CancellationB	.666	.447	-.128	.223	.305	.939
10 Problems	.460	-.079	.019	.521	.492	.757
11 Number Series	.207	.134	.218	.720	.627	.928
12 Definitions	.365	.182	.056	.328	.189	.679
13 Rep.of Digits	.400	.000	.000	.000	.159	.878

* The reliability coefficients of the tests were calculated by the split-half method.

THE INTERPRETATION OF THE FACTORS

FACTOR I: The tests with significant loadings in column I of table 10 are

(10) Problems ..... .460

(13) Repetition of Digits. .. .400

(12) Definitions ..... .365

(3) Alphabet..... .317

(4) Four Rules B ..... .277

(7) Fractions ..... .269

(2) Opposites ..... .253

(6) Four Rules C ..... .243

(11) Number Series ..... .207

It is clear that this factor is the same as the first factor found in the preliminary investigation. The fact that it emerged so clearly from this totally different battery and with different subjects, and further that the loadings of the problems and repetition of digits tests which were included in both batteries, are approximately the same, is proof of the objectivity of this factor.

Its exact nature is however still obscure. The loading of the definitions test appears to stress the retentive element, while the loadings of the alphabet and computation tests appear to stress the element of concentration.

FACTOR II: In the second column the tests with significant loadings are

(5) Cancellation A ..... .763

(2) Opposites ..... .561

(1) Four Rules A ..... .541

(9) Cancellation B..... .447

(3) Alphabet ..... .413

(4) Four Rules B..... .346

(6) Four Rules C ..... .126

This factor appears to be the same as the speed factor (III) identified in the preliminary investigation. The fact that the loadings of the tests of computation fall from .541 for test 1 to .126 for test 6 is in agreement with the hypothesis that this factor enters into the simplest arithmetical processes only.

The high loadings of Cancellation A, Opposites and Four Rules A confirm the hypothesis that this factor has a Speed characteristic. Continuous response seems to play an important part in the successful solution of these tests.

The high loadings of the Opposites and Alphabet tests are in agreement with the hypothesis that this factor is not confined to number alone. It must be remembered here that as Afrikeans, the home language of the subjects, is taught by the phonetic method the alphabet as such is never memorised and is therefore not as well-practised as the number combinations. This fact may be the explanation of the relatively low loading of the alphabet test on this factor

(7) Fractions ..... .481

(12) Definitions ..... .328

This factor is clearly similar to the reasoning factor found in the preliminary investigation. The loadings of the problems and fractions tests which were included in both batteries, are practically the same, while the loading of the number series test is higher. The loading of Definitions on this factor appears to stress the selective element in this factor. Being the residual it is probably not a pure factor here.

DISCUSSION

The results of this investigation are in agreement with the following hypotheses

(a) that achievement in arithmetic cannot be accounted for by a single specific factor.

(b) that achievement in arithmetic can be accounted for by at least four major factors.

(c) that the ability to compute accurately can be accounted for by at least two factors one of which has a speed characteristic.

(d) that this speed factor is not confined to number alone.

The factor pattern found in the preliminary investigation is consistent with that found in this battery, which shows that the factors can be accepted as being ob-

jective in the sense that these factors are not determined by the test battery. It was however necessary to test this pattern further as the number of factors in this battery was limited.

This analysis, however, did not throw much light on the nature of the factors. Several points still remained to be cleared up, namely (a) whether retention or concentration is the main characteristic of the first factor, (b) whether language or verbal ability is involved in the reasoning factor, and (c) whether the second factor in computation is of an associative or manipulative character.

Further it was necessary to show that the factors identified in these studies enter into arithmetic as tested at an actual school examination and that they are sufficient to explain the total variance of such a test after allowing for error.

CHAPTER IV

THE THIRD TEST EXPERIMENT

With the object of testing the hypotheses constructed on the results of the previous investigations and to clear up the points mentioned at the end of the last chapter a new battery was drawn up consisting of the following tests:

1. Speed of Movement: The subjects were given clean sheets of ruled foolscap paper and required to draw lines as fast as they could from the first line on the paper to the second, being careful to start and finish on the lines. Three tests lasting 15 seconds each were given and the average score taken. This test was devised to throw light on the nature of the retention and speed factors as it involved concentration and speed of movement.

2. Repetition of Digits: Taken from the previous battery. Time 2 minutes.

3. Four Rules A: Taken from the previous battery. Time: 2 min.

4. Dividing Geometrical Figures: This test consisted of 20 rows of composite geometrical figures made up out of two figures given at the beginning of each row. The subjects were required to indicate by means of drawing a line, how each of these composites should be cut up in order to obtain the separate figures at the beginning of each row. Time: 4 minutes. Score: number marked correctly.

This test was included in the battery as a non-verbal

reasoning test

5. Definitions: Taken from the previous battery. Time: 4min.
6. Fractions: " " " " " " 5min.
7. Opposites: " " " " " " 3min.
8. Four Rules B " " " " " " 8min.
9. Change " " " " " " 2½min.
10. Number Series: " " " " " " 8min.
11. Alphabet: " " " " " " 3min.

12. Three Digit Addition: Time: 4 min. Score: number correct.

13. Maze: This test consisted of 30 mazes linked up with one another and increasing in complexity. Time: 3 minutes. This test was included in the battery as a non-verbal reasoning test.

14. Problems: Taken from previous battery. Time: 7 min.

15. Reading Test: This was the silent reading test of the National Bureau of Educational and Social Research. It consists of a continuous passage in which at intervals a word has to be selected from words given in brackets in order to complete the story. Time: 3 minutes. Score: Number of words marked correctly. This test was included in the battery to determine to what extent reading ability played a part in arithmetical reasoning.

16. Reasoning: This test consisted of fifteen problems which could only be solved by eliminating the wrong answers. Time: 8 minutes. Score: number correct. It was thought

that this test would call for a different type of reasoning from that involved in arithmetical problems.

17. Part-Whole: This test consisted of two parts. In the first the subjects were asked to select from a list of five names the part of an object named at the beginning of the list. In the second part they had to indicate the object of which the part was given. Time: 2 minutes for each part. Score: number correct. The scores for the two sections were combined.

18. Cancellation of 9's: Taken from previous battery. Time: 3 minutes.

19. Compound Words: This test consisted of 90 words to each of which the subjects were asked to add a word in order to form a compound word, e.g., school(room), table(cloth), etc. Time: 3 minutes. Score: number of words formed. These word tests were devised to throw light on the nature of the manipulation factors found in the previous investigations.

20. Spelling: This test formed part of the Cape Departmental Standard VI Examination. The subjects were required to spell a set of words given separately and in sentences. The test had to be given and marked according to a carefully prepared scheme.

As Afrikaans spelling is mainly phonetic it was thought this that test would call for the manipulation of symbols, and hence throw some light on the nature of the manipulation factor.

21. Arithmetic: This test too, formed part of the Departmental StdVI Examination. It consisted of two papers lasting 30 minutes and one hour and thirty minutes respectively. The tests had to be marked strictly according to a carefully prepared memorandum and the marking was checked by the principal of each school and the circuit inspector.

THE SUBJECTS

The subjects came from the following schools:

Parow Primary School	43
Parow North Primary School	60
Goodwood Primary School	61
Epping Primary School	37
Tiger Valley Primary School	27
Bellville High School	43
Maitland High School	90
Total	361

The average age of the group was 14 years 6 months. They had all written the Std VI examination in December 1942

ADMINISTRATION OF THE TESTS AND RESULTS

The tests were given during the first half of December 1942 in two sessions lasting about 40 minutes each. The same procedure as in the previous investigations was followed. All the subjects co-operated well and enjoyed taking the tests. The distributions, means and standard deviations of the raw scores are given in table 11.

TABLE 11

DISTRIBUTION OF RAW SCORES

Speed		Repetition of Digits		Four Rules A		Figures		Definitions	
Score	f	Score	f	Score	f	Score	f	Score	f
5- 8	1	4- 6	5	29-35	12	21-30	6	3- 4	5
9-12	13	7- 9	22	36-42	28	31-40	8	5- 6	9
13-16	68	10-12	65	43-49	51	41-50	6	7- 8	28
17-20	89	13-15	75	50-56	79	51-60	35	9-10	58
21-24	70	16-18	78	57-63	85	61-70	37	11-12	100
25-28	56	19-21	47	64-70	60	71-80	54	13-14	94
29-32	40	22-24	35	71-77	26	81-90	71	15-16	36
33-36	17	25-27	21	78-84	16	91-100	63	17-18	23
37-40	7	28-30	13	85-91	2	101-110	39	19-20	5
				92-98	2	111-120	24	21-22	2
						121-130	11	23-24	1
						131-140	7		
M. 21.86		16.52		57.38		84.36		12.11	
S.D. 6.56		5.43		12.04		22.44		3.18	

Fractions		Opposites		Four Rules B		Change		Number Ser.	
Score	f	Score	f	Score	f	Score	f	Score	f
1- 2	3	26-30	1	6-10	2	3- 4	3	0- 2	32
3- 4	11	31-35	14	11-15	23	5- 6	18	3- 4	51
5- 6	23	36-40	33	16-20	61	7- 8	41	5- 6	43
7- 8	60	41-45	67	21-25	111	9-10	75	7- 8	69
9-10	72	46-50	68	26-30	76	11-12	95	9-10	52
11-12	83	51-55	77	31-35	51	13-14	101	11-12	51
13-14	64	56-60	52	36-40	24	15-16	22	13-14	39
15-16	33	61-65	32	41-45	11	17-18	5	15-16	27
17-18	11	66-70	12	46-50	2	19-20	1	17-18	14
19-20	1	71-75	5					19-20	3
M. 10.65		50.39		25.65		10.17		8.75	
S.D. 3.36		8.95		7.48		2.76		4.48	

TABLE 11 (continued)

Alphabet		3-Digit Add.		Maze		Problems		Reading	
Score	f	Score	f	Score	f	Score	f	Score	f
1- 8	3	21-30	3	20-24	3	0- 2	4	4- 5	4
9-16	36	31-40	27	25-28	16	3- 4	24	6- 7	16
17-24	54	41-50	65	29-32	61	5- 6	63	8- 9	38
25-32	69	51-60	97	33-36	88	7- 8	81	10-11	80
33-40	57	61-70	91	37-40	87	9-10	77	12-13	141
41-48	52	71-80	42	41-44	47	11-12	65	14-15	50
49-56	38	81-90	22	45-48	38	13-14	33	16-17	20
57-64	28	91-100	7	49-52	16	15-16	10	18-19	11
65-72	14	101-110	6	53-56	6	17-18	2	20-21	3
73-80	10	111-120	1	57-60	2	19-20	2		
M.	36.96	60.40		37.68		8.86		12.03	
S.D.	16.80	15.55		6.56		3.24		2.78	

Reasoning		Part-Whole		Cancellation		Compound words	
Score	f	Score	f	Score	f	Score	f
2-	1	13-18	5	9-12	3	9-16	1
3-	9	19-24	28	13-16	10	17-24	4
4-	20	25-30	65	17-20	31	25-32	18
5-	57	31-36	76	21-24	61	33-40	49
6-	81	37-42	75	25-28	87	41-48	78
7-	79	43-48	53	29-32	75	49-56	96
8-	52	49-54	25	33-36	54	57-64	81
9-	42	55-60	26	37-40	23	65-72	24
10-	12	61-66	6	41-44	13	73-80	8
11-	5	67-72	2	45-48	3	81-88	2
12-	3			49-52	1		
M.	6.75	38.19		28.03		50.47	
S.D.	1.76	10.86		6.68		11.76	

TABLE 11 (continued)

Spelling		Arithmetic	
Score	f	Score	f
5- 6	8	10-20	4
7- 8	18	21-32	9
9-10	48	33-44	26
11-12	70	45-56	38
13-14	96	57-68	49
15-16	76	69-80	65
17-18	39	81-92	62
19-20	5	93-104	49
21-22	1	105-116	27
		117-128	19
		129-140	9
		141-150	4
M.	13.05		78.68
S.D.	3.02		26.88

An inspection of this table shows that all the tests gave fairly normal distributions and satisfactory dispersions, there being no abnormal deviations. It was therefore assumed that the distributions are in the correct form for factorial analysis.

THE INTER-CORRELATIONS

The inter-correlations between the 21 variables were calculated by the product-moment method to four places of decimals. The results are given in table 12 correct to 2 figures and with the decimal point omitted.

TABLE 12

THE CORRELATION MATRIX

Test	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1																					
2	22																				
3	32	26																			
4	09	11	19																		
5	11	08	24	20																	
6	13	13	35	26	29																
7	19	19	31	23	25	28															
8	11	09	30	16	23	45	30														
9	05	12	41	20	32	48	18	44													
10	02	15	07	40	31	41	20	26	37												
11	12	24	24	26	18	29	45	23	20	23											
12	04	08	53	10	15	36	28	68	44	15	25										
13	06	05	07	45	14	11	15	13	16	32	20	14									
14	15	16	19	26	40	36	16	29	55	49	23	18	18								
15	10	19	20	15	23	25	38	21	17	18	27	24	23	16							
16	05	17	-12	14	23	21	20	15	28	29	19	05	16	31	21						
17	-04	09	18	25	41	20	43	23	22	28	30	15	32	12	45	30					
18	16	-01	31	23	01	13	13	28	01	00	11	24	14	19	06	01	15				
19	09	09	22	21	30	19	35	20	18	19	24	12	24	15	32	24	45	14			
20	21	23	22	04	24	25	25	28	21	18	32	26	00	26	31	11	13	-09	16		
21	08	18	26	19	57	52	14	34	51	46	20	27	21	62	24	22	16	-14	10	37	

As in the previous investigations the correlations were again low. Of the 210 correlations all were positive excepting 5 which were negligibly small. The average correlation coefficient was .2221 which is more than 6 times its P.E. .0338. It may therefore be assumed that there is a positive relationship between the inter-correlated variables.

The tests appear to fall together in groups, namely (a) Repetition of digits, speed of movement and Spelling alphabet, (b) Cancellation, four rules A, addition, and speed of movement; (c) Four rules, fractions, change, addition, opposites, part-whole, spelling and arithmetic; (d) Reading, opposites, part-whole, compound words, and spelling; (e) Number series, problems, reasoning, definitions, arithmetic, and figures.

THE FACTORIAL ANALYSIS

In order to determine the under-lying relationships indicated by an inspection of the correlation-coefficients the matrix was factored to six factors by Thurstone's centroid method with guessed communalities. After six factors had been removed the average residual fell to .0255 which is below the P.E. (.0338) of the average correlation coefficient, thus indicating that further factors would be unreliable. The results are given in table 12.

TABLE 12

THE CENTROID MATRIX.

Test	I	II	III	IV	V	VI
1	.2534	-.1720	.1025	.1868	.3313	-.1290
2	.2911	-.0904	-.0251	.2607	.1932	.0295
3	.5556	-.3722	.3979	-.1826	.1024	-.1011
4	.4476	.1725	-.2583	-.2773	.3641	.1501
5	.5091	.1236	-.0630	.0550	-.1504	-.2511
6	.6089	.1241	.2213	.0373	-.1076	.0715
7	.5457	-.3301	-.2421	.0868	-.0206	.0158
8	.5966	-.1846	.3839	-.2394	-.2115	.1045
9	.3981	.2709	.2940	-.0970	-.1781	-.0803
10	.5559	.4386	-.1075	-.0178	.0252	.1898
11	.5108	-.1750	-.1522	.1242	.0844	.1872
12	.5305	-.2694	.3937	-.2762	-.2172	.2345
13	.3861	.1518	-.2884	-.3032	.1596	.1337
14	.5926	.4277	.2168	.1540	.1175	-.1749
15	.4929	-.2000	-.2260	.1083	-.1596	.0370
16	.3711	.2083	-.2457	.1210	-.1532	-.0795
17	.5244	-.0690	-.4798	-.1425	-.2788	-.1424
18	.2341	-.2207	.0804	.3280	.2390	-.2019
19	.4579	-.1377	-.2989	-.0684	-.0938	-.2141
20	.4152	-.1344	.0805	.4027	-.0850	.1490
21	.5863	.4221	.2440	.2733	-.1321	.1642

THE ROTATION OF THE AXES

Considering the grouping of the tests as revealed by the correlation coefficients in conjunction with the results of the previous investigations, the first axis was passed through test 2, Repetition of digits, to locate the

retention factor. The second axis was passed through test 18, Cancellation, to locate the speed factor and the third axis through test 8, Four Rules 2, for the manipulation factor. The fourth axis was passed through test 10, Number series for the reasoning factor, and the fifth through 7, Opposites, 17, Part-Whole, and 19, Compound words, which appeared to form a group containing some sort of verbal factor. Then considering the factors together, slight adjustments were made by graphic means in order to reduce the negative loadings as far as possible.

The transformation matrix is given in table 13 and the resulting factorial matrix in table 14.

TABLE 13

THE TRANSFORMATION MATRIX

	I	II	III	IV	V	VI
I	.5190	.3781	.5340	.3377	.2457	.3578
II	-.0725	-.3749	-.0470	.8910	-.2025	-.1301
III	-.1052	.3631	.4842	-.0422	-.6167	-.4904
IV	.7877	-.2346	-.1784	-.1181	.0152	-.5278
V	.2415	.4977	-.5993	.1103	-.5158	.3272
VI	.1881	-.5748	.2955	-.2551	-.5025	.4794

TABLE 14

THE FACTORIAL MATRIX

		I	II	III	IV	V	VI	h_j^2
1	Speed of Movement	.3361	.3762	-.0770	-.0246	-.0704	.0108	.2661
2	Repetition of digits	.4154	.1387	.0000	.0000	.0000	.0647	.1960
3	Four Rules A	.1250	.6333	.4575	-.1089	-.0433	.1264	.6378
4	Figures	.1426	.1598	-.0089	.3496	-.0455	.5920	.5207
5	Definitions	.2217	.1884	.2417	.3265	.3435	-.0017	.3677
6	Fractions	.3006	.1660	.5054	.2724	.0081	.0726	.4529
7	Opposites	.3986	.2047	.1906	-.1157	.3552	.3110	.4734
8	Four Rules B	.6627	.3357	.7134	.0014	.0000	.1565	.6501
9	Change	.1568	.2114	.5426	.1220	.0416	-.0325	.3814
10	Number Series	.2855	-.0945	.2577	.5328	.0000	.2960	.5283
11	Alphabet	.4472	.1045	.1899	-.0302	.1191	.3320	.3723
12	Addition	.0276	.2773	.7354	-.0910	-.0680	.2189	.6793
13	Maze	.0447	.0501	.0574	.2971	.0879	.5362	.3913
14	Problems	.3706	.2595	.2517	.6115	-.0451	-.0767	.6499
15	Reading	.3479	.0612	.2504	-.0370	.3664	.2216	.3722
16	Reasoning	.2516	-.0697	.1037	.3126	.3116	.0811	.2803
17	Part-whole	.1213	.0404	.2014	.1582	.6119	.3476	.5772
18	Cancellat[illegible]	.0975	.5228	.0000	.0000	.0000	.2439	.3423
19	Compound W	.1612	.2133	.1114	.0969	.4796	.2311	.3767
20	Spelling	.5419	.0193	.2889	-.0776	.0537	-.0418	.3881
21	Arithmetic	.4623	-.0657	.4903	.4751	-.1022	-.0736	.7000

THE INTERPRETATION OF THE FACTORS

FACTOR I The tests with significant loadings on factor I are

(20)	Spelling	.5419
(21)	Arithmetic	.4623
(11)	Alphabet	.4472
(2)	Repetition of digits	.4154
(7)	Opposites	.3986
(14)	Problems	.3706
(15)	Reading	.3479
(1)	Speed of Movement	.3361
(6)	Fractions	.3006

This factor is obviously the same as that found in the previous investigations. It was there tentatively described as a retention factor. While span of apprehension-and-report undoubtedly plays an important part in such a test as Repetition of digits, success in the other tests listed above appears to depend chiefly upon alertness and sustained concentration. This is especially clear in such tests as Speed of Movement, Opposites and Alphabet. Further it is natural to expect that good concentration would result in better retention

Woodrow[1] identified an "Attention" factor which appears to be similar to the factor found here. This factor had a

1. Woodrow, H., The common Factors in Fifty-two Mental Tests Psychometrika, 4, 2, 1939.

loading in the Arithmetic test of .597 and on his alphabet test of .606 .

Wittenborn[1] who designed tests to measure what he conceived to be attention, isolated an independent factor. He contends that"the tests which are highly saturated with factor IV (Attention) ... appear to depend upon the application of sustained concentrated effort"

This view closely resembles that reached here. To the writer, however, concentration appears to be a better term than attention to describe this factor.

Factor II:

The tests with significant loadings on this factor are

(3)	Four Rules A	.6333
(18)	Cancellation.....	.5228
(11)	Speed of Movement.....	.3762
(8)	Four Rules B	.3357
(12)	Addition	.2773
(14)	Problems	.2595
(19)	Compound Words	.2133
(9)	Change.....	.2114
(7)	Opposites.....	.2047

This factor is clearly the same as the speed factor found in the previous investigations. As in the second in-

1. Wittenborn,J.R., Factorial Equations for Tests of Attention, Psychometrika, March 1943, pp.19 - 35

vestigation Four Rules A and Cancellation have high loadings on this factor. The considerable loadings of the Opposites and Alphabet tests in the second investigation have been greatly reduced here. This , however, is consistent with the view that this factor functions only in tests dealing with well-known and highly practised material. The subjects in the present investigation were on the average two years younger and two years behind those of the second investigation in school. It therefore seems reasonable to believe that the alphabet and opposites tests were less familiar to them. Another possible explanation is , that in the second investigation only four factors were extracted while in this case there were six, with the result that the variance of these tests have been further divided.

Fluency or continuity of response appears to be the chief characteristic of this factor as is shown by the loading of the speed of movement test. The subjects who went straight ahead without hesitation in such tests as Four Rules A, Cancellation, Speed of movement, etc., had a better c chance of putting up a high score. This view is further supported by the fact that the loadings on this factor decrease with the complexity of the tests. Four rulesA has a loading of .6533 whereas Four Rules B has a loading of .3356 and Fractions .1660 .

This factor may therefore with a fair amount of certainty be described as fluency under control involving speed and continuity of response.

The number factor identified by Woodrow[1] appears to be similar to this factor. His dot-counting test had a loading of .722 on the number factor and the digit-cancellation test .513, while two verbal tests had loadings greater than .400

Coombs[2] in a special study of number ability identified a number factor which he described as " facility in manipulating a symbolic system according to a specified set of rules". He found that " a test involving manipulation of a symbolic system is a better measure of number ability the more familiar the symbolism", and that " a test involving operations according to a set of rules becomes a better measure of number ability with practice" His number tests namely, two-digit, three-digit, four-digit and column addition, and multiplication had loadings on this number factor ranging from .74 for two-digit addition to .45 for multiplication, thus indicating that this factor has a "speed characteristic" These findings are in agreement with the results of the present investigation. The exact nature of this factor,however, did not come to light in the study

1. Woodrow, H., Op.cit.
2. Coombs,C.H., A Factorial Study of Number Ability, Psychometrika, 6, 3,1941, pp. 161 - 189.

of Coombs, for as he showed himself, several other tests in his battery, including his A B, A B C, Forms, Alphabet I, II, III and Digit Cancellation had correlations with this factor ranging from .49 to .34 although their loadings on this factor were approximately zero. This anomalous result can only be explained by the fact that he rotated the axes "blindly" without knowledge of the identity of the tests, in a purely mechanical fashion to "simple structure" thereby failing to bring to light the relationship underlying the correlations between the tests.

FACTOR III. The tests with significant loadings on this factor are

(12)	Addition	.7354
(8)	Four Rules B	.7134
(9)	Change	.5426
(6)	Fractions	.5054
(21)	Arithmetic	.4903
(3)	Four Rules A	.4375
(17)	Part-whole	.2014
(20)	Spelling	.2889
(10)	Number series	.2577
(14)	Problems	.2517
(15)	Reading	.2504
(5)	Definitions	.2417

This factor is clearly the same as that found in the previous investigations. As in the other two batteries

the more involved tests of computation have the highest loadings onthis factor. The loading of the addition test seems to contradict this, but as experienced teachers know Std.VI pupils generally do not find even 3 digit-addition a very simple matter, many of them still using the counting method

Another significant point about this factor is the fact that some of the verbal tests have substantial loadings on it. This confirms the hypothesis that this factor is not confined to number alone

The loading of the spelling test supports the view that this factor primarily measures the ability to manipulate symbols. The other verbal tests, namely Part-Whole, Reading and Definitions, were all multiple choice tests and as such involved the manipulation of word symbols.

This factor may therefore with some confidence be described as the ability to manipulate symbols according to well-practised rules.

FACTOR IV. The tests with significant loadings on the fourth factor are:

(14)	Problems	.6115
(10)	Number Series.....	.5328
(21)	Arithmetic.....	.4751
(4)	Figures.....	.3496
(5)	Definitions	.3265
(16)	Reasoning.....	.3126
(13)	Maze.....	.2971

(6) Fractions ..... .2724

This appears to be the same reasoning factor as found in the previous studies and similar to factors identified by other investigators.

Thurstone[1] points out that the "characteristic of this factor seems to be " the successful completion of a task that involves some form of restriction in the solution" Coombs[2] identifies it with deduction but does not explain on what grounds. Sutherland[3] identifies it with Spearman's "g" factor. Blackwell[4] describes it as "the power of selective quantitive thinking and of deductive reasoning, involving the ability to apply general principles to particular cases in number, symbolic and geometric work, to abstract, generalise, and use the essential features of a given complex situation, and to make deductions from these for the elucidation of other complex situations" This description is obviously too wide for the factor identified here.

Directed relational thinking clearly enters into all the tests listed above. The loading of the number series test indicates an inductive element, while the loadings of the figures and maze tests indicate an element of selection. This factor may therefore with confidence be described as

1. Thurstone, L.L., Primary Mental Abilities, p.80
2. Coombs, C.H., op. cit. , p.185
3. Sutherland, J. , An Investigation into Some Aspects of Problem Solving, Brit. J. Educ. Psych. Feb.1942, pp.35-46.
4. Blackwell, A.M., The Factors involved in Mathematical Ability, Brit. J. Educ. Psych, X,1940, p. 221

the ability to see relationships, to select essentials, to abstract relevant qualities from the given data and to apply these to the solution of a clearly conceived task.

FACTOR V The tests with significant loadings on the fifth factor are

(17)	Part-whole	.6119
(19)	Compound-words	.4796
(15)	Reading.....	.3664
(7)	Opposites	.3552
(5)	Definitions	.3435
(16)	Reasoning	.3116

These tests are all of a verbal nature. Merely to say however, that this factor involves some sort of intellectual verbal ability is unsatisfactory. The Part-whole, Compound-words and Opposites tests suggest that it has an associative character. Further there is a high degree of restriction as to the appropriate responses, while knowledge of the meaning of the words appears to be important as well.

This factor may therefore be described as fluency of verbal association under restriction.

It appears to be related to the W factor of Thurstone[1] and to be similar to the A factor identified by Carroll[2] .

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1. Thurstone, L.L., Op.Cit.
2. Carroll, J.B., A Factor Analysis of Verbal Ability, Psychometrika, 6, 5, 1941, p.296.

FACTOR VI The tests with significant loadings on this factor are

(4)	Figures	.5920
(13)	Maze	.5362
(17)	Part-whole	.3476
(11)	Alphabet	.3320
(7)	Opposites	.3110
(10)	Number Series	.2960
(18)	Cancellation	.2439
(19)	Compound-words	.2311
(15)	Reading	.2216

This is probably not a pure factor, being the residual. The high loadings of the figures and maze tests on this factor indicate that it has a spatial characteristic. This is, however, contradicted by the substantial loadings of the Part-whole, Alphabet and Opposites tests, although the latter involves direction. Visual imagery appears to be a valuable aid in solving all these tests. Hence this factor may tentatively be described as a visual imagery factor.

C H A P T E R V

DISCUSSION AND CONCLUSIONS

Considering the three experiments together the first important fact which emerges is that similar factor patterns were revealed in all three the analyses although the batteries were different and different subjects were used in each case. The structural pattern obtained in this study may therefore be regarded as having some measure of objectivity. The analyses are further coherent with those of other investigators using arithmetical tests, as has been shown in the interpretation of the factors. In addition the factor loadings of the tests which appeared in the different batteries remained constant within reasonable limits as is shown by the table below for the main arithmetical tests:

Factor	Test	Loading in 1st Battery	2nd.	3rd.
Concentration	Repetition Digits	.3918	.4000	.4154
"	Problems	.4954	.4600	.3706
Speed	Four Rules A	.....	.5410	.6333
"	Four Rules B	.....	.3460	.3557
Manipulation	Four Rules B	.....	.6950	.7134
"	Fractions	.4009	.4620	.5054
Reasoning	Problems	.5224	.5210	.6115
"	Number Series	.5114	.7200	.5328

(In comparing these loadings it should be borne in mind

that a loading of .2000, the biggest difference recorded here, represents only 4 per cent of the total variance of the test.)

The factors identified here may therefore be regarded as having some measure of objectivity in the sense that they are not dependent upon the battery of tests used or the population in which they are measured

The second fact which is clear from a comparison of the results of the different analyses, is that there is no evidence of a general common factor in Spearman's sense. On the contrary the tests in all the batteries divided their variance among a number of independent factors.

Sutherland[1] in his study of problem solving in arithmetic established a common factor which he identifies with Spearman's "g". His common factor, however, has a loading of .509 in Arithmetic Fundamentals, .497 in Arithmetical Problems, a fact which is difficult to explain on the basis of relational thinking. Another explanation for this finding may be the fact that Sutherland worked with 11 year olds, it being generally accepted that the abilities of young children are less differentiated.

The third fact which emerges from these results is that there is no specific factor common to all tests of arithme-

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1. Sutherland, J., Op. Cit.

tical ability. At least four major factors are involved in the different arithmetical tests and none of them are present in all the tests. Further none of these factors appear to be confined to arithmetic alone.

The ability to solve problems in arithmetic can be accounted for by two factors, namely, (a) the ability to concentrate and (b) the ability to see relationships. The ability to compute accurately appearsto call for two other factors, namely (a) speed in well-practised tasks of a continuous nature and (b) the ability to manipulate symbols according to a set of well-practised rules.

The Std.VI Arithmetic Test of the Cape Education Department was accounted for by three factors, namely,'concentration','manipulation' and 'reasoning'. They appear to be of approximately equal importance the loadings being .4623, .4903 and .4751 respectively. These factors accounted for approximately 70 per cent of the total variance of this test. It therefore seems unlikely that further important factors will be found to be involved in this test

SUMMARY

Object: The object of this study was to isolate the common factors involved in arithmetical ability and to determine their nature.

Procedure: Several tests of arithmetical ability together with other mental tests were given in three different batteries to (a) 112 standard VII pupils (b) 100 standard

VIII pupils and (c) to 361 standard VI pupils.

In each case the correlations between the tests were calculated by the product-moment method, the correlation matrices were factored by Thurstone's centroid method with guessed communalities, and the reference axes were rotated to significant positions by the method developed by Reyburn and Taylor.

Results: The three analyses revealed similar factor patterns with reasonably constant factor loadings for the tests which appeared in all three or two of the batteries.

In the final battery six factors were identified, namely

(a) a Concentration factor involving a high degree of sustained effort, alertness and retentive ability;

(b) a speed factor described as fluency in the performance of simple well-practised tasks involving speed and continuity of response;

(c) a manipulation factor described as facility in manipulating a symbolic system according to a set of well-practised rules

(d) a reasoning factor described as the ability to see relationships, to select essentials, to select relevant qualities from the data and to apply these to the solution of a clearly conceived task.

(e) a verbal factor described as fluency of verbal association where there is some measure of restriction in the task

imposed;

(f) a visual imagery factor, which formed the residual

The first four of these factors accounted for approximately 70 per cent of the total variance of the Standard VI Arithmetic test of the Cape Education Department and entered into one or more of the different tests of arithmetical ability used in this battery.

Conclusions:

1. There is no evidence in the results of the present study of a general common factor in Spearman's sense.

2. There are at least four independent factors involved in arithmetical ability as tested by different tests of computation and problem-solving

3. There is no evidence in the present study of a specific arithmetical ability

4. The factors involved in arithmetical ability are not confined to arithmetic alone.

5. The ability to solve problems in arithmetic can be accounted for by two main factors, namely, concentration and reasoning

6. The ability to compute accurately can be accounted for by two main factors, namely, speed and manipulative power

7. Arithmetical ability as tested by the Std.VI Examination of the Cape Education Department can be accounted for by three main factors of approximately equal importance, namely, concentration, manipulation and reasoning.

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