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INTERIOR ALGEBRAS AND TOPOLOGY

by

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To the memory of

RABBI MEIR KAHANE

TABLE OF CONTENTS

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INTRODUCTION

An interior algebra is a Boolean algebra enriched with an interior operator ^{I} and a closure operator ^{C} satisfying the identities: (i) $x^I \leq x$, (ii) $(xy)^I = x^I y^I$, (iii) $x^{II} = x^I$, (iv) $1^I = 1$, and (v) $x^C = x'^{I}{}'$. If $\langle X, \mathcal{T} \rangle$ is a topological space then the power set Boolean algebra on X together with the usual topological interior and closure operators, forms an interior algebra. (In fact, it can be easily shown that if $\mathcal{K}$ is the class of algebras obtained by appending the topological interior and closure operators to the power set Boolean algebras of topological spaces, then the variety generated by $\mathcal{K}$ is precisely the variety of interior algebras.) Interior algebras are thus an algebraic generalization of topological spaces.

Interior algebras were introduced by McKinsey and Tarski in [14] under the name 'closure algebras', as an algebraic generalization of topological spaces. What were essentially interior algebraic techniques had however already been used as aids to topology by Kuratowski and others. McKinsey and Tarski were concerned with the problem of which interior algebraic identities hold in all topological spaces and which topological spaces satisfy only the interior algebraic identities which hold in all topological spaces.

In [16] Nöbeling investigated the application of interior algebraic techniques to topology. Sikorski investigated the application of results concerning countably complete interior algebras to topology. (See for example [20].) McKinsey, Tarski, Nöbeling and Sikorski all observed that certain results in topology, not involving points, can be generalized to interior algebras by replacing sets with elements, finite intersections with meets, finite unions with joins and set–theoretic complements with Boolean algebraic complements. They also observed that results in topology involving arbitrary or countable intersections and unions could be generalized to complete or countably complete interior algebras respectively, and results involving points could be generalized to complete atomic interior algebras, points being replaced by atoms. However every complete atomic interior algebra is isomorphic to the power set interior algebra of a topological space and so we do not obtain any really new results by generalizing topological results to complete atomic interior algebras, only known results in new notation. Nöbeling and Sikorski preferred to treat topological spaces as (complete atomic) power set interior algebras and used interior algebraic formulations of topological axioms as the foundations for topology instead of set–theoretic formulations. However modern topologists do not follow this approach and nowadays topological spaces are considered to be pairs consisting of a set together with a system of open subsets. Moreover, interior algebraic techniques are generally not used by topologists with the exception perhaps of trivial computations involving the interior and closure operators.

The idea of generalizing topological results involving points (which most topological results do involve) to non–countably complete or non–atomic interior algebras has never to our knowledge been investigated before. The reason for this seems to be that most proofs of topological results do not generalize to interior algebras. In this thesis we will see that despite this, many results in topology can still be generalized to interior algebras using different proofs to those used in topology. Completeness (which corresponds to the ability to take arbitrary intersections and arbitrary unions of subsets) is very seldom required although atomicity and properties strictly weaker than atomicity (which correspond to the existence of points in certain non–empty sets) are sometimes needed without completeness. The results of this thesis, although motivated by topology, are in fact algebraic. We deal with connections between interior algebras and topology and generalizations of topological results for the sake of finding out more about interior algebras and not as an aid to topology. We point out that there have been other attempts at algebraic generalizations of topology viz. certain types of complete distributive lattice such as frames/locales [12], and neighbourhood (semi)–lattices [17], but these will not concern us. (Also, in [9] G.A. Edgar showed that a topological space can be considered to be a multi–algebra.)

In this thesis connections between categories of interior algebras and categories of topological spaces, and generalizations of topological concepts to interior algebras, are investigated. The following are some of the most significant results we obtain: The establishment of a duality between topological spaces and complete atomic interior algberas formalized in terms of a category–theoretic co–equivalence between the category of topological spaces and continuous maps and the category of complete atomic interior algebras and maps known as complete topomorphisms (Theorem 2.1.7). Under this co–equivalence, continuous open maps correspond to complete homomorphisms (Theorem 2.1.8). We also establish a duality between arbitrary interior algebras and structures known as Stone fields in terms of a co–equivalence between the category of interior algebras and topomorphisms (see Definition 1.1.8) and the category of Stone fields and their morphisms the field maps (Theorem 2.2.14). Under this co–equivalence weakly open field maps (see Definition 2.2.17) correspond to homomorphisms (Theorem 2.2.18). The well known connection between pre–ordered sets and interior algebras is shown to be a special case of topological duality (see section 4 of chapter 2). The topological concepts of neighbourhoods, convergence and accumulation are generalized to interior algebras (Chapter 3), and are used to generalize the topological separation and compactness properties to interior algebras (Chapter 4 and Chapter 5). What is particularly interesting with regard to the separation properties is that most of them are first order properties of interior algebras (see Theorem 4.5.11). This should be contrasted with the situation for frames/locales [12] and topological model theory [10]. By generalizing the concept of α–separation to interior algebras we obtain an ω chain of strictly elementary classes of interior algebras all of which have hereditarily undecidable first order theories (Theorem 4.3.14). Characterizations of irreducibility properties for interior algebras are also found. These properties (subdirect irreducibility, finite subdirect irreducibility, direct indecomposability, simplicity and semi–simplicity) can be characterized in many different ways. Characterizations in terms of open elements (fixed points of the interior operator) are found (Theorem 1.3.18 and Theorem 1.3.21) and these are used to obtain further charaterizations. In particular a characterization in terms of topological properties of Stone spaces of interior algebras is obtained (Theorem 2.3.9). We also find characterizations of the irreducibility properties in the power set interior algebras of topological spaces (Theorem 2.1.15) and in interior algebras obtained from pre–ordered sets (Theorem 2.4.16). What is particularly striking is that the irreducibility properties correspond to very natural topological properties. (Other results characterizing or related to the irreducibility properties are 2.4.11, 2.4.17, 5.1.13, and 5.1.15).

Topology is not the only branch of mathematics related to interior algebras. It is well known that interior algebras play the same role for the modal logic S4 as Boolean algebras play for ordinary propositional logic. Most of the research on interior algebras has focussed on the application of interior algebras to modal logic. There has been very little research on interior algebras for their own sake. However, the work of the algebraic logicians H. Rasiowa, W.J. Blok, L.L. Maksimova, V.V. Rybakov, W. Dziobiak, R. Goldblatt and M.P. Tropin, although motivated by modal logic, has produced many 'purely interior algebraic' results especially in connection with varieties and quasi–varieties of interior algebras which are related to extensions of the modal logic S4. There is an extensive literature dealing with interior algebras and modal logic which we will not be concerned with in this thesis. (For the reader who is unfamiliar with modal logic, [8] is still a good introduction.)

Finally we point out that in the literature various names have been used for interior algebras. These include: *closure algebras* (the original name used by McKinsey and

Tarski), *S4−algebras*, *topo−Boolean algebras* and *topological Boolean algebras* (not to be confused with the topological Boolean algebras of topological algebra). The term 'closure algebra' was originally used since in the earliest work on interior algebras only the closure operator, and not the interior operator, was considered to be a fundamental operation. (Sikorski used the term 'closure algebra' to mean a countably complete interior algebra.) The name 'interior algebra', which we use, was probably first used by Blok in whose work the interior operator, and not the closure operator, was considered to be fundamental. In this thesis we consider both the interior and closure operators to be fundamental but use the name 'interior algebra' since this has become the standard name for these algebras in English language publications.

SUMMARY

CHAPTER 1.

§1.1. We revise the definition of an interior algebra and summarize some basic results. The concept of a generalized topological space is introduced as a useful alternative description of interior algebras. We also introduce the concept of a topomorphism which is a natural generalization of the usual concept of an interior algebra homomorphism and is motivated by the equivalence of interior algebras and generalized topological spaces – the definition of a topomorphism is such that the category of interior algebras and topomorphisms is isomorphic to the category of generalized topological spaces and their homomorphisms. (See 1.1.7 and 1.1.8.) We briefly consider embeddings and quotient maps in the category of interior algebras and topomorphisms.

§1.2. An exposition of the well known connection between Heyting algebras and interior algebras is given using a category–theoretic approach. The first person to present a modern exposition like this was probably Blok in [3] although the basic ideas behind the connection between Heyting algebras and interior algebras have been known for a long time (as far back as McKinsey and Tarski's work in the 1940's) albeit in forms using now obsolete notation and terminology. (See [3] and the references given there.) One concept, related to the connection between Heyting algebras and interior algebras, which has apparently not been discussed before is that of 'clopen elements' which are elements of interior algebras that are both open and closed. (See 1.1.6 and 1.2.34 for definitions and 1.2.36.)

§1.3. We investigate congruences on interior algebras. Many of the results in this section are folkloric although we use an approach to interior algebra congruences which differs from the usual approach. (We prefer to represent congruences on interior algebras by filters of open elements instead of filters closed under interiors.) Irreducibility properties are also investigated and characterized.

§1.4. It is well known that intervals in a Boolean algebra can be turned into Boolean algebras which are then precisely the principal homomorphic images of the original Boolean algebra (up to isomorphism). We show that a similar situation holds for intervals in interior algebras. (Special cases of the concept of interval algebras of interior algebras appeared already in [14] and [3].) In general, an interval algebra of an interior algebra is not a homomorphic image, however, up to isomorphism the interval algebras of an interior algebra are precisely its principal topomorphic quotient map images (principal quotients). In fact every principal quotient of an interior algebra can be represented both as an ideal and as a filter in the interior algebra. (See Theorem 1.4.13.) We characterize the interval algebras which are homomorphic images (Theorem 1.4.6 (iii)) and show that every principal homomorphic image of an interior algebra can be represented as an ideal generated by an open element (Corollary 1.4.15). A new construction known as a 'join of interior algebras' is introduced.

§1.5. We end Chapter 1 with a brief look at certain atomicity properties. These properties ('closed', 'open' and 'residual' atomicity) guarantee the existence of 'enough atoms' in interior algebras and will be required in later chapters.

CHAPTER 2.

§2.1. We begin our investigation of topological duality for interior algebras by firstly considering complete atomic interior algebras and complete topomorphisms. We show that the category of complete atomic interior algebras and complete topomorphisms ($\mathbf{CIn}^+$) is co–equivalent to the category of topological spaces and continuous maps (**Top**). This is partly accomplished using basic properties of inverse images under continuous maps and the well known fact that a complete atomic interior algebra can be represented as a power set interior algebra. The significant part of the proof is that every complete topomorphism is obtainable from a continuous map in a co–functorial way. (See 2.1.2 – 2.1.4.) What is particulary interesting is that the co–equivalence restricts to a co–eqivalence between the category of complete atomic interior algebras and complete homomorphisms (**CIn**) and the category of topological spaces and continuous open maps (**Tco**) (Corollary 2.1.9). Many interesting constructions and morphisms in **Top** correspond to interesting constructions and morphisms in $\mathbf{CIn}^+$ under the co–equivalence. (See 2.1.10 – 2.1.14.) A particularly elegant result is that irreducibility properties for interior algebras correspond to connectedness properties for topological spaces (Theorem 2.1.15). In particular subdirectly irreducible interior algebras correspond to supercompact spaces. The usual supercompactification construction for topological spaces (see 2.1.17) can in fact be generalized to a construction of subdirectly irreducible interior algebras which we briefly investigate. (See 2.1.17 – 2.1.23.) We also show that under the co–equivalence between **Top** and $\mathbf{CIn}^+$, finitely generated topological spaces correspond to 'operator complete' interior algebras, that is, interior algebras in which the interior operator is completely multiplicative and the closure operator is completely additive (Corollary 2.1.27). The operator complete interior algebras play an important role in relating the topological duality for interior algebras with the pre–order duality for interior algebras which will be considered in §2.4.

§2.2. In this section we establish a duality between the category of interior algebras and topomorphisms and a certain category of topological fields of sets, the Stone fields, and their appropriate morphisms, the field maps (2.2.1 – 2.2.14). The usual Stone duality for Boolean algebras and the usual representation of Boolean algebras by fields of sets can be considered to be a special case of the duality between interior algebras and Stone fields. (See 2.2.15.) We characterize the field maps which correspond to homomorphisms in terms of a certain neighbourhood preserving property (weak openness). We introduce the concept of a subfield of a Stone field and characterize the subsets of a Stone field that are underlying sets of subfields (Theorem 2.2.23). Subfields are used to establish a duality between injective field maps and surjective topomorphisms. There is also a duality between surjective field maps and injective topomorphisms. (See Theorem 2.2.31.) We briefly investigate the connection between subfields and embeddings in the category of Stone fields and field maps. It turns out that up to isomorphism, inclusion maps of subfields are precisely the embeddings in the category of Stone fields and weakly open field maps.

§2.3. The Stone field dual to an interior algebra consists of a field of sets in a certain topological space, the Stone space of the interior algebra. We investigate the relationship between the Stone space of an interior algebra and the usual Stone space of its Heyting algebra of open elements (the latter is a retract of the former). (See Theorem 2.3.3.) We also investigate the relationship between a topological space and the Stone space of its dual complete atomic interior algebra (the former is a subspace of the latter). (See Theorem 2.3.5.) Other results concerning Stone spaces are investigated, in particular a correspondence between irreducibilty properties for interior algebras and connectedness properties for Stone spaces is established (Theorem 2.3.9).

§2.4. Interior algebras are a subvariety of modal algebras. (See page 51 for definition.) It is a well known result that every atomic 'operator complete' modal algebra (that is, complete with completely multiplicative 'interior operator' and completely additive 'closure operator'), can be represented as the power algebra of a frame (a set with a single binary relation). This result is in fact just a special case of a more general result concerning Boolean algebras with operators. (See [13].) The power algebra of a frame will be an interior algebra iff the frame is a pre–ordered set. Thus there is a duality between atomic operator complete interior algebras and pre–ordered sets. We show that there is a concrete isomorphism between the category of pre–ordered sets and homomorphisms and the category of finitely generated topological spaces and continuous maps. (See 2.4.2 – 2.4.6.) Now, as is shown in §2.1, the co–equivalence between **Top** and **CIn**$^+$ restricts to a co–equivalence between the category of finitely generated spaces and continuous maps (**ToF**) and the category of atomic operator complete interior algebras and complete topomorphisms (**OIn**$^+$), and so we obtain a category–theoretic formalization of the duality between atomic operator complete interior algebras and pre–ordered sets. **ToF** is a bi–reflective subcategory of Top and so OIn$^+$ is a bi–reflective subcategory of CIn$^+$ and we briefly examine the bi–reflection of a complete atomic interior algebra (2.4.9 –2.4.11). We also characterize the pre–ordered set homomorphisms that correspond to continuous open maps and hence to interior algebra homomorphisms (Theorem 2.4.13). (That these particular pre–order homomorphisms correspond to interior algebra homomorphisms in the case of finite pre–ordered sets, is a well known folkloric result among modal logicians.) We also show that the irreducibility properties for interior algebras can be characterized using pre–orders (Theorem 2.4.16). It is well known that every interior algebra can be represented as a field of subsets of a pre–ordered set (pre–order field.) (This is a special case of a result concerning Boolean algebras with operators.) The concrete isomorphism between the category of finitely generated spaces and continuous maps and the category of pre–ordered sets and homomorphisms, allows us to establish a concrete isomorphism between the category of Stone fields and field maps and a category of 'canonical' pre–order fields and their appropriate morphisms, the pre–order field maps (Theorem 2.4.23). Since the former category is co–equivalent to the category of interior algebras and topomorphisms we obtain a category–theoretic formalization of the representation of an interior algebra as a field of subsets of a pre–ordered set. The pre–order field maps that correspond to weakly open field maps and hence to interior algebra homomorphisms are characterized (Theorem 2.4.25). We end this section by showing how the pre–order obtained from an interior algebra may be described algebraically (Theorem 2.4.28). Modal logicians will recognize the relevance of the results in this section to the construction of canonical S4–frames. The results of this section show that duality theorems for interior algebras involving pre–orders can be viewed as topological duality in disguise.

CHAPTER 3.

§3.1. The concept of a neighbourhood function on a Boolean algebra and neighbourhoods in interior algebras are defined and are used to give an alternative description of interior algebras (Theorem 3.1.4).

§3.2. A very general interior algebraic generalization of system convergence in topology is introduced. The canonical pre–order on an interior algebra and the concept of an encloser of an element are introduced and are used to establish an interior algebraic generalization of accumulation in topology. The basic properties of convergence, enclosers and accumulation are investigated and a connection between the canonical pre–order and neighbourhoods is established (3.2.3 – 3.2.7). The notion of a 'section' operator is introduced and is used to establish connections between convergence and accumulation (3.2.8 – 3.2.15). The enclosers of the top element 1 in an interior algebra are called dense elements and it is shown that they can be characterized using the section operator (Theorem 3.2.17).

§3.3. This section covers the basic results concerning nets and sequences in interior algebras (which will be required in the next section) and the relationship between nets and filter bases. Almost all the results in this section only make use of the Boolean algebraic aspects of interior algebras and are thus essentially results about Boolean algebras. However, the concept of a net in a Boolean algebra is more relevant to the theory of interior algebras than to the theory of Boolean algebras and so the results have been stated for interior algebras. (In fact no generality is lost by this since a Boolean algebra may be considered to be an interior algebra.) Nets in interior algebras are important since they allow us to generalize many topological concepts and results to interior algebras, in particular, net convergence and accumulation, which are investigated in the next section.

§3.4. Convergence and accumulation of nets and sequences in interior algebras and their relationship to subset convergence and accumulation are investigated (3.4.1 – 3.4.11). A relationship between net convergence and net accumulation is established (Theorem 3.4.12). We investigate conditions under which sequence convergence and accumulation behave like net convergence and accumulation (3.4.16 – 3.4.18). We end this section with a brief look at the connection between the canonical pre–order and net and sequence convergence and accumulation (Proposition 3.4.19).

§3.5. In this section we look at bases and countability properties in interior algebras. Bases of generalized topological spaces or interior algebras are a generalization of bases of topological spaces. We characterize the subsets of Boolean algebras which are bases for generalized topologies (Theorem 3.5.3, Corollary 3.5.4). This characterization allows us to generalize results concerning complete extensions of Boolean algebras to interior algebras (Corollary 3.5.5, Remark 3.5.7). The concept of a base for an interior algebra allows us to generalize the concept of second countability to interior algebras. We also generalize first countability and the Lindelöf property. Every second countable interior algebra is both first countable and Lindelöf (Theorem 3.5.9). We investigate the relationship between the Lindelöf property and accumulation of countably complete filters (3.5.3 – 3.5.11). We also show that every first countable interior algebra is 'sequentially determined' (Definition 3.5.12, Theorem 3.5.12). We end this section with an investigation of the preservation of the countability properties under constructions (3.5.15, 3.5.17 – 3.5.22). The fact that the class of second countable interior algebras is closed under principal quotients allows us to show that an interior algebra is second countable iff any base for it contains a countable base (Theorem 3.5.16).

CHAPTER 4.

§4.1. The topic of 'seperation' in interior algebras deals with generalizations of the separation properties satisfied by metric spaces. In the first section of Chapter 4 we investigate separation properties which are related to the convergence of filters and nets to atoms and the uniqueness of such atomic limits. The Hausdorff separation property is generalized to interior algebras and various alternative characterizations of this property are found (4.1.1, 4.1.2). We also define properties related to the Hausdorff property such as 'point Hausdorff', 'sequentially Hausdorff' and 'sequentially point Hausdorff' and investigate conditions under which these properties are equivalent. In particular we show that Hausdorff and point Hausdorff are equivalent in open atomic interior algebras (Theorem 4.1.4), Hausdorff and sequentially Hausdorff are equivalent in first countable interior algebras (Theorem 4.1.6), point Hausdorff and sequentially point Hausdorff are equivalent in first countable interior algebras (Theorem 4.1.8) and thus all these properties are equivalent in first countable open atomic interior algebras (Corollary 4.1.9).

§4.2. In this section we investigate separation properties related to the canonical pre–order

§4.5. In this last section of Chapter 4 we investigate the preservation of the separation properties under constructions. Many of the separation properties are preserved under principal quotients (Theorem 4.5.3, Theorem 4.5.9). The class of normal algebras is closed under closed quotients (Theorem 4.5.4) but not open quotients. We characterize those normal algebras which have the property that all their principal quotients are normal (Theorem 4.5.6). A particularly interesting result is that many of the separation properties are definable by a single universal–existential Horn sentence (Theorem 4.5.11). Thus these classes have the usual preservation properties associated with such first order sentences, in particular they are closed under reduced products. We also show that many other classes of interior algebras are closed under products (Theorem 4.5.12, Theorem 4.5.14).

CHAPTER 5.

§5.1. Compactness properties in interior algebras are generalizations of compactness properties in topology and are thus related to the existence of atoms to which filters or nets of atoms accumulate. One generalization of the compactness property in topology is 'filter compactness' (see 5.1.1). Various characterizations of filter compactness are found (Theorem 5.1.3). Two other generalizations of topological compactness are 'cover compactness' and 'point compactness'. We show that each of these combined with closed atomicity is equivalent to filter compactness (Theorem 5.1.5). Other compactness properties are introduced: 'sequence compactness', 'strong sequence compactness' and 'countable compactness'. Every closed atomic sequence compact interior algebra is countably compact (Theorem 5.1.8). In a first countable interior algebra, sequence compactness is equivalent to strict sequence compactness (Corollary 5.1.10) and in a second countable closed atomic interior algebra all the compactness properties we have mentioned are equivalent (Corollary 5.1.11). We also generalize supercompactness to interior algebras. One generalizition is simply called 'supercompactness', other generalizations are 'point supercompactness' and 'sequence supercompactness'. In Chapter 2 it was shown that subdirect irreducibility is also a generalization of topological supercompactness. We find alternative characterizations of supercompactness and in particular we show that point supercompactness combined with closed atomicity is equivalent to supercompactness as is subdirect irreducibility combined with closed atomicity in non–trivial interior algebras (Theorem 5.1.13). In a closed atomic interior algebra with more than one but not more than $\aleph_0$ atoms, subdirect irreduciblity, (point) defined in Chapter 3. In particular we generalize the symmetric, Kolmogorov and Fréchet properties to interior algebras. Alternative characterizations of the Kolmogorov and Fréchet properties are given (Proposition 4.2.2, Theorem 4.2.4). It is shown that every sequentially point Hausdorff interior algebra or every interior algebra with 'openly generated atoms', is Kolmogorov (Theorem 4.2.7), moreover for finite interior algebras the properties: Kolmogorov, openly generated and openly generated atoms, are all equivalent (Theorem 4.2.8). A stronger form of the Fréchet property is introduced and is shown to be equivalent to the Fréchet property in the case of residually atomic interior algebras (Proposition 4.2.10).

§4.3. We generalize the concepts of α–separation, α an ordinal, to interior algebras. These properties turn out to be more interesting for interior algebras than for topology. The classes of n–separated interior algebras, $n < \omega$, form strictly elementary classes of interior algebras with hereditarily undecidable first order theories (Proposition 4.3.6, Theorem 4.3.18). Alternative characterizations of the α–separation properties are investigated (4.3.15 – 4.5.17). In particular we show that the 1–separated interior algebras are just the Fréchet algebras (Proposition 4.3.5) and that the 2–separated interior algebras are just the point Hausdorff algebras (Theorem 4.3.18). We generalize the Urysohn property to interior algebras and show that every Urysohn algebra is 3–separated and moreover these two properties are equivalent for interior algebras which are both open atomic and closed atomic. We introduce the property of 'clopen separation' and show that every clopen separated interior algebra is ∞–separated, that is, α–separated for all ordinals α (Theorem 4.3.24). It is shown that for (3–)saturated interior algebras ω–separation is equivalent to ∞–separation (Theorem 4.3.26) from which it follows that every ω–separated interior algebra is elementarily embeddable in an ∞–separated interior algebra (Corollary 4.3.27) and that if $\omega < \alpha$ or $\alpha = \infty$, then the class of α–separated interior algebras is not elementary (Corollary 4.3.29).

§4.4. We generalize the concepts of regularity and normality to interior algebras and various alternative characterizations of these properties are found (4.4.1, 4.4.2, 4.4.8, 4.4.9). We show that any regular Kolmogorov algebra is Urysohn (Theorem 4.4.3) and that any atomic regular Kolmogorov algebra is ∞–separated (Theorem 4.4.5). We show that in a regular atomic interior algebra every open element is a join of regular closed elements and that the regular open elements form a base (Theorem 4.4.6). We also find an alternative characterization of the Lindelöf property in regular atomic interior algebras. (Theorem 4.4.7).

supercompactness and sequence compactness are all equivalent (Theorem 5.1.15). We end this section with an investigation of the preservation of the compactness properties (Theorem 5.1.17, Theorem 5.1.18).

§5.2. In this section we investigate the interaction between separation and compactness. In a closed atomic interior algebra, filter, cover and point compactness coincide and so we may speak of 'compact' interior algebras. We show that an atomic compact Hausdorff algebra is normal (Theorem 5.2.2) from which it follows that the properties normal, regular, Urysohn and Hausdorff are all equivalent in atomic compact Fréchet algebras (Corollary 5.2.3). We may call an element of an interior algebra filter, cover or point compact iff the corresponding principal ideal quotient algebra has the respective property. We show that in a residually atomic Hausdorff algebra every filter compact element is closed (Theorem 5.2.5) and investigate some corollaries of this fact (5.2.6 – 5.2.9). In an atomic interior algebra compact elements are shown to behave like atoms as regards the properties regular, Urysohn and Hausdorff (Theorem 5.2.10). We generalize the concept of local compactness to interior algebras and investigate its preservation (5.2.11, 5.1.12). Alternative characterizations of local compactness in atomic Hausdorff algebras are found (Theorem 5.2.13). As corollaries to this we see that any atomic compact Hausdorff algebra is locally compact (Corollary 5.2.14) and show that an atomic locally compact Hausdorff algebra is Lindelöf iff it is a countable join of cover compact interior algebras (Corollary 5.2.15). Every residually atomic locally compact Hausdorff algebra is regular (Proposition 5.2.16). We may refer to elements of interior algebras as being locally compact iff the corresponding principal ideal quotient algebras are locally compact. We investigate locally compact elements in Hausdorff algebras (5.2.17 – 5.2.21). In particular we introduce the concept of 'residually closed' elements and show that in atomic Hausdorff algebras every locally compact element is residually closed (Theorem 5.2.20), and moreover these two concepts are equivalent in atomic locally compact Hausdorff algebras (Corollary 5.2.21).

§5.3. We briefly look at compactifications of interior algebras, a concept which generalizes compactifications and completions of topological spaces and metric spaces. We show that the constuction of S.I. interior algebras discussed in Chapter 2 may be viewed as 'compactifying' non–trivial interior algebras (Proposition 5.3.3). The rest of this section deals with a very general method of compactifying complete interior algebras (5.3.4 – 5.3.10). In particular we use this compactification construction to characterize filter compactness in complete residually atomic Hausdorff algebras (Theorem 5.3.9).

Note on terminology and notation

We assume throughout that the reader is familiar with the basic terminology for category theory, universal algebra, topology, model theory, Boolean algebras and Heyting algebras. In keeping with modern terminology, one–to–one maps will be called *injective* and onto maps will be called *surjective.* Throughout the thesis we will make use of the standard abbreviation 'iff' for the expression 'if and only if'. An index of new or uncommon definitions is provided on page 144.

As far as notation is concerned we point out the following: Bold capitals will usually be used to denote structures. If **A**, **B**, **C**,... are structures then A, B, C,... will denote the underlying sets of **A**, **B**, **C**,... respectively. We use f[S] and $f^{-1}[S]$ (with square brackets) to denote the image and pre–image respectively of a subset S under a map f. The symbol $\sqcap$ is used to denote arbitrary meets in a lattice structure while Π is reserved for products of structures. ∞ is used to denote an informal upper bound to the class of all ordinals. The remaining notational conventions are standard. A complete glossary of symbols and notation used can be found on page 136.

CHAPTER 1

BASIC THEORY OF INTERIOR ALGEBRAS

1.1 INTERIOR ALGEBRAS AND GENERALIZED TOPOLOGICAL SPACES

1.1.1 Definition

By an **interior algebra** we mean an algebraic structure $\langle \mathbf{L}, {}^I, {}^C \rangle$ where:

i) $\mathbf{L} = \langle L, \cdot, +, ', 0, 1 \rangle$ is a Boolean algebra

ii) ^{I} and ^{C} are unary operations satisfying:

a) $1^I = 1$

b) $a^I \leq a$

c) $a^{II} = a^I$

d) $(ab)^I = a^I b^I$

e) $a^C = a'^{I'}$ for all a,b $\in$ L

The operations ^{I} and ^{C} are called the **interior operator** and **closure operator** respectively. For a $\in$ L a^I and a^C are known as the **interior** and the **closure** of a respectively. □

We can generalize the principle of duality for Boolean algebras to interior algebras as follows: Given a sentence φ in a formal language for interior algebras define the dual of φ to be the sentence φ' obtained from φ by interchanging $\cdot$ and +, 0 and 1, and ^{I} and ^{C}. Then φ holds iff φ' holds.

1.1.2 Proposition

For any interior algebra **A** and all a,b $\in$ A:

i) $a^C \geq a$

ii) $a^{CC} = a^C$

iii) $(a + b)^C = a^C + b^C$

iv) $a^I = a'^{C'}$

v) $0^C = 0$, $0^I = 0$, $1^C = 1$

vi) $a^{CICI} = a^{CI}$, $a^{ICIC} = a^{IC}$

vii) $a \leq b$ implies $a^I \leq b^I$ and $a^C \leq b^C$ □

1.1.3 Definition and Remark

By a **generalized topological space** we mean a structure $\langle$ L, G $\rangle$ where:

i) $\mathbf{L} = \langle L, \cdot, +, ', 0, 1 \rangle$ is a Boolean algebra

ii) G is a unary relation satisfying:

a) $0,1 \in G$

b) G is closed under arbitrary joins.

c) G is closed under finite meets.

d) For all $a \in L$, $\Sigma \{ b \in G : b \leq a \}$ exists.

G is said to be a **generalized topology** in the Boolean algebra **L**. Note that if **L** is complete then condition (d) above holds automatically. The defining conditions for a generalized topology are in a form which emphasizes the connection with topology: If $\mathcal{T}$ is a topology on a set X then $\mathcal{T}$ is a generalized topology in the power set Boolean algebra on X. The conditions can be simplified as we shall see below. □

<u>1.1.4 Proposition</u>

Let **L** be a Boolean algebra. A subset $G \subseteq L$ is a generalized topology in **L** iff:

a) $1 \in G$

b) G is closed under binary meets.

c) For all $a \in L$, $\max \{ b \in G : b \leq a \}$ exists.

<u>Proof</u>:

If G is a generalized topology in L then (a), (b) hold by definition. Let $a \in L$. Then $\Sigma \{ b \in G : b \leq a \}$ exists and is in G since it is a join of elements of G, whence (c) holds. Now suppose (a), (b), (c) hold. $\max \{ b \in G : b \leq 0 \}$ exists and must be 0 whence $0 \in G$. Let $S \subseteq G$ such that ΣS exists. Put $d = \max \{ b \in G : b \leq \Sigma S \}$. For all $a \in S$, $a \leq \Sigma S$ whence $a \leq d$. Thus $\Sigma S \leq d$. But $d \leq \Sigma S$ and so $\Sigma S = d \in G$. Thus G is closed under arbitrary joins. (a) and (b) ensure that G is closed under finite meets. □

<u>1.1.5 Remark</u>

Let φ be the conjunction of the following sentences in the language for Boolean algebras with added unary relation:

$G(1)$

$(\forall x)(\forall y) (G(x) \wedge G(y) \Longrightarrow G(xy))$

$(\forall x)(\exists y) (G(y) \wedge y \leq x \wedge (\forall z) (G(z) \wedge z \leq x \Longrightarrow y \leq z))$

By Proposition 1.1.4 we see that $\langle L, G \rangle \models \varphi$ iff $\langle L, G \rangle$ is a generalized topological space. Thus the class of all generalized topological spaces is a finitely axiomatizable elementary subclass of the class of all Boolean algebras with added unary relation. □

We now describe the connection between interior algebras and generalized topological spaces:

1.1.6 Definition and Notation

Given an interior algebra **A**, an element $a \in A$ is called **open** iff $a^I = a$. Note that a^I is open for all $a \in A$. Put $A^O = \{ a \in A : a \text{ is open} \}$. Let $\mathbf{A}^u$ denote the underlying Boolean algebra of **A**. Put $\text{Gt}\ \mathbf{A} = \langle \mathbf{A}^u, A^O \rangle$ Given a generalized topological space $\langle \mathbf{L}, G \rangle$ define operations ^{I} and ^{C} on L by $a^I = \max \{ b \in G : b \leq a \}$ and $a^C = a'^{I}{}'$ or equivalently $a^C = \min \{ b' : b \in G \text{ and } a \leq b' \}$, for all $a \in L$. Put $\text{Alg}\ \langle \mathbf{L}, G \rangle = \langle \mathbf{L}, {}^I, {}^C \rangle$. □

A tedious but routine proof gives:

1.1.7 Theorem

Let **A** be an interior algebra and **T** a generalized topological space.

i) Gt **A** is a generalized topological space.

ii) Alg **T** is an interior algebra.

iii) Gt Alg **T** = **T**

iv) Alg Gt A = A □

Thus interior algebras and generalized topological spaces are essentually the same things. Moreover if $f : \mathbf{A} \longrightarrow \mathbf{B}$ is an interior algebra homomorphism then $f : \text{Gt}\ \mathbf{A} \longrightarrow \text{Gt}\ \mathbf{B}$ is a homomorphism of Boolean algebras with added unary relation, since if $a \in A^O$ $f(a)^I = f(a^I) = f(a)$ whence $f(a) \in B^O$. However, not every homomorphism between generalized topological spaces gives rise to an interior algebra homomorphism (See paragraph after Corollary 1.1.11.) This leads us to define:

1.1.8 Definition and Notation

Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be a map between two interior algebras. f is called a **topomorphism** iff f is a Boolean algebra homomorphism with $f[A^O] \subseteq B^O$. Let **Int** and **Int$^+$** denote the categories of interior algebras and homomorphisms, and interior algebras and topomorphisms respectively. (Up to isomorphism **Int$^+$** is just the category of generalized topological spaces and their homomorphisms.) □

1.1.9 Remark

A useful way of representing finite interior algebras (with not much more than four atoms) and portions of infinite interior algebras, is to give the lattice diagram for the underlying Boolean algebra and to indicate the open elements by circles. □

1.1.10 Theorem

Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be an injective topomorphism. f is an embedding in $\mathbf{Int}^{+}$ iff $f^{-1}[B^{O}] = A^{O}$.

Proof:

Let f be an embedding in $\mathbf{Int}^{+}$. Suppose there is an $a \in A$ with $f(a) \in B^{O}$ but $a \notin A^{O}$. Let $\mathbf{C}$ be the interior algebra given in Fig.1:

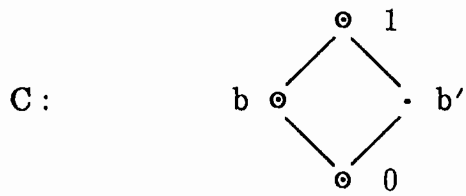

Fig. 1

Let $g : \mathbf{C} \longrightarrow \mathbf{A}$ be the Boolean algebra homomorphism with $g(b) = a$. Then fg is a topomorphism but g is not, a contradiction. Hence $f^{-1}[B^{O}] = A^{O}$. Now suppose $f^{-1}[B^{O}] = A^{O}$. Let $h : \mathbf{D} \longrightarrow \mathbf{A}$ be a map with fh a topomorphism. f is a Boolean algebra embedding so h is a Boolean algebra homomorphism. Let $a \in D^{O}$. Then $fh(a) \in B^{O}$. Hence $h(a) \in A^{O}$ and so h is a topomorphism and the result follows. □

1.1.11 Corollary

Every embedding in $\mathbf{Int}$ is an embedding in $\mathbf{Int}^{+}$.

Proof:

Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be an $\mathbf{Int}$ embedding Let $a \in A$ with $f(a) \in B^{O}$. Then $f(a^{I}) = f(a)^{I} = f(a)$ and so $a^{I} = a$, that is $a \in B^{O}$. Thus $f^{-1}[B^{O}] = A^{O}$. □

The converse of the above fails: Consider the interior algebras $\mathbf{A}$ and $\mathbf{B}$ given by Fig. 2

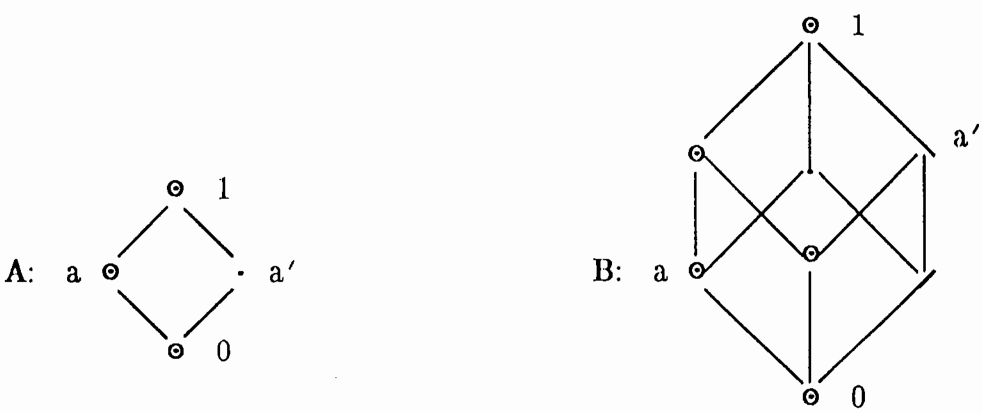

Fig. 2

The inclusion map of **A** in **B** is an embedding in $\mathbf{Int}^+$ but it is not in **Int**.

1.1.12 Remark

Isomorphisms in $\mathbf{Int}^+$ and **Int** coincide. A topomorphism $f : \mathbf{A} \longrightarrow \mathbf{B}$ is an isomorphism iff f is bijective and $f[A^O] = B^O$. □

A nice characterization of quotient maps in $\mathbf{Int}^+$ does not seem to be obtainable. However we have the following:

1.1.13 Proposition

i) Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be a surjective topomorphism. If $f[A^O] = B^O$ then f is a quotient map in $\mathbf{Int}^+$.

ii) Every quotient map in **Int** (surjective homomorphism) satisfies (i) and so is a quotient map in $\mathbf{Int}^+$.

Proof:

(i): Suppose $f[A^O] = B^O$. Let $g : \mathbf{B} \longrightarrow \mathbf{C}$ be a map such that gf is a topomorphism. Since f is a surjective Boolean algebra homomorphism g is a Boolean algebra homomorphism. Let $b \in B^O$. Then there is an $a \in A^O$ with $b = f(a)$. Then $g(b) = gf(a) \in C^O$. Hence g is a topomorphism and the result follows. (ii): Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be a surjective homomorphism. Let $b \in B^O$. There is an $a \in A$ with $b = f(a)$. Then $a^I \in A^O$ with $f(a^I) = f(a)^I = b^I = b$. Thus $f[A^O] = B^O$. □

1.2 INTERIOR ALGEBRAS AND HEYTING ALGEBRAS

1.2.1 Definition and Remark

For any interior algebra **A**, $0,1 \in A^O$ and A^O is closed under finite meets and joins. Thus $\langle A^O, \cdot, +, 0, 1 \rangle$ is a distributive 0,1–lattice. Moreover, for all $a,b \in A^O$, $a \longrightarrow b = (a' + b)^I$ is the relative pseudocomplement of a with respect to b. Thus we have a Heyting algebra $\mathbf{A}^O = \langle A^O, \cdot, +, \longrightarrow, 0, 1 \rangle$. If $f : \mathbf{A} \longrightarrow \mathbf{B}$ is an interior algebra homomorphism then $f^O = f|_{A^O} : \mathbf{A}^O \longrightarrow \mathbf{B}^O$ is a Heyting algebra homomorphism. If **Halg** denotes the category of Heyting algebras and their homomorphisms then we have a functor $^O : \mathbf{Int} \longrightarrow \mathbf{Halg}$. □

1.2.2 Remark

Let **A** and **B** be as in Fig. 3.

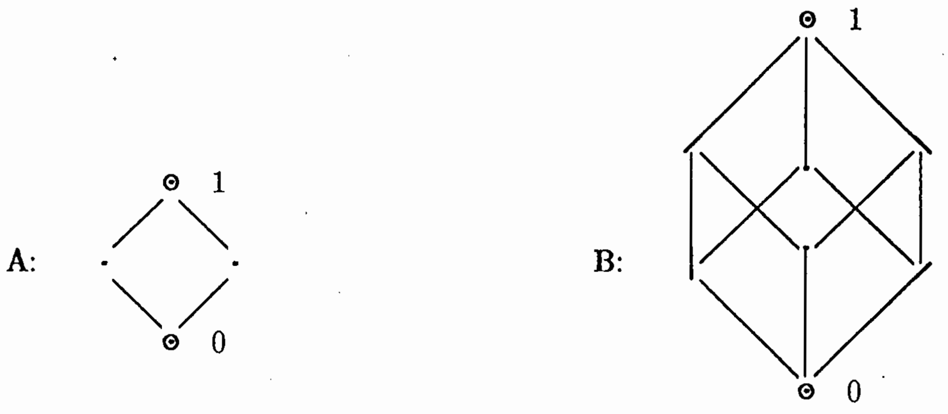

Fig. 3

The functor ^{O} is not full: There is an isomorphism from $\mathbf{B}^O$ to $\mathbf{A}^O$ but no homomorphism from **B** to **A**. ^{O} is not faithful: There are several embeddings of **A** into **B** all of which have the same image under ^{O}. □

Despite the fact that ^{O} is neither full nor faithful it still has interesting preservation properties:

<u>1.2.3 Proposition</u>
^{O} preserves embeddings, surjective homomorphisms, products and direct limits. □

Since ^{O} preserves products we see:

<u>1.2.4 Proposition</u>
If $\{ T_i : i \in I \}$ is a family of generalized topological spaces then $\Pi \{ T_i : i \in I \} = \mathrm{Gt}\, \Pi \{ \mathrm{Alg}\, T_i : i \in I \}$. □

<u>1.2.5 Corollary</u>
Products in **Int** are also products in $\mathbf{Int}^+$. □

<u>1.2.6 Proposition</u>
^{O} reflects embeddings.
<u>Proof</u>:
Suppose $f : \mathbf{A} \longrightarrow \mathrm{B}$ is not an embedding. Then there are distinct $a,b \in B$ with $f(a) = f(b)$. Then $f^O((ab + a'b')^I) = f^O(1)$ but $(ab + a'b')^I < 1$ and so f^O is not an embedding. □

The following is an important proposition:

1.2.7 Proposition

Let **A** be an interior algebra and $S \subseteq A^O$. The join of S exists in **A** iff the join of S exists in A^O. If these joins exist then they are equal.

Proof:

If b is the join of S in **A** then since A^O is a generalized topology in $\mathbf{A}^u$, $b \in A^O$ and so b is also the join of S in A^O. Conversely if b is the join of S in A^O consider an upper bound d of S in **A**. Then for all $a \in S$ $a \leq d$ and so $a = a^I \leq d^I$. Hence $b \leq d^I \leq d$ and so b is also the join of S in **A**. □

Thus when talking about a join of open elements we do not have to specify whether the join is taken in A or A^O.

1.2.8 Proposition

Let **A** be an interior algebra and $S \subseteq A$. Suppose b is the meet of S in **A**. Then b^I is the meet of $\{ a^I : a \in S \}$ in $\mathbf{A}^O$. In particular if $S \subseteq A^O$ then b^I is the meet of S in $\mathbf{A}^O$. □

1.2.9 Corollary

If A is a complete interior algebra, $\mathbf{A}^O$ is a complete Heyting algebra. Joins in $\mathbf{A}^O$ coincide with joins in A while meets in $\mathbf{A}^O$ are the interiors of meets in **A**. □

1.2.10 Definition and Remark (cf. Blok [3])

Given a Heyting algebra **L** let $\mathcal{X}$ be the set of all prime proper filters in L. Let $j : L \longrightarrow \mathcal{P}(\mathcal{X})$ be given by $j(a) = \{ F \in \mathcal{X} : a \in F \}$. Let L^H be the subset of $\mathcal{P}(\mathcal{X})$ generated as a Boolean subalgebra by j[L]. j is injective so we can replace elements of j[L] by corresponding elements of L to obtain a Boolean algebra $\langle L^H, \cdot, +, ', 0, 1 \rangle$ with $L \subseteq L^H$ as a generating set. This is in fact the free Boolean extension of L as a distibutive 0,1–lattice. Now $1 \in L$ and L is closed under finite meets. If $c \in L^H$ then there are $a_1,\dots,a_n,b_1,\dots,b_n \in L$ such that $c = (a_1' + b_1)\cdots(a_n' + b_n)$ in L^H. Then $(a_1 \longrightarrow b_1)\cdots(a_1 \longrightarrow b_n) = \max \{ d \in L : d \leq c \}$. By Proposition 1.1.4 L is a generalized topology. Thus we obtain an interior algebra L^H with $(L^H)^u = \langle L^H, \cdot, +, ', 0, 1 \rangle$ and $(L^H)^O = L$. If $f : \mathbf{L} \longrightarrow \mathbf{M}$ is a Heyting algebra homomorphism then we can consider f to be a 0,1–lattice homomorphism. Hence there is a unique extension f^H of f to a Boolean algebra homomorphism from $\mathbf{L}^H$ to $\mathbf{M}^H$. It is easily seen that f^H must then also be an interior algebra homomorphism. If **L** is a Heyting algebra $id(\mathbf{L})^H = id(\mathbf{L}^H)$ by the uniqueness of $id(\mathbf{L})^H$. If $f : \mathbf{L} \longrightarrow \mathbf{M}$ and $g : \mathbf{M} \longrightarrow \mathbf{N}$ are Heyting algebra homomorphisms then $(gf)^{H} = g^{H}f^{H}$ by the uniqueness of $(gf)^{H}$. We thus have a functor ^{H} : **Halg** $\longrightarrow$ **Int** which is a right inverse of ^{O} : **Int** $\longrightarrow$ **Halg**. □

1.2.11 Proposition (cf. Blok [3])
^{H} is a left adjoint of ^{O}. □

1.2.12 Remark
^{H} preserves anything reflected by ^{O} and reflects anything preserved by ^{O}. In particular ^{H} preserves embeddings. □

1.2.13 Proposition (cf. Blok [3])
^{H} preserves surjective homomorphisms, finite products, and direct limits. □

The above is easily proved using the fact that L generates L^{H} as a Boolean algebra.

1.2.14 Proposition
Let ϵ denote the co–unit of the adjunction $\langle\, ^{H}, ^{O} \rangle$. Then for all interior algebras A $\epsilon(A) : A^{OH} \longrightarrow A$ is an embedding.
Proof:
$\epsilon(A)^{O} = id(A^{O})$ for all interior algebras A. Since ^{O} reflects embeddings the result follows. □

1.2.15 Corollary
For all interior algebras A, ϵ gives an isomorphism between A^{OH} and the subalgebra of A generated by A^{O}. □

1.2.16 Corollary (cf. Blok [3])
Let $f : L \longrightarrow A^{O}$ be a Heyting algebra embedding, where A is an interior algebra. Let B be the subalgebra of A generated by f[L]. Then $L^{H} \cong B$. □

1.2.17 Definition and Remark
An interior algebra A is said to be openly generated iff A^{O} generates A. (Openly generated interior algebras were called '*–algebras' in [3].) Note that L^{H} is openly generated for all Heyting algebras L. Let **IntO** denote the full subcategory of **Int** consisting of the openly generated interior algebras. □

We immediately see:

1.2.18 Corollary (cf. Blok [3])

The following are equivalent:

i) $\mathbf{A}$ is openly generated.

ii) $\mathbf{A}^{O}$ generates $\mathbf{A}$ as a Boolean algebra.

iii) $\epsilon(\mathbf{A})[\mathbf{A}^{OH}] = \mathbf{A}$ □

1.2.19 Proposition

$^{O}: \mathbf{IntO} \longrightarrow \mathbf{Halg}$ and $^{H}: \mathbf{Halg} \longrightarrow \mathbf{IntO}$ form a co–equivalence system.

Proof:

$^{HO}: \mathbf{Halg} \longrightarrow \mathbf{Halg}$ is the identity functor on $\mathbf{Halg}$ while $^{OH}: \mathbf{IntO} \longrightarrow \mathbf{IntO}$ is naturally isomorphic to the identity functor on $\mathbf{IntO}$ via ϵ. □

1.2.20 Corollary

$\mathbf{IntO}$ is a co–reflective subcategory of $\mathbf{Int}$. The co–reflector is $^{OH}: \mathbf{Int} \longrightarrow \mathbf{IntO}$ and the co–reflection morphisms are the embeddings $\epsilon(\mathbf{A}) : \mathbf{A}^{OH} \longrightarrow \mathbf{A}$. □

We investigate some properties of openly generated interior algebras.

1.2.21 Definition (cf. Blok [3])

Define polynomials $\rho_n(x)$, $n < \omega$ inductively as follows: Put $\rho_0(x) = x'^{I}$ and $\rho_1(x) = (\rho_0(x) + x)^{I}$. If $\rho_{2k-2}(x)$ and $\rho_{2k-1}(x)$ have been defined for $k \geq 1$ put $\rho_{2k}(x) = (\rho_{2k-1}(x) + x')^{I}$ and $\rho_{2k+1}(x) = (\rho_{2k}(x) + x)^{I}$. For all $n < \omega$ define $\sigma_n(x)$ inductively as follows: Put $\sigma_0(x) = \rho_0(x)'\rho_1(x)$. If $\sigma_k(x)$ has been defined for $k < \omega$ put $\sigma_{k+1}(x) = \sigma_k(x) + \rho_{2k+2}(x)'\rho_{2k+3}(x)$. □

We can use the polynomials $\sigma_n(x)$, $n < \omega$ to characterize the subalgebra $\epsilon(\mathbf{A})[\mathbf{A}^{OH}]$ and hence to characterize openly generated interior algebras:

1.2.22 Lemma (cf. Blok [3])

Let $n < \omega$.

i) $\rho_{2n}(x)'\rho_{2n+1}(x) \leq x$

ii) $\rho_{2n+1}(x)'\rho_{2n+2}(x) \leq x'$

iii) $\rho_n(x) \leq \rho_{n+1}(x)$ □

1.2.23 Lemma (cf. Blok [3])

Let $n < \omega$. Then $\sigma_n(x) = x \cdot \rho_{2n+1}(x)$.

Proof:

By Lemma 1.2.22 (i) and (ii) $\sigma_n(x) \leq x \cdot \rho_{2n+1}(x)$. We prove by induction that in fact $\sigma_n(x) = x \cdot \rho_{2n+1}(x)$. For $n = 0$ we have $x \cdot \rho_1(x) \leq x^C \cdot \rho_1(x) = x'^{I\prime}\rho_1(x) = \rho_0(x)'\rho_1(x) = \sigma_0(x)$ and so $x \cdot \rho_1(x) = \sigma_0(x)$. Now suppose the result holds for $n = k < \omega$. Then $x \cdot \rho_{2k+3}(x) = x \cdot (\ \rho_{2k+1}(x) + \rho_{2k+1}(x)'\rho_{2k+2}(x) + \rho_{2k+2}(x)'\rho_{2k+3}(x)\ ) \leq x \cdot \rho_{2k+1}(x) + xx' + \rho_{2k+2}(x)'\rho_{2k+3}(x) = \sigma_k(x) + \rho_{2k+2}(x)'\rho_{2k+3}(x) = \sigma_{k+1}(x)$, using Lemma 1.2.22 (ii). Thus the result holds for $n = k + 1$. □

1.2.24 Theorem (cf. Blok [3])

Let **A** be an interior algebra and let **B** be the subalgebra of **A** generated by A^O. Then $B = \{\ a \in A : \sigma_n(a) = a \text{ for some } n < \omega\ \}$.

Proof:

Clearly if $a \in A$ with $\sigma_n(a) = a$ then $a \in B$. Suppose $a \in B$. Then there are $u_0,\ldots,u_i,v_0,\ldots,v_i \in A^O$ with $a = u_0'v_0 + \ldots + u_i'v_i$. Let **C** be the subalgebra of **A** generated by $\{\ u_0,\ldots,u_i,v_0,\ldots,v_i\ \}$. $\mathbf{C}^u$ is the free Boolean extension of $\mathbf{C}^O$ as a distributive 0,1–lattice. Now C^O is countable and so there is a chain $D \subseteq C^O$ such that D generates $\mathbf{C}^u$. (See [1] V.7.3) Then there are $d_0 < \ldots < d_{2n+1} \in D$ such that $a = d_0'd_1 + \ldots + d_{2n}'d_{2n+1}$. We show by induction that for $r = 0,\ldots,2n+1$, $d_r \leq \rho_r(a)$. For $r = 0$ we have $d_0 \cdot a = d_0 \cdot (\ d_0'd_1 + \ldots + d_{2n}'d_{2n+1}\ ) = 0$, using Lemma 1.2.22 (iii), and so $d_0 \leq a'$ whence $d_0 = d_0^I \leq a'^I = \rho_0(a)$. Now suppose $d_{2k} \leq \rho_{2k}(a)$ where $0 \leq k \leq n$. Then $d_{2k+1} = d_{2k} + d_{2k}'d_{2k+1} \leq \rho_{2k}(a) + a$ and so $d_{2k+1} \leq (\ \rho_{2k}(a) + a\ )^I = \rho_{2k+1}(a)$. Also if $k < n$ then $d_{2k+2} = d_{2k+1} + d_{2k+1}'d_{2k+2} \leq \rho_{2k+1}(a) + a'$ and so $d_{2k+2} \leq (\ \rho_{2k+1}(a) + a'\ )^I = \rho_{2k+2}(a)$. Thus for $r = 0,\ldots,2n+1$, $d_r \leq \rho_r(a)$. Then $a = ad_{2n+1} \leq a\rho_{2n+1}(a) = \sigma_n(a)$ by Lemma 1.2.23, whence $a = \sigma_n(a)$. □

1.2.25 Corollary (cf. Blok [3])

The following are equivalent for an interior algebra **A**:

i) **A** is openly generated.

ii) For all $a \in A$ there is an $n < \omega$ with $a = \sigma_n(a)$.

iii) For all $a \in A$ the subalgebra of **A** generated by a is openly generated. □

1.2.26 Proposition (cf. Blok [3])

The class of openly generated interior algebras is closed under subalgebras, topomorphic images, finite products and direct limits.

Proof:

Subalgebras: Let **A** be openly generated and let **B** be a subalgebra of **A**. Let $a \in B$. Then $a \in A$ and so by Corollary 1.2.25 there is an $n < \omega$ with $a = \sigma_n(a)$ in **A** and hence in **B**.

Thus again by Corollary 1.2.25 **B** is openly generated. Closure under topomorphic images and direct limits follows easily from Corollary 1.2.16 (ii). Finite products: Let $\mathbf{A}_1,...,\mathbf{A}_n$ be openly generated. Then $\mathbf{A}_1 \times ... \times \mathbf{A}_n \cong \mathbf{A}_1^{OH} \times ... \times \mathbf{A}_n^{OH} \cong (\mathbf{A}_1 \times ... \times \mathbf{A}_n)^{OH}$ using Corollary 1.2.18, Proposition 1.2.3, and Proposition 1.2.13. Thus $\mathbf{A}_1 \times ... \times \mathbf{A}_n$ is openly generated. □

Of course **IntO** has products since **Halg** does. From the above we see:

1.2.27 Corollary

Finite products in **IntO** are products in **Int**. □

1.2.28 Remark

The above does not hold for arbitrary products. This follows from the fact that the class of openly generated interior algebras is not closed under arbitrary products in **Int** as is shown by the following example: (c.f. 2.9 in Blok [3]) For $n < \omega$ let $\mathbf{C}_n$ be the interior algebra with C_n^u the power set Boolean algebra on 2n and C_n^O the chain $0 < ... < 2n$, (considered as subsets of 2n). Then clearly for all $n < \omega$, C_n^O generates $\mathbf{C}_n$ and so $\mathbf{C}_n$ is openly generated. Put $\mathbf{C} = \Pi \{ \mathbf{C}_n : n < \omega \}$. Consider the sequence $a = (a_n) \in C$ given inductively by $a_0 = 0$, $a_{k+1} = a_k \cup \{ 2k + 2 \}$. Then for all $n < \omega$ $\sigma_n(a) = (c_n)$ where $c_k = a_k$ for $k \leq n + 1$ and $c_k = a_{n+1}$ for $k > n + 1$. Thus for all $n < \omega$ $\sigma_n(a) \neq a$ and so by Corollary 1.2.25 **C** is not openly generated. Thus we also see that ^{H} does not preserve arbitrary products or else arguing as in Proposition 1.2.26 we get a contradiction to the above. □

1.2.29 Notation and Remark

For each class $\mathcal{K}$ of interior algebras put Halg($\mathcal{K}$) = $\{ \mathbf{A}^O : \mathbf{A} \in \mathcal{K} \}$. For each class $\mathcal{K}$ of Heyting algebras put Int($\mathcal{K}$) = $\{ \mathbf{A} : \mathbf{A}^O \in \mathcal{K} \}$. Notice that for all classes $\mathcal{K}$ of Heyting algebras Halg(Int($\mathcal{K}$)) = $\mathcal{K}$ since ^{O} has a right inverse ^{H}. In particular the assignment $\mathcal{K} \longmapsto$ Halg($\mathcal{K}$) is surjective and the assignment $\mathcal{K} \longmapsto$ Int($\mathcal{K}$) is injective. However if $\mathcal{K}$ is a class of interior algebras we need not have $\mathcal{K}$ = Int(Halg($\mathcal{K}$)). (Of course we always have $\mathcal{K} \subseteq$ Int(Halg($\mathcal{K}$)).) □

1.2.30 Definition

Let $\mathcal{K}$ be a class of interior algebras. $\mathcal{K}$ is called a **Heyting class** iff $\mathcal{K}$ = Int(Halg($\mathcal{K}$)) or equivalently $\mathcal{K}$ = Int($\mathcal{H}$) for some class $\mathcal{H}$ of Heyting algebras. □

Many interesting classes of interior algebras are Heyting classes as we shall see.

1.2.31 Definition

Let $\mathcal{LI}$ and $\mathcal{LH}$ denote the first order languages for interior algebras and Heyting algebras respectively. We define an interpretation τ from $\mathcal{LH}$ to $\mathcal{LI}$ as follows: For any variable x of $\mathcal{LH}$ put $\tau(x) := x^I$, put $\tau(0) := 0$, $\tau(1) := 1$. If t and s are terms such that $\tau(t)$ and $\tau(s)$ have been defined put $\tau(t \cdot s) := \tau(t) \cdot \tau(s)$, $\tau(t + s) := \tau(t) + \tau(s)$ and $\tau(t \rightarrow s) := (\tau(t)' + \tau(s))^I$. This defines the interpretation on terms. The interpretation extends to formulas by distributing over equality, connectives and quantification. □

An easy induction argument shows:

1.2.32 Theorem

Let **A** be an interior algebra:

i) Let $\varphi(x_1,...,x_n)$ be a Heyting algebra formula and let $a_1,...,a_n \in A^O$. Then $A^O \models \varphi[a_1,...,a_n]$ iff $A \models \tau(\varphi)[a_1,...,a_n]$.

ii) Let φ be a Heyting algebra sentence. Then $A^O \models \varphi$ iff $A \models \tau(\varphi)$. □

1.2.33 Corollary

Let $\mathcal{I}$ and $\mathcal{H}$ denote the varieties of interior algebras and Heyting algebras respectively.

i) If Γ is a set of sentences in $\mathcal{LH}$ then $\text{Int}(\mathcal{H} \cap \text{Mod}\ \Gamma) = \mathcal{I} \cap \text{Mod}\ \tau[\Gamma]$.

ii) If $\mathcal{K}$ is an elementary class of Heyting algebras defined by sentences of a certain structural type then $\text{Int}(\mathcal{K})$ is an elementary class of interior algebras defined by sentences of the same structural type.

iii) If $\mathcal{K}$ is a strictly elementary class of Heyting algebras then $\text{Int}(\mathcal{K})$ is a strictly elementary class of interior algebras. □

The above follows from Theorem 1.2.32 and the fact that although τ changes the structure of terms it does not change the way formulas are built up from atomic formulas.

1.2.34 Definition and Remark

CLOSED ELEMENTS

The dual notion to that of an open element is that of a closed element, that is, if A is an interior algebra then $a \in A$ is closed iff $a^C = a$. Notice that a is closed iff a' is open. If we put $A^\square = \{ a \in A : a \text{ is closed} \} = \{ a' : a \in A^O \}$ then in a completely analogous way to 1.2.1 we obtain a functor $^\square : \textbf{Int} \longrightarrow \textbf{Balg}$ where **Balg** is the category of Brouwerian algebras, the lattice theoretic duals of Heyting algebras. (The term 'Brouwerian algebra' was used differently in [3].) The dual relative pseudocomplementation in $A^\square$ is given by

$a * b = (a'b)^C$. All the results concerning $\mathbf{A}^O$ apply in dual form to $\mathbf{A}^\square$. In particular we have a functor ^{B} : **Balg** ⟶ **Int** dual to the functor ^{H}. If ${}^-$: **Halg** ⟶ **Balg** is the obvious isomorphism then ${}^{-B}$: **Halg** ⟶ **Int** is naturally isomorphic to ^{H}. Topomorphisms can be characterized as Boolean algebra homomorphisms that preserve closed elements. □

1.2.35 Definition and Remark

REGULAR OPEN AND REGULAR CLOSED ELEMENTS

Pseudocomplementation in $\mathbf{A}^O$ is given by a'^I for all $a \in A^O$ and dual pseudocomplementation in $\mathbf{A}^\square$ is given by a'^C for all $a \in A^\square$. The regular elements of $\mathbf{A}^O$ are called **regular open** and are characterized by $a^{CI} = a$. All elements of the form a^{CI} are regular open. The dual regular elements of $\mathbf{A}^\square$ are called **regular closed** and are characterized by $a^{IC} = a$. All elements of the form a^{IC} are regular closed. As usual the regular and dual regular elements of $\mathbf{A}^O$ and $\mathbf{A}^\square$ respectively form Boolean algebras. Let $\mathbf{A}^{RO}$ and $\mathbf{A}^{RC}$ denote the Boolean algebras of regular open and regular closed elements respectively. Operations in A^{RO} can be expressed in terms of operations in **A** as follows:

i) $1^{RO} = 1$ and $0^{RO} = 0$.

ii) If $S \subseteq A^{RO}$ then $\sqcap^{RO} S = (\sqcap S)^I$. In particular, if $a,b \in \mathbf{A}^{RO}$ then $a \cdot^{RO} b = ab$.

iii) If $S \subseteq A^{RO}$ then $\Sigma^{RO} S = (\Sigma\{ a^C : a \in S \})^I$. In particular, if $a,b \in A^{RO}$ then $a +^{RO} b = (a + b)^{CI}$.

iv) If $a \in A^{RO}$ then $a'^{RO} = a'^I$.

Operations in $\mathbf{A}^{RC}$ are dual to the above. □

1.2.36 Definition and Remark

CLOPEN ELEMENTS

Let **A** be an interior algebra. $a \in A$ is said to be **clopen** iff it is both open and closed. The clopen elements of A form a subalgebra $A^\diamond$. Call an interior algebra **Boolean** iff it satisfies the identity $x^I = x$, that is, all elements are open hence in fact clopen. $\mathbf{A}^\diamond$ is always Boolean. Let **IntB** denote the full subcategory of **Int** consisting of the Boolean interior algebras. Then **IntB** is isomorphic to the category of Boolean algebras via the functor that forgets the interior and closure operators. We can thus identify the category of Boolean algebras with **IntB**. Boolean Heyting algebras form a co–reflective subcategory of **Halg** and so we see that **IntB** is a co–reflective subcategory of **IntO** and hence of **Int**. We can choose the co–reflector to be the functor ${}^\diamond$: **Int** ⟶ **IntB** which assigns an interior algebra **A** to $\mathbf{A}^\diamond$ and a homomorphism $f : A \longrightarrow B$ to $f^\diamond = f|_{A^\diamond} : A^\diamond \longrightarrow B^\diamond$. The co–reflection morphisms are then the inclusion maps $i : \mathbf{A}^\diamond \longrightarrow \mathbf{A}$. The functor ${}^\diamond$ factors through ^{H} via the functor that assigns to each Heyting algebra its Boolean algebra of complemented elements. Notice that

$A^\diamond$ is always a Boolean subalgebra of both $A^{R.O}$ and $A^{R.C}$. □

1.3 CONGRUENCES ON INTERIOR ALGEBRAS

Interior algebras are Boolean algebras with operators [13] and so we immediately have:

1.3.1 Proposition

Interior algebras are congruence distributive, congruence permutable and congruence extensile. □

1.3.2 Remark

For all interior algebras **A**, **Con(A)** is clearly a meet complete 0,1–sublattice of $\mathbf{Con(A^u)}$. We thus see that congruences on **A** can be represented by filters in **A**: If Θ is a congruence on **A** then $1/\Theta$ is a filter in **A** which completely determines Θ since a Θ b iff ac = bc for some $c \in 1/\Theta$. However not every filter in **A** represents a congruence in this way. In fact, since a Boolean algebra congruence on **A** is an interior algebra congruence iff it respects interiors, we see that a filter in **A** represents an interior algebra congruence iff it is closed under the interior operator. □

1.3.3 Definition and Remark

Let **A** be an interior algebra. Call a subset $S \subseteq A$ an **open** subset iff it is closed under ^{I}, that is, $a \in S$ implies $a^I \in S$. Note that the set of open filters in **A** forms an algebraic meet complete 0,1–sublattice of the lattice of all filters in **A**. For every $S \subseteq A$ there is a smallest open filter containing S, the open filter generated by S. Notice that the open filter generated by S is precisely $\{ a \in A : a \geq (\sqcap R)^I$ for some finite $R \subseteq S \}$. In particular, if $S \subseteq A^O$ then the open filter generated S is just the filter generated by S. The principal open filters are those of the form [a) where $a \in A^O$ and these are the finitely generated open filters. □

1.3.4 Notation and Remark

Let **A** be an interior algebra. Let $L^O(\mathbf{A})$ denote the lattice of open filters in **A**. (It is a meet complete 0,1–sublattice of the lattice of all filters in **A**.) If **M** is a Heyting algebra let **L(M)** denote the lattice of all filters in **M**. If **A** is an interior algebra and $F \in L(\mathbf{A^O})$ we can define a congruence $\Theta(F)$ on **A** by a $\Theta(F)$ b iff there is an $c \in F$ with ac = bc. Conversely if $\Theta \in Con(A)$ putting $F(\Theta) = A^O \cap 1/\Theta$ gives a filter in $\mathbf{A^O}$. □

1.3.5 Theorem

For any interior algebra $\mathbf{A}$ the following lattices are isomorphic: $\mathbf{L}^{O}(\mathbf{A})$, $\mathbf{L}(\mathbf{A}^{O})$, $\mathbf{Con}(\mathbf{A})$, and $\mathbf{Con}(\mathbf{A}^{O})$. Isomorphisms are given by $F \longmapsto F \cap A^{O}$, $F \longmapsto \Theta(F)$, $\Theta \longmapsto \Theta \cap (A^{O} \times A^{O})$ respectively. Moreover, the map $\Theta \longmapsto F(\Theta)$ is the inverse of the isomorphism from $\mathbf{L}(\mathbf{A}^{O})$ to $\mathbf{Con}(\mathbf{A})$. □

To see the above note that $a \,\Theta\, b$ iff $(ab + a'b')^{I} \,\Theta\, 1$. The above theorem shows that we can represent congruences on $\mathbf{A}$ by filters in $\mathbf{A}^{O}$. In this case every filter represents a congruence on $\mathbf{A}$.

We will use the following abbreviations henceforth:

S.I.	subdirectly irreducible
F.S.I.	finitely subdirectly irreducible
D.I.	directly indecomposable

Recall that an algebra is called *semi–simple* iff all its S.I. homomorphic images are simple.

1.3.6 Corollary

$\mathbf{A}$ is simple, S.I., F.S.I., D.I. or semi–simple iff $\mathbf{A}^{O}$ is simple, S.I., F.S.I. D.I. or semi–simple respectively. □

Notice that the expression $(ab + a'b')^{I}$ occurs in the proof of Proposition 1.2.6 and is needed to show that the maps $\Theta \longmapsto F(\Theta)$ and $F \longmapsto \Theta(F)$ are inverses. This leads us to define:

1.3.7 Definition

Let $\mathbf{A}$ be an interior algebra. Define the operation ∇ on $\mathbf{A}$ by: for all $a,b \in A$ $a \nabla b = (ab + a'b')^{I}$. $a \nabla b$ is called the **open dual difference** of a and b. □

1.3.8 Proposition

Let $\mathbf{A}$ be an interior algebra, $\Theta \in \mathbf{Con}(\mathbf{A})$ and $a,b,c,d \in A$:

i) $a \,\Theta\, b$ iff $(a \nabla b) \,\Theta\, 1$

ii) $F(\Theta) = \{\, p \nabla q : p \,\Theta\, q \,\}$

iii) $\mathrm{con}(a,b) = \Theta(\,[a \nabla b)\,) = \mathrm{con}(a \nabla b,1)$

iv) $(c,d) \in \mathrm{con}(a,b)$ iff $\mathbf{A} \models \sigma\,[a,b,c,d]$ where $\sigma\,(x,y,z,w) := (x \nabla y)\cdot z = (x \nabla y)\cdot w$.

v) $\mathrm{con}(a,b) \cap \mathrm{con}(c,d) = \mathrm{con}(\rho\,[a,b,c,d],1)$ where $\rho\,(x,y,z,w) := (x \nabla y) + (z \nabla w)$. □

From the above we see:

1.3.9 Corollary (See [5] and [6].)
Interior algebras have equationally definable principal congruences (EDPC) and equationally definable principal meets (EDPM). □

1.3.10 Remark (See [4] and [7].)
For the reader who is familiar with the theory of EDPC we mention the following: Interior algebras are congruence permutable, 1–regular and have EDPC and so they are weak Brouwerian semi–lattices with filter preserving operations (WBSO's) with respect to 1. Note that the open dual difference is a Gödel equivalence term for interior algebras. Weak meet terms are given by $x \cdot_w y :=$ xy, $x^I y^I$, $x^I y$, or $x^I y + x^{I\prime} x$. Weak relative pseudocomplementation terms are given by $x \rightarrow_w y := (x^{I\prime} + y^I)^I$, $x^{I\prime} + y^I$, $x^{I\prime} + y^I$ or $x^{I\prime} + y$. The last mentioned weak meet and weak relative pseudocomplementation terms are obtained from the quaternary deductive term for interior algebras which is given by $q(x,y,z,w) := (x \nabla y)\cdot z + (x \nabla y)'\cdot w$. A commutative ternary deductive term, regular with respect to 1, is given by $p(x,y,z) := (x \nabla y)\cdot z$. Notice that the weak ordering and weak equality on interior algebras is given by $x \preceq y$ iff $x^I \leq y^I$ and $x \approx y$ iff $x^I = y^I$ respectively, which coincides with the usual order and equality on A^O. The weak filters of an interior algebra are just the open filters which may be seen directly or from the fact that the weak filters are just the congruence filters. □

We can consider $L(A^O)$ and hence $Con(A)$ to be Brouwerian algebras. Note that the principal filters in A^O form a subalgebra $L_p(A^O)$ of $L(A^O)$ isomorphic to $A^\square$ via the map $a \longmapsto [a')$.

1.3.11 Corollary
The principal congruences on A form a subalgebra $Con_p(A)$ of $Con(A)$ isomorphic to $A^\square$ via the map $a \longmapsto \Theta(\,[a')\,)$. □

Alternatively we may consider dual isomorphisms $a \longmapsto [a)$ and $a \longmapsto \Theta(\,[a)\,)$ from A^O.

1.3.12 Corollary
Let A be an interior algebra. The following are equivalent for a congruence $\Theta \in Con(A)$:

i) Θ is principal.

ii) Θ is compact.

iii) $\Theta = \Theta(\,[a)\,)$ for some $a \in A^O$. □

1.3.13 Corollary

If $\mathbf{A}^O$ is finite $\mathbf{Con}(\mathbf{A})$ is isomorphic to $\mathbf{A}^\square$ and dually isomorphic to $\mathbf{A}^O$. □

1.3.14 Definition

Let $\mathbf{A}$ be an interior algebra. $F \in L(A^O)$ is said to be a **full filter** of $\mathbf{A}$ iff for every endomorphism f on $\mathbf{A}$ we have $f[F] \subseteq F$. □

A straightforward argument establishes:

1.3.15 Proposition

A congruence Θ on $\mathbf{A}$ is fully invariant iff $F(\Theta)$ is a full filter. (Equivalently a filter F in $\mathbf{A}^O$ is full iff $\Theta(F)$ is fully invariant.) □

1.3.16 Remark

Note that the fully invariant congruences on $\mathbf{A}$ form an algebraic 0,1–sublattice $\mathbf{Con}_f(\mathbf{A})$ of $\mathbf{Con}(\mathbf{A}) \cong \mathbf{Con}(\mathbf{A}^O)$. Thus we see that the full filters of $\mathbf{A}$ form an algebraic 0,1–sublattice $\mathbf{L}_f(\mathbf{A})$ of $\mathbf{L}(\mathbf{A}^O)$. Moreover $\mathbf{L}_f(\mathbf{A})$ is isomorphic to $\mathbf{Con}_f(\mathbf{A})$ via the map $F \longmapsto \Theta(F)$. □

1.3.17 Remark

THE CONGRUENCE EXTENSION PROPERTY

Recall that interior algebras are congruence extensile. Using the representation of congruences on an interior algebra $\mathbf{A}$ by filters in $\mathbf{A}^O$ we can easily describe extensions of congruences: Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$. If $\Theta \in \mathrm{Con}(\mathbf{B})$ then $\Theta(F) \in \mathrm{Con}(\mathbf{A})$ is the smallest extension of Θ to a congruence on $\mathbf{A}$, where F is the filter $\{\ a \in A^O\ :\ b \leq a$ for some $b \in F(\Theta)\ \}$ in $\mathbf{A}^O$. Moreover if $\Theta = \mathrm{con}(a,b)$ in $\mathbf{B}$ then $\Theta(F) = \mathrm{con}(a,b)$ in $\mathbf{A}$. Since interior algebras have a ternary deductive term (see Remark 1.3.9) or since they are normal Boolean algebras with operators satisfying EDPC, they in fact satisfy the *Strong Congruence Extension Property.* (See [7].) □

1.3.18 Theorem

Let $\mathbf{A}$ be a non–trivial interior algebra.

i) $\mathbf{A}$ is simple iff $A^O = \{\ 0, 1\ \}$.

ii) $\mathbf{A}$ is S.I. iff 1 is completely join irreducible in $\mathbf{A}^O$.

iii) $\mathbf{A}$ is F.S.I. iff 1 is join irreducible in $\mathbf{A}^O$.

iv) A is D.I. iff $A^\diamond = \{ 0, 1 \}$.

Proof:

The map $a \longmapsto \Theta([a))$ is a dual embedding of A^O into **Con(A)**. Moreover for all non–trivial $\Theta \in \mathrm{Con}(A)$ there is an $a < 1$ in A^O with $\Theta([a)) \subseteq \Theta$. (i) – (iii) now follow easily. For (iv): Note that by congruence permutability **A** is D.I. iff **Con(A)** contains a pair of non–trivial complementary congruences. Now if Θ and Ψ are non–trivial complementary congruences in **Con(A)** then $F(\Theta)$ and $F(\Psi)$ are non–trivial and complementary in $\mathbf{L}(\mathbf{A}^O)$. Then $0 \in F(\Theta) + F(\Psi)$ and so there are $a \in F(\Theta)$ and $b \in F(\Psi)$ such that $0 = ab$. Now $[a + b) = [a) \cap [b) \subseteq F(\Theta) \cap F(\Psi) = \{ 1 \}$ and so $a + b = 1$. Hence $a \in A^\diamond$ with $0 < a < 1$. Conversely if there is an $a \in A^\diamond$ with $0 < a < 1$ then $\Theta([a))$ and $\Theta([a'))$ are complementary non–trivial congruences in **Con(A)**. □

An alternative way of seeing (i) and (ii) above, is to use Corollary 1.3.7 and the well known fact that a Heyting algebra is simple iff it is a copy of the two element chain, and a Heyting algebra is S.I. iff it is of the form $L \oplus 1$. (See [1].) We point out that (iii) and (iv) together with Corollary 1.3.7 give a proof of the less well known facts that a Heyting algebra is F.S.I. iff its top element is join irreducible and it is D.I. iff its only complemented elements are 0 and 1.

1.3.19 Definition and Remark

Notice that **A** is S.I. iff it has a largest non–1 open element. If **A** is S.I. its largest non–1 open element will be called the **monolith** of **A**, since if m is the monolith of **A** the monolithic congruence of **A** will just be $\Theta([m))$. □

1.3.20 Remark

Consider the following sentences in $\mathcal{LI}$:

φ^S : $0 \neq 1 \wedge (\forall x)\,(x < 1 \Rightarrow x^I = 0)$

φ^{SI} : $(\exists x)(\forall y)\,(x^I < 1 \wedge (y^I < 1 \Rightarrow y^I \leq x^I))$

φ^{FSI} : $0 \neq 1 \wedge (\forall x)(\forall y)\,(x^I + y^I = 1 \Rightarrow (x = 1 \vee y = 1))$

φ^{DI} : $0 \neq 1 \wedge (\forall x)\,(x^I = x^C \Rightarrow (x = 0 \vee x = 1))$

Then we see that **A** is simple iff $\mathbf{A} \models \varphi^S$, **A** is S.I. iff $\mathbf{A} \models \varphi^{SI}$, **A** is F.S.I. iff $\mathbf{A} \models \varphi^{FSI}$ and **A** is D.I. iff $\mathbf{A} \models \varphi^{DI}$ Thus the classes of simple, F.S.I. and D.I. interior algebras are strictly elementary universal classes while the class of S.I. interior algebras is a strictly elementary existential–universal class. Although we have shown this directly, this result for simple, S.I. and F.S.I. follows indirectly from the fact that interior algebras have strictly first order definable principal congruences (see [6]) and the result for D.I. follows indirectly from the latter fact as well as the fact that interior algebras are congruence permutable. The result for F.S.I. also follows from the fact that interior algebras have EDPM (see [5]). □

We can now characterize the semi–simple interior algebras. In the following theorem let S denote the interior algebra given in Fig. 4.

S :

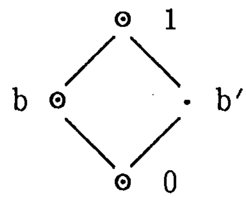

Fig. 4

<u>1.3.21 Theorem</u>

The following are equivalent for an interior algebra **A**:

i) **A** is semi–simple.

ii) Every open element of **A** is closed or equivalently every closed element of **A** is open.

iii) $\mathbf{A} \models (x^{IC} = x^{I})$

iv) $\mathbf{A}^{O}$ is a Boolean Heyting algebra.

v) S is not homomorphically embeddable in **A**.

<u>Proof</u>:

Clearly (ii), (iii), (iv) are equivalent. (i) $\Longrightarrow$ (iii): Assume (i). Then **A** is a subdirect product of simple interior algebras. Every simple interior algebra satisfies $x^{IC} = x^{I}$ since its only open elements are 0 and 1. Thus $\mathbf{A} \models (x^{IC} = x^{I})$. (ii) $\Longrightarrow$ (i): Assume (ii). Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be a surjective homomorphism with **B** S.I. Suppose **B** is not simple and let m be the monolith of **B**. Then $0 < m$. There is an $a \in A^{O}$ with $f(a) = m$. Then $a' \in A^{O}$ and so $f(a') \in B^{O}$, that is $m' \in B^{O}$. Also $m' < 1$ and so $m' \leq m$, a contradiction. Thus **B** is simple and so (i) holds. Clearly (ii) $\Longrightarrow$ (v). (v) $\Longrightarrow$ (ii): Assume (v). Suppose there is an $a \in A^{O}$ with $a \notin A^{\square}$. Put $b = a + a'^{I}$. $a > 0$ so $b > 0$. Also $b < 1$ or else $b'^{I} \geq b'$ contradicting $b \notin A^{\square}$. Also $b \in A^{O}$. Now $b'^{I} = (a'a'^{I\prime})^{I} = b'^{I}b'^{I\prime I} = 0$. Thus $\{ 0, b, b', 1 \}$ forms a subalgebra of **A** isomorphic to S, a contradiction. Thus $A^{\square} = A^{O}$ as required. □

<u>1.3.22 Remark</u>

By (iii) of Theorem 1.3.21 we see that the semi–simple interior algebras form a finitely based variety. We also see that up to notation and terminology the semi–simple interior algebras are just the monadic Boolean algebras [11] : the universal and existential quantifier operators are just the interior and closure operators respectively. □

1.3.23 Remark

The classes of simple, S.I., F.S.I., D.I. and semi–simple interior algebras are all examples of Heyting classes, they correspond to the classes of simple, S.I., F.S.I., D.I. and Boolean Heyting algebras respectively. Note that the Boolean Heyting algebras are just the semi–simple Heyting algebras. The latter fact is equivalent to the equivalence of (i) and (iv) in Theorem 1.3.21, by Corollary 1.3.6. The equivalence of (iv) and (v) in Theorem 1.3.21 is equivalent to the fact that a Heyting algebra is Boolean iff the three element Heyting algebra **3** is not embeddable in it. This follows from the fact that $\mathbf{S} = \mathbf{3}^{\mathrm{H}}$. □

Although we have used filters in A^O to represent congruences instead of open filters, the latter are still interesting in their own right. We mention the following results which follow easily from the correspondence between open filters and filters in A^O and well known results concerning filters in Heyting algebras:

1.3.24 Corollary

Let A be an interior algebra. The following are equivalent for a proper open filter F in A:

i) F is a maximal proper open filter.

ii) F is meet irreducible in $L^O(A)$.

iii) $a^I + b^I \in F$ implies $a \in F$ or $b \in F$, for all $a,b \in A$.

iv) $F \cap A^O$ is a prime filter in A^O.

v) $A / \Theta(F \cap A^O)$ is simple.

vi) For all $a \in A$, $a^I \in F$ or $a^{I\prime} \in F$ (but not both). □

1.3.25 Corollary

Let A be an interior algebra.

i) Every proper open filter in A can be extended to a maximal proper open filter in A.

ii) If F is a proper open filter and $a \in A$ with $a \notin F$ then there is maximal proper open filter G extending F with $a \notin G$.

iii) If a and b are not weakly equal in A (that is $a^I \neq b^I$) then there is a maximal proper open filter in A containing precisely one of a and b.

iv) Every proper open filter is the intersection of all maximal proper open filters containing it. □

(The equivalence of (v) and (vi) in Corollary 1.3.24 follows from Theorem 1.3.18 (iv). To see Corollary 1.3.25 (ii) and the equivalence of (iii) and (vi) in Corollary 1.3.24 note that for an open filter F in A, $a \in F$ iff $a^I \in F$, for all $a \in A$.)

1.3.26 Definition and Remark

The dual notion to that of an open subset is that of a **closed** subset. More precisely, if S is a subset of an interior algebra **A**, S is closed iff $a \in S$ implies $a^C \in S$, for all $a \in A$. The closed ideals of **A** form an algebraic meet complete 0,1–sublattice $L^C(A)$ of the lattice of all ideals in **A**. For all $S \subseteq A$ put $S^d = \{ a' : a \in S \}$. Then $F \subseteq A$ is an open filter iff F^d is a closed ideal, in fact $^d : L^O(A) \longrightarrow L^C(A)$ is an isomorphism. Also if F is a proper open filter then $F \cup F^d$ is a subalgebra of **A** in which F is a maximal proper open filter (and hence in which F^d is a maximal proper closed ideal). $F \cup F^d = A$ iff F is a maximal open filter in **A** or equivalently, F^d is a maximal proper closed ideal in **A**. □

1.4 INTERVAL ALGEBRAS

Given a Boolean algebra **B** it is well known that the intervals of **B** can be turned into Boolean algebras. We can generalize this to interior algebras:

1.4.1 Definition and Notation

Let **A** be an interior algebra. Let $a \leq b$ in **A** and consider the interval [a,b] in **A**. Define operations $'^{a,b}$, $^{I_{a,b}}$ and $^{C_{a,b}}$ on [a,b] by $r'^{a,b} = a + br'$, $r^{I_{a,b}} = a + b\cdot(b' + r)^I$ and $r^{C_{a,b}} = a + b\cdot(a'r)^C$. Put $\mathbf{[a,b]} = \langle [a,b], \cdot, +, '^{a,b}, a, b, ^{I_{a,b}}, ^{C_{a,b}} \rangle$. **[a,b]** is called the **interval algebra** on [a,b]. We use **(a]** and **[a)** to denote **[0,a]** and **[a,1]** respectively and the operations in **(a]** and **[a)** will be denoted by $'^{,a}$, ^{I,a}, ^{C,a} and $'^{a,}$, $^{I_{a,}}$, $^{C_{a,}}$ respectively. If we want to emphasize the interior algebra **A** in which [a,b] is taken we write **A[a,b]**. □

A routine but tedious argument shows:

1.4.2 Proposition

Let **A** be an interior algebra and let $a \leq b$ in **A**. Then **[a,b]** is an interior algebra. □

1.4.3 Proposition

Let **A** be an interior algebra and let $a \leq c \leq d \leq b$ in **A**. Let **B** = **A[a,b]**. Then **B[c,d]** = **A[c,d]**. □

The above can be proved by routine computations. Under certain conditions $^{I_{a,b}}$ or $^{C_{a,b}}$ is 'independent' of either a or b:

1.4.4 Proposition

Let $\mathbf{A}$ be an interior algebra.

i) $r^{I_{a,b}} = a + r^{I}$ for all $r \in [a,b]$ iff $b = a + b^{I}$.

ii) $r^{I_{a,b}} = b\cdot(b' + r)^{I}$ for all $r \in [a,b]$ iff $a = b\cdot(b' + a)^{I}$.

iii) $r^{C_{a,b}} = br^{C}$ for all $r \in [a,b]$ iff $a = ba^{C}$.

iv) $r^{C_{a,b}} = a + (a'r)^{C}$ for all $r \in [a,b]$ iff $b = a + (a'b)^{C}$.

Proof:

(i): Suppose $b = a + b^{I}$ and let $r \in [a,b]$. Then we have $r^{I_{a,b}} = a + b\cdot(b' + r)^{I} = a + (a + b^{I})(b' + r)^{I} = a + a\cdot(b' + r)^{I} + b^{I}(b' + r)^{I} = a + (br)^{I} = a + r^{I}$. (ii): Suppose $a = b\cdot(b' + a)^{I}$ and let $r \in [a,b]$. Then $a \leq b\cdot(b' + r)^{I}$ and so $r^{I_{a,b}} = a + b\cdot(b' + r)^{I} = b\cdot(b' + r)^{I}$. Renaming a and b, (iii) and (iv) are the duals of (i) and (ii) respectively. □

1.4.5 Corollary

Let $\mathbf{A}$ be an interior algebra and let $a \leq b$ in $\mathbf{A}$. Let $r \in [a,b]$.

i) If b is open $r^{I_{a,b}} = a + r^{I}$.

ii) If a is open $r^{I_{a,b}} = b\cdot(b' + r)^{I}$.

iii) If a and b are open $r^{I_{a,b}} = r^{I}$.

iv) If b is closed $r^{C_{a,b}} = a + (a'r)^{C}$.

v) If a is closed $r^{C_{a,b}} = br^{C}$.

vi) If a and b are closed $r^{C_{a,b}} = r^{C}$. □

1.4.6 Theorem

Let $\mathbf{A}$ be an interior algebra and let $a \leq b$ in $\mathbf{A}$. Let $k : \mathbf{A} \longrightarrow \mathbf{[a,b]}$ be the surjective Boolean algebra homomorphism given by $k(r) = a + br$ for all $r \in [a,b]$.

i) $k[A^{O}] = [a,b]^{O}$ (equivalently $k[A^{\square}] = [a,b]^{\square}$)

ii) k is a quotient map in $\mathbf{Int}^{+}$.

iii) k is a homomorphism iff $a = ba^{C}$ and $b = a + b^{I}$.

Proof:

(i): Let $r \in A^{O}$. Then $k(r)^{I_{a,b}} = (a + br)^{I_{a,b}} = a + b\cdot(b' + (a + br))^{I} = a + b\cdot(b' + a + r)^{I} \geq a + br^{I} = a + br = k(r)$. Thus $k(r)^{I_{a,b}} = k(r)$. Hence $k[A^{O}] \subseteq [a,b]^{O}$. Let $r \in [a,b]^{O}$. Then $r = r^{I_{a,b}} = a + b\cdot(b' + r)^{I} = k(\,(b' + r)^{I}\,)$ and so $[a,b]^{O} \subseteq k[A^{O}]$ whence the result follows. (ii): This follows from (i) and Proposition 1.1.13. (iii): Suppose k is a homomorphism. Then $a = a^{C_{a,b}} = k(a)^{C_{a,b}} = k(a^{C}) = a + ba^{C} = ba^{C}$ and $b = b^{I_{a,b}} = k(b)^{I_{a,b}} = k(b^{I}) = a + bb^{I} = a + b^{I}$. Conversely suppose $a = ba^{C}$ and $b = a + b^{I}$. To show that k is a homomorphism we have to show that for any $r \in A$, $(a + br)^{I_{a,b}} = a + br^{I}$. Now $(a + br)^{I_{a,b}} = (((a + br)'^{a,b})^{C_{a,b}})'^{a,b} = ((a + br')^{C_{a,b}})'^{a,b}$. By Proposition 1.4.4 we have

$(a + br)^{I_{a,b}} = (b\cdot(a + br')^{C})'^{a,b} = (b\cdot(a^{C} + (br')^{C}))'^{a,b} = (ba^{C} + b\cdot(br')^{C})'^{a,b} = (a + b\cdot(br')^{C})'^{a,b} = a + b\cdot(br')^{C\prime} = a + b\cdot(br')'^{I} = a + b\cdot(b' + r)^{I} = a + (br)^{I} = a + b^{I}r^{I} = (a + b^{I})(a + r^{I}) = b\cdot(a + r^{I}) = a + br^{I}$, as required. □

1.4.7 Corollary

Let A be an interior algebra and let $a \leq b$ in A.

i) If a and b are open then $[a,b]^{O} = A^{O} \cap [a,b]$.

ii) If a and b are closed then $[a,b]^{\square} = A^{\square} \cap [a,b]$. □

1.4.8 Corollary

Let $k : A \longrightarrow [a,b]$ be as in Theorem 1.4.6. If a is closed and b is open, k is a homomorphism. □

Generalized topological spaces are relational structures and so we may speak of congruences on them. Note that if A is an interior algebra then the congruences on Gt A are simply the congruences on A^{u} and hence are determined by filters.

1.4.9 Corollary

Let $a \leq b$ in A. Let Θ be the congruence on Gt A determined by the filter $F = [a'b)$ in A.

i) The quotient structure Gt A / Θ is a generalized topological space and moreover we have Alg (Gt A / Θ) $\cong [a,b] \cong (a'b] \cong [a + b')$.

ii) If $a = ba^{C}$ and $b = a + b^{I}$, Θ is the interior algebra congruence $\Theta(F \cap A^{O})$.

iii) If a is closed and b is open, Θ is a principal interior algebra congruence. □

1.4.10 Corollary

Let A be an interior algebra. Let $a_i,b_i \in A$ for $i \in I$ with $a_i = b_i a_i^{C}$ and $b_i = a_i + b_i^{I}$ for all $i \in I$. Let $f : A \longrightarrow \Pi \{ [a_i,b_i] : i \in I \}$ be given by $f(r) = (a_i + b_i r)_I$ for all $r \in A$. Then the following are equivalent:

i) f is a subdirect homomorphic embedding.

ii) $\Sigma \{ a_i' b_i : i \in I \} = 1$ □

Recall that a *partition of 1* in a Boolean algebra B is a subset $S \subseteq B$ which is pairwise disjoint (that is $a,b \in S$ and $a \neq b$ implies $ab = 0$) and has join $\Sigma S = 1$.

1.4.11 Corollary

Let A be an interior algebra. Let $a_1,\ldots,a_n,b_1,\ldots,b_n \in A$ with $a_i = b_i a_i^{C}$ and $b_i = a_i + b_i^{I}$ for

$i = 1,...,n$. Let $f : A \longrightarrow \Pi \{ [a_i,b_i] : i \in I \}$ be given by $f(r) = (a_1 + b_1 r ,..., a_n + b_n r)$ for all $r \in A$. Then the following are equivalent:

i) f is an isomorphism.

ii) $\{ a_1' b_1, \dots ,a_n' b_n \}$ is a partition of 1. □

1.4.12 Definition

Let **A** be an interior algebra. An interior algebra **B** is called a **principal quotient** of **A** iff there is a principal congruence Θ on Gt **A** such that $B \cong$ Alg (Gt **A** / Θ). □

1.4.13 Theorem

Let **A** and **B** be interior algebras. The following are equivalent:

i) **B** is a principal quotient of **A**.

ii) $B \cong (a]$ for some $a \in A$.

iii) $B \cong [a)$ for some $a \in A$.

iv) $B \cong [a,b]$ for some $a \leq b$ in **A**.

Proof:

By Corollary 1.4.9 (i) we immediately see that (ii), (iii), (iv) are equivalent and (iv) $\Longrightarrow$ (i). Now assume (i). There is a principal congruence $\Theta = con(c,d)$ on Gt **A** such that $B \cong$ Alg (Gt **A** / Θ). Put $a = cd + c'd'$. Then $\Theta = con(a,1)$. But then Θ is the congruence on Gt **A** determined by the filter $[a) = [1 \cdot a) = [0' \cdot a)$ in **A** and so again by Corollary 1.4.9 (i) we see that $B \cong (a]$, say. Thus (i) $\Longrightarrow$ (ii) and the result follows. □

1.4.14 Definition and Remark

Let **B** be a principal quotient of **A**. By Theorem 1.4.13 there is an $a \in A$ such that $B \cong (a]$. **B** is said to be an **open quotient** of **A** iff we can choose a to be an open element. Similarly we have **closed quotients, clopen quotients** etc. Notice that direct factors are always clopen quotients: If $A = \Pi \{ A_i : i \in I \}$ is a product of interior algebras and $j \in I$ then $A_j \cong (a]$ where $a = (a_i)$ is given by $a_j = 1$ and $a_i = 0$ for $i \neq j$. □

1.4.15 Corollary

Let **A** and **B** be interior algebras. The following are equivalent:

i) **B** is an open quotient of **A**.

ii) **B** is a principal homomorphic image of **A**.

Proof:

(i) $\Longrightarrow$ (ii) by Corollary 1.4.8. (ii) $\Longrightarrow$ (i): Assume (ii). There is a principal congruence Θ on **A** such that $B \cong A / \Theta$. By Corollary 1.3.11 there is an $a \in A^O$ with $\Theta = \Theta([a))$. But then by Corollary 1.4.9 we see $\mathbf{B} \cong (\mathbf{a}]$. □

1.4.16 Definition and Remark

JOINS OF INTERIOR ALGEBRAS

Let $\mathbf{A}$ be an interior algebra and let $\{ \mathbf{A}_i : i \in I \}$ be a family of interior algebras. Suppose we have elements $S = \{ a_i : i \in I \}$ in $\mathbf{A}$ such that $\Sigma S = 1$ and for all $i \in I$, $(\mathbf{a_i}] \cong \mathbf{A}_i$. Then we say that $\mathbf{A}$ is a **join** of the algebras $\{ \mathbf{A}_i : i \in I \}$. We may refer to a join as **open**, **closed** or **clopen** etc. iff we can choose S to be a set of open, closed or clopen elements etc., respectively. Notice that by the observation in 1.4.14 we see that products of interior algebras are always clopen joins. Moreover we see that $\mathbf{A}$ is a join of $\{ \mathbf{A}_i : i \in I \}$ iff $\mathbf{A}^u$ is a subdirect product of $\{ \mathbf{A}_i{}^u : i \in I \}$ and for all $i \in I$ the ith projection $k_i : \mathbf{A}^u \longrightarrow \mathbf{A}_i{}^u$ is a principal quotient topomorphism $k_i : \mathbf{A} \longrightarrow \mathbf{A}_i$. We see that if $\mathbf{A}$ is a join of $\{ \mathbf{A}_i : i \in I \}$ then every element of b of $\mathbf{A}$ can be considered to be a tuple (b_i) in $\Pi \{ \mathbf{A}_i : i \in I \}$ where for all $i \in I$, $b_i = k_i(b)$ where $k_i : \mathbf{A} \longrightarrow \mathbf{A}_i$ is the canonical quotient map. The concept of a join of interior algebras is important since many classes of interior algebras are closed under joins. □

1.4.17 Proposition

Let $\mathbf{A} = \Pi \{ \mathbf{A}_i : i \in I \}$ be a product of interior algebras. Suppose $a = (a_i) \in A$ and $b = (b_i) \in A$ with $a \leq b$. Then $\Pi \{ [\mathbf{a_i},\mathbf{b_i}] : i \in I \} = [\mathbf{a},\mathbf{b}]$. □

1.4.18 Proposition

Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be a topomorphism and let $a \leq b$ in $\mathbf{A}$. Let $g = f|[a,b] : [\mathbf{a},\mathbf{b}] \longrightarrow [\mathbf{f(a)},\mathbf{f(b)}]$.

i) If f is any of the following then so is g : (a) injective, (b) surjective, (c) a homomorphism.

ii) If $f[A^o] = B^o$ then $g[\ [a,b]^o\ ] = [f(a),f(b)]^o$.

Proof:

(i): The results for (a) and (c) are clear. For (b): Let f be surjective. Let $t \in [f(a),f(b)]$. There is an $r \in A$ with $f(r) = t$. Then $a + br \in [a,b]$ and $g(a + br) = f(a) + f(b)\cdot f(r) = f(a) + f(b)\cdot t = t$. Thus g is surjective. (ii): Let $t \in [f(a),f(b)]^o$. There is an $r \in B^o$ with $t = f(a) + f(b)\cdot r$. Then there is a $u \in A^o$ with $f(u) = r$. Then $a + bu \in [a,b]^o$ and $g(a + bu) = f(a) + f(b)\cdot f(u) = f(a) + f(b)\cdot r = t$. Thus $g[\ [a,b]^o\ ] = [f(a),f(b)]^o$. □

1.5 ATOMICITY PROPERTIES

Besides the usual concept of atomicity in interior algebras there are three particularly important atomicity properties which we will need in the subsequent chapters:

1.5.1 Definition and Remark

Let **A** be an interior algebra. **A** is called **open atomic** iff for all open $b > 0$ in **A** there is an atom a in **A** with $a \leq b$. **A** is called **closed atomic** iff for all closed $b > 0$ in **A** there is an atom a in **A** with $a \leq b$. **A** is **residually atomic** iff for all $b \in A$, $b^C b' > 0$ implies that there is an atom a in A with $a \leq b^C b'$. Notice that A is residually atomic iff for all non–closed b in A the interval algebra $[b, b^C]$ has an atom. □

Clearly any atomic interior algebra is open atomic, closed atomic and residually atomic. However each of these properties is distinct from atomicity and they are pairwise incomparable: Let **L** be the product of the free Boolean algebra on $\aleph_0$ generators and the two element Boolean algebra. Then **L** has a unique atom a. Put **A** = Alg ⟨ **L**, { 0, a, 1 } ⟩, **B** = Alg ⟨ **L**, { 0, a′, 1 } ⟩ and let **C** be any atomless Boolean interior algebra. Then **A** is clearly open atomic but not closed atomic since $a' > 0$ is closed with no atom below it. **A** is also not residually atomic since $a^C a' = 1 \cdot a' = a'$. **B** is clearly closed atomic but not open atomic since $a' > 0$ is open with no atom below it. **B** is also not residually atomic since $a'^C a'' = 1 \cdot a' = a'$. **C** is obviously not open atomic or closed atomic but it is trivially residually atomic since for all $a \in C$, $a^C a' = aa' = 0$. Notice also that none of A, B and C are atomic.

1.5.2 Notation

Let Atom(z) denote the formula $z \neq 0 \wedge (\forall w) (w \leq z \Longrightarrow (w = 0 \vee w = z))$ in $\mathcal{LI}$. In other words Atom(z) says that z is an atom. □

1.5.3 Theorem

The following classes of interior algebras are all strictly elementary classes closed under joins:

i) Atomic interior algebras.

ii) Open atomic interior algebras.

iii) Closed atomic interior algebras.

iv) Residually atomic interior algebras.

Proof:

For each class we give a sentence defining it whence it follows that the classes are all strictly elementary classes:

i) $(\forall x) (x > 0 \Longrightarrow (\exists y) (\text{Atom}(y) \wedge y \leq x))$

ii) $(\forall x)\ (x^{I} = x \wedge x > 0 \Longrightarrow (\exists y)\ (Atom(y) \wedge y \leq x))$

iii) $(\forall x)\ (x^{C} = x \wedge x > 0 \Longrightarrow (\exists y)\ (Atom(y) \wedge y \leq x))$

iv) $(\forall x)\ (x^{C}x' > 0 \wedge x > 0 \Longrightarrow (\exists y)\ (Atom(y) \wedge y \leq x))$

We now have to show that the classes are closed under joins. This will follow easily from the following two observations: Let **A** be an interior algebras and let $S \subseteq A$ with $\Sigma S = 1$. (1): If $b > 0$ in **A** then $ab > 0$ for some $a \in S$ and if b is open or closed in **A** then ab is open or closed in **(a]** respectively. (2): If $b = d^{C}d'$ for some $d \in A$ then putting $e = ad$ gives $e^{C,a}e^{\prime,a} = a(ad)^{C}a(ad)' = (ad)^{C}a(a' + d') = (ad)^{C}ad' \leq d^{C}d'$. □

1.5.4 Theorem

i) The class of atomic interior algebras is closed under principal quotients.

ii) The class of open atomic interior algebras is closed under open quotients.

iii) The class of closed atomic interior algebras is closed under closed quotients.

iv) The class of residually atomic interior algebras is closed under principal quotients.

Proof:

(i) is trivial and (ii) and (iii) follow easily from Corollary 1.4.7. We prove (iv): Let **A** be a residually atomic interior algebra and let $a \in A$. Suppose $b \in (a]$ with $b^{C,a}\cdot b^{\prime,a} > 0$. Put $d = (ab^{C}b')'$. Then $d^{C}d' = (a' + b^{C\prime} + b)^{C}(ab^{C}b') = (a'^{C} + b^{C\prime C} + b^{C})\cdot ab^{C}b' = (a'^{C} + 1)\cdot ab^{C}b' = ab^{C}b' = (ab^{C})\cdot(ab') = b^{C,a}\cdot b^{\prime,a} > 0$. Hence there is an atom $c \leq d^{C}d' = b^{C,a}\cdot b^{\prime,a}$ and the result follows. □

Let **A** and **B** be as on page 26. Then $\mathbf{A}(\mathbf{a}']$ is not open atomic and so the class of open atomic interior algebras is not closed under closed quotients. $\mathbf{B}(\mathbf{a}']$ is not closed atomic and so the class of closed atomic interior algebras is not closed under open quotients.

CHAPTER 2

TOPOLOGICAL DUALITY

2.1 COMPLETE ATOMIC INTERIOR ALGEBRAS

Consider the following categories:

Top : Topological spaces and continuous maps.

Tco : Topological spaces and continuous open maps.

CIn : Complete atomic interior algebras and complete homomorphisms.

CIn$^+$: Complete atomic interior algebras and complete topomorphisms.

We will show that the categories **CIn**$^+$ and **Top** are co–equivalent and moreover that this co–equivalence restricts to a co–equivalence between **CIn** and **Tco**.

2.1.1 Definition and Remark

Let $\mathbf{X} = \langle X, \mathcal{T} \rangle$ be a topological space. If $\mathcal{P}(X)$ is the power set Boolean algebra on X then $\langle \mathcal{P}(X), \mathcal{T} \rangle$ is a generalized topological space. Put $\mathbf{X}^A = \mathrm{Alg}\ \langle \mathcal{P}(X), \mathcal{T} \rangle$. For each continuous map $f : \mathbf{X} \longrightarrow \mathbf{Y}$ define $f^A : \mathbf{Y}^A \longrightarrow \mathbf{X}^A$ by $f^A(S) = f^{-1}[S]$ for all $S \subseteq Y$. Then f^A is a complete topomorphism. We thus have a co–functor ^{A} : **Top** $\longrightarrow$ **CIn**$^+$. □

2.1.2 Definition and Remark

For each complete atomic interior algebra **A** let A^T denote the set of atoms in **A**. Let $A^\sigma = \{ S \subseteq A^T : \Sigma S \in A^O \}$. Put $\mathbf{A}^T = \langle A^T, A^\sigma \rangle$. Then $\mathbf{A}^T$ is a topological space. If $f : \mathbf{A} \longrightarrow \mathbf{B}$ is a complete topomorphism between complete atomic interior algebras define $f^T : \mathbf{B}^T \longrightarrow \mathbf{A}^T$ as follows: If $b \in B^T$ consider $a = \sqcap f^{-1}[\,[b)\,]$. $f(a) = \sqcap ff^{-1}[\,[b)\,] \geq \sqcap [b) = b > 0$. Thus $a > 0$. Consider $e \in A$. Then either $b \leq f(e)$ or $b \leq f(e)' = f(e')$. Thus $e \in f^{-1}[\,[b)\,]$ or $e' \in f^{-1}[\,[b)\,]$ whence $a \leq e$ or $a \leq e'$. Thus $a \in A^T$. Put $f^T(b) = a$. □

2.1.3 Lemma

Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be a complete topomorphism between complete atomic interior algebras. Then $f^T : \mathbf{B}^T \longrightarrow \mathbf{A}^T$ is a continuous map.

Proof:

Let $S \subseteq B^T$ be open, that is $\Sigma S \in B^O$. Then $(\Sigma f^{T-1}[S])^I = (\Sigma f^{T-1}[\,B^T \cap (\Sigma S]\,])^I = f(\Sigma S)^I = f(\Sigma S^I) = f(\Sigma S) = \Sigma f^{T-1}[\,B^T \cap (\Sigma S]\,] = \Sigma f^{T-1}[S]$. Hence $\Sigma f^{T-1}[S] \in A^O$ and so $f^{T-1}[S]$ is open. Thus f^T is continuous. □

2.1.4 Lemma

${}^{T}: \mathbf{CIn}^{+} \longrightarrow \mathbf{Top}$ is a co-functor.

Proof:

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be in $\mathbf{CIn}^{+}$. Let $c \in C^{T}$. Then $f^{T}g^{T}(c) = \sqcap f^{-1}[\, [\, g^{T}(c)\,)\,]$. Now if $a \in A$ and $g^{T}(c) \leq f(a)$ then $g(g^{T}(c)) \leq gf(c)$. But $g(g^{T}(c)) = g(\sqcap g^{-1}[\,[c)\,]\,) = \sqcap gg^{-1}[\,[c)\,] \geq c$ and so $c \leq gf(a)$. Conversely if $c \leq gf(a)$ then $f(a) \geq \sqcap g^{-1}[\,[c)\,] = g^{T}(c)$. Thus $f^{T}g^{T}(c) = \sqcap (gf)^{-1}[\,[c)\,] = (gf)^{T}(c)$. Hence $f^{T}g^{T} = (gf)^{T}$. Now let $\mathbf{A}$ be a complete atomic interior algebra. Then for $a \in A^{T}$, $id(A)^{T}(a) = \sqcap id(A)^{-1}[\,[a)\,] = \sqcap [a) = a$ and so $id(A)^{T} = id(A^{T})$. □

2.1.5 Lemma

For each complete atomic interior algebra $\mathbf{A}$ define $\delta(A): A \longrightarrow A^{TA}$ by $\delta(A)(a) = A^{T} \cap (a]$ for all $a \in A$. Then $\delta: id \longrightarrow (\;)^{TA}$ is a natural isomorphism.

Proof:

Note that isomorphisms in $\mathbf{CIn}^{+}$ are just isomorphisms in $\mathbf{Int}^{+}$ between complete atomic interior algebras and so it is easily seen that δ gives an isomorphism. Let $f: A \longrightarrow \mathbf{B}$ be in $\mathbf{CIn}^{+}$. Let $a \in A$. $f^{TA}\delta(A)(a) = f^{TA}(\,A^{T} \cap (a]\,) = f^{T\,-1}[\,A^{T} \cap (a]\,] = \{\, b \in B^{T} : f^{T}(b) \in A^{T} \cap (a] \,\} = \{\, b \in B^{T} : f^{T}(b) \leq a \,\}$. Now if $f^{T}(b) \leq a$ we have $b \leq \sqcap ff^{-1}[\,[b)\,] = ff^{T}(b) \leq f(a)$. Conversely suppose $b \leq f(a)$. Then $a \in f^{-1}[\,[b)\,]$ whence $f^{T}(b) = \sqcap f^{-1}[\,[b)\,] \leq a$. Thus $f^{TA}\delta(a) = \{\, b \in B^{T} : b \leq f(a) \,\} = B^{T} \cap (\,f(a)\,] = \delta(B)f(a)$. Thus $f^{TA}\delta(A) = \delta(\mathbf{B})f$ and so δ is natural. □

2.1.6 Lemma

For each topological space $\mathbf{X}$ define $\zeta(X): X \longrightarrow X^{AT}$ by $\zeta(X)(x) = \{\, x \,\}$ for all $x \in X$. Then $\zeta: id \longrightarrow (\;)^{AT}$ is a natural isomorphism.

Proof:

Isomorphisms in $\mathbf{Top}$ are just homeomorphisms and so it is easily seen that ζ gives an isomorphism. Let $f: \mathbf{X} \longrightarrow \mathbf{Y}$ be in $\mathbf{Top}$. Let $x \in X$. Then $f^{AT}\zeta(\mathbf{X})(x) = f^{AT}(\,\{\, x \,\}\,) = \cap f^{A\,-1}[\,[\,\{\, x \,\}\,)\,] = \cap f^{A\,-1}[\,\{\, S \subseteq X : x \in S \,\}\,] = \cap \{\, R \subseteq Y : x \in f^{A}(R) \,\} = \cap \{\, R \subseteq Y : f(x) \in R \,\} = \{\, f(x) \,\} = \zeta(Y)f(x)$. Thus $f^{AT}\zeta(\mathbf{X}) = \zeta(Y)f$ and so ζ is natural. □

Combining the previous lemmas we have:

2.1.7 Theorem

${}^{A}: \mathbf{Top} \longrightarrow \mathbf{CIn}^{+}$ and ${}^{T}: \mathbf{CIn}^{+} \longrightarrow \mathbf{Top}$ form a co-equivalence system. □

2.1.8 Theorem

Let $f : \mathbf{X} \longrightarrow \mathbf{Y}$ be a continuous map. The following are equivalent:

i) f is an open map.

ii) f^A is a homomorphism.

Proof:

(i) $\Longrightarrow$ (ii): Assume (i). Let $S \subseteq Y$. Since f^A is a topomorphism $f^A(S^I) \subseteq f^A(S)^I$. Now $f^A(S)^I \subseteq f^A(S) = f^{-1}[S]$ whence $f[f^A(S)^I] \subseteq S$. But f is open so $f[f^A(S)^I]$ is open. Thus $f[f^A(S)^I] \subseteq S^I$. Then $f^A(S)^I \subseteq f^{-1}[S^I] = f^A(S^I)$. Thus $f^A(S)^I = f^A(S^I)$ and so f^A is a homomorphism. (ii) $\Longrightarrow$ (i): Assume (ii). Let $S \subseteq X$ be open. $S \subseteq f^{-1}f[S] = f^A(f[S])$. Hence $S \subseteq f^A(f[S])^I = f^A(f[S]^I) = f^{-1}[f[S]^I]$. Thus $f[S] \subseteq f[S]^I$ and so $f[S] = f[S]^I$, that is, f[S] is open. Thus f is open. □

2.1.9 Corollary

$^A : \mathbf{Tco} \longrightarrow \mathbf{CIn}$ and $^T : \mathbf{CIn} \longrightarrow \mathbf{Tco}$ form a co–equivalence system. □

2.1.10 Proposition

Let $f : \mathbf{X} \longrightarrow \mathbf{Y}$ be a continuous map.

i) f is surjective iff f^A is injective.

ii) f is injective iff f^A is surjective. □

(The above follows by purely set–theoretic arguments.)

Using the one point space it is easy to see that monomorphisms in **Top** are injective, and using the two point indiscrete space it is easy to see that epimorphisms in **Top** are surjective. Hence in $\mathbf{CIn}^+$ epimorphisms are surjective and monomorphisms are injective. We now show that a similar result holds for **Tco** and **CIn**.

2.1.11 Theorem

In **Tco** monomorphisms are injective, equivalently, in **CIn** epimorphisms are surjective.

Proof:

Let $f : \mathbf{X} \longrightarrow \mathbf{Y}$ be a monomorphism in **Tco**. Put $W = \{ (x,y) \in X^2 : f(x) = f(y) \}$ Let $g : W \longrightarrow X$ and $h : W \longrightarrow X$ be the projections and let **W** be the topological space obtained by supplying W with the initial topology with respect to g and h. Then $g : \mathbf{W} \longrightarrow \mathbf{X}$ and $h : \mathbf{W} \longrightarrow \mathbf{X}$ are obviously continuous. Consider an open base element $(S \times R) \cap W$ of **W** where S and R are open subsets of **X**. Then $g[(S \times R) \cap W] = \{ x \in S :$ there is a $y \in R$ with $f(x) = f(y) \} = \{ x \in S : f(x) \in f[R] \} = S \cap f^{-1}f[R]$ which is open since f is open and continuous. Thus g is open and similarly h is open. Now gf = hf and so g = h. Let x,y ∈ X with $f(x) = f(y)$. Then $(x,y) \in W$ and so $x = g(x,y) = h(x,y) = y$. Thus f is injective. □

2.1.12 Theorem

In **Tco** epimorphisms are surjective, equivalently, in **CIn** monomorphisms are injective.

Proof:

Let $f : X \longrightarrow Y$ be an epimorphism in Tco. Let Z be the quotient of Y obtained by collapsing $f[X]$ and let $p : Y \longrightarrow Z$ be the quotient map. Note that $f[X]$ is an open point in Z. Let $k : Y \longrightarrow Z$ be the constant map with value $f[X]$. Then p and k are in **Tco** with $pf = kf$. Hence $p = k$ and so $f[X] = Y$, that is, f is surjective. □

2.1.13 Remark

Note that under the co–equivalence between **Top** and **CIn**$^+$ finite topological spaces correspond to finite interior algebras. Theorems 2.1.11 and 2.1.12 still hold if we replace **Tco** and **CIn** by the full subcategories consisting of their finite members. □

2.1.14 Remark

If X is a topological space and Y is a subspace of X then we see that Y^A is just the interval algebra $(Y]$ in X^A. In fact if $f : Y \longrightarrow X$ is the inclusion map of Y in X then $f^A : X^A \longrightarrow Y^A$ is the canonical quotient map of X^A onto Y^A. (See Theorem 1.4.6.)

If $X = \amalg \{ X_i : i \in I \}$ is a topological sum and for all $i \in I$, $m_i : X_i \longrightarrow X$ is the ith injection, then the sink $(m_i : X_i \longrightarrow X)_I$ is a co–product in **Top**. Also if $A = \Pi \{ A_i : i \in I \}$ is an algebraic product of complete atomic interior algebras and for all $i \in I$, $p_i : A \longrightarrow A_i$ is the ith projection, then the source $(p_i : A \longrightarrow A_i)_I$ is easily seen to be a product in **CIn**$^+$. Thus we see that if $X = \amalg \{ X_i : i \in I \}$ is a topological sum then $X^A \cong \Pi \{ X_i^A : i \in I \}$. In fact an isomorphism is given explicitly by $S \longmapsto (S \cap X_i)_I$ for all $S \subseteq X$. Similar results hold for the direct and inverse limit constructions for topological spaces and interior algebras.

If $X = \Pi \{ X_i : i \in I \}$ is a topological product and for all $i \in I$, $p_i : X \longrightarrow X_i$ is the ith projection, then the source $(p_i : X \longrightarrow X_i)_I$ is a product in **Top**. Thus we see that **CIn**$^+$ has co–products. If A_i, $i \in I$, are complete atomic interior algebras then their co–product is $(m_i : A_i \longrightarrow X^A)_I$ where $X = \Pi \{ A_i^T : i \in I \}$ and for all $i \in I$, $m_i = p_i^A \delta(A_i)$ (see Lemma 2.1.5) where $p_i : X \longrightarrow A_i^T$ is the ith projection. □

An important feature of the co–equivalence between **Top** and **CIn**$^+$ is that under this co–equivalence many natural topological properties correspond to natural algebraic properties: for example empty spaces correspond to trivial interior algebras. What is more interesting is that connectedness properties correspond to irreducibility properties:

Recall that a topological space **X** is *supercompact* iff every open cover of **X** contains X; **X** is *ultra–connected* iff for all open sets S,R $\subseteq$ X, if X = S ∪ R then S = X or R = X. If the clopen sets of **X** form a base for **X**, **X** is *zero–dimensional.* **X** is *strongly zero–dimensional* iff every open set of **X** is in fact clopen. (Note that these are the modern standard definitions of 'zero–dimensional' and 'strongly zero–dimensional'. In the past, however, some authors have used 'zero–dimensional' to describe a topological space in which every point has a neighbourhood base of clopen sets, and they have used 'strongly zero–dimensional' to mean zero–dimensional in the modern sense.)

2.1.15 Theorem

Let **X** be a topological space.

i) $\mathbf{X}^A$ is Boolean iff **X** is discrete.

ii) $\mathbf{X}^A$ is semi–simple iff **X** is strongly zero–dimensional.

If **X** is non–empty:

iii) $\mathbf{X}^A$ is simple iff **X** is indiscrete.

iv) $\mathbf{X}^A$ is S.I. iff **X** is supercompact.

v) $\mathbf{X}^A$ is F.S.I. iff **X** is ultra–connected.

vi) $\mathbf{X}^A$ is D.I. iff **X** is connected. □

The above follows easily from Theorem 1.3.18 and Theorem 1.3.21.

2.1.16 Remark

Note that a non–empty product of topological spaces is indiscrete, supercompact, ultra–connected or connected iff each factor is indiscrete, supercompact, ultra–connected or connected respectively. Thus classes of simple, S.I., F.S.I., and D.I. interior algebras are closed under co–products and co–factors in $\mathbf{CIn}^+$. □

2.1.17 Notation and Remark

Given a topological space **X** = ⟨ X, $\mathcal{T}$ ⟩ and a set S let X[S] denote the disjoint union of X and S. Let **X**[S] denote the topological space ⟨ X[S], $\mathcal{T}$ ∪ { X[S] } ⟩. If **X** is supercompact we will refer to the monolith of $\mathbf{X}^A$ as the monolith of **X**. If S ≠ ϕ then **X**[S] is supercompact with monolith X. Moreover every supercompact space is of this form since if **Y** is supercompact with monolith X then $Y \cong X[S]$ where X is the subspace of Y induced by X and $S = Y - X$. We can generalize this to interior algebras. □

2.1.18 Lemma

Let A be an interior algebra and let B be a Boolean algebra. Then $G = (A^O \times \{ 0 \}) \cup \{ (1,1) \}$ is a generalized topology in $A^u \times B$.

Proof:

$(1,1) \in G$ and G is clearly closed under binary meets. Let $(a,b) \in A \times B$. If $b = 0$ put $c = (a^I,0)$. If $b \neq 0$ and $(a,b) \neq (1,1)$ put $c = (0,0)$. If $(a,b) = (1,1)$ put $c = (1,1)$. Then $c = \max \{ d \in G : d \leq (a,b) \}$. The result follows by Proposition 1.1.4. □

2.1.19 Notation and Remark

Let A be an interior algebra and let B be a Boolean algebra. Let A[B] denote the interior algebra Alg $\langle A^u \times B, G \rangle$ where G is as in Lemma 2.1.18. Note that if B is non–trivial then A[B] is S.I. with monolith (1,0). Moreover A is isomorphic to ((1,0)] via the map $a \longmapsto (a,0)$. □

2.1.20 Proposition

Let A be S.I. with monolith m. Put $B = (m]$ and $C = (m']^u$. Then $A \cong B[C]$ via the map $a \longmapsto (am,am')$. □

2.1.21 Proposition

Let X be a topological space and let S be a set. Then $X[S]^A \cong X^A[\mathcal{P}(S)]$ via the map $R \longmapsto (R \cap S , R \cap S')$, where $\mathcal{P}(S)$ is the power set Boolean algebra on S. □

The proofs of the above propositions are straightforward and are left to the reader. From the above we see that the results discussed in 2.1.17 may be viewed as special cases of more general results about S.I. interior algebras.

2.1.22 Corollary (cf. Lemma 3.10 of [14])

For all interior algebras A there is an S.I. interior algebra B such that $A \cong (m]$ where m is the monolith of B. We may choose B to be complete, atomic or complete and atomic if A is and we may choose B to have the same cardinality as A if A is infinite.

Proof:

Put $B = A[C]$ where C is any finite Boolean algebra. □

Recall that in a category of algebras and homomorphisms an object **A** is called *weakly projective* iff every surjective homomorphism with co–domain **A** is a retraction.

2.1.23 Corollary

Any weakly projective interior algebra in **Int** is F.S.I.

Proof:

Let **A** be weakly projective. By corollary 2.1.22 there is an S.I., hence F.S.I., interior algebra **B** such that **A** is a homomorphic image of **B**. Hence **A** is homomorphically embeddable in **B** and so, since F.S.I. interior algebras are definable by a universal sentence (see Remark 1.3.20) **A** is F.S.I. □

2.1.24 Definition and Remark

Recall that the following conditions are equivalent for a topological space **X**:

i) An arbitrary intersection of open sets in **X** is open in **X**.

ii) $(f_i : X_i \longrightarrow X)_I$ is a final sink in Top, where I is the set of finite subsets of X and for each $i \in I$ $f_i : X_i \longrightarrow X$ is the inclusion map of the subspace X_i induced by the finite subset i.

iii) There is a final sink $(f_i : X_i \longrightarrow X)_I$ in Top with X_i finite for all $i \in I$.

iv) If $S \subseteq X$ and $x \in S^C$ in **X** then there is a $y \in S$ with $x \in \{ y \}^C$ in **X**.

Following the usual terminology used by categorical topologists we will call topological spaces satisfying the above equivalent conditions **finitely generated spaces**, since they form the *final hull* of the class of finite spaces (condition (iii) above). We will show that the finitely generated spaces correspond to a natural class of interior algebras. □

2.1.25 Definition

A complete interior algebra **A** is called **operator complete** iff the interior operator of **A** is completely multiplicative (that is it distributes over arbitrary meets) or equivalently the closure operator of **A** is completely additive (that is it distributes over arbitrary joins). □

2.1.26 Theorem

Let **A** be a complete interior algebra. The following are equivalent:

i) **A** is operator complete.

ii) A^O is closed under arbitrary meets in **A**.

Proof:

(i) $\Longrightarrow$ (ii): Assume (i). Let $S \subseteq A^O$. Then $(\sqcap S)^I = \sqcap \{ a^I : a \in S \} = \sqcap \{ a : a \in S \} = \sqcap S$ and so $\sqcap S \in A^O$. (ii) $\Longrightarrow$ (i): Assume (ii). Let $S \subseteq A$. Then for all $a \in S$, $\sqcap S \leq a$ whence

$(\sqcap S)^{I} \leq a^{I}$. Thus $(\sqcap S)^{I} \leq \sqcap \{ a^{I} : a \in S \}$. But for all $b \in S$, $\sqcap \{ a^{I} : a \in S \} \leq b$ and so $\sqcap \{ a^{I} : a \in S \} \leq \sqcap S$. Now A^{O} is closed under arbitrary meets in A and so we have $\sqcap \{ a^{I} : a \in S \} \in A^{O}$ whence $\sqcap \{ a^{I} : a \in S \} \leq \sqcap S$. Thus $\sqcap \{ a^{I} : a \in S \} = \sqcap S$. □

Combining the above with condition (i) of 2.1.24 we get our desired result:

2.1.27 Corollary

The following are equivalent for a topological space X:

i) X is finitely generated.

ii) X^{A} is operator complete. □

Note that the class of finitely generated spaces is closed under topological sums, continuous open images and open subspaces. These facts may be viewed as a special cases of the following results which are easily proved using Theorem 2.1.26.

2.1.28 Proposition

The class of operator complete interior algebras is closed under products, substructures with respect to complete ($\mathbf{Int}^{+}$) embeddings and topomorphic images with respect to complete surjective topomorphisms. □

The following result is also worth noticing:

2.1.29 Proposition

Any semi–simple complete interior algebra is operator complete.

Proof:

In any interior algebra A, A^{O} is closed under arbitrary joins and so dually, $A^{\square}$ is closed under arbitrary meets. But in a semi–simple interior algebra $A^{O} = A^{\square}$ (see Theorem 1.3.21) and the result follows. □

2.2 STONE FIELDS

2.2.1 Definition and Remark

Recall that a *topological field of sets* in a topological space X is a field of sets $\mathcal{R} \subseteq \mathcal{P}(X)$ which is closed under interiors and hence under closures, equivalently $\mathcal{R}$ forms a subalgebra of X^{A}. It will be convenient to treat fields of sets as pairs $\langle X, \mathcal{R} \rangle$ where X is the set in which $\mathcal{R}$ is a field of sets, and to treat topological fields of sets as triples $\langle X, \mathcal{T}, \mathcal{R} \rangle$ where

$\langle X, \mathcal{T} \rangle$ is the topological space in which $\mathcal{R}$ is a topological field of sets. By a **field map** $f : \langle X, \mathcal{T}, \mathcal{R} \rangle \longrightarrow \langle Y, \mathcal{S}, \mathcal{W} \rangle$ between two topological fields of sets we mean a continuous map $f : \langle X, \mathcal{T} \rangle \longrightarrow \langle Y, \mathcal{S} \rangle$ with the property that $f^{-1}[S] \in \mathcal{R}$ for all $S \in \mathcal{W}$. We can thus form a category of topological fields and field maps. We will show that $\mathbf{Int}^{+}$ is co–equivalent to a full subcategory of this category. □

2.2.2 Definition

Let $X = \langle X, \mathcal{R} \rangle$ be a field of sets. X is called **separative** iff for all distinct $x,y \in X$ there is an $S \in \mathcal{R}$ with $x \in S$ but $y \notin S$. X is called **compact** iff for all proper filters $\mathcal{F}$ over X, $\bigcap (\mathcal{R} \cap \mathcal{F})$ is non–empty. □

2.2.3 Lemma

Let $X = \langle X, \mathcal{R} \rangle$ be a field of sets. Let $\mathcal{S}$ be the topology on X generated by $\mathcal{R}$.

i) $\mathcal{S}$ is zero–dimensional.

ii) $\mathcal{S}$ is Hausdorff iff X is separative.

iii) $\mathcal{S}$ is compact with compact open sets $\mathcal{R}$ iff X is compact.

iv) $\mathcal{S}$ is Boolean with clopen sets $\mathcal{R}$ iff X is separative and compact.

Proof:

(i) and (ii) are easy and (iv) follows from (ii) and (iii), we prove (iii): Suppose $\mathcal{S}$ is compact with compact open sets $\mathcal{R}$. Let $\mathcal{F}$ be a proper filter over X. Suppose $\bigcap (\mathcal{R} \cap \mathcal{F}) = \phi$. Each member of $\mathcal{R} \cap \mathcal{F}$ is closed and so there is a finite $\mathcal{W} \subseteq \mathcal{R} \cap \mathcal{F}$ with $\bigcap \mathcal{W} = \phi$. But then $\phi \in \mathcal{F}$, a contradiction. Conversely suppose X is compact. Let $\mathcal{W} \subseteq \mathcal{R}$ with $\bigcup \mathcal{W} \in \mathcal{R}$. Suppose for all finite $\mathcal{V} \subseteq \mathcal{W}$, $\bigcup \mathcal{V} \neq \bigcup \mathcal{W}$. Put $\mathcal{Y} = \{ \bigcup \mathcal{W} - S : S \in \mathcal{W} \}$. Then $\mathcal{Y}$ has the finite intersection property and so $\mathcal{F} = \{ T \subseteq X :$ there is a finite $\mathcal{U} \subseteq \mathcal{Y}$ with $\bigcap \mathcal{U} \subseteq T \}$ is a proper filter over X. But $\bigcap (\mathcal{R} \cap \mathcal{F}) = \bigcap \mathcal{Y} = \phi$, a contradiction. Thus there is a finite $\mathcal{V} \subseteq \mathcal{W}$ with $\bigcup \mathcal{V} = \bigcup \mathcal{W}$. Since $\mathcal{R}$ is a base for $\mathcal{S}$ it follows that all members of $\mathcal{R}$ are compact open in $\mathcal{S}$, in particular $\mathcal{S}$ is a compact topology. Now let $T \in \mathcal{S}$ be compact. Then there is a $\mathcal{W} \subseteq \mathcal{R}$ with $\bigcup \mathcal{W} = T$. By compactness there is a finite $\mathcal{V} \subseteq \mathcal{W}$ with $\bigcup \mathcal{V} = T$ and so $T \in \mathcal{R}$. Thus the compact members of $\mathcal{S}$ are precisely $\mathcal{R}$. □

2.2.4 Definition

Let $X = \langle X, \mathcal{T}, \mathcal{R} \rangle$ be a topological field of sets. X is called **algebraic** iff there is a base $\mathcal{A}$ of $\mathcal{T}$ with $\mathcal{A} \subseteq \mathcal{R}$. □

2.2.5 Proposition

Let $X = \langle X, \mathcal{T}, \mathcal{R} \rangle$ be a compact algebraic topological field of sets. Let $\mathcal{B}$ be the set of compact open sets of $\mathcal{T}$. Then the following hold:

i) $\mathcal{B}$ is a base for $\mathcal{T}$.

ii) $\mathcal{B} = \mathcal{R} \cap \mathcal{T}$

iii) $\mathcal{T}$ is a compact topology.

Proof:

(i): There is a base $\mathcal{A}$ of $\mathcal{T}$ with $\mathcal{A} \subseteq \mathcal{R}$. Thus if $\mathcal{S}$ is the topology generated by $\mathcal{R}$, $\mathcal{T} \subseteq \mathcal{S}$. By Lemma 2.2.3 (ii) each member of $\mathcal{A}$ is compact with respect to $\mathcal{S}$ hence with respect to $\mathcal{T}$. Thus $\mathcal{A} \subseteq \mathcal{B}$ and so $\mathcal{B}$ is a base for $\mathcal{T}$. (ii): Let $S \in \mathcal{B}$. There is a $\mathcal{W} \subseteq \mathcal{A}$ with $S = \cup \mathcal{W}$. By compactness there is a finite $\mathcal{V} \subseteq \mathcal{W}$ with $S = \cup \mathcal{V}$ and so $S \in \mathcal{R}$. Thus $\mathcal{B} \subseteq \mathcal{R} \cap \mathcal{T}$. Let $S \in \mathcal{R} \cap \mathcal{T}$. Then by Lemma 2.2.3 (ii) S is compact in $\mathcal{S}$ and hence in $\mathcal{T}$ whence $S \in \mathcal{B}$. Thus $\mathcal{B} = \mathcal{R} \cap \mathcal{T}$. (iii): This follows from Lemma 2.2.3 (ii) and the fact that $\mathcal{T} \subseteq \mathcal{S}$. □

2.2.6 Definition and Remark

Call a topological field of sets a **Stone field** iff it is separative, compact and algebraic. Let **Sfld** denote the category of Stone fields and field maps. Given a Stone field $\mathbf{X} = \langle X, \mathcal{T}, \mathcal{R} \rangle$ let $\mathbf{X}^S$ be the subalgebra of $\langle X, \mathcal{T} \rangle^A$ with underlying set $\mathcal{R}$. For each field map $f : \mathbf{X} \longrightarrow \mathbf{Y}$ between Stone fields let $f^S : \mathbf{Y}^S \longrightarrow \mathbf{X}^S$ be given by $f(S) = f^{-1}[S]$ for all $S \subseteq Y$. Then f^S is a topomorphism. We thus have a co–functor $^S : \mathbf{Sfld} \longrightarrow \mathbf{Int}^+$. □

2.2.7 Definition and Remark

For each interior algebra $\mathbf{A}$ let A^F be the set of ultrafilters in $\mathbf{A}$. Define $\alpha(\mathbf{A}) : A \longrightarrow \mathcal{P}(A^F)$ by $\alpha(\mathbf{A})(a) = \{ F \in A^F : a \in F \}$. Then $\alpha(\mathbf{A})$ is a Boolean algebra embedding. When $\mathbf{A}$ is understood we will simply write α for $\alpha(\mathbf{A})$. $\alpha[A^O]$ is closed under finite intersections and contains $A^F = \alpha(1)$ and so it is a base for a topology $\mathcal{T}(\mathbf{A})$ on A^F. Put $\mathbf{A}^F = \langle A^F, \mathcal{T}(\mathbf{A}), \alpha[A] \rangle$. $\alpha[A]$ is the field of clopen sets of the Stone topology of $\mathbf{A}^u$. By Lemma 2.2.3 and the fact that $\alpha[A^O] \subseteq \alpha[A]$, $\mathbf{A}^F$ is a Stone field. For each topomorphism $f : \mathbf{A} \longrightarrow \mathbf{B}$ define $f^F : \mathbf{B}^F \longrightarrow \mathbf{A}^F$ by $f^F(F) = f^{-1}[F]$ for all $F \in B^F$. (This is possible since the pre–image of an ultrafilter under a Boolean algebra homomorphism is an ultrafilter.) □

2.2.8 Lemma

Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be a topomorphism.

i) $f^{F\,-1}[\alpha(\mathbf{A})(a)] = \alpha(\mathbf{B})f(a)$ for all $a \in A$.

ii) $f^F : \mathbf{B}^F \longrightarrow \mathbf{A}^F$ is a field map.

iii) $^F : \mathbf{Int}^+ \longrightarrow \mathbf{Sfld}$ is a co–functor.

Proof:

(i): Let $a \in A$. Then $f^{F\,-1}[\alpha(\mathbf{A})(a)] = \{ F \in B^F : f^F(F) \in \alpha(\mathbf{A})(a) \} = \{ F \in B^F : a \in f^{-1}[F] \}$

$= \{ F \in B^F : f(a) \in F \} = \alpha(B)f(a)$. (ii) follows from (i) since $\alpha(A)[A^O]$ and $\alpha(B)[B^O]$ are bases for $\mathcal{T}(A)$ and $\mathcal{T}(B)$ respectively and $f[A^O] \subseteq B^O$, (iii) is now trivial. □

2.2.9 Lemma

$\alpha : \mathrm{id} \longrightarrow (\ )^{FS}$ is a natural isomorphism.

Proof:

Let **A** be an interior algebra Clearly $\alpha : \mathbf{A} \longrightarrow \mathbf{A}^{FS}$ is a Boolean algebra isomorphism. (Recall that $\alpha : A \longrightarrow \mathcal{P}(A^F)$ was a Boolean algebra embedding.) We show that it is an interior algebra isomorphism: Let $b \in A$. Then $\alpha(b)^I = \cup \{ \alpha(a) : a \in A^O \text{ and } \alpha(a) \subseteq \alpha(b) \}$ $= \cup \{ \alpha(a) : a \in A^O \text{ and } a \leq b \} = \cup \{ \alpha(a) : a \in A^O \text{ and } a \leq b^I \} = \cup \{ \alpha(a) : a \in A^O$ and $\alpha(a) \subseteq \alpha(b^I) \} = \alpha(b^I)$. Thus α is an interior algebra isomorphism as required. Naturality follows from Lemma 2.2.8 (i). □

2.2.10 Definition and Remark

For each Stone field $\mathbf{X} = \langle X, \mathcal{T}, \mathcal{R} \rangle$ and all $x \in X$ the set $\{ S \in \mathcal{R} : x \in S \}$ is an ultrafilter in $\mathbf{X}^S$ and so we can define a map $\beta(\mathbf{X}) : \mathbf{X} \longrightarrow \mathbf{X}^{SF}$ by $\beta(\mathbf{X})(x) = \{ S \in \mathcal{R} : x \in S \}$ for all $x \in X$. When **X** is understood we simply write β for $\beta(\mathbf{X})$. □

2.2.11 Lemma

Let $f : \langle X, \mathcal{T}, \mathcal{R} \rangle \longrightarrow \langle Y, \mathcal{S}, \mathcal{W} \rangle$ be a field map between Stone fields. f is an isomorphism in **Sfld** iff f is bijective, $f[S] \in \mathcal{W}$ for all $S \in \mathcal{R}$ and $f[S] \in \mathcal{S}$ for all $S \in \mathcal{R} \cap \mathcal{T}$. □

The above follows easily from Proposition 2.2.5.

2.2.13 Lemma

i) For each Stone field $\mathbf{X} = \langle X, \mathcal{T}, \mathcal{R} \rangle$ we have $\beta(\mathbf{X})[S] = \alpha(\mathbf{X}^S)(S)$ for all $S \in \mathcal{R}$.

ii) $\beta : \mathrm{id} \longrightarrow (\ )^{SF}$ is a natural isomorphism.

Proof:

Let $\mathbf{X} = \langle X, \mathcal{T}, \mathcal{R} \rangle$. First observe that $\beta(\mathbf{X})$ is surjective: Let $\mathcal{U}$ be an ultrafilter in $\mathbf{X}^S$. Then $\mathcal{U} = \mathcal{R} \cap \mathcal{F}$ where $\mathcal{F}$ is an extension of $\mathcal{U}$ to a proper filter over X. Since **X** is compact $\cap \mathcal{U} = \cap (\mathcal{R} \cap \mathcal{F}) \neq \phi$. Let $x \in \cap \mathcal{U}$. Then $\mathcal{U} \subseteq \beta(x)$ and since these are both ultrafilters $\mathcal{U} = \beta(x)$. Thus β is surjective as required. (i): Let $S \in \mathcal{R}$. Let $x \in S$. Then $S \in \beta(x)$ whence $\beta(x) \in \alpha(S)$. Thus $\beta[S] \subseteq \alpha(S)$. Let $\mathcal{F} \in \alpha(S)$. Then $S \in \mathcal{F}$. By surjectivity there is an $x \in X$ with $\mathcal{F} = \beta(x)$. Then $x \in S$ and so $\mathcal{F} \in \beta[S]$. Thus $\alpha(S) \subseteq \beta[S]$ and so $\alpha(S) = \beta[S]$. (ii): Since **X** is separative β is injective and hence bijective. Let $S \in \mathcal{R}$. By (i) $\beta[S] \in \alpha[\mathbf{X}^S]$. Moreover if $S \in \mathcal{R} \cap \mathcal{T}$ then $\beta[S] \in \mathcal{T}(\mathbf{X}^S)$. By Lemma 2.2.11 $\beta : \mathbf{X} \longrightarrow \mathbf{X}^{SF}$ is an isomorphism. Suppose

$f : X \longrightarrow Y$ is field map in **Sfld**. Then for all $x \in X$ we have $f^{SF}\beta(x) = f^{S-1}[\beta(x)] = \{ S \in Y^S : f^S(S) \in \beta(x) \} = \{ S \in Y^S : x \in f^{-1}[S] \} = \{ S \in Y^S : f(x) \in S \} = \beta f(x)$. Thus $f^{SF}\beta(X) = \beta(Y)f$ and so β is natural. □

Combining the previous lemmas we finally obtain the result we have been aiming for:

2.2.14 Theorem

^{S} : **Sfld** $\longrightarrow$ **Int**$^+$ and ^{F} : **Int**$^+$ $\longrightarrow$ **Sfld** form a co–equivalence system. □

2.2.15 Definition and Remark

Note that under the co–equivalence given in Theorem 2.2.14 Boolean interior algebras correspond to Stone fields of the form $\langle X, \mathcal{T}, \mathcal{R} \rangle$ where $\langle X, \mathcal{T} \rangle$ is a Boolean space and $\mathcal{R}$ is its field of clopen sets. We will call such fields **Boolean fields**. Let **Bfld** denote the full subcategory of **Sfld** consisting of the Boolean fields. The functor which drops $\mathcal{T}$ gives an isomorphism from **Bfld** to the category **Fld** whose objects are fields of sets $\langle X, \mathcal{R} \rangle$ and whose morphisms are of the form $f : \langle X, \mathcal{R} \rangle \longrightarrow \langle Y, \mathcal{W} \rangle$, where $f : X \longrightarrow Y$ is a map such that $f^{-1}[S] \in \mathcal{R}$ for all $S \in \mathcal{W}$. We thus obtain a category–theoretic formalization of the representation of Boolean algebras by fields of sets. On the other hand the functor that drops $\mathcal{R}$ gives an isomorphism from **Bfld** to the category of Boolean spaces. Moreover, since for a Boolean interior algebra A, $\mathcal{T}(A)$ is just the Stone topology of A^u, we in fact have the *Stone Duality Theorem* for Boolean algebras as a special case of Theorem 2.2.14. □

2.2.16 Remark

Consider a separative field of sets $\langle X, \mathcal{R} \rangle$. If X is finite then $\mathcal{R} = \mathcal{P}(X)$. If $\mathcal{R} = \mathcal{P}(X)$ and $\langle X, \mathcal{R} \rangle$ is also compact then X is finite. From this we see that the full subcategory of **Sfld** consisting of the finite Stone fields is isomorphic via the forgetful functor to the full subcategory of Top consisting of the finite topological spaces. It is easy to see that under this isomorphism the functor ^{S} corresponds to ^{A}. □

What sort of field maps correspond to homomorphisms?

2.2.17 Definition

Let $f : \langle X, \mathcal{T}, \mathcal{R} \rangle \longrightarrow \langle Y, \mathcal{S}, \mathcal{W} \rangle$ be a field map between topological fields of sets. Let $x \in X$. f is said to be **weakly open at** x iff for all $S \in \mathcal{W}$, if $f^{-1}[S]$ is a neighbourhood of x then S is a neighbourhood of f(x). f is said to be **weakly open** iff it is weakly open at all $x \in X$. Let **Sfwo** denote the category of Stone fields and weakly open field maps. □

2.2.18 Theorem

Let $f : \langle X, \mathcal{T}, \mathcal{R} \rangle \longrightarrow \langle Y, \mathcal{S}, \mathcal{W} \rangle$ be a field map between Stone fields. Then the following are equivalent:

i) f is weakly open.

ii) For all $S \in \mathcal{T}$ and all $R \in \mathcal{W}$, $f[S] \subseteq R$ implies $f[S] \subseteq R^I$.

iii) f^S is a homomorphism.

Proof:

(i) $\Longrightarrow$ (ii): Let f be weakly open. Let $S \in \mathcal{T}$ and $R \in \mathcal{W}$ and suppose $f[S] \subseteq R$. Then $S \subseteq f^{-1}[R]$. Let $x \in S$. Then $f^{-1}[R]$ is a neighbourhood of x and so R is a neighbourhood of f(x), that is $f(x) \in R^I$. Thus $f[S] \subseteq R^I$. (ii) $\Longrightarrow$ (iii): Assume (ii). Let $R \in \mathcal{W}$. Then $f^S(R)^I \in \mathcal{T}$ and $f[f^S(R)^I] \subseteq R$ since $f^S(R)^I \subseteq f^S(R) = f^{-1}[R]$. Thus $f[f^S(R)^I] \subseteq R^I$ and so $f^S(R)^I \subseteq f^{-1}[R^I] = f^S(R^I)$. But $f^S(R^I) \subseteq f^S(R)^I$ since f^S is a topomorphism and so $f^S(R^I) = f^S(R)^I$. Thus f^S is a homomorphism. (iii) $\Longrightarrow$ (i): Let f^S be a homomorphism. Let $x \in X$. Let $R \in \mathcal{W}$ and suppose $f^{-1}[R]$ is a neighbourhood of x. Then $x \in f^S(R)^I$. But $f^S(R)^I = f^S(R^I) = f^{-1}[R^I]$ and so $f(x) \in R^I$, that is R is a neighbourhood of f(x). □

Further charaterizations of weakly open maps will be given in Theorem 2.4.25.

2.2.19 Corollary

^{S} : **Sfwo** $\longrightarrow$ **Int** and ^{F} : **Int** $\longrightarrow$ **Sfwo** form a co–equivalence system. □

2.2.20 Corollary

Let $f : \mathbf{X} \longrightarrow \mathbf{Y}$ be a field map between Stone fields. If f is open (as a map between the underlying topological spaces) then f^S is a homomorphism. □

The above follows from the fact that an open field map is obviously weakly open. The converse fails however: Let A be an infinite Boolean interior algebra. Then there is a non–principal ultrafilter in F in A. Let 2 be the two element Boolean interior algebra. Consider the homomorphism $f : A \longrightarrow 2$ with $F = f^{-1}[\{1\}]$. Then f^F is weakly open by Theorem 2.2.18. But $2^F \in \mathcal{T}(2)$ is mapped onto $\{F\} \notin \mathcal{T}(A)$ by $f^F$and so f^F is not open. However we have the following result:

2.2.21 Proposition

Any weakly open field map between Stone fields with finite co–domain is open.

Proof:

Let $f : \langle X, \mathcal{T}, \mathcal{R} \rangle \longrightarrow \langle Y, \mathcal{S}, \mathcal{P}(Y) \rangle$ be a weakly open field map between Stone fields with finite co–domain. (See Remark 2.2.16.) Let $S \in \mathcal{T}$. Consider $x \in S$. Then $f^{-1}f[S]$ is a neighbourhood of x. Hence $f[S] \in \mathcal{P}(Y)$ is a neighbourhood of f(x). It follows that $f[S] \in \mathcal{S}$. □

2.2.22 Definition

Let $\mathbf{X} = \langle X, \mathcal{T}, \mathcal{R} \rangle$ and $\mathbf{Y} = \langle Y, \mathcal{S}, \mathcal{W} \rangle$ be Stone fields. **X** is called a **subfield** of **Y** iff $\langle X, \mathcal{T} \rangle$ is a topological subspace of $\langle Y, \mathcal{S} \rangle$ and $\mathcal{R} = \{ X \cap S : S \in \mathcal{W} \}$. **X** is said to be **induced** by $X \subseteq Y$. Given a Stone field **Y** a subset $X \subseteq Y$ is said to be **inducive in Y** iff it induces a subfield of **Y**. □

The concepts of subfield and inducive subset will be useful for establishing a duality between injective field maps and surjective topomorphisms. Before doing this we briefly investigate subfields and inducive subsets. Note that not every subset of a Stone field is inducive. The following theorem characterizes inducive subsets.

2.2.23 Theorem

Let $\mathbf{Y} = \langle Y, \mathcal{S}, \mathcal{W} \rangle$ be a Stone field and let $X \subseteq Y$. The following are equivalent:

i) X is inducive in **Y**.

ii) There is a $\mathcal{V} \subseteq \mathcal{W}$ with $X = \cap \mathcal{V}$ and for all $Z \in \mathcal{W}$ there is an $S \in \mathcal{W}$ with $X \cap S = X \cap (X' \cup Z)^{I}$.

Proof:

Let $\langle X, \mathcal{T} \rangle$ be the subspace of $\langle Y, \mathcal{S} \rangle$ induced by X. Let $\mathcal{R} = \{ X \cap S : S \in \mathcal{W} \}$. (i) $\Longrightarrow$ (ii): Assume (i). Suppose for all $\mathcal{V} \subseteq \mathcal{W}$, $X \neq \cap \mathcal{V}$. Then $X \subset \cap \{ S \in \mathcal{W} : X \subseteq S \}$. Let $x \in \cap \{ S \in \mathcal{W} : X \subseteq S \} - X$. Let $\mathcal{D} = \{ S \cap X : x \in S \in \mathcal{W} \}$. Suppose that for some $S \in \mathcal{W}$ with $x \in S$ we have $S \cap X = \phi$. Then $X \subseteq S'$ whence $x \in S'$, a contradiction. Thus $\phi \notin \mathcal{D}$. Let $x \in R, S \in \mathcal{W}$. Then $x \in R \cap S \in \mathcal{W}$ and $(X \cap R) \cap (X \cap S) = X \cap (S \cap R)$. Hence $\mathcal{D}$ has the finite intersection property and so there is a proper filter $\mathcal{F}$ over X with $\mathcal{D} \subseteq \mathcal{F}$. Then there is a $y \in \cap (\mathcal{R} \cap \mathcal{F})$. Then $y \in S$ for all $S \in \mathcal{W}$ with $x \in S$. Thus $x = y$ by separativeness, and so $x \in X$, a contradiction. Thus $X = \cap \mathcal{V}$ for some $\mathcal{V} \subseteq \mathcal{W}$. Let $Z \in \mathcal{W}$ and let U be the interior of $X \cap Z$ in $\langle X, \mathcal{T} \rangle$. Then $U \in \mathcal{R}$. Hence there is an $S \in \mathcal{W}$ with and $U = X \cap S$. Then $X \cap S = X \cap (X' \cup (X \cap Z))^{I} = X \cap (X' \cup Z)^{I}$. (ii) $\Longrightarrow$ (i): Assume (ii). Clearly $\mathcal{R}$ is a field of sets. Let $G \in \mathcal{R}$ and let U be the interior of G in $\langle X, \mathcal{T} \rangle$. There is a $Z \in \mathcal{W}$ with $G = X \cap Z$. Then $U = X \cap (X' \cup G)^{I} = X \cap (X' \cup Z)^{I}$. Hence there is an $S \in \mathcal{W}$ with $U = X \cap S$ and so $U \in \mathcal{R}$. Thus $\mathbf{X} = \langle X, \mathcal{T}, \mathcal{R} \rangle$ is a topological field of sets. Clearly **X** is separative and algebraic since **Y** is. Let $\mathcal{F}$ be a proper filter over X. There is an extension $\mathcal{E}$ of $\mathcal{F}$ to a proper filter over Y. There is a $\mathcal{V} \subseteq \mathcal{W}$ with $X = \cap \mathcal{V}$. Then $\mathcal{V} \subseteq \mathcal{W} \cap \mathcal{E}$ and so $\cap (\mathcal{W} \cap \mathcal{E}) \subseteq X$. Thus $\cap (\mathcal{W} \cap \mathcal{E}) = X \cap (\cap (\mathcal{W} \cap \mathcal{E})) = \cap (\mathcal{R} \cap \mathcal{W})$. But then $\cap (\mathcal{R} \cap \mathcal{W}) \neq \phi$ since **Y** is compact.

Thus **X** is compact and hence a Stone field. □

2.2.24 Corollary

Let $\mathbf{Y} = \langle Y, \mathcal{S}, \mathcal{W} \rangle$ be a Stone field. Then every member of $\mathcal{W}$ is inducive in **Y**. □

2.2.25 Corollary

Let $\mathbf{Y} = \langle Y, \mathcal{S}, \mathcal{W} \rangle$ be a Boolean Stone field. Then $\cap \mathcal{V}$ is inducive in **Y** for all $\mathcal{V} \subseteq \mathcal{W}$.

Proof:

Suppose $X = \cap \mathcal{V}$ for some $\mathcal{V} \subseteq \mathcal{W}$. Then X is closed with respect to $\mathcal{S}$. Hence if $Z \in \mathcal{W}$ then $X \cap (X' \cup Z)^{I} = X \cap (X' \cup Z) = X \cap Z$. The result folows by Theorem 2.2.23 □

2.2.26 Notation

For each interior algebra **A** and all $a \in A$ let $\boldsymbol{\alpha}(\mathbf{a})$ denote the subfield of $\mathbf{A}^{F}$ induced by $\alpha(a)$. Let $\mathcal{T}(a) = \{ \cup\alpha[S] : S \subseteq (a]^{O} \}$. □

Using Theorem 1.4.6 (i) we easily see:

2.2.27 Proposition

If **A** is an interior algebra and $a \in A$ then $\boldsymbol{\alpha}(\mathbf{a}) = \langle \alpha(a), \mathcal{T}(a), \alpha[(a]] \rangle$. □

2.2.28 Corollary

Let **A** be an interior algebra and $a \in A$. Then $\alpha : (\mathbf{a}] \longrightarrow \boldsymbol{\alpha}(\mathbf{a})^{S}$ is an isomorphism. Consequently $\boldsymbol{\alpha}(\mathbf{a}) \cong (\mathbf{a}]^{F}$. □

2.2.29 Lemma

Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be topomorphism. Then $f^{F}[B^{F}] = \{ F \in A^{F} : 0 \notin f[F] \} = \cap\alpha[R]$ where $R = f^{-1}[\{ 1 \}]$.

Proof:

Suppose $F \in f^{F}[B^{F}]$. Then there is a $G \in B^{F}$ with $F = f^{-1}[G]$. Then $f[F] \subseteq G$ and so $0 \notin f[F]$ since $0 \notin G$. Conversely suppose $0 \notin f[F]$. $f[F]$ is closed under finite meets since F is. Thus there is an ultrafilter G in **B** with $f[B] \subseteq G$. Then $F \subseteq f^{-1}[G]$ and since F and $f^{-1}[G]$ are both ultrafilters we in fact have $F = f^{-1}[G] = f^{F}(G)$. Thus $f^{F}[B^{F}] = \{ F \in A^{F} : 0 \notin f[F] \}$. Again suppose $0 \notin f[F]$. Suppose there is an $a \in R$ with $a \notin F$. Then $a' \in F$. Hence $0 = f(a') \in F$, a contradiction. Hence $R \subseteq F$. Conversely suppose $R \subseteq F$. Assume there is an $a \in F$ with $f(a) = 0$. Then $f(a') = 1$ and so $a' \in F$, a contradiction. Thus $0 \notin f[F]$. Hence $\{ F \in A^{F} : 0 \notin F \} = \cap\alpha[R]$. □

2.2.30 Lemma

A bijective field map between Boolean Stone fields is an isomorphism. □

2.2.31 Theorem

Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be a topomorphism.

i) f is injective iff f^F is surjective.

ii) f is surjective iff f^F is injective.

Proof:

(i): Suppose f is injective. Let $F \in A^F$. $0 \notin f[F]$ or else there is an $a \in F$ with $f(a) = 0$. Since f is injective $a = 0$, a contradiction. Thus by Lemma 2.2.29 $F \in f^F[B^F]$. Hence f^F is surjective. Conversely suppose f^F is surjective. Let $a,b \in A$ with $f(a) = f(b)$. Assume $a \neq b$. Without loss of generality $a \in F$ but $b \notin F$. There is an ultrafilter G in **B** with $F = f^F(G) = f^{-1}[G]$. Then $f(b) = f(a) \in G$ and so $b \in f^{-1}[G] = F$, a contradiction. Thus $a = b$ and so f is injective. (ii): Suppose f is surjective. Let F and G be ultrafilters in **B** with $f^F(F) = f^F(G)$. Assume $F \neq G$. Then there is a $b \in F$ with $b \notin G$. Now there is an $a \in A$ with $b = f(a)$. But then $a \in f^{-1}[F] = f^{-1}[G]$ whence $b \in G$, a contradiction. Thus $F = G$ and so f^F is injective. Conversely suppose f^F is injective. Let $b \in B$. Put $X = f^F[B^F]$. By Lemma 2.2.29 $X = \cap \mathcal{V}$ for some $\mathcal{V} \subseteq \alpha[A]$. By Corollary 2.2.25 X is inducive in the Stone field $\langle A^F, \mathcal{S}, \alpha[A] \rangle$, where $\mathcal{S}$ is the topology generated by $\alpha[A]$. (See Lemma 2.2.3 (iv).) Let $\mathbf{X} = \langle X, \mathcal{T}, \mathcal{W} \rangle$ be the subfield induced by X and let $\mathbf{Z} = \langle B^F, \mathcal{U}, \alpha[B] \rangle$ where $\mathcal{U}$ is the topology generated by $\alpha[B]$. Then $f^F : \mathbf{Z} \longrightarrow \mathbf{X}$ is obviously bijective. Let $S \in \mathcal{W}$. There is an $a \in A$ with $S = X \cap \alpha(a)$. Then $f^{F\,-1}[S] = f^{F\,-1}[\alpha(a)] = \alpha f(a) \in \alpha[B]$. It follows that $f^F : \mathbf{Z} \longrightarrow \mathbf{X}$ is a field map and hence an isomorphism by Lemma 2.2.30. Thus $f^F[\alpha(b)] \in \mathcal{W}$. Hence there is a $c \in A$ with $f^F[\alpha(b)] = X \cap \alpha(c)$. Then $\alpha(b) = f^{F\,-1}[\alpha(c)] = \alpha f(c)$. Hence $b = f(c)$. Thus f is surjective. □

The above theorem can also be 'lifted' by means of appropriate functors from a corresponding result for the Stone duality for Boolean algebras. Since we are considering the latter duality to be a special case of Theorem 2.2.14 (see Remark 2.2.15) a more direct proof has been given instead.

2.2.32 Theorem

In $\mathbf{Int}^+$ monomorphisms are injective, equivalently, in **Sfld** epimorphisms are surjective.

Proof:

Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be a monomorphism in $\mathbf{Int}^+$. Consider $a,b \in A$ with $f(a) = f(b)$. Let $\mathbf{C} = \mathbf{I}(2)^A$ where $\mathbf{I}(2) = \langle \{ 0, 1 \}, \{ \phi, \{ 0, 1 \} \} \rangle$ the two element indiscrete space. Let

$g : C \longrightarrow A$ and $h : C \longrightarrow A$ be the unique topomorphisms with $g(\{0\}) = a$ and $h(\{0\}) = b$. Then $fg = fh$ and so $g = h$. Thus $a = b$ and so f is injective. □

Of course since **Int** consists of a variety and its homomorphisms the above result still holds if we replace $\mathbf{Int}^+$ and **Sfld** by **Int** and **Sfwo** respectively.

2.2.33 Theorem

In $\mathbf{Int}^+$ epimorphisms are surjective, equivalently, in **Sfld** monomorphisms are injective.

Proof:

Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be an epimorphism in $\mathbf{Int}^+$. Let F and G be ultrafilters in **B** with $f^F(F) = f^F(G)$. Let **2** be the two element interior algebra and let $g : \mathbf{B} \longrightarrow \mathbf{2}$ and $h : \mathbf{B} \longrightarrow \mathbf{2}$ be the topomorphisms with $F = g^{-1}[\{1\}]$ and $G = h^{-1}[\{1\}]$. Then $(gf)^{-1}[\{1\}] = f^{-1}[\{1\}] = f^{-1}[\{1\}] = (hf)^{-1}[\{1\}]$. Thus $gf = hf$ and so $g = h$. Thus $F = G$ and so f^F is injective whence f is surjective. □

It is not known if the above result can be generalized to **Int** and **Sfwo**. We now examine embeddings in **Sfld** and **Sfwo**.

2.2.34 Proposition

Let $f : \langle X, \mathcal{T}, \mathcal{R} \rangle \longrightarrow \langle Y, \mathcal{S}, \mathcal{W} \rangle$ be the inclusion map of a subfield. Then f is an embedding in **Sfld**.

Proof:

Let $g : \langle Z, \mathcal{U}, \mathcal{K} \rangle \longrightarrow \langle X, \mathcal{T}, \mathcal{R} \rangle$ be a map such that fg is in **Sfld**. Let $Z \in \mathcal{R}$. Then there is an $S \in \mathcal{W}$ with $Z = X \cap S$. Thus $g^{-1}[Z] = (fg)^{-1}[S] \in \mathcal{K}$. Similarly if $Z \in \mathcal{T}$, $g^{-1}[Z] \in \mathcal{U}$ and so g is a field map. □

2.2.35 Lemma

Let $f : \langle X, \mathcal{T}, \mathcal{R} \rangle \longrightarrow \langle Y, \mathcal{S}, \mathcal{W} \rangle$ be a field map between Stone fields. Let $\mathcal{K} = \{ f^{-1}[Z] : Z \in \mathcal{W} \}$. Let $\mathcal{U}$ be the topology generated by $\mathcal{K}$. Then $\langle X, \mathcal{U}, \mathcal{K} \rangle$ is a Stone field.

Proof:

Clearly $\langle X, \mathcal{U}, \mathcal{K} \rangle$ is an algebraic topological field of sets. Let x and y are distinct in X. Then f(x) and f(y) are distinct in Y and so there is a $Z \in \mathcal{W}$ with $f(x) \in Z$ but $f(y) \notin Z$. Then $f^{-1}[Z] \in \mathcal{K}$ with $x \in f^{-1}[Z]$ but $y \notin f^{-1}[Z]$. Thus $\langle X, \mathcal{U}, \mathcal{K} \rangle$ is separative. Let $\mathcal{F}$ be a proper filter over X. Since f is a field map, $\mathcal{K} \subseteq \mathcal{R}$ and so $\phi \neq \cap(\mathcal{R} \cap \mathcal{F}) \subseteq \cap(\mathcal{U} \cap \mathcal{F})$. Thus $\langle X, \mathcal{U}, \mathcal{K} \rangle$ is compact and the result follows. □

2.2.36 Proposition

Let $f : \langle X, \mathcal{T}, \mathcal{R} \rangle \longrightarrow \langle Y, \mathcal{S}, \mathcal{W} \rangle$ be an initial map (in particular an embedding) in **Sfld**. Then $\mathcal{R} = \{ f^{-1}[Z] : Z \in \mathcal{W} \}$.

Proof:

Put $\mathcal{K} = \{ f^{-1}[Z] : Z \in \mathcal{W} \}$ and let $\mathcal{U}$ be the topology generated by $\mathcal{K}$. By Lemma 2.2.35 $\langle X, \mathcal{U}, \mathcal{K} \rangle$ is a Stone field. Let $g : \langle X, \mathcal{U}, \mathcal{K} \rangle \longrightarrow \langle X, \mathcal{T}, \mathcal{R} \rangle$ be the identity map. Then $(fg)^{-1}[Z] = f^{-1}[Z] \in \mathcal{K}$. If $Z \in \mathcal{S} \cap \mathcal{W}$ then $(fg)^{-1}[Z] = f^{-1}[Z] \in \mathcal{K} \subseteq \mathcal{U}$. Since $\mathcal{S} \cap \mathcal{W}$ is a base for $\mathcal{S}$ (see Proposition 2.2.5), fg is a field map. Hence g is a field map and so $\mathcal{R} \subseteq \mathcal{K}$. But $\mathcal{K} \subseteq \mathcal{R}$ since f is a field map and so $\mathcal{K} = \mathcal{R}$ as required. □

A 'nice' complete characterization of embeddings in **Sfld** does not seem to exist. The situation is more elegant for **Sfwo**.

2.2.37 Lemma

Let $f : \langle X, \mathcal{T}, \mathcal{R} \rangle \longrightarrow \langle Y, \mathcal{S}, \mathcal{W} \rangle$ be the inclusion map of a subfield and let $g : \langle Z, \mathcal{U}, \mathcal{K} \rangle \longrightarrow \langle X, \mathcal{T}, \mathcal{R} \rangle$ be a map such that fg is a weakly open field map. Then g is a weakly open field map.

Proof:

By Proposition 2.2.34 g is a field map. Let $G \in \mathcal{U}$ and $S \in \mathcal{R}$ with $g[G] \subseteq S$. There is a $Z \in \mathcal{W}$ with $S = X \cap Z$. Then $fg[G] \subseteq Z$ whence $fg[G] \subseteq Z^I$. Thus $g[G] \subseteq X \cap Z^I \subseteq X \cap (X' \cup Z)^I = X \cap (X' \cup S)^I$, the interior of S in $\langle X, \mathcal{T} \rangle$. The result follows by Theorem 2.2.18. □

2.2.38 Theorem

Let $f : \langle X, \mathcal{T}, \mathcal{R} \rangle \longrightarrow \langle Y, \mathcal{S}, \mathcal{W} \rangle$ be in **Sfwo**. Then the following are equivalent:

i) f is injective.

ii) f is an embedding in **Sfwo**.

iii) f is the inclusion map of a subfield (up to isomorphism).

iv) f^F is surjective.

Proof:

By Lemma 2.2.37 (iii) ⟹ (ii) and trivially (ii) ⟹ (i). (i) ⟺ (iv) by Theorem 2.2.31 (i). (iv) ⟹ (iii): Assume (iv). $\mathcal{R} = f^F[\mathcal{W}] = \{ f^{-1}[Z] : Z \in \mathcal{W} \}$. Also f^F is a homomorphism and f^{FO} is surjective. Note that $Y^{FO} = \mathcal{S} \cap \mathcal{W}$ and $X^{FO} = \mathcal{T} \cap \mathcal{R}$. Thus $\mathcal{T} \cap \mathcal{R} = f^{FO}[\mathcal{S} \cap \mathcal{W}] = \{ f^{-1}[Z] : Z \in \mathcal{S} \cap \mathcal{W} \}$. By Proposition 2.2.5 $\mathcal{T} \cap \mathcal{R}$ and $\mathcal{S} \cap \mathcal{W}$ are bases for $\mathcal{T}$ and $\mathcal{S}$ respectively and so we see that $\mathcal{T} = \{ f^{-1}[Z] : Z \in \mathcal{S} \}$. Since (iv) ⟺ (i) f is injective and so (iii) follows. □

2.2.39 Remark

The above theorem shows that under the co–equivalence between **Int** and **Sfwo**, quotient maps in **Int** (surjective homomorphisms) correspond to embeddings in **Sfwo** (injective weakly open field maps or equivalently weakly open inclusion maps of subfields). Note that all this required proof since the co–equivalence was not concrete. □

No elegant description of quotient maps in Sfld or Sfwo have been found at present.

2.3 STONE SPACES OF INTERIOR ALGEBRAS

2.3.1 Definition and Remark

By composing the forgetful functor from Sfld to Top with the functor ^{F} we obtain a faithful co–functor ^{D} : $\mathbf{Int}^+ \longrightarrow$ Top. More explicitly, if A is an interior algebra, $\mathbf{A}^D = \langle A^D, \mathcal{T}(\mathbf{A}) \rangle$ where $A^D = A^F$. If $f : \mathbf{A} \longrightarrow \mathbf{B}$ is a topomorphism $f^D : \mathbf{B}^D \longrightarrow \mathbf{A}^D$ is the continuous map given by $f^D(F) = f^{-1}[F]$. $\mathbf{A}^D$ will be called the Stone space of **A**. □

Two other topologies associated with A are the usual Stone topology of $\mathbf{A}^u$ and the Stone topology of $\mathbf{A}^O$. The relationship between $\mathcal{T}(\mathbf{A})$ and the Stone topology of $\mathbf{A}^u$ is clear. We investigate the connection between $\mathbf{A}^D$ and the Stone space of $\mathbf{A}^O$.

2.3.2 Definition and Remark

For every interior algebra A, let A^P denote the Stone space of A^O. More precisely A^P = $\langle A^P, \mathcal{S}(A) \rangle$ where A^P is the set of proper prime filters in A^O and $\mathcal{S}(A)$ is the topology on A^P with base $j[A^O]$ where $j : A^O \longrightarrow \mathcal{P}(A^P)$ is given by $j(a) = \{ F \in A^P : a \in F \}$ for all $a \in A^O$. Now let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be a topomorphism. Then if $F \in B^P$, $f^{-1}[F] \cap A^O \in A^P$ as a straightforward argument shows. We can thus define a map $f^P : \mathbf{B}^P \longrightarrow \mathbf{A}^P$ by $f^P(F) = f^{-1}[F] \cap A^O$ for all $F \in B^P$. Consider a base element j(a) of $\mathbf{A}^P$ where $a \in A^O$. $f^{P\,-1}[j(a)] = \{ F \in B^P : f^P(F) \in j(a) \} = \{ F \in B^P : a \in f^{-1}[F] \cap A^O \} = \{ F \in B^P : f(a) \in F \}$ $= jf(a)$. Since $f(a) \in B^O$, $f^{P\,-1}[j(a)]$ is open in $\mathbf{B}^P$. We thus have a co–functor ^{P} : $\mathbf{Int}^+ \longrightarrow$ Top. □

2.3.2 Lemma

Let A be an interior algebra.

i) If $F \in A^D$ then $F \cap A^O \in A^P$.

ii) Define $e : \mathbf{A}^D \longrightarrow \mathbf{A}^P$ by $e(F) = F \cap A^O$ for all $F \in A^D$. Then for all $a \in A^O$ $e^{-1}[j(a)] = \alpha(a)$. □

2.3.3 Theorem

Let $\mathbf{A}$ be an interior algebra and let $e : \mathbf{A}^D \longrightarrow \mathbf{A}^P$ be as in Lemma 2.3.2. Then:

i) e is a continuous.

ii) e is surjective.

iii) Any right inverse of e is continuous, in particular e is a retraction in **Top**.

iv) e is an open map and hence a quotient map in **Top**.

v) e is natural in $\mathbf{A}$.

Proof:

(i): Follows from Lemma 2.3.2 (ii) since $j[A^O]$ and $\alpha[A^O]$ are bases for $\mathbf{A}^P$ and $\mathbf{A}^E$ respectively. (ii): Let $G \in \mathbf{A}^P$. Put $R = G \cup \{ a' : a \in A^O - G \}$. Let S be a finite subset of R. If $S \subseteq G$ then $\Pi S \neq 0$. Suppose there are $a_1,\dots,a_n \in A^O - G$ such that $S - G = \{ a_1',\dots,a_n' \}$. Put $a = a_1 + \dots + a_n$. Suppose $\Pi S = 0$. Then $\Pi (S \cap G) \cdot a' = 0$ and so $\Pi (S \cap G) \leq a$. Now $\Pi (S \cap G) \in G$ and so $a \in G$ whence $a_i \in G$ for some $i \in \{ 1,\dots,n \}$, a contradiction. Thus $\Pi S \neq 0$. R has the finite intersection propert and so there is an $F \in \mathbf{A}^D$ with $R \subseteq F$. Then $G \subseteq F \cap A^O$. Suppose $G \neq F \cap A^O$. Let $a \in F \cap A^O - G$. Then $a' \in R \subseteq F$, a contradiction. Thus $G = F \cap A^O = e(F)$. Thus e is surjective. (iii): Let $f : \mathbf{A}^P \longrightarrow \mathbf{A}^D$ be a right inverse of e, that is $f(G) \cap A^O = G$ for all $G \in \mathbf{A}^P$. Let $\alpha(a)$ be a base element of $\mathbf{A}^D$ where $a \in A^O$. Then $f^{-1}[\alpha(a)] = \{ G \in \mathbf{A}^P : f(G) \in \alpha(a) \} = \{ G \in \mathbf{A}^P : a \in f(G) \} = \{ G \in \mathbf{A}^P : a \in f(G) \cap A^O \} = \{ G \in \mathbf{A}^P : a \in G \} = j(a)$ which is open in $\mathbf{A}^P$. Thus f is continuous. (iv): Let $\alpha(a)$ be a base element of $\mathbf{A}^D$ where $a \in A^O$. By Lemma 2.3.2 (ii) and (ii) we have $e[\alpha(a)] = j(a)$ which is open in $\mathbf{A}^P$. Thus e is open. (v): Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be a topomorphism. Let $F \in \mathbf{B}^D$. Then $f^{-1}[F \cap B^O] \cap A^O \subseteq f^{-1}[F] \cap A^O$. Let $b \in f^{-1}[F] \cap A^O$. Then $f(b) \in F$ and $f(b) \in B^O$, that is $f(b) \in F \cap B^O$, whence $b \in f^{-1}[F \cap B^O]$. Hence $f^{-1}[F] \cap A^O \subseteq f^{-1}[F \cap B^O] \cap A^O$. Thus $ef^D(F) = f^P e(F)$ and the result follows. □

Given a topological space $\mathbf{X}$, how is $\mathbf{X}$ related to $\mathbf{X}^{AD}$?

2.3.4 Lemma

Let $\mathbf{X}$ be a topological space. Define $d : \mathbf{X} \longrightarrow \mathbf{X}^{AD}$ by $d(x) = \{ R \subseteq X : x \in R \}$. Then $d^{-1}[\alpha(\mathbf{X}^A)(R)] = R$ for all $R \subseteq X$. □

2.3.5 Theorem

Let $\mathbf{X}$ be a topological space and let $d : \mathbf{X} \longrightarrow \mathbf{X}^{AD}$ be as in Lemma 2.3.4. Then:

i) d is an embedding in **Top** and so $\mathbf{X}$ is homeomorphic to a subspace of $\mathbf{X}^{AD}$.

ii) d is natural in $\mathbf{X}$.

Proof:

(i): Since $\{ \alpha(R) : R \subseteq X \text{ is open in } \mathbf{X} \}$ is a base for $\mathbf{X}^{AD}$ we see by Lemma 2.3.4 that d is continuous and initial. Clearly d is injective. (ii): Let $f : \mathbf{X} \longrightarrow \mathbf{Y}$ be a continuous map. Let $x \in X$. Then $df(x) = \{ R \subseteq Y : f(x) \in R \} = \{ R \subseteq Y : x \in f^{-1}[R] \} = \{ R \subseteq Y : x \in f^{A}(R) \} = \{ R \subseteq Y : f^{A}(R) \in d(x) \} = \{ R \subseteq Y : f^{A-1}[d(x)] \} = f^{AD}d(x)$ and the result follows. □

2.3.6 Remark

Note that if $\mathbf{X}$ is a finite space then the map d is a homeomorphism from $\mathbf{X}$ to $\mathbf{X}^{AD}$. □

2.3.7 Remark

Let **A** be an interior algebra. Then for all $a \in A$ $\alpha(a)$ is compact in the Stone space of $\mathbf{A}^{u}$ and hence in $\mathbf{A}^{D}$. □

2.3.8 Theorem

Let A be an interior algebra:

i) The compact open sets of $\mathbf{A}^{D}$ are precisely $\alpha[A^{O}]$.

ii) The clopen sets of $\mathbf{A}^{D}$ are precisely $\alpha[A^{\diamond}]$.

Proof:

(i): By Proposition 2.2.5 the compact open sets of $\mathbf{A}^{D}$ are precisely $\alpha[A] \cap \mathcal{T}(A)$. Clearly $\alpha[A^{O}] \subseteq \alpha[A] \cap \mathcal{T}(A)$. Let $b \in A$ with $\alpha(b) \in \mathcal{T}(A)$. There is an $S \subseteq A^{O}$ with $\alpha(b) = \cup\alpha[S]$. By Remark 2.3.7 $\alpha(b)$ is compact and so there is a finite $R \subseteq S$ with $\alpha(b) = \cup\alpha[R]$. Put $a = \Sigma R$. Then $\alpha(b) = \alpha(a)$ whence $b = a \in A^{O}$. Thus $\alpha[A^{O}] = \alpha[A] \cap \mathcal{T}(A)$. (ii): Clearly every member of $\alpha[A^{\diamond}]$ is clopen. Let $\mathcal{D}$ be clopen in $\mathbf{A}^{D}$. Then $\mathcal{D}$ and $\mathcal{D}'$ are closed in $\mathbf{A}^{D}$ hence compact. Thus $\mathcal{D}$ and $\mathcal{D}'$ are compact open in $\mathbf{A}^{D}$. By (i) there are $a,b \in A^{O}$ with $\mathcal{D} = \alpha(a)$ and $\mathcal{D}' = \alpha(b)$. Then $\alpha(a') = \alpha(b)$ whence $a' = b \in A^{O}$. Thus $a \in A^{\diamond}$ and so $\mathcal{D} \in \alpha[A^{\diamond}]$. □

How do properties of interior algebras correspond to properties of the corresponding Stone spaces?

2.3.9 Theorem

Let A be a interior algebra.

i) A is semi–simple iff $\mathbf{A}^{D}$ is zero–dimensional.

If A is non–trivial:

ii) A is simple iff $\mathbf{A}^{D}$ is indiscrete.

iii) A is S.I. iff $\mathbf{A}^{D}$ is supercompact and the monolith of $\mathbf{A}^{D}$ is compact.

iii) A is F.S.I. iff A^D is ultra–connected.

iv) A is D.I. iff A^D is connected.

Proof:

(i): Suppose **A** is semi–simple. By Theorem 1.3.20 the clopen sets $\alpha[A^\diamond] = \alpha[A^O]$ form a base for A^D. Conversely suppose A^D is zero–dimensional. Let $a \in A^O$. By Theorem 2.3.8 (ii) there is an $S \subseteq A^\diamond$ with $\alpha(a) = \cup\alpha[S]$. By compactness of $\alpha(a)$ there is a finite $R \subseteq S$ with $\alpha(a) = \cup\alpha[R]$. Put $b = \Sigma R$. Then $\alpha(a) = \alpha(b)$ and so $a = b \in A^\diamond$. By Theorem 1.3.20 **A** is semi–simple. Now suppose **A** is non–trivial $\alpha : \mathbf{A} \longrightarrow \mathbf{A}^{DA}$ is a homomorphic embedding and so by Theorem 2.1.15 and the fact that simple, F.S.I. and D.I. are defined by universal sentences (see Remark 1.3.20) the reverse directions in (ii), (vi) and (v) hold. For the forward directions: (ii): If **A** is simple, $A^O = \{ 0, 1 \}$ whence $\mathcal{T}(A) = \{ \alpha(0), \alpha(1) \} = \{ \phi, A^D \}$, that is A^D is indiscrete. (vi): Suppose A^D is not ultra–connected. Then there are proper open subsets $\mathcal{D}$ and $\mathcal{E}$ of A^D such that $\mathcal{D} \cup \mathcal{E} = A^D$. Now there are $R,S \subseteq A^O$ with $\mathcal{D} = \cup\alpha[R]$ and $\mathcal{E} = \cup\alpha[R]$. Then $A^D = \cup\alpha[\, R \cup S\,]$. By compactness of A^D there is a finite $W \subseteq R \cup S$ such that $A^D = \cup\alpha[W]$. Put $b = \Sigma W$. Then $A^D = \alpha(b)$ and so $b = 1$. However $1 \notin W$ since $\cup\alpha[R] \neq A^D$ and $\cup\alpha[S] \neq A^D$. Thus 1 is join reducible in A^O and so **A** is not F.S.I. The result follows by contraposition. (v): Suppose A^D is not connected. Then A^D has a proper non–empty clopen subset $\mathcal{D}$. By Theorem 2.3.8 (ii) there is a $b \in A^\diamond$ with $\mathcal{D} = \alpha(b)$. Then $0 < b < 1$. Thus $A^\diamond \neq \{ 0, 1 \}$ and so **A** is not D.I. The result follows by contraposition. Now for (iii): **A** is S.I. iff there is an $a \in A^O$ with $a < 1$ and for all $b \in A^O$ with $b < 1$, $b \leq a$; iff there is an $a \in A^O$ with $\alpha(a) \neq A^D$ and for all $b \in A^O$ with $\alpha(b) \neq A^D$, $\alpha(b) \subseteq \alpha(a)$; iff there is a proper compact open subset of A^D that contains all proper compact open subsets of A^D (see Theorem 2.3.8 (i)); iff there is a proper compact open subset of A^D that contains all proper open subsets of A^D (by Proposition 2.2.5 (i)); iff A^D is supercompact with compact monolith. □

2.3.10 Remark

The above characterizations should be compared with Theorem 2.1.15. The condition in (ii) of Theorem 2.3.9 that the monolith of A^D must be compact cannot be dropped: Consider the topological space $\omega = \langle\, \omega, \omega + 1 \,\rangle$. Now $\omega = \cup \{ n : n < \omega \}$ and so ω is not completely join irreducible in ω^{AO}. Thus ω^A is not S.I. However ω^{AD} is supercompact: Suppose it is not. Then $\omega^{AD} = \cup\alpha[\omega]$. Let F be an ultrafilter over ω extending the filter of all co–finite subsets of ω. Then there is an $n < \omega$ with $F \in \alpha(n)$, that is $n \in F$, a contradiction since $n' \in F$. Thus ω^{AD} is supercompact. ω^A is an example of an interior algebra that is homomorphically embeddable in an S.I. interior algebra but which is not S.I. itself: ω^{ADA} is S.I. since ω^{AD} is supercompact, and $\alpha : \omega^A \longrightarrow \omega^{ADA}$ is a homomorphic embedding. □

For all interior algebras A, A^D is compact and so the properties Hausdorff, Urysohn, strongly regular, and strongly normal are equivalent for A^D. We in fact have:

2.3.11 Theorem

The following are equivalent for an interior algebra A:

i) A is Boolean.

ii) A^D is a Boolean space.

iii) A^D is Hausdorff.

Proof:

If A is Boolean A^D is just the Stone space of A^u and so (i) $\Longrightarrow$ (ii). Obviously (ii) $\Longrightarrow$ (iii). Assume (iii). Let $a \in A$. Then $\alpha(a)$ is compact in A^D and so by (iii) it is closed. Thus $\alpha(a) = \alpha(a)^C = \alpha(a^C)$ and so $a = a^C$. Thus A is Boolean and so (iii) $\Longrightarrow$ (i). □

2.3.12 Remark

THE STRONG JOINT EMBEDDING PROPERTY

Using Stone spaces one can give quick proofs that certain subcategories of **Int**$^+$ or **Int** satisfy the *Strong Joint Embedding Property* (SJEP), that is, for every family of objects in the subcategory there is an object in the subcategory into which each member of the family can be embedded. Let $\{ A_i : i \in I \}$ be a family of non–trivial interior algebras. Put $X = \Pi \{ A_i^D : i \in I \}$ and for all $i \in I$ let $p_i : X \longrightarrow A_i^D$ be the ith projection. Note that p_i is open, surjective and continuous for all $i \in I$. Thus for all $i \in I$ $p_i^A\alpha : A_i \longrightarrow X^A$ is an **Int** embedding. Thus the full subcategory of **Int**$^+$ or **Int** consisting of non–trivial interior algebras satisfies the SJEP. From this we also see that the full subcategories of **Int**$^+$ or **Int** consisting of the non–trivial Boolean, simple, S.I., F.S.I., or D.I. interior algebras all satisfy the SJEP. Recall that a member of an elementary class is called *ultra–universal* iff every member of the class is embeddable in an ultrapower of it. (See [15].) An elementary class has an ultra–universal member iff it is (non–empty) and satisfies the *Joint Embedding Property* (JEP), that is, every pair of members of the class is embeddable in a third member. (See [15].) We thus see that the elementary classes of non–trivial, non–trivial Boolean, simple, S.I., F.S.I. and D.I. interior algebras all have ultra–universal members. □

2.3.13 Remark

The space A^D does not determine A uniquely up to isomorphism: Let X be the indiscrete space on a denumerable set X. Let A be the subalgebra of X^A consisting of the finite and co–finite subsets. Let B be the subalgebra of X^A generated by $A \cup \{ R \}$ where $R \subseteq X$ is neither finite nor co–finite. Then $|A^D| = |B^D| = \aleph_0$, $\mathcal{T}(A) = \{ \phi, A^D \}$ and $\mathcal{T}(B) = \{ \phi, B^D \}$. Thus $\mathbf{A}^D \cong \mathbf{B}^D$ even though we do not have $\mathbf{A} \cong \mathbf{B}$. However if $\mathbf{A}$ is a finite interior algebra then $\mathbf{A}^D \cong \mathbf{A}^T$ (see Remark 2.2.16), and so in this case $\mathbf{A}^D$ does determine $\mathbf{A}$ uniquely up to isomorphism. □

We also have:

Theorem 2.3.14

Let A and B be interior algebras. If $\mathbf{A}^D \cong \mathbf{B}^D$ then $\mathbf{A}^O \cong \mathbf{B}^O$.

Proof:

For any interior algebra $\mathbf{C}$ the compact open sets of $\mathbf{C}^D$ form a Heyting algebra $\langle \alpha[C^O], \cap, \cup, \rightarrow, \phi, C^D \rangle$ isomorphic to $\mathbf{C}^O$ via α. Now if $f : \mathbf{A}^D \rightarrow \mathbf{B}^D$ is a homeomorphism then $f^A : \mathbf{B}^{DA} \rightarrow \mathbf{A}^{DA}$ is an isomorphism. Note that relative pseudocomplementation in the Heyting algebra of compact open sets is given by $R \rightarrow S = (R' \cup S)^I$, for all compact open sets R and S, an interior algebra polynomial. Also f^A gives a bijection between the sets of compact open sets of $\mathbf{B}^D$ and $\mathbf{A}^D$. Hence f^A restricts to an isomorphism between the Heyting algebras of compact open sets of $\mathbf{B}^D$ and $\mathbf{A}^D$. But then $\mathbf{A}^O \cong \mathbf{B}^O$. □

2.4 PRE–ORDERS AND FINITELY GENERATED SPACES

A *(normal) modal algebra* [2] is an algebraic structure $\langle \mathbf{L}, ^I, ^C \rangle$ where $\mathbf{L} = \langle L, \cdot, +, ', 0, 1 \rangle$ is a Boolean algebra and ^{I}, ^{C} are unary operations satisfying: $1^I = 1$, $a^C = a'^{I'}$ and $(ab)^I = a^I b^I$ for all $a,b \in L$. (These algebras play the same role for modal logic as Boolean algebras play for classical propositional logic.) Obviously interior algebras form a subvariety of the variety of modal algebras. Given a set with a binary relation $\mathbf{W} = \langle W, \rho \rangle$ (usually called a *frame* in this context), we obtain a modal algebra $\mathbf{W}^+ = \langle \mathcal{P}(W), \cap, \cup, ', \phi, W, ^I, ^C \rangle$ where ^{I} and ^{C} are given by:

$S^I = \{ x \in W : \text{for all } y \in W, x\rho y \text{ implies } y \in S \}$

$S^C = \{ x \in W : \text{there is a } y \in S \text{ with } x\rho y \}$

for all $S \subseteq W$. Notice that $x\rho y$ iff $x \in \{ y \}^C$. Thus one easily checks that ρ is reflexive iff $S \subseteq S^C$ or dually $S^I \subseteq S$, for all $S \subseteq W$; and ρ is transitive iff $S^C = S^{CC}$ or dually $S^I = S^{II}$, for all $S \subseteq W$. Thus $\mathbf{W}^+$ is an interior algebra iff ρ is a *pre–order* (reflexive transitive relation). It was shown in [13] that every interior algebra is embeddable in an interior algebra of the form $\mathbf{W}^+$ where $\mathbf{W}$ is a pre–ordered set. The proof in [13] is concealed in very general lemmas and theorems concerning Boolean algebras with operators. It is instructive to re–examine the situation from a purely interior algebraic point of view.

Firstly we will show that the construction of an interior algebra from a pre–ordered set can be 'factorized' through the construction of an interior algebra from a topological space.

2.4.1 Definition

For each pre–ordered set $\mathbf{W} = \langle W, \ll \rangle$ let $\mathbf{W}^t = \langle W, \mathcal{O}(W) \rangle$ where $\mathcal{O}(W) = \{ S \subseteq W :$ $x \in S$ and $y \in W$ with $x \ll y$ implies $y \in S \}$. For each topological space $\mathbf{X}$ define a relation $\ll$ on X by $x \ll y$ iff y is in every neighbourhood of x or equivalently $x \ll y$ iff $x \in \{ y \}^C$ in $\mathbf{X}$. Put $\mathbf{X}^W = \langle X, \ll \rangle$. □

2.4.2 Proposition

Let $\mathbf{W} = \langle W, \ll \rangle$ be a pre–ordered set.

i) $\mathbf{W}^t$ is a finitely generated topological space.

ii) $\{ \{ y \in W : x \ll y \} : x \in W \}$ is a base for $\mathbf{W}^t$.

iii) The interior operator of $\mathbf{W}^t$ is given by:

$S^I = \{ x \in S :$ for all $y \in W$, $x \ll y$ implies $y \in S \}$

or $S^I = \{ x \in W :$ for all $y \in W$, $x \ll y$ implies $y \in S \}$

iv) The closure operator of $\mathbf{W}^t$ is given by:

$S^C = \{ x \in W :$ there is a $y \in S$ with $x \ll y$ implies $y \in S \}$

v) $\mathbf{W}^{tW} = \mathbf{W}$ □

The proof is tedious but straightforward and is left to the reader. Note that (iii) and (iv) above say that $\mathbf{W}^{tA} = \mathbf{W}^+$.

2.4.3 Proposition

Let $\mathbf{X} = \langle X, \mathcal{T} \rangle$ be a topological space.

i) $\mathbf{X}^W$ is a pre–ordered set.

ii) $\{ \cap N(x) : x \in X \}$ is a base for $\mathbf{X}^{Wt}$ where for all $x \in X$ $N(x)$ is the filter of neighbourhoods of x in $\mathbf{X}$.

iii) $\{ \cap \mathcal{A} : \mathcal{A} \subseteq \mathcal{E} \}$ is a base for $\mathbf{X}^{Wt}$ if $\mathcal{E}$ is a subbase for $\mathbf{X}$.

iv) $\mathbf{X}^{Wt} = \mathbf{X}$ iff $\mathbf{X}$ is finitely generated.

Proof:

(i) is trivial and (iii) follows from (ii) since $\cap N(x) = \cap \{ S \in \mathcal{E} : x \in S \}$ if $\mathcal{E}$ is a subbase for X. (iv) follows easily from (iii) and 2.1.24. We prove (ii): Let $S \in \mathcal{T}$. If $x \in S$ and $y \in X$ with $x \ll y$ then $y \in S$ since S is a neighbourhood of x. Hence $S \in \mathcal{O}(X^W)$. Thus $\mathcal{T} \subseteq \mathcal{O}(X^W)$. Then for all $x \in X$, $\cap N(x) = \cap \{ S \in \mathcal{T} : x \in X \} \in \mathcal{O}(X^W)$. Now consider $R \in \mathcal{O}(X^W)$. Let $x \in R$. If $y \in \cap N(x)$ then $x \ll y$ and so $y \in R$. Thus $\cap N(x) \subseteq R$. Hence $R = \cup \{ \cap N(x) : x \in R \}$ and the result follows. □

2.4.4 Definition and Remark

If $f : X \longrightarrow Y$ is a continuous map let f^W denote the same map as f, considered as a map from X^W to Y^W. Then $f^W : X^W \longrightarrow Y^W$ is clearly a homomorphism of pre–ordered sets. Let **Pro** denote the category of pre–ordered sets and their homomorphisms. Then we have a functor ^{W} : **Top** $\longrightarrow$ **Pro**. ^{W} is in fact a concrete functor over **Set** the category of sets and maps. Clearly ^{W} is faithful. If X is any Fréchet (T_1) space, then $x \ll y$ iff $x = y$ and so we see that ^{W} is not injective on objects. Let X be any space which is not finitely generated. and let f be the identity map on X. Then $f : X^W \longrightarrow X^{Wtw}$ is certainly a homomorphism since $X^{Wtw} = X^W$, but $f : X \longrightarrow X^{Wt}$ is not continuous. Thus ^{W} is not full. □

2.4.5 Definition and Remark

Let $f : \mathbf{W} \longrightarrow \mathbf{Z}$ be a homomorphism of pre–ordered sets. Let f^t denote the same map as f considered as a map from $\mathbf{W}^t$ to $\mathbf{Z}^t$. Let S be open in $\mathbf{Z}^t$. Then if $x \in f^{-1}[S]$ and $x \ll y$ in **W** we have $f(x) \in S$ and $f(x) \ll f(y)$ whence $f(y) \in S$, that is $y \in f^{-1}[S]$. Thus $f^{t\,-1}[S] = f^{-1}[S]$ is open in $\mathbf{W}^t$ and so $f^t : \mathbf{W}^t \longrightarrow \mathbf{Z}^t$ is continuous. Let **ToF** denote the full subcategory of **Top** consisting of the finitely generated spaces. Then we have a functor ^{t} : **Pro** $\longrightarrow$ **ToF** which is concrete over **Set**. □

2.4.6 Proposition

^{t} : **Pro** $\longrightarrow$ **ToF** is a concrete isomorphism over **Set** with inverse ^{W} : **ToF** $\longrightarrow$ **Pro**. □

The above follows from Proposition 2.4.2.

2.4.7 Remark

Let **OIn**$^+$ denote the full subcategory of **CIn**$^+$ consisting of the atomic operator complete interior algebras. Then by Corollary 2.1.27 we have a co–equivalence system $^{t\Lambda}$: **Pro** $\longrightarrow$ **OIn**$^+$ and ^{TW} : **OIn**$^+$ $\longrightarrow$ **Pro**. □

Note that X is not equal to X^{Wt} in general. However we have:

2.4.8 Theorem

ToF is a bico–reflective subcategory of **Top** with bico–reflector ^{Wt} : **Top** $\longrightarrow$ **ToF**.

Proof:

Let X be a topological space. Any open set of X is open in X^{Wt} and so the identity map

$f : X^{Wt} \longrightarrow X$ is continuous. Let Y be finitely generated and suppose $g : Y \longrightarrow X$ is continuous. We have to show that $g : Y \longrightarrow X^{Wt}$ is continuous. Let $\cap \mathcal{A}$ be a base element of X^{Wt} where is a set of open sets in X. Then $f^{-1}[\cap \mathcal{A}] = \cap \{ f^{-1}[S] : S \in \mathcal{A} \}$. Now for all $S \in \mathcal{A}$, $f^{-1}[S]$ is open in Y and so, since Y is finitely generated $f^{-1}[\cap \mathcal{A}]$ is open in Y and the result follows. □

2.4.9 Notation and Remark

It follows that $\mathbf{OIn}^{+}$ is a bi–reflective subcategory of $\mathbf{CIn}^{+}$ with bi–reflector given by the map $\mathbf{A} \longmapsto \delta^{-1}((\mathbf{A})^{\mathrm{TWtA}})$. (See Lemma 2.1.5.). Let $^{b} : \mathbf{CIn}^{+} \longrightarrow \mathbf{OIn}^{+}$ denote this bi–reflector. Untangling the series of operations which give ^{b} we see that for any complete atomic interior algebra A, A^{b} is the interior algebra with the same underlying Boolean algebra as A but with $A^{bo} = \{ \Sigma R : R \subseteq \{ \sqcap S : S \subseteq A^{o} \} \}$. □

2.4.10 Lemma

Let A be a complete atomic interior algebra. Then for any atom a in A, the closure of a in A^{b} is the closure of a in A. □

2.4.11 Theorem

Let A be a complete atomic interior algebra.

i) A^{b} is S.I. iff A is S.I. in which case A and A^{b} have the same monolith.

ii) A^{b} is F.S.I. iff A is F.S.I.

iii) If A^{b} is D.I. then A is D.I.

Proof:

(i): Let A^{b} be S.I. Let m be the monolith of A^{b}. Then there is a $\mathcal{B} \subseteq \mathcal{P}(A^{o})$ with $m = \Sigma \{ \sqcap S : S \in \mathcal{B} \}$. Consider $S \in \mathcal{B}$. Then $S \neq \phi$ and $1 \notin S$ since $\sqcap S \leq m < 1$. Hence there is an $a(S) \in S$ with $a(S) \leq m$. Then $m \leq \Sigma \{ a(S) : S \in \mathcal{B} \} \leq \Sigma (A^{o} - \{ 1 \})$. But $A^{o} \subseteq A^{bo}$ and so $\Sigma (A^{o} - \{ 1 \}) \leq \Sigma (A^{bo} - \{ 1 \}) = m$. Thus $m = \Sigma (A^{o} - \{ 1 \})$ and so it follows that A is S.I. with monolith m. Conversely suppose that A is S.I. with monolith m. Let $b \in A^{bo} - \{ 1 \}$. Then there is a $\mathcal{B} \subseteq \mathcal{P}(A^{o})$ with $b = \Sigma \{ \sqcap S : S \in \mathcal{B} \}$. Consider $S \in \mathcal{B}$, $\sqcap S \neq \phi$ and $1 \notin S$ since $\sqcap S \leq b < 1$. Hence there is an $a(S) \in S$ with $a(S) < m$. Then $b \leq \Sigma \{ a(S) : S \in \mathcal{B} \} \leq m$. It follows that A^{b} is S.I. with monolith m. (ii): Suppose A^{b} is F.S.I. Then 1 is join irreducible in A^{bo} hence in $A^{o} \leq A^{bo}$ and so A is also F.S.I. Conversely suppose A is F.S.I. Let $a,b \in A^{\square}$ with $ab = 0$. Suppose $a,b > 0$. Then there are atoms $c \leq a$ and $d \leq b$. Then $c^{c} \leq a$ and $d^{c} \leq b$ in A^{b} and hence in A, by Lemma 2.4.10. Then $c^{c}d^{c} = 0$ in A, a contradiction by the dual of Theorem 1.3.18 (ii). Thus $a = 0$ or $b = 0$ whence again by Theorem 1.3.18 (ii) and duality we have that A^{b} is F.S.I. (iii): Let

A^b be D.I. Then $A^{b\diamond} = \{ 0, 1 \}$ and so, since $A^\diamond \subseteq A^{b\diamond}$, $A^\diamond = \{ 0, 1 \}$ and the result follows. □

Note that the converse of (iii) above does not hold: Let **X** be the co–finite space on a set X. Let $\mathbf{A} = \dot{\mathbf{X}}^A$. Then **A** is D.I. but A^b is Boolean hence not D.I. Also note that if **A** is an atomic operator complete interior algebra then $\mathbf{A}^b = \mathbf{A}$ by Proposition 2.1.29.

What sort of pre–order homomorphisms correspond to continuous open maps and hence to interior algebra homomorphisms?

2.4.12 Definition

A homomorphism $f : \mathbf{W} \to \mathbf{Z}$ between pre–ordered sets is said to be **regular** iff for all $x \in W$ and $z \in Z$, $f(x) \ll z$ implies there is a $y \in W$ with $x \ll y$ and $z = f(y)$. □

2.4.13 Theorem

i) Let **X** be a finitely generated space and let $f : \mathbf{X} \to \mathbf{Y}$ be a continuous open map. Then $f^W : \mathbf{X}^W \to \mathbf{Y}^W$ is a regular homomorphism.

ii) Let $f : \mathbf{W} \to \mathbf{Z}$ be a regular pre–order homomorphism. Then $f^t : \mathbf{W}^t \to \mathbf{Z}^t$ is a continuous open map.

Proof:

(i): Let $x \in X$ and $z \in Y$ with $f(x) \ll z$ in $\mathbf{Y}^W$. Put $S = \{ y \in X : x \ll y \text{ in } \mathbf{X}^W \}$ Then $S = \cap N(x)$ and so S is open in **X**. Hence f[S] is open in **Y**. Now $f(x) \in f[S]$ and so $z \in f[S]$ and the result follows. (ii): Let S be open in $\mathbf{W}^t$. Let $z \in f[S]$ and let $z \ll v$ in $\mathbf{Z}^t$. There is an $x \in S$ with $f(x) = z$. Hence there is a $y \in W$ with $x \ll y$ and $f(y) = v$. Then $y \in S$ and so $v \in f[S]$. Thus f[S] is open and so f is an open map. □

Let **TcF** be the full subcategory of Tco consisting of the finitely generated spaces and let **RPro** be the category of pre–ordered sets and regular homomorphisms. Then we see:

2.4.14 Corollary

^{t} : **RPro** $\to$ **TcF** is a concrete isomorphism over Set with inverse ^{W} : **TcF** $\to$ **RPro**. □

2.4.15 Remark

If **OIn** denotes the full subcategory of **CIn** consisting of the atomic operator complete interior algebras. Then by Corollary 2.1.27 we have a co–equivalence system: ^{tA} : **RPro** $\to$ **OIn** and ^{TW} : **OIn** $\to$ **RPro**. □

We will not pursue a detailed discussion of special morphisms and consructions in **Pro** and **RPro** since these do not lead to interesting results about interior algebras. We do, however, have interesting characterizations of the irreducibility and related properties in atomic operator complete interior algebras, in terms of pre–orders:

2.4.16 Theorem

Let **W** be a pre–ordered set.

i) $\mathbf{W}^+$ is Boolean iff $x \ll y$ in **W** implies $x = y$.

ii) $\mathbf{W}^+$ is semi–simple iff $x \ll y$ in **W** implies $y \ll x$.

iii) $\mathbf{W}^+$ is S.I. iff there is an $x \in W$ with $x \ll y$ for all $y \in W$.

If **W** is non–empty:

iv) $\mathbf{W}^+$ is simple iff $x \ll y$ for all $x,y \in W$.

v) $\mathbf{W}^+$ is F.S.I. iff for all $x,y \in W$ there is a $z \in W$ with $z \ll x$ and $z \ll y$.

vi) $\mathbf{W}^+$ is D.I. iff for all $x,y \in W$ there is a finite sequence $x = z_0,\dots,z_n = y$ in W such that $z_{i-1} \ll z_i$ or $z_{i-1} \ll z_i$ for all $i = 1,\dots,n$.

Proof:

(i): Suppose $\mathbf{W}^+$ be Boolean. Let $x \ll y$. $\{ x \}$ is closed and so $y \in \{ x \}$, that is $x = y$. Conversely suppose $x \ll y$ implies $x = y$. Let $S \subseteq W$. If $x \in S$ and $x \ll y$ then $y = x \in S$ and so S is open. (ii): Suppose $\mathbf{W}^+$ is semi–simple. Let $x \ll y$. Put $S = \{ z \in W : z \ll x \}$. Then S is closed, hence open and so $y \in S$, that is $y \ll x$. Conversely suppose that $x \ll y$ implies $y \ll x$. Let S be closed. Let $x \in S$ and $x \ll y$. Then $y \ll x$ and so $y \in S$. Hence S is open. (See Theorem 1.3.21.) (iii): Suppose $\mathbf{W}^+$ S.I. Let R be the complement of the monolith of $\mathbf{W}^+$. Let $x \in R$. Consider $y \in W$. Put $S = \{ z \in W : z \ll y \}$. S is non–empty and is closed whence $R \subseteq S$. Thus $x \in S$, that is $x \ll y$. Conversely suppose there is an $x \in W$ with $x \ll y$ for all $y \in W$. Put $R = \{ z \in W : z \ll x \}$. Then R is non–empty and closed. Let $S \subseteq W$ be non–empty and closed. Let $y \in S$. Then $x \ll y$ whence $z \ll y$ for all $z \in R$. Thus $R \subseteq S$ and so R is the smallest non–empty closed subset of $\mathbf{W}^+$. Thus $\mathbf{W}^+$ is S.I. with monolith R'. Now let **W** be non–empty. (iv): Suppose $\mathbf{W}^+$ is simple. Let $x,y \in W$. Put $S = \{ z \in W : x \ll z \}$. Then S is open and non–empty whence $S = W$. In particular $y \in S$, that is $x \ll y$. Conversely suppose $x \ll y$ for all $x,y \in W$. Let $S \subseteq W$ be non–empty and open. There is an $x \in S$. For all $y \in W$ $x \ll y$ whence $y \in S$. Thus $S = W$ and so $\mathbf{W}^+$ is simple. (v): Suppose $\mathbf{W}^+$ is F.S.I. Let $x,y \in W$. Put $S = \{ z \in W : z \ll x \}$ and $R = \{ z \in W : z \ll y \}$. Then S and R are non–empty and closed and so $S \cap R \neq \phi$ (using duality and Theorem 1.3.18). Let $z \in S \cap R$. Then $z \ll x$ and $z \ll y$. Conversely suppose for all $x,y \in W$ there is a $z \in W$ with $z \ll x$ and $z \ll y$. Let $S,R \subseteq W$ be non–empty and closed. Then there are $x \in S$ and $y \in R$. Hence there is a $z \in W$ with $z \ll x$ and $z \ll y$. Then $z \in S \cap R$ whence $S \cap R$ is non–empty. It follows that W^+ is F.S.I. (vi): Suppose W^+ is D.I. Let $x,y \in W$. Put $S = \{ t \in W$: there is a finite sequence $x = z_0,\ldots,z_n = y$ with $z_{i-1} \ll z_i$ or $z_i \ll z_{i-1}$ for all $i = 1,\ldots,n$ $\}$. S is non–empty since $x \in S$. If $t \in S$ and $u \ll t$ or $t \ll u$ then obviously $u \in W$. Thus S is clopen. Thus $S = W$ whence $y \in S$ and so there is a sequence $x = z_0,\ldots,z_n = y$ with $z_{i-1} \ll z_i$ or $z_i \ll z_{i-1}$ for all $i = 1,\ldots n$. Conversely suppose that for all $x,y \in W$ there is a finite sequence $x = z_0,\ldots,z_n = y$ with $z_{i-1} \ll z_i$ or $z_i \ll z_{i-1}$ for all $i = 1,\ldots,n$. Consider a clopen $S \subseteq W$. Then if there is a sequence $z_0,\ldots,z_n$ with $z_{i-1} \ll z_i$ or $z_i \ll z_{i-1}$ for all $i = 1,\ldots,n$ then $z_0 \in S$ implies $z_n \in S$ by induction on n. Let $S \subseteq W$ be non–empty and clopen. Then there is an $x \in S$. Let $y \in W$. There is a sequence $x = z_0,\ldots,z_n = y$ with $z_{i-1} \ll z_i$ or $z_i \ll z_{i-1}$ for all $i = 1,\ldots,n$ and so $y \in S$. Thus $S = W$ and so W^+ is D.I. □

The above results as well as Theorem 2.4.11 have been stated in terms of interior algebras and irreducibility properties. We point out that they may be restated in terms of finitely generated spaces and connectedness properties.

2.4.17 Corollary

Let A be a non–trivial interior algebra and let $W = A^{DW}$.

i) A is simple iff $x \ll y$ for all $x,y \in W$.

ii) A is F.S.I. iff for all $x,y \in W$ there is a $z \in W$ with $z \ll x$ and $z \ll y$.

Proof:

Put $X = W^t$. W is the finitely generated bico–reflection of A^D. (i): A is simple iff A^D is indiscrete, iff X is indiscrete iff $x \ll y$ for all $x,y \in W$. (ii): A is F.S.I. iff A^D is ultra–connected iff X is ultra–connected iff for all $x,y \in W$ there is a $z \in W$ with $z \ll x$ and $z \ll y$. (Where we have used Theorems 2.1.15, 2.3.9, 2.4.11 and 2.4.16.) □

It was shown in [13] that every interior algebra can be represented as a field of subsets of a pre–ordered set, that is, a subalgebra of W^+ where W is a pre–ordered set. This method of representing an interior algebra can be 'factorized' through the representation of an interior algebra as a Stone field. We now formalize this:

2.4.18 Definition and Remark

We will consider a field of subsets of a pre–ordered set to be a triple $\langle W, \ll, \mathcal{T} \rangle$ where $\langle W, \ll \rangle$ is the pre–ordered set in which $\mathcal{T}$ is a field. We will use the term **pre–order field** to mean a triple of the above kind. By a **pre–order field map** we mean a map $f : \langle W, \ll, \mathcal{T} \rangle \longrightarrow \langle Z, \ll, \mathcal{S} \rangle$ between two pre–order fields such that $f : \langle W, \ll \rangle \longrightarrow \langle Z, \ll \rangle$ is a homomorphism and $f^{-1}[Z] \in \mathcal{T}$ for all $Z \in \mathcal{S}$. As with topological fields of sets we may speak of separative and compact pre–order fields. We call a pre–order field **algebraic** iff there is an $\mathcal{A} \subseteq \mathcal{R}$ such that for all x,y $\in$ W, x $\ll$ y iff for all S $\in \mathcal{A}$, x $\in$ S implies y $\in$ S. Call a pre–order field **canonical** iff it is separative , compact and algebraic. Let **Can** denote the category of canonical pre–order fields and pre–order field maps. We will show that **Can** and **Sfld** are isomorphic. □

2.4.19 Lemma

Let $\langle X, \mathcal{T}, \mathcal{R} \rangle$ be a Stone field and let $\langle X, \ll \rangle = \langle X, \mathcal{T} \rangle^{W}$.

i) For each S $\in \mathcal{R}$ the closure of S in $\langle X, \mathcal{T} \rangle$ is the closure of S in $\langle X, \ll \rangle^{+}$ (and hence for each S $\in \mathcal{R}$ the interior of S in $\langle X, \mathcal{T} \rangle$ is the interior of S in $\langle X, \ll \rangle^{+}$).

ii) $\langle X, \ll, \mathcal{R} \rangle$ is a canonical pre–order field.

Proof:

(i): Let S $\in \mathcal{R}$. Let Z be the closure of S in $\langle X, \mathcal{T} \rangle$ and let V be the closure of S in $\langle X, \ll \rangle^{+}$. Then V $\subseteq$ Z. Let x $\in$ Z. Put $\mathcal{E} = \{ S \cap U : U \in N(x) \text{ in } \langle X, \mathcal{T} \rangle \}$. Then $\mathcal{E} \neq \phi$ since X $\in \mathcal{E}$ and $\mathcal{E}$ is closed under finite intersections since N(x) is. Hence there is a filter $\mathcal{F}$ over X with $\mathcal{E} \subseteq \mathcal{F}$. By compactness $\cap (\mathcal{R} \cap \mathcal{F}) \neq \phi$. Let y $\in \cap (\mathcal{R} \cap \mathcal{F})$. Consider W $\in$ N(x). By Proposition 2.2.5 there is a Y $\in \mathcal{R} \cap \mathcal{T}$ with x $\in$ Y $\subseteq$ W. Then Y $\in \mathcal{R} \cap \mathcal{F}$ and so y $\in$ Y $\subseteq$ W. Hence x $\ll$ y in $\langle X, \ll \rangle$. Note that $\cap (\mathcal{R} \cap \mathcal{F}) \subseteq$ S and so y $\in$ S. Hence x $\in$ V. Thus V $\subseteq$ Z whence V = Z as required. (ii): By (i) $\langle X, \ll, \mathcal{R} \rangle$ is a pre–order field and it is separative and compact. Let $\mathcal{A} = \mathcal{R} \cap \mathcal{T}$. Then x $\ll$ y in $\langle X, \ll \rangle$ iff for all S $\in \mathcal{A}$, x $\in$ S implies y $\in$ S, since by Proposition 2.2.5 $\mathcal{A}$ is a base for $\mathcal{T}$. Thus $\langle X, \ll, \mathcal{R} \rangle$ is also algebraic. □

2.4.20 Definition and Remark

Given a Stone field X = $\langle X, \mathcal{T}, \mathcal{R} \rangle$ let X^{M} denote the canonical pre–order field $\langle X, \ll, \mathcal{R} \rangle$ where $\langle X, \ll \rangle = \langle X, \mathcal{T} \rangle^{W}$. Given a field map f : X ⟶ Y between two Stone fields, let f^{M} denote the same map as f considered as a map from X^{M} to Y. Then f^{M} is a pre–order field map. We thus have a functor ^{M} : **Sfld** ⟶ **Can** which is concrete over **Fld**. (See Definition and Remark 2.2.15.) □

2.4.21 Lemma

Let $\langle X, \ll, \mathcal{R} \rangle$ be a canonical pre–order field. Let $\mathcal{A}_0$ be the union of all $\mathcal{A} \subseteq \mathcal{R}$ such that x $\ll$ y in $\langle X, \ll \rangle$ iff for all S $\in \mathcal{A}$, x $\in$ S implies y $\in$ S. Let $\mathcal{T}$ be the topology generated by $\mathcal{A}_0$. Then $\langle X, \mathcal{T}, \mathcal{R} \rangle$ is a Stone field with $\langle X, \mathcal{T}, \mathcal{R} \rangle^{M} = \langle X, \ll, \mathcal{R} \rangle$.

Proof:

Let $\mathcal{B} = \{ \cap C : C$ is a finite a finite subset of $\mathcal{A}_0 \}$. Then $\mathcal{B} \subseteq \mathcal{R}$ and $\mathcal{B}$ is a base for $\mathcal{T}$. Thus $\langle X, \mathcal{T}, \mathcal{R} \rangle$ is certainly a Stone field. Let x $\ll$ y in $\langle X, \ll \rangle$. Let S $\in$ N(x) in $\langle X, \mathcal{T} \rangle$. Then there is a finite $\mathcal{C} \subseteq \mathcal{A}_0$ with $x \in \cap\mathcal{C} \subseteq S$. Then $y \in \cap\mathcal{C} \subseteq S$. Conversely suppose that for all $S \in N(x)$ in $\langle X, \mathcal{T} \rangle$, $y \in S$. Then in particular for all $S \in \mathcal{A}_0$ with $x \in S$, $y \in S$ and so $x \ll y$. Thus $\langle X, \mathcal{T}, \mathcal{R} \rangle^M = \langle X, \ll, \mathcal{R} \rangle$. □

2.4.22 Definition and Remark

Given a canonical pre–order field $\mathbf{X} = \langle X, \ll, \mathcal{R} \rangle$ let $\mathbf{X}^N$ denote the Stone field given by Lemma 2.4.21. If $f : \mathbf{X} \longrightarrow \mathbf{Y}$ is a pre–order field map between canonical pre–order fields let f^N be the same map as f considered as a map from $\mathbf{X}^N$ to $\mathbf{Y}^N$. Then by 2.4.5 we see that f^N is a field map of Stone fields. ^{N} : **Can** $\longrightarrow$ **Sfld** is thus a concrete functor over **Fld**. □

2.4.23 Theorem

^{M} : **Sfld** $\longrightarrow$ **Can** is a concrete isomorphism over **Fld** with inverse ^{N} : **Can** $\longrightarrow$ **Sfld**. □

What sort of pre–order field maps correspond to weakly open field maps and hence to interior algebra homomorphisms?

2.4.24 Definition

A pre–order field map $f : \langle X, \ll, \mathcal{R} \rangle \longrightarrow \langle Y, \ll, \mathcal{W} \rangle$ is called **regular** iff the underlying pre–order homomorphism $f : \langle X, \ll \rangle \longrightarrow \langle Y, \ll \rangle$ is regular. Let **Cnr** denote the category of canonical pre–order fields and regular pre–order field maps. □

2.4.25 Theorem

Let $\mathbf{X} = \langle X, \mathcal{T}, \mathcal{R} \rangle$ and $\mathbf{Y} = \langle Y, \mathcal{S}, \mathcal{W} \rangle$ be Stone fields and let $f : \mathbf{X} \longrightarrow \mathbf{Y}$ be a field map. The following are equivalent:

i) $f : \mathbf{X} \longrightarrow \mathbf{Y}$ is weakly open.

ii) $f^M : \mathbf{X}^M \longrightarrow \mathbf{Y}^M$ is regular.

iii) $f : \langle X, \mathcal{T} \rangle^{wt} \longrightarrow \langle Y, \mathcal{S} \rangle^{wt}$ is an open map.

Proof:

(i) $\Longrightarrow$ (ii): Assume (i). Let $x \in X$ and let $y \in Y$ with $f(x) \ll y$. Then for all $V \in \mathcal{S}$, $f(x) \in V$ implies $y \in V$. Put $\mathcal{C} = f^S[\beta(y)]$. Then $\mathcal{C}$ is clearly closed under finite intersections. Suppose $\phi \in \mathcal{C}$. Then there a $U \in \beta(y)$ with $f^S(U) = \phi$. Using Theorem 2.2.18 and (i) we see that $f^S(U^C) = \phi$. Now $y \in U \subseteq U^C$ and $U^{C\prime} \in \mathcal{S}$. It follows that $f(x) \notin U^{C\prime}$ and so $f(x) \in U^C$. Thus $U^C \in \beta f(x) = f^{S-1}[\beta(x)]$ whence $\phi = f^S(U^C) \in \beta(x)$, a contradiction. Thus $\phi \notin \mathcal{C}$ and so putting $\mathcal{E} = \{ S \in \mathcal{R} : f^S(V) \subseteq S$ for some $V \in \beta(y) \}$ gives a proper filter in $\mathbf{X}^S$. Put $\mathcal{D} = \{ S \in \mathcal{R} : x \notin S^C \}$. Then $\mathcal{D}$ is an ideal in $\mathbf{X}^S$. Suppose that there is an $S \in \mathcal{D} \cap \mathcal{E}$. There is a $V \in \beta(y)$ with $f^S(V) \subseteq S$. Hence $f^S(V) \in \mathcal{D}$ and so $x \notin f^S(V)^C = f^S(V^C)$ (using Theorem

2.2.18). Hence $x \in f^S(V^C)' = f^S(V^{C}{}') = f^{-1}[V^{C}{}']$ and so $f(x) \in V^{C}{}' \in \mathcal{S}$. Thus $y \in V^{C}{}' \subseteq V'$, a contradiction since $V \in \beta(y)$. Thus $\mathcal{D} \cap \mathcal{E} = \phi$ and so there is an ultrafilter $\mathcal{F}$ in $\mathbf{X}^S$ with $\mathcal{E} \subseteq \mathcal{F}$ and $\mathcal{D} \cap \mathcal{F} = \phi$. But then $\mathcal{F} = \beta(z)$ for some $z \in X$. Now $\mathcal{C} \subseteq \beta(z)$ and so $\beta(y) \subseteq f^{S-1}[\beta(z)] = \beta f(z)$ whence, since these are both ultrafilters, $\beta(y) = \beta f(z)$. Thus $y = f(z)$. Suppose $P \in \mathcal{T} \cap \mathcal{R}$ with $x \in P$. Then $x \notin P' = P'^C$ and $P' \in \mathcal{R}$. Thus $P' \in \mathcal{D}$ and so $P' \notin \mathcal{F}$, that is $z \notin P'$. But then $z \in P$. Thus $x \ll z$. It follows that $f : \mathbf{X}^M \longrightarrow \mathbf{Y}^M$ is regular. (ii) $\Longrightarrow$ (i): Assume (ii): Let $x \in X$ and let $V \in \mathcal{W}$ with $f^{-1}[V]$ a neighbourhood of x in $\langle X, \mathcal{T} \rangle$, that is $x \in f^{-1}[V]^I$. Note that $f^{-1}[V] \in \mathcal{R}$. Let $y \in Y$ with $y \gg f(x)$. There is a $z \gg x$ with $f(z) = y$. Thus $z \in f^{-1}[V]$ and so $y \in V$. Hence $f(x) \in V^I$, that is V is a neighbourhood of f(x) in $\langle Y, \mathcal{S} \rangle$. Thus $f : \mathbf{X} \longrightarrow \mathbf{Y}$ is weakly open. (ii) $\Leftrightarrow$ (iii) by Theorem 2.4.13 since $\langle X, \mathcal{T} \rangle^{wt} = \langle X, \ll \rangle^t$ and $\langle Y, \mathcal{S} \rangle^{wt} = \langle Y, \ll \rangle^t$. □

2.4.26 Corollary

^{M} : **Sfwo** $\longrightarrow$ **Cnr** is a concrete isomorphism over **Fld** with inverse ^{N} : **Cnr** $\longrightarrow$ **Sfwo**. □

2.4.27 Remark

Note that by Theorem 2.4.23 and Theorem 2.2.14 ^{FM} : **Int**$^+$ $\longrightarrow$ **Can** and ^{NS} : **Can** $\longrightarrow$ **Int**$^+$ form a co–equivalence system and by Corollary 2.4.26 and Corollary 2.2.19 ^{FM} : **Int** $\longrightarrow$ **Cnr** and ^{NS} : **Cnr** $\longrightarrow$ **Int** form a co–equivalence system. We thus have a category–theoretic formalization of the representation of interior algebras as fields of subsets of pre–ordered sets. We can obtain other duality results from the above: Via the functor that replaces a canonical pre–order field $\langle X, \ll, \mathcal{R} \rangle$ with the structure $\langle X, \mathcal{T}, \mathcal{R} \rangle$ where $\langle X, \mathcal{T} \rangle = \langle X, \ll \rangle^t$, we obtain a duality between interior algebras and another class of separative compact topological fields of sets besides the Stone fields. Via the the functor which replaces $\langle X, \ll, \mathcal{R} \rangle$ with the structure $\langle X, \mathcal{B}, \ll \rangle$ where $\mathcal{B}$ is the topology generated by $\mathcal{R}$, we obtain a duality between interior algebras and a certain class of pre–ordered Boolean spaces. If we restrict ourselves to working with homomorphisms and not arbitrary topomorphisms it will be noticed that this latter duality is in fact just a special case of a more general Stone duality for normal Boolean algebras with operators [13]. We leave it to the reader to work out descriptions of the appropriate morphisms and other category–theoretic details. The work of this section has shown that representations and duality theorems for interior algebras involving pre–orders may be viewed as topological duality in disguise. □

Given an interior algebra **A** how can the pre–order of $\mathbf{A}^{FM}$ be described directly?

2.4.28 Theorem

If **A** is an interior algebra and F and G are ultrafilters in **A**, then the following are equivalent:

i) $F \ll G$ in A^{FM}

ii) $F \cap A^{O} \subseteq G$

iii) $G \cap A^{\square} \subseteq F$

Proof:

(i) $\Leftrightarrow$ (ii): Since $\alpha[A^{O}]$ is a base for $\mathcal{T}(A)$ we see $F \ll G$ iff for all $a \in A^{O}$, $F \in \alpha(a)$ implies $G \in \alpha(a)$; iff $a \in F \cap A^{O}$ implies $a \in G$; iff $F \cap A^{O} \subseteq G$. (ii) $\Longrightarrow$ (iii): Assume (ii). Suppose that there is an $a \in G \cap A^{\square}$ with $a \notin F$. Then $a' \in F \cap A^{O} \subseteq G$, a contradiction. Thus (iii) holds. (iii) $\Longrightarrow$ (ii): Assume (iii). Suppose that there is an $a \in F \cap A^{O}$ with $a \notin G$. Then $a' \in G \cap A^{\square} \subseteq F$, a contradiction. Thus (ii) holds. □

CHAPTER 3

NEIGHBOURHOODS, CONVERGENCE AND ACCUMULATION

3.1 NEIGHBOURHOOD FUNCTIONS

3.1.1. Definition

Let **B** be a Boolean algebra. By a **neighbourhood function** on **B** we mean a map N from B to the set of filters in **B** satisfying:

i) For all $a \in B$, $\max \{ b \in B : a \in N(b) \}$ exists.

ii) For all $a,b \in B$, $a \in N(b)$ iff there is a $c \in B$ with $b \leq c \leq a$ and $c \in N(c)$.

For each $a \in B$, N(a) is called a **neighbourhood filter at** a. If $b \in N(a)$, b is called a **neighbourhood of** a and is said to **surround** a. □

3.1.2 Proposition

Let N be a neighbourhood function on a Boolean algebra **B**.

i) For all $a \leq b$ in **B**, $N(b) \subseteq N(a)$.

ii) For all $a \in B$, $N(a) \subseteq [a)$ and $N(a) = [a)$ iff $a \in N(a)$. □

3.1.3 Definition

Let **B** be a Boolean algebra. If N is a neighbourhood function on **B** put $\mathrm{G}(N) = \{ b \in B : b \in N(b) \}$ or equivalently $\mathrm{G}(N) = \{ b \in B : N(b) = [b) \}$. If G is a generalized topology in B define a map $\mathrm{N}(G)$ on B by $\mathrm{N}(G)(a) = \{ b \in B : a \leq c \leq b$ for some $c \in G \}$. □

3.1.4 Theorem

Let **B** be a Boolean algebra. Let N and G be a neighbourhood function on **B** and a generalized topology in **B** respectively. Then:

i) $\mathrm{G}(N)$ is a generalized topology in **B**.

ii) $\mathrm{N}(G)$ is a neighbourhood function on **B**.

iii) $\mathrm{NG}(N) = N$

iv) $\mathrm{GN}(G) = G$

Proof:

(i): Since N(1) is a filter $1 \in N(1)$ and so $1 \in \mathrm{G}(N)$. Let $a,b \in \mathrm{G}(N)$. By Proposition 3.1.2 $N(a),N(b) \subseteq N(ab)$ and so $a,b \in N(ab)$ whence $ab \in N(ab)$. Thus $ab \in \mathrm{G}(N)$. Hence $\mathrm{G}(N)$ is closed under binary meets. Let $a \in B$. Put $c = \max \{ b \in A : a \in N(b) \}$. There is a $d \in B$

with $c \leq d \leq a$ and $d \in N(d)$. Then $a \in N(d)$ and so $d \leq c$ whence $c = d$. Thus $c \in N(c)$ and $c \in \mathrm{G}(N)$. Now consider $e \in \mathrm{G}(N)$ with $e \leq a$. Then $a \in N(e)$ and so $e \leq c$. Thus $c = \max\{ e \in \mathrm{G}(N) : e \leq a \}$. Hence $\mathrm{G}(N)$ is a generalized topology in $\mathbf{B}$ by Proposition 1.4.4. (ii): Let $a \in B$. Let $b \in \mathrm{N}(G)(a)$ and $b \leq c$. Then there is a $d \in G$ with $a \leq d \leq b$ whence $a \leq d \leq c$ and so $c \in \mathrm{N}(G)(a)$. Now suppose $b,c \in \mathrm{N}(G)(a)$. There are $d,e \in G$ with $a \leq d \leq b$ and $a \leq e \leq c$. Then $de \in G$ and $a \leq de \leq bc$ whence $bc \in \mathrm{N}(G)(a)$. Thus $\mathrm{N}(G)(a)$ is a filter as required. Also $b \in \mathrm{N}(G)(a)$ iff there is a $d \in G$ with $a \leq d \leq b$, iff there is a $d \in B$ with $d \in \mathrm{N}(G)(d)$ and $a \leq d \leq b$. Lastly if $a \in B$ put $c = \max\{ d \in G : d \leq a \}$. Then $a \in \mathrm{N}(G)(c)$. Let $b \in B$ with $a \in \mathrm{N}(G)(b)$. There is a $d \in G$ with $b \leq d \leq a$. Then $d \leq c$ whence $b \leq c$. Thus $c = \max\{ b \in B : a \in \mathrm{N}(G)(b) \}$. Thus $\mathrm{N}(G)$ is a neighbourhood function on $\mathbf{B}$. (iii): Consider $a \in B$. Then for $b \in B$, $b \in \mathrm{NG}(N)(a)$ iff there is a $d \in \mathrm{G}(N)$ with $a \leq d \leq b$, iff there is a $d \in B$ with $d \in N(d)$ and $a \leq d \leq b$, iff $b \in N(a)$. Thus $\mathrm{NG}(N) = N$. (iv): For all $a \in B$, $a \in \mathrm{GN}(G)$ iff $a \in \mathrm{N}(G)(a)$, iff there is a $d \in G$ with $a \leq d \leq a$, iff $a \in G$. Thus $\mathrm{GN}(G) = G$. □

3.1.5 Remark

From the above theorem we see that pairs $\langle \mathbf{B}, N \rangle$, where $\mathbf{B}$ is a Boolean algebra and N is a neighbourhood function on B, are essentially the same as things as generalized topological spaces and hence essentially the same things as interior algebras. Given an interior algebra A, the corresponding neighbourhood function $\mathrm{N}(A^O)$ on $\mathbf{A}^u$ is given by $\mathrm{N}(A^O)(a) = \{ b \in A : a \leq b^I \}$, for all $a \in A$. Conversely if N is a neighbourhood function on $\mathbf{B}$, the interior operator of Alg $\langle \mathbf{B}, \mathrm{G}(N) \rangle$ is given by $a^I = \max\{ b \in B : a \in N(b) \}$. □

3.1.6 Notation and Remark

From now on, when working with an an interior algebra $\mathbf{A}$, we will denote the neighbourhood function $\mathrm{N}(A^O)$ simply by N. For readers interested in neighbourhood lattices (see [17]) we mention the following: If $\mathbf{A}$ is an interior algebra then it is easily seen that the structure $\langle A, \leq, N \rangle$ is a Boolean neighbourhood lattice (that is a neighbourhood lattice whose underlying lattice is Boolean) and moreover any Boolean neighbourhood lattice is of this form. Thus neighbourhood lattices are in fact a generalization of interior algebras and not just of topological spaces. However the concept of a neighbourhood lattice is too general for our purposes and we will not discuss them further. □

3.1.7 Proposition

Let $\mathbf{A}$ be an interior algebra. For all $a \in A$, N(a) is an open filter. □

However not every open filter is a neighbourhood filter: Let **X** be a discrete space on an infinite set X and let $\mathbf{A} = \mathbf{X}^A$. Let $\mathcal{F}$ be the filter of co–finite subsets of X. Then $\mathcal{F}$ is an open filter in **A** but it is not the neighbourhood filter of any element of **A**.

3.2 CONVERGENCE AND ACCUMULATION OF SUBSETS

3.2.1 Definition and Notation

Let **A** be an interior algebra. Let $R \subseteq A$ and $a \in A$. R is said to **converge** to a and a is called a **limit** of R, denoted by $R \longrightarrow a$, iff for all $b \in N(a)$ there is an $r \in R$ with $r \leq b$. For each $R \subseteq A$ let $\lim R = \{ a \in A : R \longrightarrow a \}$. □

The above definition, in the case where a is an atom, is a generalization of the concepts of limits and convergence in topology: If X is a topological space, $x \in X$ and $\mathcal{A} \subseteq \mathcal{P}(X)$ then $\mathcal{A} \longrightarrow x$ in **X** in the usual sense, iff $\mathcal{A} \longrightarrow \{ x \}$ in the interior algebra $\mathbf{X}^A$. Since we did not restrict the definition to atoms we have a broader generalization. We summarize the basic properties of convergence.

3.2.2 Remark

Let **A** be an interior algebra, let $R \subseteq A$ and let $a \in A$. Note that the following are equivalent:

i) $R \longrightarrow a$

ii) For all open $b \geq a$ there is an $r \in R$ with $r \leq b$.

iii) For all $b \in N(a)$ there is an $r \in R$ with $r \leq b^I$. □

We remind the reader of some order–theoretic concepts which we will be using. Recall that a *cone* in a poset **P**, is a subset $S \subseteq P$ such that for all $a,b \in P$, if $a \in S$ and $a \leq b$ then $b \in S$. A non–empty cone is called a *stack*. If **P** is a join semi–lattice, a stack S in **P** is a *grill* iff it satisfies the primeness condition: for all $a,b \in P$, $a + b \in S$ implies $a \in S$ or $b \in S$. (In a Boolean algebra, any filter is a stack and any ultrafilter is a grill.) Recall that if $S,R \subseteq P$ we say that S *refines* R iff for all $r \in R$ there is an $s \in S$ with $s \leq r$.

3.2.3 Theorem

Let **A** be an interior algebra.

lim is a normal weak closure operator on $\mathcal{P}(A)$, that is:

a) $R \subseteq \lim R$

b) $S \subseteq R$ implies $\lim S \subseteq \lim R$

c) lim (lim R) = lim R

d) lim R = ϕ iff R = ϕ for all S,R $\subseteq$ A.

In addition we have for all S,R $\subseteq$ A:

e) If S refines R then lim S $\subseteq$ lim R.

f) lim R is a cone. (In particular, lim R is a stack iff R $\neq \phi$.)

g) lim R = A iff 0 $\in$ R

h) lim { 1 } = { a $\in$ A : N(a) = { 1 } } = { a $\in$ A : $A^O \cap [a)$ = { 1 } }

Proof:

We only prove (c) and leave the rest to the reader. Let R $\subseteq$ A. By (a) lim R $\subseteq$ lim (lim R). Let a $\in$ lim (lim R). Let b $\in$ N(a). There is a c $\in$ lim R with c $\leq b^I$. Then b $\in$ N(c) and so there is an r $\in$ R with r $\leq$ b. Thus a $\in$ lim R and so lim (lim R) $\subseteq$ lim R. □

3.2.4 Definition and Remark

It is convenient to introduce a concept of 'accumulation' in interior algebras. Let **A** be an interior algebra. We can define a pre–order on A, called the **canonical pre–order**, by a $\ll$ b iff a $\leq b^C$, for all a,b $\in$ A. For each a $\in$ A put E(a) = { b $\in$ A : a $\ll$ b }. The members of E(a) are called **enclosers** of a and are said to **enclose** a. If R $\subseteq$ E(a) we say that R **accumulates** at a and a is an **accumulant** of R, denoted by R $\longleftarrow$ a. For all R $\subseteq$ A let acc R = { a $\in$ A : R $\longleftarrow$ a }. Note that in the case when a is an atom the definition of accumulation is a generalization of the accumulation at points in topology. □

3.2.5 Theorem

Let **A** be an interior algebra and let a and b be atoms in **A**.

i) a $\ll$ b iff N(a) $\subseteq$ N(b).

ii) $a^C = b^C$ iff N(a) = N(b).

Proof:

(i): Let a $\ll$ b. Let d $\in$ N(a). Suppose d $\notin$ N(b). Then b $\nleq d^I$ and so b $\leq d^{I\prime}$. Thus a $\leq b^C \leq d^{I\prime C} = d^{I\prime}$, a contradiction since a $\leq d^I$. Hence d $\in$ N(b) and so N(a) $\subseteq$ N(b). Conversely let N(a) $\subseteq$ N(b). Suppose that a $\nleq b^C$. Then a $\leq b^{C\prime} = b'^I$, that is b' $\in$ N(a). But then b' $\in$ N(b), a contradiction. (ii) follows from (i) since $a^C = b^C$ iff a $\ll$ b and b $\ll$ a. □

We summarize the basic properties of enclosers and accumulation leaving the straightforward proofs to the reader.

3.2.6 Proposition

Let **A** be an interior algebra.

i) For all $a \in A$, $E(a)$ is a stack.

ii) For all atoms a in **A**, $E(a)$ is a grill.

iii) For all $a \leq b$ in **A**, $E(b) \subseteq E(a)$.

iv) For all $a \in A$, $E(a) = E(a^c)$. □

Note that if $E(a)$ is a grill, a need not be an atom – consider a simple interior algebra with more than four elements.

3.2.7 Proposition

Let **A** be an interior algebra and let $S,R \subseteq A$.

i) acc R is a closed dual stack.

ii) $S \subseteq R$ implies acc $R \subseteq$ acc S.

iii) acc $R = \{ 0 \}$ iff $0 \in R$.

iv) acc $R = A$ iff $R \subseteq \{ 1 \}$. □

3.2.8 Definition and Remark

In order to establish a connection between convergence and accumulation we introduce the **section operator**: Let **L** be a meet semi–lattice with 0 and define an operation $\mathrm{Sec} : \mathcal{P}(L) \longrightarrow \mathcal{P}(L)$ by $\mathrm{Sec}\, R = \{ b \in L : ab > 0 \text{ for all } a \in R \}$ for all $R \subseteq L$. (The section operator has been used before in the literature in the special case where **L** is a power set Boolean algebra.) □

3.2.9 Proposition

Let **L** be a meet semi–lattice with 0 and let $S,R \subseteq L$.

i) $S \subseteq R$ implies Sec $R \subseteq$ Sec S.

ii) $R \subseteq$ Sec S iff $S \subseteq$ Sec R.

iii) $R \subseteq$ Sec (Sec R).

iv) Sec $R = \phi$ iff $0 \in R$.

v) Sec R is a cone. (In particular, Sec R is a stack iff $0 \notin R$.) □

We are now ready to relate convergence and accumulation.

3.2.10 Proposition

Let **A** be an interior algebra and let $a > 0$ in **A**. Then $E(a) \subseteq$ Sec $N(a)$ and $N(a) \subseteq$ Sec $E(a)$.

Proof:

Let $b \in N(a)$ and $d \in E(a)$. Suppose $bd = 0$. Then $b \leq d'$ and so $b^I \leq d'^I$. But $a \leq b^I$ and so $d^C = d'^{I\prime} \leq a'$. Then $a \leq d^C \leq a'$, a contradiction. Thus $bd > 0$ and the result follows. □

3.2.11 Proposition

Let A be an interior algebra, let a be an atom in A and let $R \subseteq A$.

i) $E(a) = \text{Sec } N(a)$ and $N(a) = \text{Sec } E(a)$.

ii) $R \longrightarrow a$ implies $\text{Sec } R \longleftarrow a$.

iii) $\text{Sec } R \longrightarrow a$ implies $R \longleftarrow a$.

Proof:

(i): It suffices to show $\text{Sec } N(a) \subseteq E(a)$. Suppose there is a $b \in \text{Sec } N(a)$ with $b \notin E(a)$. Then $a \nleq b^C$ and so $a \leq b^{C\prime} = b'^I$. Hence $b' \in N(a)$. However $bb' = 0$, contradicting $b \in \text{Sec } N(a)$. (ii): Let $R \longrightarrow a$. Suppose we do not have $\text{Sec } R \longleftarrow a$. Then there is a $b \in \text{Sec } R$ with $a \nleq b^C$. Then $a \leq b^{C\prime} = b'^I$. Hence there is an $r \in R$ with $r \leq b'$. Then $rb = 0$, contradicting $b \in \text{Sec } R$. (iii): Let $\text{Sec } R \longrightarrow a$. By (ii) $\text{Sec } (\text{Sec } R) \longleftarrow a$. But $R \subseteq \text{Sec } (\text{Sec } R)$ and so $R \longleftarrow a$. □

Note that (i) above does not characterize atoms – consider a simple interior algebra with more than four elements.

3.2.12 Remark

Note that if S is a stack in an interior algebra A (in particular a filter or grill) and $a \in A$ then $S \longrightarrow a$ iff $N(a) \subseteq S$. □

3.2.13 Proposition

Let A be an interior algebra, let $a > 0$ in A and let G be a grill in A. Then $G \longleftarrow a$ implies $G \longrightarrow a$.

Proof:

Suppose $G \longleftarrow a$. Let $b \in N(a)$. Suppose $b \notin G$. Since G is a grill $b' \in G$. Then $a \leq b'^C = b^{I\prime}$. But $a \leq b^I$ and so $a = 0$, a contradiction. Thus $b \in G$. Hence $G \longrightarrow a$. □

3.2.14 Corollary

Let A be an interior algebra. Let $F \subseteq A$ be a proper filter and let a be an atom in A.

i) $F \longrightarrow a$ implies $F \longleftarrow a$.

ii) If F is an ultrafilter, $F \longrightarrow a$ iff $F \longleftarrow a$.

Proof:

(i): Let $F \longrightarrow a$. By Proposition 3.2.11 (ii) Sec $F \longleftarrow a$. Since F is a proper filter $F \subseteq$ Sec F and so $F \longleftarrow a$. (ii) follows from (i) and Proposition 3.2.13. □

3.2.15 Proposition

Let A be an interior algebra, let a,b $\in$ A with $a > 0$. Consider the following:

i) $a \ll b$

ii) There is a proper (ultra)filter F of (b] with $F \longrightarrow a$ in A.

iii) There is an $F \subseteq (b] - \{ 0 \}$ with $F \longrightarrow a$ in A.

We always have (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). If a is an atom then (i), (ii) and (iii) are all equivalent.

Proof:

Assume (i). Put $R = \{ bd : d \in N(b) \}$. Then $R \subseteq (b]$. By Proposition 3.2.10, b $\in$ Sec N(a). Thus $0 \notin R$. Since N(a) is closed under finite meets, so is R. Thus there is an ultrafilter F of (b] with $R \subseteq F$. If d is a neighbourhood of a then $bd \in F$ and $bd \leq d$. Thus $F \longrightarrow a$ and so (i) $\Longrightarrow$ (ii). Clearly (ii) $\Longrightarrow$ (iii). Now let a be an atom. Assume (iii). Let d be a neighbourhood of a. Then there is a $c \in F$ with $c \leq d$. Then $0 < c = bc \leq bd$. Hence b $\in$ Sec N(a). By Proposition 3.2.11 (i) $a \ll b$. Thus (iii) $\Longrightarrow$ (i). □

3.2.16 Definition

In an interior algebra **A** the enclosers of 1 are called the **dense** elements of **A**. □

We can characterize dense elements using the section operator:

3.2.17 Theorem

Let **A** be an interior algebra. Sec ($A^O - \{ 0 \}$) is the set of dense elements of **A**.

Proof:

Let a be dense in **A**. Let $b \in A^O - \{ 0 \}$. Suppose $ab = 0$. Then $a \leq b'$ whence $1 = b'^C = b'$ and so $b = 0$, a contradiction. Thus $ab > 0$ and so a $\in$ Sec ($A^O - \{ 0 \}$). Conversely let a $\in$ Sec ($A^O - \{ 0 \}$). Suppose $a^C < 1$. Then $a^{C\prime} \in A^O - \{ 0 \}$ and so $aa^{C\prime} > 0$, a contradiction. Thus $a^C = 1$, that is a is dense. □

3.2.18 Remark

Let **A** be an interior algebra and let $a \in A^O$. a is dense iff $a^C = 1$, iff $a^{C\prime} = 0$, iff $a'^I = 0$. Recalling that a'^I is the pseudocomplement of a in A^O we see that the open dense elements of **A** are precisely the dense elements of the Heyting algebra A^O in the usual sense. □

Recall that a *directed set* $\mathbf{W} = \langle W, \ll \rangle$ is a pre–ordered set with the property that for all $x,y \in W$ there is a $z \in W$ with $x \ll z$ and $y \ll z$. If W is non–empty and X is a set then a *net in* X *based on* $\mathbf{W}$ is a family (x_i) of elements of X indexed by W. Note that any sequence (x_n) in X can be regarded as a net based on the directed set $\langle \omega, \leq \rangle$.

3.3.1 Definition and Notation

Let (x_i) be a net in a set X based on a directed set $\mathbf{W}$. If $j \in W$ the set $\{ x_i : i \gg j \text{ in } \mathbf{W} \}$ is called a **tail** of (x_i) and is denoted by $t_j(x_i)$. □

3.3.2 Definition and Remark

Let **A** be an interior algebra and let $B \subseteq A$. Then $\langle B, \geq \rangle$ is a directed set iff B is a filter base in **A**. If B is a filter base we thus have a net (a_i) in A based on $\langle B, \geq \rangle$ with $a_i = i$ for all $i \in B$. (a_i) will be called the **canonical net** of B. If (a_i) is a net in **A** based on a directed set **W** and if $\Sigma\, t_j(a_i)$ exists for all $j \in W$ then $\{ \Sigma\, t_j(a_i) : j \in W \}$ is a filter base. Note that that every filter base in **A** is obtained in this way from its canonical net. □

3.3.3 Notation

If **A** is an interior algebra we will denote the set of atoms in **A** by At **A**. □

Recall that a filter base B is called *proper* iff $0 \notin B$.

3.3.4 Theorem

Let B be a proper filter base in an atomic interior algebra **A**. Then there is a net (a_i) in At **A** based on a directed set **W** such that $B = \{ \Sigma\, t_j(a_i) : j \in W \}$.

Proof:

Put $W = \{ (c,b) : b \in B \text{ and } c \text{ is an atom in } A \text{ with } c \leq b \}$. There is a $b \in B$. Since B is proper $b > 0$. Since **A** is atomic there is an atom $c \leq b$. Then $(c,b) \in W$ and so $W \neq \phi$. Define $\ll$ on W by $(c,b) \ll (d,a)$ iff $b \geq a$. Then clearly $\ll$ is a pre–order. If $(c,b),(d,a) \in W$ then $a,b \in B$ and so there is an $e \in B$ with $e \leq ab$. Since B is proper, $e > 0$ and so by atomicity there is an atom $v \leq e$. Then $(v,e) \in W$, $(c,b) \ll (v,e)$ and $(d,a) \ll (v,e)$. Thus $\mathbf{W} = \langle W, \ll \rangle$ is a directed set. If $i = (c,b) \in W$ put $a_i = c$. We thus have a net (a_i) in At **A**. Consider a $j \in W$ with $j = (c,b)$. Let $i \gg j$ in **W** with $i = (d,e)$. Then $a_i \leq e \leq b$. Thus $t_j(a_i) \subseteq \text{At}\,(\mathbf{b}]$. Now suppose d is an atom with $d \leq b$. Then $(d,b) \gg j$ and so $d \in t_j(a_i)$. Thus in fact $t_j(a_i) = \text{At}\,(\mathbf{b}]$. Hence $b = \Sigma\, t_j(a_i)$ and so we see that $\{ \Sigma\, t_j(a_i) : j \in W \} \subseteq B$.

Lastly consider a $b \in B$. As before there is an atom $c \leq b$. Put $j = (c,b)$. Then $j \in W$ and as above we see that $b = \Sigma\, t_j(a_i)$. Thus in fact $B = \{ \Sigma\, t_j(a_i) : j \in W \}$. □

What sort of filter bases correspond to sequences?

3.3.5 Definition and Remark

Let **A** be an interior algebra and let B be a filter base in **A**. B is called a **sequential** base iff $\langle B, \leq \rangle \cong \langle u, \geq \rangle$ where $1 \leq u \leq \omega$. (Of course any chain is a filter base.) If (a_n) is a sequence in A such that the join $\Sigma\, t_k(a_n)$ exists for all $k < \omega$, then $\{ \Sigma\, t_k(a_n) : k < \omega \}$ is a sequential base. A sequential base B in **A** is called **elementary** iff the top element of B is a countable non–empty join of atoms and for all $a,b \in B$, a covers b in $\langle B, \leq \rangle$ iff a covers b in **A**. □

The following theorem shows that the elementary sequential bases are precisely the filter bases obtained from sequences of atoms. Although the result is fairly intuitive it is quite tricky to prove correctly and so we have given a fairly detailed proof.

3.3.6 Theorem

Let **A** be an interior algebra and let $B \subseteq A$. Then B is an elementary sequential base iff there is a sequence (a_n) in At **A** with $B = \{ \Sigma\, t_k(a_n) : k < \omega \}$.

Proof:

Let B be an elementary sequential base. There is $1 \leq u \leq \omega$ such that $\langle B, \leq \rangle \cong \langle u, \geq \rangle$. Let $\{ b_n : n < u \}$ be the enumeration of B given by the isomorphism. There are two cases: Case (1): $u < \omega$. We define atoms c_n, $n < u-1$, by finite induction such that for all $n < u-1$, $b_{n+1} = b_0 \cdot c_0' \cdots c_n'$. b_0 is a join of atoms so in fact $b_0 = \Sigma\, \mathrm{At}\,(\mathbf{b_0}]$. If $u = 1$ let c_0 be any atom below b_0. Otherwise $b_1 < b_0$ and so there is an atom $c_0 \in (b_0] - (b_1]$. Then $b_1 \leq c_0'$ and so $b_1 \leq b_0 \cdot c_0'$. Since b_0 covers b_1 we in fact have $b_1 = b_0 \cdot c_0'$. Now suppose for some $k < u-2$ we have found $c_0,\ldots,c_k$ with the desired property. We have $b_{k+1} = b_0 \cdot c_0' \cdots c_k' = \Sigma\,(\mathrm{At}\,(\mathbf{b_0}] - \{ c_0,\ldots,c_k \})$. $b_{k+2} < b_{k+1}$ and so there is a $c_{k+1} \in \mathrm{At}\,(\mathbf{b_0}] - \{ c_0,\ldots,c_k \}$ not below b_{k+2}. Then $b_{k+2} \leq c_{k+1}'$ and so $b_{k+2} \leq b_{k+1} \cdot c_{k+1}'$. But b_{k+1} covers b_{k+2} and so in fact $b_{k+2} = b_{k+1} \cdot c_{k+1}' = b_0 \cdot c_0' \cdots c_{k+1}'$ and the induction is complete. Put $c_n = c_{u-2}$ for all $n \geq u-1$ to obtain (c_n). There are two cases now: (i): $\mathrm{At}\,(\mathbf{b_0}] = \{ c_n : n < u-1 \}$. Then simply put $(a_n) = (c_n)$. Then $b_0 = \Sigma\, \mathrm{At}\,(\mathbf{b_0}] = \Sigma\, t_0(a_n)$. If $k < u-1$ then $b_{k+1} = b_0 \cdot c_0' \cdots c_k' = \Sigma\,(\mathrm{At}\,(\mathbf{b_0}] - \{ c_0,\ldots,c_k \}) = \Sigma\, t_{k+1}(a_n)$. If $k \geq u-1$ then $\Sigma\, t_k(a_n) = \Sigma\, t_{u-2}(a_n) = b_{u-2}$. Thus $B = \{ \Sigma\, t_k(a_n) : k < \omega \}$. (ii): Of course $\{ c_n : n < u-1 \} \subset \mathrm{At}\,(\mathbf{b_0}]$. $\mathrm{At}\,(\mathbf{b_0}] - \{ c_n : n < u-1 \}$ is countable and non–empty. Let

(d_n) be a sequence such that $\{ d_n : n < \omega \} = At\,(b_0] - \{ c_n : n < u - 1 \}$. Define (a_n) as follows: Consider the polynomial $f(n) = (n^2 + 3n) / 2$. Let $n < \omega$. If $k = f(n)$ for some $n < \omega$ put $a_k = c_n$. If $f(n) < k < f(n + 1)$ for some $n < \omega$ put $a_k = d_i$ where $i = k - f(n) - 1$. Observe that $f(n + 1) = f(n) + n + 2$ for all $n < \omega$. Let $k < \omega$. There is an $j < \omega$ such that $k \leq f(j)$. Consider $r < \omega$. Put $n = j + r$. Then $0 \leq r < n + 1 = f(n + 1) - f(n) - 1$. Hence $f(n) < f(n) + 1 \leq r + f(n) + 1 < f(n + 1)$ and so $a_i = d_r$ where $i = r + f(n) + 1$. Also $k < f(j) < f(n) < i$. Thus $d_r \in t_k(a_n)$. Hence for all $k < \omega$, $\{ d_n : n < \omega \} \subseteq t_k(a_n)$. Then $At\,(b_0] = \{ c_n : n < u - 1 \} \cup \{ d_n : n < \omega \} = \{ c_n : n < \omega \} \cup \{ d_n : n < \omega \} = t_0(a_n)$ and so $b_0 = \Sigma\, t_0(a_n)$. Let $k < u - 1$. Put $i = f(k + 1)$. Then $b_{k+1} = b_0 \cdot c_0' \cdots c_k' = \Sigma\, (At\,(b_0] - \{ c_0,\ldots,c_k \}) = \Sigma\, t_i(a_n)$. Let $k \geq u - 1$. Put $i = f(k + 1)$ and $j = f(u - 2)$. Then $\Sigma\, t_i(a_n) = \Sigma\, t_j(a_n)$. Lastly let $k < \omega$ and suppose $f(k) < i < f(k + 1)$. Then $\Sigma\, t_i(a_n) = \Sigma\, t_j(a_n)$ where $j = f(k)$. It follows that $B = \{ \Sigma\, t_k(a_n) : k < \omega \}$. Case (2): $u = \omega$. Then we proceed as in (1) but obtain the whole of (c_n) by induction. We proceed as in (1) but consider $k < \omega$ instead of $k < u - 1$ and no longer consider $k \geq u - 1$. This completes the forward implication. For the converse: Let (a_n) be a sequence such that $B = \{ \Sigma\, t_k(a_n) : k < \omega \}$. Then B is a sequential base with top element $\Sigma\, t_0(a_n)$, a countable non–empty join of atoms. Obviously if $a,b \in B$ and a covers b in **A** then a covers b in $\langle B, \leq \rangle$. Suppose that a covers b in $\langle B, \leq \rangle$. There are $i < j < \omega$ with $a = \Sigma\, t_j(a_n)$ and $b = \Sigma\, t_i(a_n)$. Let $k \leq j$ be least such that $a = \Sigma\, t_k(a_n)$. Then $b \leq \Sigma\, t_{k-1}(a_n) < a$ and so $b = \Sigma\, t_{k-1}(a_n)$. Thus $a = b + a_k$ and a_k is an atom whence it follows that a covers b in **A**. Thus B is elementary. □

3.3.7 Definition and Remark

Let F be a filter in an interior algebra A. If $\{ b_n : n < \omega \}$ is a countable base for F define $d_n = b_0 \cdots b_n$ for all $n < \omega$. Then $\{ d_n : n < \omega \}$ a sequential base for F. Thus a filter has a countable base iff it has a sequential base. We therefore call a filter with a countable base a **sequential** filter. Call a sequential filter **elementary** iff it has an elementary sequential base. □

Note that the intersection of two sequential filters is sequential since if F and G are filters with bases B and C respectively then $\{ b + c : b \in B \text{ and } c \in C \}$ is a base for $F \cap G$. We now show that a similar result holds for elementary filters.

3.3.8 Theorem

Let F and G be elementary filters in an interior algebra A. Then $F \cap G$ is elementary.

Proof:

There are sequences (a_n) and (b_n) in At A such that $B = \{ \Sigma t_k(a_n) : k < \omega \}$ and $C = \{ \Sigma t_k(a_n) : k < \omega \}$ are bases for F and G respectively. Define (c_n) by $c_{2n} = a_n$ and $c_{2n+1} = b_n$ for all $n < \omega$. For all $k < \omega$, $t_{2k}(c_n) = t_k(a_n) \cup t_k(b_n)$ and so $\Sigma t_{2k}(c_n) = \Sigma t_k(a_n) + \Sigma t_k(b_n)$. Thus $\{ \Sigma t_{2k}(c_n) : k < \omega \}$ is a base for $F \cap G$. Also for all $k < \omega$, $t_{2k+1}(c_n) = t_{2k+2}(c_n) \cup \{ b_k \}$ whence $\Sigma t_{2k+1}(c_n) = \Sigma t_{2k+2}(c_n) + b_k \geq \Sigma t_{2k+2}(c_n)$. It follows that $\{ \Sigma t_k(c_n) : k < \omega \}$ is a base for $F \cap G$. □

3.3.9 Notation and Remark

We can convert nets into filters: Let (a_i) be a net in an interior algebra **A** based on a directed set **W**. Put $F(a_i) = \{ b \in A : t_j(a_i) \subseteq (b] \text{ for some } j \in W \}$. Clearly $F(a_i)$ is a stack. Let $b,c \in F(a_i)$. There are $j,k \in W$ with $t_j(a_n) \subseteq (b]$ and $t_k(a_n) \subseteq (c]$. There is an $r \in W$ with $j \ll r$ and $k \ll r$. Then $t_r(a_n) \subseteq (b] \cap (c] = (bc]$. Thus $bc \in F(a_i)$ and so $F(a_i)$ is a filter. Moreover, every filter in **A** is of this form since if F is a filter in **A**, $F = F(a_i)$ where (a_i) is the canonical net of F. By Theorem 3.3.4 we see that if **A** is atomic then for every proper filter F in **A** there is a net (a_i) in At A with $F = F(a_i)$. □

3.3.10 Definition

Call a filter F in an interior algebra A, **pseudo–elementary** iff there is a sequence (a_n) in At **A** such that $F = F(a_n)$. □

3.3.11 Definition

Let **A** be an interior algebra. If λ is a cardinal we say that **A** is **λ–collectable** iff for all $R \subseteq$ At **A** with $|R| \leq \lambda$, ΣR exists. **A** is called **collectable** iff it is $|A|$–collectable. Call an $\aleph_0$–collectable interior algebra **countably collectable**. □

Recall that a net (x_i) is called *injective* iff the map $i \longmapsto x_i$ is injective.

3.3.12 Theorem

Let **A** be an interior algebra an let F be an ultrafilter in **A**. Consider the following:

i) F is principal.

ii) F is elementary.

iii) F is sequential.

iv) F is pseudo–elementary.

Then we always have (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii),(iv). If **A** is countably collectable then (i), (ii) and (iv) are all equivalent. If **A** is atomic then (iii) $\Longrightarrow$ (iv). In particular, if **A** is atomic and countably collectable then (i) – (iv) are all equivalent.

Proof:

Recall that if F is a principal ultrafilter then F = [a) for some atom a. Putting $a_n = a$ for all $n < \omega$ gives $\{ \Sigma t_k(a_n) : n < \omega \} = \{ a \}$ and so F is elementary. Thus (i) $\Longrightarrow$ (ii). Obviously (ii) $\Longrightarrow$ (iii),(iv). Let **A** be countably collectable. Suppose that there is a sequence (a_n) in At **A** with $F = F(a_n)$. Then for all $k < \omega$, $\Sigma t_k(a_n)$ exists and $\{ \Sigma t_k(a_n) : k < \omega \}$ is a base for F. Thus (iv) $\Longrightarrow$ (ii). Assume (ii). There is a sequence (a_n) in At **A** such that $B = \{ \Sigma t_k(a_n) : k < \omega \}$ is a base for F. Suppose that $\{ a_n : n < \omega \}$ is infinite. Define a subsequence (b_k) of (a_n) inductively as follows: Put $m(0) = 0$. Suppose that for some $k < \omega$ we have defined $m(0) < \ldots < m(k)$ so that $a_{m(0)},\ldots,a_{m(k)}$ are all distinct. Since $\{ a_n : n < \omega \}$ is infinite there is an $m(k+1) > m(k)$ such that $a_{m(0)},\ldots,a_{m(k+1)}$ are all distinct. We thus have a strictly increasing map $m : \omega \longrightarrow \omega$ and hence a subsequence $(b_k) = (a_{m(k)})$ of (a_n). In fact (b_k) is injective. Since **A** is countably collectable we can put $c = \Sigma \{ b_{2k} : k < \omega \}$. Suppose $c \in F$. Then there is an $i < \omega$ with $\Sigma t_i(a_n) \leq c$. Then there is a $k < \omega$ with $b_{2k+1} \in t_i(a_n)$. Thus $b_{2k+1} \leq c$ and so $b_{2k+1} = b_{2n}$ for some $n < \omega$. But then $2k + 1 = 2n$, a contradiction. Thus $c \notin F$. But F is an ultrafilter and so $c' \in F$. Thus there is an $i < \omega$ with $\Sigma t_i(a_n) \leq c'$. Then there is a $k < \omega$ with $b_{2k} \in t_i(a_n)$. Then $b_{2k} \leq c'$, a contradiction since $b_{2k} \leq c$. Hence the assumption that $\{ a_n : n < \omega \}$ is infinite is false. Thus B is finite and so F is principal. Thus (ii) $\Longrightarrow$ (i). Now let **A** be atomic. Assume (iii). Then there is a base B for F with $\langle B, \leq \rangle \cong \langle u, \geq \rangle$ where $1 \leq u \leq \omega$. If $u < \omega$ then (i) and hence (iv) holds. Suppose $u = \omega$. Let $\{ b_n : n < \omega \}$ be the enumeration of B given by the isomorphism between $\langle B, \leq \rangle$ and $\langle \omega, \geq \rangle$. Then for all $n < \omega$, $b_n > 0$ and so there is an atom $a_n \leq b_n$. Let $k < \omega$. If $n \geq k$ then $a_n \leq b_n \leq b_k$ and so $t_k(a_n) \subseteq (b_k]$. Thus $F \subseteq F(a_n)$. Since F is an ultrafilter and $F(a_n)$ is proper we have $F = F(a_n)$. Thus (iii) $\Longrightarrow$ (iv). □

3.3.13 Theorem

Let **A** be an open atomic interior algebra. If F is an open sequential ultrafilter in **A** then F is pseudo–elementary.

Proof:

There is a sequential base B for F. There is a $u \leq \omega$ with $\langle B, \leq \rangle \cong \langle u, \geq \rangle$. If $u < \omega$ then F is principal and hence pseudo–elementary. Suppose $u = \omega$. Let $\{ b_n : n < \omega \}$ be the enumeration of B given by the isomorphism between $\langle B, \leq \rangle \cong \langle u, \geq \rangle$. For all $n < \omega$, $b_n^I \in F$ and so $b_n^I > 0$ whence by open atomicity there is an atom $a_n \leq b_n^I$. Let $k < \omega$. For all $n > k$, $a_n \leq b_n^I \leq b_k^I \leq b_k$ and so $t_k(a_n) \subseteq (b_k]$. Thus $F \subseteq F(a_n)$. Since F is an ultrafilter and $F(a_n)$ is proper, $F = F(a_n)$ and so F is pseudo–elementary. □

3.3.14 Corollary

Let **A** be an open atomic countably collectable interior algebra. The following are equivalent for an open ultrafilter in **A**:

i) F is principal.

ii) F is elementary.

iii) F is sequential.

iv) F is pseudo–elementary. □

3.4 CONVERGENCE AND ACCUMULATION OF NETS AND SEQUENCES

3.4.1 Definition and Notation

Let **A** be an interior algebra, let (a_i) be a net in A based on a directed set **W** and let b ∈ A. We say that (a_i) **converges** to b and that b is a **limit** of (a_i), denoted $(a_i) \longrightarrow b$, iff for all d ∈ N(b) there is a j ∈ W such that $a_i \leq d$ for all i ≫ j in **W**. Put $\lim (a_i) = \{ b \in A : (a_i) \longrightarrow b \}$. □

3.4.2 Remark

Of course in the case of atoms, net convergence in interior algebras is a generalization of net convergence in topology. Let **A** be an interior algebra, let (a_i) be a net in **A** based on a directed set **W** and let b ∈ A. Note that the following are equivalent:

i) $(a_i) \longrightarrow b$

ii) For all open d ≥ b there is a j ∈ W such that $a_i \leq d$ for all i ≫ j in **W**.

iii) For all d ∈ N(b) there is a j ∈ W such that $a_i \leq d^I$ for all i ≫ j in **W**. □

3.4.3 Proposition

Let **A** be an interior algebra and let (a_i) be a net in A.

i) $\lim (a_i)$ is a stack.

ii) If (b_k) is a net in $\lim (a_i)$ then $\lim (b_k) \subseteq \lim (a_i)$.

Proof:

(i) is trivial, we prove (ii): Let (b_k) be a net in $\lim (a_i)$ based on a directed set **Z** and let **W** be the directed set on which (a_i) is based. Let c ∈ A with $(b_k) \longrightarrow c$. We have to show that $(a_i) \longrightarrow c$. Let d ∈ N(c). Then there is an r ∈ Z with $b_k \leq d^I$ for all k ≫ r in **Z**, in particular $b_r \leq d^I$. Then d ∈ $N(b_r)$. But then there is a j ∈ W with $a_i \leq d$ for all i ≫ j in **W**. □

How is net convergence related to subset convergence?

3.4.4 Remark

Note that if (a_i) is a net in an interior algebra $\mathbf{A}$ and $b \in A$ then $(a_i) \longrightarrow b$ iff $F(a_i) \longrightarrow b$, that is $\lim (a_i) = \lim F(a_i)$. Thus net convergence can be considered to be a special case of filter convergence. □

3.4.5 Proposition

Let $\mathbf{A}$ be an interior algebra and let (a_i) be a net in A based on a directed set $\mathbf{W}$. Suppose that for all $j \in W$ the join $\Sigma\, t_j(a_i)$ exists. Then $\lim (a_i) = \lim \{ \Sigma\, t_j(a_i) : j \in W \}$.

Proof:

For all $b \in A$ we have $(a_i) \longrightarrow b$ iff for all $d \in N(b)$ there is a $j \in W$ with $a_i \leq d$ for all $i \gg j$ in $\mathbf{W}$, iff for all $d \in W$ there is a $j \in W$ with $\Sigma\, t_j(a_i) \leq d$, and the result follows. □

The above result together with 3.3.2 shows that filter base convergence can be considered to be a special case of net convergence.

3.4.6 Theorem

Let $\mathbf{A}$ be an interior algebra, let $R \subseteq A$ and let $b \in A$. Then $R \longrightarrow b$ iff there is a net (a_i) in R with $(a_i) \longrightarrow b$.

Proof:

Suppose $R \longrightarrow b$. We define a net in R based on the directed set $\langle N(b), \geq \rangle$ as follows: For each $i \in N(b)$ let $a_i \in R$ with $a_i \leq i$. Let $d \in N(b)$. For all $i \geq d$ in $\langle N(b), \geq \rangle$, $a_i \leq i \leq d$ and so $(a_i) \longrightarrow b$. Conversely suppose there is a net (a_i) in R based on a directed set $\mathbf{W}$ with $(a_i) \longrightarrow b$. Let $d \in N(b)$. There is a $j \in W$ with $a_i \leq d$ for all $i \gg j$ in $\mathbf{W}$, in particular, $a_j \leq d$. Thus $R \longrightarrow b$. □

3.4.7 Definition and Notation

Let $\mathbf{A}$ be an interior algebra, let (a_i) be a net in A based on a directed set $\mathbf{W}$ and let $b \in A$. We say that (a_i) **accumulates** at b and b is an **accumulant** of (a_i), denoted $(a_i) \longleftarrow b$, iff for all $d \in N(b)$ and for all $j \in W$, there is an $i \gg j$ in $\mathbf{W}$ with $a_i \leq d$. Put $\text{acc}\,(a_i) = \{ b \in A : (a_i) \longleftarrow b \}$. □

3.4.8 Remark

Of course in the case of atoms, net accumulation in interior algebras generalizes net accumulation in topology. Let $\mathbf{A}$ be an interior algebra, let (a_i) be a net in A based on a directed set $\mathbf{W}$ and let $b \in A$. Note that the following are equivalent:

i) $(a_i) \longleftarrow b$

ii) For all open $d \geq b$ and for all $j \in W$, there is an $i \gg j$ with $a_i \leq d$.

iii) For all $d \in N(b)$ and for all $j \in W$, there is an $i \gg j$ with $a_i \leq d^I$. □

3.4.9 Proposition

Let $\mathbf{A}$ be an interior algebra and let (a_i) be a net in A.

i) acc (a_i) is a stack.

ii) lim $(a_i) \subseteq$ acc (a_i).

iii) If (b_k) is a net in acc (a_i) then acc $(b_k) \subseteq$ acc (a_i).

Proof:

(i) and (ii) are trivial. For (iii): Let (b_k) be a net in acc (a_i) based on a directed set $\mathbf{Z}$ and let $\mathbf{W}$ be the directed set on which (a_i) is based. Let $c \in A$ with $(b_k) \longleftarrow c$. We have to show that $(a_i) \longleftarrow c$. Let $d \in N(c)$ and let $j \in W$. There is an $r \in Z$. (By the definition of a net Z must be non–empty.) Then there is a $k \gg r$ with $b_k \leq d^I$. Then $d \in N(b_k)$. Hence there is an $i \gg j$ with $a_i \leq d$. □

Net accumulation is generally very different to subset accumulation. However we have:

3.4.10 Proposition

Let $\mathbf{A}$ be an interior algebra, let (a_i) be a net in $A - \{ 0 \}$ based on a directed set $\mathbf{W}$.

i) At $\mathbf{A} \cap$ acc $(a_i) \subseteq$ acc $F(a_i)$.

ii) Suppose that for all $j \in W$ the join $\Sigma\, t_j(a_i)$ exists. Then At $\mathbf{A} \cap$ acc $(a_i) \subseteq$ acc $\{ \Sigma\, t_j(a_i) : j \in W \}$.

Proof:

(i): Suppose b is an atom in $\mathbf{A}$ with $(a_i) \longleftarrow b$. Consider $c \in F(a_i)$. Let $d \in N(b)$. There is a $j \in W$ with $t_j(a_i) \subseteq (c]$. Then there is an $i \gg j$ in $\mathbf{W}$ with $a_i \leq d$. Then $a_i \leq cd$ and so $cd > 0$. Hence $c \in$ Sec $N(b) = E(b)$ (See Proposition 3.2.10 (i).) Thus $F(a_i) \longleftarrow b$ as required. (ii) follows from (i) since if $\Sigma\, t_j(a_i)$ exists for all $j \in W$ then $\{ \Sigma\, t_j(a_i) : j \in W \} \subseteq F(a_i)$. □

3.4.11 Proposition

Let $\mathbf{A}$ be an atomic interior algebra and let (a_i) be a net in At $\mathbf{A}$ based on a directed set $\mathbf{W}$. Suppose that for all $j \in W$ the join $\Sigma\, t_j(a_i)$ exists. Then At $\mathbf{A} \cap$ acc (a_i) = At $\mathbf{A} \cap$ acc $\{ \Sigma\, t_j(a_i) : j \in W \}$.

Proof:

Let b be an atom in $\mathbf{A}$ with $\{ \Sigma\, t_j(a_i) : j \in W \} \longleftarrow b$. Let $d \in N(b)$ and let $j \in W$. Then $\Sigma\, t_j(a_i) \in E(b)$. But $E(b) =$ Sec $N(b)$ and so $d \cdot \Sigma\, t_j(a_i) > 0$ whence there is an atom $c \leq d \cdot \Sigma\, t_j(a_i)$. But then $c = a_i$ for some $i \gg j$ in $\mathbf{W}$ and then $a_i \leq d$. Thus $(a_i) \longleftarrow b$. Hence

$\text{At } A \cap \{ \Sigma t_j(a_i) : j \in W \} \subseteq \text{At } \mathbf{A} \cap \text{acc}(a_i)$. The reverse inclusion follows from Proposition 3.4.10 (ii). □

Note that Proposition 3.4.9 (ii) already gives a relationship between net convergence and net accumulation. There is a further connection. Before we investigate this recall the following: Let (x_i) be a net in a set X based on a directed set **W**. Let **Z** be a directed set and let the map $m : \mathbf{Z} \longrightarrow \mathbf{W}$ be a co–final homomorphism. Then the net $(x_{m(k)})$ in X based on **Z** is said to be a *subnet* of (x_i). (A subsequence of a sequence is always a subnet of the sequence.)

3.4.12 Theorem

Let A be an interior algebra, let (a_i) be a net in A and let $b \in A$. Then $(a_i) \longmapsto b$ iff there is a subnet (c_k) of (a_i) with $(c_k) \longrightarrow b$.

Proof:

Let **W** be the directed set on which (a_i) is based. Suppose $(a_i) \longmapsto b$. Define a directed set **Z** as follows: Put $Z = \{ (i,d) : i \in W \text{ and } d \in N(b) \text{ with } a_i \leq d \}$. Note that $Z \neq \phi$ since there is an $i \in W$ whence $(i,1) \in Z$. Define $\ll$ on Z as by $(i,d) \ll (j,c)$ iff $i \ll j$ and $d \geq c$. Clearly $\ll$ is a pre–order on Z. Let $(i,d),(j,c) \in Z$. There is a $t \in W$ with $i \ll t$ and $j \ll t$ in **W**. Also $cd \in N(b)$. Since $(a_i) \longmapsto b$ there is a $s \in W$ with $s \gg t$ and $a_s \leq cd$. Then $(s,cd) \in Z$, $(i,d) \ll (s,cd)$ and $(j,c) \ll (s,cd)$. Thus $\mathbf{Z} = \langle Z, \ll \rangle$ is a directed set. If $k = (i,d) \in Z$ put $m(k) = i$. Then the map $m : \mathbf{Z} \longrightarrow \mathbf{W}$ is clearly a homomorphism. To see that it is co–final note that if $i \in W$ then $(i,1) \in Z$. Hence we have a subnet $(c_k) = (a_{m(k)})$ of (a_i). Let $d \in N(b)$. There is a $j \in W$. Then there is an $i \gg j$ with $a_i \leq d$. Then $(i,d) \in Z$. Let $k = (t,c) \gg (i,d)$ in Z. Then $c_k = a_t \leq c \leq d$. Thus $(c_k) \longrightarrow b$. Conversely suppose there is a subnet $(c_k) = (a_{m(k)})$ of (a_i) based on a directed set **Z**, with $(c_k) \longrightarrow b$. Let $d \in N(b)$ and let $j \in W$. There is an $r \in Z$ such that $c_k \leq d$ for all $k \gg r$ in **Z**. Since the map $m : \mathbf{Z} \longrightarrow \mathbf{W}$ is co–final there is a $v \in Z$ with $j \ll m(v)$ in **W**. There is a $k \in Z$ with $k \gg r$ and $k \gg v$. Then $m(k) \gg m(v)$ and so $m(k) \gg j$. Moreover $a_{m(k)} = c_k \leq d$. Thus $(a_i) \longmapsto b$. □

From Theorem 3.4.6 and Theorem 3.4.12 we have:

3.4.13 Corollary

Let A be an interior algebra, let $R \subseteq A$ and let $b \in A$. Then the following are equivalent:

i) $R \longrightarrow b$

ii) There is a net (a_i) in R with $(a_i) \longrightarrow b$.

iii) There is a net (a_i) in R with $(a_i) \longmapsto b$. □

Note that if (x_i) is a net in a set X, (y_k) is a subnet of (x_i) and (z_j) is a subnet of (y_k), then (z_j) is also a subnet of (x_i). Thus from Theorem 3.4.12 we get:

3.4.14 Corollary

Let **A** be an interior algebra, let (a_i) be a net in A and let (c_k) be a subnet of (a_i). Then acc $(c_k) \subseteq$ acc (a_i). □

3.4.15 Proposition

Let **A** be an interior algebra, let (a_i) be a net in A and let (c_k) be a subnet of (a_i). Then lim $(a_i) \subseteq$ lim (c_k).

Proof:

Let **W** and **Z** be the directed sets on which (a_i) and $(c_k) = (a_{m(k)})$ are respectively based. Suppose $b \in A$ with $(a_i) \longrightarrow b$. We must show $(c_k) \longrightarrow b$. Let $d \in N(b)$. There is a $j \in W$ such that $a_i \leq d$ for all $i \gg j$ in **W**. But there is a $r \in Z$ such that $m(r) \gg j$. Then for all $k \gg r$ in **Z** we have $m(k) \gg j$ in **W** whence $c_k = a_{m(k)} \leq d$. □

In certain cases Theorem 3.4.6 and Theorem 3.4.12 can be generalized to sequences.

3.4.16 Theorem

Let **A** be an interior algebra, let $R \subseteq A$ and let $b \in A$ such that N(b) is a sequential filter. Then $R \longrightarrow b$ iff there is sequence (a_n) in R with $(a_n) \longrightarrow b$.

Proof:

Suppose that $R \longrightarrow b$. There is a countable base B for N(b). Let $\{ r_n : n < \omega \}$ be an enumeration of B. We define a sequence (a_n) in R as follows. For all $n < \omega$, $r_0 \cdots r_n \in N(b)$ and so there is an $a_n \in R$ with $a_n \leq r_0 \cdots r_n$. Let $d \in N(b)$. Then for some $k < \omega$ we have $r_k \leq d$. But then for all $n \geq k$, $a_n \leq r_0 \cdots r_k \leq r_n \leq d$. Thus $(a_n) \longrightarrow b$. Conversely if we have a sequence (a_n) in R then $(a_n) \longrightarrow b$ by Theorem 3.4.6. □

3.4.17 Theorem

Let **A** be an interior algebra, (a_n) a sequence in A and $b \in A$ such that N(b) is a sequential filter. Then $(a_n) \longleftarrow b$ iff there is a subsequence (c_k) of (a_n) with $(c_k) \longrightarrow b$.

Proof:

Let $(a_n) \longleftarrow b$. There is a countable base B for N(b). Let $\{ r_n : n < \omega \}$ be an enumeration of B. We define (c_k) inductively as follows: $r_0 \in N(b)$ and so there is an $m(0) \geq 0$ such that $a_{m(0)} \leq r_0$. Suppose we have found $m(0) < \ldots < m(k)$ for some $k < \omega$ with $a_{m(n)} \leq r_0 \cdots r_n$ for $0 \leq n \leq k$. Now $r_0 \cdots r_{k+1} \in N(b)$ and so there is an $m(k+1) > m(k)$ such that

$a_{m(k+1)} \leq r_0 \cdots r_{k+1}$. Now we have a strictly increasing map $m : \omega \longrightarrow \omega$ and hence a subsequence $(c_k) = (a_{m(k)})$ of (a_n). Moreover, for all $k < \omega$, $c_k \leq r_0 \cdots r_k$. Now suppose $d \in N(b)$. Then there is a $n < \omega$ such that $r_n \leq d$. But then for all $k \geq n$ we have $a_k \leq r_0 \cdots r_k \leq r_n \leq d$. Thus $(c_k) \longrightarrow b$. The converse follows from Theorem 3.4.12. □

3.4.18 Corollary

Let A be an interior algebra, let $R \subseteq A$ and let $b \in A$ such that $N(b)$ is a sequential filter. The following are equivalent:

i) $R \longrightarrow b$

ii) There is a sequence (a_n) in R with $(a_n) \longrightarrow b$.

iii) There is a sequence (a_n) in R with $(a_n) \longleftarrow b$. □

3.4.19 Proposition

Let A be an interior algebra and let $b,c \in A$ with $b > 0$. Consider the following:

i) $b \ll c$

ii) There is a net (a_i) in $(c] - \{ 0 \}$ with $(a_i) \longrightarrow b$.

iii) There is a net (a_i) in $(c] - \{ 0 \}$ with $(a_i) \longleftarrow b$.

iv) There is a sequence (a_n) in $(c] - \{ 0 \}$ with $(a_n) \longrightarrow b$.

v) There is a sequence (a_n) in $(c] - \{ 0 \}$ with $(a_n) \longleftarrow b$.

We always have (i) $\Longrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Longleftarrow$ (v) $\Longleftarrow$ (iv). If b is an atom then (i), (ii) and (iii) are equivalent. If $N(b)$ is a sequential filter then (ii) – (v) are all equivalent. In particular if b is an atom and $N(b)$ is sequential, (i) – (v) are all equivalent.

Proof:

(i) $\Longrightarrow$ (ii): Assume (i). By Proposition 3.2.14 there is an ultrafilter F in $(c]$ with $F \longrightarrow b$ in A. But then $F \subseteq (c] - \{ 0 \}$ and F is a filter base in A. Letting (a_i) be the canonical net of F gives (ii). (ii) $\Longrightarrow$ (iii) by Proposition 3.4.9 (ii), (iii) $\Longrightarrow$ (ii) by Theorem 3.4.12 and clearly (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (iii). Suppose that b is an atom. Assume (ii). Let $d \in N(b)$. Let **W** be the directed set on which (a_i) is based. There is a $j \in W$ such that $a_i \leq d$ for all $i \gg j$ in **W**, in particular $a_j \leq d$. Thus $0 < a_j \leq cd$. Thus $c \in \text{Sec}\, N(a)$ and so, since b is an atom $c \in E(a)$, that is $b \ll c$. Thus if b is an atom (ii) $\Longrightarrow$ (i) and so in fact (i), (ii) and (iii) are all equivalent. Suppose now that $N(b)$ is sequential. Assume (ii). Put $S = \{ a_i : i \in W \}$ where **W** is the directed set on which (a_i) is based. By Theorem 3.4.6, $S \longrightarrow b$. By Theorem 3.4.16 there is a sequence (d_n) in S with $(d_n) \longrightarrow b$. Noting that (d_n) is a sequence in $(c] - \{ 0 \}$ gives (iv). Thus if $N(b)$ is sequential (ii) – (v) are all equivalent. □

3.4.20 Definition and Remark

Let A be an interior algebra and let (a_i) and (b_i) be two nets in A based on the same directed set **W**. (a_i) is said to **dominate** (b_i) iff $a_i \geq b_i$ for all $i \in W$. If (a_i) dominates (b_i) then clearly $\lim (a_i) \subseteq \lim (b_i)$ and $\mathrm{acc}\,(a_i) \subseteq \mathrm{acc}\,(b_i)$. In particular we see that if the interior algebra A in Proposition 3.4.19 is atomic, then we may replace nets and sequences in $(c] - \{ 0 \}$ by nets and sequences in $\mathrm{At}\,(c]$ in (ii) – (v) of Proposition 3.4.19. □

3.5 BASES AND COUNTABILITY PROPERTIES

3.5.1 Definition

Let G be a generalized topology in a Boolean algebra **L**. By a base for G we mean a subset $S \subseteq G$ such that every element of G is a join of elements in S. If $S \subseteq G$ we say that S is a **subbase** for G iff $\{ \sqcap R : R$ is a finite subset of $S \}$ is a base for G. By a base or subbase for an interior algebra **A** we mean a base or subbase for the generalized topology A^O. □

3.5.2 Lemma

Let A be an interior algebra and let S be a base for **A**. Then for all $a \in A$, $a^I = \Sigma \{ b \in S : b \leq a \}$.

Proof:

Let $a \in A$. Then $a^I = \max \{ b \in A^O : b \leq a \}$. Thus a^I is an upper bound for $\{ b \in S : b \leq a \}$. Let d be any upper bound for $\{ b \in S : b \leq a \}$. Suppose $b \in A^O$ with $b \leq a$. There is an $R \subseteq S$ with $b = \Sigma R$. Then for all $r \in R$, $r \leq b \leq a$ and so $r \leq d$. Thus $b \leq d$. But then $d \leq a^I$. Hence $a^I = \Sigma \{ b \in S : b \leq a \}$.

3.5.3 Theorem

Let L be a Boolean algebra and let $S \subseteq B$. Then S is a base for a generalized topology in **B** iff it satisfies the following:

i) For all $a \in L$, $\Sigma \{ b \in S : b \leq a \}$ exists.

ii) If R is a finite subset of S then $\sqcap R = \Sigma \{ b \in S : b \leq \sqcap R \}$.

Proof:

Let S be a base for a generalized topology G in L. From Lemma 3.5.2 we see that (i) holds. Now suppose R is a finite subset of S. Then $\sqcap R \in G$ and so we see that $\Sigma \{ b \in S : b \leq \sqcap R \} = \Sigma \{ b \in G : b \leq \sqcap R \} = \sqcap R$ whence (ii) holds. Conversely suppose (i) and (ii) hold. Let G be the set of all joins of members of S. We show that G is a generalized topology in L. ϕ is a finite subset of S and $1 = \sqcap \phi$ whence $1 = \Sigma \{ b \in S : b \leq 1 \} \in G$. Let $a,b \in G$. Then c,d are joins of elements in S and so we in fact have $c = \Sigma \{ b \in S : b \leq c \}$

and $d = \Sigma \{ r \in S : r \leq d \}$. Then $cd = \Sigma \{ cr : r \in S \text{ and } b \leq d \} = \Sigma \{ \Sigma \{ br : b \in S \text{ and } b \leq c \} : r \in S \text{ and } r \leq d \} = \Sigma \{ br : b,r \in S, b \leq c \text{ and } r \leq d \}$. Thus $cd \in G$. Lastly consider $a \in \mathbf{L}$. Put $c = \Sigma \{ b \in S : b \leq a \}$. Then $c \in G$ and $c \leq a$. Let $d \in G$ with $d \leq a$. Then $d = \Sigma R$ for some $R \subseteq S$. Then for all $r \in R$, $r \leq d \leq a$ whence $r \leq c$. Thus $c = \max \{ b \in G : b \leq a \}$. By Proposition 1.1.4 G is a generalized topology in **L** and so S is a base. □

3.5.4 Corollary

Let **L** be a complete Boolean algebra and let $S \subseteq \mathbf{L}$.

i) If $1 \in S$ and S is closed under binary meets then S is a base for a generalalized topology in **L**.

ii) S is always a subbase for a generalized topology in **L**. □

3.5.5 Corollary

Let **A** be an interior algebra and let **L** be a complete Boolean algebra. If $f : \mathbf{A}^u \longrightarrow \mathbf{L}$ is an embedding then there is a generalized topology G in **L** with base $f[A^O]$ such that $f : \mathbf{A} \longrightarrow \mathrm{Alg} \langle \mathbf{L}, G \rangle$ is a homomorphic embedding.

Proof:

By Corollary 3.5.4 (i), $f[A^O]$ is indeed a base for a generalized topology G in **L**. Let $a \in A$. By Lemma 3.5.2 we see that in $\mathrm{Alg} \langle \mathbf{L}, G \rangle$, $f(a)^I = \Sigma \{ f(b) : b \in A^O \text{ and } f(b) \leq f(a) \} = \Sigma \{ f(b) : b \in A^O \text{ and } b \leq a \} = \Sigma \{ f(b) : b \in A^O \text{ and } b \leq a^I \} = \Sigma \{ f(b) : b \in A^O \text{ and } f(b) \leq f(a^I) \} = f(a^I)$ and the result follows. □

3.5.6 Corollary

Let **L** be a complete Boolean algebra and let **A** be a complete interior algebra. If $f : \mathbf{L} \longrightarrow \mathbf{A}^u$ is a complete surjective homomorphism then there is a generalized topology G in **L** with base $f^{-1}[A^O]$ such that $f : \mathrm{Alg} \langle \mathbf{L}, G \rangle \longrightarrow \mathbf{A}$ is a complete surjective homomorphism.

Proof:

By Corollary 3.5.4, $f^{-1}[A^O]$ is a base for a generalized topology in **L**. Let $a \in L$. By Lemma 3.5.2 we see that in $\mathrm{Alg} \langle \mathbf{L}, G \rangle$, $a^I = \Sigma \{ b \leq a : f(b) \in A^O \}$. Thus $f(a^I) = \Sigma \{ f(b) : f(b) \leq f(a) \text{ and } f(b) \in A^O \} = \Sigma \{ c \leq f(a) : c \in A^O \} = f(a)^I$ and the result follows. □

3.5.7 Remark

Corollary 3.5.5 is particularly important since it allows us to generalize the many results concerning complete extensions of Boolean algebras, to interior algebras. For example, every interior algebra is homomorphically embeddable in a complete interior algebra via an embedding that preserves a specified set of joins and meets. □

3.5.8 Definition and Remark

Let A be an interior algebra. A is said to be **first countable** iff for each atom b in A the filter N(b) is sequential. A is said to be **second countable** iff A has a countable base. A is said to be a **Lindelöf** algebra iff for all $S \subseteq A^O$, $\Sigma S = 1$ implies that $\Sigma R = 1$ for some countable $R \subseteq S$. Note that the above concepts generalize the concepts in topology of the same name. By Proposition 1.2.7 we see that the classes of second countable and Lindelöf interior algebras are both Heyting classes. □

3.5.9 Theorem

Any second countable interior algebra is first countable and Lindelöf.

Proof:

Let A be second countable. Let B be a countable base for A. Consider an atom b in A. Put $S = \{ r \in B : b \leq r \}$. Then $S \subseteq N(b)$. Let $d \in N(b)$. There is an $R \subseteq B$ with $d^I = \Sigma R$. Now $b \leq d^I$ and b is an atom so $b \leq r$ for some $r \in R$. Then $r \in S$ and $r \leq d$. Thus S is a base for the filter N(a). Moreover S is countable since B is and so A is first countable. Now consider an $S \subseteq A^O$ with $\Sigma S = 1$. For all $a \in S$ there is an $R(a) \subseteq B$ with $a = \Sigma R(a)$. Put $R = \cup \{ R(a) : a \in S \}$. For each $t \in R$ there is an $a(t) \in S$ with $t \in R(a(t))$. Put $W = \{ a(t) : t \in R \}$. Then $|W| \leq |R| \leq |B|$ and so $W \subseteq S$ is countable. Let d be an upper bound for W. Then for all $t \in R$, $t \leq \Sigma R(a(t)) = a(t) \leq d$. Thus d is an upper bound for R. Hence for all $a \in S$, d is an upper bound for R(a) whence $a \leq \Sigma R(a) \leq d$. Thus d is an upper bound for S and so $d = 1$ whence $\Sigma W = 1$. Thus A is Lindelöf. □

Even for complete atomic interior algebras the converse of the above fails and the classes of first countable and Lindelöf algebras are incomparable as can be seen from topology. (See [19].)

3.5.10 Proposition

Let A be an interior algebra. Consider the following assertions:

i) A is a Lindelöf algebra.

ii) Every countably complete proper filter in $A^\square$ has a non–zero lower bound in $A^\square$.

iii) Every countably complete proper filter in A has a non–zero accumulant.

We always have (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). If A is countably complete then (i), (ii) and (iii) are all equivalent.

Proof:

Assume (i). Let F be a countably complete proper filter in $A^{\square}$. Suppose $\sqcap F = 0$. By duality and the definition of Lindelöf there is a countable $S \subseteq F$ with $\sqcap S = 0$. (Recall that by the dual of Proposition 1.2.7, meets in $A^{\square}$ and A coincide.) But then $0 \in F$, a, contradiction. Hence there is a lower bound $b > 0$ for F in $A^{\square}$. Thus (i) $\Longrightarrow$ (ii). Assume (ii). Let F be a countably complete proper filter in A. Then $F \cap A^{\square}$ is a countably complete proper filter in $A^{\square}$. Let b be a non–zero lower bound for $F \cap A^{\square}$. Noting that $F \cap A^{\square} = \{ a^C : a \in F \}$ we see that $F \longleftarrow b$. Thus (ii) $\Longrightarrow$ (iii). Now let A be countably complete. Assume (iii). Let $S \subseteq A^{\square}$ with $\sqcap S = 0$. Suppose that $\sqcap R > 0$ for all countable $R \subseteq S$. Let $F = \{ b \in A : b \geq \sqcap R$ for some countable $R \subseteq S \}$. Clearly F is a proper filter. Let W be a countable subset of F. For all $a \in W$ there is a countable subset $R(a) \subseteq S$ with $a \geq \sqcap R(a)$. Put $R = \cup \{ R(a) : a \in W \}$. Then R is a countable subset of S. Then for all $a \in W$, $a \geq \sqcap R(a) \geq \sqcap R$ and so $\sqcap W \geq \sqcap R$ whence $\sqcap W \in F$. Thus F is countably complete. Hence $F \longleftarrow b$ for some $b > 0$. But $S \subseteq F$ and so for all $a \in S$, $b \leq a^C = a$, a contradiction since $\sqcap S = 0$. Thus $\sqcap R = 0$ for some countable $R \subseteq S$. By duality we see that A is Lindelöf. Thus (iii) $\Longrightarrow$ (i) and so it follows that (i), (ii) and (iii) are all equivalent. □

3.5.11 Corollary

Let A be a countably complete interior algebra. The following are equivalent:

i) Every countably complete proper filter in A accumulates at an atom.

ii) A is a closed atomic Lindelöf algebra.

Proof:

Assume (i). By Theorem 3.5.10, A is Lindelöf. Let $b > 0$ be closed in A. Then $[b)$ is a countably complete (in fact complete) proper filter in A and so $[b) \longleftarrow a$ for some atom a. But then $a \leq b^C = b$. Thus A is closed atomic. Thus (i) $\Longrightarrow$ (ii). Now assume (ii). Let F be a countably complete proper filter in A. Since A is Lindelöf, $F \longleftarrow b$ for some $b > 0$. Then $F \longleftarrow b^C$. Also $b^C > 0$ and so there is an atom $a \leq b^C$ whence $F \longleftarrow a$. Thus (ii) $\Longrightarrow$ (i). □

Let A be an interior algebra. Suppose $c \in A$ and b is an atom in A. If there is sequence (a_n) in $(c] - \{ 0 \}$ with $(a_n) \longrightarrow b$ then by Proposition 3.4.19 we see that $b \ll c$. However the converse fails even for complete atomic interior algebras: Let X be the co–countable space on $\aleph_1$. Then in X we have $\{ 0 \}'^C = \aleph_1$ and so $0 \in \{ 0 \}'^C$. There is no sequence (x_n) in $Z = \{ 0 \}'$ with $(x_n) \longleftarrow 0$ in X. For suppose (x_n) is such a sequence. Then $Y = \aleph_1 - \{ x_n : n < \omega \}$ is a neighbourhood of 0 in X and so for some $k < \omega$, $x_k \in Y$, a contradiction. By 3.4.20 we see that there is no sequence (B_n) in $(Z] - \{ \phi \}$ with even $(B_n) \longleftarrow \{ 0 \}$ in X^A although $\{ 0 \} \leq \{ 0 \}'^C$ in X^A. We therefore make the following definition:

3.5.12 Definition

An interior algebra A is called sequentially determined iff for all $c \in A$ and all atoms b in A, $b \ll c$ implies that $(a_n) \longrightarrow b$ for some sequence (a_n) in $(c] - \{ 0 \}$. □

3.5.13 Theorem

Any first countable interior algebra is sequentially determined. □

The above follows from Proposition 3.4.19. However, the converse fails even for complete atomic interior algebras: Let **X** be the co–finite space on $\aleph_1$. Suppose that the neighbourhood filter of 0 has a countable base with enumeration $\{ B_n : n < \omega \}$. $C = \cap \{ B_n : n < \omega \}$ is a countable intersection of co–finite sets and so it is co–countable. In particular there is an $x \in C - \{ 0 \}$. Then $\{ x \}'$ is a neighbourhood of 0 and so $B_n \subseteq \{ x \}'$ for some $n < \omega$. Then $x \notin B_n$, a contradiction. Thus **X** is not a first countable space. However X^A is sequentially determined: Let $z \in \aleph_1$ and suppose $Y \subseteq \aleph_1$ with $z \in Y^C$ in **X**. If Y is finite then $z \in Y$ and so putting $x_n = z$ for for all $n < \omega$ gives a sequence in Y with $(x_n) \longrightarrow z$. If Y is infinite let (x_n) be any injective sequence in Y. Let U be an open neighbourhood of z. Suppose that for all $k < \omega$ there is an $m(k) \geq k$ with $x_{m(k)} \notin U$. Then $\{ x_{m(n)} : n < \omega \}$ is an infinite subset of U', a contradiction since U is co–finite. Thus there is a $k < \omega$ with $x_n \in U$ for all $n \geq k$. Thus $(x_n) \longrightarrow b$. It follows that X^A is sequentially determined.

The countability properties we have introduced are particularly interesting as regards their preservation under principal quotients and joins. We investigate this now.

3.5.14 Lemma

Let **A** be an interior algebra and let $c \in A$. If N^c denotes the neighbourhood function for **(c]** then for all $b \in (c]$, $N^c(b) = \{ ac : a \in N(b) \text{ in } \mathbf{A} \}$.

Proof:

Let $k : \mathbf{A} \longrightarrow \mathbf{(c]}$ denote the canonical topomorphism. Clearly all topomorphisms preserve neighbourhoods and so if a is a neighbourhood of b in **A** then $ac = k(a)$ is a neighbourhood of b in **(c]**. Conversely suppose that d is a neighbourhood of b in **(c]**. There is an open element e of **(c]** with $b \leq e \leq d$. Then there is an open element v of **A** with $e = k(v) = vc$. (See Theorem 1.4.6.) Put $a = v + d$. Then $a \geq v \geq e \geq b$ and so a is a neighbourhood of b in **A**. Moreover $ac = vc + dc = e + d = d$, and the result follows. □

3.5.15 Theorem

The following classes are closed under principal quotients:

i) First countable interior algebras.

ii) Second countable interior algebras.

iii) Sequentially determined interior algebras.

Proof:

(i): Let **A** be first countable and let $c \in A$. Let b be an atom in **(c]**. Then b is an atom in **A** and so there is a countable base B for the filter N(b) in **A**. But by Lemma 3.5.14, the neighbourhood filter of b in (c] is just $N^c(b) = \{ ac : a \in N(b) \}$. Now $C = \{ ac : a \in B \}$ is a countable subset of $N^c(b)$ and for all $a \in N(b)$ there is an $e \in B$ with $e \leq a$ whence $ec \in C$ and $ec \leq ac$. Thus C is a base for $N^c(b)$ and the result follows. (ii): Let **A** be a second countable interior algebra and let $c \in A$. There is a countable base B for **A**. Put $C = \{ ac : a \in B \}$. Then C is a countable subset of $(c]^O$. Let $d \in (c]^O$. There is a $b \in A^O$ with $d = bc$. Then there is an $S \subseteq B$ with $\Sigma S = b$. But then $d = \Sigma \{ ac : a \in S \}$ and $\{ ac : a \in S \} \subseteq C$. Thus C is a base for **(c]** and the result follows. (iii): Let **A** be a sequentially determined interior algebra and let $c \in A$. Let b be an atom in **(c]** and let $b \ll d$ in **(c]**. Then $b \leq d^{C,c} = cd^C \leq d^C$ and so $b \ll d$ in **A**. Hence there is a sequence (a_n) in $(d] - \{ 0 \}$ with $(a_n) \longrightarrow b$ in **A**. Let e be a neighbourhood of b in **(c]**. By Lemma 3.5.14 there is a neighbourhood v of b in **A** with $e = vc$. Hence there is a $k < \omega$ with $a_n \leq v$ for all $n \geq k$. But for all $n < \omega$, $a_n \leq d \leq c$ and so for all $n \geq k$, $a \leq vc = e$. Thus $(a_n) \longrightarrow b$ in **(c]** and the result follows. □

(ii) above allows us to prove:

3.5.16 Theorem

The following are equivalent for an interior algebra **A**:

i) **A** is second countable.

ii) Any base for **A** contains a countable base for **A**.

Proof:

Clearly (ii) ⟹ (i) since A^O is a base for A. Assume (i). Then there is a countable base B for A. Suppose S is a base for A. Let $b \in B$. There is an $R \subseteq S$ with $\Sigma R = b$. Then $R \subseteq (b]$ and so in fact $R \subseteq (b]^O$ and $\Sigma R = b$ in **(b]**. By Theorem 3.5.15 (ii), **(b]** is second countable. By Theorem 3.5.9, **(b]** is Lindelöf. Hence there is a countable $T(b) \subseteq R$ with $\Sigma T(b) = b$ in **(b]** and hence in **A**. Put $T = \cup \{ T(b) : b \in B \}$. Then T is a countable subset of S. Consider $d \in A^O$. There is a $W \subseteq B$ with $\Sigma W = d$. Then $d = \Sigma \{ \Sigma T(b) : b \in W \} = \Sigma (\cup \{ T(b) : b \in W \})$ and moreover $\cup \{ T(b) : b \in W \} \subseteq T$. Thus T is a base for **A**. Thus (i) ⟹ (ii). □

3.5.17 Theorem

The class of Lindelöf algebras is closed under closed quotients.

Proof:

Let **A** be a Lindelöf algebra and let c be closed in **A**. Let $R \subseteq (c]^O$ with $\Sigma R = c$ in **(c]** and hence in **A**. Then for all $r \in R$ there is a $b(r) \in A^O$ with $r = cb(r)$. Suppose d is an upper bound for $\{ b(r) : r \in R \} \cup \{ c' \}$. Then for all $r \in R$, $r = cb(r) \leq cd$. Hence $c = cd$, that is $c \leq d$. But $c' \leq d$ and so $d = 1$. Thus $\Sigma (\{ b(r) : r \in R \} \cup \{ c' \}) = 1$. Hence there is a countable $W \subseteq R$ with $\Sigma (\{ b(r) : r \in W \} \cup \{ c' \}) = 1$. But then $c = \Sigma (\{ cb(r) : r \in W \} \cup \{ cc' \}) = \Sigma (W \cup \{ 0 \}) = \Sigma W$. Thus (c] is a Lindelöf algebra. □

However, we know from topology that even the class of complete atomic Lindelöf algebras is not closed under open quotients (which correspond to open subspaces in topology). (See [19].) We therefore make the following definition:

3.5.18 Definition

An interior algebra **A** is called an **hereditarily Lindelöf algebra** iff every principal quotient of **A** is a Lindelöf algebra. □

3.5.19 Corollary

Every second countable interior algebra is an hereditarily Lindelöf algebra. □

The above follows from Theorem 3.5.9 and Theorem 3.5.15 (ii). Again we know from topology that the converse fails. Consider the co–finite space on $\aleph_1$. We saw on page 84 that **X** is not first countable and hence not second countable. However any subspace of **X** is a co–finite space which is compact and hence Lindelöf. Thus $\mathbf{X}^A$ is an hereditarily Lindelöf algebra. (Recall that principal quotients correspond to subspaces.)

3.5.20 Theorem

The following classes of interior algebras are closed under open joins (in particular, under products).

i) First countable interior algebras.

ii) Sequentially determined interior algebras.

Proof:

(i): Let **A** be an interior algebra and let $S \subseteq A^O$ with $\Sigma S = 1$ and **(a]** first countable for all $a \in S$. Let b be an atom in **A**. Then $b \leq a$ for some $a \in S$. There is a countable base B for the neighbourhood filter $N^a(b)$ of b in **(a]**. Note that since a is open $N^a(b) = N(b) \cap (a]$ and so in fact $B \subseteq N(b)$. Let $d \in N(b)$. Then $ad \in N^a(b)$ and so there is a $v \in W$ with $v \leq ad \leq d$. Thus B is a countable base for N(b) and the result follows. (ii): Let **A** be an interior algebra and $S \subseteq A^O$ with $\Sigma S = 1$ and (a] sequentially determined for all $a \in S$. Let b be an atom in **A** and let $c \in A$ with $b \ll c$. Then there is a $d \in S$ with $b \leq d$. Note that since d is open the quotient map from **A** onto **(d]** given by $r \longmapsto dr$ for all $r \in A$, is a homomorphism and so we see that $b \ll dc$ in **(d]**. Hence there is a sequence (a_n) in $(dc] - \{ 0 \}$ with $(a_n) \longrightarrow b$ in **(d]**. But then (a_n) is a sequence in $(d] - \{ 0 \}$ and by Lemma 3.5.14 we see that $(a_n) \longrightarrow b$ in **A** whence the result follows. □

The classes of first countable and sequentially determined interior algebras are not closed under closed joins: Let **X** be the co–countable space on $\aleph_1$. On page 83 we saw that $\mathbf{X}^A$ was not sequentially determined and hence also not finitely based. However every $x \in \aleph_1$ is closed in **X** and so it follows that $\mathbf{X}^A$ is a closed join of $\aleph_1$ copies of the unique two element interior algebra which is trivially finitely based and hence also sequentially determined.

Considering the correspondence between topological sums and products of interior algebras we see from topology that the classes of Lindelöf algebras, hereditarily Lindelöf algebras and second countable interior algebras are not closed under products. (See [19].) However we have:

<u>3.5.21 Theorem</u>

The class of second countable interior algebras is closed under countable open joins (in particular, under countable products).

<u>Proof:</u>

Let **A** be an interior algebra and let $S \subseteq A^O$ be countable with $\Sigma S = 1$ and let **(a]** be second countable for all $a \in S$. Then for all $a \in S$ there is a countable base B(a) for **(a]**. Moreover since $a \in A^O$ we have $B(a) \subseteq A^O$. Putting $B = \cup \{ B(a) : a \in S \}$ gives a countable subset of A^O. Now if $a \in S$ and $b \in (a]^O$ there is an $R(a,b) \subseteq B(a)$ with $b = \Sigma R(a,b)$ in **(a]** and hence in **A**. Now if $d \in A^O$, $ad \in (a]^O$ for all $a \in S$ and so we have $d = \Sigma \{ ad : a \in S \} = \Sigma \{ \Sigma R(a,ad) : a \in S \} = \Sigma R(d)$ where $R(d) = \cup \{ R(a,ad) : a \in S \}$. Then for all $d \in A^O$, $R(d) \subseteq B$ and so B is a base for **A** whence the result follows. □

The class of second countable interior algebras is not closed under countable closed joins: Let **X** be the Arens–Fort space. (See [19] for details). **X** is not second countable but it is T_1 (whence every point is closed) and countable [19]. Thus $\mathbf{X}^A$ is a countable closed join of $\aleph_0$ copies of the two element interior algebra which is trivially second countable.

3.5.22 Theorem

The following classes are closed under countable joins (in particular, under countable products).

i) Lindelöf algebras.

ii) Hereditarily Lindelöf algebras.

Proof:

(i): Let **A** be an interior algebra and let $S \subseteq A$ with $\Sigma S = 1$ and **(a]** a Lindelöf algebra for all $a \in S$. Let $R \subseteq A^O$ with $\Sigma R = 1$. Then for all $a \in S$, $\Sigma \{ ar : r \in R \} = a$ in **A** hence in **(a]**. But $\{ ar : r \in R \} \subseteq (a]^O$ and so there is a countable subset $T(a) \subseteq R$ with $\Sigma \{ ar : r \in T(a) \} = a$ in **(a]** hence in **A**. Put $T = \cup \{ T(a) : a \in S \}$. Then T is a countable subset of R. Suppose d is an upper bound of T. Then for all $a \in S$, d is a upper bound of T(a) and hence of $\{ ar : r \in T(a) \}$. Thus for all $a \in S$, $d \geq \Sigma \{ ar : r \in T(a) \} = a$. Thus $d = 1$ and so $\Sigma T = 1$. Thus A is a Lindelöf algebra. (ii): Let **A** be an interior algebra and let $S \subseteq A$ with $\Sigma S = 1$ and **(a]** an hereditarily Lindelöf algebra for all $a \in S$. Let $b \in A$. Then $b = \Sigma \{ ab : a \in S \}$ in **A** and hence in **(b]**. Also for all $a \in S$, **(ab]** is a Lindelöf algebra since it is a principal quotient of **(a]**. Thus **(b]** is a join of Lindelöf algebras and so it is Lindelöf by (i). Thus **A** is hereditarily Lindelöf. □

CHAPTER 4

SEPARATION PROPERTIES IN INTERIOR ALGEBRAS

The topic of 'separation' in interior algebras is concerned with the generalization of certain properties of metrizable topologies to interior algebras. It is concerned with the existence of certain neighbourhoods of distinct atoms or disjoint closed elements.

4.1 SEPARATION PROPERTIES RELATED TO UNIQUENESS OF ATOMIC LIMITS

Recall that a pair of elements a and b in a meet–semilattice with 0 are called *disjoint* iff $ab = 0$.

4.1.1 Theorem

The following are equivalent for an interior algebra A:

i) Every proper (ultra)filter in A converges to at most one atom.

ii) Every proper grill in A accumulates at at most one atom.

iii) Every net in $A - \{0\}$ converges to at most one atom.

iv) For all $R \subseteq A$ and all atoms a and b in A, $R \longrightarrow a$ and $R \longleftarrow b$ implies $a = b$.

v) For each pair of distinct atoms a and b in A there are neighbourhoods $d \in N(a)$ and $c \in N(b)$ which are disjoint.

vi) For all atoms a and b in A, $N(a) \subseteq E(b)$ implies $a = b$.

Proof:

Assume (i). Let a and b be distinct atoms in A. Suppose for all $d \in N(a)$ and all $c \in N(b)$, $cd > 0$. Then there is a proper (ultra)filter F in A with $N(a) \cup N(b) \subseteq F$. Then $F \longrightarrow a$ and $F \longrightarrow b$, a contradiction. Thus there are a disjoint pair $d \in N(a)$ and $c \in N(b)$. Thus (i) $\Longrightarrow$ (v). Assume (v). Suppose there is an $R \subseteq A$ and distinct atoms with $R \longrightarrow a$ and $R \longleftarrow b$. There is a disjoint pair $d \in N(a)$ and $c \in N(b)$. There is an $r \in R$ with $r \leq d$. Now $r \in E(b)$ whence $d \in E(b)$. But $E(b) = \text{Sec}\, N(b)$, a contradiction since $cd = 0$. Thus (v) $\Longrightarrow$ (iv). By Proposition 3.2.13 we see that (iv) $\Longrightarrow$ (ii). Assume (ii). Let F be a proper filter in A. Let a and b be atoms in A with $F \longrightarrow a$ and $F \longrightarrow b$. There is an ultrafilter G with $F \subseteq G$. Then $G \longleftarrow a$ and $G \longleftarrow b$ by Corollary 3.2.14. Thus $a = b$ and so (ii) $\Longrightarrow$ (i). Thus (i), (ii), (iv) and (v) are all equivalent. (v) $\Longleftrightarrow$ (vi) By Proposition 3.2.11. Considering the canonical net of a filter we see that (iii) $\Longrightarrow$ (i). Considering the filter $F(a_i)$ obtained from a net (a_i) we see that (i) $\Longrightarrow$ (iii). Thus (i) and (iii) are equivalent and so in fact

(i) – (vi) are all equivalent. □

4.1.2 Definition and Remark

Call an interior algebra satisfying the equivalent conditions in Theorem 4.1.1, a **Hausdorff** algebra. From condition (iv) in Theorem 4.1.1 we see that this definition is, as desired, a generalization of Hausdorff (T_2) spaces. □

4.1.3 Definition

Call an interior algebra **A** a **point Hausdorff** algebra iff every net in At **A** converges to at most one atom. □

Clearly every Hausdorff algebra is point Hausdorff. But the converse fails: Let **L** be the free Boolean algebra on $\aleph_0$ generators and let 2 be the two element Boolean interior algebra. Choose any $a,b \in L$ such that a,b are incomparable and $ab > 0$. Let **A** be the interior algebra with $A^u = L \times 2^u \times 2^u$ and $A^O = \{ (0,0,0), (ab,0,0), (a,1,0), (b,0,1), (a+b,1,1), (1,1,1) \}$. Then **A** has two atoms $x = (0,1,0)$ and $y = (0,0,1)$ but these do not have a pair of disjoint neighbourhoods. Thus **A** is not Hausdorff. Consider a net (z_i) in At **A** based on a directed set **W**. Suppose $(z_i) \longrightarrow x$ and $(z_i) \longrightarrow y$. Now $(a,1,0) \in N(x)$. Thus there is a $j \in W$ with $z_i \leq (a,1,0)$ for all $i \gg j$ in W. Also $(b,0,1) \in N(y)$ and so there is a $k \in W$ with $z_i \leq (b,0,1)$ for all $i \gg k$ in **W**. There is an $i \gg j,k$. Then we must have $x = z_i = y$, a contradiction. Thus **A** is point Hausdorff but not Hausdorff.

4.1.4 Theorem

Let A be an open atomic interior algebra. Then the following are equivalent:

i) **A** is Hausdorff.

ii) **A** is point Hausdorff.

Proof:

We know (i) $\Longrightarrow$ (ii). Assume (ii). Let a and b be distinct atoms in **A**. Suppose that for all $d \in N(a)$ and $c \in N(b)$, $cd > 0$. Then $N(a) \cup N(b)$ is a subbase for a proper filter F in **A**. Note that F is just the join of the filters N(a) and N(b) which are open and so F is itself open. Define a directed set $\mathbf{W} = \langle W, \ll \rangle$ as follows: For all $v \in F$, $v^I \in F$ and so $v^I > 0$ whence by open atomicity there is at least one atom $r \leq v^I$, that is $v \in N(r)$. Put $W = \{ (v,r) : v \in F \text{ and } r \text{ is an atom with } v \in N(r) \}$. Let $(v,r) \ll (w,s)$ in **W** iff $v \geq w$. Clearly $\ll$ is a pre–order. Let $(v,r),(w,s) \in W$. Then $vw \in F$ and so as we saw there is at least one atom t in **A** with $vw \in N(t)$. Then $(vw,t) \in W$ and $(vw,t) \gg (v,r),(w,s)$. Thus $\mathbf{W} = \langle W, \ll \rangle$ is a directed set as required. For all $i = (v,r) \in W$ put $z_i = r$. We thus have a net (z_i) in At **A**. Now consider $d \in N(a)$. Then $(d,a) \in W$. Suppose $i = (v,r) \gg (d,a)$ in **W**. $z_i = r \leq v \leq d$. Thus $(z_i) \longrightarrow a$ and similarly we see that $(z_i) \longrightarrow b$, a contradiction. Thus there is a disjoint pair $d \in N(a)$ and $c \in N(b)$. Thus (ii) $\Longrightarrow$ (i). □

4.1.5 Definition

Call an interior algebra **A**, **sequentially Hausdorff** iff every sequence in $A - \{ 0 \}$ converges to at most one atom. □

Clearly every Hausdorff algebra is sequentially Hausdorff. However the converse fails even for complete atomic interior algebras: Let X be the co–countable space on $\aleph_1$. Suppose there is a neighbourhood S of 0 and a neighbourhood R of 1 with $S \cap R = \phi$. Then $S^I \cap R^I = \phi$, a contradiction since $S^I \cap R^I$ is an intersection of two co–countable sets and hence is co–countable itself. Thus X is not a Hausdorff space. Now suppose there is a sequence (x_n) in X and distinct points y and z in X with $(x_n) \longrightarrow y$ and $(x_n) \longrightarrow z$. $\{ y \}'$ is a neighbourhood of z. Hence there is a $k < \omega$ with $x_n \in \{ y \}'$ for all $n \geq k$. $W = \{ x_n : n \geq k \}$. Then $W \subseteq \{ y \}'$. Now $y \in W'$ and so W', being co–countable, is a neighbourhood of y. Thus there is an $s < \omega$ with $x_n \in W'$ for all $n \geq s$. Then we both $x_{k+s} \in W$ and $x_{k+s} \in W'$, a contradiction. Using 3.4.20 we now see that every sequence (B_n) in $X^A - \{ \phi \}$ converges to at most one atom in X^A. Thus X^A is sequentially Hausdorff but not Hausdorff.

4.1.6 Theorem

Let **A** be a first countable interior algebra. Then the following are equivalent:

i) **A** is Hausdorff.

ii) **A** is sequentially Hausdorff.

Proof:

We know that (i) $\Longrightarrow$ (ii). Assume (ii). Let (z_i) be a net in $A - \{ 0 \}$ based on a directed set **W**. Let a and b be atoms in **A** with $(z_i) \longrightarrow a$ and $(z_i) \longrightarrow b$. There are countable bases V and W for the filters N(a) and N(b) respectively. Let $\{ v_n : n < \omega \}$ and $\{ w_n : n < \omega \}$ be enumerations of V and W respectively. For all $n < \omega$ put $s_n = (v_0 \cdots v_n)^I$ and $r_n = (w_0 \cdots w_n)^I$. Let $n < \omega$. Then $s_n \in N(a)$ and $r_n \in N(b)$. Hence there are $k,j \in$ **W** with $z_i \leq s_n$ for all $i \gg k$ and $z_i \leq r_n$ for all $i \gg j$. Then there is an $s \gg k,j$ in **W**. Put $x_n = z_s$. We thus have a sequence (x_n) in $A - \{ 0 \}$. Let $d \in N(a)$. Then there is a $k < \omega$ with $v_k \leq d$. Then for all $n \geq k$ we have $x_n \leq s_n \leq s_k \leq v_k \leq d$. Thus $(x_n) \longrightarrow a$. Similarly we see that $(x_n) \longrightarrow b$. Thus $a = b$ by (ii). It follows that **A** is Hausdorff. Thus (ii) $\Longrightarrow$ (i). □

4.1.7 Definition and Remark

Call an interior algebra A, **sequentially point Hausdorff** iff every sequence in At A converges to at most one atom. □

Clearly every sequentially Hausdorff algebra is sequentially point Hausdorff and every point Hausdorff algebra is sequentially point Hausdorff. Consider the interior algebra A discussed on page 90. Then A is not sequentially Hausdorff since the sequence (z_n) with $z_n = (ab,0,0)$ for all $n < \omega$ converges to both atoms x and y since the only open element above x is (a,1,0) and the only open element above y is (b,0,1) both of which are above (ab,0,0). However A is sequentially point Hausdorff since it is point Hausdorff. Consider the co–countable space X on $\aleph_1$. We saw on page 91 that X^A is sequentially Hausdorff and hence sequentially point Hausdorff. However X^A is not point Hausdorff since it is open atomic but not Hausdorff. However we easily see from 3.4.20 that an atomic interior algebra is sequentially Hausdorff iff it is sequentially point Hausdorff. It is still not known whether this holds for open atomic interior algebras which would be an analogue of Theorem 4.1.4. We do have analogue of Theorem 4.1.6. The proof is essentially the same as in 4.1.6 and is left to the reader.

4.1.8 Theorem

Let A be first countable. The following are equivalent:

i) A is point Hausdorff.

ii) A is sequentially point Hausdorff. □

4.1.9 Corollary

The following are equivalent for a first countable open atomic interior algebra:

i) A is Hausdorff.

ii) A is point Hausdorff.

iii) A is sequentially Hausdorff.

iv) A is sequentially point Hausdorff. □

4.2 SEPARATION PROPERTIES RELATED TO THE CANONICAL PRE–ORDER

4.2.1 Definition and Remark

For each interior algebra A let **At A** denote the pre–ordered set ⟨ At A, ≪ ⟩ where ≪ is the canonical pre–order given by a ≪ b iff $a \leq b^C$. Notice that if X is a topological space then X^W is isomorphic to **At** X^A via the map $x \longmapsto \{ x \}$. (See Definition 2.4.1.) We call an interior algebra **A** a **Fréchet** algebra iff $\ll$ is the equality relation on At **A**. We call **A** a **symmetric** algebra iff $\ll$ is symmetric and hence an equivalence relation on At **A**, and we call **A** a **Kolmogorov** algebra iff $\ll$ is anti–symmetric and hence a partial order on At **A**. Thus **A** is Fréchet iff it is both symmetric and Kolmogorov. **A** is symmetric iff for all a,b $\in$ At **A**, $N(a) \subseteq N(b)$ implies $N(a) = N(b)$, equivalently $a \leq b^C$ implies $a^C = b^C$. **A** is Kolmogorov iff for all a,b $\in$ At **A**, $N(a) = N(b)$ implies $a = b$, equivalently $a^C = b^C$ implies $a = b$. (See Theorem 3.2.5.) □

4.2.2 Proposition

Let **A** be an interior algebra.

i) **A** is Kolmogorov iff for each pair of distinct atoms a and b in **A** there is a $d \in N(a)$ with $b \leq d'$ or there is a $c \in N(b)$ with $a \leq c'$.

ii) **A** is Fréchet iff for each pair of distinct atoms a and b there there is a $d \in N(a)$ with $b \leq d'$ and there is a $c \in N(b)$ with $a \leq c'$. □

The above follows easily if we consider interiors of neighbourhoods.

4.2.3 Remark

From the above we see that Kolmogorov algebras and Fréchet algebras are generalizations of Kolmogorov (T_0) spaces and Fréchet (T_1) spaces respectively. □

4.2.4 Theorem

Let **A** be an interior algebra.

i) **A** is Kolmogorov iff for each atom b in **A**, $\cap \{ (a] : a \in A^O \cup A^\square \text{ and } b \leq a \}$ contains a unique atom, namely b.

ii) **A** is Fréchet iff for each atom b in **A**, $\cap \{ (a] : a \in A^O \text{ and } b \leq a \}$ contains a unique atom, namely b.

Proof:

(ii) is easy, we prove (i): Let **A** be Kolmogorov. Let b be an atom in **A** and suppose there is an atom c distinct from a with $c \in \cap \{ (a] : a \in A^O \cup A^\square \text{ and } b \leq a \}$. Then if $d \in N(b)$ we have $b \leq d^I \in A^O$ and so $c \leq d^I \leq d$. Since **A** is Kolmogorov we must then have by Proposition 4.2.2 (i), an $e \in N(c)$ with $b \leq e'$. But then $b \leq e'^C \in A^\square$ and so $c \leq e'^C = e^{I\prime}$, a contradiction since $e \in N(c)$. Conversely suppose that for all atoms b in **A**, $\cap \{ (a] : a \in A^O \cup A^\square \text{ and } b \leq a \}$ contains no atom other than b. Suppose b and c are distinct atoms in **A**. Then there is an $a \in A^O \cup A^\square$ with $b \leq a$ but $c \nleq a$. If $a \in A^O$ then $a \in N(b)$ and $c \leq a'$, while if $a \in A^\square$ then $c \leq a' \in A^O$ and so $a' \in N(c)$ and $b \leq a = a''$. Thus **A** is Kolmogorov by Proposition 4.2.2 (i). □

4.2.5 Theorem

Every sequentially point Hausdorff algebra is a Fréchet algebra.

Proof:

Let **A** be sequentially point Hausdorff. Suppose a and b are distinct atoms in **A**. Consider the sequence (a_n) given by $a_n = a$ for all $n < \omega$. Obviously $(a_n) \longrightarrow a$ and so we cannot have $(a_n) \longrightarrow b$. Hence there is a $c \in N(b)$ such that for all $n < \omega$ there is a $k \geq n$ with $a_k \nleq c$. In particular for some $k < \omega$ we have $a = a_k \nleq c$ whence $a \leq c'$. Similarly there is a $d \in N(a)$ with $b \leq d'$ and the result follows by Proposition 4.2.2 (ii). □

The converse to the above fails even for complete atomic interior algebras. Let **X** be the co–finite space on $\aleph_0$. Then we know that X is a Fréchet space. Consider any point $z \in X$ and any sequence (x_n) in X. Consider an open neighbourhood B of z. Then B is co–finite and so for some $k < \omega$ we have $x_n \in B$ for all $n \geq k$. Thus $(x_n) \longrightarrow z$. Thus any sequence in X converges to any point, in particular X^A is not sequentially point Hausdorff.

Besides the Fréchet algebras there is another natural class of interior algebras which is Kolmogorov:

4.2.6 Definition

Let **A** be an interior algebra. Let **B** be the subalgebra of A generated by A^O. We say that **A** **has openly generated atoms** iff At $A \subseteq B$. □

Obviously any openly generated interior algebra has openly genereated atoms. To see that the converse fails consider the free Boolean algebra on $\aleph_0$ generators, **L**. Let **A** be the interior algebra with $\mathbf{A}^u = \mathbf{L} \times \mathbf{2}^u$ and $A^O = \{ (0,0), (0,1), (1,1) \}$. Noting that (0,1) is the unique atom of **A** we see that it has openly generated atoms but the subalgebra **B** generated by A^O has $B = A^O \cup \{ (1,0) \}$ and so A is not openly generated.

4.2.7 Theorem

Every interior algebra with openly generated atoms is Kolmogorov.

Proof:

Let **A** have openly generated atoms. Suppose a and b are distinct atoms in **A**. There are elements $c_1,\dots,c_n,d_1,\dots,d_n \in A^O$ with $a = (c_1 + d_1')\dots(c_n + d_n')$. Now $b \nleq a$ and so $b \leq c_1'd_1 + \dots + c_n'd_n$. Hence there is an $i \in \{ 1,\dots,n \}$ with $b \leq c_i'd_i$. Now $a \leq c_i + d_i'$.

Thus $a \leq c_i$ or $a \leq d_i'$. In the first case $c_i \in N(a)$ with $b \leq c_i'$, in the second case $b \leq d_i$ whence $d_i \in N(b)$ and $a \leq d_i'$. The result follows from Proposition 4.2.2 (i). □

To see that the converse fails let **L** be as before and again let $\mathbf{A}^u = \mathbf{L} \times \mathbf{2}^u$ but this time put $A^o = \{ (0,0), (1,1) \}$. A^o already forms a subalgebra but it does not contain the unique atom (0,1) and so **A** does not have openly generated atoms. But having a unique atom, **A** is trivially Kolmogorov. However we have the following result:

4.2.8 Theorem

The following are equivalent for a finite interior algebra **A**:

i) **A** is Kolmogorov.

ii) **A** is openly generated.

iii) **A** has openly generated atoms.

Proof:

(ii) ⇔ (iii) since every element of **A** is a finite join of atoms. We know that (iii) ⟹ (i). Assume (i) and consider an atom b in A. Then $b = \sqcap \{ a \in A^o \cup A^\square : b \leq a \}$ or else there is an atom $c \in \cap \{ (a] : a \in A^o \cup A^\square \; b \leq a \}$ distinct from b contradicting Theorem 4.2.4 (i). Since $A^\square = \{ a' : a \in A^o \}$ we see that A has openly generated atoms. Thus (i) ⟹ (iii) and the result follows. □

Recall that if **X** is a Fréchet space then every point of **X** is closed. However if **A** is a Fréchet algebra then the atoms of **A** need not be closed: Let **A** be the interior algebra considered before Theorem 4.2.8. Then the unique atom of A is not closed but **A** is trivially Fréchet. $\mathbf{A} \times \mathbf{2}$ is easily seen to be Fréchet with two atoms only one of which is closed and other examples can be found. This leads us to make the following definition:

4.2.9 Definition

Let **A** be an interior algebra. **A** is said to be a **strongly Fréchet** algebra iff every atom of **A** is closed. If $\mathcal{K}$ is a class of interior algebras we will use the term **strongly $\mathcal{K}$** algebra to mean a member of $\mathcal{K}$ which is strongly Fréchet. For example, a strongly Hausdorff algebra is a Hausdorff algebra which is strongly Fréchet. □

Clearly every strongly Fréchet algebra is Fréchet and we have seen that converse fails. Note also that if **A** is the interior algebra discussed before the definition then **A** is sequentially point Hausdorff but not strongly Fréchet and the same is true for the interior algebra **A** discussed on page 90. However we have:

4.2.10 Proposition

Let A be a residually atomic interior algebra. The following are equivalent:

i) A is a Fréchet algebra.

ii) A is a strongly Fréchet algebra.

Proof:

We know (ii) $\Longrightarrow$ (i). Assume (i). Let a be an atom in A. Suppose a is not closed. Then $a^C a' > 0$ and so there is an atom $b \leq a^C a'$. Then $b \ll a$ but $b \neq a$, a contradiction. Thus (i) $\Longrightarrow$ (ii). □

4.3 α–SEPARATION IN INTERIOR ALGEBRAS

Recall the following definition from topology: Let X be a topological space. For each $S \subseteq X$ define S^α, α an ordinal, inductively as follows: For all $S \subseteq X$ put $S^0 = S$. Suppose $\beta > 0$ is an ordinal and S^α has been defined for all $S \subseteq X$ and all $\alpha < \beta$. If β is a limit ordinal put $S^\beta = \cup \{ S^\alpha : \alpha < \beta \}$ for all $S \subseteq X$. If $\beta = \alpha + 1$ for some α put $S^\beta = \{ x \in X : S^\alpha \cap R^\alpha \neq \phi$ for all $R \in N(x) \}$ for all $S \subseteq X$. Also consider $\alpha < \infty$ for all ordinals α and put $S^\infty = \cup \{ S^\alpha : \alpha < \infty \}$ for all $S \subseteq X$. If $\alpha \leq \infty$ we say that X is *α–separated* iff $\{ x \}^\alpha = \{ x \}$ for all $x \in X$. This motivates the following definition:

4.3.1 Definition

Let A be an interior algebra. Define $c^\alpha(a)$ for all $a \in A$ and $\alpha \leq \infty$ inductively as follows: For all $a \in A$ put $c^0(a) = \text{At}\,(a]$. Suppose $0 < \beta \leq \infty$ and $c^\alpha(a)$ has been defined for all $a \in A$ and all $\alpha < \beta$. If β is a limit ordinal or ∞ put $c^\beta(a) = \cup \{ c^\alpha(a) : \alpha < \beta \}$ for all $a \in A$. If $\beta = \alpha + 1$ for some α put $c^\beta(a) = \{ b \in \text{At}\, A : c^\alpha(a) \cap c^\alpha(d) \neq \phi$ for all $d \in N(b) \}$ for all $a \in A$. If $\alpha \leq \infty$ we call A **α–separated** iff $c^\alpha(a) = \{ a \}$ for all atoms a in A. □

It is not difficult to see that a topological space X is α–separated in the usual sense iff X^A is an α–separated interior algebra.

4.3.2 Proposition

Let A be an interior algebra and let $\beta < \alpha \leq \infty$.

i) $c^\beta(a) \subseteq c^\alpha(a)$ for all $a \in A$.

ii) If A is α–separated then A is β–separated.

iii) If for all $a \in A$, $c^\beta(a) = c^\alpha(a)$, then for all $a \in A$, $c^\beta(a) = c^\gamma(a)$ for all γ with $\beta \leq \gamma \leq \infty$. □

The above proposition can be proved using easy induction arguments which are left to the reader.

4.3.3 Proposition

Let **A** be an interior algebra and let $\alpha \leq \omega$.

i) $c^{\alpha}(0) = \phi$.

ii) $c^{\alpha}(1) = \text{At } A$.

iii) If $a \leq b$ then $c^{\alpha}(a) \subseteq c^{\alpha}(b)$.

iv) If $a,b \in A$ then $c^{\alpha}(a + b) = c^{\alpha}(a) \cup c^{\alpha}(b)$.

Proof:

(i), (ii) and (iii) are easy and are left to the reader, we prove (iv) by induction. We obviously have $c^0(a + b) = c^0(a) \cup c^0(b)$ for all $a,b \in A$. Suppose $\alpha > 0$ and that $c^{\beta}(a + b) = c^{\beta}(a) \cup c^{\beta}(b)$ for all $a,b \in A$. If α is a limit ordinal or ω we then have $c^{\alpha}(a + b) = \cup\{ c^{\beta}(a + b) : \beta < \alpha \} = \cup\{ c^{\beta}(a) \cup c^{\beta}(b) : \beta < \alpha \} = (\cup\{ c^{\beta}(a) : \beta < \alpha \}) \cup (\cup\{ c^{\beta}(b) : \beta < \alpha \}) = c^{\alpha}(a) \cup c^{\alpha}(b)$ for all $a,b \in A$. Suppose $\alpha = \gamma + 1$. Let $a,b \in A$. By (iii) we have $c^{\alpha}(a) \cup c^{\alpha}(b) \subseteq c^{\alpha}(a + b)$. Suppose that $c^{\alpha}(a) \cup c^{\alpha}(b) \neq c^{\alpha}(a + b)$. Then there is an $r \in c^{\alpha}(a + b)$ with $r \notin c^{\alpha}(a)$ and $r \notin c^{\alpha}(b)$. Then there are $d,e \in N(r)$ with $c^{\gamma}(a) \cap c^{\gamma}(d) = \phi$ and $c^{\gamma}(b) \cap c^{\gamma}(e) = \phi$. Then $ed \in N(r)$ and $c^{\alpha}(a + b) \cap c^{\gamma}(ed) = (c^{\gamma}(a) \cup c^{\gamma}(b)) \cap c^{\gamma}(ed) = (c^{\gamma}(a) \cap c^{\gamma}(ed)) \cup (c^{\gamma}(b) \cap c^{\gamma}(ed)) = \phi$, a contradiction. Thus $c^{\alpha}(a + b) = c^{\alpha}(a) \cup c^{\alpha}(b)$. □

4.3.4 Lemma

Let **A** be an interior algebra and let $a \in A$.

i) $c^1(a) \subseteq c^0(a^C)$

ii) If $(\mathbf{a}]$ is open atomic then $c^1(a) = c^0(a^C)$.

Proof:

(i): Suppose that there is a $b \in c^1(a)$ with $b \notin c^0(a^C)$. Then $b \nleq a^C$ and so $b \leq a^{C\prime} = a'^{I}$, that is $a' \in N(b)$. Thus $c^0(a) \cap c^0(a') \neq \phi$, that is, there is an atom r in **A** with $r \leq a$ and $r \leq a'$, a contradiction. Thus $c^1(a) \subseteq c^0(a^C)$. (ii): Now suppose that $(\mathbf{a}]$ is atomic. Suppose that $b \in c^0(a^C)$. Then $a \in E(b) = \text{Sec } N(b)$. Let $e \in N(b)$. Then $e^I \in N(b)$. Hence $ae^I > 0$ and ae^I is open in $(\mathbf{a}]$, and so there is an atom $r \leq ae^I$. Then $r \in c^0(a) \cap c^0(e)$. It follows that $b \in c^1(a)$. Thus $c^0(a^C) \subseteq c^1(a)$ and so the result follows from (i). □

Note that the converse of (ii) above fails as can be seen by considering an atomless interior algebra. From (ii) above we see in particular that if a is an atom then $c^0(\mathbf{a}^C) = c^1(a)$, and so we have:

4.3.5 Proposition

An interior algebra is 1–separated iff it is a Fréchet algebra. □

Note that every interior algebra is 0–separated. Recall that a (first order) theory is called *hereditarily undecidable* iff all its subtheories are undecidable. The classes of n–separated interior algebras, $n < \omega$ are interesting since they are examples of strictly elementary classes of interior algebras which have hereditarily undecidable first order theories. In particular the first order theory of interior algebras is hereditarily undecidable. First we must show that the n–separated interior algebras, $n < \omega$, do indeed form strictly elementary classes.

4.3.6 Proposition

For all $n < \omega$ the class of n–separated interior algebras is a strictly elementary class.

Proof:

Define formulas $\varphi_n(x,y)$, $n < \omega$, in $\mathcal{LI}$ inductively as follows: $\varphi_0(x,y) := x \leq y$. If $\varphi_n(x,y)$ has been defined for some $n < \omega$ put $\varphi_{n+1}(x,y) := (\forall u)\ (\ x \leq u^I \Longrightarrow (\exists z)\ (\ \mathrm{Atom}(z) \wedge \varphi_n(z,y) \wedge \varphi_n(z,u)\ )\ )$, where Atom(z) is the formula defined in 1.5.2. Thus '$a \in c^n(b)$' is expressed by $\mathrm{Atom}(a) \wedge \varphi_n(a,b)$. We thus see that **A** is n–separated iff $\mathbf{A} \models \psi_n$ where ψ_n is the sentence $(\forall y)\ (\ \mathrm{Atom}(y) \Longrightarrow (\forall x)\ (\ \mathrm{Atom}(x) \wedge \varphi_n(x,y) \Longrightarrow a = b\ )\ )$. □

The following is a known result in topology due to M. Ziegler. An elegant proof of it may be found in [10].

4.3.7 Lemma

If $1 < n < \omega$ there is an $(n + 1)$–separated topological space **X** which is not $(n + 2)$–separated and which has exactly two points $x,y \in X$ satisfying the property: for all $S \in N(x)$ and all $R \in N(y)$, $S^n \cap R^n \neq \phi$. □

4.3.8 Lemma

If α is a limit ordinal or ∞ then **A** is α–separated iff **A** is β–separated for all $\beta < \alpha$.

We now see:

4.3.9 Proposition

The class of ω–separated interior algebras is an elementary class but it is not strictly elementary. □

Recall that a *graph without isolated points* is a structure $G = \langle G, \perp \rangle$ such that $\perp$ is a reflexive symmetric relation on G and for all $a \in G$ there is a $b \neq a$ in G with $a \perp b$. To show that the classes of n–separated interior algebras, $n < \omega$, have hereditarily undecidable theories, we use the following result:

4.3.10 Lemma (Rabin [18])
The (first order) theory of graphs without isolated points is hereditarily undecidable. □

We thus have to establish a connection between interior algebras and graphs without isolated points.

4.3.11 Definition
For each interior algebra A and all $\alpha \leq \infty$ define a relation $\perp^\alpha$ on At A by $a \perp^\alpha b$ iff for all $d \in N(a)$ and $e \in N(b)$, $c^\alpha(d) \cap c^\alpha(e) \neq \phi$. Put $At^\alpha A = \{ b \in At\ A : \text{there is an } a \in At\ A \text{ with } a \neq b \text{ and } a \perp^\alpha b \}$. Put $\mathbf{At}^\alpha \mathbf{A} = \langle At^\alpha A, \perp^\alpha \rangle$. Clearly $\mathbf{At}^\alpha \mathbf{A}$ is a graph without isolated points. If $\mathbf{X}$ is a topological space then we will use $\mathbf{At}^\alpha \mathbf{X}$ to denote $\mathbf{At}^\alpha \mathbf{X}^{\mathbf{A}}$. □

4.3.12 Lemma
If $1 \leq n < \omega$ and $G = \langle G, \perp \rangle$ is a graph without isolated points then there is an $(n + 1)$–separated topological space $\mathbf{X}$ such that $\mathbf{At}^n \mathbf{X} \cong \mathbf{G}$.
Proof:
Put $\mathcal{A} = \{ \{ a, b \} \subseteq G : a \perp b \text{ in } G \text{ and } a \neq b \}$. For each $P = \{ a, b \} \in \mathcal{A}$ we have by Lemma 4.3.7, an $(n + 1)$–separated topological space $\mathbf{X}(P)$ with $a,b \in X(P)$ the only two points satisfying: for all $S \in N(a)$ and all $R \in N(b)$, $S^n \cap R^n \neq \phi$. We may assume that $X(P) \cap G = \{ a, b \}$. Put $X = \cup \{ X(P) : P \in \mathcal{A} \}$. We may define a topology $\mathcal{T}$ on X by $\mathcal{T} = \{ S \subseteq X : S \cap X(P) \text{ is open in } \mathbf{X}(P) \text{ for all } P \in \mathcal{A} \}$. Putting $\mathbf{X} = \langle X, \mathcal{T} \rangle$ gives the result. □

4.3.13 Remark
Note that the class of strongly Fréchet algebras is a strictly elementary class since an interior algebra A is strongly Fréchet iff $A \models (\forall z)\ (\text{Atom}(z) \Longrightarrow z^C = z)$. Recall also that the class of atomic interior algebras is strictly elementary. (See Theorem 1.5.3.). We thus see that for all $n < \omega$ the class of atomic strongly n–separated interior algebras is strictly elementary. (Note also that if $1 \leq \alpha \leq \infty$ then by Proposition 4.2.10 and Proposition.4.3.5, any (residually) atomic α–separated interior algebra is in fact strongly α–separated.) □

4.3.14 Theorem

For all $n < \omega$ the class of atomic strongly n–separated interior algebras has an hereditarily undecidable first order theory.

Proof:

For each graph without isolated points G, and each n with $1 \leq n < \omega$, let $\mathbf{A}(n,G)$ be an atomic (strongly) (n + 1)–separated interior algebra with $\mathbf{At}^n\, \mathbf{A}(n,G) \cong G$, the existence of which is given by Lemma 4.3.12. Now let $1 \leq n < \omega$ be fixed. We define an interpretation v from the elementary language for graphs, $\mathcal{LG}$, to $\mathcal{LI}$ such that for all sentences φ of $\mathcal{LG}$ and all graphs without isolated points G, $G \models \varphi$ iff $\mathbf{At}^n\, \mathbf{A}(n,G) \models v(\varphi)$. Let the formulas $\varphi_n(x,y)$ be as in Proposition 4.3.5. For all variables x and y of $\mathcal{LG}$ put $v(x = y) := x = y$. Put $v(x \perp y) := (\forall p)(\forall q)\ (\ x \leq p^I \wedge y \leq q^I \Longrightarrow (\exists z)\ (\ \mathrm{Atom}(z) \wedge \varphi_n(z,x) \wedge \varphi_n(z,y)\ )\ )$. This defines the interpretation on atomic formulas. The interpretation distributes over connectives. For quantification: If φ is a formula such that $v(\varphi)$ has been defined and x is a variable, put $v(\ (\exists x)\ \varphi\ ) := (\exists x)\ (\ \mathrm{Atom}(x) \wedge (\exists p)\ (\ p \neq x \wedge v(x \perp p)\ )\ \wedge v(\varphi)\ )$. An easy induction argument shows that if $\varphi\,(x_1,\ldots,x_n)$ is a formula in $\mathcal{LG}$, G is a graph without isolated points and $a_1,\ldots,a_n \in G$ then $G \models \varphi\,[a_1,\ldots,a_n]$ iff $\mathbf{A}(n,G) \models v(\varphi)\,[f(a_1),\ldots,f(a_n)]$ where $f : G \longrightarrow \mathbf{At}^n\, \mathbf{A}(n,G)$ is the isomorphism. Thus for all sentences φ of $\mathcal{LG}$, $G \models \varphi$ iff $\mathbf{A}(n,G) \models v(\varphi)$. By Lemma 4.3.8 we see that the result holds for $2 \leq n < \omega$. But any 2–separated interior algebra is 1–separated and 0–separated and so the result holds for all $n < \omega$. □

It is not known whether the class of ω–separated interior algebras has an (hereditarily) undecidable theory or not. We now investigate some alternative characterizations of α–separation:

4.3.15 Theorem

Let $\mathbf{A}$ be an interior algebra and let $\alpha < \infty$. The following are equivalent:

i) A is $(\alpha + 2)$–separated.

ii) For each pair of distinct atoms a and b in A there are $d \in N(a)$ and $e \in N(b)$ with $c^\alpha(d) \cap c^\alpha(e) = \phi$.

Proof:

Assume (i). Let A be $(\alpha + 2)$–separated. Let a and b be distinct atoms in $\mathbf{A}$. Then $a \notin c^{\alpha+2}(b)$ and so there is a $d \in N(a)$ with $c^{\alpha+1}(d) \cap c^{\alpha+1}(b) = \phi$. Then $b \notin c^{\alpha+1}(d)$ and so there is a $e \in N(b)$ with $c^\alpha(e) = \phi$ as required. Thus (i) $\Longrightarrow$ (ii). Now assume (ii) but suppose that A is not $(\alpha + 2)$–separated. Let $\beta \leq \alpha + 2$ be least such that $\mathbf{A}$ is not β–separated. Then $\beta = \gamma + 1$ for some γ. (Or else by Lemma 4.3.8, $\mathbf{A}$ is γ–separated for some $\gamma < \beta$, a contradiction.) There are distinct atoms a and b in **A** with $a \in c^\beta(b)$. Let $d \in N(a)$ and $e \in N(b)$ be such that $c^\alpha(d) \cap c^\alpha(e) = \phi$. Now $c^\gamma(d) \cap c^\gamma(e) \neq \phi$. Suppose that $\gamma \leq \alpha$. Then $c^\gamma(d) \subseteq c^\alpha(d)$ and $c^\gamma(b) \subseteq c^\gamma(e) \subseteq c^\alpha(e)$ whence $c^\gamma(d) \cap c^\gamma(e) \subseteq c^\alpha(d) \cap c^\alpha(e) = \phi$, a contradiction. Thus $\alpha < \gamma$ and so $\beta = \alpha + 2$. Then we have $c^{\alpha+1}(d) \cap c^{\alpha+1}(e) \neq \phi$ and **A** is $(\alpha + 1)$–separated. Hence $b \in c^{\alpha+1}(d)$. But then $c^\alpha(d) \cap c^\alpha(e) \neq \phi$, a contradiction. Thus (ii) $\Longrightarrow$ (i). □

4.3.16 Corollary

Let **A** be an interior algebra and let α be a limit ordinal or ∞. The following are equivalent:

i) **A** is α–separated.

ii) For each pair of distinct atoms a and b and all $\beta < \alpha$ there are $d \in N(a)$ and $e \in N(b)$ with $c^\beta(d) \cap c^\beta(e) = \phi$. □

4.3.17 Corollary

Let **A** be an interior algebra and let α be a limit ordinal or ∞. If for each pair of distinct atoms a and b in **A** there are $d \in N(a)$ and $e \in N(b)$ with $c^\alpha(d) \cap c^\alpha(e) = \phi$, then **A** is α–separated. □

Theorem 4.3.15 also allows us to show:

4.3.18 Theorem

An interior algebra is 2–separated iff it is point Hausdorff.

Proof:

Let **A** be 2–separated. Suppose there is a net (z_i) in At **A** based on a directed set **W** with $(z_i) \longrightarrow a$ and $(z_i) \longrightarrow b$ for two distinct atoms a and b in **A**. By Theorem 4.3.15 there are $d \in N(a)$ and $e \in N(b)$ with $c^0(d) \cap c^0(e) = \phi$. Then there are $j,k \in W$ with $z_i \leq d$ for all $i \gg j$ in **W** and $z_i \leq e$ for all $i \gg k$ in **W**. There is an $i \in W$ with $i \gg j,k$. But then $z_i \in c^0(d) \cap c^0(e)$, a contradiction. Thus **A** is point Hausdorff and so (i) $\Longrightarrow$ (ii). Now let **A** be point Hausdorff. Suppose that **A** is not 2–separated. By Theorem 4.3.15 there are distinct atoms a and b in **A** such that $c^0(d) \cap c^0(e) \neq \phi$ for all $d \in N(a)$ and all $e \in N(b)$. Define a directed set $\mathbf{W} = \langle W, \ll \rangle$ as follows: Put $W = \{ (d,e,r) : d \in N(a), e \in N(b)$ and $r \in c^0(d) \cap c^0(e) \}$. Let $(d,e,r) \ll (v,w,s)$ in W iff $d \geq v$ and $e \geq w$. Then $\ll$ is clearly a pre–order. Suppose $(d,e,r),(v,w,s) \in W$. Then $dv \in N(a)$ and $ew \in N(b)$ and so there is a $p \in c^0(dv) \cap c^0(ew)$. Then $(dv,ew,p) \in W$ and $(dv,ew,p) \gg (d,e,r),(v,w,p)$. Thus $\mathbf{W} = \langle W, \ll \rangle$ is a directed set. For all $i = (d,e,r) \in W$ put $z_i = r$. We thus have a net (z_i) in At **A**. Consider a neighbourhood $d \in N(a)$. Then for some atom r, $(d,1,r) \in W$. Then for all $i = (v,w,s) \gg (d,1,r)$, $z_i \leq v \leq d$. Thus $(z_i) \longrightarrow a$. Similarly $(z_i) \longrightarrow b$, a contradiction. Thus **A** is 2–separated and so (ii) $\Longrightarrow$ (i). □

4.3.19 Definition

A is called a **Urysohn** algebra iff for each pair of distinct atoms a and b there is a closed $d \in N(a)$ and a closed $e \in N(b)$ which are disjoint, equivalently there are $d \in N(a)$ and $e \in N(b)$ such that d^C and e^C are disjoint. □

4.3.20 Proposition

Any Urysohn algebra is 3–separated and Hausdorff.

Proof:

Clearly any Urysohn algebra is Hausdorff. Let A be Urysohn. Let a and b distinct atoms in A. Then there are $d \in N(a)$ and $e \in N(b)$ with $d^C e^C = 0$. Then $c^0(d^C) \cap c^0(e^C) = \phi$ and so by Lemma 4.3.4 (i) we have $c^1(d) \cap c^1(e) = \phi$. By Theorem 4.3.15, **A** is 3–separated. □

The converse to the above fails: As before let **L** be the free Boolean algebra on $\aleph_0$ generators. Let **A** be the interior algebra with $A^u = L \times 2^u \times 2^u$ and $A^o = \{ (0,0,0), (0,1,0), (0,0,1), (0,1,1), (1,1,1) \}$. The only atoms of **A** are (0,1,0) and (0,0,1) and so **A** is clearly Hausdorff and 3–separated. However the closed neighbourhoods of (0,1,0) are (1,1,0) and (1,1,1), and the closed neighbourhoods of (0,0,1) are (1,0,1) and (1,1,1), whence we see that **A** is not Urysohn. Of course a topological space **X** is a Urysohn space iff X^A is a Urysohn algebra. Since a Hausdorff space need not be Urysohn [19] we see that even a complete atomic Hausdorff interior algebra need not be Urysohn.

4.3.21 Theorem

Let **A** be an interior algebra which is open atomic and closed atomic. The following are equivalent:

i) A is an Urysohn algebra.

ii) A is 3–separated.

Proof:

We know that (i) $\Longrightarrow$ (ii). Assume (ii). Let a and b be distinct atoms in **A**. Then there are $d \in N(a)$ and $e \in N(b)$ with $c^1(d) \cap c^1(e) = \phi$. Then $d^I \in N(a)$, $e^I \in N(b)$ and $c^1(d^I) \cap c^1(e^I) = \phi$. Now by Theorem 1.5.4 and the fact that **A** is open atomic we see that $(d^I]$ and $(e^I]$ are open atomic and so by Lemma 4.3.4 (ii) we see that $c^0(d^{IC}) \cap c^0(e^{IC}) = \phi$. Then $d^{IC}e^{IC} = 0$ or else by closed atomicity there is an atom $r \leq d^{IC}e^{IC}$ whence $r \in c^0(d^{IC}) \cap c^0(e^{IC})$, a contradiction. Thus **A** is Urysohn and so (ii) $\Longrightarrow$ (i). □

4.3.22 Definition and Remark

Call an interior algebra **A**, **clopen separated** iff for each pair of distinct atoms a and b in **A** there is a clopen d in **A** with $a \leq d$ and $b \leq d'$. Recall Theorem 4.2.4. We clearly have a similar characterization of clopenly separated interior algebras: **A** is clopen separated iff for each atom b in **A**, $\bigcap \{ (a] : a \in A^{\diamond} \text{ and } b \leq a \}$ contains a unique atom, namely b. □

4.3.23 Lemma

Let **A** be an interior algebra. For all $\alpha \leq \infty$ and all $a \in A^{\diamond}$, $c^{\alpha}(a) = c^{0}(a)$.

Proof:

Let $\alpha \leq \infty$. Suppose that for all $\beta < \alpha$ and all $a \in A^{\diamond}$, $c^{\beta}(a) = c^{0}(a)$. Let $a \in A^{\diamond}$ be fixed now. Clearly if α is a limit ordinal or ∞ then $c^{\alpha}(a) = c^{0}(a)$. Suppose $\alpha = \gamma + 1$. We have $c^{0}(a) \subseteq c^{\alpha}(a)$. Suppose there is a $b \in c^{\alpha}(a)$ with $b \notin c^{0}(a)$. Then $b \leq a'$. Hence $a' \in N(b)$ and so $c^{\gamma}(a') \cap c^{\gamma}(a) \neq \phi$, that is $c^{0}(a') \cap c^{0}(a) \neq \phi$, a contradiction. Thus $c^{\alpha}(a) = c^{0}(a)$ and the result follows by induction. □

4.3.24 Theorem

Every clopen separated interior algebra is ∞–separated and Urysohn.

Proof:

Clearly every clopen separated interior algebra is Urysohn. Let **A** be a clopen separated interior algebra. Let a be an atom in A. We show by induction that for all ordinals α, $c^{\alpha}(a) = \{ a \}$. Let α be an ordinal and suppose that $c^{\beta}(a) = \{ a \}$ for all $\beta < \alpha$. Then if α is a limit ordinal we obviously have $c^{\alpha}(a) = \{ a \}$. Suppose $\alpha = \gamma + 1$. Let $b \in c^{\alpha}(a)$. Suppose $b \neq a$. Then there is a clopen d in A with $b \leq d$ and $a \leq d'$. Then $d \in N(b)$ and so $c^{\gamma}(d) \cap c^{\gamma}(a) \neq \phi$. By Lemma 4.3.23 and the inductive hypothesis, $c^{0}(d) \cap \{ a \} \neq \phi$ and so $a \leq d$, a contradiction. Thus for all $\alpha < \infty$, $c^{\alpha}(a) = \{ a \}$ and so $c^{\infty}(a) = \{ a \}$. Thus **A** is ∞–separated. □

The converse of the above fails: Let **A** be the interior algebra with $\mathbf{A}^{u} = \mathbf{L} \times \mathbf{2}^{u} \times \mathbf{2}^{u}$ and $A^{o} = \{ (0,0,0), (0,1,0), (0,1,1), (a'b,0,0), (a'b,1,0), (a'b,0,1), (a'b,1,1), (b,1,0), (a',0,1), (b,1,1), (a',1,1), (1,1,1) \}$ where $0 < a < b < 1$ in **L**. Then (a,1,0) is a closed neighbourhood of the atom (0,1,0) and (b′,0,1) is a closed neighbourhood of the atom (0,0,1) whence we see that A is Urysohn. Also, using Corollary 4.3.17 and the fact that $c^{\infty}((0,1,0)) = \{ (0,1,0) \}$ and $c^{\infty}((0,0,1)) = \{ (0,0,1) \}$ we see that **A** is ∞–separated. However the only clopen elements of A are (0,0,0) and (1,1,1) whence **A** is not clopen separated.

4.3.25 Lemma

Let **A** be a (3–)saturated interior algebra. Then for all α with $\omega \leq \alpha \leq \infty$ and all $a \in A$, $c^{\alpha}(a) = c^{\omega}(a)$.

Proof:

Let $a \in A$. Let $b \in c^{\omega+1}(a)$. Suppose $b \notin c^{\omega}(a)$. Then $b \notin c^{n}(a)$ for all $n < \omega$. Hence for all $n < \omega$ there is a $d_n \in N(b)$ with $c^n(d_n) \cap c^n(a) = \phi$. Let the formulas $\varphi_n(x,y)$, $n < \omega$, be as in Proposition 4.3.5. For all $n < \omega$ let $\psi_n(v,a)$ be the formula $\neg(\exists z)$ ($\mathrm{Atom}(z) \wedge \varphi_n(z,v) \wedge \varphi_n(z,a)$). Put $\Sigma = \{ \psi_n(v,a) : n < \omega \} \cup \{ b \leq v^I \}$. Let Γ be a finite subset of Σ. If $\Gamma = \phi$ or $\{ b \leq v^I \}$ then $\langle \mathbf{A}, a, b \rangle \models \Gamma\,[d_0]$. Otherwise put $m = \max \{ n < \omega : \psi_n(v,a) \in \Gamma \}$. Then $\langle \mathbf{A}, a, b \rangle \models \Gamma\,[d_m]$. Since **A** is (3–)saturated there is a $d \in A$ with $\langle \mathbf{A}, a, b \rangle \models \Sigma\,[d]$. Then $d \in N(b)$ and for all $n < \omega$, $c^n(d) \cap c^n(a) = \phi$. Then $c^{\omega}(d) \cap c^{\omega}(a) = \phi$ or else if $r \in c^{\omega}(d) \cap c^{\omega}(a)$ there are $j,k < \omega$ with $r \in c^j(d)$ and $r \in c^k(a)$. Putting $n = \max \{ j, k \}$ we have $r \in c^n(d) \cap c^n(a)$, a contradiction. However we now have a contradiction to the fact that $b \in c^{\omega+1}(a)$. Thus $c^{\omega+1}(a) = c^{\omega}(a)$. But then by Proposition 4.3.2 (iii) we have $c^{\alpha}(a) = c^{\omega}(a)$ for all α with $\omega \leq \alpha \leq \infty$. □

From the above we easily see:

4.3.26 Theorem

Let **A** be a (3–)saturated interior algebra. Then **A** is ω–separated iff it is ∞–separated. □

4.3.27 Corollary

Every ω–separated interior algebra is (homomorphically) elementarily embeddable in an ∞–separated interior algebra. □

4.3.28 Remark

From the above we see that if $\omega < \alpha \leq \infty$ then the elementary class generated by the α–separated interior algebras is the class of ω–separated interior algebras. □

We know from topology that even for complete atomic interior algebras, ω–separation does not imply $(\omega + 1)$–separation. (See [10].) From this we see:

4.3.29 Corollary

If $\omega < \alpha \leq \infty$ then the class of α–separated interior algebras is not an elementary class. □

4.4.1 Theorem

The following are equivalent for an interior algebra A:

i) For all $R \subseteq A$ and all atoms b in A, $R \longrightarrow b$ implies $\{ r^C : r \in R \} \longrightarrow b$.

ii) For all filters F in A and all atoms b in A, $F \longrightarrow b$ implies $F \cap A^{\square} \longrightarrow b$.

iii) For all atoms b in A, $N(b) \cap A^{\square}$ is a base for the filter N(b).

iv) For all atoms b in A, $N(b) \cap A^{RC}$ is a base for the filter N(b).

v) For all atoms b in A and all $a \in A^{\square}$ with $b \leq a'$ there are neighbourhoods $d \in N(a)$ and $c \in N(b)$ which are disjoint.

Proof:

(i) $\Longrightarrow$ (ii) since for any filter F in A, $F \cap A^{\square} = \{ r^C : r \in F \}$. (ii) $\Longrightarrow$ (iii) is clear. (iii) $\Longrightarrow$ (iv): Assume (iii). Let b be an atom in A. Let $c \in N(b)$. There is a $d \in N(b) \cap A^{\square}$ with $d \leq c$. Then $d^{IC} \leq d \leq c$ and $d^{IC} \in N(b) \cap A^{RC}$. (iv) $\Longrightarrow$ (v): Assume (iv). Let b be an atom and let $a \in A^{\square}$ with $b \leq a'$. Then $a' \in N(b)$ and so there is a $c \in N(b) \cap A^{RC}$ with $c \leq a'$. Then $a \leq c' \in A^{O}$ and so $c' \in N(a)$. Putting $d = c'$ gives (v). (v) $\Longrightarrow$ (i): Assume (v). Let $R \subseteq A$ and let b be an atom in A with $R \longrightarrow b$. Let $e \in N(b)$. Then $e^{I\prime}$ is closed and $b \leq e^{I} = e^{I\prime\prime}$. Hence there are a disjoint pair $d \in N(e^{I\prime})$ and a $c \in N(b)$. Then $c \leq d' \leq d^{I\prime}$ and so $d^{I\prime} \in N(b)$. Hence there is an $r \in R$ with $r \leq d^{I\prime}$. Hence $r^C \leq d^{I\prime C} = d^{I\prime}$. But $e^{I\prime} \leq d^{I}$ and so $d^{I\prime} \leq e^{I} \leq e$ whence $r^C \leq e$. Thus $\{ r^C : r \in R \} \longrightarrow b$ as required. □

4.4.2 Definition and Remark

We will call an interior algebra satisfying the equivalent conditions in Theorem 4.1.1, a **regular** algebra. From condition (v) in Theorem 4.1.1 we see that regular algebras are a generalization of (weakly) regular topological spaces. □

In general the class of regular algebras is not comparable with the other classes of interior algebras we have discussed in this chapter as can be seen from topology. (See [19].) However we have the following result:

4.4.3 Theorem

Any regular Kolmogorov algebra is Urysohn.

Proof:

Let A be a regular Kolmogorov algebra. Let a and b be distinct atoms in A. Without loss of generality there is a $z \in N(a)$ with $b \leq z'$. Since A is regular there is an $e \in N(a) \cap A^{\square}$ with $e \leq z$. Then $b \leq e'$. Again by regularity there are a disjoint pair $d \in N(e)$ and $c \in N(b)$. But since $a \leq e$, $N(e) \subseteq N(a)$ whence $d \in N(a)$. Once again by regularity there are $r \in N(a) \cap A^{\square}$ and $s \in N(b) \cap A^{\square}$ with $r \leq d$ and $s \leq c$. Then r and s are disjoint. It follows that A is Urysohn. □

We know from topology that the converse fails even for complete atomic interior algebras.

4.4.4 Lemma

Let **A** be atomic and regular. Then for all $a \in A$ and all α with $1 \leq \alpha \leq \infty$, $c^{\alpha}(a) = c^{1}(a)$.

Proof:

By Proposition 4.3.2 (iii) it suffices to show that $c^{2}(a) = c^{1}(a)$ for all $a \in A$. Let $a \in A$. Let $b \in c^{2}(a)$ but suppose that $b \notin c^{1}(a)$. By Lemma 4.3.4, $b \notin c^{0}(a^{C})$, that is $b \nleq a^{C}$ and so $b \leq a^{C\prime}$. Hence there is a disjoint pair $d \in N(b)$ and $c \in N(a^{C})$. There is an $e \in N(b) \cap A^{\square}$ with $e \leq d$. Then $c^{1}(e) \cap c^{1}(a) \neq \phi$. Let $r \in c^{1}(e) \cap c^{1}(a)$. By Lemma 4.3.4, $r \leq e^{C} = e$ and $r \leq a^{C}$. But then $r \leq cd$, a contradiction. Hence $b \in c^{1}(a)$ and so $c^{2}(a) = c^{1}(a)$ as required. □

4.4.5 Theorem

If **A** is atomic, regular and Kolmogorov then A is ω–separated.

Proof:

Let A be as specified and consider an atom b in A. By Lemma 4.4.4 we have $c^{\infty}(a) = c^{1}(a)$. By Theorem 4.4.3, **A** is Urysohn and hence Fréchet. By Proposition 4.3.5 we have $c^{1}(a) = \{ a \}$ and the result follows. □

4.4.6 Theorem

Let **A** be a regular atomic interior algebra. Then:

i) Every open element in A is a join of regular closed elements.

ii) The regular open elements form a base for A.

Proof:

(i): Let $b \in A^{O}$. Consider an atom $r \leq b$. $b \in N(r)$ and so by regularity there is a $d(r) \in N(r) \cap A^{RC}$ with $d(r) \leq b$. Since $b = \Sigma\, At\, (b]$ we see that $b = \Sigma \{ d(r) : r \in At\, (b] \}$ whence the result follows. (ii): Let $b \in A^{O}$. Let $\{ d(r) : r \in At\, (b] \}$ be as in (i). Then since $b = \Sigma\, At\, (b]$ we see that $b = \Sigma \{ d(r)^{I} : r \in At\, (b] \}$. Now for all $r \in At\, (b]$, $d(r)$ is closed whence $d(r)^{I}$ is regular open and the result follows. □

4.4.7 Theorem

Let **A** be a regular atomic interior algebra. Then the following are equivalent:

i) **A** is a Lindelöf algebra.

ii) If $S \subseteq A^O$ with $\Sigma S = 1$ then there is a countable $R \subseteq S$ with $\Sigma \{ r^C : r \in R \} = 1$.

Proof:

Clearly (i) $\Longrightarrow$ (ii). (ii) = (i): Assume (ii). Let $S \subseteq A^O$ with $\Sigma S = 1$. Let r be an atom in **A**. Then there is an $s(r) \in S$ with $r \leq s(r)$. Then $s(r) \in N(r)$ and so by regularity there is a regular closed neighbourhood $t(r)$ of r with $t(r) \leq s(r)$. By atomicity $1 = \Sigma \text{ At } \mathbf{A}$ and so $1 = \Sigma \{ t(r)^I : r \in \text{At } \mathbf{A} \}$. Hence there is a countable $W \subseteq \text{At } \mathbf{A}$ with $1 = \Sigma \{ t(r)^{IC} : r \in W \} = \Sigma \{ t(r) : r \in W \}$. Hence $1 = \Sigma \{ s(r) : r \in W \}$ and the result follows. □

4.4.8 Theorem

The following are equivalent for an interior algebra **A**:

i) For all $R \subseteq A$ and all closed b in **A**, $R \longrightarrow b$ implies $\{ r^C : r \in R \} \longrightarrow b$.

ii) For all filters F in **A** and all closed b in **A**, $F \longrightarrow b$ implies $F \cap A^{\square} \longrightarrow b$.

iii) For all closed b in **A**, $N(b) \cap A^{\square}$ is a base for the filter N(b).

iv) For all closed b in **A**, $N(b) \cap A^{RC}$ is a base for the filter N(b).

v) For each pair of disjoint closed elements a and b in **A** there are neighbourhoods $d \in N(a)$ and $c \in N(b)$ which are disjoint. □

The proof of the above is essentially the same as that of Theorem 4.4.1 and is left to the reader.

4.4.9 Definition and Remark

We will call an interior algebra satisfying the equivalent conditions in Theorem 4.4.8, a **normal** algebra. From Theorem 4.4.8 (v) normal algebras are a generalization of (weakly) normal spaces. Notice that any semi–simple or F.S.I interior algebra is normal. By Theorem 2.1.15 and the fact that a normal space need not be ultra–connected or strongly zero–dimensional we see that the converse fails even for complete atomic interior algebras. Any strongly normal interior algebra is clearly regular and again by topology we see that the converse fails. (See [19].) □

4.5 PRESERVATION OF SEPARATION PROPERTIES

We have already seen that the classes of α–separated interior algebras, $\alpha \leq \omega$, are elementary classes and hence have the preservatiom properties associated with such classes. We now show that many of the classes of interior algebras introduced in this chapter are in fact strictly elementary universal–existential Horn classes closed under joins of interior algebras. We also investigate preservation of separation properties under principal quotients.

4.5.1 Lemma

Let **A** be an interior algebra and let B be a principal quotient of **A**. Then **At B** is embeddable in **At A**.

Proof:

There is an a ∈ A with B ≅ **(a]**. It suffices to show that **At (a]** is a substructure of **At A**. Of course it is a subset. Let r,s ∈ At (a]. Suppose r ≪ s in **(a]**, that is $r \leq s^{C,a} = as^{C}$. Then $r \leq s^{C}$, that is r ≪ s in A. Conversely if $r \leq s^{C}$ in A then since r ≤ a we have $r \leq as^{C} = s^{C,a}$, that is r ≪ s in **(a]**. □

The following lemma follows easily from Lemma 3.5.14.

4.5.2 Lemma

Let **A** be an interior algebra and let c ∈ A. Let (a_i) be a net in **(c]** and let b ∈ (c]. Then $(a_i) \longrightarrow b$ in **A** iff $(a_i) \longrightarrow b$ in **(c]**. □

4.5.3 Theorem

The following classes of interior algebras are closed under principal quotients:

i) Symmetric algebras.
ii) Kolmogorov algebras.
iii) Fréchet algebras.
iv) Strongly Fréchet algebras.
v) Hausdorff algebras.
vi) Point Hausdorff algebras.
vii) Sequentially Hausdorff algebras.
viii) Sequentially point Hausdorff algebras.
ix) Urysohn algebras.
x) Clopen separated interior algebras.
xi) Regular algebras.

Proof:

(i), (ii) and (iii) follow from Lemma 4.5.1 and (iv) is clear. (vi), (vii) and (viii) follow easily from Lemma 4.5.2. Let **A** be an interior algebra. Let a ∈ A. Suppose r and s are distinct atoms in **(a]** and there are d ∈ N(r) and c ∈ N(s) in **A** with cd = 0. Then ad ∈ $N^a(r)$ and ac ∈ $N^a(s)$, where N^a is the neighbourhood function of **(a]**, and (ac)(ad) = acd = 0. Moreover, if c and d are closed in **A** then ac and ad are closed in **(a]**. (v) and (ix) now follow. For (x): Let **A** be a clopen separated. Let $a \in A$. Suppose r and s are distinct atoms in **(a]**. Then there is a clopen d in **A** with $r \leq d$ and $s \leq d'$. Then ad is clopen in **(a]** and $r \leq ad$ while $s \leq (ad)'^{a}$ since $(ad)'^{a} = a(ad)' \geq ad'$ and $s \leq ad'$. Thus **(a]** is clopen separated and so (x) follows. For (xi): Let A be a regular algebra and let $a \in A$. Let r be an atom in **(a]** and let s be closed in **(a]** with $r \leq s'^{a}$. Then there is a closed b in **A** with $s = ab$. Then $r \leq a(ab)' \leq (ab)' = a' + b'$. Now $r \nleq a'$ and so $r \leq b'$. Hence there are $d \in N(r)$ and $c \in N(b)$ with $cd = 0$. Then by Lemma 3.5.14, $ad \in N^{a}(r)$ and $ac \in N^{a}(s)$ and moreover, $(ac)(ad) = acd = 0$. Thus **(a]** is regular and so (xi) follows. □

We know that the class of normal spaces are not closed under (open) subspaces and hence we see that the class of (complete atomic) normal algebras is not closed under (open) principal quotients. (See [19] and Remark 2.1.14.) However we have:

<u>4.5.4 Theorem</u>

The class of normal algebras is closed under closed quotients.

<u>Proof</u>:

Let **A** be a normal algebra and let $a \in A^{\square}$. Let r and s be disjoint closed elements of **(a]**. Then by Corollary 1.4.7, r and s are closed in A. Hence there are $d \in N(r)$ and $c \in N(s)$ with $cd = 0$. Then $ad \in N^{a}(r)$ and $ac \in N^{a}(s)$, where N^{a} is the neighbourhood function for **(a]**, and $(ac)(ad) = acd = 0$. Thus **(a]** is normal and the result follows. □

Since the class of normal algebras is not closed under arbitrary principal quotients we make the following definition:

<u>4.5.5 Definition</u>

Call an interior algebra **A** an **hereditarily normal algebra** iff every principal quotient of **A** is normal. □

For example, if X is a metrizable space then X^{A} is always an hereditarily normal algebra since any subspace of a metrizable space is normal [19]. We have an interesting characterization of hereditarily normal algebras:

<u>4.5.6 Theorem</u>

The following are equivalent for an interior algebra A:

i) **A** is hereditarily normal.

ii) For all $a,b \in A$, if $a^{C}b = 0$ and $ab^{C} = 0$ then there are neighbourhoods $d \in N(a)$ and

$c \in N(b)$ which are disjoint.

iii) For all $a,b \in A$, if $a^C b^C \leq a'b'$ then there is are neighbourhoods $d \in N(a)$ and $c \in N(b)$ which are disjoint.

Proof:

Consider $a,b \in A$. Then $a^C b = 0$ and $ab^C = 0$ iff $a^C b + ab^C = 0$, iff $a^C b^C (a + b) = 0$ (since $a^C b^C (a + b) = a^C b + ab^C$), iff $a^C b^C \leq (a + b)'$, iff $a^C b^C \leq a'b'$, and so (ii) $\Leftrightarrow$ (iii). Assume (i). Let $a,b \in A$ with $a^C b = 0$ and $ab^C = 0$. Put $e = a^{C\prime} + b^{C\prime}$. Then $a,b \in (e]$. Also $a^{C,e} b^{C,e} = (ea^C)(eb^C) = (b^{C\prime} a^C)(a^{C\prime} b^C) = 0$. Now $(e]$ is normal and so there are neighbourhoods d of $a^{C,e}$ and c of $b^{C,e}$ in $(e]$ which are disjoint.Now e is open and so by Corollary 1.4.5 (i), say, we see that d is a neighbourhood of a in **A** and c is a neighbourhood of b in **A**. Thus (i) $\Longrightarrow$ (ii). Assume (ii). Let $a \in A$. Suppose r and s are closed and disjoint in $(a]$. Then $r^C s = ar^C s = r^{C,a} s = rs = 0$ and similarly $rs^C = 0$. Hence there are neighbourhoods $d \in N(a)$ and $c \in N(b)$ which are disjoint. Thus $(a]$ is normal and so **A** is hereditarily normal. Thus (ii) $\Longrightarrow$ (i). □

4.5.7 Notation

If **A** is an interior algebra, $a \in A$ and $\alpha \leq \infty$ we will use c_a^α to denote the c^α function for the interior algebra $(a]$. □

4.5.8 Lemma

Let **A** be an interior algebra and let $a \in A$. Then for all $b \in (a]$ and all $\alpha \leq \infty$, $c_a^\alpha(b) \subseteq c^\alpha(b)$.

Proof:

Let $\alpha \leq \infty$. Obviously if $\alpha = 0$, $c_a^\alpha(b) \subseteq c^\alpha(b)$ for all $b \in (a]$. Suppose that $\alpha > 0$ and that for all $\beta < \alpha$, $c_a^\alpha(b) \subseteq c^\alpha(b)$ for all $b \in (a]$. Let $b \in (a]$. If α is a limit ordinal or ∞ then $c_a^\alpha(b) = \cup\, c_a^\beta(b) \subseteq \cup\, c^\beta(b) = c^\alpha(b)$. Suppose $\alpha = \gamma + 1$. Consider $r \in c_a^\alpha(b)$. Let d be a neighbourhood of r in **A**. Then ad is a neighbourhood of r in $(a]$ by Lemma 3.5.14. Hence $c_a^\gamma(r) \cap c_a^\gamma(ad) \neq \phi$. Now $c_a^\gamma(r) \subseteq c^\gamma(r)$ and $c_a^\gamma(ad) \subseteq c^\gamma(ad) \subseteq c^\gamma(d)$. Thus $c^\gamma(r) \cap c^\gamma(d) \neq \phi$. Hence $r \in c^\alpha(b)$ and so $c_a^\alpha(b) \subseteq c^\alpha(b)$ as required and the result follows by induction. □

4.5.9 Theorem

For all $\alpha \leq \infty$ the class of α–separated interior algebras is closed under principal quotients.

Proof:

Let $\alpha \leq \infty$. Let **A** be α–separated and let $a \in A$. Let b be an atom in $(a]$. Then $c_a^\alpha(b) \subseteq c^\alpha(b)$ by Lemma 4.5.8 and so since $c^\alpha(b) = \{ b \}$ we have $c_a^\alpha(b) = \{ b \}$ and the result follows. □

4.5.10 Notation

If $\mathbf{A} = \Pi \{ \mathbf{A}_i : i \in I \}$ is a product of interior algebras let $m_j : \mathbf{A}_j \longrightarrow \mathbf{A}$ denote the map given by: For all $a \in A_j$, $m_j(a) = (b_i)$ where $b_j = a$ and $b_i = 0$ for $i \in I - \{ j \}$. □

4.5.11 Theorem

The following classes of interior algebras are strictly elementary universal–existential Horn classes:

i) Symmetric algebras.
ii) Kolmogorov algebras.
iii) Fréchet algebras.
iv) Strongly Fréchet algebras.
v) Hausdorff algebras.
vi) Urysohn algebras.
vii) Clopen separated algebras.
viii) Regular algebras.
ix) Normal algebras.
x) Hereditarily normal algebras.

Proof:

To show that these classes are strictly elementary universal–existential classes we exhibit universal–existential sentences defining these classes:

i) $(\forall x)(\forall y)(\exists z)\ ((z < x \wedge 0 < z) \vee (z < y \wedge 0 < z) \vee x = 0 \vee y = 0 \vee (x \leq y^C \Longrightarrow y \leq x^C))$

ii) $(\forall x)(\forall y)(\exists z)\ ((z < x \wedge 0 < z) \vee (z < y \wedge 0 < z) \vee x = 0 \vee y = 0 \vee (x^C = y^C \Longrightarrow x = y))$

iii) The conjunction of (i) and (ii).

iv) $(\forall x)(\exists y)\ ((y < x \wedge 0 < y) \vee x = 0 \vee x^C = x)$

v) $(\forall x)(\forall y)(\exists w)(\exists z)\ ((w < x \wedge 0 < w) \vee (z < y \wedge 0 < z) \vee x = 0 \vee y = 0 \vee x = y \vee (x \leq z^I \wedge y \leq w^I \wedge zw = 0))$

vi) $(\forall x)(\forall y)(\exists w)(\exists z)\ ((w < x \wedge 0 < w) \vee (z < y \wedge 0 < z) \vee x = 0 \vee y = 0 \vee x = y \vee (x \leq z^I \wedge y \leq w^I \wedge z^C w^C = 0))$

vii) $(\forall x)(\forall y)(\exists z)\ ((z < x \wedge 0 < z) \vee (z < y \wedge 0 < z) \vee x = 0 \vee y = 0 \vee x = y \vee (x \leq z \wedge y \leq z' \wedge z^I = z^C))$

viii) $(\forall x)(\forall y)(\exists w)(\exists z)\ ((z < x \wedge 0 < z) \vee x = 0 \vee y^C \neq y \vee x \leq y \vee (x \leq z^I \wedge y \leq w^I \wedge zw = 0))$

ix) $(\forall x)(\forall y)(\exists w)\ ((x^C \neq x \vee y^C \neq y \vee xy > 0 \vee (x \leq w^I \wedge y \leq z^I \wedge zw = 0))$

x) $(\forall x)(\forall y)(\exists w)(\exists z)\ (x^C y > 0 \vee xy^C > 0 \vee (x \leq z^I \wedge y \leq w^I \wedge zw = 0))$

We leave it to the reader to check that the above sentences do indeed define the required classes. (For (x) use Theorem 4.5.6.) Recall that a universal–existential direct product sentence is always a Horn sentence. It thus suffices to show that the classes are closed under direct products: (i): Let $\mathbf{A} = \Pi\{ \mathbf{A}_i : i \in I \}$ be a product of symmetric algebras. Suppose $a \ll b$ in $\mathbf{At}\ \mathbf{A}$. There are $j,k \in I$, $r \in \mathrm{At}\ \mathbf{A}_j$ and $s \in \mathrm{At}\ \mathbf{A}_k$ with $a = m_j(r)$ and $b = m_k(s)$. Then $r \leq b_j^C$ and so we must have $j = k$. Then $r \ll s$ in $\mathbf{At}\ \mathbf{A}_k$. Thus $s \ll r$ in $\mathbf{At}\ \mathbf{A}_k$ and so $b \ll a$ in $\mathbf{At}\ \mathbf{A}$. Thus $\mathbf{A}$ is symmetric. (ii): Let $\mathbf{A} = \Pi\{ \mathbf{A}_i : i \in I \}$ be a product of Kolmogorov algebras. Let $a \ll b$ and $b \ll c$ in $\mathbf{At}\ \mathbf{A}$. There are $j,k,l \in I$, $r \in \mathrm{At}\ \mathbf{A}_j$, $s \in \mathrm{At}\ \mathbf{A}_k$ and $t \in \mathrm{At}\ \mathbf{A}_l$ with $a = m_j(r)$, $b = m_k(s)$ and $c = m_l(t)$. Then $r \leq b_j^C$ and so we must have $j = k$. Similarly since $s \leq c_l^C$ we have $k = l$. Then $r \ll s$ and $s \ll t$ in $\mathbf{At}\ \mathbf{A}_k$ whence $r \ll t$ in $\mathbf{At}\ \mathbf{A}_k$. Thus $a \ll c$ in At A and so A is Kolmogorov. (iii): Since the classes of symmetric and Kolmogorov algebras are closed under products so is the class of Fréchet algebras. (iv): Let $\mathbf{A} = \Pi\{ \mathbf{A}_i : i \in I \}$ be a product of strongly Fréchet algebras. Let a be an atom in $\mathbf{A}$. Then $a = m_j(b)$ for some $j \in I$ and some atom b in $\mathbf{A}_j$. Then b is closed whence it follows that $m_j(b) = a$ is closed. Thus A is strongly Fréchet. (v): Let $\mathbf{A} = \Pi\{ \mathbf{A}_i : i \in I \}$ be a product of Hausdorff algebras. Let a and b be distinct atoms in $\mathbf{A}$. There are $j,k \in I$, $r \in \mathrm{At}\ \mathbf{A}_j$ and $s \in \mathrm{At}\ \mathbf{A}_k$ with $a = m_j(r)$ and $b = m_k(s)$. If $j \neq k$ let $d = m_j(1)$ and $c = m_k(1)$. If $j = k$ then $r \neq s$ and so there are disjoint neighbourhoods $v \in N(r)$ and $w \in N(s)$. Put $d = m_j(v)$ and $c = m_j(w)$. Then $d \in N(a)$, $c \in N(b)$ and $cd = 0$. Thus $\mathbf{A}$ is Hausdorff. (vi): We proceed as for (v) noting that we can choose v and w to be closed whence d and c are closed. (vii): Let $\mathbf{A} = \Pi\{ \mathbf{A}_i : i \in I \}$ be a product of clopen separated interior algebras. Let a and b be distinct atoms in a A. Then there are $j,k \in I$, $r \in \mathrm{At}\ \mathbf{A}_j$ and $s \in \mathrm{At}\ \mathbf{A}_k$ with $a = m_j(r)$ and $b = m_k(s)$. If $j \neq k$ then $m_j(1)$ is clopen in $\mathbf{A}$ with $a \leq m_j(1)$ and $b \leq m_j(1)'$. If $j = k$ then there is a clopen d in $\mathbf{A}_j$ with $r \leq d$ and $s \leq d'$. Then $m_j(d)$ is clopen in $\mathbf{A}$ with $a \leq m_j(d)$ and $b \leq m_j(d)'$. Thus $\mathbf{A}$ is clopen separated. (viii): Let $\mathbf{A} = \Pi\{ \mathbf{A}_i : i \in I \}$ be a product of regular algebras. Let b be an atom in $\mathbf{A}$ and let a be closed in $\mathbf{A}$ with $b \leq a'$. There is a $j \in I$ and an atom r in $\mathbf{A}_j$ with $b = m_j(r)$. Then $r \leq a_j'$ and a_j is closed in A. Hence there are disjoint neighbourhoods $d \in N(r)$ and $c \in N(a_j)$. For $i \in I - \{ j \}$ put $v_i = 1$ and put $v_j = c$. Put $v = (v_i) \in A$. Then $v \in N(a)$, $m_j(d) \in N(b)$ and $v \cdot m_j(d) = 0$. Thus A is regular. (ix): Let $\mathbf{A} = \Pi\{ \mathbf{A}_i : i \in I \}$ be a product of normal algebras. Let a and b be closed and disjoint in $\mathbf{A}$. Then for all $i \in I$, $a_i = b_i$ are closed and disjoint in $\mathbf{A}_i$ whence there are disjoint neighbourhoods $d_i \in N(a_i)$ and $c_i \in N(b_i)$. Let $c = (c_i) \in A$ and $d = (d_i) \in A$. Then $d \in N(a)$, $c \in N(b)$ and c and d are disjoint. Thus A is normal. (x): Let $\mathbf{A} = \Pi\{ \mathbf{A}_i : i \in I \}$ be a product of hereditarily normal algebras. Let $a,b \in A$ with $a^C b^C \leq a'b'$. Then for all $i \in I$, $a_i^C b_i^C \leq a_i' b_i'$. Thus proceeding as for (ix) we obtain disjoint neighbourhoods $d \in N(a)$ and $c \in N(b)$. □

4.5.12 Theorem

The following classes of interior algebras are closed under products:

i) Point Hausdorff algebras.

ii) Sequentially Hausdorff algebras.

iii) Sequentially point Hausdorff algebras.

Proof:

(i): Let $A = \Pi \{ A_i : i \in I \}$ be a product of point Hausdorff algebras. Let (z_i) be a net in At A based on a directed set W. Suppose a and b are atoms in A with $(z_i) \longrightarrow a$ and $(z_i) \longrightarrow b$. Then there are $j,k \in I$, $r \in$ At A_j and $s \in$ At A_k with $a = m_j(r)$ and $b = m_k(s)$. Then $m_j(1) \in N(a)$ and $m_k(1) \in N(b)$ whence there are $v,w \in W$ such that $z_i \leq m_j(1)$ for all $i \gg v$ in W and $z_i \leq m_k(1)$ for all $i \gg w$ in W. There is an $u \gg v,w$. Then $z_u \leq m_j(1)\cdot m_k(1)$ whence it follows that $j = k$. For all $i \gg u$ in W, $z_i \leq m_k(1)$ and so there is an atom x_i in A_k with $z_i = m_k(x_i)$. Let Z be the substructure of W with $Z = \{ i \in W : i \gg u \}$. Then Z is a directed set and so we have a net (x_i) in At A_k based on Z. Let $d \in N(r)$. Then $m_k(d) \in N(a)$ and so there is a $p \in W$ with $z_i \leq m_k(d)$ for all $i \gg p$ in W. There is a $q \in W$ with $q \gg p,u$. Then $q \in Z$ and for all $i \gg q$ in Z, $z_i \leq m_k(d)$ whence $x_i \leq d$. Thus $(x_i) \longrightarrow r$. Similarly $(x_i) \longrightarrow s$. Since A_k is point Hausdorff we have $r = s$ and so $a = b$. Thus A is point Hausdorff. (ii): Let $A = \Pi \{ A_i : i \in I \}$ be a product of sequentially Hausdorff algebras. Let (z_n) be a sequence in $A - \{ 0 \}$. Suppose a and b are atoms in A with $(z_n) \longrightarrow a$ and $(z_n) \longrightarrow b$. There are $j,k \in I$, $r \in$ At A_j and $s \in$ At A_k with $a = m_j(r)$ and $b = m_k(s)$. As in (i) we see that $j = k$ and that there is a $c < \omega$ with $z_n \leq m_k(1)$ for all $n \geq c$. For all $n \geq c$ there is a $y_n \in A_k$ with $z_n = m_k(y_n)$. For all $n \geq c$ put $x_n = z_{c+n}$. Then arguing as in (i) we see that $(x_n) \longrightarrow r$ and $(x_n) \longrightarrow s$ whence $r = s$ and so $a = b$. Thus A is sequentially Hausdorff. (iii) is proved similarly to (i) and (ii). □

4.5.13 Lemma

Let $A = \Pi \{ A_i : i \in I \}$ be a product of interior algebras. If $b = (b_i) \in A$ and $\alpha \leq \infty$ then $c^{\alpha}(b) = \cup \{ m_i[c^{\alpha}(b_i)] : i \in I \}$.

Proof:

Suppose $c^{\beta}(b) = \cup \{ m_i[c^{\beta}(b_i)] : i \in I \}$ for all $\beta < \alpha$ and all $b = (b_i) \in A$. Let b be fixed now. If $\alpha = 0$ then clearly $c^{\alpha}(b) = \cup \{ m_i[c^{\alpha}(b_i)] : i \in I \}$. If $\alpha > 0$ is a limit ordinal or ∞ then $c^{\alpha}(b) = \cup \{ c^{\beta}(b) : \beta < \alpha \} = \cup \{ \cup \{ m_i[c^{\beta}(b_i)] : i \in I \} : \beta < \alpha \} = \cup \{ \cup \{ m_i[c^{\beta}(b_i)] : \beta < \alpha \} : i \in I \} = \cup \{ m_i[c^{\alpha}(b_i)] : i \in I \}$. If $\alpha = \gamma + 1$ consider an $a \in c^{\alpha}(b)$. There are $k \in I$ and $r \in$ At A_k with $a = m_k(r)$. Let $d \in N(r)$. Then $m_k(d) \in N(a)$ and so $c^{\gamma}(m_k(d)) \cap c^{\gamma}(b) \neq \phi$. Let $s \in c^{\gamma}(m_k(d)) \cap c^{\gamma}(b)$. Now $c^{\gamma}(m_k(d)) = \cup \{ m_i[c^{\gamma}(m_k(d)_i)] : i \in I \}$ and so there is a $j \in I$ and a $p \in c^{\gamma}(m_k(d)_j)$ with $s = m_j(p)$. Then $j = k$ or else

$p \in c^{\gamma}(0)$, a contradiction. Thus $p \in c^{\gamma}(d)$. Also since $c^{\gamma}(b) = \cup \{ m_i[c^{\gamma}(b_i)] : i \in I \}$ there are $t \in I$ and $q \in c^{\gamma}(b_t)$ with $s = m_t(q)$. Then we must have $q = p$ and $t = k$ and so $p \in c^{\gamma}(b_k)$. Thus $c^{\gamma}(d) \cap c^{\gamma}(b_k) \neq \phi$. Hence $r \in c^{\alpha}(b_k)$ and so $a \in \cup \{ m_i[c^{\beta}(b_i)] : i \in I \}$. Conversely suppose that $a \in \cup \{ m_i[c^{\beta}(b_i)] : i \in I \}$. There is a $k \in I$ and an $r \in c^{\alpha}(b_k)$ with $a = m_k(r)$. Let $d = (d_i) \in N(a)$. Then $d_k \in N(r)$ whence $c^{\gamma}(d_k) \cap c^{\gamma}(b_k) \neq \phi$. Let $r \in c^{\gamma}(d_k) \cap c^{\gamma}(b_k)$. Then $m_k(r) \in c^{\gamma}(d) \cap c^{\gamma}(b)$. It follows that $a \in c^{\alpha}(b)$. Thus $c^{\alpha}(b) = \cup \{ m_i[c^{\alpha}(b_i)] : i \in I \}$ and the result follows by induction. □

4.5.14 Theorem

For all $\alpha \leq \infty$ the class of α–separated interior algebras is closed under direct products.

Proof:

Let $\alpha \leq \infty$ and let $A = \Pi \{ A_i : i \in I \}$ be a product of α–separated interior algebras. Let a be an atom in A. There is a $k \in I$ and an $r \in \mathrm{At}\, A_k$ with $a = m_k(r)$. Let $b \in c^{\alpha}(a)$. By Lemma 4.5.13 there is a $j \in I$ and $s \in c^{\alpha}(a_j)$ with $b = m_j(s)$. Then $j = k$ or else $s \in c^{\alpha}(0)$, a contradiction. Thus $s \in c^{\alpha}(r)$. Since A_k is α–separated we have $s = r$ and so $b = a$. Thus A is α–separated. □

Note that the above theorem gives an alternative proof of Theorem 4.5.12 (i) since by Theorem 4.3.18 an interior algebra is point Hausdorff iff it is 2–separated.

CHAPTER 5

COMPACTNESS IN INTERIOR ALGEBRAS

5.1 THE COMPACTNESS PROPERTIES

Compactness properties in interior algebras are properties guaranteeing the existence of atomic accumulants for filters and nets of atoms or properties closely related to this. They generalize compactness properties in topology.

5.1.1 Definition and Remark

An interior algebra **A** is called **filter compact** iff every proper filter in **A** accumulates at an atom. Clearly a topological space **X** is compact iff $\mathbf{X}^{\mathbf{A}}$ is filter compact. □

5.1.2 Lemma

Let **B** be a Boolean algebra and let R be a proper subset of **B**.

i) R is a filter implies Sec R is a grill.

ii) R is a grill implies Sec R is a filter.

iii) If R is a grill there is an ultrafilter F in **B** with $F \subseteq R$.

Proof:

Since R is proper Sec R is a stack. (i): Let R be a filter. Let $a,b \in B$ with $a + b \in$ Sec R. Suppose $a \notin$ Sec R and $b \notin$ Sec R. Then there are $c,d \in R$ shch that $ac = 0$ and $bd = 0$. Then $cd \in R$ and $cd \cdot (a + b) = 0$, a contradiction. Thus $a \in$ Sec R or $b \in$ Sec R and so R is a grill. (ii): Let R be a grill. Let $a,b \in$ Sec R. Suppose $ab \notin$ Sec **R**. Then there is an $r \in R$ with $abr = 0$. Then $r \leq (ab)' = a' + b'$ and so $a' + b' \in R$. But $a' \notin R$ and $b' \notin R$, a contradiction. Thus $ab \in$ Sec R and so R is a filter. (iii): Let R be a grill. By (ii) Sec R is a filter. Also Sec R is proper and so there is an ultrafilter F in **B** with Sec $R \subseteq F$. Then $F \subseteq$ Sec $F \subseteq R$. □

5.1.3 Theorem

The following are equivalent for an interior algebra A:

i) A is filter compact.

ii) Every proper grill in A converges to an atom.

iii) Every ultrafilter in A converges to an atom.

iv) Every proper filter in $A^{\square}$ is bounded below in A by an atom.

Proof:
(i) $\Longrightarrow$ (iii) by Corollary 3.2.14. (iii) $\Longrightarrow$ (ii): Assume (iii). Let R be a proper grill in **A**. By Lemma 5.1.2 (iii) there is an ultrafilter F in **A** with $F \subseteq R$. Then $F \longrightarrow a$ for some atom a in **A**. Then $R \longrightarrow a$. (ii) $\Longrightarrow$ (i): Assume (ii). Let F be a proper filter in **A**. By Lemma 5.1.2 Sec F is a grill in **A** and it is proper. Thus there is an atom a in **A** with Sec $F \longrightarrow a$. By Proposition 3.2.11 (iii) we have $F \longleftarrow a$. Thus (i), (ii) and (iii) are equivalent. (i) $\Longrightarrow$ (iv): Let **A** be filter compact. Let R be a proper filter in $A^{\square}$. Put $F = \{ b \in A : b \geq r \text{ for some } r \in R \}$. Then F is a proper filter in **A** and so $F \longleftarrow a$ for some atom a in **A**. But then a is a lower bound for R in **A**. (iv) $\Longrightarrow$ (i): Assume (iv). Let **F** be a proper filter in **A**. Then $F \cap A^{\square}$ is a proper filter in $A^{\square}$ and so there is an atom a of **A** which is a lower bound for $F \cap A^{\square}$. Noting that $\{ b^C : b \in F \} = F \cap A^{\square}$ we see that $F \longleftarrow a$. □

Besides filter compactness there are other obvious generalizations of compactness in topology:

5.1.4 Definition and Remark

A is called **cover compact** iff for all $S \subseteq A^O$, $\Sigma S = 1$ implies that $\Sigma R = 1$ for some finite $R \subseteq S$. (Equivalently, for all $S \subseteq A^{\square}$, $\sqcap S = 0$ implies $\sqcap R = 0$ for some finite $R \subseteq S$.) Every cover compact interior algebra is Lindelöf. An interior algebra is S.I. iff it is F.S.I. and cover compact. Note that the class of cover compact interior algebras is a Heyting class; it corresponds to the class of compact Heyting algebras. Call an interior algebra **A point compact** iff every net in At **A** accumulates at an atom. □

Trivially point compactness does not imply cover compactness: consider an atomless Boolean algebra. Cover compactness does not imply point compactness even in a complete open atomic F.S.I. interior algebra: Let **X** be the denumerable discrete space and let $A = X^A$. Let $\{ a_n : n < \omega \}$ be an enumeration of the atoms in **A**. Let **B** be the free Boolean algebra on $\aleph_0$ generators. Put **C** = A[B]. (See 2.1.19.) Then **C** is S.I., equivalentaly **C** is cover compact and F.S.I. (**C** is also complete and open atomic.) However, **C** is not point compact: Consider the sequence ($(a_n,0)$) in At C. Suppose ($(a_n,0)$) $\longleftarrow$ b for some atom b in C. Then $b = (a_k,0)$ for some $k < \omega$. Note that b is open in **C** whence $b \in N(b)$. But then for all $n > k + 1$, $(a_n,0) \nleq b$, a contradiction. Thus ($(a_n,0)$) does not accumulate at any atom and so **C** is not point compact.

5.1.5 Theorem

The following are equivalent for an interior algebra **A**:

i) A is filter compact.

ii) A is cover compact and closed atomic.

iii) A is point compact and closed atomic.

Proof:

(i) $\Longrightarrow$ (ii): Let A be filter compact. Let $S \subseteq A^{\square}$ with $\sqcap S = 0$. Suppose that for all finite $R \subseteq S$, $\sqcap R > 0$. Then there is a proper filter F of A with $S \subseteq F$. There is an atom a of A with $F \longleftarrow a$. But then $a > 0$ is a lower bound for S, a contradiction. Thus there is a finite $R \subseteq S$ with $\sqcap R = 0$ and so A is cover compact. Let $b > 0$ be closed in A. Then there is an atom a in A with $[b) \longleftarrow a$. Then $a \leq b^C = b$. Thus A is closed atomic. (ii) $\Longrightarrow$ (iii): Assume (ii). Let (a_i) be a net in At A based on a directed set **W**. Suppose that there is no atom b in A with $(a_i) \longleftarrow b$. Then for each atom b in A there are $d(b) \in N(b)$ and $k(b) \in W$ with $a_i \nleq d(b)$ for all $i \gg k(b)$. Suppose that $\Sigma\{ d(b)^I : b \in \text{At } A \} = 1$. Then by cover compactness there are atoms $b_1,\dots,b_n$ in A with $d(b_1)^I + \dots + d(b_n)^I = 1$. Now there is an $i \gg k(b_1),\dots,k(b_n)$. Then $a_i \nleq d(b_1),\dots,d(b_n)$ whence $a_i \nleq 1$, a contradiction. Hence there is an open upper bound $c < 1$ for $\{ d(b)^I : b \in \text{At } A \}$. But then $c' > 0$ is closed and so by closed atomicity there is an atom $b \leq c'$ in A. But then $b \leq d(b)^I \leq c$, a contradiction. Thus $(a_i) \longleftarrow b$ for some atom b in A and so A is point compact. (iii) $\Longrightarrow$ (i): Assume (iii). Let F be a proper filter in A. Then for each $a \in F$, $a^C > 0$ and so by closed atomicity there is at least one atom $b \leq a^C$ in A. Define a directed set $\mathbf{W} = \langle W, \ll \rangle$ as follows: Put $W = \{ (a,b) : a \in F$ and b is an atom in A with $b \leq a^C \}$ and let $(a,b) \ll (c,d)$ in **W** iff $a \geq c$. Then $\ll$ is clearly a pre–order on W. Let $(a,b),(c,d) \in W$. Then $a,c \in F$ whence $ac \in F$. There is an atom $r \leq (ac)^C$ in A. Then $(ac,r) \in W$ and $(ac,r) \gg (a,b),(c,d)$. Thus $\mathbf{W} = \langle W, \ll \rangle$ is a directed set. Thus we have a net (z_i) in At A based on **W** such that for all $i = (a,b) \in W$, $z_i = b$. By point compactness there is an atom r in A with $(z_i) \longleftarrow r$. Suppose that F does not accumulate at r. Then there is an $a \in F$ with $r \nleq a^C$. Then $r \leq a^{C\prime}$ and so since $a^{C\prime} \in A^O$, $a^{C\prime} \in N(r)$. There is an atom $b \leq a^C$ in A. Then $(a,b) \in W$. Hence there is an $i = (c,d) \gg (a,b)$ in **W** with $z_i \leq a^{C\prime}$, that is $d \leq a^{C\prime}$. But $d \leq c^C \leq a^C$, a contradiction. Thus $F \longleftarrow r$ and so A is filter compact. □

5.1.6 Definition and Remark

Let A be an interior algebra. A is called sequence compact iff every sequence in At A accumulates at an atom. Clearly every point compact interior algebra is sequence compact. □

A sequence compact interior algebra need not be point compact: Consider the ordinal space Ω with the interval topology. (See [19] example 42.). Ω^A is sequence compact since any sequence in Ω has an accumulation point in $\mathbf{\Omega}$. However, $\mathbf{\Omega}$ is not compact and so $\mathbf{\Omega}^A$ is not point compact.

5.1.7 Definition and Remark

Let **A** be an interior algebra. **A** is called **countably compact** iff for all countable $S \subseteq A^O$ with $\Sigma S = 1$ there is a finite $R \subseteq S$ with $\Sigma R = 1$. A topological space **X** is countably compact in the usual sense iff $\mathbf{X}^A$ is countably compact. Any cover compact interior algebra is countably compact but from topology we know that the converse fails (See [19].) In fact, an interior algebra is cover compact iff it is countably compact and Lindelöf. Notice also that the class of countably compact interior algebras is a Heyting class. □

5.1.8 Theorem

Let **A** be a closed atomic sequence compact interior algebra. Then **A** is countably compact.

Proof:

Let $\{ a_n : n < \omega \}$ be a countable set of open elements of **A** with $\Sigma \{ a_n : n < \omega \} = 1$. Suppose that for each finite subset $R \subseteq \{ a_n : n < \omega \}$, $\Sigma R < 1$. Then for all $n < \omega$, $a_0 + \ldots + a_n < 1$. Then $a_0' \cdot \ldots \cdot a_n' > 0$ and is closed whence there is an atom $r_n \leq a_0' \cdot \ldots \cdot a_n'$. Then there is an atom s in **A** with $(r_n) \longleftarrow s$. Now there is a $k < \omega$ with $s \leq a_k$. Then $a_k \in N(s)$. Hence there is an $n \geq k$ with $r_n \leq a_k$. But $r_n \leq a_0' \cdot \ldots \cdot a_n'$ whence $r_n \leq a_k'$, a contradiction. Hence $\Sigma R = 1$ for some finite $R \subseteq \{ a_n : n < \omega \}$ and so **A** is countably compact. □

Trivially sequence compactness does not imply countable compactness in the absence of closed atomicity: consider an atomless Boolean interior algebra. Also a closed atomic countably compact interior algebra need not be sequence compact: Let **B** be the interior algebra with B^u equal to the product of 2^u and the free Boolean algebra on $\aleph_0$ generators, and $B^O = \{ (0,0), (1,0), (1,1) \}$. Let $\lambda > \aleph_0$ be a cardinal and for all $i < \lambda$ let $\mathbf{B}_i = \mathbf{B}$. Put $\mathbf{A} = \Pi \{ \mathbf{B}_i : i < \lambda \}$. Then any countable $S \subseteq A^O$ with $\Sigma S = 1$ must contain (1,1) whence **A** is countably compact. **A** is also closed atomic and even open atomic but it is not sequence compact: Let (a_n) be any injective sequence in At **A**. Suppose there is an atom r with $(a_n) \longleftarrow r$. Note that At $\mathbf{A} \subseteq A^O$ and so $r \in N(r)$. There is a smallest $k < \omega$ such that for all $n \geq k$, $a_n \neq r$. Then $a_n \nleq r$ for all $n \geq k$, a contradiction.

5.1.9 Definition and Remark

Call an interior algebra **A** **strictly sequence compact** iff every sequence in At **A** has a subsequence which converges to an atom. By Theorem 3.4.12 we see that every strictly sequence compact interior algebra is sequence compact. Note that a topological space **X** is sequentially compact iff X^A is strictly sequence compact. □

Not every sequence compact interior algebra is strictly sequence compact since in fact a complete atomic point compact interior algebra need not be strictly sequence compact. (See [19].) From Theorem 3.4.16 we see:

<u>5.1.10 Corollary</u>

A first countable interior algebra is sequence compact iff it is strictly sequence compact. □

<u>5.1.11 Corollary</u>

Let **A** be a second countable closed atomic interior algebra. Then the following are equivalent:

i) **A** is strictly sequence compact.

ii) **A** is sequence compact.

iii) **A** is cover/filter/point compact.

iv) **A** is countably compact.

<u>Proof</u>:

Note that by Theorem 5.1.5 and closed atomicity, cover, filter and point compactness coincide in **A**. **A** is second and hence first countable by Theorem 3.5.9 and so by Corollary 5.1.10 (i) $\Leftrightarrow$ (ii). By Theorem 5.1.8 we have (ii) $\Longrightarrow$ (iv). Also by Theorem 3.5.9, **A** is Lindelöf. Thus (iv) $\Longrightarrow$ (iii). Clearly (iii) $\Longrightarrow$ (ii). □

<u>5.1.12 Definition and Remark</u>

The filter, point and sequence compactness properties refer to accumulation to atoms. We now strengthen these properties by replacing accumulation with convergence. Let **A** be an interior algebra. Call **A** **supercompact** iff every non–empty subset of $A - \{ 0 \}$ converges to an atom. Call **A** **point supercompact** iff every net in At **A** converges to an atom and call **A** **sequence supercompact** iff every sequence in At **A** converges to an atom. Clearly supercompactness implies point supercompactness which in turn implies sequence supercompactness which in turn implies strict sequence compactness. Clearly a topological space **X** is supercompact in the usual sense iff X^A is supercompact, iff X^A is point supercompact, and in the case where **X** is non–empty this is equivalent to X^A being S.I. □

<u>5.1.13 Theorem</u>

Let **A** be an interior algebra. The following are equivalent:

i) A is supercompact.

ii) A is closed atomic and point supercompact.

iii) A is closed atomic and either S.I. or trivial.

iv) Every net in A – { 0) converges to an atom.

Proof:

(i) ⟹ (ii) is clear since a supercompact interior algebra is filter compact. (ii) ⟹ (iii): Assume (ii). Suppose A is non–trivial. Then by closed atomicity there at least one atom $r \leq 1$. We may form a directed set $W = \langle W, \ll \rangle$ with W = At A and $r \ll s$ for all $r,s \in W$. We thus have a net (z_i) in At A with $z_i = i$ for all $i \in W$. There is an atom s in A with $(z_i) \longrightarrow s$. Suppose that there is a closed $b > 0$ in A with $s \nleq b$. Then $s \leq b' \in A^O$. Thus $b' \in N(s)$. Now there is an atom $v \leq b$. But then since $(z_i) \longrightarrow s$ we have $v \leq b'$, a contradiction. Thus s is a lower bound for $A^{\Box} - \{ 0 \}$ and so it follows that A is S.I. with monolith $s^{C\prime}$. (iii) ⟹ (iv): Assume (iii). If A is trivial there are no nets in $A - \{ 0 \}$. Suppose A is non–trivial and let m be the monolith of A. Then there is an atom $r \leq m'$ in A. Suppose there is an $a \in N(r) - \{ 1 \}$. Then $r \leq a^I \leq m$, a contradiction. Thus $N(r) = \{ 1 \}$ and so every net in $A - \{ 0 \}$ converges to r. (iv) ⟹ (i) follows easily from Theorem 3.4.6. □

5.1.14. Remark

Let A be a non–trivial supercompact interior algebra and let m be the monolith of A. Note that every subset of A and every net in A converges to all the atoms below m′ and moreover if r is any one of these atoms then $r^C = m'$. □

In general point or sequence supercompactness does not imply subdirect irreducibility in non–trivial interior algebras: consider an atomless Boolean interior algebra. Considering the interior algebra C discussed on page 116 we see that subdirect irreducibility does not imply point or sequence supercompactness. Since sequence supercompactness and point compactness are independent even in complete atomic interior algebras as we see from topology, neither of these implies point supercompactness even in a complete atomic interior algebra. Similarly strict sequence compactness does not imply sequence supercompactness even in a complete atomic interior algebra. (See [19].)

5.1.15 Theorem

Let A be a closed atomic interior algebra with $1 < |\text{At } A| \leq \aleph_0$. Then the following are equivalent:

i) A is subdirectly irreducible.

ii) **A** is (point) supercompact.

iii) **A** is sequence supercompact.

Proof:

By Theorem 5.1.13 we know that (i) $\Leftrightarrow$ (ii). Also (ii) $\Longrightarrow$ (iii). (iii) $\Longrightarrow$ (i): Assume (iii). Let $\{ b_n : n < \omega \}$ be an enumeration of At **A**. Define a sequence (a_n) in At **A** as follows: Define a function $f : \omega \longrightarrow \omega$ by $f(n) = n(n + 1) / 2$ for all $n < \omega$. Put $a_0 = b_0$. If $a_{f(n)},\dots,a_{f(n+1)-1}$ have been defined then define $a_{f(n+1)} = b_{n+1}$, $a_{f(n+1)+1} = b_n$, ... , $a_{f(n+2)-1} = b_0$. By (iii) there is an atom r in **A** with $(a_n) \longrightarrow r$. Suppose that there is a closed $c > 0$ in **A** with $r \nleq c$. Then $r \leq c' \in A^O$ and so $c' \in N(r)$. Hence there is a $k < \omega$ such that $a_n \leq c'$ for all $n \geq k$. By closed atomicity there is an atom $s \leq c$ in **A**. But then $a_n = s$ for some $n \geq k$ and so $s \leq c'$, a contradiction. Thus $r \leq c$ for all $c \in A^{\square} - \{ 0 \}$. It follows that **A** is S.I. with monolith $r^{C\prime}$. □

We now turn to the preservation of the compactness properties:

5.1.16 Lemma

Let **A** be an interior algebra and let $a \in A$. Let (z_i) be a net in $(a]$ and let $b \in (a]$. Then $(z_i) \longleftarrow b$ in **A** iff $(z_i) \longleftarrow b$ in $(\mathbf{a}]$. □

5.1.17 Theorem

The following classes of interior algebras are closed under closed quotients:

i) Cover compact interior algebras.

ii) Filter compact interior algebras.

iii) Point compact interior algebras.

iv) Countably compact interior algebras.

v) Sequence compact interior algebras.

vi) Strictly sequence compact interior algebras.

vii) Supercompact interior algebras.

viii) Point supercompact interior algebras.

ix) F.S.I. or trivial interior algebras.

x) S.I. or trivial interior algebras.

xi) Sequence supercompact interior algebras.

Proof:

(i): Let **A** be a cover compact interior algebra and let $c \in A^{\square}$. Consider an $R \subseteq (c]^O$ with $\Sigma R = c$ in $(c]$. Then for each $r \in R$ there is a $b(r) \in A^O$ with $r = cb(r)$. Put $S = \{ b(r) : r \in R \} \cup \{ c' \}$. Let d be an upper bound for S in **A**. Then for all $r \in R$, $r = cb(r) \leq cd$.

Hence $cd = c$, that is $c \leq d$. But $c' \leq d$ and so $d = 1$. Thus $\Sigma S = 1$. Hence there are $r_1,\dots,r_n \in R$ with $b(r_1) + \dots + b(r_n) + c' = 1$. Taking the meet on both sides with c gives $r_1 + \dots + r_n = c$. Thus $(c]$ is cover compact and so the class of cover compact interior algebras is closed under closed quotients. (ii): By (i), Theorem 5.1.5 and Theorem 1.5.4 (iii) it follows that the class of filter compact interior algebras is closed under closed quotients. (iii): Let A be a point compact interior algebra and let $c \in A^{\square}$. Let (z_i) be a net in $At\,(c] \subseteq At\,A$. There is an atom r in A with $(z_i) \longleftarrow r$ in A. By Proposition 3.4.19, $r \leq c^C = c$. Thus by Lemma 5.1.15 $(z_i) \longleftarrow r$ in $(c]$. Thus $(c]$ is point compact. (iv) and (v) are proved similarly to (i) and (iii) respectively. (vi): Let A be strictly sequence compact and let $c \in A^{\square}$. Let (z_n) be a sequence in $At\,(c] \subseteq At\,A$. Then there is a subsequence (x_k) of (z_n) and an atom r in A with $(x_k) \longrightarrow r$. By Proposition 3.4.19 $r \leq c^C = c$ and so by Lemma 5.1.15 $(x_k) \longrightarrow r$ in $(c]$. Thus $(c]$ is strictly sequence compact. (viii) is proved similarly to (iii) and (vii) follows from (iii), Theorem 5.1.13 and Theorem 1.5.4 (iii). (ix): It suffices to show that if A is S.I. and $c \in A^{\square} - \{ 0 \}$ then $(c]$ is F.S.I. So let A and c be as specified. Suppose $(c]$ is not F.S.I. Then there are $a,b \in (c]^{\square} - \{ 0 \}$ with $ab = 0$. By Corollary 1.4.7 $a,b \in A^{\square} - \{ 0 \}$, a contradiction. Thus $(c]$ is F.S.I. (x) follows from (ix) and (i). (xi) is proved similary to (iii). □

We know from topology that none of the classes (i) – (xi) above are closed under open quotients which correspond to open subspaces nor under (countable) products which correspond to sums. However we have:

5.1.18 Theorem

The following classes of interior algebras are closed under finite joins (in particular, under finite products).

i) Cover compact interior algebras.
ii) Filter compact interior algebras.
iii) Point compact interior algebras.
iv) Sequence compact interior algebras.
v) Strictly sequence compact interior algebras.
vi) Countably compact interior algebras.

Proof:

Let $A_1,\dots,A_n$ be interior algebras, $n < \omega$, and suppose A is a join of $A_1,\dots A_n$. Then there are $a_1,\dots,a_n \in A$ with $A_i \cong (a_i]$ for all $i = 1,\dots,n$ and $a_1 + \dots + a_n = 1$. (i): Suppose that $A_1,\dots,A_n$ are all cover compact. Let $S \subseteq A^O$ with $\Sigma S = 1$. Consider an $i \in \{ 1,\dots, n \}$. Put $T = \{ a_i c : c \in S \}$. Then $T \subseteq (a_i]^O$ and moreover $\Sigma T = 1 \cdot a_i = a_i$ in A and hence in $(a_i]$.

Thus there is a finite $X_i \subseteq S$ with $\Sigma \{ a_i c : c \in X \} = a_i$ in $(\mathbf{a_i}]$. Put $X = X_1 \dot{\cup} \ldots \cup X_n$. Then X is finite and $a_i \leq \Sigma X$ for all $i = 1,\ldots,n$ and so $b \leq \Sigma X$. However $X \subseteq S$ and so $\Sigma X \leq b$ whence $b = \Sigma X$. Thus **A** is cover compact and so the class of cover compact interior algebras is closed under finite joins. (ii): By (i), Theorem 5.1.5 and Theorem 1.5.3 we see that the class of filter compact interior algebras is closed under finite joins. (iii): Let $\mathbf{A}_1,\ldots,\mathbf{A}_n$ all be point compact. Let (z_k) be a net in At **A** based on a directed set **W**. Suppose that for all $i = 1,\ldots,n$ there is a $k(i) \in W$ such that $z_k \cdot a_i = 0$ for all $k \gg k(i)$ in **W**. There is a $k \gg k(1),\ldots,k(n)$ in **W**. Then $z_k = z_k \cdot a_1 + \ldots + z_k \cdot a_n = 0$, a contradiction. Hence there is $1 \leq i \leq n$ such that for all $j \in W$ there is a $k \gg j$ in **W** with $z_k \cdot a_i > 0$, that is $z_k \leq a_i$. Let $Z = \{ m \in W : z_m \leq a_i \}$ and let $\mathbf{Z} = \langle Z, \ll \rangle$. Then **Z** is a directed set and so we have a net (v_m) in At $(\mathbf{a_i}]$ based on **Z** with $v_m = z_m$ for all $m \in Z$. Then there is an atom r in $(\mathbf{a_i}]$ with $(v_m) \longleftarrow r$ in $(\mathbf{a_i}]$ and hence in **A** by Lemma 5.1.16. But then $(z_k) \longleftarrow r$. Thus **A** is point compact and so the class of point compact interior algebras is closed under finite joins. The proof for (iv) and (v) is similar to (iii) and that of (vi) is similar to (i). □

A product of two S.I. interior algebras is obviously not S.I. and so using Theorem 5.1.15 and Theorem 5.1.13 we easily see that none of the supercompactness properties is preserved under finite products.

5.2 COMPACTNESS AND THE SEPARATION PROPERTIES

In this section we investigate the interaction between compactness properties and separation properties.

<u>5.2.1 Definition</u>

Let **A** be a closed atomic interior algebra. We will say that **A** is **compact** iff **A** satisfies the equivalent properties: filter, cover, and point compactness. (See Theorem 5.1.5.) □

<u>5.2.2 Theorem</u>

Let **A** be atomic, compact and Hausdorff. Then **A** is normal.

<u>Proof:</u>

Let a and b be closed and disjoint in **A**. Consider an atom $r \leq a$ in **A**. Then if $s \leq b$ is an atom in **A**, $r \neq s$ and so there are disjoint neighbourhoods $v(s) \in N(s)$ and $w(s) \in N(r)$. Now we have $b = \Sigma$ At $(\mathbf{b}]$ in **A** and hence in $(\mathbf{b}]$. Thus $b = \Sigma \{ bv(s)^I : s \in \text{At } (\mathbf{b}] \}$ in $(\mathbf{b}]$. Also, for all $s \in$ At $(\mathbf{b}]$, $bv(s)^I \in (\mathbf{b}]^O$. By Theorem 5.1.17 (i), $(\mathbf{b}]$ is compact and so there are $s_1,\ldots,s_n \in$ At $(\mathbf{b}]$ with $b = b\cdot(v(s_1)^I + \ldots + v(s_n)^I)$. Put $p(r) = v(s_1) + \ldots + v(s_n)$ and put

$q(r) = w(s_1)\cdot\ldots\cdot w(s_n)$. Then $b \leq v(s_1)^I + \ldots + v(s_n)^I \leq p(r)^I$, that is $p(r) \in N(b)$. Also $q(r) \in N(r)$. Now $a = \Sigma$ At (**a**] in **A** and hence in (**a**]. Thus $a = \Sigma \{ aq(r)^I : r \in$ At (**a**] $\}$ in (**b**] and also $aq(r)^I \in (a]^O$ for all $r \in$ At (**a**]. By Theorem 5.1.17 (i), (**a**] is compact and so there are $r_1,\ldots,r_m \in$ At (**a**] with $a = a\cdot(q(r_1)^I + \ldots + q(r_m)^I)$. Put $d = q(r_1) + \ldots + q(r_n)$ and $c = p(r_1)\cdot\ldots\cdot p(r_n)$. Then $a \leq q(r_1)^I + \ldots + q(r_n)^I \leq d^I$, that is $d \in N(a)$. Also $c \in N(b)$. Note that for all $r \in$ At (**a**], $p(r)$ and $q(r)$ are disjoint whence c and d are disjoint. Thus **A** is normal. □

5.2.3 Corollary

Let **A** be an atomic compact Fréchet algebra. Then the following are equivalent:

i) **A** is normal.

ii) **A** is regular.

iii) **A** is Urysohn.

iv) **A** is Hausdorff.

Proof:

By Theorem 5.2.2 (iv) ⟹ (i). **A** is atomic and Fréchet, hence strongly Fréchet and so (i) ⟹ (ii). By Theorem 4.4.3 (ii) ⟹ (iii) and clearly (iii) ⟹ (iv). □

5.2.4 Definition and Remark

Let **A** be an interior algebra. An element $a \in A$ is called **filter**, **cover** or **point compact** iff the principal quotient (**a**] is **filter**, cover or **point compact** respectively. In atomic interior algebras these concepts clearly coincide and in this case we refer to a as being **compact.** If **B** is a Boolean algebra and $a \in B$ then a is a compact element of **B** in the traditional sense iff a is cover compact in B considered as a Boolean interior algebra. □

5.2.5 Theorem

Let **A** be a residually atomic Hausdorff algebra and let $a \in A$ be filter compact. Then a is closed.

Proof:

Suppose $a < a^C$. Then by residual atomicity we have an atom $r \leq a^C a'$ in **A**. Then $r \ll a$ and so by Proposition 3.2.15 there is an ultrafilter F of (**a**] with $F \longrightarrow r$ in **A**. Since a is filter compact there is an atom s in (**a**] with $F \longleftarrow s$ in (**a**] whence $F \longrightarrow s$ (**a**]. (See Corollary 3.2.14.) There is an ultrafilter G in **A** with $F \subseteq G$. Then if b is a neighbourhood of s in **A**, ab is a neighbourhood of s in (**a**] whence $ab \in F \subseteq G$ and hence $b \in G$. Thus $G \longrightarrow s$ in **A** and also $G \longrightarrow r$ in **A**. Since **A** is Hausdorff, $r = s$. But then $s \leq a'$, a contradiction. Hence $a^C = a$ as required. □

Using Theorem 5.1.17 (ii) we now see:

5.2.6 Corollary

Let **A** be a residually atomic filter compact Hausdorff algebra and let $a \in A$. Then a is closed in **A** iff a is filter compact. □

5.2.7 Corollary

Let **A** be a residually atomic Hausdorff algebra. If $a,b \in A$ are filter compact then ab is filter compact.

Proof:

By Theorem 5.2.5 a is closed in **A**. Hence ab is closed in **(b]**. By Corollary 5.2.6, ab is filter compact in **(b]** and hence in **A**. □

5.2.8 Corollary

Let **A** be a residually atomic Hausdorff algebra. Let **B** be a filter compact interior algebra. Then any bijective topomorphism $f : \mathbf{A} \longrightarrow \mathbf{B}$ is an isomorphism.

Proof:

Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be a bijective topomorphism. Let $b \in A$ and suppose f(b) is closed in **B**. Then by Theorem 5.1.17 (ii), **(f(b)]** is filter compact. Let F be an ultrafilter in **(b]**. Then f[F] is an ultrafilter in **(f(b)]** and so there is an atom a in **(f(b)]** with $f[F] \longrightarrow a$ in **(f(b)]**. Then a is an atom in B and so, since f is a Boolean algebra isomorphism there is an atom d in **A** with f(d) = a. Let c be a neighbourhood of d in **(b]**. Then f(c) is a neighbourhood of a in **(f(b)]** and so $f(c) \in f[F]$ whence $c \in F$. Thus $F \longrightarrow d$ in **(b]**. Thus **(b]** is filter compact. By Theorem 5.2.5 b is closed in **A**. Thus $f^{-1}[B^{\square}] = A^{\square}$ and so $f[A^{\square}] = B^{\square}$ and so it follows that f is an isomorphism. □

5.2.9 Remark

Corollary 5.2.8 is interesting as it is a generalization of the result in topology which says that a bijective continuous map from a compact space to a Hausdorff space is a homeomorphism. □

In atomic interior algebras the compact elements behave like atoms as far as certain separation properties are concerned:

5.2.10 Theorem

Let **A** be an atomic interior algebra. Then:

i) A is regular iff for each disjoint pair a,b $\in$ A with a closed and b compact, there are disjoint neighbourhoods d $\in$ N(a) and c $\in$ N(b).

ii) A is Hausdorff iff for each disjoint pair a,b $\in$ A with a and b compact, there are disjoint neighbourhooods d $\in$ N(a) and c $\in$ N(b).

iii) A is Urysohn iff for each disjoint pair a,b $\in$ A with a and b compact, there are disjoint closed neighbourhoods d $\in$ N(a) and c $\in$ N(b).

Proof:

The reverse directions are clear since atoms are compact. For the forward directions: (i): Let a,b $\in$ A be disjoint with a closed and b compact. If $r \leq b$ is an atom then $r \not\leq a$ and so $r \leq a'$, whence $a' \in N(r)$. By regularity there are disjoint neighbourhoods $c(r) \in N(r)$ and $d(r) \in N(a)$. Now $b = \Sigma$ At **(**b] in A and hence in **(**b]. Thus $b = \Sigma \{ bc(r)^I : r \in$ At **(**b] $\}$ in **(**b]. Also, for all r $\in$ At **(**b], $bc(r)^I \in (b]^O$. Thus since b is compact there are $r_1,\ldots,r_n \in$ At **(**b] with $b = b\cdot(c(r_1)^I + \ldots + c(r_n)^I)$. Put $c = c(r_1) + \ldots + c(r_n)$ and $d = d(r_1)\cdot\ldots\cdot d(r_n)$. Then $b \leq c(r_1)^I + \ldots + c(r_n)^I \leq c^I$, that is c $\in$ N(b). Also d $\in$ N(a). Moreover c and d are disjoint. (ii): Let a,b $\in$ A be disjoint with a and b compact. Let $s \leq a$ be an atom. Consider an atom $r \leq b$. Then $r \neq s$ and so there are disjoint neighbourhoods $c(r) \in N(r)$ and $d(r) \in N(s)$. Proceeding as in (i) we obtain $r_1,\ldots,r_n \in$ At **(**b] with $b = b\cdot(c(r_1)^I + \ldots + c(r_n)^I)$ in **(**b]. As in (i) putting $c(s) = c(r_1) + \ldots + c(r_n)$ and $d(s) = d(r_1)\cdot\ldots\cdot d(r_n)$ we have $c(s) \in N(b)$, $d(s) \in N(s)$ and $c(s)d(s) = 0$. Proceeding again as in (i) we now obtain $s_1,\ldots,s_m \in$ At **(**a] with $a = a\cdot(d(s_1)^I + \ldots + d(s_n)^I)$ in **(**a]. As in (i) again, putting $d = d(s_1) + \ldots + d(s_m)$ and $c = c(s_1)\cdot\ldots\cdot c(s_m)$ we have d $\in$ N(a), c $\in$ N(b) and cd = 0. (iii): As for (ii) but with the c(r)'s and d(r)'s chosen to be closed whence the c(s)'s and d(s)'s are closed and hence c and d are closed. □

5.2.11 Definition

Let A be an interior algebra. We say that A is **locally compact** iff for each atom r in A and each a $\in$ N(r) there is a filter compact b $\in$ N(r) with $b \leq a$, that is { b $\in$ N(r) : b is filter compact } is a base for the filter N(r). An element a $\in$ A is called **locally compact** iff **(**a] is a locally compact interior algebra. □

Under what constructions are locally compact interior algebras preserved?

5.2.12 Theorem

The class of locally compact interior algebras is closed under both open and closed quotients and under joins.

Proof:

Open quotients: Let A be a locally compact interior algebra. Let $a \in A^O$. Suppose r is an atom in **(a]**. Consider a neighbourhood d of r in A. By Corollary 1.4.5 (iii), say, d is a neighbourhood of r in A. There is a filter compact neighbourhood c of r in A with $c \leq d$. But then c is a neighbourhood of r in **(a]**. Thus **(a]** is locally compact. Closed quotients: Let A be locally compact. Let $a \in A^{\square}$. Suppose r is an atom in **(a]**. Let d be a neighbourhood of r in **(a]**. There is a neighbourhood c of r in A with d = ac. There is then a filter compact neighbourhood b of r in A with $b \leq c$. Now ab is closed in **(b]** and so by Theorem 5.1.17 (ii), we see that ab is compact. Moreover ab is a neighbourhood of r in **(a]** and $ab \leq b$. Thus **(a]** is locally compact. Joins: Let $\{ A_i : i \in I \}$ be a family of locally compact interior algebras and let A be a join of this family. Then there are elements $S = \{ a_i : i \in I \}$ in A with $A_i \cong (a_i]$ for all $i \in I$ and $\Sigma S = 1$. Let r be an atom in A. Then there is an $i \in I$ with $r \leq a_i$. Then r is an atom in $(a_i]$ which is locally compact. Suppose d is a neighbourhood of r in A. Then $a_i d$ is a neighbourhood of r in $(a_i]$. Hence there is a filter compact neighbourhood c of r in $(a_i]$ with $c \leq a_i d$. Now $r \leq a_i c^I \leq c^I$, that is, c is a neighbourhood of r in A. Moreover $c \leq d$. Thus A is locally compact. □

We know from topology that local compactness is not preserved under arbitrary principal quotients. (See [19].) The property of local compactness is particularly interesting as regards its interaction with the Hausdorff property. We investigate this now.

5.2.13 Theorem

Let A be an atomic Hausdorff algebra. Then the following are equivalent:

i) A is locally compact.

ii) Every atom in A has a compact neighbourhood.

iii) There is a base S for A with a^C compact for all $a \in S$.

Proof:

(i) ⟹ (iii): Assume (i). Let $a \in A^O$. Consider an atom $r \leq a$. Then $a \in N(r)$ and so there is a compact $b(a,r) \in N(r)$ with $b(a,r) \leq a$. By Theorem 5.2.5, b(a,r) is closed. Hence $b(a,r)^{IC} \leq b(a,r)^C = b(a,r)$. By Theorem 5.1.17 and the fact that $b(a,r)^{IC}$ is closed in **(b(a,r)]**, we see that $b(a,r)^{IC}$ is compact. Put $S = \{ b(a,r)^I : a \in A^O \text{ and } r \in \text{At } (a] \}$. If $a \in A^O$ then since A is atomic, $a = \Sigma \text{ At } (a]$ whence $a = \Sigma \{ b(a,r)^I : r \in \text{At } (a] \}$. Thus S is a base for A and (iii) follows. (iii) ⟹ (ii): Assume (iii). Let r be an atom in A. $\Sigma S = 1$ so $r \leq a$ for some $a \in S$. But then a^C is a compact neighbourhood of r. (ii) ⟹ (i): Assume (ii). Let r be an atom in A. There is a compact neighbourhood b of r. By Theorem 5.2.5 b is closed. Let $d \in N(r)$. Put $a = (bd)^{IC}$. Then $a \leq b$. Now a is closed and so by Corollary 5.2.6,

(a] is compact, and by Theorem 4.5.3 (v), **(a]** is Hausdorff. Of course **(a]** is atomic and so by Corollary 5.2.3 **(a]** is a regular algebra. Now $(bd)^I \leq a$ is open in **(a]** and $r \leq (bd)^I$ whence $(bd)^I$ is a neighbourhood of r in **(a]**. Hence there is a closed neighbourhood c of r in **(a]** with $c \leq (bd)^I$. Let $e = c^{I,a}$. Then $e \leq (bd)^I$ is open in **($(bd)^I$]**. By Corollary 1.4.7, say, e is open in **A**. Now $r \leq e \leq c$ and so $c \in N(r)$ in **A**. By Theorem 5.1.17 (ii), c is compact since a is. Moreover $c \leq d$. Thus **A** is locally compact. □

5.2.14 Corollary

Let **A** be an atomic compact Hausdorff algebra. Then **A** is locally compact. □

5.2.15 Corollary

Let **A** be an atomic locally compact Hausdorff algebra. Then the following are equivalent:

i) **A** is a Lindelöf algebra.

ii) **A** is a join of a countable family of cover compact interior algebras.

Proof:

By Theorem 3.5.22 we see that (ii) ⟹ (i). Now assume that **A** is Lindelöf. By Theorem 5.2.13 there is a base S for **A** with a^C compact for all $a \in S$. Now $\Sigma S = 1$ and so there is a countable $R \subseteq S$ with $\Sigma R = 1$. But then $\Sigma \{ a^C : a \in R \} = 1$ whence (ii) follows. Thus (i) ⟹ (ii). □

From Corollary 5.2.6 we see:

5.2.16 Proposition

Let **A** be a residually atomic locally compact Hausdorff algebra. Then **A** is regular. □

5.2.17 Theorem

Let **A** be a residually atomic Hausdorff algebra. If $a,b \in A$ are locally compact then ab is locally compact.

Proof:

Let r be an atom in **(ab]**. Let e be a neighbourhood of r in **(ab]**. There is a neighbourhood v of r in **A** $e = abv$. (Recall Lemma 3.5.14.) Then av and bv are neighbourhoods of r in **(a]** and **(b]** respectively. Then there is a filter compact neighbourhood c of r in **(a]** with $c \leq av$, and there is a filter compact neighbourhood d of r in **(b]** with $d \leq bv$. By Corollary 5.2.7 cd is filter compact. Also $cd \leq e$. Now $r \leq c^{I,a}$ and $r \leq d^{I,b}$ whence $r \leq a(a' + c)^I b(b' + c)^I = ab\cdot[(a' + c)\cdot(b' + c)]^I = ab(a'b' + a'd + ab'\ cd)^I \leq ab(a' + b'\ cd)^I = (cd)^{I,ab}$. Thus cd is a neighbourhood of r in **(ab]**. Thus **(ab]** is locally compact. □

5.2.18 Lemma

Let **A** be an interior algebra and let $a \in A$. The following are equivalent:

i) a is open in $(\mathbf{a^C}]$.

ii) a is open in $(\mathbf{b}]$ for some closed $b \in A$.

iii) $a^C a'$ is closed in **A**.

iv) There are $b \in A^{\square}$ and $c \in A^O$ with $a = bc$.

Proof:

Clearly (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iv). (iv) $\Longrightarrow$ (i): Assume (iv). Then $a \leq b$ whence $a^C \leq b^C = b$. Thus $a^C c \leq bc = a$. But $a \leq a^C c$ and so $a = a^C c$. Thus a is open in $(\mathbf{a^C}]$. Thus (i), (ii) and (iv) are equivalent. (i) $\Leftrightarrow$ (iii): a is open in $(\mathbf{a^C}]$ iff $a^C a'$ is closed in $(\mathbf{a^C}]$, iff $a^C a'$ is closed in **A** by Corollary 1.4.5 (vi). □

5.2.19 Definition and Remark

Let **A** be an interior algebra. We will call an element $a \in A$ **residually closed** iff a satisfies the equivalent conditions of Lemma 5.2.18. (If $a \in A$ we may call $a^C a'$ the **residue** of a. Thus $a \in A$ is residually closed iff the residue of a is closed.) If $c \in A$ is residually closed there are $a \in A^O$ and $b \in A^{\square}$ with $c = ab$. Then $(\mathbf{c}] = (\mathbf{ab}]$ is a closed quotient of the open quotient $(\mathbf{a}]$ (and an open quotient of the closed quotient $(\mathbf{b}]$). Thus a class of interior algebras is closed under residually closed quotients iff it is closed under both open and closed quotients. In particular the class of locally compact interior algebras is closed under residually closed quotients. □

5.2.20 Theorem

Let **A** be an atomic Hausdorff algebra. Then every locally compact element of **A** is residually closed.

Proof:

Let $a \in A$ be locally compact. Suppose $r \leq a$ is an atom. Then there is a compact neighbourhood c of r in $(\mathbf{a}]$. There is an open b in A with $r \leq ab \leq c$. We show that $a^C b \leq c$. Let s be an atom with $s \leq a^C b$. Then $s \ll a$ and so by Proposition 3.2.15 there is an ultrafilter F in $(\mathbf{a}]$ with $F \longrightarrow s$ in A. Then there is an ultrafilter G in **A** with $F \subseteq G$. Then $G \longrightarrow s$ in **A**. Now $s \leq b \in A^O$ and so $b \in N(s)$ in **A** whence $b \in G$. But $a \in G$ and so $ab \in G$ whence $c \in G$. Thus $G \cap (c]$ is an ultrafilter in $(\mathbf{c}]$. By Theorem 5.1.3 there is an atom z of $(\mathbf{c}]$ with $G \cap (c] \longrightarrow z$ in $(\mathbf{c}]$. Consider a neighbourhood d of z in A. Then cd is a neighbourhood of r in $(\mathbf{c}]$ and so $cd \in G \cap (c] \subseteq G$. Hence $d \in G$. Thus $G \longrightarrow z$ in **A** and so, since **A** is Hausdorff, $z = s$. Thus $s \leq c$. Since **A** is atomic it follows that $a^C b \leq c$ as required. Thus $r \leq a^C b \leq a$. Note also that $a^C b \in (a^C]^O$ since $b \in A^O$. Put $p(r) = a^C b$. Thus for each atom $r \leq a$ we have an open element $p(r)$ of $(a^C]$ with $r \leq p(r) \leq a$. Since $a = \Sigma\, At\,(a]$ in A and hence in $(a^C]$ it follows that $a = \Sigma \{ p(r) : r \in At\,(a] \}$ in $(a^C]$. Thus a is open in $(a^C]$ and so a is residually closed. □

From the above result and the fact that locally compact interior algebras are closed under residually closed quotients we obtain a nice characterization of locally compact elements in an atomic locally compact Hausdorff algebra.

5.2.21 Corollary

Let A be an atomic locally compact Hausdorff algebra. Then an element $a \in A$ is locally compact iff it is residually closed. □

5.3 COMPACTIFICATIONS OF INTERIOR ALGEBRAS

5.3.1 Definition

Let **A** be an interior algebra. By a **compactification** of **A** we mean a triple $\langle$ **B**, a, f $\rangle$ where **B** is a cover compact interior algebra, a is a dense element of **B** and $f : \mathbf{B} \longrightarrow \mathbf{A}$ is a principal quotient topomorphism with $f|(a] : (a] \longrightarrow A$ an isomorphism. □

We have already met one method of compactifying (constructing a compactification of) an interior algebra:

5.3.2 Lemma

Let A be a non–simple S.I. interior algebra. Then the monolith of A is dense.

Proof:

Let m be the monolith of **A**. Since **A** is non–simple $m > 0$. Let $b \in A^O - \{ 0 \}$. If $b = 1$, $mb = m > 0$ and if $b < 1$, $mb = b > 0$. Thus $m \in Sec\,(A^O - \{ 0 \})$ and so m is dense by Theorem 3.2.17. □

5.3.3 Proposition

Let **A** be a non–trivial interior algebra and let **B** be a non–trivial Boolean algebra. Then $\langle$ **A**[**B**], m, p $\rangle$ is a compactification of **A** where $m = (1,0)$ and $p : \mathbf{A}[\mathbf{B}] \longrightarrow \mathbf{A}$ is the projection onto **A**.

Proof:

Recall that since B is non–trivial, A[B] is S.I. (in particular, cover compact) with monolith $m = (1,0)$. Since A is non–trivial **A**[B] is non–simple whence by Lemma 5.3.2, m is dense. The result follows by 2.1.19. □

There is an interesting method of compactifying complete interior algebras:

5.3.4 Notation

Suppose P is a property of elements of an interior algebra. For each interior algebra A, P(A) will denote the set of elements of A satisfying P. □

From now on P will denote a fixed property of interior algebra elements. Consider the following conditions on P(A) for an interior algebra A:

P0: $a \in P(A)$ implies a is cover compact in A.

P1: $0 \in P(A)$.

P2: $a,b \in P(A)$ implies $a + b \in P(A)$.

P3: $a \in P(A)$ and $b \in A^{\square}$ implies $ab \in P(A)$.

5.3.5 Lemma

Let A be a complete interior algebra and let P(A) satisfy P1, P2 and P3. Put $G = \{ (a,b) \in A^{O} \times 2 : b = 1 \text{ implies } a' \in P(A) \}$. Then G is a generalized topology in the Boolean algebra $A^{u} \times 2^{u}$.

Proof:

Clearly $(0,0) \in G$ and by P1 we see that $(1,1) \in G$. Let $(a,b),(c,d) \in G$. Consider (ac,bd). $ac \in A^{O}$. If $bd = 1$ then $b = 1 = d$. Hence $a',c' \in P(A)$. By P2 $a' + c' \in P(A)$, that is $(ac)' \in P(A)$. Thus $(ac,bd) \in G$ and so G is closed under binary meets. Let $S = \{ (a_i,b_i) : i \in I \} \subseteq G$ be non–empty. If $S \subseteq A^{O} \times \{ 0 \}$ then $\Sigma S \in A^{O} \times \{ 0 \}$ and so $\Sigma S \in G$. Suppose $S \not\subseteq A^{O} \times \{ 0 \}$. Then there is a $j \in I$ with $b_j = 1$. Now $\Sigma S = (\Sigma R,1)$ where $R = \{ a_i : i \in I \}$. We have $(\Sigma R)' = \sqcap \{ a_i' : i \in I \} = a_j' \cdot \sqcap \{ a_i' : i \in I - \{ j \} \}$. Now $a_j' \in P(A)$ since $b_j = 1$, and $\sqcap \{ a_i' : i \in I - \{ j \} \}$ being a meet of closed elements is closed and so by P3 we see that $(\Sigma R)' \in P(A)$. Thus $\Sigma S \in G$. It follows that G is a generalized topology in $A^{u} \times 2^{u}$ as required. □

5.3.6 Definition and Remark

If A is a complete interior algebra and P(A) satisfies P1, P2 and P3, let A(P) denote the interior algebra $\mathrm{Alg} \langle A^{u} \times 2^{u}, G \rangle$ where G is as in Lemma 5.3.5. Notice that the projection $p : A(P) \longrightarrow A$ is a principal quotient topomorphism with $p|(a] : (\mathbf{a}] \longrightarrow A$ an isomorphism, where $a = (1,0)$. □

From now on we assume that for all interior algebras A, P(A) satisfies P0, P1, P2 and P3.

5.3.7 Theorem

Let A be a complete interior algebra. Then:

i) A(P) is cover compact.

ii) If A is not cover compact then $\langle$ A(P), a, p $\rangle$ is a compactification of A, where a = (1,0) and p : A(P) $\longrightarrow$ A is the projection onto A.

Proof:

(i): Let $S = \{ (a_i,b_i) : i \in I \} \subseteq A(P)^O$ be non-empty with $\Sigma S = (1,1)$. Then there is a $j \in I$ with $b_j = 1$. We have $\Sigma \{ a_j' \cdot a_i : i \in I \} = a_j' \cdot \Sigma \{ a_i : i \in I \} = a_j' \cdot 1 = a_j'$. For all $i \in I$ $a_j' \cdot a_i$ is open in $(a_j']$. Moreover, since $b_j = 1$, $a_j' \in P(A)$ and so it is cover compact by P0. Thus there is a finite $K \subseteq I$ with $\Sigma \{ a_j' \cdot a_i : i \in K \} = a_j'$. But then $a_j + \Sigma \{ a_i : i \in I \} = a_j + \Sigma \{ a_j' \cdot a_i : i \in I \} = a_j + a_j' = 1$. Hence $(a_j,b_j) + \Sigma \{ (a_i,b_i) : i \in I \} = (1,1)$. It follows that A(P) is cover compact. (ii): Suppose that A is not cover compact. It suffices to show that a = (1,0) is dense in A(P). Since $1 = 0'$ is not cover compact $1 \notin P(A)$ by P0 and so (0,1) is not open in A(P). Thus $(0,1)^I = (0,0)$ and so $(1,0)^C = (1,0)'^{I}{}' = (0,1)^{I}{}' = (0,0)' = (1,1)$ as required. □

5.3.8 Remark

Notice that if we choose P to be either one of the properties: cover compact or filter compact, then using Theorem 5.1.17 and Theorem 5.1.18 we see that P(A) satisfies P0, P1, P2 and P3 for all interior algebras A. Let X be a non-compact locally compact Hausdorff space and let Y be the Alexandroff compactification of X with $Y - X = \{ * \}$. Let P be either one of the properties: cover compact or filter compact. Then $(X^A)(P) \cong Y^A$ via the map $(S,0) \longmapsto S$ and $(S,1) \longmapsto S \cup \{ * \}$, for all $S \subseteq X$. □

5.3.9 Theorem

Let A be a complete residually atomic Hausdorff algebra and suppose that c is a co-atom in A. Let P be the property: filter compact. Then the following are equivalent:

i) A is filter compact.

ii) $(c](P) \cong A$.

If A is filter compact then an isomorphism from (c](P) to A is given explicitly by the map $(a,b) \longmapsto a + bc'$.

Proof:

(i) $\Longrightarrow$ (ii). A be filter compact. Let f : (c](P) $\longrightarrow$ A be the map $(a,b) \longmapsto a + bc'$. It is not difficult to check that f is a Boolean algebra isomorphism. Note that by Proposition 4.2.10, A is strongly Fréchet. Thus the atom c' is closed in A whence c is open in A. Let $(a,b) \in (c](P)^{O}$. Then $a \in (c]^{O}$ and so, since $c \in A^{O}$, $a \in A^{O}$. (See Corollary 1.4.7.) Thus if $b = 0$, $f(a,b) = a$ is open in A. Suppose $b = 1$. Then $a'^{c} = a'c$ is filter compact in (c] and hence in A. Thus $a'c$ is closed in A by Theorem 5.2.5, and so $f(a,b) = a + c' = (a'c)'$ is open in A. Thus f is a topomorphism. Now let $(a,b) \in (c](P)$ and suppose that $f(a,b)$ is open in A. Then $a = c \cdot f(a,b)$ is open in (c]. Thus if $b = 0$, (a,b) is open in (c](P). Suppose that $b = 1$. Then $a'^{c} = a'c = f(a,b)'$ is closed in A. By Theorem 5.1.17 (ii), a'^{c} is filter compact. Thus (a,b) is open in (c](P). It follows that f is an isomorphism. (ii) $\Longrightarrow$ (i): Assume (ii). By Theorem 5.3.7 (i), (c](P) is cover compact. Let (a,b) be closed in (c](P). Then a is closed in (c]. Suppose $b = 0$. Then a is filter compact in (c], in particular, closed atomic and so there is an atom r of (c] with $r \leq a$. Then $(r,0)$ is an atom in (c](P) with $(r,0) \leq (a,b)$. Suppose $b = 1$. Then $(0,1)$ is an atom in (c](P) with $(0,1) \leq (a,b)$. Thus (c](P) is closed atomic. Thus (c](P) and hence A, is filter compact. □

5.3.10 Corollary

Let A be a complete residually atomic Hausdorff algebra and let P be the property: filter compact. Suppose that A is filter compact. Then:

i) For all co–atoms c in A, $(c](P) \cong A$ via the map $(a,b) \longmapsto a + bc'$.

ii) For all non–closed co–atoms c in A, $\langle A, c, k \rangle$ is a compactification of (c], where $k : A \longrightarrow (c]$ is the canonical quotient map.

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GLOSSARY OF SYMBOLS AND NOTATION

$f : \mathbf{M} \longrightarrow \mathbf{L}$	f is a morphism (usually a map) from **M** to **L**
$f(a)$	image of element a under map f
$f[S]$	image of subset S under map f
$f^{-1}[S]$	pre–image of subset S under map f
$f\|S$	restriction of map f on subset S
id	identity transformation
$\longmapsto$	is mapped to
ϕ	empty set
$\in$	is an element of
$\notin$	is not an element of
$=$	equals
$\neq$	does not equal
$\subseteq$	is a subset of
$\nsubseteq$	is not a subset of
$\subset$	is a proper subset of
$S \cap R$	intersection of subsets S and R
$S \cup R$	union of subsets S and R
$\cap \mathcal{B}$	intersection of family $\mathcal{B}$
$\cup \mathcal{B}$	union of family $\mathcal{B}$
S'	set–theoretic complement of subset S
$S - R$	relative complement of subset R in subset S
$\mathcal{P}(S)$	power set of set S
$\mathcal{P}(S)$	power set Boolean algebra on set S
(,), [,]	brackets within expressions
$\{ a,\ldots,b \}$	set of elements a,...,b
$\{ X : Y \}$	set of X such that Y
$\langle X, \ldots \rangle$	structure with underlying set X
(a,b)	ordered pair a,b
(a,b,c)	ordered triple a,b,c
$\cong$	is isomorphic to (usually algebraic isomorphism or homeomorphism)
$\mathbf{M} \times \mathbf{L}$	direct product of objects **M** and **L**
$\Pi\, \mathcal{B}$	direct product of family $\mathcal{B}$
$\amalg\, \mathcal{B}$	co–product of family $\mathcal{B}$, only used for topological sums

$\mathbf{M} \oplus \mathbf{L}$	sum of Heyting algebras **M** and **L** (see [1])
$\mathbf{1}$	one element Heyting algebra
$\mathbf{2}$	two element chain as an interior algebra
$\mathbf{3}$	three element chain as a Heyting algebra
$\mathrm{I}(\kappa)$	indiscrete space on κ
$\boldsymbol{\Omega}$	ordinal space with interval topology
ω	smallest infinite ordinal
$\boldsymbol{\omega}$	topological space $\langle\ \omega,\ \omega + 1\ \rangle$
∞	informal upper bound to the class of all ordinals
$\aleph_0$	smallest infinite cardinal
$\aleph_1$	smallest uncountable cardinal
iff	if and only if
$\Longrightarrow$	implies
$\Leftrightarrow$	implies and is implied by, iff
$\wedge$	and
$\vee$	or
$\neg$	not
$\forall$	for all
$\exists$	there exists
$\models$	satisfies
φ'	dual of sentence φ
$\varphi(x_0,\ldots,x_n)$	the free variables of formula φ are among $x_0,\ldots,x_n$
$\varphi[a_0,\ldots,a_n]$	formula $\varphi(x_0,\ldots,x_n)$ under valuation $a_0/x_0,\ldots,a_n/x_n$
$\mathcal{LI}$	first order language for interior algebras
$\mathcal{LH}$	first order language for Heyting algebras
$\mathcal{LG}$	first order language for graphs without isolated points
τ	interpretation from $\mathcal{LH}$ to $\mathcal{LI}$
υ	interpretation from $\mathcal{LG}$ to $\mathcal{LI}$
φ^{S}	$\mathcal{LI}$ sentence defining simple
φ^{SI}	$\mathcal{LI}$ sentence defining subdirectly irreducible
φ^{FSI}	$\mathcal{LI}$ sentence defining finitely subdirectly irreducible
φ^{DI}	$\mathcal{LI}$ sentence defining directly indecomposable
Atom(z)	$\mathcal{LI}$ formula stating that z is an atom
ρ_n, σ_n	polynomials used to characterize openly generated interior algebras
$\mathcal{I}$	variety of interior algebras

$\mathcal{H}$	variety of Heyting algebras
Int	category of interior algebras and homomorphisms
Int$^+$	category of interior algebras and topomorphisms
IntO	category of openly generated interior algebras and homomorphisms
IntB	category of Boolean interior algebras and homomorphisms
Halg	category of Heyting algebras and homomorphisms
Balg	category of Brouwerian algebras and homomorphisms
CIn	category of complete atomic interior algebras and complete homomorphisms
CIn$^+$	category of complete atomic interior algebras and complete topomorphisms
OIn	category of atomic operator complete interior algebras and complete homomorphisms
OIn$^+$	category of atomic operator complete interior algebras and complete topomorphisms
Top	category of topological spaces and continuous maps
Tco	category of topological spaces and continuous open maps
ToF	category of finitely generated spaces and continuous maps
TcF	category of finitely generated spaces and continuous open maps
Sfld	category of Stone fields and field maps
Bfld	category of Boolean Stone fields and field maps
Sfwo	category of Stone fields and weakly open field maps
Fld	category of fields of sets and field maps
Pro	category of pre–ordered sets and homomorphisms
RPro	category of pre–ordered sets and regular homomorphisms
Can	category of canonical pre–order fields and pre–order field maps
Cnr	category of canonical pre–order fields and regular pre–order field maps
Set	category of sets and maps
$a \cdot b$ ab	product of a and b , usually lattice product (meet)
$a + b$	sum of a and b , usually lattice sum (join)
$a \rightarrow b$	relative pseudocomplement of a and b
$a * b$	dual relative pseudocomplement of a and b
a'	Boolean complement of element a
a^I	interior of a
a^C	closure of a

0	bottom element , zero
1	top element , one
ΣS	join of subset S
$\sqcap S$	meet of subset S
A^{RC}	Boolean algebra of regular closed elements of interior algebra **A**
A^{RO}	Boolean algebra of regular open elements of interior algebra **A**
$a \cdot^{RO} b$	meet of a and b in the Boolean algebra of regular open elements
$a +^{RO} b$	join of a and b in the Boolean algebra of regular open elements
a'^{RO}	complement of a in the Boolean algebra of regular open elements
0^{RO}	bottom element in the Boolean algebra of regular open elements
1^{RO}	top element in the Boolean algebra of regular open elements
$\Sigma^{RO} S$	join of subset S in the Boolean algebra of regular open elements
$\sqcap^{RO} S$	meet of subset S in the Boolean algebra of regular open elements
$a \nabla b$	open dual difference of a and b
$a \cdot_w b$	weak meet of a and b
$a \longrightarrow_w b$	weak relative pseudocomplement of a and b
$\preceq$	weak ordering
$\approx$	weak equality
$\leq$	less than or equal to
$\nleq$	not less than nor equal to
$\geq$	greater than or equal to
$<$	stricly less than
$>$	strictly greater than
$\ll$ $\gg$	denotes pre–order, in particular the canonical pre–order (page 65)
max S	maximum member of subset S
min S	minimum member of subset S
[a,b]	interval algebra with underlying set $[a,b] = \{ c : a \leq c \leq b \}$
(a]	interval algebra [0,a]
[a)	interval algebra [a,1]
A[a,b]	interval algebra **[a,b]** formed in interior algebra **A**
A(a]	interval algebra **(a]** formed in interior algebra **A**
A[a)	interval algebra **[a)** formed in interior algebra **A**
$r^{I_{a,b}}$	interior of r in **[a,b]**
$r^{C_{a,b}}$	closure of r in **[a,b]**
$r^{\prime_{a,b}}$	complement of r in **[a,b]**

$r^{I,a}$	interior of r in (a]
$r^{C,a}$	closure of r in (a]
$r^{\prime,a}$	complement of r in (a]
$r^{I_{a}}$	interior of r in [a)
$r^{C_{a}}$	closure of r in [a)
$r^{\prime_{a}}$	complement of r in [a)
Alg **T**	interior algebra corresponding to generalized topological space **T**
Gt **A**	generalized topological space corresponding to interior algebra **A**
S.I.	subdirectly irreducible
F.S.I.	finitely subdirectly irreducible
D.I.	directly indecomposable
EDPC	equationally definable principal congruences
EDPM	equationally definable principal meets
WBSO	weak Brouwerian semilattice with filter preserving operations
JEP	Joint Embedding Property
SJEP	Strong Joint Embedding Property
$\mathbf{A}^{u}$	underlying Boolean algebra of interior algebra **A**
$\mathbf{A}^{O}$	Heyting algebra of open elements of interior algebra **A**
f^{O}	restriction of interior algebra homomorphism f to Heyting algebra of open elements
^{O}	functor assigning interior algebras **A** to $\mathbf{A}^{O}$ and homomorphisms f to f^{O}
$\mathbf{A}^{\square}$	Brouwerian algebra of closed elements of interior algebra **A**
$^{\square}$	functor dual to ^{O}
$\mathbf{A}^{\diamond}$	subalgebra of clopen elements of interior algebra **A**
$f^{\diamond}$	restriction of interior algebra homomorphism f to subalgebra of clopen elements
$^{\diamond}$	functor assigning interior algebras **A** to $\mathbf{A}^{\diamond}$ and homomorphisms f to $f^{\diamond}$
$\mathbf{L}^{H}$	interior algebra with Heyting algebra of open elements **L** and underlying Boolean algebra equal to the free Boolean extension of **L**
f^{H}	extension of Heyting algebra homomomorphism f to free Boolean extension
^{H}	functor assigning Heyting algebras **L** to $\mathbf{L}^{H}$ and homomorphisms f to f^{H}
^{B}	functor dual to ^{H}
$^{-}$	isomorphism from **Halg** to **Balg**
$\mathbf{X}^{A}$	power set interior algebra on topological space **X**
f^{A}	inverse image topomorphism obtained from continuous map f

^{A}	functor assigning topological spaces X to X^{A} and continuous maps f to f^{A}
A^{σ}	set of all subsets of atoms of complete interior algebra **A** which have an open join
A^{T}	topological space of atoms in complete atomic interior algebra **A** with open sets A^{σ}
f^{T}	continuous map dual to complete topomorphism f between complete atomic interior algebras
^{T}	functor assigning complete atomic interior algebras **A** to A^{T} and complete topomorphisms f to f^{T}
X^{S}	interior algebra corresponding to Stone field X
f^{S}	inverse image topomorphism obtained from field map f
^{S}	functor assigning Stone fields **X** to X^{S} and field maps f to f^{S}
A^{F}	Stone field dual to interior algebra **A** with underlying set A^{F} – the set of all ultrafilters in **A**
$\mathcal{T}(A)$	topology of Stone field A^{F} with base $\alpha[A^{O}]$
f^{F}	inverse image field map obtained from topomorphism f
^{F}	functor assigning interior algebras **A** to A^{F} and topomorphisms f to f^{F}
A^{D}	Stone space of interior algebra **A**, topological space reduct of A^{F}
f^{D}	inverse image continuous map obtained from f, f^{F} as a continuous map
^{D}	functor assigning interior algebras **A** to A^{D} and topomorphisms f to f^{D}
A^{P}	Stone space of the Heyting algebra A^{O} of open elements of interior algebra **A**
W^{+}	power set modal algebra of frame **W**
W^{t}	finitely generated space obtained from pre–ordered set **W**
$\mathcal{O}(W)$	set of open sets of W^{t} where **W** is a pre–ordered set
f^{t}	pre–order homomorphism f considered as a continuous map
^{t}	concrete functor over Set assigning pre–ordered sets **W** to W^{t} and pre–order homomorphisms f to f^{t}
X^{W}	pre–ordered set obtained from topological space **X**
f^{W}	continuous map f considered as homomorphism of pre–ordered sets
^{W}	concrete functor over Set assigning topological spaces **X** to X^{W} and continuous maps f to f^{W}
X^{M}	canonical pre–order field obtained from Stone field **X**
f^{M}	field map f considered as pre–order field map
^{M}	concrete functor over Fld assigning Stone fields **X** to X^{M} and field maps f to f^{M}

X^N	Stone field obtained from canonical pre–order field **X**
f^N	pre–order field map f considered as a field map of Stone fields
^{N}	concrete functor over **Fld** assigning pre–order fields **X** to X^N and pre–order field maps f to f^N
A^b	OIn^+ bi–reflection of complete atomic interior algebra **A**
^{b}	bi–reflector from CIn^+ to OIn^+
ϵ	co–unit of adjunction $\langle\, ^H, ^O \rangle$
δ	natural isomorphism from id to ^{TA}
ζ	natural isomorphism from id to ^{AT}
$\alpha(A)$	Stone map for interior algebra **A**
α	natural isomorphism from id to ^{FS}
$\beta(X)$	'ultrafilter map' for Stone field **X**
β	natural isomorphism from id to ^{SF}
$e : A^D \longrightarrow A^P$	canonical retraction of A^D onto A^P where **A** is an interior algebra
$d : X \longrightarrow X^{AD}$	canonical embedding **X** into X^{AD} where **X** is a topological space
X[S]	disjoint union of sets X and S
X[S]	topological space $\langle$ X[S], $\mathcal{T} \cup \{$ X[S] $\}\rangle$ obtained from topological space $\mathbf{X} = \langle X, \mathcal{T} \rangle$
A[B]	interior algebra with underlying Boolean algebra $A^u \times B$ and open elements $(A^O \times \{ 0 \}) \cup \{ (1,1) \}$ where **A** is an interior algebra and **B** a Boolean algebra
Int($\mathcal{K}$)	class of all interior algebras **A** with $A^O \in \mathcal{K}$
Halg($\mathcal{K}$)	class of all Heyting algebras A^O with $A \in \mathcal{K}$
Θ(F)	congruence identifying filter of open elements F with 1
F(Θ)	filter of open elements identified with 1 by congruence Θ
a/Θ	equivalence class of element a under equivalence relation Θ
a Θ b	a is related to b by binary relation Θ
A / Θ	quotient of **A** by congruence Θ
Con(**A**)	congruence lattice of structure **A**
Con_f(**A**)	lattice of fully invarient congruences of interior algebra **A**
Con_p(**A**)	lattice of principal congruences of interior algebra **A**
con(a,b)	smallest congruence identifying a and b
L(**M**)	lattice of filters in Heyting algebra **M**
L_p(**M**)	lattice of principal filters in Heyting algebra **M**
L_f(**A**)	lattice of full filters of interior algebra **A**
L^O(**A**)	lattice of open filters of interior algebra **A**

$L^c(A)$	lattice of closed ideals of interior algebra A
S^d	set of complements of members of subset S of an interior algebra
d	isomorphism from $L^o(A)$ to $L^c(A)$ sending F to F^d
N(a)	set of neighbourhoods of a
$N^c(a)$	set of neighbourhoods of a in (c]
E(a)	set of enclosers of a
$\mathcal{N}(G)$	neighbourhood function corresponding to generalized topology G
$\mathcal{G}(N)$	generalized topology corresponding to neighbourhood function N
$R \longrightarrow a$	subset R converges to element a
$R \longleftarrow a$	subset R accumulates at element a
$(b_i) \longrightarrow a$	net (b_i) converges to element a
$(b_i) \longleftarrow a$	net (b_i) accumulates at element a
lim R	set of limits of subset R
lim (b_i)	set of limits of net (b_i)
acc R	set of accumulants of subset R
acc (b_i)	set of accumulants of net (b_i)
Sec R	set of elements having non–zero meet with all elements of R
$t_j(b_i)$	jth tail of net (b_i), $\{ b_i : i \gg j \}$
$F(b_i)$	filter $\{ a : a \geq b_i$ for some $i \in W \}$ derived from net (b_i) in interior algebra based on directed set W
S^α	'α–closure' of subset S of topological space (see page 96)
$c^\alpha(a)$	'α–closure' of element a (see page 96)
$\perp$	reflexive symmetric relation of a graph without isolated points
$\perp^\alpha$	reflexive symmetric relation on atoms of an interior algebra given by $a \perp^\alpha b$ iff there are $d \in N(a)$ and $e \in N(b)$ with $c^\alpha(d) \cap c^\alpha(e) \neq \phi$
At A	set of atoms of interior algebra A
At A	$\langle$ At A, $\ll$ $\rangle$ where $\ll$ is the canonical pre–order on interior algebra A
At^α A	$\{ b \in$ At A : there is an $a \in$ At A with $a \neq b$ and $a \perp^\alpha b \}$ where **A** is an interior algebra
At^α A	graph without isolated points $\langle At^\alpha A, \perp^\alpha \rangle$ where A is an interior algebra
At^α X	**At^α X^A** where X is a topological space
P(A)	set of elements of interior algebra A which satisfy property P
A(P)	interior algebra with underlying Boolean algbra $A^u \times 2^u$ and open elements $\{ (a,b) \in A^o \times 2 : b = 1$ implies $a' \in P(A) \}$, where A is an interior algebra and P is an appropriate property

INDEX OF DEFINITIONS