Concurrence Between the Displaced Libor Market and Hull-White Models

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy in the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.

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Abstract

The concurrence between the displaced lognormal forward-Libor model (DLFM), Gaussian Heath-Jarrow-Morton (GHJM) model and Hull-White (HW) model is explored. We briefly present the theory underpinning these models, specifically focusing on single factors. A useful volatility relation result adapted from Andersen and Piterbarg (2010) is derived. It relates the instantaneous volatility functions of the GHJM model and the DLFM model. The volatility relation allows us to state a specific GHJM model and derive a corresponding DLFM model that it is concurrent with. We take the Hull-White model and derive its corresponding GHJM model, the volatility of the GHJM model is then fed into the volatility relation in order to derive the corresponding DLFM model. This was sufficient mathematical proof of the concurrence, but numerical confirmation is also essential. The HW, GHJM and DLFM models were implemented, with applications to pricing European swaptions. Numerical results show that swaption prices are consistent across the three models. This provides good numerical evidence to support the concurrence between the DLFM and HW models.

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Chapter 1

Introduction

The aim of term structure modelling is to provide a good representation of how interest rates behave in practice (Choudhry *et al.* (2005)). Short rate, instantaneous forward rate and market models are the three classes of interest rate models that have been extensively studied in the field of interest rate modelling. These models are used for the pricing of derivatives and other instruments in the market. Extensive research has been conducted with the aim of formulating methods of implementing these models so that they can be used in practical applications. The use of a model for a particular purpose is key for implementation, depending on the problem faced, we could be interested in spot rates, instantaneous forward rates, or forward market simple rates. Interest rates are driven by many factors, some of which are stochastic in nature. Interest rate models can either be single or multi-factor, depending on the number of stochastic factors that drive the model. Multi-factor models lead to increased accuracy but this comes at the expense of a more complex modelling procedure that will involve correctly estimating the correlation structures between the various factors that affect the model. As a result of this increased complexity, these types of models will be beyond the scope of this dissertation, we will only focus on single factor models since they are easier to present, theoretically derive and numerically implement.

The aim of this dissertation is to show the concurrence between the Hull-White (HW) model (short rate), Gaussian Heath Jarrow Morton (GHJM) model (instantaneous forward rates) and Displaced Libor Forward Market (DLFM) model (simple forward rates). This concurrence will be shown theoretically and confirmed numerically. Theoretically, we show that with a suitable integration of the instantaneous volatility function that the DLFM contains a discretized GHJM model which in turn contains a discretized HW model. Numerical results are confirmed by implementing these models and pricing swaptions and assessing if these swaption prices are the same.

In the Chapter 2 we present the HW, GHJM and DLFM models. The GHJM model is also presented since it plays the key role of being the bridge between the HW and DLFM models. Since we ultimately want to apply these models to price swaptions, the theory underpinning the swaption pricing is also included. Then, in Chapter 3, we will prove the concurrence between the HW model and the DLFM model. To do this, we will derive a key volatility relation result, apply this result in order to prove the concurrence, and present a routine that can be followed in order to derive a DLFM that nests a particular affine-term-structure (ATS) HW short rate model. In Chapter 4, we provide implementation algorithms for the models. Then, the swaption pricing results are presented. Finally, in Chapter 5, we will draw conclusions.

Chapter 2

Literature Review

2.1 Hull-White Model

2.1.1 Single Factor Hull-White Model

The Vasicek model is the oldest short-rate model, dating back to 1977 (Vasicek (1977)). Under the risk neutral probability measure $\mathbb{Q}$, Brigo and Mercurio (2006) lay out the short rate dynamics, $r(t)$ as follows:

$$dr(t) = k[\theta - r(t)]dt + \sigma dW(t), \quad r(0) = r_0, \tag{2.1}$$

where $W(t)$ is a standard Brownian motion, and $k, \theta, \sigma \in \mathbb{R}^+$ represent the rate of mean reversion, the mean reversion level and the volatility respectively. The main drawback of the Vasicek model is that the current term structure of interest rates cannot be fitted exactly through calibration (Hull and White (1990)). Hull and White (1990) extended this model in order to overcome its shortcomings. The tractable dynamics of the HW model are:

$$dr(t) = [b(t) - kr(t)]dt + \sigma dW(t), \quad r(0) = r_0. \tag{2.2}$$

The observed market term structure rates are fitted to $b(t)$, this takes on the following form:

$$b(t) = \frac{\partial f(0,t)}{\partial T} + kf(0,t) + \frac{\sigma^2}{2k}(1 - e^{-2kt}), \tag{2.3}$$

where $f(0,t)$ is the instantaneous forward interest rate with maturity t at time zero. This quantity can be derived from market bond prices. We denoted the price at time zero of a bond that is trading in the market, which has a maturity of T to be $P^M(0,T)$. The instantaneous forward rates can then be infered from market bond prices using the following relation:

$$f(0,T) = -\frac{\partial \ln P^M(0,T)}{\partial T}. \tag{2.4}$$

By solving (2.2), Brigo and Mercurio (2006) give the solution to the SDE as

$$\begin{aligned} r(t) &= r(s)e^{-k(t-s)} + \sigma \int_s^t e^{-k(t-u)} b(u) du + \sigma \int_s^t e^{-k(t-u)} dW(u) \\ &= r(s)e^{-k(t-s)} + v(t) - v(s)e^{-k(t-s)} + \sigma \int_s^t e^{-k(t-u)} dW(u), \end{aligned} \tag{2.5}$$

where, $v(t) = kf(0,t) + \frac{\sigma^2}{2k^2}(1 - e^{-2kt})^2$. Then, $r(t)$ conditional on $\mathcal{F}_s$, where $\mathcal{F}_s$ denotes information available at time s and $s < t$ has a mean and variance respectively given by

$$\mathbb{E}[r(t) \mid \mathcal{F}_s] = r(s)e^{-k(t-s)} + v(t) - v(s)e^{-k(t-s)}, \tag{2.6}$$

$$\mathbb{V}\text{ar}[r(t) \mid \mathcal{F}_s] = \frac{\sigma^2}{2k}[1 - e^{-2k(t-s)}]. \tag{2.7}$$

Since $W(t) \sim \mathcal{N}(0,t)$, this implies

$$r(t) \mid \mathcal{F}_s \sim \mathcal{N}\left(r(s)e^{-k(t-s)} + v(t) - v(s)e^{-k(t-s)}, \frac{\sigma^2}{2k}[1 - e^{-2k(t-s)}] \right). \tag{2.8}$$

The distributional properties above are important when translating the theory into practical implementation.

2.1.2 Zero-Coupon Bond Pricing

One of the practical uses of short rates is the determination of prices for zero-coupon bonds, which are key instruments in the fixed income market. They can be used to price a variety of linear and non-linear products, also noting a coupon bearing bond can be stripped into a series of zero coupon bonds (ZCBs) (Burgess (2013)). This motivates the need to analyse the ZCBs.

Under $\mathbb{Q}$, Andersen and Piterbarg (2010) state the ZCB price $P(t,T)$, starting at time t and maturing at time T, where $0 < t < T$, by considering the expectation of the short rate over time

$$\begin{aligned} P(t,T) &= \mathbb{E}\left[\exp\left(-\int_t^T r(u) du \right) \middle| \mathcal{F}_t \right] \\ &= \exp\left(-\mathbb{E}\left[\int_t^T r(u) du \middle| \mathcal{F}_t \right] + \frac{1}{2}\mathbb{V}\text{ar}\left[\int_t^T r(u) du \middle| \mathcal{F}_t \right] \right), \end{aligned} \tag{2.9}$$

where (2.9) follows from the usual results for lognormal variables. Brigo and Mercurio (2006) highlight that $\int_t^T r(u)du$ has the following mean and variance of

$$\mathbb{E}\left[\int_t^T r(u)du \big| \mathcal{F}_t\right] = B(t,T)[r(t) - v(t)] + \ln\left(\frac{P^M(0,t)}{P^M(0,T)}\right) + \frac{1}{2}[V(0,T) - V(0,t)] \tag{2.10}$$

$$\mathbb{V}\text{ar}\left[\int_t^T r(u)du \big| \mathcal{F}_t\right] = V(t,T) \tag{2.11}$$

respectively, where

$$B(t,T) = \frac{1}{k}\left[1 - e^{-k(T-t)}\right], \tag{2.12}$$

$$V(t,T) = \frac{\sigma^2}{k^2}\left[(T-t) + \frac{2}{k}e^{-k(T-t)} - \frac{1}{2k}e^{-2k(T-t)} - \frac{3}{2k}\right]. \tag{2.13}$$

$P(t,T)$ generated by the Hull White Model is then given as

$$P(t,T) = e^{A(t,T) - B(t,T)r(t)}, \tag{2.14}$$

where

$$A(t,T) = \ln\left(\frac{P^M(0,T)}{P^M(0,t)}\right) + B(t,T)f(0,t) - \frac{\sigma^2}{4k}(1 - e^{-2kt})B(t,T)^2. \tag{2.15}$$

Moreover, if $r(t)$ is a short rate, with a ZCB that can be evaluated in the form of (2.14) then $r(t)$ is called an affine-term-structure model (ATS). The HW is also an ATS model (Fries (2007)). This makes the model analytically tractable and easy to manipulate when doing certain computations.

2.2 Single Factor Gaussian Heath-Jarrow-Morton Model

Having explored short rate models, it is naturally fitting to explore instantaneous forward rates since they are the expectations of future spot rates. We explore how a short rate model and instantaneous forward rate model are related to each other in the HJM framework. The framework basically postulates an Itô process as a model for the instantaneous forward rate. Fries (2007) details the GHJM model under the real world probability measure $\mathbb{P}$ as follows for $0 \leq t \leq T$:

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t) \\ f(0,T) &= f(0,T), \end{aligned} \tag{2.16}$$

where $\alpha(t,T)$ and $\sigma(t,T)$ are deterministic processes. This process gives a complete description of the interest rate curve (Fries (2007)).

Fries (2007) shows the link between the instantaneous forward rate model and short rate model, this is done by firstly writing (2.16) in integral form:

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s). \tag{2.17}$$

Further noting that $r(t) = \lim_{T\to t} f(t,T)$ so that

$$r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \sigma(s,T)dW(s), \tag{2.18}$$

and therefore the short rate process in differential notation is

$$\begin{aligned} dr(t) &= \left(\frac{\partial f(0,t)}{\partial T} + \alpha(t,t) + \int_0^t \frac{\partial \alpha}{\partial T}(s,t)ds + \int_0^t \frac{\partial \sigma}{\partial T}(s,t)dW(s)\right)dt \\ &\quad + \sigma(t,t)dW(t). \qquad (2.19) \\ dr(t) &= \mu(t)dt + \sigma(t,t)dW(t). \qquad (2.20) \end{aligned}$$

Having found the short and forward dynamics in the HJM framework, we ensure that the model has no arbitrage by satisfying certain conditions. This leads us to the HJM drift condition. To explore the origin of this condition, Fries (2007) analyses the relative price of a bond $\frac{P(t,T)}{B(t)}$ where $P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right)$ and $B(t) = \exp\left(\int_0^t r(s)ds\right)$ is a numeraire under a new measure $\mathbb{Q}$ so that

$$\frac{P(t,T)}{B(t)} = \exp(X(t)), \quad \text{where } X(t) = -\int_t^T f(t,u)du - \int_0^t r(s)ds.$$

Using (2.17) and (2.18), we have that

$$\begin{aligned} X(t) &= -\int_t^T f(t,u)du - \int_0^t r(s)ds \\ &= -\int_t^T f(0,s)ds - \int_t^T \int_0^t \alpha(s,u)dsdu - \int_t^T \int_0^t \sigma(s,u)dW(s)du \\ &\quad - \int_0^t f(0,s)ds - \int_0^t \int_0^u \alpha(s,u)dsdu - \int_0^t \int_0^u \sigma(s,u)dW(s)du \\ &= -\int_t^T f(0,s)ds - \int_0^t \int_t^T \alpha(s,u)duds - \int_0^t \int_t^T \sigma(s,u)dudW(s) \\ &\quad - \int_0^t f(0,s)ds - \int_0^t \int_s^t \alpha(s,u)duds - \int_0^t \int_s^t \sigma(s,u)dudW(s) \qquad (2.21) \\ &= -\int_0^T f(0,s)ds - \int_0^t \int_s^T \alpha(s,u)duds - \int_0^t \int_s^T \sigma(s,u)dudW(s) \end{aligned}$$

where (2.21) follows from Fubini's Theorem on change of integration bounds. We therefore have

$$dX(t) = -\left(\int_t^T \alpha(t,u)dudt + \int_t^T \sigma(t,u)dudW(t)\right). \tag{2.22}$$

Having found the dynamics of $X(t)$, we wish to explore the dynamics of the relative price of the bond $\exp(X(t))$, with the application of Itôs Lemma on $f(X(t)) = \exp(X(t))$ we have

$$\begin{aligned} & d\exp(X(t)) \\ & = \exp(X(t))dX(t) + \frac{1}{2}\exp(X(t))dX(t)^2 \\ & = \exp(X(t))\left[\left(-\int_t^T \alpha(t,u)du + \frac{1}{2}\int_t^T \sigma^2(t,u)du\right)dt - \int_t^T \sigma(t,u)dudW(t)\right]. \end{aligned} \tag{2.23}$$

For the process $\exp(X(t))$ to be a martingale under the measure $\mathbb{Q}$ we require (2.23) to be driftless, that is, we need

$$\begin{aligned} -\int_t^T \alpha(t,u)du + \frac{1}{2}\int_t^T \sigma^2(t,u)du = 0 \\ \implies \int_t^T \alpha(t,u)du = \frac{1}{2}\int_t^T \sigma^2(t,u)du. \end{aligned}$$

If this equation is valid for all T, if we differentiate both sides by $\frac{\partial}{\partial T}$ then

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du. \tag{2.24}$$

Equation (2.24) is the so called HJM drift condition, which is the condition that must be imposed on the drift of the relative bond price process $\frac{P(t,T)}{B(t)}$, to be arbitrage free under $\mathbb{Q}$, this measure is equivalent to $\mathbb{P}$.

In a nutshell, the HJM framework says that if we can specify the instantaneous volatility function, then the drift of the instantaneous forward rate process will be automatically specified through the HJM drift condition. By so doing, the entire instantaneous forward rate curve (2.16) can be modelled completely. Knowing the dynamics of a short rate model can allow us to infer a unique GHJM model that is implied, and relation (2.19) can be used to verify that the GHJM model can retrieve back this short rate model. It also turns out that the GHJM model is important because it acts as a bridge between the HW model and DLFM model but this will become clearer in Chapter 3.

2.3 Displaced Libor Model

2.3.1 Market Model Basics

Libor rates are simple rates that are observed in the market. We now investigate the last class of interest rate models, which are the forward market models. These models are focused on modelling Libor rates. Libor rates are simple rates that are observed in the market, hence the term market model. Since the DLFM model is a crucial model for this dissertation, it is important to understand its fundamental dynamics. To achieve this, we shall present the classical Libor Forward Market (LFM) model dynamics and extend this to the DLFM model dynamics. We start by defining a set of fixed tenor dates since Libor rates apply to only a set of finite maturities. Let

$$
\begin{aligned}
&0 := T_0 < T_1 < T_2 < ... < T_M < T_{M+1}, \quad \text{and} \\
&\tau_i := T_{i+1} - T_i, \quad i = 0, 1, ..., M.
\end{aligned}
$$

The τ_i values are usually approximately equal, however they might be different due to day-count conventions. We let $L_n(t)$ denote the time t simple forward rate applying to the period $[T_n, T_{n+1}]$ for $n = 0, 1, ..., M$. Then using basic arbitrage arguments, we can show, as highlighted by Haugh (2010), that

$$
L_n(t) = \frac{1}{\tau_n}\left[\frac{P(t,T_n)}{P(t,T_{n+1})} - 1\right] \quad 0 \le t \le T_n \quad n = 0, 1, ..., M \tag{2.25}
$$

$$
= \frac{1}{\tau_n}\left[\left(\frac{P(t,T_{n+1})}{P(t,T_n)}\right)^{-1} - 1\right]. \tag{2.26}
$$

Bond prices can be expressed in terms of Libor rates by inverting (2.25). The bond prices are given as

$$
P(T_i, T_n) = \prod_{j=i}^{n-1} \frac{1}{1+\tau_j L_j(T_i)} \quad n = i+1, ..., M+1, \tag{2.27}
$$

however, (2.27) only determines the bond prices at fixed maturities, for any arbitrary date t we have

$$
P(t, T_n) = P(t, \phi(t)) \prod_{j=\phi(t)}^{n-1} \frac{1}{1+\tau_j L_j(T_i)} \quad 0 \le t \le T_n, \tag{2.28}
$$

where $\phi(t)$ is the next tenor date after time t, i.e

$$
\phi(t) := \min_{i=1,...,M+1} \{i : t < T_i\}.
$$

Haugh (2010) highlights that $P(t, \phi(t))$ may make it difficult to model $L_n(t)$ at an arbitrary time t. This motivates the idea of defining an appropriate numeraire asset together with its measure. Market models are usually specified under three measures (terminal, T-forward and spot), the specific numeraire assets are then used to derive the dynamics there-in. In this dissertation we will only focus on the spot measure, under this measure we define the numeraire to be

$$B_t^* = P(t, \phi(t)) \prod_{j=0}^{\phi(t)-1} (1 + \tau_j L_j(T_i)). \tag{2.29}$$

From there, we now consider the deflated zero-coupon bond price by its spot numeraire. This deflated bond price process is

$$\begin{aligned} D_n(t) &= \frac{P(t, T_n)}{B_t^*} \\ &= \prod_{j=0}^{\phi(t)-1} \frac{1}{1 + \tau_j L_j(T_i)} \prod_{j=\phi(t)}^{n-1} \frac{1}{1 + \tau_j L_j(T_i)}. \end{aligned} \tag{2.30}$$

To this end, we have defined all the necessary basics to start unpacking the dynamics of $L_n(t)$.

2.3.2 Dynamics of the Libor Market Model Under the Spot Measure

As a starting point, Haugh (2010) states that for $\{0 \leq t \leq T_n\}$ and $\{n = 1, ..., M\}$ the Libor rates dynamics obey

$$dL_n(t) = \mu_n(t) L_n(t) dt + L_n(t) \sigma_n(t) dW(t), \tag{2.31}$$

where $W(t)$ is a Brownian motion and $\mu_n(t)$ and $\sigma_n(t)$ are adapted processes. Assuming no arbitrage and that deflated bond prices are postive, this means there is a process $\nu_n(t)$ such that

$$dD_n(t) = D_n(t) \nu_n(t) dW(t). \tag{2.32}$$

Now Itô's Lemma could be applied to (2.32) but it would be overly complicated to evaluate, instead Itô's Lemma is applied to a transformed process $\bar{D}_n(t) = \log D_n(t)$, therefore

$$d\bar{D}_n(t) = -\frac{1}{2} \|\nu_n(t)\|^2 dt + \nu_n(t) dW(t). \tag{2.33}$$

Using (2.30), the dynamics of $\bar{D}_n(t)$ are given as

$$\log D_n(t) = -\sum_{j=0}^{\phi(t)-1} \log(1+\tau_j L_j(T_i)) - \sum_{j=\phi(t)}^{n-1} \log(1+\tau_j L_j(T_i))$$

$$\begin{aligned} d\bar{D}_n(t) &= -d\sum_{j=\phi(t)}^{n-1} \log(1+\tau_j L_j(T_i)) \\ &= -\sum_{j=\phi(t)}^{n-1} \left(\frac{\tau_j \mu_j(t) L_j(t)}{1+\tau_j L_j(t)} - \frac{\tau_j^2 L_j(t)^2 \sigma_j^2(t)}{2(1+\tau_j L_j(t))^2} \right) dt \\ &\quad - \sum_{j=\phi(t)}^{n-1} \left(\frac{\tau_j L_j(t) \sigma_j(t)}{1+\tau_j L_j(t)} \right) dW(t). \end{aligned} \tag{2.34}$$

Comparing the volatility terms in (2.34) and (2.33) we find that

$$\nu_n(t) = -\sum_{j=\phi(t)}^{n-1} \left(\frac{\tau_j L_j(t) \sigma_j(t)}{1+\tau_j L_j(t)} \right). \tag{2.35}$$

To derive the expression for $\mu_j(t)$ we start by letting $n = 2$ and $\phi(t) = 1$, with some simple algebra we have

$$\mu_1(t) = -\sigma_1(t)\nu_2(t), \quad 0 \le t \le T_1.$$

We then use mathematical induction to generally deduce that

$$\begin{aligned} \mu_n(t) &= -\sigma_n(t)\nu_{n+1}(t) \\ &= \sum_{j=\phi(t)}^{n} \left(\frac{\tau_j L_j(t) \sigma_j(t) \sigma_n(t)}{1+\tau_j L_j(t)} \right). \end{aligned} \tag{2.36}$$

Therefore, the no arbitrage spot measure dynamics of the forward Libor rates where $\{0 \le t \le T_n\}$ and $\{n = 1, ..., M\}$ are given by

$$dL_n(t) = \left(\sum_{j=\phi(t)}^{n} \left(\frac{\tau_j L_j(t) \sigma_j(t) \sigma_n(t)}{1+\tau_j L_j(t)} \right) \right) L_n(t)dt + L_n(t)\sigma_n(t)dW(t). \tag{2.37}$$

2.3.3 Single Factor Displaced Forward Libor Model Dynamics

The LFM model presents a solid way of understanding the dynamics of forward market models but the model suffers from a few drawbacks. The one major drawback that Van Appel and McWalter (2020) highlight is that this model does not capture the smile or skew effects commonly seen in the market. The LFM can be extended to produce skew or smile by adapting the volatility function, in particular using a displaced diffusion. This is known as the displaced log-normal forward-Libor model (DLFM) (Van Appel and McWalter (2020)).

Having derived the dynamics of the classical Libor market model in the previous section, we can extend this model to state the dynamics of the DFLM model. For $\{0 \leq t \leq T_n\}$ and $\{n = 1, ..., M\}$, the DLFM model is represented as

$$dL_n(t) = \tilde{\mu}_n(t)(L_n(t) + a_n)dt + \sigma_n(t)(L_n(t) + a_n)dW(t), \tag{2.38}$$

where a_n is the displacement factor and $\sigma_n(t)$ is the instantaneous volatility of $L_n(t)$. The instantaneous volatility function has different parametric formulations depending on the type of modelling to be done, Brigo and Mercurio (2006) highlight a few formulations. The formulation that is frequently used is the following

$$\sigma_i(t) = [a(T_i - t) + d]e^{-b(T_i - t)} + c, \tag{2.39}$$

with a, b, c, d being constants that are selected appropriately. Using the same arguments as in the previous Section 2.3.2, we find that the arbitrage free spot measure dynamics of the displaced forward Libor rates where $\{0 \leq t \leq T_n\}$ and $\{n = 1, ..., M\}$ are given by

$$dL_n(t) = (L_n(t) + a_n)\left[\left[\sum_{j=\phi(t)}^{n}\left(\frac{\tau_j(L_j(t) + a_j)\sigma_j(t)\sigma_n(t)}{1 + \tau_j L_j(t)}\right)\right]dt + \sigma_n(t)dW(t)\right], \tag{2.40}$$

where

$$\tilde{\mu}_n(t) = \sum_{j=\phi(t)}^{n}\left(\frac{\tau_j(L_j(t) + a_j)\sigma_j(t)\sigma_n(t)}{1 + \tau_j L_j(t)}\right). \tag{2.41}$$

The advantage of the DLFM model over the LFM model is that with a suitable parametrization it allows a positive probability of obtaining negative forward rates, this is ideal because during periods of market distress interest rates can go negative (Van Appel and McWalter (2020)).

2.4 Swaption Valuation

The previous sections outlined the main theory for the HW, GHJM and DLFM models better. We shall later see in Chapter 4 how these models can be numerically implemented. Since ultimately the interest rate models that will be implemented will be applied to the valuation of interest rate derivatives, in particular a swaption, this motivates the need to explore the theory underpinning swaption valuation. To do this, we shall present a Forward Rate Agreement (FRA), define a forward starting Interest Rate Swap (IRS) as a portfolio of FRAs and then conclude with defining a swaption as an option to enter into a forward starting IRS. This section will then be concluded with a presentation of an analytical solution to the price of a swaption which involves the use of a technique called Jamshidian's trick.

FRAs are over the counter agreements to buy or sell a forward interest rate. This agreement happens at time $s \in [0, t]$, the contract settles at time t and expires at time T (Mahomed (2015)). The value of such a contract at time s on a nominal of N with strike rate $\bar{K}$ is given by

$$V_{FRA}(\omega, s, t, T) = \omega N[L(s, t, T) - \bar{K}](T - t)P(s, T), \tag{2.42}$$

where $\omega = -1$ or $\omega = 1$ for a short or long FRA respectively.

A forward starting IRS is a contract that exchanges payments between two differently indexed legs, starting from a future time instant (Brigo and Mercurio (2006)). If fixed cashflows are being exchanged for floating cashflows, this is called a payer IRS and the converse is called a receiver IRS. At every future instant T_i in pre-specified set of dates $T_{\alpha+1}, ..., T_\beta$ the fixed leg pays an amount

$$NK\tau_i,$$

where K is the strike rate. The floating leg pays an amount

$$N\tau_i L(T_{i-1}, T_i).$$

The value of such a forward starting IRS at time t that starts at T_α and has maturity T_β is given by

$$V_{FIRS}(t, T_\alpha, T_\beta, \omega) = \sum_{i=\alpha+1}^{\beta} \omega V_{FRA}(1, t, T_{i-1}, T_i) \tag{2.43}$$

$$= \sum_{i=\alpha+1}^{\beta} \omega P(t, T_i) N\tau_i \left(L(T_{i-1}, T_i) - K\right) \tag{2.44}$$

$$= \omega N \left[P(t, T_\alpha) - P(t, T_\beta) - \sum_{i=\alpha+1}^{\beta} K\tau_i P(t, T_i) \right]. \tag{2.45}$$

From (2.43), this shows that an IRS is a portfolio of FRAs, while (2.45) is just a further simplification. By the principles of no arbitrage, there exists a fair forward swap rate $K = S_{\alpha,\beta}(t)$, which is essentially the fixed swap rate that sets $V_{FIRS}(t, T_\alpha, T_\beta) = 0$. This fair forward swap rate value at time t is given by

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}. \tag{2.46}$$

Now that a forward starting IRS has been defined, we can then define what a swaption is. A European swaption is an option giving the right but no obligation to enter a forward starting IRS at a given future time, the swaption maturity T_α. Usually the swaption maturity coincides with the first reset date of the underlying IRS and the swaption has a tenor of $T_\beta - T_\alpha$. A payer swaption is an option to enter into a payer IRS, while the converse is called a receiver swaption. The value of such a swaption at time t is given by

$$\begin{aligned} V_{swaption}(t, T_\alpha, T_\beta, \omega) &= P(t, T_\alpha)\left[V_{FIRS}(T_\alpha, T_\alpha, T_\beta, \omega)\right]^+ \\ &= \omega N P(t, T_\alpha)\left[P(T_\alpha, T_\alpha) - P(T_\alpha, T_\beta) - \sum_{i=\alpha+1}^{\beta} K\tau_i P(T_\alpha, T_i)\right]^+ \\ &= \omega N P(t, T_\alpha)\left[1 - P(T_\alpha, T_\beta) - \sum_{i=\alpha+1}^{\beta} K\tau_i P(T_\alpha, T_i)\right]^+ \quad (2.47) \\ &= \omega N P(t, T_\alpha)\left[S_{\alpha,\beta}(T_\alpha) - K\right]^+ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i), \quad (2.48) \end{aligned}$$

where $\omega = 1$ and $\omega = -1$ indicate a payer and receiver swaption respectively. From, (2.47) we can observe that a swaption can be interpreted as being a European option on a coupon paying bond, that pays coupons at rate K and has a strike of one. A payer swaption is equivalent with a put option on such a coupon paying bond, while a receiver swaption being equivalent to a call option on such a bond with the strike indicated above. Expression (2.48) gives an alternative way of valuing the swaption as a form of a vanilla option on the fair swap rate.

Equations (2.47) and (2.48) give us a way of valuing a swaption using Monte Carlo approaches. We may be interested in the analytical price of the swaption to compare the accuracy of the Monte Carlo approaches that might be used in estimating the value of the swaption. The analytical price of the swaption can be valued using Jamshidian's trick. Jamshidian (1989) developed this approach in order to analytically price coupon bearing bond options, and we recall that a swaption can be valued at time $t < T_\alpha$ as an option on a coupon bearing bond option. Firstly, let the coupons be

$$c_i = K\tau_i \mathbb{I}_{(i=1,..,\beta-1)} + (1 + K\tau)\mathbb{I}_{(i=\beta)}.$$

Assuming that the short rate model is analytically tractable, and that the bond price process can be written in affine form, we must find the value of r^*, such that

$$\sum_{i=1}^{\beta} c_i e^{A(T_\alpha, T_i) - B(T_\alpha, T_i) r^*} = 1. \quad (2.49)$$

From there we set $K_i = e^{A(T_\alpha,T_i)-B(T_\alpha,T_i)r^*}$.

We then define $ZBO(t, T_\alpha, T_i, K_i, w)$ to be the time t value of an option with maturity T_α on a zero coupon bond with maturity T_i with a strike rate K_i, where $\omega = 1$ or $\omega = -1$ represents a call and put respectively. Such an option has a value given by

$$ZBO(t, T_\alpha, T_i, K_i, w) = \omega \left[P(t, T_i)\Phi(\omega d_{1,i}) - K_i P(t, T_\alpha)\Phi(\omega d_{2,i}) \right], \tag{2.50}$$

where Φ is the CDF of the normal distribution and

$$\sigma_{p,i} = \sigma \sqrt{\frac{1 - e^{-2k(T_\alpha - t)}}{2k}} A(T_\alpha, T_i) \tag{2.51}$$

$$d_{1,i} = \frac{1}{\sigma_{p,i}} \ln \left(\frac{P(t, T_i)}{P(t, T_\alpha) K_i} \right) + \frac{\sigma_{p,i}}{2} \tag{2.52}$$

$$d_{2,i} = d_{1,i} - \sigma_{p,i}. \tag{2.53}$$

The analytical value of the swaption is then given by

$$\bar{V}_{swaption}(t, T_\alpha, T_\beta, \omega) = N \sum_{i=1}^{\beta} ZBO(t, T_\alpha, T_i, K_i, \omega). \tag{2.54}$$

Chapter 3

Concurrence between the Hull White and DLFM models

In this Chapter, we shall derive a one-factor DLFM model that nests the discretized HW model. This will be done in parts, so we state the procedure that is followed in proving this concurrence. We initially prove the important volatility relation result that relates the volatility of the DLFM model, with the volatility of the GHJM model. From there we build up the concurrence by showing how to move from the HW model to the GHJM model and finally to the DLFM model. We shall take a HW model, write down its dynamics, we then notice that this is an ATS model. This makes it easy to write down the form of the bond price process generated by this model. From the bond price process, it is easy to infer the instantaneous forward rate process and its dynamics. From these dynamics we can observe the instantaneous volatility function that makes the GHJM model to be consistent with the HW model. Having observed this instantaneous volatility function, we then work in the opposite direction and apply the HJM framework to derive the short rate process that is implied by this volatility, this is more to verify that indeed this relation actually holds. Having found the appropriate volatility function, we plug this into the equation that relates the volatility of the DLFM model, with the volatility of the instantaneous forward rate model. This will yield the appropriate volatility that makes the DLFM to be consistent with the GHJM model. Now since the GHJM model is consistent with the HW model this automatically means that the DLFM is consistent with the HW model. This will then be sufficient to show that the concurrence mathematically holds.

3.1 Derivation of the Volatility Relation Between GHJM and DLFM Models

The instantaneous short and forward rate models explored in the previous chapter have their relative merits. For instance, the GHJM model is quite comprehensive for defining a term structure of instantaneous forward rates however it is quite inconvenient for use in a practical sense. Instantaneous forward rates are never quoted in the market and payoffs of traded derivatives do not directly feature these rates, this means that discretization methods would need to be used in order price certain derivatives using these rates (Brigo and Mercurio (2006)).

Brigo and Mercurio (2006) highlight that formulating a model in terms of simple Libor rates is the best way to reduce the complexities above. Therefore, for an interest rate modelling practitioner who sees it fitting to model traded financial instruments using instantaneous forward rates, it would benefit them if they could derive a relation between the instantaneous forward rate model and Libor market model, so that their valuation can proceed with a market model that mimics the underlying properties of the instantaneous forward rate model. It turns out that the two models have a relation that is governed through their volatilities.

To prove how the DLFM and GHJM models are related to each other, we need the following preliminary results adapted from Ouwehand (2019) which are highlighted as Propositions.

Proposition 3.1. *If $df(t) = \alpha(t,T)dt + \sigma(t,T)dW(t)$, then*

$$\frac{dP(t,T)}{P(t,T)} = \left[r(t) + A(t,T) + \frac{1}{2}S(t,T)^2\right]dt + S(t,T)dW(t),$$

where $A(t,T) = -\int_t^T \alpha(t,s)ds$ and $S(t,T) = -\int_t^T \sigma(t,s)ds$.

Proposition 3.2. *If $T < S$, then*

$$d\left[\frac{P(t,S)}{P(t,T)}\right] = \frac{P(t,S)}{P(t,T)}\left[(\ldots)dt + (\sigma_p(t,S) - \sigma_p(t,T))dW(t)\right],$$

where $\sigma_p(t,S)$ is the volatility of the S-bond and $\sigma_p(t,T)$ is the volatility of the T-bond.

Andersen and Piterbarg (2010) consider the following forward bond price process $P(t,T_n,T_{n+1}) = \frac{P(t,T_{n+1})}{P(t,T_n)}$ and let $O(dt)$ be the drift term. Then

$$d\left(\frac{P(t,T_{n+1})}{P(t,T_n)}\right) = \frac{P(t,T_{n+1})}{P(t,T_n)}\left[O(dt) + (\sigma_p(t,T_{n+1}) - \sigma_p(t,T_n))dW(t)\right] \tag{3.1}$$

$$\frac{dP(t,T_n,T_{n+1})}{P(t,T_n,T_{n+1})} = \left[O(dt) + \left(-\int_t^{T_{n+1}} \sigma_f(t,u)du - (-\int_t^{T_n} \sigma_f(t,u)du)\right) dW(t)\right] \tag{3.2}$$

$$= \left[O(dt) + \left(-\int_t^{T_{n+1}} \sigma_f(t,u)du + \int_t^{T_n} \sigma_f(t,u)du\right) dW(t)\right]$$

$$dP(t,T_n,T_{n+1}) = P(t,T_n,T_{n+1})\left[O(dt) - \left(\int_{T_n}^{T_{n+1}} \sigma_f(t,u)du\right) dW(t)\right],$$

where (3.1) and (3.2) follows from Proposition 3.2 and Proposition 3.1 respectively. From (2.26) we proceed to have

$$d\left[1 + L_n(t)\tau_n\right]^{-1} = \left[1 + L_n(t)\tau_n\right]^{-1}\left[O(dt) - \left(\int_{T_n}^{T_{n+1}} \sigma_f(t,u)du\right) dW(t)\right]$$

$$-\left[1 + L_n(t)\tau_n\right]^{-2}\tau_n dL_n(t) = \left[1 + L_n(t)\tau_n\right]^{-1}\left[O(dt) - \int_{T_n}^{T_{n+1}} \sigma_f(t,u)dudW(t)\right]$$

$$dL_n(t) = O(dt) + \tau_n^{-1}\left[1 + L_n(t)\tau_n\right]\int_{T_n}^{T_{n+1}} \sigma_f(t,u)dudW(t). \tag{3.3}$$

The drift term $O(dt)$ was omitted in the derivation since if we know the form of the volatility of the Libor model, then we can infer the form of the drift term, this will take the form of (2.41). Equation (3.3) implies that, the DLFM model volatility $\sigma_n(t)$ is related to the GHJM model instantaneous volatility $\sigma_f(t,T)$ by the following important result:

$$\sigma_n(t) = \int_{T_n}^{T_{n+1}} \sigma_f(t,u)du. \tag{3.4}$$

In Section 2.3.2 we derived the dynamics of a one factor LFM model, these were extended to the dynamics of the DLFM model (2.40) in Section 2.3.3 which showed clearly that the form of the volatility term specify the drift of the model. This leads us to conclude that the dynamics of the DLFM model given the knowledge of (3.4) are

$$dL_n(t)$$

$$= \tilde{\mu}_n(t)[L_n(t) + a_n]dt + \sigma_n(t)[L_n(t) + a_n]dW(t) \tag{3.5}$$

$$= \sum_{j=\phi(t)}^{n}\left(\frac{(\tau_j L_j(t) + 1)\sigma_j(t)\sigma_n(t)}{1 + \tau_j L_j(t)}\right)\left[L_n(t) + \tau_n^{-1}\right]dt + \sigma_n(t)\left[L_n(t) + \tau_n^{-1}\right]dW(t). \tag{3.6}$$

3.2 Application of the Derived Results

Now that we have proved some useful results, we apply them to derive useful relations between the models.

3.2.1 Application of the HJM Framework

Before we proceed, we highlight the following useful theorem for ATS models adapted from Ouwehand (2019).

Theorem 3.3. *A short rate model with risk-neutral dynamics $dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t)$ (with μ, σ being continuous functions) is an ATS if and only if μ and σ^2 are affine functions of r, i.e., are of the form*

$$\mu(t, r) = \alpha(t)r + \beta(t) \quad \text{and} \quad \sigma^2(t, r) = \gamma(t)r + \delta(t),$$

where $\alpha, \beta, \gamma, \delta$ are deterministic functions of t. In that case

$$P(t, T) = e^{A(t,T)-B(t,T)r(t)},$$

where $A(t, T)$,$B(t, T)$ satisfy the the weakly coupled system of ODEs

$$\begin{cases} \partial_t B(t, T) &= -\alpha(t)B(t, T) + \frac{1}{2}\gamma(t)B^2(t, T) - 1 \\ B(T, T) &= 0, \end{cases}$$

$$\begin{cases} \partial_t A(t, T) &= \beta(t)B(t, T) - \frac{1}{2}\gamma(t)B^2(t, T) \\ A(T, T) &= 0. \end{cases}$$

Consider the following HW model

$$\begin{aligned} dr(t) &= (b(t) - kr(t))dt + \sigma dW(t) \\ &= (\beta(t) + \alpha r(t))dt + \sqrt{(\gamma r(t) + \delta)}dW(t). \end{aligned}$$

This is an ATS model with $\alpha = -k, \beta = b, \gamma = 0, \delta = \sigma^2$, the corresponding system of ODEs for this HW model, with the application of Theorem 3.4 are

$$\begin{cases} \partial_t B(t, T) &= kB(t, T) - 1 \\ B(T, T) &= 0, \end{cases}$$

$$\begin{cases} \partial_t A(t, T) &= bB(t, T) - \frac{1}{2}\sigma^2 B^2(t, T) \\ A(T, T) &= 0. \end{cases}$$

This implies that the solution to the first ODE is given by

$$\begin{aligned}
\partial_t B(t,T) &= kB(t,T) - 1 \\
\partial_t B(t,T) - kB(t,T) &= -1 \\
d(e^{-kt}B(t,T)) &= -e^{-kt} \\
e^{-kT}B(T,T) - e^{-kt}B(t,T) &= \int_t^T e^{-ku}du \\
0 - e^{-kt}B(t,T) &= \frac{1}{k}(e^{-kT} - e^{-kt}) \\
B(t,T) &= \frac{1}{k}\left[1 - e^{-k(T-t)}\right].
\end{aligned}$$

Given $P(t,T) = e^{A(t,T)-B(t,T)r(t)}$, we know that the forward rate $f(t,T)$ can be calculated as follows

$$\begin{aligned}
f(t,T) &= -\frac{\partial \ln P(t,T)}{\partial T} \\
&= -\frac{\partial A(t,T)}{\partial T} + \frac{\partial B(t,T)}{\partial T} r(t) \\
\implies d(f,T) &= O(dt) + \frac{\partial B(t,T)}{\partial T} dr(t) \\
&= O(dt) + \frac{\partial B(t,T)}{\partial T}\left[(b(t) - kr(t))dt + \sigma dW(t)\right] \\
&= O(dt) + \frac{\partial B(t,T)}{\partial T}\sigma dW(t) \\
&= O(dt) + \frac{\partial}{\partial T}\frac{1}{k}\left[1 - e^{-k(T-t)}\right]\sigma dW(t) \\
&= O(dt) + \sigma e^{-k(T-t)} dW(t) \qquad (3.7) \\
&= O(dt) + \sigma_f(t,T)dW(t) \\
&= \sigma_f(t,T)\int_t^T \sigma_f(t,u)du + \sigma_f(t,T)dW(t) \\
&= \sigma^2 e^{-k(T-t)}\frac{1}{k}(1 - e^{-k(T-t)})dt + \sigma e^{-k(T-t)}dW(t) \\
&= \frac{\sigma^2}{k}(e^{-k(T-t)} - e^{-2k(T-t)})dt + \sigma e^{-k(T-t)}dW(t) \qquad (3.8)
\end{aligned}$$

Equation (3.7) gives us the volatility of the instantaneous forward rate that will be consistent with the HW model. Given this volatility, the drift term $O(dt)$ is easily calculated by the HJM drift conditions derived in (2.24). From there, (3.8) gives the complete instantaneous forward rate curve dynamics. To verify that this is correct, we need to prove this relation in the opposite direction, i.e., go from instantaneous forward rate to short rate model. To do so we invoke the results of Fries (2007) proved in Section 2.2.

Given $\sigma_f(t,T) = \sigma e^{-k(T-t)}$, we can infer the following minor results from that will be useful going forward:

$$\begin{aligned}
\alpha(t,T) &= \sigma_f(t,T)\int_t^T \sigma_f(t,u)du = \frac{\sigma^2}{k}(e^{-k(T-t)} - e^{-2k(T-t)}) \\
\alpha(t,t) &= 0 \\
\sigma_f(t,t) &= \sigma \\
\frac{\partial \alpha}{\partial T}(t,T) &= 2\sigma^2 e^{-2k(T-t)} - \sigma^2 e^{-k(T-t)} \\
\frac{\partial \sigma_f}{\partial T}(t,T) &= -k\sigma e^{-k(T-t)} = -k\sigma_f(t,T).
\end{aligned}$$

From (2.19) and (2.20) we know the following about the drift of the implied short rate from the instantaneous forward rate

$$\begin{aligned}
\mu(t) &= \frac{\partial f(0,t)}{\partial T} + \alpha(t,t) + \int_0^t \frac{\partial \alpha}{\partial T}(s,T)ds + \int_0^t \frac{\partial \sigma}{\partial T}(s,T)dW(s) \\
&= \frac{\partial f(0,t)}{\partial T} + \int_0^t \frac{\partial \alpha}{\partial T}(s,T)ds - k\int_0^t \sigma_f(s,T)dW(s).
\end{aligned} \tag{3.9}$$

To solve for $k\int_0^t \sigma_f(s,T)dW(s)$, we note the following

$$\begin{aligned}
f(t,T) &= f(0,T) + \int_0^t \alpha(u,T)du + \int_0^t \sigma_f(u,T)dW_u \\
r(t) = f(t,t) &= f(0,t) + \int_0^t \alpha(u,t)du + \int_0^t \sigma_f(u,t)dW_u \\
\implies k\int_0^t \sigma_f(u,t)dW_u &= kr(t) - kf(0,t) - k\int_0^t \alpha(u,t)du,
\end{aligned}$$

therefore, substituting this expression to (3.9) gives

$$\begin{aligned}
\mu(t) &= \frac{\partial f(0,t)}{\partial} + \int_0^t \frac{\partial \alpha}{\partial T}(s,t)ds - kr(t) + kf(0,t) + k\int_0^t \alpha(u,t)du \\
&= \frac{\partial f(0,t)}{\partial T} + kf(0,t) - kr(t) + \int_0^t \sigma^2(2e^{-2k(t-s)} - e^{-k(t-s)})ds + \\
&\int_0^t \sigma^2(e^{-k(t-s)} - e^{-2k(t-s)})ds \\
&= \frac{\partial f(0,t)}{\partial T} + kf(0,t) - kr(t) + \int_0^t \sigma^2 e^{-2k(t-s)}ds \\
&= \frac{\partial f(0,t)}{\partial T} + kf(0,t) + \frac{\sigma^2}{2k}(1 - e^{-2kt}) - kr(t).
\end{aligned}$$

Letting $b(t) = \frac{\partial f(0,t)}{\partial T} + kf(0,t) + \frac{\sigma^2}{2k}(1 - e^{-2kt})$, then $\mu(t) = b(t) - kr(t)$. The full dynamics of the implied short rate model are then given by

$$\begin{aligned} dr(t) &= \mu(t)dt + \sigma(t,t)dW(t) \\ &= [b(t) - kr(t)]\,dt + \sigma dW(t), \end{aligned}$$

which is indeed the HW model.

3.2.2 Application of the Volatility Relation Between DLFM and GHJM Models

In Section 3.1, the relationship between the volatility of the DLFM and the GHJM was shown. The question now is centered around which instantaneous volatility function should be used in order to derive a DLFM model that is consistent with the HW model. From (3.7) we saw that the appropriate instantaneous volatility function to use to achieve this is $\sigma_f(t,T) = \sigma e^{-k(T-t)}$. We now derive the volatility of the DLFM using (3.4). In particular

$$\begin{aligned} \sigma_n(t) &= \int_{T_n}^{T_{n+1}} \sigma_f(t,u)du \\ &= \int_{T_n}^{T_{n+1}} \sigma e^{-k(u-t)}du \\ &= \left[-\frac{\sigma}{k} e^{-k(u-t)} \Big|_{T_n}^{T_{n+1}} \right] \\ &= \left[\frac{\sigma}{k} \left(e^{-k(T_n-t)} - e^{-k(T_{n+1}-t)} \right) \right]. \end{aligned}$$

The full dynamics of the DLFM model under the spot measure are given by

$$\begin{aligned} &dL_n(t) \\ &= \tilde{\mu}_n(t)\,[L_n(t) + a_n]\,dt + \sigma_n(t)\,[L_n(t) + a_n]\,dW(t) \\ &= \sum_{j=\phi(t)}^{n} \left(\frac{(\tau_j L_j(t) + 1)\sigma_j(t)\sigma_n(t)}{1 + \tau_j L_j(t)} \right) \left[L_n(t) + \tau_n^{-1} \right] dt + \sigma_n(t) \left[L_n(t) + \tau_n^{-1} \right] dW(t). \end{aligned}$$

3.3 Summary of the Concurrence

We have now shown that A (HW model) is contained in B (GHJM model) and B contained in C (DLFM model), this means that A is also contained in C, thereby proving the concurrence. It is important to highlight that the converse does not hold, Brigo and Mercurio (2006) also highlights, stating that the Libor model concerns itself with discretely compounded forward rates, and we therefore cannot expect the Libor model to uniquely characterize the behavior of instantaneous forward rates and their volatilities. Importantly, we have only shown the results for a special case, by only considering a HW model with specific parameters, this will in turn yield the specific constrained parameters of the GHJM model, which will also in turn yield the specific constrained parameters of the DLFM model. In general it is not always true that any ATS short rate model is contained in a specific instantaneous forward rate model, which in turn is contained in a specific displaced Libor forward market model, so care needs to be taken when considering when the concurrence between the models holds.

Since this section is the crux of the dissertation, we shall state the steps to show the concurrence in a more neater way in Algorithm 1.

Algorithm 1: HW $\subset$ GHJM $\subset$ DLFM

1 $dr(t) = (\beta(t) + \alpha r(t))dt + \sqrt{(\gamma r(t) + \delta)}dW(t)$; *write down the HW as an ATS model*
2 *Infer the bond price process* $P(t,T) = e^{A(t,T) - r_t B(t,T)}$
3 *Infer the instantaneous forward rate process* $f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}$
4 *Infer the instantaneous forward rate dynamics* $df(t,T) = \alpha(t,T)dt + \frac{\partial B(t,T)}{\partial T}\sigma dW(t)$
5 *Solve for the instantaneous volatility function* $\sigma_f(t,T) = \frac{\partial B(t,T)}{\partial T}\sigma$
6 *Using HJM drift conditions, deduce the drift* $\alpha(t,T) = \sigma_f(t,T)\int_t^T \sigma_f(t,u)du$
7 *Verify that given* $\sigma_f(t,T)$*, we can retrieve* $dr(t)$ *in step-1 using the HJM framework*
8 *Deduce the volatility of the DLFM by* $\sigma_n(t) = \int_{T_n}^{T_{n+1}} \sigma_f(t,u)du$
9 *Deduce the dynamics of the DLFM by* $dL_n(t) = \tilde{\mu}_n(t)\left[L_n(t) + \tau_n^{-1}\right]dt + \sigma_n(t)\left[L_n(t) + \tau_n^{-1}\right]dW(t)$
10 *Under the spot measure deduce the drift* $\tilde{\mu}_n(t) = \sum_{j=\phi(t)}^{n}\left(\frac{(\tau_j L_j(t)+1)\sigma_j(t)\sigma_n(t)}{1+\tau_j L_j(t)}\right)$

Chapter 4

Implementations of the three models

4.1 Implementation Algorithms

We have explored the mathematical concurrence between the HW, GHJM and DLFM models. We now need a way to simulate them in order to check the results these models produce. This section will provide the implementation algorithms for the three models.

4.1.1 HW Model

The distributional properties of $r(t)$ and $Y(t) = \int_0^t r(u)du$ have already been explored in Sections 2.1.1 and 2.1.2. Here we explore a way of numerically implementing this short rate model. The implementation theory of the models was adapted from McWalter (2020). The implementation mainly focuses on the discretization of the distributional properties of $r(t)$ and $Y(t) = \int_0^t r(u)du$ as well as highlighting how these two quantities are related via their covariances and correlations. The short rate and its integral both follow a normal distribution, at time t_{i+1}

$$r(t_{i+1}) \sim \mathcal{N}(\mu_r(t_i, t_{i+1}), \sigma_r^2(t_i, t_{i+1}))$$

and

$$Y(t_{i+1}) \sim \mathcal{N}(\mu_Y(t_i, t_{i+1}), \sigma_Y^2(t_i, t_{i+1}))$$

respectively.

In the above cases the means and variances are discretized in the following way

$$\mu_r(t_i, t_{i+1}) = e^{-k(t_{i+1}-t_i)} r(t_i) + D(t_{i+1}) - e^{-k(t_{i+1}-t_i)} D(t_i), \tag{4.1}$$

$$\mu_Y(t_i, t_{i+1}) = Y_i + B(t_i, t_{i+1}) r(t_i) + \frac{1}{2}\sigma_Y^2(t_i, t_{i+1}) - A(t_i, t_{i+1}) \tag{4.2}$$

$$D(t) = \frac{\sigma^2}{2k^2}\left[1 - e^{-kt}(1 + kB(0,t))\right] + f(0,t) - r_0 e^{-kt}$$

$$\sigma_r^2(t_i, t_{i+1}) = \frac{\sigma^2}{2} B(2t_i, 2t_{i+1}) \tag{4.3}$$

$$\sigma_Y^2(t_i, t_{i+1}) = \frac{\sigma^2}{k^2}\left[(t_{i+1} - t_i) - B(t_i, t_{i+1}) - \frac{k}{2} B^2(t_i, t_{i+1})\right]. \tag{4.4}$$

In the above, $A(t, T)$ and $B(t, T)$ are as defined per (2.15) and (2.12) respectively. The correlation and covariance structure are given as follows:

$$\sigma_{rY}(t_i, t_{i+1}) = \frac{\sigma^2}{2} B^2(t_i, t_{i+1}) \tag{4.5}$$

$$\rho_{rY}(t_i, t_{i+1}) = \frac{\sigma_{rY}(t_i, t_{i+1})}{\sigma_r(t_i, t_{i+1})\sigma_Y(t_i, t_{i+1})}. \tag{4.6}$$

Having all the above pieces now means that the pair $(r(t_i), Y(t_i))$ can be simulated using

$$r(t_{i+1}) = \mu_r(t_i, t_{i+1}) + \sigma_r(t_i, t_{i+1}) Z_{i+1}^1 \tag{4.7}$$

$$Y(t_{i+1} = \mu_Y(t_i, t_{i+1}) + \sigma_Y(t_i, t_{i+1})\left[\rho_{rY}(t_i, t_{i+1}) Z_{i+1}^1 + \sqrt{1 - \rho_{rY}^2(t_i, t_{i+1}))} Z_{i+1}^2\right], \tag{4.8}$$

where Z^1 and Z^2 are independent standard normal random variables.

4.1.2 GHJM Model

The following algorithm, adapted from McWalter (2020) gives us a way of simulating instantaneous forward rates, where the initial term structure of these instantaneous rates is generated used market observed bonds. The algorithm is used to price a contingent claim X, with maturity T_M that produces cash-flows c_i for

$i \le M \le N$.

Algorithm 2: GHJM model

1. *Initialize* $f_i = \frac{\ln\left(\hat{B}(t_0,t_{i+1})/\hat{B}(t_0,t_i)\right)}{\tau_i}$ *for* $i = 0, 1, ..., N-1$, *with market observed bonds* $\hat{B}$
2. *Initialize numeraire* $\beta = 1$
3. *For* $i = 1, 2, ..., M$
   3.1. *Update numeraire* $\beta = \beta e^{f_{i-1}(\tau_i)}$
   3.2. *If* $i < N$
      3.2.1. *Calculate* $\sigma_j = \sigma e^{-k(t_j - t_i)}$ *for* $j = i, ..., N-1$
      3.2.2. *Calculate* $\mu_j = \frac{1}{2(\tau_j)}\left[\left(\sum_{l=i}^{j}\sigma_l(\tau_l)\right)^2 - \left(\sum_{l=i}^{j-1}\sigma_l(\tau_l)\right)^2\right]$, *for* $j = i, ..., N-1$
      3.2.3. *Generate* $W_i \sim \mathcal{N}(0,1)$
      3.2.4. *Set* $f_i = f_j + \mu_j(\tau_i) + \sigma_j\sqrt{\tau_i}W_i$, *for* $j = i, ..., N-1$.
   3.3. *Discount cash flows* c_i *using* $\frac{1}{\beta}$
4. *Return* X *as a sum of discounted cash flows.*

4.1.3 DLFM Model

Algorithm 3, adapted from McWalter (2020) gives us a way of simulating discretely compounded market forward rates, where the initial term structure of these market rates is generated used market observed bonds. This algorithm also prices a contingent claim X, with maturity T_M that produces cash flows c_i for $i \le M \le N$. Since we are discretizing a state dependent drift and valuing swaptions that are long dated with maturities, we need to use a predictor corrector Algorithm in order to correctly calculate the drift. Since we are also dealing with the DLFM model, we need to adjust the initial term structure with the displacement parameter, the final simple rates also need to scaled by the displacement parameter to undo the effect of the adjusted inital term structure.

Algorithm 3: DLFM

1. *Initialize* $L_j = L_j(t_0) = \frac{1}{\tau_j}\left[\frac{\hat{B}(t_0,t_j)}{\hat{B}(t_0,t_{j+1})} - 1\right] + \frac{1}{\tau_j}$, *for* $j = 0, 1, ..., N-1$, *with market observed bonds* $\hat{B}$

2. *Initialize* $\bar{L}_j = L_j$

3. *Initialize numeraire* $\bar{\beta} = 1$

4. *For* $i = 1, 2, ..., M$

   4.1. *Update numeraire* $\bar{\beta} = \bar{\beta}(1 + \tau_{i-1}\bar{L}_{i-1} - \frac{1}{\tau_{i-1}})$

   4.2. *If* $i < N$

   4.2.1. *Generate* $W_i \sim \mathcal{N}(0,1)$

   4.2.2. *Calculate* $\sigma_j = \frac{\sigma}{k}\left(e^{-k(t_j - t_i)} - e^{-k(t_j - t_i + 1)}\right)$, *for* $j = i, ..., N-1$.

   4.2.3. *Calculate* $\mu_j^{init} = \sum_{k=i}^{j}\left(\frac{\tau_k \bar{L}_k(t_{i-1})\sigma_j\sigma_k}{1+\tau_k\bar{L}_k(t_{i-1})}\right)$, *for* $j = i, ..., N-1$

   4.2.4. *Set* $L_j = \bar{L}_j \exp\left((\mu_j^{init} - \frac{1}{2}\sigma_j^2)\tau_{i-1} + \sigma_j\sqrt{\tau_{i-1}}W_i\right)$

   4.2.5. *Calculate* $\mu_j^{term} = \sum_{k=i}^{j}\left(\frac{\tau_k L_k(t_i)\sigma_j\sigma_k}{1+\tau_k L_k(t_i)}\right)$, *for* $j = i, ..., N-1$

   4.2.6. *Set* $\bar{L}_j = \bar{L}_j \exp\left(\frac{1}{2}(\mu_j^{init} + \mu_j^{term} - \sigma_j^2)\tau_{i-1} + \sigma_j\sqrt{\tau_{i-1}}W_i\right)$

   4.3. *Discount cash flows* c_i *using* $\frac{1}{\beta}$

5. *Calculate final matrix of rates* $L^{final} = \bar{L} - \frac{1}{\tau}$

6. *Return* X *as a sum of discounted cash flows.*

4.2 Data

The following synthetically generated zero coupon bond prices were used to fit the initial term structure of the HW model. The instantaneous forward rates were also derived using these data via relation (2.4).

Tab. 4.1: Synthetic zero coupon bond prices.

	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
Bond Price	1	0.93173	0.86734	0.80718	0.75139	0.69992	0.65263	0.60929	0.56964	0.53342	0.50033
	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}	T_{18}	T_{19}	T_{20}	T_{21}
Bond Price	0.47012	0.44252	0.4173	0.39423	0.37311	0.35376	0.336	0.31968	0.30468	0.29086	0.27812

4.3 Results

The main mathematical results of this dissertation were observed in Section 3.2. To numerically verify that the DLFM model nests the GHJM model which in turn nests the HW model, all these models will be implemented using the algorithms presented in Section 4.1. The GHJM model is included in the results presentation for the sake of completeness and since it is also the bridge that links the HW and DLFM models. All these three models were implemented and correctly parameterized. Swaption prices were calculated under these models, the goal is to confirm that if we can get prices that overlap or are approximately the same, this would give us substantial numerical evidence that the mathematical derivations hold and are correct.

The initial term structure was fitted with synthetic zero coupon bond prices in Table 4.1, with maturities spanning out to 21 years. With these data it means that we can only model swaptions with maximum maturities of 21 years but in reality longer dated swaptions can be valued as there is typically very longer dated bonds that trade on the market such as the R2048 in the South African market. Different parameters were used for different parts of the results, these will be highlighted in the different results that are to follow.

We then need to choose the parameters of these models. Under normal circumstances in order to make the models accurate for modelling, we would need to take market observed data and do a calibration on the models that will produce parameters that are appropriate. Since we are only concerned with producing numerical results that confirm that the concurrence holds, arbitrary parameters will be chosen. For all the results that are presented, the following parameters were fixed,

$$k = 0.14; \sigma = 0.01; \tau_i = 1; r_0 = 0.07.$$

We firstly consider the Monte-Carlo simulation results under the three models for payer swaptions with maturities of $T_\beta = 21$, settlement dates of $0 < T_\alpha \leq 10$ and strikes of $0 < K \leq 0.15$ on a nominal of one Rand. A surface plot of the HW model results is presented in Figure 4.1. The payoff profile is consistent with what we would expect for a payer swaption. Figure 4.2 and Figure 4.3 give surface plots of the absolute error of swaption prices for the GHJM model to the HW model and DLFM model to the HW model respectively. From this we can observe that the three models yield swaption prices that are almost identical, the surfaces overlap almost completely. The GHJM model produces results that have minimal absolute error compared to the DLFM model. This is not surprising since the GHJM model produces the expectation of the future rates of the HW model, while the DLFM model produces discretely compounded forward rates which are also computed

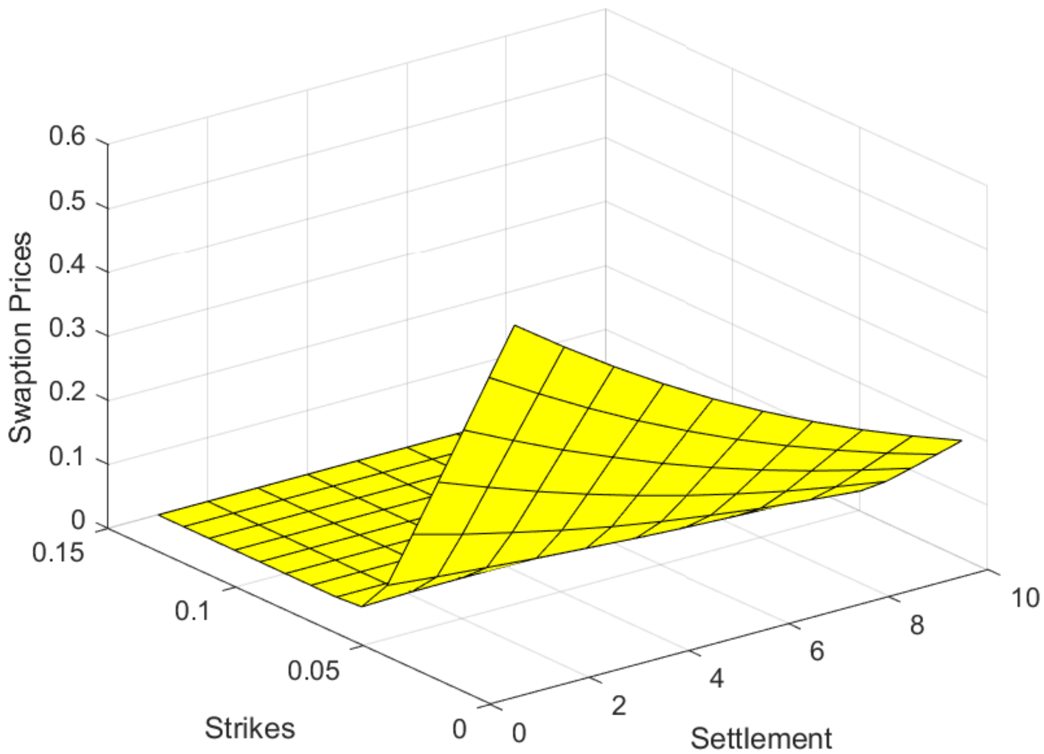

Fig. 4.1: Surface plot of payer swaption prices for different strikes and settlement dates under Hull-White model.

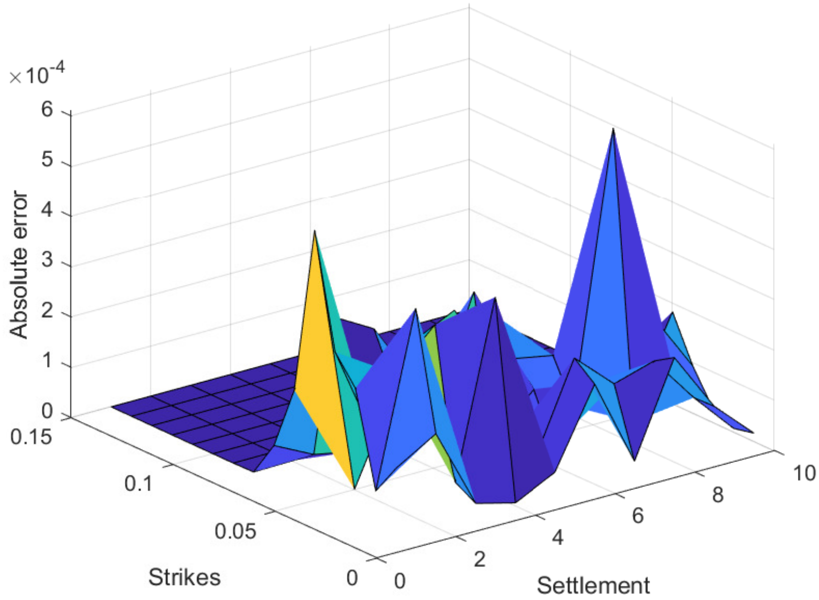

Fig. 4.2: Surface plot of payer swaption price errors, GHJM model to the HW model.

differently, this would ultimately lead to some difference in the results. What we observe is that for lower volatilites the surfaces overlap perfectly but for higher volatilities the deviations are a bit high on the prices leading to surfaces that diverge from each other.

Now we present the results for the analytical prices for the swaptions. The closed form prices were calculated using the Jamshidian's trick outlined in Section 2.4. For this, a strike of $K = 0.05$, maturities of $2 \leq T_\beta \leq 21$ and settlements of $1 \leq T_\alpha \leq 20$ and nominal of $R100$ were used. The results are a matrix of closed form payer swaption prices with different combinations of settlement dates and maturities. This matrix is seen in Table 4.2. The prices seem to intuitively make sense, for instance, a 2-for-20 payer swaption is more expensive than a 6-for-20 payer swaption because the former pays over a longer duration while the latter the payments are over a short duration. We also note that if the volatility is high, the prices tend to be higher too.

Now that we know how to calculate the analytical swaption prices, we can compare the Monte-Carlo swaption prices from the three models with the analytical prices. We present a 2-d plot of the three models coupled with the analytical prices in Figure 4.4, these were plotted using the same assumptions that were used for the

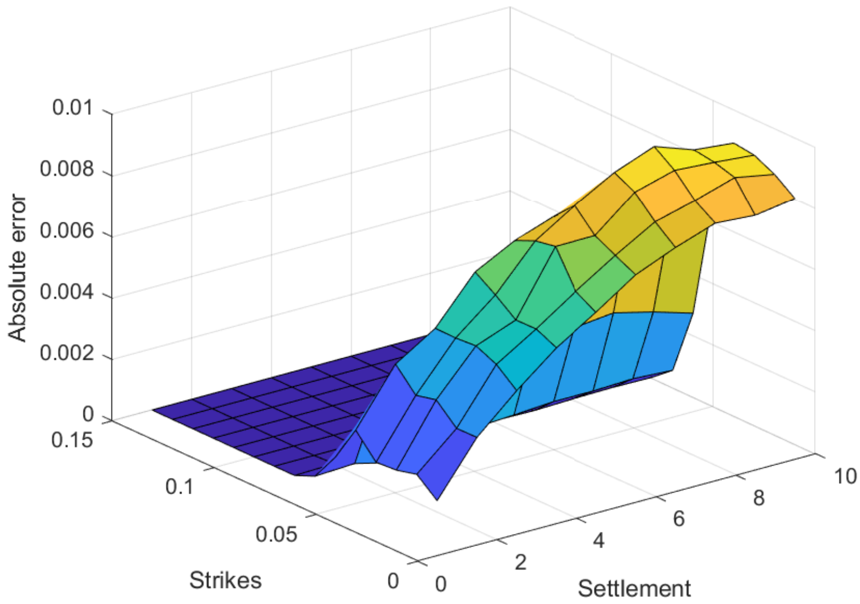

Fig. 4.3: Surface plot of payer swaption price errors, DLFM model to the HW model.

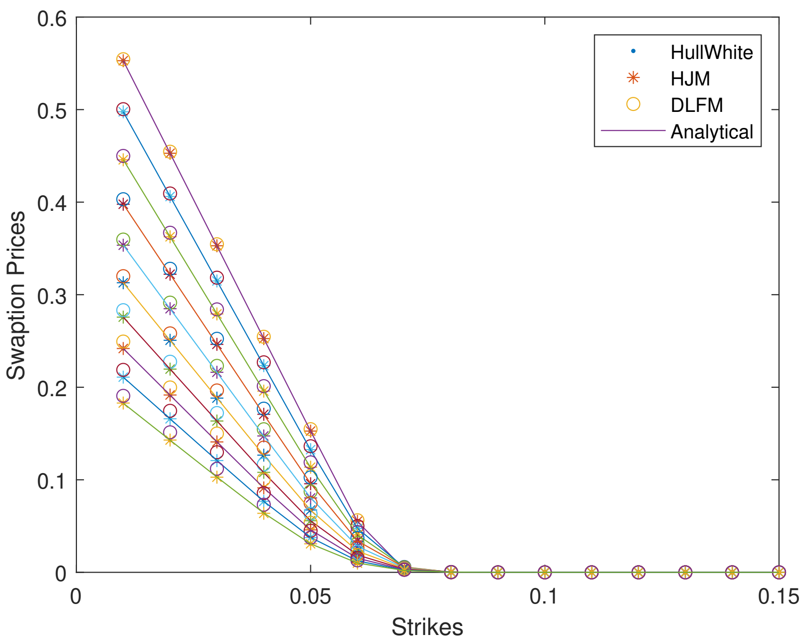

Fig. 4.4: 2-D view of the surface plot.

Tab. 4.2: Payer Swaption Prices Analytical.

$\mathbf{T}_\beta$	$\mathbf{T}_\alpha$																			
	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}	T_{18}	T_{19}	T_{20}
T_2	2.0533	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_3	3.993	1.9492	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_4	5.7835	3.7429	1.8111	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_5	7.4059	5.3655	3.4428	1.6539	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_6	8.853	6.8116	4.8932	3.1183	1.4897	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_7	10.1258	8.0832	6.1664	4.4	2.7888	1.3268	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_8	11.2308	9.187	7.2705	5.5091	3.9099	2.4681	1.1708	0	0	0	0	0	0	0	0	0	0	0	0	0
T_9	12.1775	10.1327	8.2158	6.4575	4.8666	3.4396	2.1649	1.0252	0	0	0	0	0	0	0	0	0	0	0	0
T_{10}	12.9773	10.9318	9.0143	7.2579	5.673	4.2572	2.9997	1.8845	0.89176	0	0	0	0	0	0	0	0	0	0	0
T_{11}	13.6425	11.5963	9.6783	7.9233	6.3429	4.9358	3.692	2.5963	1.6296	0.77131	0	0	0	0	0	0	0	0	0	0
T_{12}	14.1852	12.1386	10.2203	8.4664	6.8897	5.4897	4.2571	3.1773	2.232	1.4011	0.6639	0	0	0	0	0	0	0	0	0
T_{13}	14.6173	12.5705	10.6521	8.8994	7.326	5.9321	4.7091	3.6428	2.7153	1.9073	1.1985	0.56907	0	0	0	0	0	0	0	0
T_{14}	14.9501	12.9032	10.985	9.2337	7.6636	6.2754	5.0609	4.0064	3.0944	2.3058	1.621	1.0206	0.48604	0	0	0	0	0	0	0
T_{15}	15.1939	13.147	11.2294	9.4798	7.9133	6.5307	5.3242	4.2804	3.3822	2.6107	1.9467	1.371	0.8657	0.41387	0	0	0	0	0	0
T_{16}	15.3583	13.3116	11.3949	9.6476	8.0849	6.7081	5.5096	4.476	3.5906	2.8346	2.1889	1.6348	1.1546	0.73183	0.35153	0	0	0	0	0
T_{17}	15.4521	13.4057	11.4904	9.7459	8.1876	6.817	5.6265	4.6031	3.7299	2.9883	2.3594	1.8245	1.3663	0.9686	0.61686	0.29796	0	0	0	0
T_{18}	15.4831	13.4371	11.5236	9.7825	8.2293	6.8655	5.6837	4.6706	3.8094	3.0817	2.4685	1.9512	1.5127	1.1369	0.80983	0.51867	0.25214	0	0	0
T_{19}	15.4582	13.4129	11.5017	9.7648	8.2175	6.8614	5.6888	4.6865	3.8376	3.1236	2.5255	2.0247	1.604	1.2479	0.94239	0.67511	0.43523	0.2131	0	0
T_{20}	15.3839	13.3394	11.4311	9.6992	8.1588	6.8112	5.6488	4.658	3.8219	3.1218	2.5384	2.0534	1.6495	1.3113	1.0249	0.77847	0.56142	0.36462	0.17995	0
T_{21}	15.2656	13.2222	11.3175	9.5915	8.0591	6.7212	5.5699	4.5916	3.7689	3.0831	2.5146	2.0451	1.6572	1.3355	1.0665	0.83845	0.64115	0.46591	0.30509	0.15188

plotting of the 3-d surface in Figure 4.1. We see that the Monte-Carlo prices overlap with the analytical prices confirming the numerical tractability of the relations between the models. Further noting that for higher volatilities, the Monte-Carlo prices diverge to the actual price, the divergence is higher for at the money prices.

Finally, we present the three standard deviation bounds for the Monte-Carlo prices around the true swaption price. For this, a strike of $K = 0.05$, maturity of $T_\beta = 12$ and settlement of $T_\alpha = 2$ and nominal of one Rand were used. The results for the three models are tractable, the Monte-Carlo estimates of the prices lie completely within the three standard deviation bounds. For the DLFM model, we used a predictor corrector algorithm because if the standard algorithm was used then the bias would have been large. Even though we used the predictor corrector algorithm, some bias still remains and this is due to the state dependence of the drift term, this changes as a function of strike and maturity. Although more technical numerical techniques can be explored to analyse and make the numerical implementations more accurate and efficient, these were not explored in this dissertation. Overall the above results provide a reasonable good numerical confirmation that the mathematical derivations of the concurrence between the HW and DLFM models are correct.

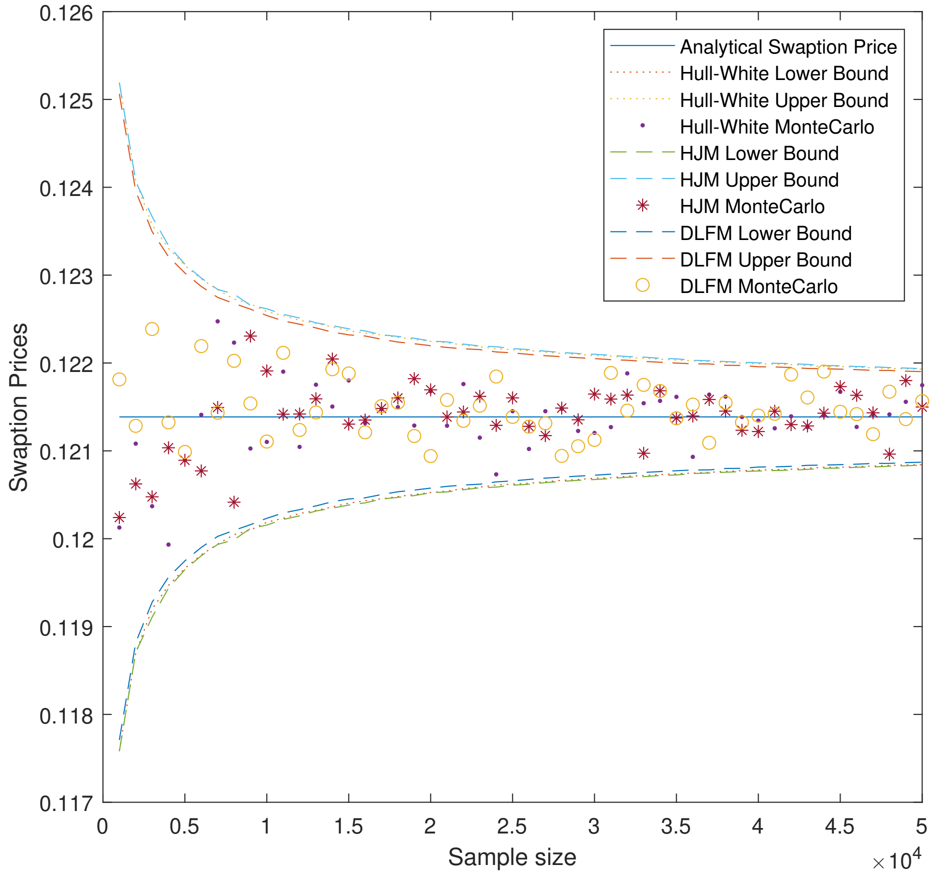

Fig. 4.5: Three standard deviation plots around the true swaption price, Hull White, GHJM and DLFM.

Chapter 5

Conclusions

Term structure modelling is imperative in financial markets for the valuation of financial instruments. With financial markets evolving rapidly, more complex interest rate derivative securities can be structured and marketed to participants in order to meet particular needs. This added complexity means market makers need to be accurate in their pricing and hedging in order to avoid unnecessary risks that could in some instances be systemic in nature. To achieve this, a thorough understanding of short, instantaneous forward and forward market models is needed. All of the different models have different advantages and disadvantages, and it is thus beneficial to know how they all relate and complement each other.

This dissertation considered the concurrence between one factor HW and DLFM models. We found that such a concurrence exists mathematically, and it is made possible by the HJM model acting as bridge between the HW and DLFM models. This concurrence was proved mathematically and is mainly driven by the dynamics of the HW model. The general procedure for this concurrence was split into three steps. Firstly, stating the dynamics of the HW model. From there, finding the GHJM model that is implied by this HW model. Lastly, we use the instantaneous volatility function of the GHJM model and plug it into a key equation that gives the volatility of the DLFM model implied by the GHJM instantaneous volatility function. Having done this, the full dynamics of the DLFM can be inferred. The DLFM model that nests the HW model would have then been found.

To confirm numerically that the concurrence holds, the three models were all implemented and swaptions were priced via Monte-Carlo simulation, the estimates were compared with the analytical swaption price calculated using Jamshidian's trick. There was enough evidence to conclude that the concurrence holds, is correct and tractable.

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