A LEARNING THEORY APPROACH TO STUDENTS' MISCONCEPTIONS IN CALCULUS

Lynn Bowie

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ABSTRACT

This study analyses students' errors in calculus through the lens of learning theories. The subjects in this study were 117 students enrolled in a calculus course for students from disadvantaged educational backgrounds at the University of Cape Town. A coding scheme to categorise the errors that these students made in the final examination was developed. This categorisation was supported by error data generated through the administration of a conceptual test and follow-up interviews.

The pattern of errors in the coding scheme suggests that the students' perception of algebra is largely that of a "game of letters". As a result of this their construction of calculus knowledge is based on the rehearsal of algorithmic procedures. Their errors indicate that they develop linking and extending mechanisms to deal with the multiplicity of rules that are generated from this process of rehearsal.

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CHAPTER 1 INTRODUCTION

1.1 The context

This research project was born out of my experiences teaching a calculus course designed to meet the needs of educationally disadvantaged students entering the University of Cape Town (UCT) under an alternative admissions programme. The history of apartheid education in South Africa has meant that a large, predominantly African, section of the population receives inadequate schooling. Despite the election of a democratic government, inequities between race groups in the type of school education received still exist. In mathematics and science these inequities are particularly pronounced. In 1993 only 26% of scholars sitting the school-leaving examinations at the African[1] high schools were taking mathematics. Of these only 25% passed. In contrast 64% of scholars from white schools were taking this subject and they achieved a 95% pass rate. (Edusource, 1994). In 1992 the pupil to teacher ratios in African schools were 46:1 in comparison to 22:1 in white schools. At the same time 72% of mathematics teachers in African schools were not qualified to teach mathematics. (Edusource, 1993). Current official statistics do not provide a breakdown on the basis of race, but the figures provided do not indicate any major change from the above. (Edusource, 1998).

UCT's response to this situation was to set up an alternative admissions programme to enable students from educationally disadvantaged backgrounds to enter UCT. These students, who do not meet the standard admission requirements for UCT, are initially enrolled in special first year courses taken over two years. The mathematics programme takes the form of a year-long basic differential and integral calculus course followed by a year-long vector algebra, complex numbers and differential equations

[1] Since the election of a democratic government in South Africa all schools are open to all pupils. However the majority of African pupils still receive their education in the schools which were designated for Africans under apartheid. These schools were vastly under-resourced in comparison to the schools designated for white pupils and thus distinctions between these schools still exist.

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course. These two courses in combination are equivalent to the standard first year mathematics course and thus enable students to proceed to senior courses in mathematics. My role as convenor of the special mathematics programme has been to oversee the development of this programme as a whole and to design and teach the calculus component. The focus of this research project is the calculus course.

A political and moral dilemma emerges from running a separate academic development course in the way described above. The majority of students entering the mainstream first-year mathematics course attain a mark of 70% or more in their school-leaving mathematics exam, whereas the majority of students entering the special academic development course attain a mark of less than 60% in their school-leaving mathematics exam. This necessitates a division in academic provision. However it is vital that the standard of mathematics taught in the academic development course is equivalent to the mainstream course so that all the students enter the senior years of their degree on the same footing. A tension thus exists between meeting this aim and simultaneously addressing the need to ensure that the academic development course articulates well with the mathematical background of the students. The impetus for the research study discussed here came from years of curricular innovation on the course which was designed to address this tension as well as from a need to reflect on the results these were achieving.

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1.2 The motivation for the research study

The evolution of the course over time has been influenced by students' evaluation of their experience on the course, the reflection of the lecturing staff on their teaching practice and their perception of students' needs, and by drawing on the experience and results of other lecturers and researchers. The Calculus Reform Movement has had a major influence on the development of the course. The Calculus Reform Movement was started in the USA in the 1980s in response to perceived weaknesses in the teaching

and learning of calculus and has generated a wealth of discussion, debate, innovative ideas and challenges for calculus education. In South Africa, collaboration with colleagues from the USA led to the establishment of an informal network (The South African Mathematics Education Reform Network, SAMERN) of mathematicians and educators with the aim of providing a forum for the discussion and dissemination of ideas relating to the teaching and learning of calculus. Although the wide diversity of ideas and innovations makes it difficult to define precisely what constitutes calculus reform, Tucker and Leitzel (1994, p1) state that

> From the start of the reform movement, there has been broad agreement that a guiding theme should be to concentrate on greater conceptual understanding, developed through extensive numerical, graphical, algebraic and modelling interpretations.

Despite this broad agreement, the forms that calculus reform has taken have been numerous and range from the incorporation of technology and the use of co-operative learning to a change in approach to specific calculus topics. These are discussed in more detail in chapter 3. The impact of the Calculus Reform Movement on the academic development course can be seen in the incorporation of a computer graphing package into the course, our use of co-operative learning workshops as a key mode of instruction and the adoption of the Harvard consortium reform textbook (Hughes-Hallett et al, 1994) as the prescribed text for the course. Our motivation for making these changes echo those outlined by Tucker and Leitzel above.

Thus of particular concern to me in surveying the literature emanating from the calculus reform movement was research that detailed the effects of the various new approaches. Although a vast literature and numerous conferences and workshops have been devoted to this area of work, a large proportion of the papers put forward either describe current initiatives, offer opinions based on anecdotal evidence or defend certain positions or decisions taken. The lack of detailed empirical research in this area is in part a product of the fact that the endeavours in this field are still fairly new and

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much energy and time has to be put into getting new courses off the ground (Cannon, Simmons and Ipina, 1994). It is also partly because opposition to reform has meant that energy has been devoted to providing empirical evidence of retention and pass rates rather than to exploring the educational outcomes in depth. Thus the evaluation component of calculus reform has tended to consider success rates of reform versus traditional calculus students (Keynes et al, 1996, Johnson, 1995) or at student evaluation of reform courses (Mittag, 1996). Most of these have shown results that favour calculus reform. There has certainly been more extensive work on the use of computers in mathematics education (Tall, 1987, Tall 1989, Breidenbach et al, 1992, Goldenberg et al, 1992), that link learning theories, practice and evaluation. However there is a need for a more thorough evaluation of reform efforts and the effects they produce. Ferrini-Mundy and Graham (1991) echo this point, saying

> most studies in calculus have been large-scale quantitative attempts to model how attitudinal and experiential variables affect overall performance.....Although several of these studies have provided valuable information, most lack a consistent perspective of student learning and have minimal influence on understanding or performance. (p629)

In South Africa a similar pattern has emerged. The SAMERN has been active, numerous workshops and conferences held, but very little research has emerged that either provides a basis for informed curriculum development or which evaluates the results of a particular programme. Within the area of calculus, research has considered the impact of the graphing calculator (Berger, 1996) and student's acquisition of calculus concepts (Bezuidenhout, 1996, Winter, 1989). Although there is enormous value in the discussion and innovation that SAMERN has stimulated, and a need to record success rates of students on reformed courses, it is also important to evaluate whether the kind of learning we want to take place is indeed taking place in these courses.

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Evaluating student learning on a mathematics course opens up a variety of potential research questions and strategies. One could, for example, analyse whether a more intuitive approach to the concept of a derivative, using multiple representations, provides a different understanding to a more rigorous limits-based approach. Or one could ask what changes the introduction of a computer graphing package might induce in students' learning strategies. Or one could analyse students' discussions in co-operative learning groups in order to probe the effectiveness of this learning method. However, in light of the paucity of research in the South African context, and observations by staff teaching on the academic development course that the students did not seem to have developed the conceptual understanding the course aimed to instil, it was decided to probe the difficulties students had in learning calculus concepts. This could then provide a background against which to evaluate teaching strategies aimed at overcoming these barriers in future studies.

Research studies on students' misconceptions have provided insights into the difficulties students have in dealing with concepts in mathematics. For example, the work of Cornu (1991) on limits demonstrates the cognitive obstacles that students face in dealing with the limit concept and the work of Vinner (1982) brings to the fore underlying misconceptions that hamper students' development of an understanding of tangents. Thus it appeared that analysing students' misconceptions in calculus could provide information about the difficulties they face in learning calculus concepts. Simply detailing students' misconceptions was not sufficient, however. The present study had to move beyond these and attempt to develop a model of student learning so as to provide the basis for curriculum development. It was for these reasons that emerging theories on student learning, particularly in the context of more advanced mathematical topics (Sfard, 1992 and Dubinsky, 1991) became on important component of the research project.

Thus, in summary, the research project focused on the following two questions:

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1. What misconceptions do students from disadvantaged educational backgrounds display in dealing with first-year university calculus concepts?
2. Can theories of student learning provide a framework or model with which to understand these misconceptions?

1.3 An outline of the research project

Since the learning theories of Sfard (1987) and Dubinsky (1991) provide the framework within which the students' misconceptions were described, chapter 2 provides an overview of their theories together with related work. In particular, the notion of the duality of mathematical objects as processes and objects and the implications of this for student learning are discussed. Although their theories seek to explain how conceptual understanding is developed, in this chapter emphasis is placed on the possible explanations their theories offer about sources of difficulty in student learning.

Chapter 3 provides an overview of relevant research literature. In this section I pay attention to the key ideas and research that have emerged from the Calculus Reform Movement. I provide a summary of the research work that has been conducted within the field of advanced mathematical thinking, with particular reference to algebra, functions and calculus. I survey research on misconceptions, focusing on the ways in which students' errors have been categorised.

Chapter 4 outlines the research design. In this chapter the motivation for the use of and details about the collection of the data used in this study are described. There were three main sets of data: students' errors on the final examination of the course, students' errors on a specially designed conceptual test and transcripts from a group discussion and interviews about this test.

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A detailed analysis of the errors that students made in the final examination is given in chapter 5. As this data was used to generate a coding scheme to categorise students' errors, a description of the generation of the coding scheme and definitions of the coding categories are given.

Chapter 6 discusses this data in relation to the learning theories of Sfard and Dubinsky and offers a model of student learning based on these theories.

The data generated through the use of the conceptual test and follow-up interviews is discussed in chapter 7. This data is used to deepen and extend the analysis provided by the data from the final examination.

The final chapter offers implications of this research for both the teaching and learning of calculus generally and for educational strategies required in a course for educationally disadvantaged students.

Copies of the final examination and conceptual test are provided in appendices.

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CHAPTER 2 LEARNING THEORIES AND THE PROCESS-OBJECT DUALITY OF MATHEMATICAL CONCEPTS

Although this study seeks to explore difficulties that students experience in learning calculus, the theoretical framework posited here considers learning theories which explain how students come to an understanding of a mathematical concept. The deliberate focus on theories of learning was taken in order to avoid producing a simple list of errors that students make and misconceptions that they have, and instead to situate the analysis of errors within a framework that provides clues to the processes students use to build knowledge.

Central to the work on advanced mathematical thinking over the last ten years has been a discussion of the duality of mathematical concepts as both "processes" and "objects". (Dubinsky, 1991, Harel and Kaput, 1991, Sfard, 1987, Dubinsky and Harel, 1992, Tall, 1991). Although the various proponents of theories differ in the details of their approach, their theoretical perspective is similar. In what follows I will give an outline of the work of Sfard, Dubinsky and Tall.

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2.1 Sfard's theory of reification

At the heart of Sfard's theory of reification is the idea that different mathematical notions can be conceived of in two fundamentally different ways: structurally (as objects) and operationally (as processes). She argues that these two approaches, despite their apparent differences, are in fact complementary and that successful learning and problem solving require a flexibility in being able to move between the two. (Sfard, 1991). She claims that an operational understanding usually precedes a structural understanding of a particular notion. The route from process to object is seen as a fundamental, but difficult one and involves three stages: interiorization,

condensation and reification. She uses the term "interiorization" in much the same way as Piaget (1970) does. During the phase of interiorization a student becomes familiar with a process and can carry it out through mental representations. Condensation is a gradual quantitative change in which a sequence of mathematical operations are dealt with "in terms of input and output without necessarily considering its component steps." (Sfard, 1992, p62). Thus although dealing with the new notion becomes more manageable it nonetheless remains a process. Reification is the most difficult of the three phases as it involves a qualitative shift in understanding. This shift occurs when the student is able to detach the notion from the processes that produced it and see it as an object. Sfard and Linchevski (1994) describe reification as follows:

> Mathematical objects are the outcome of reification - our mind's eyes' ability to envision the results of processes as permanent entities in their own right (p194)

In the case of the notion of a function one can see the phase of interiorisation occurring when the student manipulates the independent variable to find the dependent variable, condensation when the student is capable of seeing the function as a mapping, and reification when the student sees functions as objects that they are able to perform processes on. Thus Sfard's idea of conceptual development is hierarchical in nature. Processes are reified into new objects upon which new processes can be performed. These in turn are reified into objects.

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2.2 Dubinsky's action, process, object, schema theory

Dubinsky parallels Sfard in his distinction between processes and objects and in their complementary role in a "spiral" development of mathematical concepts. In the same spirit as Sfard he states: "objects are used to construct processes which are then used to construct new objects from which new processes are formed and so on." (Dubinsky, 1991a, p167). Dubinsky's model of conceptual understanding uses four terms which require definition:

Action: An action is a repeatable physical or mental manipulation that transforms objects.

Process: A process is an action that takes place entirely in the mind.

Breidenbach et al (1992, p278) state that :

> The main difference ... between an action and a process is the need in the former for an explicit recipe or formula that describes the transformation. Moreover, in an action one tends to think about the transformation in a step-by-step manner with the steps related only by recipe, and not by any relationships that exist in the mind of the subject. A process, on the other hand, represents a transformation that does not have to be very explicit, nor must a subject be absolutely certain that it exists. It is only necessary that the transformation be imagined, in the mind of the subject, as a more or less certain possibility.

Object: Dubinsky makes no distinction between mental and physical objects. The distinction between a process and an object is drawn by stating that a process becomes an object when it is perceived as an entity upon which actions and processes can be made.

Schema: A schema is a "more or less coherent collection of cognitive objects and internal processes for manipulating these objects" (Dubinsky, 1991a p 166). Any individual student will possess a vast number of schemas that are interrelated. Schemas are constantly constructed or re-constructed to deal with new problem situations.

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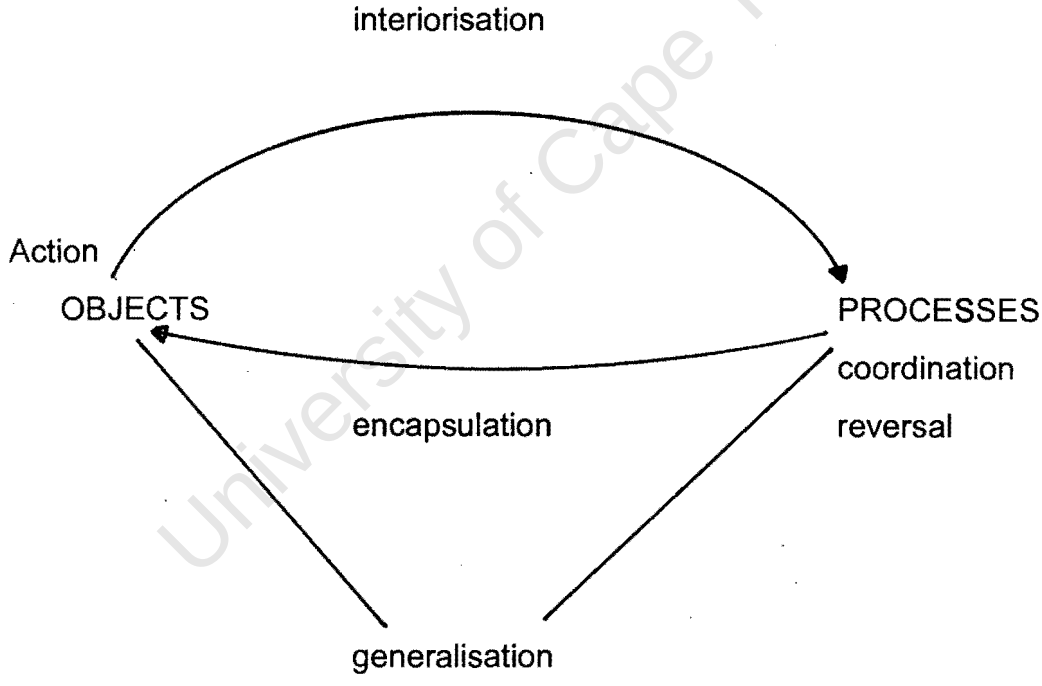
Central to Dubinsky's theory is Piaget's notion of reflective abstraction. Reflective abstraction has two components: the first, a projection of existing knowledge onto a higher plane of thought, and the second, a reorganisation of existing knowledge structures (Dubinsky, 1991). Reflective abstraction is thus a process of construction and Dubinsky (1991b) outlines five kinds of construction in reflective abstraction

1. Interiorisation: This parallels Sfard's and Piaget's notion. Actions on objects are interiorised into a system of operations.
2. Co-ordination: Two or more processes are co-ordinated in order to form a new process e.g. the chain rule for differentiation requires the co-ordination of composition of functions with derivatives.

3. Encapsulation: This echoes Sfard's notion of reification. It is the leap to seeing as an object what has previously been conceived of as a process.
4. Generalisation: An existing schema is used in a wider range of contexts. This would occur, for example, when the student is able to see functions as able to map not only numbers, but also vectors.
5. Reversal: An interiorised process can be thought of in reverse. Thus finding an antiderivative would be seen as a reversal of the process of finding the derivative.

Dubinsky (1991b, p107) summarises the above in a diagram representing schemas and their construction:

Diagram 1:



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2.3 Tall's notion of procept

In Tall's work (1991) the process/object duality is also seen as important and he has coined the term "procept" to stand for a process which is symbolised by the same symbols as the product. Thus for him $\lim_{n\to\infty}\frac{1}{n}$ is a procept as the

notation both denotes the process of tending to the limit and the value of the limit. He draws on Dubinsky in talking about "*processes* which become encapsulated as *concepts*" (Tall, 1991, p254). Tall stresses that the ambiguity of the same notation being used to represent process and product is part of the power of mathematics and allows the mathematician to use whichever is appropriate for the given task.

2.4 Implications for this research project

Although there are differences in emphasis and detail between how these three authors conceive of learning in advanced mathematics, they share the views of a process-object duality in mathematical concepts and the idea that mathematical knowledge is constructed by the student. All three stress the reification (or encapsulation) of processes into objects as a key component in the development of a student's mathematical knowledge. Following from the above brief outline of their learning theories I would like to draw out from their work some implications this theoretical stance has for my study.

There are two main reasons why this study has been situated within a framework of the kind of learning theory outlined above. The first is that the subjects were students who come from disadvantaged educational backgrounds and generally had poor results in school mathematics. Preliminary research (Bowie, 1995) suggested that difficulties these students experienced was not due to "gaps" in their knowledge, but rather to an approach to mathematics learning and mathematical concepts that did not equip them to deal with problems in advanced mathematics. The work of Sfard, Dubinsky and Tall provides a model for understanding how mathematical knowledge is constructed and thus offers a way of gaining deeper insight into what the problematic "approach" is that appeared to be the greatest stumbling block for students. The second reason is that the process-object duality provides a powerful tool for analysing the epistemological demands of concepts in advanced mathematics.

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Learning theories provide a model of how knowledge is built up. My study aimed to look at the barriers that students face, in other words, where the knowledge construction process breaks down. Thus the key implications for my work are based on what the theory has to say about deviations from the ideal model. This I will look at under three headings:

2.4.1 The value of structural understanding

One of the immediate benefits of reification for a student is that it provides coherence and structure. Sfard (1987) argues that purely operational knowledge can only be stored in unstructured, sequential cognitive schemas. Thus a purely operational understanding can make problem-solving difficult in that processing has to happen segmentally which she argues leads to a great degree of cognitive strain. This echoes the opinion of Harel and Kaput (1991) who present the first of the three roles of conceptual entities (which they use in the same way as Sfard's uses the term "object") as alleviating working memory or processing load. They argue that the mind has a limited processing capacity. A single conceptual entity is easier to store, retrieve and work with than a process. This idea is best summarised by Sfard and Linchevski:

> Although reification itself may be difficult to achieve, once it happens, its benefits become immediately obvious. The decrease in difficulty and the increase in manipulability is immense. What happens in such a transition may be compared to what takes place when a person who is carrying many different objects loose in her hands decides to put all the load in a bag. (1994, p198)

The second role Harel and Kaput outline is that of facilitating comprehension of complex concepts. They illustrate this by saying that, although a process conception of function may be adequate for dealing with a number of situations which depend on functions, when uniform operators on functions arise, these will prove inadequate. Thus to comprehend the meaning of $I(t) = \int f(x)\,dx$ one needs to be able to see the function $f(x)$ as an object

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on which the mapping $I(t)$ can act. This idea parallels the claim by Sfard and Dubinsky about how mathematics is built up.

> The whole of mathematics may therefore be thought of in terms of the construction of structures,...mathematical entities move from one level to another; an operation on such 'entities' becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by 'stronger' structures (Piaget quoted in Dubinsky, 1991b, p101).

This hierarchical notion of how mathematical knowledge is built underlines the value of reification. In order to build and develop new knowledge the student has to turn processes into objects. Being able to move from seeing $f(x) = x^2$ as an instruction to square each input value to being able to see it as an object that can be differentiated, integrated, shifted etc is a vital step in developing an understanding of algebra that allows the concepts of calculus to be built. Without such a shift in understanding, "the secondary processes will remain 'dangling in the air' - they will have to be executed ... on nothing" (Sfard and Linchevski, 1994, p221). Knowledge built on a purely operational understanding will thus lack the rich relationships that are present in structural knowledge.

The third role of conceptual entities that Harel and Kaput describe is that of "facilitating focus on those aspects of a problem representation that are most relevant to the solution of a problem." (p88). Being able to see a concept as an object with a number of properties allows a student to home in on the particular property relevant to the problem at hand. A process on the other hand is sequential and thus is carried out step by step.

Although I have concentrated here on the benefits of structural understanding, it is important to note that all of the authors name a flexible process-object conception as the most crucial ability. Sfard (1992) argues strongly that an operational understanding must precede a structural one and thus the focus on the benefits of a structural understanding is not meant

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to undermine the importance of a good operational understanding. However an operational understanding alone is not sufficient for advanced mathematics.

2.4.2 Pseudostructural conceptions

The notion of a pseudostructural conception introduced by Linchevski and Sfard (1991) offers an explanation for what can happen if the desired operational-structural cycle does not occur. The genesis of such a conception lies in the inherent difficulties of reification. For many students the mathematical objects and structures which they are meant to be able to see are not clear to them at all. Because they are expected to perform operations on such things they need to develop a way of dealing with them. The students create their own meaning. In the case of the pseudostructural approach the mathematical object is identified with its representation. A symbol, formula or graph becomes the object that is dealt with, divorced from the operations that produced it and devoid of meaning. In the words of Sfard and Linchevski (1994), the student "mistakes a signifier for the signified" (p221).

Harel and Trgalova (1996) provide an example of this when they talk about students who would think of the quadratic function $y = 2x^2 + 3x + 4$ by associating it with the equation $2x^2 + 3x + 4 = 0$ which they know how to solve. The association is not based on a deep relationship between the two but on similarity in representation. Harel and Trgalova say:

> "For these students the equation itself is no more than a collection of elements they recognize by the spatial and/or temporal relations that connect them." (p680)

Similarly Vinner and Dreyfus (1989), Vinner (1983) and Ferrini-Mundy and Graham (1994) also note the tendency of students to associate functions purely with their algebraic expressions. A similar idea, but in the context of graphical representation, is discussed by Monk (1992) where he puts forward

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the notion of iconic translation. The notion of iconic translation describes a situation where a student expects a graph to represent the physical situation literally. Thus, for example, the distance-time graph of a car travelling over a series of hills would be pictured as a series of hills. The difficulties for students in moving between algebraic and graphical representations of a function are also well documented (for example, Eisenberg, 1992, Artigue, 1992, Dreyfus, 1991) and Sfard (1992) argues that these could be as a result of a pseudostructural approach.

The consequence of a pseudostructural approach is that it appears to result in a conception of mathematics that is not coherent and lacks rich relationships. Sfard and Linchevski (1994, p224) state

> The problem is that those who adopt the pseudostructural approach and confuse the powerful abstract objects with their representations do not realise that the symbols themselves cannot perform the magic their referents are able to do: they cannot glue together lots of detailed pieces of knowledge into one powerful whole.

The ideas contained in the above quote echo the contrasts described in Hiebert and Lefevre's (1986) notions of conceptual and procedural knowledge and in Skemp's (1976) notions of instrumental and relational understanding. Sfard and Linchevski (1994), in fact, make direct reference to the fact that a pseudostructural approach may allow students to carry out the secondary processes, but, because they are detached from previously developed concepts, the students' understanding will remain instrumental.

The relationship between learning theories based on the process-object duality in mathematics and the categorisation of mathematical understanding into conceptual and procedural (or relational and instrumental) is, I believe, a crucial component of being able to build an understanding of what can go wrong in student learning. This relationship is difficult to unravel. On the one hand there seems to a clear line of argument that develops from the theory of the process-object duality, which states that if a structural understanding is

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required before the students have reified the objects needed, a pseudostructural approach can develop. Because representations are used without a link to their underlying referents, such an approach results in procedural understanding. On the other hand there appears to be a fuzzy link between procedural knowledge and an operational understanding and between conceptual knowledge and a structural understanding. The link between these conceptions and an algorithmic or rote-learning approach is also unclear and what I will attempt to do in the next section is to clarify some of the issues that surround these ideas.

2.4.3 Links to algorithmic approaches and rote-learning

Sfard (1991) considers the relationship between her process-object duality of mathematical concepts and the categorisations of mathematical understanding into conceptual and procedural (Hiebert and Lefevre, 1986). There certainly appear to be parallels between the ideas. Hiebert and Lefevre characterise conceptual knowledge as being rich in relationships. Procedural knowledge is made up of two parts: the symbol representation system and the algorithms for completing mathematical tasks. It is sequential in nature. Thus operational thinking would lead to procedural knowledge and structural thinking to conceptual knowledge. Sfard makes this point, but highlights the distinctions between her categorisation and that of Hiebert and Lefevre. The first is that in her classification Sfard has looked both at the nature of mathematical entities and at the perception of these entities by students. The second is that she sees her categorisation as a duality rather than a dichotomy. Sfard (1991 p.9) states "Whether the issue of applications or of education is concerned, the operational and structural elements cannot be separated from each other."

However, in discussing mathematical learning she makes statements such as "operational conceptions should precede the structural" (Sfard, 1991, p10) or "a structural conception should not be required as long as the student can do without it" (Sfard, 1992, p69). These kind of statements do seem to

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separate the two conceptions and I believe can give rise to problems I will describe below. Hiebert and Lefevre, Sfard and Dubinsky all stress the complementarity of the two notions.

Dubinsky (in press), in his reply to the critique by Confrey and Costa (1996) of the use of mathematical objects as the central metaphor in advanced mathematical thinking (1997), argues:

> I think that these authors are profoundly mistaken...in suggesting that we are selecting mathematical objects as central metaphor, as opposed to studying the construction, relationships, and roles of mental processes and mental objects in learning secondary and post-secondary mathematics.

All three of the authors are careful in their attempt not to privilege the one type of thinking over the other. However there are two factors which militate against this being easily achieved. The first is the hierarchical notion which has objects reified out of processes. The second lies in the terms used to describe the categorisation. If one looks at the following chain of pairs of words, each of which has parallels with the preceding pair in the list, it is not hard to see how an operational conception can come to be equated with a rote-learnt algorithm:

operational-structural
procedural-conceptual
algorithmic-abstract
knowing how to-knowing why
rote learnt-meaningfully understood

My aim in highlighting this potential confusion is to clarify the distinctions as I see them.

The operational-structural duality describes a conception of a mathematical entity. Most of the empirical work done using this approach has focused on what Dubinsky calls "the genetic decomposition" of a particular concept.

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Thus one cannot say, for example, that a student has an operational approach to mathematics, nor can a process conception be equated with an algorithmic or rote-learnt approach because the process may be well understood. However the operational-structural duality does imply that a process conception of a mathematical notion may become insufficient for dealing with the task at hand and it is at this point that links to algorithmic learning can be made. Sfard argues that, because reification often takes considerable time and effort, many students abdicate from the attempt and resign themselves to never understanding mathematics. Without understanding the only other option is rote learning. Tall (1996) gives a similar explanation. He explains that in the face of conceptual difficulties students need to learn to cope and a reasonably effective method of doing so is often to concentrate on learning the computational and manipulative skills to pass exams. He says

> if the fundamental concepts of calculus (such as the limit concept underpinning differentiation and integration) prove difficult to master, one solution is to focus on the symbolic routines of differentiation and integration

The other important issue that arises from clarifying the potential confusion outlined above is the relationship between operational and structural thinking and conceptual knowledge. Again the key to understanding this arises from the fact that the process-object duality relates to a particular mathematical concept and does not speak in broad terms about a student's mathematical knowledge. However the reification of a number of processes into a single object does provide a rich set of relationships. Moreover, the construction of processes on the new object provides a further layer of relationships at a more abstract level. Thus the continual development of layers of structural understanding provide a rich web of relationships.

The implication of the above is that a process conception of a mathematical concept cannot be linked to rote learning. Instead, if one wants to look at what goes wrong in student learning, one needs to look either at the point at which a process conception becomes inadequate for dealing with the task at

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hand or, in light of Sfard and Linchevski's discussion of the pseudostructural approach, at the type of objects upon which the students are building such processes. In addition, the importance of structural understanding in providing coherence and linking relationships between mathematical topics, begs the question of what kind of linking mechanisms are used to reduce the cognitive strain of holding vast quantities of information in the absence of a structural understanding.

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CHAPTER 3 PRECALCULUS, CALCULUS AND THE STUDY OF MISCONCEPTIONS: A REVIEW OF LITERATURE

The focus on the teaching and learning of calculus is a fairly recent phenomenon. The two key facilitators of this growing area of interest have been the calculus reform movement and the Working Group of the International Group for the Psychology of Mathematics Education on Advanced Mathematical Thinking. Dubinsky (1996) argues that we are witnessing the emergence of a new field of research that has both pure and applied components. In this survey I will delineate some of the key trends within this new field in two sections. In the first of these I will discuss the ideas that have emerged from the calculus reform movement. My intention in outlining these ideas is to provide an understanding of the challenges that have been posed to calculus teachers and the type of responses these have elicited in order to contextualise my research study. In the second section I will provide an overview of empirical research that has been conducted on both calculus and pre-calculus topics. In reviewing this research I will focus on the way in which the empirical research has been used to deepen our understanding of students' difficulties in algebra, functions and calculus.

My decision to explore student learning in calculus through their errors necessitated a review of research that has been conducted on students' misconceptions with respect to different topic areas in mathematics. The final section of this chapter thus discusses some of the research that has been done in this area with particular reference to the ways in which the researchers have categorised students' misconceptions.

3.1 The calculus reform movement

The late 1980's saw the emergence of the calculus reform movement in the USA, following a wave of general education reforms (Johnson, 1995). The

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need to look at the way calculus was being taught arose out of a dissatisfaction with the high dropout rate from calculus courses, the low grades achieved by students and complaints from client departments that the students did not have the mathematical skills they required. (Tucker and Leitzel, 1994). In addition the availability of powerful graphing calculators and computer packages provoked questions about curriculum content. The impact of this movement has been immense, especially considering the relative autonomy of university lecturers to decide how courses will be taught. By 1994 about 68% institutions in the USA were teaching reformed calculus courses (Tucker and Leitzl, 1994). In 1994 an informal network of South African mathematicians and educators was set up to discuss calculus reform in this country. This network (the South African Mathematics Education Reform Network) has been active ever since and has worked, in collaboration with American colleagues, to provide a forum for the discussion and dissemination of new approaches to calculus teaching.

The term "calculus reform" is a very general term used to cover a wide range of educational innovations at university level. What constitutes a "reformed" calculus course and the best way to reform the teaching of calculus is a matter of considerable debate within the calculus reform movement. Thus in what follows I wish to highlight some of the key trends that have emerged from the calculus reform movement, but in no way wish to imply that they are universally accepted or implemented.

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3.1.1 The role of technology

The availability of calculators and software packages that can draw graphs, rapidly calculate numerical approximations of integrals and derivatives and perform the algebraic manipulation normally learnt in a standard calculus course has provided an enormous challenge to the teaching of calculus.

Some researchers have focused on the numerical and graphing capabilities of this technology. They argue that the computer or calculator is a tool that allows students to manipulate and reflect on powerful mathematical ideas

(Dubinsky and Tall, 1991, Berger, 1996, Tall, 1989). Others highlight the role that computer programming can play in helping students construct knowledge (Dubinsky, 1997, Sfard and Leron, 1996). The evaluation of the use of technology in calculus teaching has shown that both of the above approaches can be a powerful in aiding concept formation (Tall, 1989, Tall, 1987), visualisation (Cunningham, 1991, Hughes-Hallett,1991), large scale project work and explorations (Keynes, 1997, Culotta, 1992, Dunham, 1993) and in re-sequencing the order in which topics can be taught and understood. (Heid, 1988)

Perhaps the most controversial use of technology has been witnessed with the advent of the symbolic manipulators like Maple, Mathematica or Derive. The incorporation of a symbolic manipulator into a course automatically raises questions of whether one should continue to teach the procedures for differentiation and integration which are quickly and competently handled by the computer. It also raises the question of whether students require skills in algebraic manipulation in order to do calculus.
Bennett (1996 p.2) argues

> It was our thought that use of a CAS[2] might serve to level the "algebraic playing field;" that it would be a time-saver and aid for students with adequate skills, and would be an essential tool for students with inadequate skills. Perhaps it would allow them to "hurdle" algebra difficulties and get on with calculus.

This sentiment is echoed by Tucker (1990) who says

> The de-emphasis of algebraic manipulation, common to most of the projects, may also be a help for students whose high school preparation was less than adequate. (p8)

These ideas are particularly interesting in the South African context where, for many students, inadequate school education leaves students

[2] Computer Algebra System

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underprepared for university study. Three questions immediately emerge in this context:

- What are the nature of the mathematical barriers that underprepared students face at university?
- Would a symbolic manipulator help to overcome these barriers?
- What are the long term effects of this?

There is little in the research literature that addresses these questions and the present study is an attempt to provide a contribution to answering the first question.

3.1.2 Co-operative learning

The style of teaching and learning in reform calculus courses more often than not moves away from the traditional focus on lectures. Workshops in which students work on challenging problems have become a key component of the teaching style. (Dubinsky, 1995, Culotta, 1992, Tucker, 1990, Ganter, 1994, Treisman, 1989, Keynes et al, 1996).

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3.1.3 Approach to calculus topics

The calculus reform movement has paid a great deal of attention to the goals of teaching calculus. Although there is again variety in scope and emphasis there appear to be two common trends: firstly, understanding the concepts of calculus rather than learning algorithmic manipulations and secondly, applying that understanding to solve problems in the natural sciences, and in business or social science. Many argue that achieving these takes time and there is thus a trade-off in what can be covered in a course. As Epp (1986, p58) argues "If it comes to a choice, will we settle for superficial knowledge of a lot or deeper understanding of less? Perhaps less is more", and this seems to echo the feelings of many reform proponents. What gets left out is often a source of considerable debate.

For some, as discussed under the role of technology, it is a decreased

emphasis on algebraic manipulation, rules for differentiation and methods of integration. For others, the debate has centred around intuition and rigour (McCallum, 1995). Most traditional calculus texts provide a definition-theorem-proof approach to calculus. The fact that rigour and abstraction are key elements of mathematics have made many uneasy about moving away from such an approach. However experience has shown that many students simply learn proofs by rote (Conradie and Frith, 1994). McCallum (1995) casts doubt on whether rigour was learnt in the traditional course. He says

> In fact, I suspect that most mathematicians who teach calculus either surreptitiously relax their standards of rigor and concentrate more on developing intuition, in spite of the textbooks, or employ a double standard, pretending to teach rigor without demanding any true mathematical reasoning from their students. Most of the textbook questions dealing with the abstract part of the course, are, in fact, as formulaic as drill problems about the chain rule. (p90)

Many in the calculus reform movement concentrate on developing students' intuition and providing explanations rather than proof in the hope that these will provide a strong conceptual understanding on which further abstraction and rigour can be built.

The Harvard Consortium, whose textbook is used in over 300 institutions in the USA (Tucker, 1996), use as a guiding principle the Rule of Three which states that every topic should be looked at geometrically, numerically and algebraically (Hughes-Hallett et al, 1994). This rule is often amended to include "verbally" and is thus also known as the "Rule of 4". This approach emphasises the need to be flexible in moving between different forms of representation, draws connections between the various representations, looks repeatedly at concepts from different viewpoints and underscores the need for students to make choices about the most appropriate form to use in solving problems. The "verbal" part means that students are continually asked to explain and interpret their mathematical work. This textbook, and thus this approach, formed the basis of the calculus course which is the focus of the present study.

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3.2 Research on pre-calculus and calculus topics

Over the last decade or two there has been a growing interest in studying advanced mathematical thinking, and a growing body of literature that seeks to probe the nature of learning and teaching at tertiary level. Although the scope of this field of research is broad, ranging from work on mathematical proof to the use of technology in the classroom, I have focused on the research reports that relate to calculus, and in particular at those that look at students' conceptions of calculus topics. In addition, because of the strong tie between calculus and algebra, I have also included some work on algebra and functions.

Much of the work I describe here is based on a theoretical framework that foregrounds the *process-object* duality of mathematical concepts. This framework has been discussed in depth in the previous chapter. The dominant mode of research that has been conducted from within this framework is the in-depth analysis of a particular mathematical topic to produce what Dubinsky (1991) terms a "genetic decomposition". He defines a genetic decomposition of a mathematical topic as:

> a description, in terms of our theory, and based on empirical data, of the mathematics involved and how a subject might make the constructions that would lead to an understanding of it. (p96)

Dubinsky uses observations of students learning the concept, the theory and the researcher's own mathematical understanding as the three sources for the genetic decomposition. Other researchers have also drawn on the history of the development of the concept as an additional source. (Sfard and Linchevski, 1994, Harper, 1987). Thus work within this framework provides detailed accounts of particular topics and of the difficulties students can have in learning them. These accounts have been useful in deepening my understanding of the cognitive obstacles that exist within calculus and thus in guiding my exploration of student errors. Thus, in the following discussion of

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research, which I have grouped according to topic areas, I have focused on the sources of difficulty for students that these authors present.

Another recurring theme in the research work on advanced mathematical thinking is the distinction between *concept image* and *concept definition.* A concept definition is the formal mathematical definition. The term "concept image" is used to describe an individual's construction of that concept. Tall and Vinner (1981) say "we shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes". What research (for example Vinner, 1982, and Tall, 1987, on tangents and Tall and Vinner, 1991, on limits and continuity) in this area has highlighted is that there is often a disjunction between the concept definition and a student's concept image. The student may well know the concept definition, but will react to problems using his/her concept image. Although one of the key consequences of this work has been a questioning of the practice of introducing concepts through their formal definitions, this is not the aspect of the work that I will dwell on here. Formal definitions were not emphasised because of the informal nature of the calculus course the students in the present study were enrolled on. However the concept image - concept definition distinction applies equally well in making a distinction between the intended understanding of the concept and the image the student builds up. Thus what I have drawn from this work is some of the empirical results which show the kind of concept images that students can build.

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Given that these two styles of analysis form the backbone of much of the research in this area, I now want to highlight some of the findings that these analyses have produced in relation to particular mathematical topics.

3.2.1 Algebra

Many authors (Sfard and Linchevski, 1994, Usiskin,1988, Harper, 1987 and Kuchemann, 1981) look at developmental stages in algebra, either from a

historical or psychological perspective. Although they differ in the details, they all point to a development from seeing a letter as a specific unknown to functional algebra, where the letter is seen as a variable. The process-object duality of mathematical concepts is particularly salient here. Seeing $2x+2$ purely as a process (multiply by 2 and add 2) can result in students not being able to see its equivalence to $2(x+1)$. Sfard and Linchevski (1994) report on a series of interviews conducted with students of above average ability, between the ages of 12 and 16 years. On the basis of these interviews they attribute students' difficulties in solving an equation like $15x+12=8x+47$ to their inability to see formulae as objects. The students, meeting this type of equation for the first time, made comments about the presence of two "exercises" in the question, were uncomfortable about subtracting $8x$ from both sides because they did not know what $8x$ was equal to or even whether the x's on both sides were the same. Sfard and Linchevski argue that a structural conception is a necessary prerequisite for the comprehension of the solution strategy.

Kieran (1992), in her comprehensive review of the teaching and learning of school algebra, states that

> The overall conclusion that emerges from an examination of the findings of algebra learning research is that the majority of students do not acquire any real sense of the structural aspects of algebra. (p412)

This structural approach is a necessary prerequisite for students to make meaningful use of algebra within calculus. Sfard and Linchevski (1994) argue that because of the inherent difficulty of reification many students develop a pseudostructural approach to algebra. With this approach, where signifiers come to take the place of the abstract objects they represent, algebra becomes a collection of meaningless symbol manipulations. They cite, for example, the difficulty one of the 15-year-old students in their study had in solving the equation $x^2+x+1>0$. The student used the formula for the solution of quadratic equations and pronounced the inequality to have no solution because of the presence of $\sqrt{-3}$. Here, the form of the left hand side

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of the equation provided the method of "solution". These ideas are echoed by Tall and Thomas (1991) who hypothesise

> that as soon as children are unable to give meaning to concepts, they hide their difficulties by resorting to routine activities to obtain correct answers and gain approval (p127).

The largely algorithmic approach found in research on students' understanding in algebra provides students little opportunity for developing versatile thinking in their interpretation of the role of variables in algebra. Many researchers have concluded that developing this flexible interpretation is perhaps one of the most significant cognitive difficulties in learning algebra. Tall and Thomas (1991) argue

> giving meaning to the variable concept and devising ways of overcoming the cognitive obstacles - is fundamental to laying a foundation for meaningful algebraic thinking. (p127)

This idea is repeated by Harper (1987)

> teaching should recognise and attempt to prepare pupils for the various usages of letters which they will need to assimilate. (p85)

Schoenfeld and Arcavi (1988) in their article on the meaning of variable, state

> Understanding the concept provides the basis for the transition from arithmetic to algebra and is necessary for the meaningful use of all advanced mathematics. Despite the importance of the concept, however, most mathematics curricula seem to treat variables as primitive terms that - after some practice, of course - will be understood and used in a straightforward way by most students. (p420)

Usiskin (1988) provides examples of the multifaceted ways in which variables are used. He identifies these as a formula, like $A = LW$, an equation with a single solution, $40 = 5x$, an identity, $\sin x = \cos x.\tan x$, a property, $1 = n\frac{1}{n}$ or a function, $y = kx$. Students require a flexible and

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adaptable understanding of algebra to deal meaningfully with variables across these contexts. The work of Sfard and Linchevski (1994) and Kuchemann (1981) show that this is no trivial task. Kuchemann (1981) developed an algebra test which was used in a large-scale study of pupils' understanding of mathematics. He focuses on the often used interpretation of letter as object, which he exemplifies by citing the practise of seeing $3a + 5a = 8a$ as standing for 3 apples + 5 apples = 8 apples. In his study 52% of the 1000 14 year olds interpreted $8c + 6t$ as standing for 8 cabbages and 6 turnips in response to the question

> Cabbages cost 8 pence each and turnips cost 6 pence each. If c stands for the *number* of cabbages bought and t for the *number* of turnips bought, what does $8c + 6t$ stand for.

The prevalence of this type of interpretation is seen through an array of studies which show that even college students tend to provide answers like $6S = P$ in situations where they are asked to represent the statement "there are six times as many students as professors at the university". (Clement, 1982).

This has particular implications for the teaching of calculus: families of curves mean that students need to deal with both parameters and variables, differentiation and integration rely on a sophisticated notion of relationships between variables, and applied problem solving relies on the ability to translate into and interpret from mathematical formulae.

3.2.2 Functions

The concept of a function is central to modern mathematics (Ferrini-Mundy and Lauten, 1993) and is the key underlying concept in calculus (Vinner, 1992). There has thus been considerable interest in how students acquire the concept and what idea of function they possess.

One of the perspectives from which this has been explored is via the *concept definition - concept image* distinction (Vinner, 1983, Dreyfus and Vinner,

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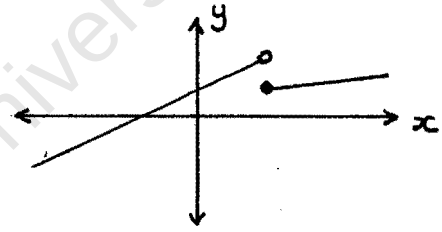
1982, Tall, 1992). The formal definition of function most commonly used is the Dirichlet-Bourbaki approach which defines a function as any correspondence between two sets which assigns to every element in the domain exactly one element in the range. The above researchers have attempted to probe to what extent the concept image of students overlaps with the concept definition. Vinner (1983), studying the conception of function of 146 high school students, used a questionnaire in which four questions asked students to decide whether or not the given examples were functions and a fifth asked them to state what they believed a function to be. Dreyfus and Vinner (1982) did a similar study on high school teachers and college students. Although there were differences between the various groups, in all cases they found that "concept definitions remained very often inactive at decision making moments when (sometimes wrong) concept images took over" (Dreyfus and Vinner, 1982, p17). This research has looked at students' conception of function in terms of the fairly complex and abstract definition. To illustrate, below are two of the questions asked:

i) Is there a function that corresponds to each number different from 0 its square and to 0 it corresponds -1?

ii) Does there exist a function the graph of which is:



From these kind of explorations the researchers highlight a number of ways in which students' concept images often differ from the concept definition. These are:

1. A function should be given by one rule
2. A function should be continuous
3. Functions which are not algebraic exist only if mathematicians officially recognise them by giving them names.
4. A graph of a function should be 'reasonable' i.e. smooth, symmetrical
5. A function is a one-to-one correspondence

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6. A graph of a function is continuous

Tall (1992) argues that when students first meet a concept via the definition, they are inevitably given only a restricted range of exemplars. This can shape their concept images in such a way as to cause cognitive conflict in the future. Thus he argues that the definition of a function is not a suitable starting point for learning about the function concept and that a more suitable starting point would be to draw on ideas that are both familiar to students and able to provide a basis for further mathematical development.

This idea is reiterated by those who base their work on the understanding of the process-object duality of mathematical concepts. Sfard (1992) equates the Dirichlet-Bourbaki definition with a structural approach. She says

> While trying to put our finger on the sources of the difficulties experienced by so many students, we made the conjecture that in some cases the key to the problem may lie in the learners' inability to create for themselves these abstract objects about which the teacher talks with unshaken confidence. To a young student, not only function, but also the more basic notions of set and element may still seem too fuzzy and amorphous to be confidently used and operated upon. (p69).

She then argues that the function concept should be introduced in operational terms, rather than structural terms. In support of this argument she looks at the history of the function concept in mathematics. Malik (1980) and Markovits, Bat-Sheva and Bruckheimer (1986) also draw on history to make suggestions about pedagogy. All of these researchers point out that the earlier definitions of function were far more operational in character. Functions were seen as composed of variables and constants (Bernoulli) or as analytic expressions relating two variables (Euler). Malik (1980) argues that these definitions suffice for a study of elementary calculus. Dirichlet's definition only emerged from his study of the convergence of series and Bourbaki's definition was influenced by the rise of abstract algebra. Sfard (1992) argues that the history of the function concept demonstrates a long struggle for reification and suggests that a similar process is necessary for

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the cognitive development of students. She argues that a structural conception of function should be delayed until the student actually requires it. Malik echoes this sentiment

> The modern definition is algebraic in its spirit. It appeals to the discrete faculty of thinking and lacks a feel for the variable. Whereas, for calculus and other practical sciences the requisite training should enable the student to develop a feel for smooth change of the variables in phenomena. (p492)

Thompson (1994) also makes this point. He distinguishes between function as correspondence and function as covariation. He argues that covariation is an important abstract concept that relies on a good understanding of variable magnitude. Making sense of differentiation and integration in "real-world" situations relies on such an understanding.

Researchers working in the process-object framework have also probed students' conceptions of function through questioning students on their definition of function and asking students to identify whether given examples were functions (Sfard, 1987, Dubinsky and Harel, 1992, Sfard, 1992, Breidenbach et al, 1992, de Marois, 1997). They describe teaching experiments, largely involving the use of computer software and based on the premise that an operational approach should precede a structural one, and look at the effect these have on student conceptions. Of particular relevance to the present study are the discussions that emerge from this research about the importance of a strong process conception and the elaboration of what the consequence are of a pseudostructural approach to the function concepts. These are thus discussed in more detail in what follows.

Dubinsky and Harel (1992) use four factors to analyse the strength of a student's process conception of function. These factors emerged from their analysis of students' work with function and are:

1. Restrictions students possess about what a function is. The three main restrictions are the manipulation restriction (you must be able to perform

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explicit manipulations), the quantity restrictions (functions work on numbers only), the continuity restrictions (graphs of functions are continuous).

2. The severity of restriction.
3. Ability to construct a process when none is explicit in the situation.
4. Confusion between 1-1 and the unique output condition of function.

Dubinsky and Harel argue that a strong process conception will help students overcome the restrictions mentioned above and enable students to construct a process in a variety of function situations. In a related paper, Breidenbach et al (1992) take students through an instructional treatment aimed at strengthening the students' understanding of function as process. In addition to analysing the development of their process conception they look at students' final examination results and argue that

> students' success on this exam supports our theoretical contention that the ability to construct a process conception of function can lead to significant improvement in mathematical performance (p275)

Sfard (1992) describes the pseudostructural approach as one in which the mathematical notion is identified with its representations. She argues that although one should clearly be able to link a mathematical concept with its representations, in a pseudostructural approach a representation becomes the concept. Thus, for many students, a formula is equivalent to a function. The result of this is that students are unable to see functions defined on a split domain as functions, nor are they able to see two formulae as representing the same function even if they differ only in the variable used. With this approach, algebraic manipulations that can be performed on the formula provide the meaning rather than the underlying numerical processes. This lack of awareness of the processes underlying the symbols means that translating between formulae and graphs and vice versa becomes difficult.

The identification of function with its graphical representation presents similar difficulties. In the same way that split domain functions are not seen

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as a single function, discontinuous curves are seen as several functions. In the pseudostructural approach the operational underpinnings of the graph are lost and thus a student's ability to interpret graphical information is impaired. Monk (1992) speaks of a particularly prevalent difficulty in this regard, which he terms iconic translation. He says

> students have an inclination to be over-literal in interpreting the visual information in a graph or its other prominent visual aspects and a real situation that the graph refers to (p176).

Essentially Sfard contrasts the semantically debased idea of identifying function with one (or more) of its representations and the mathematical idea that a function is an object which can be represented in a variety of ways. The mathematical idea develops out of having a clear idea of the processes by which those representations are constructed and she argues that these operational underpinnings are necessary for the translation between different forms of representation.

The difficulty students have in moving between different forms of representation has received a considerable amount of research attention (see for example Dreyfus and Eisenberg, 1987, Yerushalmy, 1991, Kerslake, 1981, Even, 1990, Goldenberg et al , 1992). Eisenberg (1991), Hughes-Hallett (1991), and Tall (1991) all argue that the privileging of the symbolic and downplaying of the visual has lead to a situation where symbolic manipulation is seen by students as what mathematics is about. They state that developing visualisation skills in students can promote conceptual understanding. Even (1990) argues that having the facility to move between multiple representations provides students with flexibility in problem-solving strategies. She argues that multiple representation can be used to build conceptual knowledge. She states

> When dealing with a mathematical concept in different representations, one may abstract the concept by grasping the common properties of the concept while ignoring the irrelevant characteristics that are imposed by the specific representation at hand (p 524).

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Thompson (1994) cautions that it is easy for students to see graphs, tables and symbolic expressions as simply new topics to be learnt in isolation from each other. He argues, in a similar way to Even, that the concept of function is not represented by any of the different representations and that instead it is connections between the representations that need to be foremost in students' awareness.

3.2.3 Limits, rate of change, tangents, differentiation and integration

Although the formal definition of limit was not emphasised in the introductory calculus course for students in the present study, aspects of the research literature on limits provide useful insights into student learning. Davis and Vinner (1986) studied the conceptions of limit of 15 high school students who were in the middle of a two-year calculus course. They claim that written tests and classroom discussions during the first year had shown that students understood the concept of limit in that they were able to prove standard theorems, give correct definitions and provide exemplars to counter incorrect definitions. The test administered during the second year asked students to give their own informal description of a limit of a sequence and then the precise formal definition. Their goal was to probe whether, despite the fact that students did possess the correct idea, they also possessed incorrect naïve ideas. They claim that students, when introduced to a new topic, often possess their own conceptualisations that may conflict with what is expected to be learnt. In addition, when the new topic is learnt, these old conceptualisations may exist along with the new one. They found that the students in their study did indeed have in mind many of the common misconceptions about the limit of an infinite sequence. This resonates with the findings of the work on concept image/concept definition. Of particular interest is their categorisation of the sources of these misconceptions, which I will discuss in more detail in the next section. However, one which finds resonance with other researchers in this area (Tall, 1992, Frid, 1994) is that of the influence of language. The colloquial use of the word "limit" is something that cannot be exceeded, which conflicts with the mathematical

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definition. We talk of "approaching" a limit which in the everyday sense implies the sequence gets close to but does not equal the limit. Frid (1994) argues that everyday language is likely to affect a student's learning and mentions the terms of "limit", "round off", "continuous" and "undefined" as exemplars from her research.

Another two ideas that emerge from Davis and Vinner's categorisation are:

1. mathematical ideas are often built gradually and certain parts of the idea will get adequate representation before others
2. specific examples, particularly those that dominate the early introduction to the concept, will tend to dominate the student's concept image.

Vinner (1982) shows these ideas in relation to the concept of tangent. He hypothesises that many students learn about the tangent to the circle prior to meeting any other notion of tangent. In his study of 278 college calculus students he showed that 35% produced a definition of tangent that mirrored the geometric idea of the tangent to a circle. In addition, despite the fact that 41% of the students had given a correct definition of the tangent, in all cases where students were asked to draw a tangent to a curve where it crossed the curve or touched it at more than one place, less than 18% of the students could draw it correctly.

Bezuidenhout's (1997) study of 100 university students in South Africa shows a similar pattern in students' understanding of rate of change. One of the test items he administered to first-year university students asked them to find the average rate of change of a function $g(x)$ over the interval [0,3] given information in table form about the values of $g(x)$ and $g'(x)$ at eight points in the interval. A number of students equated the idea of average with finding the arithmetic mean and used some form of adding up and dividing by the number of instances to produce their answers. Thus an old idea of "average" dominated many students' approach to average rate of change.

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Another common misconception that Bezuidenhout found in students' responses to this question related to the fact that, because students equated rate of change with derivative, they used $g'(x)$ in their calculations. This notion of linking ideas through partially remembered correct ideas is echoed in Amit and Vinner's (1990) case study. They describe how a student comes to equate the formula of the derivative with the equation of the tangent at a particular point. They offer a number of possibilities as to how this could happen. The first is that the typical drawing used to introduce the concept of derivative is that of tangent. This visual image makes it easy for a student to equate the derivative with the tangent line. The second is explained by an analysis of the language used. They offer the definition of the derivative at a point as "The derivative of a function at a certain point is the slope of the tangent to the graph of the function at this point." They argue that it is hard to memorise and thus some "omission-transformations" take place (p9) which could result in it becoming "The derivative is the tangent to the function at a certain point" .

A final example from Bezuidenhout's work highlights the influence of specific examples, and I believe, the potential influence of language. Students were asked to interpret the equation $S'(80) = 1.15$ given that $S(v)$ gave the stopping distance a vehicle covers after applying brakes as a function of the vehicle's velocity. 98% of the students provided incorrect responses, the majority of whom interpreted 1.15 as acceleration, deceleration or velocity. He hypothesises, on the basis of his interviews with students, that students' early experience with examples which show velocity as the derivative of distance and acceleration as the derivative of velocity played a role here. I would argue that the colloquial use of the word "rate", and the notion of time that accompanies it, interferes with students' ability to give meaning to derivatives (rates of change) that do not involve time.

Studies on differentiation and integration (Selden, Mason and Selden, 1989, Orton, 1983a, Orton, 1983b, Thompson, 1994) tend to show that students cope well with the routine aspects of differentiation and integration, but

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struggle with non-standard tasks and conceptual questions. Dreyfus (1990) comments on the complexity of these notions:

> Differentiation and integration assume an understanding of the concept of function, and the concept of functions assumes an understanding of the notion of variable, which in turn presupposes the number concept. This progression leads to a network of interrelated ideas, each idea integrating some of the more elementary ones into an added structure that in itself may be complex (p115)

Tall (1996) argues that the conceptual difficulties that face a student in learning calculus are likely to result in the student turning to computational and manipulative skills to pass exams. Calculus then becomes about mastering the routine procedures and techniques. Orton (1983a, 1983b), in his studies that explored 110 students' understanding of integration and differentiation through their misconceptions, provides evidence that supports this. Basically Orton found that students' had a satisfactory command of the techniques of integration and differentiation, but found difficulty in areas like seeing the tangent as the limit of a set of secants, the idea of ratio and proportion necessary for understanding rate of change, the understanding of integration as the limit of a sum and the link between integral and area under a curve.

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Tall (1996) argues that this technique-orientated approach to calculus militates against meaningful connections being made. Mundy (as quoted in Eisenberg, 1992), in her study of 973 calculus course graduates, showed that only 5.4% could correctly evaluate: $\int_{-3}^{3} |x+2|\, dx$. 48% of the students gave the answer of 12. This kind of result indicates that the algebraic manipulation took precedence over a visual understanding of integral. The connection between integral and area under the curve, which provides a easy solution to this question, was not made by the students. Eisenberg (1992) notes how much difficulty students have in achieving what he calls reverse-path-development thinking. An example of reverse-path-development thinking is encountered in integrating a function and then differentiating the result. He

argues that this kind of thinking is difficult because it relies on having a "firm foundation of a concept at least in one direction" (p173). He argues that for a number of students the Fundamental Theorem of Calculus lies outside of their understanding and as a consequence differentiation and integration become two separate, unrelated procedures. Thompson (1994), in his teaching experiment with 19 senior and graduate mathematics students, shows that students have enormous difficulties with the Fundamental Theorem of Calculus and that these stem from impoverished concepts of rate of change and inadequate development of the function concept.

3.3 Categorising misconceptions

The study of students' misconceptions in mathematics has attracted a large number of researchers (see Confrey, 1990 for a review of this work). Much of this work is premised on a constructivist view of student learning which argues that a careful study of students' misconceptions will provide insight into how students learn. Many of the papers reviewed in this literature survey have looked at students' misconceptions in relation to particular topics (limits, functions, tangents etc.). Within my research project my aim was to use errors that students made in dealing with a variety of questions in calculus as a whole to probe the conceptual barriers they face. To do this I needed to find a useful way to categorise the students' errors. Thus, in this section, I do not attempt to provide a broad overview of the field of misconceptions research, but instead focus on the type of categorisations other researchers have used.

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Orton (1983a), in looking at students' understanding of integration, first classified the errors in terms of the branches of mathematics involved (e.g. basic algebra, limits). Recognising that this led to too much detail, he then, although retaining a sense of the topics involved, simplified the classification system. He used the types of errors described by Donaldson (1963): structural, arbitrary and executive. Structural errors are those that arise out of a failure to grasp some fundamental concept which is key to the solution of

the problem or to perceive the relationship between elements in the problem. Arbitrary errors are those in which the student appears to have acted arbitrarily, disregarding given information. Executive errors are errors in manipulation. He points out that using this system to classify errors in an advanced mathematical setting, with older students, proved difficult in that it was not easy to distinguish between arbitrary errors and the other two types of error and that certain errors involved elements of more than one type. He uses the same type of classification process to probe students' understanding of differentiation (Orton, 1983b). This process led, in both cases, to a very detailed account of students' problems in calculus.

Movshovitz-Hadar et al (1987) developed a category system for exploring errors in high school mathematics. In the first part of their study they analysed the errors in a qualitative manner that they called "constructive analysis" (p5). This involved attempting to answer the questions "To what question (or questions) is the wrong solution a right answer? What logic can justify what the student in fact did?" (p5). In the process of this analysis they began to develop clusters and finally categories that were then subjected to empirical testing. They present six descriptive categories of errors as a model for classifying errors in high school mathematics:

1 Misused data
   adding in or neglecting data
   stating explicitly as a requirement something not required by the problem
   using, for example, the value given for distance for velocity
   assigning properties that are not given
2 Misinterpreted language
   translating from English to mathematics incorrectly
   using the wrong mathematical symbol
   incorrectly interpreting graphical symbols
3 Logically invalid inference

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assuming that either the converse or contrapositive of a condition holds

incorrect use of quantifiers

omitting the steps for logical inference

4 Distorted theorem or definition

imprecise recall of a theorem or definition and its conditions

applying a distributive property to a non-distributive operation

5 Unverified solution

a solution that could easily be made correct if checked against the requirements of the question

6 Technical error

computational errors

elementary manipulation errors

This model is designed to be used across mathematical topics. They argue that "grouping items by the typical errors they yield and investigating common features of items in each group may prove indicative of errors expected from similar items" (p14).

Radatz (1979) also argues that one can identify sources of errors that can be used across mathematical topics. He says "Various causes of errors that cut across mathematical content topics can be identified by examining the mechanisms used in obtaining, processing, retaining and reproducing the information contained in mathematical tasks" (p164). The categorisation he develops is thus related to an information-processing approach. (Confrey, 1990). He identifies four causes of error:

1 Errors due to difficulties in obtaining spatial information.

2 Errors due to deficient mastery of prerequisite skills, facts and concepts.

3 Errors due to incorrect associations or rigidity in thinking. Based on the work of Pippig he refines this classification to include a number of subcategories:

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Errors of preservation in which a single element of the task predominates

Errors of association involving incorrect interactions between elements

Errors of interference in which different concepts and operations interfere with one another

Errors of assimilation in which incorrect hearing causes mistakes

Errors of negative transfer from previous tasks in which ideas from a previous set of exercises are erroneously carried over into the task at hand

4 Errors due to the application of irrelevant rules or strategies

Davis and Vinner (1986) in their study of the notion of limit described earlier come up with five sources of what they call naïve misconceptions. Despite the fact that they were working with the concept of limit and Radatz's original focus was on the procedures of arithmetic there is considerable overlap between their categorisations. Davis and Vinner's categories are:

1. Language. The everyday meanings of terms used in mathematics provide misleading cues about the concept
2. Assembling mathematical representations from pre-mathematical fragments. Many of the mathematical ideas involved in a concept have their basis in experiences prior to mathematics. In a similar way to language, this everyday experience of the concept can provide erroneous ideas about the mathematical concept.
3. Mathematical ideas are built gradually. At various stages some parts of the representation will dominate.
4. The influence of specific examples. Certain examples dominate the students' image of the concept.
5. Misinterpreting one's own experience. Through experience of various exemplars, students may come to extract as an invariant property of the concept something that does not apply in all cases.

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Matz (1982) offers a process model for high school algebra errors. This account puts forward the idea that a student's problem-solving behaviour is based on two components: the base rules which are the knowledge presumed to precede the problem and extrapolation techniques which allow one to apply known rules to new problems. She also argues that because students' understanding of algebra is built onto their base knowledge of arithmetic, studying the conceptual changes needed in the transition can allow us to predict the kind of errors students will make. She puts forward three main categories of error:

1. errors generated by an incorrect choice of extrapolation technique
2. errors reflecting an impoverished (but correct) base knowledge
3. errors arising during the execution of a procedure

The way in which she uses this model is a satisfying one for although she does not attempt to catalogue frequencies of errors she uses the model to present possible sources of the misconceptions that arise and thus highlights key differences between novice and expert understandings of algebra. It seems to me that this kind of holistic view offers more in the way of potential insight into students' conceptual understanding and thus I would like to explore it in greater detail.

Under the heading of extrapolation techniques she offers three subheadings:

1) Applying a rule:

   Here she argues that although experts see rules as patterns or schemas for replacement and are comfortable substituting an expression for x in some pattern, novices tend to insist on literal-for-literal correspondence and thus often compromise the rule in complex situations by using embedded matching e.g. $\frac{ax+by}{x+y}=a+b$

2) Linearity

   An operator is employed linearly when the final result of applying it to an object is gotten by applying the operator to each subpart and then straightforwardly combining the partial results (Matz, p29).

   Linearity errors would include examples like

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$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ or $\frac{1}{3} = \frac{1}{x} + \frac{1}{7} \Rightarrow 3 = x + 7$

She argues that the assumption of linearity is probably based on its extensive use in arithmetic and at the students' introduction to algebra.

3) Generalisation

A generalisation error involves the incorrect application on the premise that the actual numbers used are arbitrary.

She points out that experts will attempt to adapt a problem so that they can apply the relevant rule, novices tend to revise the rule to fit the problem.

In looking at the impoverished (but correct) base knowledge she highlights some conceptual changes necessary in the transition from arithmetic to algebra:

1) Symbolic values

The main transition here is from arithmetic where you have clear numerical answers to dealing with the notion of operations on a variable

2) Notation

This would include the syntax of algebra and confusions caused by variables and parameters.

3) Equality

The notion of equality and the subtle differences in its use as a command to solve an equation, give a numerical result, or provide an equivalent expression is a conceptual change that is hard for many students.

In looking at executive errors she highlights both planning and processing errors. Planning errors occur when procedures are applicable, but not productive and processing errors are slips that are made by experts and novices alike.

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Although there is some overlap between the ideas for categorisation that all these authors put forward, there appear to be three distinct trends. Orton (1983a, b) uses only the three broad categories to describe the sources of error. His analysis probes the instances of these errors on a question-by-question basis. Although his work provides an impressive level of detail about the errors students made on his calculus tests, his broad categories (executive, arbitrary and structural) seemed insufficiently delicate for the aims of my study. I felt that in order to compare the categories of error with the learning theories a finer clustering than executive, arbitrary and structural was needed. Movshovitz-Hadar et al (1987) and Radatz (1979) propose a categorisation scheme that can work across mathematical topics and identifies sources of errors by looking at how information is processed and used by the student. Davis and Vinner (1986) and Matz (1982) also categorise the errors in terms of how information is processed by students. However their work is tied to a particular mathematical topic and thus the conceptual underpinnings of the tasks and the mathematical demands of the content area play a role in their choice of categories. The work of Movshovitz-Hadar et al, Radatz, Davis and Vinner, and Matz were all influential in the process of developing my coding scheme for student errors which I will outline in chapter 5.

Within this survey of literature I have looked at challenges facing us in the teaching and learning of calculus and at the insights into student learning of calculus and pre-calculus topics that have been generated through research. Of particular concern to me in reviewing the literature was the paucity of South African research about mathematics education at tertiary level. This research report thus draws on the learning theories, empirical findings and methodologies of the research reported in this chapter and makes use of them to probe students' learning in calculus in the South African context.

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CHAPTER 4 THE RESEARCH DESIGN

This research project had two broad aims. The first of these was to uncover the misconceptions that students from disadvantaged educational backgrounds display in dealing with calculus concepts. The second was to explore whether theories of student learning provided by Sfard and Dubinsky could provide a framework with which to understand these misconceptions and thereby deepen our understanding of student learning. Balancing these two aims was a crucial factor in the design of the research project. The first aim was tied to the local context. The motivation behind it was the need to build a picture of the type of difficulties experienced by disadvantaged students enrolled on the academic development calculus course at UCT in order to be able to make informed judgements about improvements to the design of that course. The second aim had a more general component to it. The desire to view the misconceptions through the lens of established theories of learning necessitated a change in focus from the specific (what do the students of this specific sample find difficult?) to the more general (what do theories of learning imply that students of calculus from similar backgrounds in different contexts might find difficult?).

In designing the research project I wanted to ensure that I did not, in locating the research within the ambit of particular theories of learning, pre-determine the kind of misconceptions that could occur through the choice of research instruments. Furthermore, although focusing on students' errors, I wanted to avoid simply producing a list of these. In reviewing the literature on misconceptions I found that those that extended the analysis of student errors by categorising them according to an underlying cause or strategy (for example, Radatz, 1979 and Davis and Vinner, 1986) provided useful information about how errors might be generated. This indicated that there was potential, through the coding of errors into categories, to provide a description based on these errors that could be articulated with the theoretical framework. For these reasons the research design needed to be

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kept relatively open to allow for a dialogic process to occur between empirical data collected on students' errors in calculus on the one hand, and the theoretical framework on the other. Data collection was thus aimed at uncovering the errors made by students on the academic development calculus course. The data analysis was aimed at providing an articulation between the empirical and theoretical fields by categorising the students' errors on the basis of common underlying misconceptions and analysing these in relation to the theoretical framework.

4.1 A brief overview of the design of the research project

The data on student errors was collected from four sources. The primary source of data was the final examination scripts of all 117 students enrolled for the academic development calculus course. This data was supplemented by scripts of 78 of these students who wrote a conceptual test designed specifically for the purposes of this research project. The data from the test was augmented by:

a) the recorded discussion of four students from the course who worked collaboratively through the test questions.
b) follow-up interviews with four groups of students.

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4.2 The selection of research instruments and data collection

4.2.1 The final examination

Students' solutions to the final examination questions were taken as the key source of data used to generate a list of errors and to extract the underlying misconceptions. The choice of the final examination as the key data source was motivated by the fact that the examination was the major determinant of a passing or failing grade for students on the course and encompassed all work studied over the year. In addition the examination was designed to cover a broad range of content and skills. The examination was designed by three members of staff of the Department of Mathematics and Applied Mathematics (including myself). In the design of the examination no

reference was made to the fact that it would be used as a research instrument and the other two members of staff were only asked for their permission to use it as such after the examination was completed. This was done in order to ensure that the design of the examination was not unduly influenced by the awareness that it would opened to scrutiny as part of a research project. As required by the standard UCT examinations policy, an external examiner, from Stellenbosch University, was asked to moderate the examination with respect to its the standard and scope. The examination was thus designed to reflect the competencies which staff believed the students should have developed during the calculus course and was similar in style to the five tests that students had written during the course of the year. I believe that these factors address the issue of validity of the instrument in terms of assessing the misconceptions and difficulties students have with this specific calculus course.

The examination was written by the students as part of the normal UCT examinations session. This meant that after the end of lectures students were given a week of study leave before writing examinations in all their subjects (usually four subjects) over a period of about three weeks. The Mathematics examination was their first examination. The examination was marked by a team of lecturers and tutors involved with the course with one of the lecturers monitoring the team's marking for consistency. The external examiner checked 10% of the scripts and reported the marking to be fair and accurate in his opinion. The students' marks reflected in this research report were taken from the official lists produced through this process. The analysis of student errors was undertaken after this process was completed.

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4.2.2 The conceptual test

The majority of students on the course live a considerable distance from UCT and return home for a three-month holiday on completion of the course and examinations. This, together with the fact that UCT's exclusion policy means that students failing more than two of their first-year subjects are

unlikely to be allowed to return to the university, made it impossible to carry out follow-up investigations with the students after the examination. In addition the examination was not designed as a research instrument and the type of questions asked in the examination were constrained by the fact that the examination provided the major part of the students' grade in the course. It was for these reasons that it was necessary to supplement the exam data with additional data gleaned from a conceptual test, specifically designed for the purpose of the research. As this test was to be administered before the end of term it could be followed up with interviews with students. The examination was to be used as the key source of data in the process of developing the coding scheme in relation to the learning theories. The conceptual test and interviews that followed it were to be used to validate the coding scheme and to deepen and enrich the explanation generated by the analysis of the exam.

Criteria used in the development of the test:

1. It was decided that the conceptual test would focus on graphical representation. Graphical representation and the interpretation and development of calculus concepts through graphical representations had played a large role in the course. Many authors (for example, Tall, 1996, Sfard, 1992 and Selden, Mason and Selden, 1989) have argued that symbolic manipulation becomes the focal point of students' experience of calculus, something they routinely apply and which can mask underlying conceptual difficulties. The focus on graphical representation in the test was intended to avoid this problem. In addition, I believed that in asking students to provide a graphical equivalent of a symbolic expression, the underlying interpretation given to the concept could be discerned.
2. As the aim of the research project was to develop an understanding of students' difficulties in calculus broadly, the test was designed to cover a range of basic calculus concepts. These were functions (and function notation), the derivative at a point and the idea of slope that underlies this, the derivative functions, the definite integral and the area notion that underlies this, and the relationship between the integral and derivative.

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3. The test was designed to contain questions of varying degrees of difficulty. Questions such as representing $f'(\frac{\pi}{3})$ on the graph of $y = \sin x$, which might be considered very easy questions for students nearing the completion of their first course in calculus were deliberately included so that very basic concepts were not assumed, but tested. Varying levels of difficulty in the framing of questions also allowed the same concepts to be tested repeatedly in slightly different forms. This also facilitated the comparison of different questions testing the same concept in the analysis of the students' solutions to the test. Questions 1 and 2 in the conceptual test examine the same concepts, but in question 1 this is achieved in the context of a well-known function ($y = \sin x$) and in question 2 this is posed in the context of a function that is only represented graphically. Question 3 asked students to draw the derivative function on the basis of a sketch of the function and question 4 asked students to reverse this process (in the context of a different function). Question 5 tested similar concepts to question 4, but was embedded in contextual information.

A first draft of the test was given to the three lecturers teaching first-year calculus courses at UCT. These lecturers were asked to comment on the standard of questions in the test and on the clarity of the questions. Their feedback suggested that the test provided a broad range of questions in terms of level of difficulty and only minor amendments to the final draft of the test were needed to improve on the clarity of the questions.

The test was administered to students during their regular weekly workshop in the second last week of term. The students had been told the previous week that they would be writing a test that would cover the main concepts learnt during the year. The purpose of the test was described to students as both a useful revision exercise for themselves in preparation for the exam and as part of research on students' difficulties with calculus that I was conducting. Students were given the option of withdrawing their work from inclusion in the research project, but no student took this up. Although attendance at the workshops is theoretically compulsory, towards the end of

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the year students, recognising that they are unlikely to be barred from the course for missing a couple, often miss one or two. The workshop in which the test was written was attended by 78 students, all of whose scripts were used in the analysis. Tutors and a lecturer were available at the workshop and answered any student's questions of clarification about the test questions.

4.2.3 The group discussion and interviews

Recognising that the data from both the examination and test would be based on the final answer that students recorded and that these might not be supported by a clear indication of the working that led to that answer, I decided to supplement this data with other forms of data that could provide more insight into the problem-solving processes they used. The constraints posed by the examination meant that these would have to be taken up in conjunction with the test.

A group of four students were asked to work collaboratively on the conceptual test so that I could observe how they set about answering the questions. These students were not selected randomly, but were chosen on the basis of the fact that they had worked collaboratively on tutorials throughout the year. I had observed lively discussions and debates between these students over the year and therefore was confident that they were comfortable discussing mathematics amongst themselves, even in my presence. I was aware that in choosing these students I was incorporating potential bias. However I believed that in order for a group discussion to provide useful data, the students in the group needed to be able to work as a group and feel free to offer opinions and contradict each other. As the group discussion was intended to reflect students' discussion, with no intervention from teaching staff, I decided against a randomly selected group which might have necessitated the intervention of a facilitator. In addition, the group chosen contained two students (Arthur and Koki) who were in the bottom third of the class, one (Simon) who was in the middle third and one (Palesa)

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who was in the top third of the class and thus I was satisfied that their lively debates were not as a result of a uniformly high level of competence amongst the group. This group was asked to write the test together in my office at the same time as the other students were writing the test. They were asked to discuss the questions on the test and agree on a solution for each question. Although the discussion was recorded on tape, I sat behind the group and took notes of their discussion and written work in order to ensure that the transcript of the tape would be intelligible.

Although I believed that the group discussion could provide useful insights into the type of reasoning behind some of the errors that students made in the test, I also recognised that this data would be limited by the fact that it could only reflect the way in which these four students approached the problems. The strength of the group discussion lay in the fact that the conversation was not guided by the researcher and thus the process of arriving at solutions was influenced only by the interactions of those four students. The weakness of the group discussion lay in the fact that a non-interventionist approach was taken; I was unable to probe interesting statements or erroneous approaches made by the students. Because of this I decided to conduct follow-up interviews after the test was written which would allow me to explore more directly common errors made by students on the test.

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The process of setting up and conducting the interviews was constrained by the consideration that the students were approaching their final examination period. This made it necessary to do an initial, quick analysis of the students' errors on the test in order to simply draw out common errors for further exploration. In addition a number of practical examinations in other subjects had been set for the last week of term which meant that I was forced to fit all the interviews into one afternoon. For these reasons the scope of the interviews was necessarily limited.

In selecting students for the interviews, I ranked students according to their scores on the conceptual test and split this ranking into quartiles. I then randomly selected 3 students from each quartile. Each of these 4 groups of students was then allocated a 40-minute time-slot for a group interview. Unfortunately two students did not arrive for their interviews and thus two of the groups consisted of only two students. The motivation for conducting the interviews in groups was twofold: The dominant mode of interaction during the course had been between staff members and groups of students. For this reason I believed that, in replicating this in the interviews, the interview situation would be less artificial and less intimidating to the students. In addition, the group discussion had shown that the interaction between students and their attempts to explain their answers to each other produced useful information.

The aim of the interviews was to provide additional information about common student errors on the test and to aid the process of understanding the underlying cause of these errors. This determined the selection of questions for the interview. The initial analysis of the errors made on the test generated three broad categories of errors

1. Errors made in representing basic calculus concepts graphically.
2. Errors which occurred in the context of an otherwise correctly answered questions.
3. Errors which appeared to be the result of an inability to make connections between two concepts.

The time constraints of the interview meant that I was unable to probe all errors and thus I selected examples from each of the three broad categories. The nature of the errors determined the form in which the questions were posed:

1 Errors in representing basic calculus concepts resulted in three types of interview questions:

In some cases a surprisingly large number of students made errors in providing the graphical representation of a basic calculus

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concept. In considering the errors it was difficult to conjecture what the cause of these errors might be. In these cases the test questions were simply repeated with the intention of probing erroneous answers provided in the context of the interview. The questions selected were to represent $f'(2), f(\frac{5\pi}{6})$ and $\frac{f(a+h)-f(a)}{h}$ on a given graph.

In other cases, although it was again difficult to conjecture about the possible causes of the errors, there were clearly a few dominant errors that occurred in that test question. In order to explore these errors I offered these erroneous answers to the students together with the correct answer and asked them to explain which of them they felt was correct. This was done with the question which asked students to represent $hf(a+h)$ on the graph of $f(x)$.

In cases where it was possible to make conjectures about the type of difficulties students experience, this conjecture was tested through asking a series of related questions. For example, students' understanding of average value was probed in the interviews by first exploring their understanding of averages.

2 Errors that indicated a particular area of difficulty in a question were probed by asking students to solve a question which focused on that area of difficulty. Thus the difficulty students had with a point of inflection in the graph of $f(x)$ when asked to draw the graph of $f'(x)$ resulted in an interview question which asked students to draw the graph of $f'(x)$, given a graph of $f(x)$ where the only feature was a point of inflection.

3 As one of the key connections in calculus is the relationship between the integral and derivative I decided to focus on the errors that appeared to indicate a difficulty in this area. Three questions, each focusing on different aspect of the relationship were asked. The first asked students to represent $\int_{\frac{\pi}{3}}^{\frac{\pi}{3}+h} f(x)\,dx$ on the graph of $f(x)$ and then

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to represent $F(\frac{\pi}{3}+h)-F(\frac{\pi}{3})$ on the graph of $f(x)$, given that $F'(x)=f(x)$. The second asked them to evaluate $\frac{1}{h}\int_{a}^{a+h} f'(x)\,dx$ and the third asked them to draw the graph of $f(x)$ based on the graph of $f'(x)$.

The questions were posed to the group as a whole. Where necessary I facilitated discussion by asking the students whether they agreed with an answer given or to explain their answer to the others in the group. The interviews were semi-structured in the sense that the question outlined above provided a framework for the interview, but there was scope for me to follow up interesting points raised by the students in discussion.

The interviews were tape-recorded and transcribed in full. Any notes or written answers provided by students during the interview were collected and used along with the transcriptions.

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4.3 Evaluation of the research design

Although the research instruments selected yielded useful data and served the aim of the research project well, a retrospective review of the research design offers possibilities for potential improvement of the design for future research projects of this nature.

Basic concepts were tested in the conceptual test by asking students to provide a graphical representation of them. The analysis of errors revealed that a number of these basic concepts were incorrectly perceived by a surprisingly large percentage of the students. The errors students made on these questions were particularly difficult to analyse and it would have been productive to probe these basic concepts more extensively in the test. I believe that asking students to provide a verbal interpretation in addition to the graphical representation would have facilitated the analysis of these

questions. Furthermore, students' tendency to record only their answers in the final two questions of the test suggests that it would have been better to ask them to motivate their answers.

Ideally the examination should have been followed up by interviews. As explained above, practical constraints prevented me from being able to do this immediately after the examination. In addition I believe the most useful time to have carried out the interviews would have been once an initial analysis of the examination had been conducted in order to probe patterns emerging from the errors in depth. However the fact that an examination is extensive in scope and length and is not structured to facilitate the analysis of errors means that even an initial analysis takes considerable time. Since examinations are mostly written at the end of a specific course, it is possible that students who wrote the examination might not be accessible to the researcher after this initial analysis. There are two possible options for overcoming this problem. The first is to use a test written earlier in the year as the key research instrument. This would allow time for follow-up interviews with the same sample of students. The drawback of using such a test is that it would not be as extensive in its coverage of topics as a final examination. The other option is to conduct interviews with students who enrol for the course in the following year at the end of that year. However the ethics of being both teacher and researcher mean that one would be unable to ignore the trends emerging from the analysis of the examination error data in teaching the course to the next cohort of students. This would affect student responses.

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CHAPTER 5 THE ANALYSIS OF STUDENTS' ERRORS IN THE EXAMINATION

5.1 Developing the coding scheme

In setting up the research project the decision was made to use the students' errors in the final examination as the key source of data. This decision was motivated by the desire to base the analysis on errors that students made in the course, MAM105H, as it stood and not to limit the errors that could arise by choosing a research instrument designed to probe aspects suggested by the theoretical framework. In so doing there was a recognition that the initial form of the data would be an unstructured list of errors. Thus the challenge in analysing the data was to build a categorisation of these errors that would allow the empirical data on students' errors to be articulated with the theoretical framework.

Miles and Huberman (1984) offer a hybrid inductive-deductive approach to the creation of a coding scheme for qualitative data. In this they suggest that the theoretical framework be used to provide broad areas into which the data can be divided. The codes within these broad areas are then developed inductively from the data. Brown and Dowling (1998), in their discussion of network analysis, argue that the development of analytic categories should be conducted via "a dialogic process which involves moving between the empirical and theoretical fields" (p69). In this way the empirical data is constantly compared with the theoretical framework and the utility of the theoretical framework in describing the data is tested.

The approaches discussed by these three authors suggested a framework for the method of coding and analysis of the students' errors used in this study. The starting point of the analysis would thus be extracting from the theoretical framework broad areas in which misconceptions could arise.

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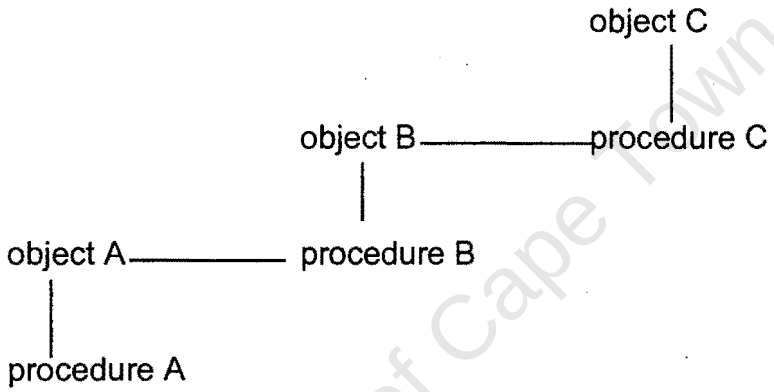
Thereafter the students' errors would be grouped according to common features. These categories would be then examined against the areas suggested by the theoretical framework.

5.1.1 Broad areas suggested by the theoretical framework

Sfard and Linchevski (1994) put forward the idea that the development of mathematical concepts involves a transition from operational to structural conceptions which they represent as follows:

Diagram 2:



A broken development chain would occur when, for example, object B has not been reified. They argue that "once the developmental chain has been broken the process of learning is doomed to collapse" (p220). As calculus relies on a versatility with algebra, it thus seemed necessary to classify separately those errors which originated from a poor conceptual understanding of algebra. Furthermore Sfard and Linchevski's discussion of the difficulties students have in the transition from algebra of a fixed value (where the letters simply stand in place of a number) to functional algebra (where the letters are regarded as variables) suggested that in categorising students' errors particular attention should be paid to their interpretation and use of symbols.

The processes through which students construct mathematical knowledge is central to the learning theories outlined in chapter 2. These learning theories suggest that when knowledge is constructed appropriately a rich web of

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relationships between coherently organised schemas develops. This suggested that in coding the students' errors in calculus it would be important to explore whether these errors provide evidence of the way in which students build their knowledge and the links students make between topics.

5.1.2 Grouping the errors

The examination scripts of the 117 students who wrote the examination were scrutinised question by question and the errors made in each question listed. In order to begin grouping the incredibly long list of errors generated through this process I developed a systematic method of probing the errors for underlying similarities. This method was born out of the necessity to provide more manageable (in terms of size) sets of data to analyse. These data sets were created by grouping questions in various ways and extracting the errors made on those questions.

The analysis was thus conducted in four phases. In the first phase I ranked the questions on the examination according to the percentage of students who had correctly answered the question. This ranking was used to compare the errors made on questions covering the same content where the facility values for those questions differed widely. The question "What made the one question more difficult than the other?" provided the starting point for analysing errors in this phase.

In the second phase this ranking was used to explore the errors made on questions which less than 50% of the students had answered correctly. The fact that the majority of students found these questions difficult suggested this group of questions would bring to light a number of common errors.

In the third phase the average mark of the students in the top, middle and bottom thirds were calculated for each question and those questions where the difference between the average mark of students in the top third and

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students in the bottom third differed by more than 30% were analysed in detail. This was necessary in order to uncover error patterns in the work of students in the bottom third.

In the final phase the groupings suggested by the analysis of errors in the first three phases were refined or clustered by reference to both the theoretical framework, the original list of errors and categorisations suggested by the work done by other researchers in the field of students' misconceptions.

Phase 1

Table 1 shows the percentage of students who got each question correct and ranks the questions in terms of this facility value.

On the basis of this table, the difference between the errors committed on questions covering the same content area where the facility values differed widely were explored. The comparison of these questions allowed me to detect the sources of error by exploring what made one question more difficult than the other. Thus, for example, question 6a and question 6b both required students to use integration by substitution. However only 21% of students managed to answer question 6b correctly, whereas 69% of the students had succeeded with question 6a. Question 6b required students to perform the manipulation $\frac{1}{x^2+2x+1} = \frac{1}{(x+1)^2}$ before they could proceed with the integration. The errors students had made in this question indicated two main sources of error: the first involved difficulties with algebraic manipulations (for example $\frac{1}{x^2+2x+1} = x^{-2} + 2x^{-1} + 1$) and the second were where the students simply gave the answer as $\ln(x^2+2x+1)$, a form of pattern matching with the rule $\int \frac{1}{x} dx = \ln|x|$. Having identified these errors I then searched the rest of the list of errors to examine whether there were other algebraic manipulation and inappropriate pattern matching errors.

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Table 1: Ranking of examination questions in terms of percentage of students who correctly answered the question

Question	Topic	% correct	% left out
Qu 14	sigma notation	0	27
Qu 17c	volume of solid of revolution	1	39
Qu 13	find maximum possible area	3	16
Qu 18c	sigma notation in context of integrals	3	21
Qu 18e	sketch the antiderivative graph	3	15
Qu 11	related rates	5	30
Qu 12	prove	5	7
Qu 6d	integral (must manipulate in order to use table of integrals)	6	8
Qu 5f	relate area under curve to antideriv.	11	6
Qu 18d	error in Riemann sums	18	37
Qu 2a	find equation of polynomial graph	21	3
Qu 6b	integral (must manipulate in order to make substitution)	21	16
Qu 18f	explain trapezoidal rule for approximating integral	21	19
Qu 5e	identify point of inflection of antiderivative	24	6
Qu 8d	differentiate (chain rule).	24	3
Qu 11b	make inferences about rate of change from context	26	44
Qu 9b	inequality with absolute value	26	11
Qu 1b	find tangent line at x=c	30	5
Qu 5d	identify local maximum of antiderivative	30	3
Qu 9c	discuss differentiability of absolute value graph	31	16
Qu 2b	find equation of trig graph	34	6
Qu 16	integration by parts	34	3
Qu 9a	sketch absolute value graph	36	6
Qu 5b	state where antiderivative is concave down	39	2
Qu 15	binomial theorem	46	3
Qu 5a	state where antiderivative is increasing	47	1
Qu 7a	differentiate using Newton Quotient	47	2
Qu 17b	find area between curves	47	12
Qu 4	sketch derivative function	48	0
Qu 18b	use error in Riemann sum to find value of $f(x)$	51	9
Qu 5c	identify local min of antiderivative	52	3
Qu 6c	integration by parts	55	2
Qu 3b	solve using logs	68	1
Qu 8c	differentiate (quotient & chain rules)	68	2
Qu 18a	find f(0) given area of 1st rectangle in Riemann sum	68	6
Qu 6a	integrate using simple substitution	69	8
Qu 1a	find tangent line at x =1	71	2
Qu 8b	differentiate (chain rule)	71	1
Qu 17a	sketch area between polynomial graphs	72	3
Qu 10b	find equation of tangent line to ellipse	73	0
Qu 10a	sketch ellipse	74	0
Qu 3a	find formula, given half-life	77	1
Qu 8a	differentiate (product rule)	87	1

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Continuing this process through all possible pairings[3] of questions generated a number of groupings of errors which were refined during the final phase and incorporated in the coding scheme.

Phase 2

In this phase the focus was on those questions that less than 50% of the students had answered correctly. Looking at each of these questions (apart from those that had been analysed with a paired question), I worked to ascertain from the students' errors those factors which rendered them difficult and then investigated whether there were other instances of similar errors in the remainder of the students' work. Thus, for example, question 12, which only 5% of the students answered correctly, highlighted the difficulty that students had in producing a logical argument. Searching through the rest of the errors yielded a number of other instances where students' difficulties in providing a justification was the key error. For example, students' motivation in question 5f for why $f(e)$ would be negative yielded responses like "because it will lie below the x-axis". Question 18f asked students to explain the trapezoidal rule for approximating definite integrals, and was surprisingly poorly answered. The main errors in this question were of three kinds: the first confused the trapezoidal and midpoint rule, the second restricted the explanation to commenting only on whether the trapezoidal rule produced an under- or over-estimate of the integral and in the third case students simply gave the formula $TRAP(N) = \frac{LEFT(N) + RIGHT(N)}{2}$. The first type of error appeared to be a case of confusing two similar ideas. Because the majority of questions encountered by students in lectures about the trapezoidal rule had focused on whether it under- or over-estimated the integral I hypothesised that the second type of error was a case of allowing well-rehearsed examples to dominate the approach to the question. The third type of error highlighted a dominance of memorised formulae. As before, the list of errors was searched to extract other instances of these types of errors. Repeating this process

[3] These pairings were questions 1a and 1b, 4 and 5d,e, 6a and 6b, 6c and 16, 8b and 8d

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through all the questions on which the students performed poorly expanded the initial grouping of errors created in phase 1.

Phase 3

As the procedures in phase 1 and 2 were based on identifying the most frequently committed error for each of the questions probed, a large number of errors remained unclassified. I anticipated differences in the kind of errors made by the top third of students and the bottom third of students in the course. Thus I decided to use questions that the bottom students found significantly more difficult than the top students to probe the errors further. In order to do this I decided to divide the students into three groups: top, middle and bottom on the basis of their total score for the examination.

Table 2: Summary of examination results

	top third	middle third	bottom third	all students in MAM105H
average mark	61%	47%	34%	46%
range of marks	53%-85%	41%-52%	18%-40%	18%-85%

Table 2 shows that the range of marks scored by both the top and bottom groups is large. However the top students' marks fall in the passing range ($\geq 50\%$) and the bottom students' all fall within the failing range ($<50\%$).

On the basis of these divisions I constructed table 3 which consists of all questions in which there was a difference between the average marks of top and bottom students which was greater than 30%.

Table 3: Questions where there is a difference of >30% between the average marks of the top and bottom third of students:

ave marks	1b	5a	5b	5c	5e	6a	6c	8d	9a	9b	9c	10b	16	17a	17b	18a	18f
bottom	34	26	27	15	10	51	41	31	30	6	21	60	28	46	30	49	21
middle	59	41	38	59	26	74	72	55	52	27	31	74	60	78	49	67	38
top	65	79	72	79	41	83	93	64	75	54	56	92	70	91	71	87	60
(top - bot)	31	53	45	64	31	32	52	33	45	47	35	32	42	45	41	38	39

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In phases 1 and 2 I had extracted the most frequently committed errors across all groups of students. In phase 3 I focused on the errors most frequently committed by the bottom third of students. This yielded further categories of error that had not been uncovered by considering only the key errors of all students. Thus, for example, in question 16 where students had to calculate $\int x \arctan x^2 dx$ many bottom-third students used a substitution to convert it to the form $\frac{1}{2}\int \arctan w\, dw$ and then simply concluded that it would be equal to $\frac{1}{1+w^2}$. The dominant error that had been picked up by probing the sources of error in this question across all students had been the omission of the chain rule in differentiating $\arctan x^2$. Thus a focus on the key source of error of the bottom third of students yielded an additional error category and I then searched the rest of the data to extract other instances where the processes of integration and differentiation were confused. Certain questions in table 3 yielded no additional categories. For example, in question 18f, the main sources of error for the bottom students were those identified in previous phases of the analysis.

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Phase 4

Integrating the information yielded on the basis of both the ranking of questions in phase 1 and 2 and the differentiation between top and bottom students in phase 3 provided the initial grouping of errors. The process from this point to the development of the eventual coding scheme was one of first refinement and then clustering. The refinement involved analysing the grouped errors to see if there were finer distinctions that could be made. Thus, for example, errors that had been grouped on the basis of the key source of error being the notions of variable and constant were divided into two groups: those in which a particular letter is endowed with an invariant meaning and those in which variables and constants are confused with one another. This process of refinement was aided by a review of the categories used by others in research on students' misconceptions. In particular the work of Davis and Vinner (1986) suggested a separate category that looked

at the effect of colloquial meanings on mathematical terms and Radatz (1979) offers the distinction between interference of elements in the question, interference between different concepts and errors that results from negative transfer from previous tasks.

The clustering of categories was informed by the theoretical framework. On the broadest level, I used the two categories suggested by the theoretical framework: pseudo-structural approach to algebra and building calculus. There were however errors that did not fit into these two categories. Apart from those which were apparently idiosyncratic or represented "slips", there appeared to be two other categories. The first relates to those categories of error which involved the use of inappropriate information to solve a problem, drawing either from the question itself or from previous experience. In grouping these I have used a term borrowed from Lave (1984) of "gap closing strategies". This term, adapted for the present study from Lave's original meaning, refers to the idea that students import information in order to close the gap between the problem the face and the solution they are required to give. The second relates to those errors which were bound up with students' difficulties expressing their mathematical ideas using language.

Within the *building calculus* category, three broad sub-categories were present. The first involved those categories of errors that indicated a student had applied an algorithm to solve a problem that was either inappropriate or incomplete. The second involved those categories that contained errors in which well-known rules or processes were generalised to situations in which they were not appropriate. The final sub-category contained those errors in which inappropriate links had been made between concepts.

These refinements and clusterings led to the overall coding scheme that is presented here.

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5.2 The coding scheme

A. PSEUDO-STRUCTURAL APPROACH TO ALGEBRA

1 ascribing an invariant meaning to letters

Part of being able to use algebra to communicate effectively relies on one being able to give meaning to the symbols used. One also needs to understand the conventions that make it easier for someone else to read and interpret one's work. However symbols do not have meaning which transcend the context in which they are used. *The errors in this category are those in which the letters are endowed with a fixed meaning, which is inappropriate in the context.*

Examples:

- *h* is frequently used to represent height. In question 11, which required the students to work with the volume of a triangular prism, students used *h* to represent both the height of the triangle as well as the height of the prism.

- *c* is encountered in equations like $y = mx + c$ at school and thus takes on a fixed meaning as the *y*-intercept.

- Angles are measured in degrees at school. The move to radian measure at university causes some confusion. π is interpreted as a unit like ^{0} rather than as a number.

2 confusing variable and constant

The notion of a variable is central to calculus. However it is difficult to define and an intuitive understanding of the concept is built up inductively. The distinction between a variable and a constant is equally elusive for many students, especially as they are both represented by letters. *Errors in which the student treats a variable as a constant or uses a variable where a constant is required (and vice versa) are included in this category.*

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Examples:

- Having found $\frac{dy}{dx} = \frac{-3x}{y}$ students write the equation of the tangent line as $y = \frac{-3x}{y}x + c$

- When asked to find the constant term in the expansion of $(x + \frac{2}{x})^{10}$ students gave an answer such as $\frac{1024}{x^{10}}$

- In question 11 where students were given a water trough in the shape of a triangular prism and required to find $\frac{dh}{dt}$, where h was the height of the triangle, given $\frac{dV}{dt}$, where V is the volume of water in the trough. A student wrote $V = \frac{1}{3}\pi r^2 h$, $\frac{dV}{dt} = \frac{dV}{dP}\frac{dP}{dt} = \frac{2}{3}\pi rh\frac{dP}{dt}$. Apart from the patently incorrect formula for the volume of a triangular trough, the student introduces of a variable P when differentiating and treats both r and h as constants.

3 ignoring process behind notation

Tall (1991) talks about the value in the ambiguity of mathematical notation which allows the same notation to represent both a process and an object. The power of notation allows us to reflect complex processes with a single symbol and to be able to use that symbol as an object that itself can have processes performed on it. However if the process that the symbol signals is "lost" the notation can become either meaningless or rigidly attached to a single idea. *The errors included in this category are those in which notation is used without reference to the process that underlies it.*

Examples:

- $\sum$ notation caused a lot of difficulty and many errors occurred when attempting to work with it or even simply to write a statement using it,

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for example $\sum(x+\frac{2}{x})^{10} = x^{n-r}(\frac{2}{x})^r$. In cases like this it appears as if the notion of $\sum$ as sum has been lost and the notation is used in an arbitrary way.

- Although sums and sigma notation are used in many contexts, they were given most prominence in Riemann sums. Many students overlooked the process which links sums with definite integrals and simply associated the two notations with each other. Thus, for example, they would write $\sum_{i=m}^{n} i = \int_{1}^{n} i\,dm$.

- The notation, $F(x)$, was used to denote the antiderivative of the function, $f(x)$. Although this notation was used in a number of contexts during the course, it was most commonly used in questions which asked students to draw the graph of the antiderivative of a function. For many students this notation appears to stand for the graph of the antiderivative function and thus the integration process which underlies it is lost.

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4 over-generalisation

Although over-generalisation also appears in the building calculus section the errors in this category refer specifically to generalising an **algebraic** rule to contexts in which it is not appropriate. *A known rule is revised and applied to a new situation. Most of the errors in this category relate to application of the distributive law in situations where it does not apply.*

Examples:

- $\frac{1}{x^2+2x+1} = \frac{1}{x^2} + \frac{1}{2x} + 1$
- $\frac{1}{(1+x^2)^2} = \frac{1}{(1+x^4)}$

- $\sum_{i=0}^{9} f(\frac{4i}{10}) = \sum_{i=0}^{9} f(\frac{2i}{10}) + \sum_{i=0}^{9} f(\frac{2i}{10})$

5 incorrect syntax

The errors in this category involve the incorrect use of syntax in the symbolic expression of mathematical ideas.

Examples:

- $x^3 = 3x^2$ in place of $\frac{d}{dx}x^3 = 3x^2$

- $(x+\frac{2}{x})^{10} = x^{10-r}(2x^{-1})^r$ in place of $(x+\frac{2}{x})^{10} = \sum_{r=0}^{10}\binom{10}{r}x^{10-r}(2x^{-1})^r$

- The use of $\Rightarrow$ in place of =

B. BUILDING CALCULUS

1. partially mastered algorithms

1.1 *inappropriate reversal*

In Dubinsky's (1991b) terms reversal occurs when an interiorised process is thought of in reverse. Within calculus, an important example of this would be seeing integration as the reversal of differentiation. *Errors of inappropriate reversal are those where the reverse process is used instead of the appropriate one.*

Examples:

- $\int \ln x \, dx = x\ln x - x + C$ is reversed to $\int x\ln x - x\, dx = \ln x$

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- The rule $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$ is well known to the students. It gets reversed to $\int \arctan x \, dx = \frac{1}{1+x^2}$

- The idea that if $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$ has a local maximum at a then $\int x \ln x - x dx = \ln x$ (or is undefined) becomes reversed to $f'(x)$ has a maximum at a implies $f(a) = 0$

1.2 *incomplete application of an algorithm*

Errors involving the execution of a known procedure without attention paid to other possible procedures or further work required to solve the problem are included here.

Examples:

- In finding the equation of the graph of the polynomial $(f(x) = -kx^2(x-1), k > 0)$ students left out the negative coefficient. This is possibly because in most cases simply using *x* and *y* intercepts can produce the equation of the polynomial. In this case the *y* intercept did not provide additional information. The students did not look at the end behaviour of the graph to determine the sign of the leading coefficient.

- In locating local maxima of a function on the basis of a graph of the derivative of that function, some students simply solved $f'(x) = 0$, without checking whether it did in fact produce a local maximum.

2. over-generalisation of known rules and processes

2.1 *assumption of continuity and regularity*

Most of the graphs dealt with in high school mathematics are smooth, continuous curves. Dealing with functions that are not continuous or

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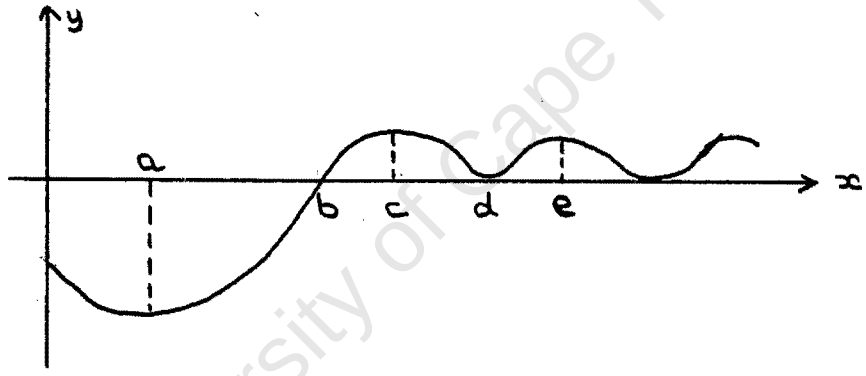
that are defined piece-wise provides difficulty. Students seemed to want to force continuity when drawing graphs. The most frequent assumption seems to be that graphs behave like polynomial graphs. *Errors included in this category were all those in which students assumed that graphs had to be continuous and regular.*

Examples:

- In question 4 the point of discontinuity in the derivative graph caused the most problems and students either made it continuous or re-started the graph at the x-axis

- In question 5 many students assumed the graph would continue in the following way:



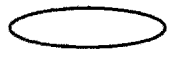
and thus included e as a point of inflection.

2.2 *assumption of standard orientation of shapes*

Shapes are seen in a standard orientation.

Examples:

- Students assumed that a triangle means a picture like this Δ even though the context (a drinking trough in the shape of a triangular prism) made it clear that it should be orientated differently ∇

- The ellipse $9x^2 + 3y^2 = 27$ was drawn this way 

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2.3 *generalisation on the basis of colloquial language cues*

Certain words carry associations with them that provide misleading cues. These cues allow students to generalise about the properties of a particular concept. Thus, for example, a maximum value is associated with "a big number" which has to be positive. *Errors where the everyday associations of a word were used inappropriately are included in this category.*

Examples:

- "$f(e)$ is positive because $f(x)$ has a local maximum at that point"

- "$f(e)$ is positive because the graph is increasing"

2.4 *generalisation on the basis of form*

Errors in this category are characterised by inappropriate pattern matching. The form, particularly of an algebraic expression, provides the cue to generalise a known rule.

Examples:

- The rule $\int \frac{1}{x}\,dx = \ln|x| + C$ is extrapolated to inappropriate situations and thus, for example $\int \frac{1}{x^2+2x+1}\,dx$ is given as $\ln|x^2+2x+1| + C$

- In the table of integrals, included in the examination paper, the rule $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + C$ was given. Most students reduced the question $\int \frac{1}{\sqrt{1-9x^2}}\,dx$ to $\int \frac{1}{\sqrt{1-(3x)^2}}\,dx$ and thus gave the answer $\arcsin 3x + C$.

3. concatenation

Ideas that are very similar and learnt together become blurred into more or less a single entity. *Errors in this category are characterised by the use*

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of a particular concept when a similar, related concept is called for.

Examples:

- Local extrema and global extrema: When asked whether there would be a local maximum on an interval, students offered the endpoint of the interval as a possible location for the local maximum.

- The midpoint rule and trapezoidal rule are both taught in a section on approximation errors in calculating definite integrals. They can both be represented in a similar form pictorially and both are affected by the concavity of the graph in terms of whether they under- or overestimate the integral. Thus when asked to describe the trapezoidal rule some students described the midpoint rule.

- The formula for the area of the circle becomes confused with the equation of a circle.

- Volume, area and perimeter (both the formulae for these and the concepts themselves) become confused with each other. In question 11 where students were required to work out the volume of a trough shaped like a triangular prism many simply used formulae like $V = \frac{1}{3}\pi r^2 h$
  In question 13 the perimeter of the arch window was given as 1.5m. Students then produced formulae like $1.5 = \frac{1}{3}\pi r^2 h + L \times B$

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C. GAP-CLOSING STRATEGIES

1 drawing on previously accessed information

Answers or information from a specific example or set of exercises, that have been worked on previously, are simply imported into the problem at hand

Example:

- A question similar to question 18 had appeared in the previous year's exam. Students used the fact that $\Delta x = 0.5$ which had been calculated in the previous year's exam instead of $\Delta x = 0.4$ which was appropriate in this year's exam.

- Also in the previous year's exam the statement RHΣ-LHΣ $= |f(b) - f(a)|\Delta x$ was written as RHΣ-LHΣ$= \frac{f(b) - f(a)}{2}$ because $\Delta x = 0.5$. Students simply transferred this as a formula in question 18 and thus stated $42.4 - 37.6 = \frac{f(4) - f(0)}{2}$.

- When asked to explain the trapezoidal rule for approximating definite integrals, many students simply answered that the trapezoidal rule gives an overestimate because the graph is concave up. Thus instead of answering the question they focused on the aspect of using the trapezoidal rule that had been the dominant feature of many previous exercises and questions.

2 drawing on information not relevant to the question

Numerical values given in the question are inappropriately used to produce equations or assumptions are made that are not given in the question.

Examples:

- In question 3a, which required students to derive the formula for the fraction of an unspecified initial amount of strontium remaining t years after 1960, some students used 1960 as the initial amount of strontium.

- In question 13, where students are given an arch window composed of a rectangle and semi-circle, many assumed the rectangular part

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was square.

- In question 18 the formula for the function was not given. Some students assumed that it was $f(x) = x^2$.

3 drawing on surface-level representation

A diagram, graphical representation or the surface-level form of the question interferes with the students' ability to tackle the question. *Errors in this category are those where surface-level features provide immediate, but incorrect cues.*

Examples:

- When trying to find the equation of cos graph that had been shifted down by ½ a unit (question 2b), the visual cue of the minimum value at $y = -1$ led students to put "-1" as opposed to "-½" in the equation.

- In question 5 where students where asked to draw conclusions about the concavity of f based on the graph of f' there was a tendency to include the concavity of the given graph.

D. LANGUAGE

1 mode of expression

Mathematical discourse requires precision in the language used. Students often wrote down explanations that lacked this rigour using words like "it" or "the function" when it was unclear which function or object they were referring to. *Errors where the imprecise use of language renders the students' answers unclear are included in this category.*

Examples:

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- "The graph is positive" in the context where there was both a graph of the derivative of the function and of the function.

- Asked to explain whether the rate at which the depth of water in a trough will increase or decrease as the amount of water in the trough increases: "It will increase since the rate has a positive value it means that it will keep on increasing as the water level rises"

2 mode of argument

Although the course does not concentrate on methods of proof the basis for this is laid by looking at what constitutes a valid argument. In the examination there was only one question that asked students to prove a statement, but a number that asked them to justify or explain their answers. *This category thus includes erroneous proofs students gave and inadequate justifications.*

Examples:

- In giving reasons as to why they had concluded a particular function was positive or negative at a certain point (on the basis of information about the derivative of the function and an initial value for the function given in question 5) students gave the following answers: "It is positive because the graph at e is positive" or "It is positive because $f(x) > 0$ at $x = e$."

- Students simply drew graphs when asked to show that $e^x > x$ for all x.

E. RESIDUAL

1 executive errors

These errors are "slips" or computational errors. These errors have not been included in the main body of the category scheme as they do not reflect a persistent, underlying misconception.

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Example:

- A student makes an error in multiplying out $(x+h)^3$ and gets $x^3+3x^2h+2xh^2+1$.

2 unclassified errors

Errors that remain unclassified fall into two broad categories. The first, and more common type of error, is where students provided an answer with no motivation or working and thus it was impossible to make any deduction about how they produced the answer. The second consists of those errors that were particular to one student only and thus considered idiosyncratic.

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5.3 Enumeration of errors in a sample of scripts using the coding scheme

The coding scheme was developed out of an exploration of all errors made by the 117 students who wrote the examination. In order to ascertain the prevalence of errors in the various categories it was necessary to re-examine the students' scripts and record the number of errors found in each category. For this enumeration the division of students into the top, middle and bottom thirds was retained in order to reflect the relative prevalence of errors across these groups. The count of errors was achieved by randomly selecting the scripts of ten students from each of the top, middle and bottom thirds of the class. These scripts were then carefully examined and each instance of an error type encoded. The enumeration of these 30 scripts is presented in table 4

Table 4: Count of errors in 10 scripts from each of the top, middle and bottom third of students

	top	middle	bottom	overall
PSEUDO-STRUCTURAL APPROACH TO ALGEBRA	**34**	**72**	**71**	**177**
1 ascribing invariant meaning to letters	4	7	6	17
2 confusing variable and constant	2	12	19	33
3 ignoring process behind notation	17	22	21	60
4 over-generalisation	3	12	11	26
5 incorrect syntax	8	19	14	41
BUILDING CALCULUS	**43**	**74**	**86**	**203**
1. partially mastered algorithms	**13**	**17**	**29**	**59**
1.1 inappropriate reversal	3	3	11	17
1.2 application of incomplete algorithm	10	14	18	42
2. over-generalisation of known rules and processes	**22**	**36**	**29**	**87**
2.1 assumption of continuity and regularity	3	8	7	18
2.2 assumption of standard orientation of shapes	0	2	2	4
2.3 generalisation on the basis of colloquial language cues	6	4	3	13
2.4 generalisation on the basis of form	13	22	17	52
3. concatenation	**8**	**21**	**28**	**57**
GAP CLOSING STRATEGIES	**27**	**40**	**35**	**102**
1. drawing on previously accessed information	9	14	13	36
2. drawing on contextual information provided in the question	9	10	8	27
3. drawing on surface-level representation	9	16	14	39
LANGUAGE	**11**	**13**	**13**	**37**
1. mode of expression	2	1	4	7
2. mode of argument	9	12	9	30
RESIDUAL	**48**	**57**	**53**	**158**
1. executive errors	13	5	7	25
2. unclassified errors	35	52	46	133
TOTAL	**163**	**256**	**258**	**677**

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The majority of errors are thus captured by the coding scheme. The 133 errors left uncategorised are made up in the main of solutions where the student had only given the answer and thus where the underlying misconception was difficult to determine.

The table gives a good idea of the relative prevalence of the errors across error categories. However the table must be read bearing in mind the following factors:

1. The examination did not provide equal opportunities for all errors to be made. In particular the small number of errors in the LANGUAGE category is in part due to the fact that there were only a few questions that required students to provide a written explanation or argument.
2. A student who provides a lengthy, partially correct answer might actually make more errors on that question than a student who simply writes down an entirely incorrect one line answer. This is shown up in the table by the fact that the total number of errors committed by students in the middle and bottom thirds of students is roughly equal.

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CHAPTER 6 DEVELOPING A MODEL FOR UNDERSTANDING MISCONCEPTIONS IN CALCULUS

The pattern of errors in the coding scheme provides a means to explore the processes students' use in constructing their knowledge. The learning theories of Sfard and Dubinsky highlight the importance of both an operational and structural understanding of mathematical concepts, arguing that encapsulation of processes into objects is an important step in being able to build further mathematical knowledge. Dubinsky (1991) describes five kinds of construction which explain how new objects, processes and schemas can be built from existing ones. In this section, I will argue on the basis of the analysis generated in the coding scheme, that the students' errors in algebra demonstrate a pseudostructural approach to algebra. As a result of this, the way in which they build their understanding of calculus is flawed. The modes of construction they employ resonate strongly with those outlined by Dubinsky, albeit in distorted form.

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6.1 Pseudostructural approach to algebra

Sfard and Linchevski (1994) argue that a pseudostructural or semantically debased conception can occur because of the inherent difficulties of reification. If primary processes are not reified, then the secondary processes will have no objects to operate on. Symbols or pictures are then substituted for the abstract entity and become the objects to be operated on. The results of this are twofold: firstly, in the words of Sfard and Linchevski, the signifier becomes the signified. Therefore the students act as if they are dealing with objects, but the semantic content of the object is either absent or distorted. Secondly, the new knowledge does not link with previous knowledge because the processes underlying the new object are lost when the object is identified only by its representation. Algebra thus becomes a game of letters and symbols.

The students in the present study have completed their studies of basic algebra at school. Ideally they should be familiar both with the use of algebraic expressions that use the letter to express a fixed unknown and those in which the letter is a variable. They should be familiar with letters used as parameters and comfortable with functions. In the words of Sfard and Linchevski, we rely, at first-year university level, on a versatile and adaptable algebraic knowledge. The pattern of errors made by students in the present study reveals a number of points at which students' thinking is neither versatile nor adaptable. This rigid thinking is indicative of a pseudostructural approach in that letters and symbols are imbued with meaning or used to facilitate manipulation through pattern matching of symbols. This takes the place of the unifying structural approach which provides links and allows students to make sense of what they are doing. In what follows I will use the errors from the coding scheme to elaborate on these ideas in more detail:

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6.1.1 Letters as objects

The errors that make up the categories *invariant meaning to letters* and *confusing variable and constant* are indicative of a tendency to treat letters as objects.

A large part of the versatility and adaptability in algebraic thinking centres on the role of letters. For example x in $x+2=3$ is simply standing in place of 1, whereas x in $y=x+2$ needs to be seen as a variable. $y=mx+c$ defines a family of functions, with m and c playing a different role to y and x.The equation of the tangent line to $y=f(x)$ at the point (x_1,y_1) is $y-y_1=f'(x_1)(x-x_1)$ where (x_1,y_1), although it can vary across all possible tuples, needs to be understood as standing for a particular point, whereas x and y represent variables. The volume of a cylinder is given by the formula $V=\pi r^2 h$, where r and h are independent variables on which V depends. However, if I refer to 5 cm high cylinders then I need to interpret h as a

constant. And if take cylinders where the height is twice the radius of the base then I need to interpret h as a function of r. Given this multiplicity of meaning the easiest shortcuts to "doing" algebra appears to be to turn the letters into objects and either attach to them their own special meaning or denude them of any role beyond being letters.

Errors in the category *invariant meaning to letters* demonstrate how letters are turned into objects, often with a clear physical representation. Thus, for example, c becomes the $y-$intercept. School textbooks use $y = mx + c$ and $y = ax^2 + bx + c$ as the general form for linear and quadratic functions. It is the process of setting $x = 0$ and evaluating the function that leads to the understanding that c is the $y-$intercept in these cases. Simply endowing the letter c with the property of being the $y-$intercept implies that the operational underpinnings have been lost.

In others instances letters lose any meaning beyond being letters. This can be seen in the errors in the category of *confusing variable and constant*. The series of examples outlined above show the complexity of the various roles letters can play. Sfard and Linchevski (1994) highlight in particular the difficulties students have in dealing with functional algebra and the role of letter as variable. They argue that:

> most of the time algebraic formulae are for some pupils not more than mere strings of symbols to which certain well-defined procedures are routinely applied. In these students' eyes, the formal manipulations are the only source from which the symbolic constructs may draw their meaning (p233).

In calculus the notions of variable and functional dependency are crucial. If these concepts are not well understood students apply processes to strings of letters without questioning their results. For example, taking the derivative is a routine procedure. Understanding that the derivative is a formula which can be evaluated at different points to provide information about the slope of the tangent to the function at those points relies on an understanding of the

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notion of variable.

6.1.2 Symbol as object

Because mathematical concepts have a dual nature, the same mathematical notation is used to represent both process and object. Notation can often summarise a complex or lengthy process and it seems as if keeping in mind the process behind the notation in order to reify it into a meaningful object is difficult for students. This can be seen in the category *ignoring process behind notation*. The students' errors indicate that the notational symbol itself can easily become the object. When this happens the process behind the notation is ignored and the meaning attached to the symbol becomes that of a cue to perform a certain routine. Thus, for example, $\sum$ notation became an indicator for either working out a term in the binomial expansion or for Riemann sums and thus integrals.

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6.1.3 Pattern matching of symbols

The errors made in the categories of *over-generalisation* and *incorrect syntax* are a manifestation of algebra as a game of letters. In part they can be seen as a result of dealing with letters and symbols as if they are objects, without regard for their context. In particular manipulating symbols whose meaning and role have been stripped away makes for rules without reason. Thus $\frac{4i}{10} = \frac{2i}{10} + \frac{2i}{10}$ can be generalised to $\sum_{i=0}^{9} f(\frac{4i}{10}) = \sum_{i=0}^{9} f(\frac{2i}{10}) + \sum_{i=0}^{9} f(\frac{2i}{10})$ because f is rendered meaningless. If $x^2 + 2x + 1$ is not seen as an object, but as a string of symbols then the rule $\frac{x^2 + 2x + 1}{a} = \frac{x^2}{a} + \frac{2x}{a} + \frac{1}{a}$ can be generalised to $\frac{1}{x^2 + 2x + 1} = \frac{1}{x^2} + \frac{1}{2x} + 1$. The use of incorrect syntax highlights the making of connections on the basis of symbols without regard to meaning. Although it is possible that incorrect syntax is simply a result of "sloppy" work, its consistent use does suggest that the student is paying more attention to recording the symbols than to making meaning. Thus, for example, writing

$(x+\frac{2}{x})^{10} = x^{10-r}(2x^{-1})^r$ in place of $(x+\frac{2}{x})^{10} = \sum_{r=0}^{10} x^{10-r}(2x^{-1})^r$ involves connecting two strings of symbols so that one can solve the problem.

The fact that errors in the section *pseudostructural approach to algebra* account for a quarter of all errors in the examination indicates that this kind of algebraic understanding plays a major role in students' performance in calculus. Furthermore, it is particularly notable that although the top students made relatively few errors in *confusing variable* and *constant, over-generalisation* and *incorrect syntax*, the middle and bottom students made a large number of these. These three categories of errors are, as explained above, those which indicate that letters are treated with no regard to their meaning or role. The top students thus appear to have a more adaptable and versatile understanding of algebra. However all three groups of students made a number of errors in which they ignored the process behind the notation. Thus there appears to be an additional difficulty in combining the understanding of the role of letters in algebra with complex notation. If one understands that notation signals a process that needs to be performed on an algebraic object, then the difficulty of errors in all three groups can be explained by what Sfard terms "the inherent difficulty of reification".

Thus the picture that emerges from an analysis of these errors is one of a group of students who all have elements of a pseudostructural approach to algebra. The bottom and middle groups of students appear to be strongly pseudostructural in their algebraic thinking. In the next section I will demonstrate the implications of this for how they build their understanding of calculus.

6.2 Building calculus

In this section I argue that the students' errors can be seen in the light of an attempt to construct a schema for calculus on the basis of a pseudostructural

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approach to algebra. In order to do this I want first to revisit Dubinsky's notion of schema and the modes of construction he outlines. Dubinsky describes a schema as "a more or less coherent collection of objects and processes" (1991, p102). He draws on Piaget to describe how the encapsulation of objects, upon which actions are subsequently performed, build structures which are subsequently incorporated into "stronger" structures. In essence the picture he paints is of a large complex of organised schemas, some of which are sub-schemas of 'stronger' schemas.

Dubinsky describes five kinds of construction in the building of schemas through reflective abstraction: interiorisation, coordination, encapsulation, generalisation and reversal. These are explained in detail in chapter 2. He conjectures that, in the light of evidence uncovered thus far, the construction of all mathematical objects can be described in these terms. However, he states that one can think of reflective abstraction as trying to tell us what needs to happen in order for conceptual understanding to be built. (Dubinsky, 1991). The present study seeks to explore the question of what happens when schemas are not built up in this way. In the analysis of errors summarised in the coding scheme, evidence emerges that students do attempt to construct their own schemas for dealing with calculus. The modes of construction they employ share similar features to those outlined by Dubinsky. This similarity is based on the notion that schemas provide an organisational structure for mathematical ideas and thus enable connections to be established between mathematical concepts. Sfard and Linchevski (1994, p198) compare the transition to structural thinking "to what takes place when a person who is carrying many different objects loose in her hands decides to put all the load in a bag" . The modes of construction that Dubinsky outlines explain how links and relationships are built up so that a collection of "loose objects" can be built into one coherent whole. I will attempt to show in the discussion that follows how the modes of construction that the students in the present study display, exist for a similar purpose. However, because they are built on a pseudostructural approach to algebra they become distorted.

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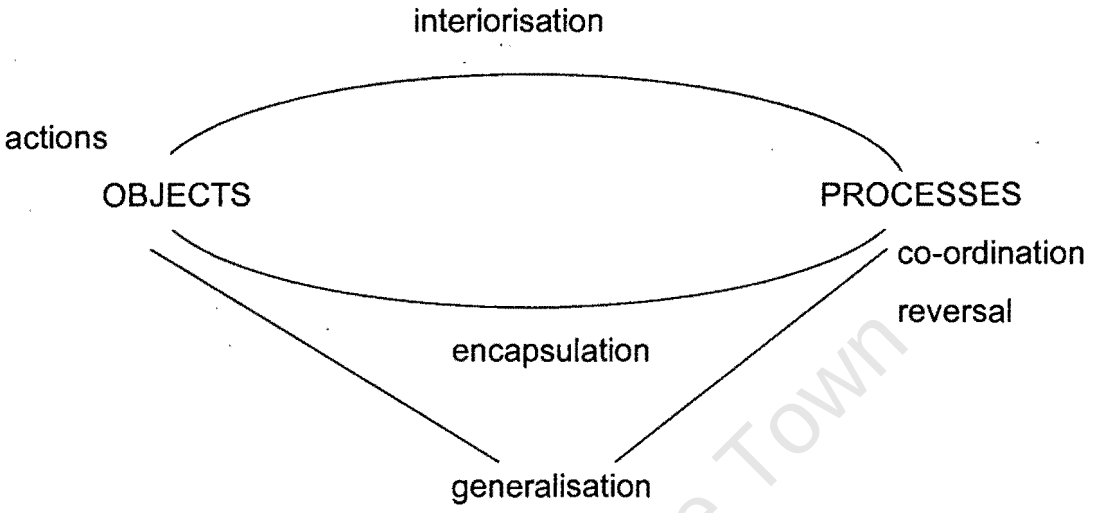
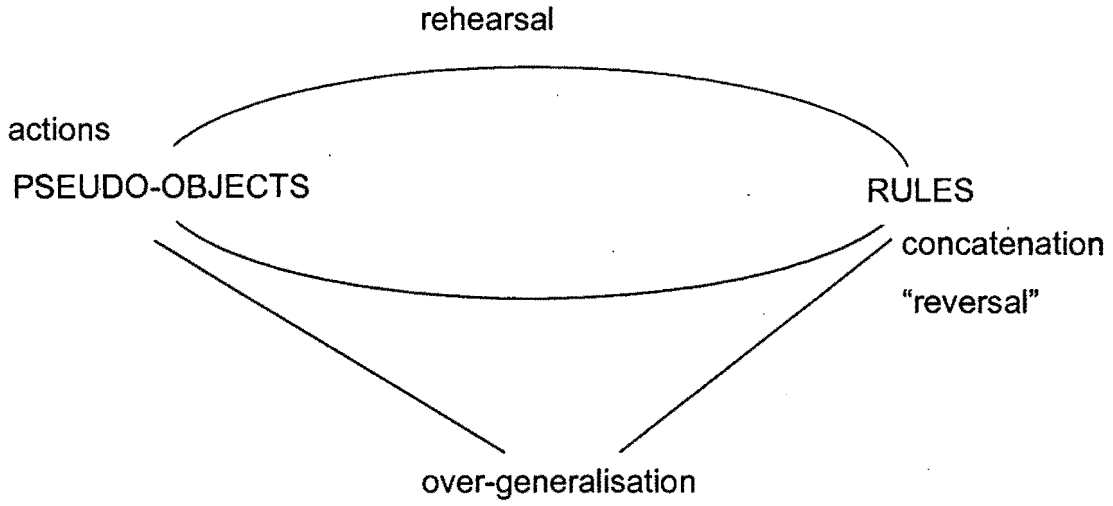
Dubinsky (1991, p107) provides the following diagrammatic representation of schemas and their construction.

Diagram 3:



My analysis of the errors students make in building calculus has led me to develop a similar diagrammatic representation of how students can construct schemas in the light of an impoverished base knowledge. The rest of this section is devoted to explaining this diagram in light of the errors in the coding scheme.

Diagram 4:



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The starting point for building calculus is students' algebraic knowledge. I have argued above that an analysis of students' errors in algebra suggests the use of a pseudostructural approach. Thus I have termed the algebraic "objects" that students employ as their building blocks, ***pseudo-objects***. Sfard and Linchevski describe the pseudostructural approach as mechanistic and argue that any secondary processes performed on these pseudo-objects must seem totally arbitrary. Because of this, it seems feasible to suggest that these secondary processes could be more accurately described as ***actions***, which, in the sense that Dubinsky uses them, are step-by-step procedures with steps only related by routines and not by any relationships that exist in the mind of the subject. The overwhelming evidence of the error data suggests that these algorithmic procedures are rehearsed so that they become rules for solving problems. Perhaps the clearest evidence of this is found in the category of errors, *application of incomplete algorithm*. Here we see, for example, the procedure for finding the equation of a polynomial which works in most cases (use the x and y intercepts) has become the rule for solving problems of this type. In solving question 2a of the examination, many students simply applied this rule even when the y intercept was 0 and thus could not provide the leading coefficient of the polynomial. Similarly, the key procedure involved in finding a local maximum is to take the derivative and set it equal to 0. It appears that this procedure is rehearsed and becomes the rule for finding local maxima. It is through these errors that we can see that an action that works in a number of cases is ***rehearsed*** into a ***rule***. Although impossible to discern from the exam data alone, it seems reasonable to assume that many correct rules emerge from this process too. The interview data, discussed later, supports this assumption.

The distinction I draw between a rule and a process is that a process is given meaning through a familiarity with the objects on which it is performed, whereas a rule is applied as a mechanistic manipulation of pseudo-objects. This distinction is reflected in Hiebert and Lefevre's discussion of procedural knowledge. They argue:

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> Procedures [...] may or may not be learned with meaning. We propose that procedures that are learned with meaning are procedures that are linked to conceptual knowledge. [...]Rote learning, on the other hand, produces knowledge that is notably absent in relationships [...]. Procedures can be acquired and executed even if they are linked tightly to the surface characteristics of the original context. (Hiebert and Lefevre (1986, p8).

One of the key properties of schemas is that they provide coherence. Dubinsky's description of the construction of a schema rests on the development of connections. Through the five modes of construction he outlines, objects at one level are connected by the processes performed on them with objects at a second level, new processes are generated and existing objects and processes are generalised to new situations. It is this coherence that "chunks" knowledge and reduces the necessity for remembering large quantities of isolated facts. If students start by building rules on the basis of pseudo-objects, they will in all likelihood end with the situation described by Hiebert and Lefevre above. However, the error data indicates that students do attempt to make connections. The large number of rules that is required in a calculus course make it difficult to memorise each individually. From the error data it appears that ***concatenation***, ***"reversal"*** and ***over-generalisation*** are three strategies that students use to reduce the number of rules required. What differentiates these strategies from Dubinsky's co-ordination, reversal and generalisation is that because they originate from a pseudostructural approach, they reflect the pattern-matching via surface level representation nature that is endemic to that approach. This idea is again reflected in Hiebert and Lefevre's discussion of procedures learnt without meaning. They argue:

> Many procedures, especially those that operate on symbol patterns, are triggered by surface features similar to those of the original context. (p8).

If we consider the errors that occur in the category *inappropriate reversal*, the disparity between the kind of connecting rules the students in the present study use and those described by Dubinsky becomes clear. For Dubinsky reversal occurs when an interiorised process can be performed in reverse.

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Being able to reverse a process requires that the process is well understood and that the relationship between the process and reversed process is apparent. Thus, for example, knowing $\frac{d}{dx}\arctan x^2 = \frac{2x}{1+x^4}$ and the relationship between derivatives and integrals allows one to assert $\int \frac{x}{1+x^4}dx = \frac{1}{2}\arctan x^2 + C$. If, however, we look at the errors in the category of inappropriate reversal it appears that the students have linked the rules of differentiation and integration, but the notion of reversing the process is overshadowed by overlearnt rules. Thus a connection is made on the basis of surface features. For example, $\int \arctan x\,dx$ is associated with $\frac{1}{1+x^2}$. It appears as if $\arctan x$ cues $\frac{1}{1+x^2}$ and that there is no monitoring process which checks whether differentiating $\frac{1}{1+x^2}$ does in fact produce $\arctan x$.

Concatenation represents the grouping of concepts on the basis of surface level cues. The errors that appear in the category of concatenation reflect how this type of connection strategy is inadequate in providing a meaningful organisation of information. Thus, for example, although students need to see a connection between the midpoint rule and trapezoidal rule for approximating integrals, they also need to be familiar with the process behind the rules so that they can recognise the distinctions between them. Students, however, appear to have collapsed these two rules into a single entity. The use of the equation of the circle in place of the area of the circle is perhaps a more extreme example of concatenation. One can see this as simply retrieving a formula that is in some way associated with the properties of a circle.

Over-generalisation is a distorted form of the mode of construction Dubinsky terms generalisation. Generalisation is the appropriate extension of processes and objects into other schemas and allows for the application of old knowledge in new situations. Thus generalisation makes connections

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between concepts and reduces the learning load. Errors in the category of *over-generalisation* suggest two approaches which explain the existence of *over-generalisation*. The first approach emerges from a rigidity in thinking, in that old knowledge is applied in new situations without modification. The sub-categories of *assumption of continuity and regularity* and *assumption of standard orientation of shapes* provide evidence of this rigidity. Assuming that all graphs are continuous and regular in their behaviour is an extension of early and familiar experiences into a new context. Sfard (1992) notes that the assumption that all functions must be continuous is a common one. She argues that students see a discontinuous curve as representing several functions rather than one and such a conviction would seem natural to students who identify a function with its graphic representation. The strength of this conviction is reflected in the kind of errors students made. For example, some students sketched the derivative function of a given graph correctly, but joined the derivative graph at the point of discontinuity.

The second approach mimics generalisation, in that it attempts to provide links between concepts. The sub-categories, *generalisation on the basis of colloquial language cues* and *generalisation on the basis of form*, resonate with the connection-making of appropriate generalisation. However the kind of connections made are qualitatively different. In *generalisation on the basis of colloquial language cues* the connotations which go with the colloquial use of a word provide the links. Overall the trends in the language errors appear to indicate a lack of familiarity with the mathematical register and thus a confusion between the colloquial and mathematical meaning of words. The connection between language use and mathematics learning, particularly in the context of second- or third-language speakers of English (the medium of instruction) from disadvantaged educational backgrounds is worthy of a study on its own. Thus although the errors in this sub-category do clearly indicate the inappropriate use of language in order to make connections, I believe that further research is needed in order to address the role of language more fully.

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The errors of over-generalisation on the basis of form connect strongly with a pseudostructural approach to algebra. Generalising a rule like $\int \frac{1}{x} dx = \ln|x|$ to $\int \frac{1}{x^2+2x+1} dx = \ln|x^2+2x+1|$ is perhaps one of the clearest examples of a symbol pattern providing the trigger for applying a known rule. Being able to see that $\int \frac{1}{x^2+2x+1} dx$ is not the same as $\int \frac{1}{x} dx$ requires an understanding of variable and functional dependency. Without this understanding the various techniques of integration used to deal with composite functions, products and quotients must seem arbitrary. In this context the pseudostructural pattern-matching approach again dominates.

The dotted line in diagram 4 represents my conjecture that rules cannot be encapsulated into meaningful objects, but will simply result in a new set of pseudo-objects. This is a conjecture not an assertion at this stage because a first-year level calculus course does not require students to operate on derivatives and integrals as if they were objects and thus the data cannot provide evidence for what the students would do when this was required.

To summarise then, the pattern of errors reveals that when students build calculus on the basis of a pseudostructural approach to algebra, algorithmic procedures are rehearsed to become rules. Rules are concatenated, reversed or generalised. These modes of construction, rehearsal, concatenation, reversal and over-generalisation, are based on establishing links on the basis of surface-level representation. The count of errors indicates that these modes of construction are present across the top, middle and bottom groups of students. The count of errors also reveals that the bottom and middle groups of students tend to make errors more frequently than the top students. These disparities echo the differences observed in the error count in the category *pseudostructural approach to algebra.*

I have used the errors made to develop a model of how students build knowledge on the basis of an impoverished background knowledge.

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Although the errors provide indicators for the modes of construction students use, the count of errors cannot be seen as indicative of the degree to which the mode of construction is used. It is entirely feasible that the top group of students use rehearsal as much as the other two groups, but simply learn the rules better and thus make fewer mistakes. The analysis of errors alone cannot bring to light whether or not students possess more appropriate schemas alongside the flawed schema described above. In other words, I cannot decide whether students' correct answers are cases of correctly rehearsed rules or whether they can be seen as indicative of a good understanding of the concept tested in that question. Both Dubinsky (1991) and Winter (1989) raise the possibility that a student may possess competing schemas. Dubinsky argues that a person will display a tendency to apply a particular schema in certain kinds of problem situations. His use of the word "tendency" conveys his belief that a person may not necessarily apply the same schema across all situations which to the outside observer appear similar. Winter argues that a problem situation will invoke a number of different schemas, but that "only those which are compatible and dominant in activation strength come to apply" (p50). In light of these arguments, the claim I make about the schema I have outlined here is that it is a schema that students possess and regularly use.

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6.3 Gap-closing strategies

The errors categorised under the title *gap-closing strategies* illustrate the mechanisms that students use to bridge the gap between the inadequate knowledge they possess and that required for the problems they are expected to solve. An analysis of these errors provides evidence of both a reliance on rote learning and of a creative use of contextual and representational information.

Rote learning is seen clearly in the errors in the sub-category *drawing on previously accessed information*. The most extreme example in this sub-category involved students using a solution from the previous year's exam to

address a similar question in their exam. Although this type of error strengthens the argument that rehearsal plays a major role in students' learning, I have chosen to categorise errors such as this outside of the building calculus section because I believe that doing so foregrounds the application of memorised solutions as a problem-solving strategy. These errors suggest that a combination of pseudo-objects and rules do not provide sufficient tools and thus memorising individual problem solutions becomes necessary. Sfard provides a second possible reason for these errors. She argues that :

> the intellectual effort required from students to create for themselves the universe of intangible objects is really immense. The learner must have a lot of determination, stamina and intellectual discipline to cope with the demanding task. (1992, p84)

Given the fact that the students in the present study come to calculus with an impoverished background knowledge, the intellectual effort required to build a meaningful understanding of calculus concepts must be especially great. If students' orientation is towards rehearsing algorithms, and if they perceive a potentially limited set of possible examination questions, then organising their knowledge by question type and memorised solution would seem easier than attempting the shift in thinking required to build a structural understanding.

The errors in the categories *drawing on contextual information provided in the question* and *drawing on surface level representation* provide examples of how students find creative ways to "close the gap" between the limitations of their knowledge and the demands of the question. If one looks at the kind of errors made in the category *drawing on surface level representation,* it appears as if students use graphical information to provide them with the cues to "solve" the problem. This has elements of a pseudostructural approach in that a tangible object (in the form of a graphical representation) is used in place of a process. Although not as easy to see, the errors in *drawing on contextual information provided in the question* originate for a

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similar reason. For example, students dealing with a question relating to Riemann sums provided themselves with a formula for the function in order to be able to perform algebraic manipulations to produce answers. Similarly, to avoid having to deal with an equation of the form $P = P_0(\frac{1}{2})^{\frac{t}{29}}$ students used another piece of numerical information (a date) from the question to replace P_0 and thus enable them to find numerical solutions to the equation.

6.4 Language

As stated previously, the relationship between language and learning mathematics is worthy of a study on its own. The present study did not seek to probe this relationship and thus the discussion here is limited. The analysis of the errors is useful in highlighting some aspects of language use in mathematics, but in the absence of a detailed study of the interplay between language and the development of mathematical knowledge it is difficult to provide anything more than some tentative explanations.

The difficulty students had in providing a valid argument is unsurprising in light of the approach to mathematics that the pattern of errors, previously discussed, suggests the students possess. A rule-bound approach does not foreground explanations for why a rule works. Thus students are either unable to provide explanations or simply reiterate the rule as an explanation. For example, the majority of errors students made in attempting to provide a justification of a statement was to use a different form of representation to restate the statement. Thus, for example, they drew the graph of $y = x$ below the graph of $y = e^x$ as an explanation for why $e^x > x$. Seeing those two forms of representation as reflecting the same abstract idea requires a structural understanding. If students' focus is on the representational forms rather than the ideas and processes that underlie them, then proof is rendered meaningless.

The lack of precision that students display in their mode of expression demonstrates a lack of familiarity with the discourse of mathematics. The fact

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that students are not first language speakers of English possibly contributes to the difficulties that students have in this regard. Research evidence (summarised in Forrest and Winberg, 1993) also points to the link between poor language use and lack of conceptual knowledge. Thus the difficulty students have in expressing themselves clearly could also be interpreted as reflecting an inadequate grasp of the subject matter.

In this chapter I have used the pattern of errors presented by the coding scheme to develop a possible explanation of how students build an understanding of calculus in the light of a pseudostructural approach to algebra and some of the consequences of this. The errors that students make appear to be a consequence of organising their learning around rules performed on pseudo-objects. The errors consistently indicate this underlying organising principle.

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CHAPTER 7 DEEPENING THE DESCRIPTION: TEST AND INTERVIEW ANALYSIS AND DISCUSSION

Seventy-eight students wrote the test (appendix B). In order to carry out the analysis the students' test scripts were broken up into three groups: top third, middle third and bottom third. These groups were chosen on the basis of the students' performance in the examination and not on the basis of their results in the conceptual test, in order to be consistent with the groups used in the analysis of the examination. There were 20 students in the bottom third (i.e. whose results for the final examination fell in the range 18%-40%), 30 students in the middle third (i.e. whose results for the examination fell in the range 41%-52%) and 28 students in the top third (i.e. whose results for the examination fell in the range 53%-85%). To enrich the test data and to gain insight into the processes students were using while solving problems presented on the conceptual test I recorded the conversation of a group of four students working together on the test (Palesa, Arthur, Koki, Simon).

After the test I interviewed four groups of students. The test was written before the final examination and thus the four groups of students were selected on the basis of their performance in the test. I grouped the students according to their results on the conceptual test and selected three students from each of the quartiles in the ranking. Unfortunately two students did not arrive for their interviews and thus two groups comprised only two students. The top quartile consisted of Isiah and Vuyisele. The 2nd quartile consisted of Sandile, Godwin and Edgar. The 3rd quartile consisted of Nixon, Dan and Mundi and the bottom quartile consisted of Patricia and Nandipha. In the interviews the interviewer is identified by L.

In the design of the research project the conceptual test was seen as a way of providing data that could support and deepen the analysis generated from the primary source of data, which was the students' errors in the final examination. An initial coding of students' errors on the conceptual test

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demonstrated that the coding scheme did adequately capture these errors. However this process provided no additional insights into either the coding scheme or the model of student learning developed out of the examination data. The test was designed to probe students' conceptual understanding of derivatives and integrals and required little in the way of calculation or algebraic manipulation. It was also designed to ensure that similar concepts were repeatedly tested in different contexts and at varying levels of complexity. The repetition of questions in the test at different levels of complexity meant that a comparison of student performance and errors on similar questions could provide a more powerful method of analysis than a count of errors. In addition, data from the interviews and group discussion could be used in combination with the test data to provide a deeper insight into how students were approaching the work. I have thus combined the results from the test, the group discussion and the interviews in the discussion that follows. In order to avoid an exposition that would basically reiterate the analysis of the examination data I have chosen to structure this section around a number of "vignettes" which highlight key themes developed in chapter 6. Exemplars from the test, group discussion and interview data are provided to lend support and give depth to the model put forward in the discussion in chapter 6.

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7.1 Pseudostructural approach to algebra

The test was designed to minimise opportunities for algebraic manipulation, but errors reflecting a pseudostructural approach to algebra clearly still occurred. These related to the meaning given to letters and symbols.

Question 2f asked students to illustrate $hf(a+h)$ on the graph of a function f where a and $a+h$ were marked on the x-axis. The table below indicates the percentage of students who answered the question correctly as well as the percentage of students who illustrated $hf(a+h)$ as a length.

Table 4: Question 2f

qu 2f. $hf(a+h)$	bottom	middle	top
correct	10%	27%	61%
length	30%	17%	11%

The poor performance of students on this seemingly simple question was probed in the interviews:

Extract 1:

1 *Sisanda: I think it's the length like mmmm. You've got like f of a plus h is this side*

2 *L: ja*

3 *Sisanda: and then multiplied by h which we don't know how much that maybe 0 4 point 0 0 0 1 and it's like making a change by multiplying by 0 0 0 point 1 or 1 which is just the height*

Sisanda appears to have attached two meanings to h. The first is that of a small number close to zero and the second is that of vertical displacement. Both of these support the idea that meanings are attached to letters on the basis of the context in which they are frequently encountered. The textbook used in the course used h in the definition of a derivative ($f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$) and h is frequently used in formulae to designate height. The combination of these two images appears to have prevented Sisanda from seeing the physical representation of h in the diagram. However, an analysis of the continuation of the discussion between the three students involved in this interview highlights issues that are not apparent from the exam analysis: Godwin had offered two other possibilities for what $hf(a+h)$ might represent. He pointed out the correct answer (that of a rectangle) and alluded to the possibility that it could also be the area under the curve. Sisanda then takes up the discussion:

Extract 2:

1 *Sisanda: Like I think if we calculate the area you have to consider the area under the graph of f prime x not the area under f of x. And like since to me this*

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is the one (Sisanda points to the picture which shows the length from $y = 0$ to $y = f(a + h)$) *that is convincing because this is f of a plus h is just the height of this thing, the length of this thing*

2 *Edgar: times h*

3 *Sisanda: and we multiply by h, which is, we don't know h, it can be 1, h can be 1*

4 *Edgar: OK*

5 *Sisanda: we don't know. h can be 1 and and this is going to just be…*

6 *Edgar: h can be 1*

7 *Sisanda: ja*

8 *Edgar: and it can be 2*

9 *Sisanda: it can be 2*

10 *Edgar: so when you start by taking the greater than..*

11 *Sisanda: if you ja it can be greater than this, ja, like but it doesn't mean you have to calculate area under this*

12 *Godwin: what's the formula for area of a rectangle? I think it's length times height*

13 *Sisanda: mm hm*

14 *Godwin: so is this the length and this the height. We're multiplying the height by the length (laughs)*

15 *Edgar: but you know that*

16 *Sisanda: no*

17 *Edgar: this distance, the whole distance is the same as a plus h minus a*

18 *Sisanda: ja*

19 *Edgar: which will be h*

20 *Sisanda: ja*

21 *Godwin: so we're multiplying h by this one*

22 *Edgar: ja which will give you this…and then the area*

23 *Godwin: f of a plus h is the length from here to here*

24 *Sisanda: but I think for area we have to …like…calculate the area under the graph of the um f prime of x. Let under the (indistinct)…that is why it's not*

25 *Edgar: oh area under the slope*

26 *Sisanda: ja… the slope graph…this is not the one that's here*

27 *Godwin: I actually (laughs) you can't be right*

28 *Sisanda: I can't?*

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Two points emerge from this discussion. The first is that Sisanda is able to agree that h is the distance between a and $a+h$ (line 20) and thus steps back from her original interpretation of h as height. The question this raises is whether Sisanda possesses two competing schemas: one in which she is capable of correctly perceiving h as the distance between a and $a+h$, and the other in which she attaches a specific meaning to the letter. The possibility of such competing schemas was raised in the previous chapter and although the coexistence of both a correct and incorrect notion cannot be seen as conclusive evidence that Sisanda possesses two different schemas, it does suggest a potentially fruitful line for investigation.

The second is Sisanda's strong conviction that area has to be associated with the area under the derivative graph. Although Sisanda never explains this conviction, there is a chain of ideas ($f(b)-f(a)=\int_a^b f'(x)\,dx$ and the integral provides a measure of the area) which provide a potential explanation of its origins. Although she is prepared to admit that h can be any value, her conviction forces her to set the value of h as 1 in order to justify her answer that $hf(a+h)$ is the length from $y=0$ to $y=f(a+h)$. It is a misconception about calculus concepts that makes her reinterpret the variable h as a constant, concrete number. This suggests that inappropriate rules reinforce a pseudostructural approach to algebra.

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Question 2a asked students to illustrate $f'(a+h)$ on the graph of $f(x)$ which was drawn for them with a and $a+h$ marked on the x-axis. Question 1b asked students to illustrate $f'(2)$ on the graph of $f(x)=\sin x$. The percentage of students answering these questions correctly is given in the table below.

Table 5: Representing the derivative at a point graphically

derivative	bottom	middle	top
qu 2a $f'(a+h)$	70%	70%	79%
qu 1b $f'(2)$	20%	27%	36%

The majority of the errors made on question 1b by the top and middle groups of students were caused by these students' inability to locate the position of $x = 2$ on the x-axis. However the bottom third of students' errors ranged from depicting an average slope to drawing a tangent line and stating that its slope was 2. The discrepancy between the success rate on the two questions, together with these observed errors, suggests that for all three groups of students the expectation that the x-values would be given as ratios of π was the chief source of error in this question. The interviews bear this out.

The discussion in extract 3 follows on from a discussion in which Patricia and Nandipha attempted to represent $f'(\frac{\pi}{3})$ on the graph of $y = \sin x$. They had struggled with this, but did eventually illustrate it correctly.

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Extract 3:

1 *L: what about then OK now we've got that idea what about f dashed of 2?*
2 *Pause*
3 *Patricia: on this graph?*
4 *L: ja*
5 *mumble*
6 *Patricia: isn't it the same idea, but it's just like this graph is in radians, pi and this one is just a simple number*
7 *L: OK so are you happy with that, that it is the same idea? Alright so what are you going to do with the fact that there's a 2.*
8 *Patricia: maybe I'll try to copy this one*
9 *L: mm hm (neutral tone)*
10 *Patricia: 2 pi, pi 2*
11 *Nandipha: 2 pi*
12 *Patricia: 2 pi*
13 *Nandipha: 2 pi ja*

This interview extract shows clearly that these students use the presence or absence of the number π as an indicator of whether or not they are "in

radians". Their solution of locating $x = 2$ in the position where $x = 2\pi$ corresponded with the majority of errors made by the students in the top and middle thirds.

Extract 4:

1 *L: The first one I want you to do is just this question here. Question 1 b. To show where f dashed of 2 is, showing it on this graph of sin x. Where would you see that, where is f dashed of 2?*

2 *Pause*

3 *Dan: I think it can be anywhere our graph is increasing*

4 *L: why do you say that?*

5 *Dan: because, ah, it's positive*

In his solutions to the conceptual test Dan had provided a correct representation of both $f'(\frac{\pi}{3})$ and of $f'(a+h)$. However here we see that the absence of π leads Dan to interpret the 2 as the slope. The meaning behind the derivative notation is distorted by the need to make sense of the 2 in the expression. This highlights the kind of distortion of calculus concepts that can be caused by the inappropriate attribution of meaning to symbols.

7.2 Rehearsed rules

In question 3 students were asked to sketch the graph of $f'(x)$ from the graph of $f(x)$.

Table 6: Summary of students' solutions to question 3

Question 3	Bottom	Middle	Top
correct	10%	40%	43%
the representation of the point of inflection at $x = 1$ on $f(x)$ in the graph of $f'(x)$ is incorrect	50%	33%	39%
the representation of the point of inflection at $x = 1$ and the linear portion of $f(x)$ in the graph of $f'(x)$ are incorrect	15%	10%	11%
only zeroes of $f'(x)$ are correct	15%	3%	0%

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In order to plot the graph of the derivative function from a graphical representation of a function one needs to understand that one is plotting the size of the slope of the graph of the function against each x- value in the domain. As indicated in table 6 the majority of students were able to draw a derivative graph that was predominantly correct. However most of these students were unable to correctly represent the point of inflection of $f(x)$ at $x = 1$ on the graph of $f'(x)$. This, together with the fact that a number of students were unable to correctly represent the linear portion of $f(x)$ as a constant slope and the fact that 15% of the bottom third of students had only correctly identified the zeroes of $f'(x)$, suggested that the majority of students had developed an incomplete algorithm for dealing with this kind of question. In the interviews I set out to test this by asking students to sketch the graph of $f'(x)$ given a graph of $f(x)$ which had no turning points and one point of inflection. Mundi, Nixon and Dan's discussion of this question are given in extract 5 and 6 below.

Extract 5:

1 *L: If I said to you this thing here is my graph f of x. Draw for me the graph f dashed of x. Someone try it*
2 *Pause*
3 *Pause*
4 *L: What's difficult?*
5 *Pause*
6 *Mundi: Um I don't see any turning points*

Dan also indicated that he would not be able to draw the graph of $f'(x)$ because of the absence of turning points in $f(x)$. I then asked whether doing question 3 would be easier for him to do.

Extract 6:

1 *Dan: This one* (points to question 3) *is easier than this one*

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2 *L: OK so how would you proceed with question 3 if it's the easier one. What would you do?*

3 *Dan: No in this ah we're having turning point here which is equal to zero here*

4 *L: mm hm* (indicating agreement)

5 *Nixon: mm* (indicating agreement)

6 *Dan: then on this one....with our turning point on f at x on this one and crosses again at zero ah and another turning point over here so, but*

7 *Nixon: the problem is that* (points to the point of inflection at $x = 1$)

8 *Dan: yeah but I think*

9 *pause*

Although the discussion continued, Dan, Nelson and Mundi were unable to provide an answer as to what happened at the point of inflection at $x = 1$ and at no point was reference made to the idea of slope. The rule these three students used to draw the graph of $f'(x)$ on the basis of the graph of $f(x)$ was to establish the zeroes of $f'(x)$ by using the turning points of $f(x)$ (extract 5, line 6 and extract 6, line 3) and then draw a graph that behaves like a polynomial graph through those points (extract 6, line 6). This rule provided a correct sketch for a portion of the graph in question 3, but is not underpinned by an understanding of the relationship between a function and its derivative.

Question 1f and question 2g both related to the average value of a function. The table below shows the percentage of students who got each of these questions correct.

Table 7: Percentage of students who answered questions on average value correctly

Average value	bottom	middle	top
qu 1f calculate average value	40%	60%	75%
qu 2g graphical depiction of ave val	10%	23%	23%

The large difference in facility values for these two questions indicates that although most students could calculate the average value of a function using

the formula, very few could depict what they were calculating. In the interviews I probed students' understanding of averages and asked specifically about the average value of a function. Vuyisela's answer, below, was typical of the type of answer I got from students to this question in the interviews.

Extract 7:

1 *L: OK so when you're thinking like that, are you thinking the average value… is it represented by area or length or slope or what?*

2 *Vuyisela: average area*

In the interviews I asked students to work out the average of three numbers and also to work out the average area of the area of two rectangles which had the same base, but different heights.

Extract 8:

1 *L: OK let's just have a look at one or two quick things around averages. If I asked you what is the average of 8 and 5 and 3, how would you work out the average of 8 and 5 and 3.*

2 *Mundi: add them divide by the number*

3 *L: and if I said to you what is the average area of those two areas, how would you work out the average area out of those two areas?*

4 *Pause*

5 *Mundi: mm…using the formula*

6 *L: which is what?*

7 *Mundi: (laughs). One over, one over x times the integral …oh oh no 1 over b minus a times the integral between a and b…of the function between a and b*

(the formula she is talking about is $\frac{1}{b-a}\int_a^b f(x)\,dx$)

8 *L: OK. What do the other two think? If I asked you to calculate the average area of those two things, what is the average of those two areas?*

9 *Pause*

10 *Dan: I think she's right.*

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Mundi resorts to "the formula" when asked to calculate the more difficult average. The precedence of the formula over prior knowledge of averaging suggests that, for Mundi, the new knowledge of the average value of a function is not linked to old knowledge of averaging. This is reflected in the difficulty students had in depicting the average value of a function graphically or even in deciding whether it was a slope, length or area on the graph of the function. It reinforces the notion that students use rehearsed rules in building their calculus knowledge.

7.3 Reversal

In question 4 students were asked to answer questions about the graph of the function on the basis of the graph of the derivative of that function. The students' discussions about this question indicated that for many students these rules were learnt without meaning.

The table below indicates that a relatively high percentage, particularly of the bottom third of students, used inappropriate reversal to provide answers in question 4e which asked the students to list three points of the function in increasing order on the basis of information provided by the graph of the derivative. They reversed the rule that a local extremum may occur where the derivative is 0 to conclude that a zero of the function occurs at the point where there is a local extremum in the derivative function (in this case at $x = x_1$).

Table 8: The percentage of students who used reversal in question 4e

Question 4e	bottom	middle	top
4e correct	0%	7%	4%
reversal $f(x_1) = 0$	35%	27%	11%

In the extract 9, taken from the group discussion, the students try to identify the points of inflection of the function whose derivative is given in question 4.

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Extract 9:

1 *Simon:* x_1, x_3 *and* x_5 *because the turning point of a derivative the turning point of a derivative*

2 *Koki: will be the inflection point for the second derivative*

3 *Simon: ha? but here we're talking about the function not the second derivative*

4 *Koki: Oh. The turning point of the function will be the inflection point of the derivative*

5 *Palesa: ho*

6 *Simon: the turning point of this is the point of inflection of the function*

Koki's comments (line 2 and line 4) suggest an attempt to recall the rule. Her second attempt (in line 4) is a reversal of the correct rule.

The extract 10 is taken from earlier in the group discussion of question 4, where they discuss where the function will be increasing.

Extract 10:

1 *Simon: the graph of the derivative of a function f of x is shown below, give intervals on which f is increasing that means the function is increasing*

2 *Arthur: ja*

3 *Simon: when the function is increasing then the derivative must be concave up*

4 *Palesa: sorry. The graph of the derivative of a function is shown below then give the intervals on which f is increasing*

5 *Simon: ja between* x_2 *and* x_4 (this is the interval on which $f'(x)$ is concave up)

6 *Koki: between* x_2

7 *Palesa: zero and* x_2 (this is an interval on which $f'(x)$ is positive, but also concave down)

8 *Koki: mm hm*

9 *Arthur: between what?* x_2 *? Zero and* x_2 *?*

10 *Koki: zero and* x_2 *mm hm*

11 *Palesa: and then*

12 *Simon: is it supposed to be concave down?*

13 *Palesa: ja, it has to be positive*

14 *Simon: but concave*

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15 *Palesa: no concavity, don't confuse the two*

Simon immediately establishes an inappropriate reversed rule (line 3) as the method to solve this problem. He remains persistent in his belief that they need to be looking at the concavity of $f'(x)$ until Palesa finally intervenes (line 15).

The final extract in this section is taken from a discussion with Patricia about her solution to question 3.

Extract 11:

1 *L: OK do you have a method that you use to do these?*
2 *Patricia: that at the, at this point* (she points to a turning point on the graph of $f'(x)$)
3 *L: yes*
4 *Patricia: it must be my, the turning point must be my point of inflection*
5 *pause*
6 *L: the turning point is your…*
7 *Patricia: this is the turning point*
8 *L: yes*
9 *Patricia: then it must be the point of inflection. Then the point of inflection must be a turning point*

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Patricia initially provides a correct rule (line 4), but reverses it inappropriately in line 9. For her the fact that a turning point of the derivative function indicates a point of inflection in the function implies that a point of inflection of the derivative indicates a turning point of the function. At a later point in the interview with Patricia she explained how she and Nandipha drew the graph of $f(x)$ given the graph of $f'(x)$ in question 4 by saying "we used this method of point of inflection and turning point". This indicates that for Patricia this reversed rule is a rule that she applies consistently. Patricia's use of inappropriate reversal however appears to be different from Simon's. In extract 10 Simon inappropriately reverses "when the derivative is increasing the function is concave up" to "when the function is increasing the derivative

must be concave up". This parallels Patricia's inappropriate reversal. However further on the discussion of question 4 he provides an explanation of how to see the concavity of the function:

Extract 12:

1 *Simon: Here we are talking about the derivative of the derivative I mean we are thinking about the derivative of*

2 *Arthur: oh are we finding the f double prime or what?*

3 *Simon: no we are… let's discuss this so we understand it always. It's like there we have the function of the derivative, it's not a function it's the derivative so what do we do? We take a derivative of a derivative ja then we only look at that derivative where is it positive where is it negative. Where that derivative is positive therefore our function is concave up and then where that derivative is negative our function is concave down*

Simon's original recourse to inappropriately reversed rules (extract 10, line 3) thus eclipsed the reasoned understanding of the link between a function and its derivative which he demonstrates in line 3 of extract 12. The coexistence of an inappropriately reversed rule with a correct understanding of the concept again raises the question of competing schemas.

7.4 Concatenation

Table 9: Average slope

Average slope	bottom	middle	top
Qu 1d) $\frac{f(\frac{\pi}{3}+h)-f(\frac{\pi}{3})}{h}$ correct	15%	27%	61%
Qu 1d) drawn as $f'(\frac{\pi}{3})$	25%	20%	18%
Qu 2c) $\frac{f(a+h)-f(a)}{h}$ correct	35%	53%	54%
Qu 2c) Say it is a slope, but don't attempt any graphical depiction	10%	7%	11%
Qu 2c) Draw as $f'(a)$	0%	7%	3%

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Qu 2c) draw a tangent line half way between a and $a+h$	5%	10%	11%

It was apparent from the poor performance of students on question 1d and question 2c that most of the students did not have an appropriate image of the average rate of change.

In the interviews, it was apparent that for most students the distinction between average rate of change and instantaneous rate of change was not clear. Patricia and Nandipha's interview provides a good example of this:

Extract 13:

1 *Patricia: No let me ask something*

2 *L: ja*

3 *Patricia: Like number d* (which asks students to represent $\frac{f(\frac{\pi}{3}+h)-f(\frac{\pi}{3})}{h}$ graphically) *were we….*

4 *L: mm hm*

5 *Patricia: were we supposed to like we know f uh f, what's this, pi by 3, it is our, pi by 3 is here, this represents a slope*

6 *L: a slope of what though?*

7 *Nandipha: isn't it the same as this thing?* (they had just looked at $f'(\frac{\pi}{3})$)

8 *Patricia: mm*

9 *L: why do you think these two are the same?*

10 *Patricia: no ah I'm trying to ask something like…no man if we used to do this we used to do this like all those Newton quotients*

11 *L: mm hm*

12 *Patricia: ja, trying to find a slope*

13 *L: mm hm*

14 *Patricia: is this, my question is, is this somehow related to this?*

15 *L: yes. Are they the same thing?*

16 *Nandipha: mm*

17 *Patricia: (mumble) 'cos that is h*

18 *pause*

19 *Patricia: same ja*

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20 *L: you think they're the same*

21 *Patricia: ja....or related*

22 *laughs*

23 (continues discussing for a while then concludes by talking about what $\frac{f(a+h)-f(a)}{h}$ represents)

24 *Patricia: no what if we just say this one is f dash a plus h*

25 *Nandipha: mm*

For Patricia it is apparent that the connection between the Newton quotient and the derivative is stronger than any sense of the process underlying the formula $\frac{f(a+h)-f(a)}{h}$. Despite her uncertainty as to whether $f'(\frac{\pi}{3})$ and $\frac{f(\frac{\pi}{3}+h)-f(\frac{\pi}{3})}{h}$ are the same or just related (line 21), she ends up concluding that $\frac{f(a+h)-f(a)}{h}$ must be $f'(a+h)$ (line 24). The discussion leading up to this conclusion indicates that the process behind the definition of the derivative at a point has been lost and that for Patricia the connection between these two ideas is based on the fact that they were learnt at the same time.

7.5 Concatenation and generalisation

In question 4c the students were asked to deduce where the local maxima and minima of $f(x)$ would be from the graph of $f'(x)$. The group discussion of this question is given in the extract below.

Extract 14:

1 *Simon: if we have a global minimum (pause)*

2 *Arthur: heh?*

3 *Simon: we have a global minimum we don't have a local max..minimum, ja*

4 *Arthur: where is the global minimum, isn't that the local minimum?*

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5 *Simon: no*

[...]

6 *Simon: hmm a local maximum, I mean, a local maximum let's define it. What is a local maximum?*

7 *Arthur: a local maximum, the highest point on the curve*

8 *Palesa: ja*

9 *Simon: the highest ne*[4]*?*

10 *Arthur: on the curve*

11 *Simon: on the curve*

12 *Arthur: ja*

13 *Simon: and then*

14 *Arthur: local minimum*

15 *Simon: local minimum*

16 *Koki: lowest turning point on the curve*

17 *Simon: mm*

18 *Arthur: and if there were like I don't know whether to say asymptotes or whatever*

19 *Palesa, Simon: mm hm*

20 *Arthur: those would represent the global maximum or minimum*

[...]

21 *Koki: they only asked if the local minimum and the maximum only we're not asked if the global*

22 *Simon: mm hm*

23 *Arthur: but I think sometimes they are similar*

[...]

24 *Simon: it's* x_4 *, ne, then put the local minimum the coordinate is* x_4 *and*

25 *Arthur: why do you say* x_4 *is the lowest point*

26 *Simon: I mean in the function for this interval that we are giving from zero to* x_6

27 *Arthur: how?*

28 *Simon: it will have the lowest y value because it's a concave down*

29 *Arthur: mm hm*

[4] "ne?" is an expression which means "isn't that so?"

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30 *Simon: and it will only have one turning point and it's concave down therefore this will be lowest value like if this was supposed to here if this was supposed to be negative 10 if this was negative 10*

31 *Arthur: ah ha*

32 *Simon: on this interval the only value of y that is the least value of y was gonna be this negative 10*

These students have concatenated the ideas of local and global extrema. However when they attempt to define the difference between the two ideas they state that the local maximum is "the highest point on the curve" (line 7), an idea they carry through their discussion of this problem. This resonates with the idea of generalisation on the basis of colloquial language cues. Simon's final explanation (lines 26-32) of local minimum reinforces this idea: he equates the local minimum incorrectly with the graph being concave down and says it is the lowest point on the curve. He uses a negative number as his example of the possible minimum value. This chain of words minimum, down, lowest and negative certainly make sense on the basis of the colloquial use of these words. Global extrema are relegated, by Arthur, to places where something odd, like asymptotes, occur (line 18). Despite the use of everyday language to give meaning to the terms "maximum" and "minimum", it is interesting to note that the students have not made any use of the meanings of "local" and "global".

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This discussion highlights the dynamic relationship between the different modes of construction set out in the discussion of the examination errors. The concatenation of the concepts of global and local extrema appears to be strongly linked to the generalisation of the notion of local maximum or minimum.

7.6 Gap-closing strategies

The group discussion about question 5 provides insight into the kind of strategies students use in producing their answers. Question 5 was

particularly difficult for students, as the table showing the percentage of correct answers demonstrates.

Table 10: Percentage of students answering question 5 correctly

Question 5	bottom	middle	top
a	5%	10%	29%
b	5%	7%	7%
c	5%	7%	7%
d	5%	10%	4%
e	0%	7%	4%
f	0%	13%	14%
g	5%	10%	7%
average	4%	9%	10%

The extract below shows the group discussing question 5a which asked the students to determine the length of a queue at 9am. The question gave them a graph depicting the rate at which people arrive at the ticket office to buy tickets for a soccer game. They were also told that people start arriving at 8am, but the ticket office only opens at 9am after which people are served at a rate of 200 people per hour.

Extract 15:

1 *Simon: you know what what is this?*

2 *Arthur: what?*

3 *Simon: this is the graph of rate, the rate (Koki: mm) which is number of people per hour (Palesa: OK) therefore to get the number we gonna divide by the hour*

4 *Palesa: no multiply*

5 *Simon: multiply by the hour whatever ja so you (pause)*

6 *Palesa, Koki (talk together)*

7 *Simon: mm?*

8 *Koki: 240 is the rate*

9 *Palesa: ja multiplied by an hour*

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10 *Simon: nine hours we're giving that these people come at 9 hours*

11 *Koki: where is that*

12 *Palesa: 9 am*

13 *Simon: ja we're gonna multiply by 9, you see*

14 *Arthur: mm hm*

15 *(continue to discuss. Palesa points out it should be 1 hour not 9)*

16 *Simon: the rate*

17 *Palesa: the take the rate and multiply by the hour*

18 *Simon: 1 800 were there. They wanted to see Bafana Bafana*[5]
[Palesa corrects Simon and explains that between 8 and 9 there is 1hour]

19 *Simon: ja in an hour there will be 200 people .. .200 people*

20 *Arthur: exactly*

21 *Palesa: maar no this doesn't say there will be 200 people. People can be served at a rate of 200 per hour. 200 per hour. They can serve 200 per hour, not that there will be 200 people arriving*

22 *Simon: arriving there, ja..so*

23 *Palesa: that is*

24 *Simon: (laughs)*

25 *Palesa: that is take this this rate multiply by an hour. This is 240 exactly.*

Simon's strategy, having established that the number of people will be equal to rate times time, is to combine the two numbers given in the question. For him the rate is 200 people per hour and the time is 9.

In the next extract they are trying to find the length of the queue at 10am.

Extract 16:

1 *Simon: if maybe we take them we make an ideal situation where number of people is fixed. Those who are come at 9 o'clock they are only one whose gonna be served. Then this is gonna be OK in here we had number how many people times only an hour.*

2 *Koki: 240*

3 *Simon: 240 people ne*

4 *Arthur, Palesa, Koki: mm*

[5] Bafana Bafana is the name of the South African soccer team

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5 *Arthur: so they started serving them*

6 *Koki: here they served about 200 people.*

7 *Simon: mm hm*

8 *Palesa: not that the queue..there are still people coming*

9 *Koki: mm*

10 *Palesa: you can't say 240 minus 200*

11 *Simon: therefore the number of people remain constant here*

12 *Arthur: ja*

13 *pause*

14 *mumble*

15 *Arthur: because here there were people coming so the graph will increase you know and they start serving them*

16 *Simon: OK ja what… when they started serving them the number is remaining at a constant. You see?*

17 *Arthur: so it is still 240 people*

18 *Simon: mm hm*

Simon introduces the idea that they fix the number of people in order to simplify the question (line 1). Arthur, Palesa and Koki appear to be attempting to grapple with the complexity of the question, but Simon then becomes insistent that there arguments simply point to his original idea that the number of people will remain constant (line 11). He uses the idea that if people are arriving and people are being served the queue will remain constant (line 16). In deciding this he makes no reference to the information given in the question.

The next extract from the question illustrates how, as the complexity of the question increased, the students begin to "make up" information not given.

Extract 17:

1 *Koki: the length of the queue at 11 am*

2 *Arthur: at 11. Ah people are now inside the stadium*

3 *Koki: ja it should be like less*

4 *Simon: most of the people now are in the stadium*

5 *Palesa,Arthur: mm, ja*

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6 *Arthur: that nobody's arriving*

7 *Simon: they are looking at Marks Maponyane[6] at zero man and also at who didn't see the first thirty minutes. At three hours, the first three hours, ey they've missed a lot. OK this is 120, 140*

[...]

8 *Palesa: like there were, they came here to watch Bafana Bafana*

9 *Simon: ja*

10 *Palesa: and the soccer takes about how many hours*

11 *Arthur: 45, 45 90 minutes*

12 *Palesa: 90 minutes (Koki: mm) this is 1 and a half hours (Arthur: mm) you see, OK we're going to sing the national anthem*

13 *Arthur: for 2 hours*

14 *Koki: (laughs)*

15 *Palesa: no. It will be minutes*

16 *Arthur: oh*

17 *Palesa: and then this is.....*

Extracts 16 and 17 were chosen because they emphasise a gap-closing strategy that did not come through strongly in the analysis of the exam errors. In this question contextual information provided in the question is used to aid the students in answering the question. It is interesting to note that Palesa, who displayed some understanding of the question in her attempt at question 5a, also eventually resorts to using this contextual information. There is a sense in the tone of the discussion, particularly in the final extract, that the students know that some of the information they are bringing in is irrelevant and thus that they are out of their depth with this question. This is borne out in their response to my challenge:

Extract 18:

1 *L: if I asked you tell me how confident you were in those answers, right, like if I said to you, um, I'll give you a hundred rand if you got those right and you must give a hundred rand if you got them wrong*

2 *Shrieks*

3 *L: would you take me up on the bet?*

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4 *Palesa: No*

5 *L: for the question, ja?*

6 *Pause*

7 *Simon: for this I was gonna take you a bet (laughs).*

However there are some serious attempts to make meaningful interpretations using the contextual information. For example, Koki's conclusion that there should be fewer people in the queue at 11 (extract 17, line 3) or Simon's conclusion that the people being served means the queue stays constant (extract 16, line 16). We do want students to make connections between the context and the mathematical solutions they are offering. However the strategy the students are using here is to ignore the mathematics in favour of their own interpretation of the context.

A comparison of question 4c and question 5e brings to light a further point about gap-closing strategies which was not immediately apparent in the exam analysis. Question 4c and question 5e both asked students to identify where the function was at its maximum, based on information about the derivative of that function.

Table 12: Comparison of question 4c and question 5e

Qu 4c and 5e where is f max	Bottom	Middle	Top
Qu 4c correct	10%	30%	54%
Qu 5e correct	0%	7%	4%
Qu 4c just chose where f' max	45%	13%	11%
Qu 5e just chose where f' max	75%	60%	50%

Question 5e was more difficult than question 4c in that the students were required to combine information about the rate from the graph and information supplied in the question. That, together with the fact that question 5e was embedded in contextual information, makes it unsurprising that the students fared worse on the question. However it is interesting to

[6] A South African soccer player

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note that there is a marked increase across all three groups in simply reading off the maximum value of the graph in question 5e. This points to the fact that using surface level representation as a gap-closing strategy is not a consistently applied strategy, but occurs when the complexity of the question prevents the students from providing a reasoned answer.

7.7 Language

As stated previously, with regard to language, the most I will do here is to point to some potential language issues for further study.

The first issue, that students had difficulty in discerning what constitutes a valid argument, corresponds with the findings of the examination analysis. In question 4e the students were asked, on the basis of the graph of $f'(x)$ to write $f(0), f(x_1)$ and $f(x_6)$ in increasing order and to provide reasons for their answers. Below are three examples, taken from the test scripts, of the reasons provided.

"because $f(x_6)$ is negative and $f(x_1) = 0$ and $f(0)$ is positive"

"because if a graph is drawn it will be clear"

"because x_6 has a small corresponding $y-$value, 0 has a small corresponding $y-$value and x_1 has a big corresponding $y-$value"

Because the students were only given the graph of $f'(x)$ none of these responses can be seen as providing a justification for their ordering. As was the case in the examination, the students repeat the statement using a different form of representation as their justification for the statement.

The following quotes, taken from the group discussions and interviews, illustrate some of the issues about how the students talk about mathematical concepts.

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In talking about an increasing graph:

Simon: "the x-axis is increasing"

In talking about the area under the graph of $f(x)$ over an interval:

Godwin: "the area of the curve"

In talking about the area under a straight line joining opposite corners of a rectangle:

Isiah: "Should be the area under the slope"

In talking about $\frac{f(a+h)-f(a)}{h}$

Nandipha: "the slope at these two points"

Dan: "it's the derivative between points"

Vuyisela: "you've got the point and that point, then the slope is a line on that point to that point"

(In all these three cases the students correctly illustrated $\frac{f(a+h)-f(a)}{h}$)

As before, because I did not specifically probe these language issues, it is not possible to conclude that they are a result of the students' attempts to express themselves in a language other than their mother tongue, or whether they betray a lack of understanding of the concepts involved or indeed whether there is a relationship between learning mathematics in a second or third language and conceptual understanding.

The "vignettes" presented in this chapter lend support to the model of student learning presented in chapter 6. This model has implications for curriculum design and further research which are discussed in the final chapter.

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CHAPTER 8 CONCLUSION

8.1 The theoretical framework

The goal of this study was to use theories of student learning to provide a framework with which to understand the misconceptions that students from disadvantaged educational backgrounds display in dealing with calculus concepts.

The learning theories of Sfard and Dubinsky have been enormously influential in this research project. Their discussion of the process-object duality of mathematical concepts, together with empirical studies that highlight the usefulness of this approach in analysing the development of students' understanding of mathematical concepts provided the motivation to use these theories as the basic framework for this research project. In looking at students' misconceptions through the lens of learning theories I have been able to develop a model for understanding these misconceptions in the light of the mechanisms students used to construct their knowledge of calculus. As the model was developed out of an analysis of student errors, its contribution is to elaborate on the work of Sfard and Dubinsky in relation to problems in the learning process.

Both Dubinsky and Sfard allude to the inherent difficulty of reifying mathematical process into objects. Sfard and Linchevski argue that this difficulty results in many students developing what they term, a "pseudostructural approach". An analysis of students' algebraic errors showed strong evidence of this pseudostructural approach. It was this issue that provided the impetus for the generation of the model. Dubinsky's model of a schema provides a coherent description of the modes of construction of mathematical knowledge. However this model has as a starting point the idea that actions on mathematical objects are interiorised into processes. My data suggested that the starting point for my students' construction of calculus concepts was what I have termed "pseudo-objects". This meant that

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Dubinsky's model could not be used without modification to provide an interpretation of student learning. Despite this, the analysis of student errors pointed to modes of construction, albeit in a distorted form, that resonated with those Dubinsky outlines. This study thus extends the work of Dubinsky and Sfard by providing a model of the construction of mathematical knowledge based on a pseudostructural conception of base knowledge.

In this model I have argued that actions on pseudo-objects become rehearsed into rules. In addition I put forward the idea that because these rules involve the mechanistic manipulation of pseudo-objects, the methods used to link, extend and apply this "knowledge" are inappropriate. The model suggests that students' errors are not arbitrary, but originate from an attempt to build a manageable set of techniques for dealing with calculus questions. Because this set of techniques is not grounded in a conceptual understanding of calculus, it provides an inadequate range of problem-solving tools. The students' creative use of gap-closing strategies is the way in which they render a given calculus problem manageable via this set of techniques.

Although the analysis of student errors demonstrated that students used strategies consistent with the model described, there were instances where the students were able to provide a correct answer when queried about their solutions. This raises the question of whether students possess competing schemas. Dubinsky describes a schema in the following way:

> An individual may form a coherent collection of actions, processes, objects and other schemas that is her or his SCHEMA for the concept in question. By coherence, we mean that there is some means, explicit or implicit, which determines what is or is not allowed as part of the schema. (posting to MATHEDU, 1997).
> All we can say is that a subject will have a propensity for responding to certain kinds of problems in a relatively (but far from totally consistent way) which we can (as far as our theory has been developed) described in terms of schemas. (Dubinsky, 1991, p103)

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In an analogous way I believe that my model shows a coherent collection of pseudo-objects, actions and rules that can be used to describe the propensity that students display in responding to certain kinds of questions. It is for these reasons I have suggested that my model describes the modes these students use to construct an inappropriate schema for dealing with calculus. There were moments in the interviews and discussions when a student, having given an incorrect answer that was consistent with the model of a schema I described, was later able to provide a correct explanation of that particular concept. As this correct answer did not belong in the coherent collection of pseudo-objects, actions and rules, the question of whether it formed part of a different schema arises. Neither Dubinsky's model nor the model developed in this research project addresses the question of whether students can possess competing schemas. I believe that this is a question which requires further research for, if this is so, how do students attempt to reconcile these competing schemas? Moreover, if such competing schemas exist, answers to the questions "What causes the inappropriate schema to dominate students' responses to certain questions?" and "Can a route to developing more appropriate schemas be aided by dismantling the inappropriate schema?" would be very useful in furthering our understanding of student learning.

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8.2 The calculus reform movement

The fact that this type of learning occurred within a mathematics course that had been heavily influenced by ideas emerging from the calculus reform movement cannot be taken as a negative evaluation of the calculus reform movement. In setting up this research project I made no attempt to compare students being taught calculus using different approaches, nor did I detail the ways in which we used ideas emerging from the calculus reform movement in the design of the course. Neither the teaching of the course nor the ways in which reforms were implemented in the course are presented for scrutiny here and thus I cannot claim that, differently taught, these ideas would not

produce more encouraging results. Despite this, I do believe this research has important implications for the calculus reform movement.

Curriculum development needs to be informed by and evaluated in light of research on student learning. Perhaps because of the debates and controversies that have surrounded the calculus reform movement, strong positions are put forward about what constitutes "good" calculus teaching. The appeal of notions like "a guiding theme should be to concentrate on greater conceptual understanding, developed through extensive numerical, graphical, algebraic and modelling interpretations " (Tucker and Leitzel, 1994, p1) make it easy to assume that simply using these multiple interpretations will enhance student learning. With this understanding, a question like "Sketch the graph of the derivative function on the basis of the graph of a function" can hold enormous appeal for teachers as an alternative to the mechanical production of derivative formulae. However the picture of student learning that emerged from my research implies that students are able to reinterpret this kind of question using mechanistic rules. It is for precisely for this reason that I believe we need to link curriculum innovations with a research-based approach to student learning. By enhancing our understanding of how students learn we are better able to assess the usefulness of new approaches and can become more critically aware of the potential pitfalls in the way we implement them.

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8.3 Disadvantaged students

The fact that the subjects of the research were students from disadvantaged educational backgrounds raises the question of whether the coding scheme for student errors and the model of learning developed out of this could be of use in exploring the misconceptions of better prepared groups of students. Although further research is clearly needed to explore this issue, I believe that the research framework outlined in this report could prove especially useful in improving our understanding of supposedly well-prepared students who fail calculus courses. At most South African institutions, preparedness

for university study is judged largely on the results students obtain in their school-leaving examinations. At UCT, students obtaining good results in their school-leaving mathematics examination are placed on the standard mathematics course, MAM100W. However, not all of these students manage to cope with the course. In 1998 of the 380 students who enrolled for MAM100W about 80 had a class mark of less than 25% by the end of the 1st quarter. I believe that the framework provided by this research project could prove useful in analysing the difficulties these students have and in making more informed decisions about placing students on courses appropriate to their educational needs. Extending the sample to include different categories of student also opens up the possibility of extending and increasing the descriptive power of the model.

In situating the research within the context of a course for disadvantaged students and in focusing on the negative aspects of student learning, I recognise there is a danger in leaving the reader of this research with the conclusion that it is futile to attempt to teach calculus to such students. The belief which motivated this research project was that these students are capable of learning mathematics and that understanding students' difficulties in learning is an important first step in designing curricula that can help students to succeed. From this view point I believe that this project raises a number of avenues for further research.

The awareness that a pseudostructural approach to algebra leads to a reinterpretation of processes in calculus as rules has provided a starting point for rethinking the design of the curriculum. This awareness has meant that in the teaching of the course we have focused on the processes through which algebraic formulae are developed and placed emphasis on the interpretation of algebraic symbols and formulae. We continue to place emphasis on multiple representations, but far more care is taken to ensure the underlying concept which links the representations is made apparent. Further research is necessary to evaluate whether these strategies are bringing us closer to our aim of improving conceptual understanding.

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In terms of curriculum design, the results of this research project do address the question of whether re-teaching the school syllabus will be beneficial for students who have had poor educational backgrounds. It is apparent from the model put forward that students build their knowledge in accordance with their previous experience of mathematics. Re-doing the school syllabus will not fundamentally challenge the pseudostructural approach the students have built unless the mathematical content is dealt with in an entirely different way. In a similar vein, allowing students to use a computer algebra system to do algebraic manipulations seems unlikely to "level the algebraic playing fields" as some of its proponents have argued (for example, Bennett). Students' difficulties in algebra are not so much in the manipulation of formulae, but in the meaning that they attribute to symbols and expressions. This is not saying that a computer algebra system might not be able to be used to help build a good conceptual understanding of both calculus and algebra, but that simply handing over the algebraic manipulations to the computer will not free students of their difficulties in algebra.

Dubinsky argues that he and his colleagues have found activities with computers to be particularly helpful in fostering reflective abstractions (Dubinsky, 1991). He argues that

> if a student implements a process on a computer [...] then the student will, as a result of the work with computers, tend to interiorise the process. If that same process, once implemented, can be treated on the computer as an object on which operations can be performed, then the student is likely to encapsulate the process (p123).

In a similar vein, the work of Tall and Thomas (1991) has provided evidence that the computer can be used as an educational aid in encouraging versatile thinking in algebra in the case of young children beginning their studies of algebra. The successes described by these authors suggest that it would be fruitful to explore whether similar strategies could aid students who appear to

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have a strongly entrenched pseudostructural approach to algebra to reconstruct this knowledge.

A final and perhaps crucially important point is raised by Sfard's argument that reification is inherently difficult. She states

> the intellectual effort required from the students to create for themselves the entire universe of intangible objects is really immense. The learner must have a lot of determination, stamina and intellectual discipline to cope with the demanding task. Because of the inherent difficulty of reification, nothing will happen without a genuine drive for understanding. (1992, p84)

Convincing students to give up a mode of learning that has successfully earned them a place at university and replace it with a struggle for genuine understanding is not an easy task. It seems evident to me that there can be no quick and easy solutions. To paraphrase Sfard, a lot of determination, stamina and intellectual discipline is required to design, test and redesign curricula in order to build a learning environment in which students are challenged to develop a conceptual understanding of calculus and are provided the tools with which to do so.

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APPENDIX A
EXAMINATION

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M12.

UNIVERSITY OF CAPE TOWN

University Examinations — October/November 1996

Department of Mathematics and Applied Mathematics

MAM105H and MAM105S

Time: 3 hours Full marks : 100

Marks available : 100

Only calculators which have been approved by the Department of Mathematics and Applied Mathematics may be used.

1. The **ANSWERS ONLY** to the questions in Section A must be entered in the blocks provided on the answer sheet for Section A.
2. The solutions to the questions in Section B must be written in your answer book and clearly numbered. Show all your work.

A table of integrals is attached. This may be used in any question involving integration.

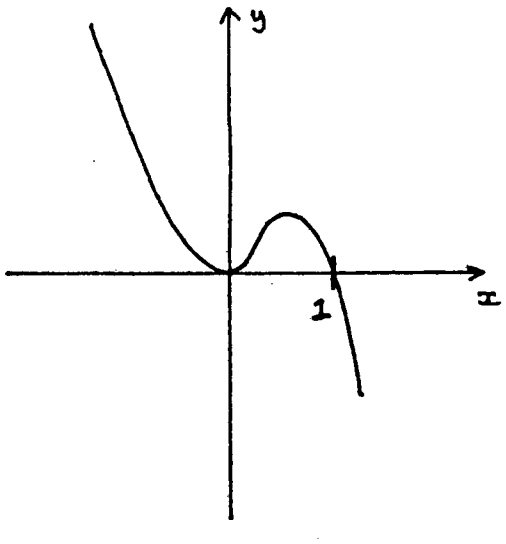
This paper consists of 5 question pages and a table of integrals.

SECTION A (33 marks)

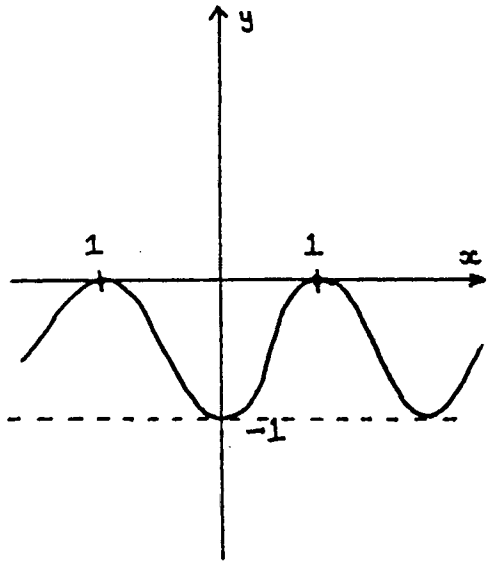
1. Find the equation of the tangent line to $f(x) = \ln x$ at
   (a) $x = 1$
   (b) $x = c$ [2, 3]

2. Find a possible equation for each of the following graphs.

(a)



(b)



[2, 2]

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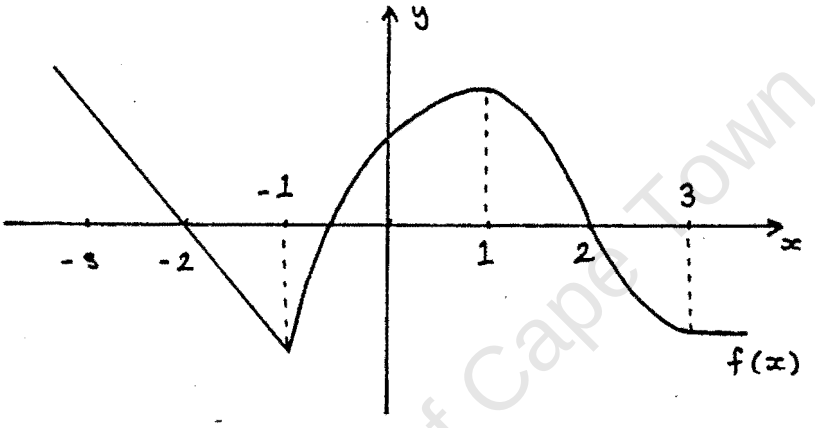
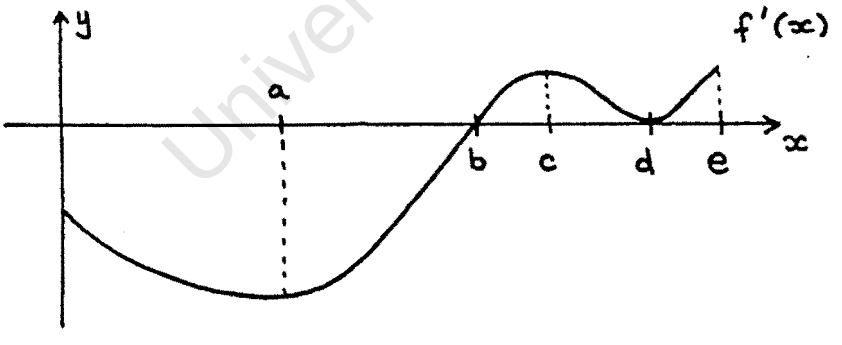
3. In 1960, radioactive strontium-90 was released during atmospheric testing of nuclear weapons and got into the bones of people alive at the time.
   (a) If the half-life of strontium-90 is 29 years, write a formula for the fraction of strontium-90 remaining in their bones t years after 1960.
   (b) How long will it take until only 10% of strontium-90 remains? [2, 1]

4. Sketch the graph of the derivative function of $f(x)$.



[3]

5. The graph of $f'(x)$ (NOT $f(x)$) is shown below



   (a) Give the intervals on which f is increasing.
   (b) Give the intervals on which f is concave down.
   (c) Does $f(x)$ have a local minimum on $[0, e]$? If so, at what x-value does it occur?
   (d) Does $f(x)$ have a local maximum on $[0, e]$? If so, at what x-value does it occur?
   (e) Does $f(x)$ have any points of inflection on $[0, e]$? If so, at what x-values do they occur?
   (f) If $f(0) = 0$ will $f(e)$ be positive or negative? Motivate your answer. [8]

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M12.

6. Evaluate the following integrals:

(a) $\int \sin^3\theta \cos\theta \; d\theta$

(b) $\int \frac{1}{x^2+2x+1} \; dx$

(c) $\int x \ln x \; dx$

(d) $\int \frac{1}{\sqrt{1-9x^2}} \; dx$ [2, 2, 3, 3]

SECTION B (67 marks)

7.(a) Use the definition of the derivative to find $f'(2)$ if $f(x) = x^3$.

(b) Check your answer using the rule for differentiating x^n. [3, 1]

8. Calculate the following derivatives:

(a) $\frac{d}{dx}((\sin x) \cdot (\ln x))$

(b) $\frac{d}{dt}(2^{(t^2)})$

(c) $f'(x)$ if $f(x) = \frac{(2-x)^4}{x^2+3}$

(d) $\frac{dP}{dt}$ if $P = P_0 \arctan(\frac{t}{c})$ [2, 2, 3, 2]

9.(a) Sketch the graph of $f(x) = |x+1| - |x|$

(b) Use (a) to solve $f(x) < \frac{1}{2}$

(c) Find all points where $f(x)$ is differentiable. [3, 2, 1]

10.(a) Sketch the curve $9x^2 + 3y^2 = 27$.

(b) Find the equation of the tangent line to the curve at $(1, \sqrt{6})$. [2, 3]

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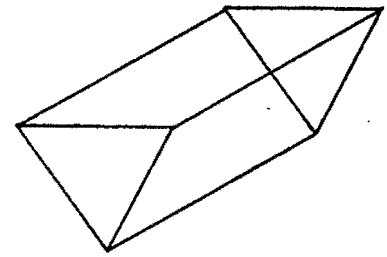
11.(a) The ends of a water trough are equilateral triangles (see diagram). If the trough is $4m$ long and water is being pumped into it at $2m^3/\text{min}$ find the rate at which the water level is rising when the depth of the water is $0,2m$.

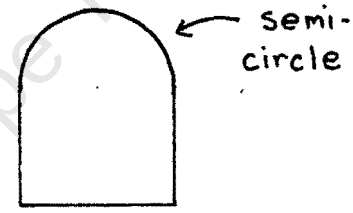
(b) As the amount of water in the trough increases will the rate at which the depth is increasing increase or decrease? Justify your answer.



[4, 2]

12. Show that $x - e^x < 0$ for all x. [4]

13. You want to make an arch window (see diagram). If the perimeter of the window has to be $1.5m$ what dimensions will give you the largest window possible? [6]



14. Given that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, find a formula for $\sum_{i=m}^{n} i$ for $1 \leq m \leq n$. [2]

15. State the Binomial theorem and use it to find the constant term of $\left(x + \frac{2}{x}\right)^{10}$.

[3]

16. Calculate the following integral. Show all your working.

$$\int_0^1 x \arctan(x^2)\, dx$$ [5]

17.(a) Sketch and shade the area enclosed by the two graphs

$$y = x^2 - 2x \quad \text{and} \quad y = x + 4$$

(b) **Calculate** the area you shaded in part (a).

(c) Write up the integral (**but do not work it out**) that gives the volume of the solid obtained when the area you sketched in (a) is rotated around the line $y = -1$.

[2, 4, 2]

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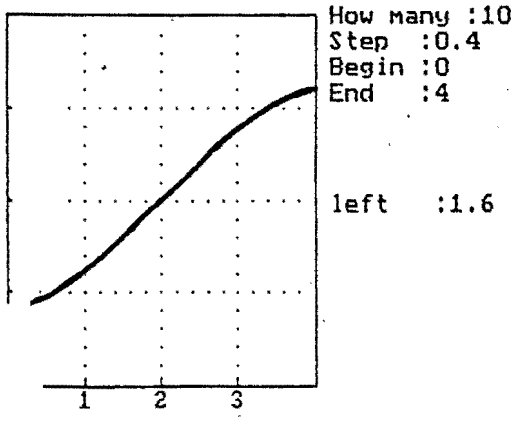
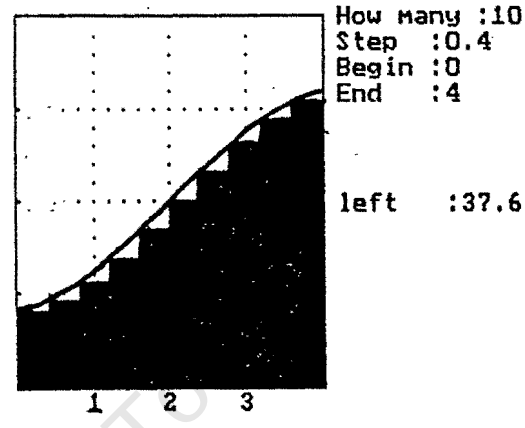
18. The printouts below show the calculation of a left-hand sum for a function $f(x)$, using "A Graphic Approach to the Calculus".





You are also told that:

- The right-hand sum for $n = 10$ is 42.4
- $\int_0^1 f(x)\,dx = 4.9$
- $\int_1^2 f(x)\,dx = 8.1$
- $\int_2^3 f(x)\,dx = 11.9$
- $\int_3^4 f(x)\,dx = 15.1$

(a) Calculate $f(0)$. (Show all your working.)

(b) Calculate $f(4)$. (Show all your working.)

(c) Calculate $\sum_{i=0}^{9} f(\frac{4i}{10})$. (Show all your working.)

(d) You are told that if we take $n = 100$ rectangles, the left-hand sum for this graph is 39.76. Calculate the right-hand sum for $n = 100$ rectangles.

(e) Draw a sketch of $F(x)$, the function such that $F'(x) = f(x)$. Assume that $F(0) = 0$ and mark the **exact** values of $F(1)$, $F(2)$, $F(3)$ and $F(4)$.

(f) Explain – using a sketch – what the **trapezoidal rule** is. For $n = 10$, what estimate of $\int_0^4 f(x)\,dx$ does the trapezoidal rule give?

[1, 2, 2, 2, 3, 3]

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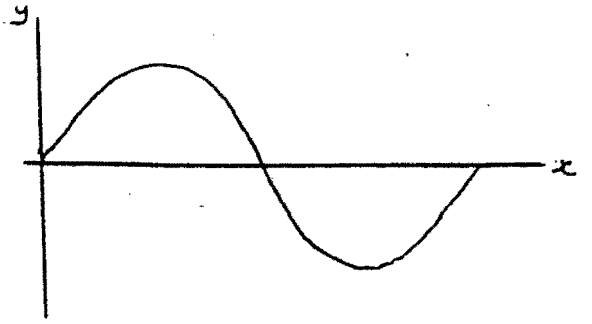
APPENDIX B
CONCEPTUAL TEST AND ANSWER SHEET

University of Cape Town

MAM105H REVIEW 1 1996

Question 1: $f(x) = \sin x$, $0 \le x \le 2\pi$. For questions a - e indicate clearly on the graph provided each of the following. State clearly whether each is represented by a length, slope or area.



a) $f'\left(\frac{\pi}{3}\right)$

b) $f'(2)$

c) $f\left(\frac{5\pi}{6}\right)$

d) $\dfrac{f\left(\frac{\pi}{3}+h\right)-f\left(\frac{\pi}{3}\right)}{h}$

e) $\int_0^{\frac{\pi}{2}} f(x)dx$

f) Find the average value of $f(x)$ on the interval $0 \le x \le \pi$.

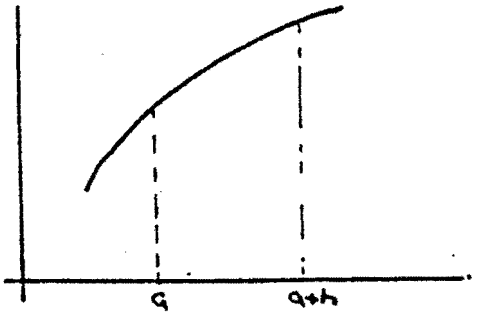
g) State all the intervals on which $f(x)$ is increasing

h) State all the intervals on which $f(x)$ is concave down

i) State all points of inflection of $f(x)$

j) If $f(x)=F'(x)$ show on the sketch $F(\frac{\pi}{3}+h)-F(\frac{\pi}{3})$

Question 2: Part of the graph of a function f is shown on the sketch below:
For questions a - f indicate clearly on the graph provided each of the following. State clearly whether each is represented by a length, slope or area.



a) $f'(a+h)$

b) $f(a)$

c) $\dfrac{f(a+h)-f(a)}{h}$

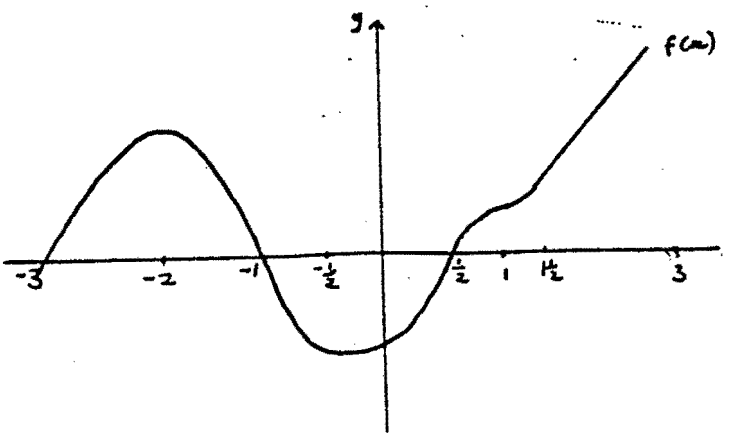
d) $\frac{1}{h}\int_a^{a+h} f'(x)\,dx$

e) $\int_a^{a+h} f'(x)\,dx$

f) $hf(a+h)$

g) Show clearly on the graph how you would see approximately where $\frac{1}{h}\int_a^{a+h} f'(x)dx$ is.

Question 3: Below is the graph of $f(x)$. Sketch the graph of $f'(x)$



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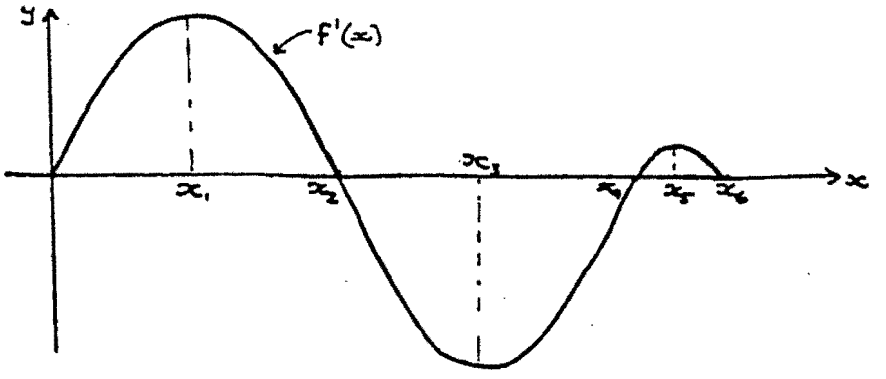
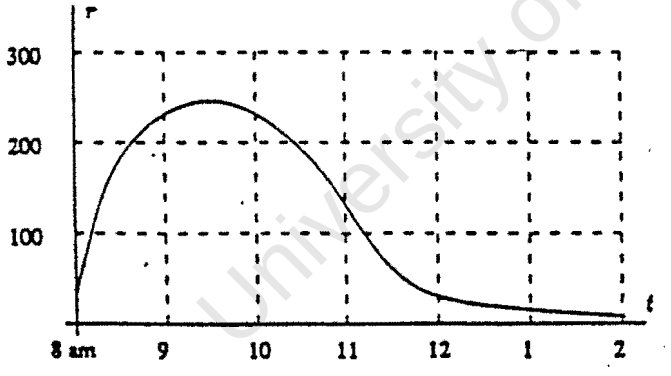
Question 4: The graph of the **derivative** of a function $f(x)$ is shown below:



a) Give the intervals on which f is increasing
b) Give the intervals on which the graph of f is concave down
c) Does f have any local maxima or minima on the interval $[0, x_6]$? If so give the x-coordinate of each such point, and say whether it is a local maximum or minimum.
d) Does f have any points of inflection on $[0, x_6]$? If so, give the x-coordinate of each such point.
e) Write the following in increasing order: $f(0)$, $f(x_1)$, $f(x_6)$. Give reasons for your answer.

Question 5: Below is the graph of the **rate** r (in number of people who arrive per hour) at which fans arrive at the ticket office in Greenpoint Stadium in order to get tickets for a Bafana Bafana game. The first people arrive at 8a.m. and the ticket windows open at 9 a.m. Suppose that once the windows open, people can be served at a rate of 200 per hour.



Use the graph to find or provide an estimate of:

a) the length of the queue at 9a.m. when the windows open.
b) the length of the queue at 10a.m.
c) the length of the queue at 11a.m.
d) the rate at which the queue is growing in length at 10a.m.
e) the time at which the length of the queue is maximum
f) the length of time a person who arrives at 9a.m. has to stand in line.
g) the time at which the queue disappears (i.e. when anyone arriving is served immediately.)

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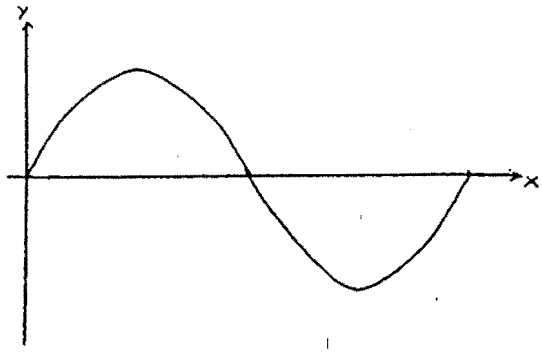
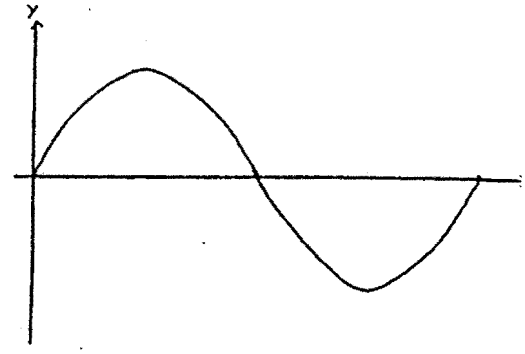
MAMIUSH ANSWER SHEET NAME:

Question 1

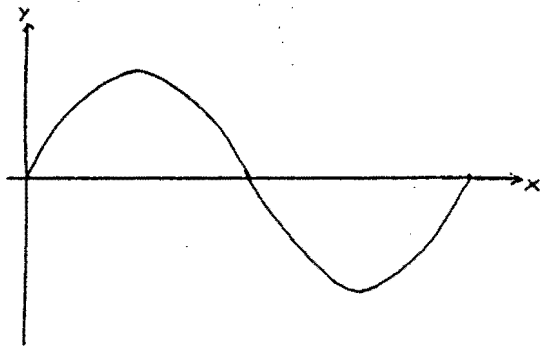
a)



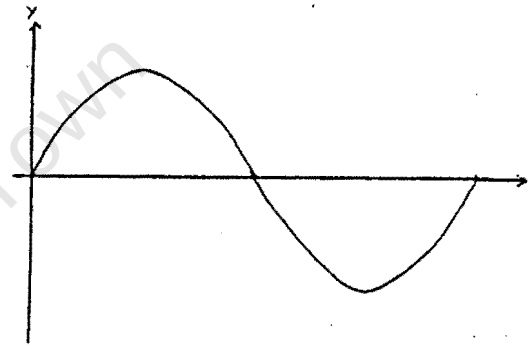
b)



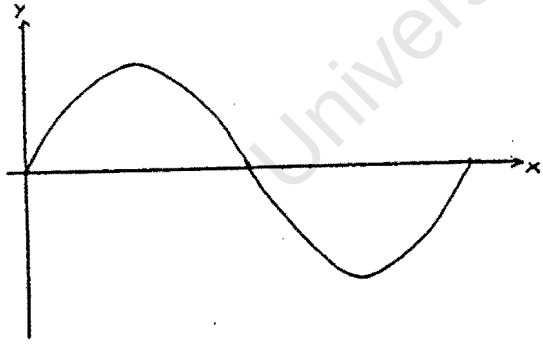
c)



d)



e)

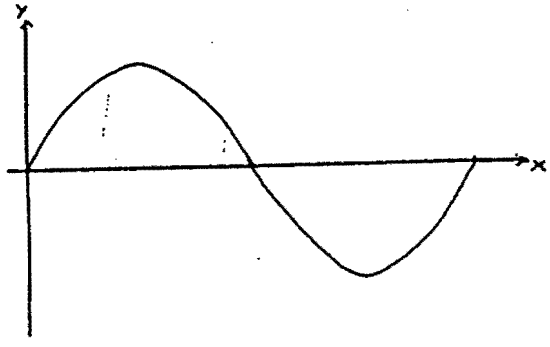


(f)

(g)

(h)

j)



(i)

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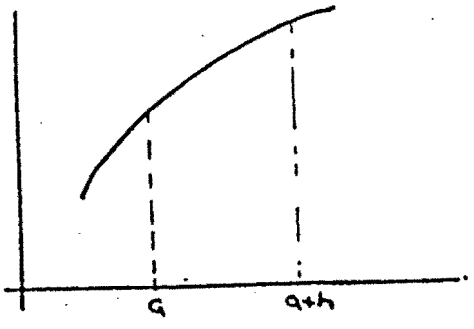
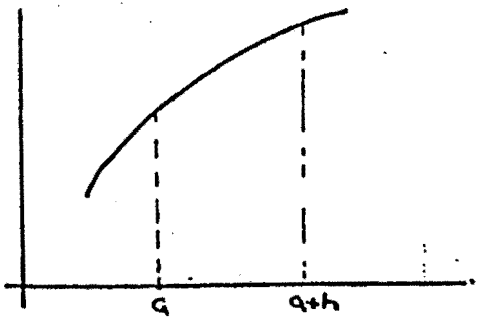
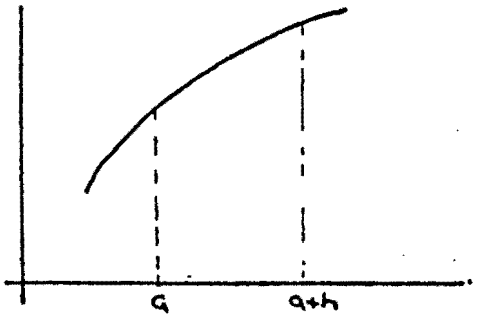
Question 4

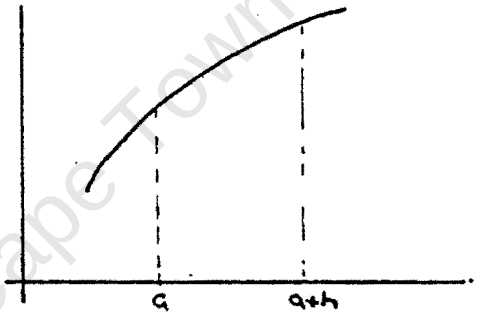


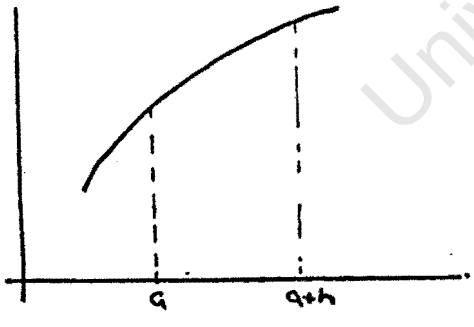
b)



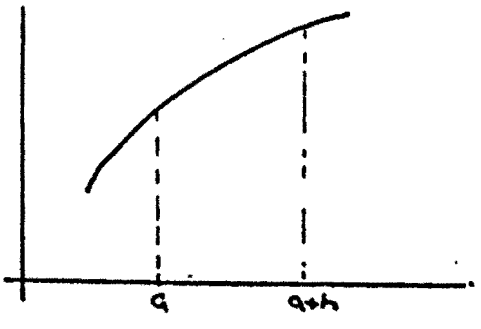


d)





f)



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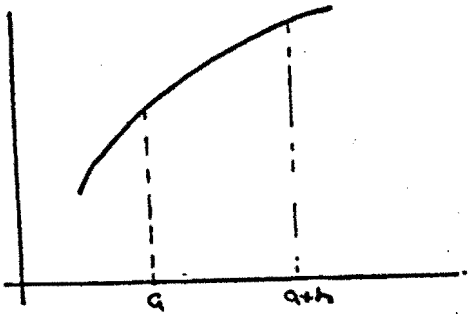

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APPENDIX C
INTERVIEW QUESTIONS

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INTERVIEW QUESTIONS

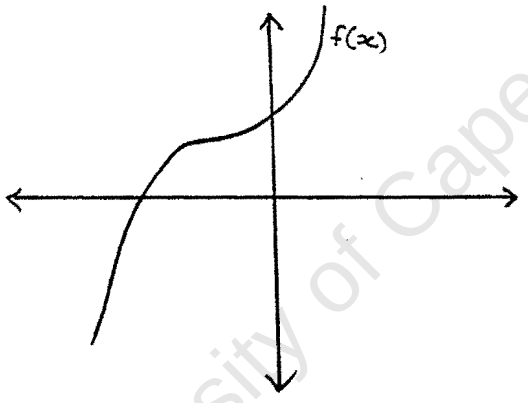
The interviews were semi-structured. The questions were based on the conceptual test. The questions presented here represent the basic questions asked to all the groups of students interviewed.

In questions 1 and 2 the students were given the graph of $f(x) = \sin x$ as in question 1 of the conceptual test.

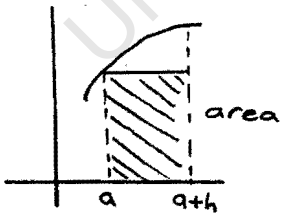
1. Indicate $f'(2)$ on the graph of $f(x) = \sin x$. State whether it is a length, slope or area.
2. Indicate $f(\frac{5\pi}{6})$ on the graph of $f(x) = \sin x$. State whether it is a length, slope or area.

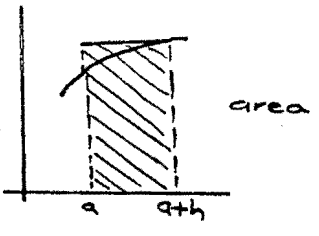
In question 3 the students were given the graph of $f(x)$ as in question 2 of the conceptual test

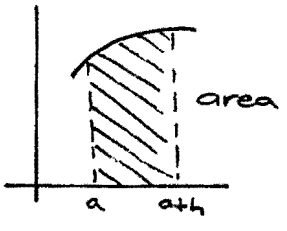
3. Indicate $\frac{f(a+h)-f(a)}{h}$ on the graph of $f(x)$. State whether it is a length, slope or area.
4. Sketch the graph of $f'(x)$ given the following graph of $f(x)$.

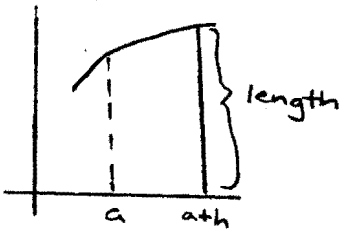


5. Below are four different answers I got to question 2f which asks students to represent $hf(a+h)$ on the graph of $f(x)$. Which of them do you think is right and why?









6. How would you work out the average of 8, 5 and 3?

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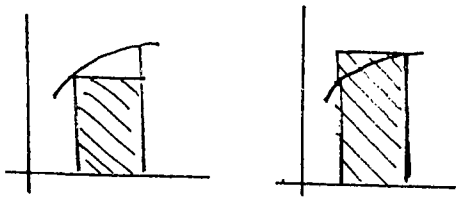
7. How would you work out the average of these two areas?



8. If I asked you to represent the average value of $f(x)$ between $x = a$ and $x = a + h$ on the graph of $f(x)$, given in question 2 of the test, would it be a length, slope or area?

In questions 9 and 10 the graph of $f(x) = \sin x$ was given, as in question 1 of the conceptual test.

9. Indicate $\int_{\frac{\pi}{3}}^{\frac{\pi}{3}+h} f(x)\ dx$ on the graph of $f(x) = \sin x$.

10. If $f(x) = F'(x)$ show $F(\frac{\pi}{3} + h) - F(\frac{\pi}{3})$ on the graph of $f(x) = \sin x$

11. Evaluate $\frac{1}{h} \int_{a}^{a+h} f'(x)\ dx$ and then represent it on the graph of $f(x)$ given in question 2 of the test.

12. Sketch a possible graph of $f(x)$, given the graph of the derivative of $f(x)$ illustrated question 4 of the test.

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