Domination in graphs: Vizing's conjecture

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I know the meaning of plagiarism and declare that all of the work in the thesis, save for that which is properly acknowledged, is my own.

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Abstract

Vizing's conjecture remains one of the biggest open problems in domination in graph theory today. The conjecture states that the domination number of the Cartesian product of two graphs is at least as large as the product of the domination numbers of the two factor graphs. The aim of this thesis is to study the various approaches implemented by researchers over the years in an attempt to prove (or disprove) Vizing's conjecture.

Graph-theoretic definitions and notations used in this paper are presented in Chapter 1, along with several theorems on domination that will be useful throughout this paper. In Chapters 2, 3 and 4, we explore some of the earliest research done in solving Vizing's conjecture. The three methods studied, namely the simple-labelling rule, the one-half argument and fair reception, all involve partitioning the vertex set of one of the factor graphs in some way and then utilising the structure of the Cartesian product to characterise large classes of graphs for which the conjecture is true.

Since Vizing's conjecture is unsolved in general, many partial results related to the conjecture have been proven over the years. One such result, the use of which has become quite widespread in the literature, is studied in Chapter 5. This partial result states that the domination number of the Cartesian product of two graphs is at least half the product of the domination numbers of the two factor graphs. We analyse the double projection argument used to prove this result and include more recent improvements of this bound.

We then consider other approaches to solving Vizing's conjecture which do not use some vertex-partitioning technique. Chapter 6 deals with proving the conjecture by minimal counterexample and we list a few properties that a possible minimal counterexample to Vizing's conjecture must satisfy. Moreover, we focus on methods of building graphs which satisfy Vizing's conjecture from other graphs in Chapter 7.

Finally, Chapter 8 covers several variations of domination and Vizing-like results for each type of domination. In particular, notable results in fractional, graph-, total, integer, paired-, upper and rainbow domination are studied in detail.

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Contents

Chapter 1

Introduction

1.1 History

Domination has been a topic of study in graph theory for over 60 years. However, the notion of domination originated much earlier in the game of chess. In 1862, Carl Friedrich de Jaenisch posed the following question in [28]: what is the minimum number of queens needed to "dominate" an 8×8 chessboard in the following sense: either (i) a queen occupies a tile, or (ii) each unoccupied tile can be occupied by a queen in one move. The answer was found to be five queens, and this problem was hence dubbed the Five Queens Problem. Nearly a century later, domination was first considered in the context of graph theory by Berge [5] in 1958, who introduced the "coefficient of external stability", which we now call the "domination number" of a graph. Later, Ore [54] first introduced the terms "domination", "dominating set" and "domination number" in 1962. Then in 1977, Cockayne and Hedetniemi [26] published a ground-breaking survey article on domination which prompted greater interest in the subject. Since then, studies on domination in graphs have become increasingly popular among graph theorists.

One of the most famous open problems in domination is Vizing's conjecture, formally posed by Vizing [61] in 1968. The conjecture posits that the domination number of the Cartesian product of two graphs is at least as large as the product of the domination numbers of these two graphs. Not much research was done in attempting to solve Vizing's conjecture until 1979, when Barcalkin and German [4] had a breakthrough. They used a "simple labelling rule" to define a large class of graphs which satisfy the conjecture (now known as BG-graphs). Later, in 1995, a larger class of graphs for which the conjecture holds true was established by Hartnell and Rall [39] using their "one-half argument", and is known as the class of Type $\mathcal{X}$ graphs. A well-known partial result of Vizing's conjecture was proven in 2000 by Clark and Suen [24], which states that the domination number of the Cartesian product of two graphs is at least as large as half the product of the domination numbers of the two factor graphs. Several researchers have applied Clark and Suen's "double-projection argument" to prove other partial results of Vizing's conjecture, as well as Vizing-like results for different variations of domination. Many other approaches have been formulated over the years, such as Brešar and Rall's [18] "fair reception" (whose class of graphs define yet another extension of the BG-graphs), Hartnell and Rall's [38] "attachable sets", and the new "cell framework" formulated by Brešar et al. [13] in 2021, to name a few.

1.2 Graph theory

We introduce the graph theoretical definitions and notations used in this paper, which are adapted from [12, 20, 32, 58]. We will only consider finite, simple graphs.

A *graph* $G = (V(G), E(G))$ consists of two sets: a non-empty finite set $V(G)$ of elements called *vertices*, and a set $E(G)$ of 2-element subsets of $V(G)$ called *edges*. If $E(G) = \emptyset$, we say that G is an *empty graph*. We denote an edge between two vertices u and v by uv, and say that u and v are *adjacent* vertices in G. Additionally, vertex u is called a *neighbour* of v (and vice versa) and the edge uv is *incident* with vertices u and v. The *open neighbourhood of* u is $N(u) = \{v \in V(G) : uv \in E(G)\}$, and the *closed neighbourhood of* u is $N[u] = N(u) \cup \{u\}$. In some cases in this paper, we will use $u \in G$ to mean $u \in V(G)$, to ease notation.

The *order* of a graph G is the number of vertices of G, i.e., $|V(G)|$, and is denoted by $|G|$ or $n(G)$. The *size* of G is the number of edges of G, i.e., $|E(G)|$, and is denoted by $m(G)$. If it is clear from context which graph we mean, we simply write n and m respectively.

The *degree* of a vertex v, $deg(v)$, is the number of edges incident with v. For any graph G, $\delta(G) = \min\{deg(v) : v \in V(G)\}$ is the minimum and $\Delta(G) = \max\{deg(v) : v \in V(G)\}$ is the maximum degree of G. A graph is *regular* if every vertex of the graph has the same degree and a graph is *r-regular* if every vertex has degree r.

A graph is *complete* if every vertex of the graph is adjacent to every other vertex. A complete graph of order n is denoted K_n.

A graph G is *bipartite* if $V(G)$ can be partitioned into two sets V_1 and V_2 such that each edge of G joins a vertex in V_1 with a vertex in V_2, and each vertex in a partition is not adjacent to any other vertex in that partition. In a *complete bipartite* graph $K_{r,s}$, each vertex in V_1 is adjacent to every vertex in V_2 and vice versa, where $|V_1| = r$ and $|V_2| = s$.

The *complement* $\overline{G}$ of a graph G is a graph with the same vertices of G and for every $u, v \in V(G)$, it holds that $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

Let G and H be two graphs. Then H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is a *spanning subgraph* of G if $E(H) \subseteq E(G)$ and $V(H) = V(G)$. Suppose we have a set of vertices S with $S \subseteq V(G)$. Then the *subgraph induced by* S is the graph $G[S]$ with $V(G[S]) = S$ and $uv \in E(G[S])$ if and only if $uv \in E(G)$ and $u, v \in S$. H is an *induced subgraph* of G if $V(H) \subseteq V(G)$ and for any $u, v \in V(H)$, $uv \in E(H)$ if and only if $uv \in E(G)$.

A *walk* is a finite alternating sequence $v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k$ where each v_i is a vertex and each e_i is an edge that joins v_{i-1} to v_i. The walk may have repeated vertices and the length of the walk is equal to the number of edges in the walk. If the start vertex and end vertex are the same, the walk is closed. A *path* is a walk that does not contain any repeated vertices (and hence no repeated edges). The *distance* $d_G(u, v)$ between two vertices u and v of a graph G is the length of a shortest path from u to v. If there is no possibility of ambiguity, then we simply write $d(u, v)$. The *diameter* $diam(G)$ is the greatest distance between any pair of vertices.

A graph G is *connected* if for every two vertices $u, v \in V(G)$, G contains a walk from u to v. If G is not connected, we say that G is *disconnected*. A *component* of a G is a maximal connected subgraph of G. A *maximal connected subgraph* of G is a connected subgraph of G that is not properly contained in any other connected subgraph of G. Note that a graph is connected if and only if it has a single component.
A *cycle* C_n of order $n \geq 3$ has size $m = n$, is connected and is 2-regular. A graph which does not contain a cycle is called *acyclic*. A *chordal* (or *triangulated*) graph is one in which every induced cycle has exactly three vertices, i.e., K_3.

A *tree* is a connected, acyclic graph. A vertex of degree 1 in a graph is known as a *leaf* or *end vertex* and an edge incident with a leaf is called a *pendant edge*; every tree of order at least 2 has at least one leaf [58]. A *forest* is an acyclic graph, and the components of a forest are trees. A *star* of order n is the complete bipartite graph $K_{1,n-1}$, which has one vertex v with $deg(v) = n - 1$ (the *internal vertex*) and $n - 1$ leaves. In particular, $K_{1,3}$ is known as a *claw*, and if G is a graph that does not contain $K_{1,3}$ as an induced subgraph, G is said to be *claw-free*.

Let G be a graph. A set $C \subseteq V(G)$ is called a *clique* in G if C induces a complete subgraph in G. A set $I \subseteq V(G)$ is an *independent set* in G if no two vertices of I are adjacent. The maximum cardinality of an independent set of G is the *independence number*, denoted $\alpha(G)$. A *maximal independent set* of G is an independent set that is not properly contained in any other independent set of G.

If G is a non-empty graph, then a *subdivision* of G is a graph obtained from G by removing an edge uv, adding a new vertex w, and then adding the edges uw and vw.

Two graphs G and H are *isomorphic* if there exists a bijection $\phi : V(G) \to V(H)$ such that for all $u, v \in V(G)$, $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$, that is, adjacency of vertices is preserved. ϕ is called an *isomorphism* of G and H and we write $G \cong H$. The *Cartesian product* of G and H, denoted $G \Box H$, is defined as follows: $V(G \Box H) = V(G) \times V(H)$ and for $(u, v), (x, y) \in V(G \Box H)$, where $u, x \in V(G)$ and $v, y \in V(H)$, are adjacent if and only if either $u = x$ and $vy \in E(H)$ **or** $v = y$ and $ux \in E(G)$.

The *power set* of a set S is the set of all subsets of S, including the empty set and S itself, and is denoted $\mathcal{P}(S)$.

1.3 Domination

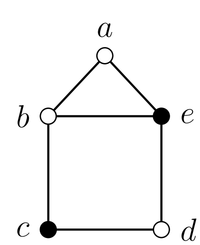
We now introduce the basic definitions and concepts related to domination in graph theory, which are adapted from [12, 21, 22, 43, 44].

Definition: A vertex v of a graph G is said to *dominate* itself and its neighbours, i.e., v dominates $N[v]$.

Definition: A set $S \subseteq V(G)$ is called a *dominating set* of a graph G if every vertex $v \in V(G)$ is either in S or adjacent to a vertex in S, i.e., every vertex in G is dominated by a vertex in S.

Definition: A dominating set of a graph G of minimum cardinality is called a *minimum dominating set*. The cardinality of a minimum dominating set of G is called the *domination number* of G, denoted $\gamma(G)$. A set $S \subseteq V(G)$ is a *minimal dominating set* of G if no proper subset of S dominates G.

Example: Consider the following graph G.



G

Here, $\{c, e\}$ is a minimum dominating set of G and so $\gamma(G) = 2$.

Notation: Let G be a graph and $S \subseteq V(G)$. We denote the minimum number of vertices in G needed to dominate $G[S]$ by $\gamma_G(S)$.

Below are a few theorems on domination that will be useful throughout this paper.

Theorem 1.1 ([54]) *Let G be a graph without any isolated vertices. If S is a minimal dominating set of G, then $V(G) - S$ is also a dominating set of G.*

Proof. We will show that every $v \in V(G)$ is dominated by some vertex in $V(G) - S$. Let v be some vertex in G. If $v \in V(G) - S$, then v dominates itself and we are done. So suppose that $v \notin V(G) - S$, that is, $v \in S$. Assume to the contrary that v is not dominated by any vertex in $V(G) - S$. Therefore, v is not adjacent to any vertex in $V(G) - S$. Since S is a dominating set, each vertex in $V(G) - S$ is dominated by some vertex in S other than v. On the other hand, since G has no isolated vertices, v must be dominated by $S - \{v\}$. This implies that $S - \{v\}$ is a dominating set of G of smaller cardinality than S, a contradiction to the minimality of S. □

Corollary 1.1.1 ([54]) *If G is a graph of order n without any isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.*

Proof. Let S be a minimum dominating set of G. Then by Theorem 1.1, $V(G) - S$ is also a dominating set of G. From $|S| + |V(G) - S| = n$ and by the minimality of S, $|S| \leq |V(G) - S|$, we have that

$$2\gamma(G) = 2|S| \leq |S| + |V(G) - S| = n \Rightarrow \gamma(G) \leq \frac{n}{2}$$

as desired. □

Theorem 1.2 ([6]) *Every maximal independent set of a graph G is a minimal dominating set of G. Therefore $\alpha(G) \geq \gamma(G)$.*

Proof. Let I be a maximal independent set of a graph G with $|I| = k$. Suppose to the contrary that there exists a vertex $u \in V(G)$ that is not dominated by I. Then u is not in I nor is it adjacent to any vertex in I. Therefore $I \cup \{u\}$ forms an independent set in G of order $k + 1$, contradicting the maximality of I. So I dominates $V(G)$.
We now show that I is a minimal dominating set. Suppose to the contrary that there exists a vertex $v \in I$ such that $I - \{v\}$ is a dominating set of G. Then $v \in N(I - \{v\})$, contradicting

the fact that I is independent. Hence, I is a minimal dominating set of G and since every maximum independent set is maximal, $\alpha(G) \geq \gamma(G)$. □

Definition: An *independent dominating set* of a graph G is both an independent set and a dominating set of G. The *independent domination number* of G is the minimum cardinality of all possible independent dominating sets of G, and is denoted by $i(G)$.

Theorem 1.3 ([6]) *Let G be a graph and $S \subseteq V(G)$. Then S is an independent dominating set of G if and only if S is a maximal independent set of G.*

Proof. ($\Leftarrow$) By Theorem 1.2, every maximal independent set of G is a dominating set of G. ($\Rightarrow$) Let S be an independent dominating set of G. Then every vertex not in S is adjacent to a vertex in S since S dominates G. Therefore S is a maximal independent set of G. □

Observe that by Theorems 1.2 and 1.3, $\gamma(G) \leq i(G)$ for any graph G.

Definitions: Let G be a graph, $S \subseteq V(G)$ and $v \in S$. We say that a vertex $w \in V(G) - S$ is an *external private neighbour of v with respect to S* if $N(w) \cap S = \{v\}$ and the *external private neighbourhood of v with respect to S*, denoted $\text{epn}(v, S)$, is the set of all external private neighbours of v with respect to S. A vertex $w \in S$ is called an *internal private neighbour of v with respect to S* if $N(w) \cap S = \{v\}$ and the *internal private neighbourhood of v with respect to S*, denoted $\text{ipn}(v, S)$, is the set of all internal private neighbours of v with respect to S. Finally, the *private neighbourhood if v with respect to S* is $\text{pn}(v, S) = \{w \in V(G) : N(w) \cap S = \{v\}\}$. Note that $\text{pn}(v, S) = \text{epn}(v, S) \cup \text{ipn}(v, S)$, $\text{epn}(v, S) = \text{pn}(v, S) \cap (V(G) - S)$ and $\text{ipn}(v, S) = \text{pn}(v, S) \cap S$.

Theorem 1.4 ([54]) *Let S be a dominating set of a graph G. S is a minimal dominating set of G if and only if for every vertex $v \in S$:*

(i) v has at least one external private neighbour with respect to S, or

(ii) v is not adjacent to any vertex of S.

Proof. Assume that every vertex $v \in S$ satisfies at least one of the two properties. Then for each $v \in S$, $S - \{v\}$ does not dominate G and therefore S is a minimal dominating set of G. Now assume that S is a minimal dominating set of G. Then as before, for each $v \in S$, $S - \{v\}$ does not dominate G. So there exists at least one vertex $w \in V(G) - (S - \{v\})$ that is not adjacent to any vertex of $S - \{v\}$. Firstly, if $w = v$, then condition (ii) holds. On the other hand, if $w \neq v$, then since S dominates G and $w \notin S$, w must be adjacent to at least one vertex of S. But since w is not adjacent to any vertex of $S - \{v\}$, $N(w) \cap S = \{v\}$ and condition (i) holds. □

Theorem 1.5 ([8]) *Every graph G without isolated vertices contains a minimum dominating set S such that every vertex in S has at least one external private neighbour.*

Proof. Of all the minimum dominating sets of G, let S be one such that $G[S]$ has maximum size. Suppose to the contrary that S contains a vertex v that does not satisfy the required property. Then by Theorem 1.4, v is not adjacent to any vertex of S, so v is isolated in $G[S]$. Furthermore, each vertex of $V(G) - S$ that is adjacent to v is also adjacent to some other vertex of S. Since G has no isolated vertices, v is adjacent to some vertex $w \in V(G) - S$. So the set $(S - \{v\}) \cup \{w\}$ is a minimum dominating set of G whose induced subgraph in G has at least one edge incident with w and therefore has greater size than $G[S]$, which is a contradiction. □

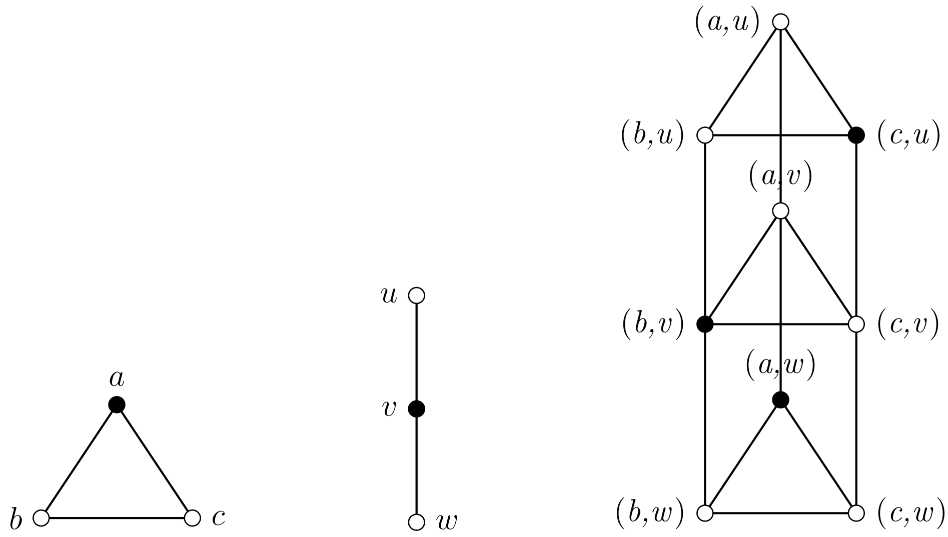
We now state *Vizing's conjecture*, the focus of our study, which was formally posed in 1968 by V.G. Vizing.

Vizing's Conjecture [61] *For every pair of finite graphs G and H,*

$$\gamma(G \Box H) \geq \gamma(G)\gamma(H).$$

We say that G "satisfies Vizing's conjecture" if this inequality holds true for every graph H.

Example: Consider the following graphs G and H and their Cartesian product. In each graph, the solid vertices form a minimum dominating set.



G $\qquad$ H $\qquad$ $G\Box H$

Here, $\gamma(G) = 1$, $\gamma(H) = 1$ and $\gamma(G\Box H) = 3$, therefore $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$.

Definition: Let G and H be two graphs with $g \in V(G)$. An H*-fibre* is the subgraph of $G\Box H$ induced by the vertex set $\{(g, h) : h \in V(H)\}$ and is denoted by ${}^{g}H$. For some fixed $h \in V(H)$, a G*-fibre*, G^h, is defined similarly.

Therefore, every vertex (g, h) of $G\Box H$ is the intersection of the two fibres G^h and ${}^{g}H$. Moreover, note that by definition of adjacency in $G\Box H$, H-fibres are isomorphic to H and G-fibres are isomorphic to G.

We now state two bounds on the domination number of the Cartesian product of two graphs that we will use later.

El-Zahar and Pareek establish the following lower bound for $\gamma(G\Box H)$ in [31].

Theorem 1.6 ([31]) *For any two graphs G and H, $\gamma(G\Box H) \geq \min\{|V(G)|, |V(H)|\}$.*

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$ and $V(H) = \{u_1, \ldots, u_m\}$ and let D be a minimum dominating set of $G\Box H$. Suppose to the contrary that $|D| < \min\{n, m\}$, then there exists G^{u_i} and ${}^{v_j}H$ such that $D \cap G^{u_i} = \emptyset$ and $D \cap {}^{v_j}H = \emptyset$. But this implies that the vertex (v_j, u_i) is not dominated by D, a contradiction. $\Box$

On the other hand, Vizing observed the following upper bound for $\gamma(G\Box H)$ in [61].

Theorem 1.7 ([61]) *For any two graphs G and H,*

$$\gamma(G\Box H) \leq \min\{\gamma(G)|V(H)|, |V(G)|\gamma(H)\}.$$

Proof. Let D_G be a minimum dominating set of G, so $|D_G| = \gamma(G)$. Then note that the set of vertices $D_G \times V(H)$ is a dominating set of $G\Box H$, therefore

$$\gamma(G\Box H) \leq |D_G \times V(H)| = \gamma(G)|V(H)|.$$

Since we can interchange G and H in our argument,

$$\gamma(G\Box H) \leq \min\{\gamma(G)|V(H)|, |V(G)|\gamma(H)\}$$

as desired. □

Even though Vizing's conjecture is still unsolved in general, there are large classes of graphs for which it is true. We will investigate the earliest to more recent research done in an attempt to prove Vizing's conjecture.

Chapter 2

BG-graphs

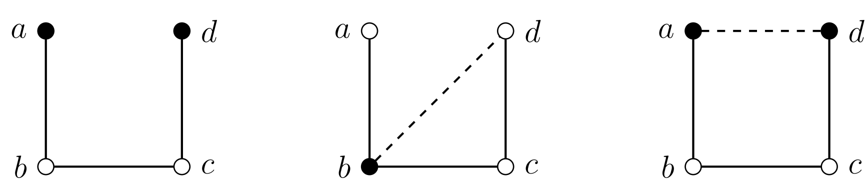
Some of the earliest work done in proving Vizing's conjecture was by Barcalkin and German in 1979 [4]. They defined a large class of graphs that satisfy Vizing's conjecture, now known as BG-graphs. To study these graphs, we first look at their theorem on *decomposable* graphs.

Definition: A graph G is said to be *decomposable* if $V(G)$ can be partitioned into $\gamma(G)$ cliques of G.

Observation: [12] Let G be a graph that is not complete, that is, there exist two vertices u and v of G that are not adjacent. Then $\gamma(G) - 1 \leq \gamma(G \cup uv) \leq \gamma(G)$.

Proof. Note that any minimum dominating set of G is also a dominating set of $G \cup uv$, so $\gamma(G \cup uv) \leq \gamma(G)$. Let D be a minimum dominating set of $G \cup uv$. If $u, v \in D$, then u and v dominate themselves in G and $\gamma(G) = \gamma(G \cup uv)$. If $u, v \notin D$, then u and v do not dominate each other in $G \cup uv$ nor in G, so $\gamma(G) = \gamma(G \cup uv)$. Finally suppose that, without loss of generality, $u \in D$ and $v \notin D$. If there exists $w \in D$, $w \neq u$, such that w dominates v in $G \cup uv$, then w also dominates v in G and $\gamma(G) = \gamma(G \cup uv)$. Otherwise, v is not dominated by D in G. In this case, $D \cup \{v\}$ dominates G and $\gamma(G) \leq \gamma(G \cup uv) + 1$. Hence $\gamma(G) - 1 \leq \gamma(G \cup uv)$, as desired. □

For example, consider the graph $G = P_4$ with dominating set $\{a, d\}$.

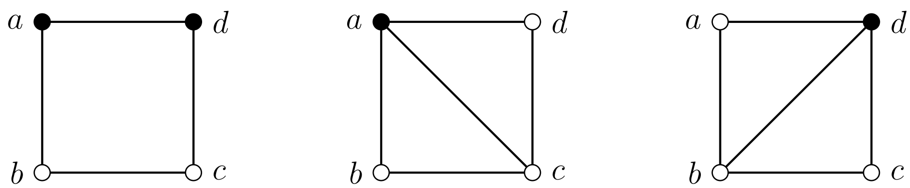


G G' G''

The graph $G' = G \cup bd$ has dominating set $\{b\}$, and $\gamma(G') = \gamma(G) - 1$. On the other hand, the graph $G'' = G \cup ad$ has dominating set $\{a, d\}$, so $\gamma(G'') = \gamma(G)$.

Definition: If G is a graph such that for all non-adjacent pairs of vertices u, v we have that $\gamma(G \cup uv) = \gamma(G) - 1$, then we say that G is *edge-critical* (with respect to domination).

C_4 is an example of an edge-critical graph:



Here, $\{a, d\}$ is a minimum dominating set of C_4 so $\gamma(C_4) = 2$. If we add either edge ac or bd, the domination number of the resulting graph is 1.

Barcalkin and German noted the following Vizing-like result for spanning subgraphs.

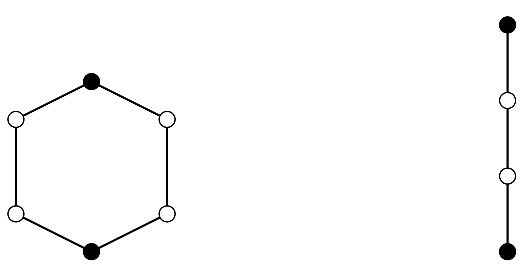
Theorem 2.1 ([4]) *Let G' be a spanning subgraph of a graph G. If G satisfies Vizing's conjecture and $\gamma(G') = \gamma(G)$, then G' satisfies Vizing's conjecture.*

Proof. Let H be any graph. Then

$$\gamma(G')\gamma(H) = \gamma(G)\gamma(H) \leq \gamma(G \square H) \leq \gamma(G' \square H).$$

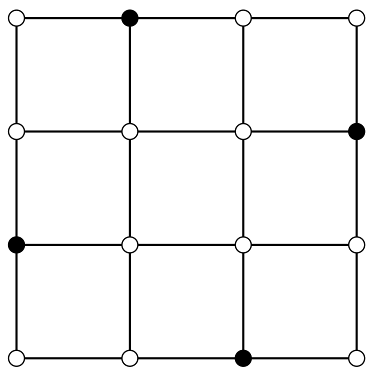
The last inequality holds since $G' \square H$ is a spanning subgraph of $G \square H$, that is, $E(G' \square H) \subseteq E(G \square H)$. So by our earlier observation, $\gamma(G' \square H) \geq \gamma(G \square H)$. $\square$

Note that it is necessary for the subgraph in the above theorem to be a spanning subgraph. Otherwise, consider for example, $G = C_6$, $G' = P_4$ and $H = P_4$ (C_6 satisfies Vizing's conjecture by Corollary 2.2.2 below). Minimum dominating sets for each graph are indicated by solid vertices in the figures below.

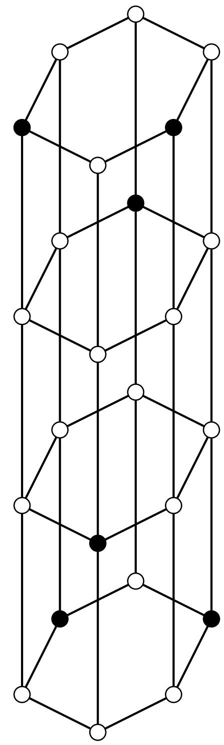


G $\qquad$ G' and H

Then $\gamma(G) = \gamma(G') = \gamma(H) = 2$.



$G' \Box H$



$G \Box H$

Note that $\gamma(G \Box H) = 6 > 4 = \gamma(G' \Box H)$. However, $\gamma(G')\gamma(H) = 4 \leq 4 = \gamma(G' \Box H)$.

Barcalkin and German use these results to extend the types of graphs which are known to satisfy Vizing's conjecture. In order to prove the next theorem, we first need to discuss projections and a property of decomposable graphs which relates to external domination.

Definition: Let G be a graph. A set $X \subseteq V(G)$ *externally dominates* a set $S \subseteq V(G)$ if $X \cap S = \emptyset$ and for every $u \in S$, there exists an $x \in X$ such that $ux \in E(G)$.

Definition: Let G and H be two graphs. The *projection to* H is the map $p_H : V(G \Box H) \to V(H)$ defined by $p_H(g, h) = h$, where $g \in V(G),\ h \in V(H)$. For some fixed $u \in V(G)$, the *projection to* ${}^uH$ maps (g, h) to (u, h). Projections to G and G-fibres are defined similarly.

External domination property: ([12]) Let G be a decomposable graph with $\gamma(G) = k$ and let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be a partition of $V(G)$ into k cliques. For $\ell < k$, let $C_{i_1}, \ldots, C_{i_\ell}$ be a collection of cliques from $\mathcal{C}$ such that each vertex in $C_{i_1} \cup \cdots \cup C_{i_\ell}$ has at least one neighbour in $G - (C_{i_1} \cup \cdots \cup C_{i_\ell})$. Let D be a minimum external dominating set of $C_{i_1} \cup \cdots \cup C_{i_\ell}$ and let $C_{j_1}, \ldots, C_{j_t}$ be all the cliques from $\mathcal{C}$ which intersect D.

<u>Claim:</u> $\displaystyle\sum_{m=1}^{t}(|C_{j_m} \cap D| - 1) \geq \ell$

Proof. Suppose to the contrary that $\displaystyle\sum_{m=1}^{t}(|C_{j_m} \cap D| - 1) < \ell$. This implies that

$$\sum_{m=1}^{t}|C_{j_m} \cap D| < \ell + t.$$

So the $\ell + t$ cliques $C_{i_1}, \ldots, C_{i_\ell}, C_{j_1}, \ldots, C_{j_t}$ are dominated by fewer than $\ell + t$ vertices. For the remaining $k - (\ell + t)$ cliques, every clique can be dominated by one vertex from each clique, so in total we count

$$\gamma(G) < \ell + t + k - (\ell + t) = k,$$

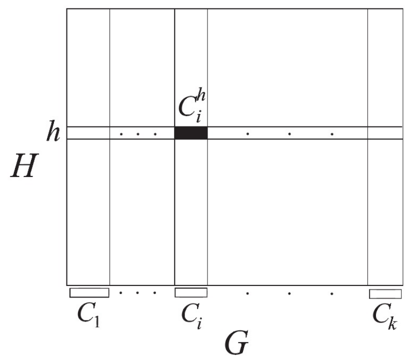
a contradiction. □

We will use this property in several proofs to come. For now, we consider an extension of Barcalkin's and German's idea of Theorem 2.1.

Theorem 2.2 ([4]) *If G is a spanning subgraph of a decomposable graph G' such that $\gamma(G) = \gamma(G')$, then for every graph H, $\gamma(G \square H) \geq \gamma(G)\gamma(H)$.*

Proof. By Theorem 2.1, we may assume that G is a decomposable graph. Let H be any graph and $\gamma(G) = k$, and consider the Cartesian product $G \square H$ with a minimum dominating set D. We want to prove that $|D| \geq k\gamma(H)$ by showing that D can be partitioned into k sets each of cardinality at least $\gamma(H)$. To do so, we will define a *simple labelling rule* which labels each vertex of D with a number from $1, \ldots, k$, then for each $i \in [1, k]$, we consider the projection of the vertices of D that are labelled i to H. Each of these k sets of vertices labelled i would be labelled in such a way that dominates H (since D dominates all H-fibres in $G \square H$ and each H-fibre is isomorphic to H), which would imply that $\gamma(G \square H) = |D| \geq k\gamma(H) = \gamma(G)\gamma(H)$, as desired.

Let $V(G) = C_1 \cup \cdots \cup C_k$, where each C_i is a clique. For each $h \in H$ and $i \in [1, k]$, define a *G-cell* as $C_i^h = V(C_i) \times \{h\}$ (note that this is a partitioning of $V(G \square H)$). Below is a diagram of this partitioning from [12]:



We define the *simple labelling rule* as follows: if $D \cap C_i^h \neq \emptyset$, label one of the vertices in $D \cap C_i^h$ with i. Then we project the vertices of $D \cap C_i^h$ to H and label h with i as well. If $|D \cap C_i^h| \geq 2$, we need to label the remaining vertices of $D \cap C_i^h$ by showing that for each i, there exists a vertex in D labelled i that is projected to the neighbourhood of h.

Case 1: there exists a vertex in D that is also in $V(C_i) \times N[h]$.
By the simple labelling rule, $D \cap C_i^h \neq \emptyset$, so one of its vertices is labelled i and is projected to h in $N[h]$ with label i.

Case 2: there is no vertex from D in $V(C_i) \times N[h]$.
In this case, we call C_i^h a *missing G-cell* for h. Let $C_{i_1}^h, \ldots, C_{i_\ell}^h$ be the $\ell < k$ missing G-cells for h. Note that $D \cap (V(C_i) \times N[h]) = \emptyset$ implies that each missing G-cell is not dominated from within the H-fibres, so it must be dominated from the G-fibre, G^h. Therefore $D \cap G^h \neq \emptyset$ and contains vertices which externally dominate $C_{i_1}^h \cup \cdots \cup C_{i_\ell}^h$. We now implement the external domination property: let $C_{j_1}^h, \ldots, C_{j_t}^h$ be the G-cells of h which intersect D. Since G^h is isomorphic to G, we have that

$$\sum_{m=1}^{t} (|C_{j_m}^h \cap D| - 1) \geq \ell.$$

So there are enough unlabelled vertices in $D \cap G^h$ remaining so that for each missing G-cell C_i^h, label i can be given to a vertex in $C_{j_m}^h \cap D$, for $|C_{j_m}^h \cap D| \geq 2$. Thus in the projection, label i will be given to h.

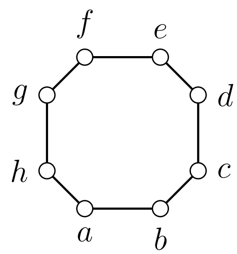
By using this method of labelling, we produce k sets of vertices labelled $1, \ldots, k$, each of which dominates H. Therefore we have partitioned D into k sets each of cardinality at least $\gamma(H)$, as desired. □

This theorem leads us to an important partial result of Vizing's conjecture.

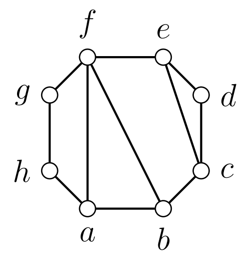
Corollary 2.2.1 ([4]) *Every decomposable graph satisfies Vizing's conjecture.*

Barcalkin and German state that such a graph G from Theorem 2.2 belongs to what they call the $\mathcal{A}$-class. This class of graphs is commonly referred to in the literature as *BG-graphs* in acknowledgement of their significant contribution to graph domination theory.

A class of common graphs which are BG-graphs are cycles C_n, $n \geq 3$. For example, consider $C_8 : a, b, c, d, e, f, g, h, a$. Note that $\gamma(C_8) = 3$ and C_8 is not decomposable, but C_8 is a BG-graph. In fact, if we add the edges af, bf and ce to C_8, the resulting graph G has domination number 3 and is decomposable. So since C_8 is a spanning subgraph of G, Theorem 2.2 says that C_8 satisfies Vizing's conjecture.



C_8



G

A similar method can be used to show that all cycles are spanning subgraphs of decomposable graphs with the same domination number. However, El-Zahar and Pareek [31] were the first to prove this by using induction on the order of the cycle.

Corollary 2.2.2 ([31]) *For any graph H and $n \geq 3$, $\gamma(C_n \Box H) \geq \gamma(C_n)\gamma(H)$.*

Note that any graph G with $\gamma(G) = 1$ satisfies Vizing's conjecture. If D is a minimum dominating set of $G \Box H$ for any graph H, then for some $v \in V(G)$, the vertices of the projection $p_H(D \cap {}^{N[v]}H)$ dominates ${}^{v}H \cong H$ and so $|D| \geq |p_H(D \cap {}^{N[v]}H)| \geq \gamma(H)$. As a consequence of Theorem 2.2, Barcalkin and German also prove that all graphs with domination number 2 are BG-graphs and therefore satisfy Vizing's conjecture. This result was independently proved by El-Zahar and Pareek [31] as well.

Theorem 2.3 ([4]) *Any graph with domination number 2 satisfies Vizing's conjecture.*

Proof. Let G be a graph such that $\gamma(G) = 2$ and let G' be a graph obtained from G by adding as many edges to G as possible such that $\gamma(G') = 2$, i.e., G' is edge-critical. We will show that G' is decomposable. Let C_1 and C_2 be two disjoint cliques in G' such that $|C_1| + |C_2|$ is as large as possible.

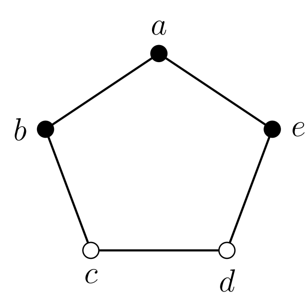
Claim: $|C_1| + |C_2| = n(G)$.

Suppose to the contrary that there exists a vertex $v \notin C_1 \cup C_2$. Then v is not adjacent to at least one vertex, say u, in C_1 and v is not adjacent to at least one vertex in C_2. Since G' is edge-critical, $\gamma(G' \cup uv) = 1$. Therefore, since v does not dominate $G' \cup uv$, u must be adjacent to every vertex in C_2. Let S be the set of vertices in C_1 that are not adjacent to v. Then $(C_1 - S) \cup \{v\}$ and $C_2 \cup S$ are two cliques of G' with more vertices than $C_1 \cup C_2$, a contradiction.

Therefore G' is decomposable and so by Theorem 2.2, G satisfies Vizing's conjecture. □

Another way of partitioning the vertex set of a graph in an attempt to prove Vizing's conjecture is considered by Faudree, Schelp and Shreve in [33]. They define *Condition CC* as follows: a graph G satisfies Condition CC if there exists a colouring (or partition) of $V(G)$ using $\gamma(G)$ colours such that any subset of $V(G)$ that has at most $\gamma(G) - 1$ vertices does not dominate some vertex in each colour that is not included in the subset. They prove in [33] that any graph which satisfies Condition CC also satisfies Vizing's conjecture.

Example: Consider $C_5 : a, b, c, d, e, a$ with $V_1 = \{a, b, e\}$ and $V_2 = \{c, d\}$. Since $\gamma(C_5) = 2$, let A be a subset of $V(G)$ of cardinality at most 1.

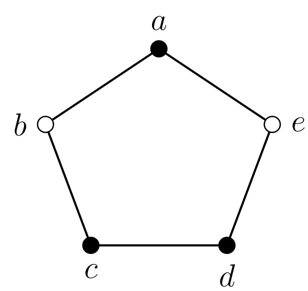


Clearly, Condition CC is satisfied if $A = \emptyset$, so suppose A is a singleton set. If $A = \{a\}$, then V_2 is not dominated by A. If $A = \{b\}$ or $A = \{e\}$, then d or c is not dominated by A. Lastly, if $A = \{c\}$ or $A = \{d\}$, then a is not dominated by A. Therefore, C_5 satisfies Condition CC for $\mathcal{V} = \{V_1, V_2\}$.

On the other hand, Chen, Piotrowski and Shreve follow a different approach to partitioning $V(G)$ in [23]. Their method is as follows: let $\mathcal{V} = \{V_1, \ldots, V_k\}$ be a partition of $V(G)$. For $i \in [1, k]$, we say that V_i is *covered* by $A \subseteq V(G)$ if $V_i \cap A \neq \emptyset$ or $V_i \subseteq N(A)$, and $\mathcal{V}$ is *extracted* if no subset A of $V(G)$ covers more than $|A|$ members of $\mathcal{V}$. Furthermore, the *extraction number of* G, denoted $x(G)$, is the largest cardinality of all extracted partitions of $V(G)$. Using these definitions, Chen et al. prove in [23] that any graph G such that $x(G) = \gamma(G)$ satisfies Vizing's conjecture as well.

Example: Let's look at $C_5 : a, b, c, d, e, a$ again with $V_1 = \{a, b, e\}$ and $V_2 = \{c, d\}$ and let $A \subseteq V(G)$. Clearly, $A = \emptyset$ covers no members of $\mathcal{V}$. If A is a singleton set, A covers the one member of $\mathcal{V}$ it has non-empty intersection with. If A is a 2-set of $V(G)$, A covers either one or both members of $\mathcal{V}$. Therefore, if A has cardinality 3, 4 or 5, A cannot cover more than $|A|$ members of $\mathcal{V}$ and so $\mathcal{V}$ is an extracted partition of C_5.

Note that the partition of the vertex set must be chosen carefully. If we had chosen $V_1 = \{a, c, d\}$ and $V_2 = \{b, e\}$ for either example, $\mathcal{V} = \{V_1, V_2\}$ would not be a suitable partition for Condition CC nor an extracted partition since $A = \{a\}$ dominates at least one vertex in each member of $\mathcal{V}$ and $A = \{a\}$ covers both members of $\mathcal{V}$.



These two approaches were formulated and proven independently to Barcalkin and German's decomposable graphs. However, the following result demonstrates the strength of the idea of BG-graphs and how they relate to graphs which satisfy Condition CC and graphs with equal extraction and domination number.

Theorem 2.4 ([40]) *If G is a graph that satisfies Condition CC or if $x(G) = \gamma(G)$, then G is a BG-graph.*

Proof. Suppose first that G satisfies Condition CC. Let $\gamma(G) = k$ and let $\mathcal{V} = \{V_1, \ldots, V_k\}$ be a partition of $V(G)$ as defined in Condition CC. Add any missing edges between the vertices in the same colour class so that each class induces a clique and denote the resulting graph by G'. If $A \subseteq V(G')$ has at most $k-1$ vertices, then at least one colour is not included in A. Suppose that $A \cap V_j = \emptyset$ for some $j \in [1, k]$. Then there exists a vertex $v \in V_j$ such that $v \notin N_G[A]$. Since no edges were added between the colour classes of $\mathcal{V}$, $x \notin N_{G'}[A]$ as well. Therefore A does not dominate G', so $\gamma(G') = k$ and G' is decomposable.
Now suppose that $x(G) = \gamma(G) = k$. Let $\mathcal{V} = \{V_1, \ldots, V_k\}$ be an extracted partition of

$V(G)$ and as before, add any missing edges to each V_i, $i \in [1,k]$, to form the graph G'. Let $A \subseteq V(G')$ be a dominating set of G'. If $A \cap V_j = \emptyset$ for some $j \in [1,k]$, then $V_j \subseteq N_{G'}[A]$ since A dominates G'. Moreover, since no edges were added between the members of $\mathcal{V}$, A covers V_j in G. Therefore, A covers the k members of $\mathcal{V}$ in G, implying that $|A| \geq k$. Since $\mathcal{V}$ is an extracted partition, $|A| \leq k$, and again we have that $\gamma(G') = k$ and G' is a decomposable graph. □

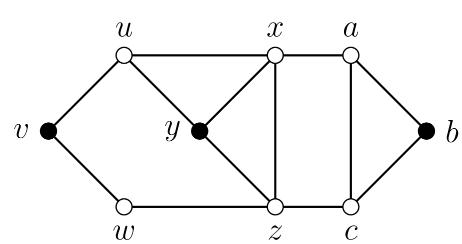
Note that this theorem shows that the class of graphs which satisfy Condition CC is a proper subset of the class of BG-graphs. Furthermore, it follows from the properties of BG-graphs and extracted partitions that every BG-graph G satisfies $x(G) = \gamma(G)$, and so the class of graphs with equal extraction and domination number is the same as the class of BG-graphs.

Chapter 3

Type $\mathcal{X}$ graphs and the one-half argument

We now look at Hartnell and Rall's [39] method of finding classes of graphs which satisfy Vizing's conjecture. Given a graph G, they partition $V(G)$ in a different way to Barcalkin and German, resulting in a new class of graphs known as Type $\mathcal{X}$ graphs. In fact, every decomposable graph is of Type $\mathcal{X}$ [12]. Before we look at the formal definition of Type $\mathcal{X}$ graphs and the "main theorem" of Hartnell and Rall regarding these graphs, we illustrate the proof of their main theorem with a working example. This proof method is known as the *one-half argument*, formulated by Hartnell and Rall in 1995 in [39].

Example: Consider the following graph G.



G

Note that $\gamma(G) = 3$ and $\{v, y, b\}$ is a minimum dominating set of G. Let H be any graph and D a minimum dominating set of $G \square H$. The method we will use to prove that G satisfies Vizing's conjecture is similar to our proof of Theorem 2.2. We want to prove that $|D| \geq 3\gamma(H)$ by showing that D can be partitioned into three disjoint sets each of cardinality at least $\gamma(H)$.

Let $V(G) = S \cup SC \cup BC$, where $SC = \{x, y, z\}$ and $BC = \{a, b, c\}$ induce cliques in G

and $S = \{u, v, w\}$ induces a P_3 in G (note that S, SC and BC are examples of the induced subgraphs described in the definition of Type $\mathcal{X}$ graphs below). Note that $b \in BC$ and $v \in S$ are two vertices whose neighbourhoods are completely contained within BC and S respectively, and each vertex of SC has at least one neighbour outside of SC.

Firstly, if ${}^vH$, ${}^yH$ and ${}^bH$ are dominated by $D \cap {}^vH$, $D \cap {}^yH$ and $D \cap {}^bH$ respectively, then

$$|D| \geq |D \cap {}^vH| + |D \cap {}^yH| + |D \cap {}^bH| \geq 3\gamma(H)$$

since each H-fibre is isomorphic to H, and we are done. So suppose this is not the case. Consider ${}^vH$ and assume that $D \cap {}^vH$ does not dominate ${}^vH$. Define the *missing fibre list* for ${}^vH$, denoted $\mathcal{L}_v$, as the set of all vertices $h \in V(H)$ such that (v, h) is not dominated by $D \cap {}^vH$. We define the missing fibre list for ${}^bH$, $\mathcal{L}_b$, similarly.

Now consider ${}^yH$ and project all the vertices of $D \cap {}^xH$ and $D \cap {}^zH$ onto ${}^yH$ using the projections

$$\begin{aligned} p_x &: D \cap {}^xH \to {}^yH \quad \text{defined by} \quad p_x(x, h) = (y, h) \\ p_z &: D \cap {}^zH \to {}^yH \quad \text{defined by} \quad p_z(z, h) = (y, h) \end{aligned}$$

Let Y be the set of all the vertices of $D \cap {}^yH$ together with the vertices of D from the projections p_x and p_z. Hence, we define the missing fibre list $\mathcal{L}_y$ as the set of vertices $h \in V(H)$ such that (y, h) is not dominated by Y. Note that each missing fibre list may be empty. Otherwise, since the vertices in $\mathcal{L}_v$, $\mathcal{L}_b$ and $\mathcal{L}_y$ are not dominated in the ${}^vH$-, ${}^bH$- and ${}^yH$-fibres respectively, they must be dominated by vertices in the corresponding G-fibres, which we will find.

Let us start with $\mathcal{L}_v$ in S. Suppose there exists an $h \in \mathcal{L}_v$. This implies that (v, h) is not dominated in ${}^vH$, so it must be dominated by either (u, h) or (w, h), that is, from within the G^h-fibre. So at least one of $(u, h), (v, h)$ is in D. Similarly for $\mathcal{L}_b$ in BC, if $h \in \mathcal{L}_b$, then (b, h) is either dominated by (a, h) or (c, h), so at least one of (a, h) and (c, h) is in D.

Note that (u, h) and (a, h) both dominate (x, h), and (w, h) and (c, h) both dominate (z, h). So when counting vertices in D, we must be sure not to double-count any vertices in order to prove that $|D| \geq 3\gamma(H)$. To this end, we project the vertices of $D \cap {}^wH$ onto ${}^uH$ using the projection $p_{{}^uH} : D \cap {}^wH \to {}^uH$. Let U be the set of all the vertices of $D \cap {}^uH$ together with the vertices of $p_{{}^uH}$, and consider the induced subgraph ${}^uH[U]$.

Firstly, if $(u, k) \notin {}^uH[U]$, then since D dominates (v, h), h cannot be in $\mathcal{L}_v$. If $(v, h) \notin D$, then (z, h) and at least one of (x, h) and (y, h) must be in D to dominate (w, h) and (u, h) respectively. On the other hand, if $(v, h) \in D$, then at least one of (x, h), (y, h) and (z, h) is in D to dominate SC. As before, if $h \in \mathcal{L}_b$, then at least one of (a, h) and (c, h) is in D. So there are at least three vertices of D in the G^h-fibre.

Now, let k be any vertex in $\mathcal{L}_y$. Then SC is not dominated by Y. Note that (u,k) must be in D to dominate (y,k), i.e., $(u,k) \in {}^uH[U]$. We consider three cases:

Case 1: (u,k) is an isolated vertex in ${}^uH[U]$ and $(u,k),(w,k) \in D$.
If $k \in \mathcal{L}_b$, then as before, (b,k) is dominated in the G^k-fibre by either (a,k) or (c,k), so at least one of them is in D. Hence we can count one of these vertices towards the missing fibre list $\mathcal{L}_b$. On the other hand, if $k \in \mathcal{L}_v$ as well, then we can count (w,k) towards $\mathcal{L}_v$ and (u,k) towards $\mathcal{L}_y$.

Case 2: (u,k) is an isolated vertex in ${}^uH[U]$ and $(u,k) \in D$, $(w,k) \notin D$.
Since (u,k) is an isolated vertex in ${}^uH[U]$, (w,k) is not dominated by $D \cap {}^wH$, so (v,k) must be in D to dominate (w,k). In this case, (c,k) must be in D to dominate (z,k). If $k \in \mathcal{L}_b$ as well, then we can count (c,k) towards $\mathcal{L}_b$ and count (u,k) towards $\mathcal{L}_y$.

Case 3: (u,k) is in a component C of order at least 2 in ${}^uH[U]$.
By Corollary 1.1.1, $\gamma(C) \leq \frac{|C|}{2}$, therefore we can count $\gamma(C)$ vertices of C towards $\mathcal{L}_v$ if necessary, and count $\gamma(C)$ vertices of C towards $\mathcal{L}_y$ if necessary. We treat BC the same as before.

In all of these cases, the vertices in $D \cap {}^vH$ together with $\mathcal{L}_v$ dominates ${}^vH$. The same is true for $D \cap {}^bH$ with $\mathcal{L}_b$ and $D \cap {}^yH$ with $\mathcal{L}_y$. Therefore, this method of counting vertices of D has allowed us to partition D into three sets each of cardinality at least $\gamma(H)$, so

$$\gamma(G \Box H) = |D| \geq 3\gamma(H) = \gamma(G)\gamma(H).$$

Hence G satisfies Vizing's conjecture.

The formal definition of a Type $\mathcal{X}$ graph is as follows.

Definition: A graph G is of Type $\mathcal{X}$ if $\gamma(G) = k+t+m+1$ and $V(G) = S \cup BC \cup C \cup SC$ such that:

(i) $S = S_1 \cup \cdots \cup S_k$ and each S_i is "star-like", that is, each S_i contains a vertex v_i that is adjacent to every vertex in $T_i = S_i - \{v_i\}$ and $N[v_i] = T_i$. Other vertices in S_i may be adjacent, but $N[v_i]$ may not induce a clique in G. Furthermore, each S_i is edge-critical.

(ii) $BC = B_1 \cup \cdots \cup B_t$ and each "buffer clique" B_i is a clique which contains a vertex b_i such that $N[b_i] \subseteq B_i$.

(iii) $C = C_1 \cup \cdots \cup C_m$ and each C_i is a clique in G. No vertex of C is adjacent to any vertex of S.

(iv) SC is called a "special clique". Each vertex in this clique SC has a neighbour outside of SC.

Graphs of Type $\mathcal{X}$ need not necessarily have a special clique; in that case, $\gamma(G) = k+t+m$. Furthermore, any of k, t and m may be 0. Note that if G has no SC and BC is empty, then G is disconnected since S and C are non-adjacent. Also note that G cannot have a special clique only, since by definition of SC, each vertex of SC must have a neighbour outside of SC.

The following theorem is the "main theorem" of Hartnell and Rall. They use the one-half argument from the previous example to prove it in [39].

Theorem 3.1 ([39]) *Every graph of Type $\mathcal{X}$ satisfies Vizing's conjecture.*

Proof sketch. Let G be a graph of Type $\mathcal{X}$, H an arbitrary graph and D a minimum dominating set of $G \Box H$. We will prove that $|D| \geq n\gamma(H)$, where $n = k+t+m+1$, by showing that D can be partitioned into n sets of vertices, each of which dominates H.
(I): We first consider S and BC in $V(G)$. Note that for each $S_i \in S$, $S_i \Box H$ contains sufficient vertices from D to dominate a copy of H (in this case, ${}^{v_i}H$). In fact, any vertex in ${}^{v_i}H$ not dominated by $D \cap {}^{v_i}H$ is dominated by $D \cap {}^{T_i}H$, since $N_G(v_i) = T_i$. The same holds true for each $B_i \in BC$ since $N_G(b_i) = B_i - \{b_i\}$. Therefore S and BC do not need any assistance from other partitions of $V(G)$ to dominate ${}^{v_i}H$ and ${}^{b_i}H$ respectively.

Since each vertex in ${}^{SC}H$ and ${}^{C_i}H$ (may) have neighbours outside of ${}^{SC}H$ and ${}^{C_i}H$, these H-fibres may not be dominated by $D \cap {}^{SC}H$ and $D \cap {}^{C_i}H$ respectively. In this case, we need to show that there are sufficient vertices from D to dominate each of $\{SC, C_1, \ldots, C_m\}$ as well as $\{S_1, \ldots, S_k, B_1, \ldots, B_t\}$ with at least $\gamma(H)$ vertices. For each clique K in $\{SC, C_1, \ldots, C_m\}$, project all the vertices of $D \cap {}^{K}H$ onto ${}^{w}H$, for some $w \in K$. Let W be the set of vertices from $D \cap {}^{w}H$ together with the vertices from the projection. If W dominates ${}^{w}H$, then there are at least $\gamma(H)$ vertices of D in ${}^{K}H$ and we're done. Otherwise, from the vertices of ${}^{w}H$ that are not dominated by W, define a *missing fibre list* for ${}^{w}H$, denoted $\mathcal{L}_w$, as the set of all $h \in V(H)$ such that (w, h) is not dominated by W. Therefore all such vertices (w, h) must be dominated by D from G^h. Consider $T_1, \ldots, T_k$. Note that by the structure of G, any vertex $(u, h) \in D \cap {}^{T_i}H$ is only needed to dominate (v_i, h) in ${}^{v_i}H$ and not any other vertex in a neighbouring G-fibre. Additionally, $D \cap {}^{T_i}H$ may help us to dominate ${}^{SC}H$. $D \cap {}^{T_i}H$ may dominate vertices in another ${}^{T_j}H$ and some ${}^{B_j}H$, but both ${}^{T_j}H$ and ${}^{B_j}H$ contain sufficient vertices of D to dominate an H-fibre by **(I)** and therefore do not require any assistance from

$D \cap {}^{T_i}H$ in the count.
(II): For each i, $1 \leq i \leq k$, project $D \cap {}^{T_i}H$ to ${}^{x}H$ for some $x \in T_i$. Let X be the set of vertices in ${}^{x}H$ together with the vertices of this projection. The induced subgraph ${}^{x}H[X]$ has components of order at least two or isolated vertices. Let F be a component of ${}^{x}H[X]$ of order at least 2. By Corollary 1.1.1, $\gamma(F)$ vertices of F can be counted towards dominating ${}^{v_i}H$ (i.e., towards $\mathcal{L}_{v_i}$) and $\gamma(F)$ vertices of F can be counted towards dominating ${}^{SC}H$ if necessary.

We now consider the components that are isolated vertices in the induced subgraphs separately. Firstly, suppose that SC is not missing in G^h.
(III): All $C_i \in C$ that are missing in G^h are dominated externally by neighbouring cliques in G^h, so by the external domination property of decomposable graphs, there are sufficient vertices of D to dominate all missing C_i in G^h.

Now suppose that SC is missing in G^h. If SC is externally dominated completely by neighbouring cliques in G^h, then **(III)** applies. If not, then some vertices of SC are dominated by $T_1, \ldots, T_k$. For each j, $1 \leq j \leq k$, select $x_j \in T_j$ and repeat **(II)**. The number of vertices of D required to dominate SC are determined by the structure of the components F_j of ${}^{x_j}H[X_j]$. There are several cases, which we do not go into detail here, that can be found in [39]. These cases are handled in a manner similar to **(II)** (if F_j has order at least 2) and **(III)** (if F_j is an isolated vertex), which implies that there are sufficient vertices from D to dominate each of $SC, C_1, \ldots, C_m, S_1, \ldots, S_k, B_1, \ldots, B_t$ with at least $\gamma(H)$ vertices. □

Note that by the definition of decomposable graphs, all BG-graphs are of Type $\mathcal{X}$ with $S = \emptyset$. Therefore this theorem is an extension of Corollary 2.2.1.

Definition: Let G be a graph and X a subset of $V(G)$. X is called a *2-packing* of G if for any two vertices $u, v \in X$, $N[u] \cap N[v] = \emptyset$. The *2-packing number* of G is the cardinality of a maximum 2-packing of G, denoted $\rho(G)$.

Jacobson and Kinch proved the following useful result in [49].

Theorem 3.2 ([49]) *For any two connected graphs G and H, $\gamma(G \square H) \geq \gamma(G)\rho(H)$.*

Proof. Let D be a minimum dominating set of $G \square H$. For any $h \in V(H)$, the G^h-fibre is dominated by $(V(G) \times N[h]) \cap D$, so

$$|(V(G) \times N[h]) \cap D| \geq \gamma(G^h) = \gamma(G) \tag{3.1}$$

since $G^h \cong G$. Also, for any $h_1, h_2 \in V(H)$ with $d_H(h_1, h_2) > 2$,

$$(V(G) \times N(h_1)) \cap (V(G) \times N(h_2)) = \emptyset \tag{3.2}$$

that is, h_1 and h_2 have no common neighbours. Therefore, for any maximum 2-packing X of H,

$$\gamma(G \square H) = |D| \overset{(2)}{\geq} \sum_{h \in X} |(V(G) \times N[h]) \cap D| \overset{(1)}{\geq} \sum_{h \in X} \gamma(G) = \gamma(G)|X| = \gamma(G)\rho(H)$$

since for all $h \in X$, each vertex in D can intersect at most one neighbourhood $N[h]$. □

We will denote graphs which have equal 2-packing and domination number as (ρ, γ)-graphs. Note that this theorem implies that any connected (ρ, γ)-graph satisfies Vizing's conjecture. In particular, Meir and Moon [52] use induction to prove that trees are (ρ, γ)-graphs.

Theorem 3.3 ([52]) *For any tree T, $\rho(T) = \gamma(T)$.*

Proof. Let P be a maximum 2-packing of T and let D be a minimum dominating set of T. Note that for each $x \in D$, $N(x)$ contains at most one vertex of P. So since D dominates T, $\rho(T) \leq \gamma(T)$. It remains to show that $\rho(T) \geq \gamma(T)$. Let $x_0, x_1, \ldots, x_m$ be the vertices of any longest path in T. Firstly, if $m \leq 2$, then $\rho(T) = 1 = \gamma(T)$. Now suppose that $m \geq 3$. Let T' be the smallest connected subgraph of T which contains all $z \in V(T)$ such that $d(x_1, z) > 1$. In particular, for each $v \in V(T')$, either $d(x_1, v) > 1$ or there exists a vertex $z_v \in V(T)$ such that $d(x_1, z_v) > 1$ and the unique path joining z_v and x_m in T contains v. Note that T' is non-empty since $d(x_1, x_m) > 1$ for $m \geq 3$. Let $P' \subseteq V(T')$ be a maximum 2-packing of T' and let $D' \subseteq V(T')$ be a minimum dominating set of T'. Since $D = D' \cup \{x_1\}$ dominates T,

$$\gamma(T) \leq \gamma(T') + 1. \tag{3.3}$$

Moreover, $P' \cup \{x_0\}$ is a 2-packing of T and so

$$\rho(T) \geq \rho(T') + 1. \tag{3.4}$$

Since $|V(T')| < |V(T)|$, we may assume that $\rho(T') \geq \gamma(T')$ as the induction hypothesis. This, together with (3.3) and (3.4), gives us

$$\rho(T) \geq \rho(T') + 1 \geq \gamma(T') + 1 \geq \gamma(T)$$

which concludes the proof. □

Corollary 3.3.1 *For any tree T and graph H, $\gamma(T\Box H) \geq \gamma(T)\gamma(H)$.*

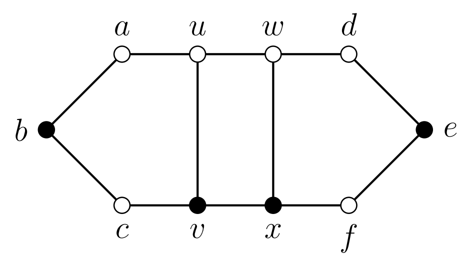
Hartnell and Rall [39] improved on Theorem 3.2 by showing that a graph G with $\gamma(G) = \rho(G) + 1$ is of Type $\mathcal{X}$, which leads to the following corollary.

Corollary 3.3.2 ([39]) *If G is a graph such that $\gamma(G) = \rho(G)+1$, then G satisfies Vizing's conjecture.*

Proof. Let $\{v_1, \ldots, v_{\rho(G)}\}$ be a maximum 2-packing of G. For each $i \in [1, \rho(G)]$, let $S_i = G[N[v_i]]$ and let $X = V(G) - \bigcup_{i=1}^{\rho(G)} N[v_i]$. Add any missing edges between the vertices of X to create a clique SC from X and add any edges to each S_i until S_i is edge-critical. Then the resulting graph G' also has domination number $\rho(G) + 1$ and is of Type $\mathcal{X}$. Since G is a spanning subgraph of G', Theorems 2.1 and 3.1 imply that G satisfies Vizing's conjecture.$\Box$

We now look at an example of how the one-half argument can be extended to show that a graph which is not of Type $\mathcal{X}$ satisfies Vizing's conjecture.

Example: The following graph G is our working example for extending the one-half argument.



G

Note that $\gamma(G) = 4$ and that any partitioning of $V(G)$ must have at least two SCs, therefore G is not of Type $\mathcal{X}$. Nonetheless, we will prove that G satisfies Vizing's conjecture.
Define the vertex 2-sets $T_1 = \{a, c\}$, $C_1 = \{u, v\}$, $C_2 = \{w, x\}$ and $T_2 = \{d, f\}$. Note that the set of solid vertices $\{b, e, v, x\}$ is a minimum dominating set of G. Let H be any graph and D a minimum dominating set of $G\Box H$. To ease notation, we will write $D \cap {}^gH$ as D_g, for some $g \in V(G)$. To use the one-half argument to show that G satisfies Vizing's conjecture, we need to prove that $|D| \geq 4\gamma(H)$, that is, D can be partitioned into 4 disjoint sets each of cardinality at least $\gamma(H)$, and dominates ${}^bH \cup {}^eH \cup {}^vH \cup {}^xH$.
If each of D_b, D_e, D_v and D_x dominates ${}^bH$, ${}^eH$, ${}^vH$ and ${}^xH$ respectively, then

$$|D| \geq |D_b| + |D_e| + |D_v| + |D_x| \geq 4\gamma(H)$$

and we are done, so suppose that there exists at least one of D_b, D_e, D_v and D_x that does not dominate its corresponding H-fibre. We associate a colour to each of these H-fibres, so assign colour 1 to ${}^bH$, 2 to ${}^eH$, 3 to ${}^vH$ and 4 to ${}^xH$. Now we will label each vertex of H with a subset of $\{1, 2, 3, 4\}$ such that each vertex subset of $V(H)$ coloured i, $1 \leq i \leq 4$, dominates H. Each subset of colours assigned to each vertex of H will increase throughout our argument, but its cardinality will not exceed $|D| = \gamma(G \Box H)$.

Firstly, we assign colour 1 to each $h \in V(H)$ with $(b, h) \in D_b$. Similarly, we assign 2, 3 and 4 to each h with $(e, h) \in D_e$, $(v, h) \in D_v$ and $(x, h) \in D_x$ respectively. Note that the neighbours of b and e are contained in T_1 and T_2 respectively, whereas v and x each have neighbours outside of C_1 and C_2 respectively.

Define the *missing fibre list* for ${}^bH$, denoted $\mathcal{L}_b$, as the set of all $h \in V(H)$ such that (b, h) is not dominated by D_b. We define the missing fibre list for ${}^eH$, $\mathcal{L}_e$, similarly.

Consider the projections

$$p_c : D_a \to {}^cH,\ p_v : D_u \to {}^vH,\ p_x : D_w \to {}^xH \text{ and } p_f : D_d \to {}^fH$$

and define the sets

$$D'_c = D_c \cup p_c(D_a),\ D'_v = D_v \cup p_v(D_u),\ D'_x = D_x \cup p_x(D_w) \text{ and } D'_f = D_f \cup p_f(D_d).$$

The missing fibre list $\mathcal{L}_v$ is then defined as the set of all $h \in V(H)$ such that (v, h) is not dominated by D'_v, and $\mathcal{L}_x$ is the set of all $h \in V(H)$ such that (x, h) is not dominated by D'_x. We now look at the components of the subgraphs of the H-fibres induced by D'_c and D'_f, each of which are either isolated, or of order at least two.

Suppose that C is a component of ${}^cH[D'_c]$.

Case 1: C has order at least 2 in ${}^cH[D'_c]$.

Let A be a minimal dominating set of C. Then by Theorem 1.1, $C - A$ is also a dominating set of C. Count the vertices of A towards $\mathcal{L}_b$ and the vertices of $C - A$ towards $\mathcal{L}_v$ if necessary. For each vertex $k \in V(H)$, if $(c, k) \in A$, then assign colour 1 to k, otherwise if $(c, k) \in C - A$, assign 3 to k.

Case 2: C is isolated in ${}^cH[D'_c]$ and $C = \{(c, k)\}$ for some $k \in V(H)$.

If both $(a, k), (c, k) \in D$, then (a, k) and (c, k) can be counted towards $\mathcal{L}_b$ and $\mathcal{L}_v$ respectively, and we assign colours 1 and 3 to k. On the other hand, without loss of generality, if $(a, k) \in D$ and $(c, k) \notin D$, then since D dominates (c, k), k cannot be on both of the missing fibre lists $\mathcal{L}_v$ and $\mathcal{L}_x$. Therefore, at least one of (b, k) and (v, k) is in D to dominate (c, k). If $k \in \mathcal{L}_b$, then assign colour 1 to k, otherwise if $k \in \mathcal{L}_v$, then assign 3 to k.

The same argument is used for the components of ${}^{f}H[D'_f]$ to colour the vertices of $\mathcal{L}_e$ and $\mathcal{L}_x$ with colours 2 and 4 respectively. Observe that if a vertex h is on the missing fibre list $\mathcal{L}_b$, then colour 1 has either been assigned to h or to a neighbour of h in H. Therefore, the set of all vertices coloured 1 in ${}^{b}H$ together with $\mathcal{L}_b$ dominates H. This holds true for all vertices h in ${}^{e}H$ together with $\mathcal{L}_e$ that are coloured 2 as well. Thus we have found two subsets of D each of cardinality at least $\gamma(H)$, and will find the next two by considering $\mathcal{L}_v$ and $\mathcal{L}_x$.
If vertex k is on the list $\mathcal{L}_v$ and $(c,k) \in D'_c$, then colour 3 has either been assigned to k or a neighbour of k in H. On the other hand, if $k \in \mathcal{L}_v$ and $(c,k) \notin D'_c$, then (x,k) is in D to dominate (v,k), so neither k nor any of its neighbours belong to $\mathcal{L}_x$ and we can colour k with 3. Similar arguments hold for vertices on the list $\mathcal{L}_x$ and regarding colour 4.
Suppose now that $k \in \mathcal{L}_v$ and colour 3 has not yet been assigned to k or to a neighbour of k in H. This implies that $(c,k) \notin D'_c$ and so $(x,k) \in D$ to dominate (v,k) and we can assign colour 3 to k. Again, the same argument holds for $\mathcal{L}_x$ and colour 4. Thus the set of all vertices coloured 3 in ${}^{v}H$ together with $\mathcal{L}_v$ dominates H, and the same holds true for the set of vertices coloured 4 in ${}^{x}H$ together with $\mathcal{L}_x$. Therefore, we have partitioned D into four sets each of cardinality at least $\gamma(H)$ and thus G satisfies Vizing's conjecture.

Chapter 4

Fair domination

We now turn our attention to another type of domination known as *fair domination*, which was introduced by Brešar and Rall in 2009 [18]. This domination also requires a partitioning of the vertices of a graph G in a way that satisfies a property similar to the earlier external domination property of Barcalkin and German [4].

Definition: Let $\{S_1, \ldots, S_k\}$ be a set of pairwise disjoint vertex subsets of a graph G with $\mathcal{S} = S_1 \cup \cdots \cup S_k$ and let $Z = V(G) - \mathcal{S}$. We say that the sets $S_1, \ldots, S_k$ form a *fair reception of size* k if for any integer ℓ, $1 \leq \ell \leq k$, and for any choice of ℓ sets $S_{i_1}, \ldots, S_{i_\ell}$ from $\mathcal{S}$, the following holds: if D externally dominates $S_{i_1} \cup \cdots \cup S_{i_\ell}$, then

$$|D \cap Z| + \sum_j (|S_j \cap D| - 1) \geq \ell$$

where the sum is taken over all j such that $S_j \cap D \neq \emptyset$.

A simple example for any graph G, is that any non-empty set $A \subseteq V(G)$ forms a fair reception of size 1. In this case $k = 1$, and since $A \cap D = \emptyset$,

$$|D \cap Z| + \sum_j (|S_j \cap D| - 1) = |D \cap Z| + 0 = |D| \geq 1 = \ell$$

where the sum is taken over all j such that $S_j \cap D \neq \emptyset$.

Definition: The largest integer k such that there exists a fair reception of size k in a graph G is called the *fair domination number of* G, denoted $\gamma_F(G)$.

The following result establishes a relationship between the 2-packing number, fair domination number and domination number of a graph.

Theorem 4.1 ([18]) *For any graph G, $\rho(G) \leq \gamma_F(G) \leq \gamma(G)$.*

Proof. (i) $\rho(G) \leq \gamma_F(G)$:
Let X be a maximum 2-packing of G, so $|X| = \rho(G)$. Let $S_1, \ldots, S_k$ be subsets of $V(G)$ such that each S_i consists of exactly one vertex of the 2-packing X, and $\mathcal{S} = S_1 \cup \cdots \cup S_k$. Then the S_i's are pairwise disjoint. Moreover, let $Z = V(G) - \mathcal{S}$ and let D be an external dominating set of $S_{i_1}, \ldots, S_{i_\ell}$, for some $1 \leq \ell \leq k$. Then since $X \cap D = \emptyset$ and no two vertices in X have common neighbours, each vertex of D can only dominate one of $S_{i_1}, \ldots, S_{i_\ell}$ and

$$|D \cap Z| + \sum_j (|S_j \cap D| - 1) = |D| + 0 = |D| \geq \ell$$

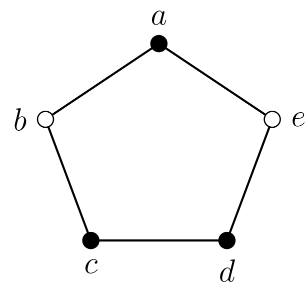
where the sum is taken over all j such that $S_j \cap D \neq \emptyset$. Therefore $k = \gamma_F(G) \geq \rho(G)$.

(ii) $\gamma_F(G) \leq \gamma(G)$:
Let A be a minimum dominating set of G with cardinality r, i.e., $\gamma(G) = r$. Let $S_1, \ldots, S_k$ form a fair reception of size k in G, i.e., $\gamma_F(G) = k$, with $\mathcal{S} = S_1 \cup \cdots \cup S_k$ and $Z = V(G) - \mathcal{S}$. Suppose to the contrary that $r = \gamma(G) < \gamma_F(G) = k$. Then A does not intersect at least one of the k sets (S_i's). Assume that $A \cap S_i = \emptyset$ for $1 \leq i \leq \ell$, and $A \cap S_j \neq \emptyset$ for $\ell + 1 \leq j \leq k$. Then A externally dominates $S_1 \cup \cdots \cup S_\ell$ and has non-empty intersection with each S_j. So by the definition of fair domination,

$$\begin{aligned}
& |A \cap Z| + \sum_{j=\ell+1}^{k} (|S_j \cap A| - 1) \geq \ell \\
\Rightarrow \quad & |A \cap Z| + \sum_{j=\ell+1}^{k} |S_j \cap A| - (k - (\ell + 1) + 1) \geq \ell \\
\Rightarrow \quad & |A \cap Z| + \sum_{j=\ell+1}^{k} |S_j \cap A| - (k - \ell) \geq \ell \\
\Rightarrow \quad & |A \cap Z| + |\mathcal{S} \cap A| - (k - \ell) \geq \ell \\
\Rightarrow \quad & |A| - k + \ell \geq \ell
\end{aligned}$$

since Z and $\mathcal{S}$ are complements in $V(G)$. Hence, $r = |A| \geq k$, a contradiction. □

For example, consider the cycle C_5. We will show that $\gamma_F(C_5) = 2$.



Consider the case $k = 2$. Let $S_1 = \{a\}$, $S_2 = \{c, d\}$ and $\mathcal{S} = S_1 \cup S_2$. Then $Z = \{b, e\} = D$ and since $\mathcal{S} \cap D = \emptyset$,

$$|D \cap Z| + \sum_j (|S_j \cap D| - 1) = |D| + 0 = 2 \geq \ell$$

where the sum is taken over all j such that $S_j \cap D \neq \emptyset$. Therefore $\gamma_F(C_5) \geq 2$, and by Theorem 4.1, $\gamma_F(C_5) \leq \gamma(C_5) = 2$. Hence, $\gamma_F(C_5) = 2$.

Brešar and Rall used the method of Theorem 2.2 by Barcalkin and German to prove the following result regarding fair domination.

Theorem 4.2 ([18]) *For any two graphs G and H,*

$$\gamma(G \Box H) \geq \max\{\gamma(G)\gamma_F(H), \gamma_F(G)\gamma(H)\}.$$

Proof. Let D be a minimum dominating set of $G \Box H$. Let $S_1, \ldots, S_k$ form a fair reception of size k of H such that $k = \gamma_F(H)$ with $\mathcal{S} = S_1 \cup \cdots \cup S_k$ and $Z = V(H) - \mathcal{S}$. For each $i \in [1, k]$, let $D_i = D \cap G^{S_i}$ and consider the projection to G, $p_G(D_i)$. Since $|D_i| \geq |p_G(D_i)|$, define $d_i = |D_i| - |p_G(D_i)|$ and let $d_Z = |D \cap G^Z|$. For each $v \in V(G)$, let

$$d_i^{\,v} = \begin{cases} |D_i \cap {}^vH| - 1 & \text{if } D_i \cap {}^vH \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and let $d^v = |D \cap (\{x\} \times Z)|$. If $D_v = D \cap {}^vH$, then $d^v + \sum_{i=1}^{k} d_i^{\,v}$ is the number of vertices in D_v that are not counted in $|p_G(D_1)| + \cdots + |p_G(D_k)|$. For each $i \in [1, k]$, let T_i be the set of vertices in G that are not dominated by $p_G(D_i)$. Then $p_G(D_i) \cup T_i$ dominates G and so

$$|p_G(D_i)| + |T_i| = |p_G(D_i) \cup T_i| \geq \gamma(G). \tag{4.1}$$

For each $v \in V(G)$, let I_v be the set of indices $i \in [1,k]$ such that $v \in T_i$. Since we counted the vertices of G that are not dominated by $p_G(D_i)$ in two ways using T_i and I_v,

$$\sum_{i=1}^{k} |T_i| = \sum_{v \in V(G)} |I_v| \tag{4.2}$$

Note that for some $v \in V(G)$ and $i \in I_v$, the set of vertices $\{(v,w) : w \in S_i\}$ is not dominated by D_i in the G-fibre, so these vertices must be dominated externally by D_v in the corresponding H-fibre. So $p_H(D_v)$ externally dominates S_i for all $i \in I_v$. By the definition of a fair reception (in H), we have that

$$d^v + \sum_{i=1}^{k} d_i{}^v \geq |I_v|. \tag{4.3}$$

Therefore,

$$\begin{aligned}
|D| = \sum_{i=1} |D_i| + d_Z &= \sum_{i=1}^{k} (|p_G(D_i)| + d_i) + d_Z \\
&= \sum_{i=1}^{k} |p_G(D_i)| + \sum_{i=1}^{k} \sum_{v \in V(G)} d_i{}^v + d_Z \\
&= \sum_{i=1}^{k} |p_G(D_i)| + \sum_{v \in V(G)} \left(\sum_{i=1}^{k} d_i{}^v + d_x \right) \\
&\overset{(4.3)}{\geq} \sum_{i=1}^{k} |p_G(D_i)| + \sum_{v \in V(G)} |I_v| \\
&\overset{(4.2)}{=} \sum_{i=1}^{k} |p_G(D_i)| + \sum_{i=1}^{k} |T_i| \\
&\overset{(4.1)}{\geq} k\gamma(G) = \gamma_F(H)\gamma(G).
\end{aligned}$$

We can interchange G and H to obtain the desired result. □

Corollary 4.2.1 ([18]) *If G is a graph such that $\gamma_F(G) = \gamma(G)$, then G satisfies Vizing's conjecture.*

The following proposition was also proven by Brešar and Rall.

Proposition: ([18]) *If G is a decomposable graph, then a partition of $V(G)$ into $\gamma(G)$ cliques forms a fair reception of size $\gamma(G)$ in G (where $\mathcal{S} = V(G)$).*

Proof. If we let $\mathcal{S} = C_1 \cup \cdots \cup C_{\gamma(G)} = V(G)$, where each C_i is a clique in G, then $Z = \emptyset$ and the result follows from the external domination property of decomposable graphs by Barcalkin and German [4]. $\square$

If a collection of sets forms a fair reception in a graph G, then the collection also forms a fair reception in any spanning subgraph of G [12]. So by the above results, the class of graphs satisfying $\gamma_F(G) = \gamma(G)$ contains the class of BG-graphs. Brešar and Rall conclude [18] with the following open question.

Question: For any graph G, is there a general lower bound for $\gamma_F(G)$ in terms of $\gamma(G)$? For instance, is it true that $\gamma_F(G) \geq \gamma(G) - 1$?

We now look at the graph invariant introduced by Aharoni and Szabó [2], the *independence-domination number*, denoted γ^i. For any graph G, $\gamma^i(G)$ is the maximum of $\gamma_G(I)$ over all independent sets I in G.

Theorem 4.3 ([18]) *For any graph G without isolated vertices, $\gamma_F(G) \geq \gamma^i(G)$.*

Proof. Let I be an independent set in G that needs $k = \gamma^i(G)$ vertices to dominate it, and let $A = \{x_1, \ldots, x_k\}$ dominate I. Since G has no isolated vertices, we may assume that A externally dominates I. Let $S_1, \ldots, S_k$ partition I such that $S_i \subseteq N(x_i)$, i.e., $I = S_1 \cup \cdots \cup S_k$ and $x_i \in A$ dominates S_i for each i.
Claim: $\{S_1, \ldots, S_k\}$ forms a fair reception in G.
To externally dominate ℓ sets from $S_1, \ldots, S_k$, $1 \leq \ell \leq k$, we need at least ℓ vertices from Z, so since $I \cap A = \emptyset$, the following inequality is satisfied:

$$|A \cap Z| + \sum_j (|S_j \cap A| - 1) = |A| + 0 = k \geq \ell$$

where the sum is taken over all j such that $S_j \cap D \neq \emptyset$. In fact, if we needed fewer than ℓ vertices to externally dominate these ℓ sets, then there would exist a set of cardinality less than k that dominates I, contradicting the minimality of A. Hence $\gamma_F(G) \geq k = \gamma^i(G)$. $\square$

The next result by Aharoni and Szabó follows from Theorems 4.2 and 4.3. However, we include their proof which proves the result directly without using fair domination.

Theorem 4.4 ([2]) *For any two graphs G and H, $\gamma(G \Box H) \geq \gamma^i(G)\gamma(H)$.*

Proof. Let I be an independent set in G such that $\gamma_G(I) = \gamma^i(G)$. We will prove that $\gamma_{G\Box H}(I \times V(H) \geq \gamma^i(G)\gamma(H)$. Since any dominating set of $G\Box H$ dominates $I \times V(H)$, this will prove the theorem.

Let $D \subseteq V(G\Box H)$ dominate $I \times V(H)$. We will show that $|D| \geq \gamma^i(G)\gamma(H)$. Let $\{h_1, \ldots, h_{\gamma(H)}\}$ be a minimum dominating set of H and choose a partitioning of $V(H)$ into $\gamma(H)$ sets, $\{\pi_1, \ldots, \pi_{\gamma(H)}\}$, such that for each $i \in [1, \gamma(H)]$, $h_i \in \pi_i$ and $\pi_i \subseteq N[h_i]$.

We call the vertex set $\{g\} \times \pi_j$, where $g \in V(G)$ and $1 \leq j \leq \gamma(H)$, an *H-cell*. Let S be the set of all *H-cells* of the form $\{v\} \times \pi_j$, where $v \in I$, that are dominated by D from within the ${}^vH$-fibre. Define the two sets

$$\begin{aligned} S_v &= \{\{v\} \times \pi_j \in S : 1 \leq j \leq \gamma(H)\} \\ S_j &= \{\{v\} \times \pi_j \in S : v \in I\} \end{aligned}$$

By the minimality of $\gamma(H)$, for any subset J of $\{1, \ldots, \gamma(H)\}$, $\gamma_H\left(\bigcup_{j\in J} \pi_j\right) \geq |J|$. Therefore, for each $v \in I$,

$$\begin{aligned} & |D \cap (\{v\} \times V(H))| \geq |S_v| \\ \Rightarrow \quad & \sum_{v\in I} |D \cap (\{v\} \times V(H))| \geq \sum_{v\in I} |S_v| \end{aligned}$$

hence,

$$|D \cap (I \times V(H))| \geq |S| \tag{4.4}$$

For some j, each H-cell, $\{v\} \times \pi_j \notin S$ contains at least one vertex (v, w) that dominated by $(u, w) \in D$ from within the corresponding G-fibre. Note that $u \notin I$ since $v \in I$ and I is an independent set. Therefore the set of all such vertices $\{u : \{v\} \times \pi_j \notin S\}$ dominates $|I| - |S_j|$ vertices in $|I|$. So if we add $|S_j|$ vertices to this set, we can form a dominating set of I. Therefore, the cardinality of $\{u : \{v\} \times \pi_j \notin S\}$ is at least $\gamma^i(G) - |S_j|$. Summing over J then gives us

$$\begin{aligned} \sum_{j=1}^{\gamma(H)} |D \cap (\{u\} \times V(H))| &\geq \sum_{j=1}^{\gamma(H)} (\gamma^i(G) - |S_j|) \\ |D \cap ((V(G) - I) \times V(H))| &\geq \gamma^i(G)\gamma(H) - |S| \end{aligned}$$

Adding in (4.4) gives us

$$|D \cap (I \times V(H))| + |D \cap ((V(G) - I) \times V(H))| \geq |S| + \gamma^i(G)\gamma(H) - |S|$$

so $|D| \geq \gamma^i(G)\gamma(H)$, as desired. □

So for any graph G in which $\gamma^i(G) = \gamma(G)$, G would satisfy Vizing's conjecture. Aharoni, Berger and Ziv [1] showed that this equality holds for chordal graphs [12]. However, it is still not known whether chordal graphs are BG-graphs or not. The class of graphs for which $\gamma^i(G) = \gamma(G)$ has not been determined yet either [12].

Corollary 4.4.1 ([2]) *Chordal graphs satisfy Vizing's conjecture.*

The following open question is posed by Brešar et al. in [12].

Question: Does there exist a constant $c > \frac{1}{2}$ such that $\gamma^i(G) \geq c\gamma(G)$ for every graph G?

Together with Theorem 4.2, the truth of this question would prove that $\gamma(G \Box H) \geq c\gamma(G)\gamma(H)$, which would be an improvement of Theorem 5.1.

Note that $\gamma^i(G) \geq \rho(G)$, since all 2-packings are independent in G, but independent sets may have common neighbours in G. However, $\gamma_F(G)$ is not easy to compute for most graphs [12], making Theorem 4.2 difficult to implement. So a different approach to Vizing's conjecture is explored in the next section.

Chapter 5

The double-projection argument

One of the common ways to approach proving Vizing's conjecture is to determine if there exists a constant $c > 0$ such that

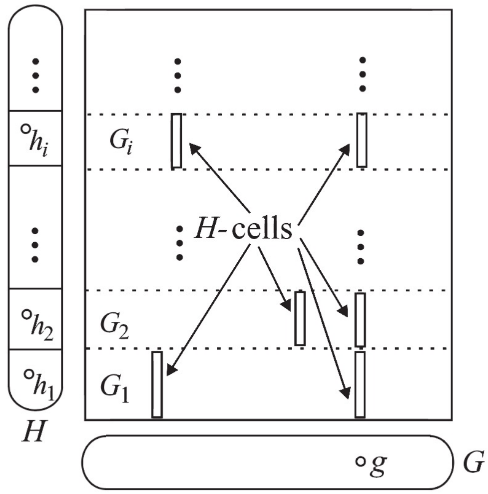
$$\gamma(G\Box H) \geq c\gamma(G)\gamma(H)$$

for any two graphs G and H [12]. The ideal goal would be that the result can eventually be proven for $c = 1$, but until then we will discuss a proof by Clark and Suen from 2009 [24] that states that such a c does indeed exist. To show this, they use a method called the *double-projection argument*, which exploits the structure of the Cartesian product.

Theorem 5.1 ([24]) *For any two graphs G and H,*

$$\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)\gamma(H).$$

Proof. Let H be a graph with $\gamma(H) = k$, $\{h_1, \ldots, h_k\}$ a minimum dominating set of H and choose a partition of the vertex set $V(H) = \pi_1 \cup \cdots \cup \pi_k$, where for each $i \in [1, k]$, $h_i \in \pi_i$ and $\pi_i \subseteq N[h]$. Let G be any graph. For each i, let $G_i = V(G) \times \pi_i$, and for some fixed $g \in V(G)$, we call the vertex set $\{g\} \times \pi_i$ an *H-cell*. Below is a diagram from [12] showing the positions of the H-cells in $G\Box H$.



Let D be a minimum dominating set of $G \Box H$ and for each i, let n_i be the number of H-cells in G_i such that all the vertices of each H-cell are dominated by D from within the corresponding H-fibre. Consider the projection to G, $p_G : D \cap G_i \to G$ defined by $p_G(g,h) = g$. Note that p_G is many-to-one, so $|D \cap G_i| \geq |p_G(D \cap G_i)|$. It follows that

$$p_G|D \cap G_i| + n_i \geq \gamma(G)$$

Therefore,

$$|D \cap G_i| + n_i \geq \gamma(G)$$

Summing over i, we get

$$\begin{aligned} \sum_{i=1}^{k}(|D \cap G_i| + n_i) &\geq \sum_{i=1}^{k}\gamma(G) \\ \sum_{i=1}^{k}(|D \cap G_i|) + \sum_{i=1}^{k} n_i &\geq \gamma(G)\sum_{i=1}^{k} 1 \\ |D| + \sum_{i=1}^{k} n_i &\geq \gamma(G) \cdot k \end{aligned}$$

so

$$|D| + \sum_{i=1}^{k} n_i \geq \gamma(G)\gamma(H) \tag{5.1}$$

Now consider the projection to H, $p_H : D \cap {}^gH \to H$ defined by $p_H(g,h) = h$. Since ${}^gH \cong H$, $|p_H(D \cap {}^gH)| = |D \cap {}^gH|$ and $\gamma({}^gH) = \gamma(H)$. Let m_g be the number of H-cells in ${}^gH$ dominated by D from within the ${}^gH$-fibre. Then

$$\gamma(H) \leq |p_H(D \cap {}^gH)| + (k - m_g)$$

that is, the number of vertices from D which dominate the H-cells from within ${}^gH$ together with the number of H-cells from ${}^gH$ dominated from outside ${}^gH$ forms a set which dominates ${}^gH \cong H$. Therefore,

$$k \leq |p_H(D \cap {}^gH)| + k - m_g \Rightarrow m_g \leq |p_H(D \cap {}^gH)|$$

and by summing over $g \in V(G)$,

$$\sum_{g \in V(G)} m_g \leq \sum_{g \in V(G)} |p_H(D \cap {}^gH)| \ \Rightarrow \sum_{g \in V(G)} m_g \leq |D|$$

Note that we have counted the number of H-cells in $G \Box H$ in two ways, using n_i and m_g, thus

$$\sum_{i=1}^{k} n_i = \sum_{g \in V(G)} m_g \ \text{ which implies that } \ |D| \geq \sum_{i=1}^{k} n_i$$

From (5.1), we therefore have

$$2\gamma(G \Box H) = 2|D| = |D| + |D| \geq |D| + \sum_{i=1}^{k} n_i \geq \gamma(G)\gamma(H)$$

as desired. □

The factor of $\frac{1}{2}$ in Theorem 5.1 is a result of double-counting the number of vertices in the dominating set D. The first improvement of this result was proven by Suen and Tarr in 2012 [59].

Theorem 5.2 *For any two graphs G and H,*

$$\gamma(G \Box H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}.$$

Proof. Let D be a minimum dominating set of $G \Box H$ and let $\{u_1, \ldots, u_{\gamma(G)}\}$ be a minimum dominating set of G. Choose a partition $\{\pi_{1\bullet}, \pi_{2\bullet}, \ldots, \pi_{\gamma(G)\bullet}\}$ of $V(G)$ such that $u_i \in \pi_{i\bullet}$ and $\pi_{i\bullet} \subseteq N_G[u_i]$ for all $i \in [1, \gamma(G)]$. Consider the projection $p_{i\bullet} : D \cap {}^{\pi_{i\bullet}}H \to H$ and define $C_{i\bullet} = V(H) - N_G[p_{i\bullet}]$. Note that $p_{i\bullet} \cup C_{i\bullet}$ dominates H and so for each $i \in [1, \gamma(G)]$,

$$|p_{i\bullet}| + |C_{i\bullet}| \geq \gamma(H).$$

For each $v \in V(H)$, let $D_{\bullet v} = D \cap G^v$ and let $S_{\bullet v} = \{i : v \in C_{i\bullet}\}$. If $i \in S_{\bullet v}$, then the G-cell $\pi_{i\bullet} \times \{v\}$ is dominated by $D_{\bullet v} \times \{v\}$ in the corresponding G^v-fibre. Let $S_H = \{(i, v) : v \in$

$C_{i\bullet}$ and $i \in [1, \gamma(G)]\}$. Then since we count the G-cells of $G\Box H$ in two ways using $S_{\bullet v}$ and $C_{i\bullet}$,

$$S_H = \sum_{v \in V(H)} |S_{\bullet v}| = \sum_{i=1}^{\gamma(G)} |C_{i\bullet}|.$$

Since $D_{\bullet v} \cup \{u_i : i \notin S_{\bullet v}\}$ dominates G,

$$|D_{\bullet v}| + (\gamma(G) - |S_{\bullet v}|) \geq \gamma(G)$$

and so $|S_{\bullet v}| \leq |D_{\bullet v}|$. Summing over all $v \in V(H)$ then gives us $S_H \leq |D|$. Now we consider two cases.

Case 1: $|p_{i\bullet}| + |C_{i\bullet}| > \gamma(H)$ for all $i \in [1, \gamma(G)]$.
Since $|D \cap {}^{\pi_{i\bullet}}H| \geq |p_{i\bullet}|$, we have that

$$\sum_{i=1}^{\gamma(G)} (|D \cap {}^{\pi_{i\bullet}}H| + |C_{i\bullet}|) \ \geq \ \sum_{i=1}^{\gamma(G)} (\gamma(H) + 1).$$

Therefore

$$S_H + |D| \geq \gamma(G)\gamma(H) + \gamma(G)$$

and since $S_H \leq |D|$,

$$\gamma(G\Box H) = |D| \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\gamma(G).$$

Case 2: $|p_{i\bullet}| + |C_{i\bullet}| = \gamma(H)$ for some $i \in [1, \gamma(G)]$.
Denote the vertices of $p_{i\bullet} \cup C_{i\bullet}$ as $v_1, \ldots, v_{\gamma(H)}$ and choose a partition $\{\pi_{\bullet 1}, \ldots, \pi_{\bullet \gamma(H)}\}$ of $V(H)$ such that $v_j \in \pi_{\bullet j}$ and $\pi_{\bullet j} \subseteq N_H[v_j]$ for all $j \in [1, \gamma(H)]$. As above, we define the projection $p_{\bullet j} : D \cap G^{\pi_{\bullet j}}$ and the set $C_{\bullet j} = V(H) - N_G[p_{\bullet j}]$. For each $u \in V(G)$, let $D_{u\bullet} = D \cap {}^{u}H$ and $S_{u\bullet} = \{j : u \in C_{\bullet j}\}$. Then, similarly to S_H,

$$S_G = \sum_{i \in V(G)} |S_{u\bullet}| = \sum_{j=1}^{\gamma(H)} |C_{\bullet j}|.$$

For each $u \in V(G)$, define the set $D'_{u\bullet} = \{v_j : (u, v_j) \in D_{u\bullet} \text{ and } j \in [1, \gamma(H)]\}$. Since $D_{u\bullet} \cup \{v_j : j \notin S_{u\bullet}\}$ dominates H and $D'_{u\bullet} = D_{u\bullet} \cap \{v_j : j \notin S_{u\bullet}\}$,

$$|D_{u\bullet}| + (\gamma(H) - |S_{u\bullet}|) - |D'_{u\bullet}| \geq \gamma(H)$$

and so

$$|S_{u\bullet}| \leq |D_{u\bullet}| - |D'_{u\bullet}|. \tag{5.2}$$

Note that for each $k \in [1, \gamma(H)]$, either $v_k \in p_{i\bullet}$ and so $(u, v_k) \in D$ for some $u \in \pi_{i\bullet}$, or $v_k \in C_{i\bullet}$ and so the G-cell $\pi_{i\bullet} \times \{v_k\}$ is dominated by some vertices $(u', v_k) \in D$ from the corresponding G-fibre. This implies that there are at least $\gamma(H)$ vertices in D that are of the form (u, v_k), therefore

$$\sum_{u \in V(G)} |D'_{u\bullet}| \geq \gamma(H).$$

So summing over all $u \in V(G)$ in (5.2) gives us $S_G \leq |D| - \gamma(H)$. Now note that similarly to our argument above, for each $j \in [1, \gamma(H)]$ we have that

$$|p_{\bullet j}| + |C_{\bullet j}| \geq \gamma(G)$$

and so by summing over all j,

$$|D| + S_G \geq \gamma(G)\gamma(H).$$

Finally, since $S_G \leq |D| - \gamma(H)$,

$$\gamma(G \Box H) = |D| \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\gamma(H).$$

Since either Case 1 or Case 2 holds, the result follows. $\square$

Brešar [11] used the 2-packing number together with the double-projection argument to formulate the following improvement of Theorem 5.1 in 2017.

Theorem 5.3 ([11]) *For any two graphs G and H,*

$$\gamma(G \Box H) \geq \max\left\{\frac{2\gamma(G) - \rho(G)}{3}\gamma(H), \frac{2\gamma(H) - \rho(H)}{3}\gamma(G)\right\}.$$

Proof. Let G and H be two graphs and $\gamma(H) = k$. Let $\{h_1, \ldots, h_k\}$ be a minimum dominating set of H and choose a partition $\{\pi_1, \ldots, \pi_k\}$ such that $h_i \in \pi_i$ and $\pi_i \subseteq N_H[h_i]$ for each $i \in [1, k]$. For some $u \in V(G)$, we will denote the H-cells of $G \Box H$ as $C_i^u = \{u\} \times \pi_i$. Let D be a minimum dominating set of $G \Box H$ and for each i, let $D_i = D \cap G^{\pi_i}$. We say that an H-cell C_i^u is *vertically dominated* by D if every vertex $(u, v) \in C_i^u$ has a neighbour $(u, v') \in D \cap ({}^uH - C_i^u)$. Let n_i be the number of vertically dominated H-cells in G^{π_i}. If an H-cell C_i^u is not vertically dominated, then there exists a vertex $(u, v) \in C_i^u$ that is either in D_i or is dominated by a vertex $(u', v) \in D_i$ in the corresponding G-fibre. If we then project the vertices of D_i onto G, the vertices in $p_G(D_i)$ together with the n_i vertices from the vertically dominated H-cells projected to G dominates G.

Claim: $|D_i| + \dfrac{n_i + \rho(G)}{2} \geq \gamma(G)$.

Define the set $S_i = \{u \in V(G) : C_i^u$ is vertically dominated by $D\}$. Then $|S_i| = n_i$. We will show that S_i can be dominated by a set of vertices with cardinality at most $\dfrac{n_i + \rho(G)}{2}$.

Firstly, if $n_i \leq \rho(G)$, then $n_i = \dfrac{n_i}{2} + \dfrac{n_i}{2} \leq \dfrac{n_i + \rho(G)}{2}$ and S_i is dominated by its n_i vertices, so we are done. Now assume that $n_i > \rho(G)$. Then S_i is not a 2-packing of G and so there exist two distinct vertices u and v in S_i such that $d_G(u,v) \leq 2$. In either case, both u and v can be dominated by one vertex. If $S_i - \{u,v\}$ is not a 2-packing of G, then we repeat this procedure successively until we obtain a 2-packing of G. This ultimately implies that $n_i - \rho(G)$ vertices of S_i can be dominated by at most $\dfrac{n_i - \rho(G)}{2}$ vertices, and the other $\rho(G)$ vertices of S_i can be dominated by at most $\rho(G)$ vertices. In total, all the vertices of S_i can be dominated by at most $\dfrac{n_i - \rho(G)}{2} + \rho(G) = \dfrac{n_i + \rho(G)}{2}$, proving the claim.

Summing over all $i \in [1,k]$, we have that

$$|D| + \frac{1}{2}\sum_{i=1}^{k} n_i + \frac{1}{2}\gamma(H)\rho(G) \geq \gamma(G)\gamma(H)$$

$$\Rightarrow \quad |D| + \frac{1}{2}\sum_{i=1}^{k} n_i \geq \frac{2\gamma(G) - \rho(G)}{2}\gamma(H).$$

We now consider the projection $p_H(D \cap {}^uH$ for some $u \in V(G)$. Let m_u be the number of vertically dominated H-cells in ${}^uH$ and note that since each H-cell C_i^u is dominated by h_i, we have that

$$|D \cap {}^uH| + (k - m_u) \geq \gamma(H) \;\;\Rightarrow\;\; |D \cap {}^uH| \geq m_u.$$

Then, summing over all $u \in V(G)$ gives us $|D| \geq \displaystyle\sum_{u \in V(G)} m_u$. Since we have counted the H-cells of $G \Box H$ in two ways using n_i and m_u, $\displaystyle\sum_{i=1}^{k} n_i = \sum_{u \in V(G)} m_u$, and so

$$\frac{2\gamma(G) - \rho(G)}{2}\gamma(H) \leq |D| + \frac{1}{2}\sum_{u \in V(G)} m_u \leq \frac{3}{2}|D| = \frac{3}{2}\gamma(G \Box H).$$

Hence

$$\gamma(G \Box H) \geq \frac{2\gamma(G) - \rho(G)}{3}\gamma(H).$$

Since we can interchange G and H in the argument, the result follows. $\square$

Suen and Tarr's result (Theorem 5.2) was further improved more recently by Zerbib [63] in 2019, who implemented a slightly different version of the double-projection argument.

Theorem 5.4 ([63]) *For any two graphs G and H,*

$$\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\max\{\gamma(G), \gamma(H)\}.$$

Proof. Let D be a minimum dominating set of $G\Box H$ and assume, without loss of generality, that $\gamma(G) \geq \gamma(H)$. Let D_G be the set of vertices of the projection $p_G(D)$. Then D_G is a dominating set of G. Let $S_G \subseteq D_G$ be a minimal dominating set of G, and so let $S_G = \{u_1, \ldots, u_k\}$, where $k \geq \gamma(G)$. Next, let $S_i = D \cap {}^{u_i}H$ and let T_i be the set of vertices from the projection $p_H(S_i)$. Note that $|T_i| \geq 1$ since $S_G \subseteq D_G$. Choose a partition $\{\pi_1, \ldots, \pi_k\}$ of $V(G)$ such that $u_i \in \pi_i$ and $\pi_i \subseteq N_G[u_i]$ for each $i \in [1, k]$. This in turn induces a partition $\{D_1, \ldots, D_k\}$ of D, where $D_i = D \cap {}^{\pi_i}H$. Let P_i denote the set of vertices in the projection $p_H(D_i)$ for each i. Then $P_i \cup (V(H) - N_H[P_i])$ is a dominating set of H, implying that for each $i \in [1, k]$,

$$|P_i| + |V(H) - N_H[P_i]| \geq \gamma(H).$$

For each $h \in V(H)$, let $D_h = D \cap G^h$ and define the following sets to count the G-cells $\pi_i \times \{h\}$ of $G\Box H$:

$$\begin{aligned} C &= \{(i, h) : \pi_i \times \{h\} \subseteq N_{G\Box H}[D_h]\} \\ n_i &= \{(i, h) \in C : h \in V(H)\} \\ m_h &= \{(i, h) \in C : i \in [1, k]\} \end{aligned}$$

Since we count the G-cells of $G\Box H$ in two ways using n_i and m_h, we have that

$$|C| = \sum_{i=1}^{k} |n_i| = \sum_{h \in V(H)} |m_h|.$$

Note that if $h \in (V(H) - N_H[P_i]) \cup T_i$, then $\pi_i \times \{h\}$ is not dominated by D from the H-fibres, and so $\pi_i \times \{h\}$ must be dominated by D_h. Hence $(i, h) \in n_i$, implying that

$$|V(H) - N_H[P_i]| + |T_i| \leq |n_i|.$$

We therefore have that

$$\begin{aligned} |C| &= \sum_{i=1}^{k} |n_i| \geq \sum_{i=1}^{k} (|V(H) - N_H[P_i]| + |T_i|) \\ &\geq \sum_{i=1}^{k} (\gamma(H) - |P_i| + 1) \\ &\geq k\gamma(H) - |D| + k \end{aligned}$$

Claim: for each $h \in V(H)$, $|m_h| \leq |D_h|$.
Proof. Suppose to the contrary that $|m_h| > |D_h|$. Then the set $S' = \{g : (g,h) \in D_h\} \cup \{u_j : (j,h) \notin m_h\}$ dominates G and has cardinality

$$|S'| = |D_h| + (k - |m_h|) = k - (|m_h| - |D_h|) < k = |S|.$$

Furthermore, $S' \subset S$ since $p_G(D_h)$ is a subset of D_G. So S' is a contradiction to the minimality of S.
Therefore,

$$|C| = \sum_{h \in V(H)} |m_h| \leq \sum_{h \in V(H)} |D_h| = |D|$$

and finally,

$$2|D| \geq |D| + k\gamma(H) - |D| + k \geq \gamma(G)\gamma(H) + \gamma(G).$$

The result follows. □

Another improvement of Theorem 5.1 was found by Wu [62] in 2013, who used the concept of *Roman domination*, which is explored in [27].

Definition: A *Roman dominating function* on a graph G is a map $f : V(G) \to \{0,1,2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The *weight* of a Roman dominating function is the value of $\sum_{v \in V(G)} f(v)$. The *Roman domination number* of a graph G, denoted $\gamma_R(G)$, is the minimum weight of a Roman dominating function on G. We call a Roman dominating function that attains $f(V(G)) = \gamma_R(G)$ a γ_R*-set.*

Let V_0 (respectively V_1 and V_2), be the set of all vertices with weight 0 (respectively weight 1 and weight 2) and let (V_0, V_1, V_2) be the ordered partition of $V(G)$ induced by the Roman dominating function f, so

$$V_i = \{v \in V(G) : f(v) = i\}, \quad \text{for } 0 \leq i \leq 2.$$

Moreover, let $n_i = |V_i|$. Note that there is a one-to-one correspondence between $\{0,1,2\}$ and (V_0, V_1, V_2), so we write $f = (V_0, V_1, V_2)$. Therefore, $f = (V_0, V_1, V_2)$ is a Roman dominating function if V_2 dominates V_0, so $V_0 \subseteq N[V_2]$. Consequently, the weight of f is

$$f(V(G)) = \sum_{v \in V(G)} f(v) = 2n_2 + n_1$$

Theorem 5.5 ([27]) *For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_R-set, so $f(V(G)) = \gamma_R(G)$, and let D be a minimum dominating set of G. Then $V_1 \cup V_2$ is a dominating set of G, since V_2 dominates V_0, so $|V_1 \cup V_2| \geq \gamma(G)$. Therefore

$$\gamma(G) \leq |V_1 \cup V_2| = |V_1| + |V_2| \leq |V_1| + 2|V_2| = f(V(G)) = \gamma_R(G)$$

On the other hand, $g = (\emptyset, \emptyset, D)$ is another Roman dominating function of G. So

$$g(V(G)) \geq \gamma_R(G) \Rightarrow 2|D| + |\emptyset| \geq \gamma_R(G) \Rightarrow 2\gamma(G) \geq \gamma_R(G)$$

Hence $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$. □

Wu used this theory to prove that $\gamma(G)\gamma(H) \leq \gamma_R(G\Box H)$, an improvement of Theorem 5.1.

Theorem 5.6 ([62]) *For any two graphs G and H, $\gamma(G)\gamma(H) \leq \gamma_R(G\Box H)$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G\Box H)$-function and let $D = V_1 \cup V_2$. Then D dominates $G\Box H$ and V_2 dominates $G\Box H - V_1$. Furthermore, let $\{g_1, \ldots, g_{\gamma(G)}\}$ be a minimum dominating set of G and choose a partitioning of the vertex set $V(G) = \pi_1 \cup \cdots \cup \pi_{\gamma(G)}$, where for each $i \in [1, \gamma(G)]$, $g_i \in \pi_i$ and $\pi_i \in N[g_i]$. This induces the partitioning of D into $\{D_1, \ldots, D_{\gamma(G)}\}$, where for each i, $D_i = (\pi_i \times V(H)) \cap D$.
For each i define the projection $p_i : D_i \to H$ by $p_i((g, h)) = h$ such that $(g, h) \in D_i$ for some $g \in \pi_i$. Since each vertex in $p_i(D_i)$ dominates its neighbourhood in H,
$p_i(D_i) \cup (V(H) - N_H[p_i(D_i)])$ dominates H. This implies that

$$|p_i(D_i)| + |V(H) - N_H[p_i(D_i)]\ | \geq \gamma(H) \tag{5.3}$$

Consider the G^h-fibre, for some $h \in V(H)$, and let $Q_h = V_2 \cap G^h$. Let the vertex set $\pi_i \times h$ be a *G-cell*, and let

$$C = \{(i, h) : \pi_i \times \{h\} \subseteq N_{G\Box H}[Q_h]\}.$$

Define

$$\begin{aligned} n_i &= \{(i, h) \in C : v \in V(H)\}\ \text{, and} \\ m_h &= \{(i, h) \in C : 1 \leq i \leq \gamma(G)\} \end{aligned}$$

Note that n_i (respectively, m_h) counts the number of G-cells dominated by Q_h in the H-fibres (respectively, the G-fibres), therefore

$$|C| = \sum_{i=1}^{\gamma(G)} |n_i| = \sum_{h \in V(H)} |m_h|$$

Note that any vertex $h \in V(H) - N_H[p_i(D_i)]$ is not dominated in the H-fibre by $p_i(D_i)$, so it must be dominated by Q_h in the G^h-fibre since V_2 dominates $G \Box H - V_1$. Hence, $(i, h) \in n_i$ and

$$|n_i| \geq |V(H) - N_H[p_i(D_i)]|$$

Therefore,

$$\begin{aligned}
|C| &\geq \sum_{i=1}^{\gamma(G)} |n_i| \\
&\geq \sum_{i=1}^{\gamma(G)} |V(H) - N_H[p_i(D_i)]| \\
&\overset{(5.3)}{\geq} \sum_{i=1}^{\gamma(G)} (\gamma(H) - |p_i(D_i)|) \\
&= \gamma(G)\gamma(H) - \sum_{i=1}^{\gamma(G)} |D_i| \\
&= \gamma(G)\gamma(H) - |D| \\
&= \gamma(G)\gamma(H) - |V_1| - |V_2|
\end{aligned}$$

We now find an upper bound for $|C|$.

Claim: for each $h \in V(H)$, $|m_h| \leq |Q_h|$.

Proof. Suppose to the contrary that $|m_h| > |Q_h|$. Then the vertices of Q_h together with the vertices g_j, $1 \leq j \leq \gamma(G)$, such that $(j, h) \notin m_h$ dominates G and has cardinality

$$|Q_h| + (\gamma(G) - |m_h|) = \gamma(G) - (|m_h| - |Q_h|) < \gamma(G),$$

a contradiction.

Therefore, we have the upper bound

$$\begin{aligned}
&|C| = \sum_{h \in V(H)} |m_h| \ \leq \sum_{h \in V(H)} |Q_h| = |V_2| \\
\Rightarrow \quad &|V_2| \geq \gamma(G)\gamma(H) - |V_1| - |V_2| \\
\Rightarrow \quad &\gamma(G)\gamma(H) \leq |V_1| + 2|V_2| = \gamma_R(G \Box H)
\end{aligned}$$

as desired. □

An improvement of Wu's result was found in 2018 by Pei, Pan and Hu [56], who use a similar argument to Suen and Tarr [59] in Theorem 5.2. We first need the following result.

Theorem 5.7 ([27]) *If G is a graph of order n, then $\gamma(G) = \gamma_R(G)$ if and only if $G = \overline{K_n}$.*

Proof. ($\Rightarrow$) If $G = \overline{K_n}$, then $\gamma(G) = \gamma_R(G) = n$ since the unique $\gamma_R(\overline{K_n})$-function assigns a weight of 1 to each vertex of $\overline{K_n}$.
($\Leftarrow$) Suppose $\gamma(G) = \gamma_R(G)$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function. Then from the proof of Theorem 5.5, $\gamma(G) = |V_1| + |V_2| = |V_1| + 2|V_2| = \gamma_R(G)$. Therefore $|V_2| = 0$ and so V_0 must be empty. This implies that $\gamma_R(G) = |V_1| = n$, so $\gamma(G) = n$. Hence $G = \overline{K_n}$. □

Theorem 5.8 ([56]) *For any two graphs G and H,*

$$\gamma_R(G\Box H) \geq \gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}.$$

Proof sketch. We consider two cases in this proof.
Case 1: G and H do not have isolated vertices.
Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G\Box H)$-function such that $|V_1|$ is as small as possible. Note that $D = V_1 \cup V_2$ dominates $G\Box H$ and Theorem 1.5 says that there exists a minimum dominating set D_G of G such that each vertex of D_G has an external private neighbour. Let $D_G = \{u_1, \ldots, u_{\gamma(G)}\}$ and choose a partition $\{\pi_1, \ldots, \pi_{\gamma(G)}\}$ of $V(G)$ such that $u_i \in \pi_i$ and $\pi_i \subseteq N_G[u_i]$ for each $i \in [1, \gamma(G)]$. This induces a partition $\{D_1, \ldots, D_{\gamma(G)}\}$ of D, where $D_i = D \cap {}^{\pi_i}H$ for each i. Consider the projection $p_i : D_i \to H$ and note that $|p_i| \leq |D_i|$. If $C_i = V(H) - N_H[p_i]$, then $p_i \cup C_i$ dominates H and so $|p_i| + |C_i| \geq \gamma(H)$ for each i. Now for each $h \in V(H)$, we define the sets $Q_h = \{g \in V(G) : (g, h) \in V_2\}$ and $R_h = \{i : h \in C_i \text{ and } i \in [1, \gamma(G)]\}$. Then by summing over all $h \in V(H)$, we obtain

$$\sum_{h \in V(H)} |Q_h| = |V_2| \quad \text{and} \quad \sum_{h \in V(H)} |R_h| = \sum_{i=1}^{\gamma(G)} |C_i|.$$

Let $i \in R_h$, then $h \in C_i$ and $\pi_i \times \{h\} \subseteq V_0$ since p_i only projects vertices from $V_1 \cup V_2$ onto H. Note that the G-cell $\pi_i \times \{h\}$ is then dominated from the G^h-fibre by $Q_h \times \{h\}$. On the other hand, if $i \notin R_h$, then (u_i, h) dominates the G-cell $\pi_i \times \{h\}$ and so $Q_h \cup \{u_i : i \notin R_h\}$ dominates G. Therefore $|Q_h| + (\gamma(G) - |R_h|) \geq \gamma(G)$, which implies that $|Q_h| \geq |R_h|$ and so summing over $h \in V(H)$ gives us

$$\sum_{h \in V(H)} |Q_h| \geq \sum_{h \in V(H)} |R_h|. \tag{5.4}$$

There are two subcases to consider, namely, (1.1) if $|p_i| + |C_i| > \gamma(H)$ for each $i \in [1, \gamma(G)]$, or (1.2) if $|p_{i_0}| + |C_{i_0}| = \gamma(H)$ for some $i_0 \in [1, \gamma(G)]$. By using a similar argument to Case 1 in Theorem 5.2, inequality (5.4) can be used to show that

$$\gamma_R(G\Box H) \geq \gamma(G)\gamma(H) + \frac{1}{2}\gamma(H)$$

in Subcase 1.1. We will outline the argument for Subcase 1.2, which uses an approach similar to Case 2 in Theorem 5.2. Let $p_{i_0} \cup C_{i_0} = \{v_1, \dots, v_{\gamma(H)}\}$ be a minimum dominating set of H and choose a partition $\{\pi'_1, \dots, \pi'_{\gamma(H)}\}$ of $V(H)$ such that $v_j \in \pi'_j$ and $\pi'_j \subseteq N_H[v_j]$ for each $j \in [1, \gamma(H)]$ and define the sets D'_j, p'_j, C'_j, Q'_g and R'_g by projecting to G, similarly to D_i, p_i, C_i, Q_h and R_h above. As before, $p'_j \cup C'_j$ dominates G and so $|p'_j| + |C'_j| \geq \gamma(G)$ for each j. We also have that

$$\sum_{g \in V(G)} |Q'_g| = |V_2| \quad \text{and} \quad \sum_{g \in V(G)} |R'_g| = \sum_{j=1}^{\gamma(H)} |C'_j|.$$

Moreover, it can be shown that $Q'_g \cup \{v_j : j \notin R'_g\}$ dominates H, and if $Q''_g = Q'_g \cap \{v_j : j \notin R'_g\}$, then

$$|Q'_g| + (\gamma(H) - |R'_g|) - |Q''_g| \geq \gamma(H) \;\Rightarrow\; |R'_g| \leq |Q'_g| - |Q''_g|.$$

In [56], the authors further prove that $\displaystyle\sum_{g \in V(G)} |Q''_g| \geq \frac{1}{2}\gamma(H)$ by showing that there exist at least $\frac{1}{2}\gamma(H)$ vertices of the form $(g, v_j) \in V_2$. Altogether, these inequalities give us

$$\gamma_R(G \Box H) \geq \gamma(G)\gamma(H) + \frac{1}{2}\gamma(H)$$

as desired. Since either subcase holds, the result follows.

Case 2: G and/or H contains isolated vertices.
Let X_G and X_H denote the set of isolated vertices in G and H respectively and let $G' = G - X_G$ and $H' = H - X_H$. Suppose that $E(G)$ is non-empty, then $|X_G| < |V(G)|$ and $|X_H| \leq |V(H)|$. We consider two subcases.

Subcase 2.2: $|X_H| < |V(H)|$.
Then by Case 1,

$$\begin{aligned}
\gamma_R(G \Box H) &= \gamma_R(G' \Box H') + |X_H|\gamma_R(G') + |X_G|\gamma_R(H') + |X_G||X_H| \\
&\geq \gamma(G')\gamma(H') + \frac{1}{2}\min\{\gamma(G'), \gamma(H')\} + |X_H|\gamma_R(G') + |X_G|\gamma_R(H') + |X_G||X_H| \\
&= (\gamma(G') + |X_G|)(\gamma(H') + |X_H|) - |X_G|\gamma(H') - |X_H|\gamma(G') - |X_G||X_H| \\
&\quad + \frac{1}{2}\min\{\gamma(G'), \gamma(H')\} + |X_H|\gamma_R(G') + |X_G|\gamma_R(H') + |X_G||X_H| \\
&= \gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G'), \gamma(H')\} + |X_H|(\gamma_R(G') - \gamma(G')) \\
&\quad + |X_G|(\gamma_R(H') - \gamma(H')).
\end{aligned}$$

Note that since G' is a subgraph of G that does not have isolated vertices, Theorem 5.7 says that $\gamma_R(G') \neq \gamma(G')$ and so $\gamma_R(G') - \gamma(G') \geq 1$. Likewise for H', $\gamma_R(H') - \gamma(H') \geq 1$. Therefore,

$$\begin{aligned}\gamma_R(G\Box H) &\geq \gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G'), \gamma(H')\} + |X_H| + |X_G| \\ &\geq \gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G') + |X_G|, \gamma(H') + |X_H|\} \\ &= \gamma(H)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}.\end{aligned}$$

<u>Subcase 2.1</u>: $|X_H| = |V(H)|$.
Then $E(H) = \emptyset$, so by Theorem 5.7, $\gamma(H) = \gamma_R(H) = |X_H| = |V(H)|$. Therefore,

$$\begin{aligned}\gamma_R(G\Box H) &= |X_H|\gamma_R(G') + |X_G||X_H| \\ &\geq |X_H|(\gamma(G') + 1) + |X_G||X_H| \\ &= \gamma(H)(\gamma(G') + |X_G| + 1) \\ &= \gamma(H)\gamma(G) + \gamma(H) \\ &\geq \gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}\end{aligned}$$

which concludes the proof. □

We now look at how Clark and Suen used the double-projection argument to prove some similar results for claw-free graphs.

Theorem 5.9 ([12]) *If G is a claw-free graph, then for any graph H without isolated vertices,*

$$\gamma(G\Box H) \geq \frac{1}{2}\alpha(G)(\gamma(H) + 1)$$

Proof. Let $A = \{g_1, \ldots, g_{\alpha(G)}\}$ be a maximum independent set in G and $\{h_1, \ldots, h_{\gamma(H)}\}$ a minimum dominating set of H. Let $\Pi = \{\pi_1, \ldots, \pi_{\gamma(H)}\}$ be a partitioning of $V(H)$ with $h_j \in \pi_j$ and $\pi_j \subseteq N[h_j]$ for each $1 \leq j \leq \gamma(H)$. Let D be a minimum dominating set for $G\Box H$ and for each $1 \leq i \leq \alpha(G)$, let $x_i = |D \cap {}^{g_i}H|$. For each i, j, let $d_{i,j} = 1$ if all vertices of the H-cell $\{g_i\} \times \pi_j$ are dominated by D from within the ${}^{g_i}H$-fibre, and let $d_{i,j} = 0$ if at least one vertex from $\{g_i\} \times \pi_j$ is dominated by D from outside ${}^{g_i}H$. We claim that

$$x_i \geq \sum_{j=1}^{\gamma(H)} d_{i,j} \tag{5.5}$$

In fact, if this was not true, we could form a smaller dominating set of H by taking the vertices of the projection $p_H(D \cap {}^{g_i}H)$ together with the vertices h_j such that $d_{i,j} = 0$, contradicting the minimality of $\{h_1, \ldots, h_{\gamma(H)}\}$.

Define a set of indices $I = \{i_1, \ldots, i_r\}$ from $1 \leq i \leq \alpha(G)$ such that $x_i = 0$, i.e., there are no vertices from D in the corresponding ${}^{g_i}H$-fibres. Note that I may be empty, in which case all H-fibres contain vertices from D. For each $i \in [1, \alpha(G)]$, consider the projection $p_G(D \cap {}^{g_i}H)$. Note that this projection is many-to-one, so $x_i \geq |p_G(D \cap {}^{g_i}H)|$. Summing over i, we have that

$$\sum_{i=1}^{\alpha(G)} |p_G(D \cap {}^{g_i}H)| = \alpha(G) - r$$

that is, the number of independent ${}^{g_i}H$-fibres containing vertices from D. Therefore,

$$\sum_{i=1}^{\alpha(G)} x_i \geq \alpha(G) - r \tag{5.6}$$

Let $B_j = (V(G) - A) \times \pi_j$, that is, all the H-cells of $G \Box H$ that are not in the independent ${}^{g_i}H$-fibres. Consider the H-cell $\{g_i\} \times \pi_j$. If $d_{i,j} = 0$, then by definition, at least one vertex of $\{g_i\} \times \pi_j$ is dominated from outside ${}^{g_i}H$. Since $g_i \in A$ and A is independent, this vertex from $\{g_i\} \times \pi_j$ must be dominated from B_j. The number of such vertices is

$$\sum_{i \notin I} \sum_{j=1}^{\gamma(H)} (1 - d_{i,j})$$

On the other hand, if $i \in I$ (so there are no vertices from D in the independent ${}^{g_i}H$-fibre), then every vertex of $\{g_i\} \times \pi_j$ must also be dominated from B_j. The number of such vertices is

$$\sum_{i \in I} |H|$$

Note that since G is claw-free and A is independent, a vertex from B_j can dominate at most two vertices from $A \times V(H)$. Altogether, we therefore have that

$$\begin{aligned}
\sum_{j=1}^{\gamma(H)} |D \cap B_j| &\geq \frac{1}{2}\left(\sum_{i \notin I} \sum_{j=1}^{\gamma(H)} (1 - d_{i,j}) + \sum_{i \in I} |H|\right) \\
\sum_{j=1}^{\gamma(H)} |D \cap B_j| &\geq \frac{1}{2}\left(\sum_{i \notin I} \sum_{j=1}^{\gamma(H)} 1 - \sum_{i \notin I} \sum_{j=1}^{\gamma(H)} d_{i,j} + r|H|\right) \\
&= \frac{1}{2}\left(\sum_{i \notin I} \gamma(H) - \sum_{i \notin I} \sum_{j=1}^{\gamma(H)} d_{i,j} + r|H|\right)
\end{aligned}$$

So

$$\sum_{j=1}^{\gamma(H)} |D \cap B_j| \ \geq \ \frac{1}{2}\left((\alpha(G)-r)\gamma(H) - \sum_{i \notin I}\sum_{j=1}^{\gamma(H)} d_{i,j} + r|H|\right) \tag{5.7}$$

For all $i \in I$, $d_{i,j} = 0$, so

$$\sum_{i=1}^{\alpha(G)} d_{i,j} = \sum_{i \in I} d_{i,j} + \sum_{i \notin I} d_{i,j} = \sum_{i \notin I} d_{i,j}$$

and from (5.5), we have that $x_i \geq \sum_{j=1}^{\gamma(H)} d_{i,j}$. Therefore,

$$\sum_{i \notin I}\sum_{j=1}^{\gamma(H)} d_{i,j} = \sum_{i=1}^{\alpha(G)}\sum_{j=1}^{\gamma(H)} d_{i,j} \ \leq \ \sum_{i=1}^{\alpha(G)} x_i \tag{5.8}$$

Now from inequalities (5.6), (5.7) and (5.8), we have

$$\begin{aligned}
|D| &= \sum_{j=1}^{\gamma(H)} |D \cap B_j| + \sum_{i=1}^{\alpha(G)} |D \cap {}^{g_i}H| \\
&\overset{(5.7)}{\geq} \frac{1}{2}\left((\alpha(G)-r)\gamma(H) - \sum_{i \notin I}\sum_{j=1}^{\gamma(H)} d_{i,j} + r|H|\right) + \sum_{i=1}^{\alpha(G)} x_i \\
&\overset{(5.8)}{\geq} \frac{1}{2}\left((\alpha(G)-r)\gamma(H) - \sum_{i=1}^{\alpha(G)} x_i + r|H|\right) + \sum_{i=1}^{\alpha(G)} x_i \\
&= \frac{1}{2}(\alpha(G)-r)\gamma(H) + \frac{r}{2}|H| + \frac{1}{2}\sum_{i=1}^{\alpha(G)} x_i \\
&\overset{(5.6)}{\geq} \frac{1}{2}(\alpha(G)-r)\gamma(H) + \frac{r}{2}|H| + \frac{1}{2}(\alpha(G)-r) \\
&= \frac{1}{2}\alpha(G)\gamma(H) - \frac{r}{2}\gamma(H) + \frac{r}{2}|H| + \frac{1}{2}\alpha(G) - \frac{r}{2} \\
&= \frac{1}{2}\alpha(G)(\gamma(H)+1) + \frac{r}{2}(|H| - \gamma(H) - 1)
\end{aligned}$$

Note that $|H| \geq \gamma(H) + 1$ since H has no isolated vertices. Hence $\frac{r}{2}(|H| - \gamma(H) - 1) \geq 0$ and

$$\gamma(G \square H) = |D| \geq \frac{1}{2}\alpha(G)(\gamma(H)+1)$$

as desired. □

By Theorem 1.2, the subsequent corollary follows.

Corollary 5.9.1 ([12]) *If G is a claw-free graph, then for any graph H without isolated vertices,*

$$\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)(\gamma(H)+1).$$

By imposing different conditions on the order of H, we can extend Theorem 5.9 in many ways using the same arguments. Here is one such result.

Theorem 5.10 ([12]) *If G is a claw-free graph, then for any graph H without isolated vertices such that $|H| \geq \Delta(H)+\gamma(H)+2$,*

$$\gamma(G\Box H) \geq \frac{1}{2}\alpha(G)(\gamma(H)+2).$$

Proof. Let $A=\{g_1,\ldots,g_{\alpha(G)}\}$, $\{h_1,\ldots,h_{\gamma(H)}\}$, $\Pi=\{\pi_1,\ldots,\pi_{\gamma(H)}\}$, D, x_i, $d_{i,j}$, I and B_j be defined as in the proof of Theorem 5.9. Then again, we have that

$$x_i \geq \sum_{j=1}^{\gamma(H)} d_{i,j} \tag{5.9}$$

Furthermore, define I' as the set of indices i for which $x_i = 1$, i.e., there is exactly one vertex from D in ${}^{g_i}H$, and let $|I'| = s$. Then

$$\sum_{i=1}^{\alpha(G)} x_i \geq 2(\alpha(G)-r-s)+s \tag{5.10}$$

Now consider the vertices of $A\times V(H)$ (i.e., independent H-fibres). We have that the number of vertices in $A\times V(H)$ not dominated from within the H-fibres is at least

$$\sum_{i\notin(I\cup I')}\sum_{j=1}^{\gamma(H)}(1-d_{i,j}) + \sum_{i\in I}|H| + \sum_{i\in I'}(|H|-\Delta(H)-1)$$

The last summation over I' in this inequality considers the H-fibres that each have one vertex from D. Such a vertex dominates at most $\Delta(H)+1$ vertices in the H-fibre. So at least $|H|-\Delta(H)-1$ vertices are dominated from outside the H-fibre. We therefore have

$$\begin{aligned}
\sum_{j=1}^{\gamma(H)}|D\cap B_j| &\geq \frac{1}{2}\left(\sum_{i\notin(I\cup I')}\sum_{j=1}^{\gamma(H)}(1-d_{i,j}) + \sum_{i\in I}|H| + \sum_{i\in I'}(|H|-\Delta(H)-1)\right)\\
&= \frac{1}{2}\left(\sum_{i\notin(I\cup I')}\sum_{j=1}^{\gamma(H)}1 - \sum_{i\notin(I\cup I')}\sum_{j=1}^{\gamma(H)}d_{i,j} + r|H| + s(|H|-\Delta(H)-1)\right)\\
&= \frac{1}{2}\left(\sum_{i\notin(I\cup I')}\gamma(H) - \sum_{i\notin(I\cup I')}\sum_{j=1}^{\gamma(H)}d_{i,j} + r|H| + s(|H|-\Delta(H)-1)\right)
\end{aligned}$$

So

$$\sum_{j=1}^{\gamma(H)} |D \cap B_j| \ \geq \ \frac{1}{2}\left((\alpha(G) - r - s)\gamma(H) - \sum_{i \notin (I \cup I')} \sum_{j=1}^{\gamma(H)} d_{i,j} + r|H| + s(|H| - \Delta(H) - 1)\right) \tag{5.11}$$

For all $i \in I$, $d_{i,j} = 0$, and for all $i \in I'$, $d_{i,j} \geq 0$, so

$$\sum_{i=1}^{\alpha(G)} d_{i,j} = \sum_{i \in I} d_{i,j} + \sum_{i \in I'} d_{i,j} + \sum_{i \notin (I \cup I')} d_{i,j} = \sum_{i \in I'} d_{i,j} + \sum_{i \notin (I \cup I')} d_{i,j}$$

and from (5.9), we have that $x_i \geq \sum_{j=1}^{\gamma(H)} d_{i,j}$. Therefore,

$$\sum_{j=1}^{\gamma(H)} \sum_{i \notin (I \cup I')} d_{i,j} = \sum_{j=1}^{\gamma(H)} \sum_{i=1}^{\alpha(G)} d_{i,j} - \sum_{j=1}^{\gamma(H)} \sum_{i \in I'} d_{i,j} \ \leq \ \sum_{i=1}^{\alpha(G)} x_i \tag{5.12}$$

Now from inequalities (5.10), (5.11) and (5.12), we have

$$\begin{aligned}
|D| \quad &= \quad \sum_{j=1}^{\gamma(H)} |D \cap B_j| + \sum_{i=1}^{\alpha(G)} |D \cap {}^{g_i}H| \\
&\overset{(5.11)}{\geq} \quad \frac{1}{2}\left((\alpha(G) - r - s)\gamma(H) - \sum_{i \notin (I \cup I')} \sum_{j=1}^{\gamma(H)} d_{i,j} + r|H| + s(|H| - \Delta(H) - 1)\right) + \sum_{i=1}^{\alpha(G)} x_i \\
&\overset{(5.12)}{\geq} \quad \frac{1}{2}\left((\alpha(G) - r - s)\gamma(H) - \sum_{i=1}^{\alpha(G)} x_i + r|H| + s(|H| - \Delta(H) - 1)\right) + \sum_{i=1}^{\alpha(G)} x_i \\
&= \quad \frac{1}{2}(\alpha(G) - r - s)\gamma(H) + \frac{r}{2}|H| + \frac{1}{2}\sum_{i=1}^{\alpha(G)} x_i + \frac{s}{2}(|H| - \Delta(H) - 1) \\
&\overset{(5.10)}{\geq} \quad \frac{1}{2}(\alpha(G) - r - s)\gamma(H) + \frac{r}{2}|H| + \frac{1}{2}(2(\alpha(G) - r - s) + s) + \frac{s}{2}(|H| - \Delta(H) - 1) \\
&= \quad \frac{1}{2}\alpha(G)(\gamma(H) + 2) + \frac{r}{2}(|H| - \gamma(H) - 2) + \frac{s}{2}(|H| - \Delta(H) - \gamma(H) - 1)
\end{aligned}$$

By assumption, $|H| \geq \Delta(H) + \gamma(H) + 2$, so the r and s terms are non-negative, hence

$$\gamma(G \Box H) = |D| \geq \frac{1}{2}\alpha(G)(\gamma(H) + 2)$$

as desired. □

In 2017, Clark and Suen's results on claw-free graphs were improved by Krop [50] as follows.

Theorem 5.11 ([50]) *For any claw-free graph G and arbitrary graph H,*

$$\gamma(G\Box H) \geq \frac{2}{3}\gamma(G)\gamma(H).$$

We will not reproduce Krop's proof here, but we will give an outline of his argument. Krop implements a labelling method, similar to Barcalkin and German's [4] technique in Theorem 2.2, which labels each vertex in a minimum dominating set D of $G\Box H$ with numbers from $\{1, \ldots, \gamma(G)\}$ such that each set of vertices labelled with the same integer has cardinality at least $\gamma(H)$. His labelling method consists of three labelling rules for each vertex of D. Rule 1 labels each vertex in D with either one or two integers, then Rules 2 and 3 reduce the number of integers on each label successively (depending on how each G-fibre is dominated by D) while ensuring that each set of vertices labelled with the same integer, when projected to H, dominates H. Ultimately, the vertices of D are relabelled in such a way that each G-fibre that contains N vertices of D labelled with two integers also contains at least N vertices of D labelled with one integer, which provides a way of proving the bound on $|D| = \gamma(G\Box H)$.

Instead, we will look at Brešar and Henning's [14] improvement of Krop's result in more detail, which they proved recently in 2020. We first need the following result on independent domination in claw-free graphs by Allan and Laskar [3].

Theorem 5.12 ([3]) *For any claw-free graph G, $\gamma(G) = i(G)$.*

Proof. ([7]) By Theorems 1.2 and 1.3, we only need to show that $i(G) \leq \gamma(G)$. Let D be a minimum dominating set of G and let I_D be a maximal independent set of $G[D]$. Moreover, let $X = V(G) - N_G[I_D]$ and let I_X be a maximal independent set of $G[X]$. Then $I_D \cup I_X$ is an independent dominating set of G. Note that every vertex $v \in D - I_D$ is adjacent to at most one vertex of I_X, otherwise v will be adjacent to two vertices of I_X and one vertex of I_D which will induce a claw in G. Therefore, $|D - I_D| \geq |I_X|$ and finally

$$i(G) \leq |I_X| + |I_D| \leq (\gamma(G) - |I_D|) + |I_D| = \gamma(G)$$

which concludes the proof. □

We can now examine Brešar and Henning's improvement of Theorem 5.11.

Theorem 5.13 ([14]) *For any claw-free graph G and arbitrary graph H,*

$$\gamma(G\Box H) \geq \frac{3}{4}\gamma(G)\gamma(H).$$

Proof sketch. We may assume that G is a connected claw-free graph and let $\gamma(G) = k$. Theorem 5.12 says that $i(G) = \gamma(G)$, so let $I = \{u_1, \ldots, u_k\}$ be an independent dominating set of G. For each $i \in [1, k]$, let $S_i = \text{epn}(u_i, I) \cup \{u_i\}$ and $S = \bigcup_{i=1}^{k} S_i$. Moreover, let $X \subseteq \{1, \ldots, k\}$ such that $|X| \geq 2$ and define the set

$$T_X = \{v \in V(G) - S : N_G(v) \cap I = \{u_i : i \in X\}\}.$$

Note that any vertex $v \in V(G) - I$ is adjacent to at most two vertices in I, otherwise v along with its neighbours in I will induce a claw in G. Therefore, for each set X with $|X| > 2$, T_X is empty. To ease notation, if $X = \{i, j\}$, then we will write T_X as T_{ij}, for $1 \leq i < j \leq k$. Observe that $\{I, \text{epn}(u_1, I), \ldots, \text{epn}(u_k, I), T_{12}, \ldots, T_{1k}, T_{23}, \ldots, T_{k-1,k}\}$ is a partition of $V(G)$. Now let D be a minimum dominating set of $G \square H$. Our goal is to project the vertices of D onto k copies of H, denoted $H_1, \ldots, H_k$, as follows: each vertex of D is projected onto one vertex in exactly one copy of H. Let $D_i = p_{H_i}(D)$ for each $i \in [1, k]$. If D_i dominates H_i, then $|D_i| \geq \gamma(H)$ and

$$\gamma(G \square H) = |D| = \sum_{i=1}^{k} |D_i| \geq \sum_{i=1}^{k} \gamma(H) = \gamma(G)\gamma(H).$$

So G satisfies Vizing's conjecture. On the other hand, if D_i does not dominate H_i, then we add a set of vertices, say D'_i, to D_i such that $D_i \cup D'_i$ dominates H_i. Let $D' = \bigcup_{i=1}^{k} D'_i$ and suppose that $|D'| \leq \frac{1}{t}|D|$ for some positive integer t. Then we have that

$$\gamma(G)\gamma(H) = \sum_{i=1}^{k} \gamma(H) \leq \sum_{i=1}^{k} (|D_i| + |D'_i|) = |D| + |D'| \leq |D| + \frac{1}{t}|D| = \frac{t+1}{t}\gamma(G \square H).$$

It can be shown that we may choose $t = 3$ by considering several cases of how the vertices of D are projected onto the copies of H and how for each vertex v that is added to D', we can find at least three vertices in D that are uniquely identified with v, but we will not go into more detail here. $\square$

5.1 The cell framework

In 2021, Brešar, Hartnell, Henning, Kuenzel and Rall introduced a new method of approaching Vizing's conjecture in [13]. Their *cell framework* method combines the cell-labelling technique of Barcalkin and German [4] and the double-projection argument of Clark and

Suen [24]. We give a description of this cell framework below and highlight the strength of this idea by showing how it can be used to prove yet another improvement of Theorem 5.1.

Let G and H be two graphs with $\gamma(G) = k$ and let $\{u_1, \ldots, u_k\}$ be a minimum dominating set of G. Choose a partition $\{\pi_1, \ldots, \pi_k\}$ of $V(G)$ such that $u_i \in \pi_i$ and $\pi_i \subseteq N_G[u_i]$ for all $i \in [1, k]$. For each $h \in V(H)$, define the G-cell $C_i^h = \pi_i \times \{h\}$ and let D be a minimum dominating set of $G \Box H$. Finally, let $D_i = D \cap {}^{\pi_i}H$, $i \in [1, k]$, and let $D^h = D \cap G^h$, $h \in V(H)$.
We assign one of four colours to each G-cell in $G \Box H$ according to how each G-cell is dominated by D. Firstly, C_i^h is coloured *blue* if $C_i^h \cap D \neq \emptyset$ and C_i^h is dominated by vertices in D^h in the G^h-fibre. Secondly, C_i^h is coloured *green* if $C_i^h \cap D \neq \emptyset$ and C_i^h is not dominated by D^h, and therefore must be dominated by D_i in the corresponding H-fibres in ${}^{\pi_i}H$. Next, C_i^h is coloured *red* if C_i^h is dominated by D^h and no vertex of C_i^h is dominated by D_i. The remaining G-cells of $G \Box H$ are coloured *white*.
Observe that all the vertices of D are contained in the blue and green G-cells. So we colour each vertex of D the same colour as the G-cell which contains it (blue or green). Note that the vertices of D are the only vertices in $G \Box H$ that are assigned a colour.
We now consider the projection $p_H : {}^{\pi_i}H \to H$ for each i. In the projection, the vertex $h \in p_H(C_i^h)$ is given the colour of C_i^h. So each vertex of H is assigned one of the four colours. Let B_i, G_i and R_i denote the sets of blue, green and red vertices in H respectively for each i.
Finally, we introduce some notation for the cardinalities of the sets of G-cells and blue and green vertices of $G \Box H$. Let b'_h denote the number of blue G-cells in G^h, let b'_i denote the number of blue G-cells in ${}^{\pi_i}H$, and let b' denote the total number of blue G-cells in $G \Box H$. Define g'_h, g'_i and g' similarly for the green G-cells and define r'_h, r'_i and r' similarly for the red G-cells. Next, let b_h denote the number of blue vertices in G^h, let b_i denote the number of blue vertices in ${}^{\pi_i}H$, and let b denote the total number of blue vertices in $G \Box H$. Define g_h, g_i and g for the green vertices of D similarly.
Since the blue and green vertices partition D, we have that $|D| = b + g$. Moreover, since each blue (respectively, green) G-cell contains at least one blue (respectively, green) vertex, we have the following inequalities:

$$b_h \geq b'_h,\ b_i \geq b'_i,\ b \geq b' \ \text{ and } \ g_h \geq g'_h,\ g_i \geq g'_i,\ g \geq g'.$$

Brešar et al. use this framework to reproduce several results we have studied from Barcalkin and German [4], Clark and Suen [24], Suen and Tarr [59], Brešar [11] and Zerbib [63], in particular, Theorems 2.2, 5.1, 5.2, 5.3 and 5.4. We now look at their improvement of Theorem 5.1 which involves *total domination*. First, we need two lemmas.

Lemma 5.14 ([13]) *$r' \leq b - b' + g$.*

Proof. Let $h \in V(H)$ and observe that the $b_h + g_h$ blue and green vertices of $G \Box H$ dominates all of the blue and red G-cells of G^h. Consider the projection $p_G(G^h)$. Then the vertices in G projected from the blue and green vertices of G^h together with the vertices u_i projected from each green and white G-cell C_i^h in G^h forms a dominating set of G, since u_i dominates C_i^h for each i. Therefore, the number of green and white G-cells in G^h is $\gamma(G) - b'_h - g'_h$ and so $\gamma(G) \leq b_h + g_h + (\gamma(G) - b'_h - r'_h)$, implying that $r'_h \leq b_h - b'_h + g_y$. Finally, summing over all $h \in V(H)$, we have that $r' \leq b - b' + g$. □

Definition: For any graph G with no isolated vertices, a set $D \subseteq V(G)$ is a *total dominating set of* G if every vertex of G is adjacent to at least one vertex of D (so a vertex does not dominate itself). The *total domination number* of G is the minimum cardinality of a total dominating set, denoted $\gamma_t(G)$.

Observe that since any total dominating set of a graph G is a dominating set of G, $\gamma(G) \leq \gamma_t(G)$.

Lemma 5.15 ([13]) *$2b' + g' + r' \geq \gamma(G)\gamma_t(H)$.*

Proof. From the description of the framework, note that for each $i \in [1, k]$, $|B_i| = b'_i$, $|G_i| = g'_i$ and $|R_i| = r'_i$ since each vertex $h \in p_H(C_i^h)$ is assigned the colour of C_i^h. Let $h \in B_i \cup R_i$ and let $n_h \in N_H(h)$. Then we have the set

$$N_i = \bigcup_{h \in B_i \cup R_i} \{n_h\}$$

with cardinality $N_i| \leq |B_i| + |R_i| = b'_i + r'_i$.

<u>Claim:</u> $B_i \cup G_i \cup N_i$ is a total dominating set of H.

Firstly, the white vertices of H are totally dominated by $B_i \cup G_i$ and the blue and red vertices are totally dominated by N_i, so we only need to consider the green vertices of H. Let $h \in G_i$ and consider the corresponding green G-cell C_i^h. Then since C_i^h is not dominated by D^h in the G^h-fibre, there exists a vertex $(u, h) \in C_i^h$ that is dominated by $(u, h') \in D_i$, where $h' \in N_H(h)$. Therefore, C_i^h is either blue or green since (u, h') is in D, which implies that

h' receives the colour blue or green in the projection to H and so h is totally dominated by $h' \in B_i \cup G_i$. Thus, all the green vertices of H are totally dominated by $B_i \cup G_i$, proving the claim.

So we therefore have that

$$\gamma_t(H) \leq |B_i \cup G_i \cup N_i| \leq b'_i + g'_i + (b'_i + r'_i) = 2b'_i + g'_i + r'_i$$

and finally, summing over all $i \in [1, k]$ gives us $2b' + g' + r' \geq \gamma(G)\gamma_t(H)$. □

Theorem 5.16 ([13]) *For any two graphs G and H,*

$$\gamma(G\Box H) \geq \max\left\{\frac{1}{2}\gamma(G)\gamma_t(H), \frac{1}{2}\gamma_t(G)\gamma(H)\right\}.$$

Proof. We use the same notation defined in the cell framework description. By Lemma 5.14 (L7.13) and Lemma 5.15 (L7.14), we have that

$$\begin{aligned}
\gamma(G)\gamma_t(H) &\overset{(L7.14)}{\leq} 2b' + g' + r' \\
&\overset{(L7.13)}{\leq} 2b' + g' + b - b' + g \\
&= (b + g) + (b' + g') \\
&\leq (b + g) + (b + g) \\
&= 2(b + g) = 2|D| = 2\gamma(G\Box H).
\end{aligned}$$

Since we can switch the roles of G and H in the argument, the result follows. □

Chapter 6

Proof by minimal counterexample

In proving Vizing's conjecture, one approach is to attempt to disprove the existence of a minimal counterexample. To do this, we assume that Vizing's conjecture is false. Therefore, there exists a graph G such that $\gamma(G \Box H) < \gamma(G)\gamma(H)$, for some graph H. Of all such graphs G, let G be one of smallest order. Then G is a *minimal counterexample* to Vizing's conjecture. A minimal counterexample to the conjecture must satisfy certain properties which we will discuss. If it can be shown that all graphs which satisfy these properties also satisfy Vizing's conjecture, then the conjecture will be true for all graphs G.

Theorem 6.1 ([12]) *Every minimal counterexample to Vizing's conjecture must be connected.*

Proof. Let G be a minimal counterexample to Vizing's conjecture and assume G is disconnected. Since G is minimal, each of its components $G_1, \ldots, G_k$ satisfies Vizing's conjecture. So for any graph H and $1 \leq i \leq k$,

$$\gamma(G_i \Box H) \geq \gamma(G_i)\gamma(H) \quad \text{and} \quad \gamma(G) = \sum_{i=1}^{k}(G_i)$$

Therefore,

$$\gamma(G \Box H) = \sum_{i=1}^{k} \gamma(G_i \Box H) \geq \sum_{i=1}^{k} \gamma(G_i)\gamma(H) = \gamma(G)\gamma(H)$$

which is a contradiction. So G must be connected. □

Recall that a graph G is edge-critical (with respect to domination) if for every pair of non-adjacent vertices u and v in G, $\gamma(G \cup uv) = \gamma(G) - 1$. Now consider Theorem 2.1: *Let G' be a spanning subgraph of a graph G. If G satisfies Vizing's conjecture and $\gamma(G') = \gamma(G)$,* *then G' satisfies Vizing's conjecture.* The condition $\gamma(G') = \gamma(G)$ in Theorem 2.1 implies that G' is not edge-critical. Now consider the contrapositive to Theorem 2.1: *Let G' be a spanning subgraph of a graph G. If G' does not satisfy Vizing's conjecture, then G satisfies Vizing's conjecture or $\gamma(G') \neq \gamma(G)$.* Note that the condition $\gamma(G') \neq \gamma(G)$ implies that $\gamma(G') > \gamma(G)$, so there exists a spanning subgraph G'' of G such that $E(G') \subseteq E(G'')$ and is edge-critical. Since every graph of order n is a spanning subgraph of K_n (and K_n satisfies Vizing's conjecture), we may assume that a minimal counterexample to Vizing's conjecture is edge-critical.

Definition: Let G be a graph and $u, v \in V(G)$. If we remove u and v from G and insert a new vertex w in G such that w is adjacent to all vertices in $N(u) \cup N(v) - \{u, v\}$, then we call w the *vertex identification* of u and v. In the case where u and v are adjacent in G, this operation is known as an *edge-contraction* of G, and the resulting graph is denoted G_{uv}.

Theorem 6.2 ([12]) *For any graph G, $\gamma(G_{uv}) \leq \gamma(G)$.*

Proof. Let D be a minimum dominating set of G and let w be the vertex identification of u and v in G_{uv}. If at least one of u and v is in D, then $D' = (D - \{u, v\}) \cup \{w\}$ dominates G_{uv} and $|D'| \leq |D|$. If both u and v are not in D, then D dominates G_{uv} and $\gamma(G_{uv}) \leq \gamma(G)$. $\square$

Also note that for any graph H, $\gamma(G_{uv} \Box H) \leq \gamma(G \Box H)$, since for any minimum dominating set D of $G \Box H$,

$$D' = (D - ({}^{u}H \cup {}^{v}H)) \cup \{(w, h) : (u, h) \in D \text{ or } (v, h) \in D\}$$

dominates $G_{uv} \Box H$.

Definition: [19] A graph G is called *dot-critical* if for any two adjacent vertices u and v of G, $\gamma(G_{uv}) < \gamma(G)$. And G is called *totally dot-critical* if for any two vertices u and v, $\gamma(G_{uv}) < \gamma(G)$.

Theorem 6.3 ([12]) *If G is a minimal counterexample to Vizing's conjecture, then G is totally dot-critical.*

Proof. Let H be a graph such that $\gamma(G \Box H) < \gamma(G)\gamma(H)$. Since G is a minimal counterexample to Vizing's conjecture and $|V(G_{uv})| < |V(G)|$, G_{uv} satisfies Vizing's conjecture. Therefore,

$$\gamma(G_{uv})\gamma(H) \leq \gamma(G_{uv} \Box H) \leq \gamma(G \Box H) < \gamma(G)\gamma(H)$$

Hence, $\gamma(G_{uv}) < \gamma(G)$. $\square$

Corollary 6.3.1 ([12]) *Let G be a minimal counterexample to Vizing's conjecture. Then for any $u \in V(G)$, there exists a minimum dominating set D of G that contains u. Furthermore, for any edge $uv \in E(G)$, there exists a minimum dominating set D such that either $u, v \in D$, or $u \in D$ and one of u, v is the only vertex of G not dominated by $D - \{u\}$.*

Proof. Let $uv \in E(G)$, let w be the vertex identification of u and v in G_{uv} and let D' be a minimum dominating set of G_{uv}.

Case 1: $w \in D'$

Then let $D = (D' - \{w\}) \cup \{u, v\}$. Then u and v dominate the neighbours of w in G and thus D dominates G.

Case 2: $w \notin D'$

Then let $D = D' \cup \{u\}$ so that both u and v are dominated in G. Therefore D dominates G.

Note that $|D| = |D'| + 1$, and from Theorem 6.3, $\gamma(G_{uv}) < \gamma(G) = |D'|$, which implies that

$$\gamma(G) > |D'| = |D| - 1 \Rightarrow \gamma(G) \geq |D|$$

so $\gamma(G) = |D|$ and D is a minimum dominating set of G containing u. Note that in Case 1, $u, v \in D$, and in Case 2, all the vertices of G except perhaps u or v are dominated by $D - \{u\}$. There exists a vertex $x \in D'$ that dominates w, which is adjacent to either u or v in G. Therefore one of u, v is the only vertex of G not dominated by $D - \{u\}$. □

The following result is another consequence of the totally dot-critical property of a minimal counterexample to Vizing's conjecture.

Theorem 6.4 ([57]) *If G is a minimal counterexample to Vizing's conjecture, then G does not contain two adjacent vertices each of degree 2.*

Proof. Let G be a minimal counterexample to Vizing's conjecture and let H be a graph such that $\gamma(G \Box H) < \gamma(G)\gamma(H)$. Suppose to the contrary that G contains two adjacent vertices x and y each of degree 2, and let a ($a \neq y$) and b ($b \neq x$) be the other neighbours of x and y respectively. We consider two cases. Firstly, assume that $a = b$. Then $\gamma(G_{xy}) = \gamma(G)$, which is a contradiction to Theorem 6.3. Now assume that $a \neq b$. Let G' be the graph obtained from G by contracting the edges ax, xy and yb and let z be the vertex identification of a, x, y and b in G'. We will show that G' is a counterexample to Vizing's conjecture, contradicting the minimality of G.

Claim: $\gamma(G') + 1 = \gamma(G)$.

From Theorem 6.3, we have that $\gamma(G') + 1 \le \gamma(G)$. Now let D' be a dominating set of G'. If $z \in D'$, then $D = (D' - \{z\}) \cup \{a, b\}$ dominates G and so $\gamma(G') + 1 \ge \gamma(G)$. On the other hand, if $z \notin D'$, then z has a neighbour $z' \in D'$ to dominate z in G'. This implies that z' is a neighbour of either a or b in G. Without loss of generality, suppose that $z' \in N_G(a)$. Then $D = D' - \{y\}$ dominates G and again we have that $\gamma(G') + 1 \ge \gamma(G)$.

Let D be a dominating set of $G \Box H$. We will now prove that $\gamma(G' \Box H) \le \gamma(G \Box H) - \gamma(H)$. Note that since G is a counterexample to Vizing's conjecture,

$$\gamma(G \Box H) - \gamma(H) < \gamma(G)\gamma(H) - \gamma(H) = (\gamma(G) - 1)\gamma(H) = \gamma(G')\gamma(H) \ ,$$

which will imply that G' is a counterexample to Vizing's conjecture. To this end, we construct a set D' that dominates $G' \Box H$ such that $|D'| \le |D| - \gamma(H)$. For any $v \in V(G)$, let D_v be the set of vertices in the projection $p_H(D \cap {}^vH)$.

Claim: $D_x \cup D_y \cup (D_a \cap D_b)$ dominates H.

Since D dominates $G \Box H$, D_x dominates $V(H) - (D_y \cup D_a)$ and D_y dominates $V(H) - (D_x \cup D_b)$. So $D_x \cup D_y$ dominates $V(H) - (D_a \cap D_b)$, proving the claim.

Therefore, $|D_x \cup D_y \cup (D_a \cap D_b)| \ge \gamma(H)$. Furthermore, we have that

$$\begin{aligned} |D_a \cup D_b \cup (D_x \cap D_y)| &\le |D_a| + |D_b| - |D_a \cap D_b| + |D_x| + |D_y| - |D_x \cup D_y| \\ &\le |D_a| + |D_b| + |D_x| + |D_y| - |D_x \cup D_y \cup (D_a \cap D_b)| \\ &\le |D_a| + |D_b| + |D_x| + |D_y| - \gamma(H) \end{aligned}$$

Let $D' = (D - ({}^aH \cup {}^xH \cup {}^yH \cup {}^bH)) \cup (\{z\} \times (D_a \cup D_b \cup (D_x \cap D_y))$. Then

$$|D'| \le |D| - (|D_a| + |D_b| + |D_x| + D_y|) + |D_a \cup D_b \cup (D_x \cap D_y)| \le |D| - \gamma(H).$$

It remains to show that D' dominates $G' \Box H$. Note that D' dominates the vertices which are dominated by D in $(V(G) - \{a, x, y, b\}) \times V(H)$. Therefore, since $\{z\} \times (D_a \cup D_b) \subseteq D'$, we have that D' dominates $V(G' \Box H) - {}^zH$. Now let $(z, h) \in {}^zH$. Firstly, if (a, h) or (b, h) is dominated by $D \cap ((V(G) - \{a, x, y, b\}) \times V(H)$, then (z, h) is dominated by $D' - {}^zH$ in the corresponding G^h-fibre. Next, if (a, h) is dominated by $D \cap {}^aH$ or (b, h) is dominated by $D \cap {}^bH$, then (z, h) is dominated by $D' \cap {}^zH$, since $\{z\} \times (D_a \cup D_b) \subseteq D'$. Lastly, if (a, u) is dominated by $D \cap {}^xH$ and (b, h) is dominated by $D \cap {}^yH$, then $h \in D_x \cap D_y$ and so $(z, h) \in D'$. Therefore D' dominates $G' \Box H$, as desired. □

Recall that in Chapter 2, we discussed two results by Barcalkin and German [4], Theorems 2.2 and 2.3, which conclude that every graph G with $\gamma(G) \le 2$ satisfies Vizing's conjecture.

More recently, Sun [60] proved in 2004 that every graph G with $\gamma(G) = 3$ also satisfies Vizing's conjecture.

Theorem 6.5 ([60]) *Let G and H be two graphs. If $\gamma(G) = 3$, then $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$.*

Proof sketch. Let $V(G) = \{1, \ldots, m\}$ and $V(H) = \{1, \ldots, n\}$. Since every graph contains a spanning subgraph that is edge-critical (which can be obtained by a finite number of edge-deletions), we may assume that G is edge-critical without loss of generality. So we can find a dominating set $\{1, p, q\}$ of G such that $N[p] \cup N[q] = V(G) - \{1\}$, since adding either edge $1p$ or $1q$ to G will result in a graph dominated by $\{p, q\}$ and hence have domination number 2. Let D be a dominating set of $G \Box H$. We will show that $|D| \geq 3\gamma(H)$.
Let $A_1 = D \cap {}^1H$, let A_2 be the neighbours of A_1 in the ${}^1H$-fibre and let A_3 be the vertices in ${}^1H$ not dominated by A_1. Furthermore, let A_4 be a maximal independent set of the subgraph of $G \Box H$ induced by A_3. Then each vertex $(1, j)$ in A_4 is dominated by a vertex $(i_j, j) \in D \cap G^j$. Let A_5 be the set of all such vertices (i_j, j) and let $D_1 = A_1 \cup A_5$. Then the vertices of the projection $p_{^1H}(D_1)$ dominates ${}^1H$ and so $|D_1| \geq \gamma(H)$.
Let $(p, k) \in {}^pH \cap N(A_5)$. Since $\gamma(G) = 3$, there exists a vertex $(t_k, k) \in G^k$ that is adjacent to (p, k) but not adjacent to (q, k) nor $(i_k, k) \in A_5$, otherwise $\{(q, k), (i_k, k)\}$ will dominate G^k, a contradiction. Let C_1 be the set of all such vertices (t_k, k) and let C_2 be the set of all vertices $(p, k) \notin {}^pH \cap N(A_5)$. Then let $F = C_1 \cup C_2$ and note that no vertex in F is dominated by D_1, F is contained in $N[{}^pH]$ and $F \cap G^j \neq \emptyset$ for each $j \in [1, n]$. Let $L_1 = {}^pH \cap D$ and let Y be the set of vertices $(i, j) \in C_1 - D$ which can only be dominated by $D \cap {}^iH$ and not by any vertex in L_1. Therefore, for each vertex $(i, j) \in Y$, there exists a vertex $(i, j') \in D \cap {}^iH$ that dominates (i, j). Let L_2 be the set of all such vertices $(i, j') \in D$.
There are many possible cases to consider based on whether or not p and q are adjacent in G, so we do not go into more detail here. In each case, it is possible to find two disjoint sets $D_2 \subseteq D$ that dominates ${}^pH$ and $D_3 \subseteq D$ which dominates ${}^qH$ by considering a maximal independent set of each of these H-fibres, then handling each case in a manner similar to ${}^1H$, where F and $L_1 \cup L_2$ are used analogously to A_5 and $A_1 \cup A_2$ respectively. Therefore $|D| = |D_1| + |D_2| + |D_3| \geq 3\gamma(H)$. □

We can now put together a list of properties that a minimal counterexample G to Vizing's conjecture must satisfy:

- $\gamma(G) \geq 4$
- G is not of Type $\mathcal{X}$

- $\gamma_F(G) < \gamma(G)$
- G is edge-critical
- for any $u, v \in V(G)$, $\gamma(G_{uv}) < \gamma(G)$
- every vertex of G is contained in a minimal dominating set of G
- G does not contain two adjacent vertices each of degree 2

Chapter 7

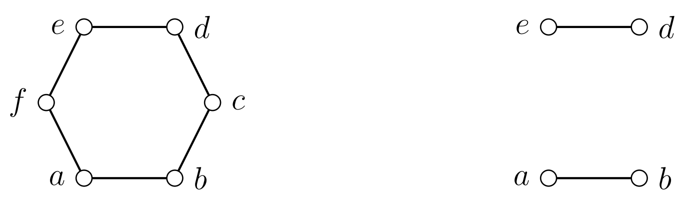
Alternative methods

So far we have looked at methods of tackling Vizing's conjecture which considers a graph G which satisfies some property and then proving that G satisfies the conjecture, usually by means of partitioning the vertex set of G in different ways. We now look at another method of proving Vizing's conjecture which starts with a class of graphs $\mathcal{C}$ which we know satisfies the conjecture and then build new graphs from graphs in $\mathcal{C}$ which also satisfy the conjecture, by applying some graph operations. The last (and most difficult) step is to show that every graph can be constructed this way, thus proving Vizing's conjecture (which, of course, has not been done yet). One such method uses *attachable sets*, which is explored by Hartnell and Rall in [38].

7.1 Attachable sets

Definition: Let G be a graph and $S \subseteq V(G)$ such that for every graph H and for every subset D of $V(G \Box H)$ that dominates $(V(G) - S) \times V(H)$, we have that $|D| \geq \gamma(G)\gamma(H)$. Then S is an *attachable set* of G.
Note that by this definition, such a graph G satisfies Vizing's conjecture, since any dominating set of $G \Box H$ (in particular, a minimum dominating set) dominates $(V(G) - S) \times V(H)$ as well.

Example: Consider $G = C_6 : a, b, c, d, e, f, a$ with $S = \{c, f\}$. Let H be any graph and let $D \subseteq V(G \Box H)$ such that D dominates $(V(G) - S) \times V(H)$.



G $\qquad$ $G[V(G) - S]$

Consider the H-fibres of $G[V(G) - S]\Box H$. Project the vertices of $D \cap {}^bH$ onto ${}^aH$ using the projection $p_{{}^aH} : D \cap {}^bH \to {}^aH$. Since b is the only neighbour of a in $G[V(G) - S]$, ${}^aH$ is dominated by $D \cap {}^aH$ together with the vertices of $p_{{}^aH}(D \cap {}^bH)$. Similarly, ${}^eH$ is dominated by $D \cap {}^eH$ together with the vertices of $p_{{}^eH}(D \cap {}^dH)$. So D dominates two copies of H and $|D| \geq 2\gamma(H) = \gamma(G)\gamma(H)$. Therefore, S is an attachable set of C_6.

The following results illustrate the usefulness of attachable sets by using them to construct graphs which satisfy Vizing's conjecture.

Theorem 7.1 ([38]) *Let G_1 and G_2 be two graphs with attachable sets S_1 and S_2 respectively. Let G be the graph obtained by taking the disjoint union of G_1 and G_2 and adding any subset of the edges that join a vertex in S_1 with a vertex in S_2. Then $S = S_1 \cup S_2$ is an attachable set of G.*

Proof. Since $|E(G)| \geq |E(G_1)| + |E(G_2)|$, we have that $\gamma(G) \leq \gamma(G_1) + \gamma(G_2)$. Let H be any graph and D any subset of $V(G\Box H)$ which dominates $(V(G) - S) \times V(H)$. Then for $i = 1, 2$, $D_i = D \cap V(G_i\Box H)$ dominates $(V(G_i) - S_i) \times V(H)$. Furthermore, since S_i is attachable in G_i, we have that

$$|D| = |D_1| + |D_2| \geq \gamma(G_1)\gamma(H) + \gamma(G_2)\gamma(H) = (\gamma(G_1) + \gamma(G_2))\gamma(H) \geq \gamma(G)\gamma(H)$$

so S is an attachable set of G and therefore G satisfies Vizing's conjecture. $\square$

Below we have two more useful properties of attachable sets.

Lemma 7.2 (see [40]) *Let S be an attachable set in a graph G and let $F \subseteq \{xy : x, y \in S\}$. Then S is an attachable set in $G \cup F$.*

Proof. Let $D \subseteq V(G\Box H)$ such that D dominates $(V(G) - S) \times V(H)$ and $|D| \geq \gamma(G)\gamma(H)$. Then D dominates $(V(G \cup F) - S) \times V(H)$ as well and since G is a spanning subgraph of $G \cup F$, we have that $|D| \geq \gamma(G)\gamma(H) \geq \gamma(G \cup F)\gamma(H)$. $\square$

Lemma 7.3 (see [40]) *Let S be an attachable set in a graph G and let $F \subseteq E(G)$ such that $\gamma(G - F) = \gamma(G)$. Then S is an attachable set in $G - F$.*

Proof. Let $D \subseteq V(G \Box H)$ such that D dominates $(V(G-F)-S) \times V(H)$. Then D dominates $(V(G)-S) \times H$ and since S is an attachable set in G, $|D| \geq \gamma(G)\gamma(H) = \gamma(G-F)\gamma(H)$. □

The following result demonstrates how to produce an attachable set in a (ρ, γ)-graph.

Theorem 7.4 ([38]) *Let G be a (ρ, γ)-graph and A a maximum 2-packing of G. Then $N(A)$ is an attachable set in G.*

Proof. Let H be any graph and let $D \subseteq V(G \Box H)$ that dominates $(V(G) - N(A)) \times V(H)$. Let $A = \{x_1, \ldots, x_{\gamma(G)}\}$, then the sets $N[x_i]$ are pairwise disjoint and partition $V(G)$, where $i \in [1, \gamma(G)]$. We will show that for each i, $N(x_i)$ is an attachable set in $G[N[x_i]]$. For each i, let $D_i = D \cap {}^{N[x_i]}H$. Note that $\gamma(G[N[x_i]]) = 1$ and D_i dominates ${}^{x_i}H = (V(G[N[x_i]]) - N(x_i)) \times V(H)$, so $|D_i| \geq \gamma(H) = \gamma(G[N[x_i]])\gamma(H)$. Therefore $N(x_i)$ is an attachable set in $G[N[x_i]]$ for all $i \in [1, \gamma(G)]$. Taking the union over i, Theorem 7.1 implies that $N(A)$ is an attachable set in G. □

We now consider another way of constructing graphs with attachable sets by defining the operation of *vertex cloning* introduced by Hartnell and Rall in [38].

Definition: Let v be a vertex of a graph G. Define the graph G_1 with $V(G_1) = V(G) \cup \{v'\}$ and $E(G_1) = E(G) \cup \{vv'\} \cup F$, where $F \subseteq \{wv' : w \in N(v)\}$. We say that G_1 is obtained from G by a *cloning of type 1* and that v' is a *type 1 clone of* v in G_1. If $F \neq \emptyset$, let $G_2 = G_1 - \{vv'\}$. Then we say that G_2 is obtained from G by a *cloning of type 2* and that v is a *type 2 clone of* v in G_2.

The following result demonstrates how vertex cloning can be used to show that graphs which satisfy Vizing's conjecture are induced subgraphs of many larger graphs which also satisfy the conjecture and also have attachable sets.

Lemma 7.5 ([38]) *Let G be a graph that satisfies Vizing's conjecture and let $v \in V(G)$. If v belongs to at least one minimum dominating set of G and v' is a type 1 clone of v in G_1, then the set $\{v'\}$ is an attachable set in G_1. If v' is a type 2 clone of v in G_2 and there exists $v'w \in E(G_2)$, where w belongs to some minimum dominating set of G, then $\{v'\}$ is an attachable set in G_2. Furthermore, if S is an attachable set in G, then $S \cup \{v'\}$ is an attachable set in the new graph.*

Proof. Let $D_1 \subseteq V(G_1 \Box H)$ that dominates $(V(G_1) - \{v'\}) \times V(H) = V(G) \times V(H)$. Since v dominates v' in G_1, $\gamma(G_1) = \gamma(G)$, and since G satisfies Vizing's conjecture,

$|D_1| \geq \gamma(G)\gamma(H) = \gamma(G_1)\gamma(H)$, so $\{v'\}$ is an attachable set in G_1. On the other hand, let $D_2 \subseteq V(G_2 \Box H)$ such that D_2 dominates $(V(G_2) - \{v'\}) \times V(H) = V(G) \times V(H)$. Since w dominates v' in G_2, $\gamma(G_2) = \gamma(G)$, and since G satisfies Vizing's conjecture, $|D_2| \geq \gamma(G)\gamma(H) = \gamma(G_2)\gamma(H)$, so $\{v'\}$ is an attachable set in G_2. Finally, if S is an attachable set in G, then $S \cup \{v'\}$ is an attachable set in the new graph (G_1 or G_2) by Theorem 7.1. □

This lemma gives us a way of constructing complete bipartite graphs with attachable sets, therefore satisfying Vizing's conjecture [12].

Corollary 7.5.1 ([38]) *For $2 \leq r \leq s$ and $s \geq 3$, $K_{r,s}$ has an attachable set of vertices.*

Proof. We start with $K_{2,2} \cong C_4$ which satisfies Vizing's conjecture. Make $s-2$ clones of type 2 of a vertex in one partition, adding in all possible edges to form $K_{2,s}$, and then make $r-2$ clones of type 2 of a vertex in the other partition, adding in all possible edges to form $K_{r,s}$. Note that the set of $s+r-4$ vertices that were added to $K_{2,2}$ forms an attachable set of $K_{r,s}$. □

We state one more way of building larger graphs which satisfy Vizing's conjecture.

Theorem 7.6 ([40]) *Let G_1 and G_2 be two graphs with $u \in V(G_1)$ and $v \in V(G_2)$ such that:*

(i) G_2 satisfies Vizing's conjecture,

(ii) $\gamma(G_1 - \{u\}) \geq \gamma(G_1)$, and

(iii) u belongs to some minimum dominating set of G_1 and is attachable in G_1.

Let G be a graph with $V(G) = V(G_1) \cup V(G_2) \cup \{x\}$ and $E(G) = E(G_1) \cup E(G_2) \cup \{ux, vx\}$. Then $\{u, x\}$ is an attachable set in G.

Proof. Let A be a minimum dominating set of G_1 with $u \in A$ and let B be a minimum dominating set for G_2. Since $A \cup B$ dominates G, $\gamma(G) \leq \gamma(G_1) + \gamma(G_2)$. Suppose that $\gamma(G) < \gamma(G_1) + \gamma(G_2)$ and let D be a minimum dominating set of G. Then x must be in D, otherwise $D \cap V(G_i)$ dominates G_i for $i = 1, 2$, and G as well, a contradiction. Define $D_i = D \cap V(G_i)$, then $D = D_1 \cup D_2 \cup \{x\}$. Note that since $|D| < \gamma(G_1) + \gamma(G_2)$, either $|D_1| < \gamma(G_1)$ or $|D_2| < \gamma(G_2)$. If $u \in D_1$, then D_1 dominates G_1, so $|D_1| \geq \gamma(G_1)$ and $|D_2| < \gamma(G_2)$. Otherwise, if $u \notin D_1$, then D_1 dominates $G - \{u\}$ and so by (ii), $|D_1| \geq \gamma(G_1 - \{u\}) \geq \gamma(G_1)$ which implies that $|D_2| < \gamma(G_2)$. But since $|D| = |D_1| + |D_2| + 1 <$

$\gamma(G_1) + \gamma(G_2)$, we have that $|D_2| \leq \gamma(G_2) - 2$. However, $D_2 \cup \{v\}$ dominates G_2 and yet $|D_2 \cup \{v\}| \leq \gamma(G_2) - 1$, which is a contradiction. Therefore $\gamma(G) = \gamma(G_1) + \gamma(G_2)$. Let H be any graph and let $X \subseteq V(G \Box H)$ such that X dominates $(V(G) - \{u, x\}) \times V(H)$. Note that $X_1 = X \cap V(G_1 \Box H)$ dominates $(G_1 - \{u\}) \Box H$ and by (iii), $\{u\}$ is attachable in G_1, so $|X_1| \geq \gamma(G_1)\gamma(H)$. Now let $X_2 = (X \cap V(G_2 \Box H)) \cup \{(v, h) : (x, h) \in X\}$. Then X_2 dominates $G_2 \Box H$ and since G_2 satisfies Vizing's conjecture by (i), $|X_2| \geq \gamma(G_2)\gamma(H)$. Hence,

$$|X| \geq |X_1| + |X_2| \geq \gamma(G_1)\gamma(H) + \gamma(G_2)\gamma(H) = \gamma(G)\gamma(H)$$

and so $\{u, x\}$ is an attachable set in G. $\Box$

We end this section with two open questions posed in [40].

Question: If G is a graph which satisfies Vizing's conjecture and $u \in V(G)$ such that $\gamma(G - \{u\}) \geq \gamma(G)$, then is $\{u\}$ an attachable set of G?

Question: Which classes of common graphs (trees, cycles, etc.) can be constructed using attachable sets?

7.2 Pairs of graphs attaining equality

We now turn our attention to another alternative method of solving Vizing's conjecture. This approach involves proving the conjecture holds for *pairs* of graphs G and H. In particular, we will discuss pairs of graphs which prove the sharpness of the bound in the conjecture, that is

$$\gamma(G \Box H) = \gamma(G)\gamma(H).$$

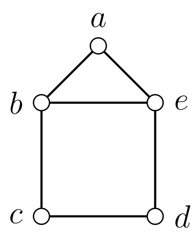
Many classes of such pairs of graphs have already been determined and can be found in [43], several of which use the *corona* of a graph.

Definition: Let G' be a graph. The *corona* of G', $cor(G')$, is the graph obtained by adding an end-vertex to each vertex of G'. If $G = cor(G')$, then we call G' the *interior* of G and the vertices of G' are the *interior vertices* of G.

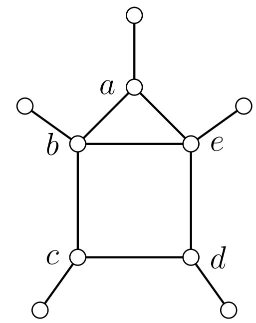
Note that if $G = cor(G')$ for some graph G' and G is a graph of order n, then by Theorem 1.1, $\gamma(G) = \dfrac{n}{2}$ (proving sharpness of Corollary 1.1.1). Furthermore, each interior vertex of

G is adjacent to exactly one end-vertex.

Example: Consider the following graph G' and its corona G.



G'



$G = cor(G')$

Note that $n(G) = 10 = 2n(G')$ and $\gamma(G) = 5 = \frac{n(G)}{2}$. Also, each interior vertex of G is adjacent to exactly one end-vertex.

Payan and Xuong [55] and Fink, Jacobson, Kinch and Roberts [34] proved independently that the only graphs which attain the bound of Corollary 1.1.1 are C_4 and the corona of some connected graph. Before we look at this result, we first introduce *edge covers* and *matchings* in a graph.

Definition: An *edge cover* of a graph G is a set of edges C such that each vertex of G is incident with at least one edge of C. An edge cover of G is *minimal* if it is not properly contained in another edge cover of G. The *edge covering number* of G is the cardinality of a minimum edge cover of G and is denoted by $\beta'(G)$.

Observe that if C is a minimum edge cover of a graph G, then since every vertex of G is incident with at least one edge in C, $\gamma(G) \leq \beta'(G)$.

Definition: A *matching* in a graph G is an independent set of edges in G (i.e., no two edges are incident with a common vertex). A matching is *maximal* if it is not properly contained in another matching of G. The *matching number* (or *edge-independence number*) of G is the cardinality of a maximum matching of G and is denoted by $\alpha'(G)$.

The following famous theorem by Gallai [37] relates the edge covering number and matching number of any graph G.

Theorem 7.7 ([37]) *Let G be a graph of order n with no isolated vertices. Then*

$$\alpha'(G) + \beta'(G) = n.$$

Proof. Let C_1 be a maximum matching of G, so $|C_1| = \alpha'(G)$. Then since C_1 is an independent set, C_1 covers $2\alpha'(G)$ vertices of G. The remaining $n - 2\alpha'(G)$ vertices can be covered by $n - 2\alpha'(G)$ edges of G that are not in C_1. Therefore

$$\beta'(G) \leq \alpha'(G) + (n - 2\alpha'(G)) = n - \alpha'(G)\ ,$$

which implies that $\alpha'(G) + \beta'(G) \leq n$.
Now let C_2 be an edge cover of G such that $|C_2| = \beta'(G)$. Note that each component of $G[C_2]$ is a tree, otherwise, if a component contains a cycle, then removing any edge of that cycle will produce an edge cover of that component with one fewer edge, and consequently an edge cover of G with smaller cardinality than C_2, a contradiction. Choose one edge from each component of $G[C_2]$ and let C_2' be the set of all such edges. Then C_2' is a matching of G, so $|C_2'| \leq \alpha'(G)$. Using the fact that the size of a tree is one less than the order of the tree, if $G[C_2]$ is a forest with k components, then the size of $G[C_2]$ is $n - k$. Hence,

$$\alpha'(G) + \beta'(G) \geq |C_2'| + |C_2| = k + (n - k) = n.$$

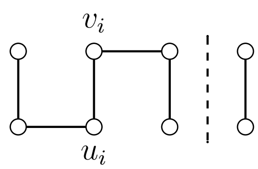
The result follows. □

Observe that a minimum edge cover C of a graph G of order n exists if and only if n is even and each vertex of G is incident with exactly one edge of C. Therefore C is an independent set of edges, i.e., a maximum matching of G, with $\beta'(G) = \frac{n}{2}$.

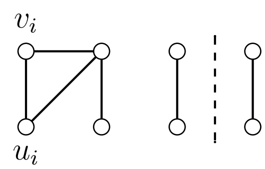
Theorem 7.8 ([55]) *Let G be a graph of order n with no isolated vertices. Then $\gamma(G) = \frac{n}{2}$ if and only if the components of G are C_4 or the corona of some connected graph H.*

Proof. ($\Leftarrow$) Since $\gamma(C_4) = \frac{n(C_4)}{2}$ and $\gamma(cor(H)) = \frac{n(cor(H))}{2}$, summing up the domination numbers of each component of G gives us $\gamma(G) = \frac{n}{2}$.
($\Rightarrow$) Suppose that $\gamma(G) = \frac{n}{2}$. For each $i \in [1, k]$, let s_i be a set of edges that induces a star in G, such that $C = \{s_1, \ldots, s_k\}$ is a minimal set of stars that covers G. Note that $|C| \leq \frac{n}{2}$ and so $\gamma(G) \leq \frac{n}{2}$. Therefore, $\gamma(G) = \frac{n}{2}$ if and only if the domination number of each component of G is $\frac{n_i}{2}$, where n_i is the order of the component. So without loss of generality, we may assume that G is connected. Since $\gamma(G) = \frac{n}{2}$, $|C| = \frac{n}{2}$, so by Theorem 7.7 and our previous

observation, each s_i is an edge in G and C is a maximum matching of G with $k = \frac{n}{2}$ edges. For each $i \in [1, k]$, let $s_i = u_i v_i$. Firstly, we will show that if $k \geq 3$, then u_i or v_i has degree 1 for each i. Suppose to the contrary that there exists an i for which both u_i and v_i have degree at least 2. Then G must contain one of the following spanning subgraphs (where to the right-hand side of the dashed line indicates copies of K_2).



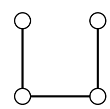
(i)



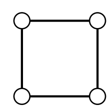
(ii)

However, in both cases we can find a minimum dominating set of cardinality $k - 1 < \frac{n}{2}$, a contradiction. Therefore, G is the corona of some connected graph. On the other hand, if $k \leq 2$, then G is isomorphic to one of the following connected graphs of order at most 4 having domination number $\frac{n}{2}$.

(i)



(ii)



(iii)

The first two graphs are coronas of connected graphs and the third graph is C_4. $\square$

Using this result, Fink et al. presented the first class of graphs which attains the bound of Vizing's conjecture in the following theorem.

Theorem 7.9 ([34]) *Let G and H be two connected graphs of order at least 4 such that G and H are the coronas of two graphs. Then $\gamma(G \square H) = \gamma(G)\gamma(H)$.*

Proof. Since G and H are the coronas of two graphs, they have even order, so let $n(G) = 2n_1$ and $n(H) = 2n_2$. Then by Theorem 7.8, $\gamma(G) = n_1$ and $\gamma(H) = n_2$. We must therefore show that $\gamma(G \square H) = n_1 n_2$. Note that G and H have n_1 and n_2 end-vertices respectively, no two end-vertices are adjacent and no two end-vertices have a common neighbour (by the structure of the corona of a graph) in both graphs. Therefore, there are $n_1 n_2$ vertices of degree 2 in $G \square H$, no two are adjacent nor do any two have a common neighbour. So each vertex of $G \square H$ can dominate at most one vertex of degree 2, therefore any dominating set of $G \square H$ must have at least $n_1 n_2$ vertices, i.e., $\gamma(G \square H) \geq n_1 n_2$. To show that $\gamma(G \square H) \leq n_1 n_2$, we will construct a dominating set of $G \square H$ with cardinality $n_1 n_2$. Let T_1 and T_2 be spanning trees of G and H respectively. We will show that $\gamma(T_1 \Box T_2) \leq n_1 n_2$, which implies that $\gamma(G \Box H) \leq n_1 n_2$ since $T_1 \Box T_2$ is a spanning subgraph of $G \Box H$. For each $i = 1, 2$, partition the set of end-vertices of T_i into two sets E_{i1} and E_{i2} such that two end-vertices are in the same set if the distance between them is even. For $j = 1, 2$, let $I_{ij} = N(E_{ij})$. Note that the vertices in each I_{ij} are interior vertices of $T_1 \Box T_2$. Let $D \subseteq V(T_1 \Box T_2)$ such that $D = (E_{11} \times I_{21}) \cup (E_{12} \times I_{22}) \cup (I_{11} \times E_{22}) \cup (I_{12} \times E_{21})$. Since each interior vertex of T_1 and T_2 is adjacent to exactly one end-vertex, $|E_{ij}| = |I_{ij}|$ and $|E_{i1} \cup |E_{i2}| = n_i$ for $i, j = 1, 2$. So $|D| = n_1 n_2$. To show that D dominates $T_1 \Box T_2$, we consider the following cases. Let $(u, v) \in V(T_1 \Box T_2)$.

Case 1: u and v are end-vertices in T_1 and T_2 respectively.
If $u \in E_{11}$ and $v \in E_{21}$, then v has a neighbour $w \in I_{21}$. Therefore (u, w) is in D and dominates (u, v). If $u \in E_{11}$ and $v \in E_{22}$, then u has a neighbour $w \in I_{11}$. Therefore (w, v) is in D and dominates (u, v). Similar arguments hold for $u \in E_{12}$ and $v \in E_{21}$ or $v \in E_{22}$.

Case 2: u and v are interior vertices in T_1 and T_2 respectively.
If $u \in I_{11}$ and $v \in I_{21}$, then u has a neighbour $w \in E_{11}$. Therefore (w, v) is in D and dominates (u, v). If $u \in I_{11}$ and $v \in I_{22}$, then v has a neighbour $w \in E_{22}$. Therefore (u, w) is in D and dominates (u, v). Similar arguments hold for $u \in I_{12}$ and $v \in I_{21}$ or $v \in I_{22}$.

Case 3: one of u and v is an end-vertex and the other is an interior vertex of T_1 and T_2 respectively.
Without loss of generality, assume that u is an end-vertex in T_1 and v is an interior vertex in T_2. If $u \in E_{11}$ and $v \in I_{21}$, then $(u, v) \in D$. If $u \in E_{11}$ and $v \in I_{22}$, then since each interior vertex of T_2 is adjacent to exactly one end-vertex and $n_2 \geq 2$ (since $n(H) \geq 4$), v has a neighbour $w \in I_{21}$ since the distance between v and w is odd. Therefore (u, w) is in D and dominates (u, v). Similar arguments hold for $u \in E_{12}$ and $v \in I_{21}$ or $v \in I_{22}$.

Hence D dominates $T_1 \Box T_2$ and the result follows. □

Finally, we can use Theorem 7.8 to extend Theorem 7.9 to the following result.

Theorem 7.10 ([34]) *Let G and H be two graphs of order at least 4 with domination number half their order. Then $\gamma(G \Box H) = \gamma(G)\gamma(H)$.*

Proof. By Theorem 7.8, we only need to consider the cases when G and H are C_4 or the corona of some graph. Since Theorem 7.9 proves the result for when both G and H are coronas of two graphs, we need to check the two remaining cases.
First, we show that $\gamma(C_4 \Box C_4) = 4$. Since $P_4 \Box P_4$ is a spanning subgraph of $C_4 \Box C_4$, $\gamma(C_4 \Box C_4) \leq \gamma(P_4 \Box P_4) = 4$. Now note that since C_4 is 2-regular, $C_4 \Box C_4$ is 4-regular with 16 vertices. Each vertex in $C_4 \Box C_4$ dominates 5 vertices, so 3 vertices can dominate at most 15 vertices of $C_4 \Box C_4$. Therefore we need at least 4 vertices to dominate $C_4 \Box C_4$, i.e., $\gamma(C_4 \Box C_4) \geq 4$.
Finally, we show that $\gamma(C_4 \Box H) = \gamma(C_4)\gamma(H)$, where H is the corona of some graph. Let $V(C_4) = \{a, b, c, d\}$ and $H = cor(H')$. Then $D = \{(a, x) : x \in H'\} \cup \{(c, y) : y \in H - H'\}$ is a dominating set of $C_4 \Box H$ with $|D| = n(H)$, since $\{a, c\}$ dominates C_4, $\{(a, x) : x \in H'\}$ dominates the vertices of the H-fibres obtained from the interior of H and $\{(c, y) : y \in H - H'\}$ dominates the vertices of the H-fibres obtained from the end-vertices of H. In fact, by Theorem 1.1, D is a minimum dominating set of $C_4 \Box H$. This, together with the assumption that $\gamma(H) = \frac{n(H)}{2}$, implies that

$$\gamma(C_4 \Box H) = |D| = n(H) = 2 \cdot \frac{n(H)}{2} = \gamma(C_4)\gamma(H)$$

as desired. □

Jacobson and Kinch use the corona to characterise a family of trees which attain equality in Vizing's conjecture in [49]. Firstly, note that if G is the corona of some graph, then the end-vertices of G form a unique maximum 2-packing of G. Therefore $\gamma(G) = \frac{n(G)}{2} = \rho(G)$ and, together with Theorem 3.2, proves that G satisfies Vizing's conjecture.

Theorem 7.11 ([49]) *Let G be any graph and H the corona of some graph. Then $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$.*

We now use this fact to prove Jacobson and Kinch's result.

Theorem 7.12 ([49]) *Let H be any graph such that $\gamma(P_4 \Box H) = 2\gamma(H)$ and let T be a tree such that $T = cor(T')$ for some tree T' of order $n \geq 2$. Then $\gamma(T \Box H) = \gamma(T)\gamma(H)$.*

Proof. Let x_1 be an end-vertex of T and u_1 the unique neighbour of x_1. Also, let u_2 be any neighbour of u_1 and x_2 the end-vertex adjacent to u_2. Then the induced subgraph $T_1 = T[\{x_1, u_1, u_2, x_2\}] \cong P_4$ and so by the hypothesis, there exists a dominating set D_1 of $T_1 \Box H$ such that $|D_1| = 2\gamma(H)$. For $i = 1, 2$, define the sets $D_{1i} = D_1 \cap {}^{N[x_i]}H$. Since D_1 dominates ${}^{x_i}H$ and $N[x_i] = \{x_i, u_i\}$, we have that $|D_{1i}| \geq \gamma(H)$. Furthermore, since

$$2\gamma(H) = |D_1| = |D_{11}| + |D_{12}| \geq 2\gamma(H) \ ,$$

we have that for $i = 1, 2$, $|D_{1i}| = \gamma(H)$.

Now we partition $V(T)$ into two sets V_1 and V_2. Place all the end-vertices of T which are an even distance from x_1, along with their neighbours, in V_1 and let $V_2 = V(T) - V_1$. Note that $x_2 \in V_2$ since $d(x_1, x_2) = 3$, and all the end-vertices which are an even distance from x_2, along with their neighbours, are in V_2. Let $D \subseteq V(T \Box H)$ such that $(t, h) \in D$ if for $i = 1, 2$, if $t \in V_i$, then either t is an end-vertex of T and $(x_i, h) \in D_{1i}$, or t is an interior vertex of T and $(u_i, h) \in D_{1i}$. Note that for any end-vertex x of T, if $x \in V_i$, then since D_{1i} dominates ${}^{x_i}H$, $|D \cap {}^{N[x]}H| = |D_{1i}| = \gamma(H)$. So since $T = cor(T')$ and has n end-vertices,

$$|D| = n\gamma(H) = \gamma(T)\gamma(H).$$

It remains to show that D dominates $T \Box H$. Let $(v, g) \in V(T \Box H) - D$ and, without loss of generality, assume that $v \in V_2$. Keeping in mind that D_{12} dominates ${}^{x_2}H$, we consider the following two cases.

Case 1: v is an end-vertex of T, so $(x_2, g) \notin D_{12}$.
Then either

(i) there exists a neighbour $h \in V(H)$ of g such that $(x_2, h) \in D_{12}$, implying that (v, h) is in D and dominates (v, g); or

(ii) $(u_2, g) \in D_{12}$ and dominates (x_2, g). If w is the unique neighbour of v in T, then $w \in V_2$ and $(w, g) \in D$ which dominates (v, g).

Case 2: v is an interior vertex of T, so $(u_2, g) \notin D_{12}$.
Since $N_{T_1}[u_2] = \{u_1, u_2, v_2\}$, either

(i) there exists a neighbour $h \in V(H)$ of g such that $(u_2, h) \in D_{12}$, implying that (v, h) is in D and dominated (v, g); or

(ii) $(x_2, g) \in D_{12}$ such that, if w is the end-vertex adjacent to v, (w, g) is in D and dominates (v, g); or

(iii) $(u_1, g) \in D_{11}$ such that, if y is any interior vertex adjacent to v (so $y \in V_1$), (y, g) is in D and dominates (v, g).

Therefore D dominates $T \Box H$ and so $\gamma(T \Box H) \leq |D| = \gamma(T)\gamma(H)$. By Theorem 7.11, $\gamma(T \Box H) \geq \gamma(T)\gamma(H)$, thus proving the theorem. □

For completeness, we generalise this result in the following corollary.

Corollary 7.12.1 ([49]) *Let H be any graph such that $\gamma(P_4 \Box H) = 2\gamma(H)$ and let G be a graph of order at least 4 such that G is the corona of some graph. Then $\gamma(G \Box H) = \gamma(G)\gamma(H)$.*

Proof. Let T be a spanning subgraph of G. Then T is a corona with $\gamma(T) = \gamma(G) = \frac{n(G)}{2}$. Since $T \Box H$ is a spanning subgraph of $G \Box H$ and applying Theorem 7.12, we therefore have that

$$\gamma(G \Box H) \leq \gamma(T \Box H) = \gamma(T)\gamma(H) = \gamma(G)\gamma(H) \ .$$

Equality follows from Theorem 7.11. □

Hartnell and Rall pose the following question on pairs of graphs achieving equality in [40].

Question: Is there a structural characterisation of all graphs G such that there exists a graph H that satisfies $\gamma(G \Box H) = \gamma(G)\gamma(H)$?

Chapter 8

Variations of domination

In this section we study other types of domination and determine Vizing-like results for each type.

8.1 Fractional domination

Definition: Let G be a graph and $f : V(G) \to [0, \infty)$ a function that assigns to every vertex of G a non-negative weight. If the sum of the function values of f over any closed neighbourhood of G is at least 1, i.e.,

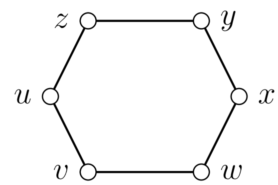
$$\sum_{v \in N[u]} f(v) \geq 1 \text{ for any } u \in V(G), \tag{8.1}$$

then f is called a *fractional-dominating function* of G.
The *weight* of a fractional-dominating function is the sum of all the function values for every vertex in G and the *fractional-domination number* of G, denoted $\gamma_f(G)$, is the minimum weight of a fractional-dominating function of G, i.e.,

$$\gamma_f(G) = \min_f \left\{ \sum_{v \in V(G)} f(v) \right\}$$

where f satisfies (8.1).

Example: Consider $G = C_6 : u, v, w, x, y, z, u$.



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Define the function $f : V(G) \to [0, \infty)$ as follows: let

$$f(u) = \frac{1}{2},\ f(v) = \frac{1}{2},\ f(w) = 0,\ f(x) = \frac{1}{2},\ f(y) = \frac{1}{2} \text{ and } f(z) = 0.$$

Note that the weight of each closed neighbourhood of G is (at least) 1 and $\gamma_f(G) = 2$.

Observe that for a graph G with dominating set D, the characteristic function of D (where each $v \in V(G)$ has weight 1 if $v \in D$ and has weight 0 otherwise) is a fractional-dominating function of G since every closed neighbourhood of G must contain at least one vertex from D, therefore $\gamma_f(G) \leq \gamma(G)$.

The corresponding Vizing-like result for fractional domination was proved in [36] using a linear algebra and strong direct products argument which we will not reproduce here.

Theorem 8.1 ([36]) *For any graphs G and H, $\gamma_f(G \Box H) \geq \gamma_f(G)\gamma_f(H)$.*

The following theorem was first proven by Fisher [35] in 1994 using a strong direct product argument similar to Theorem 8.1. Brešar [9] later formulated a variation of domination known as *graph-domination* in 2001 to present a much simpler proof of Theorem 8.2.

Theorem 8.2 ([35]) *Let G and H be two connected graphs. Then $\gamma(G \Box H) \geq \gamma_f(G)\gamma(H)$.*

Definition: Let G and H be two graphs and let $f : V(G) \to \mathcal{P}(V(H))$ be a function that assigns each vertex in G to a vertex subset of H such that for each $v \in V(G)$

$$\left(\bigcup_{w \in f(v)} N_H[w] \right) \cup \left(\bigcup_{u \in N_G(v)} f(u) \right) = V(H). \tag{8.2}$$

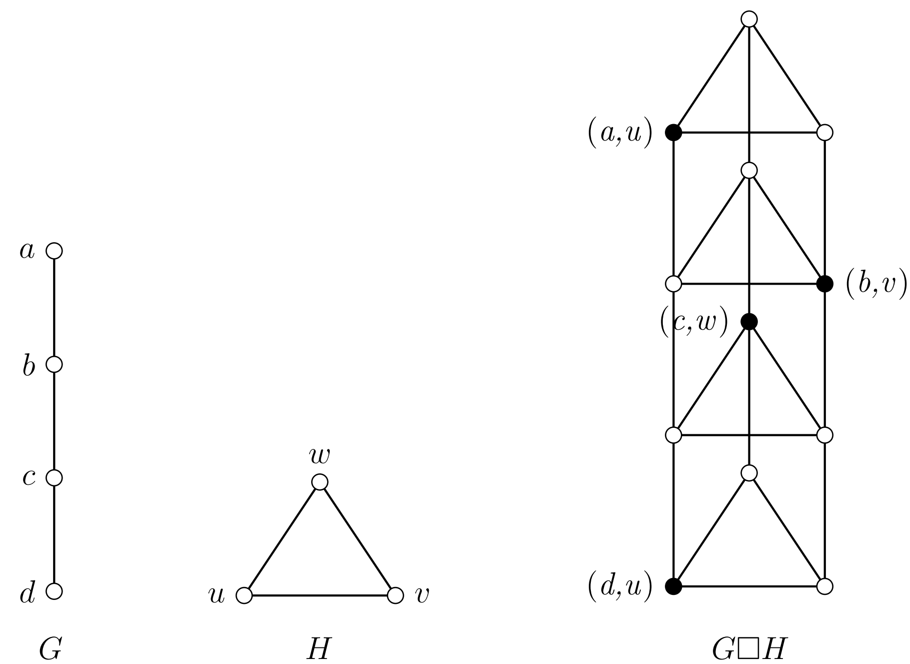
The *graph-domination number of G with respect to H*, denoted $\gamma_H(G)$, is

$$\gamma_H(G) = \min_f \left\{ \sum_{v \in V(G)} |f(v)| \right\},$$

where the minimum is taken over all such functions f satisfying (8.2).

Note that for each $v \in V(G)$, we need to take the union of all H-cells $\{v\} \times f(v)$ to dominate $G \Box H$, so f corresponds to a dominating set of $G \Box H$ (and this is true for any such f which satisfies (8.2)). Any f which attains this minimum $\gamma_H(G)$, corresponds to a minimum dominating set of $G \Box H$ with cardinality $\gamma(G \Box H)$, which implies that $\gamma_H(G) = \gamma(G \Box H)$. Therefore, G satisfies Vizing's conjecture if and only if $\gamma_H(G) \geq \gamma(G)\gamma(H)$ [12].

Example: Consider the Cartesian product of the graphs $G = P_4$ and $H = C_3$.



Define the function $f : V(G) \to \mathcal{P}(V(H))$ as follows:

$$f(a) = \{u\}, \quad f(b) = \{v\}, \quad f(c) = \{w\}, \quad f(d) = \{u\}.$$

This function satisfies condition (8.2) and

$$\sum_{x \in V(G)} |f(x)| = 4 = \gamma_H(G)$$

since $\gamma(G \Box H) = 4$. Furthermore, note that the union of the H-cells $x \times f(x)$ is the set $\{(a, u), (b, v), (c, w), (d, u)\}$ which is a minimum dominating set of $G \Box H$.

Proof of Theorem 8.2 ([9]). Let $f : V(G) \to \mathcal{P}(V(H))$ be a map that satisfies condition (8.2) of graph-domination, and assume that f attains the minimum $\gamma_H(G)$. Define $g : V(G) \to$

$[0, \infty)$ by $g(v) = \dfrac{|f(v)|}{\gamma(H)}$. Since f satisfies condition (8.2), for each $v \in V(G)$, we have that

$$\begin{aligned} & \left(\bigcup_{w \in f(v)} N_H[w]\right) \cup \left(\bigcup_{u \in N_G(v)} f(u)\right) = V(H) \\ \Rightarrow \quad & \sum_{u \in N[v]} |f(u)| \geq \gamma(H) \\ \Rightarrow \quad & \sum_{u \in N[v]} \frac{|f(u)|}{\gamma(H)} \geq 1 \end{aligned}$$

so g satisfies condition (8.1), hence g is a fractional-dominating function of G. Therefore,

$$\begin{aligned} \sum_{v \in V(G)} g(v) &\geq \gamma_f(G) \\ \sum_{v \in V(G)} \frac{|f(v)|}{\gamma(H)} &\geq \gamma_f(G) \\ \sum_{v \in V(G)} |f(v)| &\geq \gamma_f(G)\gamma(H) \\ \gamma_H(G) &\geq \gamma_f(G)\gamma(H) \end{aligned}$$

since f attains the minimum weight $\gamma_H(G)$ by assumption. □

8.2 Graph-domination

In this section, our aim is to show how Brešar's concept of graph-domination can be used to prove that all pairs of graphs with domination number 3 satisfy Vizing's conjecture.

Firstly, observe that given a graph G, if $\gamma(G) \leq \rho(G) + 1$, then G satisfies Vizing's conjecture (this follows immediately from Corollary 3.3.2 and Theorem 3.2). Therefore, we have the following result:

Corollary 8.2.1 ([9]) *If G is a graph with $\gamma(G) = 3$ and $diam(G) \geq 3$, then G satisfies Vizing's conjecture.*

Proof. Since $diam(G) \geq 3$, a maximum 2-packing of G has at least two vertices. Therefore $\gamma(G) = 3 \leq \rho(G) + 1$, so by the above observation, G satisfies Vizing's conjecture. □

This result shows us that we need to prove Vizing's conjecture holds for all pairs of graphs G and H with domination number 3 and diameter 2. Note that since the diameter of each graph is 2, we need more vertices in G and H to ensure that $\gamma(G) = \gamma(H) = 3$.

We now prove a few useful lemmas.

Lemma 8.3 ([9]) *If G is a graph with $\gamma(G) = 3$ and $diam(G) = 2$, then $\delta(G) \geq 3$.*

Proof. Since $diam(G) = 2$, each neighbourhood of a vertex in G is a dominating set of G. In particular, if v is a vertex with $deg(v) = \delta(G)$, then $N(v)$ dominates G and $|N(v)| \geq \delta(G) \geq \gamma(G) = 3$. □

Observe that for any graph G of order n, we have the upper bound $\Delta(G) \leq n - \gamma(G) + 1$. In fact, consider a vertex $v \in V(G)$ such that $deg(v) = \Delta(G)$, then v dominates its $\Delta(G)$ neighbours and $\gamma(G) \leq n - \Delta(G) + 1$. We can improve this bound for graphs with diameter 2.

Lemma 8.4 ([9]) *If G is a graph of order n with $diam(G) = 2$, then $\Delta(G) \leq n - 2\gamma(G) + 2$.*

Proof. Let $v \in V(G)$ such that $deg(v) = \Delta(G)$, let D be a minimum dominating set of G and define the vertex set $A = V(G) - N[v]$. Since v dominates $N[v]$, we need at least $\gamma(G) - 1$ vertices of G to dominate $G[A]$, so $\gamma_G(A) \geq \gamma(G) - 1$. Furthermore, since $diam(G) = 2$, every pair of vertices in A are either adjacent or have at least one common neighbour in $V(G) - \{v\}$. Since $\Delta(G) = n - |A| - 1$, we need to minimise $|A|$ to maximise $\Delta(G)$. By Corollary 1.1.1, we have that $\gamma_G(A) \leq \frac{|A|}{2}$, therefore $\gamma(G) - 1 \leq \frac{|A|}{2}$ implies that $|A| \geq 2(\gamma(G) - 1)$. For fixed n and $\gamma(G)$, if $|A|$ is even, then $|A| \geq 2(\gamma(G) - 1)$. However, if $|A|$ is odd, then $|A| \geq 2(\gamma(G) - 1) - 1 \Rightarrow |A| \geq 2\gamma(G) - 3 = 2(\gamma(G) - 2) + 1$. So $|A|$ attains this minimum bound if it is odd such that $\gamma(G) - 2$ vertices of $D \cap A$ dominates another $\gamma(G) - 2$ vertices of A as pairs of adjacent vertices, and the remaining vertex u of A is dominated by $D \cap N(v)$, where u has a common neighbour in $N(v)$ with every other vertex in A. Therefore,

$$\Delta(G) = n - |A| - 1 \leq n - 2\gamma(G) + 3 - 1 = n - 2\gamma(G) + 2$$

as desired. □

Lemma 8.5 ([9]) *If G is a graph of order n with $\gamma(G) = 3$ and $diam(G) = 2$, then $n \geq 8$.*

Proof. Firstly, by Lemma 8.3, $\delta(G) \geq 3$. Now suppose to the contrary that $n = 7$. Then by Lemma 8.4,

$$\Delta(G) \leq n - 2\gamma(G) + 2 = 7 - 2(3) + 2 = 3.$$

So G is a 3-regular graph of order 7, which contradicts the handshaking lemma.
Now suppose that $n \leq 6$. Again by Lemma 8.4,

$$\Delta(G) \leq n - 2\gamma(G) + 2 \leq 6 - 2(3) + 2 = 2.$$

So $\Delta(G) = 2$ and $\delta(G) = 3$, a contradiction. Therefore $n \geq 8$. □

The following result is an improvement on Theorem 1.6, proven by Brešar in [9] for graphs with domination number at least three.

Theorem 8.6 ([9]) *Let G and H be two graphs such that $\gamma(G)$ and $\gamma(H)$ are at least 3 and $|V(G)| \neq |V(H)|$. Then*

$$\gamma(G \Box H) \geq \min\{|V(G)|, |V(H)|\} + 1.$$

Proof. Without loss of generality, suppose that $|V(H)| < |V(G)|$ and suppose to the contrary that $\gamma(G \Box H) < \min\{|V(G)|, |V(H)|\} + 1$. Then $\gamma(G \Box H) \leq |V(H)|$, and by Theorem 1.6 above, a minimum dominating set D of $G \Box H$ has $|V(H)|$ vertices. Note that every G^h-fibre contains exactly one vertex from D, otherwise there exists a vertex $(g', h') \in V(G \Box H)$ such that $G^{h'} \cap D = \emptyset$ and ${}^{g'}H \cap D = \emptyset$. Furthermore, there exists at least one H-fibre which does not contain a vertex from D since $|D| = |V(H)| < |V(G)|$. Let $v_1, \ldots, v_k$ be all the vertices in G such that ${}^{v_i}H \cap D \neq \emptyset$ and let $\{w_1, \ldots, w_\ell\} = V(G) - \{v_1, \ldots, v_k\}$. Then for some $h \in V(H)$, every vertex (w_i, h) is dominated from within the G^h-fibre by a particular vertex (v_j, h), and each (v_j, h) must dominate all such vertices (w_i, h). This implies that any pair of vertices $v_i, w_j \in V(G)$ dominates G, so $\gamma(G) \leq 2$, a contradiction. □

Theorem 8.7 ([44]) *For any graph H, $\gamma(\overline{H})\gamma(H) \leq |V(H)|$.*

Proof. For any $S \subseteq V(H)$, define $D_e(S)$ as the set of all vertices in $V(H) - S$ that are adjacent to all the vertices in S, and $D_i(S)$ as the set of all vertices in S that are adjacent to every other vertex in S. Let $X = \{x_1, \ldots, x_{\gamma(H)}\}$ be a minimum dominating set of H and partition $V(H)$ into $\gamma(H)$ subsets $\pi_1, \ldots, \pi_{\gamma(H)}$ such that $x_j \in \pi_j$ and $\pi_j \subseteq N[x_j]$ for each $j \in [1, \gamma(H)]$. Choose such a partition P of $V(H)$ for which the sum over all j of $|D_i(\pi_j)|$ is a maximum. Suppose that $|D_e(\pi_j)| \geq 1$, then there exists a vertex $x \in \pi_k$, $k \neq j$, such that x is adjacent to every vertex of π_j. If $x \in D_i(\pi_k)$, then x is adjacent to every vertex in π_j and π_k, so $X - \{x_j, x_k\} \cup \{x\}$ is a dominating set of H with smaller cardinality than $\gamma(H)$, a contradiction. Therefore $x \notin D_i(\pi_k)$. We can now define a new partition P' with $\pi'_l = \pi_l$

for $l \neq j$ and $l \neq k$, $\pi'_j = \pi_j \cup \{x\}$ and $\pi'_k = \pi_k - \{x\}$ with cardinalities $|D_i(\pi'_l)| = |D_i(\pi_l)|$, $|D_i(\pi'_j)| = |D_i(\pi_j)| + 1$ and $|D_i(\pi'_k)| \geq |D_i(\pi_k)|$. This contradicts the maximality of $|D_i(\pi_j)|$ in partition P, so $|D_e(\pi_j)| = 0$ for all $j \in [1, \gamma(H)]$. Note that any set S with $|D_e(S)| = 0$ dominates $\overline{H}$, therefore each π_j in P dominates $\overline{H}$, so $|\pi_j| \geq \gamma(\overline{H})$. Hence,

$$|V(H)| = \sum_{j=1}^{\gamma(H)} |\pi_j| \;\geq\; \sum_{j=1}^{\gamma(H)} \gamma(\overline{H}) = \gamma(G)\gamma(H)$$

as desired. □

We are now able to prove the final result of this section.

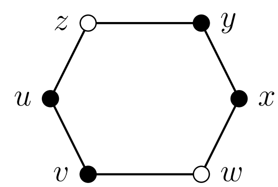
Theorem 8.8 ([9]) *If G and H are two graphs such that $\gamma(G) = \gamma(H) = 3$, then G satisfies Vizing's conjecture, i.e., $\gamma(G \Box H) \geq 9$.*

Proof. Suppose to the contrary that $\gamma(G \Box H) \leq 8$. Since Corollary 8.2.1 implies that we only need to prove Vizing's conjecture holds for graphs with diameter 2, we may assume both G and H have diameter 2 and we can apply Lemma 8.5 to both graphs. Therefore both G and H must have at least 8 vertices. This, together with Theorem 1.6, implies that $\gamma(G \Box H) = 8$ and one of these graphs must have 8 vertices. Lastly, the contrapositive of Theorem 8.6 implies that $|V(G)| = |V(H)| = 8$.
Let D be a minimum dominating set of $G \Box H$, so $|D| = 8$. As in the proof of Theorem 8.6, each G- and H-fibre must contain at least one vertex from D (otherwise one of these graphs will have domination number at most 2). We now implement the graph-domination of G with respect to H, by assigning each vertex of G to exactly one vertex of H. So define the function $f : V(G) \to \mathcal{P}(V(H))$ by $f(g) = \{h\}$, $g \in V(G)$ and $h \in V(H)$, such that $(g, h) \in D$. Note that f is bijective.
Let $a, b \in V(H)$ such that $ab \notin E(H)$. Since f is bijective, there exist vertices $r, s \in V(G)$ such that $f(r) = \{a\}$ and $f(s) = \{b\}$, so $(r, a), (s, b) \in D$. Since a and b are not adjacent in H, (r, b) is not dominated by (r, a) from $D \cap {}^{r}H$, so (r, b) must be dominated from the corresponding G^b-fibre by $D \cap G^b$. But $D \cap G^b = \{(s, b)\}$, which implies that $rs \in E(G)$. Therefore $f^{-1}(\{a\})f^{-1}(\{b\}) \in E(G)$. Thus $\overline{H}$ is a spanning subgraph of G (up to isomorphism), so $\gamma(G) \leq \gamma(\overline{H})$. By Theorem 8.7, $\gamma(\overline{H})\gamma(H) \leq |V(H)|$, therefore $\gamma(G)\gamma(H) \leq \gamma(\overline{H})\gamma(H) \leq 8$, which contradicts $\gamma(G) = \gamma(H) = 3$. □

8.3 Total domination

Recall the definition of total domination; for any graph G with no isolated vertices, a set $D \subseteq V(G)$ is a *total dominating set of* G if every vertex of G is adjacent to at least one vertex of D. The *total domination number* of G is the minimum cardinality of a total dominating set, denoted $\gamma_t(G)$.

Example: Consider $G = C_6 : u, v, w, x, y, z, u$.



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Here, $\{u, v, x, y\}$ is a minimum total dominating set of G and so $\gamma_t(G) = 4$.

We start with a simple yet useful result.

Theorem 8.9 ([22]) *For any graph G with no isolated vertices, $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$.*

Proof. Since every total dominating set of G dominates G, $\gamma(G) \leq \gamma_t(G)$. Now let $D = \{v_1, \ldots, v_{\gamma(G)}\}$ be a minimum dominating set of G. Note that each vertex of $V(G) - D$ is adjacent to some vertex in D and since G has no isolated vertices, $N(v_i)$ is non-empty for each $i \in [1, \gamma(G)]$. So there exists $x_i \in N(v_i)$ for each i, and define $D' = \{x_1, \ldots, x_{\gamma(G)}\}$. Then D is dominated by D' and $D \cup D'$ is a total dominating set of G, hence
$\gamma_t(G) \leq |D \cup D'| = 2|D| = 2\gamma(G)$. □

Henning presents a construction of trees of order at least three which attain the upper bound of Theorem 8.9 in [45].

Theorem 8.10 ([45]) *A tree G of order at least 3 satisfies $\gamma_t(G) = 2\gamma(G)$ if and only of the following conditions hold:*

(i) G has a unique minimum dominating set D,

(ii) every vertex of D is adjacent to a leaf of G, and

(iii) D is a 2-packing of G.

We now look at the total domination version of the Clark-Suen inequality (Theorem 5.1). Henning and Rall proved this inequality in [46] for one class of graphs in the following theorem.

Theorem 8.11 ([46]) *Let G and H be two graphs with no isolated vertices, at least one of which is a (ρ,γ)-graph. Then*

$$\gamma_t(G\Box H) \geq \frac{1}{2}\gamma_t(G)\gamma_t(H)$$

and this bound is sharp.

Proof. Without loss of generality, assume that G is a (ρ,γ)-graph, so $\rho(G) = \gamma(G)$. Let $S = \{v_1, \ldots, v_{\gamma(G)}\}$ be a maximum 2-packing of G, then for each $i \in [1, \gamma(G)]$, the sets $N[v_i]$ are pairwise disjoint. Let $\{\pi_1, \ldots, \pi_{\gamma(G)}\}$ be a partition of $V(G)$ into $\gamma(G)$ sets such that for each i, $v_i \in \pi_i$ and $\pi_i \in N[v_i]$. Let D be a minimum dominating set of $G\Box H$ and for each i, let $D_i = D \cap {}^{\pi_i}H$. Let S_i be a total minimum dominating set of ${}^{v_i}H$ in $G\Box H$ which contains as many vertices of ${}^{v_i}H$ as possible.Then $S_i \subseteq {}^{\pi_i}H$. If S_i contains a vertex v not in ${}^{v_i}H$, then v totally dominates exactly one vertex v' in ${}^{v_i}H$. We can now define a new total dominating set S_i' of ${}^{v_i}H$ by replacing v in S_i with a neighbour of v' in ${}^{v_i}H$, which contains more vertices of ${}^{v_i}H$ than S_i, a contradiction. Therefore $S_i \subseteq {}^{v_i}H$, so S_i is a minimum total dominating set of a copy of H, which implies that $|S_i| = \gamma_t(H)$. Furthermore, since D_i totally dominates ${}^{v_i}H$, $|D_i| \geq |S_i|$. Therefore

$$\gamma_t(G\Box H) = |D| = \sum_{i=1}^{\gamma(G)} |D_i| \geq \sum_{i=1}^{\gamma(G)} |S_i| = \sum_{i=1}^{\gamma(G)} \gamma_t(H) = \gamma(G)\gamma_t(H) \geq \frac{1}{2}\gamma_t(G)\gamma_t(H). \quad (8.3)$$

Sharpness of bound. Let G' be a graph with $V(G') = \{v_1, \ldots, v_n\}$. We construct G as follows: first take the corona of G', and then subdivide each edge in $E(G')$ exactly twice. Then $V(G')$ is both a 2-packing and a minimum dominating set of G, so $\rho(G) = \gamma(G) = n$, and $\gamma_t(G) = 2n$ since the set of end vertices and their unique neighbours in G form a minimum total dominating set of G. Now let $H = P_2 : a, b$. Then the set $\{(v_1, a), (v_1, b), (v_2, a), (v_2, b), \ldots, (v_n, a), (v_n, b)\}$ is a total dominating set of $G\Box H$, hence $\gamma_t(G\Box H) \leq 2n$. Finally, $\gamma_t(G)\gamma_t(H) = (2n)(2) \geq 2\gamma_t(G\Box H)$. So by Theorem 8.11, $\gamma_t(G\Box H) = \frac{1}{2}\gamma_t(G)\gamma_t(H)$. □

Henning and Rall ended this theorem with an open question: is this result true for any two graphs with no isolated vertices? Later, Ho showed the total domination version of the Clark-Suen inequality is indeed true in general in [47], using the same double-projection argument as in Theorem 5.1.

Theorem 8.12 ([47]) *For any graphs G and H without isolated vertices,*

$$\gamma_t(G\Box H) \geq \frac{1}{2}\gamma_t(G)\gamma_t(H).$$

Proof. Let H be a graph with $\gamma_t(H) = k$, $\{h_1, \ldots, h_k\}$ a minimum total dominating set of H and choose a partition $\{\pi_1, \ldots, \pi_k\}$ of $V(H)$ such that for each $i \in [1, k]$, $\pi_i \subseteq N(h_i)$, i.e., h_i totally dominates π_i for each i. Furthermore, for each i, let $G_i = V(G) \times \{\pi_i\}$ and for some $g \in V(G)$ we define the H-cell $\{g\} \times \pi_i$. Let D be a minimum total dominating set of $G\Box H$. Note that if there exists a vertex in an H-cell that is not totally dominated by $D \cap {}^gH$, then that vertex is totally dominated by a neighbour of g in the projection $p_G(D \cap G_i)$. We can now extend $p_G(D \cap G_i)$ to a total dominating set of G. For each i, let n_i be the number of H-cells in G_i dominated by $D \cap G_i$. Then for each $g \in V(G)$ such that the H-cell $\{g\} \times \pi_i$ is totally dominated by $D \cap {}^gH$, add any neighbour of g in G to $p_G(D \cap G_i)$. This will give us a totally dominating set of G, therefore $|p_G(D \cap G_i)| + n_i \geq \gamma_t(G)$ and so

$$|D| + \sum_{i=1}^{k} n_i \;\geq\; \sum_{i=1}^{k}(|p_G(D \cap G_i)| + n_i) \;\geq\; \sum_{i=1}^{k}\gamma_t(G) = \gamma_t(G)\gamma_t(H). \tag{8.4}$$

Now we consider an H-cell $\{g\} \times \pi_j$, where $j \in [1, k]$. Note that every vertex in this H-cell is totally dominated by (g, h_j) since $\pi_j \subseteq N(h_j)$. For each $g \in V(G)$, let m_g be the number of H-cells in ${}^gH$ totally dominated by $D \cap {}^gH$. If there exists an H-cell that is not totally dominated by $D \cap {}^gH$ for some j, then the H-cell is totally dominated by (g, h_j). Define the index set J_g as the set of all j such that the H-cell $\{g\} \times \pi_j$ is not totally dominated by $D \cap {}^gH$. Then $|J_g| = k - m_g$. Similarly, we can extend the projection $p_H(D \cap {}^gH)$ to a total dominating set of H by adding the vertices of $\bigcup_{j \in J_g}\{h_j\}$ to the projection. From this, we have that

$$\gamma_t(H) = k \leq |p_H(D \cap {}^gH)| + k - m_g \Rightarrow m_g \leq |p_H(D \cap {}^gH)|$$

and by summing over $g \in V(G)$,

$$\sum_{g \in V(G)} m_g \leq \sum_{g \in V(G)} |p_H(D \cap {}^gH)| \;\Rightarrow\; \sum_{g \in V(G)} m_g \leq |D|.$$

Since we counted the number of H-cells dominated by $D \cap {}^gH$ in two ways using n_i and m_g, $\sum_{i=1}^{k} n_i = \sum_{g \in V(G)} m_g$. From (8.4), we therefore have

$$2\gamma_t(G\Box H) = |D| + |D| \geq |D| + \sum_{i=1}^{k} n_i \geq \gamma_t(G)\gamma_t(H)$$

as desired. $\square$

Equality is achieved in Theorem 8.12 when at least one of G and H is a non-trivial tree.

Theorem 8.13 ([46]) *Let G be a non-trivial tree and H a graph with no isolated vertices. Then $\gamma_t(G\Box H) = \dfrac{1}{2}\gamma_t(G)\gamma_t(H)$ if and only if $\gamma_t(G) = 2\gamma(G)$ and H consists of disjoint copies of K_2.*

Proof. Suppose that $\gamma_t(G\Box H) = \frac{1}{2}\gamma_t(G)\gamma_t(H)$. Let D_i and S_i be defined as in the proof of Theorem 8.11. Then from (8.3), we have that $|D_i| = |S_i| = \gamma_t(H)$ for each $i \in [1, \gamma(G)]$, therefore $\gamma_t(G) = 2\gamma(G)$. Now by Theorem 8.10, G has a unique minimum dominating set $S = \{v_1, \ldots, v_{\gamma(G)}\}$ such that every vertex of S is adjacent to a leaf of G and S is a 2-packing of G. For some i, let u_i be a leaf adjacent to v_i and let w_i be any other neighbour of v_i in G. Suppose there exists a vertex $y \in D_i - ({}^{v_i}H \cup {}^{u_i}H)$. Then $D_i - \{y\}$ totally dominates ${}^{u_i}H$, implying $\gamma_t(H) < |D_i|$, a contradiction. Therefore $D_i \subseteq ({}^{v_i}H \cup {}^{u_i}H)$ for each i. On the other hand, to totally dominate ${}^{w_i}H$ we must have ${}^{v_i}H \subseteq D_i$. Hence, ${}^{v_i}H = D_i$ and $|V(H)| = |D_i|$. Keeping in mind that H has no isolated vertices, if H has a component of order at least 3, then $\gamma_t(H) < |V(H)| = |D_i|$, which is a contradiction. Therefore, H must consist of disjoint copies of K_2. $\square$

Brešar, Henning and Rall [16] considered the 2-packing number to prove the following Vizing-related result.

Theorem 8.14 ([16]) *Let G and H be two graphs with no isolated vertices. Then*

$$\gamma_t(G\Box H) \geq \max\{\gamma_t(G)\rho(H), \rho(G)\gamma_t(H)\}.$$

Proof. Let S be a maximum 2-packing of G and let D be a minimum total dominating set of $G\Box H$. Take $a \in S$ and define the set $D_a = D \cap {}^{N[a]}H$. Note that since S is a 2-packing, the sets D_a are pairwise disjoint, so $|D| = \sum_{a \in S} |D_a|$. We will show that $|D_a| \geq \gamma_t(H)$ by projecting the vertices of D_a onto H. Define a mapping $p : D_a \to V(H)$ as follows. Firstly, for each $(a, x) \in D$, let $p(a, x) = x$. Now we consider the neighbours of a in G. For some $b \in N(a)$ and $(b, y) \in D$, if $(a, y) \in D$, then let $p(b, y) = z$, where z is some neighbour of y in H. Now suppose that $(a, y) \notin D$. If there exists a vertex $w \in N(y)$ such that $|(N[a] \times \{w\}) \cap D| \geq 1$, then let $p(b, y) = y$. On the other hand, if no such w exists, then let $p(b, y) = u$, where u is a common neighbour of y and $v \in V(H)$ such that $(a, v) \in D$ (such a v exists since D totally dominates $G\Box H$ and $\{a\} \times N(y)$ must be totally dominated by ${}^{a}H$).

Therefore $p(D_a)$ totally dominates a copy of H, so $\gamma_t(H) \le |p(D_a)| \le |D_a|$. Finally,

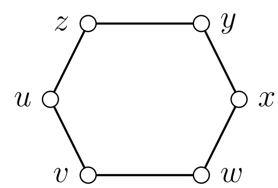
$$\gamma_t(G\Box H) = |D| = \sum_{a\in S} |D_a| \ge \sum_{a\in S} \gamma_t(H) = \rho(G)\gamma_t(H).$$

The interchangeability of G and H in the argument gives us the desired result. □

8.4 Integer domination

Definition: Let G be a graph and k a positive integer. A function $f : V(G) \to \{0, 1, \dots, k\}$ is a *k-dominating function* (kDF) if the sum of all function values over any closed neighbourhood is at least k, i.e., $\sum_{v\in N[u]} f(v) \ge k$ for any $u \in V(G)$.
The *weight* of a kDF is the sum of all the function values for every vertex in G, i.e., $f(V(G))$, and the $\{k\}$-domination number, denoted $\gamma_{\{k\}}(G)$, is the minimum weight of a kDF.

Example: Consider $G = C_6 : u, v, w, x, y, z, u$ and suppose $k = 4$.



Define the function $f : V(G) \to \{0, 1, 2, 3, 4\}$ as follows: let

$$f(u) = 4,\ f(v) = 0,\ f(w) = 0,\ f(x) = 4,\ f(y) = 0 \text{ and } f(z) = 0.$$

Note that the weight of each closed neighbourhood of G is (at least) 4 and $\gamma_f(G) = 8$.

Similarly to fractional domination, the characteristic function of a dominating set D of a graph G (where each $v \in V(G)$ has weight 1 if $v \in D$ and has weight 0 otherwise) is a $\{1\}$-dominating function of G, since every closed neighbourhood of G must contain at least one vertex from D. Therefore $\gamma_{\{1\}}(G) = \gamma(G)$. Domke et al. [29] proved the following useful fact.

Theorem 8.15 ([29]) *For any graph G and positive integer k, $\gamma_{\{k\}}(G) \le k\gamma(G)$.*

Proof. Let D be a minimum dominating set of G and define the function
$f : V(G) \to \{0, 1, \dots, k\}$ as follows: for each $v \in V(G)$, let $f(v) = k$ if $v \in D$ and let $f(v) = 0$ otherwise. Then f is a kDF of G, therefore

$$\gamma_{\{k\}}(G) \le f(V(G)) = k|D| = k\gamma(G)$$

as desired. $\square$

The bound of Theorem 8.15 is in fact sharp. For example, let G be a graph such that $\rho(G) = \gamma(G)$, let $k \geq 1$ and let X be a 2-packing of G with $\gamma(G)$ vertices. If f is a kDF of G, then for every $x \in X$, $f(N[x]) \geq k$. Since $N[x] \cap N[y] = \emptyset$ for every $x, y \in X$, summing over all $x \in X$ gives us $\gamma_{\{k\}}(G) \geq k\rho(G) = k\gamma(G)$. Equality follows from Theorem 8.15.

Domke et al. also related the fractional domination number of a graph to its $\{k\}$-domination number.

Theorem 8.16 ([29]) *For any graph G, $\gamma_f(G) = \min_{k \in \mathbb{N}} \left\{ \frac{\gamma_{\{k\}}(G)}{k} \right\}$.*

Note that the simplest integer domination version of Vizing's result, i.e., for any two graphs G and H, $\gamma_{\{k\}}(G \Box H) \geq \gamma_{\{k\}}(G)\gamma_{\{k\}}(H)$, is not true. For a counterexample, consider $H = K_1$ and $k > 1$. Then $\gamma_{\{k\}}(G \Box K_1) = \gamma_{\{k\}}(G)$ and $\gamma_{\{k\}}(G)\gamma_{\{k\}}(K_1) = k\gamma_{\{k\}}(G)$, so $\gamma_{\{k\}}(G \Box H) < \gamma_{\{k\}}(G)\gamma_{\{k\}}(H)$. This leads us to search for a "normalisation factor" that would give us a Vizing-like result similar to the factor of $c = \frac{1}{2}$ in the Clark-Suen inequality (Theorem 5.1). Hou and Lu state the following conjecture in [48].

Conjecture ([48]): *For any two graphs G and H and positive integer k,*

$$\gamma_{\{k\}}(G \Box H) \geq \frac{1}{k}\gamma_{\{k\}}(G)\gamma_{\{k\}}(H).$$

Note that this conjecture is stronger than Vizing's conjecture. In particular, the truth of this conjecture will prove Vizing's conjecture when $k = 1$.

The following theorem proven by Brešar, Henning and Klavžar in [15] shows that the inequality holds true for $c = \frac{1}{k(k+1)}$.

Theorem 8.17 ([15]) *For any two graphs G and H and positive integer k,*

$$\gamma_{\{k\}}(G \Box H) \geq \frac{1}{k(k+1)}\gamma_{\{k\}}(G)\gamma_{\{k\}}(H).$$

Proof. Let $f : V(G \Box H) \to \{0, 1, \ldots, k\}$ be a minimum kDF of $G \Box H$, therefore $f(V(G \Box H)) = \gamma_{\{k\}}(G \Box H)$. Let $D = \{v \in V(G \Box H) : f(v) \geq 1\}$. Then since f is a kDF, D is a dominating set of $G \Box H$. Furthermore, let $S = \{u_1, \ldots, u_{\gamma(G)}\}$ be a minimum dominating set of G and choose a partition of $V(G)$, $\{\pi_1, \ldots, \pi_{\gamma(G)}\}$, such that $u_i \in \pi_i$ and $\pi_i \subseteq N[u_i]$ for each $i \in [1, \gamma(G)]$. For some $h \in V(H)$, we will denote the G-cell $\pi_i \times \{h\}$ as C_i^h. For each G-cell, define it's *vertical neighbourhood* as $V_i^h = \pi_i \times N_H[h]$. We say that a G-cell is *vertically k-undominated* if $f(V_i^h) \leq k-1$ and *vertically k-dominated* if $f(V_i^h) \geq k$. For each i, let n_i denote the number of G-cells that are vertically k-undominated in ${}^{\pi_i}H$ and define a function $f_i : V(H) \to \{0, 1, \ldots, k\}$ as follows. For each $h \in V(H)$, let $f_i(h) = \min\{k, f(C_i^h)\}$ if C_i^h is vertically k-dominated, otherwise let $f_i(h) = k$. Then $\displaystyle\sum_{v \in N[h]} f_i(v) \geq k$ and so f_i is a kDF of H. Therefore,

$$\gamma_{\{k\}}(H) \leq f_i(V(H)) \leq f({}^{\pi_i}H) + kn_i$$

Summing over i, we get that

$$\gamma(G)\gamma_{\{k\}}(H) \leq f(V(G \Box H)) + k \sum_{i=1}^{\gamma(G)} n_i \tag{8.5}$$

Since f is a kDF of $G \Box H$, at least one vertex of D is in each closed neighbourhood of $G \Box H$. So for any G-cell C_i^h that is vertically k-undominated, $C_i^h \subseteq N[D \cap G^h]$. Therefore each vertex in a G-cell that is vertically k-undominated is dominated by $D \cap G^h$. However, if C_i^h is a vertically k-dominated G-cell, then every vertex in C_i^h is dominated by the vertex (u_i, h). Let m_h denote the number of G-cells that are vertically k-undominated in G^h. Then

$$\gamma(G) \leq |D \cap G^h| + (\gamma(G) - m_h) \;\Rightarrow\; m_h \leq |D \cap G^h|.$$

Since we counted the vertically k-undominated G-cells in two ways using n_i and m_h, we have that

$$\sum_{i=1}^{\gamma(G)} n_i = \sum_{h \in V(H)} m_h \leq \sum_{h \in V(H)} |D \cap G^h| = |D| \leq f(V(G \Box H)).$$

From (8.5),

$$\begin{aligned}
\gamma(G)\gamma_{\{k\}}(H) &\leq f(V(G \Box H)) + k \sum_{h \in V(H)} m_h \\
&\leq f(V(G \Box H)) + k|D| \\
&\leq f(V(G \Box H) + kf(V(G \Box H)) \\
&= (k+1)f(V(G \Box H))
\end{aligned}$$

Finally, by Theorem 8.15,

$$\frac{1}{k}\gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq (k+1)\gamma_{\{k\}}(G \Box H)$$

as desired. □

Note that for $k = 1$, we have

$$\begin{aligned} \gamma_{\{1\}}(G\Box H) &\geq \frac{1}{1(1+1)}\gamma_{\{1\}}(G)\gamma_{\{1\}}(H) \\ \Rightarrow \quad \gamma(G\Box H) &\geq \frac{1}{2}\gamma(G)\gamma(H) \end{aligned}$$

which is indeed the Clark-Suen inequality (Theorem 5.1).

Another generalisation of the Clark-Suen inequality in terms of integer domination is discussed by Brešar, Henning and Klavžar in [15]. For any two graphs G and H and positive integer k, they define a function

$$\psi(G, H) = \min\{(k\gamma(G) - \gamma_{\{k\}}(G))|V(H)|, |V(G)|(k\gamma(H) - \gamma_{\{k\}}(H))\}$$

to prove the following result. Note that $\psi(G, H) = 0$ if $\gamma_{\{k\}}(G) = k\gamma(G)$, which occurs, for example, when $\rho(G) = \gamma(G)$.

Theorem 8.18 ([15]) *For any two graphs G and H and positive integer k,*

$$2k\gamma_{\{k\}}(G\Box H) + k\psi(G, H) \geq \gamma_{\{k\}}(G)\gamma_{\{k\}}(H).$$

Proof. Let f, $S = \{u_1, \ldots, u_{\gamma(G)}\}$, $\{\pi_1, \ldots, \pi_{\gamma(G)}\}$, the G-cells C_i^w and vertical neighbourhoods V_i^w, for some $w \in V(H)$, be defined as in the proof of Theorem 8.17. For each $i \in [1, \gamma(G)]$ and $w \in V(H)$, we define the *horizontal need* $n(C_i^w)$ of a G-cell C_i^w as $k - f(V_i^w)$. For each $j \in [0, k-1]$, let n_i^j denote the number of vertically k-undominated G-cells in ${}^{\pi_i}H$ such that $f(V_i^w) = j < k$. Furthermore, let N be the sum of the horizontal needs of all the vertically k-undominated G-cells. Then,

$$N = \sum_{i=1}^{\gamma(G)} \sum_{j=0}^{k-1} n_i^j (k - j).$$

For each $i \in [1, \gamma(G)]$ and $w \in V(H)$, define the function $g_i : V(H) \to \{0, 1, \ldots, k\}$ as follows: let $g_i(w) = \min\{k, f(C_i^w)\}$ if C_i^w is vertically k-dominated, and let $g_i(w) = f(C_i^w) + n(C_i^w) = f(C_i^w) + k - f(V_i^w)$ if C_i^w is vertically k-undominated. Then $\sum_{x \in N_H[w]} g_i(x) \geq k$ since f is a kDF of $G\Box H$, so g_i is a kDF of H. Therefore, summing over all $w \in V(H)$, we have

$$\gamma_{\{k\}}(H) \leq g_i(V(H)) \leq f({}^{\pi_i}H) + \sum_{j=0}^{k-1} n_i^j (k - j)$$

and then summing over all $i \in [1, \gamma(G)]$, we have

$$\gamma(G)\gamma_{\{k\}}(H) \leq f(V(G\Box H)) + N. \tag{8.6}$$

Now, since f is a kDF of $G\Box H$, if C_i^w is a vertically k-undominated G-cell with $f(V_i^w) = j < k$, then $f(N[(v,w)] \cap G^w) \geq k - j$ for each vertex $(v,w) \in C_i^w$. However, note that u_i dominates π_i in G for each i, so by adding an additional weight of $f(V_i^w) = j$ to (u_i, w) in $G\Box H$ will ensure that $f(N[(v,w)] \cap G^w) \geq k$ for each $(v,w) \in C_i^w$ with $f(V_i^w) = j < k$. So for each $i \in [1, \gamma(G)]$ and $w \in V(H)$, we define the function $h_w(v) : V(H) \to \{0, 1, \ldots, k\}$ as follows: for each $v \in V(G)$, let

$$h_w(v) = \begin{cases} \min\{k, f((v,w)) + j\} & \text{if } v = u_i \text{ and } f(V_i^w) = j < k \\ k & \text{if } v = u_i \text{ and } f(V_i^w) \geq k \\ f((v,w)) & \text{otherwise} \end{cases}$$

Then $\displaystyle\sum_{y \in N_G[v]} h_w(y) \geq k$ and so h_w is a kDF of G. Let m_w^j denote the number of vertically k-undominated G cells in G^w with $f(V_i^w) = j < k$ for each $i \in [1, \gamma(G)]$ and $w \in V(H)$. Then, summing over all $v \in V(G)$, we have

$$\gamma_{\{k\}}(G) \leq h_w(V(G)) \leq f(G^w) + k\left(\gamma(G) - \sum_{j=0}^{k-1} m_w^j\right) + \sum_{j=0}^{k-1} m_w^j j$$

$$\Rightarrow \sum_{j=0}^{k-1} m_w^j (k - j) \leq f(G^w) + (k\gamma(G) - \gamma_{\{k\}}(G))$$

and therefore, summing over all $w \in V(H)$, we obtain

$$N = \sum_{w \in V(H)} \sum_{j=0}^{k-1} m_w^j (k - j) \leq f(V(G\Box H)) + |V(H)|(k\gamma(G) - \gamma_{\{k\}}(G)). \tag{8.7}$$

Putting together inequalities (8.6) and (8.7), we have that

$$\gamma(G)\gamma_{\{k\}}(H) \leq 2f(V(G\Box H)) + |V(H)|(k\gamma(G) - \gamma_{\{k\}}(G)).$$

Finally, since $f(V(G\Box H)) = \gamma_{\{k\}}(G\Box H)$ and $\gamma_{\{k\}}(G) \leq k\gamma(G)$ by Theorem 8.15,

$$\gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq 2k\gamma_{\{k\}}(G\Box H) + k|V(H)|(k\gamma(G) - \gamma_{\{k\}}(G)).$$

We can interchange G and H to obtain the desired result. $\Box$

Again, note that when $k = 1$, Theorem 8.18 simplifies to the Clark-Suen inequality $\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)\gamma(H)$, since $\gamma_{\{1\}}(G) = \gamma(G)$ and so $\psi(G, H) = 0$. The following related question in [15] remains open (and is, in fact, weaker that Vizing's conjecture).

Question: Does there exist a constant k such that for any two graphs G and H, $\gamma_{\{k\}}(G\Box H) \geq \gamma(G)\gamma(H)$?

We can combine the theory of Sections 8.3 and 8.4 to consider *integer total domination.*

Definition: Let k be a positive integer. A function $f : V(G) \to \{0, 1, \ldots, k\}$ is a *total k-dominating function* (TkDF) if the sum of all function values over any open neighbourhood is at least k, i.e., $\sum_{v\in N(u)} f(v) \geq k$ for any $u \in V(G)$.
The *weight* of a TkDF is the sum of all the function values for every vertex in G and the total $\{k\}$-domination number, denoted $\gamma_t^{\{k\}}(G)$, is the minimum weight of a TkDF.

As before, the characteristic function of a total dominating set D of a graph G is a total $\{1\}$-dominating function of G, so $\gamma_t^{\{1\}}(G) = \gamma_t(G)$. Similarly, we have the integer total domination version of Theorem 8.15.

Theorem 8.19 ([51]) *Let G be a graph with no isolated vertices and let k be a positive integer. Then $\gamma_t^{\{k\}}(G) \leq k\gamma_t(G)$.*

Proof. Let D be a minimum total dominating set of G and define the function
$f : V(G) \to \{0, 1, \ldots k\}$ as follows: for each $v \in V(G)$, let $f(v) = k$ if $v \in D$ and let $f(v) = 0$ otherwise. Then f is a TkDF of G and so

$$\gamma_t^{\{k\}}(G) \leq f(V(G)) = k|D| = k\gamma_t(G)$$

as desired. □

The integer total domination version of Theorem 8.17 is proven by Li and Hou in [51].

Theorem 8.20 ([51]) *For any two graphs G and H with no isolated vertices and positive integer k,*

$$\gamma_t^{\{k\}}(G\Box H) \geq \frac{1}{k(k+1)}\gamma_t^{\{k\}}(G)\gamma_t^{\{k\}}(H).$$

Proof. Let $\{x_1, \ldots, x_{\gamma_t(G)}\}$ be a minimum total dominating set of G and choose a partitioning $\{\pi_1, \ldots, \pi_{\gamma_t(G)}\}$ of $V(G)$ such that $x_i \in \pi_i$ and $\pi_i \subseteq N(x_i)$ for each $i \in [1, \gamma_t(G)]$. Let

$f : V(G\Box H) \to \{0,1,\ldots,k\}$ be a TkDF of $G\Box H$ such that $f(V(G\Box H)) = \gamma_t^{\{k\}}(G\Box H)$ and let $D = \{v \in V(G\Box H) : f(v) \geq 1\}$. Then D is a total dominating set of $G\Box H$. For each $i \in [1, \gamma_t(G)]$ and $w \in V(H)$, let $D_i = D \cap {}^{\pi_i}H$, let $C_i^w = \pi_i \times \{w\}$ denote a G-cell of $G\Box H$ and let $V_i^w = \pi_i \times N_H[w]$ denote the vertical neighbourhood of C_i^w. Finally, for each $i \in [1, \gamma_t(G)]$, let $L_i = \{(i,w) : f(V_i^w) < k\}$, i.e., L_i counts the number of vertically k-undominated G-cells in $G\Box H$, and let $W_i = \{w \in V(H) : (i,w) \in L_i\}$. Since H has no isolated vertices, we can choose a neighbour u of w in H for each $w \in W_i$ and let U_i be the set of all such neighbours. Then $|U_i| \leq |W_i| \leq |L_i|$. Let $f_i : V(H) \to \{0,1,\ldots,k\}$ be the function defined by:

$$f_i(w) = \begin{cases} k & \text{if } w \in W_i \\ \min\{k, f(V_i^w)\} & \text{if } w \notin W_i \end{cases}$$

We now show that f_i is a TkDF of H. Firstly, if $w \in W_i$, then w has a neighbour $u \in U_i$ in H and so

$$f_i(N_H(w)) = \sum_{v \in N_H(w)} f_i(v) \geq f_i(u) = k.$$

On the other hand, if $w \notin W_i$, then $(i,w) \notin L_i$ and so $f(V_i^w) \geq k$. If w has a neighbour u in H such that $f_i(u) = k$, then $f_i(N_H(w)) \geq f_i(u) = k$. If no such u exists, then $f_i(v) < k$ for each $v \in N_H(w)$. Therefore, since f is a TkDF of $G\Box H$,

$$f_i(N_H(w)) = \sum_{v \in N_H(w)} f_i(v) = \sum f(C_i^v) = f(V_i^w) \geq k.$$

Hence, f_i is a TkDF of H and so

$$\begin{aligned} \gamma_t^{\{k\}}(H) &\leq f_i(V(H)) \\ &= f_i(U_i) + f_i(V(H) - U_i) \\ &= k|U_i| + \sum_{u \in V(H)-U_i} f_i(u) \\ &\leq k|L_i| + \sum_{u \in V(H)-U_i} f(C_i^u) \\ &\leq k|L_i| + f({}^{\pi_i}H) \end{aligned}$$

Therefore, by Theorem 8.19,

$$\begin{aligned}
\gamma_t^{\{k\}}(G)\gamma_t^{\{k\}}(H) &\leq k\gamma_t(G)\gamma_t^{\{k\}}(H) \\
&= k\sum_{i=1}^{\gamma_t(G)} \gamma_t^{\{k\}}(H) \\
&\leq k\sum_{i=1}^{\gamma_t(G)} (k|L_i| + f(^{\pi_i}H)) \\
&\leq k^2\sum_{i=1}^{\gamma_t(G)} |L_i| + kf(V(G\Box H)).
\end{aligned}$$

Now for each $w \in V(H)$, let $D_w = D \cap G^w$. If $(i, w) \in L_i$, then C_i^w is vertically k-undominated and $f(V_i^w) < k$. Therefore, since f is a TkDF of $G\Box H$, C_i^w must be totally dominated by D_w. We now consider the vertically k-dominated G-cells of $G\Box H$. Note that (x_j, w) totally dominates each vertex of C_i^w in G^w, so let $M_w = \bigcup_j (x_j, w)$, where the union is taken over all j such that C_j^w is a vertically k-dominated G-cell. Then $D_w \cup M_w$ totally dominates G^w. If m_w denotes the number of vertically k-dominated G-cells in G^w, then

$$\gamma_t(G) \leq |D_w| + (\gamma_t(G) - m_w) \;\Rightarrow\; m_w \leq |D_w|.$$

Therefore, since we counted the vertically k-undominated G-cells in two ways using L_i and m_w,

$$\sum_{i=1}^{\gamma_t(G)} |L_i| = \sum_{w\in V(H)} m_w \leq \sum_{w\in V(H)} |D_w| = |D| \leq f(V(G\Box H).$$

Hence,

$$\gamma_t^{\{k\}}(G)\gamma_t^{\{k\}}(H) \leq k^2 f(V(G\Box H)) + kf(V(G\Box H)) = k(k+1)\gamma_t^{\{k\}}(G\Box H)\ ,$$

proving the theorem. □

As before, note that for $k = 1$, we have

$$\begin{aligned}
\gamma_t^{\{1\}}(G\Box H) &\geq \frac{1}{1(1+1)}\gamma_t^{\{1\}}(G)\gamma_t^{\{1\}}(H) \\
\Rightarrow \quad \gamma_t(G\Box H) &\geq \frac{1}{2}\gamma_t(G)\gamma_t(H)
\end{aligned}$$

which is the total domination version of the Clark-Suen inequality (Theorem 8.12).

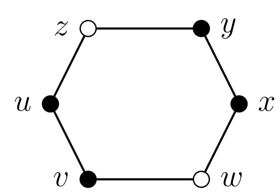
8.5 Paired-domination

Paired-domination was first introduced by Haynes and Slater in [42] as a way of modelling the following scenario. In the police, officers are assigned partners so that each officer can back up the other while on patrol. They model this scenario as follows: each vertex in the dominating set represents the location of an officer capable of guarding each vertex it dominates and their partner's location is adjacent to their location since they must be able to back up each other if necessary. For paired-domination, we consider graph matchings.

Definition: An edge-set M of a graph G is a *perfect matching* of G if every vertex in G is incident with an edge in M.

Definition: Let S be a dominating set of a graph G. Then S is a *paired-dominating set* (PDS) of G if $G[S]$ contains a perfect matching M. The *paired-domination number* of G, denoted $\gamma_{\rm pr}(G)$, is the minimum cardinality of a paired-dominating set of G. We say that two vertices joined by an edge of M are *paired* in S and are called *partners* (with respect to M).

Example: Consider $G = C_6 : u, v, w, x, y, z, u$.



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Here, $\{u, v, x, y\}$ is a minimum PDS of G in which u, v and x, y are paired in M and so $\gamma_{\rm pr}(G) = 4$.

Note that any graph G with no isolated vertices has a PDS: let M be a maximal matching of G, then the vertices of M form a dominating set S of G such that $G[S]$ contains a perfect matching. For example, in the following graph G, $S = \{a, c, d, e\}$ is a dominating set of G such that $G[S]$ has a perfect matching $M = \{ae, cd\}$.

We will use the two following bounds on the paired-domination number in later results.

Theorem 8.21 ([42]) *For any graph G without isolated vertices, $\gamma_{pr}(G) \leq 2\gamma(G)$.*

Proof. Let S be a minimum dominating set of G. By Theorem 1.5, for each $v \in S$, there exists a vertex $w \in V(G) - S$ such that $N(w) \cap S = \{v\}$. Since w is not unique, $\gamma_{\mathrm{pr}}(G) \leq 2|S| = 2\gamma(G)$. □

Theorem 8.22 *If D is a minimum PDS of a graph G with no isolated vertices, then $|D| \geq 2\alpha(G[D])$.*

Proof. Let $S \subseteq D$ such that S is a maximum independent set of $G[D]$. Since each vertex in S has a unique partner in D, $\gamma_{\mathrm{pr}}(G) = |D| \geq 2|S| = 2\alpha(G[D])$. □

To discuss some Vizing-like results for paired-domination, we need to expand the idea of 2-packings in a graph. In 1975, Meir and Moon [52] defined a *k-packing* of a graph, for some integer $k \geq 1$.

Definition: For some positive integer k, a *k-packing* of a graph G is a set X of vertices such that for any two distinct vertices $u, v \in X$, $d(u, v) > k$. The *k-packing number* of G, denoted $\rho_k(G)$, is the maximum cardinality of a k-packing of G.
(Note that in this paper, we write $\rho_2(G)$ simply as $\rho(G)$.)

A similar result to Theorem 8.14 was proven by Brešar, Henning and Rall [16] for paired-domination.

Theorem 8.23 ([16]) *Let G and H be two graphs with no isolated vertices. Then*

$$\gamma_{pr}(G \Box H) \geq \max\{\gamma_{pr}(G)\rho_3(H), \rho_3(G)\gamma_{pr}(H)\}.$$

Proof. Let $S = \{v_1, \ldots, v_{\rho_3(G)}\}$ be a maximum 3-packing of G. For each $i \in [1, \rho_3(G)]$, let $\pi_i = N[v_i]$. Then the sets π_i are pairwise disjoint. Let D be a minimum PDS of $G \Box H$ and for each i, define D_i as the set of all vertices of D in ${}^{\pi_i}H$ together with their partners. Note that the sets D_i are pairwise disjoint as well since S is a 3-packing in G and every vertex in a PDS has a unique partner. Therefore $|D| \geq \sum_{i=1}^{\rho_3(G)} |D_i|$. For each i, let S_i be a minimum set of vertices of $G \Box H$ which contains as many vertices of ${}^{v_i}H$ as possible such that (i) S_i dominates ${}^{v_i}H$ and (ii) $(G \Box H)[S_i]$ contains a perfect matching M_i. Note that for each i, D_i satisfies conditions (i) and (ii) as well, so $|D_i| \geq |S_i|$. Suppose that S_i contains a vertex x that is not in ${}^{\pi_i}H$. Then x has a partner x' (with respect to M_i) which is not in ${}^{v_i}H$ and dominates a vertex $y \in {}^{v_i}H$. We can replace x with y in S_i to form a set which satisfies conditions (i) and (ii) and has more vertices of ${}^{v_i}H$ than S_i, a contradiction. So $S_i \subseteq {}^{\pi_i}H$. We will now show that $S_i \subseteq {}^{v_i}H$. Suppose to the contrary that that S_i contains a vertex $u \notin {}^{v_i}H$, and let u' be the partner of u (with respect to M_i). If $u' \notin {}^{v_i}H$, then let w and w' be the neighbours of u and u' respectively that belong to ${}^{v_i}H$. Then we can replace u and u' with w and w' in S_i to form a set which satisfies (i) and (ii) containing more vertices of ${}^{v_i}H$ than S_i, a contradiction. So $u' \in {}^{v_i}H$. If S_i contains every neighbour of u' in ${}^{v_i}H$, then $S_i - \{u, u'\}$ dominates ${}^{v_i}H$ and has a smaller cardinality than S_i, a contradiction. Therefore there exists at least one neighbour u'' of u' in ${}^{v_i}H$ that is not in S_i. We can form a new set by replacing u in S_i with u'' and then pair u'' with u' that satisfies conditions (i) and (ii) containing more vertices of ${}^{v_i}H$ than S_i, a contradiction. Therefore $S_i \subseteq {}^{v_i}H$, implying that S_i is a PDS of a copy of H, so $|S_i| \geq \gamma_{\mathrm{pr}}(H)$. Finally,

$$\gamma_{\mathrm{pr}}(G \Box H) = |D| \geq \sum_{i=1}^{\rho_3(G)} |D_i| \geq \sum_{i=1}^{\rho_3(G)} |S_i| \geq \sum_{i=1}^{\rho_3(G)} \gamma_{\mathrm{pr}}(H) = \rho_3(G)\gamma_{\mathrm{pr}}(H).$$

We can interchange G and H to obtain the result. □

From this theorem, we obtain the following paired-domination version of the Clark-Suen inequality.

Corollary 8.23.1 ([16]) *Let G and H be two graphs with no isolated vertices, at least one of which has equal paired-domination number and 3-packing number. Then*

$$\gamma_{pr}(G \Box H) \geq \frac{1}{2}\gamma_{pr}(G)\gamma_{pr}(H)$$

and this bound is sharp.

Sharpness of bound. Let G' be a graph with $V(G') = \{v_1, \ldots, v_n\}$. We construct G as follows: first take the corona of G', and then subdivide each edge in $E(G')$ twice. Then $V(G')$ must be contained in every PDS of G to pair with each end vertex added from taking the corona of G'. Note that since $V(G')$ is an independent set of vertices in G, Theorem 8.22 says that $\gamma_{\text{pr}}(G) \geq 2|V(G')| = 2n$. On the other hand, from the way we constructed G, $V(G')$ is the unique minimum dominating set of G. So $|V(G')| = \gamma(G)$ and therefore Theorem 8.21 says that $\gamma_{\text{pr}}(G) \leq 2\gamma(G) = 2n$. Hence $\gamma_{\text{pr}}(G) = 2n$. Now note that for each $i \in [1, n]$, any 3-packing of G contains at most one vertex of $N[v_i]$. Since the n sets $N[v_i]$ partition $V(G)$, we have that $\rho_3(G) \leq n$. However, the set of n end vertices added from the subdivision step in the construction of G is a 3-packing of G, so $\rho_3(G) \geq n$. Therefore $\rho_3(G) = n$ and so G has equal paired-domination number and 3-packing number. Let $H = P_2 : a, b$. Then $\gamma_{\text{pr}}(H) = 2$ and Corollary 8.23.1 says that $\gamma_{\text{pr}}(G \Box H) \geq \frac{1}{2}\gamma_{\text{pr}}(G)\gamma_{\text{pr}}(H)$. On the other hand, note that the vertex set $\{(v_1, a), (v_1, b), (v_2, a), (v_2, b), \ldots, (v_n, a), (v_n, b)\}$ is a PDS of $G \Box H$, so $\gamma_{\text{pr}}(G \Box H) \leq 2n$. Therefore $\gamma_{\text{pr}}(G)\gamma_{\text{pr}}(H) = (2n)(2) \geq 2\gamma_{\text{pr}}(G \Box H)$, proving sharpness of the bound. □

Before we state the next result, we first need a lemma.

Lemma 8.24 ([16]) *Let G be a graph without isolated vertices. Then $\gamma_{pr}(G) \geq 2\rho_3(G)$.*

Proof. Let D be a minimum PDS of G and let S be a maximum 3-packing of G. For each $v \in S$, let u be a vertex in D that dominates v and let D' be the set of all such vertices u. Note that since the vertices of S are pairwise a distance of at least 4 apart, the vertices u are distinct and D' is an independent set of vertices in $G[D]$. So by Lemma 8.22, $\gamma_{\text{pr}}(G) = |D| \geq 2|D'| = 2\rho_3(G)$. □

The consequent Vizing-like result follows directly from Corollary 8.23.1 and Lemma 8.24.

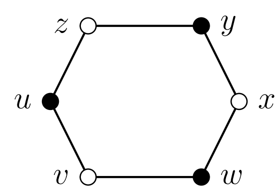
Theorem 8.25 ([16]) *Let G and H be two graphs with no isolated vertices. Then*

$$\gamma_{pr}(G \Box H) \geq 2\rho_3(G)\rho_3(H).$$

8.6 Upper domination

Definition: The *upper domination number* of a graph G is the maximum cardinality of a minimal dominating set of G, and is denoted $\Gamma(G)$.

Example: Consider $G = C_6 : u, v, w, x, y, z, u$.



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Here, $\{u, w, y\}$ is a minimal dominating set of G since no proper subset of $\{u, w, y\}$ dominates G, and any 4-set of $V(G)$ contains a dominating set of G of cardinality 3. Therefore $\{u, w, y\}$ is a minimal dominating set of G with maximum cardinality and so $\Gamma(G) = 3$, whereas $\gamma(G) = 2$.

The upper domination version of Vizing's conjecture was initially conjectured by Nowakowski and Rall in [53], and later proven by Brešar in [10].

Theorem 8.26 ([10]) *For any two graphs G and H, $\Gamma(G \Box H) \geq \Gamma(G)\Gamma(H)$.*

To prove this theorem, we first introduce some notation used by Brešar in [10]. Keeping in mind the two possible conditions for minimality of a dominating set of G from Theorem 1.4, we define the following sets: let D_G be a minimal dominating set of G, then each vertex in D_G has an external private neighbour or is not adjacent to any other vertex of D_G, or both. Let D'_G be the set of vertices in D_G that have at least one external private neighbour and let P_G be the set of vertices in $V(G) - D'_G$ which are private neighbours of some vertex in D'_G. Let N_G be the set of vertices in $V(G) - D'_G$ that are adjacent to some vertex in D'_G but not external private neighbours of any vertex in D'_G. Let $D''_G = D_G - D'_G$, then each vertex in D''_G has no external private neighbours and is not adjacent to any other vertex of D_G. Finally, let $R_G = V(G) - (D_G \cup P_G \cup N_G)$. Therefore $V(G) = D'_G \cup P_G \cup N_G \cup D''_G \cup R_G$. If it is clear from context which graph we mean, we will simply write these sets as D, D', P, N, D'' and R.
Brešar also defines two operations SP and SP' as follows: let $I \subseteq R$. Then $SP(D', I)$ is a subset of vertices of D' such that $SP(D', I) \cup I$ dominates $P \cup N$ and is "minimal" in the following sense: for each $u \in SP(D', I)$, there exists a vertex $v \in P \cup N$ such that u is the only neighbour of v from $SP(D', I) \cup I$. Next, let $J \subseteq (D'' \cup R)$. Then $SP'(D', J)$ is a subset of vertices of D' such that $P \cup N$ is dominated by $SP'(D', J) \cup J$ and is minimal in a similar sense to SP.

Sketch of proof of Theorem 8.26. ([10]) We present Brešar's proof of a slightly stronger result, i.e., for any two non-trivial graphs G and H,

$$\Gamma(G\Box H) \geq \Gamma(G)\Gamma(H) + 1.$$

To prove the result, we will construct a minimal dominating set D of $G\Box H$ with at least $\Gamma(G)\Gamma(H)+1$ vertices. Let D_G and D_H be minimal dominating sets of maximum cardinality of G and H respectively. Then $|D_G| = \Gamma(G)$ and $|D_H| = \Gamma(H)$. Firstly, suppose without loss of generality that $D''_G = \emptyset$. Then $D'_G = D_G$ and so every vertex in D_G has an external private neighbour. Therefore, $D = D'_G \times V(H)$ is a minimal dominating set of $G\Box H$ with $|D| = |D'_G||V(H)| \geq \Gamma(G)\Gamma(H) + 1$ since H is non-trivial.
Now suppose that both D'_G and D'_H are empty. Then $D''_G = D_G$ and $D''_H = D_H$ and so both sets are independent by Theorem 1.4. Let $D = (D''_G \times D''_H) \cup I$, where I is a maximal independent set of the subgraph of $G\Box H$ induced by $(V(G) - D''_G) \times (V(H) - D''_H)$. Then D is a maximal independent set of $G\Box H$, so by Theorem 1.2, D is a minimal dominating set of $G\Box H$ with $|D| = \Gamma(G)\Gamma(H) + |I| \geq \Gamma(G)\Gamma(H) + 1$ since $I \neq \emptyset$.
Finally, suppose without loss of generality that $D'_H \neq \emptyset$, $D''_H \neq \emptyset$ and $D''_G \neq \emptyset$. We will construct D as the union of six pairwise disjoint sets that we define as follows. Let $D_1 = D''_G \times D_H$, then $|D_1| = |D''_G|\Gamma(H)$. Let $D_2 = I$ be a maximum independent set of the subgraph of $G\Box H$ induced by $R_G \times R_H$. Also, for each vertex $x \in R_G$, let $I_x = I \cap {}^xH$. Then from the first operation defined above, consider $SP(D'_H, p_H(I_x)) \cup p_H(I_x) \subseteq D'_H$ and let

$$D_3 = \bigcup_{x \in R_G} \{x\} \times SP(D'_H, p_H(I_x)) \subseteq R_G \times D'_H.$$

Now we interchange G and H to obtain the next two sets. For each $y \in R_H$, let $I_y = I \cap G^y$ and consider the set $SP(D'_G, p_G(I_y)) \subseteq D'_G$. Then define

$$D_4 = \bigcup_{y \in R_H} SP(D'_G, p_G(I_y)) \times \{y\} \subseteq D'_G \times R_H.$$

For each $y \in D'_H$, let $J_y = (D_1 \cup D_3) \cap G^y$, i.e., the vertices of G^y already included in D, and for each J_y we use the second operation defined above to add vertices from G^y to D. So let

$$D_5 = \bigcup_{y \in D'_H} SP'(D'_G, p_H(J_y)) \times \{y\} \subseteq D'_G \times D'_H.$$

Finally, let $D_6 = D'_G \times (V(H) - (D'_H \cup R_H))$. Then since $|P_H| \geq |D'_H|$,

$$|D_6| = |D'_G||V(H) - (D'_H \cup R_H)| \geq |D'_G||V(H) - (P_H \cup R_H)| \geq |D'_G||D_H| \geq |D'_G|\Gamma(H).$$

Let $D = D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6$. Then we have that

$$|D| \geq |D_1| + |D_6| \geq |D''_G|\Gamma(H) + |D'_G|\Gamma(H) = |D_G|\Gamma(H) = \Gamma(G)\Gamma(H).$$

We now show that D has enough vertices. Note that $D''_G \neq \emptyset$ and $D''_H \neq \emptyset$ implies that $R_G \cup N_G$ and $R_H \cup N_H$ are non-empty. If $R_G = \emptyset$, then D_5 is non-empty. If $R_G \neq \emptyset$ and $R_H = \emptyset$, then D_3 is non-empty. Lastly, if $R_G \neq \emptyset$ and $R_H \neq \emptyset$, then D_2 is non-empty. Therefore, $|D| \geq \Gamma(G)\Gamma(H) + 1$.
Next we show that D dominates $G \Box H$. To do so, we partition $V(G \Box H)$ into sets of H-fibres and show that each set is dominated by D. Firstly, ${}^{D''_G}H$ is dominated by D_1. Next, we consider ${}^{R_G}H$. $R_G \times R_H$ is dominated by $I = D_2$ since I is a maximum independent set of $R_G \times R_H$, $R_G \times D_H$ is dominated by D_1 and the remaining vertices in ${}^{R_G}H$ are dominated by $D_3 \cup D_2$ by applying the first operation SP. ${}^{D'_G}H$ is dominated by D_6 since $V(H) - (D'_H \cup R_H) = P_H \cup N_H \cup D''_H$ dominates H. We consider the vertices of ${}^{P_G}H$ and ${}^{N_G}H$ together. If $y \in R_H$, then $(P_G \cup N_G) \times \{y\}$ is dominated by $D_4 \cup D_2$ by applying operation SP. If $y \in D'_H$, then $(P_G \cup N_G) \times \{y\}$ is dominated by $D_5 \cup (D_1 \cup D_3))$ by applying the second operation SP'. Lastly, if $y \notin R_H \cup D'_H$, then $(P_G \cup N_G) \times \{y\}$ is dominated by D_6 since D'_G dominates $P_G \cup N_G$. Therefore D dominates $G \Box H$.
Finally, the fact that D is a minimal dominating set of $G \Box H$ can be checked using Theorem 1.4. There are several possible cases which we will not consider here. In each case it can be shown that each vertex from D_1, D_2, D_3, D_4, D_5 and D_6 either has an external private neighbour or is not adjacent to any other vertex in D. For D_3, D_4 and D_5 in particular, the minimality conditions of SP and SP' are used. □

Brešar closes [10] with the following open question.

Question: For any two non-trivial graphs G and H, is it true that

$$\Gamma(G \Box H) \geq \Gamma(G)\Gamma(H) + \min\{|V(G)| - \Gamma(G), |V(H)| - \Gamma(H)\} \ ?$$

We now look at a variation of upper domination.

Definition: The *upper total domination number* of a graph G is the maximum cardinality of a minimal total dominating set of G, denoted $\Gamma_t(G)$.

Cockayne, Dawes and Hedetniemi [25] made the following useful observation on upper total domination, which is analogous to Theorem 1.4.

Theorem 8.27 ([25]) *Let S be a total dominating set of a graph G with no isolated vertices. Then S is a minimal total dominating set of G if and only if for each $v \in S$, $epn(v, S) \neq \emptyset$ or $pn(v, S) = ipn(v, S) \neq \emptyset$.*

We first state two Lemmas proved by Dorbec, Henning and Rall in [30], before we consider their Vizing-like result for upper total domination which is analogous to Corollary 5.9.1 on claw-free graphs.

Lemma 8.28 ([30]) *If G is a graph with no isolated vertices, then*

$$\Gamma_t(G \Box K_2) \geq \frac{1}{2}\Gamma_t(G)\Gamma_t(K_2)$$

with equality if and only if G consists of disjoint copies of K_2.

Proof. If $V(K_2) = \{u, v\}$, then G^u is a minimal total dominating set of $G \Box K_2$. Therefore, since $\Gamma_t(K_2) = 2$,

$$\Gamma_t(G \Box K_2) \geq |V(G)| \geq \Gamma_t(G) = \frac{1}{2}\Gamma_t(G)\Gamma_t(K_2).$$

If $\Gamma_t(G \Box K_2) = \frac{1}{2}\Gamma_t(G)\Gamma_t(K_2)$, then $|V(G)| = \Gamma_t(G)$ by the above chain of inequalities. Since G has no isolated vertices, this occurs if and only if G consists of disjoint copies of K_2. $\square$

Lemma 8.29 ([30]) *Every upper total dominating set of a graph G contains as a subset a minimal dominating set S such that $|S| \geq \frac{1}{2}\Gamma_t(G)$ and $|epn(v, S)| \geq 1$ for each $v \in S$.*

Proof. For any set $S \subseteq V(G)$, let $d_S(v)$ denote the number of vertices in S that are adjacent to v for some $v \in V(G)$. Let D be an upper total dominating set of G and partition D into the following three sets:

$$\begin{aligned} A &= \{v \in D : |\text{epn}(v, D)| \geq 1\} \\ B &= \{v \in D - A : d_A(v) \geq 1\} \\ C &= D - (A \cup B) \end{aligned}$$

Let $v \in B \cup C$. Then $\text{epn}(v, D) = \emptyset$ and so by Theorem 8.27, $|\text{ipn}(v, D)| \geq 1$. Let $v' \in \text{ipn}(v, D)$, then $v' \in D$ and $N(v') \cap D = \{v\}$ implies that $d_D(v') = 1$. Since v' is an internal private neighbour of v and v has no external private neighbours, $v' \notin B$.
First, suppose that $v \in C$. Then since $d_A(v) = 0$, $v' \notin A$ and so $v' \in C$. Therefore, $\text{epn}(v', D) = \emptyset$ and Theorem 8.27 says that $|\text{ipn}(v', D)| \geq 1$. But v is the only neighbour of v' in D, so $\text{ipn}(v', D) = \{v\}$, that is, $N(v) \cap D = \{v'\}$, and $d_D(v) = 1$. Therefore, if $C \neq \emptyset$, then $G[C] = \frac{|C|}{2}K_2$ and for each $v \in C$, $d_D(v) = 1$. For each $x \in C$, we call its unique neighbour $y_x \in G[C]$ the *partner* of x in C. Partition C into two sets X and Y such that each vertex in X (respectively Y) is adjacent in $G[D]$ only to its partner in Y (respectively X).
Now suppose that $v \in B$. Since $G[C] = \frac{|C|}{2}K_2$ and each vertex of C has only one neighbour in $G[D]$, v is not adjacent to any vertex of C and $v' \in A \cup B$. By our above argument, $v' \notin B$,

so $v' \in A$, and since v has no external private neighbour and $v' \in \operatorname{ipn}(v, D)$, $\operatorname{pn}(v, D) \subseteq A$ for each $v \in B$. So we have that

$$|A| \geq \sum_{v \in B} |\operatorname{ipn}(v, D)| \geq |B|.$$

Let $U = V(G) - (D \cup N(A) \cup N(X))$ be the set of vertices in $V(G) - D$ that are not dominated by $A \cup X$ in G. Therefore, since D is a total dominating set of G, U must be dominated by $B \cup Y$. Let B_Y be a minimum set of vertices in $B \cup Y$ that dominates U. So by Theorem 1.4, $|\operatorname{epn}(v, B_Y) \cap U| \geq 1$ for each $v \in B_Y$. Let $S = A \cup B_Y \cup X$. Note that S dominates D since $B \subseteq N(A)$ and $Y \subseteq N(X)$, and S dominates $V(G) - D$ by the construction of B_Y. Therefore, S dominates G. However, S may not be a minimal dominating set of G, so we will construct a minimal dominating set of G from S as follows. For each $x \in X$, if $\operatorname{epn}(x, S) = \emptyset$, then delete x from S. Continue such successive vertex deletions until each vertex in S has at least one external private neighbour. Note that if $y_x \notin S$ for some x, then $y_x \in \operatorname{epn}(x, S)$ and so x is not deleted from S. Let X' be the resulting set of vertices from X that belong to S after the process is completed. Then $S = A \cup B_Y \cup X'$ and $|\operatorname{epn}(v, S)| \geq 1$ for each $v \in S$. If $x \in X - X'$, then $y_x \in S$ and so S dominates C. As before, S dominates B since $B \subseteq N(A)$. Therefore S dominates D and S dominates $V(G) - D$ by the construction of S. So S is a minimal dominating set of G by Theorem 1.4. It remains to prove that $|S| \geq \frac{1}{2}\Gamma_t(G)$. For each $x \in X$, S contains at least one of x and y_x, so $|S \cap C| \geq |X| = \frac{1}{2}|C|$. Hence,

$$|S| \geq |A| + |S \cap C| \geq |A| + \frac{1}{2}|C| \geq \frac{1}{2}(|A| + |B| + |C|) \geq \frac{1}{2}|D| = \frac{1}{2}\Gamma_t(G)$$

as desired. □

Theorem 8.30 ([30]) *If G and H are two connected graphs of order at least 3 and $\Gamma_t(G) \geq \Gamma_t(H)$, then*

$$\Gamma_t(G \Box H) \geq \frac{1}{2}\Gamma_t(G)(\Gamma_t(H) + 1).$$

Proof. By Lemma 8.29, there exists a minimal dominating set S of G such that $|S| \geq \frac{1}{2}\Gamma_t(G)$ and $|\operatorname{epn}(v, S)| \geq 1$ for every $v \in S$. Let $D = {}^S H$, then D dominates $G \Box H$ since S dominates G. Also, for each $u \in S$, each vertex of ${}^u H$ is totally dominated by a neighbour in ${}^u H$ in D, so D is a total dominating set of $G \Box H$. We now show that D is a minimal total dominating set of $G \Box H$. If $v \in D$, then $v = (u, w)$ for some $u \in S$ and $w \in V(H)$. Let $v' = (u', w)$, where $u' \in \operatorname{epn}(u, S)$ in G. It follows that $v' \in \operatorname{epn}(v, D)$ in $G \Box H$, therefore $|\operatorname{epn}(v, D)| \geq 1$ for all $v \in D$. Thus, by Theorem 8.27, D is a minimal total dominating set of $G\Box H$ and so $\Gamma_t(G\Box H) \geq |D|$. Finally, since H is a connected graph of order at least 3, $|V(H)| \geq \Gamma_t(H) + 1$. Hence,

$$\Gamma_t(G\Box H) \geq |D| = |S||V(H)| \geq \frac{1}{2}\Gamma_t(G)(\Gamma_t(H) + 1)$$

completing the proof. □

The upper total domination of the Clark-Suen inequality follows directly from Lemma 8.28 and Theorem 8.30.

Theorem 8.31 ([30]) *If G and H are any two graphs with no isolated vertices, then*

$$\Gamma_t(G\Box H) \geq \frac{1}{2}\Gamma_t(G)\Gamma_t(H)$$

with equality if and only if both G and H consist of disjoint copies of K_2.

8.7 Rainbow domination

Let G be a graph and k a positive integer. Define a function $f : V(G) \to \mathcal{P}(\{1, \dots, k\})$ which assigns each vertex of G a set of colours from $\{1, \dots, k\}$. If for every $v \in V(G)$ such that $f(v) = \emptyset$ we have that

$$\bigcup_{u \in N(v)} f(u) = \{1, \dots, k\}, \tag{8.8}$$

then we call f a *k-rainbow dominating function* (kRDF) of G. The *weight* of a kRDF is $\sum_{v \in V(G)} |f(v)|$ and the minimum weight of a kDRF of G is the *k-rainbow domination number* of G, denoted $\gamma_{rk}(G)$.

Note that for $k = 1$, a 1RDF of a graph G is a usual dominating set of G since any $v \in V(G)$ with $f(v) = \emptyset$ is adjacent to a vertex coloured 1, so the set of vertices coloured 1 dominates G.

Brešar, Henning and Rall discuss the relationship between rainbow domination and $G\Box K_k$ in [17] for some graph G and integer $k \geq 1$. In particular, there is a one-to-one correspondence between the kRDFs of G and the dominating sets of $G\Box K_k$.

Theorem 8.32 ([17]) *Let G be any graph and k any positive integer. Then*

$$\gamma_{rk}(G) = \gamma(G\Box K_k).$$

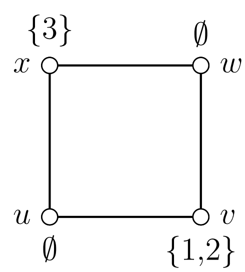
Proof. Let f be a kRDF of G and let $V(K_k) = \{v_1, \ldots, v_k\}$. Then the set

$$D_f = \bigcup_{u \in V(G)} \left(\bigcup_{i \in f(u)} \{(u, v_i)\} \right)$$

dominates $G \Box K_k$ and corresponds to the function f. Conversely, given a dominating set D of $G \Box K_k$, we define the function $f : V(G) \to \mathcal{P}(\{1, \ldots, k\})$ by $f(u) = \{i_1, \ldots, i_l\}$, where $(u, v_{i_1}), \ldots, (u, v_{i_l}) \in D \cap {}^uK_k$ and $l \in [1, k]$. Suppose there exists a vertex $w \in V(G)$ such that $f(w) = \emptyset$, i.e., ${}^wK_k$ contains no vertices from D. Then since D dominates $G \Box K_k$, each vertex in ${}^wK_k$ must be dominated by D from neighbouring K_k-fibres, and so f satisfies condition (8.8). Therefore f is a kRDF of G and corresponds to the dominating set D. $\Box$

Example: Consider $C_4 : u, v, w, x, u$ and suppose $k = 3$. Define the function $f : V(G) \to \{1, 2, 3\}$ as follows: let

$$f(u) = \emptyset, \ f(v) = \{1, 2\}, \ f(w) = \emptyset \ \text{ and } \ f(x) = \{3\}.$$



Then f is a 3RDF of C_4 with weight 3, so $\gamma_{r3}(C_4) \leq 3$. By Theorem 8.32, $\gamma_{r3}(C_4) = \gamma(C_4 \Box K_3)$. Since both C_4 and K_3 are 2-regular, $C_4 \Box K_3$ is 4-regular. Therefore, each vertex of $C_4 \Box K_3$ dominates 5 vertices, so we need at least 3 vertices to dominate all 12 vertices of $C_4 \Box K_3$. Hence $\gamma(C_4 \Box K_3) \geq 3$ and so $\gamma_{r3}(C_4) = 3$.

We therefore have the following bounds on the k-rainbow domination number in terms of the usual domination number.

Theorem 8.33 ([41]) *For any two graphs G and H and any integer $k \geq 1$,*

$$\gamma(G \Box H) \leq \gamma_{rk}(G \Box H) \leq k\gamma(G \Box H).$$

Proof. By Theorem 3.2, we obtain the lower bound

$$\gamma_{rk}(G \Box H) = \gamma((G \Box H) \Box K_k) \geq \gamma(G \Box H)\rho(K_k) = \gamma(G \Box H).$$

Next, by Theorem 1.7, we obtain the upper bound

$$\gamma_{\mathrm{r}k}(G\Box H) = \gamma((G\Box H)\Box K_k) \leq \gamma(G\Box H)|V(K_k)| = k\gamma(G\Box H)$$

as desired. □

In particular, for $k = 2$, Brešar, Henning and Rall posed the following open question on rainbow domination in [17].

Question: Is it true that for any two graphs G and H, $\gamma_{\mathrm{r}2}(G\Box H) \geq \gamma(G)\gamma(H)$?

Note that by Theorem 8.33, $2\gamma(G\Box H) \geq \gamma_{\mathrm{r}2}(G\Box H)$, which implies that this conjecture is stronger than the Clark-Suen inequality (Theorem 5.1), and since $\gamma_{\mathrm{r}2}(G\Box H) \geq \gamma(G\Box H)$, this conjecture is weaker than Vizing's conjecture.

Chapter 9

Conclusion

Vizing's conjecture has been the focus of extensive study over the last 50 years and will probably continue as such until the conjecture is either proved or disproved.

After Vizing [61] posed the conjecture in 1968, the first breakthrough in solving it was found by Barcalkin and German [4] in 1979. They defined decomposable graphs and used these graphs to characterise the first large class of graphs which satisfy Vizing's conjecture, namely, BG-graphs. Chen, Piotrowski and Shreve [23] rediscovered the class of BG-graphs using their extracted partition approach and Faudree, Schelp and Shreve [33] defined a proper subclass of BG-graphs independently by using Condition CC.

In 1995, Hartnell and Rall [39] defined Type $\mathcal{X}$ graphs and implemented the one-half argument to prove that all Type $\mathcal{X}$ graphs satisfy Vizing's conjecture and contain the class of BG-graphs. Several common graphs are of Type $\mathcal{X}$, including trees, cycles, (ρ, γ)-graphs and graphs with domination number less than 3.

Later, Brešar and Rall [18] formulated the concept of fair domination and fair reception in 2009 and showed that any graph with equal domination and fair domination numbers satisfies Vizing's conjecture. This class of graphs is yet another extension of the class of BG-graphs and Aharoni and Szabó [2] showed, using the independence-domination number, that all chordal graphs have equal domination and fair domination numbers.

Another common approach to Vizing's conjecture is to prove the existence of a constant $c > 0$ such that $\gamma(G \Box H) \geq c\gamma(G)\gamma(H)$ for any two graphs G and H and try to prove the inequality for $c = 1$. Clark and Suen [24] used another vertex-partitioning approach to prove the inequality for $c = \frac{1}{2}$, a well-known partial result of Vizing's conjecture, in 2000. Their double-projection argument was used to prove several improvements of this partial result, including some results concerning Roman domination and claw-free graphs. Brešar et al. [13] used a combination of Barcalkin and German's simple labelling rule and the double-projection argument to formulate a new approach to Vizing's conjecture in their recent 2021 paper that reproduces simple proofs of previous partial results and allowed them to present yet another improvement of the Clark-Suen inequality.

These first four methods all partition the vertex set of a graph G in some manner, impose some conditions on G and then prove that G satisfies Vizing's conjecture. Another approach to Vizing's conjecture is proof by minimal counterexample, discussed in the 2012 survey [12]. Research on this method brought a list of properties to light that a minimal counterexample to Vizing's conjecture must satisfy. Furthermore, Hartnell and Rall outlined two more approaches to Vizing's conjecture which involve building new graphs which satisfy the conjecture from a class of graphs that we already know satisfies the conjecture in [40] from 1998. These are the attachable sets method and finding pairs of graphs which prove that the bound in Vizing's conjecture is indeed sharp. All graphs which attain this bound have been characterised by Hartnell and Rall in [40], and the existence of these classes of graphs is the reason why some researchers believe that Vizing's conjecture may be false.

Finally, we summarised several Vizing-like results for various types of domination, as well as open questions and stronger and weaker conjectures. In particular, the corresponding Vizing-like conjectures were proved for fractional domination by Fisher et al. [36] and for upper domination by Brešar [10]. Many of the partial Vizing-like results for variations of domination owe their success to some method of partitioning the vertex set of a graph and/or utilising some form of projection argument. Whether or not these common approaches are the key to solving Vizing's conjecture remains unknown. However, given the continued success of these methods in characterising large classes of graphs for which the conjecture is true in both usual domination and varieties of domination, these approaches still hold some promise of proving (or disproving) Vizing's conjecture.

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