Distances in planar graphs

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- B. Du Preez, Plane graphs with large faces and small diameter [19], published in the Australasian Journal of Combinatorics.
- B. Du Preez, Planar graphs with maximal planar centers [18], which has been submitted to and is currently under review at a DHET accredited journal.

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Due to the inclusion of publications, Chapters 4 and 7 contain some repeated lemmas and definitions.

Folklore theorems, simple lemmas and statements that are obviously true

There are a number of results proven in this thesis that have not been directly attributed to any previous author, but that never-the-less I do not claim any credit for discovering. These results are indicated by a campfire symbol —[🔥]— and largely fall into one of three categories:
(1) Theorems that are stated in the literature, for which I have not been able to find a satisfactory proof,
(2) Easily derived lemmas that, in some cases, are immediate corollaries of well known results,
(3) Results that I have not been able to find stated in the literature, that are sufficiently 'obvious' that it is likely they have been noticed and mentioned before.

For results of type (1), I have provided references to where the result (or some variation there of) is stated without proof in the literature. If a result of type (2) follows immediately from a known theorem, I have stated this fact and referenced the theorem in the preceding paragraph. If a result is of type (3), I have mentioned in the paragraph preceding the result that it seems probable the result has been found before.

Abstract

In graph theory, the degree diameter problem asks for the maximum number of vertices a graph with given maximum degree and diameter can have. The face-degree of a face in plane graph is the length of the shortest closed walk traversing the boundary of the face. A plane graph is ρ-face-degree regular if every face has face-degree ρ. This thesis begins with a literature review outlining the results and methods of papers studying planar graphs, particularly those solving the degree diameter problem for various kinds of ρ-face-degree regular graphs. In this review, we provide a correction to an error in *The degree/diameter problem in maximal planar bipartite graphs* by Dalfó, Huemer and Salas.

We investigate plane graphs in which the minimum face-degree is large compared to the graph's diameter, obtain a sharp upper bound for minimum face-degree in terms of diameter, and characterise the extremal graphs for this bound. We demonstrate that if a plane graph has sufficiently large minimum face degree, then it has equally large girth. We use this result to characterise planar generalised polygons (bipartite graphs whose girth is twice their diameter), and solve the degree diameter problem for plane graphs with diameter D that are $2D$-face-degree regular.

We solve the degree diameter problem for 2-connected, 5-face-degree regular graphs of diameter 3, and demonstrate that if the maximum degree of such a graph is sufficiently large, then the graph necessarily has girth 4.

We briefly investigate the distance and connectivity properties of maximal planar graphs. We show there are no non-trivial restrictions on the radius and diameter of a maximal planar graph, give an elementary proof of the well known characterisation of minimal separators in maximal planar graphs, characterise non-separating sets of these graphs, and demonstrate that the centre of a maximal planar graph may have arbitrarily many components. We then study centres of planar and maximal planar graphs in greater depth, and characterise maximal planar graphs that are centres of planar graphs. This characterisation depends upon a new concept we introduce, which is a generalisation of eccentricity, called *quasi-eccentricity*.

We conclude with a list of ideas for further research, open questions and conjectures.

Contents

Chapter 1

Introduction and Basic Definitions

1.1 Preamble

In this thesis, we explore distances in planar graphs. More specifically, we will consider planar graphs with some additional constraints on both their connectivity and the sizes of their faces when they are embedded in the plane.

This chapter is dedicated to presenting a bouquet of standard definitions, the majority of which can be found in at least one of the following excellent textbooks: *Graph Theory* by R. Diestel [16], *Graph Theory* by J. Bondy and U. Murty [4] and *Graphs and Digraphs* by G. Chartrand and L. Lesniak [11].

In Chapter 2, we state and briefly discuss a handful of classic results in graph theory, and a few in plane topology, that are used throughout the thesis.

In Chapter 3, we give a literature review. We discuss papers that address problems similar to those solved in this thesis — making sure to point out which methods and arguments we adapt and use here. The two papers that play the most central role are *Maximal planar graphs of diameter two* [35] by K. Seyffarth and *The degree/diameter problem in maximal planar bipartite graphs* [12] by C. Dalfó, C. Huemer and J. Salas.

In Chapter 4 we consider plane graphs in which the faces are, in a sense, as large as possible. We determine a bound for how large the faces of a plane graph can be given its diameter, and investigate both the structure of, and the solution to the degree-diameter problem for plane graphs which are extremal for this bound.

The results of Chapter 4 leave us with one gap in the literature on the degree-diameter problem: the case of 5-face-degree regular plane graphs of diameter 3. We fill this gap in Chapter 5.

In Chapters 6 and 7, we consider maximal planar graphs, i.e., planar graphs in which the faces are as small as possible. Chapter 6 describes basic features of maximal planar graphs, while in Chapter 7 we consider the problem of characterising which maximal planar graphs can be realised as the centres of planar graphs.

1.2 Graph-theoretic definitions

A **graph** $G = (V, E)$ is a pair of sets: a set V whose elements are **vertices**, and a set E of unordered pairs of distinct vertices, called **edges**. The **order** of G is the cardinality of V, which will always be finite for the graphs discussed in this thesis. We sometimes use the notations $V(G)$ and $E(G)$ to refer to the vertex and edge set, respectively, of G. If $G = (V, E)$ and $H = (V', E')$ are graphs such that $V' \subseteq V$ and $E' \subseteq E$, then H is a **subgraph** of G and G is a **supergraph** of H. If we further have that $V(G) = V(H)$, then H **spans** G. If u and v are vertices of G, then the edge $e = \{u, v\}$ is said to be **incident** with u and v, and conversely u and v are incident with e. For convenience, we write the edge e incident with vertices u and v as uv, rather than the more cumbersome notation $\{u, v\}$. If two vertices are incident with the same edge, they are **adjacent** and said to be **neighbours**. The **neighbourhood** of a vertex u, denoted $\boldsymbol{N(u)}$, is the set of all neighbours of u, and the **closed neighbourhood** is $\boldsymbol{N[u]} = N(u) \cup \{u\}$. The **degree** of a vertex u, denoted $\boldsymbol{d(u)}$, is the number of neighbours of u. The **maximum degree** and **minimum degree** of the graph G are $\boldsymbol{\Delta(G)} = \max\{d(u) : u \in V\}$ and $\boldsymbol{\delta(G)} = \min\{d(u) : u \in V\}$, respectively, and the **average degree** is $(1/|V|)\left(\sum_{u \in V} d(u)\right)$. A graph is said to be **regular** if every vertex has the same degree. If A is a set of vertices, then $\boldsymbol{N[A]} = \bigcup\{N[u] : u \in A\}$, and $\boldsymbol{N(A)} = N[A] - A$. If B is a set of vertices such that $B \subseteq N[A]$, then A **dominates** B, and B is dominated by A.

A sequence $W : v_1, v_2, \ldots, v_k$ of vertices in G, such that v_i and v_{i+1} are adjacent for all i in $\{1, \ldots, k-1\}$, is called a **walk**. The walk is **closed** if $v_1 = v_k$. The walk W is a $v_1 - v_k$ **path** if it contains no repeated vertices. If $P : v_1, v_2, \ldots, v_k$ is a path, and integers i and j satisfy $1 \leq i \leq j \leq k$, then we use the notation $P[v_i, v_j]$ to indicate the subsequence $v_i, v_{i+1}, \ldots, v_j$ of P. If A and B are sets of vertices in G, then an $A - B$ path is a path starting at a vertex of A and ending at a vertex of B. Two paths $P : v_1, \ldots, v_k$ and $Q : w_1, \ldots, w_j$ are **internally disjoint** if they share no vertices, except possibly common starting and / or ending vertices. A closed walk with no repeated edges is a **circuit**, and a closed walk with no repeated vertices (apart from the starting and ending vertices) is a **cycle**. The **length** of a walk (path, circuit, cycle) W, denoted $\boldsymbol{\ell(W)}$ is the number of edges traversed by the walk (path, circuit, cycle). We can consider the walk $W : v_1, v_2, \ldots, v_k$ as a subgraph of G. The vertex set of the walk is $V(W) = \{v_1, v_2, \ldots, v_k\}$, and the edge set $E(W) = \{v_1v_2, v_2v_3, \ldots, v_kv_k\}$ is all the edges traversed by W. If W contains no repeated edges, then $\ell(W) = |E(W)|$. The **girth** of G, denoted $\boldsymbol{g(G)}$, is the length of a shortest cycle in G.

A graph $G = (V, E)$ is **connected** if for any pair u and v of vertices in G, there exists a $u - v$ path in G. If G is not connected, it is **disconnected**. A **component** (or connected component) of a graph is a maximal connected subgraph. Given two sets A and B of vertices in a connected graph G, an $A - B$ **geodesic** is an $A - B$ path of minimum length (geodesics are not, in general, unique). The **distance** between A and B, denoted $\boldsymbol{d(A, B)}$, is the length of an $A - B$ geodesic, and $d(A, B) = 0$ if and only if $A \cap B \neq \emptyset$. For convenience, we use the notation $d(u, v) = d(\{u\}, \{v\})$. When restricted to single vertices, the distance $d : V \times V \to \mathbb{N}$ is a metric, which we will refer to as **the metric** on G. If u is a vertex of G, the ***i*'th distance neighbourhood** of u is $\boldsymbol{N_i(u)} = \{v \in V : d(u, v) = i\}$, and the **ball of radius *i*** around u is $\boldsymbol{N_i[u]} = \{v \in V : d(u, v) \leq i\}$. The **eccentricity** of a vertex u in G is $\boldsymbol{e(u)} = \max\{d(u, v) : v \in V\}$. The **radius** and **diameter** of G are $\boldsymbol{r(G)} = \min\{e(u) : u \in V\}$ and $\boldsymbol{D(G)} = \max\{e(u) : u \in V\}$, respectively. The **centre** of G is the set of vertices that have minimum eccentricity in G, and G is **self-centred** if every vertex of G is in its centre.

Let $G = (V, E)$ be a graph, A a subset of V, and u a vertex in V. The subgraph **induced** by A is the graph $\boldsymbol{G[A]} = (A, E_A)$, where $E_A = \{uv \in E : u \in A \text{ and } v \in A\}$. A subgraph H of G is an **induced subgraph** if $H = G[V(H)]$. We use the notations $\boldsymbol{G - A} = G[V - A]$ and $\boldsymbol{G - u} = G[V - \{u\}]$. Similarly, if X is a set of edges, then $G - X$ is the graph $(V, E - X)$, and if e is an edge of G, then $G - e = G - \{e\}$. Further, if vertices u and v are not adjacent in G, then $G + uv$ denotes the graph with vertex set V and edge set $E \cup \{uv\}$. As a convention for the sake of convenience, we sometimes discuss a subset of vertices, and its induced subgraph, interchangeably. For example, if we say that the set A is connected in G, we mean that the induced subgraph $G[A]$ is connected. If C is a cycle in G, then a **chord** of C is any edge of $E(G[V(C)]) - E(C)$, so a cycle has no chords (or is **chordless**) in G if and only if it is an induced subgraph of G. Another notational convention we adopt is to use subscripts on our notations introduced thus far to indicate which graph we are discussing. For example, if u is a vertex of H, and G is a supergraph of H, then the degree of u in H is denoted $d_H(u)$, while the degree of u in G is denoted $d_G(u)$. A subgraph H of G is **isometric** in G if, for all pairs of vertices u and v in H, we have that $d_H(u, v) = d_G(u, v)$.

Let $G = (V, E)$ be a connected graph. A subset S of V is a **separator** (or **vertex-cut**) of G if the induced graph $G - S$ is disconnected, and we say S **separates** G. A cycle C in G such that $V(C)$ is a separator of G is a **separating cycle**. If the single vertex $\{u\}$ separates G, then u is a **cut-vertex**. The set S is a **minimal** separator of G if $G - T$ is connected for all proper subsets $T \subset S$. Given three pairwise disjoint subsets A, B and S of V, the set S is an $\boldsymbol{A - B}$ **separator** if there is no $A - B$ path in $G - S$. The set S is a **minimal** $A - B$ separator if there is an $A - B$ path in $G - T$ for all $T \subset S$. An **edge-cut** of G is a set X of edges such that $G - X$ is disconnected. If $X = \{e\}$ is an edge-cut, then e is a **bridge** of G. The **connectivity** of G, denoted $\boldsymbol{\kappa(G)}$, is the minimum cardinality of a separator of G (and is defined to be $n - 1$ for the complete graph K_n). The **edge-connectivity** of G, $\boldsymbol{\lambda(G)}$, is the minimum cardinality of an edge-cut of G (and is defined to be 0 for the trivial graph K_1). If $\kappa(G) \geq k$, then G is said to be $\boldsymbol{k}$**-connected**, and if $\lambda(G) \geq k$, then G is $\boldsymbol{k}$**-edge-connected**.

Let $G = (V, E)$ be a graph, and let $\mathcal{V} = \{V_1, V_2, \ldots, V_k\}$ be a partition of V such that each set V_i is connected in G. The graph $G/\mathcal{V}$ with vertex set $\mathcal{V}$ and edge set $\{V_iV_j : \exists x \in V_i, y \in V_j \text{ such that } xy \in E\}$ is a **contraction** of G. The graph H is a **minor** of G if some subgraph of G can be contracted to obtain H. In similar vein, let $G = (V, E)$ be a graph and $e = uv$ an edge of G. The graph formed from G by removing e, and adding a new vertex w and the edges uw and wv, is an **elementary subdivision** of G. We say that the edge uv has been **subdivided**. The graph H is a **subdivision** of G if it can be obtained from G by a sequence of elementary subdivisions.

1.3 Some topology

Topological aspects of graphs are not the focus of this thesis, and we will use very little of the topology outlined in this section for anything beyond building further definitions. Diestel [16] discusses the basics of planar graphs and their embeddings, and *Graphs on Surfaces* by B. Mohar and C. Thomassen [33] gives a detailed account of graph embeddings. We assume familiarity with the topology of the plane $\mathbb{R}^2$, circle S^1, sphere S^2 and unit interval $[0, 1]$. We further assume that the reader knows the definition of a continuous function, homeomorphism, embedding, compactness, connectedness (of a topological space), path connectedness and topological boundary (of a subset of a topological space). All of these definitions can be found in S. Willard's *General Topology* [40].

The image $f([0,1])$ of an embedding $f : [0,1] \to X$ of the unit interval into a topological space X is an **arc** in X, with **endpoints** $f(0)$ and $f(1)$. The **interior** of the arc is the set $f([0,1]) - \{f(0), f(1)\}$. The image $f(S^1)$ of an embedding of the circle into a topological space is a **Jordan curve**. If X is a compact set in either $Y = \mathbb{R}^2$ or $Y = S^2$, then a maximal path-connected subset of $Y - X$ is a **region** of $Y - X$. Visibly, regions are themselves open sets of $Y - X$. We note that an open set of $\mathbb{R}^2$ or S^2 is connected if and only if it is path-connected [40].

To motivate further definitions, we will need a classic result in plane topology, The Jordan Curve Theorem.

Theorem 1.1. [33] *If C is a Jordan curve in the plane, then $\mathbb{R}^2 - C$ has exactly two regions, both of which have C as their boundary.*

If C is a Jordan curve in $\mathbb{R}^2$, then one of the regions of $\mathbb{R}^2 - C$ is bounded, and the other unbounded. We call the bounded region of C the **interior**, denoted $\mathbf{Int}(C)$, and the unbounded region the **exterior**, denoted $\mathbf{Ext}(C)$. We also let $\mathbf{Int}[C] = \mathrm{Int}(C) \cup C$ and $\mathbf{Ext}[C] = \mathrm{Ext}(C) \cup C$.

1.4 Plane and planar graphs

Let X be either the plane $\mathbb{R}^2$ or sphere S^2, and let $G = (V, E)$ be a graph. An **embedding** of G into X is a map $\varphi : G \to X$ that acts as follows:

- φ assigns to each vertex u of V a point $\varphi(u)$ of X,
- φ assigns to each edge uv of E an arc $\varphi(uv)$ with endpoints $\varphi(u)$ and $\varphi(v)$,
- φ is injective on the vertices of G, so $\varphi(u) = \varphi(v)$ implies that $u = v$, and
- The interior of the arc $\varphi(e)$, where e is some edge, does not intersect either the image of a vertex, or the interior of the arc corresponding to any other edge.

Any subset of X that can arise as the image $\varphi(G)$ of some graph G under some embedding φ is a **plane graph**. Not every finite graph can be embedded in the plane (or, equivalently, the sphere [4]). Those graphs that can be embedded in the plane (sphere) are **planar graphs**. The same plane graph $S \subset X$ can, in principle, arise as the image of many different graphs, but we will usually have a particular graph G and a fixed embedding φ in mind. In this case, where $S = \varphi(G)$, we call G the **abstract graph** of the plane graph S. As we will be considering fixed embeddings of graphs, there is little risk of ambiguity in using the same notation to refer to both the edges and vertices of an abstract graph, and their images in X under a fixed embedding. For example, we will often use the name G to refer to both the abstract graph and the associated plane graph, and if we refer to the edge e, it will be clear from context whether this is an edge of the abstract graph, or the arc $\varphi(e)$ in X. If H is a subgraph of the abstract graph G, we can embed it as a plane graph by restricting the embedding $\varphi|_H : H \to X$ to obtain a **plane subgraph** of the plane graph G.

Any time we mention a subgraph H of a plane graph G, we mean the plane subgraph obtained in this way by restricting the embedding of G. If C is any cycle of a plane graph G, then C induces a Jordan curve in the plane. If R is some subset of X, then we denote by $\boldsymbol{G[R]}$ the plane subgraph of G of edges and vertices that lie entirely in R. For example, $G[\text{Int}[C]]$ is the subgraph of G consisting of the cycle C, and all the edges and vertices contained in $\text{Int}[C]$. A cycle C such that both $G[\text{Int}(C)]$ and $G[\text{Ext}(C)]$ contain at least one vertex is a **Jordan separating cycle**.

Let G be a connected plane graph in X. The regions of $X - G$ are **faces** of G. If $X = \mathbb{R}^2$ then exactly one face of G will be unbounded. Call this unbounded face the **outer face**, and the other faces of G **inner faces**. An edge or vertex of G is **incident** with the face f if it is a subset of the topological boundary $\partial(f)$ of f. The notation $\boldsymbol{G[f]}$ is used for the subgraph of G that forms the boundary $\partial(f)$ of f. We call $G[f]$ the subgraph that **bounds** f, and say that the face f is **bounded by** the subgraph $G[f]$. We denote the **face set** of G, the set of abstract graphs that embed as face boundaries of G, by $\boldsymbol{F(G)} = \{G[f] : f \text{ is a face of } G\}$. Different plane graphs that arise as different embeddings of the same abstract graph may have different face sets. Two embeddings of the same abstract graph that have the same face sets are **equivalent embeddings**. If $G[f]$ is a circuit or cycle, it is a **face-circuit** or **face-cycle** respectively. The length of a shortest closed walk traversing all the edges of $G[f]$ is the **face-degree** $\boldsymbol{\mathcal{E}(f)}$ of f. Note that if $G[f]$ is a circuit or cycle, then $\mathcal{E}(f) = |E(G[f])|$. If every face of G has face-degree ρ, then it is $\boldsymbol{\rho}$**-face-degree regular**. Consider a subgraph H of G, and let f' be a face of H. We say that a $u-v$ path P of G **crosses** f' if $H[f'] \cap P = \{u, v\}$, and $f' \cap P \neq \emptyset$. It is possible for a path consisting of a single edge to cross a face. For an example, consider Figure 1.1: The plane graph G has a subgraph H, and f is a face of H. Both the path P and the edge e of G cross f.

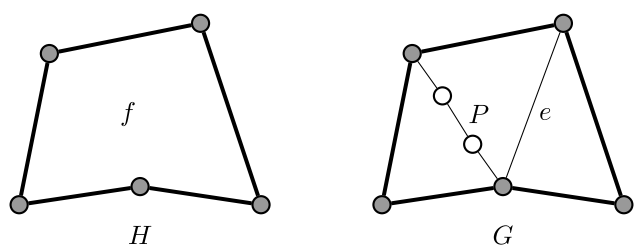


Figure 1.1: On the left is a plane graph H. On the right, the plane supergraph G, which contains H as a subgraph.

Let G be a plane graph, and let u be a vertex of G with at least two neighbours. Let D be a disk centred on u, of small enough radius that u is the only vertex in the disk, and let S be the boundary circle of D. If v is a neighbour of u, then the edge uv is an arc $A_v : [0, 1] \to X$ with $A_v(0) = u$ and $A_v(1) = v$. We denote by v^* the first point of the arc A_v that meets S, i.e., $v^* = A_v(\tau)$, where $\tau = \min\{t \in [0, 1] : A_v(t) \in S\}$. The set $S - \{v^* : v \in N(u)\}$ consists of a finite number of segments of the circle S, and the boundary of each segment consists of exactly two points like v^*. If x and y are two neighbours of u such that x^* and y^* form the boundary of a segment of $S - \{v^* : v \in N(u)\}$, then we say that x and y are **consecutive** neighbours of u. If z is a neighbour of u such that x and y are not consecutive neighbours of u in the plane graph G, but x and y are consecutive neighbours of u in $G - uz$, then we say that z lies **between** x and y. For example, consider Figure 1.2. The vertex v has neighbours v, w, x and y, and is covered by a grey disk D with boundary circle S. The circle S is divided into four arcs by the points v^*, w^*, x^* and y^*. The vertices

v and w are consecutive neighbours of u, since v^* and w^* bound a segment of $S - \{v^*, w^*, x^*, y^*\}$. Further, w is between the neighbours x and v, since x^* and v^* bound a segment of $S - \{v^*, x^*, y^*\}$, but do not bound a segment of $S - \{v^*, w^*, x^*, y^*\}$.

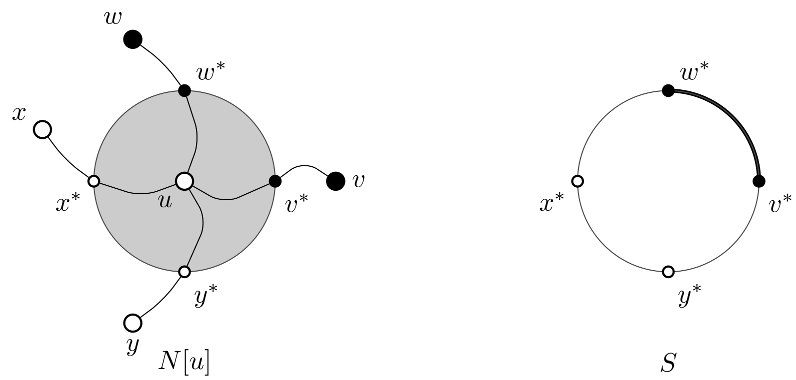


Figure 1.2: The left figure shows a neighbourhood $N[u] = \{v, w, x, y\}$ of a vertex u in a plane graph, and the right figure is the circle S.

A **maximal plane graph** is a plane graph to which no edge can be added to yield a plane supergraph, and a **maximal planar graph** is graph that is planar, but to which the addition of any edge destroys planarity. A graph that can be embedded in the plane such that every vertex is incident with the outer face is an **outerplanar** graph, and a graph is **maximal outerplanar** if it is outerplanar, but the addition of any edge to the graph destroys outerplanarity.

1.5 Common graphs

In this section, we present some common classes of graphs to which we will refer regularly. The **complete graph** of order n, denoted $\boldsymbol{K_n}$, is the graph with n vertices and every possible edge between them. An **empty graph** of order n is a graph with n vertices and no edges. If G is a graph and S is a set of vertices of G, then S is a **clique** if $G[S]$ is a complete graph, and it is an **independent set** if $G[S]$ is an empty graph. A graph $G = (V, E)$ is **bipartite** if V can be partitioned as $V = W \sqcup B$ such that every edge is incident with one vertex of W and one vertex of B (where $W \sqcup B$ denotes the disjoint union of W and B). The sets W and B are the **partite sets** of G. The **complete bipartite graph** $\boldsymbol{K_{m,n}}$ is the bipartite graph with vertex set $V = W \sqcup B$ such that $|W| = m$, $|B| = n$ and edge set $E = \{wb : w \in W, b \in B\}$. We denote by $\boldsymbol{C_n}$ and $\boldsymbol{P_n}$ the cycle of order n and the path of order n, respectively.

A **tree** is a connected graph that does not contain any cycles. If G and H are graphs, their **Cartesian Product** $\boldsymbol{G \times H}$ is the graph with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H) = \{(u, v)(u', v') : (u = u' \text{ and } vv' \in E(H)) \text{ or } (v = v' \text{ and } uu' \in E(G))\}$ (u and u' are vertices of G, and v and v' are vertices of H).

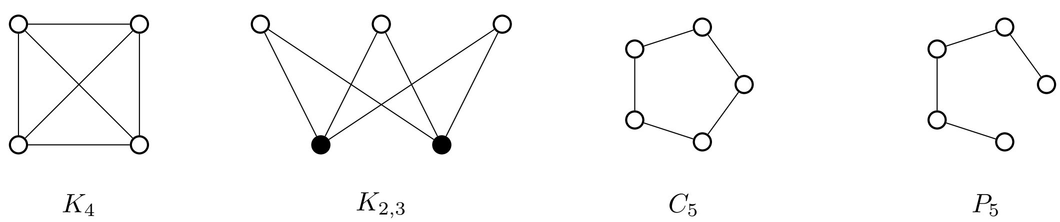


Figure 1.3: From left to right: The complete graph K_4, the complete bipartite graph $K_{2,3}$, the cycle C_5 and the path P_5.

Chapter 2

Standard results and classic theorems

In this chapter, we present a collection of foundational concepts and known results in graph theory. Some of these are essential and used throughout the thesis, while others provide context and sketch a map of the existing theory upon which this thesis is built. A number of well-known results are also presented later in the thesis where they are relevant, particularly in Chapters 4 and 6.

2.1 Elementary results

We begin with the well-known *Handshaking Lemma*, which can be proven by a simple double-counting argument.

Lemma 2.1. **[16]** *Let $G = (V, E)$ be a graph. Then $2|E| = \sum_{u \in V} d(u)$.*

The following result is the standard characterisation of bipartite graphs.

Proposition 2.2. **[16]** *A graph is bipartite if and only if it does not have any cycle of odd length as a subgraph.*

Every connected bipartite graph G has a unique bipartition, and this bipartition can be found using the metric on G. To construct this unique bipartition, pick any vertex u of G and let $V_o = \{v \in V : d(u, v) \text{ is odd}\}$ and $V_e = \{v \in V : d(u, v) \text{ is even}\}$ be the two partite sets (see, for example, Figure 2.1).

We recall the following characterisation of trees.

Proposition 2.3. **[16]** *A graph T is a tree if and only if any of the following hold:*

- *For any pair of vertices u and v, there is a unique $u - v$ path in T,*
- *Every edge of T is a bridge,*
- *There are no cycles in T, but for any vertices u and v that are not adjacent in T, the edge uv lies on a cycle of the graph $T + uv$.*

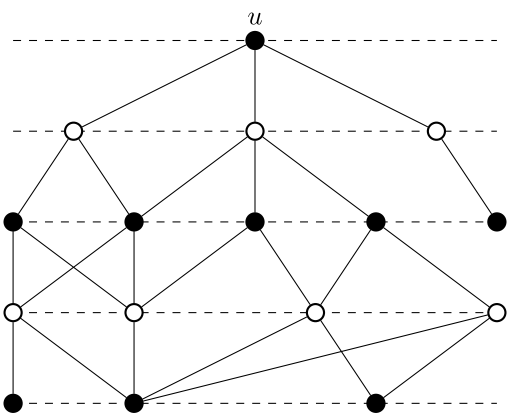


Figure 2.1: In the above bipartite graph, the vertices at even distance from u are coloured black, and those at odd distance from u are coloured white.

It is worth noting that since $u-v$ paths in a tree are unique, every path in a tree is a geodesic.

We now turn our attention to distances in graphs, starting with two well-known bounds.

Proposition 2.4. **[16]** *If G is a connected graph of radius r and diameter D, then $r \leq D \leq 2r$.*

The fact that $D \leq 2r$ follows at once by considering a central vertex of G, and using the triangle inequality. It is mentioned in [11] that given any positive integers a and b such that $a \leq b \leq 2a$, there exists a graph of radius a and diameter b. One common way of constructing such a graph is by attaching one end of a path of length $b-a$ to a cycle of radius a.

Proposition 2.5. **[16]** *If G is a connected graph of girth g and diameter D that contains at least one cycle, then $g \leq 2D+1$ and this bound is sharp.*

To see that the bound of Proposition 2.5 is sharp, it suffices to consider the cycle C_{2D+1}. The regular graphs that attain this bound are known as **Moore graphs**. Moore graphs are discussed in both [4] and [11].

2.2 Connectivity and Menger's Theorem

We frequently appeal to the first proposition of this section, which is commonly known as *Whitney's Theorem.*

Theorem 2.6. **[16]** *If G is a connected graph of order at least 2, with connectivity κ, edge-connectivity λ and minimum degree δ, then $\kappa \leq \lambda \leq \delta$.*

The majority of the graphs we consider in this thesis are, at the bare minimum, 2-edge-connected, and thus have no vertices of degree 1. In [11], it is shown by construction that for any positive integers a, b and c such that $a \leq b \leq c$, there exists a graph with connectivity a, edge-connectivity b and minimum degree c. Such a graph can be constructed by taking the disjoint union of two complete graphs K_{c+1}, and joining them with b edges such that the b edges are incident with only a vertices in one of the complete graphs.

Proposition 2.7. [4] *An edge of a connected graph is a bridge if and only if it is not an edge of some cycle.*

With Proposition 2.7, it is easy to see that every edge of a 2-edge-connected graph belongs to some cycle of the graph.

The next result is of central importance in the study of graph connectivity, and is used multiple times in this thesis. It goes by the name *Menger's Theorem*. Three different proofs of the theorem are given in [16] (as well as some consequences of it), and every graph theory textbook cited in this thesis discusses Menger's Theorem.

Theorem 2.8 (Menger's Theorem)**.** **[4, 11, 16]** *A graph G is k-connected if and only if, for any pair u and v of vertices in G, there exist k internally disjoint $u - v$ paths.*

One immediate consequence of Menger's Theorem is that a graph of order at least 3 is 2-connected exactly when any two vertices of the graph lie on a common cycle [11].

2.3 Euler, Wagner and Kuratowski

Most of the results we need concern plane and planar graphs. The first result is an analogue of the Handshaking Lemma for the face-degrees in a plane graph — recall that if f is a face of a plane graph, then $\mathcal{E}(f)$ is the face-degree of f.

Lemma 2.9. [4] *If $G = (V, E)$ is a plane graph with face-set F, then $\sum_{f \in F} \mathcal{E}(f) = 2|E|$.*

It is no coincidence that Lemmas 2.1 and 2.9 look so similar. They are in fact **dual** statements. Aside from Lemma 2.9 and the next result, Proposition 2.10, we make little use or mention of plane duality in this thesis — but the topic is explored at length in [4] and [33].

Proposition 2.10. [4] *A plane graph is bipartite if and only if every face has even face-degree.*

Note that the dual version of Proposition 2.10 also holds: every vertex of a plane graph has even degree if and only if each of its faces can be assigned one of two colours, such that no two faces which share an edge receive the same colour [4].

The next proposition yields a powerful invariant of plane graphs (and, more generally, graphs embedded on any fixed surface).

Proposition 2.11. **[16]** *If G is a connected graph embedded in the plane or sphere with vertex set V, edge set E and face set F, then $|V| - |E| + |F| = 2$.*

The quantity $|V| - |E| + |F|$ is the **Euler characteristic** of the embedded graph. We can interpret Proposition 2.11 as saying that the Euler Characteristic of any graph embedded in the sphere or plane is exactly 2, and the Euler Characteristic is thus a property of the topological space into which the graph is embedded. In fact, every surface has a well-defined Euler characteristic, and this fact is explored in [4], [33] and [11].

A number of basic results concerning plane and planar graphs follow from Proposition 2.11 and Lemmas 2.9 and 2.1, the first of which is a direct consequence of the Euler Characteristic of the plane / sphere.

Proposition 2.12. [4] *If G is a planar graph, then any two embeddings of G in the plane (sphere) have the same number of faces.*

We can also use these results to bound the number of edges in a plane graph:

Proposition 2.13. [16] *If a connected planar graph containing at least one cycle has n vertices, m edges and girth g, then*

$$m \leq \frac{g(n-2)}{g-2}.$$

And in general:

Proposition 2.14. [16] *If a graph with n vertices and m edges is planar, then $m \leq 3n - 6$.*

Proposition 2.14 gives a necessary condition for a graph to be planar, but this condition is easily seen to not be sufficient. We end this section with a pair of theorems providing sufficient and necessary conditions (in terms of subdivisions and minors) for a graph to be planar, as well as an observation that allows us to use these theorems effectively.

Proposition 2.15. [4] *Every subgraph, subdivision and minor of a planar graph is itself planar.*

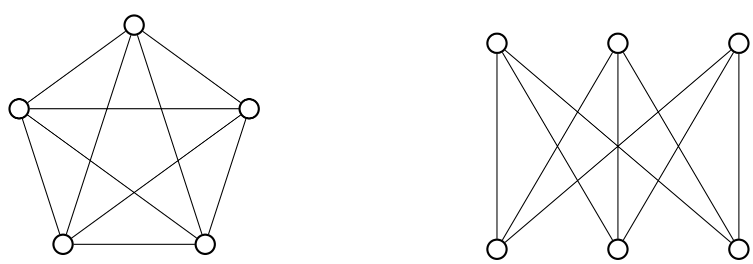


Figure 2.2: On the left is the graph K_5, and on the right is $K_{3,3}$.

Before presenting the theorems, we remind the reader that K_5 is the complete graph of order 5 and $K_{3,3}$ is the complete bipartite graph with two partite sets of three vertices (see Figure 2.2). It is easy to see (for example, by using Proposition 2.13) that neither K_5 nor $K_{3,3}$ are planar. Thus, per Proposition 2.15, a graph is not planar if it either contains a subdivision of K_5 or $K_{3,3}$, or has K_5 or $K_{3,3}$ as a minor. Astoundingly, both converses hold.

Theorem 2.16 (Kuratowski's Theorem)**. [4, 11, 16]** *A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.*

Theorem 2.17 (Wagner's Theorem)**. [4, 11, 16]** *A graph is planar if and only if it does not have either K_5 or $K_{3,3}$ as a minor.*

Full proofs of these two theorems are given in [16], [4] and [11]. They are a cornerstone of the theory of planar graphs, and we will make frequent use of them in Chapter 6.

2.4 Faces and embeddings

It is often useful to choose a given face of a plane graph as the outer face. The next Proposition shows that we may do this without loss of generality.

Proposition 2.18. **[4, 16]** *Let G be a planar graph with an embedding $\varphi : G \to \mathbb{R}^2$, let f be a face of $\varphi(G)$, and let $H = \varphi(G)[f]$ be the abstract subgraph of G that φ maps to the boundary of f. There exists an embedding $\psi : G \to \mathbb{R}^2$ that is equivalent to φ such that $\psi(H)$ is the boundary of the outer face of $\psi(G)$.*

Proposition 2.18 is discussed in [16] and can be proven using stereographic projection, which is a particular homeomorphism between the plane $\mathbb{R}^2$ and the punctured sphere $S^2 - \{x\}$, where x is any point of S^2.

As demonstrated by the next results, there is a strong relationship between the connectivity properties of a plane graph and the structure of its faces.

Proposition 2.19. **[16]** *If an edge of a plane graph is contained in some cycle, then it lies on the boundary of exactly two faces of the plane graph. If the edge is not contained in any cycle, then it lies on the boundary of exactly one face of the graph.*

From Propositions 2.7 and 2.19, we deduce that every edge of a 2-edge-connected plane graph lies on the boundary of exactly two faces.

Proposition 2.20. **[32]** *A plane graph is 2-connected if and only if each face of the graph is bounded by a cycle.*

We conclude by noting that under sufficiently strong connectivity conditions, there is effectively only one way to embed a planar graph as a plane graph.

Theorem 2.21. [4] *If a planar graph is 3-connected, then any two embeddings of the graph into the plane / sphere are equivalent.*

Different notions of 'equivalent embeddings', including the one we use (given in Chapter 1), are discussed in [16]. In Chapter 6, we mention and make use of a well-known strengthening of Theorem 2.21.

Chapter 3

Literature Review

In this chapter, we summarise and review a sample of the literature relevant to this thesis. Due to the foundational importance of the Jordan Curve Theorem in the theory of planar graphs, we begin by discussing Thomassen's proof of this result. We then consider planar separator theorems, which tackle the problem of breaking up a planar graph into connected parts of roughly equal size. These separator theorems are powerful tools for bounding the order of a planar graph. We then focus on the degree diameter problem for planar graphs. This is a well-studied problem (as shown by Miller and Širáň's extensive survey [31]), and we review some of the papers addressing it in detail. At the start of Chapter 7, we discuss a handful of papers concerning graph centres and self-centred graphs, and thus will not review these here. We remark that Buckley and Harary's textbook *Distance in Graphs* [8] covers graph centres and related topics in detail.

Throughout this section we give sketches and rough outlines of literature proofs. We indicate the start of such a proof sketch by an italicised '*Proof sketch*', and we mark the end of the proof sketch by the symbol '*QED*'.

3.1 The Jordan Curve Theorem

In this section, we give an overview of Thomassen's proof of the Jordan Curve Theorem, as is presented in *The Jordan-Schonflies Theorem and the classification of surfaces* [36] (this proof can also be found in [33]). There are many proofs of this theorem — what makes the proof we discuss here remarkable is that it uses only elementary arguments in plane topology and graph theory. Every result that appears in this section is from [36], unless stated otherwise.

We will need a few new definitions. Remember that an arc A in the plane is the image of an embedding $f : [0,1] \to \mathbb{R}^2$. A set X in the plane is **arc-connected** if, for any x and y in X, there is an arc in X with endpoints x and y. We say that A is a **polygonal arc** if it is the union of a finite number of straight line segments (i.e., sets of the form $(1-t)x + ty$ where t is in $[0,1]$ and x, y are in $\mathbb{R}^2$). Similarly, if a Jordan Curve is the union of finitely many line segments, then it is a **polygonal curve**.

Thomassen's first result demonstrates that, for the purposes of arc-connectivity, it suffices to consider polygonal arcs.

Lemma 3.1. **[36]** *For any pair x and y of points in an open, arc-connected set $X \subseteq \mathbb{R}^2$, there exists a polygonal arc in X with endpoints x and y.*

At this point, we have many different types of connectedness for subsets of $\mathbb{R}$ being considered, so it is time to simplify this situation. For open subsets of the plane, being connected, path-connected and arc-connected are all equivalent [40] (and these equivalences are not hard to establish). Thus Lemma 3.1 shows that if $\mathcal{O}$ is an open set of $\mathbb{R}^2$, and $X \subseteq \mathcal{O}$, the following are all equivalent:

- X is a maximal connected subset of $\mathcal{O}$,
- X is a region of $\mathcal{O}$ (per our definition in Chapter 1),
- X is a maximal arc-connected subset of $\mathcal{O}$,
- X is a maximal subset of $\mathcal{O}$ whose points are all connected by polygonal arcs.

In particular, we note that any two points in a face of a plane graph can be connected by a polygonal arc lying entirely in that face — a fact which allows us to redraw any plane graph as one whose edges are all polygonal arcs. We call an embedding whose image is such a plane graph a **polygonal embedding**.

Lemma 3.2. **[36]** *Every planar graph has a polygonal embedding.*

Proof sketch. Consider any embedding of the planar graph as a plane graph. Around each vertex, draw a small disk such that no two disks intersect (this can be done by compactness of plane graphs). The arcs of edges inside the disk can be replaced by straight line segments, and the arcs between disks can be replaced with polygonal arcs by Lemma 3.1, completing the proof. *QED.*

While Lemma 3.2 suffices for our purposes, it can be strengthened. In particular, Fáry's Theorem states that a planar graph can be embedded such that every edge is a straight line [33].

It will be useful to establish that cycles in (polygonally embedded) plane graphs separate the plane into two regions. To this end, Thomassen establishes a weak version of the Jordan Curve Theorem — which states that if C is a polygonal curve, then $\mathbb{R}^2 - C$ has exactly two regions, both with boundary C. The proof of this version of the theorem presented relies on two features specific to polygonal curves: it is easy to construct a path 'following' one side of a polygonal curve, and the intersection of a straight line and a polygonal curve is a (possibly empty) finite union of points and line segments. This theorem suffices to demonstrate, without using any of the literature results presented in Chapter 2, the following useful lemma:

Lemma 3.3. **[36]** *The complete bipartite graph $K_{3,3}$ is not planar.*

Proof sketch. Notice that $K_{3,3}$ can be considered as a 6-cycle with three mutually crossing chords. In any polygonal embedding of $K_{3,3}$, this 6-cycle is a polygonal curve. At least two chords cross the interior (or exterior) region of the 6-cycle, which is not possible by the polygonal version of the Jordan Curve Theorem. *QED.*

The non-planarity of $K_{3,3}$ suffices to prove the first part of the full Jordan Curve Theorem:

Lemma 3.4. [36] *The complement of a Jordan curve in the plane is not connected (i.e., the complement of the curve has at least two regions).*

Proof sketch. (With some minor modifications for ease of presentation). Let C be a Jordan curve in the plane, and assume to the contrary that $R^2 - C$ is connected. Let R be the smallest rectangle in the plane containing C. Since C is compact, such a rectangle exists, and C intersects all four sides of R. Let v_1 and v_2 be two points on C on the left and right sides of R, respectively. Note that C consists of two different arcs, A_1 and A_2, both with endpoints v_1 and v_2. Let A_3 be a polygonal arc from v_1 to v_2 that lies outside R. In effect, the arc A_3 crosses what we know should be the 'exterior' of C. We now aim to find a point that should lie in the 'interior' of C. To this end, consider a vertical line that passes through the midpoint of R. Some closed interval A_4 of this line joins a point of A_1 to a point of A_2, and is internally disjoint from C. Denote by u_1 and u_2 the endpoints of A_4. By assumption, $\mathbb{R}^2 - C$ is connected, so there is a polygonal arc A_5 from some point v_3 in $A_4 - \{u_1, u_2\}$ to a point u_3 on $A_3 - \{v_1, v_2\}$ (u_3 is the desired point in the 'interior' of C). By considering the segments of the arcs A_i, i in $\{1, \ldots, 5\}$, as edges and the points u_i and v_i, i in $\{1, 2, 3\}$, as vertices, we see that we have embedded $K_{3,3}$ in the plane — which is impossible, completing the proof. *QED.*

Thomassen further proves a number of results about 2-connected (planar) graphs. In particular, it is demonstrated that every 2-connected graph can be obtained in a sequence of steps by starting with a single cycle, and successively adding paths to the 2-connected graph of the previous step. Note that a decomposition of a 2-connected graph into such a cycle and collection of paths is called an **ear decomposition** of the graph, and these decompositions are discussed in [4]. Using the ear decomposition and the polygonal Jordan Curve Theorem, it is easy to prove the Euler Characteristic equation (Proposition 2.11) holds for polygonally embedded, 2-connected plane graphs, and that the boundary of every face of such a graph is a cycle. Using these results, we can prove a useful technical lemma:

Lemma 3.5. [36] *Let $G_1, G_2, \ldots, G_k$ be a family of polygonally embedded plane graphs that satisfy the following conditions:*

- *G_i is 2-connected for all i in $\{1, \ldots, k\}$,*
- *$|G_i \cap G_{i+1}| \geq 2$ for all i in $\{1, \ldots, k-1\}$,*
- *$G_i \cap G_j = \emptyset$ for all i and j such that $|j - i| > 1$.*

If a point x lies in the outer face of each of the plane graphs $G_1 \cup G_2$, $G_2 \cup G_3$, $\ldots$, $G_{k-1} \cup G_k$, then it lies in the outer face of $G_1 \cup G_2 \cup \cdots \cup G_k$.

Using this lemma, Thomassen shows the following:

Lemma 3.6. [36] *If A is an arc in the plane, then $\mathbb{R}^2 - A$ is a single connected region.*

Proof sketch. Consider two points x and y in $\mathbb{R}^2 - A$, and let d be the minimum distance between A and $\{x, y\}$. By compactness, the arc A can be covered by a finite number of squares, each with side length much smaller than d. The squares can be chosen such that the boundaries of two consecutive squares overlap in at least two places, and so that each square intersects A. The square boundaries are then 2-connected plane graphs such that the union of any two adjacent squares contains x and y in its outer face. Thus, we can use Lemma 3.5 to show that x and y are in the outer face of the plane graph formed by the union of all the square boundaries. Since x and y are arbitrary, any pair of points in $\mathbb{R}^2 - A$ lie in the same region, so we are done. *QED.*

The above proof technique can also be used to prove that if X and Y are disjoint plane graphs, then adding an arc between X and Y does not disconnect any region of $\mathbb{R}^2 - (X \cup Y)$ (this result is mentioned without proof in [16]). In particular, it is straightforward to use this 'covering by squares' proof technique, or the afore-mentioned result in [16], to show that a plane tree has only a single face.

There is an important consequence of the fact that every Jordan curve disconnects the plane, but no arc does. Let C be a Jordan Curve, let A be an arc of C with endpoints u and v, and let A' denote the arc $(C - A) \cup \{u, v\}$ of C. If x is any point in a region f of $\mathbb{R}^2 - C$, then there is a point y in a different region of $\mathbb{R}^2 - C$, by Lemma 3.4. Since $\mathbb{R}^2 - A'$ has a single region, per Lemma 3.6, there is an $x - y$ arc P in $\mathbb{R}^2 - A'$, and this arc P intersects A. By choosing A to be a sufficiently small arc containing whatever point of C we like, the points of C that can be reached from u by an arc in f are seen to be dense in C. This notion of **curve accessible** points is discussed further in [33]. For our purposes, this discussion is the last piece needed to explain Thomassen's proof of the Jordan Curve Theorem (note that this discussion shows C will be the boundary of every region of $\mathbb{R}^2 - C$).

Lemma 3.7. [36] *The complement of a Jordan curve in the plane has at most two regions.*

Proof sketch. Assume to the contrary that C is a Jordan curve whose complement has (at least) three regions f_1, f_2 and f_3. Let C be divided into three distinct arcs A_1, A_2 and A_3, and let u_1, u_2 and u_3 be points in f_1, f_2 and f_3 respectively. By the discussion prior to Lemma 3.7, there is a point v_1 in A_1 that is connected to u_i by a polygonal arc in f_i for all i in $\{1, 2, 3\}$. Similarly, there are points v_2 and v_3 in A_2 and A_3 connected by polygonal arcs to all the points u_i. These polygonal arcs can be modified to avoid eachother. By considering the points u_i and v_i as vertices of a plane graph, we have found an embedding of $K_{3,3}$ in the plane, which is impossible, completing the proof. *QED.*

Combining Lemmas 3.4 and 3.7, and the discussion prior to Lemma 3.7, we deduce the full Jordan Curve Theorem.

In [36], Thomassen also presents proofs of the Jordan-Schonflies Theorem and the Classification of Surfaces, but we do not discuss these here.

3.2 Planar separators

In this section, we briefly discuss a result from Lipton and Tarjan's paper *A separator theorem for planar graphs* [30].

A separator theorem is one that demonstrates a type of graph of order n can be separated into components of order at most cn, for some c in $(0, 1)$, by a separator that also has its order bounded above in some way. We can generalise this idea by giving different weighting or cost to each vertex, and then looking for a separator that bounds the total cost of any of its components. To this end, given a graph $G = (V, E)$, we call any function $\omega : V \to \mathbb{R}_{\geq 0}$ a **cost function**, and define the **cost** of a subset $A \subseteq V$ by $\omega(A) = \sum_{v \in A} \omega(v)$. Given a cost function ω that is not 0 on every vertex, we can always scale ω by a factor of $\frac{1}{\omega(V)}$ to obtain an equivalent cost function with $\omega(V) = 1$.

For our purposes, the first result of [30] suffices:

Theorem 3.8. **[30]** *Let $G = (V, E)$ be a planar graph that has a spanning tree of radius r and is equipped with a cost function ω such that $\omega(V) \leq 1$. The set V can be partitioned as $V = A \sqcup S \sqcup B$ such that:*

- *the set S is an $A - B$ separator,*
- $|S| \leq 2r + 1$*,*
- $\max\{\omega(A), \omega(B)\} \leq \frac{2}{3}$*.*

Proof sketch. Let T be a spanning tree of radius r in G. Embed the graph G in the plane, and triangulate it to create a plane graph G^* (i.e., add edges to G until every face of G is bounded by a 3-cycle). Note that if e is any edge of G^* that does not belong to T, then $T + e$ contains exactly one cycle C_e. We show that if we pick the 'best' cycle C_e, then we get the desired separator. Among all the edges of G^* not in T, let uw be an edge that minimises the quantity $\max\{\omega(\text{Int}(C_{uw})), \omega(\text{Ext}(C_{uw}))\}$. That is to say, we pick uw to try make the cost inside and outside of the cycle C_{uw} as small as possible. Since G^* is triangulated, there is a face bounded by vertices u, v and w in the interior of C. If the vertices of $\text{Int}(C_{uw})$ have a total cost of more than $\frac{2}{3}$, then by considering the cycles C_{uv} and C_{vw}, a contradiction can be derived (by Proposition 2.18, we can consider only the case where the interior of C_{vw} has greater cost without loss of generality). There are many possible cases here, based on whether the edges uv and vw belong to C_{uw} or T, so we do not go into further detail. *QED.*

Note that in Chapter 4 we use a related idea — of considering minimum-length cycles formed by adding an edge to a spanning tree — to bound the minimum face-degree of a 2-edge-connected plane graph.

Corollary 3.9. **[30]** *Let $G = (V, E)$ be a planar graph of radius r and order n. The vertex set V can be partitioned as $V = A \sqcup S \sqcup B$ such that:*

- *the set S is an $A - B$ separator,*
- $|S| \leq 2r + 1$*,*
- $\max\{|A|, |B|\} \leq \frac{2}{3}n$*.*

Proof sketch. To prove Corollary 3.9, notice that if G has radius r, then it has a spanning tree of radius r, and apply Theorem 3.8 to G with the cost function $\omega(v) = \frac{1}{n}$ for all v in V. *QED.*

We note that in *N-separators in planar graphs* [37], Tishchenko proves a generalisation of Theorem 3.8 (which we will not state here). While Theorem 3.8 demonstrates that a planar graph can be divided by a separator with few vertices into two components of low cost, Tishchenko considers the problem of separating a planar graph into an arbitrary number of components, each of which has a high portion of the total cost.

3.3 The degree diameter problem

Let us call a connected graph of maximum degree Δ and diameter D a $\boldsymbol{(\Delta, D)}$**-graph**. The degree diameter problem asks us to find (or bound above) the largest possible order of a (Δ, D)-graph. In *Moore graphs and beyond: A survey of the degree/diameter problem* [31], Miller and Širáň motivate this problem, and survey results of the degree diameter problem for many different kinds of graphs.

3.3.1 The trivial bound

We note that there is a well-known trivial upper bound on the number of vertices that any (Δ, D)-graph may have, known as the **Moore Bound**.

Theorem 3.10 (The Moore Bound)**.** [🔥] *The order n of a (Δ, D)-graph is bounded above by:*

$$n \leq \begin{cases} 1 + \Delta\frac{(\Delta-1)^D - 1}{\Delta - 2} & \text{if } \Delta > 2 \\ 2D + 1 & \text{if } \Delta = 2 \end{cases}$$

We outline the proof of this bound given in [31].

Proof sketch. Let $G = (V, E)$ be a (Δ, D)-graph and let u be a vertex of G. Note that the distance layers give a partition of V: $V = \{u\} \sqcup N_1(u) \sqcup \cdots \sqcup N_D(v)$ (it is possible that some of the $N_i(u)$ are empty). Further, each vertex of $N_i(u)$ is adjacent to at least one vertex of $N_{i-1}(u)$, and so has at most $\Delta - 1$ neighbours in $N_{i+1}(u)$. Thus $|N_{i+1}(u)| \leq (\Delta - 1)|N_i(u)|$, and so we can bound n from above by:

$$\begin{aligned} n &\leq |\{u\}| + \sum_{i=1}^{D} |N_i(u)| \\ &\leq 1 + \sum_{i=1}^{D} \Delta(\Delta - 1)^{i-1} \end{aligned}$$

The result follows by applying the formula for the sum of a geometric series. *QED.*

The (Δ, D)-graphs that meet the Moore Bound are known as **Moore Graphs**. Results about these graphs are surveyed in [31], and these graphs are discussed in both [4] and [11].

3.3.2 Planar graphs of diameter two

We say that a connected planar graph of maximum degree Δ and diameter D is a **planar** $\boldsymbol{(\Delta, D)}$**-graph** (and a **maximal planar** $\boldsymbol{(\Delta, D)}$**-graph** is defined similarly). In *Maximal planar graphs of diameter two* [35], Seyffarth proves that a maximal planar $(\Delta, 2)$-graph has at most $\frac{3}{2}\Delta + 1$ vertices, provided $\Delta \geq 8$. In this subsection, we give a brief overview of some of the ideas used to prove this bound. We mention the well-known fact that every face of a maximal planar graph of order at least 3 is bounded by a 3-cycle.

A recurring idea in [35] is to look for subgraphs that no maximal planar $(\Delta, 2)$-graph can have. One particular such subgraph is worth mentioning. A subgraph H of a maximal plane graph G is a ***k*-eye** if both the following conditions hold (see Figure 3.1):

- The subgraph H consists of two internally disjoint $x - y$ paths, A and B, of length at least $k+1$ (i.e., A and B each have at least k vertices besides x and y), and two vertices u_A and u_B such that u_A is adjacent to every vertex of A and u_B is adjacent to every vertex of B, and
- The interior of the cycle $A \cup B$ does not contain any vertex of G.

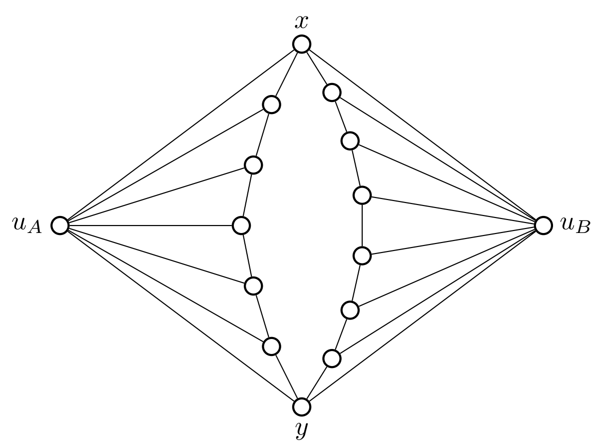


Figure 3.1: A 5-eye in a maximal planar graph.

If H is a k-eye, we say the vertices u_A and u_B are the **ends** of the k-eye. The next lemma illustrates why it is useful to consider k-eyes when trying to bound the order of a maximal planar $(\Delta, 2)$-graph.

Lemma 3.11. [35] *Let G be a maximal planar graph of maximum degree $\Delta \geq 6$. If G has a $(\frac{\Delta}{2}+1)$-eye, or a $\frac{\Delta}{2}$-eye whose ends are adjacent, then the diameter of G is strictly greater than 2.*

Proof sketch. Let H be a k-eye in a maximal plane graph G of diameter 2. Let u_A and u_B be the ends of H, let $A : x, a_1, \ldots a_p, y$ and $B : x, b_1, \ldots, b_q, y$ $(p, q \geq k)$ be the paths of the k-eye, and let C denote the cycle $A \cup B$. Note that by planarity, if P is an $a_i - b_j$ path of length at most 2 (for some $1 \leq i \leq p$ and $1 \leq j \leq q$), then P lies in $\mathrm{Int}[C]$, so we need only consider the geodesics that can be formed by adding edges in the interior of C. The first claim of the proof is that if i is in $\{2, \ldots, p-1\}$ and j is in $\{2, \ldots, q-1\}$, then a_i and b_j are not adjacent. If a_i and b_j were adjacent, then the only possible $a_1 - b_q$ paths of length 2 are a_1, a_i, b_q and a_1, b_j, b_q, so we assume without loss of generality that a_i is adjacent to both a_1 and b_q. But now the only possible $a_p - b_1$ geodesic is a_p, a_i, b_1. By similar arguments, it can be seen that every vertex of $C - \{a_i, x, y\}$ is adjacent to a_i. If H is a k-eye for k sufficiently large compared to Δ (certainly if $k \geq \frac{\Delta}{2}+1$), then a_i has degree strictly great than Δ, which is impossible. Similar arguments show that no matter what edges lie across $\mathrm{Int}(C)$, either two vertices end up distance three apart, or some vertex has degree greater than Δ, concluding the proof. *QED.*

There is another technique for bounding the order of a graph G of fixed maximum degree that is used in [35], namely, considering a small subgraph of G that dominates G. For example, we have the following result:

Lemma 3.12. **[35]** *Let G be a maximal planar $(\Delta, 2)$-graph of order n, with $\Delta \geq 6$. If a pair of adjacent vertices x and y dominate G, then $n \leq \frac{3}{2}\Delta + 1$.*

Proof sketch. Assume to the contrary that $\{x, y\}$ dominates G and $n > \frac{3}{2}\Delta + 1$. Let $S = N(x) \cap N(y)$, $A = N(x) - N(y)$ and $B = N(y) - N(x)$. By a counting argument on the degrees of x and y, it is shown that $|A| > \frac{\Delta}{2}$ and $|B| > \frac{\Delta}{2}$. Since the edge xy lies on two faces of G, there exists at least one vertex z in S. Thus we denote the 3-cycle $C : x, y, z$, and note that, without loss of generality, some vertex of A lies in $\text{Int}(C)$. Any geodesic (necessarily of length 2) between a vertex of A in $\text{Int}(C)$ and a vertex of B in $\text{Ext}(C)$ contains the vertex z (similarly, a geodesic between a vertex of A in $\text{Ext}(C)$ and a vertex of B in $\text{Int}(C)$ contains z). By the assumptions that $d(z) \leq \Delta$ and that $n > \frac{3}{2}\Delta + 1$, we can thus show that every vertex of both A and B lie in $\text{Int}(C)$. As the edge xy lies on a triangular face in the interior of C, there is a vertex w in $S - \{z\}$, such that all the vertices of $A \cup B$ lie in the interior of the 4-cycle $C' : x, z, y, w$. Using the fact that every face incident with either x or y is a triangle, we see that every vertex of A lies on one $w - z$ path, and every vertex of B lies on another $w - z$ path. Thus G contains a k-eye with ends x and y, for some $k > \frac{\Delta}{2}$, contradicting Lemma 3.11. *QED.*

Small dominating subgraphs are particularly easy to find in (maximal) planar graphs of diameter 2. It is well known that the minimum degree of a maximal planar graph with at least four vertices is 3, 4, or 5 — and that the neighbours of any vertex in a maximal planar graph are spanned by a cycle. Seyffarth further shows that a maximal planar graph of diameter 2 cannot have minimum degree 5. Thus, every maximal planar graph of diameter two is dominated by a 3-cycle or a 4-cycle. This provides a starting point for the proof of the main result of the paper:

Theorem 3.13. **[35]** *A maximal planar $(\Delta, 2)$-graph, $\Delta \geq 8$, has at most $\frac{3}{2}\Delta + 1$ vertices.*

The proof of this theorem is non-trivial, and we do not explore it here any further. We also note that the result is sharp: for all $\Delta \geq 8$, maximal planar $(\Delta, 2)$-graphs of order $\lfloor \frac{3}{2}\Delta + 1 \rfloor$ are constructed in [35].

The result of Theorem 3.13 can be generalised, as shown by Hell and Seyffarth in *Largest planar graphs of diameter two and fixed maximum degree* [23]:

Theorem 3.14. **[23]** *A planar $(\Delta, 2)$-graph, $\Delta \geq 8$, has order at most $\frac{3}{2}\Delta + 1$.*

The proof of Theorem 3.14 given in [23] makes use of Theorem 3.13.

Proof sketch. Assume to the contrary that there exists a planar $(\Delta, 2)$-graph with more than $\frac{3}{2}\Delta + 1$ vertices and $\Delta \geq 8$. Let G be such a graph with the maximum number of edges, and notice by Theorem 3.13 that G is not a maximal planar graph. Thus G has some face f, and vertices u and v incident with f, such that u and v are not adjacent in $G[f]$. By the maximality of G, there is a reason that u and v are not adjacent despite being incident to the same face. In particular, at least one of the following is true: at least one of u or v has degree Δ, or u and v are adjacent (so uv is an edge of G, but not $G[f]$). In both cases, Hell and Seyffarth derive a contradiction. *QED.*

Theorems 3.13 and 3.14 both require that the graphs in question have maximum degree $\Delta \geq 8$. In *Largest planar graphs and largest maximal planar graphs of diameter two* [41], Yang, Lin and Dai determine exact upper bounds for the orders of planar $(\Delta, 2)$-graphs and maximal planar $(\Delta, 2)$-graphs, for all $\Delta < 8$.

3.3.3 Maximal planar bipartite graphs and uses of separator theorems

A **maximal planar bipartite graph** is a planar bipartite graph to which the addition of any edge results in a graph that is not planar or not bipartite. It is well known that every face of a maximal planar bipartite graph is bounded by a 4-cycle (apart from the star graphs $K_{1,n}$, where n is any positive integer). In this subsection, we discuss some of the results in *The degree/diameter problem in maximal planar bipartite graphs* [12] by Dalfó, Huemer and Salas. We note that this paper begins with an excellent survey of the degree diameter problem in planar graphs, and a number of related classes of graphs.

Let u and v be vertices in a connected bipartite graph $G = (W \sqcup B, E)$ with partite sets W and B. Note that u and v are in the same partite set if and only if $d(u,v) = 0 \pmod 2$. In particular, if G has diameter 2, then any two vertices in different partite sets of G are adjacent — an observation that allows us to prove the first result of [12]:

Theorem 3.15. [12] *Let G be a maximal planar bipartite $(\Delta, 2)$-graph of order n. Then $n \leq 2\Delta + 2$, and $n = 2\Delta + 2$ if and only if $G = K_{2,\Delta}$.*

The graphs considered in this theorem are 4-face-degree regular, and have diameter 2. In Chapter 4, we show that a generalisation of Theorem 3.15 holds for all 2-edge-connected planar graphs of diameter D that are $2D$-face-degree regular.

We now turn our attention to maximal planar bipartite $(\Delta, 3)$-graphs, for which Dalfó, Huemer and Salas prove the following:

Theorem 3.16. [12] *If a maximal planar bipartite $(\Delta, 3)$-graph has order n, then:*

$$n \leq \begin{cases} 3\Delta - 1 & \textit{if } \Delta \textit{ is odd} \\ 3\Delta - 2 & \textit{if } \Delta \textit{ is even} \end{cases}$$

The proof of this theorem is spread across a number of different cases and lemmas. For example, in the case where $\Delta = 3$, the result follows by a simple counting argument using the Euler Characteristic Equation (Proposition 2.11) and both Handshaking Lemmas (Lemmas 2.1 and 2.9). We pay particular attention to one of the lemmas in [12]. The reason being that this lemma is false — so we illustrate a counter-example and show how the lemma can be patched.

Lemma 3.17. [12] *Let G be a maximal planar bipartite $(\Delta, 3)$-graph of order n. If some vertex of G has at least four neighbours of degree at least 3, then $n \leq 2\Delta + 2$.*

We sketch the proof, up to the point where we find issues. Embed G in the plane, and let W and B be partite sets of G. Let v be a vertex in G that has $k \geq 4$ neighbours of degree at least 3, and assume without loss of generality that v is in W. Let R be the region of the plane consisting of the faces of G incident with v and their boundaries (i.e., $R = \bigcup\{G[f] \cup f : f \text{ incident with } v\}$). In [12], it is claimed that the region R is delimited by a cycle C having at least $2k$ vertices, and that this cycle is chordless. It is here that we

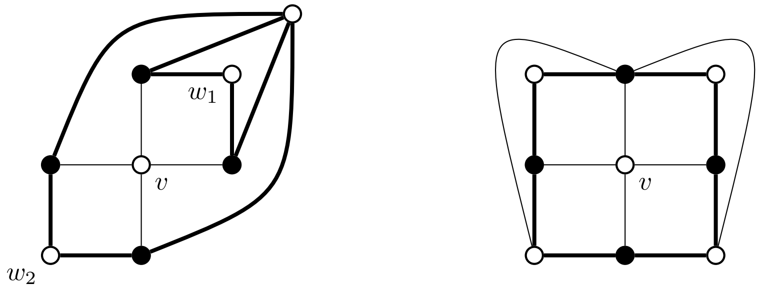


Figure 3.2: On the left, we show that the boundary of the region R (bold edges) may be a circuit that is not a cycle. On the right, the boundary-cycle (bold edges) of R has two chords.

find two issues. The first issue is a minor oversight, but the second is more severe and pursuing it reveals a counter-example to the lemma.

Problem 1. In a general maximal planar bipartite graph, it is possible for the region R to be bounded by a circuit that is not a cycle (see the left diagram in Figure 3.2). We show that under the specific conditions of Lemma 3.17, this cannot occur. First observe that if b is a vertex of degree 2 incident with v, then b lies in R, and neither edge incident with b lies on the boundary of R. In order for R to be bounded by a circuit that is not a cycle, there are two faces f_1 and f_2 incident with v, that are not both incident with any common edge, but that are both incident with some vertex w in $W - \{v\}$. Further, the vertices on the boundary of f_1 and f_2 all have degree at least 3 by the prior observation. Thus $\{v, w\} \subset W$ is a separator of G such that at least two components of $G - \{v, w\}$ contain some vertex of W. It is not possible that vertices of W in different components of $G - \{v, w\}$ have a common neighbour in B, so the diameter of G is greater than 3, a contradiction (for example, in Figure 3.2, we have $d(w_1, w_2) > 3$). This resolves the first omission and demonstrates that R is bounded by a cycle C, as is claimed in the original proof.

Problem 2. If the boundary of R is a cycle, then this cycle may have chords (see the right diagram in Figure 3.2), contrary to the claim made in [12] that C is chordless. By considering the case where v has $k = 4$ neighbours of degree at least 3, and the cycle C has two chords, we are able to find a counterexample to the statement of Lemma 3.17. We construct a counterexample G_Δ for each $\Delta \geq 9$ as follows: Let v be a vertex of G_Δ with exactly four neighbours b_1, b_2, b_3 and b_4. Add four new vertices w_1, w_2, w_3 and w_4 such that w_i is adjacent to both b_i and b_{i+1}, and let b_1 be adjacent to both w_2 and w_3 (subscripts taken mod 4). Further, add a vertex b_5 that is adjacent to w_2 and w_3, and call the bipartite graph of order 10 obtained thus far H. To H we add four additional sets of vertices B_1, B_2, B_3 and W_1. Let $|B_1| = |B_2| = \lfloor \frac{\Delta-4}{2} \rfloor$, $|B_3| = \lceil \frac{\Delta-4}{2} \rceil$ and $|W_1| = \Delta - 5$. Let each vertex of B_1 be adjacent to both v and w_3, let each vertex of B_2 be adjacent to v and w_2, make each vertex of B_3 adjacent to w_2 and w_3, and let the vertices of W_1 be adjacent to b_1 and b_5. This concludes the construction of G_Δ (see Figure 3.3). By inspection, G_Δ is a maximal planar bipartite graph of diameter 3, and the vertex v has four neighbours with degree at least 3. However, the order n_Δ of G_Δ is:

$$\begin{aligned} n_\Delta &= |V(H)| + |B_1| + |B_2| + |B_3| + |W_1| \\ &= 10 + \left\lfloor \frac{\Delta - 4}{2} \right\rfloor + \left\lfloor \frac{\Delta - 4}{2} \right\rfloor + \left\lceil \frac{\Delta - 4}{2} \right\rceil + (\Delta - 5) \\ &\geq \frac{5}{2}\Delta - 2 \end{aligned}$$

We conclude the presentation of the counterexample to Lemma 3.17 by noticing that $\frac{5}{2}\Delta - 2 > 2\Delta + 2$ for all $\Delta > 8$.

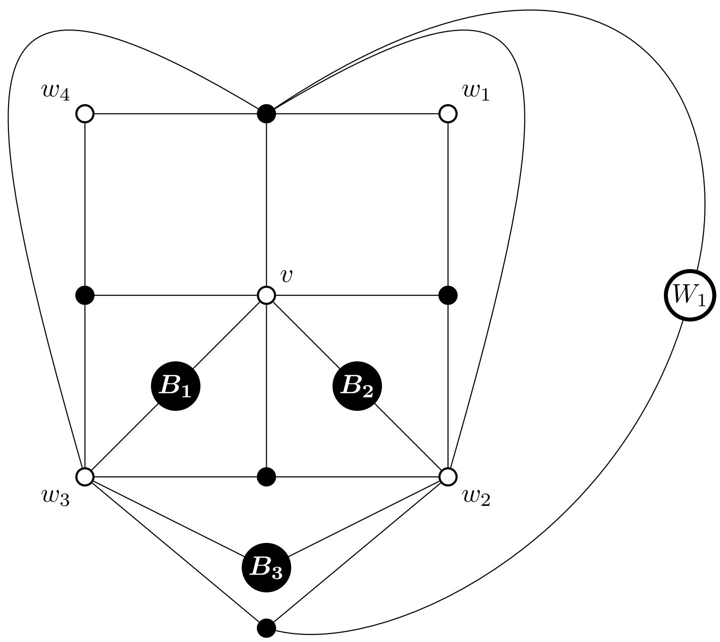


Figure 3.3: The graph G_Δ shown in the diagram provides a counterexample to the statement of Lemma 3.17.

Although Lemma 3.17 is false, the main result, Theorem 3.16, still holds. To show this, we replace Lemma 3.17 by the modified version below, Lemma 3.18, for which we give our own proof.

Lemma 3.18. *Let G be a maximal planar bipartite $(\Delta, 3)$-graph on n vertices. If G has a vertex with at least four neighbours having degree 3 or more, then $n \leq 3\Delta - 3$.*

Proof. We modify the approach used in [12]. Let the partite sets of G be W and B. Let v be a vertex of G, in W, with $k \geq 4$ neighbours of degree at least 3. Label these neighbours $b_1, b_2, \ldots, b_k$ so that they appear in clockwise order around v, and note that $N(v) \subseteq B$. Let R be the region consisting of all the faces incident with v and their boundaries. By the part of the discussion after Lemma 3.17 titled 'Problem 1', we see that R is bounded by a cycle, C, of length $2k$. We assume without loss of generality that v lies in the interior of C. Label the vertices of $V(C) \cap W$ as $w_1, w_2, \ldots, w_k$ such that w_i is adjacent to b_i and b_{i+1} (subscripts taken mod k).

If the cycle C is chordless, then the proof of Lemma 3.17 given in [12] demonstrates that $n \leq 2\Delta + 2$, so we assume that C has at least one chord. We consider two cases.

Case 1: $k = 4$.
We assume without loss of generality that the chord is b_1w_2. Since G has diameter 3 and w_1 and w_3 are both in W, we have that $d(w_1, w_3) = 2$. Thus w_1 and w_3 have a neighbour (from B) in common, so w_3 is adjacent

to b_1. Let H denote the graph with vertex set $V(C) \cup \{v\}$ obtained thus far, and note that $d_H(v) = 4$, $d_H(b_1) = 5$ and $d_H(w_2) = d_H(w_3) = 3$. In order to have a neighbour in B in common with each of v, w_1 and w_3, any vertex of $W - V(H)$ is adjacent to b_1. Similarly, in order to have a neighbour in W in common with b_1, b_2, b_3 and b_4, any vertex of $B - V(H)$ is adjacent to v, or to both w_2 and w_3. Thus the set $\{b_1, v, w_2\}$ dominates $V(G) - V(H)$, so we can bound the order of G:

$$\begin{aligned} n &= |V(H)| + |V(G) - V(H)| \\ &\leq 9 + |N(b_1) - V(H)| + |N(v) - V(H)| + |N(w_2) - V(H)| \\ &\leq 9 + (\Delta - 5) + (\Delta - 4) + (\Delta - 3) \\ &= 3\Delta - 3 \end{aligned}$$

This completes the proof of Case 1.

Case 2: $k > 4$.
Among all the chords of C, choose a chord b_iw_j minimising the distance $d_C(b_i, w_j)$. We may assume without loss of generality that the chord is of the form b_1w_j for some j in $\{2, 3, \ldots, \lfloor \frac{k}{2} \rfloor\}$.

We claim that $j = 2$. Assume to the contrary that $j > 2$. Let $P_1 : b_3, w_3, b_4, \ldots, w_j$ be the shorter of the two $b_3 - w_j$ paths in C and let $P_2 : w_j, b_1, v, b_3$ be a path in G. Note that the cycle $C' = P_1 \cup P_2$ separates w_k and w_2, and that w_2 and w_k have a common neighbour in B. The only vertex of C' to which both w_k and w_2 can be adjacent is b_1. But then b_1w_2 is a chord of C, contradicting the minimality of b_1w_2 and proving the claim.

Observe that the cycle $A : b_1, v, b_2, w_2$ separates w_1 from the set $\{w_3, w_4, \ldots, w_k\}$. Since w_1 has a neighbour (in B) in common with all these w_i ($3 \leq i \leq k$), the vertices w_i are all adjacent to b_1. In particular, we have shown that C has all $k - 2$ chords of the form b_1w_i where $2 \leq i \leq k - 1$. Let K denote the subgraph of G obtained thus far with vertex set $V(C) \cup \{v\}$, and note that $d_K(v) = k$ and $d_K(b_1) = k + 1$. The graph K is itself a maximal planar bipartite graph in which all the faces in the interior of C are bounded by cycles v, b_i, w_i, b_{i+1}, and all the faces in the exterior of C are bounded by cycles $b_1, w_i, b_{i+1}, w_{i+1}$ ($1 \leq i \leq k$, subscripts taken mod k). If w is a vertex of $W - V(K)$, then w has a neighbour in B in common with each vertex of $W \cap V(C)$. By planarity, this necessitates that w is adjacent to b_1. Similarly, any vertex b of $B - V(K)$ has a neighbour in W in common with all the b_i vertices in $B \cap V(C)$. Since G is planar, any such vertex b is adjacent to v. Thus we can bound the order of G:

$$\begin{aligned} n &= |V(K)| + |V(G) - V(K)| \\ &\leq (2k + 1) + |N(b_1) - V(K)| + |N(v) - V(K)| \\ &\leq (2k + 1) + (\Delta - (k + 1)) + (\Delta - k) \\ &= 2\Delta. \end{aligned}$$

We conclude the proof by observing that $3\Delta - 3 \geq 2\Delta$ whenever $\Delta \geq 3$. □

Lemma 3.18 does not give the best bound possible. However, for the purpose of correcting the proof of Theorem 3.16 in [12], Lemma 3.18 is a sufficient replacement for Lemma 3.17.

Dalfó, Huemer and Salas also consider the more general problem of bounding the order of maximal planar bipartite (Δ, D)-graphs for arbitrary diameter D. Using Lipton and Tarjan's separator theorem (Theorem 3.8), they show the following:

Theorem 3.19. [12] *The order n of a maximal planar bipartite (Δ, D)-graph is bounded above by approximately*

$$n \leq 3(2D+1)\left((\Delta-2)^{\lfloor D/2 \rfloor}+1\right).$$

Proof sketch. Let G be a maximal planar bipartite (Δ, D)-graph of order n. By Theorem 3.8, the vertices of G can be divided into sets A, B and S such that S separates A and B, $|A| \leq \frac{2}{3}n$, $|B| \leq \frac{2}{3}n$ and $|S| \leq 2D+1$. Since S separates A and B, and G has diameter D, either every vertex of A or every vertex of B is distance at most $\frac{D}{2}$ from S — let us assume without loss of generality that this holds for A. The rest of the proof involves finding an upper bound on the maximum number k of vertices within distance $\frac{D}{2}$ of some vertex in a maximal planar bipartite graph of maximum degree Δ (i.e., finding the maximum order of a ball of radius $\frac{D}{2}$), by explicitly constructing such a maximal ball. In particular, k is approximately $(\Delta-2)^{\lfloor D/2 \rfloor}+1$. Finally, since $\frac{1}{3}n \leq |A \cup S|$, and every vertex of $A \cup S$ is within distance at most $\frac{D}{2}$ of one of the $2D+1$ or fewer vertices of S, we deduce that $|A \cup S| \leq k(2D+1)$, from which the result follows. *QED.*

An improvement on the bound in Theorem 3.19 is also made using both Tishchenko's N-separator theorem [37], and the well-known Four Colour Theorem, but we will not explore this further. We remark that the technique used in the proof of Theorem 3.19 was first used in *Large planar graphs with given diameter and maximum degree* [20] by Fellows, Hell and Seyffarth — where it was used to prove the following result:

Theorem 3.20. [20] *If G is a planar (Δ, D)-graph with order n and $\Delta \geq 4$, then*

$$n \leq 3(2D+1)\left(2\Delta^{\lfloor D/2 \rfloor}+1\right).$$

Note that the proof Theorem 3.19 in [12] makes explicit use of the structure of maximal planar bipartite graphs, and so the bound obtained for these graphs is much smaller than the bound for arbitrary planar graphs given by Theorem 3.20.

In *The degree-diameter problem for outerplanar graphs* [15], Dankelmann, Jonck and Vetrik also use a separator theorem to address the degree diameter problem for outerplanar graphs. They demonstrate that every maximal outerplanar graph G of order n has a separator S with only two vertices, such that each component of $G-S$ has at least $\frac{n}{3}-1$ vertices, and prove the following bound:

Theorem 3.21. [15] *If an outerplanar (Δ, D)-graph has order n, $D \geq 2$ and Δ sufficiently large, then there exist constants C_1 and C_2 such that:*

$$n \leq \begin{cases} \Delta^{\frac{D}{2}} + C_1\Delta^{\frac{D}{2}-1} & \text{if } D \text{ is even} \\ 3\Delta^{\frac{D-1}{2}} + C_2\Delta^{\frac{D-1}{2}-1} & \text{if } D \text{ is odd} \end{cases}.$$

Dankelmann et al. show by construction that this bound is asymptotically sharp, and cannot be improved if we assume that the graph in question is maximal outerplanar.

Chapter 4

Plane graphs with large faces and small diameter

This chapter contains the content of the paper *Plane graphs with large faces and small diameter* [19] by B. Du Preez, as it was submitted to the Australasian Journal of Combinatorics. The paper has been published in the Australasian Journal of Combinatorics, and the journal's version of the paper (which has undergone some changes during review) can be found at
`https://ajc.maths.uq.edu.au/pdf/80/ajc_v80_p401.pdf`
The only differences between this chapter and the submitted version are the omission of title page, acknowledgements and bibliography (the full acknowledgements and bibliography are given at the end of the thesis), changes in the numbering of sections, figures, references and results to fit the numbering scheme of the overall thesis, minor formatting changes (including the inclusion of the [🔥] symbol) and minor changes to the style and grammar suggested by Examiners.

Rationale for the inclusion of this publication

In this thesis, we investigate the metric properties of ρ-face-degree regular plane graphs with some additional connectivity constraints (all graphs we consider are at least 2-edge-connected). This paper opens our investigation by determining how large ρ can be as a function of diameter for these graphs. The bound obtained provides context for both the existing literature on the degree diameter problem in face-degree regular plane graphs, and the main result of Chapter 5 of this thesis.

Face-boundaries and short cycles (defined in Section 1 below) in 2-edge-connected plane graphs will be of great importance throughout the thesis. Thus, Sections 3 and 4 of this paper — which give a number of basic results and observations about these objects — provide part of the foundation upon which the rest of the thesis is built.

Abstract

The face-degree of a face in a connected plane graph is the minimum length of a closed walk that spans all the vertices and edges of the boundary of the face. A plane graph is ρ-face-degree regular if every face has face-degree ρ. In this paper, the structure of 2-edge-connected plane graphs with large minimum face-degree is studied. We give an upper bound on the minimum face-degree of a plane graph with given radius, and characterise the graphs meeting this bound. We show that the girth and minimum face-degree of a plane graph coincide if either of these parameters are at least twice the diameter of the graph. Further, we characterise all planar generalised polygons (bipartite graphs whose girth is twice their diameter). The well-studied degree diameter problem is the problem of determining the maximum possible order of a graph given both its maximum degree and diameter. The structural results in this paper solve the degree diameter problem for plane graphs of diameter D which are either $2D$-face-degree regular or $(2D+1)$-face-degree regular.

4.1 Definitions

Most of the definitions and conventions we use can be found in Diestel's *Graph Theory* [16]. All graphs in this paper are finite and simple. We assume the reader has some familiarity with the topology of the plane.

Let G be a graph, let u and v be vertices of G, and let W be a walk in G. If a walk starts and ends at the same vertex and contains no edge repetitions, we call it a **circuit** (Figure **??** illustrates five circuits). The **length** of the walk W, which we denote $\ell(W)$, is the total number of edges that appear in the walk, counting repeated edges. A $u-v$ **geodesic** is a $u-v$ path of minimum length. The **distance** between two vertices u and v in G, denoted $d_G(u,v)$, is the length of a $u-v$ geodesic in G. We will omit the subscript if the graph in question is clear from context. The **girth** of G, denoted $g(G)$, is the minimum length of any cycle in G. The **eccentricity** of a vertex is the maximum distance between it and any other vertex of the graph. The **radius** and **diameter** of a graph are the maximum and minimum eccentricities of any vertex, respectively. A graph is **self-centred** if its radius and diameter are equal.

A **separator** of a connected graph is a subset of the vertex set whose removal disconnects the graph. A **separating cycle** is a cycle, the vertex set of which is a separator.

If $G = (V,E)$ is a connected graph with S and T subsets of V, then the distance between S and T is $d(S,T) = \min\{d(u,v) : u \in S, v \in T\}$. Given a vertex v, we write $d(v,S)$ for the distance $d(\{v\},S)$. It follows directly from these definitions that $d(S,T) = 0$ if and only if $S \cap T$ is non-empty, and that $d(S,T) > 1$ implies the induced subgraph $G[S \cup T]$ is disconnected.

Throughout the paper, we implicitly make use of the Jordan Curve Theorem. If X is an open subset of the plane, then a **region** of X is a maximal connected subset of X. We use the notation $||x-y||$ for the euclidean distance between points x and y in the plane. A graph is **planar** if it can be embedded in the plane. An embedding of a planar graph is called a **plane graph**, and the regions of the complement of the plane graph are called **faces**. Different embeddings of the same planar graph can create plane graphs with different faces (see Figure 4.1), so we will work with a fixed embedding wherever ambiguity can arise.

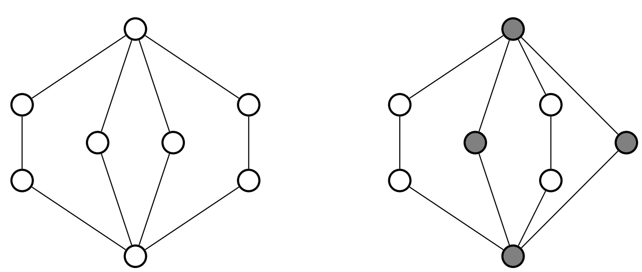


Figure 4.1: Two different embeddings of the same planar graph yield plane graphs with different faces. In the graph on the left, there is a face bounded by a 4-cycle, but every face of the graph on the right is bouned by a 5-cycle.

For the following definitions, let $G = (V,E,F)$ be a connected plane graph (where F is the set of faces of G) and f a face of G. If v is a vertex of G, we let $d_G(v)$ denote the degree of v in G. An edge or vertex of G is **incident** with the face f if it is contained in the topological boundary of f. We denote by $G[f]$ the subgraph consisting of the edges and vertices incident with f, and say that $G[f]$ **bounds** the face f. If some circuit or cycle bounds a face in a plane graph, we call it a **face-circuit** or **face-cycle** respectively. The subgraph $G[f]$ bounding f can be traversed by a closed walk. The length of the shortest such closed walk traversing $G[f]$ is the **face-degree** $\mathcal{E}(f)$ of f. We denote the **minimum face-degree** of G by $\mu(G) = \min\{\mathcal{E}(f) : f \in F\}$. We say G is ρ-**face-degree regular** if every face of the graph has face-degree ρ.

Let G be a 2-edge-connected plane graph of diameter D and minimum face-degree μ. A cycle C of G is a **short-cycle** if $\ell(C) < \mu$ (in Figure 4.1, the 4-cycle on the grey vertices is a short-cycle of the plane graph on the right).

Given a Jordan curve C in the plane (i.e., C is the image of an injective, continuous map from the circle to the plane), we denote the bounded region of $\mathbb{R}^2 - C$ by $\mathrm{Int}(C)$, the unbounded region by $\mathrm{Ext}(C)$, and let $\mathrm{Int}[C] = \mathrm{Int}(C) \cup C$ and $\mathrm{Ext}[C] = \mathrm{Ext}(C) \cup C$. Note that any cycle of a plane graph induces a Jordan curve. If C is a cycle of a plane graph G, then $G[\mathrm{Int}[C]]$ is the subgraph of G that consists of all the edges and vertices contained in $\mathrm{Int}[C]$, and $G[\mathrm{Ext}[C]]$ is defined similarly as the subgraph of all edges and vertices in $\mathrm{Ext}[C]$. If a cycle has vertices in both its interior and its exterior, we call it a **Jordan separating cycle**. Clearly a Jordan separating cycle is itself a separating cycle, but not every separating cycle of a plane graph is a Jordan separating cycle.

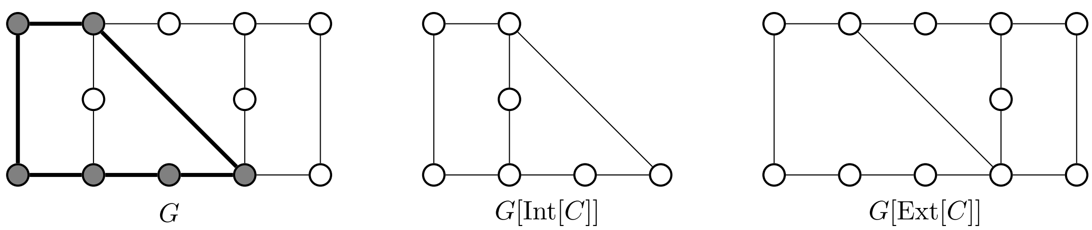


Figure 4.2: The bold cycle C in G and the subgraph $G[\mathrm{Int}[C]]$. Note that C is a Jordan separating cycle.

4.2 Background

In this paper, we investigate plane graphs that are extremal with respect to their face-degree and diameter, and demonstrate the close relationship between girth and face-degree in these extremal graphs. This investigation is motivated by two well-studied topics in graph theory: Moore graphs and the degree diameter problem. The degree diameter problem is the problem of determining the maximum possible order n of a graph with diameter D and maximum degree Δ, and Moore graphs are those graphs that obtain the 'trivial upper bound' for the degree diameter problem (this upper bound is known as the Moore Bound). Miller and Širáň have written a comprehensive survey of both topics in [31]. For plane graphs with $D = 2$ and $\Delta \geq 8$ in which every face is a triangle, Seyffarth has shown that $n \leq \frac{3}{2}\Delta + 1$. In [12], Dalfó, Huemer and Salas demonstrated that plane graphs with $D = 2$ in which every face is a quadrangle satisfy $n \leq \Delta + 2$. They also solved the degree diameter problem for plane graphs with $D = 3$ in which every face is a quadrangle, showing that $n \leq 3\Delta - 1$ for Δ odd and $n \leq 3\Delta - 2$ for Δ even.

Extremal plane graphs have also been considered outside the framework of the degree diameter problem. The relationship between the radius, maximum face-degree and order of a plane graph was investigated by Ali, Dankelmann and Mukwembi in [1], where they determined that a 3-connected plane graph of radius

r, order n and maximum face-degree M satisfies $r \leq \frac{n+5M+4}{6}$. In [17], Dowden considered extremal plane graphs that do not contain either C_4 or C_5 as a subgraph, showing that a C_4-free planar graph of order n has at most $\frac{15}{7}(n-2)$ edges, and that a C_5-free planar graph has at most $\frac{12n-33}{5}$ edges. Lan, Shi and Song consider the more general problem of bounding the maximum number of edges in a planar graph that does not contain some graph H as a subgraph in [28], and demonstrate a number of conditions under which an H-free planar graph of order n can be maximal planar.

4.3 Faces of 2-edge-connected plane graphs

The following Observation is well known (see, for example, [16]).

Observation 4.1. [ ] A plane graph is 2-connected if and only if each face is bounded by a cycle.

We prove a similar result: that every face of a 2-edge-connected plane graph is bounded by a circuit, and hence the closed walk traversing the boundary of any face in a 2-edge-connected plane graph contains no repeated edges. Recall the following results:

Lemma 4.2. [16] *Let G be a plane graph consisting of three internally disjoint $u-v$ paths P_1, P_2 and P_3 (i.e., G is a theta graph). Then G has exactly three faces, bounded by the cycles $P_1 \cup P_2$, $P_1 \cup P_3$, and $P_2 \cup P_3$.*

Lemma 4.3. [16] *Let G be a plane graph, H a subgraph of G, and f a face of G. The face f (considered as a region of the plane) is contained in some face f' of H. Further, if H contains $G[f]$ as a subgraph, then $f' = f$.*

Lemma 4.4. [16] *Let G be a plane graph, and e an edge of G. If e is in a cycle C of G, then e is incident with exactly two faces of G, one of which is contained in the interior of C, and the other of which is contained in the exterior. If e is not in any cycle of G, then e is incident with exactly one face of G.*

Lemma 4.5. [16] *Let G_1 and G_2 be two disjoint plane graphs. Let e be an edge incident with a vertex v_1 of G_1 and a vertex v_2 of G_2, and let f be the face of $G_1 \cup G_2$ containing e. Then $f - e$ is a face of $G_1 \cup G_2 \cup \{e\}$.*

A graph is 2-edge-connected if and only if every edge of the graph lies on some cycle, and hence Lemma 4.4 yields the following well-known observation:

Observation 4.6. [ ] Every edge of a 2-edge-connected plane graph lies on the boundary of exactly two faces.

The following result and its corollary are known (a strengthening of both is given as an exercise in [4]), but a literature proof is elusive and so we include a proof here for completeness.

Theorem 4.7. [ ] *A plane graph is 2-edge-connected if and only if every face of the graph is bounded by a circuit.*

Proof. If every face of a plane graph is bounded by a circuit, then every edge of the graph lies on a circuit, and hence also on a cycle. Thus the graph is 2-edge-connected.

Let $G = (V, E, F)$ be a 2-edge-connected plane graph, let f be a face of G, and let $H = G[f]$ be the subgraph bounding f. We prove that H is a circuit by showing that H is connected, and can be formed as a union of cycles, none of which have any edges in common.

By Lemma 4.3, the region f is a face of the subgraph H.

Claim 1: H is connected.
Assume to the contrary that H is not connected. Let X and Y be two components of H, and let x be a vertex of X, and y a vertex of Y. Since X and Y are compact disjoint sets in the plane, there exists a positive real number $\delta = \min\{||s - t|| : s \in X,\ t \in Y\}$. Create a new plane graph by adding the edge $e = xy$ to H, such that $e - \{x, y\}$ is contained entirely in f. This can be done since vertices x and y are both incident with f. We can consider e to be a continuous, injective function $e : [0, 1] \to \mathbb{R}^2$ such that $e(0) = x$ and $e(1) = y$.

We can choose e such that it contains some straight line segment S as follows. Pick a point q in e such that $||q - z|| \geq \frac{\delta}{4}$ for all points (on edges and vertices) z in $X \cup Y$. By the choice of δ, such a point q exists. Let D_q be the closed disk of radius $\frac{\delta}{4}$, centred at q. Since D_q is closed, the set $e^{-1}(D_q)$ is a compact subset of $[0, 1]$ with minimum value $m > 0$ and maximum value $M < 1$. We modify the edge e by replacing the curve $e([m, M])$ by the straight line segment from $e(m)$ to $e(M)$, yielding a new edge e' from x to y, which also satisfies $(e' - \{x, y\}) \subseteq f$. Thus we assume e contains some straight line segment S. We have by Lemma 4.5 that $f - e$ is a face of $H \cup e$.

Let p be the midpoint of the straight line segment S, and let D_p be an open disk centred at p of radius sufficiently small that $D_p \cap (H \cup e) \subseteq S$. Such a radius exists by compactness of e and H. The set $D_p - e$ has exactly two regions, both of which are contained in f. Let a be a point of one region of $D_p - S$, and b be a point of the other region. We can consider the points a, b and p as vertices of some plane graph containing $H \cup e$, and the straight line segments from p to a, and p to b, as edges ap and bp of this plane graph. Repeatedly applying Lemma 4.5 shows that $f' = f - (e \cup ap \cup bp)$ is a face of the plane graph H' formed by adding to H the vertices a, b and p as well as the edges e, ap and bp.

Both a and b are incident with the face f', and hence an edge ab can be added to H' such that every point of $ab - \{a, b\}$ is lies in f'. The cycle $C = a, b, p, a$ of $H' \cup ab$ is contained entirely in f, and the edge e crosses C exactly once (at p). Hence x and y are in different regions of $\mathbb{R}^2 - C$, so any $x - y$ path in G crosses C, but this is impossible as C is contained in a face f of G, contradicting that G is connected.

Claim 2: H is a union of cycles.
It suffices to show that every edge of H lies on a cycle in H. Assume for the sake of contradiction that e is an edge of H which does not lie on any cycle of H. By Lemma 4.4, the edge e is incident with a single face of H. This single face of H with which e is incident is f, by Lemma 4.3. Since f is also a face of G, it is the only face of G with which e is incident. But then e cannot be contained in any cycle of G, contradicting Observation 4.6.

Claim 3: No two distinct cycles of H share an edge.

Assume to the contrary that C_1 and C_2 are two distinct cycles of H such that there exists an edge uv in $C_1 \cap C_2$. Choose two vertices u' and v' of $C_1 \cap C_2$, such that the $u' - v'$ path in C_2 not containing the edge uv does not contain any vertex of C_1 other than u' and v', and call this path P (it is possible that $u' = u$ and $v' = v$). Let Q be the $u' - v'$ path in C_1 containing the edge uv, and let Q' be the $u' - v'$ path in C_1 not containing uv. By Lemma 4.2, the graph $C_1 \cup P$ has three faces, and by Lemma 4.3, one of these faces contains f. Assume without loss of generality that f is contained in the face of $C_1 \cup P$ bounded by $P \cup Q$. The face containing f is not incident with any edge of Q', and so f is not incident with any edge of Q', contradicting the fact that Q' is contained in $H = G[f]$ and proving the claim.

Since H is a connected graph which can be formed as the union of edge-disjoint cycles, H is a circuit. □

Corollary 4.8. [ ] *Let f be a face of a plane graph G. If G is 2-edge-connected, then the face-degree of f is the number of edges in the subgraph $G[f]$.*

Note that any circuit of length 5 or less is a cycle.

Corollary 4.9. [ ] *If every face of a 2-edge-connected plane graph has face-degree at most 5, then the graph is 2-connected.*

Proof. Let $G = (V, E, F)$ be a 2-edge-connected plane graph such that $\max\{\mathcal{E}(f) : f \in F\} \leq 5$. By Theorem 4.7, every face of G is bounded by a circuit. Any circuit of length at most 5 is a cycle, and so every face of G is bounded by a cycle. Thus, by Observation 4.1, we have that G is 2-connected. □

4.4 Cycle length and minimum face-degree of plane graphs

In a 2-edge-connected plane graph, the girth is bounded above by the minimum face-degree, as every face is bounded by either a cycle, or a circuit (and every circuit contains a cycle). However, the difference between the minimum face-degree and the girth can be arbitrarily large, as the two graphs in Figure 4.3 show. Given any positive integer $\rho \geq 3$, there is a ρ-face-degree regular 2-edge-connected graph containing a 4-cycle. Given any odd positive integer $\rho \geq 3$, there is a ρ-face-degree regular 2-edge-connected graph containing a 3-cycle.

However, if a 2-edge-connected plane graph does have a cycle of length strictly less than its minimum face-degree, that cycle is a Jordan separating cycle (this is not a new result, but I have not been able to find a reference for it).

Lemma 4.10. [ ] *Every short-cycle of a plane graph is a Jordan separating cycle.*

Proof. Let G be a plane graph and let $C = v_1, v_1, \ldots, v_k, v_1$ be a short-cycle of length $k < \mu$ in G. Certainly C is not a face-cycle as every face-cycle has length μ or greater.

We claim that $\text{Int}(C)$ contains at least one vertex. Assume to the contrary that it does not, and consider the induced subgraph $H = G[\text{Int}[C]]$. All the faces of H, except the external face bounded by C, are also

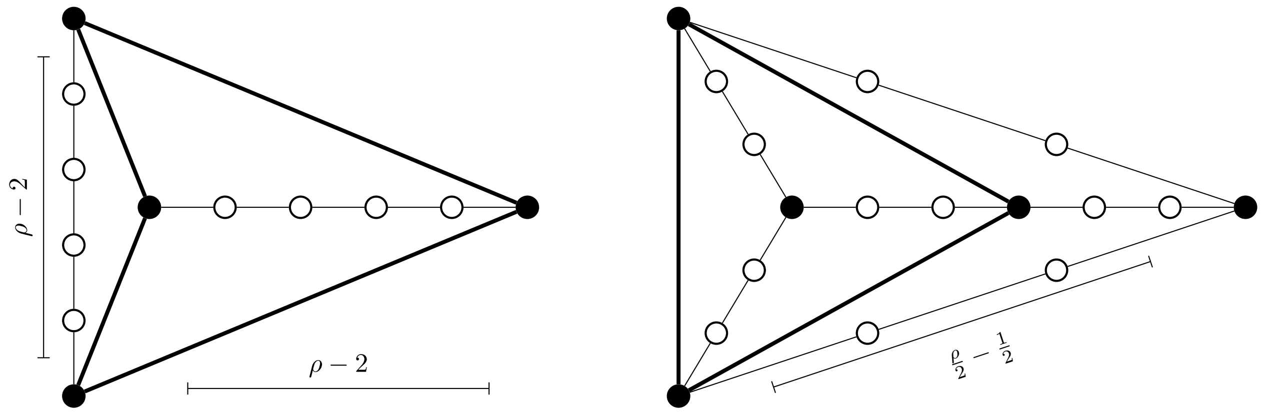


Figure 4.3: Left: let all paths of white internal vertices, starting and ending at a black vertex, have length $(\rho - 2)$. This creates a ρ-face-degree-regular graph with girth ≤ 4 for any $\rho \geq 3$. Right: Let all paths between black vertices having only white internal vertices have length $(\frac{\rho}{2} - \frac{1}{2})$. Then we obtain a ρ-face-degree-regular graph with girth 3 for any odd ρ.

faces of G, and hence all faces of H except the external face have degree at least μ. Since $V(H) = V(C)$ and C is a subgraph of H, the subgraph H is 2-connected. By Observation 4.1, every face of H is bounded by a cycle. But there are only k vertices in H with which to construct a cycle. Thus every face of H, including all the interior faces, which are faces of G, is bounded by a cycle of length at most k. This contradicts the fact that every face of G has degree at least μ.

The same argument shows that the exterior of C also contains a vertex, so C is a Jordan separating cycle. $\square$

Before presenting the main result of this section, we need some extra machinery. Given a connected graph G, with any spanning tree T of G, there is a (possibly empty) collection of cycles of G called **fundamental cycles** (with respect to T). A fundamental cycle is a cycle of G formed by the addition of a single edge of $E(G) - E(T)$ to T. Given an edge e of $E(G) - E(T)$, denote by C_e the fundamental cycle induced in $T + e$. Note that if G has any cycles at all, it has at least one fundamental cycle. Further discussions of fundamental cycles in plane graphs can be found in both Bondy and Murty's *Graph Theory* [4] and Mohar and Thomassen's *Graphs on Surfaces* [33]. A **radius-preserving spanning tree** of G is a spanning tree T of G such that both T and G have the same radius. We use the following well-known lemma, which follows from the discussion on breadth-first-search in [4].

Lemma 4.11. [4] *Every connected graph has a radius-preserving spanning tree.*

The following simple lemma has almost certainly appeared in the literature before, but we give a proof here for completeness.

Lemma 4.12. [🔥] *Let be G a connected graph and T be tree that spans G and has radius r. If C_e is a fundamental cycle of G with respect to T, then the length of C_e is at most $2r + 1$.*

Proof. Every path in T is a geodesic. Thus any path in T has length at most $2r$. Every fundamental cycle is formed by the addition of a single edge to a path in T, and hence has length at most $2r + 1$. $\square$

It is well known and easy to see that if a graph has diameter D and girth g, then the diameter bounds the girth from above by the inequality $g \leq 2D + 1$. The next result shows that the same constraint holds if we replace the girth by the minimum face-degree.

Theorem 4.13. *If G is a 2-edge-connected plane graph of radius r, then $\mu(G) \leq 2r + 1$. This bound is sharp.*

Proof. Assume for the sake of contradiction that $G = (V, E, F)$ is a 2-edge-connected plane graph with minimum face-degree μ and radius r, and that $\mu > 2r + 1$. By Lemma 4.11, the graph has a spanning tree T of radius r. By Lemma 4.12, every fundamental cycle of G with respect to T has length at most $2r + 1$. By Observation 4.10, every fundamental cycle with respect to T is a Jordan separating cycle.

Choose an edge uv in $E(G) - E(T)$ that minimises the number of vertices in $\text{Int}(C_{uv})$. Since T is spanning and connected, and G is planar, any vertex in the interior of C_{uv} is connected to C_{uv} by some path of T in $\text{Int}[C_{uv}]$. Since the only cycle in $T + uv$ is C_{uv}, there is some vertex in the interior of C_{uv}, say x, such that $d_{T+uv}(x) = 1$ (see part (1) of Figure 4.4).

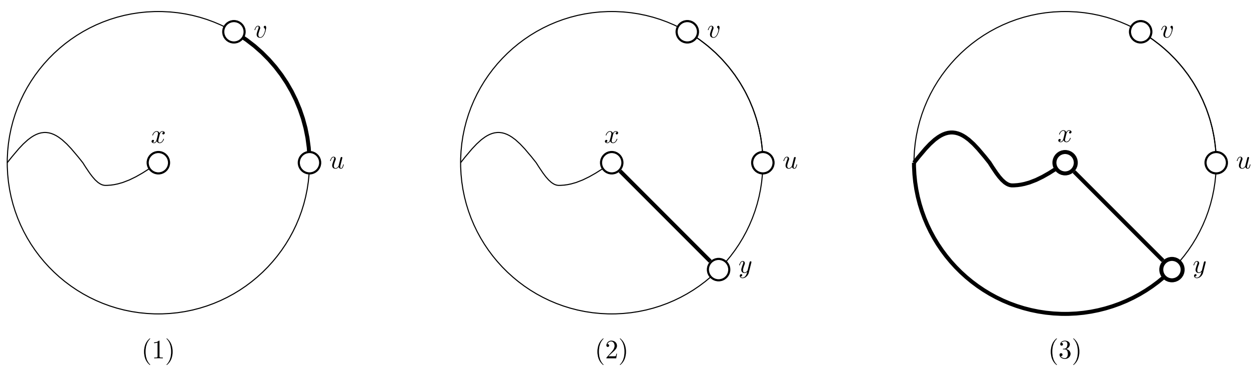


Figure 4.4: (1) The fundamental cycle C_{uv} in $T + uv$, with end vertex x in its interior. The edge uv is bold. (2) The end vertex in $T + uv$ is adjacent to some vertex y in $\text{Int}[C_{uv}]$. The vertex y is not necessarily part of the cycle itself and may lie in $\text{Int}(C)$. The edge xy is bold. (3) One of the regions induced by adding the edge xy to $T + uv$ is bounded by the fundamental cycle C_{xy}, which is bold.

Since G is 2-edge-connected, the vertex x has degree at least two in G. Thus there is some edge xy in $E(G) - E(T)$ that lies inside $\text{Int}[C_{uv}]$ (see part (2) of Figure 4.4).

As the induced subgraph $(T + uv)[\text{Int}[C_{uv}]]$ is connected, the addition of xy to $T + uv$ divides the interior of C_{uv} into two regions - exactly one of which has the edge uv on its boundary (see part (3) of Figure 4.4).

The region not containing the edge uv on its boundary contains only edges of T, and the edge xy, on its boundary, and hence is bounded by a fundamental cycle. Thus $T + xy$ contains a fundamental cycle C_{xy} that has fewer vertices in its interior than C_{uv} does, contradicting the minimality of C_{uv}.

The bound is sharp as the cycle C_{2k+1} has radius and diameter k, and both faces of the cycle have face-degree $2k + 1$. □

4.5 Extremal graphs for Theorem 4.13

In this section we will show that the odd cycle C_{2D+1} is the only graph that is 2-edge-connected, has diameter D and has minimum face-degree $2D+1$.

Lemma 4.14. *Let G be a 2-edge-connected plane graph with diameter D and minimum face-degree μ. If $\mu = 2D+1$, then G is self-centred.*

Proof. Assume to the contrary that G is a 2-edge-connected plane graph with minimum face-degree $\mu = 2D+1$, diameter D and radius $r < D$. By Lemma 4.11, the graph G has a radius-preserving spanning tree of radius at most $D-1$. Thus by Theorem 4.13, the minimum face-degree satisfies $\mu \leq 2D-1 < 2D+1$, a contradiction. □

We will need two lemmas, the first was originally proven by Buckley [6], and a proof can be found in a paper by Jarry and Laugier [25]. The second was proven by Harary and Norman [22], and a proof is given in Buckley and Harary's *Distance in Graphs* [8].

Lemma 4.15. [6, 25] *If $G = (V, E)$ is a 2-connected graph of diameter D, then:*

$$\left\lceil \frac{(|V|-2)D-1}{D-1} \right\rceil \leq |E|$$

.

Lemma 4.16. [8, 22] *The centre of a graph is contained within a single maximal non-separable subgraph (i.e., a block of the graph).*

Theorem 4.17. *If G is a 2-edge-connected plane graph with diameter D and minimum face-degree $2D+1$, then G is the odd cycle C_{2D+1}.*

Proof. Let $G = (V, E)$ be a 2-edge-connected plane graph of diameter D and minimum face-degree $2D+1$. The graph G is self-centred by Lemma 4.14. Thus by Lemma 4.16, the graph G lies entirely within a single maximal non-separable subgraph (of itself), and so is 2-connected. By Lemma 4.15, we obtain the following inequalities:

$$\frac{(|V|-2)D-1}{D-1} \leq \left\lceil \frac{(|V|-2)D-1}{D-1} \right\rceil \leq |E|. \tag{4.1}$$

Remember that the symbol $\mathcal{E}(f)$ denotes the face-degree of the face f. Noting that the minimum number of edges bounding any face is μ, we see that:

$$\mu|F| \leq \sum_{f \in F(G)} \mathcal{E}(f) = 2|E|,$$

from which we deduce:

$$|F| \leq \frac{2}{\mu}|E|. \tag{4.2}$$

Substituting inequality (4.2) into the equation for the Euler characteristic of G, i.e., $|V| - |E| + |F| = 2$, we get another inequality:

$$|E| + 2 \leq |V| + \frac{2}{\mu}|E|.$$

And hence:

$$|E| \leq \frac{\mu(|V| - 2)}{\mu - 2}. \tag{4.3}$$

Combining the inequalities (4.1) and (4.3), and substituting $\mu = 2D + 1$, we see that G satisfies the following inequality:

$$\frac{(|V| - 2)D - 1}{D - 1} \leq \frac{(|V| - 2)(2D + 1)}{2D - 1}.$$

With some rearrangement, we finally bound the order of G:

$$|V| \leq 2D + 1.$$

Since G is 2-connected, every face of G is bounded by a cycle per Observation 4.1. As $\mu = 2D + 1$, the graph G contains the cycle C_{2D+1} as a subgraph. Since $|V| \leq 2D + 1$, we conclude that G is the cycle C_{2D+1}. □

4.6 Planar generalised polygons

A graph of diameter D is a **generalised polygon** if it is bipartite and has girth $2D$. The structure of generalised polygons is explored in Godsil and Royle's *Algebraic Graph Theory* [21]. In this section we characterise planar generalised polygons. Much like the cycle C_{2D+1} is the only 2-edge-connected planar graph that has diameter D and is $(2D + 1)$-face-degree regular (Theorem 4.17), planar generalised polygons are exactly the 2-edge-connected planar graphs that have diameter D and are $2D$-face-degree regular (Corollary 4.32). Thus the results of this section demonstrate that planar generalised polygons are a useful class of 'nearly extremal' planar graphs.

As Figure 4.3 illustrates, we cannot normally use face-degrees to bound the girth of a graph from below. However, we show that if the face-degrees are all sufficiently high, then the girth is bounded below by the minimum face-degree. First, we need some lemmas. In the 1870's, Kempe observed that if every vertex of a plane graph has even degree, then the faces of the graph may be 2-coloured such that no two faces of the same colour share an edge [27]. Hence, by plane duality, we obtain the next well-known observation.

Observation 4.18. [ ] If every face of a plane graph has even face-degree, then the graph is bipartite.

The next two lemmas are familiar to many.

Lemma 4.19. [ ] *Let $G = (V, E)$ be a connected graph, and $S \subset V$ a separator of G. If two vertices u and v of G are in different components of $G - S$, then $d(u, v) \geq d(u, S) + d(v, S)$.*

Proof. Let P be a $u - v$ geodesic. As S separates u and v, there exists some vertex s in $S \cap P$, so the edges of P can be partitioned into a $u - s$ path $P[u, s]$ and an $s - v$ path $P[s, v]$. Hence we obtain the following sequence of inequalities:

$$d(u,v) = \ell(P) = \ell(P[u,s]) + \ell(P[s,v]) \geq d(u,S) + d(v,S)$$

□

Lemma 4.20. [♣] *If $G = (V,E)$ is a connected graph of diameter D, and there are subsets A, B and S of V such that $\{A,S,B\}$ is a partition of V and $d(A,B) > 1$, then either $\max_{v\in A}\{d(v,S)\} \leq \lfloor \frac{D}{2} \rfloor$ or $\max_{v\in B}\{d(v,S)\} \leq \lfloor \frac{D}{2} \rfloor$.*

Proof. Assume to the contrary that there exist vertices u in A and v in B such that both $d(u,S) > \lfloor \frac{D}{2} \rfloor$ and $d(v,S) > \lfloor \frac{D}{2} \rfloor$. The set S separates u and v, so by Lemma 4.19, we have the following inequalities:

$$d(u,v) \geq d(u,S) + d(v,S) \geq 2\left(\left\lfloor \frac{D}{2} \right\rfloor + 1\right) > D.$$

□

Observation 4.21. If B and C are two cycles of a plane graph such that C lies in the interior of B, then all vertices v in the interior of C satisfy $d(v,C) \leq d(v,B)$.

The prior observation follows from the fact that any $v - B$ geodesic contains some vertex of C.

The next theorem shows that if the minimum face-degree of a plane graph with diameter D is large enough, then the graph contains no short cycles.

Theorem 4.22. *Let G be a 2-edge-connected plane graph of diameter D. If $\mu(G) = 2D$ then $g(G) = 2D$.*

Proof. Assume for the sake of contradiction that $g < 2D$, and let B be a short-cycle in G. By Lemma 4.10, the cycle B is a Jordan separating cycle. We know by Lemma 4.20 that, without loss of generality, all vertices v in the interior of B satisfy $d(v,B) \leq \lfloor \frac{D}{2} \rfloor$. Choose C to be an interior-minimal short-cycle in $\mathrm{Int}[B]$, i.e., choose C such that there does not exist a short-cycle C' having $\mathrm{Int}(C') \subset \mathrm{Int}(C)$ (it is possible that $C = B$). Clearly C is itself a Jordan separating cycle, and by Observation 4.21, if v is a vertex in the interior of C, then $d(v,C) \leq \lfloor \frac{D}{2} \rfloor$.

Among all vertices in the interior of C, let v be one of maximum distance from C. By Lemma 4.10, such a vertex v exists. Let P be a $v - C$ geodesic, and let u be the vertex of $C \cap P$. Since G is 2-edge-connected, we have that $d(v) \geq 2$. Thus there is some vertex v' in $\mathrm{Int}[C] - P$ that is adjacent to v.

Let P' be a $v' - C$ geodesic and u' the vertex of $C \cap P'$ (it is possible that $u' = v'$). If u and u' are distinct, then the cycle C can be divided into two $u - u'$ paths. Let Q denote the shorter of these two paths and note that $\ell(Q) \leq D - 1$. If $u = u'$, then let Q be the trivial path containing only the vertex u. The maximality of v and the choice of v' ensures that the closed walk on $P \cup Q \cup P' \cup \{vv'\}$ contains some cycle C'. There are two cases to consider.

Case 1: The diameter D is odd.
The paths P and P' both have length at most $\lfloor \frac{D}{2} \rfloor = \frac{D-1}{2}$, so C' satisfies the following inequalities:

$$\begin{aligned}\ell(C') &\leq \ell(P \cup Q \cup P' \cup \{vv'\}) \\ &\leq \frac{D-1}{2} + (D-1) + \frac{D-1}{2} + 1 \\ &< 2D.\end{aligned}$$

Thus C' is a short-cycle contained entirely in the interior of C, contradicting the minimality of C and completing the proof in the case that D is odd.

Case 2: The diameter D is even.
We claim that $d(v, C) = d(v', C) = \frac{D}{2}$, and that $\ell(Q) = D - 1$. Certainly these values are all upper bounds. Assume to the contrary that $d(v, C) < \frac{D}{2}$, or that $d(v', C) < \frac{D}{2}$, or that $\ell(Q) < D-1$. In any of these cases, we get that:

$$\begin{aligned}\ell(C') &\leq \ell(P \cup Q \cup P' \cup \{vv'\}) \\ &< \frac{D}{2} + (D-1) + \frac{D}{2} + 1 = 2D.\end{aligned}$$

This contradicts the minimality of C, and thus proves the claim.

We further claim that u is the only vertex of C such that $d(u, v) \leq \frac{D}{2}$, and that u' is the only vertex of C such that $d(u', v') \leq \frac{D}{2}$. Assume to the contrary that there exists a vertex u^* in $C - u$ such that $d(u^*, v) \leq \frac{D}{2}$. Let P^* be a $v - u*$ geodesic, and let Q^* be a $u - u^*$ geodesic in the cycle C. The closed walk $P \cup Q^* \cup P^*$ contains a cycle of length at most $\frac{D}{2} + \frac{D}{2} + (D-1) = 2D - 1$, contradicting the minimality of C. The case in which u' is not the only vertex of C with $d(u', v') \leq \frac{D}{2}$ follows similarly, completing the proof of the second claim.

For a cycle C^* of G which does not contain v, define the v**-exterior** of C^*, denoted $v\text{Ext}(C^*)$, to be the region of $\mathbb{R}^2 - C^*$ that does not contain the vertex v. We also define $v\text{Ext}[C^*] = v\text{Ext}(C^*) \cup C^*$. Let $\mathfrak{S}$ be the set of all short-cycles of G that are contained in $\text{Ext}[C]$. Since C is in $\mathfrak{S}$ (with $v\text{Ext}(C) = \text{Ext}(C)$), this set $\mathfrak{S}$ is nonempty. Choose a short-cycle A in $\mathfrak{S}$ that is v-exterior minimal, i.e., choose A in $\mathfrak{S}$ such that there does not exist any short-cycle A' in $\mathfrak{S}$ having $v\text{Ext}(A') \subset v\text{Ext}(A)$.

Because A is a short-cycle, the v-exterior of A contains some vertex of G by Lemma 4.10. Let w be any vertex in $v\text{Ext}(A)$. Since the cycle C separates w from v, and $d(v, C) = \frac{D}{2}$, we have that $d(w, C) \leq \frac{D}{2}$. The cycle A either is itself C or separates w from C, so by Observation 4.21 we have that $d(w, A) \leq \frac{D}{2}$. Repeat the entire first part of the proof, replacing the cycle C with A, and the region $\text{Int}(C)$ with the region $v\text{Ext}(A)$, to show the existence of four distinct vertices x, x', y and y' (analogous to u, u', v and v' respectively) in $v\text{Ext}[A]$ that satisfy the following conditions:
(1) both x and x' lie on A, and a shortest $x - x'$ path in A has length $D - 1$,
(2) $d(x, y) = \frac{D}{2}$, and every vertex w in $A - x$ satisfies $d(y, w) > \frac{D}{2}$,
(3) $d(x', y') = \frac{D}{2}$, and every vertex w in $A - x'$ satisfies $d(y', w) > \frac{D}{2}$.

Note that both the cycles C and A (which are possibly the same) separate v from y, and separate v from y'. For it to be possible that $d(v, y) \leq D$, it is the case that $u = x$, since u is the unique vertex of C such that

$d(u,v) \leq \frac{D}{2}$, and x is the unique vertex of A such that $d(x,y) \leq \frac{D}{2}$. Similarly, since $d(v,y') \leq D$, it is the case that $u = x'$. Because x and x' are distinct, this yields a contradiction, completing the proof. ☐

Corollary 4.23. *Let G be a 2-edge-connected plane graph of diameter D. If either $g(G) \geq 2D$ or $\mu(G) \geq 2D$, then $g(G) = \mu(G)$.*

Proof. This corollary follows from Theorems 4.22 and 4.17, as well as the fact that $g(G) \leq \mu(G)$ in a 2-edge-connected plane graph. ☐

Theorems 4.13 and 4.17 demonstrate that if a 2-edge-connected plane graph G with diameter D has $\mu(G) \geq 2D+1$, then G is the cycle $2D+1$ and thus has girth $2D+1$. As such, there exists a function f of the graph's diameter D such that if $\mu(G) \geq f(D)$, then G contains no short cycle, and Theorem 4.22 illustrates that $f(D) \leq 2D$. The next theorem demonstrates that the result given by Theorem 4.22 cannot be improved.

Theorem 4.24. *For each integer $D \geq 3$, there exists a 2-edge-connected plane graph G_D of diameter D such that $\mu(G_D) = 2D - 1$ but $g(G_D) = 2D - 2$.*

Proof. Let G_D be the graph consisting of two vertices u and v, and four internally disjoint $u - v$ paths. Let two paths have length D, while the other two have length $D - 1$. Embed G_D in the plane such that every face is bounded by one path of length D, and one path of length $D - 1$ (see Figure 4.5).

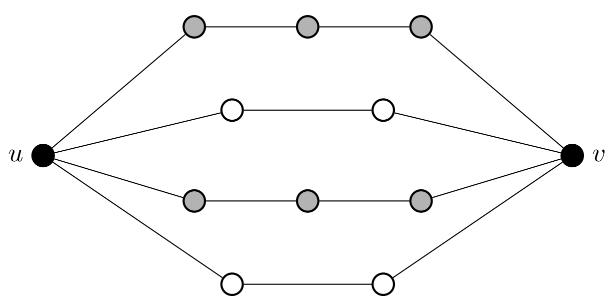


Figure 4.5: The graph G_4 of diameter 4, described in the proof of Theorem 4.24. Observe that $\mu(G_4) = 7$ but $g(G_4) = 6$.

Noting that any two vertices of G_D lie on a cycle of length at most $2D$, we see that the diameter of G_D is at most D. On the cycle formed by the two paths of length D, there exist two vertices distance D apart, so the diameter of G_D is exactly D. Each face of G_D is bounded by a cycle formed by the union of a path of length D and a path of length $D - 1$, so every face of G_D has face-degree $2D - 1$. The cycle C formed by the union of the two paths of length $D - 1$ has length $2D - 2$, and is thus a short-cycle. It is easy to see that C is the shortest cycle of G_D, and hence that $g(G_D) = 2D - 2$. ☐

The next series of lemmas and observations (that are likely well known to those working with generalised polygons) culminate in a characterisation of planar generalised polygons.

Lemma 4.25. [🔥] *Generalised polygons are self-centred.*

Proof. Assume to the contrary that G is a generalised polygon with radius r, diameter D, and that $r < D$. Let T_r be a radius-preserving spanning tree of G. Since G has girth $2D$, it contains some cycle, and so there exists at least one fundamental cycle of G with respect to T_r. By Lemma 4.12, this fundamental cycle has length at most $2r + 1$, which is strictly less than $2D$, contradicting that G is a generalised polygon. $\square$

The next observation follows from Lemmas 4.25 and 4.16.

Observation 4.26. [ ] Generalised polygons are 2-connected.

In order to further characterise planar generalised polygons, we will make use of two lemmas from *Algebraic Graph Theory* [21].

Lemma 4.27. [21] *Let G be a graph of diameter D and girth $2D$, and let u and v be vertices of G. If $d(u, v) = k < D$, then there is a unique $u - v$ path of length k in G.*

Lemma 4.28. [21] *Let G be a graph of diameter D and girth $2D$, and let u and v be vertices of G. If $d(u, v) = D$, then $d(u) = d(v)$.*

We will also need the following two simple observations.

Observation 4.29. [ ] Let G be a graph of diameter D and girth $2D$, and let H be a subgraph of G. If $d_H(u, v) = D$, then $d_G(u, v) = D$.

Proof. Assume to the contrary that $d_G(u, v) < D$. Let P be a $u-v$ geodesic in H and Q a $u-v$ geodesic in G. The closed walk $P \cup Q$ has length less than $2D$ and contains some cycle, contradicting that $g(G) = 2D$. $\square$

The next observation follows from the well-known fact that if G is a bipartite graph and v is a vertex of G, then the vertices at odd and even distance from v form partite sets of G. Nevertheless, we include a short proof for completeness.

Observation 4.30. [ ] Let G be a generalised polygon of diameter D, and let u, v and w be vertices of G. If u and v are adjacent, and $d(u, w) = D$, then $d(v, w) = D - 1$.

Proof. Certainly $d(v, w) > D - 2$, so it is suffices to show that $d(v, w) < D$. Assume to the contrary that $d(v, w) = D$, and let P be a $w - u$ geodesic, and Q a $w - v$ geodesic. Let x be the vertex of $P \cap Q$ which is farthest from the vertex w (it is possible that $x = w$). The union $P[x, u] \cup Q[x, v] \cup \{uv\}$ is an odd cycle, contradicting the fact that G is bipartite. $\square$

The next result, Theorem 4.31, demonstrates that for each pair (Δ, D) of integers with $\Delta \geq 2$ and $D \geq 2$, there exists a unique planar generalised polygon with maximum degree Δ and diameter D.

Theorem 4.31. *If G is a planar generalised polygon of maximum degree $\Delta \geq 2$ and diameter $D \geq 2$, then G consists of two vertices of degree Δ joined by Δ internally disjoint paths of length D.*

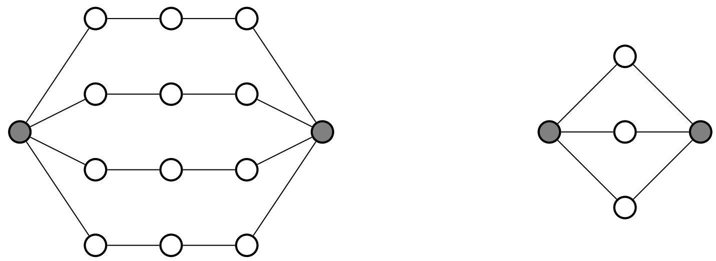


Figure 4.6: On the left is the unique planar generalised polygon with maximum degree and diameter both equal to 4. On the right is the unique planar generalised polygon with maximum degree 3 and diameter 2. The vertices of maximum degree are grey.

Proof. Let G be a planar generalised polygon of maximum degree Δ and diameter D, with a fixed embedding as a plane graph. We may assume that $\Delta > 2$, as the cycle C_{2D} is the only generalised polygon with $\Delta = 2$ and diameter D. Let u be a vertex of degree Δ. By Corollary 4.25, there is some vertex v such that $d(u, v) = D$. By Lemma 4.28, the vertex v satisfies $d(v) = \Delta$.

Label the vertices of $N(u) = \{a_1, a_2, \dots a_\Delta\}$ such that any pair a_i and a_{i+1} of vertices (subscripts taken mod Δ) are on the boundary of the same face. By Observation 4.30, we have that $d(a_i, v) = D - 1$ for all i in $\{1, 2, \dots, \Delta\}$. By Lemma 4.27, there is a unique $a_i - v$ path of length $D - 1$ for all i. Each $a_i - v$ path of length $D-1$ can be extended to a $u - v$ path of length D. Let P_i be the extended $u - v$ path containing the vertex a_i, and let b_i be the vertex of P_i which is adjacent to v.

The paths P_i and P_j are internally disjoint whenever $i \neq j$. Were they not, the union $P_i \cup P_j$ would contain some cycle of length less than $2D$.

We now have that G contains as a subgraph Δ internally disjoint $u - v$ paths of length D (the paths $P_1, P_2, \dots, P_\Delta$). Let H be the subgraph containing only the union of all the P_i's, and denote by C_i the cycle of length $2D$ on $P_i \cup P_{i+1}$ (subscripts taken mod Δ). The graph H divides the plane into Δ regions, each bounded by a cycle C_i.

What remains is to show that $G = H$. Since $g(G) = 2D$, any two vertices of H lie on a cycle of length $2D$. No edge can be added between two vertices of H, so it suffices to show that $V(G) = V(H)$. Thus we assume to the contrary that G contains some vertex not in H. Since G is connected and $d(u) = d(v) = \Delta$, an internal vertex of some P_i has a neighbour in $G - H$. Let x be the internal vertex of P_i and let y be its neighbour in $G - H$. Without loss of generality, the vertex y is in the region bounded by C_i. Since $\Delta \geq 3$, there is some path P_j of H such that $P_j \cap C_i = \{u, v\}$, and the internal vertices of P_j are not in the same region of $\mathbb{R}^2 - C_i$ as the vertex y.

Let z be the vertex of P_j that satisfies $d_H(x, z) = D$. We know per Observation 4.29 that $d_G(x, z) = D$, and from Observation 4.30 that $d_G(y, z) = D - 1$. Thus there is a $y - z$ geodesic Q of length $D - 1$ in G, which does not contain x, but contains some other vertex of C_i. Let w be the vertex of $Q \cap C_i$ which is closest to y, and note that since w and z are distinct, the path segment $Q[y, w]$ has length at most $D - 2$. Since x and w are distinct, the cycle C_i is divided into two internally disjoint $x - w$ paths. Let R be the shorter of these two paths, and note that $\ell(R) \leq D$. The union of paths $R \cup Q[y, w] \cup \{xy\}$ forms a cycle that has length at most $D + (D-2) + 1 < 2D$, a contradiction since $g(G) = 2D$. □

Corollary 4.32. *A plane graph of diameter D is a generalised polygon if and only if it is 2-edge-connected and $2D$-face-degree regular.*

Proof. By Theorem 4.31 and Observation 4.26, it is clear that a plane graph that is a generalised polygon is 2-edge-connected and $2D$-face-degree regular. The converse follows from Theorem 4.22 and Observation 4.18. □

4.7 The degree diameter problem for face-degree regular plane graphs

We obtain some new results on the degree diameter problem for face-degree regular graphs as corollaries of structural results obtained thus far.

Corollary 4.33. *If G is a 2-edge-connected plane graph of diameter D and order n in which every face has degree $2D+1$, then $n = 2D+1$.*

Proof. This follows from Theorem 4.17. □

Corollary 4.34. *If G is a 2-edge-connected plane graph of diameter D, maximum degree Δ and order n in which every face has degree $2D$, then $n = \Delta(D-1)+2$.*

Proof. This follows from Corollary 4.32 and Theorem 4.31. □

Note that in Corollary 4.34, we cannot replace the condition that G is $2D$-face-degree regular with the condition that G has minimum face-degree $2D$. To see this, we create a plane graph $G(\Delta, D)$ with maximum degree $\Delta \geq 3$, diameter $D \geq 2$, minimum face-degree $2D$ and order $\Delta(D-1)+3$ as follows. Let u, v and w be three vertices, and let u and v be adjacent. Create two internally disjoint paths of length D between v and w, and $\Delta - 2$ internally disjoint paths of length D between u and w. This completes the construction of $G(\Delta, D)$ (see Figure 4.7).

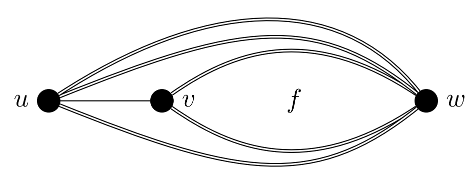


Figure 4.7: Consider every double-line in the diagram to be a path of length D. The above graph is $G(5, D)$, with diameter D and order $5(D-1)+3$. Observe that the face f has degree $2D$.

4.8 Further questions

Question 1. Corollary 4.23 gives a sufficient condition for the minimum face-degree and girth of a 2-edge-connected plane graph to agree. What other sufficient / necessary conditions can be found for these two parameters to agree?

Question 2. If a plane graph has minimum face-degree μ and girth g, what can be said about the quantity $\mu - g$?

We raise the following conjectures as possible answers to Question 2:

Conjecture 4.35. *Consider a 2-edge-connected plane graph with minimum face-degree μ, girth g and diameter D, and let $k \geq 0$ be an integer. If $\mu \geq 2D - k$, then $\mu - g \leq k$.*

Conjecture 4.36. *A 2-edge-connected plane graph with diameter D, girth g and minimum face-degree μ satisfies*

$$D \geq \frac{2\mu - g - 1}{2}.$$

Notice that the first conjecture yields Corollary 4.23 by setting $k = 0$. When $\mu = g$, the second conjecture gives $\frac{\mu-1}{2} \leq D$, which is a weakening of Theorem 4.13. As such, both conjectures certainly hold when μ is sufficiently large.

Question 3. We resolved the degree diameter problem for 2-edge-connected, $2D$-face-degree regular graphs of diameter D with Corollary 4.34. Can we obtain a similar bound for 2-edge-connected plane graphs with minimum face-degree $2D$?

In light of the example in Figure 4.7, the author has the following conjecture regarding this question.

Conjecture 4.37. *If G is a 2-edge-connected plane graph of diameter D, maximum degree Δ, order n and minimum face-degree $2D$, then $n \leq \Delta(D-1) + 3$, and this bound is sharp.*

Question 4. The degree diameter problem in planar graphs that have every face bounded by a cycle (or circuit) of length ρ has only been studied in depth for $\rho = 3$ and $\rho = 4$. What bounds can be found for the case where ρ is an arbitrary integer?

The author believes that proof of Theorem 10 in [12] may be adapted to solve the degree diameter problem in ρ-face-degree regular graphs when ρ is even.

Chapter 5

The degree diameter problem in 5-face-degree regular graphs of diameter 3

Let G be a plane graph of order n, diameter D and maximum degree Δ in which every face-boundary is a cycle of length ρ. Sharp upper bounds on n as a function of Δ are known for the following cases:

- $\rho = 3$ and $D = 2$ [35, 41],
- $\rho = 4$ and $D = 2$ [12],
- $\rho = 4$ and $D = 3$ [12],
- $\rho = 5$ and $D = 2$ (Corollary 4.33),
- $\rho = 6$ and $D = 3$ (Corollary 4.34),
- $\rho \geq 7$ and $D = 3$ (Corollary 4.33).

Graphs with large diameter have less constrained structure than those with small diameter: the difficulty in sharply bounding the number of vertices in a graph increases very quickly as the diameter increases (provided the graph is not extremal, like the graphs in the previous chapter). This makes it difficult to solve the degree diameter problem sharply for graphs with $D \geq 4$. However, it is clear that the list above has a gap: there is no sharp bound to be found when $\rho = 5$ and $D = 3$. In this chapter, we show that the number of vertices n in a pentagulation of diameter 3, maximum degree $\Delta \geq 8$ and order n satisfies $n \leq 3\Delta - 1$.

5.1 Basics

For brevity, we refer to a plane graph in which every face-boundary is a 5-cycle as a **pentagulation**. Note by Proposition 2.20 that pentagulations are 2-connected. We see by Lemma 4.20 that if some cycle separates a plane graph of diameter 3, then that cycle dominates either its interior or exterior. By Lemma 4.10, any cycle of length less than ρ in a ρ-face-degree-regular graph is a Jordan separating cycle. We can further show that any cycle slightly longer than a face cycle is also a Jordan separating cycle.

Lemma 5.1. *Let C be a cycle of length ℓ in a ρ-face-degree-regular graph. If $\rho < \ell < 2\rho - 2$, then C is a Jordan separating cycle.*

Proof. Let $C : v_1, v_2, \ldots, v_\ell$ be a cycle of length ℓ in a ρ-face-degree-regular plane graph G, with $\rho < \ell < 2\rho - 2$. Since $\rho < \ell$, the cycle C does not bound any face of G. We claim that the interior of C contains a vertex. Assume to the contrary that it does not. Since C is not a face cycle, the interior of C contains some edge, specifically, a chord v_iv_j of C. Because the length of C is at most $2\rho - 3$, one of the $v_i - v_j$ paths of C has length at most $\rho - 2$. Call this path P. The cycle $P \cup v_iv_j$ has length at most $\rho - 1$, and thus its interior contains some vertex u by Lemma 4.10. Since the interior of $P \cup v_iv_j$ is contained in the interior of C, the vertex u lies in the interior of C.

By the same argument, the exterior of C contains a vertex, and thus C is a Jordan separating cycle. □

Note that if C is a cycle in a pentagulation G and $\ell(C) = 5$, then there are three distinct possibilities:

1. The cycle C Jordan separates G,
2. C is a face-cycle that separates G, but necessarily does not Jordan separate G,
3. C is a face-cycle that does not separate G.

It's clear that this list of possibilities is exhaustive (for a cycle cannot both Jordan separate G and be a face cycle), and examples of all three kinds of 5-cycle can be found in the graph $\mathcal{I}$ of Figure 5.3.

5.2 Cycles of length 3

The following lemmas show that a 3-cycle in a pentagulation never dominates its interior (or exterior). Although we phrase our lemmas in terms of cycle interiors, the same results hold with the same proofs if we exchange the words 'interior' and 'exterior'.

Lemma 5.2. *Let G be a pentagulation. If C is a 3-cycle in G, then no single vertex of C dominates Int(C).*

Proof. For the sake of contradiction, let $C : v_1, v_2, v_3$ be a 3-cycle, the interior of which is dominated by the single vertex v_1. Choose C to be minimal, so that there does not exist a 3-cycle C' such that v_1 dominates the interior of C', and for which $\text{Int}(C') \subset \text{Int}(C)$.

By Lemma 4.10, the cycle C separates G, so there exists a vertex u in the interior of C. By assumption, u and v_1 are adjacent. As G is a pentagulation, and thus 2-connected, the vertex u has some neighbour other than v_1 in $\text{Int}[C]$. This neighbour cannot be v_2, as then v_1, v_2, u is a 3-cycle, contradicting the minimality of C. Similarly, the vertices u and v_3 are not adjacent. (see Figure 5.1).

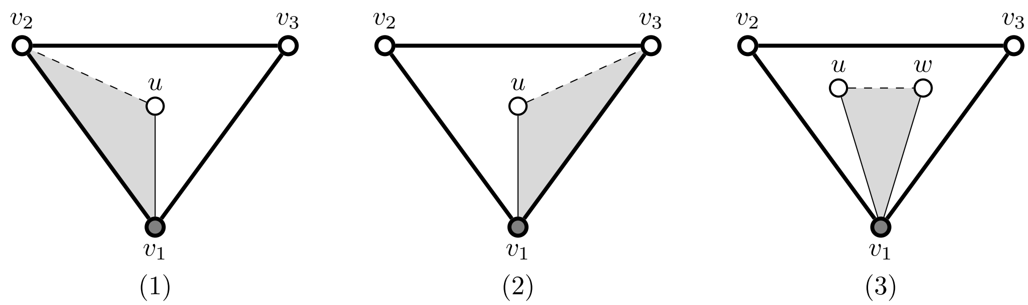


Figure 5.1: Some steps in the proof of Lemma 5.2.

Thus there is some other vertex w in $\text{Int}(C)$ that is adjacent to u. But since v_1 dominates $\text{Int}(C)$, the vertices v_1, u and w form a 3-cycle, contradicting the minimality of C. □

Lemma 5.3. *Let G be a pentagulation, and let C be a 3-cycle in G. The interior of C is not dominated by any two vertices of C.*

Proof. Let $C = v_1, v_2, v_3$ be a 3-cycle in a pentagulation. Assume to the contrary and without loss of generality that every vertex in $\text{Int}(C)$ is dominated by $\{v_1, v_2\}$. We claim that no vertex in $\text{Int}(C)$ is adjacent to v_3. Assume to the contrary there is a vertex v adjacent to v_3. Without loss of generality, we assert that v is adjacent to v_1 as well, since $\{v_1, v_2\}$ dominates $\text{Int}(C)$. Thus the triangle v_1, v, v_3 is dominated by v_1, contradicting Lemma 5.2 and proving the claim.

The edge v_1v_2 lies on the boundary of two faces, one of which is in the interior of C. Call this interior face f, and note that the boundary of f is a 5-cycle. Per Lemma 5.2, the interior of C is not dominated by a single vertex, so both v_1 and v_2 have some neighbour in $\text{Int}(C)$. Thus the cycle bounding f is of the form u, v_1, v_2, w, x, where u, w and x are vertices in the interior of C. As $\{v_1, v_2\}$ dominates $\text{Int}(C)$, the vertex x is adjacent to either v_1 or v_2. If x is adjacent to v_1, then u, x, v_1 is a triangle whose interior is dominated by the single vertex v_1, and similarly if x is adjacent to v_2 then w, x, v_2 is a triangle whose interior is dominated by v_2. Both possibilities contradict Lemma 5.2, completing the proof. □

Lemma 5.4. *Let G be a pentagulation and let C be a 4-cycle in G. Then no single vertex of C dominates Int(C).*

Proof. Let $C = v_1, v_2, v_3, v_4$ be a 4-cycle in a pentagulation. Assume for the sake of contradiction that v_1 dominates $\text{Int}(C)$, and choose C to be minimal, i.e., there is no 4-cycle C' dominated by v_1 such that

$\text{Int}(C') \subset \text{Int}(C)$. By Lemma 4.10, there exist vertices in $\text{Int}(C)$. Let u be the neighbour of v_1 in the interior of C such that uv_1 and v_1v_2 both lie on the boundary of some common face. Since u is not an end-vertex, it is adjacent to some vertex w in $\text{Int}[C] - v_1$. Up to symmetry, there are three possible choices for the vertex w.

Case 1: $w = v_2$ or $w = v_4$.
If $w = v_2$, we obtain a 3-cycle v_1, u, v_2, the interior of which is dominated by v_1, contradicting Lemma 5.2. The situation is similar if u is adjacent to v_4.

Case 2: $w = v_3$.
In this case we get a 4-cycle v_1, u, v_3, v_2, the interior of which is dominated by v_1, contradicting the minimality of C.

Case 3: w is a vertex in $\text{Int}(C)$.
By assumption, the vertex v_1 dominates $\text{Int}(C)$, so v_1 and w are adjacent. Thus v_1, u, w is a 3-cycle whose interior is dominated by v_1, contradicting Lemma 5.2. □

Lemma 5.5. *Let C be a 4-cycle in a pentagulation. No pair of vertices of C, that are adjacent in C, dominate Int(C).*

Proof. Assume for the sake of contradiction that $C = v_1, v_2, v_3, v_4$ is a 4-cycle in a pentagulation whose interior is dominated by $\{v_1, v_2\}$. By Lemma 5.4, both v_1 and v_2 each have at least one neighbour in $\text{Int}(C)$. Thus there is a face f in the interior of C, bounded by a 5-cycle of the form u, v_1, v_2, w, x, where u and v are vertices in $\text{Int}(C)$ and x is a vertex in $\text{Int}[C]$.

If x is either v_3 or v_4, then $\text{Int}[C]$ contains a triangle whose interior is dominated by v_1 or v_2 respectively, contradicting Lemma 5.2. If x lies in $\text{Int}(C)$, then it is adjacent to either v_1 or v_2. If x is adjacent to v_1, then v_1, u, x is a triangle whose interior is dominated by v_1, and if x is adjacent to v_2, then the interior of the triangle v_2, w, x is dominated by v_2. In any case, we obtain a triangle whose interior is dominated by a single vertex, contradicting Lemma 5.2. □

Lemma 5.6. *A 3-cycle in a pentagulation does not dominate its interior (or exterior).*

Proof. Let $C : v_1, v_2, v_3$ be a 3-cycle in a pentagulation G. Assume for the sake of contradiction that C dominates its interior (the exterior case can be proven in the same fashion). By Lemmas 5.2 and 5.3, no proper subset of $V(C)$ dominates $\text{Int}(C)$, so every vertex of C has at least one neighbour in $\text{Int}(C)$. Thus there exists a neighbour u of v_1 in $\text{Int}(C)$. Since G is 2-connected, the vertex u has some neighbour w in $\text{Int}[C] - v_1$.

By Lemma 5.3, the vertex w is neither v_2 nor v_3, as this induces a 3-cycle whose interior is dominated by just two vertices. We have per Lemma 5.2 that w is not adjacent to v_1, as this creates a 3-cycle whose interior is dominated by v_1. We see by Lemma 5.5 that neither v_2 nor v_3 is adjacent to w, since this induces a 4-cycle, the interior of which is dominated by two adjacent vertices. Thus u does not have a neighbour in $\text{Int}[C] - v_1$, and so we have the desired contradiction. □

Lemma 5.6 yields the following corollary, which we make extensive use of.

Corollary 5.7. *Pentagulations of diameter 3 contain no 3-cycles.*

Proof. Per Lemma 4.10, any 3-cycle in a pentagulation is a Jordan separating cycle, and is thus a separator of the pentagulation. By Lemma 4.20, any Jordan separating cycle in a graph of diameter 3 dominates either its interior or its exterior. Thus any 3-cycle in a pentagulation dominates either its interior or exterior, contradicting Lemma 5.6. □

5.3 Cycles of length 4 and 5

We have shown that diameter 3 pentagulations do not contain 3-cycles (and, hence, that any 4-cycle or 5-cycle in a such a pentagulation is chordless). They may, however contain both 4-cycles and separating 5-cycles. Thus we dedicate the next series of lemmas to characterising the structure of 4-cycles in diameter 3 pentagulations.

Lemma 5.8. *If a pentagulation contains a Jordan separating 5-cycle C, then the interior of C is dominated by neither a single vertex of C, nor by a pair of vertices which are adjacent in C.*

Proof. Let $C = v_1, v_2, v_3, v_4, v_5$ be a Jordan separating cycle of a pentagulation G. We first prove that $\text{Int}(C)$ is not dominated by a single vertex of C. Assume to the contrary that v_1 dominates $\text{Int}(C)$, and let u be a neighbour of v_1 in $\text{Int}(C)$. Since G is 2-connected, the vertex u has some neighbour in $\text{Int}[C] - v_1$.

If u is adjacent to any neighbour of v_1 (including v_2 and v_5), then G contains a triangle, contradicting Corollary 5.7. If u is adjacent to v_3 or v_4, we obtain a 4-cycle whose interior is dominated by the single vertex v_1, contradicting Lemma 5.4. Thus u has no neighbour in $\text{Int}[C] - \{v_1\}$, a contradiction, which proves that $\text{Int}(C)$ is not dominated by a single vertex of C.

We now prove that no pair of vertices that are adjacent in C dominate the cycle's interior. Assume to the contrary that $\{v_1, v_2\}$ dominates $\text{Int}(C)$. Let u be a neighbour of v_1 in the interior of C, and note again that u has some neighbour in $\text{Int}[C] - v_1$. As in the previous argument, the vertex u is not adjacent to any neighbour of v_1. If u is adjacent to either v_3 or v_4, then G contains a 4-cycle whose interior is either dominated by the single vertex v_1, or by the adjacent pair $\{v_1, v_2\}$, contradicting Lemma 5.4 or Lemma 5.5, respectively. If u is adjacent to some neighbour w of v_2, then again G contains a 4-cycle whose interior is dominated by the adjacent pair $\{v_1, v_2\}$, again yielding a contradiction. Thus there is no possible neighbour for u in $\text{Int}[C] - v_1$, which is a contradiction. □

Lemma 5.9. *Let C be a 4-cycle of a pentagulation. If C dominates its interior, then no two vertices which are adjacent in C both have neighbours in Int(C).*

Proof. Let $C = v_1, v_2, v_3, v_4$ be a 4-cycle in a pentagulation, and suppose that C dominates its interior. Assume to the contrary, and without loss of generality, that each of v_1 and v_2 has some neighbour in $\text{Int}(C)$.

The edge v_1v_2 lies on some face in the interior of C. This face is bounded by a 5-cycle of the form u, v_1, v_2, w, x, where u and w are neighbours of v_1 and v_2 respectively, and x is some vertex of Int$[C]$. Since C dominates its interior, the vertex x is either a vertex of C, or is adjacent to a vertex of C. If x is any vertex of C, or if x is adjacent to v_1 or v_2, then there is some 3-cycle in Int$[C]$ that dominates its interior, contradicting Lemma 5.6. If x is adjacent to v_3 or v_4, then there is some 4-cycle in Int$[C]$ whose interior is dominated by two adjacent vertices of the 4-cycle, contradicting Lemma 5.5. In any case, we derive a contradiction, completing the proof. □

Lemma 5.10. *Let C be a 6-cycle in a pentagulation. If the interior of C is dominated by two vertices u and v of C such that $d_C(u, v) = 3$, then no chord of C lies in the interior of C.*

Proof. Let $C = v_1, v_2, v_3, v_4, v_5, v_6$ be a 6-cycle in a pentagulation, the interior of which is dominated by $\{v_1, v_4\}$. Assume to the contrary that $e = v_iv_j$, with $|j - i| > 1 \pmod 6$, is a chord of C contained in Int$[C]$. If $|j - i| = 2$, then the chord induces a 3-cycle in C that dominates its interior, contradicting Lemma 5.6. If $|j - i| = 3$, there are two cases to consider.

Case 1: The chord e has ends v_1 and v_4.
In this case the chord induces a 4-cycle whose interior is dominated by two consecutive vertices, contradicting Lemma 5.5.

Case 2: The chord is not the edge v_1v_4.
In this case the chord induces a 4-cycle dominated by only one vertex, contradicting Lemma 5.4. □

Lemma 5.11. *Let C be a 6-cycle in a pentagulation. If Int(C) is dominated by two vertices u and v, such that $d_C(u, v) = 3$, then there exists some vertex in Int(C) that is adjacent to both u and v.*

Proof. Let G be a pentagulation. Assume to the contrary that $C = v_1, v_2, v_3, v_4, v_5, v_6$ is a 6-cycle in G whose interior is dominated by $\{v_1, v_4\}$, and that no vertex in Int(C) is adjacent to both v_1 and v_4. Choose C to be a minimal counterexample, in the sense that there does not exist any 6-cycle C' that has its interior dominated by $\{v_1, v_4\}$, and that does not contain any neighbour of both v_1 and v_4 in Int(C'), and that satisfies Int$(C') \subset$ Int(C). The cycle C is chordless by Lemma 5.10, and is a Jordan separating cycle by Lemma 5.1, so there exists some vertex w in Int(C). Without loss of generality, the vertex w is adjacent to v_1. Since G is 2-connected, there is some neighbour x of w in Int$[C] - w$, and by assumption this neighbour is not v_4.

The vertex x is neither v_2 nor v_6, as this would create a triangle v_1, w, x, v_1 that dominates its interior, contradicting Lemma 5.6. Further, x is neither v_3 nor v_5 as either case induces a 4-cycle whose interior is dominated by v_1, contradicting Lemma 5.4. We conclude that x lies in Int(C), and hence be adjacent to either v_1 or v_4. However if x is adjacent to v_1, then v_1, x, w is a triangle, the interior of which is dominated by v_1, contradicting Lemma 5.2. If x is adjacent to v_4, then the two internally disjoint paths v_1, v_2, v_3, v_4 and v_1, w, x, v_4, induce a 6-cycle in Int$[C]$. The interior of this 6-cycle is dominated by $\{v_1, v_4\}$, and by assumption, there is not a common neighbour of both v_1 and v_4 in the interior of this cycle, contradicting the minimality of C. In conclusion, the vertex w has some neighbour in Int$[C] - w$, but any possible neighbour we may choose for w yields the desired contradiction. □

Corollary 5.12. *Let C be a Jordan separating 5-cycle in a pentagulation. If Int(C) is dominated by two non-adjacent vertices u and v of C, then there is some vertex in Int(C) that is adjacent to both v and u.*

Proof. Let G be a pentagulation, and let $C = v_1, v_2, v_3, v_4, v_5$ be a Jordan separating 5-cycle in G whose interior is dominated by $\{v_1, v_3\}$.

Since C is Jordan separating, there exists a vertex w in Int(C) that is, without loss of generality, adjacent to v_1. If w is adjacent to v_3, we are done. Thus we assume w is not adjacent to v_3, and observe that since G is 2-connected, it has some neighbour x in Int$[C] - v_1$. The vertex x is not any neighbour of v_1, as then v_1, w, x is a triangle that dominates its interior, contradicting Lemma 5.6. The vertex x cannot be v_4, as this would induce a 4-cycle dominated by v_1, contradicting Lemma 5.4. The only remaining possibility is that x is a vertex in Int(C) that is adjacent to v_3. In this case, the internally disjoint paths v_1, v_5, v_4, v_3 and v_1, w, x, v_3 induce a 6-cycle whose interior is dominated by $\{v_1, v_3\}$. By Lemma 5.11, the interior of this 6-cycle contains some vertex that is adjacent to both v_1 and v_3, completing the proof. □

Lemma 5.13. *Let C be a 4-cycle in some pentagulation. If Int(C) is dominated by two vertices u and v of C, then there exist two vertices w_1 and w_2 in the interior of C such that u, w_1, w_2, v is a path of length 3 in G.*

Proof. Let G be a pentagulation. Assume to the contrary that G contains some 4-cycle $C = v_1, v_2, v_3, v_4$ whose interior is dominated by $\{v_1, v_3\}$, but that there does not exist a $v_1 - v_3$ path of length 3 through the interior of C. Further, choose C to be a minimal counterexample. That is, there does not exist a 4-cycle C' whose interior is dominated by $\{v_1, v_3\}$, and that satisfies Int$(C') \subset$ Int(C), and that also does not have a $v_1 - v_3$ path of length 3 through its interior.

By Lemma 4.10, the cycle C is a Jordan separating cycle and so there exists some vertex w_1 in the interior of C. Since $\{v_1, v_3\}$ dominates Int(C), we assume with loss of generality that w_1 is adjacent to v_1. Because G is 2-connected, the vertex w_1 is adjacent to some vertex w_2 of Int$[C] - v_1$. By assumption, the vertex w_2 is not both in Int(C) and adjacent to v_3, so w_2 is either a vertex of C, or is adjacent to v_1. The neighbour w_2 is not adjacent to v_1, as this would induce a 3-cycle that dominates its interior, contradicting Lemma 5.6. Further, w_2 is not v_3, as this induces two 4-cycles in Int$[C]$, both of which have their interiors dominated by $\{v_1, v_3\}$, neither of which can contain a $v_1 - v_3$ path of length 3 in their interiors, contradicting the minimality of C. Thus all possibilities for the vertex w_2 lead to a contradiction, completing the proof. □

Lemma 5.14. *Let G be a pentagulation. If C is a 4-cycle that dominates its interior, then every vertex u in Int(C) satisfies $d_G(u) = 2$.*

Proof. Let G be a pentagulation, let $C = v_1, v_2, v_3, v_4$ be a 4-cycle in G that dominates its interior, and let w be a vertex in Int(C). Since C dominates its interior, we assume without loss of generality that w is adjacent to v_1. Because G is 2-connected, w has at least one neighbour other than v_1 in Int$[C]$. We assume contrary to the statement of the lemmas that w has at least two distinct such neighbours, call them x_1 and x_2.

Neither x_1 nor x_2 is adjacent to v_1, as this would induce a triangle in Int$[C]$ that dominates its interior, contrary to Lemma 5.6. Further, neither x_1 nor x_2 are both adjacent to either of v_2 or v_4, and in Int(C), per

Lemma 5.9. Thus x_1 is either v_3, or is adjacent to v_3, and likewise for x_2. If $x_1 = v_3$, then x_2 is not v_3 (x_1 and x_2 are distinct by assumption), and hence is adjacent to v_3. However this induces a triangle v_3, w, x_2 that dominates its interior, contrary to Lemma 5.6. We conclude that both x_1 and x_2 are neighbours of v_3 in Int(C). But this induces a 4-cycle x_1, w, x_2, v_3 that is dominated by v_3, contradicting Lemma 5.4. □

We know by Lemmas 4.10 and 4.20 and that any 4-cycle in a pentagulation of diameter 3 dominates either its interior or exterior. The next theorem gives a complete description of the structure of this dominated region, and an example of such a region is given by Figure 5.2.

Theorem 5.15. *Let G be a pentagulation, and C a 4-cycle in G. If C dominates its interior, then there exist two non-adjacent vertices u and v of C, and a positive integer k such that the induced subgraph $G[Int[C]]$ consists of exactly:*
(1) the cycle C,
(2) k $u - v$ paths of length 3, and
(3) $k - 1$ $u - v$ paths of length 2.
All the paths in (2) and (3) are internally disjoint, and do not contain any vertices of $C - \{u, v\}$. Further, the paths of length 2 and 3 alternate, i.e., between each pair of $u - v$ paths of length 2 in $Int[C]$, there is a $u - v$ path of length 3, and vice-versa.

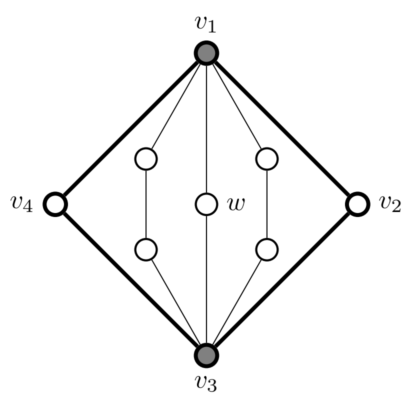


Figure 5.2: A 4-cycle dominating its interior which has $k = 2$ paths of length 3 and $k - 1 = 1$ paths of length 2 between two non-cycle-adjacent vertices v_1 and v_3, illustrating Theorem 5.15.

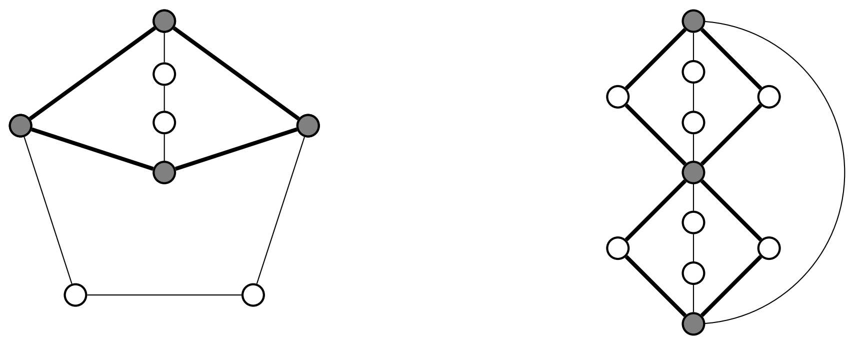
Proof. Let G be a pentagulation, and $C : v_1, v_2, v_3, v_4$ a 4-cycle in G that dominates its interior. By Lemma 5.4, at least two vertices of C have neighbours in Int(C). Per Lemma, 5.5 at most two vertices of C have such neighbours in Int(C), and these two vertices are not both be incident with a single edge of C. Thus, without loss of generality, the interior of C is dominated by the set $\{v_1, v_3\}$, and neither v_2 nor v_4 has any neighbours in Int(C).

We know that C does not have any chords in its interior, as a chord would induce a 3-cycle that dominates its interior, contradicting Lemma 5.6. By Lemma 5.13, there is at least one path of length 3 in the interior of C of the form v_1, w_1, w_2, v_3 for some vertices w_1 and w_2 in $int(C)$.

Per Lemma 5.14, any vertex in the interior of C has degree 2. Further, any vertex in Int(C) is adjacent to either v_1 or v_3, and there is no 3-cycle in the interior of C by Lemma 5.6. Putting these facts together, we deduce that every vertex in $\text{Int}(C)$ lies on a $v_1 - v_3$ path of length either 2 or 3, and that these paths are internally disjoint.

It remains only to show that paths of different lengths alternate. If P and Q are two (necessarily internally disjoint) $v_1 - v_3$ paths of length 2 in $\text{Int}[C]$, then they induce a 4-cycle $P \cup Q$ whose interior is dominated by $\{v_1, v_3\}$, and thus the existence of a $v_1 - v_3$ path of length 3 between them is guaranteed by Lemma 5.13. If R and S are two $v_1 - v_3$ paths of length 3 in $\text{Int}[C]$, they induce a 6-cycle $R \cup S$ whose interior is dominated by $\{v_1, v_3\}$. Since $d_{R\cup S}(v_1, v_3) = 3$, we obtain our desired path of length 2 between R and S by Lemma 5.11. □

We know by Corollary 5.7 that no diameter 3 pentagulation contains a triangle. The two graphs in Figure 5.3 show that there exist such graphs that contain 4-cycles.



$\mathcal{H}$ $\mathcal{I}$

Figure 5.3: Two diameter three pentagulations that contain 4-cycles, $\mathcal{H}$ and $\mathcal{I}$. Pairs of non-adjacent grey vertices dominate regions bounded by bold 4-cycles.

5.4 One of these 4-cycles is not like the other

Our overarching goal in this chapter is to show that if G is a diameter 3 pentagulation of order n and maximum degree $\Delta \geq 8$, then $n \leq 3\Delta - 1$. In order to split the problem into reasonable cases, we need a new concept. This notion will be of central importance in the next two sections, and is our primary way of investigating the structure of diameter 3 pentagulations. In Figure 5.2, consider the three 4-cycles $C_1 : v_1, v_2, v_3, v_4$; $C_2 : v_1, w, v_3, v_4$ and $C_3 : v_1, w, v_3, v_2$. Although these three cycles are all distinct, they appear to be part of the same structure in any diameter 3 pentagulation in which they may be found. Theorem 5.15 allows us to formalise this: both C_2 and C_3 are just 'substructures' of C_1, formed by C_1 and the alternating paths in its interior. For the purposes of vertex-counting, all we need to know is there is one 4-cycle, C_1, that dominates its interior and has three alternating paths inside of it. Thus it is advantageous to have a description of when two 4-cycles in a pentagulation cannot be considered part of the same collection of alternating paths. Such a description is sufficiently captured by the definition of dislocated 4-cycles introduced below.

Consider two distinct 4-cycles, C_1 and C_2, in a pentagulation G. We say that C_1 and C_2 are **dislocated** 4-cycles if there exist two regions $R_1 \in \{\text{Int}(C_1), \text{Ext}(C_1)\}$ and $R_2 \in \{\text{Int}(C_2), \text{Ext}(C_2)\}$, as well as two pairs

of vertices $\{u_1, v_1\} \subset V(C_1)$ and $\{u_2, v_2\} \subset V(C_2)$, such that all three of the following conditions hold:
(1) The regions R_1 and R_2 are dominated by $\{u_1, v_1\}$ and $\{u_2, v_2\}$, respectively,
(2) The sets $\{u_1, v_1\}$ and $\{u_2, v_2\}$ are not equal.
(3) The intersection $R_1 \cap R_2$ is empty.

It is worth observing that, as a consequence of Lemma 5.5, the edge u_1v_1 is not in $E(C_1)$, and u_2v_2 is not in $E(C_2)$.

To give an example of two 4-cycles that are not dislocated, consider Figure 5.4 (1). There are three 4-cycles, $C_1 : u_1, u_2, u_3, u_4$, $C_2 : u_1, u_5, u_3, u_4$ and $C_3 : u_1, u_5, u_3, u_2$. No cycle dominates its exterior, all three cycles have their interiors dominated by the pair $\{u_1, u_3\}$, and the pairwise intersection of these interiors is non-empty. Thus no pairs of these cycles are dislocated, as they will always fail both conditions (2) and (3) of the definition.

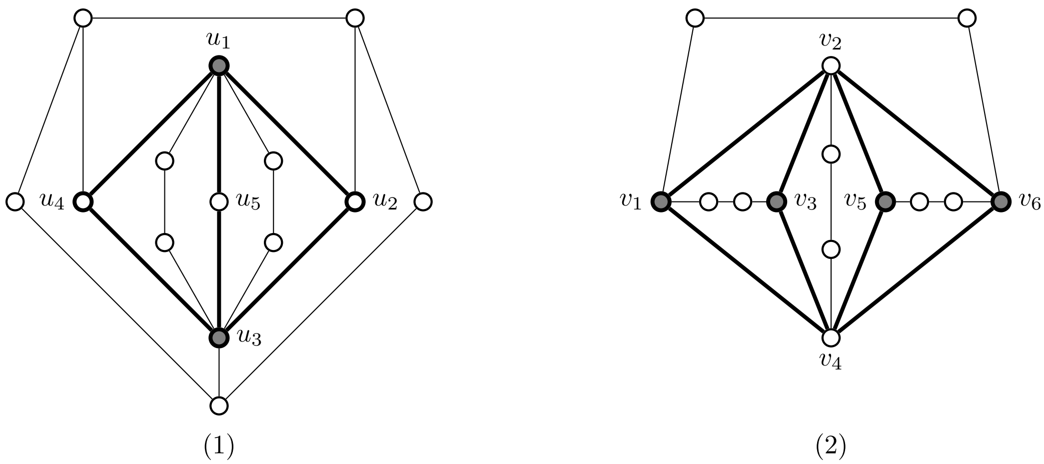


Figure 5.4: In (1), there is no pair of dislocated 4-cycles. In (2), the cycles $A_1 : v_1, v_2, v_3, v_4$ and $A_2 : v_5, v_4, v_6, v_2$ are dislocated.

For an example of two 4-cycles that are dislocated, consider Figure 5.4 (2), and let $A_1 : v_1, v_2, v_3, v_4$ and $A_2 : v_5, v_4, v_6, v_2$ be the two bold cycles. The vertices $\{v_1, v_3\}$ dominate $\text{Int}(A_1)$, and the vertices $\{v_5, v_6\}$ dominate $\text{Int}(A_2)$. Further, the intersection $\text{Int}(A_1) \cap \text{Int}(A_2)$ is empty, so A_1 and A_2 are dislocated.

5.5 Bounding the order, part I: An abundance of 4-cycles

In this section, we mostly consider what happens when a pentagulation contains two or more dislocated 4-cycles. But first, we handle a simple case, for which we recall the well-known theorem stating that if a graph of order n and maximum degree Δ is dominated by γ vertices, then $n \leq \gamma(\Delta + 1)$ (see, for example, [11]).

Lemma 5.16. *Let G be a pentagulation of order n and maximum degree $\Delta \geq 3$. If any 4-cycle of G dominates G, then $n \leq 3\Delta - 1$.*

Proof. Let G be a pentagulation of order n and maximum degree Δ that is dominated by the 4-cycle $C : v_1, v_2, v_3, v_4$.

Since Int(C) is dominated by C, we have without loss of generality, per Theorem 5.15, that every vertex of Int(C) lies on a $v_1 - v_3$ path of length 2 or 3. Further, there are at most $\frac{\Delta-1}{2}$ paths of length 3 in Int(C), and at most $\frac{\Delta-3}{2}$ paths of length 2 in Int(C).

Because Ext(C) is dominated by C, we have by Theorem 5.15 that every vertex of Ext(C) lies on either a $v_1 - v_3$ path, or a $v_2 - v_4$ path, and any such path has length 2 or 3. If the vertices of Ext(C) lies on $v_1 - v_3$ paths, then $\{v_1, v_3\}$ dominates Ext(C), and hence dominates all of G. In this case, we have that G is dominated by two vertices, so $n \leq 2\Delta + 2$, completing the proof.

Thus, we assume the vertices of Ext(C) lie on $v_2 - v_4$ paths. By Theorem 5.15 and the fact that the maximum degree of G is Δ, the number of paths of length 3 is bounded above by $\frac{\Delta-1}{2}$, and the number of paths of length 2 is less than or equal to $\frac{\Delta-3}{2}$.

Observe that each path of length 3 in Int(C) (Ext(C)) contributes 2 to the number $|V(\text{Int}(C))|$ ($|V(\text{Ext}(C))|$), and each path of length 2 contributes 1 to $|V(\text{Int}(C))|$ ($|V(\text{Int}(C))|$).

Thus, summing the total number of vertices in G yields:

$$\begin{aligned} n &= |V(C)| + |V(\text{Int}(C))| + |V(\text{Ext}(C))| \\ &\leq 4 + 2\left[2\left(\frac{\Delta-1}{2}\right) + 1\left(\frac{\Delta-3}{2}\right)\right] \\ &= 3\Delta - 1. \end{aligned}$$

□

The next few results show that if a pentagulation of diameter 3, order n and maximum degree Δ has either of the graphs $\mathcal{H}$ or $\mathcal{I}$ in Figure 5.3 as a subgraph, then $n \leq 3\Delta - 1$. In the proofs of Lemmas 5.17 and 5.19 to follow, we refer to specific vertices and faces of the graphs $\mathcal{H}$ and $\mathcal{I}$. In order to do so, we give a labelling of these vertices and faces, shown in Figure 5.5.

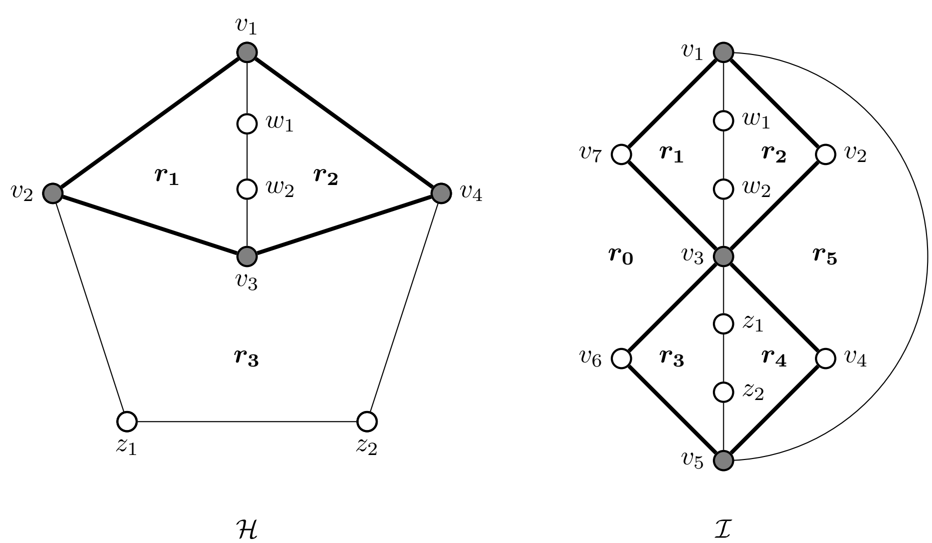


Figure 5.5: The graphs $\mathcal{H}$ and $\mathcal{I}$, with the labels used in the proofs of Lemmas 5.17 and 5.19.

Lemma 5.17. *Let G be a pentagulation of diameter 3, order n and maximum degree Δ. If G contains $\mathcal{H}$ as a subgraph, then:*
(1) G contains a dominating 4-cycle, and
(2) $n \leq 3\Delta - 1$.

Proof. Assume G contains $\mathcal{H}$ (Figure 5.3) as a subgraph, and let $C : v_1, v_2, v_3, v_4$ be the 4-cycle of H. Label the remaining vertices of $\mathcal{H}$ so that v_1, w_1, w_2, v_3 and v_2, z_1, z_2, v_4 are paths of length 3 (see Figure 5.5), with w_1 and w_2 lying in $\text{Int}(C)$ and z_1 and z_2 lying in $\text{Ext}(C)$.

Since G has diameter 3, we know that, without loss of generality, the cycle C dominates its interior by Lemma 4.20. Assume contrary to the statement of the Lemma that C does not dominate its exterior. Then there is a vertex u such that $d(u, C) \geq 2$. If u lies in the region of $\text{Ext}(C)$ that contains v_1 in its boundary, then any $u - w_2$ path has length at least 4. If u lies in the region of $\text{Ext}(C)$ that contains v_3 in its boundary, then $d(u, w_1) \geq 4$. In either case, we obtain a contradiction, so C also dominates its exterior, proving part (1).

Part (2) follow immediately from part (1) and Lemma 5.16. □

Theorem 5.18. *Let G be a pentagulation of diameter 3, order n, and maximum degree $\Delta \geq 3$. If G contains two dislocated 4-cycles, C_1 and C_2, then G contains $\mathcal{I}$ as a subgraph (see Figure 5.3), or $n \leq 3\Delta - 1$. Specifically, if the two 4-cycles have exactly one vertex u in common, and there exists a vertex v in C_1 and a vertex w in C_2 such that $\{u, v\}$ and $\{v, w\}$ dominate disjoint regions of C_1 and C_2 respectively, then G contains $\mathcal{I}$ as a subgraph. In any other case, the order of G satisfies $n \leq 3\Delta - 1$.*

Proof. Let G be a pentagulation of diameter 3, order n and maximum degree $\Delta \geq 3$. Suppose that G contains two dislocated 4-cycles $C_1 : v_1, v_2, v_3, v_4$ and $C_2 : u_1, u_2, u_3, u_4$. We prove the theorem by considering all the different configurations that two dislocated 4-cycles could have. Note that if any 4-cycle of G dominates both its interior and exterior regions, or if G contains an $\mathcal{H}$ subgraph, then $n \leq 3\Delta - 1$ by Lemmas 5.16 and 5.17. We assume without loss of generality that C_1 dominates its interior. Per Theorem 5.15, and without loss of generality, the region $\text{Int}(C_1)$ is dominated by $\{v_1, v_3\}$, and there exist two distinct vertices w_1 and w_2 in $\text{Int}(C_1)$ such that $P_1 : v_1, w_1, w_2, v_3$ is a path in G.

Case 1: The dislocated cycles C_1 and C_2 have exactly two adjacent vertices in common.
By symmetry, we may assume without loss of generality that $v_2 = u_1$ and $v_3 = u_4$ (See Figure 5.6, (1)).

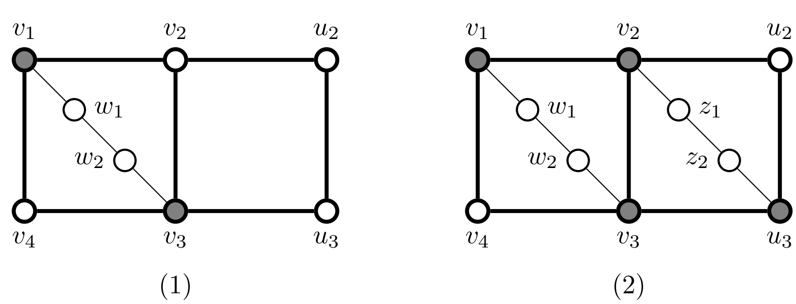


Figure 5.6: Two dislocated 4-cycles, C_1 and C_2, that share an edge, as in Case 1 of the proof of Theorem 5.18.

Since C_1 and C_2 are dislocated, both u_2 and u_3 lie in $\text{Ext}(C_1)$. By Corollary 5.7, the pentagulation G is triangle-free. Thus $d_G(w_1, C_2) = 2$, as there is no way to make w_1 adjacent to a vertex of C_2 without inducing a triangle, or a crossing edge (which is forbidden, since G is a plane graph by assumption). Since C_2 dominates either its interior or exterior, we have that C_2 dominates its interior. By an application of Theorem 5.15, there exist vertices z_1 and z_2 in $\text{Int}(C_2)$ such that either $P_1 : v_2, z_1, z_2, u_3$ is a path in G, or $P_2' : u_2, z_1, z_2, v_3$ is a path in G.

If G contains the path P_2', then there is a $z_1 - w_1$ path R of length at most 3 in G. Since G is triangle free, the vertex w_1 is only adjacent to v_1 and w_2, and z_1 is only adjacent to u_2 and z_2. Thus, since G is a plane graph and $d_G(w_1, z_1) \leq 3$, v_1 and u_2 are adjacent. This induces a triangle, which is impossible as G is triangle-free. Therefore G contains the path P_2, not the path P_2' (see Figure 5.6, (2)).

Since G has diameter 3, there exists some $w_1 - z_2$ path of length at most 3. By the same argument as in the prior paragraph, and noting that G is a triangle-free plane graph, we deduce that v_1 and u_3 are adjacent (i.e., the only possible path of length at most 3 is w_1, v_1, u_3, z_2). But now we have induced $\mathcal{H}$ as a subgraph of G, with the 4-cycle of $\mathcal{H}$ corresponding to the 4-cycle of G on $v_1, v_2 = u_1, v_3 = u_4, u_3$. Since G contains $\mathcal{H}$ as a subgraph, we conclude this case by noting that Lemma 5.17 yields $n \leq 3\Delta - 1$.

Case 2: The dislocated cycles C_1 and C_2 have exactly three vertices in common.
Up to symmetry, there are two different ways that C_1 could share three vertices with C_2: the cycles may share both the dominating vertices v_1 and v_3, or only one of them.

Case 2.1: The vertices v_1 and v_3 are in both C_1 and C_2.
Assume without loss of generality that $v_1 = u_1$, $v_2 = u_4$ and $v_3 = u_3$ (see Figure 5.7 (1)).

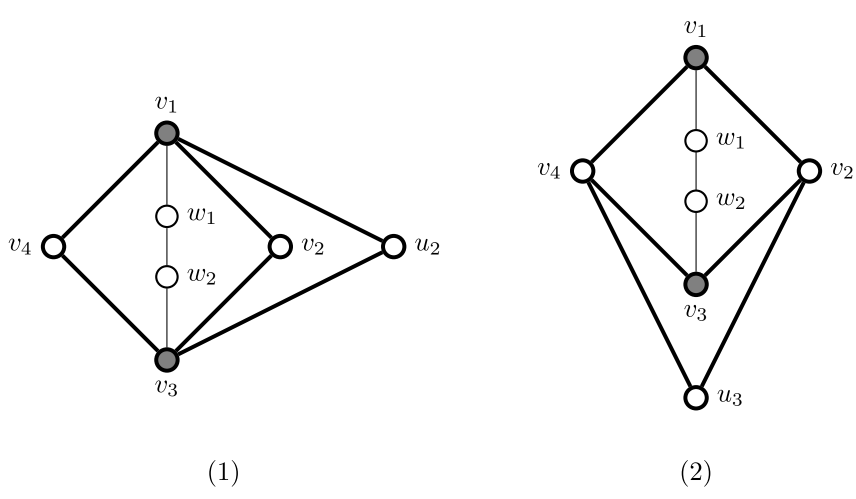


Figure 5.7: Case 2.1 in the proof of Theorem 5.18 has the two dislocated 4-cycles C_1 and C_2 sharing v_1, v_2 and v_3. Case 2.2 has the cycles sharing v_2, v_3 and v_4.

Since C_1 and C_2 are dislocated, the set $\{u_2, v_2\}$ dominates either the interior or exterior of C_2. We claim the set dominates the interior, not the exterior, of C_2. By Lemma 5.9, the vertex v_2 does not have any neighbour in $\text{Int}(C_1)$, and thus has no neighbours in $\text{Ext}(C_2)$. Per Lemma 5.4, no single vertex of C_2 dominates the exterior of C_2, so the set $\{v_2, u_2\}$ does not dominate $\text{Ext}(C_2)$, proving the claim.

Since $\{u_2, v_2\}$ dominates $\mathrm{Int}(C_2)$, there are two vertices z_1 and z_2 in $\mathrm{Int}(C_2)$ such that $P_2 : v_2, z_1, z_2, u_2$ is a path in G. The vertices of $C_1 \cup C_2 \cup P_1 \cup P_2$ induce an $\mathcal{H}$ subgraph in G.

Case 2.2: Only one of v_1 and v_3 is common to both C_1 and C_2.
Assume without loss of generality that $v_2 = u_2$, $v_3 = u_1$ and $v_4 = u_4$ (see Figure 5.7 (2)). Since G is triangle-free, the distance $d_G(w_1, C_2) = 2$, so C_2 does not dominate its exterior and thus dominates its interior. By Theorem 5.15, there are vertices z_1 and z_2 in $\mathrm{Int}(C_2)$ such that either $P_2 : v_3, z_1, z_2, u_3$ is a path of G, or $P_2' : v_2, z_1, z_2, v_4$ is a path of G. In the latter case, we obtain an $\mathcal{H}$ subgraph on $C_1 \cup C_2 \cup P_1 \cup P_2'$. The former case, in which we have the path P_2, is not possible. This is because any way of creating a $w_1 - z_2$ path of length at most 3 necessarily creates either a triangle or an edge crossing in G.

Case 3: The cycles C_1 and C_2 have exactly one vertex in common.
Since C_1 and C_2 only share one vertex, and G is triangle-free, either $d(w_1, V(C_2)) \geq 2$ or $d(w_2, V(C_2)) \geq 2$. As such, C_2 does dominate its exterior, and thus dominates its interior. Up to symmetry, there are four possible cases.

Case 3.1: The dislocated cycles C_1 and C_2 share the vertex $v_2 = u_4$ and $\mathrm{Int}(C_2)$ is dominated by $\{u_1, u_3\}$ (see Figure 5.8 (1)).

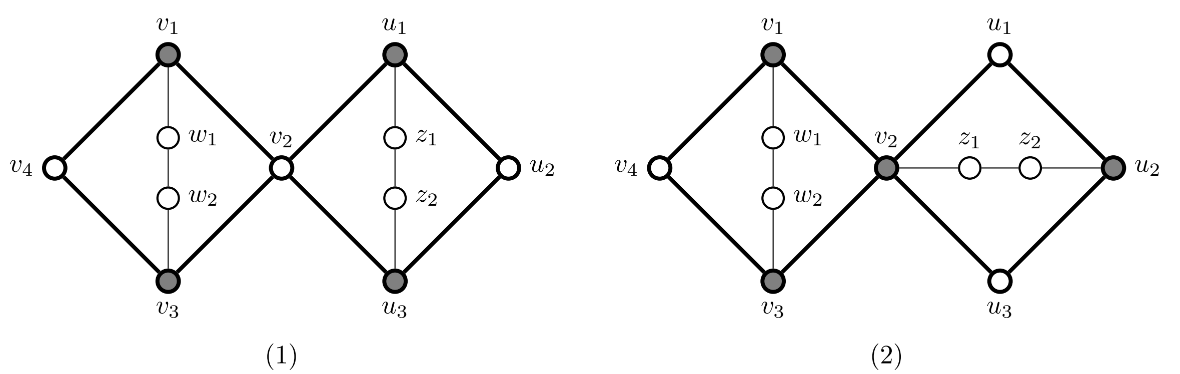


Figure 5.8: In both figures, the dislocated 4-cycles C_1 and C_2 share the vertex $v_2 = u_4$. We have (1) when the interior of C_2 is dominated by $\{u_1, u_3\}$, as in Case 3.1, and we have (2) when the interior of C_2 is dominated by $\{u_2, u_4\}$, as in Case 3.2 of the proof of Theorem 5.18.

By Theorem 5.15 and the fact that G is triangle-free, there is a vertex z_1 in $\mathrm{Int}(C_2)$ that is adjacent to u_1, but not to any other vertex of C_2. But since $d_G(w_1, z_1) \leq 3$, there is either a triangle or a pair of crossing edges in G, so this case is not possible.

Case 3.2: The dislocated cycles C_1 and C_2 share the vertex $v_2 = u_4$ and $\mathrm{Int}(C_2)$ is dominated by $\{u_2, u_4\}$ (see Figure 5.8 (2)).
By Theorem 5.15, there are two vertices z_1 and z_2 in the interior of C_2 such that $P_2 : v_2, z_1, z_2, u_2$ is a path in G. Since G is a triangle-free plane graph, and both $d_G(z_2, w_1) \leq 3$ and $d_G(z_2, w_w) \leq 3$, we have that u_2 is adjacent to both v_1 and v_3. Thus, by considering the vertices of $P_1 \cup P_2$, we observe that G contains $\mathcal{H}$ as a subgraph.

Case 3.3: The dislocated cycles C_1 and C_2 share the vertex $v_3 = u_1$ and $\text{Int}(C_2)$ is dominated by $\{u_2, u_4\}$ (see Figure 5.9 (1)).

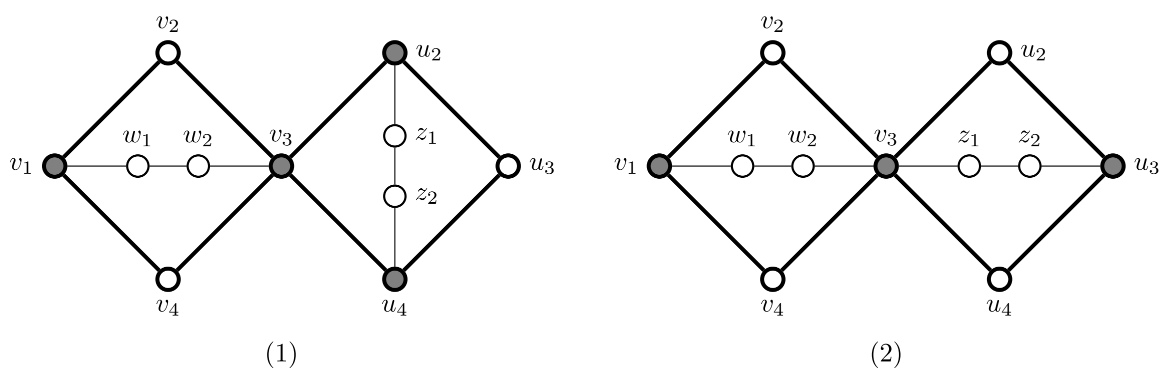


Figure 5.9: In both figures, the dislocated 4-cycles C_1 and C_2 share the vertex $v_3 = u_1$. When the interior of C_2 is dominated by u_2 and u_4, as in Case 3.3 of the proof of Theorem 5.18, (1) occurs. When the interior of C_2 is dominated by u_1 and u_3, as in Case 3.4, (2) occurs.

Reversing the roles of the cycles C_1 and C_2, we observe that this case is identical to Case 3.2, hence G contains $\mathcal{H}$ as a subgraph, and so $n \leq 3\Delta - 1$.

Case 3.4: The dislocated cycles C_1 and C_2 share the vertex $v_3 = u_1$ and $\text{Int}(C_2)$ is dominated by $\{u_1, u_3\}$ (see Figure 5.9 (2)).
By Theorem 5.15, there are vertices z_1 and z_2 in $\text{Int}(C_2)$ such that $P_2 : v_3, z_1, z_2, u_3$ is a path in G. It is only possible to find a $w_1 - z_2$ path of length at most 3 in G if v_1 and u_3 are adjacent. But then $\mathcal{I}$ is a subgraph of G, completing the proof of this case.

Case 4: The dislocated cycles C_1 and C_2 are disjoint.
In this case, it is not possible that any vertex of C_2 is adjacent to w_1, so C_2 dominates its interior. Per Theorem 5.15, and without loss of generality, there are two vertices z_1 and z_2 in the interior of C_2 and edges u_1z_1, z_1z_2 and z_2u_3. Since G has diameter 3, we have that $d_G(w_i, z_j) \leq 3$ for any indices i and j in $\{1, 2\}$. Since G is triangle-free, it contains all four edges of the form u_iw_k, where i and k are in $\{1, 3\}$. However, noting the 4-cycle on v_1, u_1, v_3, u_3, we see that G contains $\mathcal{H}$ as a subgraph.

Case 5: The dislocated cycles C_1 and C_2 share exactly two non-adjacent vertices.

Up to symmetry, there are two subcases to consider. Either $v_1 = u_1$ and $v_3 = u_3$ are common to both C_1 and C_2, or the vertices $v_2 = u_2$ and $v_4 = u_4$ are. In both cases, since C_1 and C_2 are dislocated, the set $\{u_2, u_4\}$ of vertices dominates the interior of C_2 (it is certainly not possible for them to dominate the exterior, as neither is adjacent to w_1). Thus, in both cases, per Theorem 5.15, there are vertices z_1 and z_2 in $\text{Int}(C_2)$ such that $P_2 : u_2, z_1, z_2, u_4$ is a path in G.

Case 5.1: The vertices v_1 and v_3 are common to C_1 and C_2 (See Figure 5.10 (1)).
Consider the cycle $C : v_1, v_2, v_3, u_4$. Since z_1 cannot possibly be adjacent to a vertex of C, the cycle C

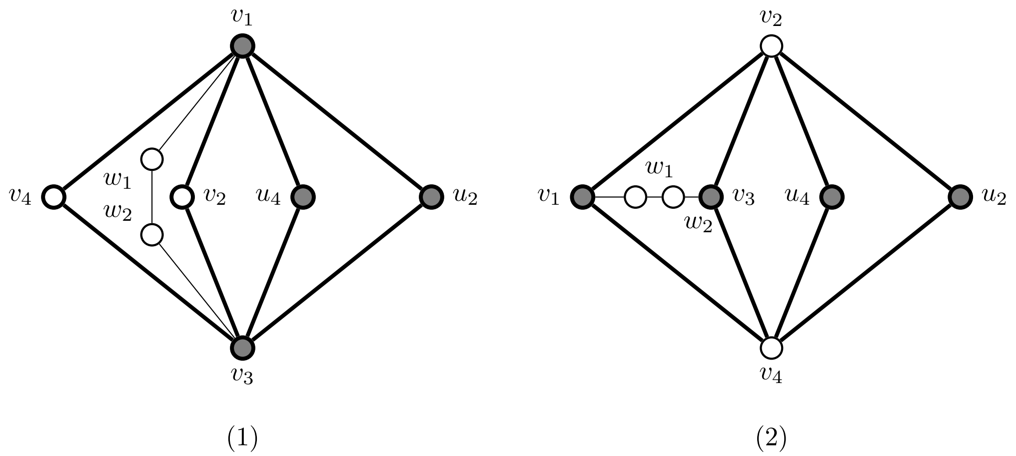


Figure 5.10: In (1), the dislocated 4-cycles C_1 and C_2 share vertices $v_1 = u_1$ and $v_3 = u_3$, as in Case 5.1 of Theorem 5.18. In Figure (2), the cycles share vertices $v_2 = u_1$ and $v_4 = u_3$, as in Case 5.2.

dominates its interior. If $\{v_1, v_3\}$ dominates $\text{Int}(C)$, then C and C_2 are dislocated 4-cycles sharing three vertices, and by Case 2 we have that $n \leq 3\Delta - 1$. Similarly, if $\{v_2, u_4\}$ dominates $\text{Int}(C)$, then C and C_1 are dislocated.

Case 5.2: The vertices v_2 and v_4 are common to C_1 and C_2 (See Figure 5.10 (2)).
Denote by C' the cycle on v_2, v_3, v_4, u_4. By the argument of the preceding paragraph, C' and C_1 are dislocated 4-cycles. Thus, by Case 2, $n \leq 3\Delta - 1$. □

Lemma 5.19. *Let G be a pentagulation of diameter 3, order n and maximum degree Δ. If G contains $\mathcal{I}$ as a subgraph, then $n \leq 3\Delta - 1$.*

Proof. Let G be a pentagulation of diameter 3, order n and maximum degree Δ that contains $\mathcal{I}$ as a subgraph. Let the vertices of $\mathcal{I}$ be labelled as they are in Figure 5.5, such that the vertices w_1 and z_1 lie in the interiors of the 4-cycles $C_1 : v_1, v_2, v_3, v_7$ and $C_2 : v_3, v_4, v_5, v_6$, respectively.

Notice that every face of $\mathcal{I}$ is bounded by a 5-cycle. Thus, by Corollary **??**, the subgraph $\mathcal{I}$ is an induced subgraph of G. Therefore, $d_G(z_2, C_1) = 2$, and by a similar argument, $d_G(w_1, C_2) = 2$. Hence, by Lemma 4.20, we deduce that the cycles C_1 and C_2 dominate their interiors. In particular, the set $\{v_1, v_3\}$ dominates $\text{Int}(C_1)$, and $\{v_3, v_5\}$ dominates $\text{Int}(C_2)$.

We refine our choice of embedding of G (or equivalently, our choice of subgraph isomorphic to $\mathcal{I}$), so that the interiors of the cycles C_1 and C_2 are maximal. In other words, there does not exist a 4-cycle C_1' such that $\text{Int}(C_1) \subset \text{Int}(C_1')$ and $\text{Int}(C_1')$ is dominated by $\{v_1, v_3\}$, and likewise for C_2.

Assume for the sake of contradiction that $n > 3\Delta - 1$.

Suppose that every vertex of $V(G) - V(\mathcal{I})$ is adjacent to at least one of v_1, v_3 or v_5. Then:

$$\begin{aligned} n &= |V(\mathcal{I})| + |V(G) - V(\mathcal{I})| \\ &\leq 11 + (d(v_1) - 4) + (d(v_3) - 6) + (d(v_5) - 4) \\ &\leq 11 + 3\Delta - 14 \\ &\leq 3\Delta - 3, \end{aligned}$$

and so $n \leq 3\Delta - 1$.

We thus assume that G contains vertices in $V(G) - V(\mathcal{I})$ that are not adjacent to any of v_1, v_3 or v_5. Let x be such a vertex, and label the faces $r_0, r_1, \ldots, r_5$ of $\mathcal{I}$ as they are labelled in Figure 5.5.

The regions $r_1 \cup r_2$, and $r_3 \cup r_4$ are dominated by the 4-cycles C_1 and C_2, respectively, and as such any vertex added to these regions is adjacent to a vertex in the set $\{v_1, v_3, v_5\}$. Thus we may assume that x is not in any of the regions r_1, r_2, r_3 or r_4.

By the symmetry of r_0 and r_5, we assume without loss of generality that x is in r_5. If x is adjacent to v_2 and v_4, then we induce a 4-cycle $C : v_2, x, v_4, v_3$, that shares an edge with the cycle C_1. Further, since $d(w_1, C) = 2$, C dominates its interior. Thus C and C_1 are dislocated 4-cycles that share an edge, so we have that $n \leq 3\Delta - 1$ by Theorem 5.18, a contradiction. Hence we may assume that x is not adjacent to both v_2 and v_4.

There are two cases to consider.

Case 1: The vertex x is not adjacent to either v_2 or v_4.
In this case, the vertex x is not adjacent to any of $v_1, \ldots, v_5$. Since the diameter of G is 3, we have that x is within distance 3 of each of w_1, w_2, z_1, z_2. Thus x has neighbours y_1, y_2 and y_3 in r_5 such that y_1v_1, y_2v_3 and y_3v_5 are all edges in G. Note that $y_1 \neq y_3$ as this induces a triangle with vertex set $\{v_1, y_1, v_5\}$. We further claim that $y_1 \neq y_2$. Assume to the contrary that $y_1 = y_2$, and let C be the 4-cycle on v_1, v_2, v_3, y_1, v_1. Since G is a triangle-free plane graph, we have that $d_G(z_2, C) = 2$, so C dominates its interior. By the maximality of C_1, we deduce that C and C_1 are dislocated 4-cycles that share more than one vertex. Thus we conclude that $n \leq 3\Delta - 1$ by Theorem 5.18, proving the claim. By symmetry, we also deduce that $y_2 \neq y_3$, so the three vertices y_1, y_2 and y_3 are distinct.

The paths $Q_1 : v_1, y_1, x$, $Q_2 : v_3, y_2, x$ and $Q_3 : v_5, y_3, x$ divide the region r_5 up into three sub-regions. Let r_6 denote the region with vertices $v_1, v_2, v_3, y_2, x, y_1$ on its boundary, let r_7 be bounded by $v_3, y_2, x, y_3, v_5, v_4$, and let r_8 be bounded by v_1, y_1, x, y_3, v_5.

We claim that the subgraph $\mathcal{I}' = \mathcal{I} \cup Q_1 \cup Q_2 \cup Q_3$ of G is an induced subgraph. Any edge between two vertices on the boundary of any region $r_0, \ldots, r_4$ necessarily induces a triangle, which is not possible since G is triangle-free. Similarly, there is no edge crossing the region r_8. Any edge crossing r_6 either creates a triangle, which is not possible, or a 4-cycle C such that C_1 and C are two dislocated 4-cycles which share at least two vertices. By Theorem 5.18, we conclude that should there be such a 4-cycle C, then $n \leq 3\Delta - 1$, contrary to assumption. The argument that no edges crosses the region r_7 proceeds identically to the argument used for r_6, by replacing the role of C_1 with C_2.

If there exists a vertex in r_6, it is adjacent to v_1 or v_3 since it is within distance 3 of z_2. Similarly, any vertex in r_7 is adjacent to v_3 or v_5 as it is within distance 3 of w_1. No vertex lies in r_8, as it would need to be adjacent to both v_1 and v_5 to be within distance 3 of w_2 and z_1 respectively, which would induce a triangle on y_4, v_1, v_5. Any vertex of r_0 is adjacent to one of v_1, v_3, v_5 to be within distance 3 of x.

The subgraph $\mathcal{I}'$ has 15 vertices, and we have shown that every vertex of $G - \mathcal{I}'$ is adjacent to one of v_1, v_3 or v_5. Noting that $d_{\mathcal{I}'}(v_1) = 5$, $d_{\mathcal{I}'}(v_3) = 7$ and $d_{\mathcal{I}'}(v_5) = 5$, we can bound the order of G:

$$\begin{aligned} n &\leq 15 + (d(v_1) - 5) + (d(v_3) - 7) + (d(v_5) - 5) \\ &\leq 3\Delta - 2 \\ &< 3\Delta - 1. \end{aligned}$$

Case 2: The vertex x is adjacent to v_2.
By assumption, the vertex x is not adjacent to any of v_1, v_3, v_4 or v_5, however there is an $x - z_2$ path of length at most 3. Noting that no two vertices on the boundary of r_5 are adjacent, we see that there exists some vertex y_1 in r_5 such that there is a path $S_1 : v_2, x, y_1, v_5$ in G.

We claim that $\mathcal{I} \cup S_1$ is an induced subgraph of G. Since G is triangle-free, there is not an edge crossing a region bounded by a 5-cycle. Thus the only region of $\mathcal{I} \cup S_1$ that may contain any additional edge is the region bounded by the two paths S_1 and v_2, v_3, v_4, v_5. However, any edge added between the vertices bounding this region creates either a triangle, which is impossible, or two 4-cycles A_1 and A_2. No matter how they are formed, every vertex of A_1 and A_2 is distance at least 2 from w_1, so both cycles dominate their interiors. Thus, for some indices i and j in $\{1, 2\}$, the cycles C_i and A_j are a pair of dislocated 4-cycles that share at least two vertices. By Theorem 5.18, we obtain that $n \leq 3\Delta - 1$, proving the claim.

Because the distance $d_G(y_1, w_2) \leq 3$, and since $\mathcal{I} \cup S_1$ is an induced subgraph of G, there exists some vertex y_2 in $r_5 - \{x, y_1\}$ such that G contains the path $S_2 : y_1, y_2, v_3$.

Let $\mathcal{I}'' = \mathcal{I} \cup S_1 \cup S_2$, and note that the paths S_1 and S_2 divide r_5 into three sub-regions: $r_6 = \mathrm{Int}(v_1, v_2, x, y_1, v_5)$, $r_7 = \mathrm{Int}(v_2, v_3, y_2, y_1, x)$ and $r_8 = \mathrm{Int}(v_3, y_2, y_1, v_5, v_4)$. We show that any vertex in G which is not in $\mathcal{I}''$ is adjacent to one of v_1, v_3 or v_5.

Since G is triangle-free, and every face of $\mathcal{I}''$ is bounded by a 5-cycle, we see that $\mathcal{I}''$ is an induced subgraph of G. As such, the only vertices on the boundary of r_6 within distance 2 of w_2 are v_1 and v_2. The region r_6 does not contain any vertices per Lemma 5.8, as it is dominated by two adjacent vertices. Similarly, the region r_7 contains no vertices, as the only vertices on the boundary of r_7 within distance 2 of w_1 are the adjacent pair v_2 and v_3.

Any vertex of $G - \mathcal{I}''$ in r_8 is adjacent to either v_3 or v_5, in order to be distance at most 3 from w_1, and any vertex in r_0 is adjacent to one of v_1, v_3 or v_5 to ensure that it is distance at most 3 from x.

Note that $\mathcal{I}''$ has 14 vertices, and that $d_{\mathcal{I}''}(v_1) = 4$, $d_{\mathcal{I}''}(v_3) = 7$ and $d_{\mathcal{I}''}(v_5) = 5$. Since we know that any vertex of $G - \mathcal{I}''$ is adjacent to one of v_1, v_2 or v_3, we can bound the order of G:

$$n \leq 14 + (d(v_1) - 4) + (d(v_3) - 7) + (d(v_5) - 5)$$
$$n \leq 3\Delta - 2.$$

In every case, we have derived a contradiction, completing the proof. □

In Theorem 5.18, we demonstrated that a pentagulation of diameter 3 with two dislocated 4-cycles either satisfies the desired bound $n \leq 3\Delta - 1$, or contains $\mathcal{I}$ as a subgraph. In Lemma 5.19, we showed that if our pentagulation contains $\mathcal{I}$ as a subgraph, then it again satisfies $n \leq 3\Delta - 1$. Thus, we conclude this section with Theorem 5.20 below.

Theorem 5.20. *Let G be a pentagulation of diameter 3, order n and maximum degree $\Delta \geq 8$. If G contains either a dominating 4-cycle, or two dislocated 4-cycles, then $n \leq 3\Delta - 1$.*

Proof. This follows immediately from Lemma 5.16, Theorem 5.18 and Lemma 5.19. □

5.6 Bounding the order, part II: The lonely 4-cycle

We have shown that if G has two dislocated 4-cycles, then it satisfies $n \leq 3\Delta - 1$. We will now show the same bound holds if G contains some 4-cycle, but no dislocated pair of them. Throughout this section, we will be working with pentagulations of diameter 3 that contain some 4-cycle C. We can assume without loss of generality that our 4-cycle dominates its interior. This motivates the following terminology. In this section, a 4-cycle C of a plane graph is **interior maximal** if it dominates its interior, and there does not exist any other 4-cycle C' such that C' dominates its interior, and $\text{Int}(C) \subset \text{Int}(C')$.

Not containing two dislocated 4-cycles is a very strong structural restriction for a pentagulation, as the following two lemmas will show.

Lemma 5.21. *Let G be a pentagulation of diameter 3 that does not contain two dislocated 4-cycles, and let C be an interior maximal 4-cycle of G. If D is any cycle in G such that $V(D) \subset Ext[C]$ and $\ell(D) \leq 7$, then D is chordless.*

Proof. Assume to the contrary D has some chord e. The subgraph $D \cup \{e\}$ either contains a 3-cycle, or a 4-cycle. If it contains a 3-cycle, then we obtain a contradiction, since G is triangle-free per Corollary 5.7. If it contains a 4-cycle, then either this 4-cycle contradicts the maximality of C, or is dislocated from C, and we derive a contradiction in both cases. □

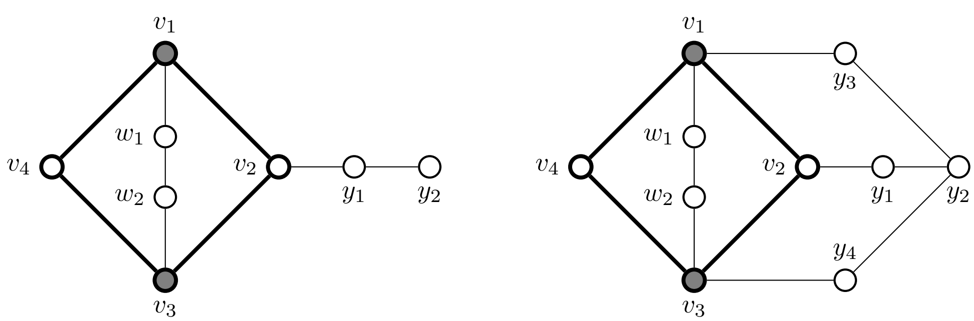
Lemma 5.22. *Let G be a pentagulation of diameter 3 that does not contain two dislocated 4-cycles, and let C be an interior maximal 4-cycle of G. If D is any 5-cycle in G such that both $Int(D) \subset Ext(C)$ and $Int(D)$ is dominated by two or fewer vertices of D, then $Int(D)$ does not contain any vertex of G.*

Proof. By Lemma 5.8, the interior of D is not dominated by either a single vertex, or a pair of vertices that are adjacent in D. Assume contrary to the statement of the lemma that there is a vertex w in $\text{Int}(D)$, and let u and v be two non-adjacent vertices of D that dominate $\text{Int}(D)$. Per Corollary 5.12, the vertex w is adjacent to both u and v. Thus, there exists some 4-cycle in $\text{Int}[D]$ that dominates its interior. The existence of this 4-cycle either contradicts the maximality of C, or the fact that G does not contain two dislocated 4-cycles. □

We now present the main theorem of this section.

Theorem 5.23. *Let G be a pentagulation of diameter 3, order n and maximum degree $\Delta \geq 8$. If G contains a 4-cycle, then $n \leq 3\Delta - 1$.*

Proof. Assume to the contrary that G contains a 4-cycle $C_1 = v_1, v_2, v_3, v_4$, and has order $n > 3\Delta - 1$. By Theorem 5.20, we may assume that there do not exist any two dislocated 4-cycles in G. Assume without loss of generality that C_1 is interior maximal, and that $\text{Int}(C_1)$ is dominated by $\{v_1, v_3\}$. By Theorem 5.15, there exist vertices w_1 and w_2 in $\text{Int}(C_1)$ such that $P_1 : v_1, w_1, w_2, v_3$ is a path in G. If every vertex of G is adjacent to either v_1 or v_3, then $n \leq 2\Delta < 3\Delta - 1$, so there exists some vertex of G in $\text{Ext}(C_1)$ which is not adjacent to v_1 or to v_3. We consider two cases, according to whether or not the vertices v_2 and v_4 have neighbours in $\text{Ext}(C_1)$.



(1) (2)

Figure 5.11: In Case 1, since the vertex y_1 is not an end-vertex, there exists some neighbour y_2 of y_1 (1). Since the diameter of G is 3, it contains $y_2 - w_1$ and $y_2 - w_2$ paths, forcing the subgraph $\mathcal{G}$ (2).

Case 1: The vertex v_2 has at least one neighbour in $\text{Ext}(C_1)$.
Let y_1 be a vertex in the exterior of C_1 that is adjacent to v_2. The vertex y_1 is not adjacent to either v_1 or v_3 as this would induce a triangle, contradicting Corollary 5.7. Further, y_1 is not adjacent to v_4 as this induces a 4-cycle on the vertices v_2, y_1, v_3, v_4, contradicting the fact that G does not contain two dislocated 4-cycles. Since y_1 is not an end vertex (G is 2-connected), there is some other vertex y_2 in $\text{Ext}(C_1)$ to which y_1 is adjacent (See Figure 5.11 (1)).

Note that y_2 has degree at least 2, and there exist $y_2 - w_1$ and $y_2 - w_2$ paths of length at most 3. Since G is triangle-free, the vertices y_2 and v_2 are not adjacent. Further, y_2 is not adjacent to either v_1 or v_3, as this induces a 4-cycle on the vertices v_1, v_2, y_1, y_2 or v_3, v_2, y_1, y_2 respectively. In either case, the induced 4-cycle would be dislocated from C_1, contradicting our assumption. Finally, y_1 is not adjacent to v_4, as this induces

$\mathcal{G}$ $\mathcal{K}$

Figure 5.12: If G is a diameter 3 pentagulation that contains some 4-cycle, but no two dislocated 4-cycles, it must contain one of $\mathcal{G}$ or $\mathcal{K}$ as a subgraph, per Cases 1 and 2 respectively in the proof of Theorem 5.23. The black vertices of $\mathcal{K}$ are not adjacent to any vertices of $G - \mathcal{K}$.

$\mathcal{H}$ as a subgraph of G, which yields a contradiction by Lemma 5.17. Since no $y_2 - w_1$ or $y_2 - w_2$ geodesic can be formed with the vertices mentioned thus far, there exist vertices y_3 and y_4 in $\text{Ext}(C_1)$ such that y_2y_3, y_3v_1, y_2y_4 and y_4v_3 are edges in G (See Figure 5.11 (2)). Note that $y_3 \neq y_4$, as this would again induce $\mathcal{H}$ as a subgraph of G.

Let $\mathcal{G}$ denote the subgraph of G constructed thus far (See Figure 5.12). By applying Lemma 5.21 to the cycles bounding the faces of $\mathcal{G}$, we deduce that $\mathcal{G}$ is an induced subgraph of G. Thus, the only two vertices of the 5-cycle $C_3 : v_1, v_2, y_1, y_2, y_3$ within distance 2 of w_2 are v_1 and v_2, so these two vertices dominate $\text{Int}(C_2)$. Hence, by Lemma 5.8, there is no vertex in $\text{Int}(C)$. Similarly, there is also no vertex in the region bounded by the cycle $C_4 : v_2, y_1, y_2, y_4, v_3$.

Any vertex of G not adjacent to v_1 or v_3 for which we have not yet accounted lies in the external region of the cycle $C_2 = v_1, y_3, y_2, y_4, v_3, v_4$. There are four subcases to consider.

Case 1.1: There exists some vertex u_1 in $\text{Ext}(C_2)$ adjacent to v_4.
Since G is triangle-free, u_1 is not adjacent to either v_1 or v_3. Because G does not contain two dislocated 4-cycles, u_1 is adjacent to neither y_3 nor y_4. As a result of the prior two sentences, and the fact that $\mathcal{G}$ is an induced subgraph of G, we deduce that a $u_1 - y_1$ geodesic contains the vertex y_2. Either u_1 is adjacent to y_2, or there exists a vertex u_2 in the exterior of C_2 such that $P_2 : u_1, u_2, y_2, y_1$ is a geodesic in G.

If u_1 and y_2 are adjacent, then $\text{Ext}(C_2)$ is subdivided into 2 regions: the region r_1 with vertices u_1, v_4, v_1, y_3 and y_2 on its boundary, and the region r_2 with u_1, v_4, v_3, y_4 and y_2 on its boundary. Since G is triangle-free, the subgraph $\mathcal{G} \cup \{u_1, u_1v_4, u_1y_2\}$ is an induced subgraph of G. Thus, the only vertices on the boundary of r_1 within distance 2 of w_2 are the adjacent pair v_1 and v_4. We see that r_1 is dominated by two adjacent vertices of the 5-cycle bounding it, so by Lemma 5.8, the region r_1 does not contain any vertices. Similarly, the region r_2 is also empty, so every vertex of G not yet mentioned is adjacent to either v_1 or v_3, and we can bound the order of G.

$$
\begin{aligned}
n &= |V(\mathcal{G}) \cup \{u_1\}| + |V(G) - V(\mathcal{G}) - \{u_1\}| \\
&\leq 11 + (d(v_1) - 4) + (d(v_3) - 4) \\
&\leq 2\Delta + 3 \\
&\leq 3\Delta - 1 \text{ when } \Delta \geq 4.
\end{aligned}
$$

This contradicts our assumption, and so the geodesic contains the new vertex u_2 (see Figure 5.13).

Let $\mathcal{G}' = \mathcal{G} \cup P_2 \cup \{u_1 v_4\}$, and note that we can see, by multiple applications of Lemma 5.21, that $\mathcal{G}'$ is an induced subgraph of G. Thus, the only way there can exist a $u_2 - w_1$ path of length at most 3 is if there is some vertex u_3 that is adjacent to both u_2 and v_1. Similarly, to ensure that $d_G(u_2, w_2) \leq 3$, there exists a vertex u_4 that is adjacent to u_2 and v_4 (see Figure 5.13). Since G is a plane graph, the vertices u_3 and u_4 are distinct.

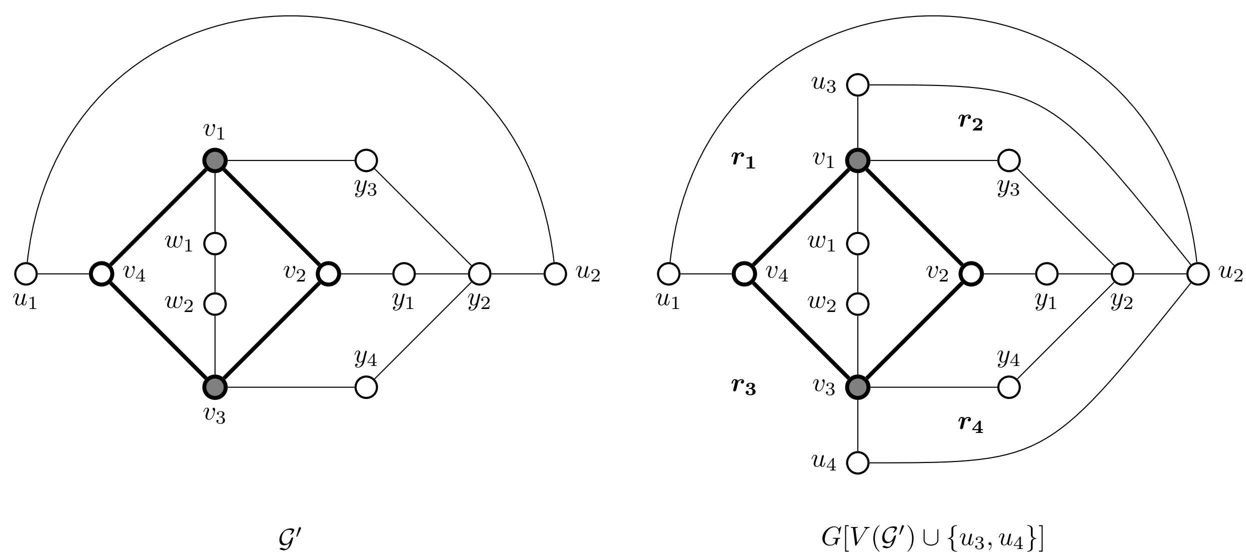


Figure 5.13: In Case 1.1 of the proof of Theorem 5.23, we assume that there is a vertex u_1 adjacent to v_4. As a result, we obtain first that $\mathcal{G}'$ is a subgraph of G (left), and then that G also contains the vertices u_3 and u_4 (right).

The region $\text{Ext}(C_2)$ is divided into four subregions, all of which are bounded by 5-cycles. Label these regions as follows: $r_1 = \text{Int}(u_1, v_4, v_1, u_3, u_2)$, $r_2 = \text{Int}(v_1, u_3, u_2, y_2, y_3)$, $r_3 = \text{Ext}(u_1, v_4, v_3, u_4, u_2)$, $r_4 = \text{Int}(v_3, y_4, y_2, u_2, u_4)$ (see Figure 5.13).

The only two vertices on the boundary of r_1 within distance 2 of w_2 are v_1 and v_4. Thus the adjacent pair $\{v_1, v_4\}$ dominates r_1, and by Lemma 5.8, the region r_1 does not contain any vertices. By a similar argument, the region r_3 does not contain any vertices. The only vertex on the boundary of r_2 within distance 2 of w_2 is v_1, and so r_2 is dominated by v_1. Per Lemma 5.8, the region r_2 contains no vertex of G, and neither does

r_4 by a similar argument. We deduce that all vertices of G not yet mentioned lie in the interior of C_1, and hence are adjacent to either v_1 or v_3. This allows us to bound the order of G:

$$\begin{aligned} n &= |V(\mathcal{G}') \cup \{u_3, u_4\}| + |V(G) - V(\mathcal{G})' - \{u_3, u_4\}| \\ &\leq 14 + (d(v_1) - 5) + (d(v_3) - 5) \\ &\leq 2\Delta + 4 \\ &\leq 3\Delta - 1 \text{ when } \Delta \geq 5. \end{aligned}$$

Case 1.2: There is some vertex u_1 in $\mathrm{Ext}(C_2)$ that is adjacent to y_2, but there does not exist any vertex in $\mathrm{Ext}(C_2)$ adjacent to v_4.
Since G is triangle-free, the vertex u_1 is adjacent to neither y_3 nor y_4. Because G does not contain two dislocated 4-cycles, the vertex u_1 is adjacent to neither v_1 nor v_3. We thus deduce that, in order for there to exist $u_1 - w_1$ and $u_1 - w_2$ geodesics of length at most 3, there are vertices u_2 and u_3 in $\mathrm{Ext}(C_2)$ such that $Q_1 : u_1, u_2, v_1$ and $Q_2 : u_1, u_3, v_3$ are paths in G.

Note that $u_2 \neq u_3$, as this would induce a 4-cycle on the vertex set $\{u_2, v_1, v_4, v_3\}$. This 4-cycle is either dislocated from C_1, contradicting our assumption, or it is not dislocated from C_1, contradicting the maximality of C_1. Denote by $\mathcal{G}^*$ the graph $\mathcal{G} \cup Q_1 \cup Q_2 \cup \{y_2u_1\}$, and observe that $\mathcal{G}^*$ is chordless per Lemma 5.21 (See Figure 5.14).

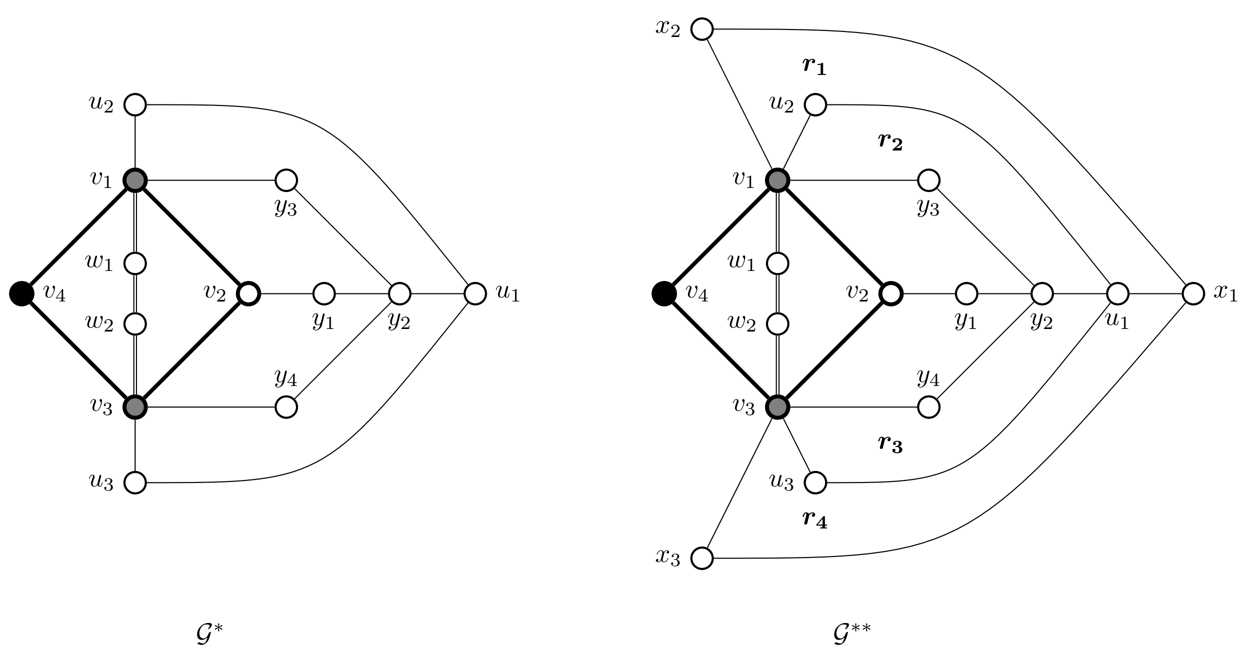


Figure 5.14: In Case 1.2, we obtain first that $\mathcal{G}^*$, and then $\mathcal{G}^{**}$, are subgraphs of G. The black vertex v_4 does not have any neighbours in G besides v_1 and v_3.

We claim that the region bounded the cycle $C_5 : v_1, u_2, u_1, y_2, y_3$ is empty. The only vertex on the boundary of this region that is within distance 2 of w_2 is v_1, so v_1 dominates $\mathrm{Int}(C_5)$. By Lemma 5.8, we conclude that

$\text{Int}(C_5)$ is empty as claimed. Similarly, the interior of the cycle $C_6 : v_3, u_3, u_1, y_2, y_4$ is empty.

Observe that if every vertex of $G - \mathcal{G}^*$ were adjacent to v_1 or v_3, then the order of G would be bounded as follows:

$$\begin{aligned}
n &= |V(\mathcal{G}^*)| + |V(G) - V(\mathcal{G}^*)| \\
n &\leq 13 + (d(v_1) - 5) + (d(v_3) - 5) \\
&\leq 2\Delta + 3 \\
&\leq 3\Delta - 1 \text{ when } \Delta \geq 4.
\end{aligned}$$

This contradicts our assumption, and thus there is a vertex x_1 of $G - \mathcal{G}^*$ not adjacent to v_1 or v_3. This vertex lies in the face of $\mathcal{G}^*$ bounded by $C_7 = u_2, u_1, u_3, v_3, v_4, v_1$, which we will refer to, without loss of generality, as the exterior of C_7. Since $\mathcal{G}^*$ is an induced subgraph of G, the distance $d_G(y_1, C_7) = 2$, and $\{v_1, v_3, u_1\}$ is the set of all vertices of C_7 that are distance exactly 2 from y_1. Because G has diameter 3, we conclude that x_1 is adjacent to u_1.

By the assumptions of Case 1.2, and since G is both triangle-free and does not contain a pair of dislocated 4-cycles, the vertex x_1 is not adjacent to any of the vertices of $V(C_7) - \{u_1\}$. Since $d_G(x_1, w_1) \leq 3$ and $d_G(x_1, w_2) \leq 3$, there exist vertices x_2 and x_3 in $\text{Ext}(C_7)$ such that $Q_3 : x_1, x_2, v_1$ and $Q_4 : x_1, x_3, v_3$ are paths in G. These two vertices are distinct, for if they were not, the 4-cycle on x_2, v_1, v_4, v_3 would be dislocated from C_1, a contradiction. Let $\mathcal{G}^{**} = \mathcal{G}^* \cup Q_3 \cup Q_4$ (See Figure 5.14 ($\mathcal{G}^{**}$)).

We now label the regions of $\mathcal{G}^{**}$ as follows. Let $r_1 = \text{Int}(v_1, x_2, x_1, u_1, u_2)$, $r_2 = \text{Int}(v_1, u_2, u_1, y_2, y_3)$, $r_3 = \text{Int}(v_3, u_3, u_1, y_2, y_4)$, $r_4 = \text{Int}(v_3, x_3, x_1, u_1, u_3)$ and $r_0 = \text{Ext}(v_1, x_2, x_1, x_3, v_3, v_4)$. Note that other than r_0, all of these regions are bounded by 5-cycles.

The regions r_1 and r_2 are both empty, as the only vertex on either of their boundaries within distance 2 of w_2 is v_1, and by Lemma 5.8, no one vertex of a Jordan separating 5-cycle dominates that cycle's interior. Similarly, the regions r_3 and r_4 are empty as the only vertex on their boundaries within distance 2 of w_1 is v_3. Any vertex of r_0 is adjacent to one of v_1 or v_3, as these are the only two vertices on the boundary of r_0 within distance 2 of y_1. Thus all vertices of $G - \mathcal{G}^{**}$ are adjacent to either v_1 or v_3, and we can constrain the order of G.

$$\begin{aligned}
n &= |V(\mathcal{G}^{**})| + |V(G) - V(\mathcal{G}^{**})| \\
&\leq 16 + (d(v_1) - 6) + (d(v_3) - 6)) \\
&\leq 2\Delta - 4 \\
&\leq 3\Delta - 1 \text{ when } \Delta \geq 5.
\end{aligned}$$

This contradicts our assumption, and so no vertex of $\text{Ext}(C_2)$ is adjacent to y_2.

Case 1.3: There exists some vertex u_1 in $\text{Ext}(C_2)$ that is adjacent to y_3, and no vertex of $\text{Ext}(C_2)$ is adjacent

to either y_2 or v_4.
Since G contains neither any 3-cycles, nor any pair of dislocated 4-cycles, the vertex u_1 is not adjacent to any vertex of $C_2 - v_3$.

As such, there are only two ways we can have a $u_1 - w_2$ geodesic with length at most 3. Either G contains the edge u_1v_3, or there is some vertex u_2 in $\text{Ext}(C_2)$ such that $S_1 : y_3, u_1, u_2, v_3$ is a path in G (see Figure 5.15).

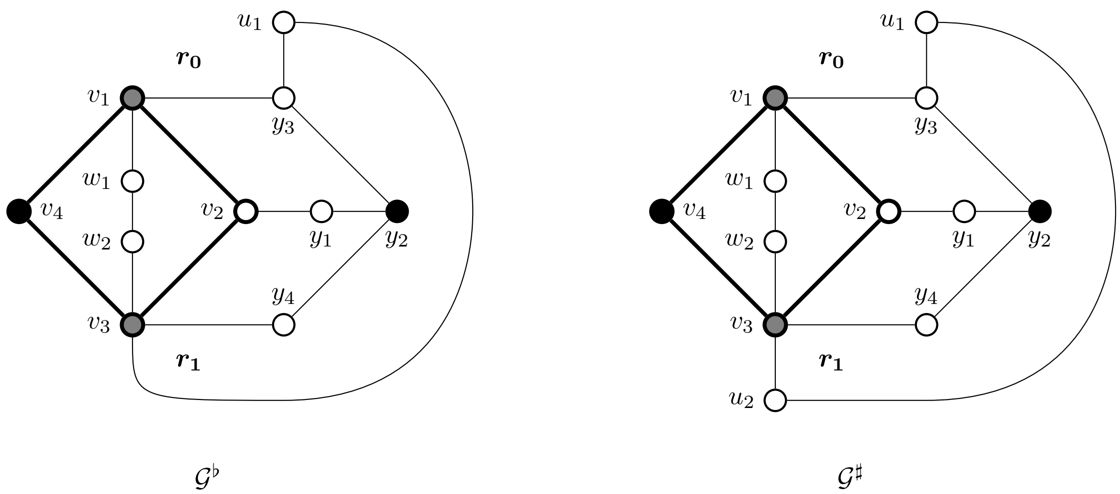


Figure 5.15: Case 1.3 assumes that there is a vertex u_1 adjacent to y_1. In this case, either $\mathcal{G}^\flat$ or $\mathcal{G}^\sharp$ is a subgraph of G. The black vertices may not have neighbours in G not shown in the diagrams.

Let us first assume that u_1 and v_3 are adjacent. Denote by S_2 the path y_3, u_1, v_3, and let $\mathcal{G}^\flat = \mathcal{G} \cup S_2$. By applications of Lemma 5.21, we see that $\mathcal{G}^\flat$ is an induced subgraph of G. The path S_2 divides $\text{Ext}(C_2)$ into two regions bounded by 5-cycles, $r_0 = \text{Ext}(v_1, y_3, u_1, v_3, v_4)$ and $r_1 = \text{Int}(y_3, u_1, v_3, y_4, y_2)$. The only vertices on the boundary of r_0 within distance 2 of y_1 are v_1, v_3 and y_3, so any vertex in r_0 is adjacent to one of these three. The only vertices on the boundary of r_1 within distance 2 of w_1 are v_3 and y_3, so the set $\{v_3, y_3\}$ dominates r_1, and we can bound the order of G.

$$
\begin{aligned}
n &= |V(\mathcal{G}^\flat)| + |V(G) - (V(\mathcal{G}^\flat)| \\
&\leq 11 + (d(v_1) - 4) + (d(v_3) - 5) + (d(y_3) - 3) \\
&\leq 3\Delta - 1.
\end{aligned}
$$

Since this contradicts our assumption, the graph G contains the path S_1. Let $\mathcal{G}^\sharp = \mathcal{G} \cup S_1$, and observe per Lemma 5.21 that $\mathcal{G}^\sharp$ is an induced subgraph of G. The region $\text{Ext}(C_2)$ is divided into two sub-regions bounded by 6-cycles, $r_0 = \text{Ext}(v_1, y_3, u_1, u_2, v_3, v_4)$ and $r_1 = \text{Int}(y_3, u_1, u_2, v_3, y_4, y_2)$. The are only two vertices, y_3 and v_3, on the 6-cycle bounding r_1 within distance 2 of w_1. Thus $\{y_3, v_3\}$ dominates r_1, and so by Lemma 5.11, there is some vertex u_3 in r_1 that is adjacent to both y_3 and v_3. Let $\mathcal{G}^{\sharp\sharp} = \mathcal{G}^\sharp \cup \{u_3, u_3y_3, u_3v_3\}$.

The only vertices on the boundary of r_0 within distance 2 of y_1 are v_1, v_3 and y_3, so every vertex of r_0 is adjacent to one of these three vertices. We can thus bound the order of G from above.

$$\begin{aligned}
n &= |V(\mathcal{G}^{\sharp\sharp})| + |V(G) - (V(\mathcal{G}^{\sharp\sharp})| \\
&\leq 13 + (d(v_1) - 4) + (d(v_3) - 6) + (d(y_3) - 4) \\
&\leq 3\Delta - 1.
\end{aligned}$$

This contradicts our assumption, and hence y_3 does not have a neighbour in $\mathrm{Ext}(C_2)$. By the same argument, the vertex y_4 also does not have a neighbour in $\mathrm{Ext}(C_2)$.

Case 1.4: The vertices v_4, y_2, y_3 and y_4 do not have any neighbours in $\mathrm{Ext}(C_2)$.
By cases 1.1 to 1.3, we have shown that the only vertices of C_2 which can have neighbours in $\mathrm{Ext}(C_2)$ are v_1 and v_3. Further, both v_1 and v_3 are at distance 2 from y_1, so any vertex in $\mathrm{Ext}(C_2)$ is adjacent to either v_1 or v_3 in order to be within distance 3 of y_1. Hence we get the following bound on n:

$$\begin{aligned}
n &= |V(\mathcal{G})| + |V(G) - V(\mathcal{G})| \\
&\leq 10 + (d(v_1) - 4) + (d(v_3) - 4) \\
&\leq 2\Delta + 2 \\
&\leq 3\Delta - 1 \text{ when } \Delta \geq 3.
\end{aligned}$$

In all subcases, $n \leq 3\Delta - 1$, and so the vertex v_2 does not have a neighbour in $\mathrm{Ext}(C_1)$. By symmetry, we further conclude that v_4 does not have any neighbours in $\mathrm{Ext}(C_1)$.

Case 2: Neither v_2 nor v_4 have any neighbours in G besides v_1 and v_3.
By the assumption that $n > 3\Delta - 1$, there is some vertex y_1 in G that is not adjacent to either v_1 or v_3. Note that $d_G(y_1, C_1) > 1$, but $d_G(y_1, w_1) \leq 3$ and $d_G(y_1, w_2) \leq 3$. Therefore, there exist vertices y_2 and y_3 in the exterior of C_1 such that $P_2 : y_1, y_2, v_1$ and $P_3 : y_1, y_3, v_3$ are paths in G (See Figure 5.16 ($\mathcal{K}$)).

Note that $y_2 \neq y_3$. If we assume these vertices are not distinct, then there is a 4-cycle on y_2, v_1, v_2, v_3, contradicting either the maximality of C_1, or the assumption that G does not contain two dislocated 4-cycles. Let $\mathcal{K} = C_1 \cup P_1 \cup P_2 \cup P_3$, and name the cycle $C_2 : v_1, y_2, y_1, y_3, v_3, v_4$ (See Figure 5.16). Observe that, by Lemma 5.21, the subgraph $\mathcal{K}$ is an induced subgraph of G.

Since $n > 3\Delta - 1$ by assumption, there exists some vertex u_1 in $G - \mathcal{K}$ that is not adjacent to either v_1 or v_3. We may assume without loss of generality that u_1 is in $\mathrm{Ext}(C_2)$. The vertex u_1 is not adjacent to both of y_2 and y_3 as this creates a 4-cycle dislocated from C_1, contradicting our assumption. There are a number of cases to consider.

Case 2.1: The vertex u_1 is adjacent to y_2.
Since G contains neither triangles nor dislocated 4-cycles, the vertex u_1 is not adjacent to any vertex of $C_2 - \{y\}$. Since $d_G(u_1, w_2) \leq 3$, there is some vertex u_2 in $\mathrm{Ext}(C_2)$ such that $Q_1 : y_2, u_1, u_2, v_3$ is a path in G. Per Lemma 5.21, the graph $\mathcal{K} \cup Q_1$ is an induced subgraph of G. Thus the interior of the 6-cycle $C_3 : y_2, u_1, u_2, v_3, y_3, y_1$ is dominated by y_2 and v_3, as these are the only vertices of the cycle within distance

2 of w_1. By Lemma 5.11, there exists a vertex u_3 in the cycle's interior such that $Q_2 : y_2, u_3, v_3$ is a path in G. The path Q_2 divides the region bounded by C_3 into two regions, each bounded by a 5-cycle. By Corollary 5.12, neither region contains any vertex of G. Let $\mathcal{K}'$ denote the graph $\mathcal{K} \cup Q_1 \cup Q_2$ (See Figure 5.16), and observe by Lemma 5.21 that it is an induced subgraph of G.

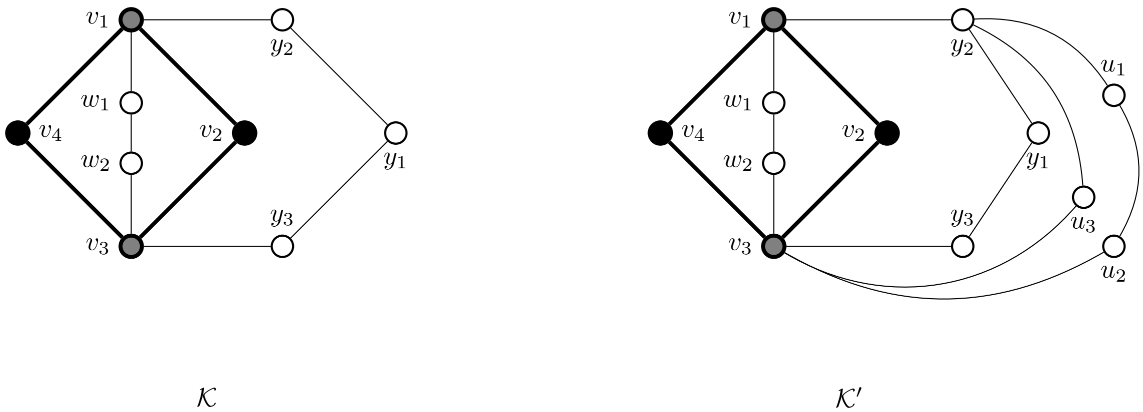


Figure 5.16: In Case 2, neither v_4 nor v_2 have neighbours other than v_1 and v_3. In this Case, G contains $\mathcal{K}$ as a subgraph. In Case 2.1, G contains $\mathcal{K}'$ as a subgraph.

If every vertex of $G - \mathcal{K}'$ is adjacent to one of v_1, v_3 or y_2, then we can bound the order of G by:

$$\begin{aligned} n &\leq 12 + (d(v_1) - 4) + (d(v_3) - 6) + (d(y_2) - 4) \\ &\leq 3\Delta - 2, \end{aligned}$$

contradicting our assumption. So there exists some vertex x_1 not adjacent to any of v_1, v_3 or y_2. Noting the symmetry between the interior of the cycle $C_4 : v_1, y_2, y_1, y_3, v_3, v_2$ and the exterior of the cycle $C_5 : v_1, y_2, u_1, u_2, v_3, v_4$, we may assume without loss of generality that x_1 is in the interior of C_4.

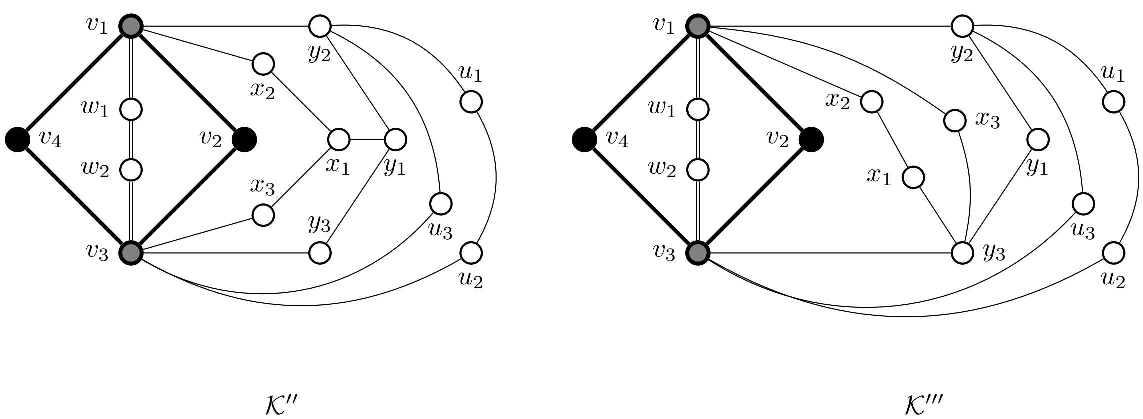


Figure 5.17: In Case 2.1.1, G has the graph $\mathcal{K}''$ as a subgraph. In Case 2.1.2, the graph $\mathcal{K}'''$ is a subgraph of G.

Case 2.1.1: The vertex x_1 is adjacent to y_1.

Since G contains neither triangles nor dislocated 4-cycles, x_1 has no neighbours in $C_4 - \{y_1\}$. Since there exist $x_1 - w_1$ and $x_1 - w_2$ geodesics, there are vertices x_2 and x_3 in $\mathrm{Int}(C_4)$ such that $Q_3 : y_1, x_1, x_2, v_1$

and $Q_4 : y_1, x_1, x_3, v_3$ are paths in G. Since C_1 is maximal and G does not contain dislocated 4-cycles, the vertices x_2 and x_3 are distinct. Denote $\mathcal{K}'' = \mathcal{K}' \cup Q_3 \cup Q_4$ (See Figure 5.17).

The exterior of the cycle on $v_1, y_2, u_1, u_2, v_3, v_4$ is dominated by $\{v_1, v_3, y_2\}$, as these are the only vertices of the cycle within distance 2 of x_1. The two regions bounded by the 5-cycles on v_1, y_2, y_1, x_1, x_2 and v_3, y_3, y_1, x_1, x_3 do not contain any vertices by Lemma 5.8, as only v_1 of the former cycle is within distance 2 of w_2, and only v_3 of the latter is within distance 2 of w_1. Finally, the 6-cycle on the vertices $v_1, x_2, x_1, x_3, v_3, v_2$ is dominated by v_1 and v_3, as these are the only two vertices of the cycle within distance 2 of u_1. Thus every vertex of $G - \mathcal{K}''$ is adjacent to v_1, v_3 or y_2, and we can bound the order of G to obtain a contradiction:

$$\begin{aligned} n &= |V(\mathcal{K}'')| + |V(G) - V(\mathcal{K}'')| \\ &\leq 15 + (d(v_1) - 5) + (d(v_3) - 7) + (d(y_2) - 4) \\ &\leq 3\Delta - 1. \end{aligned}$$

Case 2.1.2: The vertex x_1 is adjacent to y_3.
By the assumptions on both G and the vertex itself, x_1 is not adjacent to any vertex of $\mathcal{K} - \{y_3\}$. Since $d_G(x_1, w_1) \leq 3$, there exists a vertex x_2 such that $Q_5 : y_3, x_1, x_2, v_1$ a path in G. Consider the 6-cycle $C_6 : v_1, y_2, y_1, y_3, x_1, x_2$. The only vertices of C_6 within distance 2 of w_2 are v_1 and y_3. So by Lemma 5.11, there is a vertex x_3 in $\text{Int}(C_6)$ such that $Q_6 : v_1, x_3, y_3$ is a path in G. The path Q_6 divides $\text{Int}(C_6)$ into two regions bounded by 5-cycles, both dominated by $\{v_1, y_3\}$.

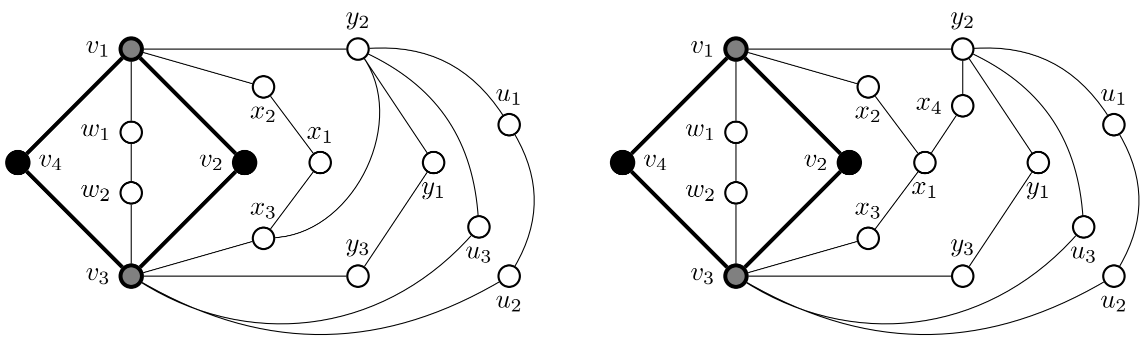
Denote by C_7 the cycle on $v_1, y_2, u_1, u_2, v_3, v_4$. The only vertices of the cycle within distance 2 of x_1 are v_1 and v_3, so $\text{Ext}(C_7)$ is dominated by $\{v_1, v_3\}$. Finally, the interior of the 6-cycle on $v_1, x_2, x_1, y_3, v_3, v_2$ is dominated by v_1 and v_3, as these are the only two vertices of the cycle within distance 2 of u_1. Thus, letting $\mathcal{K}''' = \mathcal{K}' \cup Q_5 \cup Q_6$ (See Figure 5.17), we can bound the order of G to derive a contradiction:

$$\begin{aligned} n &= |V(\mathcal{K}''')| + |V(G) - V(\mathcal{K}''')| \\ &\leq 15 + (d(v_1) + 6) + (d(v_3) - 6) + (d(y_3) - 4) \\ &\leq 3\Delta - 1. \end{aligned}$$

Case 2.1.3: The vertex x_1 is not adjacent to any vertex of $\mathcal{K}'$.
By the same argument as in Case 2.1.1, there are distinct vertices x_1 and x_2 in $\text{Int}(C_2)$ such that $Q_7 : x_1, x_2, v_1$ and $Q_8 : x_1, x_3, v_3$ are paths in G. Denote by $\mathcal{K}^*$ the subgraph $\mathcal{K}' \cup Q_7 \cup Q_8$ of G and consider the following cycle:

$$C_8 : x_1, x_2, v_1, y_2, y_1, y_3, v_3, x_3.$$

By Lemma 5.21, the interior of C_8 is the only region of $\mathcal{K}^*$ that may contain a chord of $\mathcal{K}^*$ across it. Because G contains neither triangles nor dislocated 4-cycles, and x_1 is not adjacent to y_1, the only chords that $\mathcal{K}^*$ may have in G are x_2y_3 and x_3y_2. Thus there are only two possible ways that G may exhibit a $u_1 - x_1$ path of length at most 3. Either x_3 is adjacent to y_2, or there is some new vertex y_4 that is adjacent to both x_1 and y_2.
Subcase 2.1.3 - 1: The vertices x_3 and y_2 are adjacent.
Observe by Lemma 5.21 that $\mathcal{K}^* \cup \{x_3y_2\}$ is an induced subgraph of G. Thus the only way to construct an $x_2 - y_3$ geodesic in G is by adding a vertex x_4 that is adjacent to both v_3 and x_2. Denote $\mathcal{K}^\flat = \mathcal{K}^* \cup \{x_4, x_3y_2, v_3x_4, x_2x_4\}$.



$\mathcal{K}^\flat$ $\mathcal{K}^\sharp$

Figure 5.18: In the first sub-case of 2.1.3, the vertices y_2 and x_3 are adjacent, and G contains the subgraph $\mathcal{K}^\flat$. In the second sub-case, there is a vertex x_4 adjacent to both x_1 and y_2, and G contains the subgraph $\mathcal{K}^\sharp$.

The exterior of the cycle on $v_1, y_2, u_1, u_2, v_3, v_4$ is dominated by v_1, v_3 and y_2, as these are the only vertices of the cycle within distance 2 of x_1. The interior of the 5-cycle on v_1, x_2, x_4, v_3, v_2 is dominated by v_1 and v_3, as only these vertices of the cycle are within distance 2 of u_1. The cycle on x_2, x_1, x_3, v_3, x_4 is dominated by x_3 and v_3 as these are the only two vertices within distance 2 of u_1, and so by Lemma 5.8 the interior of this cycle contains no vertices. The interior of the 5-cycle on y_2, y_1, y_3, v_3, x_3 is dominated by v_3 and y_2, as only these vertices of the cycle are distance 2 from w_1. Finally, the interior of 5-cycle on v_1, y_2, x_3, x_1, x_2 is also empty by Lemma 5.8, as only y_2 and x_3 are within distance 2 of y_3. However, since the vertices of G not in $\mathcal{K}^\flat$ are all adjacent to one of v_1, v_3 or y_2, we can bound the order of G.

$$
\begin{aligned}
n &= |V(\mathcal{K}^\flat)| + |V(G) - V(\mathcal{K}^\flat)| \\
&\leq 16 + (d(v_1) - 5) + (d(v_3) - 8) + (d(y_2) - 5) \\
&\leq 3\Delta - 2.
\end{aligned}
$$

Subcase 2.1.3 - 2: The graph G contains a vertex x_4 that is adjacent to x_1 and y_2.
Let $\mathcal{K}^\sharp$ be the subgraph $\mathcal{K}^* \cup \{x_4, x_1x_4, y_2x_4\}$ of G, and observe per Lemma 5.21 that $\mathcal{K}^\sharp$ is an induced subgraph of G.

The exterior of the cycle on $v_1, y_2, u_1, u_2, v_3, v_4$ is dominated by v_1, v_3 and y_2, as these are the only vertices of the cycle within distance 2 of x_1. The 7-cycle on $y_2, y_1, y_3, v_3, x_3, x_1, x_4$ is dominated by y_2 and v_3 as these are the only vertices within distance 2 of w_1. The interior of the 5-cycle on v_1, y_2, x_4, x_1, x_2 is empty by Lemma 5.8, as it is dominated by v_1, the only vertex of the cycle within distance 2 of w_2. The interior of the 6-cycle on $v_1, x_2, x_1, x_3, v_3, v_2$ is dominated by v_1 and v_3, the only vertices of the cycle within distance 2 of u_1. Every vertex of G that is not in $\mathcal{K}^\sharp$ is adjacent to one of v_1, v_3 or y_2, so the order of G is bounded above:

$$\begin{aligned} n &= |V(\mathcal{K}^\sharp)| + |V(G) - V(\mathcal{K}^\sharp)| \\ &\leq 16 + (d(v_1) - 5) + (d(v_3) - 7) + (d(y_2) - 5) \\ &\leq 3\Delta - 1. \end{aligned}$$

Case 2.2: The vertex u_1 is not adjacent to y_2 or y_3.
Since $d_G(u_1, w_1) \leq 3$ and $d_G(u_1, w_2) \leq 3$, there exist vertices u_2 and u_3 in G such that $S_1 : u_1, u_2, v_1$ and $S_2 : u_1, u_3, v_3$ are paths in G. The vertices u_2 and u_3 are distinct, by the maximality of C_1 and the fact that G contains no dislocated 4-cycles.

By Case 2.1, neither y_2 nor y_3 can have a neighbour in $G - \mathcal{K}$ which is not adjacent to v_1 or to v_3. By symmetry, neither u_2 nor u_3 can have any neighbour in $G - \{u_1\}$ that is not adjacent to v_1 or to v_3. Since G contains neither triangles nor dislocated 4-cycles, the only chords that may exist in the cycle on $v_1, u_2, u_1, u_3, v_3, y_3, y_1, y_2$ are y_1u_1, y_2u_3 and y_3u_2. Up to symmetry, this leaves only three possible ways to construct a $u_1 - y_1$ geodesic in G: with the edge y_2u_3, with the edge u_1y_1, or by (possibly repeated) subdivision of the edge u_1y_1. We let $\mathcal{L} = \mathcal{K} \cup S_1 \cup S_2$ (See Figure 5.19).

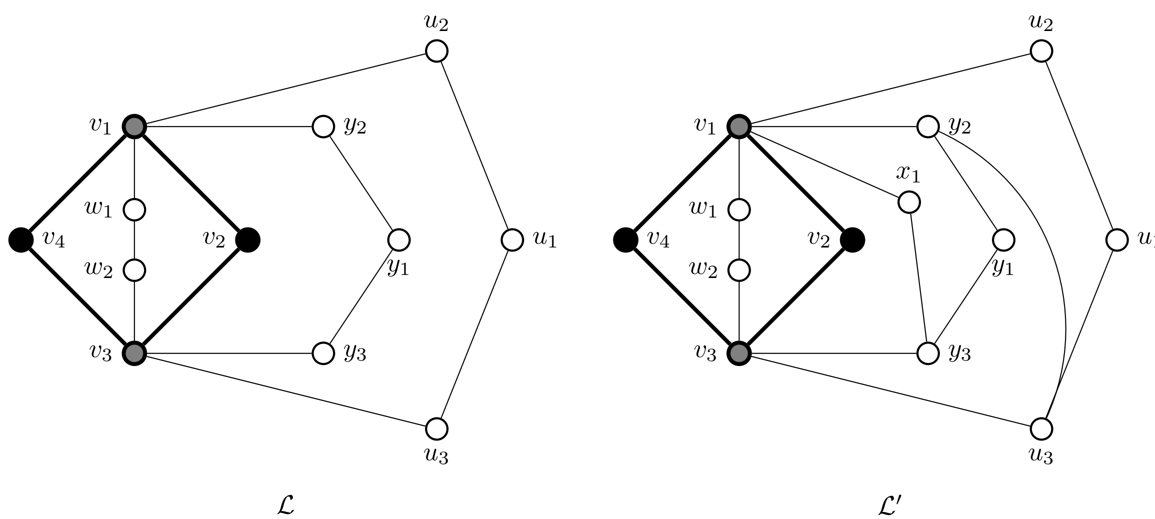


Figure 5.19: The graph G contains the subgraph $\mathcal{L}$ in Case 2.2. It contains the subgraph $\mathcal{L}'$ in Case 2.2.1.

Case 2.2.1: The vertices y_2 and u_3 are adjacent.
Per Lemma 5.21, the subgraph $\mathcal{L} \cup \{y_2u_3\}$ is an induced subgraph of G. Since $d_G(y_3, u_2) \leq 3$, there exists some vertex x_1 in G such that either $S_3 : y_3, x_1, v_1$ or $S_4 : y_3, v_3, x_1, u_2$ is a path in G. Up to a relabelling of the vertices and choosing the region bounded by $v_1, y_2, y_1, y_3, v_3, v_2$ to be the exterior region of our subgraph (i.e., actions which give us an equivalent plane graph), these possibilities are the same. Hence we assume without loss of generality that S_3 is a $y_3 - u_2$ geodesic, and we denote by $\mathcal{L}'$ the graph $\mathcal{L} \cup \{y_2u_3\} \cup S_3$ (See Figure 5.19).

The interior of the 5-cycle on v_1, v_2, v_3, y_3, x_1 is dominated by v_1 and v_3 as these are the only vertices of the cycle within distance 2 of u_1. The interiors of the two 5-cycles on v_1, y_2, y_1, y_3, x_1 and v_1, u_2, u_1, u_3, y_2

are dominated by the pairs v_1, y_3 and v_1, u_3 respectively, as these are the only vertices on the cycles within distance 2 of w_2. The interior of the 5-cycle on y_2, u_3, v_3, y_3, y_1 is dominated by y_2 and v_3, these being the only vertices of the cycle within distance 2 of w_1. By Lemma 5.22, all four of the regions mentioned in this paragraph do not contain any vertex of G.

All vertices of G not in $\mathcal{L}'$ lie in the exterior of the cycle on $v_1, u_2, u_1, u_3, v_3, v_4$. The only vertices of this cycle within distance 2 of y_1 are v_1, v_3 and u_3. Hence we can bound the number of vertices of G:

$$\begin{aligned} n &= |V(\mathcal{L}')| + |V(G) - (V(\mathcal{L}'))| \\ &\leq 13 + (d(v_1) - 6) + (d(v_3) - 5) + (d(u_3) - 3) \\ &\leq 3\Delta - 1. \end{aligned}$$

This contradicts our assumption, so y_2 and u_3 are not adjacent. By symmetry, y_3 and u_2 are not adjacent.

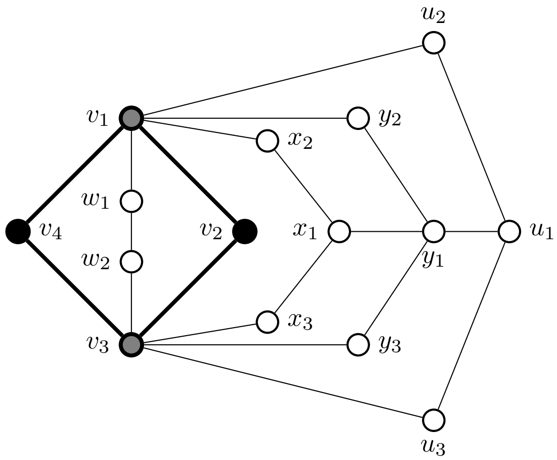
Case 2.2.2: The vertices u_1 and y_1 are adjacent.
Note the interiors of the two 5-cycles on v_1, u_2, u_1, y_1, y_2 and v_3, u_3, u_1, y_1, y_3 are dominated by only the vertices v_1 and v_3 respectively, these being the only vertices of the cycles within distance 2 of w_2 and w_1 respectively. Thus by Lemma 5.8, both interiors are empty.

Since $n > 3\Delta - 1$, there exists some vertex x_1 in $G - \mathcal{L}$ that is not adjacent to v_1 or v_3. By symmetry between the exterior of the cycle on $v_1, u_2, u_1, u_3, v_3, v_4$ and the interior of the cycle on $v_1, y_2, y_1, y_3, v_3, v_2$, we assume without loss of generality that x_1 is in the interior of the latter cycle. By Case 2.1, the vertex x_1 is not adjacent to y_2 or y_3.

By the same argument as the one at the start of Case 2.2, there exist distinct vertices x_2 and x_3 in G such that $S_5 : x_1, x_2, v_1$ and $S_6 : x_1, x_3, v_3$ are paths in G. Let $\mathcal{L}''$ denote the graph $\mathcal{L} \cup \{y_1u_1\} \cup S_5 \cup S_6$. Using both Lemma 5.21, and the fact that G contains neither triangles nor dislocated 4-cycles, we see that the only possible chords of G across the subgraph $\mathcal{L}''$ are x_1y_1, x_2y_3 and x_3y_2. As such, the only possibilities for an $x_1 - u_1$ geodesic of length at most 3 require that G contains the edge x_1y_1, or path of length 2, x_1, z_1, y_1, containing some new vertex z_1. Let $\mathcal{L}^\flat = \mathcal{L}'' \cup \{x_1y_1\}$ and $\mathcal{L}^\sharp = \mathcal{L}'' \cup \{z_1, x_1z_1, z_1y_1\}$ (See Figure 5.20).
Let us assume that G contains the path x_1, z_1, y_1. By Lemma 5.21, the subgraph $\mathcal{L}^\sharp$ is an induced subgraph of G. Since $d_G(z_1, w_1) \leq 3$ and $d_G(z_1, w_2) \leq 3$, there exist vertices z_2 and z_3 such that $S_7 : z_1, z_2, v_1$ and $S_8 : z_1, z_3, v_3$ are paths in G. By swapping the labels $z_1 \leftrightarrow x_1$, $z_2 \leftrightarrow x_2$ and $z_3 \leftrightarrow x_3$, we obtain $\mathcal{L}^\flat$ as a subgraph of G. Thus to complete the proof of Case 2.2.2, it suffices to prove the following claim.

Claim: If G contains $\mathcal{L}^\flat$ as a subgraph, then $n \leq 3\Delta - 1$.
Consider the subgraph $\mathcal{L}^\flat$, and note that it is an induced subgraph of G by Lemma 5.21.

There exist $x_2 - u_3$ and $x_3 - u_2$ geodesics of length at most 3 in G. Since $\mathcal{L}^\flat$ is an induced subgraph of G, there are only two possible $x_2 - u_3$ geodesics, both of which use a vertex t_1 in $G - \mathcal{L}^\flat$. These possible geodesics are $X_1 : x_2, v_1, t_1, u_3$ and $X_2 : x_2, t_1, v_3, u_3$. Up to relabelling of the vertices, and making the face of $\mathcal{L}^\flat$ bounded by $v_1, x_2, x_1, x_3, v_3, v_2$ the outer face of the graph, the two plane graphs $\mathcal{L}^\flat \cup X_1$ and $\mathcal{L}^\flat \cup X_2$ are the same. Thus we will assume without loss of generality that X_1 is a geodesic in G. Per Lemma 5.21, the subgraph $\mathcal{L}^\flat \cup X_1$ is an induced subgraph of G. Thus the only possible $x_3 - u_2$ geodesic in G is of the form $X_3 : x_3, t_2, v_1, u_2$, where t_2 is not among the vertices mentioned thus far. Let $\mathcal{L}^* = \mathcal{L}^\flat \cup X_1 \cup X_2$, and



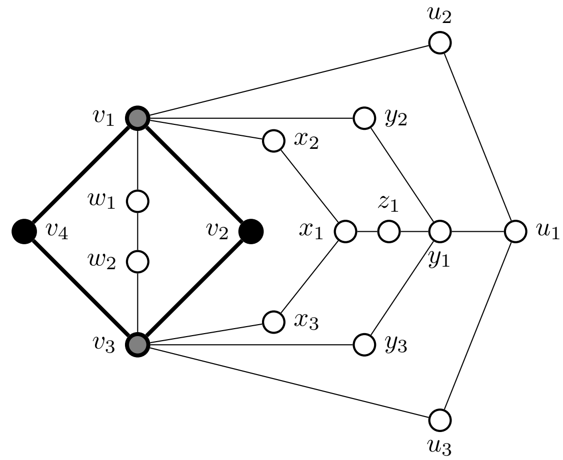


Figure 5.20: In Cases 2.2.2 and 2.2.3, the graph G always contains $\mathcal{L}^\flat$ as a subgraph. If, in Case 2.2.2, G contains $\mathcal{L}^\sharp$ as a subgraph, it will inevitably also have a $\mathcal{L}^\flat$ subgraph.

observe that it is an induced subgraph of G by Lemma 5.21.

The interior of the 5-cycle on v_1, t_2, x_3, v_3, v_2 is dominated by v_1 and v_3, these being the only vertices of the cycle within distance 2 of u_1. The interior of the 5-cycle on v_1, x_2, x_1, x_3, t_2 is dominated by v_1 and x_3, as these are the only vertices of the cycle within distance 2 of w_2. Similarly, the two regions bounded by 5-cycles that contain the vertex t_1 are also dominated by just two vertices. The interiors of the two 5-cycles on v_1, y_2, y_1, x_1, x_2 and v_3, y_3, y_1, x_1, x_3 are dominated by only v_1 and v_3 respectively, these being the only vertices of each cycle within distance 2 of w_1 and w_1, respectively. Thus, all the regions mentioned above are empty by Lemma 5.22.

As such, every vertex of $G - \mathcal{L}^*$ is in the interior of C_1, and hence adjacent to v_1 or to v_3, and we may bound the order of G:

$$
\begin{aligned}
n &= |V(\mathcal{L}^*)| + |V(G) - V(\mathcal{L}^*)| \\
&\leq 17 + (d(v_1) - 8) + (d(v_3) - 6) \\
&\leq 2\Delta + 3 \\
&\leq 3\Delta - 1 \text{ when } \Delta \geq 4
\end{aligned}
$$

This contradicts our assumption that $n > 3\Delta - 1$.

Case 2.2.3: The $y_1 - u_1$ geodesic is the single edge y_1u_1, subdivided either once or twice into a path of length 2 or 3 respectively.

Assume there exists some vertex x_1 in $G - \mathcal{L}$ on the path $Y_1 : y_1, x_1, u_1$ in G, and note that $\mathcal{L} \cup Y_1$ is an induced subgraph of G by Lemma 5.21. Since the distance between x_1 and the vertices w_1 and w_2 is at most

3, there are paths x_1, x_2, v_1 and x_1, x_3, v_3 in G. But now we see that $\mathcal{L}$ is a subgraph of G, and $n \le 3\Delta - 1$ by the claim in Case 2.2.2.

If we instead assume that there are vertices x_1 and z_1 on the path $Y_2 : y_1, x_1, z_1, u_1$, we again see that $\mathcal{L} \cup Y_2$ is an induced subgraph of G, and that G contains paths x_1, x_2, v_1 and x_1, x_3, v_3. Similarly, the graph G will also have paths z_1, z_2, v_1 and z_1, z_3, v_3, and we see that G contains $\mathcal{L}$ as a subgraph. Thus we may again invoke the claim in Case 2.2.2, completing the proof. □

5.7 Bounding the order, part III: Not a 4-cycle in sight

In the previous section, we established that if a pentagulation G of diameter 3, order n and maximum degree $\Delta \ge 8$ has any 4-cycles at all, then $n \le 3\Delta - 1$. In this section, we show that such a pentagulation G contains at least one 4-cycle. The restriction to pentagulations of maximum degree at least 8 will be critical in the proofs to follow. As demonstrated by Figure 5.21, the result is false if $\Delta \le 3$. The author does not know whether or not $n \le 3\Delta - 1$ when Δ is between 4 and 7.

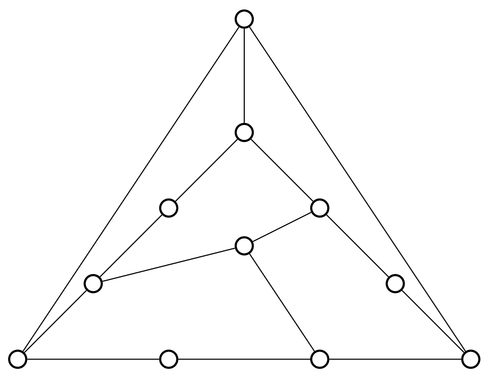


Figure 5.21: A pentagulation of diameter 3, girth 5 and maximum degree less than 8.

We first begin with some basic lemmas. Remember that $N_i(v)$ is the set of vertices at distance i from v (so $N_1(v)$ is just the usual neighbourhood $N(v)$).

Lemma 5.24. *Let G be a pentagulation with $g(G) \ge 5$ and let v be a vertex of G. Then $N_1(v)$ is an independent set, every vertex of $N_2(v)$ has a unique neighbour in $N_1(v)$, and every vertex of $N_1(v)$ has at least one neighbour in $N_2(v)$.*

Proof. Since G contains no triangles, $N_1(v)$ is an independent set. Because G contains no 4-cycles, a vertex of $N_2(v)$ may not be adjacent to two distinct vertices of $N_1(v)$. As G is 2-connected (and hence satisfies $\delta(G) \ge 2$), every vertex of $N_1(u)$ has a neighbour other than v. Since G is triangle-free, this neighbour belongs to $N_2(v)$. □

Lemma 5.25. *If G is a pentagulation of girth 5, then G is either the cycle C_5, or G does not contain two adjacent vertices of degree 2.*

Proof. Assume to the contrary that G is a pentagulation of girth 5 other than C_5 that contains two adjacent vertices x and y of degree 2. Let w be the single vertex of $N_1(x) - \{y\}$ and z the vertex of $N_1(y) - \{x\}$. Since G is not a cycle, the path $P : w, x, y, z$ lies on the boundary of two distinct faces f_1 and f_2 of G, each bounded by 5-cycles. Thus there exist two distinct vertices u and v that are both adjacent to w and z, such that the cycle on u, w, x, y, z bounds f_1, and the cycle on v, w, x, y, z bounds f_2. However, there is a 4-cycle on u, w, v, x, contradicting the assumption that $g(G) = 5$. □

Consider a vertex v in a pentagulation G. Let $\mathcal{F}$ be the subgraph of G consisting of the edges and vertices that lie on the boundary of any face incident with v. Given two vertices x and y of $N_2(v)$, call an $x - y$ path Q of length k a ***k*-chord** (with respect to v) if both $(Q - \{x, y\}) \cap N_2(v) = \emptyset$ and $E(Q) \cap E(\mathcal{F}) = \emptyset$.

For example, consider the subgraph of a girth 5 pentagulation shown in Figure 5.22. The path $P : w_1, w_5$ is a 1-chord with respect to v, while $Q : w_5, z, w_8$ is a 2-chord. The edge w_1w_2 is not a 1-chord, since it belongs to $\mathcal{F}$. Notice that $\mathcal{F} \cup P$ contains a cycle $C_P : w_1, w_5, u_3, v, u_1$ formed by taking the union of the $w_1 - w_5$ 1-chord P and the two unique $v - w_1$ and $v - w_5$ geodesics. One can construct another cycle $C_Q : w_5, z, w_8, u_5, v, u_3$ in the same fashion.

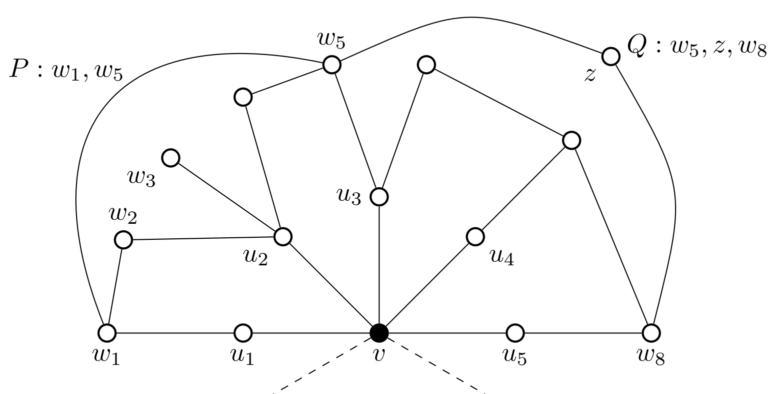


Figure 5.22: A vertex v in a pentagulation of girth five, and some of the edges and vertices near it. The dashed lines indicate some edges to parts of the graph not shown.

As the next lemma demonstrates, 1-chords and 2-chords with respect to some vertex will always induce cycles in the same manner that P and Q induce C_P and C_Q.

Lemma 5.26. *Let G be a pentagulation with $g(G) = 5$, and let v be a vertex of G such that $d(v) \geq 8$. Given distinct vertices x and y of $N_2(v)$, let $P : x, \ldots, y$ be a k-chord of v, and let u_x and u_y denote the unique vertices in $N_1(v) \cap N_1(x)$ and $N_1(v) \cap N_1(y)$ respectively. If $k \leq 2$, then u_x and u_y are distinct, and P, u_y, v, u_x is a Jordan separating cycle.*

Proof. First note that there are unique vertices u_x and u_y as described, per Lemma 5.24. Assume to the contrary that $k \leq 2$, but that $u_x = u_y$. The cycle P, u_y has length $k + 2 < 5$, which contradicts the fact that $g(G) = 5$. Thus $u_x \neq u_y$, and so $C_P : P, u_y, v, u_x$ is a cycle, and all that remains is to show that C_P is Jordan separating.

Since C_P is a cycle of length 5 or 6, and $E(P) \cap E(\mathcal{F}) = \emptyset$, the cycle C_P can neither be a face-cycle (P would need to share an edge with a face incident to v), nor can it have any chords (as the girth of G is 5). Thus C_P is a Jordan separating cycle. $\square$

In light of Lemma 5.26, we give a new definition. Let v be a vertex of a girth 5 pentagulation, and let the path $Q : x, \dots, y$ be a k-chord, for $k = 1$ or $k = 2$, with respect to v. If u_x and u_y are the unique vertices of $N_1(v)$ adjacent to x and y respectively, then the cycle $C_Q : Q, u_y, v, u_x$ is the **cycle under Q**. The chord Q is said to be **minimal** if C_Q dominates its interior, and there does not exist any k-chord (of the same length) Q' such that $\text{Int}(C_{Q'}) \subset \text{Int}(C_Q)$ (i.e., the chord Q is minimal if the interior of the cycle under Q is minimal amidst all cycles under chords of length k).

The proof that there do not exist girth 5 diameter 3 pentagulations of maximum degree at least 8 is split into three parts, the first of which we are now ready to present.

Theorem 5.27. *Let G be a diameter 3, girth 5 pentagulation of maximum degree Δ, and let v be a vertex of G with maximum degree. If $\Delta \geq 8$, then there do not exist any 1-chords with respect to v.*

Proof. We assume to the contrary that there exist vertices w'_0 and w'_j in $N_2(v)$, and some 1-chord $Q' : w'_1, w'_j$ with respect to v. Label the vertices of $N_1(v) = \{u'_0, u'_1, \dots, u'_{\Delta-1}\}$ in clockwise order, so that u'_i and u'_{i+1} always lie on the boundary of the same face (subscripts taken modulo Δ). Let u'_0 and u'_j be the unique, distinct neighbours of w'_0 and w'_j respectively (these exist by Lemmas 5.24 and 5.26).

Let $C_{Q'}$ denote the cycle under Q' with respect to v, and observe that $C_{Q'}$ is a Jordan separating cycle by Lemma 5.26. Since the diameter of G is 3, the cycle $C_{Q'}$ dominates either its interior or its exterior. We may choose to embed G in such a manner as to ensure that $C_{Q'}$ dominates its interior. Let Q be a minimal 1-chord in $\text{Int}[C_{Q'}]$ (it is possible that $Q = Q'$).

Relabel (if necessary) the vertices of $N_1(v)$ and $N_2(v)$ so that the start and end vertices of Q are labelled w_0 and w_j respectively, the neighbours u_i of $N_1(v)$ are still in clockwise order, and w_0u_0, w_ju_j are edges of $E(G)$. Let f_i be the face incident with v that has vertices u_i and u_{i+1} on its boundary.

Claim 1: The inequality $j < 3$ holds (i.e., the interior of C_Q contains at most two faces incident with v).
We first assume to the contrary that $j \geq 4$ (See Figure 5.23). Note that the neighbours u_1, u_2, $\dots$, u_{j-1} of v lie in the interior of C_Q. Let w_2 be a vertex of $N_2(v) \cap N_1(u_2)$ (such a vertex exists by Lemma 5.24). Since C_Q dominates its interior, the vertex w_2 is adjacent to some vertex of C_Q. Because G contains no triangles or 4-cycles, the vertex w_2 is not adjacent to any of u_0, v or u_j. By the minimality of the chord Q, the vertex w_2 is not adjacent to either w_0 or w_j, and thus we have a contradiction.

Having shown that $j < 4$, we now assume for the sake of contradiction that $j = 3$. Let w_1 be a vertex of $N_1(u_1) \cap N_2(v)$, and w_2 a vertex of $N_1(u_2) \cap N_2(v)$. Note by minimality of Q that w_1 is not adjacent to w_j. Since G has girth 5, the vertex w_1 is not adjacent to u_0, v or u_j. Because C_Q dominates its interior, we thus conclude that w_1 is adjacent to w_0. Similarly, the vertex w_2 is adjacent to w_j, but not to w_0. This leaves two cases to consider.

Claim 1 Case 1: The degrees of u_1 and u_2 satisfy $d(u_1) = d(u_2) = 2$.
In this case, the path w_1, u_1, v, u_2, w_2 lies along the boundary of a face of G, and so w_1 and w_2 are adjacent

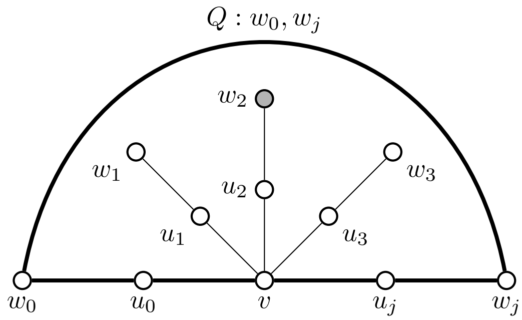


Figure 5.23: This figure shows Claim 1 of Theorem 5.27. The cycle C_Q under the 1-chord Q is bold, and the unique $N_2(v)$ neighbour w_2 of u_2 is grey.

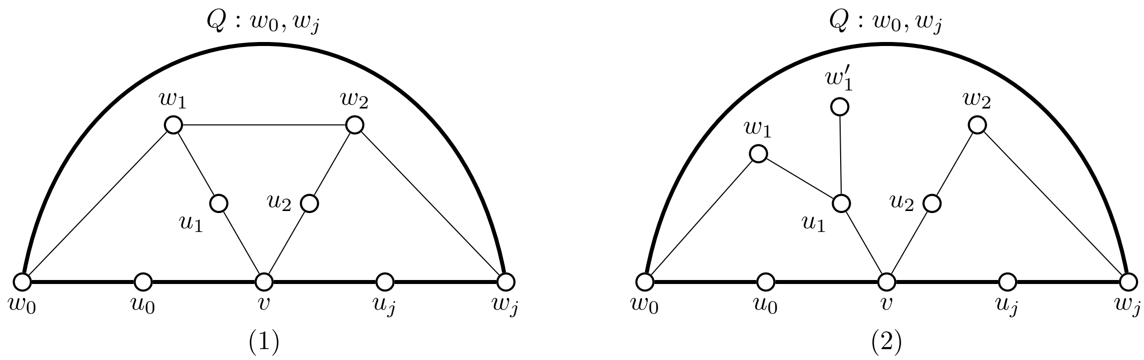


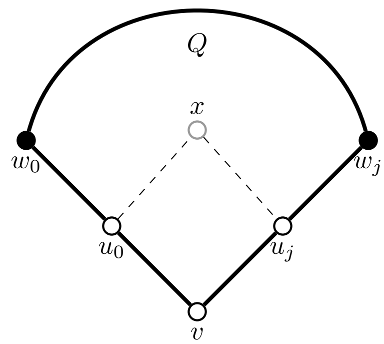
Figure 5.24: If $j = 3$ in the proof of Claim 1, there are two possibilities. Either both u_1 and u_2 have degree two (1), as in Claim 1 Case 1, or one of them has degree at least three (2), as in Claim 1 Case 2.

(See Figure 5.24 (1)). But then the vertices w_0, w_1, w_2, w_j lie on a 4-cycle, contradicting the fact that $g(G) = 5$.

Claim 1 Case 2: either u_1 or u_2 has degree at least three.
Assume without loss of generality that u_1 has a vertex w'_1 of $N_1(u_1) \cap N_2(v)$ other than w_1 (See Figure 5.24 (2)). Since C_Q dominates its interior and G has no cycles of length 3 or 4, the vertex w'_1 is adjacent to either w_0 or w_j. But then the cycle under the chord $w_0w'_1$, or the chord $w_jw'_1$, is contained strictly in $\text{Int}[C_Q]$, contradicting the minimality of Q. This completes the proof of Claim 1.

Since $j < 3$, there are at least five neighbours $u_3, u_4, \ldots, u_{\Delta-1}$ of v in $\text{Ext}(C_Q)$. There are a number of cases to consider, according to whether or not w_0 and w_j have neighbours in $\text{Int}(C_Q)$.

Case 1: Neither w_0 nor w_j have any neighbours in $\text{Int}(C_Q)$.
In $\text{Int}[C_Q]$, the only neighbours of w_0 are u_0 and w_j, and the only neighbours of w_j are u_j and w_0. Thus the path $P : u_0, w_0, w_j, u_j$ lies on the boundary of a face that is contained in $\text{Int}(C_Q)$, and so there is a vertex x such that the cycle P, x is a face-cycle. By the assumption that w_0w_j is a 1-chord with respect to v, we have $x \neq v$. But then there is a 4-cycle on v, u_0, x, u_j, which is a contradiction (See Figure 5.25).



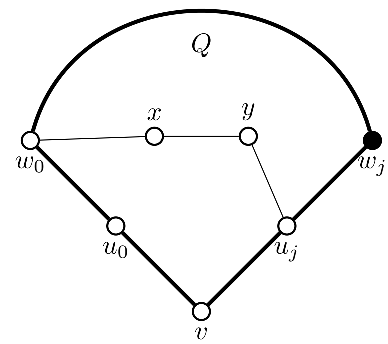


Figure 5.25: In Case 1, we assume that neither w_0 nor w_j has neighbours in $\text{Int}(C_Q)$ (and colour these vertices black to indicate this). In Case 2, we assume that w_0 has a neighbour in $\text{Int}(C_Q)$, but w_j does not.

Case 2: Either w_0 or w_j has a neighbour in $\text{Int}(C_Q)$, but not both.
Assume without loss of generality that there is a vertex x in $\text{Int}(C_Q)$ that is adjacent to w_0. If there are multiple vertices in $N_1(w_0) \cap \text{Int}(C_Q)$, choose x such that the edges w_0w_j and w_0x lie on the boundary of a common face. Because w_j has no neighbour in $\text{Int}(C_Q)$, the path $P : u_j, w_j, w_0, x$ lies on the boundary of some face f in the interior of C_Q. Thus there is some vertex y in $\text{Int}[C_Q]$ such that the cycle P, y bounds f. As G contains neither 3-cycles nor 4-cycles, the vertex y is in $N_2(v)$ (as opposed to either being v itself, or being in $N_1(v)$) (See Figure 5.25).

There are a number of cases to consider, based on the structure of the faces f_j and f_{j+1}.

Case 2.1: There is some vertex s in $N_1(w_j) \cap N_1(u_{j+1})$, and $d(u_{j+1}) = 2$.
Let t be the neighbour of s on the boundary of the face f_{j+1}, and observe that t and u_{j+2} are adjacent (see Figure 5.26). Since the girth of G is 5, we observe the following:
(1) the vertex w_j does not have any neighbours in the cycle v, u_j, w_j, s, u_{j+1} besides v and w_j,
(2) the vertex t is not adjacent to either w_0 or w_j,
(3) the vertex y is not adjacent to u_0, w_0 or w_j. Thus there is no possible $y - t$ path of length 3 or less, which is a contradiction.

Case 2.2: There is a vertex s in $N_1(w_j) \cap N_1(u_{j+1})$, and $d(u_{j+1}) \geq 3$.
Since u_{j+1} has at least two neighbours in $N_2(v)$, the neighbour of u_{j+1} on the boundary of f_{j+1} that is at distance 2 from v is distinct from s. Call this neighbour t, and let z be the vertex $N_2(v) - \{t\}$ incident with f_{j+1} (see Figure 5.26). Since G has girth 5, the vertex t is not adjacent to w_j. Because the diameter of G is 3, the vertices t and w_0 are adjacent to ensure that $d(t, y) \leq 3$. The vertex z is not adjacent to any vertex within distance 2 of y by planarity, and the fact that G contains no cycles of length 3 or 4. Thus $d(z, y) > 3$, which contradicts the fact that G has diameter 3.

Case 2.3: There is no vertex in $N_1(w_j) \cap N_1(u_{j+1})$.
Let s and t be the vertices of $N_2(v)$, incident with f_j, and adjacent to u_j and u_{j+1} respectively. Note that s and t are adjacent. If t is incident with the face f_{j+1}, then t has a neighbour z in $N_1(u_{j+2})$ that is also incident with f_{j+1} (see Figure 5.27 (1)). If t is not incident with f_{j+1}, then there is a vertex z' in

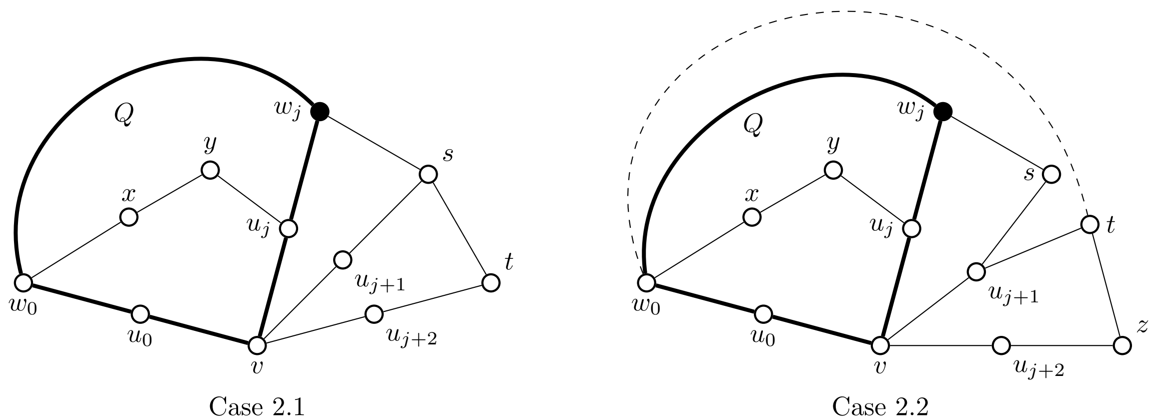


Figure 5.26: The diagram on the left illustrates Case 2.1, in which $d(u_{j+1}) = 2$ and the vertex of $N_2(v) \cap N_1(u_{j+1})$ is adjacent to w_j. On the right is Case 2.2, in which $d(u_{j+1}) > 2$, and some vertex of $N_2(v) \cap N_1(u_{j+1})$ is adjacent to w_j.

$N_1(u_{j+1}) - \{t\}$ that is incident with f_{j+1} (see Figure 5.27 (2)).

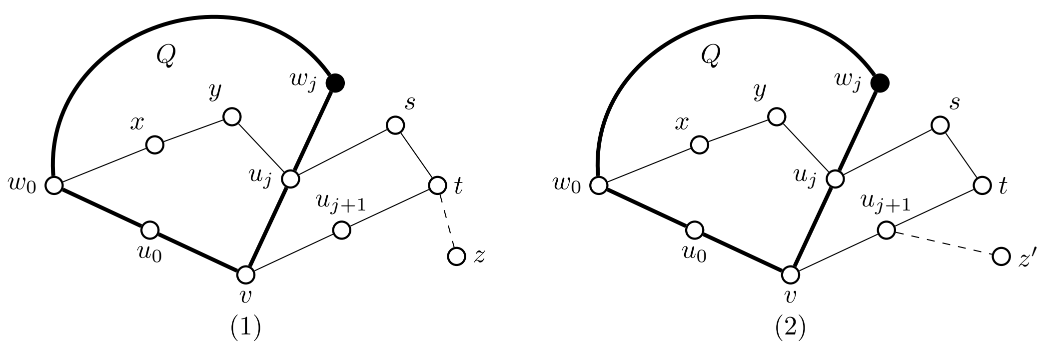


Figure 5.27: In Case 2.3, either $d(u_{j+1}) = 2$, and t has some neighbour z incident with f_{j+1} (1), or $d(u_{j+1}) > 2$, and u_{j+1} has some neighbour z' other than t that is incident with f_{j+1}.

There are three ways to construct a $t - x$ geodesic of length at most 3 that don't immediately contradict either the planarity or girth of G.

Case 2.3.1: The vertices t and w_0 are adjacent.
In this case, t, w_0, x is the needed geodesic. The graph G contains one of the vertices z or z' described above, and cannot contain any 3-cycle or 4-cycle, and so either $d(z, y) > 3$ or $d(z', y) > 3$, respectively.

Case 2.3.2: There is a vertex $w_{\Delta-1}$ that is adjacent to t, w_0 and $u_{\Delta-1}$.
The path $t, w_{\Delta-1}, w_0, x$ is a $t - x$ geodesic (see Figure 5.28). One of z or z' is present in G, so by the planarity and girth constraints of G, we see that either $d(z, y) > 3$ or $d(z', y) > 3$.

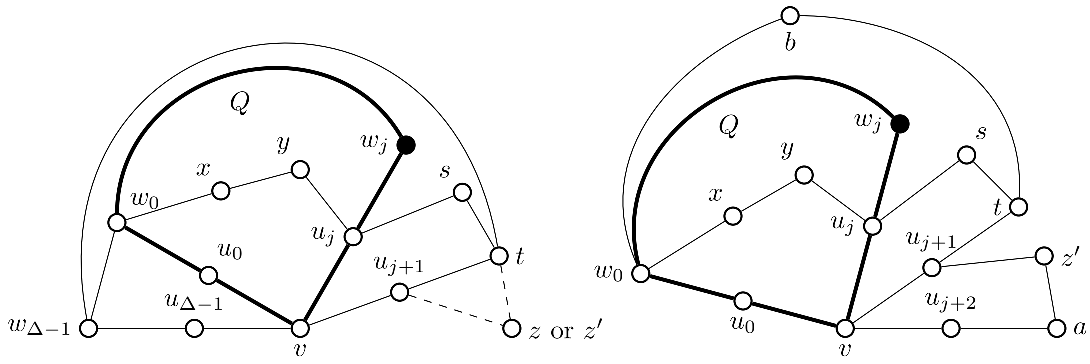


Figure 5.28: The left figure illustrates Case 2.3.2 in which t and $w_{\Delta-1}$ are adjacent. The right figure shows Case 2.3.3, under the assumption that G contains the vertex z' that is not adjacent to t.

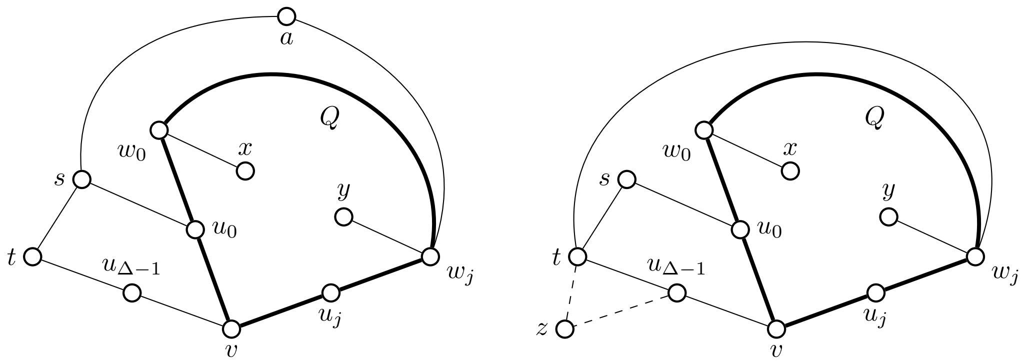
Case 2.3.3: There is some vertex b, that is not adjacent to $u_{\Delta-1}$, but that is adjacent to both t and w_0. Our desired $t - x$ geodesic in this case is the path t, b, w_0, x. If G contains the vertex z, which is adjacent to t, then z is not adjacent to w_0 as this would induce a 4-cycle on the vertices z, w_0, b and t. Thus, if G contains z, we obtain our desired contradiction since $d(z, y) > 3$. So we may assume that z' is a vertex of G, and we let a be the vertex of $N_2(v) \cap N_1(z')$ that is incident with f_{j+1} (see Figure 5.28, Case 2.3.3). The only $y - z'$ geodesic that can exist in G is the path z', w_0, x, y, so the vertices z' and w_0 are adjacent. As G may not contain any 3-cycles, the vertices a and w_0 are not adjacent, so $d(a, y) > 3$. With this contradiction, we conclude the proof of Case 2.

Case 3: The vertices w_0 and w_j each has a neighbour in $\mathrm{Int}(C_Q)$.
Let x and y be vertices in $\mathrm{Int}(C_Q)$ that are adjacent to w_0 and w_j, respectively. Observe that, since G has no shortcycles, the vertex x is not adjacent to any vertex of C_Q apart from w_0, and y is not adjacent to any vertex of C_Q besides w_j. There are two different subcases to consider.

Case 3.1: At least one of vertices u_0 and u_j has a neighbour in $\mathrm{Ext}(C_Q)$.
We assume without loss of generality that the vertex u_0 is adjacent to at least one vertex of $\mathrm{Ext}(C_Q)$. Let s be the neighbour of u_0 in $\mathrm{Ext}(C_Q)$ that is incident with the face $f_{\Delta-1}$, and let t be the other neighbour of s that is also incident with $f_{\Delta-1}$. Note that s is not adjacent to w_j, as this induces a 4-cycle on the vertices s, w_j, w_0 and u_0. Thus there are only two ways that G may contain an $s - y$ path of length at most 3, and we consider both as subcases.

Case 3.1.1: There is some vertex $a \neq t$ that is adjacent to both s and w_j.
In this case, the path s, a, w_j, y is the desired $s - y$ path (see Figure 5.29). However, there is no way to create a $t - x$ path of length 3 or less without inducing a 3-cycle or a 4-cycle. Thus the diameter of G is greater than 3, a contradiction.

Case 3.1.2: The vertices t and w_j are adjacent.
The $s - y$ path of length 3 in this case is s, t, w_j, y (see Figure 5.29). We consider the face $f_{\Delta-2}$. Either the vertex t is incident with this face, and there is a vertex z in $N_1(t) \cap N_1(u_{\Delta-2})$, or t is not incident with this



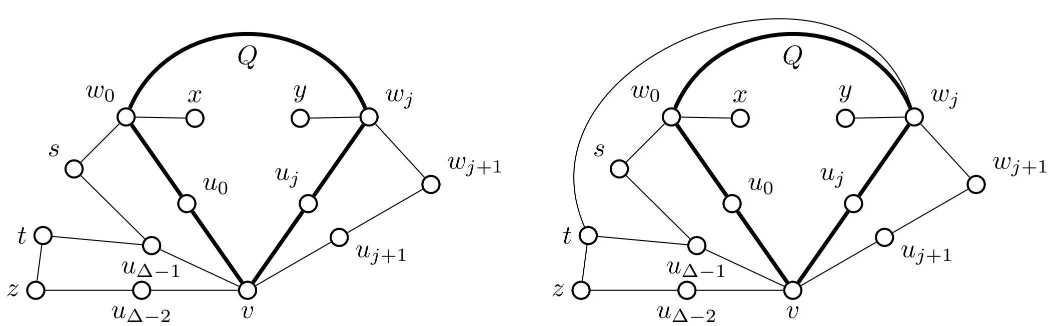
Case 3.1.1 Case 3.1.2

Figure 5.29: In Case 3.1.1, there is an $s-y$ path s, a, w_j, y containing some vertex a in $N_2(v) \cup N_3(v) - \{t\}$. In Case 3.1.2, the vertex t is adjacent to w_j, and s, t, w_j, y is an $s-y$ path of length 3.

face, and there is a vertex z' in $N_1(u_{\Delta-1}) \cap N_2(v)$. In either case we derive a contradiction, as there is no way to create a $z-x$ or $z'-x$ path without inducing a shortcycle in G. This completes the proof of Case 3.1.

Since Case 3.1 yields a contradiction, we may assume that neither u_0 nor u_j has a neighbour in $\mathrm{Ext}(C_Q)$. Since u_0 has no neighbour in $\mathrm{Ext}(C_Q)$, the vertex w_0 is incident with the face $f_{\Delta-1}$. Similarly, the vertex w_j is incident with the face f_j. Let s be the vertex of $N_1(w_0) - \{u_0\}$ that is incident with $f_{\Delta-1}$, and let w_{j+1} be the vertex of $N_1(w_j) - \{u_j\}$ that is incident with f_j.

Case 3.2: The vertex $u_{\Delta-1}$ has degree at least 3.
In this case, the vertex s is only incident with the face $f_{\Delta-1}$, and not the face $f_{\Delta-2}$. Thus we let t denote the neighbour of $u_{\Delta-1}$ that is incident with $f_{\Delta-2}$, and we let z be the vertex of $N_1(t) - u_{\Delta-1}$ that is incident with $f_{\Delta-2}$ (See Figure 5.30).



Case 3.2 Case 3.2.1

Figure 5.30: If $d(u_{\Delta-1}) > 2$, then distinct neighbours s and t of $u_{\Delta-1}$ are incident with the faces $f_{\Delta-1}$ and $f_{\Delta-2}$, respectively (Case 3.2). In Case 3.2.1, we consider the possibility that there is a $t-y$ path of the form t, w_j, y.

Given the girth and planarity constraints on G, there are only three different ways that G may have a $t-y$

path of length at most 3.

Case 3.2.1: The vertices t and w_j are adjacent.
In this case, the desired $t - y$ path is t, w_j, y (See Figure 5.30). However, since G contains no 3-cycles or 4-cycles, there does not exist a $z - x$ path of length 3 or less, which contradicts the fact that G has diameter 3.

Case 3.2.2: There is some vertex $a \neq z$ that is adjacent to both t and w_j.
Note that it is possible that $a = w_{j+1}$, but this possibility does not affect the argument. The $t - y$ path under consideration in Case 3.2.2 is t, a, w_j, y. Similarly to Case 3.2.1, the distance between z and x cannot be less than 4 without creating a shortcycle.

Case 3.2.3: The vertices z and w_j are adjacent.
The $t-y$ path in question for Case 3.2.3 is t, z, w_j, y. Consider the vertex u_{j+1}. If it has degree 2, then there is a vertex $b \neq u_{j+1}$ that adjacent to w_{j+1} and incident with the face f_{j+1}. If $d(u_{j+1}) \geq 3$, then there exists a vertex $b' \neq w_{j+1}$ that is adjacent to u_{j+1} and incident with f_{j+1}. In either case, the vertex b or b' is not adjacent to w_j since G contains neither 3-cycles nor 4-cycles respectively. Regardless of whether G contains b or b', we obtain a contradiction, since either $d(b, x) > 3$ or $d(b', x) > 3$.

Case 3.3: The vertex $u_{\Delta-1}$ has degree 2.
In this case, the vertex s is the only neighbour of $u_{\Delta-1}$ besides v, and we denote by t the vertex of $N_1(s) - \{u_{\Delta-1}\}$ that is incident with the face $f_{\Delta-2}$. Since G is a plane graph with no shortcycles, the vertex t is not adjacent to either w_0 or w_j, so there are only two subcases to consider: one for each way that G can exhibit a $t - y$ path of length at most 3.

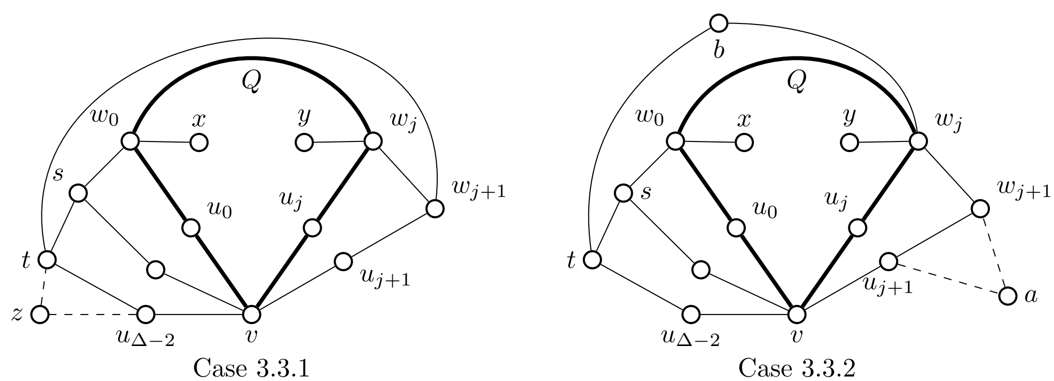


Figure 5.31: In Case 3.3, we assume that $u_{\Delta-1}$ has only two neighbours. In subcase 3.3.1, we consider what happens when the vertices t and w_{j+1} are adjacent.

Case 3.3.1: The vertices t and w_{j+1} are adjacent.
The desired $t - y$ path is t, w_{j+1}, w_j, y. Either t or $u_{\Delta-2}$ has some neighbour in $N_2(v)$ that has not yet been mentioned. Call this neighbour z, and observe that there is no way to construct a $z - x$ path of length 3 or less without contradicting either the planarity or girth constraints on G (see Figure 5.31).

Case 3.3.2: There is some vertex $b \neq w_{j+1}$ that is adjacent to both t and w_j.

Notice that t, b, w_j, y is a sufficiently short $t - y$ geodesic in G. Either $d(u_{j+1}) = 2$, and so w_{j+1} has a neighbour in $N_2(v)$ incident with f_{j+1}, or $d(u_{j+1}) \geq 3$ and u_{j+1} has a neighbour in $N_2(v) - \{w_{j+1}\}$ incident with f_{j+1}. In either case, call this neighbour a, and note that there does exist an $a - x$ path of length 3 or less in G.

In all cases, we derive a contradiction, and so there does not exist a minimal 1-chord in $N_2(v)$. Thus, there does not exist any 1-chord in $N_2(v)$, completing the proof.

□

To prove Theorem 5.27, we derived a contradiction by considering a minimal 1-chord. We now use the same proof strategy to show that there also cannot be any 2-chords with respect to a maximum degree vertex.

Theorem 5.28. *Let G be a pentagulation of diameter 3, girth 5 and maximum degree Δ, and let v be a vertex of G with maximum degree. If $\Delta \geq 8$, then G does not have any 2-chords with respect to v.*

Proof. Assume for the sake of contradiction that there does exist some 2-chord with respect to v.

Repeat the argument used at the start of the proof of Theorem 5.27, and adopt the same labelling convention for the vertices of $N_1(v)$ and $N_2(v)$, and for the faces incident with the vertex v. We deduce that there is a minimal 2-chord $Q : w_0, a, w_j$, where w_0 and w_j are vertices of $N_2(v)$, the vertex a lies in $N_3(v)$, and the cycle C_Q under Q dominates its interior. Further, the vertices u_0 and u_j are the unique vertices of $N_1(v) \cap N_1(w_0)$ and $N_1(v) \cap N_1(w_j)$, respectively.

Claim 1: The index j satisfies $j < 4$ (so $\text{Int}(C_Q)$ contains three or fewer faces incident with v).
Assume to the contrary that $j \geq 4$, and observe per Lemma 5.24 that the vertex u_2 has some neighbour w_2 in $N_2(v)$. By Theorem 5.27, the vertex w_2 is adjacent to neither w_0 nor w_j (since $j \geq 4$ by assumption). Since G contains no shortcycles, the vertex w_2 is not adjacent to either u_0 or u_j. Since C_Q dominates its interior, we conclude that w_2 is adjacent to a. However, this creates a new 2-chord w_2, a, w_0, which contradicts the minimality of Q and proves Claim 1.

Claim 2: It is not possible that both w_0 and w_j have neighbours in $\text{Int}(C_Q)$.
Assume to the contrary that w_0 has some neighbour x in $\text{Int}(C_Q)$ and w_j has a neighbour y in $\text{Int}(C_Q)$. The vertices x and y are distinct: were they not, there would be a 4-cycle on the vertices w_0, x, w_j, a. Since G has no shortcycles, the vertex x is not adjacent to a or w_j, and y is not adjacent to a or w_0.

Consider the face f_{j+2}. This face is bounded by the 5-cycle $v, u_{j+2}, s, t, u_{j+3}$, where s and t are vertices of $N_2(v)$. Since the diameter of G is 3, there exists a $t - y$ path of length at most 3. Per Theorem 5.27, the vertex t is not adjacent to any vertices of $N_2(v)$ apart from s, and possibly one other vertex that is incident with the face f_{j+3}. Hence G can only exhibit a $t - y$ path in one of two possible ways (see Figure 5.32):
(1) the vertices a and t are adjacent, and the desired path is t, a, w_j, y, or
(2) there is some vertex b in $N_3(v)$ that is adjacent to both t and w_j, forming a path t, b, w_j, y.

Since G has girth 5, and by Theorem 5.27, G contains neither shortcycles nor 1-chords across $N_2(v)$. Thus, in both case (1) and (2), there is no way to create an $s - x$ path of length 3 or less. As the diameter of G is

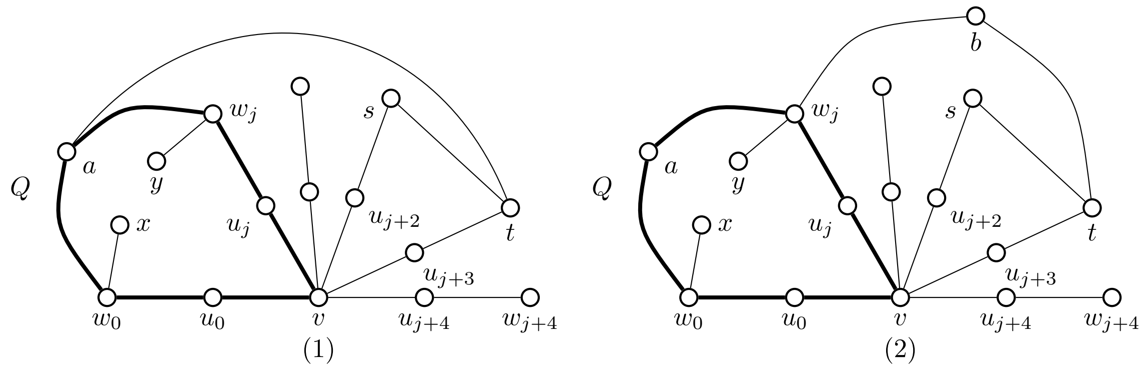


Figure 5.32: In Claim 2, since $N_2(v)$ has no 1-chords but G has diameter 3, either t, a, w_j, y is a $t - y$ path (1), or t, b, w_j, y is (2).

3, this yields a contradiction, completing the proof of Claim 2.

Claim 3: $j < 3$.
By Claim 1, we need only show that $j \neq 3$. Thus we assume for the sake of contradiction that $j = 3$. By Lemma 5.24, the vertices u_1 and u_2 each have some neighbour, say w_1 and w_2 respectively, in $\text{Int}(C_Q)$. Per Theorem 5.27, there are no 1-chords across v, so w_1 is not adjacent to w_j. By the minimality of Q, the vertices w_1 and a are not adjacent, and since G has girth 5, the vertex w_1 is not adjacent to v, u_0 or u_j. Similarly, the vertex w_2 is not adjacent to any of w_0, a, v, u_0 or u_j. Since C_Q dominates its interior, we have that w_1 is adjacent to w_0 and w_2 is adjacent to w_3. However, by Claim 2, this is not possible, and so we have obtained the contradiction needed to prove Claim 3.

Claim 2 demonstrated that it is not possible for both w_0 and w_j to have neighbours in $\text{Int}(C_Q)$. This leaves two cases to consider.

Case 1: Either w_0 or w_j has a neighbour in $\text{Int}(C_Q)$, but not both.
Assume without loss of generality that w_0 has some neighbour, call it x, in $\text{Int}(C_Q)$. By assumption, the vertex v has $d(v) \geq 8$, and by Claim 3, at most one neighbour of v is contained in $\text{Int}(C_Q)$, so v has at least five neighbours in the exterior of C_Q. Consider the face f_{j+2} incident with v. This face is bounded by a 5-cycle $v, u_{j+2}, s, t, u_{j+3}$, where s and t are vertices of $N_2(v)$. Since G has diameter 3, both s and t are within distance 3 of x. It is possible that x is adjacent to u_j, but it is not possible that x is adjacent to any other vertex of $V(C_Q) - \{w_0\}$, since G may not contain any shortcycles. From both the previous sentence, and the fact that there are no 1-chords across v per Theorem 5.27, we deduce that there are two ways that G may exhibit a $t - x$ path of length at most 3.

Case 1.1: The vertices t and a are adjacent.
This case yields the path t, a, w_0, x (see Figure 5.33). Since G contains no shortcycles, the vertex s is not adjacent to a, b or w_j. Because there are no 1-chords across v by Theorem 5.27, the vertex t cannot be adjacent to w_0 (more generally, no neighbour of s is adjacent to w_0). Thus there is no $s - x$ path of length at most 3 in G, which contradicts the diameter of G.

Case 1.2: There is a vertex b in $N_3(v)$ that is adjacent to both w_0 and t.

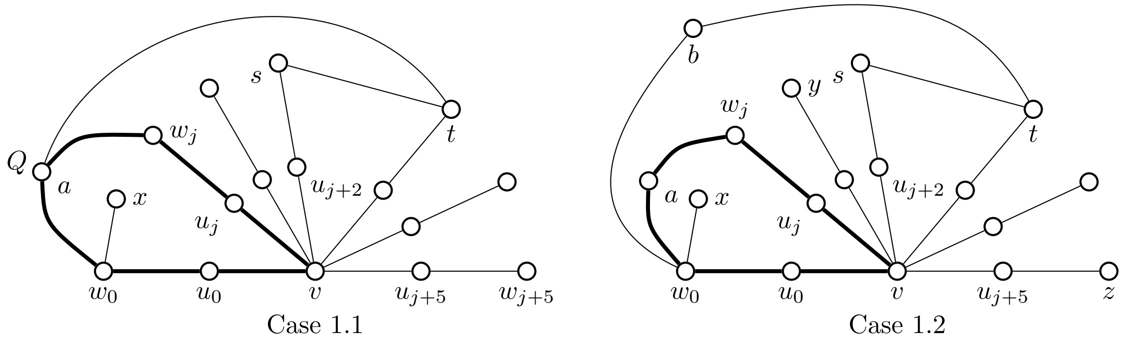


Figure 5.33: There are two possibilities in Case 1, either t, a, w_0, x is a $t - x$ path, as in subcase 1.1, or t, b, w_0, x is, as in subcase 1.2.

The $t - x$ path in question is t, b, w_0, x (see Figure 5.33). There exists an $s - x$ path of length at most 3. The graph G may exhibit such a path in one of two ways:
(1) either s and a are adjacent, or
(2) there is some vertex c in $N_3(v)$ that is adjacent to both s and w_0.
In either case, letting y and z be vertices of $N_1(u_{j+1}) \cap N_2(v)$ and $N_1(u_{j+5}) \cap N_2(v)$, respectively, we observe that it is not possible to construct a $y - z$ path in G of length at most 3. This completes the proof of Case 1.

Case 2: Neither w_0 nor w_j has a neighbour in $\mathrm{Int}(C_Q)$.
We claim that both u_0 and u_j have neighbours in $\mathrm{Int}(C_Q)$. Assume to the contrary and without loss of generality that u_0 does not have any neighbour in $\mathrm{Int}(C_Q)$. Since w_0 also does not have a neighbour in $\mathrm{Int}(C_Q)$, the path a, w_0, u_0, v lies on the boundary of some face f in $\mathrm{Int}(C_Q)$. Since f is bounded by a 5-cycle, there is some vertex z that is adjacent to both a and v. However, this means that v, z, a is a $v - a$ path of length 2, which contradicts the fact that $Q : w_0, a, w_j$ is a 2-chord (i.e., a is in $N_3(v)$). Thus there exist vertices x and y in $\mathrm{Int}(C_Q)$ that are adjacent to u_0 and u_j respectively. Since G contains no 4-cycles, the vertices x and y are distinct, and neither vertex is adjacent to a. Consider the face f_{j+2}. This face is bounded by a 5-cycle $v, u_{j+2}, s, t, u_{j+3}$, where s and t are vertices of $N_2(v)$. Because there are no 1-chords across v (per Theorem 5.27), the vertex s is not adjacent to any vertex of $N_2(v) \cap N_1(u_0)$ or $N_2(v) \cap N_1(u_j)$. Thus the only way there could exist an $s - x$ path of length 3 or less is if s is adjacent to a (and there is some vertex adjacent to both a and x). Similarly, since $d(t, x) \leq 3$, the vertex t is also adjacent to a. However, this creates a triangle on a, s and t, which contradicts the girth of G and completes the proof.

□

The penultimate result of this chapter follows with some use of Theorems 5.27 and 5.28.

Theorem 5.29. *There does not exist a pentagulation with diameter 3, girth 5 and maximum degree greater than or equal to 8.*

Proof. Assume to the contrary that G is a pentagulation of girth 5, diameter 3 and maximum degree Δ, where $\Delta \geq 8$. Let v be a vertex of G with maximum degree, and label the neighbours $u_1, u_2, \ldots, u_\Delta$ of v such that each path u_i, v, u_{i+1} lies on the boundary of a face (subscripts taken mod Δ). By Lemma 5.24, for each i in $\{1, 2, \dots, \Delta\}$, there is a vertex w_i in $N_1(u_i) \cap N_2(v)$. Each vertex w_i is not adjacent to u_j for any $j \neq i$. Since G has diameter 3, there is a path of length at most 3 from w_0 to w_4.

We claim that such a path, call it Q, is a 3-chord across v, i.e., the path Q is of the form w_0, a, b, w_4, where a and b are vertices of $N_3(v)$. By Theorem 5.27, there are no 1-chords across v, so it is not possible that w_0 and w_4 are adjacent. Similarly, there are no 2-chords across v per Theorem 5.28, so it is not possible that Q is of the form w_0, c, w_4, where c is some vertex of $N_3(v)$. The vertex v is not in Q, since $\ell(Q) \leq 3$ and $d(v, x_0) = d(v, x_4) = 2$. The path Q does not contain any vertex of $N_1(v)$: If Q did contain a vertex u_j of $N_1(v)$, and Q had length 2, then Q would be of the form $Q : w_0, u_j, w_4$, which is impossible by the previous paragraph. If Q contains u_j and has length 3, it is either of the form w_0, u_j, x, w_4 or w_0, x, u_j, w_4, where x is some vertex of $N_2(v)$ (x is not in $N_1(v)$ since G is triangle-free). But then either xw_4 or w_4x is a 1-chord across v, which is impossible, so Q contains no vertices of $N_1(v)$.

To complete the proof of the claim, it suffices to show that $V(Q) \cap N_2(v) = \{w_0, w_4\}$. Assume to the contrary that there is a vertex x of Q, that is not w_0 or w_4, in $N_2(v)$. If Q has length 2, then it is of the form w_0, x, w_4. Since there are no 1-chords across v, the vertex x is adjacent to u_1 or $u_{\Delta-1}$, and so xw_4 is a 1-chord across v, which is a contradiction. If Q has length 3, then it is either of the form w_0, x, y, w_4 or the form w_0, y, x, w_4, where y is a vertex of $N_2(v)$ (y cannot be in $N_3(v)$, since there are no 2-chords across v). By symmetry, we may assume without loss of generality that $Q : w_0, x, y, w_4$. Since there are no 1-chords across v, the vertex x is a neighbour of u_1 or $u_{\Delta-1}$, and y is a neighbour of u_3 or u_5. In all possible cases, the edge xy is a 1-chord across v, which yields the desired contradiction and proves the claim.

The cycle $C_Q : w_0, a, b, w_4, u_4, v, u_0$ under $Q : w_0, a, b, w_4$ is a separating cycle that dominates either its interior or exterior. Thus either the vertex w_2, or the vertex w_6, is adjacent to a vertex of C_Q. Let us assume w_2 is adjacent to a vertex of C_Q (the proof for w_6 is identical). As G contains no shortcycles, the vertex w_2 is not adjacent to any of u_0, v or u_4. Because G contains no 1-chords across v, w_2 is not adjacent to either w_0 or w_4, so w_2 is adjacent to a or b. If w_2 is adjacent to a, then w_2, a, w_0 is a 2-chord across v, and if w_2 is adjacent to b, then w_2, b, w_4 is a 2-chord. In either case we obtain a contradiction, completing the proof. □

We now conclude this chapter by stating and proving the full solution to the degree-diameter problem for diameter 3 pentagulations.

Theorem 5.30. *Let G be a pentagulation of diameter 3, order n and maximum degree $\Delta \geq 8$. The order of G satisfies $n \leq 3\Delta - 1$.*

Proof. By Theorem 5.29 and Corollary 5.7, the pentagulation G has neither girth 5, nor any 3-cycles, and thus has girth 4. The result then follows from Theorem 5.23. □

Chapter 6

Distances and separators in maximal planar graphs

6.1 Foundations

In this chapter, we explore the structure of maximal planar graphs, with emphasis on their distance-theoretic properties. Although obvious, it is worth highlighting that every plane (planar) graph is a spanning subgraph of some maximal plane (planar) graph. We recall a handful of well-known results, the first four of which can be found in Diestel's *Graph Theory* [16].

Theorem 6.1. **[16]** *A plane graph of order $n \geq 3$ is a maximal plane graph if and only if every face of the graph is bounded by a 3-cycle.*

Proposition 6.2. **[16]** *If a maximal plane graph has order $n \geq 3$, then it has $3n - 6$ edges.*

Proposition 6.3. **[16]** *A planar graph of order $n \geq 3$ is a maximal planar graph if and only if it has $3n - 6$ edges.*

Proposition 6.4. **[16]** *If a maximal planar graph has order $n \geq 4$, then it is 3-connected.*

Proposition 6.5. [ ] *If G is a maximal planar graph of order at least 4, and v is a vertex of G, then the subgraph $G[N(v)]$ has a spanning cycle.*

A proof of Proposition 6.5 can be found in [24].

The above propositions and Theorem 2.21 imply that any maximal planar graph has a unique (up to the choice of outer face) embedding in the plane / sphere.

6.2 No constraints for radius and diameter

Recall from the discussion following Proposition 2.4 that given any two positive integers r and d such that $r \le d \le 2r$, there exists a connected graph with radius r and diameter d. In this section, we show by construction that the same is true for maximal planar graphs.

Theorem 6.6. *If r and d are positive integers satisfying $r \le d \le 2r$, then there exists a maximal planar graph of radius r and diameter d.*

Proof. Note that K_4 has radius and diameter 1, and $K_5 - \{e\}$ has radius 1 and diameter 2. For $r \ge 2$, $r \le d \le 2r$, we construct a maximal planar graph G_r^d with radius r and diameter d as follows:

The graph G_r^d has $d+1$ layers, label these $L_0, L_1, \ldots, L_d$. Let layers L_0 and L_d each consist of a single vertex: $L_0 = \{v_0\}$ and $L_d = \{v_d\}$. Layers L_1, L_2, $\ldots$, L_{d-1} are all cycles of length $2r$. Label the vertices of each layer so that $L_i : v_{i,0}, v_{i,1}, \ldots, v_{i,2r-1}$. Call an edge *horizontal* if it is contained in some cycle L_i, and denote $c = v_{\lfloor d/2 \rfloor,0}$ (See Figure 6.1).

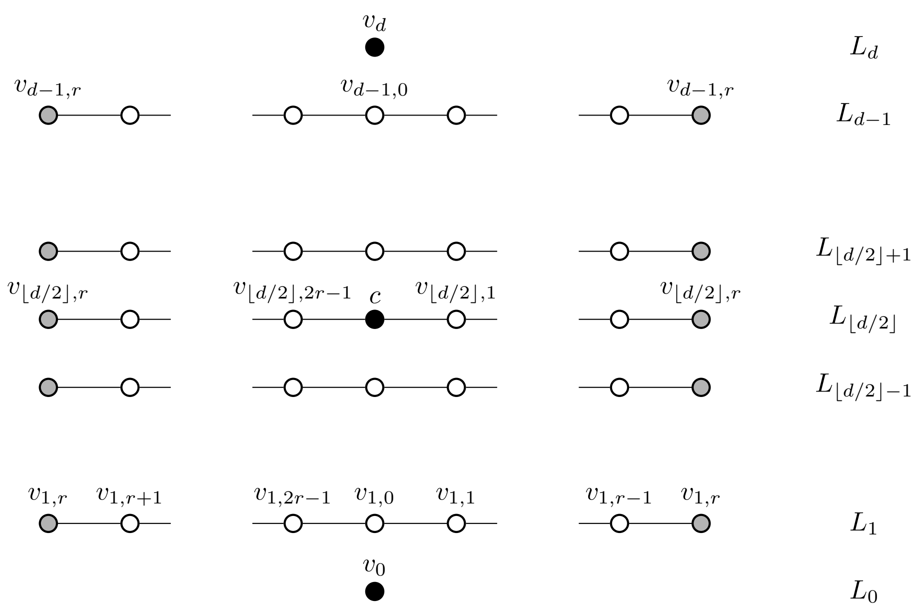


Figure 6.1: This diagram demonstrates the layers L_0 to L_d of the graph G_r^d in the proof of Theorem 6.6. The vertices c, v_0 and v_d are all coloured black. The two grey vertices of each layer are identified with each other.

Append the following edges to the graph (See Figure 6.2):
For all i in $[0, 2r-1]$, add the edges $v_0 v_{1,i}$ and $v_d v_{d-1,i}$. For each $i \in [1, d-2]$, add the edge $v_{i,j} v_{i+1,j}$. We say these edges are *vertical*. In the graph obtained thus far, all the faces of the graph incident with v_0 or v_d are triangles, and the other faces are squares.

The remaining edges to add will radiate diagonally away from c, and divide the remaining square faces into triangular faces:
For $i \in [\lfloor d/2 \rfloor, d-2]$ and $j \in [0, r-1]$, add all edges $v_{i,j} v_{i+1,j+1}$.

For $i \in [\lfloor d/2 \rfloor, d-2]$ and $j \in [r+1, 2r]$ (taken modulo $2r$), add all edges $v_{i,j}v_{i+1,j-1}$.
For $i \in [2, \lfloor d/2 \rfloor]$ and $j \in [0, r-1]$, add all edges $v_{i,j}v_{i-1,j+1}$.
For $i \in [2, \lfloor d/2 \rfloor]$ and $j \in [r+1, 2r]$ (taken modulo $2r$), add all edges $v_{i,j}v_{i-1,j-1}$.
Call these edges *diagonal*.

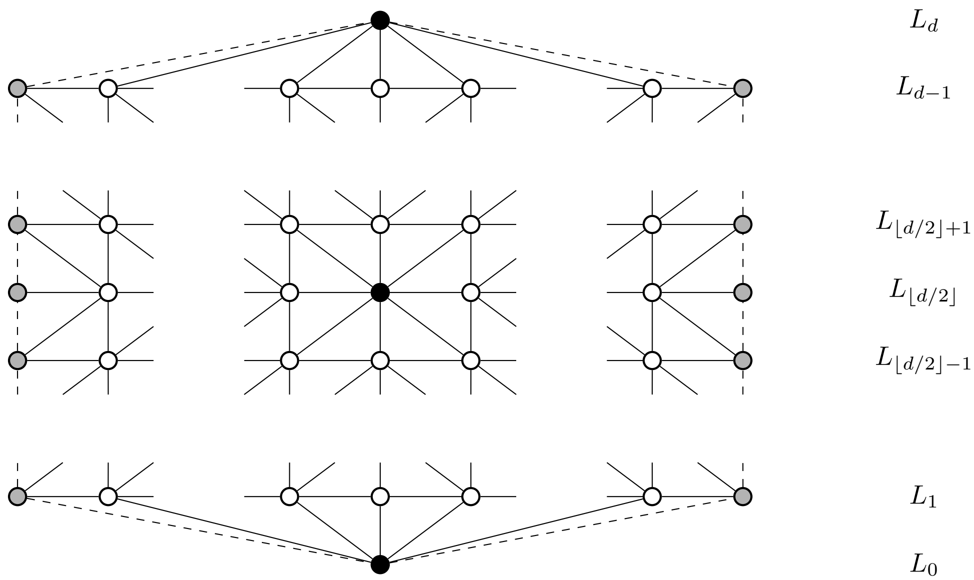


Figure 6.2: The graph G_r^d from the proof of Theorem 6.6. Identify the two grey vertices of each layer to obtain a triangulation of the sphere.

It's clear the construction yields a plane graph with triangular faces, and hence, a maximal planar graph. We must still prove that the constructed graph, G_r^d, has radius r and diameter d. Let $G = G_r^d$ and $H = G_r^d - \{v_0, v_d\}$. When no subscript is given, assume the graph is question is G.

Two vertices $v_{i,j}$ and $v_{k,l}$ are adjacent only if $|k-i| \leq 1$ and $|l-j| \leq 1$ (latter subscripts taken module $2r$). As such, $d_H(v_{i,j}, v_{k,l}) \geq \max\{|k-i|, |l-j|\}$. Note that the inequality does not hold in G.

The diameter of G_r^d is d:
Certainly $d(v_0, v_d) = d$ exactly, so the diameter of G_r^d is at least d. It is easy to see that $d(v_0, v_{i,j}) = i$, and $d(v_d, v_{i,j}) = d - i$, and that $d(v_{i,j}, v_{k,j}) = |k-i|$. We need only show that $d(v_{i,j}, v_{k,l}) \leq d$ for $j \neq l$. Let $P_j = v_0, v_{1,j}, v_{2,j}, \ldots, v_{d-1,j}, v_d$ and $P_l = v_0, v_{1,l}, v_{2,l}, \ldots, v_{d-1,l}, v_d$ be two paths in G. The paths P_j and P_l are internally disjoint $v_0 - v_d$ paths of length d, and $P_j \cup P_l$ is a cycle of length $2d$ containing both $v_{i,j}$ and $v_{k,l}$. Hence $d(v_{i,j}, v_{k,l}) \leq d$.

The radius of G_r^d is r:
We first prove the eccentricity of any vertex of G is at least r: Clearly $e(v_0) = e(v_d) = d \geq r$. Given $v = v_{i,j}$ in H, consider the vertex $v' = v_{d-i,j+r}$ (with the second subscript taken modulo $2r$). By the inequality derived above, $d_H(v, v') \geq \max\{|d-2i|, r\} \geq r$. Since $d(v, v_0) = d(v', v_d) = i$, and $d(v, v_d) = d(v', v_0) = d-i$, any $v - v'$ path in G containing v_0 or v_d has length at least $i + (d-i) = d \geq r$. Thus, any $v - v'$ geodesic either lies in H or has length at least d, and so $e(v) \geq r$.

Now we show the eccentricity of $c = v_{\lfloor d/2 \rfloor, 0}$ is exactly r. From the prior argument, $e(c) \geq r$. Further,

$d(c, v_0) = \lfloor d/2 \rfloor$ and $d(c, v_d) = \lceil d/2 \rceil$, both of which are at most r as $r \le d \le 2r$ by assumption.
Consider a vertex $v_{i,j}$. We need that $d(c, v_{i,j}) \le r$. We will assume that $i \in [\lfloor d/2 \rfloor, d-1]$ and $j \in [0, r-1]$, as the other three possible cases follow similarly (The cases are exactly those defined in the construction of the diagonal edges radiating outward from c.) Let $k = \min\{i - \lfloor d/2 \rfloor, j\}$. The path formed by following diagonal edges $c, v_{\lfloor d/2 \rfloor+1,1}, \ldots, v_{\lfloor d/2 \rfloor+k,k}$, and then either vertical or horizontal edges from $v_{\lfloor d/2 \rfloor+k,k}$ to $v_{i,j}$ has length at most $\max\{d - \lfloor d/2 \rfloor - 1, r - 0\} \le r$. Thus $e(c) = r$ as $e(c) \ge r$ and $d_G(c, v) \le r$ for all vertices v of G.

□

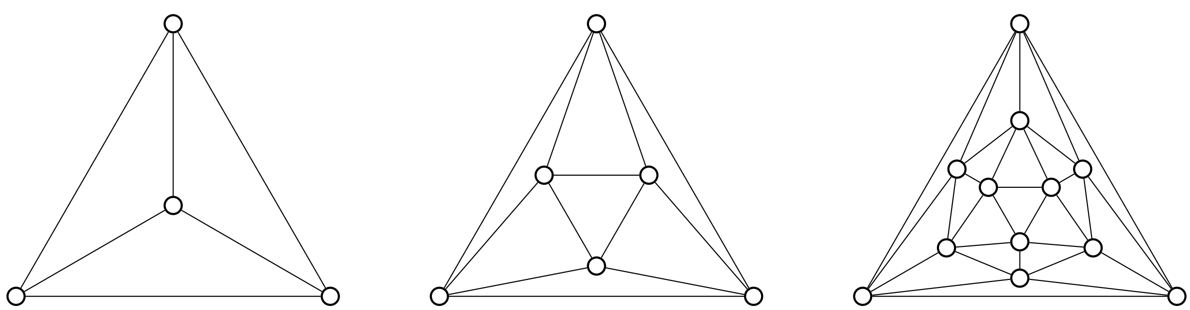
6.3 Degree constraints

Let $G = (V, E)$ be a maximal planar graph with $n \ge 3$ vertices, m edges and minimum degree δ. By Proposition 6.2 and the Handshaking Lemma (Lemma 2.1), we see that the sum of the degrees of G is at most $6n - 12$ and so $\delta \le 5$. By Whitney's Theorem (Theorem 2.6) and Proposition 6.4, we have $\delta \ge 3$. The maximum degree Δ cannot be bounded from above, but the number of vertices of degree $k \ge 7$ in G is bounded above. We let n_k denote the number of vertices of G with degree k ($k \ge 7$) and observe that

$$k \cdot n_k \le \sum_{u \in V} d(u) \le 6n - 12 < 6n.$$

From this inequality, it is clear that $\frac{n_k}{n} < \frac{6}{k}$ (this bound is substantially improved in [38]). In this section, we investigate similar bounds for k in $\{3, 4, 5, 6\}$.

As a corollary to the centuries-known result that there are exactly five regular polyhedra (see [11] for a proof), there are only three regular maximal planar graphs of order at least 4 (see Figure 6.3).



(1) (2) (2)

Figure 6.3: The three regular maximal planar graphs are the tetrahedral graph (1), the octahedral graph (2) and the icosahedral graph (3).

The next result is well known. However, it is difficult to find a proof of the result in full generality (In [26], it is proven under some additional conditions).

Lemma 6.7. [ ] *If G is a maximal planar graph of order 5 or more, then no two vertices of degree 3 are adjacent in G.*

Proof. Let $n \geq 5$ denote the order of G, and let m be the number of edges of G. Assume to the contrary that u and v are two adjacent degree 3 vertices of G, and let $G' = G - \{u, v\}$. Further, let n' and m' be the number of vertices and edges of G' respectively. Since G' is a subgraph of G, it is planar. It's clear that $n' = n - 2$. Since u and v are adjacent vertices of degree 3 we have that $m' = m - 5$. By Proposition 6.2, we see that $m = 3n - 6$. Thus:

$$m' = m - 5 = 3n - 11 = 3n' - 5$$

Per Proposition 6.2, this contradicts the planarity of G', completing the proof. $\square$

Lemma 6.7 gives us enough leverage to find the best possible bound for $\frac{n_3}{n}$, as shown in Theorem 6.8 below. The construction used in the proof of Theorem 6.8 also appears in [3], where it is used to demonstrate that an independent set of a maximal planar graph of order n can have up to $\frac{2n-4}{3}$ vertices (as such, this result has likely been noticed before, although I have not been able to find it in the literature).

Theorem 6.8. [ ] *Let G be a maximal planar graph of order $n \geq 5$, and let n_3 denote the number of vertices in G of degree 3. Then $\frac{n_3}{n} < \frac{2}{3}$, and this bound is asymptotically sharp.*

Proof. Embed G as a maximal plane graph. Let V_3 denote the set of vertices with degree 3 in G (so $n_3 = |V_3|$). Further, let $V' = V(G) - V_3$, let $n' = |V'|$, denote $G' = G[V']$ and let $m' = |E(G')|$. We consider G' as a plane subgraph of G.

Claim: The subgraph G' is a maximal plane graph.
By Proposition 6.3, it suffices to show that $m' = 3n' - 6$. Per Lemma 6.7, no two vertices of V_3 are adjacent in G. As such, every vertex of V_3 is incident with three edges, all three of which are also incident with a vertex of V'. Thus $m' = m - 3n_3$, so $m = m' + 3n_3$. It is also clear that $n = n' + n_3$. Substituting these expressions for m and n into the equation $m = 3n - 6$ (which hold by Proposition 6.2), we obtain the equation $m' = 3n' - 6$, completing the proof of the claim.

Let σ denote the number of faces of G'. Using the Euler Characteristic equation $n' - m' + \sigma = 2$, and the identity $m' = 3n' - 6$, we deduce that $\sigma = 2n' - 4$. Since each face of G' contains at most one vertex of V_3, we can bound n_3 above by $n_3 \leq \sigma = 2n' - 4$. Thus:

$$\frac{n_3}{n} = \frac{n_3}{n' + n_3} \leq \frac{2n' - 4}{3n' - 4} < \frac{2}{3}$$

It remains to show that this bound is asymptotically sharp. Let H_n be any maximal plane graph of order $n \geq 4$. This graph has $2n - 4$ triangular faces. Create a new maximal plane graph H'_n by adding a vertex to each face of H_n, and making the new vertex in each face adjacent to the three vertices of H_n on the boundary of that face. The graph H'_n has $n + (2n - 4) = 3n - 4$ vertices, at least $2n - 4$ of which have degree 3. We can use this construction for arbitrarily large n, obtaining graphs for which $\frac{2}{3} - \frac{|V_3|}{|V|}$ is arbitrarily small. $\square$

Note that Theorem 6.8 does not hold for general planar graphs, even 3-connected ones, and examples illustrating this are plentiful (consider, for example, the product graph $C_n \times K_2$). The next proposition, which has surely been noticed before, shows that the analogues of Theorem 6.8 for degrees 4, 5 and 6 do not hold.

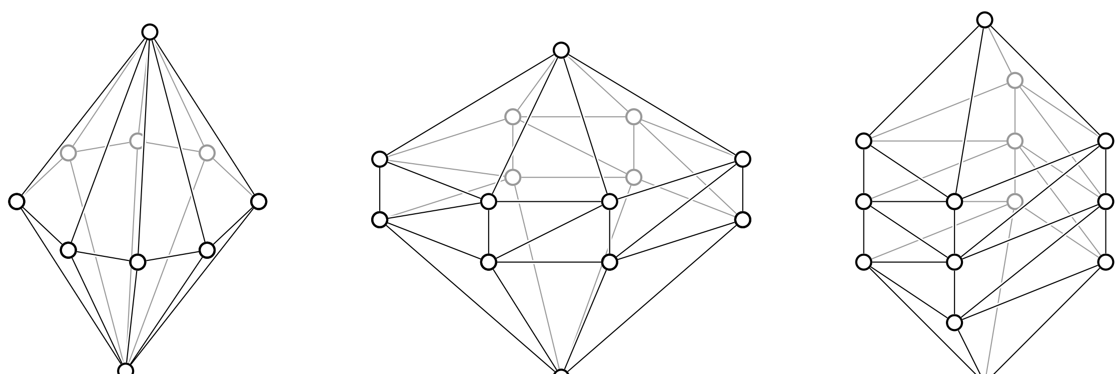
Proposition 6.9. [ ] *Let k be an integer in $\{4, 5, 6\}$, let $N > 0$ be an integer and let $\epsilon > 0$ be a real number. There exists a maximal planar graph of order $n > N$, with n_k vertices of degree k, such that $1 - \frac{n_k}{n} < \epsilon$.*

Proof. We prove the theorem by exhibiting three infinite families X_n, Y_n and Z_n of maximal planar graphs (see Figure 6.4 for an example of a graph from each family). There are three cases to consider.

Case 1: $k = 4$
Let X_n ($n \geq 3$) be the graph formed by adding two vertices to the cycle C_n of length n, and making these two vertices adjacent to each vertex of the cycle (i.e., X_n is a double wheel). The graph X_n has $n+2$ vertices, at least n of which have degree 4. We close Case 1 by observing that $\frac{n}{n+2} \to 1$ as $n \to \infty$.

Case 2: $k = 5$
We construct the graph Y_n ($n \geq 3$). Consider the union of two cycles $C_1 : u_0, \ldots, u_{n-1}$ and $C_2 : v_0, \ldots, v_{n-1}$ of length n, and two vertices u and v. Make u adjacent to every vertex of C_1, and v adjacent to every vertex of C_2. To complete the construction of Y_n, add all edges of the forms $u_j v_j$ and $u_j v_{j+1}$ (subscripts taken mod n). The maximal planar graph Y_n has at least $2n$ vertices of degree 5, out of a total $2n + 2$ vertices. Since $\frac{2n}{2n+2} \to 1$ as $n \to \infty$, we are done with Case 2.

Case 3: $k = 6$
The construction of Z_n ($n \geq 1$) is as follows. We begin with a family $C_1, \ldots C_{n+2}$ of $n + 2$ disjoint 4-cycles, and two vertices u and v (there is nothing special about 4-cycles, any cycle length works here). For i in $\{1, \ldots, n + 2\}$, label the vertices of C_i such that $C_i : v_{i,0}, v_{i,1}, v_{i,2}, v_{i,3}$. For $i \leq n + 1$, and j in $\{0, 1, 2, 3\}$, add all edges of the forms $v_{i,j} v_{i+1,j}$ and $v_{i,j} v_{i+1,j+1}$ (subscripts j taken mod 4). Finally, make u adjacent to each vertex of C_1 and v adjacent to each vertex of C_{n+2}. If $i \in \{2, 3, \ldots, n - 1\}$, then each vertex of C_i has degree 6. Thus Z_n has order $4(n + 2) + 2 = 4n + 10$, and at least $4n$ of these vertices have degree 6. We conclude the proof by observing that $\frac{4n}{4n+10} \to 1$ as $n \to \infty$. □



X_8 Y_6 Z_1

Figure 6.4: One graph of each infinite family constructed in the proof of Proposition 6.9.

6.4 Separators of maximal planar graphs

It is well known that the minimal separators of maximal planar graphs are induced cycles (see, for example, [2]), but simple proofs of this basic fact are elusive in the literature. In this section, we provide such a proof, and show that the minimal sets separating any two connected subgraphs of a maximal planar graph are also induced cycles.

We begin with an observation made by many graph theorists in the past, and a slight variation of a well-known lemma (that can be found in the exercises of [16], among other places).

Observation 6.10. [ ] Let G be a connected graph and v a vertex of G. If $G[N(v)]$ is connected then v is not a cut-vertex of G.

Lemma 6.11. [16] *Let $G = (V, E)$ be a graph, let S a separator of G, and let $C_1, \ldots, C_k$ be the components of $G - S$. If S is minimal, then for every vertex s in S, and for all i in $\{1, \ldots, k\}$, there is some vertex v_i in C_i such that s and v_i are adjacent.*

Proof. Assume to the contrary and without loss of generality that no vertex of C_1 is adjacent to some vertex s in S. Since C_1 and C_2 are components of $G - S$, and no vertex of C_1 is adjacent to s, there is no $C_1 - C_2$ path in $G - (S - s)$. Thus $S - s$ is a separator of G, contradicting the minimality of S. □

Note that the converse of Lemma 6.11 also holds. Let S be a separator such that every vertex of S is adjacent to each component of $G - S$, and let $T \subset S$ be a proper subset of S. The vertices of $S - T$ are adjacent to all the components of $G - S$, so $G - T$ is connected, and thus S is a minimal separator.

We need one more lemma before presenting our result. It is not original, but the proof is short and provided for completeness.

Lemma 6.12. [5] *Let $G = (V, E)$ be a 3-connected planar graph, and let S a minimal separator of G. Then $G - S$ has exactly two components.*

Proof. Assume to the contrary that C_1, C_2 and C_3 are distinct components of $G - S$. As G is 3-connected, we have that $|S| \geq 3$. Let s_1, s_2 and s_3 be three vertices of S. From Lemma 6.11, each vertex s_i is adjacent to some vertex of each component C_i. Thus we deduce that $K_{3,3}$ is a minor of G (contract each component C_i to a single vertex), contradicting the planarity of G. □

Theorem 6.13. [ ] *Let $G = (V, E)$ be a maximal planar graph. If S is a minimal separator of G, then S is a chordless Jordan separating cycle.*

Proof. Let S be a minimal separator of G. By Lemma 6.12, $G - S$ has exactly two components, call them A and B. We first show that $\delta(G[S]) \geq 2$ (And hence, $G[S]$ contains some cycle). We then show that any cycle C of $G[S]$ is itself a Jordan separating cycle, from which we deduce that $G[S] = C$ by minimality of S. Finally, we will show that $\Delta(G[S]) \leq 2$, which implies that $G[S]$ is chordless.

Claim 1: $\delta(S) \geq 2$

Assume to the contrary that there exists a vertex s in S such that $d_S(s) \leq 1$. As $G[N(s)]$ is spanned by a cycle (G is an mpg), $G[N(s) - S]$ is connected. Since S is minimal, by Lemma 6.11, both $N(s) \cap A$ and $N(s) \cap B$ are non-empty. However this implies that some vertex of $A \cap N(s)$ is adjacent to some vertex of $B \cap N(s)$, which contradicts that A and B are distinct components of $G - S$.

Since $\delta(S) \geq 2$, the induced graph $G[S]$ contains some cycle C as a subgraph.

Claim 2: C is a Jordan separating cycle of G.

Assume to the contrary that C is not a Jordan separating cycle. As C is not Jordan separating, without loss of generality, both A and B lie in the exterior of C, and the interior of C does not contain any vertex of G. Let s_1, s_2 and s_3 be three vertices of C lying on the same triangular face, f, of the interior of C. By Lemma 6.11, for all i in $\{1, 2, 3\}$, there exist vertices a_i in A and b_i in B such that s_i is adjacent to both a_i and b_i. Since G is planar, the graph G' formed by adding to f a new vertex, x, and the edges xs_1, xs_2 and xs_3, is also planar. But then $K_{3,3}$ is a minor of G' (contract A and B each to a single vertex), contradicting planarity of G'.

By the minimality of S, we deduce that $S = V(C)$. Hence $G[S]$ induces a Jordan separating cycle. To prove that $G[S]$ is chordless, it is sufficient to prove that $\Delta(S) \leq 2$.

Claim 3: $\Delta(S) \leq 2$

Assume to the contrary that some vertex s in S has $d_G[S](s) \geq 3$. Let s_1, s_2 and s_3 be three vertices of $N(s) \cap S$. By Lemma 6.11, for all i in $\{1, 2, 3\}$, there exist vertices a_i in A and b_i in B such that s_i is adjacent to both a_i and b_i. But then $K_{3,3}$ is a minor of G, contradicting the planarity of G. □

The graph in Figure 6.5 demonstrates that the hypothesis of Theorem 6.13 cannot be weakened from 'maximal planar' to '3-connected planar'. The graph in the figure is a 3-connected plane graph containing a minimal separator that does not induce a cycle.

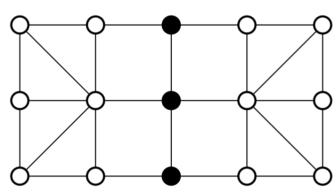


Figure 6.5: In this 3-connected plane graph, the three black vertices are a minimal separator that does not induce a cycle.

We need one more celebrated result in the study of planar graphs before we can completely describe the relationship between chordless cycles and minimal separators in maximal planar graphs.

Theorem 6.14. [16] *The face-cycles of a 3-connected plane graph are exactly the non-separating induced (chordless) cycles.*

Much like Theorem 6.13, the next result, Theorem 6.15, is certainly not a new discovery. We prove it here for completeness.

Theorem 6.15. [🔥] *A chordless cycle in a maximal planar graph is either a minimal separator or it is face-cycle.*

Proof. Let G be a maximal planar graph, and let C be a chordless cycle of G that is not a face-cycle. By Theorem 6.14, the cycle C separates G. Since C separates G, the set $V(C)$ contains some minimal separator S. Per Theorem 6.13, the graph $G[S]$ is a chordless Jordan separating cycle. Since C is chordless, it does not contain any cycle as a proper subgraph, so $C = G[S]$. □

With Theorem 6.14 in mind, we may be tempted to think that Theorem 6.15 may hold if we replace the condition that the graph is maximal planar with the condition that it is 3-connected and planar. The 3-connected plane graph in Figure 6.6 shows that we cannot do this, since the bold cycle is an induced cycle that is neither a face-cycle nor a minimal separator.

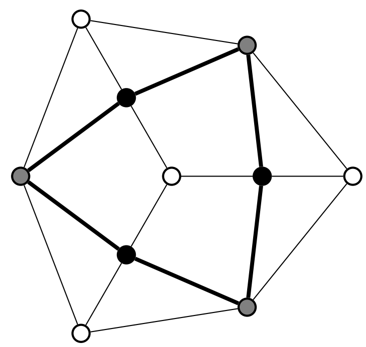


Figure 6.6: In this 3-connected plane graph, the chordless bold cycle is an induced cycle that is neither a minimal separator nor a face-cycle. Note that the three black vertices are a minimal separator of the graph.

6.5 Separating connected subsets

If A and B are two connected subgraphs of a graph G, then a minimal $A - B$ separator is not necessarily a minimal separator (for example, consider the graph in Figure 6.7). However the situation is different in 3-connected planar graphs, where a minimal separator of two connected subgraphs is itself a minimal separator (see Lemma 6.17). Hence we deduce by Theorem 6.13 that, in maximal planar graphs, minimal separators of connected subgraphs are also chordless Jordan separating cycles.

Lemma 6.16. [5, 16] *Let G be a graph, let A and B be sets of vertices of G such that $G[A]$ and $G[B]$ are connected, and let S be a minimal $A - B$ separator such that S, A and B are pairwise disjoint. If G_A and G_B are the components of $G - S$ containing A and B respectively, then every vertex of S is adjacent to a vertex of G_A and to a vertex of G_B.*

Lemma 6.16 is a mild strengthening of the corresponding result in [5,16]. The original Lemma is stated only for the case where A and B both consist of a single vertex, but the result still follows easily from an argument similar to that in the proof of Lemma 6.11. Similarly, Proposition 4 in [5] states that if u and v are vertices

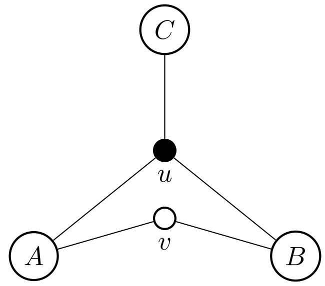


Figure 6.7: A, B and C are connected subgraphs of a graph G. Note that $\{u, v\}$ is a minimal $A - B$ separator, but not a minimal separator (since $G - u$ is disconnected).

of a 3-connected planar graph G, then any minimal $u - v$ separator is a minimal separator of G. Using a similar proof technique, we give a mild strengthening of this result with Lemma 6.17.

Lemma 6.17. [ ] *Let $G = (V, E)$ be a 3-connected planar graph and let A, B and S be pairwise disjoint subsets of V such that both $G[A]$ and $G[B]$ are connected, and the set S separates A and B. If S is a minimal $A - B$ separator, then it is also a minimal separator of G.*

Proof. Let S be a minimal $A - B$ separator, and denote by G_A and G_B the components of $G - S$ containing A and B, respectively.

Claim: The components G_A and G_B are the only components of $G - S$.
Assume to the contrary that $G - S$ contains a third component, call it G_C. Let $T \subseteq S$ denote the vertices of S that are adjacent to any vertex of G_C, and note that $|T| \geq 3$ since G is 3-connected. By Lemma 6.16, every vertex of T is adjacent to a vertex of G_A and a vertex of G_B. However, by contracting G_A, G_B and G_C down to three single vertices, we deduce that G has a $K_{3,3}$ minor, contradicting the planarity of G and proving the claim.

Since $G - S$ has only two components G_A and G_B, and every vertex of S is adjacent to a vertex of both G_A and G_B (per Lemma 6.16), we conclude that S is itself a minimal separator of G. □

The three graphs G, H and I in Figure 6.8 demonstrate that all the hypotheses in the statement of Lemma 6.17 are necessary. In particular, the graph G is 3-connected and planar, but the set $G[B]$ is not connected, and the black vertices form a minimal $A - B$ separator that is not a minimal separator of G (the square vertices are a minimal separator). The graph H is 2-connected, 3-edge-connected and planar, and the black vertices form a minimal $C - D$ separator that is not a minimal separator of H. Finally the graph I is 3-connected but not planar, and the black vertices form a minimal $\{u\} - \{v\}$ separator that is not a minimal separator of I.

Theorem 6.18. [ ] *Let $G = (V, E)$ be a maximal planar graph of order at least 4 and let A, B and S be pairwise disjoint subsets of V such that $G[A]$ and $G[B]$ are connected, and S is an $A - B$ separator. If S is a minimal $A - B$ separator, then it induces a chordless separating cycle that Jordan separates A and B.*

Proof. Let G, A, B and S be as described in the Theorem statement. By Proposition 6.4, the graph G is 3-connected, and so per Lemma 6.17, the $A - B$ separator S is a minimal separator of G. Hence S induces

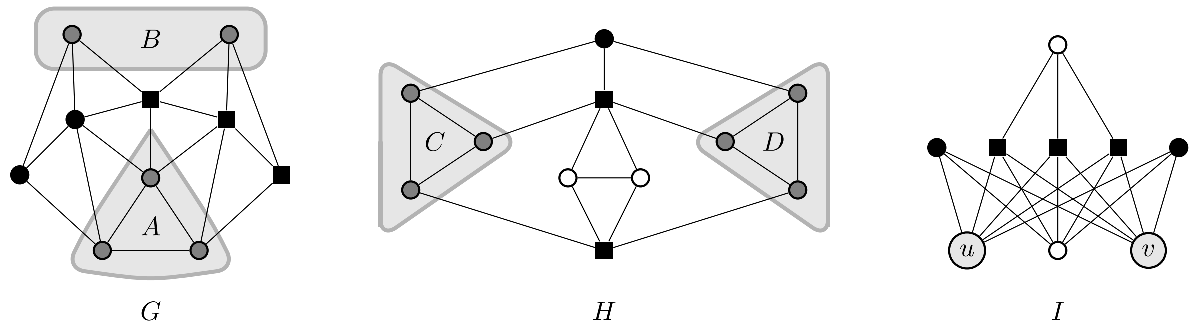


Figure 6.8: The graph G is 3-connected and planar, the graph H is 2-connected, 3-edge connected and planar, and the graph I is 3-connected. These three graphs illustrate the necessity of all three hypotheses in the statement of Lemma 6.17.

a chordless Jordan separating cycle by Theorem 6.13. Since $G - S$ has exactly two components by Lemma 6.12, and $G[S]$ Jordan separates G, one component of $G - S$ lies in the interior of $G[S]$ and the other in the exterior of $G[S]$. Thus $G[S]$ Jordan separates A and B. □

6.6 Non-separating subsets of maximal planar graphs

Theorems 6.13 and 6.18 describe the structure of minimal separators in maximal planar graphs, and Theorem 6.15 tells us when a set of vertices in a maximal planar graph is itself a minimal separator. With these theorems in mind, there is a natural question to pose: what do the non-separating sets of vertices in a maximal planar graph look like? In light of Observation 6.10, it is reasonable to assume that if A is a set of vertices in a connected graph G such that $N(A)$ is connected, then A is not a separator of G (Theorem 6.21 below justifies this assumption). In arbitrary graphs, the converse does not hold, but we will see that it does in maximal planar graphs, provided $G[A]$ is connected. In further pursuit of a description of non-separating sets, we examine the structure of a very simple class of non-separating sets in maximal planar graphs: single vertices. Per Proposition 6.5, if G is a maximal planar graph of order at least 4, and v is a vertex of G, then $G - v$ is 2-connected and $N(v)$ is spanned by a cycle. We show that in maximal planar graphs, these properties are closely related, even when the non-separating set we consider is not a single vertex.

We need two lemmas from Bondy and Murty's *Graph Theory* [4]. Both lemmas are straightforward consequences of Menger's Theorem (Theorem 2.8), and Lemma 6.20 is known as *The Fan Lemma*.

Lemma 6.19. **[4]** *Let $G = (V, E)$ be a k-connected graph and let A and B be subsets of V such that $|X| \geq k$ and $|Y| \geq k$. Then there is a collection of k disjoint $A - B$ paths in G.*

Lemma 6.20. **[4]** *Let $G = (V, E)$ be a k-connected graph, let u be a vertex of V, and let $A \subseteq V - u$ be a set of vertices such that $|A| \geq k$. Then there exist k distinct internally disjoint $u - A$ paths in G.*

While I have not been able to locate the following theorem in the literature, it is hard to imagine that it has not been stated before. The theorem allows us to remove sets from a k-connected graph without harming the connectivity of the graph, provided we know that the neighbourhood of the set we are removing is itself k-connected.

Theorem 6.21. [♣] *Let $G = (V, E)$ be a k-connected graph, and let $A \subset V$ be a set of vertices of G. If the induced graph $G[N(A)]$ is k-connected, then $G - A$ is also k-connected.*

Proof. Let $B = G - N[A]$ (i.e, B is the set of vertices of G that are neither in A nor $N(A)$). It is clear that, if B is non-empty, then $N(A)$ is an $A - B$ separator. Per Menger's Theorem, to prove the theorem, it suffices to demonstrate that if u and v are two arbitrary vertices of $G - A$, then there exist k internally disjoint $u - v$ paths in $G - A$.

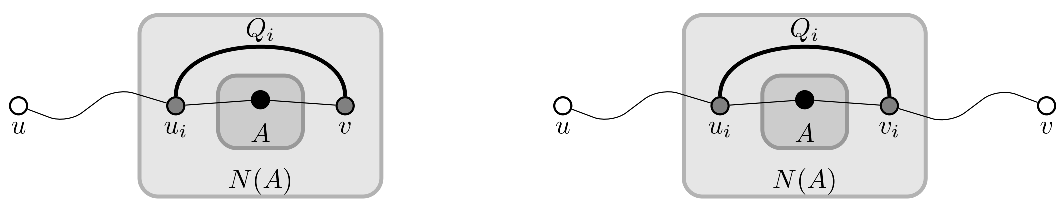
There are three cases to consider.

Case 1: Both u and v are in $N(A)$.
Since $G[N(A)]$ is k-connected, we obtain the k internally disjoint $u - v$ paths desired as an immediate consequence of Menger's Theorem.

Case 2: The vertex u is in B, but v is in $N(A)$.
As G is k-connected, there exists a family $P_1, P_2, \ldots, P_k$ of internally disjoint $u - v$ paths in G by Menger's Theorem. If none of these paths contain a vertex of $N(A)$ or A other than v, then the family of paths also exists in $G - A$, and we are done. Thus we assume that, up to relabelling, the paths $P_1, P_2, \ldots, P_j$ ($j \leq k$) intersect $N(A) - v$. Observe that if a path $P_i : u, x_1, \ldots x_s, a, x_t, \ldots, v$ contains a vertex a of A, then there is some vertex u_i in $N(A)$ that appears before a in P_i. Thus, for each i in $\{1, \ldots, j\}$, we denote by u_i the first vertex of P_i that belongs to $N(A)$. Invoking Lemma 6.20 and the fact that $G[N(A)]$ is k-connected, there is a family $Q_1, Q_2, \ldots, Q_j$ of internally disjoint $\{u_1, \ldots, u_j\} - v$ paths in $G[N(A)]$ (see Figure 6.9). For each integer i in $\{1, \ldots, k\}$, define a path R_i as follows:

$$R_i = \begin{cases} P[u, u_i] \cup Q_i & \text{if } i \leq j \\ P_i & \text{if } i > j \end{cases}$$

The family $R_1, R_2, \ldots, R_k$ is a collection of k internally disjoint $u - v$ paths, completing the proof of Case 2.



Case 2 Case 3

Figure 6.9: This figure illustrates one of the paths Q_i contained in $N(A)$, from the proof of Theorem 6.21.

Case 3: Both u and v are vertices of B.
As in Case 2, there exists a family $P_1, P_2, \ldots, P_k$ of internally disjoint $u - v$ paths in G. If no paths intersect $N[A]$, we are done. Thus, with possible relabelling of the paths, we let $P_1, P_2, \ldots, P_j$ be the paths that do intersect $N[A]$. For $i \leq j$, let u_i denote the first vertex of the path P_i that lies in $N(A)$, and let v_i denote the last vertex of P_i that lies in $N(A)$. It is possible that $|V(P_i) \cap N[A]| = 1$, in which case u_i and v_i will coincide. Let ℓ be the number of paths P_i such that $|V(P_i) \cap N[A]| = 1$, and relabel the paths P_i such that the paths $P_1, \ldots, P_{j-\ell}$ each contain at least two vertices of $N[A]$, the paths $P_{j-\ell+1}, \ldots P_j$ each contain exactly one vertex of $N[A]$, and the paths $P_{j+1}, \ldots, P_k$ do not intersect $N[A]$ at all. Note that for i in $\{1, \ldots, j - \ell\}$,

the vertices u_i and v_i are distinct. Since $G[N(A)]$ is k-connected, the graph $G[N(A)] - \{u_{j-\ell+1}, \ldots, u_j\}$ is $(k-\ell)$-connected, and is thus $(j-\ell)$-connected. Hence there exists a family $Q_1, Q_2, \ldots Q_{j-\ell}$ of $j-\ell$ disjoint $\{u_1, \ldots, u_{j-\ell}\} - \{v_1, \ldots, v_{j-\ell}\}$ paths in $N(A)$ per Lemma 6.19 (see Figure 6.9). As such, we can construct a family $R_1, R_2, \ldots, R_k$ of internally disjoint $u - v$ paths in $G - A$ as follows:

$$R_i = \begin{cases} P[u, u_i] \cup Q_i \cup P[v_i, v] & \text{if } i \leq j - \ell \\ P_i & \text{if } i > j - \ell \end{cases}$$

In all cases, there is a family of at least k internally disjoint $u-v$ paths in $G-A$, so $G-A$ is k-connected. □

We can now present the promised description of non-separating sets of vertices in maximal planar graphs. Note that we only consider non-separating sets that are themselves connected. The characterisation provided by Theorem 6.22 can be seen as both a generalisation of Proposition 6.5 and a restricted form of the converse of Theorem 6.21 for maximal planar graphs.

Theorem 6.22. *Let $G = (V, E)$ be a maximal planar graph of order at least 4, and let $A \subset V$ be a connected set of vertices. Then:*
(1) The graph $G - A$ is connected if and only if $N(A)$ is connected,
(2) The graph $G - A$ is 2-connected if and only if $N(A)$ is spanned by a cycle.

Proof. Per Theorem 6.21, if $N(A)$ is connected, then $G - A$ is connected. Further, if $N(A)$ is spanned by a cycle, then it is 2-connected and by the same result we deduce that $G - A$ is 2-connected.

(Part 1) We prove that if $G - A$ is connected, then $N(A)$ is also connected.
Assume to the contrary that $G - A$ is connected but $N(A)$ is not. Let $B = G - N[A]$, and observe that $B \neq \emptyset$, for if B were empty, then $G - A$ would be the graph $G[N(A)]$, which is not connected.

Claim: There exists a component of $G[B]$ that is adjacent to at least two components of $G[N(A)]$.
Assume to the contrary that every component of B is adjacent to exactly one component of $N(A)$, and let X_1 and X_2 be two components of $N(A)$. Since $G - A$ is connected, there exists an $X_1 - X_2$ path in $G - A$. However, this is impossible, since any such path contains an edge from X_1 to a component B_1 of B, and another edge from B_1 to a different component X_i of $N(A)$. This contradicts the assumption that each component of B is adjacent to only one component of $N(A)$, proving the claim.

Let B_1 denote a component of B that is adjacent to multiple components of $N(A)$. Since $N(A)$ is an $A - B_1$ separator, there exists a chordless cycle C of $G[N(A)]$ that Jordan separates A and B_1 per Theorem 6.18. Since C is connected, it lies in a single component, say X_1, of $N(A)$. However, since B_1 is adjacent to multiple components of $N(A)$, there exists a component X_2 of $N(A)$ that is adjacent to B_1 and satisfies $V(C) \cap V(X_2) = \emptyset$. Because every vertex of the connected graph $G[X_2]$ is adjacent to A, and A is adjacent to B_1, there is an $A - B_1$ path P such that $V(P) \cap V(X_1) = \emptyset$. However this contradicts the fact that C is an $A - B_1$ separator, so $N(A)$ is connected.

(Part 2) We prove that if $G - A$ is 2-connected, then $N(A)$ is spanned by a cycle.
Embed G as a maximal plane graph, and consider $G - A$ as a plane subgraph of G. Because A is connected,

it lies in a single face, call it f, of $G - A$. Since $G - A$ is 2-connected, the boundary of f is a cycle C by Proposition 2.20. The face f is either the interior of C, or it is the exterior of C, so we will assume without loss of generality that $f = \mathrm{Int}(C)$. To find a cycle spanning $N(A)$, it suffices to show that $V(C) = N(A)$. As the set A lies in the face f, we have that $N(A) \subseteq V(C)$ by the plane embedding of G. So all that remains is to prove that $V(C) \subseteq N(A)$.

To this end, we construct an 'auxiliary' maximal plane graph G^* as follows:
Take the plane graph G, and remove from it every edge and vertex that does not lie in $\mathrm{Int}[C]$ (i.e., form the graph $G - \mathrm{Ext}(C)$). Note that $\mathrm{Ext}(C)$ is a face of $G - \mathrm{Ext}(C)$, and that this face is bounded by the cycle C. Create the plane graph G^* by adding a new vertex v to the region $\mathrm{Ext}(C)$, and making this vertex v adjacent to every vertex of C. Observe that the plane graphs $G \cap \mathrm{Int}[C]$ and $G^* \cap \mathrm{Int}[C]$ are identical. So the faces of G^* contained in $\mathrm{Int}(C)$ are the same as the faces of G contained in $\mathrm{Int}(C)$, and are thus triangles. Further, it is clear that each face of G^* contained in the region $\mathrm{Ext}(C)$ is a triangle of the form v, x, y, where xy is an edge of C. So G^* is a maximal plane graph with vertex set $A \cup V(C) \cup \{v\}$ (see Figure 6.10).

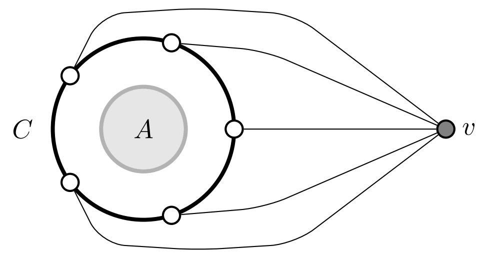
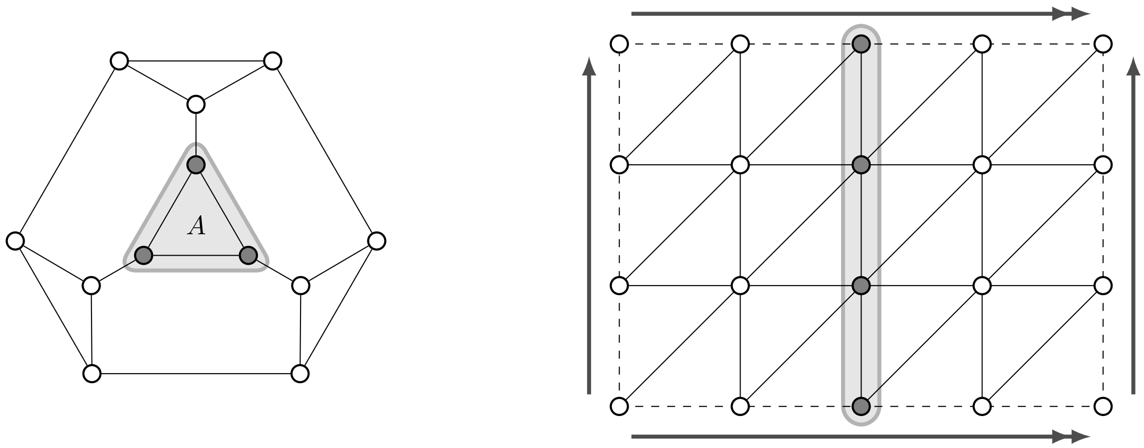


Figure 6.10: In the proof of Theorem 6.22, the graph maximal plane graph G^* agrees with G on $\mathrm{Int}[C]$, and the exterior of C contains a single vertex v that is adjacent to each vertex of C.

Because the interior of C is a face of $G - A$, the only edges of G that can intersect $\mathrm{Int}(C)$ are edges incident with at least one vertex of A, and thus no chord of C crosses $\mathrm{Int}(C)$ in G. Since G and G^* agree on $\mathrm{Int}(C)$, there is no chord of C that crosses $\mathrm{Int}(C)$ in G^*. By construction, there is also no chord of C that crosses the region $\mathrm{Ext}(C)$ in G. As C does not have any chords that cross either its interior or its exterior, it is an induced cycle in G^*. Further, the cycle C Jordan separates the components $\{v\}$ and A of $G^* - C$, and is this not a face-cycle of G^*. By Theorem 6.15, the cycle C is a minimal separator of G^*. Per Lemma 6.11, every vertex of C is adjacent to a vertex of A in G^*. As A lies in $\mathrm{Int}(C)$ and the graphs $G \cap \mathrm{Int}[C]$ and $G^* \cap \mathrm{Int}[C]$ are the same, every vertex of C is adjacent to a vertex of A in the graph G. Thus $V(C) \subseteq N(A)$, completing the proof. □

There is little we can do to generalise Theorem 6.22. For example, the theorem statement is not true if we replace the hypothesis 'G is a maximal planar graph' with the hypothesis 'G is a 3-connected planar graph', as demonstrated by the graph G in Figure 6.11. The statement of Theorem 6.22 is also false if we let G be a triangulation of an arbitrary surface (as opposed to restricting G to being a triangulation of a sphere), as shown by the triangulation H of a torus in Figure 6.11.



G H

Figure 6.11: The set $A \subset V(G)$ induces a connected subgraph of the 3-connected graph G, and $G - A$ is 2-connected, but $N(A)$ is not connected. Similarly, the set $B \subset V(H)$ of grey vertices induces a connected, non-separating subgraph of the toroidal triangulation H, yet $N(B)$ is not connected.

6.7 Preserving distances

In this section, we explore isometric subgraphs of (maximal) planar graphs. This section's first result is that a 2-connected subgraph H of a plane graph G is isometric in G if no path of G creates a 'shortcut' through any face of H. Thus, using only 'local' information about whether face-boundaries of H are isometric, we can deduce that the whole subgraph is isometric. An immediate consequence of this result is the fact that any maximal planar subgraph of a planar graph is always an isometric subgraph. Discussing faces requires us to embed these graphs onto the plane or sphere, but whether or not a subgraph is isometric does not depend on the chosen embedding. Given its simplicity, it is likely that this theorem has appeared in the literature (or it has been used implicitly), but I have not been able to find it.

Theorem 6.23. [ ] *Let H be a 2-connected subgraph of a plane graph G. If for every face f of H and every pair x and y of vertices on the boundary of f, we have that $d_{H[f]}(x,y) = d_{G[f]}(x,y)$, then H is isometric in G.*

Proof. Assume the hypotheses of the theorem, let u and v be vertices of H, and let $d_G(u,v) = k$. It suffices to show that the distance in H between u and v is at most k.

Let $P : u = x_0, x_1, \ldots, x_k = v$ be a $u - v$ geodesic in G. Observe that every edge $x_j x_{j+1}$ of P lies either in, or on the boundary of, some face of H. We can thus partition the edges of P into a number of smaller paths $P = Q_0 \cup Q_1 \cup \cdots \cup Q_i$ such that the start and end vertices of each Q_j are contained in H, and each of the Q_j's is either strictly contained in H, or crosses exactly one face of H.

For each $l \in [0, i]$, we find a path R_l in H as follows:
If Q_l is a path in H, then let $R_l = Q_l$. If not, then Q_l lies across some face f of H. By assumption, there is some geodesic R in $H[f]$ between the starting and ending points of Q_l which satisfies $\ell(R) \leq \ell(Q_l)$. In this case, we choose $R_l = R$. In all cases, the paths R_l and Q_l have the same endpoints, but R_l is a path of H

that is never longer than Q_l.

Thus $R_0 \cup R_1 \cup \cdots \cup R_i$ is a $u - v$ path in H of length at most k, completing the proof. □

Corollary 6.24. **[10]** *A maximal planar subgraph of a planar graph is an isometric subgraph.*

Proof. Let H be a maximal planar subgraph of a planar graph G, and embed G as a plane graph. The corollary follows trivially if H has fewer than three vertices, so we may assume without loss of generality that the order of H is at least 3. By Proposition 6.3, the boundary of every face of H is a triangle. So if f is a face of H, and x and y are two vertices of $H[f]$, then x and y are adjacent in $H[f]$. Thus it is not possible that $d_{G[f]}(x, y) < d_{H[f]}(x, y)$, and the corollary follows from Theorem 6.23. □

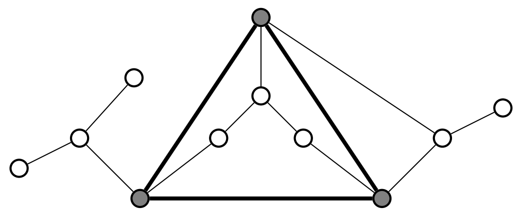


Figure 6.12: There is no way to create a 'shortcut' path through a triangle in a planar graph.

Visibly, any finite graph G of order n is a subgraph of a complete graph K_n. However, even the path of length 2 is not an isometric subgraph of any complete graph. When a graph H is an isometric subgraph of a graph G, it indicates that G has 'enough space' to fit the subgraph H without shortening the distances in H. For example, if k is a nonnegative integer, the path P_{k+1} is an isometric subgraph of G if and only if the diameter of G is at least k.

We demonstrate constructively that every tree is an isometric subgraph of some maximal planar graph.

Theorem 6.25. *Every tree is an isometric subgraph of some maximal planar graph.*

Proof. Let T be a tree with radius r. We may assume without loss of generality that T is neither K_1 nor K_2. Let v be a central vertex of T, and let S_i be the circle in $\mathbb{R}^2$ of radius i centred at the origin.

Observe that each maximal path of T that has v as its starting vertex has a leaf u as its end vertex. Create a new tree T' from T as follows:
For each leaf u of T, attach to u a path of length $r - d(u, v)$. The graph T' formed in this way is a tree with centre v, in which every leaf is distance r from v.

We now embed T' into the plane such that v is placed at the origin, every vertex at distance i from v lies on the circle S_i, and all edges are straight line segments. (See, for example, Figure 6.13).

We modify T' to obtain a new plane tree G_0. There are two possible cases:

Case 1: v has degree 3 or greater.
Between each pair of consecutive neighbours of v, attach a path of length r having v as an endpoint. Place the vertices of these paths such that each vertex at distance i from v lies on S_i.

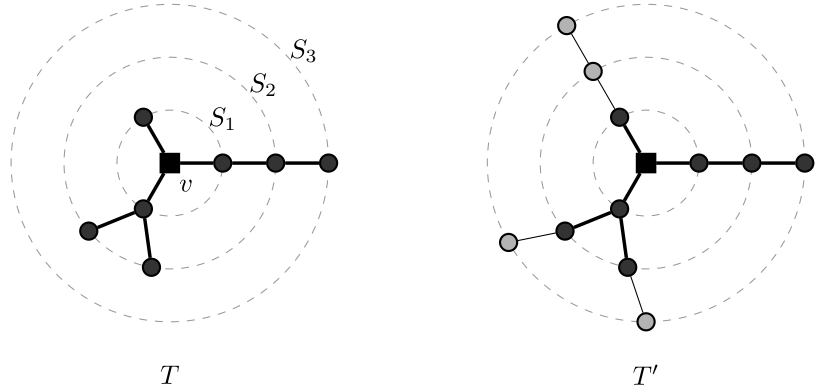


Figure 6.13: Two steps of the proof of Theorem 6.25. Left: The tree T, in which the vertex v is indicated by a square, and the dashed circles S_1, S_2 and S_3. The vertices of T are black. Right: The tree T'. The vertices of $T' - T$ are grey.

Case 2: v has degree 2.
The construction in Case 2 is the same as that in Case 1, except instead of attaching one path between each pair of consecutive neighbours of v, two paths of length r are attached to v such that the two vertices of $N_1^{T'}(v)$ are no longer consecutive neighbours of v.

In any case, observe that as we move clockwise around the circle S_1, we alternate between encountering vertices of T' and vertices of $G_0 - T'$.

From G_0, we create a plane tree G_1.
Let u be a vertex of $N_1^{T'}(v)$. There are two cases to consider:

Case 1: u has degree 3 or greater.
Between each pair of consecutive neighbours of u, attach a path of length $r-1$ having u as an endpoint. Place the vertices of each path such that any vertex at distance i from v lies on S_i. (See Figure 6.14).

Case 2: u has degree 2.
Attach two paths of length $r-1$ to u such that the two neighbours of u are no longer consecutive neighbours of u. As in Case 1, the vertices of $N_i^{G_1}(v)$ are placed on S_i.

For $i < r$, we create G_i from G_{i-1}:
To each vertex u of $T' \cap S_i$, attach paths of length $r-i$ between each pair of consecutive neighbours of $N^{T'}(u)$. If u has degree 2, attach a second path of length $r-i$ such that the two vertices of $N^{T'}(u)$ are no longer consecutive neighbours of u. Place the vertices of all these paths such that they lie on S_i when they are distance i from v.

From G_{r-1}, create G_r by making all pairs of vertices which are consecutive on some circle S_i adjacent. There is now a collection of $r-1$ disjoint cycles which separate v from the vertices of $N_r^{G_r}(v)$ in G_r. Indeed for any $i \in [2, r]$, there are $i-1$ disjoint cycles separating v from $N_i^{G_r}(v)$. (See Figure 6.15).

Let G be the graph obtained by taking two copies of G_r, denote them H_1 and H_2, and attaching H_1 and

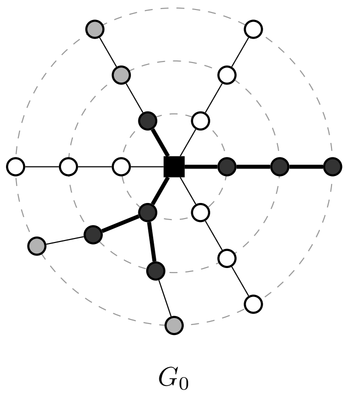


Figure 6.14: The trees G_0 (left) and G_1 (right) in the proof of Theorem 6.25. The vertices of $G_i - T'$ are white.

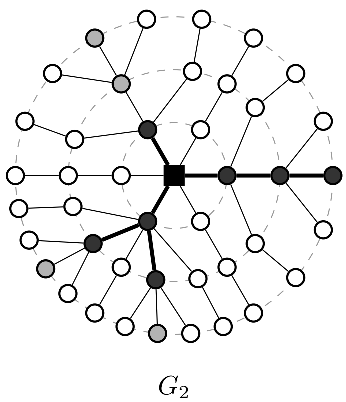


Figure 6.15: The final tree G_2 (left) and the graph G_3, as in the proof of Theorem 6.25.

H_2 by identifying the vertices of S_r in H_1 with the corresponding vertices of S_r in H_2. Every face of G is either a triangle or a square. The reader may finish the construction of G by triangulating the square faces in whatever way they find most pleasing.

By our observation that v is separated from $N_i^G(v)$ by $i-1$ disjoint cycles in the plane, we know that distances from v are the same in G as they are in T. Now consider vertices u and w of $T - v$. We may assume that $d_T(u, w) = k > 1$ without loss of generality. We need to show that $d_G(u, w) = d_T(u, w)$. To this end, let $P : u = x_0, \ldots, x_k = w$ be the unique $u - w$ geodesic in T. Observe that for each $i \in [1, k-1]$, there are two paths Q_1 and Q_2 of G_{r-1}, extending from x_i to S_r, such that the paths Q_1 and Q_2, and the 'reflections' of Q_1 and Q_2 in the second copy of G_r, form a cycle. While it may be possible to make many possible choices for the paths Q_1 and Q_2, we can always choose these paths such that the cycle we create separates u from w (further, it separates x_{i-1} from x_{i+1}). The $k-1$ cycles we get for each x_i are disjoint, and Jordan-separate u from w. Thus, in G, the distance between u and w is at least k, completing the proof. □

6.8 Centres — a prelude

In this chapter, we have sketched out some basic features of maximal planar graphs, none in very much depth. There is one particular aspect of these graphs that we will explore in much greater detail in the next chapter: their centres. As a simple example in the next chapter demonstrates (see Figure 7.3), the centre of a maximal planar graph need not even be connected. This begs the question: is this example a deeply unusual pathology, or can we easily find more maximal planar graphs with disconnected centres? As the lone theorem in the section illustrates, the centre of a maximal planar graph can fail to be connected quite spectacularly: we can find centres with arbitrarily many components.

Theorem 6.26. *Let $k \geq 3$ be an integer. There exists a maximal planar graph whose centre induces an independent set with k vertices.*

Proof. Consider the graph Γ_0 with vertex set $V(\Gamma_0) = \mathbb{Z} \times \mathbb{Z}$, such that vertices (i_1, j_1) and (i_2, j_2) are adjacent when $|i_1 - i_2| + |j_1 - j_2| = 1$. Note that the distance in Γ_0 between vertices (i_1, j_1) and (i_2, j_2) is $|i_1 - i_2| + |j_1 - j_2|$. Let Γ_1 be the subgraph of Γ_0 induced by $\{(i,j) : |j - i| \leq k \text{ and } |j + i| \leq k - 1\}$. Observe that $(0,0)$ is in $V(\Gamma_1)$, and that every other vertex of Γ_1 is distance at most k from $(0,0)$ (see Figure 6.16 for an example).

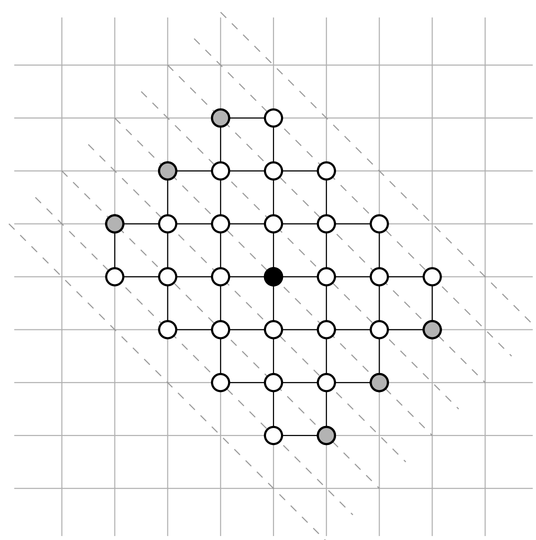


Figure 6.16: The graph Γ_1 for $k = 4$, as in the proof of Theorem 6.26. The vertex is $(0,0)$ is the black vertex in the middle, and two vertices are opposite if they are both grey and on the same dashed line.

Consider the lines $\lambda_c = \{(i,j) \in V(\Gamma_1) : j = -i + c\}$, where c is an integer in the interval $[-k+1, k-1]$. Note that these lines form a partition of the set $V(\Gamma_1)$, and that a vertex in λ_c only has neighbours in λ_{c+1} and λ_{c-1}. We say that a pair of vertices are *opposite* each other when they lie on the same line λ_c, and both are distance k from $(0,0)$. Note that a line λ_c has a pair of opposite vertices when $k \equiv c \pmod 2$, so exactly every second line of vertices has ends which we consider 'opposite'.

We construct a new graph Γ_2 from Γ_1:
First, identify all opposite vertices (ie, if u and v are opposite in Γ_2, then they are replaced by a single vertex w in Γ_1 that is adjacent to every neighbour of both u and v). Denote by L_c the set obtained from λ_c after

performing the identification. Add vertices a and b to Γ_2 and make a adjacent to every vertex of L_{k-1}, and b adjacent to every vertex of L_{-k+1}. We now define $L_k = \{a\}$ and $L_{-k} = \{b\}$. Note that Γ_2 can be embedded on the sphere (uniquely, since it is 3-connected) such that every face is a quadrangle. Each quadrangular face of Γ_2 has two vertices in L_c and one in each of L_{c+1} and L_{c-1}, for some $c \in [k-1, -k+1]$.

We now construct our final graph G from Γ_2:
Let f be a face of Γ_2. Let, without loss of generality, $\Gamma_2[f] = \Gamma_2[\{u, v, w, x\}]$ with vertices u and w in L_c and vertices v and x in L_{c+1} and L_{c-1} respectively. We say that the face f is **traversing** the layer L_c. Place new vertices s and t in f, and add to G the edges us, st, tw, vs, xs, vt and xt. Repeat this process of adding two vertices and seven edges in every face of Γ_2 to obtain the graph G. Denote by L'_c the union of L_c and all vertices that have been added to the faces traversing L_c. Further, fix $L'_k = L_k$ and $L'_{-k} = L_{-k}$ (See Figure 6.17). The graph G is maximal planar as the process of constructing G from Γ_2 divided every quadrangular face of Γ_2 into triangles. Further, the graph Γ_2 is an isometric subgraph of G.

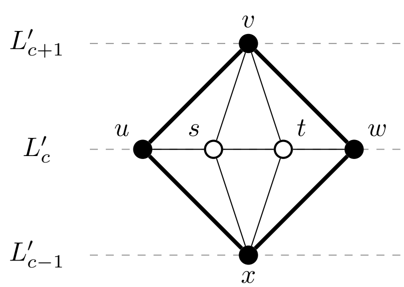


Figure 6.17: The bold cycle u, v, w, x (bold edges and vertices) is the boundary of a face of Γ_2, in the proof of Theorem 6.25.

What remains is to show is that the centre of G is precisely the set L_0, and that L_0 is an independent set of k vertices.

As before, the sets $\{L'_c : c \in [-k, k]\}$ form a partition of $V(G)$. Since the vertices of L'_c are only adjacent to vertices of L'_m for $m \in [c-1, c+1]$, we deduce that $\max\{d(u, a), d(u, b)\} \geq |c| + k$ for all vertices u in L'_c. So if the vertex u is in L'_c, then $e(u) \geq |c| + k$. Thus every vertex of L'_c, $c \neq 0$, has eccentricity greater than k. On the other hand, the vertex $(0, 0)$ in L_0 is within distance $k-1$ of at least two vertices of the boundary of every face of Γ_2, and so $e((0, 0)) = k$. By symmetry, every vertex of L_0 has eccentricity k.

Note that L_0 is an independent set in G (unlike L'_0, which induces a cycle). Letting $k = 2l$ or $2l + 1$ if k is even or odd respectively, we can represent L_0 as:

$$L_0 = \{(i, -i) : 2|i| \leq k\} = \{(-l, l), (-l + 1, l - 1), \ldots, (0, 0), \ldots (l - 1, -l + 1), (l, -l)\}.$$

If k is even, then $(-l, l)$ and $(l, -l)$ are opposite and hence identified, so $|L_0| = (2l + 1) - 1 = k$. If k is odd, then no pair of vertices were identified in L_0, and so $|L_0| = 2l + 1 = k$. In any case, L_0 is an independent set containing k vertices.

We now show that the vertices of $L'_0 - L_0$ have eccentricity greater than k. By symmetry, it suffices to show this for a single vertex of $L'_0 - L_0$.

Case 1: $k = 2l$ is even.

Let s and t in $L_0'-L_0$ be the vertices internal to the face f_1 of Γ_2 bounded by the cycle $\Gamma_2[\{(-1,1),(-1,0),(0,0),(0,1)\}]$, such that s is adjacent to $(-1,-1)$ and t is adjacent to $(0,0)$. Similarly, let p and q be vertices in the face f_2 of Γ_2 bounded by $\Gamma_2[\{(l-1,-l+1),(l-1,-l),(l,-l),(l,-l+1)\}]$ with p adjacent to $(l-1,-l+1)$ and q adjacent to $(l,-l)$. Any $t-q$ geodesic contains a vertex of each of $\Gamma_2[f_1]$ and $\Gamma_2[f_2]$, and t and q are themselves distance 1 from $\Gamma_2[f_1]$ and $\Gamma_2[f_2]$, respectively. The only pairs of vertices of $\Gamma_2[f_1]$ and $\Gamma_2[f_2]$ within distance $2l-2$ of each other in Γ_2 (which is isometric in G) are $(0,0),(l-1,-l+1)$ and $(-1,1),(l,-l)$. However q is distance 2 from $(l-1,-l+1)$ and t is distance 2 from $(-1,1)$, so any $t-q$ path has length at least $2l+1$ as desired.

Case 2: $k=2l+1$ is odd.
This follows similarly, only we use vertices s and t in the face $\Gamma_2[\{(-1,1),(-1,0),(0,0),(0,1)\}]$ with s and $(-1,1)$ adjacent, and vertices p and q on the face $\Gamma_2[\{(l,-l),(l,-l-1),(-l,l),(l+1,-l)\}]$ with p and $(l,-l)$ adjacent. In this case, the vertices t and p will be distance at least $k+1$ from each other by a similar argument to Case 1.

In conclusion, $L_0 \subseteq V(G)$ is the centre of G, and induces an independent set containing k vertices. $\square$

Chapter 7

Planar graphs with maximal planar centres

This chapter contains the content of the preprint *Planar graphs with maximal planar centers*, by B. Du Preez, which has been submitted for publication and is currently under review. The only modifications made are the omission of title page, acknowledgements and bibliography (the full acknowledgements and bibliography are given at the end of the thesis), changes in the numbering of sections, figures, references and results to fit the numbering scheme of the overall thesis, minor formatting changes (including the inclusion of the [🕯] symbol) and minor changes to the style and grammar suggested by Examiners.

Rationale for the inclusion of this publication

While Chapter 4 examines the face-degree regular plane graphs in which faces are as large as possible, this paper, together with Chapter 6, studies plane graphs in which the faces are as small as possible. Thus, with the inclusion of this paper, we are able to give a report, that encompasses both extremes, of the different ways in which restricting the face size of a plane graph impacts its metric properties. Further, due to the inclusion of both this paper and the material in Chapter 6, the thesis gives a broad account of distances in maximal planar graphs — an important and well-studied class of (face-degree regular) graphs.

Abstract

A maximal planar graph is a graph which can be embedded in the plane such that every face of the graph is a triangle. The centre of a graph is the subgraph induced by the vertices of minimum eccentricity. We characterise maximal planar graphs that are subgraphs of the centre of some maximal planar graph and show these are exactly the maximal planar graphs that are the centre of some planar graph.

7.1 Definitions and introduction

A graph is **maximal planar** if it is planar, but the addition of any edge destroys planarity. An embedding of a maximal planar graph into the plane is a **maximal plane graph**. A plane graph of order at least 3 is maximal plane if and only if every face of the graph is bounded by a 3-cycle.

If H is a path or cycle in some graph, let $\ell(H) = |E(H)|$ denote the **length** of H. If G is a graph, we use $V(G)$ and $E(G)$ to refer to the sets of vertices and edges of G, respectively. Let G and H be graphs. The **Cartesian Product** $G \times H$ is the graph with vertex set $V(G \times H) = \{(u, v) : u \in V(G), v \in V(H)\}$ and edge set $E(G \times H) = \{(u, v)(u', v') : (u = u' \text{ and } vv' \in E(H)) \text{ or } (v = v' \text{ and } uu' \in E(G))\}$. The **union** of G and H is the graph $V(G) \cup V(H) = (V(G) \cup V(H), E(G) \cup E(H))$, and the **intersection** is the graph $G \cap H = (V(G) \cap V(H), E(G) \cap E(H))$. If G is a plane graph and f is a face of G, then $G[f]$ denotes the graph consisting of all the edges and vertices of G that lie on the boundary of f.

Let $G = (V, E)$ be a simple graph, let u and v be vertices of G, and let S be a subset of V. For all the definitions to follow, we omit the subscript G if the graph in question is unambiguous. The **induced subgraph** $G[S]$ is the subgraph of G with vertex set S, such that two vertices of S are adjacent in $G[S]$ if and only if they are adjacent in G. If the induced subgraph $G[V - S]$ is disconnected, then S **separates** G, and we call S a **separating set**. The **distance** between u and v in G, $d_G(u, v)$, is the length of a shortest $u - v$ path in G. Such a path is a $u - v$ **geodesic**. If A and B are subsets of V, the distance between these sets is given by:

$$d_G(A, B) = \min\{d(a, b) : a \in A, b \in B\}.$$

We let $d_G(v, A) = d_G(\{v\}, A)$. If H and K are subgraphs of G, we use the notation $d_G(H, K)$ to refer to the distance $d_G(V(H), V(K))$. The **eccentricity** of u in G is $e_G(u) = \max\{d_G(u, x) : x \in V\}$. The **radius** and **diameter** of G, denoted $\text{rad}(G)$ and $\text{diam}(G)$, are the minimum and maximum eccentricities among the vertices of G, respectively. The **centre** of G is the subgraph induced by the vertices of minimum eccentricity. A **peripheral vertex** is a vertex whose eccentricity is equal to the graph's diameter, and a **central vertex** is a vertex whose eccentricity is equal to the graph's radius.

The i^{th} **eccentricity layer** of G, $\mathcal{E}_G(i)$, is the set of all vertices of G with eccentricity i. A subgraph H of G is **equi-eccentric** in G if there is some integer i such that $V(H) \subseteq \mathcal{E}_G(i)$. Clearly, the centre of a graph is an equi-eccentric subgraph. A subgraph H of G is **isometric** if, for all pairs of vertices u and v in H, we have $d_H(u, v) = d_G(u, v)$. If G is a planar graph, and H is a subgraph of G which is maximal planar, then H is always isometric in G [10].

Lemma 7.1. [10] *Every maximal planar subgraph of a planar graph is isometric.*

Proof. Assume to the contrary that H is a maximal planar subgraph of a planar graph G and that H contains vertices u and v with $d_G(u, v) < d_H(u, v)$, and let P be a $u - v$ geodesic in G. Because H is maximal planar, it is an induced subgraph of G; so P contains a vertex w in $V(G) - V(H)$. Since the vertex w lies in a face $f : x, y, z$ of H, the path P contains at least two of the vertices on the boundary of f, say x and y. We can thus replace the segment of P from x to y with the edge xy to obtain a shorter $u - v$ path, which yields a contradiction. □

It is well known that every graph is the centre of some graph [9]. However, even if G is planar, the graph $H(G)$ constructed in [9] having G as its centre is not planar if G contains any vertex of degree at least three [10]. To construct $H(G)$, add two vertices a and b to G, and make each of these adjacent to every vertex in G. Then, add another two vertices c and d, and make c adjacent to a, and d adjacent to b (See Figure 7.1). In fact, there exist (maximal) planar graphs which cannot be the centre of any planar graph. For example, graph in Figure 7.6 is not contained in the centre of any planar graph.

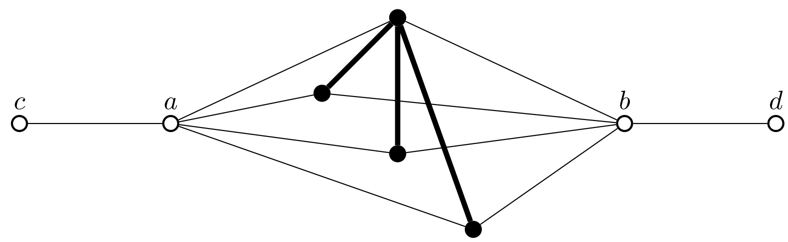


Figure 7.1: Given any graph G, the Hedetniemi construction yields a graph $H(G)$ with G as its centre. In the example above, the vertices and edges of G are bold.

A natural starting point for investigating centres of graphs is to consider graphs which are their own centres. In [7], Buckley gives a survey of results and topics concerning such graphs. Jarry and Laugier give a proof of a strengthening of Buckley's theorem bounding the number of edges in a self-centred graph in [25].

The centres of a number of graph classes, including maximal outerplanar graphs and chordal graphs, have already been described. An **outerplanar graph** is a planar graph which can be embedded in the plane such that every vertex is on the boundary of the outer face of the graph. A **maximal outerplanar graph** is an outerplanar graph to which the addition of any edge results in a graph that is not outerplanar. In [34], Proskurowski shows that the centre of any maximal outerplanar graph is one of seven graphs, all of which are themselves maximal outerplanar (See Figure 7.2).

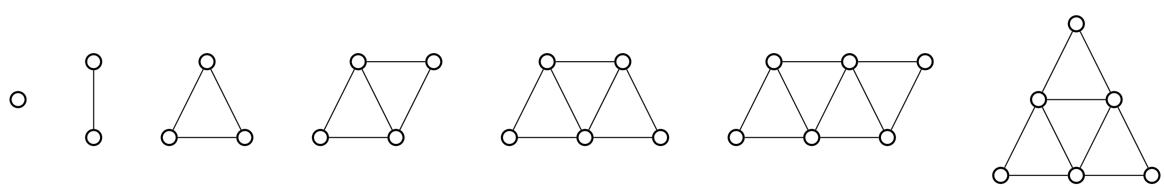


Figure 7.2: The seven possible centres of a maximal outerplanar graph.

A graph is **chordal** if the only induced cycles are 3-cycles. Laskar and Shier showed in [29] that the centre of a connected chordal graph is itself a connected chordal graph. The centre of a planar graph, or even a maximal planar graph, is not necessarily connected as Figure 7.3 from [10] illustrates.

A similar problem to that of finding centres of graphs is describing the collection of eccentricities that a graph has. When written as an ordered sequence of positive integers, this collection is called the **eccentric sequence** of the graph. In [14], Dankelmann, Erwin, Goddard, Mukwembi and Swart characterise eccentric sequences of maximal outerplanar graphs.

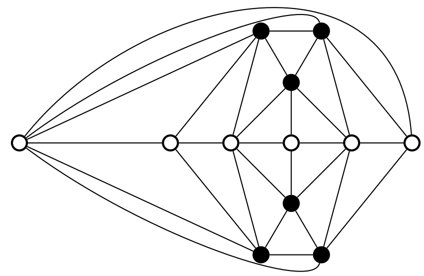


Figure 7.3: A maximal planar graph with centre $2K_3$. The central vertices are black.

7.2 Quasi-eccentricity

Consider a (not necessarily planar) graph G. Given a vertex v in G, we say that u is an **eccentric vertex** of v if $d(u, v) = e(v)$. Denote the set of vertices eccentric to v by $\mathrm{Ecc}(v)$. Given a subset S of $V(G)$, we can similarly define $\mathrm{Ecc}(S)$ as the set of vertices at maximum distance from S. The eccentricity of the set S, $e(S)$, can be realised as the distance $d(S, \mathrm{Ecc}(S))$.

We now introduce a similar concept. Given a vertex u and a subset S of vertices of G, we say that u is a **quasi-eccentric vertex** of S in G if, for any vertex v of G, there exists a vertex s in S such that $d(u, s) \geq d(v, s)$. We denote the set of quasi-eccentric vertices of S by:

$$\mathrm{Qcc}_G(S) = \{u \in V(G) : (\forall v \in V(G))(\exists s \in S) \text{ such that } d(u, s) \geq d(v, s)\}.$$

If the graph in question is clear, we omit the subscript G. If H is a subgraph of G, we use the notation $\mathrm{Qcc}(H)$ to refer to the set $\mathrm{Qcc}(V(H))$. Define the **quasi-eccentricity** $q(S)$ of S as $q(S) = d(S, \mathrm{Qcc}(S))$. Observe that quasi-eccentricity generalises eccentricity:

Observation 7.2. Let H be a graph and S a set of vertices of H. If the vertex u is an eccentric vertex of S, it is also a quasi-eccentric vertex of S.

We illustrate the concept with an example. Consider the path $G : v_1, \ldots, v_5$ shown in Figure 7.4. Observe that $\mathrm{Ecc}(S) = \{v_5\}$, while $\mathrm{Qcc}(S) = \{v_1, v_5\}$. Thus $\mathrm{Ecc}(S)$ is properly contained in $\mathrm{Qcc}(S)$. Also, the eccentricity $e(S) = 2$, while the quasi-eccentricity $q(S) = 1$.

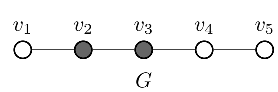


Figure 7.4: The path graph $G : v_1, v_2, v_3, v_4, v_5$. The vertices of the set $S = \{v_2, v_3\}$ are coloured grey.

7.3 The quasi-eccentric face criterion

The question of whether a planar graph H can be embedded into the centre of some planar (or maximal planar) graph G has a natural generalisation. We ask whether it is possible to embed H into G such that H is an equi-eccentric subgraph of G, and give a necessary condition for this.

Theorem 7.3 (The quasi-eccentric face criterion)**.** *Let H be a plane graph of diameter d, and let $\alpha \geq d$ be an integer. If there exists a plane graph G such that H is an isometric subgraph of G, and for which $V(H) \subseteq \mathscr{E}_G(\alpha)$, then for all vertices u in H that satisfy $e_H(u) < \alpha$, there exists a face f of H such that $u \in Qcc_H(H[f])$.*

Proof. Assume that there exists a vertex u in H with $e_H(u) < \alpha$ that is not quasi-eccentric to any face of H, and assume to the contrary that H is an isometric subgraph of some plane graph G such that $V(H) \subseteq \mathscr{E}_G(\alpha)$. Since the eccentricity of u is less than α in H, but is exactly α in G, there is some vertex s in $G - H$ with $d(u, s) = \alpha$. This vertex s lies in some face f of H.

By the assumption that u is not quasi-eccentric to $H[f]$ in H, there exists a vertex v of H such that $d(v, x) > d(u, x)$ for all vertices x in $H[f]$. Let $P : v = x_0, x_1, \ldots, x_i = w, \ldots, x_j = s$ be a $v - s$ geodesic in G, where w is the last vertex of P which belongs to $H[f]$. Such a vertex w exists: the path P starts outside of f and ends in f. Further, v does not lie in f since v is a vertex of H, and v does not lie on the boundary of f because $d(v, x) > d(u, x) \geq 0$ for all vertices x in $H[f]$.

Let Q be a $u - w$ geodesic in G, and observe that Q is shorter than $P[v, w]$. Thus the $u - s$ path $Q \cup P[w, s]$ is strictly shorter than the $v - s$ geodesic P, and so the eccentricity of v in G is strictly greater than the eccentricity of u in G. This contradicts the assumption that u and v are in $\mathscr{E}_G(\alpha)$, completing the proof. □

The next corollary follows immediately from Theorem 7.3 and Lemma 7.1.

Corollary 7.4. *Let H be a maximal plane graph of diameter d, and let $\alpha \geq d$ be an integer. If there exists a plane graph G containing H as a subgraph, and for which $V(H) \subseteq \mathscr{E}_G(\alpha)$, then for all vertices u in H that satisfy $e_H(u) < \alpha$, there exists a face f of H such that $u \in Qcc_H(H[f])$.*

7.4 Other necessary conditions

In this section, we explore another necessary condition for a plane graph to be an isometric, equi-eccentric subgraph of some plane graph. We show that this necessary condition is implied by the condition of Theorem 7.3, but that the converse does not hold. We first need two simple and well-known lemmas, whose proofs we include for completeness.

Lemma 7.5. [🕯] *Let $G = (V, E)$ be a connected graph, and S a separator of G. If vertices u and v in G are in different components of $G - S$, then $d(u, v) \geq d(u, S) + d(v, S)$.*

Proof. Let P be a $u - v$ geodesic. Because G is connected and S separates the vertices u and v, there exists a vertex s in $S \cap P$. The geodesic P can be split into two paths, $P[u, s]$ and $P[s, v]$, which have no edges in common. Since $P[u, s]$ is a $u - S$ path and $P[s, v]$ is an $S - v$ path, it follows that $\ell(P[u, s]) \geq d(u, S)$ and $\ell(P[s, v]) \geq d(S, v)$, and thus we obtain the following chain of inequalities:

$$d(u, v) = \ell(P) = \ell(P[u, s]) + \ell(P[s, v]) \geq d(u, S) + d(v, S)$$

□

Lemma 7.6. [🔥] *Let G be a connected graph, and S a connected subgraph of G. If u is a vertex of S, and v is a vertex of $G - S$, then $d(u, v) \leq diam(S) + d(S, v)$.*

Proof. Let w be a vertex of S such that $d(w, v) = d(S, v)$, and let P be a $v - w$ geodesic. Let Q be a $w - u$ geodesic in S. Since the length of P is $d(S, v)$ and the length of Q is at most $\text{diam}(S)$, the walk $P \cup Q$ is a $v - u$ walk of length of at most $\text{diam}(S) + d(S, v)$. □

A cycle in a plane graph is a **Jordan separating cycle** if there are vertices in both its interior and exterior. Not all separating cycles are necessarily Jordan separating, but all Jordan separating cycles are separators (See Figure 7.4).

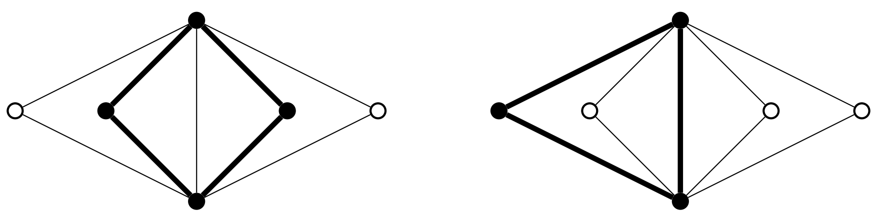


Figure 7.5: The bold cycle on the left is a separating cycle which is not a Jordan separating cycle. The bold cycle on the right is a Jordan separating cycle.

Lemma 7.7. *Let H be a plane graph of diameter d, and let $\alpha \geq d$ be an integer. If there exists a plane graph G such that H is an isometric subgraph of G and $V(H) \subseteq \mathscr{E}_G(\alpha)$, then for all cycles C of H, and all vertices a and b of H which C Jordan separates, either $d(a, C) \leq diam(C)$ or $d(b, C) \leq diam(C)$.*

Proof. Assume to the contrary that there exists an embedding of H into G as described in the hypothesis of the lemma, but also that there exist, in H, vertices a and b, and a cycle C, such that both $d(a, C) > \text{diam}(C)$ and $d(b, C) > \text{diam}(C)$. We may assume without loss of generality that a lies in the interior of C and b lies in the exterior of C. Let u be a vertex of C, and let v be an eccentric vertex of u in G. Since $d(a, C) > \text{diam}(C)$, the vertex v is not a vertex of C. Thus v lies in either the interior or exterior of C. Assume without loss of generality that v lies in the interior of C. Since C Jordan-separates the vertices v and b, Lemmas 7.5 and 7.6 imply the following chain of inequalities:

$$e(u) = d(u, v) \leq \text{diam}(C) + d(C, v) < d(b, C) + d(C, v) \leq d(b, v) \leq e(b).$$

Thus the eccentricity of u in G is less than the eccentricity of b in G, a contradiction. □

The following corollary of Lemma 7.7, which appears in [10], follows immediately from Lemma 7.7 and Lemma 7.1.

Lemma 7.8. [10] *Suppose that H is a maximal plane graph, C is a Jordan separating cycle of H, and that a and b are vertices in the interior and exterior, respectively, of C. If $d(a,C) > diam(C)$ and $d(b,C) > diam(C)$, then H is not the centre of any planar graph G.*

We now show that if some plane graph H satisfies the quasi-eccentric face criterion of Theorem 7.3, it also satisfies the condition of Lemma 7.7.

Lemma 7.9. *Let H be a plane graph of diameter d. If for all vertices u of H, there exists a face f such that u is in $Qcc(H[f])$, then for all cycles C of H, and all vertices a and b of H which C Jordan separates, either $d(a,C) \leq diam(C)$ or $d(b,C) \leq diam(C)$.*

Proof. We prove the contrapositive. Assume that H is a plane graph with a cycle C and vertices a and b such that C Jordan separates a and b, and both $d(a,C) > \text{diam}(C)$ and $d(b,C) > \text{diam}(C)$. We show that there exists a non-peripheral vertex which is not quasi-eccentric to any face of H.

First, we find a non-peripheral vertex. Let u be a vertex of the cycle C, and let v be an eccentric vertex of u. As $d(u,b) > \text{diam}(C)$, the vertex v is not contained in the cycle C. We can assume without loss of generality that v is in the region of C containing the vertex b. Thus, by Lemmas 7.5 and 7.6, we obtain the following chain of inequalities:

$$d(a,v) \geq d(a,C) + d(C,v) > \text{diam}(C) + d(C,v) \geq d(u,v).$$

Since v is eccentric to u, we conclude that $e(u) < d$.

Now, we show that u is not quasi-eccentric to any face. Observe that every face of H is either in the region of C in which a resides, or the region of C in which b resides. Consider a face f of H, and let f be contained in the region of C in which b resides (it is possible that $H[f] \cap C$ is nonempty). If w is a vertex of $H[f]$, then by Lemmas 7.5 and 7.6, we deduce that the following inequalities hold:

$$d(a,w) \geq d(a,C) + d(C,w) > \text{diam}(C) + d(C,w) \geq d(u,w).$$

Consequently, every vertex of $H[f]$ is strictly further from a than it is from u, so u is not quasi-eccentric to $H[f]$. Similarly, if f is a face in the region of C in which a lies, then $d(b,w) > d(u,w)$ for every vertex w on the boundary of the face f. In any case, the vertex u is not quasi-eccentric to the face f. $\square$

We illustrate, in Figure 7.6, a maximal plane graph G which satisfies the condition of Lemma 7.7, but does not satisfy the quasi-eccentric face criterion of Theorem 7.3. Observe that if C is a separating 3-cycle in G (of which there are exactly eight), then every vertex in the interior of C is distance 1 from C. If C is not a 3-cycle, then the criterion of Lemma 7.7 is not broken as G has diameter 4, and so no cycle C of diameter 2 or greater can separate a pair of vertices a and b such that both $d(a,C) > 2$ and $d(b,C) > 2$. Thus G satisfies the condition of Lemma 7.7.

To see that G does not satisfy the quasi-eccentric face criterion, observe that the black vertex is not quasi-eccentric to any face. Given any face in the interior of the bold cycle, there is a grey vertex outside the cycle

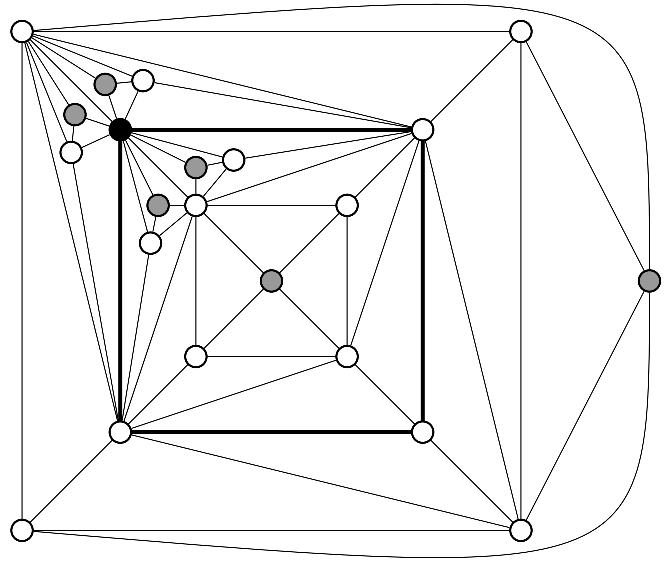


Figure 7.6: The graph G shows that the converse of Lemma 7.9 does no hold. Note the symmetry around the bold cycle.

which is strictly further from every vertex of the face than the black vertex is. The situation is similar for faces in the exterior of the bold cycle.

7.5 The curious case of maximal planar graphs

In this section, we will explore the quasi-eccentric face criterion of Theorem 7.3 when the graph H, which we are embedding into some supergraph, is a maximal planar graph of order at least 4. We show in this case that the criterion is not only necessary, but also sufficient. That is to say, we show that for a maximal planar graph H and an integer $\alpha \geq \operatorname{diam}(H)$, we can embed H into some (maximal) planar graph G such that $V(H) \subseteq \mathscr{E}_G(\alpha)$ if and only if, for all vertices u in H, there exists a face f of H such that u is in $\operatorname{Qcc}_H(H[f])$. To discuss faces of H, we will need to commit to an embedding of H as a maximal plane graph, but the particular choice of embedding is unimportant. By a theorem of Whitney [39], any two embeddings of a 3-connected planar graph have the same faces. Diestel gives a proof of and context for this theorem in [16].

Let H be a maximal plane graph of order at least 4, and let f be a face of H. We can uniquely identify f by the three vertices on its boundary. Thus we use the notation $f : x, y, z$ to indicate that f is the face with vertices x, y and z on its boundary. Given a vertex u and a face $f : x, y, z$ of H, the **distance vector** of u relative to $H[f]$ is the ordered list:

$$(d(u,x), d(u,y), d(u,z))$$

Note that two distinct vertices u and v may have the same distance vector relative to $H[f]$. If $f : x, y, z$ is a face of H, then the **configuration** of the set $\operatorname{Qcc}(H[f])$ is the set of distance vectors of vertices in $\operatorname{Qcc}(H[f])$ relative to $H[f]$. Symbolically, the configuration of $\operatorname{Qcc}(H[f])$ is the set:

$$\{(d(u,x), d(u,y), d(u,z)) : u \in \operatorname{Qcc}(H[f])\}.$$

The proof that the criterion of Theorem 7.3 is sufficient if H is a maximal planar graph, to which this entire section is devoted, will unfold as follows.
Given a face f of a maximal plane graph H, we will show that for any configuration of $\mathrm{Qcc}_H(H[f])$, there exists a maximal plane graph G_f with the following properties:

(1) H is a subgraph of G_f,
(2) Every edge and vertex which belongs to G_f, but not H, lies in f,
(3) Every vertex of $\mathrm{Qcc}_H(H[f])$ has eccentricity α in G_f,
(4) Every vertex of H has eccentricity at most α in G_f.

The proof will conclude by constructing the graph G as the union over all faces f of H of the graphs G_f.

We begin by finding constraints on the configuration of the set $\mathrm{Qcc}_H(H[f])$ for a given face f of a maximal plane graph. The constraints we determine here are what allow us to guarantee that, for any possible configuration of $\mathrm{Qcc}_H(H[f])$, we can construct the desired graph G_f.

Lemma 7.10. *Let H be a maximal plane graph and f a face of H. If u and v are vertices of $Qcc(H[f])$, then $|d(u, H[f]) - d(v, H[f])| \leq 1$.*

Proof. Assume to the contrary, and without loss of generality, that $d(u, H[f]) - d(v, H[f]) \geq 2$. Since all three vertices of $H[f]$ are mutually adjacent, we have that $d(w, x) \leq d(w, H[f]) + 1$ for all vertices x in $H[f]$. Thus for all vertices x in $H[f]$, we have that $d(u, x) > d(v, x)$, which contradicts the fact that v is quasi-eccentric to f. □

We deduce from Lemma 7.10 that if the quasi-eccentricity $q(H[f]) = k$, then the distance between $H[f]$ and any vertex of $\mathrm{Qcc}(H[f])$ is either k or $k + 1$.

In order to establish stronger constraints on the relationship between f and $\mathrm{Qcc}(H[f])$, we need to begin describing the configuration of $\mathrm{Qcc}(H[f])$ in more detail. If u is a vertex of H, and $f : x, y, z$ is a face of H, observe that $d(u, H[f])$ is the minimum of the three distances $d(u, x)$, $d(u, y)$ and $d(u, z)$.

Throughout the rest of this section, we will assume that H is a maximal plane graph, and that $f : x, y, z$ is a face of H. Let $(f, k) \subseteq \mathrm{Qcc}(H[f])$ denote the set of all quasi-eccentric vertices of $H[f]$ that are distance k from $H[f]$. If the context makes it clear that we are referring to the face f, we refer to this set as (k). For a vertex t in $\{x, y, z\}$, we add a subscript $(k)_t$ to denote the subset of (k) consisting of the vertices u satisfying $d(u, t) = d(u, H[f]) = k$. We add a superscript $(k)^t$ to indicate the set of vertices u in (k) satisfying $d(u, t) = d(u, H[f]) + 1 = k + 1$. Thus, if we say that u is in $(k)_x^{y,z}$, it means that u is a quasi-eccentric vertex of f satisfying both $k = d(u, H[f]) = d(u, x)$ and $k + 1 = d(u, y) = d(u, z)$.

For example, consider the maximal plane graph G^* in Figure 7.7. The face $f^* : x, y, z$ is shaded grey. All of the vertices x, y and z have eccentricity 2, and every vertex of $G^* - \{x, y, z\}$ is distance 2 from at least one of x, y or z. We thus deduce that every vertex of $G^* - \{x, y, z\}$ is a quasi-eccentric vertex of $G^*[f^*]$. Since the distances $d(a, x) = d(a, y) = 1$, and $d(a, z) = 2$, the vertex a is in $(1)_{x,y}^z$, and the quasi-eccentricity $q(G^*[f^*]) = 1$. Similarly, vertex b is in $(2)_{x,y,z}$, vertex c is in $(1)_x^{y,z}$, and vertex d is in $(1)_y^{x,z}$.

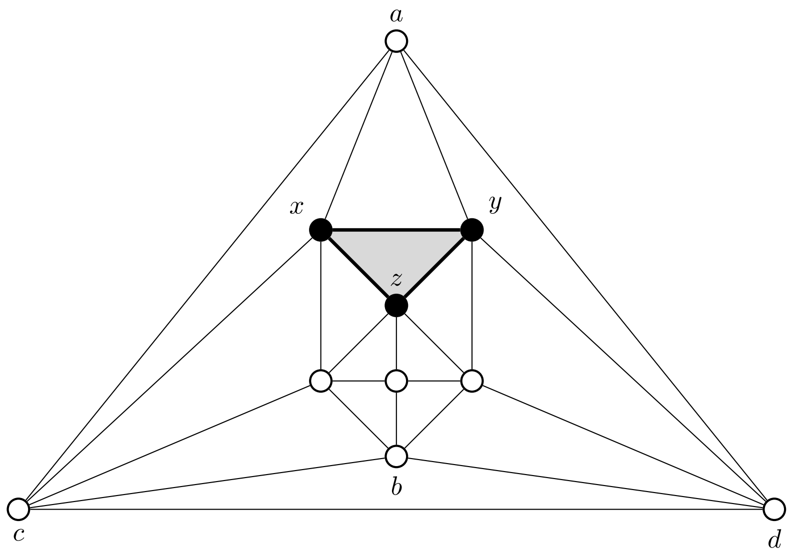


Figure 7.7: The graph G^* is a maximal plane graph. The face $f^* : x, y, z$ is shaded grey.

If $q(H[f]) = k$, then there is a vertex in (k), and by Lemma 7.10, all vertices in $\mathrm{Qcc}(H[f])$ are vertices of (k) or $(k+1)$. Further, if a vertex u is in (k), then for all vertices t in $\{x, y, z\}$, we have that $d(u,t) = k$ or $d(u,t) = k+1$, since the vertices x, y and z are all mutually adjacent. Similarly, if u is in $(k+1)$, then $d(u,t) = k+1$ or $d(u,t) = k+2$ for all t in $\{x, y, z\}$. The next series of lemmas will establish other constraints on the configuration of $\mathrm{Qcc}(H[f])$. For all of these lemmas, assume that H is a maximal plane graph and that $f : x, y, z$ is a face of H. Unless explicitly stated otherwise, we make no assumptions about the value of $q(H[f])$.

Lemma 7.11. *If $q(H[f]) = k$ and $(k)_{x,y,z}$ is non-empty, then every vertex of $Qcc(H[f])$ is in (k).*

Proof. By Lemma 7.10, and since $q(H[f]) = k$, every vertex of $\mathrm{Qcc}(H[f])$ is either distance k or distance $k+1$ from $H[f]$. Assume to the contrary that there exists a vertex u in $(k)_{x,y,z}$, and a vertex v in $(k+1)$. Then $d(v,t) > d(u,t)$ for all t in $\{x, y, z\}$, contradicting that u is quasi-eccentric to $H[f]$. □

Lemma 7.12. *If $(k)^{z}_{x,y}$ is non-empty, then $(k+1)^{z}_{x,y}$ is empty, and vice versa. Similarly, at most one of $(k)^{y,z}_{x}$ and $(k+1)^{y,z}_{x}$ is non-empty.*

Proof. If $\mathrm{Qcc}(H[f])$ did contain a vertex u in $(k)^{z}_{x,y}$, and a vertex v in $(k+1)^{z}_{x,y}$, then v would be farther from each of x, y and z than u, contradicting the quasi-eccentricity of u. The case for $(k)^{y,z}_{x}$ and $(k+1)^{y,z}_{x}$ is similar. □

Lemma 7.13. *If $(k+1)^{y,z}_{x}$ is non-empty, then $(k)_{x}$ is empty.*

Proof. Assume to the contrary that $(k)_{x}$ contains a vertex u and $(k+1)^{y,z}_{x}$ contains a vertex v. Then v is farther from each of x, y and z than u is. This contradicts the fact that u is quasi-eccentric to $H[f]$. □

We observe the following consequence of Lemma 7.13:

Corollary 7.14. *If both $(k+1)_x^{y,z}$ and $(k+1)_y^{x,z}$ are non-empty, then $(k)_{s,t}$ is empty for any pair of distinct vertices s and t in $\{x, y, z\}$. Further, if it is also true that $q(H[f]) = k$, then $(k)_z^{x,y}$ is non-empty.*

Proof. By Lemma 7.13, if both $(k+1)_x^{y,z}$ and $(k+1)_y^{x,z}$ are non-empty, then both $(k)_x$ and $(k)_y$ are empty. Since $\{s, t\}$ contains either x or y, the set $(k)_{s,t}$ is a subset of either $(k)_x$ or $(k)_y$. Thus $(k)_{s,t}$ is empty.

If $q(H[f]) = k$, then there exists some vertex u in (k). By Lemma 7.13, the vertex u is not in $(k)_x$ or $(k)_y$, thus u is in $(k)_z^{x,z}$. □

As a consequence of Lemmas 7.11, 7.12 and 7.13, we deduce another corollary:

Corollary 7.15. *If $q(H[f]) = k$, then at most two of the sets $(k+1)_x^{y,z}$, $(k+1)_y^{x,z}$ and $(k+1)_z^{x,y}$ are non-empty.*

Proof. If all three sets listed in the hypothesis are non-empty, then all of the sets $(k)_x$, $(k)_y$ and $(k)_z$ are empty, by Lemma 7.13. Thus no vertex of $\mathrm{Qcc}(H[f])$ is distance k from any of the vertices x, y or z in $H[f]$, contradicting the assumption that $q(H[f]) = k$. □

We are ready to begin proving that the quasi-eccentric face criterion of Theorem 7.3 is sufficient when H is a maximal plane graph. We show that given a face $f : x, y, z$ of H, and an integer $\alpha \geq \mathrm{diam}(H)$, we can construct a plane supergraph G_f such that $G_f[f]$ has, for each vertex u in $\mathrm{Qcc}_H(H[f])$, a vertex at distance α from u. We will further show that no vertex of $G_f[f]$ is further than α from any vertex of H.

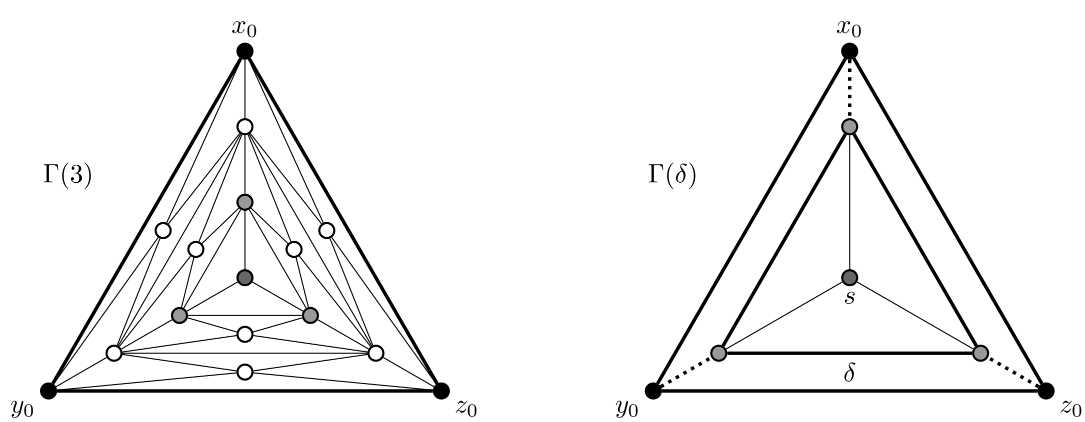


Figure 7.8: Left: the graph $\Gamma(3)$. The vertices x_0, y_0 and z_0 are shaded black. The innermost vertex s and the vertices x_2, y_2 and z_2 are all shaded grey. Right: the way we will normally display the graph $\Gamma(\delta)$.

Let $\delta \geq 1$ be an integer, which we call the **depth** of the construction. Create a maximal plane graph $\Gamma(\delta)(f)$ (when the face f in question is clear from context, we will just refer to the graph as $\Gamma(\delta)$) as follows:
Create the graph $P_\delta \times C_3$ and denote by T_i the i^{th} copy of C_3, with vertices $\{x_i, y_i, z_i\}$, where i is in $[0, \delta - 1]$. In the triangle $T_{\delta-1}$, place a vertex s, and make s adjacent to the vertices $x_{\delta-1}$, $y_{\delta-1}$ and $z_{\delta-1}$. In the interior of each face bounded by a 4-cycle, add a single vertex, and make it adjacent to each vertex contained in the boundary of the face. (see Figure 7.8). For $\delta = 0$, let $\Gamma(\delta)$ be the triangle T_0 with vertex set $\{x_0, y_0, z_0\}$. In order to build the desired maximal plane supergraph G_f from H, we identify the vertices x, y and z of

H with the vertices x_0, y_0 and z_0 of $\Gamma(\delta)$. We will call this operation of taking the union $H \cup \Gamma(\delta)$ and identifying x, y and z with x_0, y_0 and z_0 respectively the **glueing** of $\Gamma(\delta)$ along the face f.

Observation 7.16. Consider the graph $\Gamma(\delta)$, where $\delta \geq 0$. The vertex x_0 is distance at most δ from every vertex of $\Gamma(\delta)$, is distance exactly δ from s, $y_{\delta-1}$ and $z_{\delta-1}$, and is distance $\delta - 1$ from $x_{\delta-1}$.

We will normally only be interested in what is happening in the faces $s, x_{\delta-1}, y_{\delta-1}$; $s, y_{\delta-1}, z_{\delta-1}$ and $s, x_{\delta-1}, z_{\delta-1}$ of $\Gamma(\delta)$, and will add vertices and edges inside these faces as needed. As such, we will leave out the additional clutter in our diagrams (see Figure 7.8).

For the following lemmas, let H be a maximal plane graph, let $f : x, y, z$ be a face of H, and let $\alpha \geq \text{diam}(H)$ be an integer. These lemmas will demonstrate how to construct the graph G_f for some of the possible configurations of $\text{Qcc}_H(H[f])$. By Lemma 7.1, the graph H will always be an isometric subgraph of the graph G_f. Therefore, if u and v are vertices of H, there is no ambiguity in referring to 'the distance $d(u, v)$'.

Lemma 7.17. *Assume there is some integer $k < \alpha$ such that every vertex of $Qcc_H(H[f])$ is distance k from $H[f]$. Let $\delta = \alpha - k$, and let G_f be the graph formed by gluing $\Gamma(\delta)$ to H along f. Then every vertex of $Qcc_H(H[f])$ has eccentricity α in G_f, and no vertex of H has eccentricity greater than α in G_f.*

Proof. Per Observation 7.2, the set $\text{Ecc}_H(H[f])$ is a subset of $\text{Qcc}_H(H[f])$, so every vertex of H is within distance k of $H[f]$. By Observation 7.16, every vertex of $\Gamma(\delta)$ is within distance $\delta = \alpha - k$ of each vertex of $H[f]$, and so every vertex of H is within distance α of every vertex of $\Gamma(\delta)$. Consequently, every vertex u in H satisfies the inequality $e_{G_f}(u) \leq \alpha$.

Let s be the unique vertex of $\Gamma(\delta)$ that is distance $\delta = \alpha - k$ from $H[f]$ and observe that $H[f]$ separates s from $\text{Qcc}_H(H[f])$. Since the vertices of $\text{Qcc}_H(H[f])$ are distance k from $H[f]$, and all the vertices of $H[f]$ are distance $\delta = \alpha - k$ from s, every vertex of $\text{Qcc}_H(H[f])$ is distance α from s. Thus every vertex of $\text{Qcc}_H(H[f])$ has eccentricity α in G_f. □

In Lemma 7.17, we assumed that k was strictly less than α. If k is equal to α, i.e., every vertex of $\text{Qcc}_H(H[f])$ is distance α from some vertex of $H[f]$, then every vertex of $\text{Qcc}_H(H[f])$ has eccentricity α in H. Therefore, if k is equal to α, choosing G_f to be the graph H will suffice.

We consider a small modification to the construction of $\Gamma(\delta)$. For a face $f : x, y, z$ of H and positive integer δ, construct $\Gamma(\delta)$ as usual. Denote by $\Gamma_x(\delta)$ the graph obtained as follows:
Place one additional vertex, call it t_x, in the face $f_x : s, y_{\delta-1}, z_{\delta-1}$ of $\Gamma(\delta)$. Then, make the vertex t_x adjacent to all three vertices on the boundary of f_x. (Figure 7.9)

Observation 7.18. The vertex $x = x_0$ is distance at most δ from every vertex of $\Gamma_x(\delta)$ except for t_x, from which it is distance $\delta + 1$. Further, both y and z are distance exactly δ from s, and distance at most δ from every other vertex of $\Gamma_x(\delta)$.

Lemma 7.19. *Assume that $q(H[f]) = k$, and that both $(k+1)_y^{x,z}$ and $(k+1)_z^{x,y}$ are non-empty. Let $\delta = \alpha - k - 1$. If G_f is the graph formed by gluing $\Gamma_x(\delta)$ to H along f, then every vertex of $Qcc_H(H[f])$ has eccentricity α in G, and no vertex of H has eccentricity greater than α in G_f.*

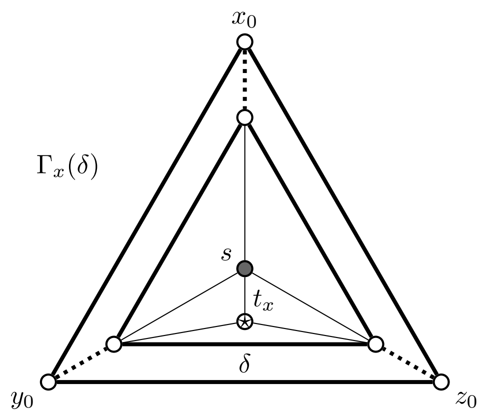


Figure 7.9: The graph $\Gamma_x(\delta)$. The vertex t_x, lies in the face $s, y_{\delta-1}, z_{\delta-1}$ of $\Gamma(\delta)$ and satisfies the equation $d(x_0, t_x) = \delta + 1$.

Proof. Since the set $(k+1)_y^{x,z}$ is non-empty, some vertex of H is distance $k+2$ from x. Thus $\alpha \geq \text{diam}(H) \geq k+2$, so $\delta = \alpha - k - 1 \geq 1$. By Corollary 7.14, the set $(k+1)_x^{y,z}$ is empty, and $(k)_x^{y,z}$ is non-empty. Further, by Lemmas 7.11 and 7.12, all the vertices in $\text{Qcc}_H(H[f]) - (k)_x^{y,z}$ are distance $k+1$ from $H[f]$. Since $\text{Ecc}_H(H[f])$ is a subset of $\text{Qcc}_H(H[f])$, all the vertices of $H - \text{Qcc}_H(H[f])$ are within distance k of $H[f]$.

From Observation 7.18, and the facts listed in the previous paragraph, we can deduce that every vertex of H has eccentricity at most α in G_f. Every vertex of $(k+1)$ is exactly distance α from s, and the vertices of $(k)_x^{y,z}$ are distance α from t_x. Thus every vertex of $\text{Qcc}_H(H[f])$ has eccentricity α in G_f. □

In much the same way that we constructed $\Gamma_x(\delta)$, we construct $\Gamma_{x,y}(\delta)$ from $\Gamma(\delta)$ by placing a vertex t_x in the face $f_x : s, y_{\delta-1}, z_{\delta-1}$ of $\Gamma(\delta)$, and placing a vertex t_y in the face $f_y : s, x_{\delta-1}, z_{\delta-1}$. Then, we add the three edges incident with each of t_x and t_y needed to ensure that the resulting graph is a maximal plane graph. See, for example, the left side of Figure 7.10.

We can also construct $\Gamma_{x,y,z}(\delta)$ in the same way by adding a third vertex t_z in the face $f_z : s, x_{\delta-1}, y_{\delta-1}$. See the right side of Figure 7.10 for an example of this construction. The vertices t_x, t_y and t_z are labelled with $\star$ symbols.

In the spirit of Observations 7.16 and 7.18, we note that for any vertex p in $\{x, y, z\}$, we have that $d(p, t_p) = \delta + 1$, but p is within distance δ of every other vertex of $\Gamma_{x,y,z}(\delta)$.

We will need one more type of modification of $\Gamma(\delta)$. We want to modify $\Gamma(\delta)$ in such a way as to have some vertex t_{xy} which is distance δ from z and distance $\delta+1$ from x and y. To this end, we first construct $\Gamma_{x,y}(\delta)$. To the face $f_{xy} : s, z_{\delta-1}, t_x$, add a vertex t_{xy}, and let t_{xy} be adjacent to s, $z_{\delta-1}$ and t_x. Call the resulting graph $\Gamma_{x,y}^{xy}(\delta)$. The superscript xy indicates that there exists a vertex distance $\delta + 1$ from both x and y. In a similar fashion, we can construct $\Gamma_{x,y,z}^{xy}(\delta)$, $\Gamma_{x,y,z}^{xy,yz}(\delta)$ and $\Gamma_{x,y,z}^{xy,yz,zx}(\delta)$. See Figure 7.11 for some of these constructions. The vertices t_{xy}, t_{yz} and t_{xz} are marked with $\bullet$ symbols.

Observation 7.20 (Distances in Γ graphs)**.** Consider any graph $\Gamma(\delta)$ (with possible subscripts and superscripts) of depth $\delta \geq 1$ that we have constructed thus far, which has vertices x_0, y_0 and z_0 on the boundary of its outermost triangle. The vertex s inside the innermost triangle is distance δ from each of x_0, y_0 and

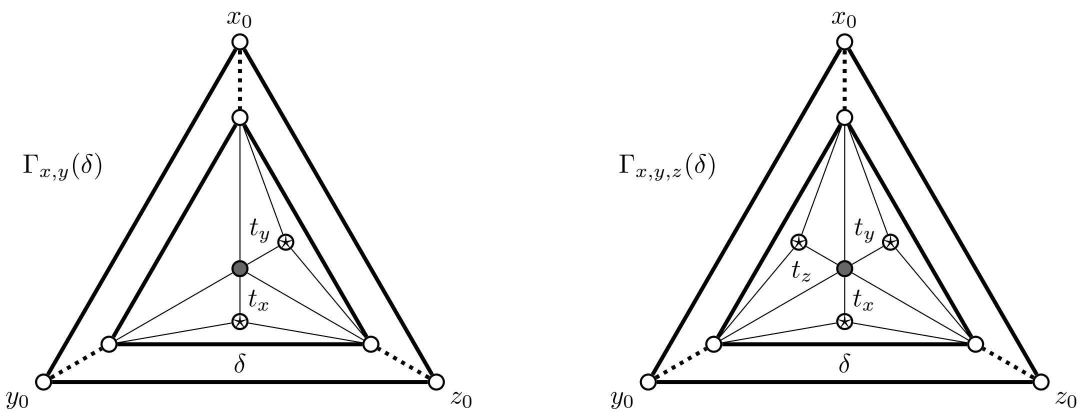


Figure 7.10: Left: the graph $\Gamma_{x,y}(\delta)$. Right: the graph $\Gamma_{x,y,z}(\delta)$. The vertex s is grey, and the vertices t_x, t_y and t_z contain star $\star$ symbols.

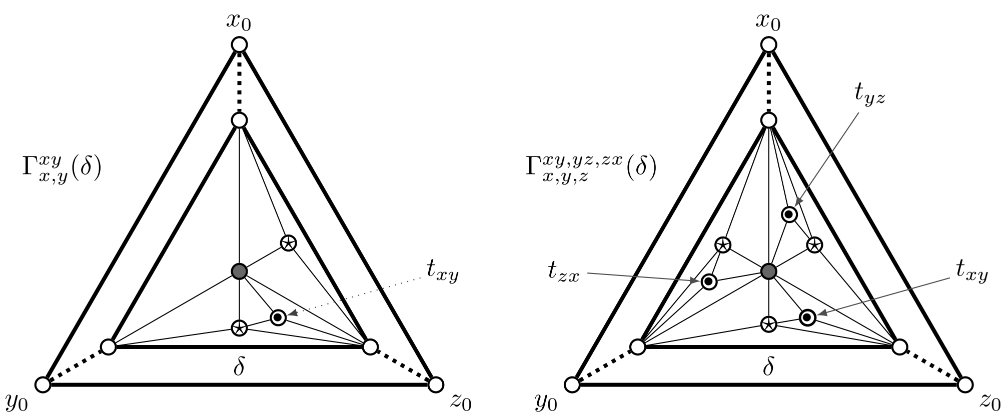


Figure 7.11: Left: the graph $\Gamma^{xy}_{x,y}(\delta)$. Right: the graph $\Gamma^{xy,yz,zx}_{x,y,z}(\delta)$.

z_0. Any vertex t_p, where p is in $\{x, y, z\}$, is distance $\delta + 1$ from p and distance δ from each of the other two vertices in $\{x, y, z\}$. Any vertex t_{pq}, where p and q are distinct elements of $\{x, y, z\}$, is distance $\delta + 1$ from both p and q, and distance δ from the remaining vertex of $\{x, y, z\}$.

In the proof of the main theorem (Theorem 7.22), we will construct, for each face f of H, a plane supergraph G_f of H such that the quasi-eccentric vertices of $H[f]$ have eccentricity α in G_f. In order to construct a plane supergraph G of H in which every vertex of H has eccentricity α, we will take the union of all the graphs G_f, where f is a face of H. Theorem 7.21 demonstrates that this union has the properties we desire.

Theorem 7.21. *Let $\{G_m = (V_m, E_m) : m \in \{1, \dots, n\}\}$ be a finite collection of connected graphs such that for any distinct integers m and l in $\{1, \dots, n\}$, we have that the pairwise intersection $V_m \cap V_l$ is the same nonempty set S (that is to say, the set S does not depend on the choice of integers m and l). Define the graphs $G = \bigcup\{G_m : m \in \{1, \dots, n\}\}$ and $H = \bigcap\{G_m : m \in \{1, \dots, n\}\}$ (Note that $V(H) = S$). If both of the conditions*

(1) H is connected, and

(2) H is an isometric subgraph of G_m for all m in $\{1, \dots, n\}$,

are satisfied, then each graph G_m is an isometric subgraph of G, and every vertex u in H satisfies:

$$e_G(u) = \max\{e_{G_m}(u) : m \in \{1, \ldots, n\}\}.$$

Proof. We begin by proving that G_1 is an isometric subgraph of G. Assume that $\{G_m : m \in \{1, \ldots, n\}\}$ satisfies the two conditions of the theorem, and let u and v be vertices of G_1. By assumption, the set $V(H) = S$ is non-empty. Since each graph G_m is connected, and the intersection H of the family of graphs is non-empty, the graph G is connected. Let P be a $u - v$ geodesic in G. In order to prove that G_1 is isometric in G, it suffices to prove that there exists a path P' in G_1 such that $\ell(P') \leq \ell(P)$. If P itself is contained in G_1, then we set $P' = P$. We thus assume without loss of generality that P is not contained in G_1. Observe that if xy is an edge of G, then it is not possible that x is contained in $G_m - H$ and y is contained in $G_l - H$ for distinct integers m and l. Thus we can partition the edges of P into a number of smaller paths $Q_0, Q_1, \ldots, Q_i$ such that $P = Q_0 \cup Q_1 \cup \cdots \cup Q_i$ satisfies the following:

(1) the path Q_0 starts at u and ends in H, and is contained in G_1,
(2) the path Q_i starts in H and ends at v, and is contained in G_1,
(3) for each integer j in $\{1, 2, \ldots, i-1\}$, the path Q_j starts and ends in H, and is contained entirely in some graph G_{m_j}.

Since for each j in $\{1, \ldots, i-1\}$ the path Q_j starts and ends in H, and H is isometric in each G_m, there exists some path R_j in H with the same starting and ending vertices as Q_j such that $\ell(R_j) \leq \ell(Q_j)$. We can thus let $P' = Q_0 \cup R_1 \cup \ldots R_{i-1} \cup Q_i$, completing the proof that G_1 is isometric in G. In the same way, every graph G_m is also isometric in G.

We now prove that the eccentricity in G of a vertex u in H is the maximum of the eccentricities $e_{G_1}(u)$, $e_{G_2}(u)$, $\ldots$, $e_{G_n}(u)$. Assume that $\{G_m : m \in \{1, \ldots, n\}\}$ satisfies the conditions of the theorem, and let u be a vertex of H. If v is a eccentric vertex of u in G, then since $V(G) = \bigcup\{V_m : m \in \{1, \ldots, n\}\}$, the vertex v belongs to G_m for some integer m in $\{1, \ldots, n\}$. Any $u - v$ path in G_m is itself a $u - v$ path in G, so $d_G(u, v) \leq d_{G_m}(u, v)$. Consequently,

$$e_G(u) \leq \max\{e_{G_m}(u) : m \in \{1, \ldots, n\}\}.$$

For each integer m in $\{1, \ldots, n\}$, let v_m be an eccentric vertex of u in G_m. Because each G_m is isometric in G, we have that $d_G(u, v_m) = d_{G_m}(u, v_m)$, and so

$$e_G(u) \geq \max\{e_{G_m}(u) : m \in \{1, \ldots, n\}\},$$

completing the proof. □

Theorem 7.22 (The quasi-eccentric face criterion in maximal planar graphs)**.** *Let H be a maximal plane graph of diameter d, and let $\alpha \geq d$ be an integer. The graph H is a subgraph of some maximal plane graph G, such that $V(H)$ is a subset of $\mathscr{E}_G(\alpha)$, if and only if for all u in H that satisfy $e_H(u) < \alpha$, there exists a face f of H such that u is in $Qcc_H(H[f])$.*

Proof. The necessity of the condition follows from Theorem 7.3. It only remains to prove sufficiency. Assume that for all vertices u of H that have eccentricity less than α in H, there exists a face f of H such that u is in $\mathrm{Qcc}_H(H[f])$.

We will construct the graph G. For each face f of H, we will create a plane supergraph G_f of H using one of the Γ constructions described in this section, and glueing the Γ graph to H along the boundary of f. The graph G_f will be chosen such that each vertex of $\mathrm{Qcc}_H(H[f])$ will have eccentricity exactly α in G_f, and each vertex of H will have eccentricity at most α in G_f. Further, given any two faces f_1 and f_2 of H, the intersection $G_{f_1} \cap G_{f_2}$ will be the graph H, which is isometric in any plane graph which contains it. By Theorem 7.21, setting $G = \bigcup\{G_f : f \text{ a face of } H\}$ will complete the proof. As such, all that we need to do now is to construct the graphs G_f, where f is a face of H.

Let $f : x, y, z$ be a face of H, and let $q(H[f]) = k$. By Lemma 7.10, every vertex of $\mathrm{Qcc}_H(H[f])$ is either distance k from $H[f]$, or distance $k+1$ from $H[f]$. Up to relabelling of the vertices x, y and z, the following list describes all possible configurations of $\mathrm{Qcc}_H(H[f])$:

- Case 1: Every vertex of $\mathrm{Qcc}_H(H[f])$ is distance k from $H[f]$. That is to say, the sets $\mathrm{Qcc}_H(H[f])$ and (k) are equal.
- Case 2: The sets $(k+1)_x^{y,z}$, $(k+1)_y^{x,z}$ and $(k+1)_z^{x,y}$ are all non-empty.
- Case 3: The sets $(k+1)_x^{y,z}$ and $(k+1)_y^{x,z}$ are non-empty, but $(k+1)_z^{x,y}$ is empty.
- Case 4.1: The set $(k+1)_x^{y,z}$ is non-empty, and the sets $(k+1)_y^{x,z}$, $(k+1)_z^{x,y}$ and $(k)_{y,z}^{x}$ are empty.
- Case 4.2: The sets $(k+1)_x^{y,z}$ and $(k)_{y,z}^{x}$ are non-empty, but the sets $(k+1)_y^{x,z}$ and $(k+1)_z^{x,y}$ are empty.
- Case 5.1: The sets $(k+1)_x^{y,z}$, $(k+1)_y^{x,z}$ and $(k+1)_z^{x,y}$ are all empty. However, each of the sets $(k+1)_{x,y}^{z}$, $(k+1)_{x,z}^{y}$ and $(k+1)_{z,y}^{x}$ is non-empty.
- Case 5.2: The sets $(k+1)_x^{y,z}$, $(k+1)_y^{x,z}$ and $(k+1)_z^{x,y}$ are all empty. Further, the sets $(k+1)_{x,y}^{z}$ and $(k+1)_{x,z}^{y}$ are non-empty, but the set $(k+1)_{z,y}^{x}$ is empty.
- Case 5.3: The sets $(k+1)_x^{y,z}$, $(k+1)_y^{x,z}$ and $(k+1)_z^{x,y}$ are all empty. On the other hand, the set $(k+1)_{x,y}^{z}$ is non-empty, but the sets $(k+1)_{z,y}^{x}$ and $(k+1)_{x,z}^{y}$ are empty.
- Case 6: The set $(k+1)_{x,y,z}$ is non-empty. Further, every vertex of $\mathrm{Qcc}_H(H[f])$ at distance $k+1$ from $H[f]$ is contained in the set $(k+1)_{x,y,z}$.

We show that for each configuration in the list presented above, we can construct the maximal plane graph G_f by placing another maximal plane graph Γ_f inside the face f, and glueing Γ_f to H along f. If f_1 and f_2 are two distinct faces of H, it is clear that H will be a subgraph of both G_{f_1} and G_{f_2}. Since H is a maximal plane graph, it will be an isometric subgraph of both G_{f_1} and G_{f_2}, per Lemma 7.1. Because the graphs Γ_{f_1} and Γ_{f_2} lie in different faces of H, the intersection $V(G_{f_1}) \cap V(G_{f_2})$ is exactly the set $V(H)$.

Case 1:

If all vertices of $\mathrm{Qcc}_H(H[f])$ are distance k from $H[f]$, then define $\delta = \alpha - k$. If $\delta = 0$, then $k = \alpha$ and every vertex of $\text{Qcc}_H(H[f])$ is distance $k = \alpha$ from $H[f]$. Thus every vertex of $\text{Qcc}_H(H[f])$ has eccentricity α in H, so choose $\Gamma_f = H$. If $\delta > 0$, then by Lemma 7.17, we can choose $\Gamma_f = \Gamma(\delta)$.

Case 2:
By Corollary 7.15, this case is not possible.

Case 3:
Let $\delta = \alpha - k - 1$. Since the set $(k+1)^z$ is non-empty, there exists a vertex in H which is distance $k+2$ from y. Thus $\alpha \geq \text{diam}(H) \geq k+2$, and so $\delta \geq 1$. By Lemma 7.19, We can choose $\Gamma_f = \Gamma_z(\delta)$.

Case 4.1:
Let $\delta = \alpha - k - 1$ and $\Gamma_f = \Gamma_{y,z}(\delta)$. By the same reasoning used in Case 3, we have $\delta \geq 1$. Vertices in the sets $(k)_y^{x,z}$ and $(k)_z^{x,y}$ are distance α from the vertices t_y and t_z respectively. By Lemmas 7.11, 7.12 and 7.13, all the vertices in $\text{Qcc}_H(H[f]) - ((k)_y^{x,z} \cup (k)_z^{x,y})$ are in the set $(k+1)$, and thus distance $\delta + k + 1 = \alpha$ from s. Since the sets $(k+1)_y^{x,z}$ and $(k+1)_z^{x,y}$ are empty, every vertex of $\text{Qcc}_H(H[f])$ has eccentricity at most α in G_f, by Observation 7.20. As $\text{Ecc}(H[f])$ is a subset of $\text{Qcc}_H(H[f])$, we have that $e(H[f]) = k+1$, so every vertex of $H - \text{Qcc}_H(H[f])$ is within distance k of $H[f]$. Thus each vertex of $H - \text{Qcc}_H(H[f])$ is within distance α of every vertex of G_f. As such, every vertex of H has eccentricity at most α in G_f by Observation 7.20.

Case 4.2:
Let $\delta = \alpha - k - 1$ and $\Gamma_f = \Gamma_{y,z}^{yz}(\delta)$. As in Case 3, we have that $\delta \geq 1$. By Observation 7.20, the vertices in the set $(k)_{y,z}^x$ are distance α from the vertex t_{xy}, and distance at most α from every other vertex of G_f. The rest of this case follows in the same way as Case 4.1.

Case 5.1:
Let $\delta = \alpha - k - 1$ (hence $\delta \geq 1$) and $\Gamma_f = \Gamma_{x,y,z}(\delta)$. By Lemmas 7.11 and 7.12, every vertex in the set (k) is contained in $(k)_x^{y,z} \cup (k)_y^{x,z} \cup (k)_z^{x,y}$. By Lemma 7.10, every vertex of $\text{Qcc}_H(H[f]) - (k)$ is in the set $(k+1)$. As a consequence of the previous two sentences, every vertex of $\text{Qcc}_H(H[f])$ is distance α from some vertex in the set $\{s, t_x, t_y, t_z\}$. By Observation 7.20, every vertex of $\text{Qcc}_H(H[f])$ is distance at most α from each of the vertices t_x, t_y and t_z. The argument that every other vertex of H is distance at most α from any vertex of Γ_f is the same as the argument used in Case 4.1.

Case 5.2:
Let $\delta = \alpha - k - 1$ (and note again that we have $\delta \geq 1$) and $\Gamma_f = \Gamma_{x,y,z}^{yz}(\delta)$. This case is the same as Case 5.1, apart from the fact that the vertices in the set $(k)_{y,z}^x$ are distance α from the vertex t_{yz}.

Case 5.3:
Let $\delta = \alpha - k - 1$ (and observe that $\delta \geq 1$) and $\Gamma_f = \Gamma_{x,y,z}^{yz,zx}(\delta)$. This case is similar to Case 5.2, noting that any vertices in the set $(k)_{x,z}^y$ is distance α from the vertex t_{zx}.

Case 6:
Let $\delta = \alpha - k - 1$. If $\delta = 0$, then $\alpha = k+1$. By Lemma 7.11, every vertex of $\text{Qcc}_H(H[f])$ is distance $\alpha = k+1$ from some vertex of $H[f]$, so it suffices to let $G_f = H$. On the other hand, if $\delta \geq 1$, let $\Gamma_f = \Gamma_{x,y,z}^{xy,yz,zx}(\delta)$. By Observation 7.20, every vertex of $\text{Qcc}_H(H[f])$ is distance at least α from some vertex

in the set $\{s, t_{xy}, t_{yz}, t_{xz}\}$. Note that the only vertices distance $k+1$ from $H[f]$ are those in $(k+1)_{x,y,z}$, and every other vertex of H is distance at most k from $H[f]$. Thus every vertex of $\mathrm{Qcc}_H(H[f])$ is within distance at most α of each vertex of Γ_f.

To complete the construction, let $G = \bigcup\{G_f : f \text{ a face of } H\}$. It is clear that G is a maximal plane supergraph of H. In the case analysis presented above, we have shown that regardless of the configuration of $\mathrm{Qcc}_H(H[f])$, the vertices of $\mathrm{Qcc}_H(H[f])$ have eccentricity α in G_f, and that every vertex of H has eccentricity at most α in G_f. Thus, by Theorem 7.21, every vertex which is quasi-eccentric to some face of H has eccentricity α in G. By assumption, every vertex of H is quasi-eccentric to some face of H. Thus, every vertex of H has eccentricity α in G. □

Observation 7.23. Let $G = (V, E)$ be a connected graph, and S a subset of V. Then the following inclusion holds:

$$\{u \in V : u \text{ is an eccentric vertex of some vertex in } S\} \subseteq \mathrm{Qcc}(S).$$

Corollary 7.24. *If H is a maximal planar graph in which every vertex is an eccentric vertex, then for any integer $\alpha \geq diam(H)$, there exists a maximal planar graph G into which H embeds such that $H \subseteq \mathscr{E}_G(\alpha)$.*

Proof. We prove that every vertex of H is quasi-eccentric to some face of H. Let u be a vertex of H, and v a vertex to which u is eccentric. There exists some face f such that v is on the boundary of f, so by Observation 7.23, the vertex u is quasi-eccentric to $H[f]$. The result now follows from Theorem 7.22. □

Corollary 7.25. *If H is a maximal planar graph such that every vertex is eccentric to $H[f]$ for some face f of H, then for any integer $\alpha \geq diam(H)$, there exists a maximal planar graph G into which H embeds such that $H \subseteq \mathscr{E}_G(\alpha)$.*

Proof. By Observation 7.2, every vertex is quasi-eccentric to some face (in particular, the face to which it is eccentric). The result follows from Theorem 7.22. □

7.6 Refinements and corollaries of Theorem 7.22

Theorem 7.22 gives a necessary condition for a maximal planar graph H to be a subgraph of the centre of some (maximal) planar graph G (as the centre of a graph is always an equi-eccentric subgraph), and a sufficient condition for H to be equi-eccentric, with eccentricity α, in G. Theorem 7.27 shows that the construction used in Theorem 7.22 ensures that α is the lowest eccentricity present in G, and hence that Theorem 7.22 exactly characterises maximal planar graphs which are contained in the centre of some (maximal) planar graph. But first, we need a lemma.

Lemma 7.26. *Let H be a maximal plane graph of order at least 4 such that every vertex of H is quasi-eccentric to some face of H. Then every vertex u of H is quasi-eccentric to some face f such that u is not contained in the subgraph $H[f]$ induced by the boundary of f.*

Proof. By the assumption, there exists a face to which the vertex u is quasi-eccentric. If this face does not contain u in its boundary, we are done. Assume that the face to which u is quasi-eccentric does contain u in its boundary. Call this face $f_1 : u, y, z$. Since H is a maximal plane graph of order 4 or more, the edge yz lies on two faces, the face $f_1 : u, y, z$ and another face $f_2 : x, y, z$. The vertex u is quasi-eccentric to the vertex set $\{u, y, z\}$ of $H[f_1]$, and $d(u, x) > d(u, u) = 0$, so u is also quasi-eccentric to the set $\{x, y, z\}$. Thus u is quasi-eccentric to $H[f_2]$, completing the proof. □

Theorem 7.27. *Let H be a maximal plane graph of order at least 4 satisfying the condition of Theorem 7.22, and let $\alpha \geq diam(H)$ be an integer. Denote by G the graph constructed in Theorem 7.22 such that $V(H) \subseteq \mathcal{E}_G(\alpha)$. Then the radius of G is α, and so H is contained in the centre of G.*

Proof. Let H be a maximal plane graph satisfying the hypotheses of the theorem. We will need the following claim.

Claim: If u is a vertex of H, there is a vertex v of $G - H$ that is not contained in any face of H that has u in its boundary, and such that $d(u, v) = \alpha$.

Proof of Claim: By construction of G, any face f of H such that u is in $\text{Qcc}_H(H[f])$ contains an eccentric vertex of u. Thus it suffices to find a face $f : x, y, z$ such that u is in $\text{Qcc}_H(H[f])$, but for which u is not in $\{x, y, z\}$. By Lemma 7.26, the desired face f exists, proving the claim.

We now prove the theorem. Since every vertex of H has eccentricity α in G, it suffices to prove that any vertex of $G - H$ has eccentricity at least α in G. Let w be a vertex of $G - H$, and let $f : x, y, z$ be the face of H containing w. By the prior claim, there is some vertex v which does not lie inside the face f, such that $d(x, v) = \alpha$. Since $H[f]$ separates w and v, by Lemma 7.5 any $w - v$ path P has length:

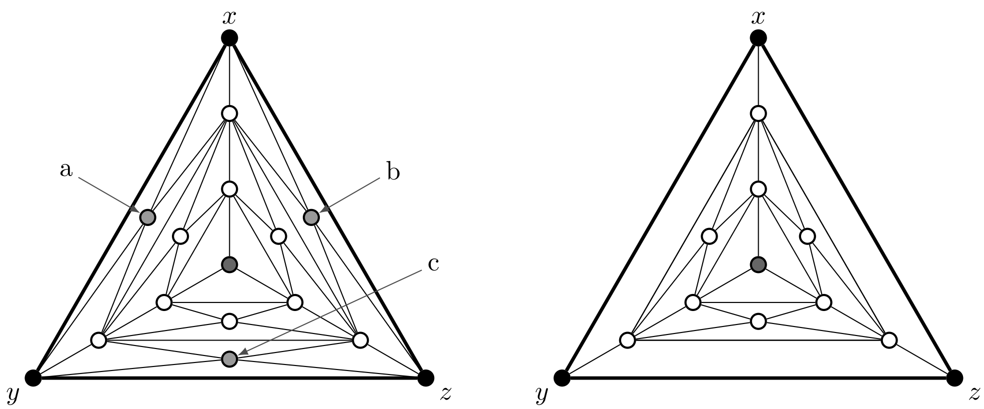
$$\ell(P) \geq d(w, H[f]) + d(v, H[f]) \geq 1 + (\alpha - 1) = \alpha$$

Note that the distance $d(v, H[f]) \geq (\alpha - 1)$ since $d(x, v) = \alpha$ and x is adjacent to every vertex of $H[f]$. Since $d(w, v) \geq \alpha$, we have $e(w) \geq \alpha$ in G, completing the proof. □

We can refine Theorems 7.22 and 7.27 further. Given a maximal planar graph H which satisfies the hypotheses of Theorem 7.22, we can find a planar (but not maximal planar) graph PG such that H is exactly the centre of PG (rather than just being contained in the centre, as we obtain in Theorem 7.27).

To do so, we need only make two small modifications to the proof of Theorem 7.22, the latter of which was suggested by Dankelmann [13]. First, we insist that the integer α satisfies $\alpha \geq \text{diam(H)} + 3$. Note that this forces any occurrence of the integer δ in the proof of Theorem 7.22 to satisfy $\delta \geq 2$. Second, find all the vertices of $G - H$ which are adjacent to exactly two vertices of H. Remove these vertices from the maximal planar graph G to obtain the planar graph PG (See Figure 7.12).

Theorem 7.28. *Let H be a maximal plane graph of order at least 4 satisfying the hypotheses of Theorem 7.22, and let $\alpha \geq diam(H) + 3$ be an integer. Let G be the maximal planar graph constructed in Theorem 7.22, and let PG be the planar graph formed by removing from G all vertices of $G - H$ which are adjacent to exactly two vertices of H. Then H is the centre of PG.*



The subgraph $G[f]$ of G The subgraph $PG[f]$ of PG

Figure 7.12: On the left is a possible subgraph $G[f]$ of G, obtained in the proof of Theorem 7.22, where f is a face of H. The vertices a, b and c are each adjacent to exactly two vertices of H. On the right is $PG[f] = G[f] - \{a, b, c\}$, which is a subgraph of PG.

Proof. By Lemma 7.1, the graph H is isometric in PG.

Claim: The graph PG is an isometric subgraph of G.

Proof of claim: Note we obtained PG from the maximal planar graph G by only removing vertices of degree 4, none of which are adjacent in G (by construction of G). Let u_1 and u_k be vertices of PG and let P be a $u_1 - u_k$ geodesic in G. Among all such geodesics, let P contain the minimum number of vertices of $G - PG$. If P contains no vertex of $G - PG$, we have proven the claim. So assume to the contrary that P has the form $P : u_1, \ldots, a, b, c, \ldots, u_k$, where b is a vertex removed in constructing PG. Letting b^* be either of the vertices in $N(b) - \{a, c\}$, we see that the path $P^* : u_1, \ldots, a, b^*, c, \ldots, u_k$ is also a $u_1 - u_k$ geodesic in G, and one that contains fewer vertices of $G - PG$, contradicting the minimality of P and thus proving the claim.

By the construction of PG, and the fact that PG is isometric in G, every vertex of H has eccentricity α in PG (by the same arguments as those outlined in the proof of Theorem 7.22). Thus to prove the theorem, it suffices to prove that every vertex of $PG - H$ has eccentricity strictly greater than α.

First, we establish that each vertex u of H has an eccentric vertex which does not lie inside a face containing u in its boundary. This follows from Lemma 7.26 by the same argument as was used in Theorem 7.27 (note that since $\alpha \geq \text{diam}(H) + 3$, the desired eccentric vertex is not adjacent to any vertex of H, and hence is in PG).

Now let w be a vertex of $PG - H$, and denote by $f : x, y, z$ the face of H containing w. There are two cases to consider.

Case 1: the vertex w is adjacent to one of x, y or z. Assume without loss of generality that w is adjacent to x. By the construction of PG, the vertex w is not adjacent to y or z. By the preceding paragraph, there is a vertex v that is eccentric to x, such that v does not lie in a face of H containing x on its boundary. Because the set $V(H[f]) = \{x, y, z\}$ separates w and v, and w is not adjacent to y or z, any $w - v$ path P satisfies either:

$$\ell(P) = d(w, x) + d(x, v) = 1 + \alpha > \alpha$$

or, for some $t \in \{y, z\}$:

$$\ell(P) = d(w, t) + d(t, v) \geq 2 + (\alpha - 1) > \alpha.$$

Where $d(t, v) \geq (\alpha - 1)$, since $d(x, v) = \alpha$ and x is adjacent to t. Since the distance $d(w, v) > \alpha$, the eccentricity $e(w) > \alpha$ in PG.

Case 2: the vertex w is not adjacent to any of x, y or z, and thus the distance $d(w, H[f]) \geq 2$. Similarly to Case 1, we have a vertex v that does not lie in a face of H that has x on its boundary, and such that $d(x, v) = \alpha$. The set $H[f]$ separates w and v, so by Lemma 7.5:

$$d(w, v) \geq d(w, H[f]) + d(H[f], v) \geq 2 + (\alpha - 1) > \alpha.$$

Thus w has eccentricity strictly greater than α, completing the proof. □

The following corollary is immediate, but worth stating.

Corollary 7.29. *A maximal plane graph H is the centre of some plane graph if and only if, for each vertex u of H, there exists some face f of H such that u is in $Qcc_H(H[f])$.*

By Theorems 7.27 and 7.28, Corollaries 7.24 and 7.25 also give sufficient conditions for a maximal planar graph to be contained in the centre of some maximal planar graph, and to be exactly the centre of some planar graph.

We end off by illustrating that the results of Theorems 7.27 and 7.28 should not be surprising, and do not depend strongly on the specific construction used in the proof of Theorem 7.22.

Theorem 7.30. *Let H be a maximal plane graph which does not contain any face f such that every vertex of H is adjacent to some vertex of $H[f]$. If H is an equi-eccentric subgraph of a plane graph G, then H is a subgraph of the centre of G.*

Proof. Let α be the integer such that $V(H) \subseteq \mathcal{E}_G(\alpha)$. Assume to the contrary that there exists a vertex u of $G - H$ such that $e(u) < \alpha$. Let $f : x, y, z$ be the face of H containing u, and let w be an eccentric vertex of x in G. We consider two cases.

Case 1: The vertex w lies inside the face f. Let v be a vertex of H which is not adjacent to any vertex of $H[f]$ (such a vertex exists by assumption). Then, since $H[f]$ separates v and w, we have by Lemma 7.5 that:

$$d(v, w) \geq d(v, H[f]) + d(w, H[f]) \geq 2 + (\alpha - 1) > \alpha.$$

This contradicts the assumption that all vertices of H have the same eccentricity in G.

Case 2: The vertex w lies outside the face f. Because w is not in f, the subgraph $H[f]$ separates u and w, so by Lemma 7.5:

$$d(u, w) \geq d(u, H[f]) + d(w, H[f]) \geq 1 + (\alpha - 1) \geq \alpha.$$

This contradicts the assumption that $e(u) < \alpha$, completing the proof. □

7.7 Further questions

Corollary 7.29 gives a sufficient and necessary condition for a maximal planar graph to be the centre of a planar graph. Does this result hold for a larger class of graphs than just maximal planar graphs? We present the following conjecture as a possible answer to this question:

Conjecture 7.31. *Let H be a 2-connected plane graph such that each vertex of H is quasi-eccentric to $H[f]$ for some face f of H. There exists a plane graph G such that H is both the centre of G and an isometric subgraph of G.*

Figure 7.3 demonstrates that the centre of a maximal planar graph may be disconnected. What conditions are sufficient and / or necessary for a (maximal) planar graph to have a connected centre?

Chapter 8

Conclusion

In this thesis, we have explored and expanded upon the known theory of distances in planar graphs. In particular, we have presented novel results detailing the size and structure of face-degree regular plane graphs, reviewed and summarised a collection of literature investigating these graphs and given a consolidated account of the basic properties of maximal planar graphs (which are themselves a special class of face-degree regular graphs). We conclude the thesis by first giving an overview of the results proven here and how they contribute to the existing literature, and we then present conjectures and unanswered questions raised by the work done thus far.

8.1 Overview

8.1.1 Introductory chapters

In Chapter 1, we introduced the basic definitions used throughout the thesis. In Chapter 2, we presented standard results we needed in the theory of plane and planar graphs — in particular, the Jordan Curve Theorem, the Euler Characteristic Equation and the theorems of Wagner and Kuratowski.

We gave a literature review in Chapter 3. In this review, we summarised Thomassen's proof of the Jordan Curve Theorem [36], discussed the celebrated planar separator theorem of Lipton and Tarjan [30] and explored the main results of multiple papers on the degree diameter problem in planar graphs. Particular attention was paid to those papers which solved the degree diameter problem for classes of face-degree regular plane graphs of low diameter — namely Seyffarth's bound for diameter 2 maximal planar graphs [35] and Dalfó et al.'s bound for diameter 2 and 3 maximal planar bipartite graphs [12]. Note that we also presented a brief review of the literature on graph centres in Chapter 7, drawing attention to Laskar and Shier's description of the centres of chordal graphs [29] and Proskurowski's characterisation of the centres of maximal outerplanar graphs [34].

8.1.2 Chapter 4

In Chapter 4, we considered ρ-face-degree regular plane graphs that are extremal with respect to their face-degree and diameter. In particular, we deduced that the minimum face-degree μ of a 2-edge-connected plane graph of radius r and diameter D is $\mu = 2r+1$ (Theorem 4.13), and hence that $\mu \leq 2D+1$. We demonstrated that the only 2-edge-connected graph attaining the latter bound is the odd cycle C_{2D+1} (Theorem 4.17). The former result can be considered a planar analogue of the well-known fact that every graph with girth g and diameter D satisfies $g \leq 2D+1$, and the latter result demonstrates that the odd cycle is the only 'Moore type graph' for this bound (complementing the known fact that it is also the only planar Moore graph in the usual sense [31]).

We further explained the link between diameter, girth and minimum face-degree in plane graphs, showing that if one of the girth or minimum face-degree of a plane graph is at least twice the diameter, then these two parameters are equal (Corollary 4.23), and that this result could not be improved (Theorem 4.24).

Due to their value in studying finite geometries, generalised polygons are a well-studied class of graphs [21]. We have contributed to this theory by showing there is a unique planar generalised polygon of given maximum degree and diameter (Theorem 4.31), and we give a further characterisation of planar generalised polygons as 2-edge-connected plane graphs with diameter D that are $2D$-face-degree regular (Corollary 4.32). Using these results, we proved two extremal cases of the degree-diameter problem in 2-edge-connected face-degree regular plane graphs: if such a graph has diameter D and is $2D+1$-face-degree regular, then the order is exactly $2D+1$, and if it is $2D$-face-degree regular and has maximum degree Δ, then $n = \Delta(D-1)+2$ (Corollaries 4.33 and 4.34).

8.1.3 Chapter 5

The literature results in [12], [35] and [20], together with our results in Chapter 4, solve the degree diameter problem for all ρ-face-degree regular 2-edge-connected plane graphs of diameter $D \leq 3$, where $\rho \geq 3$, except for two cases: one in which $D = 3$ and $\rho = 5$ and another in which $\rho = D = 3$. Hence, in Chapter 5, we solved this problem for $D = 3$ and $\rho = 5$. Specifically, we proved that if such a graph has order n and maximum degree $\Delta \geq 8$, then $n \leq 3\Delta - 1$.

8.1.4 Chapter 6

In Chapter 6, we gave an account of the basic distance and separation properties of maximal planar graphs (per Theorem 6.1, these are precisely the planar graphs that embed as 3-face-degree regular graphs, provided the order of the graph in question is at least 3). It is well known that for every pair r and D of positive integers with $r \leq d \leq 2D$, there is a graph with radius r and diameter D. We demonstrated that this is also true of maximal planar graphs, and explored rough bounds on how many vertices of each degree a maximal planar graph may contain. It is common knowledge that is used in the literature that every minimal separator of a maximal planar graph is a chordless cycle. However, it is not easy to find proof of this fact, so we presented an elementary proof. We extended this result by showing that the same characterisation held for minimal $A - B$ separators (where A and B are connected sets of vertices). Further, we also characterised connected sets of maximal planar graphs that are non-separating, and connected sets whose complement is 2-connected. We presented a sufficient condition for a subgraph of a plane graph to be an isometric subgraph, and used this to prove the known fact that every maximal planar subgraph of a planar graph is isometric. Finally, we constructed maximal planar graphs whose centres had arbitrarily many components.

8.1.5 Chapter 7

We began Chapter 7 by giving a brief survey of results concerning the centres of graphs. We then presented a necessary condition for a planar graph to be an isometric and equi-eccentric subgraph of some planar graph, and showed that this condition completely characterised which maximal planar graphs are centres of planar graphs. In order to state this characterisation, we introduced a new notion, that of *quasi-eccentricity*, which may be a valuable concept to consider for similar problems, and is an interesting distance-theoretic property of a set of vertices in a graph in its own right. We further demonstrated that whenever a sufficiently large maximal planar graph was an equi-eccentric subgraph of a planar graph, it was necessarily contained in the centre of that planar graph.

8.2 Open questions and conjectures

There are a number of questions related to — but not fully answered by — the existing literature we have discussed and the work of this thesis. Here we present such questions, as well as some conjectures by the author concerning those questions.

In a 2-edge-connected plane graph G of diameter D, the girth g and minimum face-degree μ satisfy $g \leq \mu$, and in general the difference between these parameters can be arbitrarily large (see, for example, Figure 4.3). However, Corollary 4.23 shows that if $\mu \geq 2D$ or $g \geq 2D$, then $\mu = g$. This is very convenient, since classes of graphs that are in some sense extremal with respect to their girth given their diameter are well studied (in this thesis, two such classes we have encountered are Moore graphs and generalised polygons). Unfortunately, the most obvious extension of Corollary 4.23 does not hold, as demonstrated by Theorem 4.24 (which shows the existence of a 2-edge-connected plane graph with diameter D having minimum face-degree $\mu = 2D - 1$ and girth $g = 2D - 2$). There may, however, be other conditions that are sufficient for the girth and minimum face degree to agree, or at least for the difference $\mu - g$ to be small.

Question 1: Other than the condition given by Corollary 4.23, are there other sufficient or necessary conditions for the minimum face-degree μ and girth g of a plane (or planar) graph to agree?

Question 2: What bounds can be found for quantity $\mu - g$?

We raise the following conjecture in an attempt to answer Question 2:

Conjecture 8.1. *Let G be a 2-edge-connected plane graph with minimum face-degree μ, girth g and diameter D, and let k be any non-negative integer. If $\mu \geq 2D - k$, then $\mu - g \leq k$.*

This conjecture extends Corollary 4.23 (consider $k = 0$), and is consistent with the result of Chapter 5 stating that a plane pentagulation of diameter 3 has girth at least 4 (Corollary 5.7).

Question 3: How do minimum face-degree, girth and diameter relate to each other in general?

Keeping Question 3 in mind, we conjecture the following strengthening of Theorem 4.13:

Conjecture 8.2. *If a 2-edge-connected plane graph has diameter D, girth g and minimum face-degree μ, then:*

$$D \geq \frac{2\mu - g - 1}{2},$$

and this bound is the best possible.

Conjecture 8.2 holds trivially for graphs in which $\mu - g$ is sufficiently small, and thus is most interesting in the context of graphs in which μ is large but g is small. Heuristically, if a 2-edge-connected plane graph has a cycle C of short length, but μ is large, then in order to divide the interior and exterior of C into faces, some vertices must be placed far away from C in both the interior and exterior of C.

To see that this bound cannot be improved, it suffices to consider the odd cycle C_{2k+1}. There is a class of plane graphs $\{G_k : k \in \mathbb{N}, k \geq 2\}$, for which $g = 4$ and μ is arbitrarily large, that provide motivation for this conjecture. The graph G_k is constructed as follows: Let C be a cycle of length $2k$ embedded in the plane, and let u and v be vertices of C such that $d(u, v) = k$. To form G_k from C, add a vertex x to the interior of C, and a vertex y to the exterior of C, and make x and y adjacent to both u and v. Note that the graph G_k has girth 4, minimum face-degree $k + 2$ and diameter $k = \lceil \frac{2(k+2)-4-1}{2} \rceil$.

In Chapter 4, we solved the degree diameter problem for 2-edge-connected plane graphs that are $2D$-face-degree regular. However, as demonstrated by the discussion preceding Figure 4.7, this results does not hold if we replace the condition that the graph is $2D$-face-degree regular with the condition that the minimum face-degree is $2D$, which brings us to our next question.

Question 4: What is the maximum number of vertices in a 2-edge-connected plane graph of diameter D and minimum face-degree $2D$?

Keeping the example of Figure 4.7 in mind, we restate Conjecture 4.37:

Conjecture 8.3. *If a plane graph is 2-edge-connected, has diameter D, maximum degree Δ, order n and minimum face-degree $2D$, then:*

$$n \leq \Delta(D - 1) + 3,$$

and this bound is sharp.

In order to study the degree diameter problem for plane graphs in which each face is bounded by a cycle with constrained length, we introduce two functions $\mathcal{F}$ and $\mathcal{G}$. Define $\mathcal{F}(\rho, \Delta, D)$ as the maximum number of vertices in a 2-connected ρ-face-degree regular plane graph of maximum degree Δ and diameter D, and $\mathcal{G}(\mu, \Delta, D)$ to be the maximum number of vertices in a 2-connected plane graph of maximum degree Δ, diameter D and minimum face-degree μ. Note that $\mathcal{F}(3, \Delta, 2)$ has been determined completely by [35] and [41], and that $\mathcal{F}(4, \Delta, D)$ is studied in [12]. The results of Chapter 4 of this thesis have also determined $\mathcal{F}(2D, \Delta, D)$ and $\mathcal{G}(2D+1, \Delta, D)$.

Question 5: In general, what upper and lower bounds can be found for $\mathcal{F}(\rho, \Delta, D)$ and $\mathcal{G}(\mu, \Delta, D)$? The approach used to prove Theorem 10 in [12] appears promising — particularly if one considers ρ to be an even number.

If the diameter and maximum degree are fixed by $D = D_0$ and $\Delta = \Delta_0$, then $\mathcal{F}(\rho, \Delta_0, D_0)$ and $\mathcal{G}(\mu, \Delta_0, D_0)$ are functions of ρ and μ respectively.

Question 6: How do the functions $\mathcal{F}(\rho, \Delta_0, D_0)$ and $\mathcal{G}(\mu, \Delta_0, D_0)$ behave? In other words, how does the maximum possible order of the graph depend on the (minimum) face-degree?

Many edges are required to ensure that every face of a plane graph is bounded by a small cycle (for example, a plane graph of order n is a triangulation if and only if it has $3n-6$ edges), which in turn may raise the maximum degree of the graph. On the other hand, if every face is large compared to the graph's diameter, then the graph cannot have many faces, which may also bound the order of the graph. Combining these two heuristic observations, we present the following conjecture as a possible answer to Question 6:

Conjecture 8.4. *The functions* $\mathcal{F}(\rho, \Delta_0, D_0) : \{3, 4, \ldots, 2D_0+1\} \to \mathbb{N}$ *and* $\mathcal{G}(\mu, \Delta_0, D_0) : \{3, 4, \ldots, 2D_0+1\} \to \mathbb{N}$ *are unimodal (i.e., for some* k*, they increase on the interval* $[3, k]$ *and decrease on the interval* $[k, 2D_0+1]$*). Further, the value* k *at which they obtain their maximum depends only on* D_0*, not* Δ_0*, for* Δ_0 *sufficiently large.*

Theorem 6.18 describes the minimal $A-B$ separators in maximal planar graphs when A and B are connected subsets of the graph (these separators are chordless cycles that Jordan separate A and B). The proof of this relies on the fact that both A and B are connected: the minimal separators of disconnected subsets of maximal planar graphs are likely more complicated.

Question 7: Can we find a description of the minimal $A-B$ separators in a maximal planar graph when A and B are not connected?

Maximal planar graphs correspond to triangulations of spheres, and so we may also consider the minimal separators in triangulations of other surfaces.

Question 8: Given non-adjacent vertices u and v in a triangulation G of some surface, can we describe the minimal $u-v$ separators in G?

Theorem 6.21 in Chapter 6 shows that if a graph G is k-connected, and A is a set of vertices of the graph such that $N(A)$ is also k-connected, then $G-A$ is k-connected. Theorem 6.22 shows that the converse also holds for $k=1$ and $k=2$ in maximal planar graphs — thus we consider the following question:

Question 8: What conditions can be found for the converse of Theorem 6.21 to hold? In other words, for A a set of vertices of a graph G, when does $G-A$ being k-connected imply $N(A)$ is k-connected?

Theorem 6.26 demonstrates that the centre of a maximal planar graph may have arbitrarily many connected components, however there may be conditions that we can impose on the graph to ensure that its centre is connected.

Question 9: What are sufficient / necessary conditions for the centre of a (maximal) planar graph to be connected?

In Chapter 7, we introduced quasi-eccentricity, and showed that it generalised the notion of eccentricity. Given the central role of the concept of quasi-eccentricity in characterising which maximal planar graphs were centres of planar graphs, it may be worth investigating this notion further, which leads us to our final three questions:

Question 10: What else can be said about quasi-eccentricity?

Question 11: Theorem 7.3 states that H is an isometric, equi-eccentric plane subgraph of some plane graph only if each vertex of H is quasi-eccentric to the set of vertices on the boundary of some face of H. Can we prove similar results for classes of graphs other than plane graphs, by replacing the 'set of vertices on the boundary of some face' with some other, appropriately chosen subset of vertices?

Question 12: Corollary 7.29 characterises maximal planar graphs that are centres of planar graphs. Can this result be improved (for example, does the result hold for a larger class of planar graphs than just those that are maximal planar)?

In particular, we have the following conjecture regarding Question 12:

Conjecture 8.5. *Let H be a 2-connected plane graph. If every vertex of H is quasi-eccentric to the set of vertices on the boundary of some face of H, then there exists a plane graph G having H as both its centre and an isometric subgraph.*

Note that this conjecture is a generalisation of Corollary 7.29, and thus the conjecture holds if every face of H is bounded by a 3-cycle.

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Chapter 9

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