UNIVERSITY OF CAPE TOWN

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ASCENT, DESCENT, NULLITY AND DEFECT OF LINEAR OPERATORS

by

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Ascent, Descent, Nullity and Defect of Linear Operators.

PREFACE

The main object of this thesis is to study the relationships between the ascent, descent, nullity and defect of linear operators, denoted respectively by $\alpha(T)$, $\delta(T)$, $n(T)$ and $d(T)$. These concepts will be explained later. The known theory is largely contained in two papers, one by A.E. Taylor (13) and the other by M.A. Kaashoek (5). It was thus one of my tasks, in the preparation of this thesis, to integrate these two papers. However, a fair amount of material was also taken from Taylor's book (12) and from van Dulst (15). All this material was welded into a composite whole.

The proof of Lemma 9.3 is my own work, whilst the alternative proof of Lemma 4.5, pointed out in the remark after that lemma, is original. Also original are the remarks before Theorem 6.5(b), after Theorem 6.8 and after Theorem 9.7, as well as the examples in the Appendix to Chapter IV. The summary and discussion of the functional-analytic proof of Rouché's Theorem in the last chapter is my own work too.

The following results are improved versions of those in the original papers: Lemma 3.3, Theorem $5.3^{\#}$, Theorem 5.5, Theorem 6.8 and Theorem 7.1. Corollary 6.6 was also considerably shortened.

Important corrections were made in Lemma 3.4, Lemma 4.10 and Theorem $5.2^{\#}$, as well as in the definition of "completely reduced" in Chapter VII, whilst minor corrections were also made in Theorem 6.1, Theorem 7.1, Theorem 8.1 and Lemma 9.2.

Details - sometimes quite numerous - were added to the proofs of various results, notably Lemma 3.5, Theorem 4.8(b), Theorem 5.1, Theorem 6.5, Theorem 7.1(g), Theorem 8.1, Theorem 8.2, Lemma 9.2 and Theorem 9.7.

CHAPTER 1

Introduction

This thesis is intended to be a survey of nullity and defect of linear operators on the one hand, and ascent and descent on the other, and the relationships between these concepts. These quantities are of considerable use in the discussion of linear operators, e.g. compact operators. See Riesz and Sz.-Nagy (10), pages 217-219.

It is first necessary to explain the terms mentioned above. Let X and Y be vector spaces over the field of real or complex numbers. Let T be a linear operator with domain D(T) in X and range R(T) in Y. The null space of T, denoted by N(T), is the set $\{x \in D(T): Tx = 0.\}$ The dimension of N(T) is denoted by n(T) and called the nullity of T. The dimension of the quotient space Y/R(T) is called the defect of T, and denoted by d(T). In general, the dimension of a linear subspace M is denoted by dim M; it is 0 if $M = \{0\}$, a positive integer p if M contains exactly p linearly independent elements. Otherwise M is not finite-dimensional and we write dim $M = \infty$. We do not distinguish different infinite dimensions in this thesis. With this proviso, the dimension of a vector space X can also be defined as the cardinal number of any Hamel basis of X (all of which have the same cardinality). We now define Hamel basis:

Let H be a subset of the vector space X. H is a Hamel basis of X if

(i) H is a linearly independent set

(ii) H spans X.

To define ascent and descent we consider the case in which D(T) and R(T) are in the same linear space X. Once again, the considerations are all algebraic. We define T^n by induction, with $T^0 = I$, $T^1 = T$, and $T^n x = T(T^{n-1}x)$ where n is any positive integer.

If $n \geqslant 1$, $D(T^n) = \{x : x, Tx, \ldots, T^{n-1}x \in D(T)\}$, and

if $n \geqslant 2$, $D(T^n) = \{x : x \text{ and } Tx \in D(T^{n-1})\}$.

Let n and m be integers, $n \geqslant 0$, $m > 0$. Then $x \in D(T^{n+m})$ if and only if

$T^n x \in D(T^m)$, and in this case $T^m(T^n x) = T^{n+m}x$.

Observe $D(T^o) = X$.

In general we have $D(T^n) \subset D(T^{n-1})$; the inclusion may be proper.

We can then consider $R(T^n)$ and $N(T^n) = \{x \in D(T^n) : T^n x = 0\}$.

Clearly $N(T^o) = \{0\}$ and $R(T^o) = X$.

Theorem 1.1 On Ascent.

$N(T^n) \subset N(T^{n+1})$, $n = 0,1,2,$ ---

If equality holds for $n = k$, then it holds for all $n \geqslant k$.

Proof:

$$\begin{aligned} x \in N(T^n) &\Rightarrow T^n x = 0 \\ &\Rightarrow T^{n+1}x = 0 \\ &\Rightarrow x \in N(T^{n+1}) \rightarrow \end{aligned}$$

Suppose that there exists an integer k such that $N(T^k) = N(T^{k+1})$.

Let $x \in N(T^{k+2})$. Then $x \in D(T^{k+2})$ and $T^{k+2}x = 0$.

Hence $Tx \in D(T^{k+1})$ and $T^{k+1}(Tx) = 0$,

so that $Tx \in N(T^{k+1}) = N(T^k)$

Then $T^k(Tx) = T^{k+1}x = 0$

Therefore $x \in N(T^{k+1})$.

Therefore $N(T^{k+2}) \subset N(T^{k+1})$

Since the reverse inclusion also holds, $N(T^{k+1}) = N(T^{k+2})$.

The second assertion of the theorem now follows by induction.

Definition. If there is some integer $n \geqslant 0$ such that $N(T^n) = N(T^{n+1})$, the smallest such integer is called the ascent of T and denoted by $\alpha(T)$. We say T has finite ascent. If no such integer exists, we set $\alpha(T) = \infty$ and say that T has infinite ascent.

Theorem 1.2: On Descent.

$R(T^{n+1}) \subset R(T^n)$, $n = 0,1,2,---$.

If equality holds for $n = k$, then it holds for all $n \geqslant k$.

Proof:

Let $y \in R(T^{n+1})$. Then $y = T^{n+1}x$, where $x \in D(T^{n+1})$

Then $Tx \in D(T^n)$ and $y = T^n(Tx)$, so that $y \in R(T^n)$.

Therefore $R(T^{n+1}) \subset R(T^n)$ and the first statement is proved.

Before proving the second statement, we observe that

(1-1) $R(T^{n+1}) = T\{R(T^n) \cap D(T)\}$, $n = 0,1,2, ---$.

Now suppose that $R(T^{k+1}) = R(T^k)$ for some k, and suppose $y \in R(T^{k+1})$.

By (1-1) we can write $y = Tx$, where $x \in R(T^k) \cap D(T) = R(T^{k+1}) \cap D(T)$.

Thus again by (1-1), $y \in R(T^{k+2})$.

Therefore $R(T^{k+1}) \subset R(T^{k+2})$.

Since the reverse inclusion also holds, $R(T^{k+1}) = R(T^{k+2})$

The second assertion of the theorem now follows by induction.

Definition: If there is some integer $n \geqslant 0$ such that $R(T^{n+1}) = R(T^n)$, the smallest such integer is called the descent of T and denoted by $\delta(T)$. We say T has finite descent. If no such integer exists, we set $\delta(T) = \infty$ and say that T has infinite descent. In this case $R(T^{n+1})$ is always a proper subspace of $R(T^n)$.

Remark: $\alpha(T) = 0$ if and only if T^{-1} exists.

$\delta(T) = 0$ if and only if $R(T) = X$.

It is now necessary to outline the scope of this thesis, and to say what will, and what will not, be included. Our purpose is to explain and clarify the theory relating to $n(T)$, $d(T)$, $\alpha(T)$, and $\delta(T)$, especially for the cases in which these quantities are finite. We follow mainly the researches of A.E. Taylor (13) and M.A. Kaashoek (5).

Chapters 1 to 8 are entirely algebraic, while Chapter 9 does involve topology. In Chapter 2 we gather a few simple facts on subspaces of a linear space. These facts are presented because they will be frequently used later. Chapter 3 is devoted to the study of the relationships between the linear subspaces $N(T^k)$, $D(T^k)$, and $R(T^k)$ for $k = 0,1,2,$ ---. A number of lemmas is presented to show how these subspaces are situated in the space X. This leads to a better understanding of the relationships between the numbers $n(T)$ and $d(T)$ on the one hand, and $\alpha(T)$ and $\delta(T)$ on the other, in the later chapters.

The emphasis is placed on the word "relationships", and thus well-known results, such as $n(T^*) = d(T)$ and $n(T) \leq d(T^*)$, (where T^* is the adjoint operator), are not discussed. Anyway, these results require a topology, and a slightly different definition of $d(T)$.

In general we do not require $D(T) = X$. This enables the theory to be applied to the case of closed operators which need not be bounded - a topic beyond the scope of this thesis.

The numbers $n(T)$, $d(T)$, $\alpha(T)$ and $\delta(T)$ have extensive applications to Spectral Theory, and if this thesis were more ambitious, Chapter 9 would be followed by chapters on Spectral Theory for closed linear operators in Banach Spaces. However, Spectral Theory is too extensive a topic to be covered in a thesis as modest as this one, but references to relevant research papers will be given at the end.

Likewise, Perturbation Theory for nullity and defect, due to its ramifications, especially in locally convex topological vector spaces, (See van Dulst (14)) has had to be avoided as being beyond the scope of this thesis, except for van Dulst's proof of Rouché's Theorem by functional-analytic methods (15), which may serve at the end of this thesis to whet the reader's appetite for more.

CHAPTER II

Complementary subspaces and quotient spaces.

Let X be a linear space.

Two linear subspaces M_1, M_2 in X are called complementary if

(i) $M_1 \cap M_2 = \{0\}$

(ii) span $(M_1 \cup M_2) = X$ [Equivalently, $M_1+M_2 = X$ where $M_1+M_2 = \{x_1+x_2 : x_1 \in M_1 \& x_2 \in M_2.\}$]

In this case every element x in X has a unique representation $x = x_1+x_2$ where $x_i \in M_i$; the uniqueness follows from the fact that $M_1 \cap M_2 = \{0\}$. We write $X = M_1 \oplus M_2$ and call X the direct sum of M_1 and M_2. We also say M_1 is a complement of M_2, and conversely.

If we define operators P_i by setting $P_i x = x_i$ (where $x = x_1+x_2$ as above), then P_i is a linear operator which is a projection ($P_i^2 = P_i$). We call P_1 the projection of X on M_1 along M_2, and P_2 the projection of X on M_2 along M_1.

Note that $M_1 = R(P_1)$ and $M_2 = N(P_1)$.

If P is any linear operator which is a projection, R(P) and N(P) are complementary subspaces. If M is any subspace of X, it can be shown that there exists a projection P such that R(P) = M. Hence there exists a subspace, namely N(P), which is complementary to M. (see Theorem 4.8-A in Taylor (12)).

If M is a subspace of X, the quotient space X/M, also denoted by $\frac{X}{M}$ (X modulo M) is defined in a well-known manner. (see §3.14 in Taylor (12)). The quotient space is also a linear space. Its elements are certain equivalence classes x+M in X. If $x \in X$, the equivalence class which contains x is denoted by $[x]$. The mapping $\varphi : X \rightarrow X/M$ defined by $\varphi(x) = [x]$ is called the canonical mapping of X onto the quotient space.

Definition: The linear spaces X_1 and X_2 are said to be isomorphic whenever there exists a one-one linear mapping from X_1 onto X_2. For abbreviation we use the symbol $X_1 \cong X_2$ to denote that X_1 and X_2 are isomorphic.

If M and N are complementary subspaces, it is easily seen that N and X/M are isomorphic as linear spaces. For, let P be the projection of X onto M along N. If f is the restriction of φ to N, we will prove that f maps N isomorphically onto X/M. The mapping is onto, because if $[x]$ is any element of X/M, $x-Px \in N$ and $f(x-Px) = [x]$. To verify this we observe that $x-(x-Px) = Px \in M$, and hence $[x-Px] = [x]$. But then $f(x-Px) = [x-Px] = [x]$. The mapping is one-one, because if $x \in N$ and $f(x) = 0$, this means $[x] = 0$, or $x \in M$, whence $x = 0$, since $M \cap N = \{0\}$.

Observe that $M = \{0\} \Rightarrow \frac{X}{M} \cong X$.

and $M = X \Rightarrow \frac{X}{M} \cong \{0\}$.

Obviously, two isomorphic linear spaces have the same dimension. This statement has a partial converse as follows: If the linear spaces X_1 and X_2 have the same finite dimension, then X_1 and X_2 are isomorphic. In passing we may mention that if two linear spaces have bases of the same transfinite cardinality, then they are also isomorphic.

Lemma 2.1: Suppose X_1 and X_2 are subspaces of X such that $X_1 \subset X_2$. Then

$$\text{(2-1)} \qquad \dim \frac{X}{X_1} = \dim \frac{X}{X_2} + \dim \frac{X_2}{X_1}$$

(The understanding, in case a dimension is infinite, is that $\infty + p = p + \infty = \infty$, if p is a non-negative integer or ∞).

Proof:

Let Y_1 be a complement to X_1 in X_2, so that $X_2 = X_1 \oplus Y_1$.

Let Y_2 be a complement to X_2 in X, so that $X = X_2 \oplus Y_2$.

Then it is clear that $Y_1 \oplus Y_2$ is a complement to X_1 in X.

Therefore $\frac{X}{X_1} \cong Y_1 \oplus Y_2$, $\frac{X}{X_2} \cong Y_2$ and $\frac{X_2}{X_1} \cong Y_1$.

Clearly $\dim (Y_1 \oplus Y_2) = \dim Y_1 + \dim Y_2$, with the indicated understanding in case any of the dimensions is infinite, viz $\infty + p = p + \infty = \infty$ if $p = 0,1,2, ---, \infty$.

This gives the required result.

Lemma 2.2: Let Y and Z be subspaces of X such that $Y \cap Z = \{0\}$ and $\dim \frac{X}{Z} \leqslant \dim Y < \infty$. Then $X = Y \oplus Z$.

Proof:

Let $W = Y \oplus Z$. Then $Z \subset W \subset X$, so that, by Lemma 2.1,

$$\dim \frac{X}{Z} = \dim \frac{X}{W} + \dim \frac{W}{Z} . \tag{2-2}$$

Also, $\dim Y = \dim \frac{W}{Z}$ and from (2-2) it appears that

$$\dim \frac{W}{Z} \leq \dim \frac{X}{Z} \quad \text{i.e. } \dim Y \leqslant \dim \frac{X}{Z}$$

Hence $\dim \frac{X}{Z} = \dim Y = \dim \frac{W}{Z}$.

Thus, again from (2-2), $\dim \frac{X}{W} = 0$, whence $X = W = Y \oplus Z$.

Lemma 2.3: Let M and N be linear subspaces in the linear space X.

Then $\frac{M}{M \cap N} \cong \frac{M+N}{N}$

Proof:

Let $[x]$ denote a coset in the quotient space $(M+N)/N$.

Define for each $m \in M$

$$Jm = [m].$$

Then J is a linear mapping from M into $(M+N)/N$.

If $[x]$ is an element in $(M+N)/N$, then $x = m+z$ where $m \in M$ and $z \in N$, and hence $[x] = [m]$. This shows that J is a linear mapping onto $(M+N)/N$. Combining this fact with the fact that the kernel of J is the subspace $M \cap N$, we arrive at the required result.

Lemma 2.4: Let M_1, M_2 and N be linear subspaces of the linear space X. Suppose that $M_1 \subset M_2$. Then

$$\dim \frac{M_1}{M_1 \cap N} \leqslant \dim \frac{M_2}{M_2 \cap N} . \tag{2-3}$$

Proof:

Let $[x]$ denote a coset in the quotient space $M_2/(M_2 \cap N)$. Then the mapping J defined by

$$Jm = [m]$$

for each $m \in M_1$, is a linear mapping from M_1 into $M_2/(M_2 \cap N)$. The kernel of J is the subspace $M_1 \cap N$. Hence $M_1/(M_1 \cap N)$ is isomorphic with a subspace in $M_2/(M_2 \cap N)$. But then formula (2-3) is true.

Lemma 2.5: Let M_1, M_2 and N be linear subspaces of the linear space X. Suppose that $M_1 \subset M_2$, and that

$$\dim \frac{M_1}{M_1 \cap N} = \dim \frac{M_2}{M_2 \cap N} < \infty .$$

Then $M_1+N = M_2+N$.

Proof:

Let J be defined as in the proof of the preceding lemma.

Then $\dim \{M_1/(M_1 \cap N)\} = \dim JM_1$

and hence $\dim \{M_2/(M_2 \cap N)\} = \dim JM_1 < \infty$.

But then J is a mapping onto $M_2/(M_2 \cap N)$.

Take an element $x \in M_2$. Since J is a mapping onto $M_2/(M_2 \cap N)$, there exists an element $m \in M_1$ such that $Jm = [m] = [x]$. But then $x = m+z$, with $z \in N$. This shows that $M_1+N \subset M_2+N \subset M_1+N$, and hence $M_1+N = M_2+N$.

CHAPTER III

The subspaces $N(T^k)$, $D(T^k)$ and $R(T^k)$

In this chapter T will be a linear operator with domain D(T) and range R(T) in the same linear space X.

The chapter is devoted to the study of the relationships between the subspaces $N(T^k)$, $D(T^k)$ and $R(T^k)$ $(k = 0,1,2,---)$. First of all, we recall the following formulae, used in Chapter I:

$$N(T^k)\subset N(T^{k+1}),\ D(T^k)\supset D(T^{k+1}),\ R(T^k)\supset R(T^{k+1}),\ \text{for } k = 0,1,2,---$$

We prove the following lemmas :

Lemma 3.1: For $k = 0,1,2,---$ and $i = 0,1,2,---$, we have

$$\frac{N(T^{i+k})}{N(T^i)} \cong N(T^k)\cap R(T^i)$$

Proof:

Define for each x in $N(T^{i+k})\subset D(T^i)$

$$Jx = T^i x.$$

Then J is a linear operator from $N(T^{i+k})$ into the linear space $N(T^k)\cap R(T^i)$.

Next, we prove that J is onto. Let $y\in N(T^k)\cap R(T^i)$

Then $y = T^i x$ for some $x\in D(T^i)$, and $T^i x \in N(T^k)\subset D(T^k)$.

This implies that $x\in D(T^{i+k})$ and $T^{i+k}x = T^k(T^i x) = 0$.

Hence $x\in N(T^{i+k})$, and $Jx = y$. But then we have proved that J is a mapping onto $N(T^k)\cap R(T^i)$.

Since the kernel of J is $N(T^i)$, the last fact implies that

$$\frac{N(T^{i+k})}{N(T^i)} \cong N(T^k)\cap R(T^i).$$

Lemma 3.2: For $k = 0,1,2,---$ and $i = 0,1,2,---$, we have

$$\frac{R(T^i)}{R(T^{i+k})} \cong \frac{D(T^i)}{\{R(T^k)+N(T^i)\}\cap D(T^i)}.$$

Proof:

Let $[y]$ denote any coset in the quotient space $R(T^i)/R(T^{i+k})$. Define for each x in $D(T^i)$

$$Jx = [T^i x].$$

Obviously, J is a linear operator from $D(T^i)$ onto $R(T^i)/R(T^{i+k})$.

If $Jx = 0$, then $T^i x = T^{i+k} z$ for some $z \in D(T^{i+k})$, and hence $x - T^k z \in N(T^i)$.

This shows that

$$N(J) \subset \{R(T^k) + N(T^i)\} \cap D(T^i).$$

Conversely, if $x \in \{R(T^k) + N(T^i)\} \cap D(T^i)$, then $T^i x \in R(T^{i+k})$ and hence $Jx = 0$. This shows

$$N(J) \supset \{R(T^k) + N(T^i)\} \cap D(T^i).$$

But then $N(J) = \{R(T^k) + N(T^i)\} \cap D(T^i)$ and

$$R(T^i)/R(T^{i+k}) \cong D(T^i)/N(J).$$

This completes the proof.

Lemma 3.3: For $k = 0,1,2,\text{---}$ and $i = 0,1,2,\text{---}$, we have

$$\frac{D(T^i)}{D(T^{i+k})} \cong \frac{R(T^i)}{D(T^k) \cap R(T^i)}$$

Proof:

Let $[y]$ denote any coset in $R(T^i)/\{D(T^k) \cap R(T^i)\}$. Define for each x in $D(T^i)$

$$Jx = [T^i x]$$

It then follows that J is a linear operator from $D(T^i)$ onto $R(T^i)/\{D(T^k) \cap R(T^i)\}$ with $N(J) = D(T^{i+k})$. Hence

$$\frac{D(T^i)}{D(T^{i+k})} = \frac{D(T^i)}{N(J)} \cong R(J) = \frac{R(T^i)}{D(T^k) \cap R(T^i)}$$

Lemma 3.4: For $i = 0,1,2,\text{---}$, we have

$$\frac{N(T^{i+1})}{\{N(T^i)+R(T)\} \cap N(T^{i+1})} \cong \frac{N(T) \cap R(T^i)}{N(T) \cap R(T^{i+1})} \tag{3-1}$$

Proof:

Let $[y]$ denote any coset in the quotient space

$$\frac{N(T)\cap R(T^i)}{N(T)\cap R(T^{i+1})} \tag{3-2}$$

Define for each $x \in N(T^{i+1})$

$$Jx = [T^i x]$$

Then J is a linear operator into the space (3-2). Hence, to prove (3-1), it will suffice to show that J is a mapping onto (3-2), and that

$$N(J) = \{N(T^i) + R(T)\} \cap N(T^{i+1}) \ .$$

If $[y]$ is in (3-2), then $y = T^i x \in N(T)$ for some $x \in D(T^i)$ But then $x \in N(T^{i+1}) = D(J)$ and $Jx = [T^i x] = [y]$. This shows that J is a mapping onto (3-2).

Let $x \in N(J)$. Then $T^i x \in N(T)\cap R(T^{i+1})$, and $T^i x = T^{i+1} z$ for some $z \in D(T^{i+1})$. But then $x - Tz \in N(T^i)$, and so $x \in N(T^i) + R(T)$. This shows

$$N(J) \subset \{N(T^i) + R(T)\} \cap N(T^{i+1}). \tag{3-3}$$

Conversely, let $x \in \{N(T^i) + R(T)\} \cap N(T^{i+1})$.
Then $x = n + Tz$ for some $n \in N(T^i)$ and some $z \in D(T)$.
Since $Tz = x - n \in N(T^{i+1}) \subset D(T^{i+1}) \subset D(T^i)$, we have $z \in D(T^{i+1})$.
But then $T^i x = T^i n + T^{i+1} z = T^{i+1} z$ and hence $Jx = 0$.
Combining this with (3-3), we obtain $N(J) = \{N(T^i) + R(T)\} \cap N(T^{i+1})$.
This completes the proof.

Lemma 3.5: For i = 0,1,2,---, we have

$$\dim \frac{N(T)}{N(T)\cap R(T^i)} = \dim \frac{N(T^i)}{R(T)\cap N(T^i)}$$

Proof:

Firstly, we observe that it follows from Lemmas 2.3 and 3.4 that

$$\frac{N(T^{i+1}) + R(T)}{N(T^i) \quad + R(T)} \cong \frac{N(T)\cap R(T^i)}{N(T)\cap R(T^{i+1})} \quad \text{for } i = 0,1,2,---$$

But then $$\dim \frac{N(T)}{N(T)\cap R(T^i)} = \sum_{k=o}^{i-1} \dim \frac{N(T)\cap R(T^k)}{N(T)\cap R(T^{k+1})}$$ by Lemma 2.1 extended

$$= \sum_{k=o}^{i-1} \dim \frac{N(T^{k+1}) + R(T)}{N(T^k) + R(T)}$$

$$= \dim \frac{N(T^i) + R(T)}{R(T)}$$ by Lemma 2.1 again.

And so, once again by Lemma 2.3,

$$\dim \frac{N(T)}{N(T)\cap R(T^i)} = \dim \frac{N(T^i)}{R(T)\cap N(T^i)} \qquad \text{Q.E.D.}$$

Remark:

If for some non-negative integer i, either side of the last equation is finite, then Lemma 3.5 implies that

$$\frac{N(T)}{N(T)\cap R(T^i)} \cong \frac{N(T^i)}{R(T)\cap N(T^i)} \tag{3-4}$$

(3-4) is also true if the dimension of each side equals the same transfinite cardinal number.

CHAPTER IV

Ascent and Descent.

In this chapter we follow mainly Taylor's paper (13).

Throughout this chapter X denotes a linear space and T denotes a linear operator with domain and range in X. Except where we explicitly assume that $D(T) = X$ it is possible that $D(T) \neq X$. There is no topology; all considerations are algebraic.

Lemma 4.1: Suppose there exists a non-negative integer N such that

$n(T^k) \leqslant N$ when $k = 0,1,2,\text{---}$. Then $\alpha(T) \leqslant N$.

Proof:

The conclusion is obvious if $\alpha(T) = 0$, so we can suppose $\alpha(T) > 0$. Suppose, for some k, that $N(T^k) \neq N(T^{k+1})$. Then

$$0 = n(T^0) < n(T) < \dots < n(T^{k+1})$$

Then $1 + k \leqslant n(T^{k+1}) \leqslant N$.

It follows from this that $\alpha(T) \leqslant N$. Q.E.D.

Lemma 4.2: Suppose there exists a non-negative integer N such that

$d(T^k) \leqslant N$ when $k = 0,1,2,\text{---}$. Then $\delta(T) \leqslant N$.

Proof:

We can assume that $\delta(T) > 0$.

Suppose, for some k, that $R(T^k) \neq R(T^{k+1})$

Then we see from Lemma 2.1 that

$$0 = d(T^0) < d(T) < \dots < d(T^{k+1})$$

and hence $1+k \leqslant d(T^{k+1}) \leqslant N$

This implies that $\delta(T) \leqslant N$. Q.E.D.

Lemma 4.3: (a) If $n(T)<\infty$, then $n(T^k) \leqslant kn(T)$, $k = 1,2,3,\text{---}$.

(b) If $d(T)<\infty$, then $d(T^k) \leqslant kd(T)$, $k = 1,2,3,\text{---}$.

Proof of (a):

Since $N(T^k) \subset N(T^{k+1})$, there is a subspace Y which is complementary to $N(T^k)$ in $N(T^{k+1})$, so that $N(T^{k+1}) = N(T^k) \oplus Y$ and $n(T^{k+1}) = n(T^k) + \dim Y$. We shall prove that $\dim Y \leqslant n(T)$. From this we obtain the desired conclusion by induction.

Now let $x_1,\ldots,x_p$ be linearly independent elements of Y. Since $Y \subset N(T^{k+1})$, the elements $T^k x_1,\ldots,T^k x_p$ are in $N(T)$. They are linearly independent. For, suppose that

$$\sum_{i=1}^{p} c_i T^k x_i = 0$$

Then $\sum_{i=1}^{p} c_i x_i \in Y \cap N(T^k) = \{0\}$, so that $\sum_{i=1}^{p} c_i x_i = 0$

But then $c_1 = \ldots = c_p = 0$ whence $p \leqslant n(T)$ whence $\dim Y \leqslant n(T)$.

Proof of (b):

Let $X = R(T) \oplus N$, so that $d(T) = \dim N$. For given k let

$$D(T^k) = \left[D(T^k) \cap R(T)\right] \oplus M_k .$$

We shall prove that $\dim M_k \leqslant \dim N$. Suppose, to the contrary, that $m = \dim M_k > \dim N$. Let $x_1,\ldots, x_m$ be linearly independent elements of M_k. Write $x_i = u_i+v_i$ where $u_i \in R(T)$, $v_i \in N$. Then $v_1,\ldots,v_m$ are linearly dependent, so that there exist constants $a_1,\ldots,a_m$ (not all zero) such that $\sum_{i=1}^{m} a_i v_i = 0$. Hence $\sum_{i=1}^{m} a_i x_i = \sum_{i=1}^{m} a_i u_i \in M_k \cap R(T)$.

But $M_k \cap R(T) = \{0\}$, from the definition of M_k as the complement to $D(T^k) \cap R(T)$ in $D(T^k)$. Therefore $\sum_{i=1}^{m} a_i x_i = 0$, in contradiction to the choice of the x_i's.

Now let Y_k be a subspace complementary to $R(T^{k+1})$ in $R(T^k)$. It follows from Lemma 2.1 that $d(T^{k+1}) = d(T^k) + \dim Y_k$. If we prove that $\dim Y_k \leqslant d(T)$, we can use induction to conclude that $d(T^j) \leqslant jd(T)$ for every j.

Let $y_1,\dots,y_p$ be linearly independent elements in Y_k. We can write $y_i = T^k w_i$, because $Y_k \subset R(T^k)$. Then, since $w_i \in D(T^k)$, we can write $w_i = Tr_i + s_i$ where $Tr_i \in D(T^k) \cap R(T)$ and $s_i \in M_k$. Thus $y_i = T^{k+1}r_i + T^k s_i$. We see from this that $[y_i]$ and $[T^k s_i]$ are the same element of $R(T^k)/R(T^{k+1})$. These elements (for $i = 1,2,\text{---},p$) are linearly independent, as a consequence of the fact that $y_1,\dots,y_p$ are linearly independent elements of Y_k. Now, the elements $[T^k s_1]$,..., $[T^k s_p]$ are obtained by applying a linear mapping to the elements $s_1,\dots,s_p$ of M_k. Hence $s_1,\dots,s_p$ are linearly independent, and $p \leqslant \dim M_k$. It follows that $\dim Y_k \leqslant \dim M_k \leqslant d(T)$, and our proof is complete.

<u>Corollary 4.4</u>: For any non-negative integer k

(a) $n(T^k) \leqslant \alpha(T)n(T)$

(b) $d(T^k) \leqslant \delta(T)d(T)$.

<u>Proof of (a)</u>:

We firstly observe that $\alpha(T) = 0$ if and only if $n(T) = 0$. Hence the product $\alpha(T)n(T)$ is well defined. We need only consider the case where both $\alpha(T)$ and $n(T)$ are finite. Let $\alpha(T) = p$. Then $n(T^k) \leqslant n(T^p)$ for any k, and if we show $n(T^k) \leqslant kn(T)$ for every non-negative integer k, the result will follow,

since $\quad n(T^k) \leqslant n(T^p) \leqslant pn(T) = \alpha(T)n(T)$,

and $\quad n(T^k) \leqslant kn(T)$ is precisely what we have already proved in Lemma 4.3.

Proof of (b):

Since $\delta(T) = 0$ if and only if $d(T) = 0$, the product $\delta(T)d(T)$ is well defined and we need only consider the case when $\delta(T)$ and $d(T)$ are finite. Again it suffices to invoke Lemma 4.3, viz. for each positive integer k, $d(T^k) \leq kd(T)$.

For, let $\delta(T) = q$. Then for any k, $d(T^k) \leq d(T^q) \leq qd(T) = \delta(T)d(T)$. Q.E.D.

Lemma 4.5: (A necessary and sufficient condition for $\alpha(T) \leq p$.)

(a) If $N(T) \cap R(T^p) = \{0\}$ for a certain non-negative integer p, then $\alpha(T) \leq p$.

(b) If $\alpha(T) \leq p$, then $N(T^k) \cap R(T^p) = \{0\}$ when $k = 1,2,---$.

Proof of (a):

Assuming $N(T) \cap R(T^p) = \{0\}$, we shall show that $N(T^{p+1}) \subset N(T^p)$. This will imply that $\alpha(T) \leq p$.

If $x \in N(T^{p+1})$, then $T^{p+1}x = T(T^p x) = 0$, so that $T^p x \in N(T) \cap R(T^p) = \{0\}$, and hence $T^p x = 0$ or $x \in N(T^p)$.

Proof of (b):

Suppose that $\alpha(T) \leq p$, and let $x \in N(T^k) \cap R(T^p)$ where k is a positive integer. Then $x = T^p u$ for some u, and $T^k x = 0$, so that $u \in N(T^{p+k})$. But $N(T^{p+k}) = N(T^p)$, and therefore $T^p u = 0$. But $T^p u = x$. This proves that $N(T^k) \cap R(T^p) = \{0\}$. Q.E.D.

Remark: This lemma can also be deduced from Lemma 3.1.

Lemma 4.6: (a) Suppose that, for some $q \geq 0$ and some $k \geq 1$, there exists a subspace M_k such that $M_k \subset N(T^q)$, $M_k \cap R(T^k) = \{0\}$, and $D(T^q) = [D(T^q) \cap R(T^k)] \oplus M_k$. Then $\delta(T) \leq q$.

(b) If $\delta(T) \leq q$, then to each $k \geq 1$ corresponds a subspace M_k such that $M_k \subset N(T^q)$, $M_k \cap R(T^k) = \{0\}$ and $D(T^q) = [D(T^q) \cap R(T^k)] \oplus M_k$.

Proof of (a):

It will suffice to prove that $R(T^q) \subset R(T^{q+k})$, for this implies that $R(T^{q+1}) = R(T^q)$.

An element of $R(T^q)$ has the form $T^q x$, where $x \in D(T^q)$.
By hypothesis we can write $x = T^k u+v$, where $T^k u \in D(T^q)$ and $v \in M_k$.
Since $M_k \subset N(T^q)$, it follows that $T^q x = T^{q+k} u \in R(T^{q+k})$.

Proof of (b):

Let N_k be a complement to $D(T^q) \cap R(T^k)$ in $D(T^q)$, so that $D(T^q) = \left[D(T^q) \cap R(T^k)\right] \oplus N_k$. Let H be a Hamel basis for N_k.
Now, $N_k \subset D(T^q)$; hence, if $v \in H$, $T^q v \in R(T^q)$.
But $R(T^q) = R(T^{q+k})$, and so $T^q v = T^{q+k} w$ for some w.
It follows that $v-T^k w \in N(T^q)$. For each v in H let one corresponding w be chosen in the manner just indicated, and let M_k be the subspace generated by the set of all the elements $v-T^k w$.
Observe that $M_k \subset N(T^q)$.

To verify that $M_k \cap R(T^k) = \{0\}$, suppose $y \in M_k \cap R(T^k)$.
Then y can be expressed as $T^k z$ and also as a finite linear combination

$$\sum_i c_i(v_i - T^k w_i)$$

Thus $\sum_i c_i v_i = T^k(\sum_i c_i w_i + z) \in N_k \cap R(T^k) = \{0\}$, by definition of N_k.

Hence $\sum_i c_i v_i = 0$. But the v_i's are from the basis H, and hence the c_i's are all 0, and it follows that $y = 0$.

We have now to show that $D(T^q) = \left[D(T^q) \cap R(T^k)\right] \oplus M_k$.
Suppose $x \in D(T^q)$. We can write $x = T^k u+v$, where $T^k u \in D(T^q)$ and $v \in N_k$.
We can express v as a finite linear combination $\sum_i a_i v_i$, where each $v_i \in H$.
Let w_i correspond to v_i in the manner indicated earlier.

Then $$x = T^k u + \sum_i a_i(v_i - T^k w_i) + \sum_i a_i T^k w_i$$

$$x = T^k(u + \sum_i a_i w_i) + \sum_i a_i(v_i - T^k w_i)$$

This representation has the proper form to show that

$$D(T^q) \subset \left[D(T^q) \cap R(T^k)\right] \oplus M_k$$

Since the inclusion relation the other way is evident, the proof is complete. Q.E.D.

Theorem 4.7: Suppose that $p = \alpha(T)$ and $q = \delta(T)$ are finite. Then

(4-1) $\alpha(T) \leq \delta(T)$

and we have equality in (4-1) if and only if T has the additional property

(4-2) $D(T^p) \subset R(T) + D(T^q)$.

Remark: (1) In equation (4-1) we need $\alpha(T) < \infty$. See the example on page 32 of Taylor (13) where $\delta(T) < \infty$ but $\alpha(T) = \infty$.

(2) If $D(T) = X$, the linear operator T always satisfies the condition (4-2).

Proof:

Suppose that $p = \alpha(T) > \delta(T) = q$. Then $R(T^p) = R(T^q)$, and hence we have by Lemma 3.1

$$\begin{aligned} 0 &= \dim \left\{N(T^{p+1})/N(T^p)\right\} = \dim \left\{N(T) \cap R(T^p)\right\} \\ &= \dim \left\{N(T) \cap R(T^q)\right\} = \dim \left\{N(T^{q+1})/N(T^q)\right\} . \end{aligned}$$

But this implies $p = \alpha(T) \leq q$, contradicting the assumption $p > q$. Hence we must have $p \leq q$.

If $p = q$, then trivially $D(T^p) \subset R(T) + D(T^q)$.

Conversely, suppose that T has the additional property (4-2). Since $q = \delta(T) < \infty$, Lemma 3.2 implies that

$$D(T^q) \subset R(T) + N(T^q).$$

Combining this fact with (4-2) we obtain

$$D(T^p) \subset R(T) + N(T^q)$$

Since $p \leq q$, the null space $N(T^p) = N(T^q)$, and so

$$D(T^p) \subset R(T) + N(T^p).$$

But then, by Lemma 3.2,

$$\frac{R(T^p)}{R(T^{p+1})} \cong \frac{D(T^p)}{\{R(T) + N(T^p)\} \cap D(T^p)} = [0]$$

and hence $R(T^p) = R(T^{p+1})$. This implies $p \geq q$.

Combining this fact with $p \leq q$, we obtain $\alpha(T) = p = q = \delta(T)$.

Theorem 4.8:

(a) Suppose, for the positive integer r, that $N(T^r) \cap R(T^r) = \{0\}$ and that $D(T^r) = [D(T^r) \cap R(T^r)] \oplus N(T^r)$. Then $\alpha(T) \leq r$ and $\delta(T) \leq r$.

(b) Suppose that $\alpha(T)$ and $\delta(T)$ are finite. Let $\delta(T) = q$. Then

$$N(T^q) \cap R(T^q) = \{0\} \text{ and}$$

$$D(T^q) = [D(T^q) \cap R(T^q)] \oplus N(T^q). \tag{4-3}$$

Let T_1 be the operator in $R(T^q)$ defined by $D(T_1) = D(T) \cap R(T^q)$, $T_1x = Tx$ if $x \in D(T_1)$. Then T_1 is a one-one mapping of $D(T_1)$ onto all of $R(T^q)$.

Proof of (a):

Since $N(T) \subset N(T^r)$, the hypothesis implies that $\alpha(T) \leq r$, because of Lemma 4.5(a).

To prove that $\delta(T) \leq r$ it will suffice to show that $R(T^r) \subset R(T^{2r})$. Consider $T^r x \in R(T^r)$ where $x \in D(T^r)$.

Write $x = x_1 + x_2$, where $x_1 \in D(T^r) \cap R(T^r)$ and $x_2 \in N(T^r)$.

Then $x_1 = T^r u$ for some u, and $T^r x = T^{2r} u \in R(T^{2r})$.

Proof of (b):

Let $p = \alpha(T)$. We know that $p \leqslant q$ (Theorem 4.7), and hence $N(T^q) \cap R(T^q) \subset N(T^q) \cap R(T^p)$. Lemma 4.5(b) then shows that $N(T^q) \cap R(T^q) = \{0\}$. To prove (4-3) it now suffices to show that $D(T^q) \subset [D(T^q) \cap R(T^q)] \oplus N(T^q)$. This is clear if $q = 0$. If $q \geqslant 1$ it follows from Lemma 4.6(b) with $k = q$.

Now consider T_1. To see that T_1^{-1} exists, suppose $y = T^q x \in D(T) \cap R(T^q) = D(T_1)$ and $T_1 y = Ty = 0$.
Then $T^{q+1}x = 0$, so $x \in N(T^{q+1})$.
But $\alpha(T) \leq q$, so $N(T^{q+1}) = N(T^q)$ and thus $y = T^q x = 0$.
Hence the mapping by T_1 is one-one.

To see that $R(T_1) = R(T^q)$, suppose $y = T^q x$ is any element of $R(T^q)$. Since $R(T^q) = R(T^{q+1})$, we can write $y = T^{q+1}u$, or $y = T(T^q u)$, where $T^q u \in D(T) \cap R(T^q)$. Hence $y = T_1(T^q u) \in R(T_1)$. So $R(T^q) \subset R(T_1)$ and clearly

$$R(T_1) = T_1 D(T_1) = T_1 [D(T) \cap R(T^q)] \subset T [D(T) \cap R(T^q)] = R(T^{q+1}) = R(T^q).$$

Remark: As a special case of Theorem 4.8(b), we observe that if $D(T) = X$, and if both $\alpha(T)$ and $\delta(T)$ are finite, they are equal (Theorem 4.7 and remark) and (4-3) becomes, with $q = \delta(T)$,

$$X = R(T^q) \oplus N(T^q). \tag{4-4}$$

In this case T, when restricted to $R(T^q)$, is a one-one mapping of $R(T^q)$ onto all of itself.

In the next two lemmas we consider powers of the operator $\lambda I - T$, where λ is a complex number and I is the identity operator. For simplicity we shall write $\lambda - T$ in place of $\lambda I - T$. It is understood that $D(\lambda - T) = D(T)$.

Lemma 4.9: Suppose $\lambda_1 \neq \lambda_2$ and let j, k be positive integers. Then

$$N\left[(\lambda_1-T)^j\right] \cap N\left[(\lambda_2-T)^k\right] = \{0\}$$

Proof:

We observe first of all that if $(\lambda_1-T)^j x = 0$, then $x \in D(T^n)$ for all $n \geq 1$. This is evident if $\lambda_1 = 0$. If $\lambda_1 \neq 0$, we observe that $x \in D(T^j)$ and

(4-5) $\lambda_1^j x - j\lambda_1^{j-1}Tx + \dots + (-1)^j T^j x = 0$

so that $T^j x$ is a linear combination of x, Tx, ..., $T^{j-1}x$. Hence $T^j x \in D(T)$. By applying T to (4-5) we see that $T^{j+1}x$ is a linear combination of Tx, ..., $T^j x$, and hence that $T^{j+1}x \in D(T)$. In this way we see by induction that $x \in D(T^n)$ for all $n \geq 1$.

Now let $p_1(\lambda) = (\lambda_1 - \lambda)^j$, $p_2(\lambda) = (\lambda_2 - \lambda)^k$. Since $\lambda_1 \neq \lambda_2$, the polynomials p_1, p_2 are relatively prime, and hence there exist polynomials $q_1(\lambda)$, $q_2(\lambda)$ such that

(4-6) $q_1(\lambda)p_1(\lambda) + q_2(\lambda)p_2(\lambda) \equiv 1.$

We deduce that if $x \in D(T^n)$ for sufficiently large n, then

(4-7) $q_1(T)p_1(T)x + q_2(T)p_2(T)x = x$

In particular, if $x \in N\left[(\lambda_1-T)^j\right] \cap N\left[(\lambda_2-T)^k\right]$, we see that $p_1(T)x = p_2(T)x = 0$, and hence, from (4-7), x = 0.

This completes the proof.

Lemma 4.10: Suppose that, for some λ_0, $\alpha(\lambda_0-T)$ and $\delta(\lambda_0-T)$ are finite. Let $q = \delta(\lambda_0-T)$. Then $N\left[(\lambda-T)^k\right] \subset R\left[(\lambda_0-T)^q\right]$ for k = 0,1,2,--- if $\lambda \neq \lambda_0$.

Proof:

We can assume $k \geq 1$ and $q \geq 1$, for the result is obvious if k = 0 or q = 0 . By Theorem 4.8(b) we can write

$$D\left[(\lambda_0-T)^q\right]=\left\{D\left[(\lambda_0-T)^q\right]\cap R\left[(\lambda_0-T)^q\right]\right\}\oplus N\left[(\lambda_0-T)^q\right].$$

Suppose that $x\in N\left[(\lambda-T)^k\right]$. Then $x\in D(T^n)$ for all $n\geqslant 1$, and hence $x\in D\left[(\lambda_0-T)^q\right]$, by an argument given in the proof of Lemma 4.9. We can write $x=(\lambda_0-T)^q u+v$ where $(\lambda_0-T)^q u\in D\left[(\lambda_0-T)^q\right]$ and $v\in N\left[(\lambda_0-T)^q\right]$. Then v, and hence also $(\lambda_0-T)^q u$, belongs to $D(T^n)$ for all $n\geqslant 1$. Applying $(\lambda-T)^k$ to x, we see that

(4-8) $\qquad (\lambda-T)^k(\lambda_0-T)^q u=-(\lambda-T)^k v.$

Now, we can express $(\lambda-T)^k$ in the form

$$(\lambda-T)^k=\sum_{i=0}^{k}c_i(\lambda_0-T)^i$$

From this we see that

$$(\lambda-T)^k(\lambda_0-T)^q u=\sum_{i=0}^{k}c_i(\lambda_0-T)^{i+q}u\in R\left[(\lambda_0-T)^q\right].$$

It is clear also that $(\lambda-T)^k v\in N\left[(\lambda_0-T)^q\right]$. Hence, since $R\left[(\lambda_0-T)^q\right]\cap N\left[(\lambda_0-T)^q\right]=\{0\}$, we see from (4-8) that $(\lambda-T)^k v=0$. But then $v\in N\left[(\lambda-T)^k\right]\cap N\left[(\lambda_0-T)^q\right]$ and hence $v=0$, by Lemma 4.9. It now follows that $x=(\lambda_0-T)^q u\in R\left[(\lambda_0-T)^q\right]$, as was to be proved.

Appendix to Chapter IV, relating to Corollary 4.4.

Example 1: We show that we can get equality in Corollary 4.4.

Consider the kxk matrix $T = \begin{bmatrix} 01 & & & & & & \\ & 01 & & & & & \\ & & 01 & & & & 0 \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ 0 & & & & 01 & & \\ & & & & & 01 & \\ & & & & & & 0 \end{bmatrix}$

Since nullity T + rank T = k, we have

$$n(T) = d(T) = k\text{-rank } T = k-(k-1) = 1.$$

Now by row-column multiplication

$$T^2 = \begin{bmatrix} 001 & & & & & & \\ & 001 & & & & & \\ & & 001 & & & & 0 \\ & & & \cdots & & & \\ & & & & \cdots & & \\ & & & & & \cdots & \\ 0 & & & & 001 & & \\ & & & & & 00 & \\ & & & & & & 0 \end{bmatrix}, \ldots, \quad T^{k-2} = \begin{bmatrix} & & 010 \\ & & 01 \\ & & 0 \\ & 0 & \end{bmatrix}, \quad T^{k-1} = \begin{bmatrix} & & 01 \\ & & 0 \\ & & \\ & 0 & \end{bmatrix}, \quad T^k = 0.$$

We next deduce that $n(T^k) = d(T^k) = k$, either by the same reasoning as for T, or from the fact that $N(T^k) =$ whole space X, $R(T^k) = \{0\}$.

We must next find $\alpha(T)$ and $\delta(T)$.

Let $1 \leqslant i \leqslant k$.

$n(T^i) = k\text{-rank } T^i = k-(k-i) = i$ which increases (strictly) to k.

Therefore $\alpha(T) = k$.

$\dim R(T^i) = \text{rank } T^i = k-i \ (1 \leqslant i \leqslant k)$ which decreases (strictly) to 0.

Therefore $\delta(T) = k$.

Hence $n(T^k) = \alpha(T)n(T)$ and $d(T^k) = \delta(T)d(T)$.

Note: Equality is impossible in the first case if $k < \alpha(T)$, and in the second if $k < \delta(T)$.

Example 2:

We have in Corollary 4.4, $n(T^k) \leq \alpha(T)n(T)$ for $k = 1,2,3,\ldots,$

and $d(T^k) \leq \delta(T)d(T)$ for $k = 1,2,3,\ldots$.

Assume $\alpha(T) < \infty$ and $\delta(T) < \infty$.

We have in particular $n(T^{\alpha(T)}) \leq \alpha(T)n(T)$ and $d(T^{\delta(T)}) \leq \delta(T)d(T)$.

We show that if $n(T) = 1$ in the first inequality, we always have equality, and if $d(T) = 1$ in the second inequality, we always have equality.

Firstly; let $n(T) = 1$

Then $n(T^{\alpha(T)}) \leq \alpha(T)$ by Corollary 4.4.

R.T.P. $n(T^{\alpha(T)}) \geq \alpha(T)$.

Clearly if $n(T) = 1$, then $n(T^2) \geq 2$ if $2 \leq \alpha(T)$ since $N(T^2) \supsetneqq N(T)$

whence $n(T^3) \geq 3$ if $3 \leq \alpha(T)$ since $N(T^3) \supsetneqq N(T^2)$,

whence by induction $n(T^s) \geq s$ if $s \leq \alpha(T)$.

Put $s = \alpha(T)$. Then $n(T^{\alpha(T)}) \geq \alpha(T)$.

Hence $n(T^k) = \alpha(T)n(T)$ if $k = \alpha(T)$ and $n(T) = 1$.

Secondly, let $d(T) = 1$.

Then $d(T^{\delta(T)}) \leq \delta(T)$ by Corollary 4.4.

R.T.P. $d(T^{\delta(T)}) \geq \delta(T)$

If $d(T) = 1$, then $d(T^2) \geq 2$ if $2 \leq \delta(T)$ since $d(T) = \operatorname{codim} R(T)$ and

$R(T^2) \subsetneqq R(T)$,

whence $d(T^3) \geq 3$ if $3 \leq \delta(T)$ since $R(T^3) \subsetneqq R(T^2)$,

whence by induction $d(T^t) \geq t$ if $t \leq \delta(T)$.

Put $t = \delta(T)$. Then $d(T^{\delta(T)}) \geq \delta(T)$.

Hence $d(T^k) = \delta(T)d(T)$ if $k = \delta(T)$ and $d(T) = 1$.

Example 3:

We show that we can get equality in $n(T^k) \leq \alpha(T)n(T)$ with $n(T) > 1$, and in $d(T^k) \leq \delta(T)d(T)$ with $d(T) > 1$.

We first observe that if T is the block matrix

$$T = \begin{bmatrix} T_1 & & & & & & \\ & T_2 & & & & & \\ & & T_3 & & & & \\ & & & \ddots & & & \\ & & & & T_{r-2} & & \\ & & & & & T_{r-1} & \\ & & & & & & T_r \end{bmatrix}$$

where T_i is a square matrix for $i = 1,2,\ldots, r$, $(r > 1)$,

then

$$T^n = \begin{bmatrix} T_1^n & & & & & & \\ & T_2^n & & & & & \\ & & T_3^n & & & & \\ & & & \ddots & & & \\ & & & & T_{r-2}^n & & \\ & & & & & T_{r-1}^n & \\ & & & & & & T_r^n \end{bmatrix}$$

for all non-negative integers n.

The above statement can be checked by row-column multiplication and induction.

For each $i = 1,2,3,\ldots, r$, let T_i be the $k \times k$ matrix

$$T_i = \begin{bmatrix} 0 & 1 & & & & & 0 \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & \ddots & \ddots & & \\ & & & & 0 & 1 & \\ & & & & & 0 & 1 \\ 0 & & & & & & 0 \end{bmatrix}$$

as in Example 1.

We have seen the nature of $T_i^2, \ldots, T_i^{k-2}, T_i^{k-1}, T_i^k = 0$ in Example 1.

Note that T is a $kr \times kr$ matrix.

We have $n(T) = d(T) = kr - \text{rank } T = kr-(k-1)r = r$.

Example 3 (contd)

Next,

$$T^k = \begin{bmatrix} T_1{}^k & & & & \\ & T_2{}^k & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \; T_r{}^k \end{bmatrix} = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \; 0 \end{bmatrix} = 0$$

and we have seen in Example 1 that $T_i{}^{k-1} \neq 0$ for all $i = 1,2,\dots,r$, whence $T^{k-1} \neq 0$.

Therefore $N(T^k) = X$ and $n(T^k) = kr$.

$R(T^k) = \{0\}$ and $d(T^k) = \operatorname{codim} R(T^k) = kr$.

Since $N(T^{k-1}) \neq X$, $\alpha(T) = k$

and since $R(T^{k-1}) \neq \{0\}$, $\delta(T) = k$.

Therefore $n(T^k) = kr = \alpha(T)n(T)$ where $n(T) = r > 1$,

and $d(T^k) = kr = \delta(T)d(T)$ where $d(T) = r > 1$.

Example 4:

We produce some examples where $n(T^{\alpha(T)}) < n(T)\alpha(T)$

and $d(T^{\delta(T)}) < d(T)\,\delta(T)$.

The inequalities must be strict.

(a) Let T be the (2k-1) x (2k-1) matrix

$$\begin{bmatrix} 001 & & & & & & 0 \\ & 001 & & & & & \\ & & 001 & & & & \\ & & & \cdots & & & \\ & & & & \cdots & & \\ & & & & & \cdots & \\ 0 & & & & & 001 & \\ & & & & & 00 & \\ & & & & & & 0 \end{bmatrix}$$

(With the 1's in the diagonal as shown, the following argument does not work for a 2kx2k matrix, as can be checked.)

$$T^2 = \begin{bmatrix} 00001 & & & & & & 0 \\ & 00001 & & & & & \\ & & 00001 & & & & \\ & & & \cdots\cdots & & & \\ & & & & \cdots\cdots & & \\ & & & & & \cdots\cdots & \\ & & & & & & \cdots\cdots \\ & & & & & & 00001 \\ 0 & & & & & & 0000 \\ & & & & & & 000 \\ & & & & & & 00 \\ & & & & & & 0 \end{bmatrix}, \ldots, \quad T^{k-1} = \begin{bmatrix} \overbrace{00\ldots00}^{2k-2\text{ noughts}}1 \\ 0 \\ \\ 0 \\ \\ \end{bmatrix}, \quad T^k = 0$$

$$n(T) = d(T) = (2k-1) - \text{rank } T = (2k-1) - (2k-3) = 2.$$

Since $X = N(T^k) \supsetneqq N(T^{k-1})$, $\alpha(T) = k$.

Since $R(T^k) = \{0\} \subsetneqq R(T^{k-1})$, $\delta(T) = k$.

$$n(T^{\alpha(T)}) = n(T^k) = n(0) = 2k-1$$

$$d(T^{\delta(T)}) = d(T^k) = d(0) = 2k-1.$$

Hence $n(T^{\alpha(T)}) = 2k-1 < 2k = n(T)\alpha(T)$

$$d(T^{\delta(T)}) = 2k-1 < 2k = d(T)\,\delta(T)$$

Example 4 (contd)

(b) Let T be the kxk matrix $(k > 2)$ $T = \begin{bmatrix} & & 0\,1 \\ & & 0 \\ 0 & & \end{bmatrix}$, $T^2 = 0$

$$n(T) = d(T) = k\text{-rank } T = k-1.$$

$$\alpha(T) = \delta(T) = 2$$

$$n(T^{\alpha(T)}) = n(T^2) = n(0) = k$$

$$d(T^{\delta(T)}) = d(T^2) = d(0) = k.$$

Now $2(k-1) > k$ since $k > 2$. Hence

$$n(T^{\alpha(T)}) < n(T)\alpha(T) \quad \text{and} \quad d(T^{\delta(T)}) < d(T)\delta(T).$$

(c) Consider this block matrix

$$T = \begin{bmatrix} T_{k_1} & & & & & \\ & T_{k_2} & & & & \\ & & T_{k_3} & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & T_{k_r} \end{bmatrix}$$

where $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_r$ and, $T_k = \begin{bmatrix} 0\,1 & & & & & 0 \\ & 0\,1 & & & & \\ & & 0\,1 & & & \\ & & & \ddots & & \\ 0 & & & & 0\,1 & \\ & & & & & 0\,1 \\ & & & & & 0 \end{bmatrix}$ (the kxk matrix of Example 1.)

Since $T^{k_1 - 1} \neq 0$ and $T^{k_1} = 0$, we have $\alpha(T) = \delta(T) = k_1$.

Also $n(T) = d(T) = r$.

$$n(T^{\alpha(T)}) = n(T^{k_1}) = n(0) = k_1 + k_2 + \ldots + k_r .$$

$$d(T^{\delta(T)}) = d(T^{k_1}) = d(0) = k_1 + k_2 + \ldots + k_r .$$

Now to say that one of the k's is less than k_1 is equivalent to saying $k_r < k_1$.

So if $k_r < k_1$ then $n(T^{\alpha(T)}) < n(T)\alpha(T)$

and $d(T^{\delta(T)}) < d(T)\delta(T)$.

Example 5:

We know that $n(T^{\alpha(T)}) \leq \alpha(T)n(T)$ and $d(T^{\delta(T)}) \leq \delta(T)d(T)$ and that these maxima can be attained. We now find the minimum of $n(T^{\alpha(T)})$ for fixed $n(T)$ and $\alpha(T)$, and the minimum of $d(T^{\delta(T)})$ for fixed $d(T)$ and $\delta(T)$.

Consider first $n(T^{\alpha(T)})$.

Case 1: (trivial) $\alpha(T) = 0$ (Recall: $\alpha(T) = 0 \Leftrightarrow n(T) = 0 \Leftrightarrow T$ is one-to-one.)

$$n(T^{\alpha(T)}) = n(T^0) = n(I) = 0 .$$

Case 2: $\alpha(T) \geq 1$ (whence $n(T) \geq 1$)

Clearly $n(T) \leq n(T^{\alpha(T)}) \leq \alpha(T)n(T)$.

We consider the left-hand inequality.

If $2 \leq \alpha(T)$, then $N(T^2) \supsetneqq N(T)$ and $\min\limits_{T} n(T^2) \geq n(T) + 1$.

If $3 \leq \alpha(T)$, then $N(T^3) \supsetneqq N(T^2)$ and $\min\limits_{T} n(T^3) \geq \min\limits_{T} n(T^2) + 1 \geq n(T) + 2$

and so on.

We find that for fixed $n(T)$ and $\alpha(T)$

$$\min_{T} n(T^{\alpha(T)}) \geq n(T) + \alpha(T) - 1$$

Next, we check that $n(T) + \alpha(T) - 1 \leq \alpha(T)n(T)$.

Now $n(T) + \alpha(T) - 1 \leq \alpha(T)n(T) \Leftrightarrow (n(T)-1)(\alpha(T)-1) \geq 0$

and we see that Case 2 requires $n(T) \geq 1$, $\alpha(T) \geq 1$.

Secondly, we consider $d(T^{\delta(T)})$.

By an argument similar to that above,

$$\min_{T} d(T^{\delta(T)}) \geq d(T) + \delta(T) - 1 \text{ for fixed } d(T) \text{ and } \delta(T) \geq 1.$$

We have already seen two examples of operators which attain these minima, namely Example 1 and Example 4(b). But Example 1 had $n(T) = d(T) = 1$, and Example 4(b) had $\alpha(T) = \delta(T) = 2$. We want these numbers to be chosen arbitrarily.

Example 5 (contd)

So, let $r = n(T)$ and $k = \alpha(T)$ be given. Consider the block matrix:

$$T = \begin{bmatrix} T_k & & & & \\ & T_1 & & & \\ & & T_1 & & \\ r \text{ matrices} & & & \ddots & \\ & & & & T_1 \end{bmatrix}$$

where T_k is the $k \times k$ matrix of Ex.1, viz.

$$\begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & 0 \\ & & 0 & 1 & & & \\ & & & \ddots & \ddots & & \\ & & & & 0 & 1 & \\ 0 & & & & & 0 & 1 \\ & & & & & & 0 \end{bmatrix}$$

and $T_1 = (0)$.

Here $n(T) = r$ and $\alpha(T) = k$,

and $n(T^{\alpha(T)}) = n(T^k) = n(0) = r+k-1 = n(T) + \alpha(T)-1$.

Also $d(T) = r$ and $\delta(T) = k$,

and $d(T^{\delta(T)}) = d(T^k) = d(0) = r+k-1 = d(T) + \delta(T)-1$.

Thus, given $n(T)$, $d(T)$, $\alpha(T)$ and $\delta(T)$, we have produced an operator with $n(T^{\alpha(T)})$ and $d(T^{\delta(T)})$ equal to the minimum possible value.

CHAPTER V

Nullity and defect, as related to ascent and descent.

In this chapter we again follow Taylor (13) and Kaashoek (5). The general assumptions about X and T are the same as in Chapter IV, namely: X is a linear space and T a linear operator with $D(T) \subset X$ and $R(T) \subset X$. Except where we explicitly assume that $D(T) = X$ it is possible that $D(T) \neq X$. There is no topology; all considerations are algebraic. We pursue here the study of the relationships between the numbers $n(T)$, $d(T)$, $\alpha(T)$ and $\delta(T)$.

We begin by recalling the following results from Chapter IV:-

Lemma 4.1: If there exists a non-negative integer N such that $n(T^k) \leq N$ when $k = 0,1,2,---$, then $\alpha(T) \leq N$.

Lemma 4.2: If there exists a non-negative integer N such that $d(T^k) \leq N$ when $k = 0,1,2,---$, then $\delta(T) \leq N$.

Corollary 4.4: For any non-negative integer k

(a) $n(T^k) \leq \alpha(T)n(T)$ (b) $d(T^k) \leq \delta(T)d(T)$.

<u>Definitions</u>: A subspace M of X is called invariant under T if $T[M \cap D(T)] \subset M$. By the restriction of T to M we then mean the operator T_1 in the space M, defined as follows :-

$D(T_1) = M \cap D(T)$, $T_1x = Tx$ if $x \in D(T_1)$.

If M is invariant under T and if $T[M \cap D(T)] = M$, we say that M is <u>exactly invariant</u> under T.

<u>Theorem 5.1</u>: Let $n(T) < \infty$. Then $\alpha(T) < \infty$ if and only if, for each subspace M which is exactly invariant under T, the restriction T_1 of T to M is such that $\alpha(T_1) = \delta(T_1) = 0$. i.e. $n(T_1) = d(T_1) = 0$.

Proof:

Suppose that $\alpha(T) < \infty$. Let $\mathcal{M}$ be exactly invariant under T, and let T_1 be the restriction of T to $\mathcal{M}$. It is easily verified that

$$D(T_1^n) = \mathcal{M} \cap D(T^n) \text{ and } N(T_1^n) = \mathcal{M} \cap N(T^n).$$

We see then that $\alpha(T_1) \leq \alpha(T) < \infty$. Now $R(T_1) = \mathcal{M}$, and hence $\delta(T_1) = 0$. But then $\alpha(T_1) = 0$ by Theorem 4.7.

Suppose, conversely, that $\alpha(T_1) = \delta(T_1) = 0$ for each T_1 associated in the indicated manner with an exactly invariant subspace. Consider the sequence $\{N(T) \cap R(T^n)\}$, $n = 0,1,2,---$. Since $R(T^{n+1}) \subset R(T^n)$ and $n(T) < \infty$, there is some non-negative integer r such that

$$N(T) \cap R(T^r) = N(T) \cap R(T^n) \text{ if } n \geq r.$$

Let $\mathcal{M} = \bigcap_{i=0}^{\infty} R(T^{r+i})$. Observe that $\mathcal{M} = \bigcap_{i=j}^{\infty} R(T^{r+i})$ if $j \geq 0$, and therefore that $\mathcal{M}$ is invariant under T. The proof runs as follows:

$$\begin{aligned} T\left[D(T) \cap \mathcal{M}\right] &= T\left[D(T) \cap \bigcap_{i=j}^{\infty} R(T^{r+i})\right] \\ &= T\left[\bigcap_{i=j}^{\infty} \{D(T) \cap R(T^{r+i})\}\right] \\ &\subset \bigcap_{i=j}^{\infty} T\left[D(T) \cap R(T^{r+i})\right] \\ &= \bigcap_{i=j}^{\infty} R(T^{r+i+1}) \quad \text{since it is easy to prove that for all } k \\ &\qquad\qquad R(T^{k+1}) = T\left[R(T^k) \cap D(T)\right] \\ &= \bigcap_{i=j+1}^{\infty} R(T^{r+i}) \\ &= \mathcal{M}. \end{aligned}$$

Observe also that $N(T) \cap \mathcal{M} = N(T) \cap R(T^r)$.

We shall show that $\mathcal{M}$ is <u>exactly</u> invariant under T. Suppose $y \in \mathcal{M}$. Then there exists $x_i \in D(T^{r+i})$ such that $y = T^{r+i}x_i$, $i \geq 0$. Let

$$u_i = T^r x_1 - T^{r+i-1}x_i = T^r(x_1 - T^{i-1}x_i), \quad i \geq 1.$$

Then $0 = T^{r+1}x_1 - T^{r+i}x_i = T(T^r x_1 - T^{r+i-1}x_i) = Tu_i$.

Clearly $u_i \in N(T) \cap R(T^r)$. We shall see that $T^r x_1 \in \mathcal{M}$. In fact $T^r x_1 = u_i + T^{r+i-1}x_i$. Since $T^{r+i-1}x_i \in R(T^{r+i-1})$ and

$u_i \in N(T) \cap R(T^r) = N(T) \cap R(T^{r+i-1})$, we see that $T^r x_1 \in R(T^{r+i-1})$ when $i \geq 1$, and hence $T^r x_1 \in M$.

Since $T(T^r x_1) = y$, it follows that $T\left[M \cap D(T)\right] = M$.

By hypothesis, if T_1 is the restriction of T to M, $\alpha(T_1) = 0$; that is if $x \in M \cap D(T)$ and $Tx = 0$, then $x = 0$. But this means that $N(T) \cap M = \{0\}$, and hence $N(T) \cap R(T^r) = \{0\}$. This implies $\alpha(T) \leq r$, by Lemma 4.5(a). Q.E.D.

Theorem 5.2: Suppose that either $n(T)$ or $d(T)$ is finite, and that $p = \alpha(T) < \infty$. Then

(5-1) $\quad n(T) \leq d(T)$

and we have equality in (5-1) if and only if T has the additional property

(5-2) $\quad X = R(T) + N(T^p)$.

Remark: (5-1) is not true in general if $\alpha(T) = \infty$. On page 32 of his paper (13), Taylor produces an example where $\alpha(T) = \infty$, $d(T) = 1 < 2 = n(T)$.

Proof:

Since $p = \alpha(T) < \infty$, it follows from Lemma 3.1 that

$$N(T) \cap R(T^p) = \{0\}.$$

But then Lemma 3.5 implies

$$n(T) = \dim \frac{N(T)}{N(T) \cap R(T^p)} = \dim \frac{N(T^p)}{R(T) \cap N(T^p)}$$

and so, by Lemma 2.4

(5-3) $$n(T) = \dim \frac{N(T^p)}{R(T) \cap N(T^p)} \leq \dim \frac{X}{R(T) \cap X} = d(T).$$

This shows that $n(T) \leq d(T)$.

Now suppose we have equality in (5-1). Then we also have equality in (5-3), and hence

$$\dim \frac{N(T^p)}{R(T) \cap N(T^p)} = \dim \frac{X}{R(T) \cap X} < \infty .$$

But then Lemma 2.5 implies that $X = R(T) + N(T^p)$.

Conversely, suppose that T has the additional property (5-2). Then by Lemma 2.3,

$$\frac{X}{R(T)} = \frac{R(T) + N(T^p)}{R(T)} \cong \frac{N(T^p)}{R(T) \cap N(T^p)}$$

But then we have equality in (5-3), and hence $n(T) = d(T)$. Q.E.D.

<u>Theorem 5.3</u>: Suppose that either $n(T)$ or $d(T)$ is finite, and that $q = \delta(T) < \infty$. Then

(5-4) $d(T) \leq n(T) + \dim X/\{D(T^q) + R(T)\}$

and we have equality in (5-4) if T has the additional property

(5-5) $N(T) \cap R(T^q) = \{0\}$.

In the particular case when also $d(T) < \infty$, we have equality in (5-4) if and only if T has the additional property (5-5).

<u>Proof</u>:

Since $X \supset D(T^q) + R(T) \supset N(T^q) + R(T) \supset R(T)$, we have by Lemma 2.1,

$$d(T) = \dim \frac{X}{R(T)} = \dim \frac{X}{D(T^q)+R(T)} + \dim \frac{D(T^q)+R(T)}{N(T^q)+R(T)} + \dim \frac{N(T^q)+R(T)}{R(T)}$$

By Lemma 3.2, $q = \delta(T) < \infty$ implies that $D(T^q) \subset R(T) + N(T^q)$ and so $\dim \frac{D(T^q) + R(T)}{N(T^q) + R(T)} = 0$.

Furthermore, it follows from Lemmas 2.3 and 3.5 that

$$\dim \frac{R(T) + N(T^q)}{R(T)} = \dim \frac{N(T^q)}{R(T) \cap N(T^q)} = \dim \frac{N(T)}{R(T^q) \cap N(T)}$$

Combining these facts, we obtain

(5-6) $$d(T) = \dim \frac{X}{D(T^q) + R(T)} + \dim \frac{N(T)}{R(T^q) \cap N(T)}$$

and hence $d(T) \leq n(T) + \dim X/\{D(T^q) + R(T)\}$.

If, in addition, $N(T) \cap R(T^q) = \{0\}$, then

$$n(T) = \dim N(T)/\{R(T^q) \cap N(T)\}.$$

But then formula (5-6) implies that we have equality in (5-4).

Conversely, suppose that $\infty > d(T) = n(T) + \dim X/\left\{D(T^q) + R(T)\right\}$. Then by formula (5-6), $\infty > n(T) = \dim N(T)/\left\{R(T^q) \cap N(T)\right\}$, and hence $R(T^q) \cap N(T) = \{0\}$. Q.E.D.

<u>Remark:</u> Formula (5-4) implies that, under the conditions of the theorem, $d(T) \leq n(T)$ if $D(T) = X$.

<u>Remark:</u> The proofs of Theorems 5.2 and 5.3 given above are due to M.A. Kaashoek (5). These proofs are more elegant, and the results stronger, than those of A.E. Taylor in his paper (13). However, subsequent results of Taylor's (especially Theorem 5.5 (a), (b), (c), (d) of this thesis - see later), which I wish to reproduce, depend heavily on the techniques used by Taylor in his proofs of Theorems 5.2 and 5.3. It is thus both necessary, and, I think, instructive, to reproduce at this point Taylor's proof of Theorems 5.2 and 5.3. I shall call these two theorems, Theorems $5.2^{\#}$ and $5.3^{\#}$.

<u>Theorem $5.2^{\#}$</u> (Taylor's proof of Theorem 5.2).

Suppose that either $n(T)$ or $d(T)$ is finite, and that $\alpha(T)$ is finite. Then $n(T) \leq d(T)$.

<u>Proof:</u>

Let $\alpha(T) = p$. Then $N(T) \cap R(T^p) = \{0\}$ by Lemma 4.5(b). We shall see that from this we can conclude that $n(T) \leq d(T^p)$. Suppose $x_1, \ldots, x_k$ are linearly independent elements in $N(T)$, and let $[x_1], \ldots, [x_k]$ be the elements (cosets) in $X/R(T^p)$ which contain $x_1, \ldots, x_k$, respectively. Then these are linearly independent, for $c_1[x_1] + \ldots + c_k[x_k] = 0$ implies $c_1x_1 + \ldots + c_kx_k \in R(T^p)$, whence $c_1x_1 + \ldots + c_kx_k = 0$ (because $N(T) \cap R(T^p) = \{0\}$) and therefore, $c_1 = \ldots = c_k = 0$. Hence $n(T) \leq d(T^p)$. It now follows from Lemma 4.3(b) that $n(T) \leq pd(T)$ if $d(T) < \infty$.

We now see that the assumptions of the theorem imply that $n(T) < \infty$. We still have to prove that $n(T) \leq d(T)$. This is certainly true if $d(T) = \infty$, and it is true as a consequence of the relation $n(T) \leq pd(T)$ if $d(T) < \infty$ and $p = 0$ or $p = 1$. Hence we can assume $d(T) < \infty$ and $p \geq 2$.

Let $M_k = N(T) \cap R(T^k)$, $k = 0, 1, ---, p$. Observe that $M_k \subset M_{k-1}$, that $M_0 = N(T)$, and that $M_p = \{0\}$. We apply Lemma 2.1, observing that $M_k/M_p = M_k$, $k = 0, 1, ---, p-1$. Thus

(5-7) $\dim M_k = \dim M_k/M_{k+1} + \dim M_{k+1}$, $k = 0, 1, 2, ---, p-1$.

Let us write

(5-8) $m_k = \dim M_k/M_{k+1}$

Then, on combining the equations which result from (5-7) for $k = 0, 1, 2, ---, p-1$, we obtain the result

(5-9) $n(T) = m_0 + m_1 + \dots + m_{p-1}$.

Now, if $0 \leq k \leq p-1$, let $y_1^{(k)}, \dots, y_{m_k}^{(k)}$ be elements of M_k such that the corresponding cosets $[y_1^{(k)}], \dots, [y_{m_k}^{(k)}]$ belonging to M_k/M_{k+1} are linearly independent as elements of the quotient space. Then $y_j^{(k)} = T^k x_j^{(k)}$, where $x_j^{(k)} \in D(T^k)$. Consider the elements

$$x_1^{(0)}, \dots, x_{m_0}^{(0)}, x_1^{(1)}, \dots, x_{m_1}^{(1)}, \dots, x_1^{(p-1)}, \dots, x_{m_{p-1}}^{(p-1)}$$

(there are $n(T)$ of them), and the corresponding elements $[x_j^{(k)}]$ of the quotient space $X/R(T)$. If we show that the latter elements $[x_j^{(k)}]$ of the quotient space are linearly independent, we shall have proved that $n(T) \leq d(T)$ as required. To put the matter in another way, we shall suppose that c_{ij} are constants such that

(5-10) $$\sum_{i=0}^{p-1} \sum_{j=1}^{m_i} c_{ij} x_j^{(i)} = Tu \text{ (an element of } R(T))$$

and from this we shall deduce that $c_{ij} = 0$ for each relevant i and j.

Let $$w_i = \sum_{j=1}^{m_i} c_{ij} x_j^{(i)}$$

Then $w_i \in D(T^i)$ and

$$(5\text{-}11)\qquad T^i w_i = \sum_{j=1}^{m_i} c_{ij} y_j^{(i)} \in M_i \subset N(T)$$

Observe that $x_j^{(o)} = y_j^{(o)} \in N(T)$, so that $w_o \in N(T) \subset D(T)$. We rewrite (5-10) as

$$(5\text{-}12)\qquad \sum_{i=0}^{p-1} w_i = Tu.$$

We see that each w_i (and hence Tu) belongs to D(T), and

$$\sum_{i=1}^{p-1} Tw_i = T^2u.$$

Since $T^i w_i \in N(T)$, we can continue until we obtain

$$T^{p-1} w_{p-1} = T^p u \in N(T) \cap R(T^p) = \{0\} .$$

whence $T^{p-1}w_{p-1} = 0$. Referring now to (5-11) and recalling the original requirements placed on the elements $y_j^{(p-1)}$, we see that $c_{p-1,j} = 0$ if $j = 1,\dots,m_{p-1}$. Therefore $w_{p-1} = 0$. Going back to (5-12), we now have

$$\sum_{i=o}^{p-2} w_i = Tu,$$

from which we conclude

$$(5\text{-}13)\qquad T^{p-2} w_{p-2} = T^{p-1} u \in M_{p-1}$$

But the elements $\left[y_1^{(p-2)}\right],\dots,\left[y_{m_{p-2}}^{(p-2)}\right]$ of M_{p-2}/M_{p-1} are linearly independent. From this fact, along with (5-11) and (5-13), we conclude that $c_{p-2,j} = 0$, $j = 1,\dots, m_{p-2}$, and hence $w_{p-2} = 0$. Continuing in this way, we are able to prove that $c_{ij} = 0$ for every i and j . Thus the proof is complete.

Theorem 5.3# (Taylor's proof of Theorem 5.3)

Suppose that either n(T) or d(T) is finite, and that $\delta(T) = q$ is finite. Then

$$(5\text{-}14)\qquad d(T) \leq n(T) + \dim X/D(T^q).$$

In particular, $d(T) \leq n(T)$ if $D(T) = X$.

Proof:

We can assume $q \geqslant 1$, for $d(T) = 0$ if $q = 0$, and then (5-14) is certainly true.

Let $Q_i = N(T^i) + R(T)$ $\quad i = 0,1,---, q-1.$

Observe that $Q_i \subset Q_{i+1}$. We shall show that

(5-15) $$p_i = \dim \frac{N(T^{i+1})}{Q_i \cap N(T^{i+1})} < \infty.$$

Let $x_1^{(i)}, x_2^{(i)}, \ldots, x_{n_i}^{(i)}$ be a finite set of elements of $N(T^{i+1})$ such that the corresponding cosets in the quotient space $N(T^{i+1}) / \left[Q_i \cap N(T^{i+1})\right]$ are linearly independent. We claim that the elements of $X/R(T)$ corresponding to the elements $x_j^{(i)}$ ($j = 1, \ldots, n_i$, $i = o, \ldots, q-1$) are linearly independent, so that

(5-16) $$n_o + n_1 + \ldots + n_{q-1} \leqslant d(T)$$

In fact, suppose that

$$\sum_{i=o}^{q-1} \sum_{j=1}^{n_i} c_{ij} x_j^{(i)} \in R(T)$$

Then $$\sum_{j=1}^{n_{q-1}} c_{q-1,j}\, x_j^{(q-1)} \in N(T^{q-1}) + R(T) = Q_{q-1}$$

and hence $c_{q-1,\, j} = 0$ if $j = 1, \ldots, n_{q-1}$. We then have

$$\sum_{i=o}^{q-2} \sum_{j=1}^{n_i} c_{ij} x_j^{(i)} \in R(T)$$

$$\sum_{j=1}^{n_{q-2}} c_{q-2,j}\, x_j^{(q-2)} \in N(T^{q-2}) + R(T) = Q_{q-2}$$

and we can continue the argument to show that all the c_{ij}'s are 0 . This proves (5-16).

Now let $y_j^{(i)} = T^i x_j^{(i)}$, and note that $y_j^{(i)} \in N(T)$. We shall prove that the elements $y_j^{(i)}$ are linearly independent, so that

(5-17) $$n_o + n_1 + \ldots + n_{q-1} \leqslant n(T).$$

In fact we shall prove somewhat more, namely

$$(5\text{-}18) \qquad n_o + n_1 + \dots + n_{q-1} \leqslant \dim \frac{N(T)}{N(T) \cap R(T^q)}$$

For this purpose we observe first of all that $y_j^{(o)} = x_j^{(o)}$ and that $x_1^{(o)}, \dots, x_{n_o}^{(o)}$ are elements of $N(T)$ which were chosen so that if

$$a_1 x_1^{(o)} + \dots + a_{n_o} x_{n_o}^{(o)} \in Q_o = R(T)$$

then $a_1 = \dots = a_{n_o} = 0$

Consider the following proposition P_m :

$$\text{If } \sum_{i=o}^{m-1} \sum_{j=1}^{n_i} a_{ij} y_j^{(i)} \in R(T^m)$$

for a fixed m (where $1 \leqslant m \leqslant q$) then each a_{ij} appearing here is 0. We know this is true if m = 1. We shall deduce that P_{m+1} is true if P_m is true and $m \leqslant q-1$, and thus prove that P_q is true. We therefore suppose P_m is true and assume

$$\sum_{i=o}^{m} \sum_{j=1}^{n_i} a_{ij} y_j^{(i)} = T^{m+1} u \, .$$

Then $$\sum_{i=o}^{m-1} \sum_{j=1}^{n_i} a_{ij} \, y_j^{(i)} = T^{m+1} u - \sum_{j=1}^{n_m} a_{mj} \, T^m x_j^{(m)} \in R(T^m)$$

Therefore $a_{ij} = 0$ of $0 \leqslant i \leqslant m-1$. But then

$$\sum_{j=1}^{n_m} a_{mj} \, T^m x_j^{(m)} = T^{m+1} u \, .$$

Let $$x = \sum_{j=1}^{n_m} a_{mj} x_j^{(m)} - Tu$$

Note that $T^m x = 0$. Then

$$\sum_{j=1}^{n_m} a_{mj} x_j^{(m)} = x + Tu \in N(T^m) + R(T) = Q_m$$

By the way in which the elements $x_j^{(m)}$ were chosen, it follows that $a_{mj} = 0$ for each j . Thus the proposition P_q is true. This implies the truth of (5-18) and also that of (5-17).

Our assumption is that either $n(T)$ or $d(T)$ is finite. Hence, from either (5-16) or (5-17) we see that (5-15) is true for each i. Now let the numbers n_i be chosen maximally (i.e. $n_i = p_i$, where p_i is given by (5-15)). Then we see that

$$(5\text{-}19) \qquad p_o + p_1 + \dots + p_{q-1} \leq \min\left[n(T), d(T)\right],$$

and also

$$(5\text{-}20) \qquad p_o + p_1 + \dots + p_{q-1} \leq \dim \frac{N(T)}{N(T) \cap R(T^q)}$$

Next we shall prove that

$$(5\text{-}21) \qquad p_o + \dots + p_{q-1} \quad = \quad \dim \frac{N(T^q)}{R(T) \cap N(T^q)}$$

The elements $x_j^{(i)}$ all belong to $N(T^q)$. Therefore the argument which proved (5-16) also proves that

$$p_o + \dots + p_{q-1} \quad \leq \quad \dim \frac{N(T^q)}{R(T) \cap N(T^q)}$$

To prove the inequality in the opposite direction, observe that any element of $N(T^{i+1})$ is expressible as a linear combination of $x_1^{(i)}, \dots, x_{p_i}^{(i)}$, plus an element of Q_i. When we recall that $Q_i = N(T^i) + R(T)$ and apply this observation repeatedly, with $i = q-1, q-2, \dots, 0$, we see that an element x of $N(T^q)$ is expressible in the form

$$x = \sum_{i=o}^{q-1} \sum_{j=1}^{p_i} a_{ij} x_j^{(i)} + w$$

where $w \in R(T)$. This shows that

$$\dim \frac{N(T^q)}{R(T) \cap N(T^q)} \leq p_o + \dots + p_{q-1} .$$

We have now proved (5-21).

We now proceed to prove (5-14). Since $R(T) \cap D(T^q) \subset D(T^q) \subset X$, we see by Lemma 2.1 that

$$\dim \frac{X}{R(T) \cap D(T^q)} = \dim \frac{X}{D(T^q)} + \dim \frac{D(T^q)}{R(T) \cap D(T^q)}$$

In the same way we see that

$$\dim \frac{X}{R(T) \cap D(T^q)} = \dim \frac{X}{R(T)} + \dim \frac{R(T)}{R(T) \cap D(T^q)}$$

and therefore

(5-22) $$\dim \frac{X}{R(T)} + \dim \frac{R(T)}{R(T) \cap D(T^q)} = \dim \frac{X}{D(T^q)} + \dim \frac{D(T^q)}{R(T) \cap D(T^q)} .$$

By Lemma 4.6(b) we know that

$$D(T^q) = \left[R(T) \cap D(T^q)\right] \oplus M_1$$

where

(5-23) $$M_1 \subset N(T^q) \quad \text{and} \quad M_1 \cap R(T) = \{0\} .$$

Thus

(5-24) $$\dim \frac{D(T^q)}{R(T) \cap D(T^q)} = \dim M_1$$

From (5-23) we see that

(5-25) $$\dim M_1 = \dim \frac{N(T^q)}{R(T) \cap N(T^q)}$$

Therefore, by (5-22), (5-24), (5-25), (5-21) and (5-19) we see that

$$\dim \frac{X}{R(T)} = d(T) \leq \dim \frac{X}{D(T^q)} + n(T) .$$

This proves (5-14).

Remark: This concludes the proofs of theorems 5.2 and 5.3 given by Taylor (13). At a later stage it will be necessary to refer back to the techniques he employs.

Corollary 5.4: If $\alpha(T)$ and $\delta(T)$ are both finite, and if either $n(T)$ or $d(T)$ is finite, then

(5-26) $$n(T) \leq d(T) \leq n(T) + \dim X/\{D(T^q) + R(T)\} \text{ where } q = \delta(T).$$

Hence, if $X = D(T)$, we have $n(T) = d(T)$ under these circumstances.

Proof:

This is merely a combination of Theorems 5.2 and 5.3.

Theorem 5.5:

(a) Suppose that $n(T)$ and $p = \alpha(T)$ are finite. Suppose also that

(5-27) $$\dim \frac{D(T^p)}{R(T) \cap D(T^p)} \leq n(T)$$

Then $\delta(T) = \alpha(T)$. Furthermore, we actually have equality, rather than inequality, in (5-27). As a consequence, it follows that $n(T) = d(T)$ if $D(T) = X$.

(b) Suppose that $n(T)$ and $q = \delta(T)$ are finite. Suppose also that

(5-28) $$n(T) \leq \dim \frac{N(T^q)}{R(T) \cap N(T^q)}$$

Then $\alpha(T) \leq q = \delta(T)$. Furthermore, we actually have equality, rather than inequality, in (5-28). Also

(5-29) $$\dim \frac{D(T^q)}{R(T) \cap D(T^q)} = n(T)$$

In the case $D(T) = X$ we can conclude that $n(T) = d(T)$.

(c) Suppose that $\alpha(T)$ is finite and that $n(T) = d(T) < \infty$. Then $\delta(T) = \alpha(T)$.

(d) Suppose that $D(T) = X$, that $\delta(T)$ is finite, and that $n(T) = d(T) < \infty$. Then $\alpha(T) = \delta(T)$.

Proof of (a):

We see that the hypotheses of Theorem $5.2^{\#}$ are satisfied. Consider the elements $x_j^{(i)}$ $(i = 0,1,\dots,p-1,\ j = 1,\dots, m_i)$ introduced in the proof of Theorem $5.2^{\#}$. There are $n(T)$ of these elements (see 5-9)). These elements all belong to $N(T^p)$, and the corresponding elements $\left[x_j^{(i)}\right]$ of the quotient space $X/R(T)$ are linearly independent.

Let M be the subspace of X generated by this set of $n(T)$ elements. Then $M \cap R(T) = \{0\}$ and $\dim M = n(T)$. Thus we conclude from (5-27) and Lemma 2.2 that $D(T^p) = \left[R(T) \cap D(T^p)\right] \oplus M$.

Since $M \subset N(T^p)$, we can appeal to Lemma 4.6(a) to conclude that $\delta(T) \leq p$. But then $\alpha(T)$ and $\delta(T)$ are both finite, and hence $\alpha(T) \leq \delta(T)$ (Theorem 4.7). Hence $\alpha(T) = \delta(T)$.

The proof shows that $\dim M = \dim D(T^p)/\{R(T) \cap D(T^p)\}$. Hence we actually have equality, rather than inequality, in (5-27). We also observe that, if $D(T) = X$, $\dim D(T^p)/\{R(T) \cap D(T^p)\} = \dim X/R(T) = d(T)$. Hence $n(T) = d(T)$ in this case.

Proof of (b):

We see that the hypotheses of Theorem $5.3^{\#}$ are satisfied. Consider the elements $y_j^{(i)}$ ($i = 0,\dots, q-1$, $j = 1,\dots p_i$) introduced in the proof of Theorem $5.3^{\#}$. There are $p_0 + \dots + p_{q-1}$ of these elements. They belong to $N(T)$, and if a linear combination of them lies in $R(T^q)$, all the coefficients in this linear combination are 0 (by proposition P_q, which was established in the course of the proof of Theorem $5.3^{\#}$). In view of (5-21) and the hypothesis (5-28), it then follows that the subspace generated by the elements $y_j^{(i)}$ is $N(T)$, and therefore $N(T) \cap R(T^q) = \{0\}$. This implies $\alpha(T) \leq q$ (see Lemma 4.5(a)). Moreover, $p_0 + \dots + p_{q-1} = n(T)$, so that the inequality in (5-28) becomes an equality.

We now obtain (5-29) from (5-24) and (5-25). In the case $D(T) = X$, (5-29) implies that $d(T) = n(T)$.

Proof of (c):

We observe to begin with that if $p = \alpha(T)$, then

$$\dim \frac{D(T^p)}{R(T) \cap D(T^p)} \leq \dim \frac{X}{R(T)} = d(T)$$

merely because $D(T^p) \subset X$ (Lemma 2.4). As a consequence of the hypothesis in (c), it then follows that (5-27) is valid, and hence we can apply Theorem 5.5(a) to conclude that $\alpha(T) = \delta(T)$.

Proof of (d):

Let $q = \delta(T)$. Since $D(T) = X$, $D(T^q) = X$ also, and we can use Lemma 4.6(b) to write $X = R(T) \oplus M$, where M is a subspace such that $M \cap R(T) = \{0\}$ and $M \subset N(T^q)$. We see from this situation that

$$(5\text{-}30) \qquad d(T) = \dim M \leq \dim \frac{N(T^q)}{R(T) \cap N(T^q)}$$

Now the hypotheses of (d) enable us to use the results of Theorem 5.3# and its proof. From (5-30), (5-19) and (5-21), we conclude that

$$d(T) = \dim \frac{N(T^q)}{R(T) \cap N(T^q)}$$

Since $d(T) = n(T)$, it follows that Theorem 5.5(b) is applicable, and we conclude $\alpha(T) \leq \delta(T)$. But then $\alpha(T) = \delta(T)$ by Theorem 4.7 and the remark which followed it.

Remark 1: A variation of the proof of (d) would be as follows: Having established (5-30), we could prove $d(T) = \dim N(T^q)/\{R(T) \cap N(T^q)\}$ by noting $N(T^q) \subset X$ and invoking Lemma 2.4, thus:

$$d(T) = \dim \frac{X}{R(T)} = \dim \frac{X}{X \cap R(T)} \geq \dim \frac{N(T^q)}{N(T^q) \cap R(T)}$$

Remark 2 on Theorem 5.5(d): We shall later be able to drop the condition $D(T) = X$ and replace it with a weaker condition.

Lemma 5.6: Suppose $D(T) = X$ and $\lambda \neq 0$. Let $X_1 = R(T)$, and let T_1 be the restriction of T to X_1. (Observe that $R(T_1) \subset X_1$). Then:

(a) $\alpha(\lambda - T) = \alpha(\lambda - T_1)$ and $n(\lambda - T) = n(\lambda - T_1)$.

(b) If M is a subspace of X_1 which is a complement to $R(\lambda - T_1)$ in X_1, it is also a complement to $R(\lambda - T)$ in X. Therefore $d(\lambda - T) = d(\lambda - T_1)$.

(c) If $n(\lambda - T_1) = d(\lambda - T_1) < \infty$, and if either $\alpha(\lambda - T_1)$ or $\delta(\lambda - T_1)$ is finite, then $\alpha(\lambda - T) = \delta(\lambda - T) < \infty$.

Proof of (a):

We shall prove that $N\left[(\lambda - T_1)^m\right] = N\left[(\lambda - T)^m\right]$ if $m = 0,1,2,---$. This is clearly true if $m = 0$. It is also evident that $N\left[(\lambda - T_1)^m\right] \subset N\left[(\lambda - T)^m\right]$ if $m \geqslant 1$. Suppose $m \geqslant 1$ and $x \in N\left[(\lambda - T)^m\right]$ (m fixed). Then $0 = (\lambda - T)^m x = \lambda^m x - m\lambda^{m-1}Tx + \ldots + (-1)^m T^m x$. Since $\lambda \neq 0$, we see that $x \in R(T) = X_1$, and therefore $0 = (\lambda - T)^m x = (\lambda - T_1)^m x$, so that $N\left[(\lambda - T)^m\right] \subset N\left[(\lambda - T_1)^m\right]$. The truth of (a) follows at once.

Proof of (b):

We begin by observing that $R(\lambda - T_1) = R(\lambda - T) \cap X_1$. For this it suffices to show that $R(\lambda - T) \cap X_1 \subset R(\lambda - T_1)$, the inclusion in the other direction being evident. Suppose $y = (\lambda - T)x \in X_1$. Then $y = Tu$ for some u, and $\lambda x = Tx + Tu$, whence $x \in X_1$ and hence $y \in R(\lambda - T_1)$. Next, we see that $M \cap R(\lambda - T) = \{0\}$. This is because $M \cap R(\lambda - T_1) = \{0\}$ and $M \cap R(\lambda - T) = M \cap X_1 \cap R(\lambda - T) = M \cap R(\lambda - T_1)$.
To complete the proof of (b), it will now suffice to prove that $X \subset R(\lambda - T) \oplus M$. Given x, let $y = (\lambda - T)x$. Since $Tx \in X_1$ and $X_1 = R(\lambda - T_1) \oplus M$, we can write $Tx = (\lambda - T_1)u + v$ where $u \in X_1$ and $v \in M$. Then

$$x = \frac{1}{\lambda}(Tx + y) = \frac{1}{\lambda}\left[(\lambda - T_1)u + v + (\lambda - T)x\right],$$

$$x = (\lambda - T)\left(\frac{u+x}{\lambda}\right) + \frac{v}{\lambda} \in R(\lambda - T) \oplus M.$$

Proof of (c)

If $\delta(\lambda - T_1)$ is finite, it follows by applying Theorem 5.5(d) to $\lambda - T_1$ that $\alpha(\lambda - T_1) = \delta(\lambda - T_1)$. Then $\alpha(\lambda - T) < \infty$ by part (a) of the present lemma. Moreover, $n(\lambda - T) = d(\lambda - T) < \infty$ as a consequence of parts (a) and (b) of the present lemma. We now use Theorem 5.5(c) to conclude that $\delta(\lambda - T) = \alpha(\lambda - T)$.

If $\alpha(\lambda - T_1)$ is finite, then we see that $\delta(\lambda - T_1) = \alpha(\lambda - T_1)$, by an application of Theorem 5.5(c) to $\lambda - T_1$. The preceding argument then shows $\alpha(\lambda - T) = \delta(\lambda - T) < \infty$.

Theorem 5.7: Suppose $D(T) = X$, $\lambda \neq 0$, and $\dim R(T) < \infty$. Then $n(\lambda - T) = d(\lambda - T) < \infty$ and $\alpha(\lambda - T) = \delta(\lambda - T) < \infty$.

Proof:

Let X_1 and T_1 be as in Lemma 5.6. Here $\dim X_1 < \infty$, and therefore, necessarily, $n(\lambda - T_1)$ and $d(\lambda - T_1)$ are finite. Also, $\alpha(\lambda - T_1)$ and $\delta(\lambda - T_1)$ are finite. Then $n(\lambda - T_1) = d(\lambda - T_1)$ by Corollary 5.4. It now follows from Lemma 5.6 that $n(\lambda - T) = d(\lambda - T) < \infty$ and $\alpha(\lambda - T) = \delta(\lambda - T) < \infty$.

CHAPTER VI

Decomposition Theorems and related theorems

In this chapter we consider theorems dealing with expression of the operator T in the form $T = A + B$ where A and B are linear operators such that $D(A) = D(T)$, $D(B) = X$, and A, B have properties which are related to properties of T. The general aim is to be able to study T more effectively by means of the structure revealed by the properties of A and B. As in the preceding chapters, T is a linear operator in a linear space X. Our considerations here are entirely algebraic. We do not assume $D(T) = X$ except where explicitly stated.

Theorem 6.1:

(a) Suppose that $n(T) \leq d(T)$ and $n(T) < \infty$. Then there exists a linear operator B with $D(B) = X$, $R(B) \subset X$, and $\dim R(B) \leq n(T)$ such that, if $Ax = Tx - Bx$, with $D(A) = D(T)$, then A^{-1} exists (i.e. $Ax = 0$ implies $x = 0$).

(b) If $d(T) \leq n(T)$ and $d(T) < \infty$, there exists a linear operator B with $D(B) = X$, $R(B) \subset X$, $\dim R(B) \leq d(T)$ such that, if $Ax = Tx - Bx$, with $D(A) = D(T)$, then $R(A) = X$.

(c) If $n(T) = d(T) < \infty$, the assertions about A and B in (a) and (b) can be made with simultaneous validity; in this case, therefore, A is a one-one mapping of $D(T)$ onto all of X.

Proof of (a):

If $n(T) = 0$, we take $B = 0$, $A = T$. If $1 \leq p = n(T) < \infty$, let $x_1, \dots, x_p$ be a basis for $N(T)$. Choose linear functionals $x_1', \dots, x_p'$ such that $x_i'(x_j) = \delta_{ij}$ (the Kronecker delta) $i, j = 1, \dots, p$. Choose element $u_1, \dots, u_p$ in X such that the corresponding elements

$[u_1], \ldots, [u_p]$ of $X/R(T)$ are linearly independent. This is possible because of the assumption that $p \leqslant d(T)$. Now define B on X by

$$Bx = \sum_{i=1}^{p} x_i'(x)u_i$$

and let $Ax = Tx - Bx$ if $x \in D(T)$. Clearly $\dim R(B) \leqslant p$.

If $x \in D(T)$ and $Ax = 0$, we see that $Bx = Tx \in R(T)$. From the nature of the u_i's we then infer that $x_i'(x) = 0$ for each i, whence $Bx = 0$, and so $Tx = 0$, i.e. $x \in N(T)$. But then x is a linear combination of $x_1, \ldots, x_p$, say

$$x = \sum_{i=1}^{p} a_i x_i$$

Since $0 = x_j'(x) = a_j$, we see that $x = 0$. Hence A^{-1} exists.

<u>Proof of (b)</u>:

If $d(T) = 0$, we take $B = 0$, $A = T$.

If $1 \leqslant q = d(T) < \infty$, let $x_1, \ldots, x_q$ be a set of q linearly independent elements of $N(T)$ (such elements exist, because $q \leqslant n(T)$).

Let $x_1', \ldots, x_q'$ be linear functionals such that $x_i'(x_j) = \delta_{ij}$ (the Kronecker delta), $i, j = 1, \ldots, q$. Choose $u_1, \ldots, u_q$ in X so that $[u_1], \ldots, [u_q]$ is a basis for $X/R(T)$. Define B by

$$Bx = \sum_{i=1}^{q} x_i'(x)u_i$$

and let $Ax = Tx - Bx$ if $x \in D(T)$. Clearly $\dim R(B) \leqslant d(T)$.

To show that $R(A) = X$, suppose $y \in X$. The subspace generated by $u_1, \ldots, u_q$ is a complement to $R(T)$ in X, so that we can write y in the form $y = \sum_{i=1}^{q} a_i u_i + Tx$ where $x \in D(T)$. Now let

$$z = x - \sum_{i=1}^{q} \left[a_i + x_i'(x)\right] x_i .$$

We see that $z \in D(T)$. Since $Bx_j = u_j$, we find by a direct calculation that $Az = y$. Thus $R(A) = X$.

Proof of (c):

If $n(T) = d(T) < \infty$, we observe that the constructions of B and A in parts (a) and (b) are identical. This justifies the assertion in part (c).

Theorem 6.2: Suppose that T is given, and suppose that there exists a linear operator B in X, with $D(B) = X$ and $\dim R(B) < \infty$, such that, if we define A by $Ax = Tx - Bx$, $x \in D(T)$, then $n(A) = d(A) = 0$ (which means that A is a one-one mapping of D(T) onto all of X).
Let the operator S be defined by $S = I + A^{-1}B$ (with $D(S) = X$).
Then $n(T) = n(S) = d(S) = d(T) < \infty$ and $\alpha(S) = \delta(S) < \infty$.

Proof:

We observe that $A^{-1}B$ maps X into D(A) (= D(T)) and that $AA^{-1}B = B$. Note also that $S\left[D(A)\right] \subset D(A)$. We can write

(6-1) $\quad Tx = Ax + AA^{-1}Bx = ASx, \; x \in D(T).$

Observe that $\dim R(A^{-1}B) \leq \dim R(B) < \infty$. It follows from Theorem 5.7 and the definition of S that $n(S) = d(S) < \infty$ and $\alpha(S) = \delta(S) < \infty$.

We shall prove that $N(T) = N(S)$ and thus that $n(T) = n(S)$. The fact that $N(T) \subset N(S)$ follows from (6-1) and the fact that A^{-1} exists. We see that $N(S) \subset D(T)$, because, if $0 = Sx = x + A^{-1}Bx$, we have $x = -A^{-1}Bx \in D(T)$. It is then clear from (6-1) that $N(S) \subset N(T)$. Hence $N(S) = N(T)$.

Next we prove that $d(T) \leq d(S)$. We can assume $1 \leq d(T)$. Let N be a subspace complementary to R(T) in X, and let $x_1, \ldots, x_n$ be linearly independent elements of N. Since $R(A) = X$, we can write $x_i = Au_i$ where $u_i \in D(T)$. We are going to prove that the elements $\left[u_1\right], \ldots, \left[u_n\right]$ of $X/R(S)$ are linearly independent, and therefore that $n \leq d(S)$, whence $d(T) = \dim N \leq d(S)$. If we suppose that $\sum_{i=1}^{n} a_i \left[u_i\right] = \left[0\right]$, this means that $\sum_{i=1}^{n} a_i u_i$ is some element Sv in $R(S)$. Hence

$$\sum_{i=1}^{n} a_i u_i = v + A^{-1}Bv,$$

and from this it is clear that $v \in D(T)$. Then

$$A\left(\sum_{i=1}^{n} a_i u_i\right) = \sum_{i=1}^{n} a_i x_i = Av + Bv = Tv \in R(T).$$

Since $N \cap R(T) = \{0\}$, it follows that $\sum_{i=1}^{n} a_i x_i = 0$, and hence that all the a_i's are 0. This completes the argument that $d(T) \leq d(S)$.

Finally, we prove that $d(S) \leq d(T)$. Let $q = d(S)$, $m = \delta(S)$. There exists a subspace M such that $M \cap R(S) = \{0\}$, $M \subset N(S^m)$, and $X = R(S) \oplus M$ (by Lemma 4.6(b)). Let $u_1, \ldots, u_q$ be a basis for M. Since

$$0 = S^m u_i = (I + V)^m u_i = u_i + mVu_i + \ldots + V^m u_i$$

where $V = A^{-1}B$, we see that $u_i \in R(V) \subset D(T)$. Let $x_i = Au_i$. We shall show that the elements $[x_1], \ldots, [x_q]$ of $X/R(T)$ are linearly independent, so that $d(T) \geq d(S)$. In fact, suppose that $\sum_{i=1}^{q} c_i x_i$ is some element Tu of $R(T)$. Then $\sum_{i=1}^{q} c_i Au_i = Tu$, and so (see (6-1))

$$\sum_{i=1}^{q} c_i u_i = A^{-1}Tu = Su \in R(S) \cap M = \{0\},$$

whence $\sum_{i=1}^{q} c_i u_i = 0$, and the c_i's are all 0. This completes the proof of Theorem 6.2.

Remark: The next theorem relates a certain type of splitting of the operator T in the form $T = A + B$ to a direct-sum decomposition of the space in the form $X = M \oplus N$, related in a suitable way to T.

Theorem 6.3: Suppose that M and N are subspaces of X such that $M \cap N = \{0\}$, $N \subset D(T)$, $T(N) \subset N$, $\dim N < \infty$, $X = M \oplus N$, and such that T maps $M \cap D(T)$ in a one-to-one manner onto all of M. Let P and Q be the projections of X onto M along N, and onto N along M, respectively.

Suppose that $\lambda \neq 0$, and define

(6-2) $\quad Ax = TPx - \lambda Qx, \quad x \in D(T)$

(6-3) $\quad Bx = TQx + \lambda Qx, \quad x \in X.$

Then $Tx = Ax+Bx$ and $ABx = BAx$ if $x \in D(T)$; $R(B) \subset N$, and hence $\dim R(B) < \infty$. Also

(6-4) $\quad Tx = \frac{1}{\lambda}(\lambda - B)Ax$ if $x \in D(T)$,

(6-5) $\quad R(\lambda - B) = R(T)$,

and A is a one-to-one mapping of $D(T)$ onto all of X.

Proof:

Note that $I = P+Q$, $R(P) = N(Q) = M$, $N(P) = R(Q) = N$, $P^2 = P$, $Q^2 = Q$, $PQ = QP = 0$. Observe that $R(Q) \subset D(T)$. It follows from this that $P\left[D(T)\right] \subset D(T)$. We see from these remarks that A and B are well defined by (6-2) and (6-3) with $D(A) = D(T)$ and $D(B) = X$. It is clear that $Ax+Bx = T(Px+Qx) = Tx$ if $x \in D(T)$. The relation $R(B) \subset N$ is a consequence of $R(Q) = N$ and $T(N) \subset N$.

The assumptions on T guarantee that if $x \in D(T)$, then $TPx \in M$ and hence $QTPx = 0$. It follows that

$$BAx = (TQ + \lambda Q)(TPx - \lambda Qx) = -\lambda TQx - \lambda^2 Qx.$$

Likewise $TQx \in N$, so $PTQx = 0$ and $QTQx = TQx$, whence

$$ABx = (TP - \lambda Q)(TQx + \lambda Qx) = -\lambda TQx - \lambda^2 Qx$$

and so $ABx = BAx$.

Next we prove that, if $y \in X$, there is a unique $x \in D(T)$ such that $Ax = y$. Supposing that we have one such x, the equation $Ax = y$ is equivalent to

$$(TP - \lambda Q)(Px + Qx) = Py + Qy$$

or (in view of $PQ = QP = 0$) to

(6-6) $\quad TPx - \lambda Qx = Py + Qy.$

In view of the uniqueness implied in $X = M \oplus N$, (6-6) is equivalent to the two equations

(6-7) $\quad TPx = Py, \qquad -\lambda Qx = Qy.$

Since $\lambda \neq 0$ and T maps $M \cap D(T)$ in a one-to-one manner onto M, we see that Px and Qx are uniquely determined, and hence $x = Px + Qx$ is uniquely determined. The unique solution x of $Ax = y$ is

(6-8) $\quad x = A^{-1}y = u - \frac{1}{\lambda}Qy ,$

where u is the unique element of $M \cap D(T)$ such that $Tu = Py$.

Finally, we prove (6-4) and (6-5). If $x \in D(T)$, we can verify (6-4) by direct use of (6-2) and (6-3), noting that $QTPx = 0$. From (6-4) we see that $R(T) \subset R(\lambda - B)$. On the other hand, if $y = (\lambda - B)x$, we can write $\lambda x = Au$ for some $u \in D(T)$. Then $y = (\lambda - B)(\frac{1}{\lambda}Au) = Tu$, by (6-4), and so $R(\lambda - B) \subset R(T)$. Q.E.D.

Remark: The next theorem continues the investigation, using the conclusions of Theorem 6.3 as hypotheses.

Theorem 6.4: Suppose, given T, there exist linear operators A, B in X, with $D(A) = D(T)$, $D(B) = X$, $\dim R(B) < \infty$, $R(B) \subset D(T)$, $Tx = Ax + Bx$ and $ABx = BAx$ if $x \in D(T)$, and such that A maps $D(T)$ in a one-to-one manner onto all of X. Then $n(T) = d(T) < \infty$ and $\alpha(T) = \delta(T) < \infty$.

Proof:

Theorem 6.2 implies that $n(T) = d(T) < \infty$. We shall prove that $N(T^k) \subset R(B)$ and hence $n(T^k) \leq \dim R(B) < \infty$ when $k > 0$. It will then follow from Lemma 4.1 that $\alpha(T) < \infty$, and thereafter from Theorem 5.5(c) that $\alpha(T) = \delta(T)$.

We begin by proving that $A^{-1}Bx = BA^{-1}x$ for each x in X. Given x, choose $u \in D(T)$ so that $Au = x$. Then $u = A^{-1}x$. Now $ABu = BAu = Bx$ and therefore $Bu = A^{-1}Bx$, or $BA^{-1}x = A^{-1}Bx$.

We now prove that $N(T^k) \subset R(B)$ if $k \geq 0$. This is certainly true if $k = 0$. We proceed by induction, assuming $N(T^k) \subset R(B)$ for a certain k. If $x \in N(T^{k+1})$ we see that $Tx \in N(T^k)$, so that $Ax + Bx \in R(B)$. Let $Ax + Bx = Bv$. Then $Ax + B(x-v) = 0$, or $A\left[x + A^{-1}B(x-v)\right] = 0$, whence (since A^{-1} exists), $x = -A^{-1}B(x-v) = -BA^{-1}(x-v) \in R(B)$. This proves that $N(T^{k+1}) \subset R(B)$ and completes the induction.

Theorem 6.5: Suppose that $n(T) = d(T) < \infty$, and that $p = \alpha(T) < \infty$.
Then (a) $\delta(T) = \alpha(T)$
(b) $n(T^i) = d(T^i) < \infty$ for $i = 0,1,2,---$,
(c) $X = R(T^p) \oplus N(T^p)$

Remark: Part (a) of this theorem has already been proved by Taylor's methods (see (13)) as Theorem 5.5(c). It required a fairly long preamble, depending as it did on Theorem 5.5(a). I feel that it will be instructive to compare Taylor's approach with the brevity and elegance of Kaashoek's proof (given here - from (5)), which invokes a result from Chapter III. Later - after the present theorem, we will be able to give Kaashoek's improved version of the last part of Theorem 5.5, namely Theorem 5.5(d).

Proof of (a):

From Theorem 5.2 it follows that

$$X = R(T) + N(T^p)$$

This implies, by Lemma 3.2, that

$$\frac{R(T^p)}{R(T^{p+1})} \cong \frac{D(T^p)}{\{R(T) + N(T^p)\} \cap D(T^p)} = \{[0]\}$$

and hence $\delta(T) \leq \alpha(T) < \infty$. Combining this fact with the result of Theorem 4.7, we obtain $\delta(T) = \alpha(T)$.

Remark: Part (b) of this theorem does not require $\alpha(T)<\infty$, provided $D(T) = X$. This is a consequence of Theorem 8.1 (see later), which says that if $D(T_k) = X$ $(k = 1,2)$, then

$$n(T_k) = d(T_k) < \infty \; (k = 1,2) \Rightarrow n(T_1T_2) = d(T_1T_2) < \infty .$$

By induction we find

$$n(T_k) = d(T_k) < \infty \; (k = 1,2,\ldots,i) \Rightarrow n(T_1T_2\ldots T_i) = d(T_1T_2\ldots T_i) < \infty .$$

Putting $T_1 = T_2 = \ldots = T_i = T$, we get

$$n(T) = d(T) < \infty \Rightarrow n(T^i) = d(T^i) < \infty \text{ provided } D(T) = X,$$

i.e. we have dispensed with $p = \alpha(T) < \infty$.

Proof of (b):

Let k be a non-negative integer. Since $X \supset R(T^k) \supset R(T^{k+1})$ it follows from Lemma 2.1 that

$$d(T^{k+1}) = \dim \frac{X}{R(T^{k+1})} = \dim \frac{X}{R(T^k)} + \dim \frac{R(T^k)}{R(T^{k+1})}$$

$$= d(T^k) + \dim \{R(T^k)/R(T^{k+1})\} .$$

Since $d(T) < \infty$, we have by Lemma 4.3(b) that $d(T^k) \leq d(T^{k+1}) < \infty$, and so $d(T^{k+1}) - d(T^k) = \dim \{R(T^k)/R(T^{k+1})\}$.

Then we deduce from Lemma 3.2 that

$$d(T^{k+1}) - d(T^k) = \dim \frac{D(T^k)}{\{R(T) + N(T^k)\} \cap D(T^k)}$$

Observe that $R(T) + N(T^k) + D(T^k) = R(T) + D(T^k)$, and apply Lemma 2.3.

Then $$d(T^{k+1}) - d(T^k) = \dim \frac{R(T) + D(T^k)}{R(T) + N(T^k)}$$

Since $n(T) = d(T) < \infty$, and since $p = \alpha(T) < \infty$, Theorem 5.2 implies that $X = N(T^p) + R(T)$. Observe that $N(T^p) \subset D(T^k)$. Then

$$X = R(T) + N(T^p) \subset R(T) + D(T^k) \subset X ,$$

and hence

$$d(T^{k+1}) - d(T^k) = \dim X/\{R(T) + N(T^k)\} . \tag{6-9}$$

From $X \supset N(T^k) + R(T) \supset R(T)$ and Lemma 2.1, it follows that

$$\infty > d(T) = \dim \frac{X}{N(T^k) + R(T)} + \dim \frac{N(T^k) + R(T)}{R(T)}$$

Combining this with formula (6-9), we obtain

$$(6\text{-}10) \qquad d(T^{k+1}) - d(T^k) = d(T) - \dim \left\{ N(T^k) + R(T) \right\} / R(T).$$

Observe that Lemmas 2.3 and 3.5 imply that

$$\dim \frac{N(T^k) + R(T)}{R(T)} = \dim \frac{N(T^k)}{R(T) \cap N(T^k)} = \dim \frac{N(T)}{N(T) \cap R(T^k)} \quad .$$

Since $\dim N(T) = n(T) < \infty$, we have

$$\dim \frac{N(T)}{N(T) \cap R(T^k)} = n(T) - \dim \left\{ N(T) \cap R(T^k) \right\} .$$

Combining these facts with formula (6-10) and using the hypothesis $n(T) = d(T) < \infty$, we obtain

$$d(T^{k+1}) - d(T^k) = \dim \left\{ N(T) \cap R(T^k) \right\} ,$$

and hence, by Lemma 3.1

$$d(T^{k+1}) - d(T^k) = \dim \left\{ N(T^{k+1}) / N(T^k) \right\} .$$

Since $N(T^k) \subset N(T^{k+1})$, and since $\dim N(T^{k+1}) = n(T^{k+1}) < \infty$ (by Lemma 4.3(a)) it follows that

$$(6\text{-}11) \qquad d(T^{k+1}) - d(T^k) = n(T^{k+1}) - n(T^k).$$

Formula (6-11) holds for each non-negative integer k. Hence $d(T^0) = n(T^0) = 0$ implies

$$d(T^i) = \sum_{k=0}^{i-1} \left\{ d(T^{k+1}) - d(T^k) \right\} = \sum_{k=0}^{i-1} \left\{ n(T^{k+1}) - n(T^k) \right\} = n(T^i)$$

By Lemma 4.3 both $d(T^i)$ and $n(T^i)$ are finite.

This completes the proof of (b).

Proof of (c):

Since $p = \alpha(T) < \infty$, Lemma 3.1 implies

$$(6\text{-}12) \qquad N(T^p) \cap R(T^p) = \{0\} .$$

But then, invoking Lemma 2.4,

$$n(T^p) = \dim \frac{N(T^p)}{R(T^p) \cap N(T^p)} \leqslant \dim \frac{X}{R(T^p)} = d(T^p)$$

By (b) we have $n(T^p) = d(T^p) < \infty$. Hence

$$\dim \frac{N(T^p)}{R(T^p) \cap N(T^p)} = \dim \frac{X}{R(T^p)} < \infty$$

and so, by Lemma 2.5, we have $X = R(T^p) + N(T^p)$. Combining this with (6-12), we obtain

$$X = R(T^p) \oplus N(T^p). \qquad \text{Q.E.D.}$$

<u>Remark</u>: In the particular case $D(T) = X$ and $p = \alpha(T) = \delta(T) < \infty$, it is possible to prove that

$$X = R(T^p) \oplus N(T^p)$$

without using the hypothesis $n(T) = d(T) < \infty$. (Kaashoek mentions a proof of this due to Heuser). But in general Theorem 6.5(c) does not hold if we omit the assumption that $n(T) = d(T) < \infty$, as is seen from the following example.

Let D be a subspace in the infinite-dimensional linear space X such that $\dim X/D = 1$. Suppose that T is the restriction of the null operator from X into X, to D. Then $D = N(T)$ and $R(T) = \{0\}$, and hence

$$n(T) = d(T) = +\infty, \quad \alpha(T) = \delta(T) = 1 .$$

But $X \neq D = N(T) \oplus R(T)$.

<u>Corollary 6.6</u>: Let T, M and N satisfy the hypotheses of Theorem 6.3. Then $n(T) = d(T) < \infty$ and $\alpha(T) = \delta(T) < \infty$. If $\alpha(T) = p$, we have $R(T^p) \cap N(T^p) = \{0\}$ and $X = R(T^p) \oplus N(T^p)$.

<u>Proof</u>:

The conclusions about $n(T)$, $d(T)$, $\alpha(T)$ and $\delta(T)$ follow from Theorems 6.3 and 6.4. The fact that $X = R(T^p) \oplus N(T^p)$ follows from Theorem 6.5(c).

Remark: In Theorem 5.5(d) we proved that if $D(T) = X$, $\delta(T) < \infty$, and $n(T) = d(T) < \infty$ then $\alpha(T) = \delta(T)$. We are now in a position to prove a stronger version of this theorem by weakening the condition $D(T) = X$. We replace it by $X = D(T^q) + R(T)$, where $q = \delta(T) < \infty$. Clearly $X = D(T) \Rightarrow X = D(T^q) \Rightarrow X = D(T^q) + R(T)$.

Theorem 6.7: Suppose that $n(T) = d(T) < \infty$ and that $q = \delta(T) < \infty$. Then

$$\alpha(T) = \delta(T)$$

if and only if $X = D(T^q) + R(T)$.

Proof:

In the case $X = D(T^q) + R(T)$, our hypotheses imply that we have equality in formula (5-4) (see Theorem 5.3), and hence, since $d(T) < \infty$, we have $N(T) \cap R(T^q) = \{0\}$. But then, as a consequence of Lemma 3.1, we have $p = \alpha(T) \leq q = \delta(T) < \infty$.

Since $D(T^p) \subset X = D(T^q) + R(T)$,

Theorem 4.7 implies that $\alpha(T) = \delta(T)$.

Conversely, suppose that $n(T) = d(T) < \infty$ and $q = \alpha(T) = \delta(T) < \infty$. Then, by Theorem 6.5(c), we have $X = N(T^q) \oplus R(T^q)$. Since $N(T^q) \subset D(T^q)$ and $R(T^q) \subset R(T)$, this implies $X = D(T^q) + R(T)$. Q.E.D.

Remark: It is interesting to note that in the case $n(T) = d(T) < \infty$ and $p = \alpha(T) = \delta(T) < \infty$, it is not necessary that $D(T^p) = D(T^{p+1})$. In order to show this, as a matter of general interest, we prove the following theorem.

Theorem 6.8: Suppose that $n(T) = d(T) < \infty$ and that $\alpha(T) = \delta(T) < \infty$. Then, for $i = 0,1,2,---$,

$$\frac{D(T^i)}{D(T^{i+1})} \cong \frac{X}{D(T)} ,$$

Remark: If X were a finite-dimensional space and $D(T) \neq X$, this result would be clearly false. But this can never happen, and the apparent problem is easily resolved when we notice that $n(T) = d(T) \Rightarrow D(T) = X$ if X is finite-dimensional. The proof runs as follows:

$$\dim N(T) + \dim R(T) = \dim D(T). \quad \text{(a well-known result)}$$

Let $d_1(T)$ be the defect of $D(T)$, i.e. $d_1(T) = \operatorname{codim} D(T)$.

Then $n(T) + \left[\dim X - d(T)\right] = \left[\dim X - d_1(T)\right]$ where $\dim X < \infty$.

i.e. $n(T) = d(T) - d_1(T)$

whence $n(T) = d(T) \Rightarrow d_1(T) = 0 \Rightarrow D(T) = X$.

Proof:

Let i be some non-negative integer. Since $p = \delta(T) < \infty$, we have $R(T^p) \subset R(T^i)$. By Theorem 6.5, $p = \alpha(T) = \delta(T) < \infty$ implies that $X = R(T^p) \oplus N(T^p)$. Combining these facts with $N(T^p) \subset D(T)$, we obtain

$$X = R(T^p) + N(T^p) \subset R(T^i) + N(T^p) \subset R(T^i) + D(T) \subset X,$$

and hence $X = R(T^i) + D(T)$. Then, by Lemma 2.3,

$$\frac{X}{D(T)} = \frac{D(T) + R(T^i)}{D(T)} \cong \frac{R(T^i)}{D(T) \cap R(T^i)}$$

But then, as a consequence of Lemma 3.3,

$$\frac{X}{D(T)} \cong \frac{D(T^i)}{D(T^{i+1})}$$

Q.E.D.

CHAPTER VII

Reducibility

As usual, T is a linear operator with domain and range in the linear space X.

Suppose that M_1 and M_2 are subspaces of X such that $M_1 \cap M_2 = \{0\}$ and $X = M_1 \oplus M_2$. Let P_1 and P_2 be the projections of X on M_1 along M_2 and on M_2 along M_1, respectively. T is said to be <u>completely reduced</u> by the pair (M_1, M_2) if

(i) M_1 and M_2 are invariant under T, i.e. $T\left[D(T) \cap M_i\right] \subset M_i$ for i = 1, 2.

(ii) $P_i\left[D(T)\right] \subset D(T)$ for i = 1, 2.

A set of necessary and sufficient conditions that this be so, is that $P_1\left[D(T)\right] \subset D(T)$ and $P_1Tx = TP_1x$ if $x \in D(T)$. The corresponding conditions are then satisfied by P_2. The verification of these assertions is simple. It is also easy to check that if T is completely reduced by (M_1,M_2), then so is T^{-1}, if it exists.

Assuming that T is completely reduced by (M_1,M_2), let T_i be the restriction of T to M_i (with $D(T_i) = M_i \cap D(T)$) where i = 1,2. The following facts may be noted (we omit the proofs, which are straightforward and easy):

(7-1) $D(T) = D(T_1) \oplus D(T_2)$

whence $P_i\left[D(T)\right] = M_i \cap D(T)$, i = 1,2.

(7-2) $N(T) = N(T_1) \oplus N(T_2)$.

(7-3) $R(T) = R(T_1) \oplus R(T_2)$.

One sees that T^{-1} exists if and only if T_1^{-1} and T_2^{-1} both exist. Also, $R(T) = X$ if and only if $R(T_i) = M_i$ for i = 1,2. (Straightforward).

Theorem 7.1: Let T be a linear operator in X which is completely reduced by (M_1,M_2), and let T_i be the restriction of T to M_i $(i = 1,2)$. Then:

(a) $n(T) = n(T_1) + n(T_2)$ (The understanding, in the case where a dimension is infinite, is that $\infty + p = p + \infty = \infty$ if p is a non-negative integer or ∞).

Hence $n(T)<\infty$ if and only if both $n(T_1)$ and $n(T_2)$ are finite.

(b) $d(T) = d(T_1) + d(T_2)$ (with the same understanding as in (a)).

Hence $d(T)<\infty$ if and only if both $d(T_1)$ and $d(T_2)$ are finite.

(c) $\alpha(T_i) \leqslant \alpha(T)$ when $i = 1,2$.

If $\alpha(T_i) = p_i$ when $i = 1,2$, then $\alpha(T) = \max(p_1,p_2)$.

(d) $\delta(T_i) \leqslant \delta(T)$ when $i = 1,2$.

If $\delta(T_i) = q_i$ when $i = 1,2$, then $\delta(T) = \max(q_1, q_2)$.

(e) If $n(T) = d(T)<\infty$ and $\alpha(T)<\infty$, then $n(T_i) = d(T_i)<\infty$ and $\alpha(T_i) = \delta(T_i)<\infty$ when $i = 1,2$.

(f) If $D(T) = X$, if $n(T) = d(T)<\infty$, and if $\delta(T)<\infty$, then $n(T_i) = d(T_i)<\infty$ and $\alpha(T_i) = \delta(T_i)<\infty$ when $i = 1,2$.

(g) If $n(T_i) = d(T_i)<\infty$ and $\alpha(T_i)<\infty$ for $i = 1,2$, then $n(T) = d(T)<\infty$ and $\alpha(T) = \delta(T)<\infty$.

Proof of (a): This follows from (7-2).

Proof of (b): Let M be a complement of $R(T)$ in X, and let A_i be a complement of $R(T_i)$ in M_i $(i = 1,2)$. In view of (7-3) and the fact that $X = M_1 \oplus M_2$, we have

$$R(T_1) \oplus R(T_2) \oplus M = R(T_1) \oplus R(T_2) \oplus A_1 \oplus A_2$$

Hence $A_1 \oplus A_2$ is also a complement of $R(T)$ in X; (b) is now evident.

Proof of (c): It can be checked that T^n is completely reduced by (M_1, M_2) and that the restriction of T^n to M_i is $(T_i)^n$. Hence we can apply (7-2) and (7-3) with T^n in place of T. Suppose now that $\alpha(T) = p < \infty$.

If $x \in N(T_1^{p+1})$, we see that

$$x \in N(T^{p+1}) = N(T^p) = N(T_1^p) \oplus N(T_2^p).$$

We can write $x = x_1 + x_2$, where $x_i \in N(T_i^p)$ $(i = 1,2)$. Then $x - x_1 = x_2$. Here $x - x_1 \in N(T_1^{p+1})$ and $x_2 \in N(T_2^p) \subset N(T_2^{p+1})$

But then $x - x_1 = x_2 = 0$, and so $x = x_1 \in N(T_1^p)$. Thus

$\alpha(T_1) \leqslant p$. By symmetry we see that $\alpha(T_2) \leq p$.

And if $\alpha(T) = \infty$, then clearly $\alpha(T_i) \leqslant \alpha(T)$ when $i = 1,2$.

So in all cases $\alpha(T_i) \leqslant \alpha(T)$ when $i = 1,2$.

Conversely, suppose that $\alpha(T_1) = p_1 < \infty$, $\alpha(T_2) = p_2 < \infty$, and let $p = \max(p_1, p_2)$. Then

$$N(T^{p+1}) = N(T_1^{p+1}) \oplus N(T_2^{p+1})$$
$$= N(T_1^p) \oplus N(T_2^p) = N(T^p)$$

so that $\alpha(T) \leqslant p = \max(p_1, p_2)$. The first part of (c) now shows that $\max(p_1, p_2) \leqslant \alpha(T)$. Hence $\alpha(T) = \max(p_1, p_2)$.

And if $\alpha(T_1) = p_1 = \infty$ or $\alpha(T_2) = p_2 = \infty$ then $\max(p_1, p_2) = \infty$, and $\alpha(T) = \infty$ by the first part of (c). So $\alpha(T) = \max(p_1, p_2)$.

So in all cases, i.e. if $p_1 \leqslant \infty$ or $p_2 \leqslant \infty$, we have $\alpha(T) = \max(p_1, p_2)$.

Proof of (d): This is similar to the proof of (c) and I omit the argument.

Proof of (e): Theorem 5.5(c) shows that $\delta(T) = \alpha(T)$. Then $n(T_i)$, $d(T_i)$, $\alpha(T_i)$ and $\delta(T_i)$ are all finite, $(i = 1,2)$, by parts (a) - (d) of the present theorem. By Theorem 5.2 we infer that

$n(T_i) \leqslant d(T_i)$. Now

$$n(T_1) + n(T_2) = n(T) = d(T) = d(T_1) + d(T_2)$$

so $n(T_1) - d(T_1) = d(T_2) - n(T_2)$.

The left side here is not positive, and the right side is not negative. Hence both sides are equal to zero. So $n(T_i) = d(T_i) < \infty$ for $i = 1,2$. We can now apply Theorem 5.5(c) to conclude that $\alpha(T_i) = \delta(T_i)$ $(i = 1,2)$. This completes the proof.

Proof of (f): The argument is like that in proving (e), except that we use Theorem 5.5(d).

Proof of (g): By Theorem 5.5(c), $\delta(T_i) = \alpha(T_i) < \infty$ for $i = 1,2$.

Now $n(T) = n(T_1) + n(T_2) < \infty$ by (a) of the present theorem

$= d(T_1) + d(T_2) < \infty$

$= d(T)$ by (b) of the present theorem.

Also $\alpha(T) = \max\left[\alpha(T_1), \alpha(T_2)\right] < \infty$ by (c) of the present theorem

$= \max\left[\delta(T_1), \delta(T_2)\right] < \infty$ by Theorem 5.5(c).

$= \delta(T) < \infty$ by (d) of the present theorem.

CHAPTER VIII

Products of operators.

In this chapter we deal with linear operators in X which have all of X as their domain. No topology is required. We follow Taylor (13).

Theorem 8.1: Suppose that T_1 and T_2 are linear operators in X , each defined on all of X. Suppose $n(T_i) = d(T_i) < \infty$, $i = 1,2$.

(a) Then $n(T_1T_2) = d(T_1T_2) < \infty$.

(b) If we also assume that $T_1T_2 = T_2T_1$ and that $\alpha(T_i)$ and $\delta(T_i)$ are finite for $i = 1,2$, then $\alpha(T_1T_2)$ and $\delta(T_1T_2)$ are finite and equal.

Proof of (a): By Theorem 6.1(c) we can write $T_i = A_i + B_i$, where A_i and B_i are linear, defined on all of X, dim $R(B_i) \leq n(T_i)$, and A_i is a one-to-one mapping of X onto X. Write

$$T_1T_2 = A_1A_2 + (A_1B_2 + B_1A_2 + B_1B_2).$$

We see that A_1A_2 is a one-to-one mapping of X onto X, and $A_1B_2 + B_1A_2 + B_1B_2$ has finite-dimensional range. Hence $n(T_1T_2) = d(T_1T_2) < \infty$, by Theorem 6.2.

Proof of (b): Observe that $(T_1T_2)^n = T_1^nT_2^n$. We see inductively from (a) that $n(T^k) = d(T^k) < \infty$, $k = 2,3\ldots$, and that $n\left[(T_1T_2)^k\right] = d\left[(T_1T_2)^k\right] < \infty$, $k = 1,2,3,\ldots$. Let $\alpha(T_i) = p_i$. Then $N(T_i^k) \subset N(T_i^{p_i})$ if $k = 0,1,2,\ldots$, and hence $n(T_i^k) \leq n(T_i^{p_i})$.

Now $$N(T_1^kT_2^k) = \left\{x:\ T_2^kx \in N(T_1^k)\right\} ;$$

that is, $N(T_1^kT_2^k) = (T_2^k)^{-1}N(T_1^k)$, and so $T_2^k\left\{N(T_1^kT_2^k)\right\} \subset N(T_1^k)$.

Also, $N(T_1^k) \subset N(T_1^kT_2^k)$, because $T_1^kT_2^k = T_2^kT_1^k$. Hence $T_2^k\left\{N(T_1^kT_2^k)\right\} \subset N(T_1^kT_2^k)$. If we let T be the restriction of T_2^k to

$N(T_1^kT_2^k)$, we know from algebra that

$$\dim R(T) + n(T) = n(T_1^kT_2^k).$$

Clearly $\dim R(T) \leqslant n(T_1^k)$ and $n(T) \leqslant n(T_2^k)$.

Therefore $n(T_1^kT_2^k) \leqslant n(T_1^k) + n(T_2^k)$,

and hence $n(T_1^kT_2^k) \leqslant n(T_1^{p_1}) + n(T_2^{p_2})$.

It now follows from Lemma 4.1 that $\alpha(T_1T_2) \leqslant n(T_1^{p_1}) + n(T_2^{p_2}) < \infty$.

It then follows from Theorem 5.5(c) that $\delta(T_1T_2) = \alpha(T_1T_2) < \infty$. Q.E.D.

Remark: The next theorem is a sort of converse.

Theorem 8.2: Suppose that T_1 and T_2 are linear operators in X, each defined on all of X. Suppose that $T_1T_2 = T_2T_1$, $n(T_1T_2) = d(T_1T_2) < \infty$, and that $\alpha(T_1T_2) < \infty$. Then $n(T_i) = d(T_i) < \infty$ and $\alpha(T_i) = \delta(T_i) < \infty$. $(i = 1,2)$.

Proof: By Theorem 5.5(c) $\alpha(T_1T_2) = \delta(T_1T_2) < \infty$. Let $p = \alpha(T_1T_2) = \delta(T_1T_2)$. Since $(T_1T_2)^k = T_1^kT_2^k$, it follows from Theorem 8.1(a) that $n(T_1^pT_2^p) = d(T_1^pT_2^p) < \infty$. Now,

$$N(T_1^k) \subset N(T_2^kT_1^k) = N(T_1^kT_2^k) \subset N(T_1^pT_2^p)$$

if $k \geqslant 0$, and hence $n(T_1^k) \leqslant n(T_1^pT_2^p) < \infty$. Also,

$$R(T_1^k) \supset R(T_1^kT_2^k) \supset R(T_1^pT_2^p)$$

if $k \geqslant 0$, so that $d(T_1^k) \leqslant d(T_1^pT_2^p) < \infty$. It follows from Lemmas 4.1 and 4.2 that $\alpha(T_1)$ and $\delta(T_1)$ are finite, and therefore equal by Theorem 4.7 and the subsequent remark. It follows by Corollary 5.4 that $n(T_1) = d(T_1) < \infty$. The same considerations apply to T_2, because $T_1T_2 = T_2T_1$. Q.E.D.

CHAPTER IX

Closed linear operators in normed linear spaces.

In this chapter we deal with closed linear mappings from one normed linear space to another. In some cases we need completeness of one or more spaces. Some proofs are omitted.

Lemma 9.1: Let T be a closed linear mapping with domain in X and range in Y, where X and Y are normed linear spaces. Suppose that, for each conditionally compact (relatively compact) set S in Y, the bounded subsets of $T^{-1}(S)$ are conditionally compact. Then $n(T) < \infty$, and R(T) is closed.

Proof: It is clear that $n(T) < \infty$, for $N(T) = T^{-1}\{0\}$ and $\{0\}$ is compact in Y, so that the bounded subsets of N(T) are conditionally compact.

In proving that R(T) is closed, we observe, firstly, that if $\{x_n\}$ is a sequence in D(T) such that $Tx_n \to y$ and $y \notin R(T)$, then $\{x_n\}$ is not bounded. For, the set consisting of the Tx_n's is conditionally compact, and so if $\{x_n\}$ were bounded it would contain a convergent subsequence, say $\lim_{i\to\infty} x_{n_i} = x$, because the set of x_n's would be conditionally compact. But then, from $x_{n_i} \to x$, $Tx_{n_i} \to y$ and the fact that T is closed, we could conclude that $x \in D(T)$ and $Tx = y$, contradicting $y \notin R(T)$.

We now suppose that R(T) is not closed. Then there is a sequence $\{x_n\}$ in D(T) such that $Tx_n \to y$, where $y \notin R(T)$. The fact that T is closed implies that N(T) is closed. Certainly $y \neq 0$, and hence $Tx_n \neq 0$ i.e. $x_n \notin N(T)$, if n is sufficiently large. We can assume that $x_n \notin N(T)$ for every n. Then the distance d_n from x_n to N(T) is positive. Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\varepsilon_n \to 0$. Choose $u_n \in N(T)$ in such a way that $\|x_n - u_n\| < (1 + \varepsilon_n)d_n$. We observe that $T(x_n - u_n) = Tx_n \rightarrow y$, so that $\{x_n - u_n\}$ is not a bounded sequence. Choose $n_1, n_2, \ldots$ so that $\lim_{i\to\infty} \|x_{n_i} - u_{n_i}\| = \infty$. Let $w_i = \dfrac{x_{n_i} - u_{n_i}}{\|x_{n_i} - u_{n_i}\|}$.

Then $\|w_i\| = 1$ and $Tw_i = \dfrac{Tx_{n_i}}{\|x_{n_i} - u_{n_i}\|} \rightarrow 0$. The Tw_i's form a conditionally compact set and the w_i's form a bounded set. Therefore the w_i's form a conditionally compact set. Let $i_1, i_2, \ldots$ be a subsequence such that $w_{i_k} \rightarrow w$ as $k \rightarrow \infty$. Since T is closed and $Tw_{i_k} \rightarrow 0$, we infer that $w \in D(T)$ and $Tw = 0$, ie $w \in N(T)$. Then $u_{n_i} + \| x_{n_i} - u_{n_i}\| w \in N(T)$, and so

$$d_{n_i} \leq \left\| x_{n_i} - u_{n_i} - \|x_{n_i} - u_{n_i}\| w \right\| = \|x_{n_i} - u_{n_i}\| \cdot \| w_i - w\|$$

$$d_{n_i} \leq (1 + \varepsilon_{n_i}) d_{n_i} \|w_i - w\|, \text{ and } 1 \leq (1 + \varepsilon_{n_i}) \|w_i - w\| .$$

This contradicts the fact that $w_{i_k} \rightarrow w$. Thus R(T) must be closed.

Remark: For the converse we need complete spaces.

Lemma 9.2: Let X and Y be Banach spaces. Let T be a closed linear operator with domain in X and range in Y. Suppose that $n(T) < \infty$ and that R(T) is closed. Then, if S is a conditionally compact set in Y, each bounded subset of $T^{-1}(S)$ is conditionally compact.

Proof: Since $n(T) < \infty$, there exists a continuous projection P of X onto N(T). We can then write $X = R(P) \oplus N(P)$, where $R(P) = N(T)$. Suppose $x \in D(T)$. Then $Px \in N(T) \subset D(T)$, and hence $x - Px \in D(T)$. Note also that $x - Px \in N(P)$. We see that $Tx = T(x - Px)$, and hence $R(T) = T\left[D(T) \cap N(P)\right]$. If $x \in D(T) \cap N(P)$ and $Tx = 0$, then $x \in N(T) = R(P)$, so $x = 0$. So on $D(T) \cap N(P)$, T is one-to-one. Hence, if we define T_1 as follows: $D(T_1) = D(T) \cap N(P)$, $T_1 x = Tx$ when $x \in D(T_1)$, we see that T_1

is a one-to-one mapping of $D(T_1)$ onto $R(T)$. The fact that T is closed implies that T_1 is closed, and thus also that T_1^{-1} is closed. Since X and Y are complete, the closed subspaces $N(P)$, $R(T)$ can be regarded as complete spaces in themselves. But then, by the Closed Graph Theorem, T_1^{-1} is continuous, for it is a closed linear mapping of all of $R(T)$ into $N(P)$.

Now let S be a conditionally compact set in Y, and let Q be a bounded subset of $T^{-1}(S)$. Suppose $\{x_n\}$ is a sequence in Q. Then $\{Px_n\}$ is a bounded sequence in $N(T)$. Hence, because $\dim N(T) < \infty$, there is a subsequence $\{v_n\}$ of $\{x_n\}$ such that $\{Pv_n\}$ is convergent. Now, $Tv_n \in S$, and therefore there is a subsequence $\{w_n\}$ of $\{v_n\}$ such that $\{Tw_n\}$ is convergent. But

$Tw_n = TPw_n + T(w_n - Pw_n) = T(w_n - Pw_n) = T_1(w_n - Pw_n)$, or

$w_n - Pw_n = T_1^{-1}(Tw_n)$. From this we see that $\{w_n\}$ is convergent. Since $\{w_n\}$ is a subsequence of $\{x_n\}$, we have proved that Q is conditionally compact.

Remarks:

1. In Lemma 9.2 the only use of completeness of X and Y was in establishing that $N(P)$ and $R(T)$ are complete spaces.
2. The next lemma deals with a reformulation of the property which occurs in Lemmas 9.1 and 9.2.

Lemma 9.3: Let X and Y be normed linear spaces, and let T be a closed linear mapping with domain in X and range in Y. Then

(a) If T has either of the following two properties (i), (ii), it has the other:

Property (i) : If S is a conditionally compact set in Y and Q is a bounded subset of $T^{-1}(S)$, Q is conditionally compact.

Property (ii): If G is a compact set in Y and F is a closed and bounded subset of $T^{-1}(G)$, then F is compact.

(b) If T has property (i) and F is a closed and bounded subset of $D(T)$, then $T(F)$ is a closed set.

Remark: We do not insist that S and G be contained in $R(T)$, but our statements are valid in the special cases when $S \subset R(T)$ and $G \subset R(T)$. When, however, we talk of $T^{-1}(S)$ and $T^{-1}(G)$, we mean $T^{-1}(S \cap R(T))$ and $T^{-1}(G \cap R(T))$.

Proof of (a): Suppose T has property (i). We are required to prove that T has property (ii).

Let G be a compact set in Y.

Then G is a conditionally compact set in Y (since $\overline{G} = G$).

Therefore, by property (i), if F is a bounded subset of $T^{-1}(G)$, F is conditionally compact.

Therefore, if F is also closed (ie closed and bounded), F is closed and conditionally compact.

ie $F = \overline{F}$ and $\overline{F}$ is compact.

Therefore F is compact.

Hence Property (i) implies Property (ii).

Now suppose that T has property (ii). We are required to prove that T has property (i).

We first show that if H is a compact set in Y and T is a closed linear operator, then $T^{-1}(H)$ is closed.

Select $\{x_n\}$ in $T^{-1}(H)$, $x_n \to x$. We must show $x \in T^{-1}(H)$.

Now for all natural numbers n, $Tx_n \in T(T^{-1}(H)) = H$ and H is compact in Y.

Therefore from the infinite sequence $\{Tx_n\}$ we can select a convergent subsequence $\{Tx_{n_i}\}$ with $\lim Tx_{n_i} \in H$.

So $Tx_{n_i} \longrightarrow y \in H$ and $x_{n_i} \longrightarrow x$.

Since T is closed, $y = Tx$. Hence $Tx \in H$, whence $x \in T^{-1}(H)$.

Therefore $T^{-1}(H)$ is closed.

Now let S be a conditionally compact set in Y and let Q be an arbitrary bounded subset of $T^{-1}(S)$. We must show that Q is conditionally compact.

Let $G = \overline{S}$. Then G is compact in Y.

Let $F = \overline{Q}$. Then F is bounded (Q bounded implies $\overline{Q}$ bounded in any topological vector space) and closed as a subset of the space X.

In fact, F is a closed and bounded subset of the set $T^{-1}(G)$ for, putting G = H in the proof above, we see that $T^{-1}(G)$ is closed, whence $Q \subset T^{-1}(S) \subset T^{-1}(G) \Rightarrow F = \overline{Q} \subset T^{-1}(G)$.

Therefore, by property (ii), F is compact and Q is conditionally compact.

<u>Proof of (b)</u>: Recall that if T is a linear map from a normed space X into a normed space Y, then T is closed if and only if:

for all sequences $\{x_n\} \subset D(T), \left.\begin{matrix} x_n \to x \\ Tx_n \to y \end{matrix}\right\} \Rightarrow x \in D(T)$ and $Tx = y$

We are required to prove that T(F) is closed, i.e. if we take $\{y_n\} \subset T(F)$, $y_n \to y$, then $y \in T(F)$.

Since $F \subset D(T)$ and $y_n \in T(F)$, there exists $x_n \in F$ such that $y_n = Tx_n$.

So $Tx_n \longrightarrow y$.

Now, since $x_n \in F$ and F is bounded, $\{x_n\}$ is bounded.

Also since $\{y_n\}$ is a convergent sequence, $S = \{Tx_n\}$ is conditionally compact.

So we can apply property (i) with $Q = \{x_n\}$, a bounded set of points in $T^{-1}(S)$. By property (i), Q is conditionally compact. Therefore we can select a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Let $x_{n_i} \longrightarrow x$.

Now $\{x_{n_i}\} \subset D(T)$ and $Tx_{n_i} \rightarrow y$. Further, T is closed. So, since $x_{n_i} \rightarrow x$ and $Tx_{n_i} \rightarrow y$ we have $x \in D(T)$ and $Tx = y$ by our initial comment.

Also $x_{n_i} \in F$ and F is closed, whence $x \in F$.

Therefore $y = Tx \in T(F)$ as required. Q.E.D.

Lemma 9.4: Let X, Y, Z be normed linear spaces. Let A and B be closed linear mappings, with $D(A) \subset X$, $R(A) \subset Y$, $D(B) \subset Y$, $R(B) \subset Z$. Let C be defined as follows:

$$D(C) = \{x: x \in D(A),\ Ax \in D(B)\}, \quad Cx = BAx.$$

Suppose that A and B have property (ii) as in Lemma 9.3, and suppose that C is closed. Then C has property (ii).

Proof: Let G be a compact set in Z and let F be a closed and bounded subset of $C^{-1}(G)$. We wish to prove that F is compact. Observe that $A(F) \subset B^{-1}(G)$, because $x \in F$ implies $Cx = BAx \in G$, and so $Ax \in B^{-1}(G)$. We know that A(F) is closed (see Lemma 9.3). If we can show that A(F) is bounded, it will be compact, because B has property (ii). But $F \subset A^{-1}(A(F))$, and it will then follow that F is compact, by the property (ii) of A.

Proof that A(F) is bounded;

Suppose the contrary. Then there is a sequence $\{x_n\}$ in F such that $\|Ax_n\| \rightarrow \infty$. We can assume that $Ax_n \neq 0$. Now $C(F) \subset G$, so $BAx_n \in G$, and G is a bounded set (being compact). Let $y_n = \frac{Ax_n}{\|Ax_n\|}$. Then $By_n = \frac{BAx_n}{\|Ax_n\|} \rightarrow 0$ because $\{BAx_n\}$ is bounded.

Now, $\|y_n\| = 1$, so the y_n's form a bounded set which is mapped by B into a conditionally compact set. Hence there is a convergent subsequence $\{y_{n_i}\}$ with limit y. Observe now that

$\frac{x_{n_i}}{\|Ax_{n_i}\|} \to 0$ as $i \to \infty$ (because $x_{n_i} \in F$ and F is bounded), and

$$A\left(\frac{x_{n_i}}{\|Ax_{n_i}\|}\right) = y_{n_i} \to y.$$

Since A is closed, it follows that $0 = A(0) = y$. This contradicts $\|y\| = 1$, and the proof is finished.

Theorem 9.5: Let X be a Banach space, and let T be a closed linear operator with domain and range in X. Suppose also that all powers $T^2, T^3, \ldots$ are closed. Then, if $n(T) < \infty$ and $R(T)$ is closed, it follows that $n(T^k) < \infty$ and $R(T^k)$ is closed, $k = 2, 3, \ldots$.

Proof: We appeal to Lemmas 9.2, 9.3, 9.4, 9.1, and use induction. The proof is straightforward.

Lemma 9.6: (Riesz's Lemma) Suppose that X is a normed linear space. Let X_0 be a subspace of X such that X_0 is a closed and proper subset of X. Then for each θ such that $0 < \theta < 1$, there exists a vector $x_\theta \in X$ such that $\|x_\theta\| = 1$ and $\|x - x_\theta\| \geq \theta$ for all $x \in X_0$. i.e. $\rho(x_\theta, X_0) \geq \theta$ where $\rho(x_\theta, X_0) = \inf_{x \in X_0} \|x - x_\theta\|$.

Proof: See Taylor (12), Theorem 3.12-E.

Theorem 9.7: Let T be a compact linear operator from the normed linear space X into X, and let $\lambda \neq 0$. Then

$$\alpha(\lambda I - T) = \delta(\lambda I - T) < \infty.$$

Proof: Suppose that $\alpha(\lambda I - T) = \infty$.
Then $N\left[(\lambda I - T)^{n-1}\right]$ is a proper closed subspace of $N\left[(\lambda I - T)^n\right] = N_n$ for $n = 1, 2, \ldots$. By Lemma 9.6 there exists $x_n \in N_n$ such that $\|x_n\| = 1$ and $\|x_n - x\| \geq \frac{1}{2}$ if $x \in N_{n-1}$. Assume $1 \leq m < n$ and let

$$z = x_m + \lambda^{-1}(\lambda I - T)(x_n - x_m)$$

Then $(\lambda I - T)^{n-1}z = (\lambda I - T)^{n-1}x_m + \lambda^{-1}(\lambda I - T)^n(x_n - x_m) = 0$ and so $z \in N_{n-1}$. Hence $\|x_n - z\| \geqslant \frac{1}{2}$. But we easily calculate that $Tx_n - Tx_m = \lambda(x_n - z)$ and so $\|Tx_n - Tx_m\| \geqslant \frac{|\lambda|}{2} > 0$. This shows that $\{Tx_n\}$ can have no convergent subsequence even though $\{x_n\}$ is bounded. This contradicts the fact that T is compact. Therefore $\alpha(\lambda I - T) < \infty$.

The proof that $\delta(\lambda I - T)$ is finite is similar. Suppose $\delta(\lambda I - T) = \infty$. Then $R[(\lambda I - T)^n] = R_n$ would be a proper closed subspace of R_{n-1} for $n = 1,2,\ldots$. By Riesz's Lemma we choose $y_n \in R_n$ such that $\|y_n\| = 1$ and $\|y_n - y\| \geqslant \frac{1}{2}$ for all $y \in R_{n+1}$. If $1 \leqslant m < n$ let

$$w = y_n - \lambda^{-1}(\lambda I - T)y_n + \lambda^{-1}(\lambda I - T)y_m.$$

We can write $y_k = (\lambda I - T)^k x_k$ for some x_k. Thus

$$w = (\lambda I - T)^n x_n - \lambda^{-1}(\lambda I - T)^{n+1}x_n + \lambda^{-1}(\lambda I - T)^{m+1}x_m$$

from which we see that $w \in R_{m+1}$. Therefore $\|y_m - w\| \geqslant \frac{1}{2}$.

But $Ty_m - Ty_n = \lambda(y_m - w)$, and so $\|Ty_m - Ty_n\| \geqslant \frac{|\lambda|}{2} > 0$

So $\{Ty_n\}$ has no convergent subsequence. This contradicts the compactness of T. Therefore $\delta(\lambda I - T) < \infty$.

Hence $\alpha(\lambda I - T) = \delta(\lambda I - T) < \infty$ by Theorem 4.7 and the remark which followed. Q.E.D.

<u>Remark</u>: If T is closed and X complete there is a connection between Theorem 9.7 above and the purely algebraic Theorem 5.7, since by the Closed Graph Theorem $D(T) = X$ implies T is bounded. Also, every bounded operator with finite-dimensional range is compact, and in Theorem 5.7 we have $\dim R(T) < \infty$. Both theorems require $\lambda \neq 0$ and prove $\alpha(\lambda I - T) = \delta(\lambda I - T) < \infty$.

CHAPTER X

Conclusion

The stage is now set for the study of the applications of nullity, defect, ascent and descent to Spectral Theory for closed linear operators in Banach space. These quantities have wide applications to both Spectral Theory and Perturbation Theory, which are, unfortunately, beyond the scope of this thesis. References will be given a little later. In the meanwhile, in order to whet the reader's appetite for more, I present a summary of a proof to illustrate how nullity and defect can be applied to Perturbation Theory. The proof is due to D. van Dulst (15), who entitled it a "Functional Analytic Proof of Rouché's Theorem". He shows how Rouché's theorem (cf Saks and Zygmund (11)) can be derived from a perturbation theorem for linear operators in Banach spaces which is due to T. Kato (8).

Theorem 10.1 Rouché's Theorem

Let f and g be analytic on a bounded open set G in the complex plane, and continuous on the closure $\overline{G}$. If $|g(z)| < |f(z)|$ for all $z \in \overline{G} \setminus G$, then f and f + g have the same number of zeros in G (a zero of order p being counted p times).

Summary of proof:

Firstly, $B(\overline{G})$ is defined as the Banach space of all functions h continuous on $\overline{G}$ and analytic on G, with norm $\|h\| = \sup_{z \in \overline{G}} |h(z)|$.

Our object is to prove that the number of zeros of f equals the number of zeros of f + g. To this end van Dulst produces an operator T_f on $B(\overline{G})$ such that $d(T_f)$ equals the number of zeros of f, and which with T_g satisfies the conditions of Kato's perturbation theorem, whence $d(T_{f+g}) = d(T_f)$ i.e. the required result.

We note in passing that T_f is nothing more than the multiplicative operator defined by $T_f h = fh$.

To show that the codimension of $R(T_f)$, viz $d(T_f)$, equals the number of zeros of f, van Dulst produces a subspace N of $B(\overline{G})$ which is a complement (i.e. a "direct sum" complement) of $R(T_f)$ and has dimension equal to the number of zeros of f. The rest of the proof is taken up with showing that T_f and T_g satisfy the conditions of Kato's theorem.

Remarks:

This proof is longer than the usual Complex Variable proof of Rouché's Theorem, but van Dulst claims for it the advantages that "using this approach one needs only two facts concerning analytic functions, namely the possibility of power series expansion and the maximum modulus principle. No integration is involved in this proof". It is true that the proof avoids contour integration, but if this is the advantage one is claiming for it, one must prove all one's subsidiary results without using integration, and, for example, the process of power series expansion, as normally done, does require complex integration.

The following is an instance where this problem can be rectified. We have at one stage

$$(10\text{-}1)\quad F(z) = a_0+a_1(z-z_1) + a_2(z-z_1)^2 +\ldots+ a_{p_1-1}(z-z_1)^{p_1-1}+(z-z_1)^{p_1} H(z)$$

where $F(z) \in B(\overline{G})$, and we need to know that H(z) is analytic in a neighbourhood of $z_2 (z_2 \neq z_1)$, so that we can expand H(z) in a neighbourhood of z_2 as an analytic function.

It is tempting to invoke Theorem 8 on page 101 of L.V. Ahlfors' "Complex Analysis" (International Student Edition), (McGraw-Hill, Kōgakusha) (1953), but this theorem depends upon the preceding theorem (Theorem 7) which does require integration. So we must find a different proof. The difficulty is that z_2 may not lie in $F(z)$'s circle of convergence about z_1. We overcome the problem as follows:

Let $g(z) = a_0 + a_1(z-z_1) + a_2(z-z_1)^2 + \dots + a_{p_1-1}(z-z_1)^{p_1-1}$

Then $H(z) = \dfrac{F(z) - g(z)}{(z-z_1)^{p_1}}$ (derived from (10-1))

Now $F(z)$ is analytic in a neighbourhood of z_2 (since we are given that it is analytic in G). And $g(z)$ is analytic in a neighbourhood of z_2. Also $\dfrac{1}{(z-z_1)^{p_1}}$ is analytic in a small enough neighbourhood of z_2 (given $z_2 \neq z_1$). Therefore $H(z)$ is analytic in a neighbourhood of z_2.

In conclusion it should be pointed out that one definite advantage of van Dulst's proof is that the boundary need not consist of a finite number of rectifiable simple closed curves.

We conclude by listing a few research papers which deal with further developments in the theory. The applications of $n(T)$, $d(T)$, $\alpha(T)$ and $\delta(T)$ to Spectral Theory are discussed in Kaashoek (6), Kaashoek and Lay (7), Caradus (3) and (2), Lay (9) and, of course, Taylor (13) Section 9. The applications of $n(T)$ and $d(T)$ to Perturbation Theory are discussed in Caradus (2), Kaashoek (4) and Beals (1), whilst van Dulst generalises certain perturbation theorems in Banach spaces to locally convex spaces in (14).

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Then $(\lambda I - T)^{n-1}z = (\lambda I - T)^{n-1}x_m + \lambda^{-1}(\lambda I - T)^n(x_n - x_m) = 0$ and so $z \in N_{n-1}$. Hence $\|x_n - z\| \geqslant \frac{1}{2}$. But we easily calculate that $Tx_n - Tx_m = \lambda(x_n - z)$ and so $\|Tx_n - Tx_m\| \geqslant \frac{|\lambda|}{2} > 0$. This shows that $\{Tx_n\}$ can have no convergent subsequence even though $\{x_n\}$ is bounded. This contradicts the fact that T is compact. Therefore $\alpha(\lambda I - T) < \infty$.

The proof that $\delta(\lambda I - T)$ is finite is similar. Suppose $\delta(\lambda I - T) = \infty$. Then $R\left[(\lambda I - T)^n\right] = R_n$ would be a proper closed subspace of R_{n-1} for $n = 1,2,\ldots$. By Riesz's Lemma we choose $y_n \in R_n$ such that $\|y_n\| = 1$ and $\|y_n - y\| \geqslant \frac{1}{2}$ for all $y \in R_{n+1}$. If $1 \leqslant m < n$ let

$$w = y_n - \lambda^{-1}(\lambda I - T)y_n + \lambda^{-1}(\lambda I - T)y_m.$$

We can write $y_k = (\lambda I - T)^k x_k$ for some x_k. Thus

$$w = (\lambda I - T)^n x_n - \lambda^{-1}(\lambda I - T)^{n+1}x_n + \lambda^{-1}(\lambda I - T)^{m+1}x_m$$

from which we see that $w \in R_{m+1}$. Therefore $\|y_m - w\| \geqslant \frac{1}{2}$.

But $Ty_m - Ty_n = \lambda(y_m - w)$, and so $\|Ty_m - Ty_n\| \geqslant \frac{|\lambda|}{2} > 0$

So $\{Ty_n\}$ has no convergent subsequence. This contradicts the compactness of T. Therefore $\delta(\lambda I - T) < \infty$.

Hence $\alpha(\lambda I - T) = \delta(\lambda I - T) < \infty$ by Theorem 4.7 and the remark which followed. Q.E.D.

<u>Remark</u>: If T is closed and X complete there is a connection between Theorem 9.7 above and the purely algebraic Theorem 5.7, since by the Closed Graph Theorem $D(T) = X$ implies T is bounded. Also, every bounded operator with finite-dimensional range is compact, and in Theorem 5.7 we have $\dim R(T) < \infty$. Both theorems require $\lambda \neq 0$ and prove $\alpha(\lambda I - T) = \delta(\lambda I - T) < \infty$.

CHAPTER X

Conclusion

The stage is now set for the study of the applications of nullity, defect, ascent and descent to Spectral Theory for closed linear operators in Banach space. These quantities have wide applications to both Spectral Theory and Perturbation Theory, which are, unfortunately, beyond the scope of this thesis. References will be given a little later. In the meanwhile, in order to whet the reader's appetite for more, I present a summary of a proof to illustrate how nullity and defect can be applied to Perturbation Theory. The proof is due to D. van Dulst (15), who entitled it a "Functional Analytic Proof of Rouché's Theorem". He shows how Rouché's theorem (cf Saks and Zygmund (11)) can be derived from a perturbation theorem for linear operators in Banach spaces which is due to T. Kato (8).

Theorem 10.1 Rouché's Theorem

Let f and g be analytic on a bounded open set G in the complex plane, and continuous on the closure $\overline{G}$. If $|g(z)| < |f(z)|$ for all $z \in \overline{G}\setminus G$, then f and f + g have the same number of zeros in G (a zero of order p being counted p times).

Summary of proof:

Firstly, $B(\overline{G})$ is defined as the Banach space of all functions h continuous on $\overline{G}$ and analytic on G, with norm $\|h\| = \sup_{z \in \overline{G}} |h(z)|$.

Our object is to prove that the number of zeros of f equals the number of zeros of f + g. To this end van Dulst produces an operator T_f on $B(\overline{G})$ such that $d(T_f)$ equals the number of zeros of f, and which with T_g satisfies the conditions of Kato's perturbation theorem, whence $d(T_{f+g}) = d(T_f)$ i.e. the required result.

We note in passing that T_f is nothing more than the multiplicative operator defined by $T_f h = fh$.

To show that the codimension of $R(T_f)$, viz $d(T_f)$, equals the number of zeros of f, van Dulst produces a subspace N of $B(\overline{G})$ which is a complement (i.e. a "direct sum" complement) of $R(T_f)$ and has dimension equal to the number of zeros of f. The rest of the proof is taken up with showing that T_f and T_g satisfy the conditions of Kato's theorem.

Remarks:

This proof is longer than the usual Complex Variable proof of Rouché's Theorem, but van Dulst claims for it the advantages that "using this approach one needs only two facts concerning analytic functions, namely the possibility of power series expansion and the maximum modulus principle. No integration is involved in this proof". It is true that the proof avoids contour integration, but if this is the advantage one is claiming for it, one must prove all one's subsidiary results without using integration, and, for example, the process of power series expansion, as normally done, does require complex integration.

The following is an instance where this problem can be rectified. We have at one stage

(10-1) $F(z) = a_0+a_1(z-z_1) + a_2(z-z_1)^2 +\ldots+ a_{p_1-1}(z-z_1)^{p_1-1}+(z-z_1)^{p_1} H(z)$

where $F(z) \in B(\overline{G})$, and we need to know that $H(z)$ is analytic in a neighbourhood of $z_2 (z_2 \neq z_1)$, so that we can expand $H(z)$ in a neighbourhood of z_2 as an analytic function.

It is tempting to invoke Theorem 8 on page 101 of L.V. Ahlfors' "Complex Analysis" (International Student Edition), (McGraw-Hill, Kōgakusha) (1953), but this theorem depends upon the preceding theorem (Theorem 7) which does require integration. So we must find a different proof. The difficulty is that z_2 may not lie in $F(z)$'s circle of convergence about z_1. We overcome the problem as follows:

Let $g(z) = a_0 + a_1(z-z_1) + a_2(z-z_1)^2 + \dots + a_{p_1-1}(z-z_1)^{p_1-1}$

Then $H(z) = \dfrac{F(z) - g(z)}{(z-z_1)^{p_1}}$ (derived from (10-1))

Now $F(z)$ is analytic in a neighbourhood of z_2 (since we are given that it is analytic in G). And $g(z)$ is analytic in a neighbourhood of z_2. Also $\dfrac{1}{(z-z_1)^{p_1}}$ is analytic in a small enough neighbourhood of z_2 (given $z_2 \neq z_1$). Therefore $H(z)$ is analytic in a neighbourhood of z_2.

In conclusion it should be pointed out that one definite advantage of van Dulst's proof is that the boundary need not consist of a finite number of rectifiable simple closed curves.

We conclude by listing a few research papers which deal with further developments in the theory. The applications of $n(T)$, $d(T)$, $\alpha(T)$ and $\delta(T)$ to Spectral Theory are discussed in Kaashoek (6), Kaashoek and Lay (7), Caradus (3) and (2), Lay (9) and, of course, Taylor (13) Section 9. The applications of $n(T)$ and $d(T)$ to Perturbation Theory are discussed in Caradus (2), Kaashoek (4) and Beals (1), whilst van Dulst generalises certain perturbation theorems in Banach spaces to locally convex spaces in (14).

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