



Ideals in von Neumann algebras  
and in  
associated operator algebras

by

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## 0.1 Introduction

The compact operators on a Hilbert space are those operators for which the image of the unit ball is relatively compact in the norm topology. These operators form an ideal, in the algebra of all continuous linear operators on the Hilbert space, which is closed in the uniform norm. In the case that the underlying Hilbert space is separable this is the only such ideal, while for non-separable Hilbert spaces the norm-closed ideals are easily characterised by means of cardinal numbers.

The algebra of all continuous linear operators on a Hilbert space is a specific example of a von Neumann algebra, and the theory of compact operators and the ideal they form admit certain generalisations to von Neumann algebras. One of the characterisations of the ideal of compact operators is that it is the closure of the ideal of finite rank operators, and hence the closed ideal generated by the finite dimensional projections. Kaftal has considered the ideal of so called algebraically compact operators, which is defined to be the closed ideal generated by the algebraically finite projections in the von Neumann algebra, and has shown that this ideal consists of those operators which map the unit ball to sets which have compact-like properties. This characterisation was generalised to arbitrary norm-closed ideals by Ströh.

In this thesis we explore the extent to which norm-closed ideals in von Neumann algebras resemble the ideal of compact operators on a Hilbert space. We extend the theory developed by Kaftal and Ströh, and show that arbitrary ideals in von Neumann algebras can be characterised in terms of bornologies and topologies.

We also consider continuity characterisations of norm-closed ideals in von Neumann algebras, generalising the characterisation of the compact operators as being those that are continuous from the unit ball equipped with the weak topology, to the Hilbert space equipped with the norm topology. Furthermore we briefly consider sequential continuity characterisations as first analysed by Kaftal in the case of the algebraically compact ideal.

Finally, in the case of a semifinite von Neumann algebra equipped with a faithful semifinite normal trace  $\tau$ , we generalise the characterisation of the compact operators given in terms of the singular value sequence, by showing that the ideal of  $\tau$ -measurable operators whose generalised singular function decreases to 0 possess many of the same properties as the ideal of compact operators.

We now give a more detailed discussion of the contents of each of the chapters.

In Chapter 1 we analyse the properties enjoyed by the set of projections that belong to a given ideal. Conversely, one can ask to what extent these projections determine the ideal in question. This old problem has been fully solved by Wright via the study of p-

ideals; he showed that every closed ideal is the closed ideal generated by its projections. This generalises the fact that the ideal of compact operators is generated by the finite rank projections. In Chapter 1 we give a brief summary of the theory of p-ideals as a proper understanding of this theory is crucial for later developments. Furthermore we derive some new results, some of which are used extensively in subsequent chapters. We also give a number of straightforward examples of p-ideals, and introduce a class of p-ideals which we call  $\sigma$ p-ideals.

In Chapter 2 we consider the quotient map

$$\pi : \mathcal{M} \rightarrow \mathcal{M}/\mathcal{I}$$

where  $\mathcal{I}$  is a norm-closed ideal in a von Neumann algebra  $\mathcal{M}$ , and spectral theory in this quotient algebra (which is in fact a  $C^*$ -algebra). We also examine Fredholm theory in the quotient, as developed by Olsen for example, and show that the quotient algebra behaves very much like the classical Calkin algebra. Some of the material here is the work of others (although in many cases we offer new or improved proofs) but is necessarily included as it forms a basis for subsequent developments in the thesis.

In Chapter 3 we show that for any ideal  $\mathcal{I}$  in a von Neumann algebra,  $\mathcal{I}$  not necessarily norm closed, there exists a topology  $t(\mathcal{I})$  on  $\mathcal{H}$  which characterises  $\mathcal{I}$  in that for  $x \in \mathcal{M}$ ,

$$x \in \mathcal{I} \Leftrightarrow x : (\mathcal{H}, t(\mathcal{I})) \rightarrow (\mathcal{H}, \|\cdot\|)$$

is continuous. A dual bornological characterisation is also derived : we consider a bornology  $b(\mathcal{I})$ , which is in fact the equicontinuous bornology of  $t(\mathcal{I})$ , which characterises  $\mathcal{I}$  in that for  $x \in \mathcal{M}$ ,

$$x \in \mathcal{I} \Leftrightarrow x\mathcal{B}_{\mathcal{H}} \in b(\mathcal{I})$$

These results apply in particular to the smallest ideal generated by a p-ideal, and we spend some time discussing this case, in order to prepare for subsequent developments. There are numerous parallels in this chapter to the theory of ideals of operators between Banach spaces as developed by Pietsch, and in particular to the characterisations due to Stephani of injective and surjective operator ideals.

In Chapter 4 we study a total boundedness characterisations of norm-closed ideals by considering a generalisation of the measure of non-compactness. This function, called  $q_{\mathcal{I}}$ , is seen to satisfy all of the usual properties of the standard measure of non-compactness and moreover determines  $\mathcal{I}$  in that

$$x \in \mathcal{I} \Leftrightarrow q_{\mathcal{I}}(x\mathcal{B}_{\mathcal{H}}) = 0$$

In the case that the ideal in question is the ideal of compact operators on  $\mathcal{H}$ , the function  $q_{\mathcal{I}}$  does in fact reduce to the standard measure of non-compactness.

In the last section of Chapter 4 we study an application of the function  $q_{\mathcal{I}}$  to left-Fredholm operators.

In Chapter 5 we generalise the characterisation of the compact operators, as being those which are continuous from the unit ball equipped with the weak topology to the Hilbert space equipped with the norm topology, to any norm-closed ideal in a von Neumann algebra. Observing that the weak topology is determined by the p-ideal of the compact operators, we find that the topology which is determined by the p-ideal of the given ideal leads to an analogous characterisation. In this chapter we make substantial use of the theory of mixed topologies.

In the last section of Chapter 5 we look at some ideals determined by sequential continuity criteria. It is known that the compact operators are those which are sequentially continuous from the Hilbert space equipped with the weak topology to the Hilbert space equipped with the norm topology. This has been generalised for the ideal of algebraically compact operators by Kaftal but substantial hurdles remain in the general case.

In Chapter 6 we suppose  $\mathcal{M}$  is a semifinite von Neumann algebra equipped with a faithful semifinite normal trace  $\tau$ , and consider the algebra  $\widetilde{\mathcal{M}}$  of  $\tau$ -measurable operators affiliated to  $\mathcal{M}$ . This is of course an algebra of unbounded operators. We briefly analyse the ideals in  $\widetilde{\mathcal{M}}$  that are closed in the topology of convergence in measure and see that there are a number of connections between measure-closed ideals in  $\widetilde{\mathcal{M}}$  and norm-closed ideals in  $\mathcal{M}$ , some of which are established by means of further work with p-ideals.

For most of Chapter 6 we study the ideal of  $\tau$ -compact operators, which consists of those operators whose generalised singular function decreases to 0. This gives an appropriate generalisation of the characterisation of the compact operators as being those operators whose singular value sequence decreases to 0. Furthermore, we show that this ideal satisfies many of the compactness-like properties of ideals of bounded operators already discussed.

In the last section of this Chapter we use properties of the  $\tau$ -compact operators previously established to prove a commutativity result for the trace  $\tau$ .

## 0.2 Index of notation

Our conventions for Hilbert spaces are as follows: points in Hilbert space will be denoted by  $\xi, \zeta, \eta$ , etc. Furthermore:

$\mathcal{H}$	Hilbert space
$\mathcal{B}_{\mathcal{H}}$	unit ball of the Hilbert space $\mathcal{H}$
$co B$	convex hull of $B$
$\overline{co} B$	closed convex hull of $B$
$ac B$	absolutely convex hull of $B$
$\overline{ac} B$	closed absolutely convex hull of $B$
$[B]$	closed subspace generated by $B$
$\mathcal{B}(\mathcal{H})$	algebra of all bounded operators on $\mathcal{H}$
$\mathcal{K}(\mathcal{H})$	compact operators in $\mathcal{B}(\mathcal{H})$
$\mathcal{F}(\mathcal{H})$	finite dimensional operators in $\mathcal{B}(\mathcal{H})$
$\mathcal{P}(\mathcal{H})$	power set of $\mathcal{H}$

There is a one-to-one correspondence between closed subspaces of a Hilbert space  $\mathcal{H}$  and selfadjoint projections on  $\mathcal{H}$ . On occasion, to prevent notation becoming cumbersome, we do not distinguish between a closed subspace and the projection onto that subspace. Remember that  $[B]$  denotes the closed subspace generated by  $B$ .

Our conventions concerning linear operators are as follows: we use  $x, y, z$  for bounded operators and also densely defined unbounded operators in a Hilbert space. Bounded operators are understood to be everywhere defined. The symbol  $u$  will usually denote a unitary operator and  $v$  a partial isometry, although this will always be made clear in the context. Furthermore:

$D(x)$	domain of $x$ , only used for unbounded operators
$x \subset y$	$D(x) \subset D(y)$ and $x\xi = y\xi$ for all $\xi \in D(x)$
$1$	the identity operator on $\mathcal{H}$
$x_{\lambda}$	$x - \lambda 1$
$\sigma(x)$	spectrum of $x$
$\rho(x)$	resolvent set of $x$
$x^{+}$	positive part of the self-adjoint operator $x$
$x^{-}$	negative part of the self-adjoint operator $x$
$Ker(x)$	null space of $x$
$N(x)$	projection onto $Ker(x)$
$r(x)$	right support projection of $x$ , also called ...
$supp(x)$	... the support of $x$ . It coincides with $1 - N(x)$
$range(x)$	range of $x$
$l(x)$	left support projection of $x$ , which coincides with ...
$R(x)$	... the projection on the closure of the $range(x)$

We follow the following notational conventions concerning von Neumann algebras and their subsets:

$\mathcal{M}$	A von Neumann algebra, understood to be a subalgebra of $\mathcal{B}(\mathcal{H})$
$\mathcal{B}_{\mathcal{M}}$	unit ball of $\mathcal{M}$
$\mathcal{M}^u$	set of unitary operators in $\mathcal{M}$
$\mathcal{M}^{sa}$	set of self-adjoint operators in $\mathcal{M}$
$\mathcal{M}^p$	lattice of self-adjoint projections in $\mathcal{M}$
$\eta(\mathcal{M})$	algebra of closed densely defined operators affiliated with $\mathcal{M}$
$\mathcal{M}'$	commutant of $\mathcal{M}$ in $\mathcal{B}(\mathcal{H})$
$\mathcal{Z}(\mathcal{M})$	centre of $\mathcal{M}$ , that is, $\mathcal{M} \cap \mathcal{M}'$
$A^p$	$A \cap \mathcal{M}^p$ , where it is usually understood that $A \subset \mathcal{M}$
$A^{sa}$	$A \cap \mathcal{M}^{sa}$ , where it is usually understood that $A \subset \mathcal{M}$
$((B))$	two sided ideal in $\mathcal{M}$ generated by $B \subset \mathcal{M}$
$[[B]]$	closed two sided ideal in $\mathcal{M}$ generated by $B \subset \mathcal{M}$
$p \sim q$	$p$ and $q$ are equivalent projections
$\mathcal{P}_{\sim}$	algebraically finite projections in $\mathcal{M}^p$
$\mathcal{M}_{\sim}$	ideal of algebraically compact operators

If  $p$  and  $q$  are equivalent projections in a von Neumann algebra  $\mathcal{M}$ , and this equivalence is implemented by the partial isometry  $v$  in the sense that  $v^*v = p$  and  $vv^* = q$ , then we will notate this as  $p \overset{v}{\sim} q$ .

Topologies will be denoted by  $\mu, \nu$ , etc. This applies both to topologies on Hilbert spaces or on von Neumann algebras. In either case the symbol  $\|\cdot\|$  denotes both the norm, and the topology induced by the norm. Furthermore:

$\overset{\mu}{\rightarrow}$	convergence in the topology $\mu$
$wo$	the weak-operator topology
$so$	the strong-operator topology
$\sigma$	the weak topology on $\mathcal{H}$

## 0.3 Prerequisites

### The Spectral Theorem

We will need some advanced spectral theory and operational calculus in the course of our work; it is appropriate to state in some detail the versions and results that we will be using.

Much of our work is with bounded operators, but we will also make a substantial amount of use of the spectral theory of unbounded operators. When dealing with unbounded operators, we always consider closed densely defined operators. In particular, the sum and product operations are the strong operations i.e. the closure of the ordinary sum and product. This makes sense since the sum and product of closed operators are always preclosed. Furthermore, we follow the usual convention that the operator norm of an unbounded operator is infinite.

Suppose  $\mathcal{M}$  is a von Neumann algebra; denote by  $\mathcal{M}^p$  the lattice of all self-adjoint projections in  $\mathcal{M}$ . The following definition will more than suffice for our purposes.

**Definition 0.3.1** *A spectral measure is a Boolean algebra homomorphism*

$$B(\mathbb{C}) \rightarrow \mathcal{M}^p : B \rightarrow e_B$$

such that  $e_{\mathbb{C}} = 1$ . (Here  $B(\mathbb{C})$  denotes the algebra of Borel measurable subsets of  $\mathbb{C}$ .)

A spectral measure is said to be countably additive if

$$\sum_{i=1}^{\infty} e_{B_i} = e_{\bigcup_{i=1}^{\infty} B_i}$$

for  $\{B_i\}$  a disjoint sequence in  $B(\mathbb{C})$ .

**Theorem 0.3.2** [DS88] Theorem XII 2.3, [KR86] Theorems 5.2.1, 5.2.2, 5.6.18

Suppose  $\mathcal{M}_x$  is the abelian von Neumann algebra generated by a self-adjoint operator  $x$  acting on a Hilbert space  $\mathcal{H}$ . (For von Neumann algebras generated by unbounded operators, see [KR86] Remark 5.6.11.) Then  $\sigma(x) \subset \mathbb{R}$ , and there exists a countably additive spectral measure  $e(x)$  with range in  $\mathcal{M}_x$  that vanishes off  $\sigma(x)$ .

We put

$$e_t(x) = e_{(-\infty, t] \cap \sigma(x)}(x)$$

The family  $\{e_t(x) : t \in \mathbb{R}\}$  is called the spectral family for  $x$ , or the spectral resolution of  $x$ , and satisfies the following properties :-

(a) If  $t_1 \leq t_2$  then  $e_{t_1}(x) \leq e_{t_2}(x)$

(b) The family is right continuous, that is,  $e_s(x) \downarrow e_t(x)$  as  $s \downarrow t$

(c)  $e_t(x) \uparrow 1$  as  $t \uparrow \infty$

(d)  $e_t(x) \downarrow 0$  as  $t \downarrow -\infty$

(e)  $x e_t(x) \leq t e_t(x)$  for  $t \in \mathbf{R}$

(f)  $t(1 - e_t(x)) \leq x(1 - e_t(x))$  for  $t \in \mathbf{R}$

The spectral measure  $e.(x)$  is uniquely determined subject to the above conditions.

Suppose  $\mathcal{M}_x \cong \mathcal{C}(X)$ , where  $X$  is an extremely disconnected compact Hausdorff space. Suppose further that  $x \in \mathcal{M}_x$  i.e.  $x$  is bounded. If  $x$  corresponds to  $f \in \mathcal{C}(X)$  then  $e_t(x)$  corresponds to the characteristic function of

$$X_t = \overline{\{x \in X : f(x) > t\}}$$

which is the largest clopen set on which  $f \leq t$ .

(If  $x$  is unbounded then a similar characterisation holds, although this is technically somewhat more complicated - and we will not need to make use of it - so we will not deal with it here. For details see [KR86] Theorem 5.6.12, and §5.6 in general.)

The spectral family determines  $x$  in the following manner:

(a)

$$D(x) = \{\xi \in \mathcal{H} : \int_{-\infty}^{\infty} t^2 d\|e_t(x)\xi\|^2 < \infty\} \quad (0.1)$$

(b) For  $\xi \in D(x)$  and  $\zeta \in \mathcal{H}$

$$\langle x\xi, \zeta \rangle = \int_{-\infty}^{\infty} t d\langle e_t(x)\xi, \zeta \rangle \quad (0.2)$$

A closed operator  $x$  with domain  $D(x)$  dense in  $\mathcal{H}$  is said to be affiliated to  $\mathcal{M}$  if  $yx \subset xy$  for all  $y$  in the commutant  $\mathcal{M}'$  of  $\mathcal{M}$ . The collection of affiliated operators is a  $*$ -algebra with respect to strong sum, strong product and the adjoint operation. This algebra will be denoted  $\eta(\mathcal{M})$ .

The following theorem deals with the operational calculus.

**Theorem 0.3.3** [DS88] Theorems XII 2.6 and 2.9, [KR86] Theorem 5.6.26

If  $\mathcal{M}_x$  is the abelian von Neumann algebra generated by a self-adjoint operator  $x$  acting on a Hilbert space  $\mathcal{H}$ , then there exists a homomorphism  $f \rightarrow f(x)$  of the algebra  $\mathcal{B}(\sigma(x))$  into  $\eta(\mathcal{M}_x)$ , which maps the constant function 1 onto the identity operator 1 and the identity function  $id_{\sigma(x)}$  onto  $x$ . For  $f \in \mathcal{B}(\sigma(x))$ , the operator  $f(x)$  has the following properties:

(a)

$$D(f(x)) = \left\{ \xi \in \mathcal{H} : \int_{-\infty}^{\infty} |f(t)|^2 d\|e_t(x)\xi\|^2 < \infty \right\} \quad (0.3)$$

(b) For  $\xi \in D(f(x))$  and  $\zeta \in \mathcal{H}$

$$\langle f(x)\xi, \zeta \rangle = \int_{-\infty}^{\infty} f(t) d\langle e_t(x)\xi, \zeta \rangle \quad (0.4)$$

(c)  $f(x)$  commutes with  $e_B(x)$  for any  $B \in \mathcal{B}(\mathbb{C})$ .

(d)

$$\|f(x)\| = e.(x) - \text{ess sup}_{t \in \sigma(x)} |f(t)| = \inf_{\substack{B \in \mathcal{B}(\mathbb{R}) \\ e_B(x)=1}} \sup_{t \in B \cap \sigma(x)} |f(t)| \quad (0.5)$$

and so  $f(x) \in \mathcal{M}$  iff  $f$  is  $e.(x)$ -essentially bounded.

(e)

$$\sigma(f(x)) = \bigcap_{\substack{B \in \mathcal{B}(\mathbb{C}) \\ e_B(x)=1}} \overline{f(B)} \quad (0.6)$$

(f) If  $f$  is real valued then  $f(x)$  is self adjoint and

$$e_B(f(x)) = e_{f^{-1}(B)}(x) \quad (0.7)$$

for every  $B \in \mathcal{B}(\mathbb{C})$

(g) The map  $f \rightarrow f(x)$  is  $\sigma$ -normal, that is, if  $f_n(t) \uparrow f(t)$  for  $e.(x)$ -a.e.  $t$  then  $f_n(x)$  is increasing in the partial order and so-convergent to  $f(x)$ .

(h) If  $f = \chi_B$ , where  $B$  is a Borel set, then  $f(x) = e_B(x)$

(i) If  $f(t) = \begin{cases} t - \lambda & \text{for } t > \lambda \\ 0 & \text{for } t \leq \lambda \end{cases}$  then  $f(x) = x_\lambda^+$  and  $e_\lambda(x) = N(x_\lambda^+)$ .

## Duality theory

We assume some basic knowledge of duality theory for topological vector spaces and in particular the properties of polars, as found, for example, in [RR64].

First recall that any member of  $(\mathcal{H}, \|\cdot\|)'$  can be represented as  $\langle \cdot, y \rangle$  for  $y \in \mathcal{H}$ , by the Riesz Representation Theorem, and it follows that  $(\mathcal{H}, \|\cdot\|)'$  is conjugate-isomorphic to  $\mathcal{H}$ .

Since polars are invariant under conjugation, it suffices to consider polars to be taken in  $\mathcal{H}$ , rather than in  $(\mathcal{H}, \|\cdot\|)'$ . Therefore we work throughout with the pairing  $\langle \mathcal{H}, \mathcal{H} \rangle$ , with the inner product replacing the usual bilinear form, and are able to apply all the

standard results concerning dual pairs to this pairing. Expressed less formally : we think of  $\mathcal{H}$  as being its own dual. Thus we feel justified in using certain notation from classical duality theory, for example, we define for  $B \subset \mathcal{H}$ ,

$$B^\circ = \{\xi \in \mathcal{H} : |\langle \xi, \zeta \rangle| \leq 1 \forall \zeta \in B\} \quad (0.8)$$

From [RR64] Chapter II Lemma 6 we have that

$$x^{-1}(B^\circ) = (x^*B)^\circ \quad (0.9)$$

and from this it follows in particular that

$$x^{-1}\mathcal{B}_{\mathcal{H}} = (x^*\mathcal{B}_{\mathcal{H}})^\circ \quad (0.10)$$

$$(x^{-1}\mathcal{B}_{\mathcal{H}})^\circ = x^*\mathcal{B}_{\mathcal{H}} \quad (0.11)$$

$$xB \subset \mathcal{B}_{\mathcal{H}} \Leftrightarrow x^*\mathcal{B}_{\mathcal{H}} \subset B^\circ \quad (0.12)$$

for  $x \in \mathcal{B}(\mathcal{H})$ ,  $B \subset \mathcal{H}$ .

We will also make some use of the properties of bornologies, as found in [HN77]. By a vector bornology on  $\mathcal{H}$  we mean a family  $\mathcal{B}$  of subsets of  $\mathcal{H}$  satisfying the following conditions:

- (a) If  $B \in \mathcal{B}$  and  $A \subset B$  then  $A \in \mathcal{B}$ ;
- (b) if  $B_1, B_2 \in \mathcal{B}$  then  $B_1 + B_2 \in \mathcal{B}$ ;
- (c) if  $B \in \mathcal{B}$  and  $\lambda \in \mathbb{C}$  then  $\lambda B \in \mathcal{B}$ ;
- (d) if  $B \in \mathcal{B}$  then  $co B \in \mathcal{B}$ .

A vector bornology is said to be convex (or simply a convex bornology) if in (d) we have  $ac B$  instead of just  $co B$ .

Most sources require that a vector bornology covers the vector space in question, but this condition is technical and much of the theory stands without this condition. We will not make this requirement.

A base for a vector bornology  $\mathcal{B}$  is any subfamily  $\mathcal{B}_0$  of  $\mathcal{B}$  such that any set in  $\mathcal{B}$  is contained in some set in  $\mathcal{B}_0$ . A collection  $\mathcal{B}_0$  of sets is a base for a vector bornology (called the generated bornology) if and only if it satisfies the following conditions:

- (a) For  $B_1, \dots, B_n \in \mathcal{B}_0$  there is a  $B \in \mathcal{B}_0$  such that  $B_1 + \dots + B_n \subset B$ ;
- (b) for  $B \in \mathcal{B}_0$  and  $\lambda \in \mathbb{C}$  there is a  $B' \in \mathcal{B}_0$  such that  $\lambda B \subset B'$ ;
- (c) for any  $B \in \mathcal{B}_0$  we have  $co B \in \mathcal{B}_0$ .

The system will be a base for a convex bornology if in (c) we have  $acB$  instead of just  $coB$ .

### Reduced von Neumann algebras

For  $x \in \mathcal{M}$ ,  $Q \in \mathcal{M}^p$ , we denote by  $x_Q$  the restriction of  $Qx$  to  $Q\mathcal{H}$ , and by  $\mathcal{M}_Q$  the set  $\{x_Q : x \in \mathcal{M}\}$ .  $\mathcal{M}_Q$  is usually called the reduction of  $\mathcal{M}$  by  $Q$ .

The commutant of  $\mathcal{M}$  is preserved under reductions: that is,

$$(\mathcal{M}')_Q = (\mathcal{M}_Q)'$$

For a proof of this the reader may consult [Dix81] I.2.1. It follows that  $\mathcal{M}_Q$  is a von Neumann algebra acting on the Hilbert space  $Q\mathcal{H}$ .

It follows from the definition that the algebra  $\mathcal{M}_Q$  is isomorphic to the algebra  $Q\mathcal{M}Q$ . Denote this isomorphism by  $\simeq$ . We note that composition in  $\mathcal{M}_Q$  is given by

$$x_Q y_Q = (x_Q y)_Q$$

The following result is well known.

#### Lemma 0.3.4

$$(\mathcal{M}_Q)^p = \{p_Q : p \in \mathcal{M}^p, p \leq Q\} \tag{0.13}$$

**Proof:** Suppose  $p \in \mathcal{M}^p$  and  $p \leq Q$ . Then

$$p_Q p_Q \simeq QpQQpQ = QpQ \simeq p_Q$$

and

$$p_Q \simeq QpQ = Qp^*Q \simeq p^*_Q$$

and so  $p_Q \in (\mathcal{M}_Q)^p$ .

Conversely, suppose  $x_Q \in (\mathcal{M}_Q)^p$ . Then, in the same way as argued above,

$$Qx_Q = (Qx_Q)^* = (Qx_Q)^2$$

Thus  $Qx_Q \in \mathcal{M}^p$  and certainly  $Qx_Q \leq Q$ . Since  $(Qx_Q)_Q = x_Q$ , the result follows. ■

We can therefore adopt the following convention concerning projections in  $\mathcal{M}_Q$ : while in general a member of  $\mathcal{M}_Q$  will be denoted  $x_Q$ , where  $x$  is in the obvious sense arbitrary, projections in  $\mathcal{M}_Q$  will always be denoted as  $p_Q$ , where  $p$  is itself a projection in  $\mathcal{M}$ , and  $p \leq Q$ .

## Ideals in von Neumann algebras

There are certain elementary results on ideals in von Neumann algebras that we shall need. These are summarised in this section.

It is well known that the two-sidedness of an ideal in a von Neumann algebra is equivalent to the self-adjointness of the ideal - see [KR86] Proposition 6.8.9. Furthermore,

$$x \in \mathcal{I} \Leftrightarrow |x| \in \mathcal{I} \quad (0.14)$$

since if  $x = v|x|$  is the polar decomposition then  $|x| = v^*x$ .

The following result concerning the spectral family will prove very important. The reader should note that this result coincides with [KR86] Theorem 6.8.1. There is in reality no difference between this proof and the one appearing there, which 'constructs' the operator  $f(x)$  from topological first principles.

**Theorem 0.3.5** *Suppose  $\mathcal{I}$  is an ideal in  $\mathcal{M}$  and  $x \in \mathcal{I}^{\text{sa}}$ . Then*

$$e_{(t,\infty)}(x) \in \mathcal{I}^p \text{ for every } t > 0 \quad (0.15)$$

and

$$e_{(-\infty,t)}(x) \in \mathcal{I}^p \text{ for every } t < 0 \quad (0.16)$$

**Proof:** Suppose  $\lambda > 0$ . Consider the function

$$f(t) = \begin{cases} 0 & \text{for } t \leq \lambda \\ \frac{1}{t} & \text{for } t > \lambda \end{cases}$$

Then  $f(x) \in \mathcal{M}$  since  $f$  is bounded and

$$f(x)x = (f \cdot id)(x) = (\chi_{(\lambda,\infty)})(x) = e_{(\lambda,\infty)}(x)$$

and so  $e_{(\lambda,\infty)}(x) \in \mathcal{I}$ . The case where  $\lambda < 0$  is quite similar. ■

**Corollary 0.3.6** *Suppose  $x \in \mathcal{I}$ . Then for every  $\epsilon > 0$  there exists  $p \in \mathcal{I}^p$  such that  $\|x - xp\| \leq \epsilon$  or  $\|x - px\| \leq \epsilon$ .*

**Proof:** Suppose  $x = v|x|$  is the polar decomposition. Note that  $|x| = v^*x \in \mathcal{I}$ , and hence  $p = e_{(\epsilon,\infty)}(|x|) \in \mathcal{I}^p$  by Theorem 0.3.5. Furthermore

$$\|x - xp\| = \|v|x| - v|x|p\| \leq \| |x| - |x|p \| = \| |x| e_{\epsilon}(|x|) \| \leq \epsilon$$

and this completes the proof of the first case. The other case is similar. ■

## $\tau$ -measurable operators

In Chapter 6,  $\mathcal{M}$  will denote a semifinite von Neumann algebra of operators on a Hilbert space  $\mathcal{H}$  and  $\tau$  will be a distinguished faithful semifinite normal trace. In other places where a trace is mentioned, this assumption will also be in force.

An affiliated operator  $x$  is called  $\tau$ -measurable if for every  $t > 0$  there exists a  $p \in \mathcal{M}^p$  such that  $p\mathcal{H} \subset D(x)$  and  $\tau(1-p) \leq t$ . The set of all  $\tau$ -measurable operators, denoted by  $\widetilde{\mathcal{M}}$ , is a  $*$ -algebra of operators on  $\mathcal{H}$  where the sum and product operation is the closure of the ordinary sum and product. The sets

$$\widetilde{\mathcal{M}}(\epsilon, t) = \{x \in \widetilde{\mathcal{M}} : \exists p \in \mathcal{M}^p \text{ such that } \|xp\| \leq \epsilon \text{ and } \tau(1-p) \leq t\} \quad (0.17)$$

form a base at 0 (as  $\epsilon, t > 0$  vary) for a metrisable vector topology  $\tau_{cm}$  on  $\widetilde{\mathcal{M}}$ , called the topology of convergence in measure. Equipped with this topology,  $\widetilde{\mathcal{M}}$  is a complete topological  $*$ -algebra in which  $\mathcal{M}$  is dense. In fact, if

$$x = v|x| = v \int_0^\infty \lambda \, de_\lambda(|x|) \in \widetilde{\mathcal{M}}$$

then

$$x_n = v \int_{\frac{1}{n}}^\infty \lambda \, de_\lambda(|x|) \in \mathcal{M}$$

for all  $n \in \mathbb{N}$ , and  $x_n \xrightarrow{\tau_{cm}} x$ . For proofs of these facts the reader may consult Nelson ([Nel74]), Terp ([Ter81]), or Fack and Kosaki ([FK86]).

Suppose

$$x = v|x| = v \int_0^\infty \lambda \, de_\lambda(|x|) \in \widetilde{\mathcal{M}}$$

By [Ter81] Proposition 21,

$$d_t(x) = \tau(e_{(t, \infty)}(|x|)) \quad (0.18)$$

the *distribution function* of  $x$ , is eventually finite valued, and  $d_t(x) \rightarrow 0$  as  $t \rightarrow \infty$ .

Hence if  $t > 0$  there exists  $s \geq 0$  such that  $d_s(x) \leq t$ . Thus

$$\mu_t(x) = \inf\{s \geq 0 : d_s(x) \leq t\} \quad (0.19)$$

the *generalised singular function* of  $x$ , is finite valued. Moreover, this function admits the alternative characterisation

$$\mu_t(x) = \inf\{\|xp\| : p \in \mathcal{M}^p, \tau(1-p) \leq t\} \quad (0.20)$$

Furthermore the generalised singular function and the topology of convergence in measure are related in the following manner:

$$x \in \widetilde{\mathcal{M}}(\epsilon, t) \Leftrightarrow \mu_t(x) \leq \epsilon \quad (0.21)$$

These results are due to Fack and Kosaki; for full details the reader is referred to [FK86].

It is interesting to note that the infimum mentioned above is actually attained when choosing the spectral projection  $e_{\mu_t(x)}(|x|)$ . This follows from the fact that

$$\tau(1 - e_{\mu_t(x)}(|x|)) = d_{\mu_t(x)}(|x|) \leq t$$

(by [FK86] Proposition 2.2) and thus

$$\mu_t(x) = \inf\{\|xp\| : p \in \mathcal{M}^p, \tau(1 - p) \leq t\} \leq \|xe_{\mu_t(x)}(|x|)\| \leq \mu_t(x)$$

In particular, we stress the interesting fact that the inequality

$$\|xe_{\theta}(|x|)\| \leq \theta$$

is in fact an equality when  $\theta$  is in the range of  $\mu_t(x)$ .

For  $p \in \mathcal{M}^p$ , it is easy to verify that

$$\mu_t(p) = \chi_{(0, \tau(p))}(t) \tag{0.22}$$

For the terminology appearing in the following items the reader is referred to [KPS82] and [DDP89b]. If  $\mathcal{E}(0, \infty)$  is a rearrangement invariant symmetric Banach function space then the set

$$\mathcal{E}(\mathcal{M}) = \{x \in \mathcal{M} : \mu_t(x) \in \mathcal{E}(0, \infty)\} \tag{0.23}$$

is a Banach space of  $\tau$ -measurable operators when equipped with the norm

$$\|x\|_{\mathcal{E}(\mathcal{M})} = \|\mu_t(x)\|_{\mathcal{E}(0, \infty)} \tag{0.24}$$

and is a normed  $\mathcal{M}$ -module with this norm. See [DDP89b] and [DDP89a].

In particular, for  $1 \leq p \leq \infty$ ,

$$\mathcal{L}^p(\mathcal{M}) = \{x \in \widetilde{\mathcal{M}} : \mu_t(x) \in \mathcal{L}^p(0, \infty)\} \tag{0.25}$$

is a normed module with norm

$$\|x\|_p = \sqrt[p]{\int_0^\infty \mu_t(x)^p dt} = \sqrt[p]{\tau(|x|^p)} \tag{0.26}$$

As other examples of such spaces, we can mention the Lorentz and Marcinkiewicz spaces. For details, see [KPS82] and [DDP90].

# Chapter 1

## p-Ideals

In the study of norm-closed ideals in a von Neumann algebra, attention can be focused on the set of projections that belong to the ideal. As we shall see, the reason for this is that Wright showed in [Wri54] that such ideals are fully determined by their projections. In this chapter we first give a brief summary of the theory of p-ideals, as a proper understanding of this theory is crucial for later developments. Often we present proofs even for known results, both for the convenience of the reader and because it is necessary to become familiar with the applicable techniques. Furthermore we derive a few new results, some of which we shall use extensively. Finally we examine some elementary examples of p-ideals.

It is important to note that p-ideals are by no means the only known method of characterising ideals. Wils has considered the problem of determining invariants for ideals in von Neumann algebras in [Wil70b] by making use of the dimension function of Tomiyama and the notion of ‘ideal bases’. In fact Wils developed an extensive theory of characterising ideals via the dimension function. To consider these issues in any detail would take us too far from our aim of reviewing the elementary theory of p-ideals and giving some examples of p-ideals. We refer the interested reader to [Wil70b] and [Ols84] for more information.

### 1.1 p-ideals

Wright uses the following definition of a p-ideal which is due to Kawada, Higuti and Matusima.

**Definition 1.1.1** *A p-ideal  $\mathcal{P} \subset \mathcal{M}^p$  is a set satisfying*

(a) *If  $p, q \in \mathcal{P}$  then  $p \vee q \in \mathcal{P}$*

(b) If  $\mathcal{P} \ni p \geq q \in \mathcal{M}^p$  then  $q \in \mathcal{P}$

(c) If  $\mathcal{P} \ni p \sim q \in \mathcal{M}^p$  then  $q \in \mathcal{P}$

It is clear that condition (b) is equivalent to  $\mathcal{P}$  being closed under meets. Thus a  $p$ -ideal is a (not necessarily complete) sublattice of  $\mathcal{M}^p$  which also satisfies the extra condition (c). Note that if  $p \sim q$  then  $q = vpv^*$ , so condition (c) really is an ideal type condition.

The following result will be useful in applications:

**Proposition 1.1.2** *In the definition of a  $p$ -ideal, condition (a) can be replaced with the equivalent condition*

(a') If  $p, q \in \mathcal{P}$  and  $p \perp q$  then  $p + q \in \mathcal{P}$

**Proof:** It is clear that condition (a) implies condition (a').

Conversely, suppose  $p, q \in \mathcal{P}$ . Then

$$p - p \wedge q \in \mathcal{P}$$

from (b), and hence

$$p \vee q - q \in \mathcal{P}$$

from (c), since the parallelogram identity states that

$$p - p \wedge q \sim p \vee q - q$$

Hence

$$p \vee q = (p \vee q - q) + q \in \mathcal{P}$$

from (a'). ■

An immediate consequence is the following proposition.

**Proposition 1.1.3** [Wri54] *Lemma 2.1*

*Suppose  $\mathcal{I}$  is an ideal in  $\mathcal{M}$ . Then  $\mathcal{I}^p$  is a  $p$ -ideal.*

We will denote the ideal in  $\mathcal{M}$  generated by a set  $B$  by  $((B))$ , and the norm-closed ideal generated by  $[[B]]$ . Of course we have that

$$[[B]] = ((B))^{-\|\cdot\|} \tag{1.1}$$

With the exception of the following technical result, we will have no use for one-sided ideals. All ideals are understood to be two-sided unless specifically stated otherwise. Denote by  $((B)$  the left ideal generated by a set  $B$ , and by  $B))$  the right ideal generated. We include the proof of the following Lemma as it illustrates some important techniques which will be used again later.

**Lemma 1.1.4** [Wri54] *Lemma 2.3*

Suppose  $\mathcal{P}$  is a  $p$ -ideal. Then

$$\mathcal{P}) = ((\mathcal{P} = ((\mathcal{P}))$$

**Proof:** Suppose  $p \in \mathcal{P}$  and  $x \in \mathcal{M}$ ; we show that  $xp \in \mathcal{P}$ ). We have that  $r(xp) \leq p$ , so  $r(xp) \in \mathcal{P}$ . Hence  $l(xp) \in \mathcal{P}$ , since  $r(xp) \sim l(xp)$ . Thus

$$xp = l(xp) xp \in \mathcal{P})$$

This shows that  $((\mathcal{P}$  is included in  $\mathcal{P})$ ).

The converse inclusion follows likewise, and this completes the proof. ■

As a consequence we have the following result of which we will make later use:

**Proposition 1.1.5** cf. [Wil70b] *Lemma 2.4*

Suppose  $\mathcal{P}$  is a  $p$ -ideal. Then

$$((\mathcal{P})) = \{x \in \mathcal{M} : l(x) \in \mathcal{P}\} \tag{1.2}$$

$$= \{x \in \mathcal{M} : r(x) \in \mathcal{P}\} \tag{1.3}$$

**Proof:** Suppose  $x \in ((\mathcal{P}))$ , then by Lemma 1.1.4,  $x$  may be expressed in the form  $\sum_{i=1}^n e_i x_i$ , where  $e_1, \dots, e_n \in \mathcal{P}$  and  $x_1, \dots, x_n \in \mathcal{M}$ . Then  $l(x) \leq \bigvee_{i=1}^n e_i \in \mathcal{P}$ , and so  $l(x) \in \mathcal{P}$ .

The converse inclusion is immediate, while the second equality follows likewise. ■

**Definition 1.1.6** A vector topology  $\mu$  on  $\mathcal{M}$  is said to be left or right solid if there exists a base  $\mathcal{U}$  for the neighbourhoods of 0 such that

$$B_{\mathcal{M}} U = U$$

or

$$U B_{\mathcal{M}} = U$$

respectively, for every  $U \in \mathcal{U}$ . The topology is said to be solid if

$$B_{\mathcal{M}} U B_{\mathcal{M}} = U$$

for all  $U \in \mathcal{U}$  i.e. it is both left and right solid.

A vector topology on  $\mathcal{M}$  which is locally convex with a defining family of seminorms  $\{f_i : i \in I\}$  is solid iff

$$f_i(xyz) \leq \|x\|_{\infty} f_i(y) \|z\|_{\infty}$$

for  $i \in I$  and  $x, y, z \in \mathcal{M}$ .

Similar (one sided) formulations hold for left or right solid topologies.

**Examples 1.1.7** (1) The norm topology is solid.

(2) The  $so$ -topology is left solid but not right solid.

(3) The topology of convergence in measure, restricted to  $\mathcal{M}$ , is solid.

We will make extensive use of the following proposition. In a certain sense it provides a tool for studying a large class of topologies on  $\mathcal{M}$  by restricting attention to the behaviour of the topology on the projections.

**Proposition 1.1.8** *Suppose  $\mu$  is a vector topology on  $\mathcal{M}$  and that*

$$\mathcal{M} \supset x_\alpha \xrightarrow{\mu} p \in \mathcal{M}^p$$

(a) *If  $\mu$  is right solid then  $r(x_\alpha p) \xrightarrow{\mu} p$*

(b) *If  $\mu$  is left solid then  $l(p x_\alpha) \xrightarrow{\mu} p$*

**Proof:** (a) Since  $x_\alpha \xrightarrow{\mu} p$ , we have that  $x_\alpha p \xrightarrow{\mu} p^2 = p$ , since  $\mu$  is right solid. Furthermore, note that  $r(x_\alpha p) \leq p$ . Then

$$\begin{aligned} p - r(x_\alpha p) &= p - x_\alpha p + x_\alpha p - r(x_\alpha p) \\ &= p - x_\alpha p + (x_\alpha p - p) r(x_\alpha p) \\ &\xrightarrow{\mu} 0 \end{aligned}$$

since  $r(x_\alpha p) \in \mathcal{B}_{\mathcal{M}}$ , and  $\mu$  is right solid.

(b) is of course entirely similar. ■

The above result was suggested by part of the argument in [Wri54] Theorem 2.4. In the case that  $\mu = \|\cdot\|$  it is fundamental in proving the original characterisation theorem of Wright, which reads as follows:-

**Theorem 1.1.9** [Wri54] *Theorem 2.4*

*There is a one-to-one correspondence between norm-closed ideals and  $p$ -ideals, that is*

(a) *If  $\mathcal{I}$  is a norm closed ideal then  $\mathcal{I} \cap \mathcal{M}^p$  is a  $p$ -ideal and  $[[\mathcal{I} \cap \mathcal{M}^p]] = \mathcal{I}$ .*

(b) *If  $\mathcal{P}$  is a  $p$ -ideal then  $[[\mathcal{P}]] \cap \mathcal{M}^p = \mathcal{P}$ .*

Thus closed ideals in von Neumann algebras are fully determined by their projections. Consequently, in the study of closed ideals, it often suffices to focus attention on the projections that belong to the ideals of interest. And certainly we can anticipate that studying the behaviour of projections will be simpler than studying the behaviour of

arbitrary operators. Of course, one of the reasons for this is the availability of the lattice structure on the set of projections.

The theorem has a corollary which is obvious but deserves emphasis : it is noted in [Wri54] and proved in [Wil70b] Lemma 2.4 and Theorem 1.4. We give a proof for the convenience of the reader.

**Corollary 1.1.10** *Consider the collection of all ideals in  $\mathcal{M}$  and consider the equivalence relation determined by their  $p$ -ideals i.e. ideals are equivalent if they have the same  $p$ -ideal.*

*Then each equivalence class so determined has, with respect to set inclusion, a least member*

$$\{x \in \mathcal{M} : l(x) \in \mathcal{P}\}$$

*(where  $\mathcal{P}$  denotes the  $p$ -ideal under consideration) and a greatest member*

$$[[\mathcal{P}]]$$

*and the latter coincides with the norm closure of any ideal in the equivalence class.*

**Proof:** That  $\{x \in \mathcal{M} : l(x) \in \mathcal{P}\}$  is the least member follows from Proposition 1.1.5.

Suppose  $\mathcal{I}$  is any ideal in the equivalence class. Suppose  $x \in \mathcal{I}$ , then by Corollary 0.3.6 we can for any given  $\epsilon > 0$  choose  $p \in \mathcal{I} \cap \mathcal{M}^p$  such that  $\|x - xp\| < \epsilon$ . Since  $xp \in ((\mathcal{I} \cap \mathcal{M}^p))$  we thus have that  $x \in [[\mathcal{I} \cap \mathcal{M}^p]]$ , since  $\epsilon$  was arbitrary. Thus  $\mathcal{I} \subset [[\mathcal{P}]]$  for any  $\mathcal{I}$  in the equivalence class.

Since  $\mathcal{I} \supset ((\mathcal{P}))$  we have that  $\bar{\mathcal{I}} \supset ((\mathcal{P}))^{-\|\cdot\|} = [[\mathcal{P}]]$ . ■

## 1.2 Examples of $p$ -ideals

We now turn to some examples of  $p$ -ideals.

**Examples 1.2.1** (1) Consider the collection  $\mathcal{P}$  of all projections of finite rank in the von Neumann algebra  $\mathcal{B}(\mathcal{H})$ . It is clear that  $\mathcal{P}$  is a  $p$ -ideal. The ideal generated is the set of finite rank operators, often denoted  $\mathcal{F}(\mathcal{H})$ , and the closed ideal generated is exactly  $\mathcal{K}(\mathcal{H})$ .

(2) A projection  $p$  is said to be algebraically finite relative to  $\mathcal{M}$  if whenever  $p \sim q \leq p$  then  $p = q$ . Consider the collection  $\mathcal{P}_{\sim}$  of projections that are algebraically finite relative to  $\mathcal{M}$ . It is fairly easy to show that this collection is closed under sub-projections and similar projections - [KR86] Proposition 6.3.2. It can also be shown - although this substantially more difficult - that it is closed under finite

joins - [KR86] Theorem 6.3.8. Thus  $\mathcal{P}_\sim$  is a p-ideal. The ideal generated is usually called the algebraically finite ideal, and the closed ideal generated the algebraically compact ideal. We will denote the latter ideal by  $\mathcal{M}_\sim$ .

- (3) Suppose  $\{\mathcal{H}_i : i \in I\}$  is a family of Hilbert spaces, and  $\mathcal{M}_i \subset \mathcal{B}(\mathcal{H}_i)$  are von Neumann algebras.

Recall that

$$\bigoplus_{i \in I} \mathcal{H}_i = \{ \{ \xi_i \} : \xi_i \in \mathcal{H}_i, \sum_{i \in I} \|\xi_i\|^2 < \infty \} \quad (1.4)$$

equipped with the norm

$$\| \{ \xi_i \} \| = \sqrt{ \sum_{i \in I} \|\xi_i\|^2 } \quad (1.5)$$

is a Hilbert space and

$$\bigoplus_{i \in I} \mathcal{M}_i = \{ \{ x_i \} : x_i \in \mathcal{M}_i, \sup_{i \in I} \|x_i\| < \infty \} \quad (1.6)$$

equipped with the norm

$$\| \{ x_i \} \| = \sup_{i \in I} \|x_i\| \quad (1.7)$$

is a von Neumann algebra, with action

$$\{ x_i \} \{ \xi_i \} = \{ x_i \xi_i \} \quad (1.8)$$

on  $\mathcal{B}(\bigoplus_{i \in I} \mathcal{H}_i)$ . See [KR86] §2.6 and p. 336.

Suppose  $\mathcal{I}_i$  is an ideal in  $\mathcal{M}_i$ . Then it is easy to see that

$$\bigoplus_{i \in I} \mathcal{I}_i = \{ \{ x_i \} : x_i \in \mathcal{I}_i, \sup_{i \in I} \|x_i\| < \infty \} \quad (1.9)$$

is an ideal in  $\bigoplus_{i \in I} \mathcal{M}_i$ , which is norm-closed if each of the components  $\mathcal{I}_i$  is norm-closed. Furthermore,  $\bigoplus_{i \in I} \mathcal{I}_i$  has p-ideal

$$\{ \{ p_i \} : p_i \in \mathcal{I}_i^p \} \quad (1.10)$$

- (4) Suppose  $\mathcal{M}$  is a commutative von Neumann algebra. We may identify  $\mathcal{M}$  with some space  $\mathcal{L}^\infty(X, \Sigma, \mu)$ , the algebra of multiplication operators acting on the Hilbert space  $\mathcal{L}^2(X, \Sigma, \mu)$ .

Then of course

$$\mathcal{M}^p = \{ \chi_A : A \in \Sigma \}$$

which we identify with  $\Sigma$  itself. A subcollection  $\mathcal{A}$  of  $\Sigma$  is a p-ideal iff it is closed under finite unions and whenever  $A \in \mathcal{A}$  and  $A \supset B \in \Sigma$  then  $B \in \mathcal{A}$ . The equivalence condition in the definition of a p-ideal is satisfied vacuously, since the  $\sim$ -relation reduces to the trivial equality relation in a commutative von Neumann algebra.

- (5) Suppose  $\tau$  is a numerical trace on  $\mathcal{M}$ , and consider the collection  $\mathcal{P}_\tau$  of projections with finite trace. If  $p \sim q$  then

$$\tau(p) = \tau(v^*v) = \tau(vv^*) = \tau(q)$$

and so  $\mathcal{P}_\tau$  is closed under similar projections. By the monotonicity of the trace it is closed under subprojections. Finally, if  $p, q \in \mathcal{P}_\tau$  then

$$\tau(p + q) = \tau(p) + \tau(q)$$

and so  $\mathcal{P}_\tau$  is closed under finite orthogonal sums. It follows that  $\mathcal{P}_\tau$  is a p-ideal.

Usually one supposes that  $\mathcal{M}$  is semifinite, in which case it admits a faithful semifinite normal trace  $\tau$ . If  $\mathcal{M}$  is a semifinite factor, then  $\tau$  is uniquely determined up to scalar multiples, and in this case  $\mathcal{P}_\tau$  and  $\mathcal{P}_\sim$  coincide - see [KR86] Proposition 8.5.2.

The generated ideal

$$((\mathcal{P}_\tau)) = \{x \in \mathcal{M} : \tau(l(x)) < \infty\} \quad (1.11)$$

has been called the ideal of operators with finite rank relative to  $\tau$ , and the norm closed ideal  $[[\mathcal{P}_\tau]]$  the ideal of operators that are compact relative to  $\tau$ , by Fack in [Fac83] p 317. This terminology is quite illuminating. When considering the von Neumann algebra  $\mathcal{B}(\mathcal{H})$  equipped with the canonical trace, it can be verified directly that  $\mathcal{P}_\tau$  is exactly the set of projections with finite dimensional range - since

$$\tau(p) = \dim p\mathcal{H}$$

(here  $\dim$  denotes the Hilbert dimension) - hence  $((\mathcal{P}_\tau))$  is indeed the collection of finite rank operators and  $[[\mathcal{P}_\tau]]$  the compact operators. Alternatively this follows from the fact, mentioned previously, that  $\mathcal{P}_\tau$  and  $\mathcal{P}_\sim$  coincide, and the latter is clearly the p-ideal of finite dimensional projections.

We will denote  $[[\mathcal{P}_\tau]]$  by  $\mathcal{M}_0$ . We will see more of this ideal, and its unbounded analogue, called the  $\tau$ -compact operators and denoted  $\widetilde{\mathcal{M}}_0$ , in Chapter 6.

Besides  $((\mathcal{P}_\tau))$  and  $\mathcal{M}_0$ , there are a number of intermediate ideals that share the p-ideal  $\mathcal{P}_\tau$ .

A consideration of the ideal

$$\mathcal{L}^1(\mathcal{M}) \cap \mathcal{M} = \{x \in \mathcal{M} : \tau(|x|) < \infty\} \quad (1.12)$$

is very often the starting point for a study of non-commutative integration theory. It is the linear span of  $\{x \in \mathcal{M}^+ : \tau(x) < \infty\}$ . Traditionally this ideal is denoted  $\mathcal{M}_\tau$  and referred to as the trace ideal or trace class. In [DDP90] and [DDP91] this ideal is denoted  $H(\mathcal{M})$ .

The ideal  $\mathcal{L}^2(\mathcal{M}) \cap \mathcal{M}$  is also of some interest historically, and is often denoted  $\mathcal{N}_\tau$ . See [Tak79] for more details.

Suppose  $\mathcal{E}(0, \infty)$  is a rearrangement invariant Banach function space which is minimal, in the sense that  $H(0, \infty)$  is dense in  $\mathcal{E}(0, \infty)$  - see [KPS82] II §4.5 and [DDP90] §2. It can be shown that  $\mathcal{E}(0, \infty)$  is minimal iff

$$f\chi_{(0,n]} \rightarrow f, \quad (|f| - n1)^+ \rightarrow |f|$$

for every  $f \in \mathcal{E}(0, \infty)$ . In particular, any rearrangement invariant Banach function space for which the norm is order continuous is a minimal space. Thus the spaces  $\mathcal{L}^p(0, \infty)$ , where  $1 \leq p < \infty$ , are minimal.

It is easy to verify that for any minimal space  $\mathcal{E}(0, \infty)$ ,

$$H(0, \infty) \subset \mathcal{E}(0, \infty) \subset \{f : \mu_t(f) \rightarrow 0\}$$

and so

$$H(\mathcal{M}) \subset \mathcal{E}(\mathcal{M}) \subset \{x \in \widetilde{\mathcal{M}} : \mu_t(x) \rightarrow 0\}$$

and thus

$$H(\mathcal{M}) \subset \mathcal{E}(\mathcal{M}) \cap \mathcal{M} \subset \{x \in \mathcal{M} : \mu_t(x) \rightarrow 0\}$$

Thus

$$\mathcal{P}_\tau = H(\mathcal{M})^p \subset \mathcal{E}(\mathcal{M})^p \subset \{x \in \mathcal{M}^p : \mu_t(x) \rightarrow 0\} = \mathcal{P}_\tau$$

and the  $p$ -ideal of  $\mathcal{E}(\mathcal{M}) \cap \mathcal{M}$  is exactly  $\mathcal{P}_\tau$ .

It is known that  $\mu_t(x)$  is a bounded function for each  $x \in \mathcal{M}$ . It is then easy to verify that

$$p_1 \leq p_2 \Rightarrow \mathcal{L}^{p_1}(\mathcal{M}) \cap \mathcal{M} \subset \mathcal{L}^{p_2}(\mathcal{M}) \cap \mathcal{M} \quad (1.13)$$

Therefore we have the inclusion of ideals

$$((\mathcal{P}_\tau)) \subset H(\mathcal{M}) \subset \mathcal{L}^{p_1}(\mathcal{M}) \cap \mathcal{M} \subset \mathcal{L}^{p_2}(\mathcal{M}) \cap \mathcal{M} \subset \mathcal{M}_0 \quad (1.14)$$

for  $1 < p_1 < p_2 < \infty$ ; and these inclusions are in general strict.

For the purposes of finding further examples of  $p$ -ideals we briefly consider the concept of 'generated projections'. For any  $B \subset \mathcal{H}$  recall that we denote its closed linear span by  $[B]$ .

**Definition 1.2.2** [KR86] *Definition 5.5.8*

A projection  $p \in \mathcal{M}^p$  is said to be generated by a set  $B \subset \mathcal{H}$  (or,  $B$  is generating for  $p$ ) if  $p$  is the projection onto  $[\mathcal{M}'B]$ . A projection is said to be cyclic if it is generated by a singleton.

Suppose now  $p \in \mathcal{M}^p$  is generated by a set  $B$ .

If  $\mathcal{M}^p \supset \mathcal{q} \leq p$ , then

$$\mathcal{q}\mathcal{H} = \mathcal{q}p\mathcal{H} = \mathcal{q}[\mathcal{M}'B] = [\mathcal{q}\mathcal{M}'B] = [\mathcal{M}'\mathcal{q}B]$$

and so  $q$  is generated by  $q\mathcal{B}$ .

If on the other hand  $p \sim^v q$ , then

$$q\mathcal{H} = vpv^*\mathcal{H} \subset vp\mathcal{H} = qv\mathcal{H} \subset q\mathcal{H}$$

and so

$$q\mathcal{H} = vp\mathcal{H} = v[\mathcal{M}'B] = [v\mathcal{M}'B] = [\mathcal{M}'vB]$$

that is,  $q$  is generated by  $vB$ .

Lastly, if  $p, q$  are orthogonal members of  $\mathcal{M}^p$ , generated by  $B_1$  and  $B_2$  respectively, then from [KR86] Proposition 2.5.3

$$(p + q)\mathcal{H} = p\mathcal{H} + q\mathcal{H} = [\mathcal{M}'B_1] + [\mathcal{M}'B_2] = [\mathcal{M}'(B_1 \cup B_2)]$$

that is,  $p + q$  is generated by  $B_1 \cup B_2$ .

We now have the following proposition:

**Proposition 1.2.3** *Let  $\mathcal{P}_f$  denote the set of projections that are finitely generated, and, for any cardinal  $\alpha \geq \aleph_0$ , let  $\mathcal{P}_\alpha$  denote the set of projections that are generated by some subset of  $\mathcal{H}$  with cardinality less than or equal to  $\alpha$ .*

*Then all the sets mentioned are  $p$ -ideals, and*

$$\mathcal{P}_f \subset \mathcal{P}_\alpha \subset \mathcal{P}_\beta \subset \mathcal{P}_{\dim \mathcal{H}} = \mathcal{M}^p \quad (1.15)$$

*for any infinite cardinals  $\alpha \leq \beta$ .*

**Proof:** It follows from the previous calculations that all the sets appearing are indeed  $p$ -ideals.

It suffices now to show that  $\mathcal{P}_{\dim \mathcal{H}} = \mathcal{M}^p$ . Let  $\{\xi_i : i \in I\}$  be an orthonormal base for  $\mathcal{H}$  which of course has cardinality  $\dim \mathcal{H}$ . Then

$$\mathcal{H} = [\{\xi_i : i \in I\}] = [1\{\xi_i : i \in I\}] \subset [\mathcal{M}'\{\xi_i : i \in I\}]$$

and so  $1 \in \mathcal{P}_{\dim \mathcal{H}}$ . ■

At this point it should be noted that Wright has considered families of projections which are mutually both orthogonal and equivalent, and analysed the cardinality properties of such families. His chief purpose in so doing was to derive a characterisation of the  $p$ -ideals in a factor; for details the reader is referred to [Wri58].

**Proposition 1.2.4** *For  $\alpha \geq \aleph_0$ , the following conditions are equivalent:-*

(a)  $p \in \mathcal{P}_\alpha$

- (b)  $p = \bigvee_{i \in I} p_i$  where each  $p_i$  is cyclic and  $I$  is an indexing set of cardinality less than or equal to  $\alpha$ .
- (c) Each orthogonal family of non-zero subprojections of  $p$  is of cardinality at most  $\alpha$ .
- (d)  $p = \sum_{i \in I} p_i$  where the  $p_i$  are orthogonal and cyclic and  $I$  is an indexing set of cardinality less than or equal to  $\alpha$ .

**Proof:** (a)  $\Rightarrow$  (b) Suppose  $p$  is generated by a set  $B \subset \mathcal{H}$  of cardinality less than or equal to  $\alpha$ . For  $\xi \in B$  let  $p_\xi$  be the projection onto  $[\mathcal{M}'\xi]$ . Then

$$p\mathcal{H} = [\mathcal{M}'B] \supset [\mathcal{M}'\xi] = p_\xi\mathcal{H}$$

so  $p \geq p_\xi$  for all  $\xi \in B$ , and thus  $p \geq \bigvee_{\xi \in B} p_\xi$ .

On the other hand, if  $\xi \in \text{span}(\mathcal{M}'B)$  then

$$\xi = \sum_{i \in F} x'_i \xi_i$$

for some finite  $F \subset I$ ,  $x'_i \in \mathcal{M}'$  and  $\xi_i \in B$ . Then

$$\xi \in \bigvee_{i \in F} p_{\xi_i} \mathcal{H} \subset \bigvee_{\xi \in B} p_\xi \mathcal{H}$$

and so  $\text{span}(\mathcal{M}'B) \subset \bigvee_{\xi \in B} p_\xi \mathcal{H}$ . Thus  $[\mathcal{M}'B] \subset \bigvee_{\xi \in B} p_\xi \mathcal{H}$ , and so  $p \leq \bigvee_{\xi \in B} p_\xi$ .

(b)  $\Rightarrow$  (c) The following argument is basically identical to [KR86] Proposition 5.5.19, which argues the case  $\alpha = \aleph_0$ .

For each  $i \in I$  find  $\xi_i \in \mathcal{H}$  that is generating for  $p_i$ .

Suppose  $\{q_a : a \in A\}$  is an orthogonal family of non-zero subprojections of  $p$ . For  $i \in I$  let

$$A_i = \{a \in A : q_a \xi_i \neq 0\}$$

Since

$$\sum_{a \in A} \|q_a \xi_i\|^2 \leq \|p \xi_i\|^2 < \infty$$

we have that  $A_i$  is countable for  $i \in I$ .

For any  $a \in A$ ,  $0 \neq q_a \leq p \leq \bigvee_{i \in I} p_i$ , and so  $q_a p_i \neq 0$  for some  $i \in I$ . We show that  $a \in A_i$  for such  $i$ . If  $a \notin A_i$ , then  $q_a \xi_i = 0$ , and so

$$\{0\} = [\mathcal{M}' q_a \xi_i] = [q_a \mathcal{M}' \xi_i] = q_a [\mathcal{M}' \xi_i]$$

and so  $q_a p_i = 0$ , giving the required contradiction.

Thus  $A \subset \bigcup_{i \in I} A_i$ , and so

$$\text{card}(A) \leq \text{card}(I) \cdot \aleph_0 = \text{card}(I)$$

since each  $A_i$  is countable. The result follows.

(c)  $\Rightarrow$  (d) It is clear (via Zorn's lemma; see [KR86] Proposition 5.5.9 for a proof) that any projection is the union of an orthogonal family of cyclic projections. In the case that  $p \in \mathcal{P}_\alpha$ , such an expression must have at most  $\alpha$  terms.

(d)  $\Rightarrow$  (a) Suppose  $p = \sum_{i \in I} p_i$  where  $p_i$  is the projection onto  $[\mathcal{M}'\xi_i]$  for some  $\xi_i \in \mathcal{H}$ , and  $I$  is an indexing set of cardinality less than or equal to  $\alpha$ .

Then

$$p\mathcal{H} \supset \bigcup_{i \in I} p_i\mathcal{H} \supset \bigcup_{i \in I} \mathcal{M}'\{\xi_i\} = \mathcal{M}'\{\xi_i : i \in I\}$$

and so  $\{\xi_i : i \in I\}$  generates a subprojection of  $p$ .

On the other hand, suppose  $\epsilon > 0$  is given and  $\xi \in p\mathcal{H}$  is given. Since the net of finite subsums of  $\{p_i : i \in I\}$  is  $so$ -convergent to  $p$ , we can find some finite  $F \subset I$  such that

$$\left\| \xi - \sum_{i \in F} p_i \xi \right\| \leq \frac{\epsilon}{2}$$

For  $i \in F$  find  $x'_i \in \mathcal{M}'$  such that

$$\|p_i \xi - x'_i \xi_i\| \leq \frac{\epsilon}{2n}$$

where  $n$  is the cardinality of  $F$ . Then  $\sum_{i \in F} x'_i \xi_i \in \text{span}(\mathcal{M}'\{\xi_i\})$  and

$$\begin{aligned} \left\| \xi - \sum_{i \in F} x'_i \xi_i \right\| &\leq \left\| \xi - \sum_{i \in F} p_i \xi + \sum_{i \in F} p_i \xi - \sum_{i \in F} x'_i \xi_i \right\| \\ &\leq \left\| \xi - \sum_{i \in F} p_i \xi \right\| + \sum_{i \in F} \|p_i \xi - x'_i \xi_i\| \\ &\leq \frac{\epsilon}{2} + n \frac{\epsilon}{2n} \\ &= \epsilon \end{aligned}$$

which shows that  $p\mathcal{H} \subset [\mathcal{M}'\{\xi_i\}]$ . Thus  $p\mathcal{H} = [\mathcal{M}'\{\xi_i\}]$ , and  $p \in \mathcal{P}_\alpha$ . ■

For convenience we will denote the  $p$ -ideal  $\mathcal{P}_{\aleph_0}$ , i.e. the set of projections that are countably generated, by  $\mathcal{P}_\sigma$ . Condition (c) above shows that  $\mathcal{P}_\sigma$  consists exactly of the projections that are countably decomposable or  $\sigma$ -finite (in the terminology of [KR86], Definition 5.5.14); or of countable type (in the terminology of [SZ79] §4.13).

Note that in the case  $\mathcal{M}$  is abelian, any countably decomposable projection is in fact cyclic - [KR86] Proposition 5.5.10. Thus if  $\mathcal{M}$  is abelian then  $\mathcal{P}_f = \mathcal{P}_\sigma$ .

At the other extreme, in the case  $\mathcal{M} = \mathcal{B}(\mathcal{H})$ , for any  $\xi \in \mathcal{H}$  we have that

$$[\mathcal{M}'\xi] = \text{span}(\xi)$$

It follows that

$$\mathcal{P}_\alpha = \{p \in \mathcal{B}(\mathcal{H})^p : \dim p\mathcal{H} \leq \alpha\}$$

We now abstract one of the properties that the  $p$ -ideals  $\mathcal{P}_\alpha$  enjoy.

**Definition 1.2.5** A  $p$ -ideal  $\mathcal{P}$  is said to be a  $\sigma p$ -ideal if it is closed under countable join.

**Examples 1.2.6** (a) It is clear from Proposition 1.2.4 (b) that  $\mathcal{P}_\alpha$  is a  $\sigma p$ -ideal for any  $\alpha \geq \aleph_0$ . This follows from the fact that for any such  $\alpha$ ,  $\alpha \cdot \aleph_0 = \alpha$ .

(b) We will see in Proposition 1.2.9 that for  $p \in \mathcal{Z}(\mathcal{M})^p$ ,

$$\mathcal{P} = \{q \in \mathcal{M}^p : q \leq p\}$$

is a  $p$ -ideal. It is clear that such a  $p$ -ideal would be a  $\sigma p$ -ideal. Indeed, it is closed not only under countable join, but under arbitrary join.

(c)  $\mathcal{P}_\tau$  is not a  $\sigma p$ -ideal.

**Proposition 1.2.7** For a  $p$ -ideal  $\mathcal{P}$ , the following are equivalent :-

(a)  $((\mathcal{P})) = [[\mathcal{P}]]$

(b)  $\mathcal{P}$  is closed under countable orthogonal sums.

(c)  $\mathcal{P}$  is a  $\sigma p$ -ideal.

**Proof:** (a)  $\Rightarrow$  (b) Suppose  $((\mathcal{P})) = [[\mathcal{P}]]$ . Suppose  $\{p_i : i \in \mathbb{N}\}$  is an orthogonal family of projections from  $\mathcal{P}$ . Now

$$((\mathcal{P})) \ni \sum_{i=1}^n \frac{1}{i} p_i \xrightarrow{\|\cdot\|} \sum_{i=1}^{\infty} \frac{1}{i} p_i$$

and so by hypothesis

$$\sum_{i=1}^{\infty} \frac{1}{i} p_i \in [[\mathcal{P}]] = ((\mathcal{P}))$$

Since

$$1 \left( \sum_{i=1}^{\infty} \frac{1}{i} p_i \right) = \bigvee_{i=1}^{\infty} p_i$$

we have that  $\bigvee_{i=1}^{\infty} p_i \in \mathcal{P}$ , as required.

(b)  $\Rightarrow$  (c) Suppose  $\mathcal{P}$  is closed under orthogonal join, and let  $\{p_i : i \in \mathbb{N}\} \subset \mathcal{P}$ .

Let  $q_1 = p_1$  and for  $i \geq 2$  let

$$q_i = \bigvee_{j=1}^{i+1} p_j - \bigvee_{j=1}^i p_j$$

Now  $q_i \in \mathcal{P}$  for all  $i \in \mathbb{N}$ , and  $\{q_i\}$  is an orthogonal family, so  $\bigvee_{i=1}^{\infty} q_i \in \mathcal{P}$  by hypothesis.

Since  $\bigvee_{i=1}^n q_i = \bigvee_{i=1}^n p_i$  for all  $n \in \mathbb{N}$ , we have that  $\bigvee_{i=1}^{\infty} q_i = \bigvee_{i=1}^{\infty} p_i$ ,  $\bigvee_{i=1}^{\infty} p_i \in \mathcal{P}$ , and  $\mathcal{P}$  is closed under countable join.

(c)  $\Rightarrow$  (a) Suppose  $\mathcal{P}$  is a  $\sigma$ p-ideal, and suppose  $((\mathcal{P})) \supset x_i \xrightarrow{\|\cdot\|} x \in [[\mathcal{P}]]$ . Then

$$\left( \bigvee_{i=1}^{\infty} l(x_i) \right) x_i = x_i \xrightarrow{\|\cdot\|} \left( \bigvee_{i=1}^{\infty} l(x_i) \right) x$$

and so  $(\bigvee_{i=1}^{\infty} l(x_i)) x = x$ , from which it follows that

$$l(x) \leq \bigvee_{i=1}^{\infty} l(x_i)$$

Now  $l(x_i) \in \mathcal{P}$  for all  $i \in \mathbb{N}$ , and so  $\bigvee_{i=1}^{\infty} l(x_i) \in \mathcal{P}$  since  $\mathcal{P}$  is a  $\sigma$ p-ideal. Thus  $l(x) \in \mathcal{P}$ , and  $x \in ((\mathcal{P}))$ , which shows that  $[[\mathcal{P}]] \subset ((\mathcal{P}))$ .  $\blacksquare$

It follows from Corollary 1.1.10 that the previous result characterises the  $\sigma$ p-ideals as being exactly those p-ideals which generate a unique ideal. We will find a topological condition characterising  $\sigma$ p-ideals in Corollary 5.3.6.

Combining some of the previous observations we recover the following well known result:

**Corollary 1.2.8** *For each  $\aleph_0 \leq \alpha \leq \dim \mathcal{H}$ , there exists a unique ideal in  $\mathcal{B}(\mathcal{H})$  whose elements have ranges of dimension less than or equal to  $\alpha$ .*

Here we say nothing about the existence or otherwise of ideals in  $\mathcal{B}(\mathcal{H})$  which are not determined by cardinal numbers. In fact such ideals do not exist : it is a consequence of the work of Wright and Wils that in a factor the closed ideals are totally ordered by inclusion and are indexed by means of the cardinal numbers - see [Wri58] §4 and [Wil70b] Proposition 2.6 and Corollary 2.8. In fact it is shown that the lattice structure of the closed ideals in a factor is isomorphic to the lattice structure of the infinite cardinals which are less than or equal to the dimension of the identity operator - see the remarks in [Wil70b] §2. Here again 'dimension' refers to the dimension function of Tomiyama.

We have seen that there is a one-to-one correspondence between norm closed ideals and p-ideals. Note that  $s\sigma$ -closed ideals are norm closed, since the  $s\sigma$ -topology is weaker than the norm topology. We are therefore motivated to ask if we can characterise those p-ideals that correspond to the  $s\sigma$ -closed ideals. As a consequence we do in fact get a characterisation of the  $s\sigma$ -closed ideals. This result is essentially the same as [KR86] Theorem 6.8.8, where  $w\sigma$ -closed ideals are characterised; of course the  $w\sigma$ -closed and  $s\sigma$ -closed ideals coincide. Our proof is completely different, and is not claimed to be simpler; but it is interesting in that it makes full use of the technique of p-ideals.

**Proposition 1.2.9** Suppose  $p \in \mathcal{M}^p$ . Then

$$\mathcal{P} = \{q \in \mathcal{M}^p : q \leq p\} \quad (1.16)$$

is a  $p$ -ideal iff  $p \in \mathcal{Z}(\mathcal{M})^p$ .

**Proof:** Suppose  $p \in \mathcal{Z}(\mathcal{M})^p$ . Clearly  $\mathcal{P}$  is closed under subprojections. To show that  $\mathcal{P}$  is  $\sim$ -closed, it clearly suffices to show that if  $\mathcal{M}^p \ni e \check{\sim} q \leq p$  then  $e \leq p$ . Now

$$e = v^*qv = v^*pqv = pv^*qv$$

so  $e\mathcal{H} \subset p\mathcal{H}$ , and  $e \leq p$ . Finally, if  $e \perp f$  and  $e, f \leq p$  then by [KR86] Proposition 2.5.3 we have that

$$(e + f)\mathcal{H} = e\mathcal{H} + f\mathcal{H} \subset p\mathcal{H}$$

and so  $e + f \leq p$ .

Conversely, suppose  $\mathcal{P}$  is a  $p$ -ideal. Suppose we have any  $x \in \mathcal{M}$ . Then  $xp \in ((\mathcal{P}))$ , and so  $l(xp) \leq p$ . This shows that  $xp = pxp$ . Likewise  $x^*p = px^*p$  and so  $px = pxp = xp$ . Thus  $p \in \mathcal{Z}(\mathcal{M})^p$ . ■

**Theorem 1.2.10** The  $so$ -closed ideals are exactly of the form  $\mathcal{M}_p$  for  $p \in \mathcal{Z}(\mathcal{M})^p$ , and have corresponding  $p$ -ideals  $\mathcal{P} = \{q \in \mathcal{M}^p : q \leq p\}$ .

**Proof:** We first note that  $\mathcal{M}^p$  is  $so$ -closed. If  $\mathcal{M}^p \supset p_\alpha \xrightarrow{so} x$ , then  $p_\alpha = p_\alpha p_\alpha \xrightarrow{so} xx$ , since multiplication is  $so$ -continuous when the first factor is norm-bounded. Thus  $x = x^2$  since the  $so$ -topology is Hausdorff. Further,  $p_\alpha \xrightarrow{wo} x$ , and so  $p_\alpha = p_\alpha^* \xrightarrow{wo} x^*$ , since adjunction is  $wo$ -continuous. Again  $x = x^*$  since the  $wo$ -topology is Hausdorff.

It follows that if an ideal  $\mathcal{I}$  is  $so$ -closed then, as the intersection of  $so$ -closed sets, so is  $\mathcal{I}^p$ . Let  $p = \bigvee \mathcal{I}^p$ . From the elementary theory of the lattice of projections we have that the net of finite joins of members of  $\mathcal{I}^p$  is  $so$ -convergent to  $p$  - see [KR86] Proposition 2.5.6, for example. Thus  $p \in \mathcal{I}^p$ .

It is then immediate that  $\mathcal{I}^p = \{q \in \mathcal{M}^p : q \leq p\}$ . It then follows from Proposition 1.2.9 that  $p \in \mathcal{Z}(\mathcal{M})^p$ . Furthermore as previously mentioned  $\mathcal{I}^p$  is a  $\sigma$ -ideal, and so  $((\mathcal{I}^p)) = [[\mathcal{I}^p]]$  by Proposition 1.2.7.

Finally, if  $x \in \mathcal{M}$  then  $xp \in ((\mathcal{I}^p))$  and so  $\mathcal{M}_p \subset ((\mathcal{I}^p)) = [[\mathcal{I}^p]]$ . On the other hand, if  $x \in [[\mathcal{I}^p]] = ((\mathcal{I}^p))$ , then  $r(x) \in \mathcal{I}^p$  and so  $r(x) \leq p$ . Therefore  $x = x r(x) = xp \in \mathcal{M}_p$ . Thus  $\mathcal{I} = \mathcal{M}_p$ .

Conversely,  $\mathcal{M}_p$  is clearly a  $so$ -closed ideal if  $p \in \mathcal{Z}(\mathcal{M})^p$ . ■

We now briefly consider the behaviour of  $p$ -ideals and ideals under the taking of reductions. Suppose  $Q \in \mathcal{M}^p$ , we consider the reduced von Neumann algebra  $\mathcal{M}_Q$ . Recall that we follow the convention that any projection in  $\mathcal{M}_Q$  is represented as  $p_Q$  where  $p$  is a projection in  $\mathcal{M}$ .

**Definition 1.2.11** Suppose  $\mathcal{P}$  is a  $p$ -ideal and  $Q \in \mathcal{M}^p$ . The  $p$ -ideal reduction of  $\mathcal{P}$  by  $Q$  is

$$\mathcal{P}_{\leq Q} = \{p_Q : p \in \mathcal{P}, p \leq Q\} \quad (1.17)$$

It follows from Lemma 0.3.4 that  $\mathcal{P}_{\leq Q}$  coincides with  $\{p_Q : p \in \mathcal{P}\} \cap (\mathcal{M}_Q)^p$ . It is important to note that  $\mathcal{P}_Q = \{p_Q : p \in \mathcal{P}\}$  is not necessarily a  $p$ -ideal, since it is not necessarily a collection of projections.

Suppose  $\mathcal{I}$  is an ideal in  $\mathcal{M}$ . Let

$$\mathcal{I}_Q = \{x_Q : x \in \mathcal{I}\} \quad (1.18)$$

Then it is easy to verify that  $\mathcal{I}_Q$  is an ideal in the reduced von Neumann algebra  $\mathcal{M}_Q$ , and that  $\mathcal{I}_Q$  is closed if  $\mathcal{I}$  is closed.

The suitability of the definition of the reduction of a  $p$ -ideal is borne out by the following result:

**Proposition 1.2.12** Suppose  $\mathcal{I}$  is an ideal of  $\mathcal{M}$ . The  $p$ -ideal of  $\mathcal{I}_Q$  is  $(\mathcal{I}^p)_{\leq Q}$ .

**Proof:** It suffices to show that  $\mathcal{I}_Q \cap (\mathcal{M}_Q)^p = \{p_Q : p \in \mathcal{I}^p, p \leq Q\}$ .

So suppose  $p_Q \in \mathcal{I}_Q \cap (\mathcal{M}_Q)^p$ , so  $p \in \mathcal{M}^p, p \leq Q$  and there exists  $x \in \mathcal{I}$  such that  $QxQ = QpQ = p$ . Thus  $p \in \mathcal{I}$ .

The converse is even simpler. ■

It follows via Theorem 1.1.9 that if  $\mathcal{P}$  is a  $p$ -ideal then  $\mathcal{P}_{\leq Q}$  is indeed a  $p$ -ideal in the von Neumann algebra  $\mathcal{M}_Q$ .

**Proposition 1.2.13**  $\sigma p$ -ideals are preserved under reductions.

**Proof:** Suppose  $\mathcal{P}$  is a  $\sigma p$ -ideal, and  $\{p_i\} \subset \mathcal{P}_{\leq Q}$ . Let  $p = \bigvee_{i=1}^{\infty} p_i \in \mathcal{P}$ . Since  $p_i \leq Q$  for all  $i \in \mathbb{N}$ , we have that  $p \leq Q$ . Thus  $p_Q \in \mathcal{P}_{\leq Q}$  and

$$\bigvee_{i=1}^{\infty} p_{iQ} Q\mathcal{H} = \left[ \bigcup_{i=1}^{\infty} p_{iQ} Q\mathcal{H} \right] = \left[ \bigcup_{i=1}^{\infty} p_i \mathcal{H} \right] = p\mathcal{H}$$

and so

$$\bigvee_{i=1}^{\infty} p_{iQ} = p_Q$$
■

## Chapter 2

# Quotient algebras and Fredholm theory

In this chapter we suppose  $\mathcal{I}$  is a fixed norm-closed two sided ideal of  $\mathcal{M}$ . Let

$$\pi : \mathcal{M} \rightarrow \mathcal{M}/\mathcal{I} : x \rightarrow x + \mathcal{I} \quad (2.1)$$

be the canonical quotient map.

It is well known that  $\mathcal{M}/\mathcal{I}$  is an abstract  $C^*$ -algebra when equipped with the canonical quotient norm and involution  $\pi(x)^* = \pi(x^*)$ . A more or less direct proof of this fact may be found in [Tak79] I.8.1; this proof makes use of two-sided approximate identities in  $\mathcal{I}$ . In this chapter we will consider a concrete realisation of the  $C^*$ -algebra  $\mathcal{M}/\mathcal{I}$  by use of the Gelfand-Neumark theorem, and extend it to a von Neumann algebra by taking the double commutant. We follow the convention of identifying members of the quotient algebra with their images under this representation; so the elements  $\pi(x)$  are considered when necessary to be operators.

We denote by  $\alpha$  the seminorm on  $\mathcal{M}$  induced by the canonical quotient norm on  $\mathcal{M}/\mathcal{I}$ , i.e.

$$\alpha(x) = \|\pi(x)\| = \inf_{k \in \mathcal{I}} \|x - k\| \quad (2.2)$$

In this chapter we will analyse the properties of  $\alpha$  and use these to derive results on the essential spectrum of self adjoint operators.

By analogy with the standard theory of Fredholm operators, an operator  $x \in \mathcal{M}$  is said to be (left, right) Fredholm if  $\pi(x)$  is (left, right) invertible in  $\mathcal{M}/\mathcal{I}$ . We will consider a quantity  $m(x)$  first considered by Olsen in [Ols84] which characterises Fredholm operators. Here we derive a slightly improved characterisation and show, in analogy with Atkinson's result on Fredholm operators, that it can always be arranged that the product of the operator and its Fredholm inverse be a projection. Furthermore we establish a new relationship between  $\alpha(x)$  and  $m(x)$ .

## 2.1 The Quotient Map and the Essential Spectrum

We begin by examining the properties of  $\alpha$  and the quotient map  $\pi$ . Note that

$$\alpha \text{ has kernel } \mathcal{I} \quad (2.3)$$

$$\alpha(x) \leq \|x\| \quad (2.4)$$

directly from the definition;

$$\alpha(xy) \leq \alpha(x) \alpha(y) \quad (2.5)$$

$$\alpha(x^*x) = \alpha(x)^2 \quad (2.6)$$

$$\alpha(x^*) = \alpha(x) \quad (2.7)$$

since  $\mathcal{M}/\mathcal{I}$  is a  $C^*$ -algebra;

$$\alpha(x) = \alpha(|x|) \quad (2.8)$$

since if  $x = v|x|$  is the polar decomposition, then

$$\alpha(x) = \alpha(v|x|) \leq \alpha(v) \alpha(|x|) \leq \alpha(|x|) = \alpha(v^*x) \leq \alpha(v^*) \alpha(x) \leq \alpha(x)$$

Also, for  $p \in \mathcal{M}^p$ ,

$$0 \leq \alpha(p) = \alpha(p^2) \leq \alpha(p)^2 \leq \|p\|^2 \leq 1$$

and so  $\alpha(p) \in \{0, 1\}$ . Hence

$$\alpha(p) = \begin{cases} 0 & \text{if } p \in \mathcal{I} \\ 1 & \text{if } p \notin \mathcal{I} \end{cases} \quad (2.9)$$

We note that  $\pi$  preserves self-adjointness, normality, projections, positive operators, the order structure, the positive-negative decomposition of self-adjoint operators, and absolute values. That  $\pi$  preserves self-adjointness and normality is clear from the definition of involution in the quotient. That  $\pi$  preserves projections is clear. Since the positivity of an operator is equivalent to it being the square of some operator, it follows that  $\pi$  preserves positive operators, and hence the order structure. That it preserves the positive-negative decomposition then follows from the uniqueness condition attached to that decomposition - [KR86] Proposition 4.2.3. Finally, if  $x \in \mathcal{M}$ , then

$$\pi(|x|)^2 = \pi(|x|^2) = \pi(x^*x) = \pi(x)^* \pi(x) = |\pi(x)|^2$$

and so

$$\pi(|x|) = |\pi(x)| \quad (2.10)$$

by the uniqueness of positive square roots.

We now mention some of the deficiencies of  $\pi$ , which we shall see are not too severe. Remember that the elements  $\pi(x)$  are considered to be operators. We have that

$$R(\pi(x)) \leq \pi(R(x)) \quad (2.11)$$

since by definition  $R(\pi(x))$  is the smallest projection  $\Psi$  in the von Neumann algebra generated by  $\mathcal{M}/\mathcal{I}$  such that  $\Psi\pi(x) = \pi(x)$ , and certainly  $\pi(R(x))$  does the job. One then derives the equivalent inequality

$$N(\pi(x)) \geq \pi(N(x)) \quad (2.12)$$

However, these inequalities may be strict. For example, if we have an operator  $x \in \mathcal{I}$  which is injective, then  $N(\pi(x)) = 1$  and  $\pi(N(x)) = 0$ .

Now if  $v|x|$  is the polar decomposition of  $x$  then

$$\pi(x) = \pi(v)\pi(|x|) = \pi(v)|\pi(x)|$$

and by direct calculation we find that  $\pi(v)$  is a partial isometry with initial space  $\pi(R(x^*))$  and final space  $\pi(R(x))$ . On the other hand the partial isometry appearing in the polar decomposition of  $\pi(x)$  will have initial space  $R(\pi(x^*))$  and final space  $R(\pi(x))$ . So the polar decomposition is not preserved by  $\pi$ , but not in a serious manner : the only 'error' is that the partial isometry we get is 'too large'.

Suppose now that  $x \in \mathcal{M}^{sa}$ , and

$$x = \int t de_t(x)$$

is the spectral decomposition of  $x$ , this being a Riemann-Stieltjes integral, understood in the sense of norm convergence. Since  $x$  is self-adjoint,  $\pi(x)$  is self-adjoint, and so  $\sigma(\pi(x)) \subset \mathbf{R}$ . Furthermore, by the norm continuity of  $\pi$ , we have that

$$\pi(x) = \int t d\pi(e_t(x)) \quad (2.13)$$

once again in the Riemann-Stieltjes sense. On the other hand we can consider the canonical spectral decomposition

$$\pi(x) = \int t de_t(\pi(x)) \quad (2.14)$$

in the von Neumann algebra generated by  $\mathcal{M}/\mathcal{I}$ . We want to relate (2.13) and (2.14). Of course  $\{\pi(e_t(x))\}$  is a set of projections, but it is not necessarily  $\{e_t(\pi(x))\}$ , the spectral family of  $\pi(x)$ . We do have that

$$\pi(e_t(x)) = \pi(N(x_t^+)) \leq N(\pi(x_t^+)) = N(\pi(x)_t^+) = e_t(\pi(x)) \quad (2.15)$$

for any  $t \in \mathbf{R}$ , but once again we have that this inequality may be strict. As before, if we take an operator  $0 \leq x \in \mathcal{I}$  which is injective, then  $N(\pi(x)) = 1$  but  $\pi(N(x)) = 0$ . Since the 0-th spectral projection of a positive operator is exactly the null space, it follows that  $\pi$  does not preserve the spectral family.

We can examine this example in some more detail. Note that by Theorem 0.3.5 we have that  $e_t(x) \in 1 - \mathcal{I}^p$  for  $t > 0$ . Hence

$$e_t(\pi(x)) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

but

$$\pi(e_t(x)) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } t > 0 \end{cases}$$

and so  $\{\pi(e_t(x))\}$  is not even a spectral family, since it is not right continuous at 0. In particular, this shows that  $\pi$  does not preserve countable meets, and thus joins, in the projection lattice.

Nevertheless, the sets  $\{\pi(e_t(x))\}$  and  $\{e_t(\pi(x))\}$  are very closely related and furthermore the former fully determines  $\sigma(\pi(x))$ , as we shall now see.

We will need the following technical lemma, the proof of which is almost identical to that of [KR86] Theorem 5.2.3.

**Lemma 2.1.1** *Suppose  $x \in \mathcal{M}^{sa}$  and  $p \in \mathcal{M}^p$  is such that*

$$(i) \quad xp \leq tp$$

$$(ii) \quad x(1-p) \geq t(1-p)$$

for some  $t \in \mathbb{R}$ . Then

$$e_{(-\infty, t)}(x) \leq p \leq e_t(x)$$

**Proof:** Implicit in (i) is the understanding that  $xp$  is self adjoint, and it follows that  $p$  commutes with  $x$ . Let  $\mathcal{M}_{x,p}$  be the abelian von Neumann algebra generated by  $x$  and  $p$ , and let  $X$  be the extremely disconnected compact Hausdorff space for which  $\mathcal{M}_{x,p} \cong \mathcal{C}(X)$ . As usual  $\{e_t(x)\}$  denotes the spectral family for  $x$ ; this family is included in  $\mathcal{M}_{x,p}$ . Recall that if  $x$  corresponds to  $f \in \mathcal{C}(X)$  then  $e_t(x)$  corresponds to the characteristic function of

$$X_t = \overline{\{x \in X : f(x) > t\}}$$

which is the largest clopen set on which  $f \leq t$ . ([KR86] Theorem 5.2.2.) Now  $p$  corresponds to the characteristic function of a clopen set  $Y_t$ , and  $f \leq t$  on  $Y_t$ , since  $xp \leq tp$ . Thus  $Y_t \subset X_t$  and so  $p \leq e_t(x)$ .

For the other inequality, it suffices to show that if  $\delta > 0$  then  $p \geq e_{t-\delta}(x)$ , and for this we need show that  $X_{t-\delta} \subset Y_t$ . Now since  $x(1-p) \geq t(1-p)$  we have that  $f \geq t$  on  $Y_t'$ , and so  $f > t - \delta$  on  $Y_t'$ . Thus

$$Y_t' \subset \{x \in X : f(x) > t - \delta\} \subset \text{clos}\{x \in X : f(x) > t - \delta\} = X_{t-\delta}'$$

Thus  $X_{t-\delta} \subset Y_t$ . ■

**Theorem 2.1.2** *Suppose  $x \in \mathcal{M}^{sa}$ . Consider a concrete realisation of the  $C^*$ -algebra  $\mathcal{M}/\mathcal{I}$  by implementing the Gelfand-Neumark theorem, and extend it to a von Neumann algebra by taking the double commutant, and consider the canonical spectral decomposition of  $\pi(x)$  there. Then*

$$e_{(-\infty, t)}(\pi(x)) \leq \pi(e_t(x)) \leq e_t(\pi(x)) \quad (2.16)$$

for all  $t \in \mathbf{R}$ .

**Proof:** We know from [KR86] Theorem 5.2.2 that

$$\begin{aligned} x e_t(x) &\leq t e_t(x) \\ x(1 - e_t(x)) &\geq t(1 - e_t(x)) \end{aligned}$$

and hence

$$\begin{aligned} \pi(x) \pi(e_t(x)) &\leq t \pi(e_t(x)) \\ \pi(x)(1 - \pi(e_t(x))) &\geq t(1 - \pi(e_t(x))) \end{aligned}$$

since  $\pi$  preserves the order structure. Thus for each  $t \in \mathbf{R}$ ,  $\pi(e_t(x))$  satisfies the conditions of Lemma 2.1.1, and so  $e_{(-\infty, t)}(\pi(x)) \leq \pi(e_t(x)) \leq e_t(\pi(x))$ .  $\blacksquare$

**Corollary 2.1.3** *Suppose  $x \in \mathcal{M}^{sa}$ . If at  $\lambda$  we have that*

(i)  $\{e_t(\pi(x))\}$  is continuous then  $\{\pi(e_t(x))\}$  is continuous, and

$$e_\lambda(\pi(x)) = \pi(e_\lambda(x))$$

(ii)  $\{\pi(e_t(x))\}$  is right continuous then

$$e_\lambda(\pi(x)) = \pi(e_\lambda(x))$$

(iii)  $\{\pi(e_t(x))\}$  is left continuous then

$$e_{(-\infty, \lambda)}(\pi(x)) = \pi(e_\lambda(x))$$

(iv)  $\{\pi(e_t(x))\}$  is continuous then  $\{e_t(\pi(x))\}$  is continuous, and

$$e_\lambda(\pi(x)) = \pi(e_\lambda(x))$$

*The statements also hold with 'continuous' replaced by 'locally constant'.*

**Proof:** From (2.16) we have that

$$e_{t-2\delta}(\pi(x)) \leq \pi(e_{t-\delta}(x)) \leq e_{t-\delta}(\pi(x)) \leq \pi(e_t(x)) \leq e_t(\pi(x)) \leq \pi(e_{t+\delta}(x)) \leq e_{t+\delta}(\pi(x))$$

for any  $\delta > 0$ , and the results then follow easily.  $\blacksquare$

It can be shown that for  $x \in \mathcal{B}(\mathcal{H})^{sa}$ ,  $\lambda \in \rho(x) \cap \mathbf{R}$  iff  $\lambda$  is a point of constancy of  $\{e_t(x)\}$ . Proofs of this fact are usually quite complicated, see, for example, [Kre78] Theorem 9.11-2 or [GGK90] V Corollary 5.3.

It should be noted that the spectrum (and thus the resolvent) of an operator is ‘algebra-independent’, that is, the spectrum is the same regardless of the  $\mathcal{C}^*$ -algebra the operator is considered to be a member of ([KR86] Proposition 4.1.5). This is of course due to the fact that inverses, if they exist, lie in the smallest  $\mathcal{C}^*$ -algebra that the operator itself lies in.

Thus if we implement the Gelfand-Neumark Theorem and consider  $\mathcal{M}/\mathcal{I}$  as a concrete  $\mathcal{C}^*$ -algebra, we have that  $\lambda \in \rho(\pi(x)) \cap \mathbf{R}$  iff  $\lambda$  is a point of constancy of  $\{e_t(\pi(x))\}$ , and from Corollary 2.1.3 (ii) this occurs iff  $\lambda$  is a point of constancy of  $\{\pi(e_t(x))\}$ . This is equivalent to there existing some  $n \in \mathbf{N}$  such that  $e_{(\lambda-\frac{1}{n}, \lambda+\frac{1}{n}]}(x) \in \mathcal{I}^p$ .

We now prove the last-mentioned result in a straightforward manner. This gives an independent and simpler proof of the first mentioned result than that appearing in [Kre78] or [GGK90], by putting  $\mathcal{M} = \mathcal{B}(\mathcal{H})$  and  $\mathcal{I} = \{0\}$ .

**Theorem 2.1.4** *Suppose  $x \in \mathcal{M}^{sa}$ . Then*

$$\rho(\pi(x)) \cap \mathbf{R} = \left\{ \lambda \in \mathbf{R} : e_{(\lambda-\frac{1}{n}, \lambda+\frac{1}{n}]}(x) \in \mathcal{I}^p \text{ for some } n \in \mathbf{N} \right\} \quad (2.17)$$

$$\sigma(\pi(x)) = \left\{ \lambda \in \mathbf{R} : e_{(\lambda-\frac{1}{n}, \lambda+\frac{1}{n}]}(x) \notin \mathcal{I}^p \text{ for all } n \in \mathbf{N} \right\} \quad (2.18)$$

**Proof:** It is clear that (2.17) and (2.18) are equivalent; we prove the first. Suppose then that  $\lambda \in \rho(\pi(x)) \cap \mathbf{R}$ , so  $\pi(x_\lambda)$  is invertible, with inverse  $\pi(y)$ , for some  $y \in \mathcal{M}$ . Thus for all  $n \in \mathbf{N}$

$$\begin{aligned} \alpha \left( e_{(\lambda-\frac{1}{n}, \lambda+\frac{1}{n}]}(x) \right) &= \alpha \left( e_{(\lambda-\frac{1}{n}, \lambda+\frac{1}{n}]}(x) x_\lambda y \right) \\ &\leq \alpha \left( e_{(\lambda-\frac{1}{n}, \lambda+\frac{1}{n}]}(x) x_\lambda \right) \alpha(y) \\ &\leq \left\| e_{(\lambda-\frac{1}{n}, \lambda+\frac{1}{n}]}(x) x_\lambda \right\| \alpha(y) \\ &\leq \frac{1}{n} \alpha(y) \end{aligned}$$

It follows from (2.9) that there exists  $n \in \mathbf{N}$  such that  $\alpha(e_{(\lambda-\frac{1}{n}, \lambda+\frac{1}{n}]}(x)) = 0$ , i.e.  $e_{(\lambda-\frac{1}{n}, \lambda+\frac{1}{n}]}(x) \in \mathcal{I}^p$ .

Conversely, suppose that there exists  $n \in \mathbb{N}$  such that  $e_{(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}]}(x) \in \mathcal{I}^p$ . Let

$$f(t) = \begin{cases} (t - \lambda)^{-1} & \text{for } t \in (-\infty, \lambda - \frac{1}{n}] \\ 0 & \text{for } t \in (\lambda - \frac{1}{n}, \lambda + \frac{1}{n}] \\ (t - \lambda)^{-1} & \text{for } t \in (\lambda + \frac{1}{n}, \infty) \end{cases}$$

Implementing the Functional Calculus, it is easy to see that

$$x_\lambda f(x) = f(x) x_\lambda = 1 - e_{(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}]}(x)$$

So

$$\pi(x)_\lambda \pi(f(x)) = \pi(f(x)) \pi(x)_\lambda = \pi\left(1 - e_{(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}]}(x)\right) = \pi(1)$$

This shows that  $\lambda \in \rho(\pi(x))$ . ■

It is clear that in Equation (2.18) the intervals  $(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}]$  could be replaced by the other corresponding half-open, open, or closed intervals.

As already mentioned, by putting  $\mathcal{M} = \mathcal{B}(\mathcal{H})$  and  $\mathcal{I} = \{0\}$ , we have the following corollary:

**Corollary 2.1.5** *Suppose  $x \in \mathcal{B}(\mathcal{H})^{sa}$ . Then*

$$\sigma(x) = \left\{ \lambda \in \mathbb{R} : e_{(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}]}(x) \neq 0 \text{ for all } n \in \mathbb{N} \right\} \quad (2.19)$$

This corollary would also follow by explicitly repeating the proof of Theorem 2.1.4 in the case that  $\pi = id$  and  $\alpha(x) = \|x\|$  for all  $x \in \mathcal{M}$ . In that case it is easy to see that the corollary will generalise to any closed densely defined operator affiliated with a von Neumann algebra. As can be seen by inspecting the proof, the finiteness of the norm of the operator is not required. Indeed, the finiteness of the norm of the inverse is used; there is no problem with this, since it is a well known consequence of the closed graph theorem that if a closed unbounded operator has an everywhere defined inverse then this inverse is bounded.

In the remainder of this section we consider some further applications of Theorem 2.1.4.

**Corollary 2.1.6**

$$\{t \in \mathbb{R} : e_t(\pi(x)) \neq \pi(e_t(x))\} \subset \sigma(\pi(x)) \quad (2.20)$$

**Proof:** Suppose  $\lambda \in \rho(\pi(x)) \cap \mathbb{R}$ . Then by (2.17) we have that  $\lambda$  is a point of constancy of  $\{\pi(e_t(x))\}$  and so by Corollary 2.1.3 we have that  $e_\lambda(\pi(x)) = \pi(e_\lambda(x))$ . ■

It should be pointed out that this inclusion is strict, for example, if  $\mathcal{I} = \{0\}$  then  $\{t \in \mathbb{R} : e_t(\pi(x)) \neq \pi(e_t(x))\} = \emptyset$  for any  $x \in \mathcal{M}^{sa}$ .

The following quantity, sometimes called the lower bound of  $x$  relative to  $\mathcal{I}$ , was considered by Olsen - [Ols84] §4.

**Definition 2.1.7** For  $x \in \mathcal{M}$ , put

$$m(x) = \inf \sigma(\pi(|x|)) \quad (2.21)$$

At this point we can also note that since  $\alpha$  induces a  $\mathcal{C}^*$ -norm,

$$\alpha(x) = \alpha(|x|) = \sup \sigma(\pi(|x|)) \quad (2.22)$$

It follows that  $[m(x), \alpha(x)]$  is the smallest interval containing  $\sigma(\pi(|x|))$ . Also these endpoints are elements of  $\sigma(\pi(|x|))$ , since the spectrum is closed.

In the following, (2.23) improves [Ols84] Proposition 4.2. Furthermore, (2.24) and (2.25) first appeared as [Str89] Corollary 3.8 and Theorem 3.7 respectively. Theorem 2.1.4 has made substantial simplification of the proofs possible.

**Corollary 2.1.8**

$$m(x) = \sup \{ \lambda \geq 0 : e_\lambda(|x|) \in \mathcal{I}^p \} \quad (2.23)$$

$$\alpha(x) = \inf \{ \lambda \geq 0 : e_{(\lambda, \infty)}(|x|) \in \mathcal{I}^p \} \quad (2.24)$$

**Proof:** Since  $m(x) \in \sigma(\pi(|x|))$ , we have that  $e_{(m(x) - \frac{1}{n}, m(x) + \frac{1}{n})}(|x|) \notin \mathcal{I}^p$  by Theorem 2.1.4, and so  $e_{m(x) + \frac{1}{n}}(|x|) \notin \mathcal{I}^p$ , for all  $n \in \mathbb{N}$ .

Thus

$$\sup \{ \lambda \geq 0 : e_\lambda(|x|) \in \mathcal{I}^p \} \leq m(x)$$

Similarly we deduce that

$$\inf \{ \lambda \geq 0 : e_{(\lambda, \infty)}(|x|) \in \mathcal{I}^p \} \geq \alpha(x)$$

Conversely, for any small  $\delta > 0$  we have that

$$[-\delta, m(x) - \delta] \subset \rho(\pi(|x|))$$

Thus by Theorem 2.1.4 we have that for every  $\lambda \in [-\delta, m(x) - \delta]$  there exists  $n(\lambda) \in \mathbb{N}$  such that  $e_{(\lambda - \frac{1}{n(\lambda)}, \lambda + \frac{1}{n(\lambda)})}(|x|) \in \mathcal{I}^p$ . Then the intervals  $(\lambda - \frac{1}{n(\lambda)}, \lambda + \frac{1}{n(\lambda)})$  form an open cover of the compact interval  $[-\delta, m(x) - \delta]$ , which thus has a finite subcover. By taking the finite join of the spectral projections that correspond to these open intervals, we conclude via Proposition 1.1.3 that  $e_{m(x) - \delta}(|x|) \in \mathcal{I}^p$ . Thus

$$\sup \{ \lambda \geq 0 : e_\lambda(|x|) \in \mathcal{I}^p \} \geq m(x) - \delta$$

This concludes the proof of (2.23), since  $\delta$  was arbitrarily small.

By treating the interval

$$[\alpha(x) + \delta, \|x\| + \delta]$$

in a like manner, we also conclude the proof of (2.24). ■

It is now easy to see that in Equations (2.23) and (2.24) we could also have taken the corresponding open, respectively closed, intervals.

Note that (2.24) does to a certain extent generalise Theorem 0.3.5.

**Corollary 2.1.9**

$$\alpha(x) = \inf_{p \in 1 - \mathcal{I}^p} \|xp\| \tag{2.25}$$

**Proof:** If  $p \in 1 - \mathcal{I}^p$ , then by the definition of  $\alpha$  we have that

$$\alpha(x) \leq \|x - x(1 - p)\| = \|xp\|$$

Conversely, for any  $\lambda > \alpha(x)$ , we have that  $\|xe_\lambda(|x|)\| \leq \lambda$  and  $e_\lambda(|x|) \in 1 - \mathcal{I}^p$  by (2.24). ■

We now briefly consider the question of the uniqueness of  $\alpha$ . It is known that there can only be one norm on a  $*$ -algebra making it a  $C^*$ -algebra - [Mur90] Corollary 2.1.2. Hence  $\alpha$  is unique amongst the seminorms  $\delta$  on  $\mathcal{M}$  satisfying the conditions:

- (a)  $\delta$  has kernel  $\mathcal{I}$
- (b)  $\mathcal{M}/\mathcal{I}$  with the induced norm (also denoted  $\delta$ ) is a  $C^*$ -algebra

which are of course equivalent to the conditions:

- (a)  $\delta$  has kernel  $\mathcal{I}$
- (b)  $\mathcal{M}/\mathcal{I}$  is complete under  $\delta$
- (c)  $\delta(yx) \leq \delta(y) \delta(x)$  for  $y, x \in \mathcal{M}$
- (d)  $\delta(x)^2 \leq \delta(x^*x)$  for  $x \in \mathcal{M}$

Note that these conditions automatically imply that  $\delta$  is invariant under involution, since

$$\delta(x)^2 \leq \delta(x^*x) \leq \delta(x^*) \delta(x)$$

for  $x \in \mathcal{M}$ . Furthermore since  $\delta$  has kernel  $\mathcal{I}$  which is self-adjoint, we have that  $\delta(x) = 0$  iff  $\delta(x^*) = 0$ , and so it follows that  $\delta(x) \leq \delta(x^*)$ .

The following theorem proves uniqueness under a simpler set of conditions : retaining only conditions (a) and (c) and requiring in addition that  $\delta$  be majorised by the norm. This new set of conditions also imply that  $\delta$  is invariant under involution, although this

is now shown in a completely different manner. If  $x = v|x|$  is the polar decomposition of  $x \in \mathcal{M}$ , then

$$x = vx^*v \text{ and } x^* = v^*xv^* \quad (2.26)$$

and so

$$\delta(x) \leq \delta(v) \delta(x^*) \delta(v) \leq \|v\| \delta(x^*) \|v\| \leq \delta(x^*)$$

and similarly

$$\delta(x^*) \leq \delta(v^*) \delta(x) \delta(v^*) \leq \|v^*\| \delta(x) \|v^*\| \leq \delta(x)$$

This theorem was first proved by Ströh ([Str89] Theorem 3.3). We are able to give a simpler proof than the one appearing there by making use of Theorem 2.1.4.

**Theorem 2.1.10**  $\alpha$  is unique amongst the seminorms  $\delta$  on  $\mathcal{M}$  satisfying

(a)  $\delta$  has kernel  $\mathcal{I}$

(b)  $\delta(x) \leq \|x\|$  for  $x \in \mathcal{M}$

(c)  $\delta(yx) \leq \delta(y) \delta(x)$  for  $y, x \in \mathcal{M}$

**Proof:** We have already seen that  $\alpha$  satisfies the stated properties.

Suppose  $\delta$  is given, satisfying the stated properties. Then for any  $x \in \mathcal{M}$ ,

$$\delta(x) = \delta(x - k) \leq \|x - k\|$$

for every  $k \in \mathcal{I}$ . Hence  $\delta(x) \leq \alpha(x)$ .

Since  $\alpha(x) = \alpha(|x|)$  and  $\delta(x) = \delta(|x|)$  - the latter is shown in same way as for  $\alpha$  - it suffices to show the reverse inequality for  $x \geq 0$ .

Recall from (2.22) that  $\alpha(x) = \sup \sigma(\pi(x))$ . Thus it suffices to show that if  $\lambda \in \sigma(\pi(x))$  then  $\lambda \leq \delta(x)$ . So suppose  $\lambda \in \sigma(\pi(x))$ ; of course  $\lambda \geq 0$ .

Note that as before,

$$\delta(p) = \begin{cases} 0 & \text{if } p \in \mathcal{I} \\ 1 & \text{if } p \notin \mathcal{I} \end{cases}$$

Then for  $n \in \mathbb{N}$ , we have by Theorem 2.1.4 that  $\delta(e_{(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}]}(x))) = 1$ . Thus

$$\begin{aligned} \lambda &= \delta\left(\lambda e_{(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}]}(x)\right) \\ &= \delta\left(-x\lambda e_{(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}]}(x) + x e_{(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}]}(x)\right) \\ &\leq \delta\left(-x\lambda e_{(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}]}(x)\right) + \delta(x) \delta\left(e_{(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}]}(x)\right) \end{aligned}$$

$$\begin{aligned} &\leq \left\| -x\lambda e_{(\lambda-\frac{1}{n}, \lambda+\frac{1}{n})}(x) \right\| + \delta(x) \\ &\leq \frac{1}{n} + \delta(x) \end{aligned}$$

So  $\lambda \leq \delta(x)$ , as required. ■

## 2.2 Fredholm Operators

For future reference, we set apart the following:

**Lemma 2.2.1** *Suppose  $x, y$  are members of any  $C^*$ -algebra. Suppose  $x$  is self-adjoint (positive), and  $y$  is a left inverse for  $x$ . Then  $y$  is a self-adjoint (positive) two sided inverse for  $x$ .*

**Proof:** By the Gelfand-Neumark Theorem we may suppose that  $x, y$  are operators on a Hilbert Space  $\mathcal{H}$ . Since  $yx = 1$ , we have that  $xy^* = 1$ . It thus suffices to show that  $y$  is self-adjoint (positive). This follows from the fact that

$$\langle y\xi, \xi \rangle = \langle yxy^*\xi, \xi \rangle = \langle xy^*\xi, y^*\xi \rangle$$

for every  $\xi \in \mathcal{H}$ . ■

In Theorem 2.2.2, Corollary 2.2.3 and Theorem 2.2.5 we provide a synopsis of the theory of Fredholm operators that we shall need. These results are due to Olsen, see [Ols84] §4; for convenience we give brief outlines of the proofs.

**Theorem 2.2.2** [Ols84] *Theorem 4.5*  
*The following are equivalent:*

- (a)  $x$  is left Fredholm
- (b)  $|x|$  is Fredholm
- (c)  $m(x) > 0$

**Proof:** (a)  $\Rightarrow$  (b) follows from Lemma 2.2.1. All the other implications are immediate. ■

By using the fact that  $x$  is right Fredholm iff  $x^*$  is left Fredholm, a dual theorem holds characterising right Fredholm operators - [Ols84] Theorem 4.6.

**Corollary 2.2.3** [Ols84] *Theorem 4.5*  
*If  $x$  is left Fredholm then  $N(x) \in \mathcal{I}$ .*

**Proof:** If  $x$  is left Fredholm then  $m(x) > 0$  and so by (2.23) we have that

$$N(x) = N(|x|) = e_0(|x|) \in \mathcal{I}^p$$

■

**Note 2.2.4** Suppose  $x \in \mathcal{M}$  and  $x = v|x|$  is the polar decomposition of  $x$ . Note that

$$|x^*| = v|x|v^*$$

This implies that  $|x|$  acting on  $v^*v\mathcal{H} = \text{supp}(|x|)$  is unitarily equivalent to  $|x^*|$  acting on  $vv^*\mathcal{H} = \text{supp}(|x^*|)$ . It follows that

$$\sigma(|x|) \cup \{0\} = \sigma(|x^*|) \cup \{0\} \quad (2.27)$$

These comments also apply to the images of  $\pi$ , by making a simple appeal to the Gelfand-Neumark theorem. We know from 2.10 that  $\pi(|x|) = |\pi(x)|$ , and likewise that  $\pi(|x^*|) = |\pi(x^*)|$ , and hence

$$\sigma(\pi(|x|)) \cup \{0\} = \sigma(\pi(|x^*|)) \cup \{0\} \quad (2.28)$$

In particular, if  $m(x)$  and  $m(x^*)$  are both positive (i.e.  $x$  is Fredholm), then they are equal - [Ols84] Proposition 4.3.

Even more than (2.27) is true: suppose  $B$  is a spectral set that does not include 0. Then

$$e_B(|x^*|) = ve_B(|x|)v^* \quad (2.29)$$

from which it follows that corresponding spectral projections that are orthogonal to the respective null spaces are  $\sim$ -equivalent.

**Theorem 2.2.5** [Ols84] Theorem 4.7

*The following are equivalent:*

- (a)  $x$  is Fredholm
- (b)  $m(x) = m(x^*) > 0$
- (c)  $x$  is left Fredholm and  $N(x^*) \in \mathcal{I}$
- (d)  $x$  is right Fredholm and  $N(x) \in \mathcal{I}$

**Proof:** Suppose  $x$  is left Fredholm and  $N(x^*) \in \mathcal{I}$ . Then for any  $0 < \epsilon < m(x)$ ,  $e_{(0,\epsilon]}(|x|) \in \mathcal{I}^p$ , and so  $e_{(0,\epsilon]}(|x^*|) \in \mathcal{I}^p$  by (2.29). Thus by hypothesis we have that

$$e_\epsilon(|x^*|) = N(x^*) + e_{(0,\epsilon]}(|x^*|) \in \mathcal{I}^p$$

This shows (c)  $\Rightarrow$  (b).

(d)  $\Rightarrow$  (b) follows likewise and the other implications are clear.  $\blacksquare$

This result has a very simple and interesting corollary which does not seem to have been previously noticed.

**Corollary 2.2.6** *If  $\mathcal{M}$  is algebraically finite then any left (right) Fredholm operator is Fredholm.*

**Proof:** For any operator  $x \in \mathcal{M}$  we have that  $R(x) \sim R(x^*)$ . Since  $N(x) = 1 - R(x^*)$  and  $N(x^*) = 1 - R(x)$ , we have by the algebraic finiteness of  $\mathcal{M}$  that  $N(x) \sim N(x^*)$ , by [Tak79] V Proposition 1.38.

Suppose now that  $x$  is left Fredholm. Then by Corollary 2.2.3 we have that  $N(x) \in \mathcal{I}$ . Thus  $N(x^*) \in \mathcal{I}$ , and the result follows by Theorem 2.2.5.  $\blacksquare$

Recall that a commutative von Neumann algebra is finite, and of course in this setting left and right Fredholm-ness are equivalent. So the above result extends this observation.

By definition we have that if  $x$  is left Fredholm then there exists  $y \in \mathcal{M}$  such that  $yx \in 1 - \mathcal{I}$ . Similar statements hold for right Fredholm and Fredholm operators. In the classical characterisation theorem for operators that are Fredholm w.r.t. the compact operators, one can arrange for these members of  $1 - \mathcal{K}(\mathcal{H})$  to be projections. We generalise this result in the following theorem.

**Theorem 2.2.7** *(Atkinson-type characterisation of Fredholm operators)*

- (a) *If  $x$  is left Fredholm then there exists  $y \in \mathcal{M}$ ,  $e \in 1 - \mathcal{I}^p$  such that  $yx = e$*
- (b) *If  $x$  is right Fredholm then there exists  $y \in \mathcal{M}$ ,  $e \in 1 - \mathcal{I}^p$  such that  $xy = e$*
- (c) *If  $x$  is Fredholm then there exists  $y \in \mathcal{M}$ ,  $e_1, e_2 \in 1 - \mathcal{I}^p$  such that  $yx = e_1$ ,  $xy = e_2$ , and  $e_1 \sim e_2$*

**Proof:** (a) Suppose  $x$  is left Fredholm. Then  $|x|$  is Fredholm, and so by Theorem 2.1.4 we can find  $n \in \mathbb{N}$  such that  $e_{\frac{1}{n}}(|x|) \in \mathcal{I}^p$ , and as in the proof of that Theorem we have

$$f(|x|) |x| = e_{(\frac{1}{n}, \infty)}(|x|)$$

where

$$f(t) = \begin{cases} 0 & \text{for } t \leq \frac{1}{n} \\ \frac{1}{t} & \text{for } t > \frac{1}{n} \end{cases}$$

Thus

$$f(|x|) v^* x = e_{(\frac{1}{n}, \infty)}(|x|)$$

and we put  $y = f(|x|) v^*$ .

(b) follows by taking adjoints.

(c) Suppose now that  $x$  is Fredholm. By (2.29) we have that  $e_{\frac{1}{n}}(|x^*|) \in \mathcal{I}^p$  where  $n$  is the same value as considered in (a). Then

$$vf(|x|)x^* = vf(|x|)v^*xv^* = ve_{(\frac{1}{n}, \infty)}(|x|)v^* = e_{(\frac{1}{n}, \infty)}(|x^*|)$$

from (2.26), (a) and (2.29); and so

$$xy = xf(|x|)v^* = e_{(\frac{1}{n}, \infty)}(|x^*|)$$

by taking adjoints. Finally by (2.29) we have that

$$e_{(\frac{1}{n}, \infty)}(|x|) \sim e_{(\frac{1}{n}, \infty)}(|x^*|)$$

and this completes the proof. ■

As we have seen the quantity  $m(\cdot)$  is very useful in the study of Fredholm operators. Recall that this quantity is defined in terms of the spectral family of the absolute value. The following result is of interest because it characterises  $m(\cdot)$  without reference to the spectrum.

**Theorem 2.2.8** *Suppose  $x = v|x|$  and  $y \in \mathcal{M}$ . If  $y$  is a left inverse for  $x$  modulo  $\mathcal{I}$ , then*

$$m(x) = \frac{1}{\alpha(yv)} \tag{2.30}$$

*If  $y$  is an inverse for  $x$  modulo  $\mathcal{I}$ , then*

$$m(x) = \frac{1}{\alpha(y)} \tag{2.31}$$

**Proof:** First suppose that  $x \geq 0$  and  $y$  is an inverse for  $x$  modulo  $\mathcal{I}$ . Since  $x$  is positive, it follows that  $\pi(x)$  is positive, and by Lemma 2.2.1 that  $\pi(y)$  is positive. Thus

$$\begin{aligned} m(x) &= \inf \sigma(\pi(x)) \\ &= \inf \left\{ \lambda : \frac{1}{\lambda} \in \sigma(\pi(y)) \right\} \\ &= \inf \left\{ \frac{1}{\lambda} : \lambda \in \sigma(\pi(y)) \right\} \\ &= \frac{1}{\sup \sigma(\pi(y))} \\ &= \frac{1}{\alpha(y)} \end{aligned}$$

Now suppose that  $y$  is a left inverse for  $x$  modulo  $\mathcal{I}$ . Then  $yv$  is a left inverse for  $|x|$  modulo  $\mathcal{I}$ . It now follows from Lemma 2.2.1 that  $\pi(yv)$  is positive and  $yv$  is a two-sided inverse for  $|x|$  modulo  $\mathcal{I}$ . It follows as before that

$$m(x) = m(|x|) = \frac{1}{\alpha(yv)}$$

which establishes (2.30).

Finally, if  $x$  is Fredholm, then in particular  $x^*$  is left Fredholm and so

$$vv^* = 1 - N(x^*) \in 1 - \mathcal{I}^p$$

by Corollary 2.2.3. Hence

$$\alpha(y) = \alpha(yvv^*) \leq \alpha(yv) \leq \alpha(y)$$

which, together with (2.30), establishes (2.31). ■

The following result was first shown in [Str89] Proposition 3.20. Here we prove the result using Theorem 2.2.8.

**Corollary 2.2.9** *For any  $x \in \mathcal{M}$ ,*

$$m(x) = \inf \{ \alpha(xp) : p \in \mathcal{M}^p \setminus \mathcal{I}^p \} \quad (2.32)$$

**Proof:** Suppose  $\delta > 0$ . Let  $p = e_{m(x)+\delta}(|x|)$ . Then by (2.23) we have that  $p \notin \mathcal{I}^p$  and

$$\alpha(xp) \leq \|xp\| = \||x|p\| \leq m(x) + \delta$$

Thus  $\inf \{ \alpha(xp) : p \in \mathcal{M}^p \setminus \mathcal{I}^p \} \leq m(x)$ , since  $\delta > 0$  was arbitrary.

Conversely, we may suppose  $m(x) > 0$ , i.e.  $x$  is left Fredholm, otherwise there is nothing to show. Suppose  $v|x|$  is the polar decomposition of  $x$ , and  $y$  is a left inverse for  $x$  modulo  $\mathcal{I}$ . Then for any  $p \in \mathcal{M}^p \setminus \mathcal{I}^p$  we have

$$1 = \alpha(p) = \alpha(yv|x|p) \leq \alpha(yv) \alpha(|x|p) = \frac{1}{m(x)} \alpha(xp)$$

So  $m(x) \leq \alpha(xp)$ . ■

By combining (2.25) and (2.32) we get

**Corollary 2.2.10**

$$m(x) = \inf \{ \||xpq\| : p \in \mathcal{M}^p \setminus \mathcal{I}^p, q \in 1 - \mathcal{I}^p \} \quad (2.33)$$

# Chapter 3

## Topological and bornological characterisations of ideals

In this chapter we are going to see that any ideal in a von Neumann algebra can be characterised in a very natural way both by topologies and bornologies on  $\mathcal{H}$ .

### 3.1 The injectivity and surjectivity properties

We borrow some terminology from the theory of operator ideals, as developed for example in the monograph of Pietsch ([Pie80]). The injective and surjective operator ideals are of some interest in that they are essentially those operator ideals which are invariant under enlargement of the codomain or enlargement of the domain, respectively. We will not be any more precise than this here, simply referring the reader to [Pie80]. The special properties of such operator ideals (and injective and surjective hulls of operator ideals) were first studied by Stephani ([Ste80] and [Ste83]).

Of course the setting we are considering is somewhat different to the setting of operator ideals, in that the domain and codomain of the operators of interest are always the same Hilbert space.

However, Stephani has shown that an operator ideal  $\mathcal{A}$  is injective if and only if

$$x \in \mathcal{A}(\mathcal{E}, \mathcal{F}), y \in B(\mathcal{E}, \mathcal{G}), \|y\xi\| \leq \|x\xi\| \forall \xi \in \mathcal{E} \Rightarrow y \in \mathcal{A}(\mathcal{E}, \mathcal{G})$$

and  $\mathcal{A}$  is surjective if and only if

$$x \in \mathcal{A}(\mathcal{E}, \mathcal{F}), y \in B(\mathcal{G}, \mathcal{F}), yB_{\mathcal{G}} \subset xB_{\mathcal{E}} \Rightarrow y \in \mathcal{A}(\mathcal{G}, \mathcal{F})$$

(See [Ste83] and also [WW88] Lemma 2.1.) We will now show that ideals in a von Neumann algebra satisfy properties analogous to the above two characterisations, which we naturally call injectivity and surjectivity properties.

We would like to thank C.L. Olsen for pointing out the article of Douglas ([Dou66]) in which the equivalence of conditions (a), (d) and (f) of the following proposition are established in the case  $\mathcal{M} = \mathcal{B}(\mathcal{H})$ . This has enabled us to improve our original result which simply showed the implications (e)  $\Rightarrow$  (b)  $\Rightarrow$  (c) - which suffices to establish the injectivity and surjectivity properties for ideals in von Neumann algebras - see Corollary 3.1.2.

**Proposition 3.1.1** *Suppose  $\mathcal{M}$  is a von Neumann algebra and  $x, y \in \mathcal{M}$ . The following are equivalent:*

- (a) *There exists  $\lambda \geq 0$  such that  $y^*y \leq \lambda^2 x^*x$*
- (b) *There exists  $\lambda \geq 0$  such that  $\|y\xi\| \leq \lambda\|x\xi\|$  for all  $\xi \in \mathcal{H}$*
- (c) *There exists  $z \in \mathcal{M}$  such that  $y = zx$*
- (d) *There exists  $z \in \mathcal{B}(\mathcal{H})$  such that  $y = zx$*
- (e) *There exists  $\lambda \geq 0$  such that  $y^*\mathcal{B}_{\mathcal{H}} \subset \lambda x^*\mathcal{B}_{\mathcal{H}}$*
- (f)  *$\text{range}(y^*) \subset \text{range}(x^*)$*

**Proof:** Suppose (a) holds. Then for any  $\xi \in \mathcal{H}$ , we have

$$\|y\xi\|^2 = \langle y\xi, y\xi \rangle = \langle y^*y\xi, \xi \rangle \leq \langle \lambda^2 x^*x\xi, \xi \rangle = \|\lambda x\xi\|^2 = \lambda^2 \|x\xi\|^2$$

and so (b) follows.

Now suppose (b) holds. Define the function

$$z : \text{range}(x) \rightarrow \mathcal{H} : x\xi \rightarrow y\xi$$

which is well defined by the hypothesis. Obviously it is linear, and by hypothesis this function is norm continuous. Therefore it has a uniquely determined extension, also denoted  $z$ , to  $R(x)$ . We now define  $z$  to be 0 on  $1 - R(x)$ , and then by considering the decomposition

$$\mathcal{H} = R(x)\mathcal{H} \oplus (1 - R(x))\mathcal{H}$$

we can define  $z$  on the whole of  $\mathcal{H}$ . It is clear from the construction that

$$y = zx$$

It is clear that  $z \in \mathcal{B}(\mathcal{H})$ , in fact,  $\|z\| \leq 1$ .

(To this point the proof of this implication coincides with that of [Dou66] Theorem 1 (2)  $\Rightarrow$  (3).) To complete the proof it will suffice to show that  $z \in \mathcal{M}$ .

Suppose  $u \in \mathcal{M}'^u$ . Suppose  $\xi \in \mathcal{H}$ . Then there exists  $(\zeta_n) \subset \mathcal{H}$  such that

$$x\zeta_n \xrightarrow{\|\cdot\|} R(x)\xi$$

Note that by the hypothesis  $y\zeta_n$  is also a convergent sequence. It follows that all of the limits appearing in the following calculation exist:

$$\begin{aligned} uz\xi &= uz[R(x)\xi + (1 - R(x))\xi] \\ &= uz \lim_n x\zeta_n \\ &= \lim_n uzx\zeta_n \\ &= \lim_n uyz\zeta_n \\ &= \lim_n yuz\zeta_n \\ &= \lim_n zxy\zeta_n \\ &= \lim_n zux\zeta_n \\ &= zuR(x)\xi \\ &= zR(x)u\xi \\ &= zu\xi \end{aligned}$$

Thus  $zu = uz$ , and so  $z \in \mathcal{M}'' = \mathcal{M}$ .

(c)  $\Rightarrow$  (d) is clear, while if (d) holds then  $y^* = x^*z^*$  and so

$$y^*\mathcal{B}_{\mathcal{H}} = x^*z^*\mathcal{B}_{\mathcal{H}} \subset x^*\|z^*\|\mathcal{B}_{\mathcal{H}} = \|z^*\|x^*\mathcal{B}_{\mathcal{H}}$$

and (e) follows. If (e) holds then

$$\text{range}(y^*) = \bigcup_{n=1}^{\infty} y^*n\mathcal{B}_{\mathcal{H}} = \bigcup_{n=1}^{\infty} ny^*\mathcal{B}_{\mathcal{H}} \subset \bigcup_{n=1}^{\infty} n\lambda x^*n\mathcal{B}_{\mathcal{H}} \subset \text{range}(x^*)$$

and (f) follows.

(f)  $\Rightarrow$  (a) is [Dou66] Theorem 1 (1)  $\Rightarrow$  (2). ■

**Corollary 3.1.2** *Any ideal  $\mathcal{I}$  in a von Neumann algebra  $\mathcal{M}$  satisfies*

- (a) *If  $x \in \mathcal{I}$ ,  $y \in \mathcal{M}$  and  $\|y\xi\| \leq \|x\xi\|$  for all  $\xi \in \mathcal{H}$  then  $y \in \mathcal{I}$  (injectivity property)*
- (b) *If  $x \in \mathcal{I}$ ,  $y \in \mathcal{M}$  and  $y\mathcal{B}_{\mathcal{H}} \subset x\mathcal{B}_{\mathcal{H}}$  then  $y \in \mathcal{I}$  (surjectivity property)*

**Proof:** Immediate from the implications (e)  $\Rightarrow$  (b)  $\Rightarrow$  (c) of Proposition 3.1.1 and the self-adjointness of  $\mathcal{I}$ . ■

The argument of the following lemma was suggested by the usual proof of the von Neumann double commutant theorem.

**Lemma 3.1.3** Suppose  $\mathcal{I}$  is an ideal in  $\mathcal{M}$ . Suppose  $x \in \mathcal{M}$ ,  $x_1, \dots, x_n \in \mathcal{I}$ . If

$$\|x\xi\| \leq \max_{1 \leq i \leq n} \|x_i \xi\|$$

for all  $\xi \in \mathcal{H}$ , then  $x \in \mathcal{I}$ .

**Proof:** Consider the algebra  $M_n(\mathcal{M})$  acting on the Hilbert space  $\bigoplus_{i=1}^n \mathcal{H}$ , where

$$M_n(\mathcal{M}) = \{[y_{ij}] : y_{ij} \in \mathcal{M} \text{ for } 1 \leq i, j \leq n\} \quad (3.1)$$

and where the usual rules of matrix multiplication apply. It is easy to see that  $M_n(\mathcal{M})$  is a von Neumann algebra; in any case this is a special case of the result proved in [SZ79] Lemma 3.16.

We consider

$$M_n(\mathcal{I}) = \{[y_{ij}] \in M_n(\mathcal{M}) : y_{ij} \in \mathcal{I} \text{ for } 1 \leq i, j \leq n\} \quad (3.2)$$

which by the nature of matrix multiplication is clearly an ideal in  $M_n(\mathcal{M})$ .

Now for any  $(\xi_1, \xi_2, \dots, \xi_n) \in \bigoplus_{i=1}^n \mathcal{H}$  we have that

$$\begin{aligned} \left\| \begin{bmatrix} x & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} x\xi_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\|^2 \\ &= \|x\xi_1\|^2 \\ &\leq \max_{1 \leq i \leq n} \|x_i \xi_1\|^2 \\ &\leq \sum_{i=1}^n \|x_i \xi_1\|^2 \\ &= \left\| \begin{pmatrix} x_1 \xi_1 \\ x_2 \xi_1 \\ \vdots \\ x_n \xi_1 \end{pmatrix} \right\|^2 \\ &= \left\| \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x_n & 0 & \cdots & 0 \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} \right\|^2 \end{aligned}$$

Now  $\begin{bmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x_n & 0 & \cdots & 0 \end{bmatrix} \in M_n(\mathcal{I})$  and so by the injectivity property enjoyed by the ideal

$M_n(\mathcal{I})$  we have that  $\begin{bmatrix} x & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \in M_n(\mathcal{I})$ . Therefore  $x \in \mathcal{I}$ . ■

## 3.2 Topological and bornological characterisations

We are going to see that any ideal  $\mathcal{I}$  in a von Neumann algebra can be characterised by means of a topology  $t(\mathcal{I})$  in that for  $x \in \mathcal{M}$

$$x \in \mathcal{I} \Leftrightarrow x : (\mathcal{H}, t(\mathcal{I})) \rightarrow (\mathcal{H}, \|\cdot\|) \text{ is continuous}$$

This can be compared with similar questions raised in the theory of operator ideals. For example, Stephani shows in [Ste83] that if an operator ideal is injective then it can be characterised in such a fashion, by means of ‘ideal topologies’; see also [WW88]. Considering that ideals in a von Neumann algebra satisfy the so-called injectivity property, the above-mentioned result should be expected.

Furthermore there is a notion dual to ‘ideal topologies’, namely that of ‘ideal bornologies’. (Stephani calls these ‘generating systems of sets’ in [Ste80]; see also [WW88] for a treatment of such bornologies in the setting of locally convex topological vector spaces.) It is shown that if an operator ideal is surjective then it can be characterised in terms of these bornologies. Again, since ideals in von Neumann algebras satisfy the surjectivity property, we should not be surprised to find that bornologies can be used to characterise them.

**Definition 3.2.1** *Suppose  $\mathcal{I}$  is an ideal in  $\mathcal{M}$ . Let  $t(\mathcal{I})$  be the initial vector topology on  $\mathcal{H}$  with respect to the source*

$$\mathcal{I} \ni x : \mathcal{H} \rightarrow (\mathcal{H}, \|\cdot\|)$$

*i.e.  $t(\mathcal{I})$  is the weakest vector topology on  $\mathcal{H}$  making all the operators in  $\mathcal{I}$  continuous (where the codomain is equipped with the norm topology). Thus  $t(\mathcal{I})$  has as basic neighbourhoods the sets*

$$\left\{ \bigcap_{i=1}^n x_i^{-1} B_{\mathcal{H}} : n \in \mathbb{N}, x_i \in \mathcal{I} \right\} \quad (3.3)$$

*Alternatively,  $t(\mathcal{I})$  is determined by the seminorms*

$$\{ \|x \cdot \| : x \in \mathcal{I} \} \quad (3.4)$$

**Note 3.2.2** (a) For any  $0 \neq x \in \mathcal{I}$  we have that

$$\frac{1}{\|x\|} B_{\mathcal{H}} \subset \{ \xi : x\xi \in B_{\mathcal{H}} \} = x^{-1} B_{\mathcal{H}}$$

Therefore if  $x_1, \dots, x_n \in \mathcal{I} \setminus \{0\}$  then

$$\frac{1}{\max_{1 \leq i \leq n} \|x_i\|} \mathcal{B}_{\mathcal{H}} \subset \bigcap_{i=1}^n x_i^{-1} \mathcal{B}_{\mathcal{H}} \quad (3.5)$$

from which it follows that  $t(\mathcal{I}) \leq \|\cdot\|$ .

- (b) Recall that any bounded linear operator is weakly continuous, in particular the sets  $x^{-1}\mathcal{B}_{\mathcal{H}}$  are weakly closed and of course convex for any  $x \in \mathcal{M}$ . This then is also true for intersections of such sets. It follows in particular that the basic neighbourhoods describing the topology  $t(\mathcal{I})$  are convex and weakly closed, and hence invariant under the taking of bipolars.

**Theorem 3.2.3** *Suppose  $x \in \mathcal{M}$ . Then*

$$x : (\mathcal{H}, t(\mathcal{I})) \rightarrow (\mathcal{H}, \|\cdot\|)$$

*is continuous iff  $x \in \mathcal{I}$ .*

**Proof:** If  $x \in \mathcal{I}$  then obviously  $x : (\mathcal{H}, t(\mathcal{I})) \rightarrow (\mathcal{H}, \|\cdot\|)$  is continuous. If, on the other hand,  $x : (\mathcal{H}, t(\mathcal{I})) \rightarrow (\mathcal{H}, \|\cdot\|)$  is continuous, then there exist  $x_1, \dots, x_n \in \mathcal{I}$  such that

$$\|x\xi\| \leq \max_{1 \leq i \leq n} \|x_i\xi\|$$

for all  $\xi \in \mathcal{H}$ . Therefore  $x \in \mathcal{I}$  by Lemma 3.1.3. ■

**Corollary 3.2.4** *Two ideals  $\mathcal{I}_1$  and  $\mathcal{I}_2$  coincide iff  $t(\mathcal{I}_1)$  and  $t(\mathcal{I}_2)$  coincide.*

**Corollary 3.2.5** *Suppose  $x \in \mathcal{M}$ . Then*

$$x : (\mathcal{H}, t(\mathcal{I})) \rightarrow (\mathcal{H}, t(\mathcal{I}))$$

*is continuous.*

**Proof:** Suppose  $\xi_\alpha \xrightarrow{t(\mathcal{I})} 0$ . Now  $y\xi \in \mathcal{I}$  for every  $y \in \mathcal{I}$ , and so  $y\xi_\alpha \xrightarrow{\|\cdot\|} 0$  for all  $y \in \mathcal{I}$ , by Theorem 3.2.3. By definition this shows that  $x\xi_\alpha \xrightarrow{t(\mathcal{I})} 0$ , and this completes the proof. ■

**Note 3.2.6** We can view these constructions from a different perspective and ask: what topologies on  $\mathcal{H}$  determine an ideal in  $\mathcal{M}$ ? Similar questions have been asked in the setting of operator ideals with the concept of ‘generating topology’. See [Ste83] for this, and [WW88] for the convex topological vector space case.

Suppose that  $\tau$  is a vector topology on  $\mathcal{H}$  and that

$$x : (\mathcal{H}, \tau) \rightarrow (\mathcal{H}, \tau)$$

is continuous for every  $x \in \mathcal{M}$ . Let

$$i(\tau) = \{x \in \mathcal{M} : x : (\mathcal{H}, \tau) \rightarrow (\mathcal{H}, \|\cdot\|) \text{ is continuous}\} \quad (3.6)$$

It is easy to see that  $i(\tau)$  is an ideal of  $\mathcal{M}$ .

By definition  $i(\tau)$  consists of operators that are  $\tau - \|\cdot\|$  continuous, while  $t(i(\tau))$  is the weakest topology making the members of  $i(\tau)$  continuous as maps to  $(\mathcal{H}, \|\cdot\|)$ . It follows that  $t(i(\tau)) \leq \tau$ .

Since topological and bornological constructions are in general dual to each other, it is perhaps not surprising that while the topological characterisation of ideals involves initial topologies, the bornological characterisation involves final bornologies. For additional information on bornologies the reader should consult [HN77].

**Definition 3.2.7** Suppose  $\mathcal{I}$  is an ideal in  $\mathcal{M}$ . Let  $b(\mathcal{I})$  be the convex bornology which has as a base the collection

$$\{co \bigcup_{i=1}^n x_i B_{\mathcal{H}} : n \in \mathbb{N}, x_i \in \mathcal{I}\}$$

It is easy to see that this collection does in fact satisfy the requisite properties of being a base for a convex bornology.

According to [HN77] Theorem 2:6(1), given a source of linear maps

$$f_j : \mathcal{E}_j \rightarrow \mathcal{E}$$

from a family of vector spaces  $\{\mathcal{E}_j : j \in \mathcal{J}\}$  each equipped with a convex bornology  $B_j$ , to a vector space  $\mathcal{E}$ , it is possible to characterise the finest convex bornology on  $\mathcal{E}$  making all the maps  $(f_j)$  bounded. This is called the final convex bornology on  $\mathcal{E}$  with respect to  $(f_j)$  and is generated by the collection

$$\{co \bigcup_{j \in \mathcal{J}} f_j(B_j) : B_j \in B_j\}$$

Specialising to the case under consideration, the finest convex bornology on  $\mathcal{H}$  making all the maps

$$\mathcal{I} \ni x : (\mathcal{H}, \|\cdot\|) \rightarrow \mathcal{H}$$

bounded will be generated by the collection

$$\{co \bigcup_{x \in \mathcal{I}} x B_x : B_x \text{ is } \|\cdot\| \text{-bounded}\}$$

It follows from this that  $b(\mathcal{I})$  is finer than the final bornology. It is clear, however, that the family

$$\mathcal{I} \ni x : (\mathcal{H}, \|\cdot\|) \rightarrow (\mathcal{H}, b(\mathcal{I}))$$

comprises bounded maps. It follows that  $b(\mathcal{I})$  is in fact the final convex bornology with respect to

$$\mathcal{I} \ni x : (\mathcal{H}, \|\cdot\|) \rightarrow \mathcal{H}$$

The following result should not come as a surprise:

**Proposition 3.2.8** *Suppose  $\mathcal{I}$  is an ideal. Then  $b(\mathcal{I})$  is the equicontinuous bornology for the topology  $t(\mathcal{I})$ .*

**Proof:** For every  $x_1, \dots, x_n \in \mathcal{I}$  we have by [RR64] II Corollary 3 that

$$\left( \bigcap_{i=1}^n x_i^{-1} \mathcal{B}_{\mathcal{H}} \right)^{\circ} = \text{co} \bigcup_{i=1}^n (x_i^{-1} \mathcal{B}_{\mathcal{H}})^{\circ} = \text{co} \bigcup_{i=1}^n x_i^* \mathcal{B}_{\mathcal{H}}$$

The result now follows from the self-adjointness of  $\mathcal{I}$ . ■

**Corollary 3.2.9**

$$x \in \mathcal{I} \Leftrightarrow x^{-1} \mathcal{B}_{\mathcal{H}} \in t(\mathcal{I}) \Leftrightarrow x \mathcal{B}_{\mathcal{H}} \in b(\mathcal{I})$$

**Proof:**

$$\begin{aligned} x \in \mathcal{I} &\Leftrightarrow x^{-1} \mathcal{B}_{\mathcal{H}} \in t(\mathcal{I}) \\ &\Leftrightarrow (x^{-1} \mathcal{B}_{\mathcal{H}})^{\circ} \in b(\mathcal{I}) \\ &\Leftrightarrow x^* \mathcal{B}_{\mathcal{H}} \in b(\mathcal{I}) \end{aligned}$$

The result now follows from the self-adjointness of  $\mathcal{I}$ . ■

**Corollary 3.2.10**

$$\mathcal{M} b(\mathcal{I}) \subset b(\mathcal{I}) \tag{3.7}$$

**Proof:** Suppose  $B \in b(\mathcal{I})$  and  $x \in \mathcal{M}$ . Then from Proposition 3.2.8 and Note 3.2.2 (b) we may suppose that  $B^{\circ}$  is a basic neighbourhood in  $t(\mathcal{I})$ . Therefore

$$(x^* B)^{\circ} = x^{-1}(B^{\circ}) \in t(\mathcal{I})$$

by Corollary 3.2.5, and so  $(x^* B)^{\circ\circ} \in b(\mathcal{I})$  again by Proposition 3.2.8. Thus  $x^* B \in b(\mathcal{I})$  and the result follows. ■

**Note 3.2.11** Once again we can ask what bornologies on  $\mathcal{H}$  determine an ideal in  $\mathcal{M}$ . This question, in the context of operator ideals, has led to the introduction of ‘generating systems of sets’ by Stephani ([Ste80]) and ‘generating bornologies’ by Wong and Wong ([WW88]).

**Proposition 3.2.12** *Suppose  $\mathcal{B}$  is a bornology on  $\mathcal{H}$  such that  $\mathcal{M}\mathcal{B} \subset \mathcal{B}$ . Then*

$$j(\mathcal{B}) = \{x \in \mathcal{M} : x\mathcal{B}_{\mathcal{H}} \in \mathcal{B}\} \quad (3.8)$$

*is an ideal in  $\mathcal{M}$ .*

**Proof:** Suppose  $x, y \in j(\mathcal{B})$ ,  $z \in \mathcal{M}$ ,  $\lambda \in \mathbb{C}$ . Then

$$(x + y)\mathcal{B}_{\mathcal{H}} \subset x\mathcal{B}_{\mathcal{H}} + y\mathcal{B}_{\mathcal{H}} \in \mathcal{B} + \mathcal{B} = \mathcal{B}$$

$$\lambda x\mathcal{B}_{\mathcal{H}} \in \mathbb{C}\mathcal{B} = \mathcal{B}$$

$$xz\mathcal{B}_{\mathcal{H}} \subset x\|z\|\mathcal{B}_{\mathcal{H}} = \|z\|x\mathcal{B}_{\mathcal{H}} \in \mathbb{C}\mathcal{B} = \mathcal{B}$$

and finally

$$zx\mathcal{B}_{\mathcal{H}} \in z\mathcal{B} \subset \mathcal{M}\mathcal{B} = \mathcal{B}$$

and so  $j(\mathcal{B})$  is an ideal in  $\mathcal{M}$ . ■

We will see more on bornologically determined ideals in Chapter 4.

### 3.3 $t(\mathcal{I}^p)$ and $b(\mathcal{I}^p)$

In this section we are going to examine the initial topology and the final bornology for the ideal  $((\mathcal{I}^p))$ , where  $\mathcal{I}$  is an arbitrary ideal in a von Neumann algebra, and show that they have simple characterisations purely in terms of  $\mathcal{I}^p$ . This will prove to be very useful later because of the lattice structure on  $\mathcal{I}^p$ , lacking in general on  $\mathcal{I}$  or  $((\mathcal{I}^p))$ .

**Definition 3.3.1** *Suppose  $\mathcal{I}$  is an ideal in  $\mathcal{M}$ . Let  $t(\mathcal{I}^p)$  be the initial topology with respect to the source*

$$\mathcal{I}^p \ni p : \mathcal{H} \rightarrow (\mathcal{H}, \|\cdot\|)$$

*i.e.  $t(\mathcal{I}^p)$  has the sets*

$$p^{-1}(\epsilon\mathcal{B}_{\mathcal{H}})$$

*where  $p \in \mathcal{I}^p$ ,  $\epsilon > 0$  vary, as subbasic neighbourhoods of 0.*

We will set

$$p^{-1}(\epsilon \mathcal{B}_{\mathcal{H}}) = \{\xi \in \mathcal{H} : \|p\xi\| \leq \epsilon\} = U(p, \epsilon) \quad (3.9)$$

In fact the given system is basic, for, if  $p_1, \dots, p_n \in \mathcal{I}^p$  and  $\epsilon_1, \dots, \epsilon_n > 0$  then it is easy to see that

$$U\left(\bigvee_{i=1}^n p_i, \bigwedge_{i=1}^n \epsilon_i\right) \subset \bigcap_{i=1}^n U(p_i, \epsilon_i)$$

while  $\bigvee_{i=1}^n p_i \in \mathcal{I}^p$ .

Therefore  $t(\mathcal{I}^p)$  is determined by the basic seminorms

$$\{\|p \cdot\| : p \in \mathcal{I}^p\} \quad (3.10)$$

It is easy to verify that

$$U(p, \epsilon)^{\circ} = p \frac{1}{\epsilon} \mathcal{B}_{\mathcal{H}} \quad (3.11)$$

where the polar is taken in  $\mathcal{H}$ .

**Examples 3.3.2** (a) Recall that the  $\sigma$ -topology has as a neighbourhood base of 0 the sets

$$U_{\zeta_1, \zeta_2, \dots, \zeta_n, \epsilon} = \{\xi \in \mathcal{H} : |(\xi, \zeta_i)| \leq \epsilon, 1 \leq i \leq n\}$$

where  $\zeta_1, \zeta_2, \dots, \zeta_n$  are arbitrary unit vectors in  $\mathcal{H}$ , and  $\epsilon > 0$ .

We claim that

$$U(p, \epsilon) = \{\xi \in \mathcal{H} : \|p\xi\| \leq \epsilon\}$$

where  $p \in \mathcal{K}(\mathcal{H})^p, \epsilon > 0$  vary, form a basic neighbourhood system of 0 for  $\sigma$ . Therefore

$$\sigma = t(\mathcal{K}(\mathcal{H})^p) \quad (3.12)$$

To see this: if  $p \in \mathcal{K}(\mathcal{H})^p$  then we find an orthonormal system  $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$  which form a base for the range of  $p$ . Then

$$U(p, \epsilon) \supset U_{\zeta_1, \zeta_2, \dots, \zeta_n, \frac{\epsilon}{\sqrt{n}}}$$

On the other hand, if  $\zeta_1, \zeta_2, \dots, \zeta_n$  are unit vectors in  $\mathcal{H}$ , then let  $p$  be the projection generated by them, and so we have that

$$U(p, \epsilon) \subset U_{\zeta_1, \zeta_2, \dots, \zeta_n, \epsilon}$$

(b) Consider the von Neumann algebra  $\ell^\infty$  acting on the Hilbert space  $\ell^2$ . Consider the ideal

$$\mathcal{K}(\ell^2) \cap \ell^\infty = c_0$$

Note that  $c_0^p$  consists of those sequences consisting only of finitely many 1's, and 0's otherwise; it follows that  $t(c_0^p)$  is the topology of pointwise convergence.

**Proposition 3.3.3** Suppose  $x \in \mathcal{M}$ . Then

$$x : (\mathcal{H}, t(\mathcal{I}^p)) \rightarrow (\mathcal{H}, \|\cdot\|)$$

is continuous iff  $x \in ((\mathcal{I}^p))$ .

**Proof:**

$$\begin{aligned} x : (\mathcal{H}, t(\mathcal{I}^p)) \rightarrow (\mathcal{H}, \|\cdot\|) \text{ is continuous} &\Leftrightarrow \exists p \in \mathcal{I}^p, \epsilon > 0 : xU(p, \epsilon) \subset \mathcal{B}_{\mathcal{H}} \\ &\Leftrightarrow \exists p \in \mathcal{I}^p, \epsilon > 0 : x^* \mathcal{B}_{\mathcal{H}} \subset U(p, \epsilon)^{\circ} = p \frac{1}{\epsilon} \mathcal{B}_{\mathcal{H}} \\ &\Leftrightarrow \exists p \in \mathcal{I}^p : l(x^*) \leq p \\ &\Leftrightarrow l(x^*) \in \mathcal{I}^p \\ &\Leftrightarrow x^* \in ((\mathcal{I}^p)) \\ &\Leftrightarrow x \in ((\mathcal{I}^p)) \end{aligned}$$

Here we use Proposition 1.1.5 and the fact that any two sided ideal in a von Neumann algebra is self adjoint. ■

The above result can in fact be improved, as follows.

**Corollary 3.3.4**

$$t(\mathcal{I}^p) = t(((\mathcal{I}^p))) \tag{3.13}$$

**Proof:** It is clear that the initial vector topology with respect to  $((\mathcal{I}^p))$  must be stronger than  $t(\mathcal{I}^p)$ .

Conversely, suppose  $x \in ((\mathcal{I}^p))$ . Then by Proposition 1.1.5  $r(x) \in \mathcal{I}^p$ , and of course we may suppose  $x \neq 0$ . If  $\xi \in U(r(x), \frac{1}{\|x\|})$  then

$$\|x\xi\| = \|xr(x)\xi\| \leq \|x\| \|r(x)\xi\| \leq \|x\| \frac{1}{\|x\|} = 1$$

and so  $U(r(x), \frac{1}{\|x\|}) \subset x^{-1}\mathcal{B}_{\mathcal{H}}$ . It follows that

$$x : (\mathcal{H}, t(\mathcal{I}^p)) \rightarrow (\mathcal{H}, \|\cdot\|)$$

is continuous. This completes the proof, as  $t(((\mathcal{I}^p)))$  is the weakest vector topology with this property. ■

**Note 3.3.5** Suppose  $\mathcal{I}$  is an ideal. Let  $b(\mathcal{I}^p)$  be the convex bornology which has as a base the sets

$$\{p\delta\mathcal{B}_{\mathcal{H}} : p \in \mathcal{I}^p, \delta > 0\}$$

It is easy to see that this collection does in fact satisfy the requisite properties of being a base for a convex bornology, since if  $p_1, \dots, x_n \in \mathcal{I}^p$  and  $\delta_1, \dots, \delta_n > 0$  then

$$p_1 \delta_1 \mathcal{B}_{\mathcal{H}} + \dots + p_n \delta_n \mathcal{B}_{\mathcal{H}} \subset \left( \bigvee_{i=1}^n p_i \right) \left( \max_{1 \leq i \leq n} \delta_i \right) \mathcal{B}_{\mathcal{H}}$$

and of course  $\bigvee_{i=1}^n p_i \in \mathcal{I}^p$ .

Since  $U(p, \epsilon)^\circ = p \frac{1}{\epsilon} \mathcal{B}_{\mathcal{H}}$ , it is clear that  $b(\mathcal{I}^p)$  is the equicontinuous bornology for  $t(\mathcal{I}^p)$  and is therefore the final bornology for  $((\mathcal{I}^p))$ . Thus

$$b(\mathcal{I}^p) = b((\mathcal{I}^p)) \tag{3.14}$$

In particular

$$x \in ((\mathcal{I}^p)) \Leftrightarrow x \mathcal{B}_{\mathcal{H}} \in b(\mathcal{I}^p) \tag{3.15}$$

# Chapter 4

## Bornological characterisations of closed ideals

Recall that the ball measure of non-compactness of a set  $B$  in a Hilbert space  $\mathcal{H}$  is given by

$$\tilde{\Psi}(B) = \inf\{\epsilon > 0 : \exists \text{ finite } F \subset \mathcal{H} \text{ such that } B \subset F + \epsilon\mathcal{B}_{\mathcal{H}}\} \quad (4.1)$$

This quantity is called a measure of non-compactness because

$$\tilde{\Psi}(B) = 0 \Leftrightarrow B \text{ is relatively compact}$$

See [EE90] for other properties. (Note that some sources, such as the one cited, require the  $\epsilon$ -balls in the definition to be open, however, it is easy to see that this makes no difference.)

It follows that for  $x \in \mathcal{B}(\mathcal{H})$ ,

$$x \in \mathcal{K}(\mathcal{H}) \Leftrightarrow \tilde{\Psi}(x\mathcal{B}_{\mathcal{H}}) = 0 \quad (4.2)$$

Thus the ideal of compact operators is characterised by the measure of non-compactness. In this chapter we are going to analyse how ideals in von Neumann algebras can be characterised by generalisations of measures of non-compactness which are determined by bornologies.

### 4.1 Generating bornologies and $q$ -functions

The process discussed above can be seen from a bornological point of view. Consider the bornology  $\mathcal{F}$  in  $\mathcal{H}$  generated by the finite sets. Then the collection  $\tilde{\mathcal{F}}$  comprising

$$\{B \subset \mathcal{H} : \forall \epsilon > 0 \exists F \in \mathcal{F} \text{ such that } B \subset F + \epsilon\mathcal{B}_{\mathcal{H}}\} \quad (4.3)$$

is the bornology of relatively compact sets, and

$$\tilde{\Psi}(B) = 0 \Leftrightarrow B \in \tilde{\mathcal{F}}$$

and

$$x \in \mathcal{K}(\mathcal{H}) \Leftrightarrow x\mathcal{B}_{\mathcal{H}} \in \tilde{\mathcal{F}}$$

This process admits a simple generalisation to a wide class of vector bornologies on  $\mathcal{H}$ . Suppose  $\mathcal{M}$  is a von Neumann algebra, and  $\mathcal{B}$  is a vector bornology on  $\mathcal{H}$  such that  $\mathcal{M}\mathcal{B} \subset \mathcal{B}$ . Let

$$\tilde{\mathcal{B}} = \{B \subset \mathcal{H} : \forall \epsilon > 0 \exists B' \in \mathcal{B} \text{ such that } B \subset B' + \epsilon\mathcal{B}_{\mathcal{H}}\} \quad (4.4)$$

It is clear that  $\tilde{\mathcal{B}}$  is a vector bornology; furthermore if  $B \subset B' + \epsilon\mathcal{B}_{\mathcal{H}}$  then  $B \subset acB' + \epsilon\mathcal{B}_{\mathcal{H}}$  and so  $acB \subset acB' + \epsilon\mathcal{B}_{\mathcal{H}}$  since the latter is absolutely convex. This shows that if  $\mathcal{B}$  is a convex bornology then so is  $\tilde{\mathcal{B}}$ .

Finally, if  $B \subset B' + \epsilon\mathcal{B}_{\mathcal{H}}$  and  $z \in \mathcal{M}$  then

$$zB \subset zB' + z\epsilon\mathcal{B}_{\mathcal{H}} \subset zB' + \|z\|\epsilon\mathcal{B}_{\mathcal{H}}$$

which shows that if  $\mathcal{M}\mathcal{B} \subset \mathcal{B}$  then  $\mathcal{M}\tilde{\mathcal{B}} \subset \tilde{\mathcal{B}}$ .

Now consider the function

$$q_{\mathcal{B}}(B) = \inf\{\epsilon > 0 : \exists B' \in \mathcal{B} \text{ such that } B \subset B' + \epsilon\mathcal{B}_{\mathcal{H}}\} \quad (4.5)$$

We will verify that this function satisfies the properties appearing in the following definition of a so-called  $q$ -function, and moreover

$$q_{\mathcal{B}}(B) = 0 \Leftrightarrow B \in \tilde{\mathcal{B}}$$

A  $q$ -function is to be thought of as a function which satisfies the properties typical of the canonical measure of non-compactness. (See [EE90] I Lemma 2.2 and 2.5, [LS71] (3.1), for example.)

**Definition 4.1.1** *A  $q$ -function relative to  $\mathcal{M}$  is a function*

$$q : \mathbf{P}(\mathcal{H}) \rightarrow [0, \infty]$$

*satisfying*

- (i)  $q(B_1) \leq q(B_2)$  for  $B_1 \subset B_2$ ;
- (ii)  $q(\lambda B) = |\lambda|q(B)$  for  $\lambda \in \mathbb{C}$ ,  $B \subset \mathcal{H}$  (with the usual convention that  $0 \cdot \infty = 0$ );
- (iii)  $q(B_1 + B_2) \leq q(B_1) + q(B_2)$  for  $B_1, B_2 \subset \mathcal{H}$ ;

(iv)  $q(\mathcal{B}_{\mathcal{H}}) < \infty$  - the case  $q(\mathcal{B}_{\mathcal{H}}) = 0$  leads to trivialities, so upon normalisation we usually suppose that  $q(\mathcal{B}_{\mathcal{H}}) = 1$ ;

(v)  $q(xB) \leq \|x\|q(B)$  for  $x \in \mathcal{M}$ ,  $B \subset \mathcal{H}$ .

$q$  is said to be convex if  $q(ac B) = q(B)$  for  $B \subset \mathcal{H}$ .

**Proposition 4.1.2** Suppose  $\mathcal{M}$  is a von Neumann algebra, and  $\mathcal{B}$  is a vector bornology on  $\mathcal{H}$  such that  $\mathcal{M}\mathcal{B} \subset \mathcal{B}$ . Then  $q_{\mathcal{B}}$  is a  $q$ -function. If  $\mathcal{B}$  is a convex bornology then  $q_{\mathcal{B}}$  is convex.

**Proof:** Clearly  $q_{\mathcal{B}}$  satisfies properties (i) to (iv). If  $B' \in \mathcal{B}$  and

$$B \subset B' + \epsilon\mathcal{B}_{\mathcal{H}}$$

and  $x \in \mathcal{M}$  then

$$xB \subset xB' + x\epsilon\mathcal{B}_{\mathcal{H}} \subset xB' + \|x\|\epsilon\mathcal{B}_{\mathcal{H}}$$

from which (v) follows since  $xB' \in \mathcal{B}$ .

Finally if  $\mathcal{B}$  is convex then as we have seen previously we have  $ac B \subset ac B' + \epsilon\mathcal{B}_{\mathcal{H}}$ . Therefore  $q_{\mathcal{B}}(ac B) \leq q_{\mathcal{B}}(B)$  since  $ac B' \in \mathcal{B}$ , while the reverse inequality follows from (i). ■

One immediate consequence of the definition of a  $q$ -function is that

$$q(xB) = q(|x|B) \tag{4.6}$$

for  $x \in \mathcal{M}$ ,  $B \subset \mathcal{H}$ . To see this, suppose  $x = v|x|$  is the polar decomposition. Then

$$q(|x|B) = q(v^*xB) \leq \|v^*\| q(xB) \leq q(xB) = q(v|x|B) \leq \|v\| q(|x|B) \leq q(|x|B)$$

**Proposition 4.1.3** Suppose  $\mathcal{B}$  is a bornology on  $\mathcal{H}$  such that  $\mathcal{M}\mathcal{B} \subset \mathcal{B}$ . Then

$$i(\mathcal{B}) = \{x \in \mathcal{M} : x\mathcal{B}_{\mathcal{H}} \in \tilde{\mathcal{B}}\} \tag{4.7}$$

$$= \{x \in \mathcal{M} : q_{\mathcal{B}}(x\mathcal{B}_{\mathcal{H}}) = 0\} \tag{4.8}$$

is a norm-closed ideal in  $\mathcal{M}$ .

**Proof:** It is clear that

$$j(\tilde{\mathcal{B}}) = i(\mathcal{B}) \tag{4.9}$$

and so it follows from Proposition 3.2.12 that  $i(\mathcal{B})$  is an ideal. We show that it is closed. Suppose  $i(\mathcal{B}) \supset x_n \xrightarrow{\|\cdot\|} x$ . Find  $n$  sufficiently large so that  $\|x - x_n\| \leq \epsilon$  and then find

$B \in \mathcal{B}$  such that  $x_n \mathcal{B}_{\mathcal{H}} \subset B + \epsilon \mathcal{B}_{\mathcal{H}}$ . Then

$$\begin{aligned} x \mathcal{B}_{\mathcal{H}} &= (x - x_n + x_n) \mathcal{B}_{\mathcal{H}} \\ &\subset (x - x_n) \mathcal{B}_{\mathcal{H}} + x_n \mathcal{B}_{\mathcal{H}} \\ &\subset \|x - x_n\| \mathcal{B}_{\mathcal{H}} + B' + \epsilon \mathcal{B}_{\mathcal{H}} \\ &\subset B' + 2\epsilon \mathcal{B}_{\mathcal{H}} \end{aligned}$$

which shows that  $x \in i(\mathcal{B})$ , as required. ■

**Note 4.1.4** It is clear that  $j(\mathcal{B}) \subset i(\mathcal{B})$ . We have been unable to find a satisfactory answer to the following question: under what conditions is  $i(\mathcal{B})$  the norm closure of  $j(\mathcal{B})$ ?

**Note 4.1.5** There are some other constructions in the mathematical literature which can be regarded as generalised measures of non-compactness and which broadly fall into the above framework. We mention two examples.

Firstly, de Pagter and Schep ([PS88]) used the bornology of order bounded sets in a Banach lattice in an analogous construction that characterises almost order-bounded sets and semi-compact maps.

Secondly, recall that it was shown by Grothendieck that in a Banach space  $\mathcal{E}$ , a set  $K$  is relatively weakly compact iff for every  $\epsilon > 0$  there exists a weakly compact set  $K_\epsilon$  such that

$$K \subset K_\epsilon + \epsilon \mathcal{B}_{\mathcal{E}}$$

For a proof see [Die84] Chapter XIII Lemma 2. With this result in mind, de Blasi ([dB77]) has introduced a measure of weak non-compactness; the definition is again analogous to that of the function  $q_{\mathcal{B}}$ .

## 4.2 The function $q_{\mathcal{I}}$

Suppose throughout this section that  $\mathcal{I}$  is a norm-closed ideal in a von Neumann algebra  $\mathcal{M}$ . We would like to generalise the characterisation of  $\mathcal{K}(\mathcal{H})$ , to a characterisation for  $\mathcal{I}$ , by finding a suitable bornology that will determine  $\mathcal{I}$  in the manner described in Proposition 4.1.3.

Of course  $b(\mathcal{I})$  characterises  $\mathcal{I}$ , but we shall see that  $b(\mathcal{I}^p)$  is a far more useful bornology which can also be used to characterise  $\mathcal{I}$ .

In order to motivate the use of  $b(\mathcal{I}^p)$  we again return to the example of the ideal  $\mathcal{K}(\mathcal{H})$  and the measure of non-compactness  $\tilde{\Psi}$ . Instead of considering the bornology of finite sets we consider the convex bornology  $b(\mathcal{K}(\mathcal{H})^p)$ . Consider the function

$$q_{b(\mathcal{K}(\mathcal{H})^p)}(B) = \inf\{\epsilon > 0 : \exists p \in \mathcal{K}(\mathcal{H})^p, \delta > 0 \text{ such that } B \subset p\delta \mathcal{B}_{\mathcal{H}} + \epsilon \mathcal{B}_{\mathcal{H}}\}$$

as defined in (4.5). This is a convex  $q$ -function by Proposition 4.1.2. We claim that

$$q_{b(\mathcal{K}(\mathcal{H})^p)}(B) = \tilde{\Psi}(B)$$

**Proof:** Suppose  $B \subset F + \epsilon\mathcal{B}_{\mathcal{H}}$  where  $F$  is finite. Let  $p$  be the projection onto  $\text{span } F$ , then  $p \in \mathcal{K}(\mathcal{H})^p$ . Now clearly  $F$  is bounded and included in  $p\mathcal{H}$ , and so we have that  $\tilde{\Psi}(B) \geq q_{b(\mathcal{K}(\mathcal{H})^p)}(B)$ .

Conversely, suppose  $B \subset p\delta\mathcal{B}_{\mathcal{H}} + \epsilon\mathcal{B}_{\mathcal{H}}$  for some  $p \in \mathcal{K}(\mathcal{H})^p$ ,  $\delta > 0$ . Then  $p\delta\mathcal{B}_{\mathcal{H}}$  is a bounded set in the finite dimensional space  $p\mathcal{H}$ , and is thus relatively compact, and hence totally bounded. Suppose  $\eta > 0$  is any given constant. Find a finite set  $F \subset p\mathcal{H}$  such that  $p\delta\mathcal{B}_{\mathcal{H}} \subset F + \eta\mathcal{B}_{\mathcal{H}}$ . Then

$$B \subset p\delta\mathcal{B}_{\mathcal{H}} + \epsilon\mathcal{B}_{\mathcal{H}} \subset F + \eta\mathcal{B}_{\mathcal{H}} + \epsilon\mathcal{B}_{\mathcal{H}} = F + (\epsilon + \eta)\mathcal{B}_{\mathcal{H}}$$

so  $\tilde{\Psi}(B) \leq \epsilon + \eta$ . Thus  $\tilde{\Psi}(B) \leq \epsilon$ , since  $\eta$  was arbitrary, and so  $\tilde{\Psi}(B) \leq q_{b(\mathcal{K}(\mathcal{H})^p)}(B)$ . ■

The above discussion has amounted to a reformulation of the definition of  $\tilde{\Psi}$  in an algebraic rather than a geometric manner. This result is fairly obvious if we realise that  $\tilde{\mathcal{F}}$  and  $b(\mathcal{K}(\mathcal{H})^p)$  are the same bornology, namely the bornology of relatively compact sets.

In this way the measure of non-compactness,  $\tilde{\Psi}$ , is determined by  $b(\mathcal{K}(\mathcal{H})^p)$ . This suggests that we should use  $b(\mathcal{I}^p)$  in a like manner to define a so-called measure of non- $\mathcal{I}$ -compactness. Now by Corollary 3.2.10 we have that  $\mathcal{M} b(\mathcal{I}^p) \subset b(\mathcal{I}^p)$  and so from Proposition 4.1.2 the function  $q_{b(\mathcal{I}^p)}$  is indeed a convex  $q$ -function. For convenience this function will be denoted  $q_{\mathcal{I}}$ .

The definition of the function  $q_{\mathcal{I}}$  is due to Ströh; see [Str89]. This function was first considered by Kaftal in [Kaf77] in the special case where  $\mathcal{M}$  is semifinite and  $\mathcal{I} = \mathcal{M}_{\sim}$ ; see also [SS89] in this regard. (Recall that  $\mathcal{M}_{\sim}$  is the ideal of algebraically compact operators.) In turn, these ideas were first suggested by Sonis in [Son71]. These authors considered collections known as  $\mathcal{I}$ -finite  $\epsilon$ -nets. It seems that by viewing the function  $q_{\mathcal{I}}$  as being derived via bornologies enables one to simplify the theory and derive some powerful new results.

For convenience we give a full definition of the function  $q_{\mathcal{I}}$ .

**Definition 4.2.1** *Suppose  $B \subset \mathcal{H}$ . The measure of non  $\mathcal{I}$ -compactness of  $B$  is given by*

$$q_{\mathcal{I}}(B) = \inf\{\epsilon > 0 : \exists p \in \mathcal{I}^p, \delta > 0 \text{ such that } B \subset p\delta\mathcal{B}_{\mathcal{H}} + \epsilon\mathcal{B}_{\mathcal{H}}\} \quad (4.10)$$

*$B$  is said to be  $\mathcal{I}$ -totally bounded or relatively  $\mathcal{I}$ -compact if  $q_{\mathcal{I}}(B) = 0$ .*

The following Proposition summarises the properties of  $q_{\mathcal{I}}$  that follow by virtue of it being a convex  $q$ -function.

**Proposition 4.2.2** Suppose  $B, B_1, B_2 \subset \mathcal{H}$ ,  $\lambda \in \mathbb{C}$ ,  $x \in \mathcal{M}$ .

$$B_1 \subset B_2 \Rightarrow q_{\mathcal{I}}(B_1) \leq q_{\mathcal{I}}(B_2) \quad (4.11)$$

$$q_{\mathcal{I}}(\lambda B) = |\lambda| q_{\mathcal{I}}(B) \quad (4.12)$$

$$q_{\mathcal{I}}(B_1 + B_2) \leq q_{\mathcal{I}}(B_1) + q_{\mathcal{I}}(B_2) \quad (4.13)$$

$$q_{\mathcal{I}}(xB) \leq \|x\| q_{\mathcal{I}}(B) \quad (4.14)$$

$$q_{\mathcal{I}}(B) = q_{\mathcal{I}}(acB) \quad (4.15)$$

$$q_{\mathcal{I}}(xB) = q_{\mathcal{I}}(|x|B) \quad (4.16)$$

It is clear that  $q_{\mathcal{I}}(B) < \infty$  iff  $B$  is bounded in norm. For sets which are norm-bounded, we have the following useful result.

**Proposition 4.2.3** If  $B$  is bounded in norm then  $q_{\mathcal{I}}(B) \leq \epsilon$  iff there exists  $p \in \mathcal{I}^p$  such that  $B \subset p(B) + \epsilon \mathcal{B}_{\mathcal{H}}$ .

**Proof:** Suppose  $q_{\mathcal{I}}(B) \leq \epsilon$ , so there exists  $p \in \mathcal{I}^p$  and  $\delta > 0$  such that  $B \subset p\delta \mathcal{B}_{\mathcal{H}} + \epsilon \mathcal{B}_{\mathcal{H}}$ . Then indeed

$$\|\xi - p\xi\| = \inf_{\eta \in p\mathcal{H}} \|\xi - \eta\| \leq \inf_{\eta \in p\delta \mathcal{B}_{\mathcal{H}}} \|\xi - \eta\| \leq \epsilon$$

for all  $\xi \in B$ . Thus  $B \subset p(B) + \epsilon \mathcal{B}_{\mathcal{H}}$ .

The converse is clear :  $p(B)$  is the required member of  $b(\mathcal{I}^p)$  - this set is bounded in norm since  $B$  is bounded in norm. ■

The following new characterisation of the function  $q_{\mathcal{I}}$  follows from the previous Proposition. For any  $B \subset \mathcal{H}$

$$q_{\mathcal{I}}(B) = \begin{cases} \inf_{p \in \mathcal{I}^p} \sup_{\xi \in B} \|\xi - p\xi\| = \inf_{p \in 1-\mathcal{I}^p} \sup_{\xi \in B} \|p\xi\| & \text{if } B \text{ is norm bounded} \\ \infty & \text{otherwise} \end{cases} \quad (4.17)$$

We now establish some further elementary properties of  $q_{\mathcal{I}}$  which are also analogous to the properties enjoyed by the canonical measure of non-compactness : see, for example, [LS71] Proposition 4.13 and [EE90] I Lemma 2.2.

**Proposition 4.2.4** Suppose  $B, B_1, B_2 \subset \mathcal{H}$ ,  $\lambda \in \mathbb{C}$ .

$$q_{\mathcal{I}}(\mathcal{B}_{\mathcal{H}}) = 1 \quad (4.18)$$

$$q_{\mathcal{I}}(B_1 \cup B_2) = \max\{q_{\mathcal{I}}(B_1); q_{\mathcal{I}}(B_2)\} \quad (4.19)$$

$$q_{\mathcal{I}}(\overline{B}) = q_{\mathcal{I}}(B) \quad (4.20)$$

**Proof:** (4.18) follows easily from (4.17). (Here we are implicitly assuming that  $\mathcal{I} \neq \mathcal{M}$ . In the case that  $\mathcal{I} = \mathcal{M}$ ,  $q_{\mathcal{I}}$  vanishes on bounded sets.)

We may suppose that the quantities involved are finite, otherwise it is easy to see that there is nothing to prove.

It follows from (4.11) that  $q_{\mathcal{I}}(B_1 \cup B_2) \geq \max\{q_{\mathcal{I}}(B_1); q_{\mathcal{I}}(B_2)\}$ . Conversely, suppose that for  $i = 1, 2$  we have  $p_i \in \mathcal{I}^p$  and  $\delta_i > 0$  such that  $B_i \subset p_i \delta_i \mathcal{B}_{\mathcal{H}} + \epsilon_i \mathcal{B}_{\mathcal{H}}$ . Then

$$\begin{aligned} B_1 \cup B_2 &\subset (p_1 \delta_1 \mathcal{B}_{\mathcal{H}} + \epsilon_1 \mathcal{B}_{\mathcal{H}}) \cup (p_2 \delta_2 \mathcal{B}_{\mathcal{H}} + \epsilon_2 \mathcal{B}_{\mathcal{H}}) \\ &\subset (p_1 \delta_1 \mathcal{B}_{\mathcal{H}} \cup p_2 \delta_2 \mathcal{B}_{\mathcal{H}}) + \max\{\epsilon_1, \epsilon_2\} \mathcal{B}_{\mathcal{H}} \\ &\subset (p_1 \vee p_2) \max\{\delta_1, \delta_2\} \mathcal{B}_{\mathcal{H}} + \max\{\epsilon_1, \epsilon_2\} \mathcal{B}_{\mathcal{H}} \end{aligned}$$

Hence  $q_{\mathcal{I}}(B_1 \cup B_2) \leq \max\{\epsilon_1, \epsilon_2\}$ , and (4.19) follows.

Since  $B \subset \overline{B}$ , we have from (4.11) that  $q_{\mathcal{I}}(B) \leq q_{\mathcal{I}}(\overline{B})$ . Conversely, if  $B \subset p \delta \mathcal{B}_{\mathcal{H}} + \epsilon \mathcal{B}_{\mathcal{H}}$ , then

$$\overline{B} \subset p \delta \mathcal{B}_{\mathcal{H}} + \epsilon \mathcal{B}_{\mathcal{H}} + \eta \mathcal{B}_{\mathcal{H}} = p \delta \mathcal{B}_{\mathcal{H}} + (\epsilon + \eta) \mathcal{B}_{\mathcal{H}}$$

for any  $\eta > 0$ . Thus (4.20) follows. ■

We now consider the behaviour of the function  $q_{\mathcal{I}}$  when applied to the images of sets under elements of  $\mathcal{M}$ . Recall that the canonical measure of non-compactness of an operator  $x$  is defined as

$$\overline{\beta}(x) = \tilde{\Psi}(x \mathcal{B}_{\mathcal{H}}) \quad (4.21)$$

It does have other definitions; see [EE90] I §2 for these alternatives and other properties. This quantity is called the  $q$ -seminorm by Lebow and Schechter - see [LS71] §3. We would anticipate that  $q_{\mathcal{I}}(x \mathcal{B}_{\mathcal{H}})$  would be a measure of non- $\mathcal{I}$ -ness, this is indeed the case - it is just  $\alpha(x)$ , the measure with which we are familiar.

**Theorem 4.2.5** *Suppose  $B \subset \mathcal{H}$  is bounded in norm and  $x \in \mathcal{M}, k \in \mathcal{I}$ .*

$$q_{\mathcal{I}}(x \mathcal{B}_{\mathcal{H}}) = \alpha(x) \quad (4.22)$$

$$q_{\mathcal{I}}(xB) \leq \alpha(x) q_{\mathcal{I}}(B) \quad (4.23)$$

$$q_{\mathcal{I}}(xB) = q_{\mathcal{I}}((x+k)B) \quad (4.24)$$

**Proof:**

$$\begin{aligned} q_{\mathcal{I}}(x \mathcal{B}_{\mathcal{H}}) &= \inf_{p \in 1-\mathcal{I}^p} \sup_{\xi \in \mathcal{B}_{\mathcal{H}}} \|p x \xi\| \\ &= \inf_{p \in 1-\mathcal{I}^p} \|p x\| \\ &= \inf_{p \in 1-\mathcal{I}^p} \|x^* p\| \\ &= \alpha(x^*) \\ &= \alpha(x) \end{aligned}$$

by equation (2.25), which establishes (4.22).

Suppose  $\delta > 0$ . Let  $e = e_{\alpha(x^*) + \delta}(|x^*|)$ . Then

$$\|ex\| = \|x^*e\| \leq \alpha(x^*) + \delta = \alpha(x) + \delta$$

and by (2.24) we have that  $1 - e \in \mathcal{I}^p$ .

Suppose  $f \in 1 - \mathcal{I}^p$ . Let

$$q = N((1 - f)x^*) = 1 - R(x(1 - f))$$

Note that we have that

$$qx = qxf$$

Put

$$p = e \wedge q$$

Now we have that

$$1 - q = R(x(1 - f)) \sim R((1 - f)x^*) \leq 1 - f$$

and so  $1 - q \in \mathcal{I}^p$ . Thus  $1 - p \in \mathcal{I}^p$ , and so

$$\begin{aligned} q(xB) &\leq \sup_{\xi \in B} \|px\xi\| \\ &= \sup_{\xi \in B} \|pqx\xi\| \\ &= \sup_{\xi \in B} \|pqxf\xi\| \\ &= \sup_{\xi \in B} \|pexf\xi\| \\ &\leq \sup_{\xi \in B} \|p\| \|ex\| \|f\xi\| \\ &\leq [\alpha(x) + \delta] \sup_{\xi \in B} \|f\xi\| \end{aligned}$$

Since  $f \in 1 - \mathcal{I}^p$  was arbitrary and  $\delta > 0$  was arbitrary, (4.23) follows.

The above argument is inspired by one of Nelson's : see [Nel74] Theorem 1, the proof of (17'); or [Ter81], the proof of Proposition 5(ii).

Finally

$$\begin{aligned} q_{\mathcal{I}}((x + k)B) &\leq q_{\mathcal{I}}(xB + kB) \\ &\leq q_{\mathcal{I}}(xB) + q_{\mathcal{I}}(kB) \\ &= q_{\mathcal{I}}(xB) \\ &= q_{\mathcal{I}}((x + k - k)B) \\ &\leq q_{\mathcal{I}}((x + k)B + (-k)B) \\ &\leq q_{\mathcal{I}}((x + k)B) + q_{\mathcal{I}}(-kB) \\ &= q_{\mathcal{I}}((x + k)B) \end{aligned}$$

and this proves (4.24). ■

Under the convention that  $0 \cdot \infty = 0$ , the requirement that  $B$  be bounded is necessary in (4.23) : it is possible to find an unbounded set  $B$  and an operator  $x$  belonging to the ideal in question such that  $q_{\mathcal{I}}(x(B)) = \infty$  - see Example 6.4.10. Therefore in proving (4.24), having used (4.11) and (4.13), and also (4.23), we need the assumption that  $B$  be bounded.

Finally we note that  $\mathcal{I}$  is indeed characterised by  $q_{\mathcal{I}}$ .

### Corollary 4.2.6

$$x \in \mathcal{I} \Leftrightarrow xB_{\mathcal{H}} \in b(\widetilde{\mathcal{I}^p}) \Leftrightarrow q_{\mathcal{I}}(xB_{\mathcal{H}}) = 0 \Leftrightarrow q_{\mathcal{I}}(xB) = 0 \quad (4.25)$$

for  $B \subset \mathcal{H}$  bounded.

## 4.3 $q_{\mathcal{I}}$ and left-Fredholm operators

In this section we consider an application of  $q_{\mathcal{I}}$  to left Fredholm operators. The first statement of the following theorem has already been proved in [Str89] Theorem 3.12. We give a complete proof as the method here is substantially different and allows us to derive the second statement of the theorem, which is new. This in turn enables us to derive a new characterisation of  $m(x)$  for any operator  $x \in \mathcal{M}$ .

**Theorem 4.3.1**  $x \in \mathcal{M}$  is left Fredholm w.r.t.  $\mathcal{I}$  iff there exists  $c > 0$  such that

$$q_{\mathcal{I}}(xB) \geq c q_{\mathcal{I}}(B)$$

for all sets  $B$  that are bounded in norm. The largest possible value of  $c$  is  $m(x)$ .

**Proof:** Suppose  $x \in \mathcal{M}$  is left Fredholm w.r.t.  $\mathcal{I}$ . Then  $m(x) > 0$ . Let

$$f(t) = \begin{cases} 0 & \text{for } t \leq \frac{m(x)}{2} \\ \frac{1}{t} & \text{for } t > \frac{m(x)}{2} \end{cases}$$

Then

$$f(|x|)|x| = e_{(\frac{m(x)}{2}, \infty)}(|x|) \in 1 - \mathcal{I}^p$$

so  $f(|x|)$  is an inverse for  $|x|$  modulo  $\mathcal{I}$ .

We then have for any  $B$  which is norm bounded that

$$\begin{aligned} q_{\mathcal{I}}(B) &= q_{\mathcal{I}}(e_{(\frac{m(x)}{2}, \infty)}(|x|) B) \\ &= q_{\mathcal{I}}(f(|x|)|x| B) \end{aligned}$$

$$\begin{aligned}
&\leq \alpha(f(|x|)) q_I(|x|B) \\
&= \frac{1}{m(x)} q_I(|x|B) \\
&= \frac{1}{m(x)} q_I(xB)
\end{aligned}$$

So we take  $c = m(x)$ . Note that the first equality above follows from (4.24); (4.23), Theorem 2.2.8 and (4.16) are also used here.

Conversely, suppose we have some  $c > 0$  such that  $q_I(xB) \geq c q_I(B)$  for all  $B$  that are bounded in norm. Then for any  $0 < \delta < c$ ,

$$\begin{aligned}
c &> \delta \\
&\geq \|xe_\delta(|x|)\| \\
&\geq \alpha(xe_\delta(|x|)) \\
&= q_I(xe_\delta(|x|)\mathcal{B}_\mathcal{H}) \\
&\geq c q_I(e_\delta(|x|)\mathcal{B}_\mathcal{H}) \\
&= c \alpha(e_\delta(|x|))
\end{aligned}$$

So  $1 > \alpha(e_\delta(|x|))$ , and thus from (2.2) we have that  $\alpha(e_\delta(|x|)) = 0$ .

Thus  $e_\delta(|x|) \in \mathcal{I}$ , and so  $m(x) \geq \delta > 0$ . This means that  $x$  is left Fredholm.

From the first part of the proof we have that  $m(x)$  is admissible as a value of  $c$ . From the second part of the proof we have that  $m(x)$  is the largest possible such value, since  $m(x) \geq \delta$  with  $\delta < c$  arbitrary. ■

The following result should be compared to the definition appearing on p 42 of [EE90], and Theorem 4.6 there.

**Corollary 4.3.2** *For any  $x \in \mathcal{M}$ ,*

$$m(x) = \inf_{0 < q_I(B) < \infty} \frac{q_I(xB)}{q_I(B)} \quad (4.26)$$

**Proof:** For the one direction we have that

$$\begin{aligned}
\inf_{0 < q_I(B) < \infty} \frac{q_I(xB)}{q_I(B)} &\leq \inf_{p \in \mathcal{M}^p \setminus \mathcal{I}^p} \frac{q_I(xp\mathcal{B}_\mathcal{H})}{q_I(p\mathcal{B}_\mathcal{H})} \\
&= \inf_{p \in \mathcal{M}^p \setminus \mathcal{I}^p} \frac{\alpha(xp)}{\alpha(p)} \\
&= \inf_{p \in \mathcal{M}^p \setminus \mathcal{I}^p} \alpha(xp) \\
&= m(x)
\end{aligned}$$

Conversely, we may suppose  $m(x) > 0$ , i.e.  $x$  is left Fredholm, otherwise there is nothing to show. From the proof of Theorem 4.3.1 we have that  $m(x) q_I(B) \leq q_I(xB)$  for any  $B$  which is bounded in norm.

The result follows. ■

## Chapter 5

# Topological characterisations of closed ideals

Throughout this chapter we suppose, unless specifically stated otherwise, that  $\mathcal{I}$  is a norm closed ideal in a von Neumann algebra  $\mathcal{M}$  acting on a Hilbert space  $\mathcal{H}$ .

We aim to generalise the following known topological characterisations of  $\mathcal{K}(\mathcal{H})$  : for  $x \in \mathcal{B}(\mathcal{H})$ , the following are equivalent:

- (a)  $x \in \mathcal{K}(\mathcal{H})$  i.e.  $x\mathcal{B}_{\mathcal{H}}$  is relatively compact for the norm topology on  $\mathcal{H}$ ;
- (b)  $x|_{\mathcal{B}_{\mathcal{H}}} : (\mathcal{B}_{\mathcal{H}}, \sigma|_{\mathcal{B}_{\mathcal{H}}}) \rightarrow (\mathcal{H}, \|\cdot\|)$  is continuous;
- (c)  $x|_{\mathcal{B}_{\mathcal{H}}} : (\mathcal{B}_{\mathcal{H}}, \sigma|_{\mathcal{B}_{\mathcal{H}}}) \rightarrow (\mathcal{H}, \|\cdot\|)$  is sequentially continuous;
- (d)  $x : (\mathcal{H}, \sigma) \rightarrow (\mathcal{H}, \|\cdot\|)$  is sequentially continuous.

Here  $\sigma$  denotes the weak topology on  $\mathcal{H}$ . Condition (d) is usually known as the Hilbert condition. In general Banach spaces the operators with this property are known as the completely continuous operators.

As motivation for what will follow, and also to illustrate some points that will be made later, we offer an outline of the proof of this result.

**Proof:** (a)  $\Rightarrow$  (b) Suppose  $\mathcal{B}_{\mathcal{H}} \supset (\xi_{\alpha}) \xrightarrow{\sigma} 0$ . Then, since any bounded operator is weakly continuous, we have that  $x\xi_{\alpha} \xrightarrow{\sigma} 0$ . Now suppose for a contradiction that it is not the case that  $x\xi_{\alpha} \xrightarrow{\|\cdot\|} 0$ , so for some  $\epsilon > 0$  we can find a subnet  $\xi_{\alpha_{\beta}}$  such that  $x\xi_{\alpha_{\beta}} \notin \epsilon\mathcal{B}_{\mathcal{H}}$  for all  $\beta$ . But  $x\mathcal{B}_{\mathcal{H}}$  is relatively norm compact, and therefore  $(x\xi_{\alpha_{\beta}})$  has a convergent subnet, whose limit must be 0 since  $x\xi_{\alpha} \xrightarrow{\sigma} 0$ . This gives us the required contradiction.

(b)  $\Rightarrow$  (c) is clear, while (c)  $\Rightarrow$  (d) follows from the fact that  $\sigma$ -convergent sequences must be bounded in norm.

(d)  $\Rightarrow$  (a) First note that  $\mathcal{B}_{\mathcal{H}}$  is relatively  $\sigma$ -compact by the reflexivity of  $\mathcal{H}$ , and therefore relatively sequentially  $\sigma$ -compact by the Eberlein-Smulian theorem. Therefore if  $(\xi_n)$  is a sequence in  $\mathcal{B}_{\mathcal{H}}$  it has a convergent subsequence  $(\xi_{n_k})$ . By hypothesis  $(x\xi_{n_k})$  is norm convergent. Hence  $x\mathcal{B}_{\mathcal{H}}$  is relatively norm compact.  $\blacksquare$

As has been noted in Examples 3.3.2, we have that

$$\sigma = t(\mathcal{K}(\mathcal{H})^p)$$

which is the initial convex topology with respect to the  $p$ -ideal of projections with finite rank. We attempt to generalise this observation as follows: given a norm closed ideal  $\mathcal{I}$ , we consider the topology  $t(\mathcal{I}^p)$  discussed in Section 3.3. We will show that for  $x \in \mathcal{M}$ ,

$$x \in \mathcal{I} \Leftrightarrow x|_{\mathcal{B}_{\mathcal{H}}} : (\mathcal{B}_{\mathcal{H}}, t(\mathcal{I}^p)|_{\mathcal{B}_{\mathcal{H}}}) \rightarrow (\mathcal{H}, \|\cdot\|) \text{ is continuous}$$

thus generalising the equivalence of (a) and (b). This will be done by using some of the theory of mixed topologies.

In the last section of this chapter we will discuss generalisations of conditions (c) and (d), with specific reference to the work of Kaftal ([Kaf82]).

## 5.1 Strong-operator dense ideals

In the following proposition we characterise the class of norm-closed ideals which are strong-operator dense in  $\mathcal{M}$ . As will prove apparent later these ideals are important when considering mixed topologies.

**Proposition 5.1.1** *The following conditions are equivalent:*

- (a)  $\mathcal{I}$  is wo-dense in  $\mathcal{M}$
- (b)  $\mathcal{I}$  is so-dense in  $\mathcal{M}$
- (c) There exists  $(p_\alpha) \subset \mathcal{I}^p$  such that  $p_\alpha \xrightarrow{so} 1$
- (d)  $\bigvee \mathcal{I}^p = 1$
- (e) For every  $0 \neq q \in \mathcal{M}^p$  there exists  $0 \neq p \in \mathcal{I}^p$  such that  $p \leq q$
- (f)  $\|\xi\| = \sup_{p \in \mathcal{I}^p} \|p\xi\|$  for every  $\xi \in \mathcal{H}$
- (g)  $\|x\| = \sup_{p \in \mathcal{I}^p} \|px\|$  for every  $x \in \mathcal{M}$
- (h)  $\mathcal{B}_{\mathcal{H}}$  is  $t(\mathcal{I}^p)$ -closed
- (i)  $t(\mathcal{I}^p)$  is Hausdorff

**Proof:** (a)  $\Leftrightarrow$  (b)  $\Leftarrow$  (c)  $\Leftrightarrow$  (d)  $\Leftrightarrow$  (e)  $\Rightarrow$  (f) are clear.

(b)  $\Rightarrow$  (c) Since  $\mathcal{I} = [[\mathcal{I}^p]] = ((\mathcal{I}^p))^{-\|\cdot\|}$ , we have that

$$\mathcal{M} = \mathcal{I}^{-s_0} = ((\mathcal{I}^p))^{-\|\cdot\|^{-s_0}} = ((\mathcal{I}^p))^{-s_0}$$

Thus we can find  $((\mathcal{I}^p)) \supset a_\alpha \xrightarrow{s_0} 1$ , and so  $\mathcal{I}^p \supset r(a_\alpha) \xrightarrow{s_0} 1$  by Proposition 1.1.5 and Proposition 1.1.8.

(f)  $\Rightarrow$  (g) If  $x \in \mathcal{M}$  then

$$\|x\| = \sup_{\|\xi\|=1} \|x\xi\| = \sup_{\|\xi\|=1} \sup_{p \in \mathcal{I}^p} \|px\xi\| = \sup_{p \in \mathcal{I}^p} \sup_{\|\xi\|=1} \|px\xi\| = \sup_{p \in \mathcal{I}^p} \|px\|$$

(g)  $\Rightarrow$  (d) Let  $q = 1 - \bigvee \mathcal{I}^p \in \mathcal{M}^p$ . Then

$$\|q\| = \sup_{p \in \mathcal{I}^p} \|pq\| = 0$$

(f)  $\Rightarrow$  (h) Suppose  $\mathcal{B}_{\mathcal{H}} \supset \xi_\alpha \xrightarrow{t(\mathcal{I}^p)} \xi$ . Then  $\|p\xi_\alpha\| \rightarrow \|p\xi\|$  for all  $p \in \mathcal{I}^p$ . Thus

$$\|\xi\| = \sup_{p \in \mathcal{I}^p} \|p\xi\| = \sup_{p \in \mathcal{I}^p} \lim_{\alpha} \|p\xi_\alpha\| \leq 1$$

(h)  $\Rightarrow$  (i) Suppose  $\xi \neq 0$ . Since  $\mathcal{B}_{\mathcal{H}}$  is  $t(\mathcal{I}^p)$ -closed, so is  $\frac{1}{2}\|\xi\|\mathcal{B}_{\mathcal{H}}$ . Then  $\mathcal{H} \setminus \frac{1}{2}\|\xi\|\mathcal{B}_{\mathcal{H}}$  separates  $\xi$  from 0, and so  $t(\mathcal{I}^p)$  is Hausdorff.

(i)  $\Rightarrow$  (d) Let  $q = 1 - \bigvee \mathcal{I}^p \in \mathcal{M}^p$ . For  $\xi \in q\mathcal{H}$ ,  $\|p\xi\| = 0$  for all  $p \in \mathcal{I}^p$ . Thus  $\xi = 0$  by Hausdorffness, and so  $q = 0$ . ■

**Examples 5.1.2** Recall that if  $\mathcal{M} = \mathcal{B}(\mathcal{H})$  then  $t(\mathcal{K}(\mathcal{H})^p) = \sigma$ . Thus  $t(\mathcal{I}^p)$  is a 'weak' topology, in the sense that it is constructed as a generalisation of  $\sigma$ . Therefore we are motivated to examine the general relationship between  $t(\mathcal{I}^p)$  and  $\sigma$ .

(1) If  $t(\mathcal{I}^p) \geq \sigma$ , then  $\mathcal{B}_{\mathcal{H}}$  is  $t(\mathcal{I}^p)$ -closed since it is  $\sigma$ -closed, and so the conditions of Proposition 5.1.1 are satisfied.

(2) We show that  $t(\mathcal{I}^p) \leq \sigma$  iff  $\mathcal{I} \subset \mathcal{K}(\mathcal{H})$ .

**Proof:** If  $\mathcal{I} \subset \mathcal{K}(\mathcal{H})$  then  $t(\mathcal{I}^p) \leq \sigma$  since the set of seminorms determining  $t(\mathcal{I}^p)$  is a subset of those determining  $\sigma$ .

Conversely, suppose  $t(\mathcal{I}^p) \leq \sigma$  and assume for a contradiction that there exists  $p \in \mathcal{I}^p$  such that  $p\mathcal{H}$  is infinite dimensional. Then any orthonormal sequence  $(\xi_n) \subset p\mathcal{H}$  is  $\sigma$ -convergent to 0, but  $\|p\xi_n\| = \|\xi_n\| = 1$ , so not  $t(\mathcal{I}^p)$ -convergent to 0. This gives us the required contradiction. Thus  $\mathcal{I}^p \subset \mathcal{K}(\mathcal{H})^p$ , and so from the theory of  $p$ -ideals we have that  $\mathcal{I} \subset \mathcal{K}(\mathcal{H})$ . ■

In general we can show that if  $t(\mathcal{I}^p) = \sigma$  then  $\mathcal{I} = \mathcal{M} \cap \mathcal{K}(\mathcal{H})$ .

**Proof:** Suppose  $t(\mathcal{I}^p) = \sigma$ . That  $\mathcal{I} \subset \mathcal{M} \cap \mathcal{K}(\mathcal{H})$  follows from (2).

Clearly  $\mathcal{M} \cap \mathcal{K}(\mathcal{H})$  is a norm-closed ideal in  $\mathcal{M}$ , so to show the reverse inclusion we may once again use the theory of  $p$ -ideals. Suppose  $p \in \mathcal{M}^p$  is of finite Euclidean dimension; we may suppose  $p$  is minimal in  $\mathcal{M}^p$ . Since  $t(\mathcal{I}^p) \geq \sigma$ , we have by (1) that  $\mathcal{I}$  satisfies the equivalent conditions of Proposition 5.1.1. By condition (e) there we have that  $p \in \mathcal{I}^p$ , by its minimality. ■

It is clear that the converse of this result is false - as a trivial example, if  $\mathcal{M}$  does not have a Type I direct summand in its type decomposition, we have that

$$\mathcal{M} \cap \mathcal{K}(\mathcal{H}) = \{0\}$$

since  $\mathcal{M}$  contains no minimal projections, and hence no projections with finite Euclidean dimension.

As a more interesting example, consider again the von Neumann algebra  $\ell^\infty$  acting on the Hilbert space  $\ell^2$ , and the ideal

$$\mathcal{K}(\ell^2) \cap \ell^\infty = c_0$$

as seen in Examples 3.3.2 (b).

Since  $c_0 \subset \mathcal{K}(\ell^2)$  we have that  $t(c_0^p) \leq \sigma$ . In fact this follows directly since  $t(c_0^p)$  is the topology of pointwise convergence. Now  $\sigma$  and the topology of pointwise convergence coincide on bounded sets, thus  $t(c_0^p)$  and  $\sigma$  agree on bounded sets. But certainly  $t(c_0^p) < \sigma$ , since the sequence

$$\xi_n = (0, 0, \dots, 0, n, 0, 0, \dots) \xrightarrow{t(c_0^p)} 0$$

(where the term 'n' appears in the  $n$ th position) but clearly

$$\xi_n \not\xrightarrow{\sigma} 0$$

since weakly convergent sequences have to be bounded in norm.

- (3) It is worthwhile to note that the  $t(\mathcal{I}^p)$  and  $\sigma$  topologies need not be comparable. As a simple example, let  $0 < p < 1$  be a central projection with infinite Euclidean dimension, and let  $\mathcal{I} = \mathcal{M}p$ . Then by (2) we have that  $t(\mathcal{I}^p) \not\leq \sigma$ . On the other hand, it is apparent from (1) that  $\sigma \not\leq t(\mathcal{I}^p)$ .
- (4) More interestingly, we can find an example where the conditions of Proposition 5.1.1 are satisfied but where the  $t(\mathcal{I}^p)$  and  $\sigma$  topologies are incomparable. This is taken from Example 2 of [Kaf82]; we sketch the details for the convenience of the reader.

Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be separable Hilbert spaces (with respective orthonormal bases  $\{\zeta_i\}$  and  $\{\eta_i\}$ ) and let

$$\mathcal{M} = \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{C}_{\mathcal{H}_2}$$

where  $\mathbf{C}_{\mathcal{H}_2}$  denotes the von Neumann algebra of all scalar multiples of the identity acting on  $\mathcal{H}_2$ . Then from [KR86] Example 11.1.4 we have that

$$\mathcal{M} = \{x \otimes 1 : x \in \mathcal{B}(\mathcal{H}_1)\}$$

Then it is easy to see that

$$\mathcal{M}_{\sim} = \{x \otimes 1 : x \in \mathcal{K}(\mathcal{H}_1)\}$$

and

$$\mathcal{M}_{\sim}^p = \mathcal{P}_{\sim} = \{p \otimes 1 : p \in \mathcal{K}(\mathcal{H}_1)^p\}$$

Thus  $\vee \mathcal{P}_{\sim} = 1$ , and so the conditions of Proposition 5.1.1 are satisfied. Since  $\mathcal{M}_{\sim} \not\subseteq \mathcal{K}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ , we have that  $t(\mathcal{P}_{\sim}) \not\subseteq \sigma$ .

We now consider the sequence

$$\xi_n = \sum_{i=n}^{\psi(n)} \zeta_i \otimes \eta_i$$

where  $\psi : \mathbf{N} \rightarrow \mathbf{N}$  is any function for which  $\psi(n) \geq n$ . Suppose we have some  $p \otimes 1 \in \mathcal{P}_{\sim}$ ; we may suppose  $p$  is minimal i.e.

$$p \cdot = \langle \cdot, \zeta \rangle \zeta$$

for some unit vector  $\zeta \in \mathcal{H}_1$ . Then

$$\begin{aligned} \|(p \otimes 1)\xi_n\|^2 &= \left\| \sum_{i=n}^{\psi(n)} p\zeta_i \otimes \eta_i \right\|^2 \\ &= \left\| \sum_{i=n}^{\psi(n)} \langle \zeta_i, \zeta \rangle \zeta \otimes \eta_i \right\|^2 \\ &= \sum_{i=n}^{\psi(n)} |\langle \zeta_i, \zeta \rangle|^2 \\ &\leq \sum_{i=n}^{\infty} |\langle \zeta_i, \zeta \rangle|^2 \\ &\rightarrow 0 \end{aligned}$$

by the Bessel inequality. Thus  $\xi_n \xrightarrow{t(\mathcal{P}_{\sim})} 0$ . Now we choose  $\psi(n) = 2n$ , for example, and then  $\|\xi_n\|^2 = n+1$ . Then the sequence  $\xi_n$  is unbounded, and therefore cannot be  $\sigma$ -convergent. Thus  $t(\mathcal{P}_{\sim}) \not\subseteq \sigma$ .

## 5.2 The closed ideal $\mathcal{W}(\mathcal{I})$ and the mixed topology

For convenience we make the following definition:

**Definition 5.2.1**

$$\mathcal{W}(\mathcal{I}) = \{x \in \mathcal{M} : x|_{\mathcal{B}_{\mathcal{H}}} : (\mathcal{B}_{\mathcal{H}}, t(\mathcal{I}^p)|_{\mathcal{B}_{\mathcal{H}}}) \rightarrow (\mathcal{H}, \|\cdot\|) \text{ is continuous} \} \quad (5.1)$$

We want to show that  $\mathcal{I} = \mathcal{W}(\mathcal{I})$ . One of the directions is easy:

**Proposition 5.2.2**  $\mathcal{I} \subset \mathcal{W}(\mathcal{I})$

**Proof:** Suppose  $x \in \mathcal{I}$ , and suppose  $\epsilon > 0$  is given. By Corollary 0.3.6, we can choose  $p \in 1 - \mathcal{I}^p$  such that  $\|xp\| \leq \epsilon$ . Suppose  $\mathcal{B}_{\mathcal{H}} \supset \xi_{\alpha} \xrightarrow{t(\mathcal{I}^p)} 0$ , then by definition we have that  $\|(1-p)\xi_{\alpha}\| \rightarrow 0$ , and thus

$$\begin{aligned} \|x\xi_{\alpha}\| &= \|[x(1-p) + xp]\xi_{\alpha}\| \\ &\leq \|x\| \|(1-p)\xi_{\alpha}\| + \|xp\| \|\xi_{\alpha}\| \\ &\leq \|x\| \|(1-p)\xi_{\alpha}\| + \epsilon \end{aligned}$$

since  $\|\xi_{\alpha}\| \leq 1$ . Thus  $x\xi_{\alpha} \xrightarrow{\|\cdot\|} 0$ . ■

**Theorem 5.2.3**  $\mathcal{W}(\mathcal{I})$  is a closed ideal.

**Proof:**  $\mathcal{W}(\mathcal{I})$  is clearly a vector space, and also a left ideal. That it is a right ideal follows from Corollary 3.2.5.

Suppose  $\mathcal{W}(\mathcal{I}) \supset x_n \xrightarrow{\|\cdot\|} x$ , and suppose  $\mathcal{B}_{\mathcal{H}} \supset \xi_{\alpha} \xrightarrow{t(\mathcal{I}^p)} 0$ . Then

$$\|x\xi_{\alpha}\| = \|x\xi_{\alpha} - x_n\xi_{\alpha} + x_n\xi_{\alpha}\| \leq \|x - x_n\| \|\xi_{\alpha}\| + \|x_n\xi_{\alpha}\| \leq \|x - x_n\| + \|x_n\xi_{\alpha}\|$$

and hence  $x\xi_{\alpha} \xrightarrow{\|\cdot\|} 0$ . Thus  $x \in \mathcal{W}(\mathcal{I})$ , and  $\mathcal{W}(\mathcal{I})$  is norm-closed. ■

We now find a topology  $t$  for which the members of  $\mathcal{W}(\mathcal{I})$  are exactly those members of  $\mathcal{M}$  that are  $(\mathcal{H}, t) \rightarrow (\mathcal{H}, \|\cdot\|)$  continuous. In order to define this topology we need to introduce the theory of mixed topological spaces. Our basic reference is [Coo78]. For the convenience of the reader we briefly state the few results that we will need.

**Definition 5.2.4** [Coo78] I Definition 1.4 and I Proposition 1.5

A triple  $\{\mathcal{E}, \mu, \mathcal{B}\}$  of a vector space  $\mathcal{E}$ , a topology  $\mu$  and a vector bornology  $\mathcal{B}$ , such that all of the members of  $\mathcal{B}$  are  $\mu$ -bounded, is called a mixed space. The finest locally convex topology which is identical with  $\mu$  on all the members of  $\mathcal{B}$  is called the mixed topology, and denoted  $\mu^{\mathcal{B}}$ .

A mixed space  $\{\mathcal{E}, \mu, \mathcal{B}\}$  will be said to be normal if  $\mathcal{B}$  has a basis consisting of  $\mu$ -closed sets.

Note that the mixed spaces of Cooper are tacitly assumed to be normal; see the comments before [Coo78] I Definition 1.4. Most other sources made a distinction between mixed spaces and normal mixed spaces. Indeed there are a number of different possible formulations for normality, see for example [Per63].

Since  $t(\mathcal{I}^p) \leq \|\cdot\|$ , we have that all of the norm bounded sets are  $t(\mathcal{I}^p)$ -bounded. Thus  $\{\mathcal{H}, t(\mathcal{I}^p), \mathcal{B}\}$  is a mixed space; here  $\mathcal{B}$  denotes the bornology of norm bounded sets. By a slight abuse of notation, but one which we believe should assist the reader, we denote this mixed space by  $\{\mathcal{H}, t(\mathcal{I}^p), \|\cdot\|\}$  and the mixed topology by  $t(\mathcal{I}^p)^{\|\cdot\|}$ . (In fact the latter notation is consistent with [Per63], for example, who only considers the case where the bornology is the von Neumann bornology of some topology.) It is clear, since

$$\{n\mathcal{B}_{\mathcal{H}} : n \in \mathbf{N}\}$$

forms a basic system for the bornology of norm bounded sets, that  $t(\mathcal{I}^p)^{\|\cdot\|}$  is exactly the finest locally convex topology that is identical with  $t(\mathcal{I}^p)$  on  $\mathcal{B}_{\mathcal{H}}$ .

If the ideal  $\mathcal{I}$  under consideration satisfies the equivalent conditions of Proposition 5.1.1 then via condition (h) of that proposition we have that the collection of norm-bounded sets have a basis of  $t(\mathcal{I}^p)$ -closed sets. Thus the mixed space  $\{\mathcal{H}, t(\mathcal{I}^p), \|\cdot\|\}$  is normal. Furthermore,  $\{\mathcal{H}, t(\mathcal{I}^p), \|\cdot\|\}$  is a Saks space in the terminology of Cooper - see [Coo78] I Lemma 3.1 (which should be compared to Proposition 5.1.1) and I Definition 3.2.

**Proposition 5.2.5** [Per63] Proposition 1.2, [Coo78] I Corollary 1.7

*Suppose  $\{\mathcal{E}, \mu, \mathcal{B}\}$  is a mixed space. If  $(\mathcal{F}, \alpha)$  is any locally convex space then a linear map*

$$T : (\mathcal{E}, \mu^{\mathcal{B}}) \rightarrow (\mathcal{F}, \alpha)$$

*is continuous iff*

$$T|_B : (B, \mu|_B) \rightarrow (\mathcal{F}, \alpha)$$

*is continuous for every  $B \in \mathcal{B}$ .*

In the case we are considering, this implies that for  $x \in \mathcal{M}$ ,

$$x : (\mathcal{H}, t(\mathcal{I}^p)^{\|\cdot\|}) \rightarrow (\mathcal{H}, \|\cdot\|)$$

is continuous iff

$$x|_{\mathcal{B}_{\mathcal{H}}} : (\mathcal{B}_{\mathcal{H}}, t(\mathcal{I}^p)|_{\mathcal{B}_{\mathcal{H}}}) \rightarrow (\mathcal{H}, \|\cdot\|)$$

is continuous. In other words, for  $x \in \mathcal{M}$

$$x \in \mathcal{W}(\mathcal{I}) \Leftrightarrow x : (\mathcal{H}, t(\mathcal{I}^p)^{\|\cdot\|}) \rightarrow (\mathcal{H}, \|\cdot\|) \text{ is continuous} \quad (5.2)$$

Thus we have the topological characterisation of  $\mathcal{W}(\mathcal{I})$  that was promised previously.

We will show that if the equivalent conditions of Proposition 5.1.1 are satisfied then  $\mathcal{W}(\mathcal{I}) = \mathcal{I}$ . The proof will make some use of the theory of polars and duality. Thus

we first we determine the dual spaces of  $\mathcal{H}$  when equipped with the topologies under consideration.

We will make use of the following well known result known as the Grothendieck Completeness Theorem.

**Lemma 5.2.6** [RR64] Chapter VI Theorem 2

Let  $\mu$  be a topology of the dual pair  $(\mathcal{E}, \mathcal{E}')$  and let  $\mathcal{B}$  be any system of closed absolutely convex bounded subsets of  $\mathcal{E}$  for which

$$(a) \text{ span } \bigcup_{B \in \mathcal{B}} B = \mathcal{E}$$

$$(b) \text{ if } B \in \mathcal{B} \text{ and } \lambda \in \mathbb{C} \text{ then } \lambda B \in \mathcal{B}$$

$$(c) \text{ if } B_1, B_2 \in \mathcal{B} \text{ then there exists some } B_3 \in \mathcal{B} \text{ such that } B_1 \cup B_2 \subset B_3$$

Then the completion of  $\mathcal{E}'$  under the topology of  $\mathcal{B}$ -convergence (i.e. the topology of uniform convergence on the polars of the members of  $\mathcal{B}$ ) is the set of all linear functionals which are  $\mu|_{\mathcal{B}}$ -continuous on each  $B \in \mathcal{B}$ .

**Proposition 5.2.7** If  $\mathcal{I}$  satisfies the conditions of Proposition 5.1.1 then

$$(\mathcal{H}, t(\mathcal{I}^p)\|\cdot\|)' = \mathcal{H} \quad (5.3)$$

(Here the usual identification of  $\mathcal{H}$  as a dual space is being made, as noted in Section 0.3.)

**Proof:** Since  $t(\mathcal{I}^p) \leq \|\cdot\|$ , we have that

$$(\mathcal{H}, t(\mathcal{I}^p))' \subset (\mathcal{H}, \|\cdot\|)' = \mathcal{H}$$

Since  $t(\mathcal{I}^p)$  is a Hausdorff topology, we have from [RR64] Chapter II Proposition 10 that

$$(\mathcal{H}, t(\mathcal{I}^p))' = \bigcup_{\substack{p \in \mathcal{I}^p \\ \epsilon > 0}} U(p, \epsilon)^\circ$$

where the polar is taken in  $\mathcal{H}^*$ , the algebraic dual of  $\mathcal{H}$ . But since we have already shown that  $(\mathcal{H}, t(\mathcal{I}^p))' \subset \mathcal{H}$ , it follows that the above polars may as well be taken in  $\mathcal{H}$ . It follows from (3.11) that

$$(\mathcal{H}, t(\mathcal{I}^p))' = \bigcup_{\substack{p \in \mathcal{I}^p \\ \epsilon > 0}} U(p, \epsilon)^\circ = \bigcup_{\substack{p \in \mathcal{I}^p \\ \epsilon > 0}} p \frac{1}{\epsilon} \mathcal{B}_{\mathcal{H}} = \bigcup_{p \in \mathcal{I}^p} p \mathcal{H}$$

By definition of the operations on the lattice of projections,  $\bigcup_{p \in \mathcal{I}^p} p \mathcal{H}$  is norm dense in  $(\bigvee \mathcal{I}^p) \mathcal{H}$ , which is equal to  $\mathcal{H}$  by the hypothesis on the ideal.

We consider  $t(\mathcal{I}^p)$  as a topology of the pair  $(\mathcal{H}, \bigcup_{p \in \mathcal{I}^p} p\mathcal{H})$  and take  $\mathcal{B}$  to be the collection of scalar multiples of the unit ball. Then by Proposition 5.2.5 the set of linear functionals which are  $t(\mathcal{I}^p)|_{\mathcal{B}}$ -continuous on each  $B \in \mathcal{B}$  is exactly  $(\mathcal{H}, t(\mathcal{I}^p)^{\|\cdot\|})'$ . On the other hand it is clear that the topology of  $\mathcal{B}$ -convergence is just the norm topology on the dual space. As previously indicated the completion of  $\bigcup_{p \in \mathcal{I}^p} p\mathcal{H}$  under the norm topology is  $\mathcal{H}$ , since  $\bigvee \mathcal{I}^p = 1$ .

Thus we have from Lemma 5.2.6 that  $(\mathcal{H}, t(\mathcal{I}^p)^{\|\cdot\|})' = \mathcal{H}$ . ■

**Note 5.2.8** In particular,  $t(\mathcal{I}^p)^{\|\cdot\|}$  is a topology compatible with the pairing  $(\mathcal{H}, \mathcal{H})$ , and so  $\sigma \leq t(\mathcal{I}^p)^{\|\cdot\|}$ , since  $\sigma$  is the weakest such topology. Furthermore, in this setting the norm topology is the Mackey topology with respect to this pairing, and so  $t(\mathcal{I}^p)^{\|\cdot\|}$  is coarser than the norm topology.

We now state three important results which can be deduced from more general results in the theory of mixed spaces and Saks spaces. All of these results require that the mixed topology be normal, so in each case we suppose that  $\mathcal{I}$  satisfies the conditions of Proposition 5.1.1.

**Proposition 5.2.9** *Deduced from [Gar64] Proposition 1.*

*A basis for the  $t(\mathcal{I}^p)^{\|\cdot\|}$  neighbourhoods of 0 is given by the sets*

$$ac \left[ \bigcup_{n=1}^{\infty} n\mathcal{B}_{\mathcal{H}} \cap U(p_n, \epsilon_n) \right] \quad (5.4)$$

where  $(p_n)$  is an arbitrary sequence from  $\mathcal{I}^p$  and  $(\epsilon_n)$  is an arbitrary sequence of positive numbers.

**Proposition 5.2.10** *Deduced from [Coo78] I Proposition 1.2.2.*

*A set  $B \subset \mathcal{H}$  is  $t(\mathcal{I}^p)^{\|\cdot\|}$ -equicontinuous iff for every  $\epsilon > 0$  there exists a  $t(\mathcal{I}^p)$ -equicontinuous set  $B'$  such that*

$$B \subset B' + \epsilon\mathcal{B}_{\mathcal{H}}$$

We know that the  $t(\mathcal{I}^p)$ -equicontinuous sets are the members of  $b(\mathcal{I}^p)$ , that is, are those sets included in  $p\delta\mathcal{B}_{\mathcal{H}}$  for some  $p \in \mathcal{I}^p$  and  $\delta > 0$ .

Thus  $B \subset \mathcal{H}$  is  $t(\mathcal{I}^p)^{\|\cdot\|}$ -equicontinuous iff  $B \in \widetilde{b(\mathcal{I}^p)}$ .

**Proposition 5.2.11** *Deduced from [Coo78] I Proposition 4.5(a).*

*The topology  $t(\mathcal{I}^p)^{\|\cdot\|}$  is defined by the collection of seminorms*

$$\phi(\xi) = \sup_n \frac{\|p_n \xi\|}{\lambda_n} \quad (5.5)$$

where  $(p_n) \subset \mathcal{I}^p$  and  $(\lambda_n) \uparrow \infty$  are arbitrary.

The proof of the following lemma is basically the same as part of the proof of [RR64] Chapter VI Theorem 2.

**Lemma 5.2.12** *Suppose  $n \in \mathbb{N}, p \in \mathcal{I}^p, \epsilon > 0$ . Then*

$$[n\mathcal{B}_{\mathcal{H}} \cap U(p, \epsilon)]^\circ \subset (n\mathcal{B}_{\mathcal{H}})^\circ + U(p, \epsilon)^\circ \quad (5.6)$$

**Proof:**  $(n\mathcal{B}_{\mathcal{H}})^\circ = \frac{1}{n}\mathcal{B}_{\mathcal{H}}$  is  $\sigma$ -compact, by the Alaoglu-Bourbaki theorem. Furthermore, any polar is  $\sigma$ -closed, in particular,  $U(p, \epsilon)^\circ$  is  $\sigma$ -closed.

Therefore,  $(n\mathcal{B}_{\mathcal{H}})^\circ + U(p, \epsilon)^\circ$  is  $\sigma$ -closed, by [RR64] Chapter III Lemma 7(iii). Thus  $(n\mathcal{B}_{\mathcal{H}})^\circ + U(p, \epsilon)^\circ$  is a  $\sigma$ -closed absolutely convex set which contains  $(n\mathcal{B}_{\mathcal{H}})^\circ \cup U(p, \epsilon)^\circ$ .

Now  $[n\mathcal{B}_{\mathcal{H}} \cap U(p, \epsilon)]^\circ$  is the  $\sigma$ -closed absolutely convex hull of  $(n\mathcal{B}_{\mathcal{H}})^\circ \cup U(p, \epsilon)^\circ$ , by [RR64] Chapter II Corollary 3, and so

$$[n\mathcal{B}_{\mathcal{H}} \cap U(p, \epsilon)]^\circ \subset (n\mathcal{B}_{\mathcal{H}})^\circ + U(p, \epsilon)^\circ$$

as required. ■

**Theorem 5.2.13** *If  $\mathcal{I}$  satisfies the conditions of Proposition 5.1.1 then  $\mathcal{W}(\mathcal{I}) = \mathcal{I}$ .*

**Proof:** It follows from Proposition 5.2.2 that  $\mathcal{I} \subset \mathcal{W}(\mathcal{I})$ .

Suppose  $x \in \mathcal{W}(\mathcal{I})$ , then  $x : (\mathcal{H}, t(\mathcal{I}^p)\|\cdot\|) \rightarrow (\mathcal{H}, \|\cdot\|)$  is continuous. Let  $\mathcal{V}$  denote the system of basic convex closed neighbourhoods of 0 for  $t(\mathcal{I}^p)\|\cdot\|$  defined in Proposition 5.2.9, then there exists  $V \in \mathcal{V}$  such that  $x[V] \subset \mathcal{B}_{\mathcal{H}}$ . Suppose

$$V = ac \left[ \bigcup_{n=1}^{\infty} n\mathcal{B}_{\mathcal{H}} \cap U(p_n, \epsilon_n) \right]$$

where  $(p_n)$  is a sequence from  $\mathcal{I}^p$  and  $(\epsilon_n)$  is a sequence of positive numbers. Then

$$\begin{aligned} x^*\mathcal{B}_{\mathcal{H}} &\subset V^\circ \\ &= \left\{ ac \left[ \bigcup_{n=1}^{\infty} n\mathcal{B}_{\mathcal{H}} \cap U(p_n, \epsilon_n) \right] \right\}^\circ \\ &= \left\{ \bigcup_{n=1}^{\infty} n\mathcal{B}_{\mathcal{H}} \cap U(p_n, \epsilon_n) \right\}^\circ \\ &= \bigcap_{n=1}^{\infty} [n\mathcal{B}_{\mathcal{H}} \cap U(p_n, \epsilon_n)]^\circ \\ &\subset \bigcap_{n=1}^{\infty} (n\mathcal{B}_{\mathcal{H}})^\circ + U(p_n, \epsilon_n)^\circ \\ &= \bigcap_{n=1}^{\infty} \frac{1}{n}\mathcal{B}_{\mathcal{H}} + p_n \frac{1}{\epsilon_n}\mathcal{B}_{\mathcal{H}} \end{aligned}$$

Therefore, given any  $n \in \mathbb{N}$ , we can find  $p_n \in \mathcal{I}^p$  and  $\delta_n > 0$  such that

$$x^* \mathcal{B}_{\mathcal{H}} \subset p_n \delta_n \mathcal{B}_{\mathcal{H}} + \frac{1}{n} \mathcal{B}_{\mathcal{H}}$$

Thus  $q_{\mathcal{I}}(x^* \mathcal{B}_{\mathcal{H}}) = 0$ , and so by Corollary 4.2.6, this means that  $x^* \in \mathcal{I}$ , and so  $x \in \mathcal{I}$ . ■

**Note 5.2.14** It is easy to see that a proof of this result can be given using the bornological characterisation of the mixed topology given in Proposition 5.2.10 rather than the ‘direct’ characterisation given in Proposition 5.2.9. In fact, it is very easy to see (by making use of Lemma 5.2.12 and the usual properties of polars) that each of Proposition 5.2.10 and Proposition 5.2.9 imply the other.

### 5.3 A reduction procedure

In this section we show that the assumption that a norm-closed ideal  $\mathcal{I}$  be strong-operator dense in  $\mathcal{M}$  is not necessary, and set out to establish that even without this assumption we still have that  $\mathcal{W}(\mathcal{I}) = \mathcal{I}$ . We achieve this by using reduced von Neumann algebras to reduce the general case to the one already considered.

In what follows, let  $\mathcal{I}$  be any ideal in  $\mathcal{M}$ , not necessarily norm-closed. Let

$$\bigvee \mathcal{I}^p = \mathcal{Q} \in \mathcal{M}^p$$

As usual we denote by  $\mathcal{M}_{\mathcal{Q}}$  the reduction of  $\mathcal{M}$  by  $\mathcal{Q}$ , which is a von Neumann algebra acting on the Hilbert space  $\mathcal{Q}\mathcal{H}$ . Recall that  $\mathcal{I}_{\mathcal{Q}} = \{x_{\mathcal{Q}} : x \in \mathcal{I}\}$  is a norm closed ideal in  $\mathcal{M}_{\mathcal{Q}}$ . Since  $p \leq \mathcal{Q}$  for all  $p \in \mathcal{I}^p$  we have from Proposition 1.2.12 that

$$\mathcal{I}_{\mathcal{Q}}^p = \{p_{\mathcal{Q}} : p \in \mathcal{I}^p\}$$

We denote by  $t(\mathcal{I}_{\mathcal{Q}}^p)$  the topology on  $\mathcal{Q}\mathcal{H}$  induced as in Definition 3.3.1 by  $\mathcal{I}_{\mathcal{Q}}^p$ .

**Lemma 5.3.1** *Let  $\mathcal{I}$  be any ideal in  $\mathcal{M}$ . Suppose  $x \in \mathcal{I}$  and let  $\bigvee \mathcal{I}^p = \mathcal{Q}$  be as before. Then  $R(x) \leq \mathcal{Q}$ .*

**Proof:**

$$\begin{aligned} R(x) &= R(xx^*) \\ &= e_{(0, \infty)}(xx^*) \\ &= \bigvee_{\epsilon > 0} e_{(\epsilon, \infty)}(xx^*) \\ &\leq \mathcal{Q} \end{aligned}$$

Here we have used the right continuity of the spectral family, and that  $e_{(\epsilon, \infty)}(xx^*) \in \mathcal{I}^p$  for all  $\epsilon > 0$ , as established in Theorem 0.3.5. ■

**Proposition 5.3.2** *Let  $\mathcal{I}$  be any ideal in  $\mathcal{M}$ . The subspace topology of  $t(\mathcal{I})$*

(a) *on  $(1 - Q)\mathcal{H}$  is the indiscrete topology;*

(b) *on  $Q\mathcal{H}$  is  $t(\mathcal{I}_Q)$ .*

**Proof:** (a) Suppose  $x \in \mathcal{I}$ . Since  $x^* \in \mathcal{I}$ , we have  $R(x^*) \leq Q$ , and so  $(1 - Q)x^* = 0$ . Thus  $x(1 - Q) = 0$ . Therefore

$$\{\xi \in (1 - Q)\mathcal{H} : x\xi \in \mathcal{B}_{\mathcal{H}}\} = (1 - Q)\mathcal{H}$$

and the result follows.

(b) Suppose  $x \in \mathcal{I}$ . A basic neighbourhood of 0 in  $Q\mathcal{H}$  for  $t(\mathcal{I}_Q)$  is

$$\begin{aligned} \{\xi \in Q\mathcal{H} : x_Q \xi \in \mathcal{B}_{Q\mathcal{H}}\} &= \{\xi \in Q\mathcal{H} : Qx\xi \in \mathcal{B}_{\mathcal{H}}\} \\ &= \{\xi \in Q\mathcal{H} : x\xi \in \mathcal{B}_{\mathcal{H}}\} \\ &= x^{-1}(\mathcal{B}_{\mathcal{H}}) \cap Q\mathcal{H} \end{aligned}$$

and the result follows. ■

**Proposition 5.3.3** *Suppose  $\mathcal{I}$  is a norm-closed ideal, and let  $\vee \mathcal{I}^p = Q$ . Then*

$$\mathcal{W}(\mathcal{I})_Q = \mathcal{W}(\mathcal{I}_Q) \tag{5.7}$$

**Proof:** Suppose  $x \in \mathcal{W}(\mathcal{I})$ , and suppose

$$\mathcal{B}_{Q\mathcal{H}} \supset \xi_\alpha \xrightarrow{t(\mathcal{I}_Q^p)} \xi$$

We specialise Proposition 5.3.2 to the topology  $t(\mathcal{I}^p)$ ; this makes sense since it is the initial topology for the ideal  $((\mathcal{I}^p))$ . By (b) we have that the subspace topology of  $t(\mathcal{I}^p)$  on  $Q\mathcal{B}_{\mathcal{H}}$  is  $t(\mathcal{I}_Q^p)$ , and so

$$\mathcal{B}_{\mathcal{H}} \supset \xi_\alpha \xrightarrow{t(\mathcal{I}^p)} \xi$$

considered as a net in  $\mathcal{H}$ . Thus  $x\xi_\alpha \xrightarrow{\|\cdot\|} x\xi$ , by the hypothesis on  $x$ . From this it follows that

$$QxQ\xi_\alpha \xrightarrow{\|\cdot\|} QxQ\xi$$

or equivalently, that

$$x_Q\xi_\alpha \xrightarrow{\|\cdot\|} x_Q\xi$$

Thus  $x_Q \in \mathcal{W}(\mathcal{I}_Q)$ , which shows that  $\mathcal{W}(\mathcal{I})_Q \subset \mathcal{W}(\mathcal{I}_Q)$ .

Conversely, suppose  $x_Q \in \mathcal{W}(\mathcal{I}_Q)$ , and suppose  $\mathcal{B}_{\mathcal{H}} \supset \xi_\alpha \xrightarrow{t(\mathcal{I}^p)} \xi$ .

Then by Proposition 3.2.5 we have that  $Q\xi_\alpha \xrightarrow{t(\mathcal{I}^p)} Q\xi$ , and so again by Proposition 5.3.2 (b) we have

$$\mathcal{B}_{Q\mathcal{H}} \supset Q\xi_\alpha \xrightarrow{t(\mathcal{I}_Q^p)} Q\xi$$

Thus  $x_Q Q\xi_\alpha \xrightarrow{\|\cdot\|} x_Q Q\xi$  by hypothesis. Then clearly  $Qx_Q\xi_\alpha \xrightarrow{\|\cdot\|} Qx_Q\xi$ , which means that  $Qx_Q \in \mathcal{W}(\mathcal{I})$ . Thus

$$x_Q = (Qx_Q)_Q \in \mathcal{W}(\mathcal{I})_Q$$

**Theorem 5.3.4** For any norm closed ideal  $\mathcal{I}$ ,  $\mathcal{W}(\mathcal{I}) = \mathcal{I}$

**Proof:** First note that  $\forall \mathcal{I}_Q^p = Q$ , the identity in  $\mathcal{M}_Q$ . Thus by Theorem 5.2.13 we have that  $\mathcal{W}(\mathcal{I}_Q) = \mathcal{I}_Q$ .

Because both  $\mathcal{I}$  and  $\mathcal{W}(\mathcal{I})$  are norm-closed ideals, it suffices by the theory of  $p$ -ideals to show that if  $p \in \mathcal{W}(\mathcal{I})^p$  then  $p \in \mathcal{I}^p$ .

If  $p \in \mathcal{W}(\mathcal{I})^p$  then

$$p_Q \in \mathcal{W}(\mathcal{I})_Q = \mathcal{W}(\mathcal{I}_Q) = \mathcal{I}_Q$$

and so  $Qp_Q \in \mathcal{I}$ .

To complete the proof, it suffices to show that  $p \leq Q$ . So suppose  $\xi \in (1 - Q)\mathcal{H}$ . Then from Proposition 5.3.2 (a) the sequence  $(\xi, 0, \xi, 0, \dots)$  is  $t(\mathcal{I}^p)$ -convergent, and so the sequence  $(p\xi, 0, p\xi, 0, \dots)$  is norm-convergent, since  $p \in \mathcal{W}(\mathcal{I})$ . This shows that  $p\xi = 0$ , and so  $p \leq Q$ . ■

**Corollary 5.3.5**  $t(\mathcal{I}^p) = \|\cdot\| \Leftrightarrow \mathcal{I} = \mathcal{M}$

**Proof:** It is clear that if  $\mathcal{I} = \mathcal{M}$  then  $t(\mathcal{I}^p) = \|\cdot\|$ . Conversely, if  $t(\mathcal{I}^p) = \|\cdot\|$  then it is immediate that  $1 \in \mathcal{W}(\mathcal{I})$ . The result then follows by Theorem 5.3.4. ■

**Corollary 5.3.6**  $\mathcal{I}^p$  is a  $\sigma p$ -ideal iff for  $x \in \mathcal{M}$ , the following are equivalent:

(a)  $x : (\mathcal{H}, t(\mathcal{I}^p)) \rightarrow (\mathcal{H}, \|\cdot\|)$  is continuous.

(b)  $x|_{\mathcal{B}_\mathcal{H}} : (\mathcal{B}_\mathcal{H}, t(\mathcal{I}^p)|_{\mathcal{B}_\mathcal{H}}) \rightarrow (\mathcal{H}, \|\cdot\|)$  is continuous.

## 5.4 Identifying the mixed topology

**Theorem 5.4.1** *Suppose  $\mathcal{I}$  is a norm-closed ideal. Then*

$$t(\mathcal{I}) \leq t(\mathcal{I}^p)^{\|\cdot\|} \quad (5.8)$$

*Suppose further that  $\mathcal{I}$  satisfies the conditions of Proposition 5.1.1. Then*

$$t(\mathcal{I}) = t(\mathcal{I}^p)^{\|\cdot\|} \quad (5.9)$$

*that is, the mixed topology is the initial topology of  $\mathcal{I}$ .*

**Proof:** Since  $\mathcal{I} \subset \mathcal{W}(\mathcal{I})$  we have that if  $x \in \mathcal{I}$  then  $x : (\mathcal{H}, t(\mathcal{I}^p)^{\|\cdot\|}) \rightarrow (\mathcal{H}, \|\cdot\|)$  is continuous. Therefore  $t(\mathcal{I}) \leq t(\mathcal{I}^p)^{\|\cdot\|}$  since  $t(\mathcal{I})$  is by definition the weakest vector topology with this property.

Now suppose  $\mathcal{I}$  satisfies the conditions of Proposition 5.1.1. Recall from Proposition 5.2.11 that in this case the topology  $t(\mathcal{I}^p)^{\|\cdot\|}$  is defined by the collection of seminorms

$$\phi(\xi) = \sup_n \frac{\|p_n \xi\|}{\lambda_n}$$

where  $(p_n) \subset \mathcal{I}^p$  and  $(\lambda_n) \uparrow \infty$  are arbitrary.

Suppose we are given such a seminorm. To show that  $t(\mathcal{I}) \geq t(\mathcal{I}^p)^{\|\cdot\|}$  it will suffice to show that there exists  $x \in \mathcal{I}$  such that  $\phi(\xi) \leq \|x\xi\|$  for all  $\xi \in \mathcal{H}$ . We may suppose without loss of generality that  $p_n$  is a sequence of increasing projections. We then put  $q_1 = p_1$  and  $q_{n+1} = p_{n+1} - p_n$  for  $n \in \mathbb{N}$ . Further we define

$$x = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} q_n$$

Then  $x \in \mathcal{I}$  since  $\mathcal{I}$  is closed. Now for all  $n \in \mathbb{N}$ ,

$$\begin{aligned} \left( \frac{\|p_n \xi\|}{\lambda_n} \right)^2 &= \left\| \frac{p_n}{\lambda_n} \xi \right\|^2 \\ &= \frac{1}{\lambda_n^2} \sum_{i=1}^n \|q_i \xi\|^2 \\ &\leq \sum_{i=1}^n \frac{1}{\lambda_i^2} \|q_i \xi\|^2 \\ &= \sum_{i=1}^n \left\| \frac{q_i}{\lambda_i} \xi \right\|^2 \\ &= \left\| \sum_{i=1}^n \frac{q_i}{\lambda_i} \xi \right\|^2 \\ &= \|p_n x \xi\|^2 \\ &\leq \|x \xi\|^2 \end{aligned}$$

and thus

$$\phi(\xi) = \sup_n \frac{\|p_n \xi\|}{\lambda_n} \leq \|x\xi\|$$

As previously indicated this completes the proof. ■

**Note 5.4.2** It seems plausible that the topologies considered above should always be equal, however, the behaviour of mixed topologies with respect to subspaces is not well understood except in the case that the mixed topology is normal. In the normal case, the behaviour under taking subspaces is in certain cases as one would hope for: see [Coo78] Proposition I 4.6.

Suppose now that  $\mathcal{I}$  is any ideal. It is clear from the fact that  $t(\mathcal{I}) \leq \|\cdot\|$  that we can consider the mixed space  $(\mathcal{H}, t(\mathcal{I}), \|\cdot\|)$ .

**Corollary 5.4.3** *Suppose  $\mathcal{P}$  is a  $p$ -ideal. Then the initial topologies  $t(\mathcal{I})$  of all of the ideals  $\mathcal{I}$  for which  $\mathcal{I}^p = \mathcal{P}$  coincide on  $\mathcal{B}_{\mathcal{H}}$ , and the mixed topologies  $t(\mathcal{I})^{\|\cdot\|}$  all coincide.*

**Proof:** Suppose  $\mathcal{I}$  is an ideal for which  $\mathcal{I}^p = \mathcal{P}$ . Then

$$t(\mathcal{I}^p) \leq t(\mathcal{I}) \leq t(\overline{\mathcal{I}}) \leq t(\mathcal{I}^p)^{\|\cdot\|}$$

Therefore

$$t(\mathcal{I}^p)^{\|\cdot\|} \leq t(\mathcal{I})^{\|\cdot\|} \leq (t(\mathcal{I}^p)^{\|\cdot\|})^{\|\cdot\|} = t(\mathcal{I}^p)^{\|\cdot\|}$$

Thus the mixed topologies all coincide with  $t(\mathcal{I}^p)^{\|\cdot\|}$ . By definition of the mixed topology this means that  $t(\mathcal{I})$  coincides with  $t(\mathcal{I}^p)$  on  $\mathcal{B}_{\mathcal{H}}$ , and the result follows. ■

**Corollary 5.4.4** *An ideal is closed if its initial topology is invariant under mixing, that is,  $t(\mathcal{I}) = t(\mathcal{I})^{\|\cdot\|}$ .*

**Proof:** If the topology  $t(\mathcal{I})$  is invariant under mixing then

$$t(\mathcal{I}) = t(\mathcal{I})^{\|\cdot\|} \geq t(\mathcal{I}^p)^{\|\cdot\|} \geq t(\overline{\mathcal{I}}) \geq t(\mathcal{I})$$

and so  $\mathcal{I} = \overline{\mathcal{I}}$  by Corollary 3.2.4. ■

The next result follows from Proposition 5.2.10 and Theorem 5.4.1.

**Corollary 5.4.5** *Suppose  $\mathcal{I}$  is a norm-closed ideal satisfying the conditions of Proposition 5.1.1. Then*

$$b(\overline{\mathcal{I}^p}) = b(\mathcal{I}) \tag{5.10}$$

It is of some interest to show that this property can be proved directly. Furthermore, note that the following proof is valid for any ideal, and not just for those characterised by Proposition 5.1.1. However, this general result does *not* imply a general form of Theorem 5.4.1 since Proposition 5.2.10 requires the normality of the mixed space in question.

**Theorem 5.4.6** *Suppose  $\mathcal{I}$  is a norm-closed ideal. Then*

$$b(\widetilde{\mathcal{I}^p}) = b(\mathcal{I}) \quad (5.11)$$

**Proof:** As previously noted  $b(\mathcal{I})$  is the finest convex bornology for which all the operators  $x \in \mathcal{I}$  are bounded. Thus, since  $b(\widetilde{\mathcal{I}^p})$  also has this property by Corollary 4.2.6, we have that  $b(\mathcal{I}) \subset b(\widetilde{\mathcal{I}^p})$ .

For the converse inclusion, suppose  $B \in b(\widetilde{\mathcal{I}^p})$ . Now  $B$  must be norm bounded, so we may suppose without loss of generality that  $B \subset \mathcal{B}_{\mathcal{H}}$ . Since  $B \in b(\widetilde{\mathcal{I}^p})$ , we can find an increasing sequence of projections  $p_n \in \mathcal{I}^p$  such that

$$B \subset p_n \mathcal{B}_{\mathcal{H}} + \frac{1}{4^n} \mathcal{B}_{\mathcal{H}}$$

for all  $n \in \mathbb{N}$ . Define  $q_1 = p_1$ , and  $q_{n+1} = p_{n+1} - p_n$  for  $n \in \mathbb{N}$ .

Suppose  $\xi \in B$ . We claim that for every  $n \in \mathbb{N}$  we have that

$$\begin{aligned} \xi &= \sum_{i=1}^n q_i \xi + (1 - p_n) \xi \\ \|q_i \xi\| &\leq \frac{1}{4^{i-1}} \text{ for } 1 \leq i \leq n \\ \|(1 - p_n) \xi\| &\leq \frac{1}{4^n} \end{aligned}$$

The proof of this result proceeds by induction. We rely on the following observation: if  $\xi \in \mathcal{H}$  and  $q \in \mathcal{M}^p$ , then the best approximation to  $\xi$  in  $q\mathcal{H}$  is  $q\xi$ . Suppose  $\xi \in B$ . Since

$$B \subset q_1 \mathcal{B}_{\mathcal{H}} + \frac{1}{4} \mathcal{B}_{\mathcal{H}}$$

we have that  $\|q_1 \xi\| \leq 1$  and  $\|(1 - p_1) \xi\| \leq \frac{1}{4}$ . This shows that the result holds true in the case  $n = 1$ . Now suppose the result holds for some  $n \in \mathbb{N}$ . Considering the decomposition

$$\xi = \sum_{i=1}^{n+1} q_i \xi + (1 - p_{n+1}) \xi$$

we have that

$$\|q_{n+1} \xi\| \leq \|(1 - p_n) \xi\| \leq \frac{1}{4^n}$$

via the induction hypothesis. Now

$$B \subset p_{n+1}\mathcal{B}_{\mathcal{H}} + \frac{1}{4^{n+1}}\mathcal{B}_{\mathcal{H}}$$

and so

$$\|(1 - p_{n+1})\xi\| \leq \frac{1}{4^{n+1}}$$

which completes the induction step. It follows that  $\xi$  has the expansion

$$\xi = \sum_{n=1}^{\infty} q_n \xi$$

where  $\|q_n \xi\| \leq \frac{1}{4^{n-1}}$  for all  $n \in \mathbb{N}$ . Put

$$x = \frac{4}{\sqrt{3}} \sum_{n=1}^{\infty} \frac{1}{2^n} q_n$$

Then  $x \in \mathcal{I}$  since  $\mathcal{I}$  is norm-closed. Now  $\sum_{m=1}^{\infty} 2^m q_m \xi$  is a convergent series since

$$\begin{aligned} \left\| \sum_{m=1}^{\infty} 2^m q_m \xi \right\|^2 &= \sum_{m=1}^{\infty} \|2^m q_m \xi\|^2 \\ &= \sum_{m=1}^{\infty} 4^m \|q_m \xi\|^2 \\ &\leq \sum_{m=1}^{\infty} \frac{4^m}{4^{2m-2}} \\ &= \sum_{m=1}^{\infty} \frac{1}{4^{m-2}} \\ &= \frac{16}{3} \end{aligned}$$

and so

$$\frac{\sqrt{3}}{4} \sum_{m=1}^{\infty} 2^m q_m \xi \in \mathcal{B}_{\mathcal{H}}$$

Then from the orthogonality of  $\{q_n\}$  we have that

$$\xi = \left( \frac{4}{\sqrt{3}} \sum_{n=1}^{\infty} \frac{1}{2^n} q_n \right) \left( \frac{\sqrt{3}}{4} \sum_{m=1}^{\infty} 2^m q_m \xi \right) \in x\mathcal{B}_{\mathcal{H}}$$

and so  $B \subset x\mathcal{B}_{\mathcal{H}}$ , as required. ■

## 5.5 Ideals determined by sequential convergence

In this section we are going to analyse ideals determined by sequential continuity criteria. We consider some of the work of Kaftal, in particular the paper [Kaf82]. To facilitate our discussion we introduce the following notation:

**Definition 5.5.1**

$$\mathcal{W}_{bc}(\mathcal{I}) = \{x \in \mathcal{M} : x|_{\mathcal{B}_{\mathcal{H}}} : (\mathcal{B}_{\mathcal{H}}, t(\mathcal{I}^p)|_{\mathcal{B}_{\mathcal{H}}}) \rightarrow (\mathcal{H}, \|\cdot\|) \text{ is continuous} \} \quad (5.12)$$

$$\mathcal{W}_c(\mathcal{I}) = \{x \in \mathcal{M} : x : (\mathcal{H}, t(\mathcal{I}^p)) \rightarrow (\mathcal{H}, \|\cdot\|) \text{ is continuous} \} \quad (5.13)$$

$$\mathcal{W}_{b\sigma}(\mathcal{I}) = \{x \in \mathcal{M} : x|_{\mathcal{B}_{\mathcal{H}}} : (\mathcal{B}_{\mathcal{H}}, t(\mathcal{I}^p)|_{\mathcal{B}_{\mathcal{H}}}) \rightarrow (\mathcal{H}, \|\cdot\|) \text{ is seq. continuous} \} \quad (5.14)$$

$$\mathcal{W}_\sigma(\mathcal{I}) = \{x \in \mathcal{M} : x : (\mathcal{H}, t(\mathcal{I}^p)) \rightarrow (\mathcal{H}, \|\cdot\|) \text{ is seq. continuous} \} \quad (5.15)$$

Of course  $\mathcal{W}_{bc}(\mathcal{I})$  has until now been denoted  $\mathcal{W}(\mathcal{I})$ .

Recall that via Proposition 3.3.3 we have that

$$\mathcal{W}_c(\mathcal{I}) = ((\mathcal{I}^p)) \quad (5.16)$$

while via Theorem 5.3.4 we have that

$$\mathcal{W}_{bc}(\mathcal{I}) = \mathcal{I} \quad (5.17)$$

Thus we have the following diagram of inclusions:

$$\begin{array}{ccccc} \mathcal{I} & = & \mathcal{W}_{bc}(\mathcal{I}) & \subset & \mathcal{W}_{b\sigma}(\mathcal{I}) \\ \cup & & \cup & & \cup \\ ((\mathcal{I}^p)) & = & \mathcal{W}_c(\mathcal{I}) & \subset & \mathcal{W}_\sigma(\mathcal{I}) \end{array} \quad (5.18)$$

**Proposition 5.5.2**  $\mathcal{W}_{b\sigma}(\mathcal{I})$  is a norm-closed ideal.  $\mathcal{W}_\sigma(\mathcal{I})$  is an ideal.

**Proof:** It is clear that both these sets are left ideals since the norm topology is metrisable. It follows from Corollary 3.2.5 that they are right ideals.

The proof that  $\mathcal{W}_{b\sigma}(\mathcal{I})$  is norm-closed is similar to Theorem 5.2.3 and so is omitted. ■

**Example 5.5.3** It is clear that if any of

$$t(\mathcal{I}^p), t(\mathcal{I}^p)|_{\mathcal{B}_{\mathcal{H}}}, \text{ or } t(\mathcal{I}^p)^{\|\cdot\|}$$

are metrisable then

$$\mathcal{W}_{bc}(\mathcal{I}) = \mathcal{W}_{b\sigma}(\mathcal{I})$$

while if  $t(\mathcal{I}^p)$  is metrisable then

$$\mathcal{W}_c(\mathcal{I}) = \mathcal{W}_\sigma(\mathcal{I})$$

We take an example that has been considered before : the von Neumann algebra  $\ell^\infty$  acting on the Hilbert space  $\ell^2$ , with the ideal

$$\mathcal{K}(\ell^2) \cap \ell^\infty = c_0$$

Recall that  $c_0^{\mathcal{P}}$  consists of those sequences consisting only of finitely many 1's, and 0's otherwise, and of course  $((\mathcal{I}^{\mathcal{P}})) = \psi$ , the space of sequences with only finitely many non-zero terms. Also recall that  $t(c_0^{\mathcal{P}})$  is in fact the topology of pointwise convergence, which is metrisable. Therefore we have the diagram

$$\begin{array}{ccccc} c_0 & = & \mathcal{W}_{bc}(c_0) & = & \mathcal{W}_{b\sigma}(c_0) \\ \cup \neq & & \cup & & \cup \neq \\ \psi & = & \mathcal{W}_c(c_0) & = & \mathcal{W}_\sigma(c_0) \end{array} \quad (5.19)$$

In any case, it is easy to verify directly that the inclusion  $\mathcal{W}_\sigma(c_0) \subset \mathcal{W}_{b\sigma}(c_0)$  is proper. Certainly  $\underline{x} = (\frac{1}{n}) \in c_0 \subset \mathcal{W}_{b\sigma}(c_0)$ . Now  $\xi_n = (0, 0, \dots, 0, n, 0, 0, \dots) \xrightarrow{t(c_0^{\mathcal{P}})} 0$ , where the term 'n' appears in the  $n$ th position, while  $x\xi_n = (0, 0, \dots, 0, 1, 0, 0, \dots)$  and this sequence is not norm convergent to 0. Thus  $\underline{x} \notin \mathcal{W}_\sigma(c_0)$ .

**Example 5.5.4** Consider the case  $\mathcal{M} = \mathcal{B}(\mathcal{H})$ . As we have seen before,

$$\begin{array}{ccccc} \mathcal{K}(\mathcal{H}) & = & \mathcal{W}_{bc}(\mathcal{K}(\mathcal{H})) & = & \mathcal{W}_{b\sigma}(\mathcal{K}(\mathcal{H})) \\ \cup & & \cup & & \parallel \\ \mathcal{F}(\mathcal{H}) & = & \mathcal{W}_c(\mathcal{K}(\mathcal{H})) & \subset & \mathcal{W}_\sigma(\mathcal{K}(\mathcal{H})) \end{array} \quad (5.20)$$

In particular this shows that the inclusion  $\mathcal{W}_c(\mathcal{I}) \subset \mathcal{W}_\sigma(\mathcal{I})$  can be strict.

**Example 5.5.5** Consider the case where  $\mathcal{M}$  is a semifinite von Neumann algebra. It has been shown by Kaftal ([Kaf82] Proposition 3) that if  $q \in \mathcal{M}^{\mathcal{P}}$  is algebraically infinite then there exists an orthonormal sequence  $\{\xi_n\} \subset q\mathcal{H}$  such that  $\xi_n \xrightarrow{t(\mathcal{P}_{\sim})} 0$ .

It follows from this that  $\mathcal{W}_{b\sigma}(\mathcal{M}_{\sim}) = \mathcal{M}_{\sim}$  - ([Kaf82] Theorem 7).

$$\begin{array}{ccccc} \mathcal{M}_{\sim} & = & \mathcal{W}_{bc}(\mathcal{M}_{\sim}) & = & \mathcal{W}_{b\sigma}(\mathcal{M}_{\sim}) \\ \cup & & \cup & & \cup \\ ((\mathcal{P}_{\sim})) & = & \mathcal{W}_c(\mathcal{M}_{\sim}) & \subset & \mathcal{W}_\sigma(\mathcal{M}_{\sim}) \end{array} \quad (5.21)$$

**Example 5.5.6** We again consider the case where  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are separable Hilbert spaces and the von Neumann algebra

$$\mathcal{M} = \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{C}_{\mathcal{H}_2}$$

We are going to show that

$$\mathcal{W}_\sigma(\mathcal{M}_{\sim}) \subset \mathcal{L}^2(\mathcal{B}(\mathcal{H}_1)) \otimes 1 \quad (5.22)$$

Since both of the above sets are ideals in  $\mathcal{M}$  it suffices to consider positive operators. If  $x \otimes 1 \in \mathcal{W}_\sigma(\mathcal{M}_{\sim})$  then from (5.21) we have that  $x \otimes 1 \in \mathcal{M}_{\sim}$ , and so  $x \in \mathcal{K}(\mathcal{H}_1)$ . Thus  $x$  has a Schmidt decomposition

$$x \cdot = \sum_{i=1}^{\infty} \lambda_i \langle \cdot, \zeta_i \rangle \zeta_i$$

where  $(\lambda_i)$  is the  $s$ -number sequence of  $x$  and  $\zeta_i$  is an orthonormal sequence of eigenvectors for  $x$  each with corresponding eigenvalue  $\lambda_i$ .

Now suppose  $x \otimes 1 \notin \mathcal{L}^2(\mathcal{B}(\mathcal{H}_1)) \otimes 1$ , so  $x \notin \mathcal{L}^2(\mathcal{B}(\mathcal{H}_1))$  and thus

$$\sum_{i=1}^{\infty} \lambda_i^2 = \infty$$

Therefore for each  $n \in \mathbb{N}$ ,  $\sum_{i=n}^{\infty} \lambda_i^2 = \infty$ , so we can construct a function  $\psi(n)$  such that

$$\sum_{i=n}^{\psi(n)} \lambda_i^2 > 1$$

for all  $n \in \mathbb{N}$ .

Consider the sequence

$$\xi_n = \sum_{i=n}^{\psi(n)} \zeta_i \otimes \eta_i$$

where  $\eta_i$  is any fixed orthonormal base for  $\mathcal{H}_2$ . As shown in Examples 5.1.2 (4) we have that  $\xi_n \xrightarrow{t(\mathcal{P}_{\sim})} 0$ .

However, we have that

$$\begin{aligned} (x \otimes 1)\xi_n &= \sum_{i=n}^{\psi(n)} x\zeta_i \otimes \eta_i \\ &= \sum_{i=n}^{\psi(n)} \lambda_i \zeta_i \otimes \eta_i \end{aligned}$$

and so

$$\|(x \otimes 1)\xi_n\|^2 = \sum_{i=n}^{\psi(n)} \lambda_i^2 > 1$$

which shows that  $(x \otimes 1)\xi_n$  does not norm-converge to 0. Thus  $x \otimes 1 \notin \mathcal{W}_\sigma(\mathcal{M}_{\sim})$ . This gives us the required contradiction, and so  $\mathcal{W}_\sigma(\mathcal{M}_{\sim}) \subset \mathcal{L}^2(\mathcal{B}(\mathcal{H}_1)) \otimes 1$ .

Therefore in this case we have the following diagram:

$$\begin{array}{ccc} \mathcal{M}_{\sim} & = & \mathcal{W}_{bc}(\mathcal{M}_{\sim}) = \mathcal{W}_{b\sigma}(\mathcal{M}_{\sim}) \\ & & \cup \neq \\ \cup \neq & & \cup \mathcal{L}^2(\mathcal{B}(\mathcal{H}_1)) \otimes 1 \\ & & \cup \\ ((\mathcal{P}_{\sim})) & = & \mathcal{W}_c(\mathcal{M}_{\sim}) \subset \mathcal{W}_\sigma(\mathcal{M}_{\sim}) \end{array} \quad (5.23)$$

We have now seen some ideals  $\mathcal{I}$  for which we have  $\mathcal{I} = \mathcal{W}_{b\sigma}(\mathcal{I})$ . This condition has been called the generalised Hilbert condition. A fundamental question that needs to be

asked is whether or not the generalised Hilbert condition will hold for any closed ideal  $\mathcal{I}$  in  $\mathcal{M}$ .

We have seen that it was shown by Kaftal in [Kaf82] that the generalised Hilbert condition holds in the case of a semifinite von Neumann algebra and the ideal of algebraically compact operators. However, the proof uses specific properties of the algebraically finite projections, and so there is no hope for this proof being extended to other ideals.

On the other hand, recall that the proof of the classical Hilbert condition relies on two fundamental results in functional analysis: that the unit ball is  $\sigma$ -compact (due to the reflexivity of  $\mathcal{H}$ ), and the Eberlein-Smulian theorem.

If we attempt to imitate this proof in the general case, we can bypass the Eberlein-Smulian theorem by making use of the theory of angelic spaces. For the definition of angelic spaces see [Flo80] Definition 3.3. One of the most important properties of angelic spaces is that relative compactness and relative sequential compactness coincide, as seen in [Flo80] Theorem 3.3 (1). By [Flo80] 3.10 (1)  $(\mathcal{H}, \sigma)$  is an angelic space; furthermore from [Flo80] Theorem 3.3 (2) any vector topology stronger than an angelic topology is again angelic.

Therefore, at least in the case where  $\forall \mathcal{I}^p = 1$ ,  $t(\mathcal{I}^p)^{\|\cdot\|}$  is angelic since it is stronger than  $\sigma$ , by Note 5.2.8. Therefore relative  $t(\mathcal{I}^p)$ -compactness and relative sequential  $t(\mathcal{I}^p)$ -compactness will coincide on  $\mathcal{B}_{\mathcal{H}}$ . The restriction that  $\forall \mathcal{I}^p = 1$  is not severe as it is clear that the reduction process described in Section 5.3 will work for  $\mathcal{W}_{b\sigma}(\mathcal{I})$  too.

However, the other problem posed seems unsurmountable : in the case that  $\forall \mathcal{I}^p = 1$  there is no hope that  $\mathcal{B}_{\mathcal{H}}$  be compact in the topology  $t(\mathcal{I}^p)$ , for then the bijection

$$id : (\mathcal{B}_{\mathcal{H}}, t(\mathcal{I}^p)|_{\mathcal{B}_{\mathcal{H}}}) \rightarrow (\mathcal{B}_{\mathcal{H}}, \sigma|_{\mathcal{B}_{\mathcal{H}}})$$

is continuous by Note 5.2.8, and therefore a homeomorphism, by [Wil70a] Theorem 17.14. This is impossible except in trivial cases, as shown by the following proposition (which improves the result noted in Examples 5.1.2 (2)).

**Proposition 5.5.7** *Suppose  $t(\mathcal{I}^p)|_{\mathcal{B}_{\mathcal{H}}} = \sigma|_{\mathcal{B}_{\mathcal{H}}}$ . Then*

$$\mathcal{I} = \mathcal{M} \cap \mathcal{K}(\mathcal{H})$$

**Proof:** Suppose  $t(\mathcal{I}^p)|_{\mathcal{B}_{\mathcal{H}}} = \sigma|_{\mathcal{B}_{\mathcal{H}}}$ . Then

$$\begin{aligned} \mathcal{I} &= \mathcal{W}_{bc}(\mathcal{I}) \\ &= \{x \in \mathcal{M} : x : (\mathcal{B}_{\mathcal{H}}, t(\mathcal{I}^p)|_{\mathcal{B}_{\mathcal{H}}}) \rightarrow (\mathcal{H}, \|\cdot\|) \text{ is continuous}\} \\ &= \{x \in \mathcal{M} : x : (\mathcal{B}_{\mathcal{H}}, \sigma|_{\mathcal{B}_{\mathcal{H}}}) \rightarrow (\mathcal{H}, \|\cdot\|) \text{ is continuous}\} \\ &= \{x \in \mathcal{M} : x \in \mathcal{K}(\mathcal{H})\} \\ &= \mathcal{M} \cap \mathcal{K}(\mathcal{H}) \end{aligned}$$

■

# Chapter 6

## Ideals of $\tau$ -measurable operators

Throughout this chapter,  $\mathcal{M}$  denotes a semifinite von Neumann algebra of operators on a Hilbert space  $\mathcal{H}$  equipped with a distinguished faithful semifinite normal trace  $\tau$ . We consider the algebra  $\widetilde{\mathcal{M}}$  of  $\tau$ -measurable operators affiliated to  $\mathcal{M}$ , equipped with the topology of convergence in measure.

In Section 6.1 we look at ideals of  $\tau$ -measurable operators that are closed in the topology of convergence in measure. We show there is a satisfactory relationship between the class of norm-closed ideals in  $\mathcal{M}$  and the class of measure-closed ideals in  $\widetilde{\mathcal{M}}$ . This enables us to use some of the theory of p-ideals in our study. The situation is especially satisfactory in the case that the p-ideal under consideration includes the p-ideal  $\mathcal{P}_\tau$ . As indicated in Chapter 1, this p-ideal behaves very much like the p-ideal  $\mathcal{K}(\mathcal{H})^p$ , and in fact coincides with it in the case where  $\mathcal{M} = \mathcal{B}(\mathcal{H})$  and  $\tau$  is the canonical trace. In the remainder of this chapter (Section 6.2 onwards) we analyse the measure-closed ideal determined by this p-ideal. This ideal has come to be known as the ideal of  $\tau$ -compact operators.

It is well known that the compact operators are exactly those whose  $s$ -number sequence decreases to zero. Following the lead of Fack and Kosaki, the  $\tau$ -compact operators are defined to be those  $\tau$ -measurable operators whose generalised singular function decreases to zero. Indeed, in the case where we consider  $\mathcal{M} = \mathcal{B}(\mathcal{H})$  with the canonical trace, we have that  $\widetilde{\mathcal{M}} = \mathcal{M}$  and the  $\tau$ -compact operators are exactly the compact ones. The given definition seems to be an appropriate generalisation of the notion of compactness of an operator in the setting of the  $\tau$ -measurable operators, as we are able to show that the  $\tau$ -compact operators possess a number of properties typical of compact operators, as well as many of the compact-like properties, considered previously in this thesis, enjoyed by norm-closed ideals in  $\mathcal{M}$ .

These operators were first considered by Fack in [Fac82] and [Fac83] and then by Fack and Kosaki ([FK86]). More recently Dodds, Dodds, and de Pagter have made some use of this class in their study of symmetric non-commutative Banach function spaces

([DDP89b], [DDP90] and [DDP91]). Furthermore Hiai and Nakamura have used this class in their study of unitary orbits in von Neumann algebras ([HN89] and [HN91]), and Chilin, Krygin and Sukochev in their study of convexity of symmetric sets ([CKS92a]) and spaces of measurable operators ([CKS92b]).

The material appearing in Sections 6.2, 6.3 and 6.5 is an expanded and improved version of a paper ([SW]) written jointly with Anton Ströh. The material on the index function in Section 6.6 also appears there. Furthermore a small proportion of the material appearing in Section 6.4 was anticipated in [SW]. For the sake of coherence full details are given.

## 6.1 Norm-closed and measure-closed ideals

In this first section we want to establish the relationship between the class of norm closed ideals in  $\mathcal{M}$  and the class of measure closed ideals in  $\widetilde{\mathcal{M}}$ . We begin with the following theorem:

**Theorem 6.1.1** *Suppose  $\mathcal{I}$  is a norm-closed ideal in  $\mathcal{M}$  and  $\widetilde{\mathcal{I}}$  a measure-closed ideal in  $\widetilde{\mathcal{M}}$ . Then*

(a)  $\mathcal{I}^{-\tau_{cm}}$  is a measure-closed ideal in  $\widetilde{\mathcal{M}}$ .

(b)  $\widetilde{\mathcal{I}} \cap \mathcal{M}$  is a norm-closed ideal in  $\mathcal{M}$ , and  $(\widetilde{\mathcal{I}} \cap \mathcal{M})^{-\tau_{cm}} = \widetilde{\mathcal{I}}$ .

**Proof:** (a) is clear since  $\tau_{cm}$  is compatible with the algebraic structure.

(b) Certainly  $\widetilde{\mathcal{I}} \cap \mathcal{M}$  is an ideal in  $\mathcal{M}$ . Suppose

$$\widetilde{\mathcal{I}} \cap \mathcal{M} \supset x_n \xrightarrow{\|\cdot\|} x.$$

Then  $x_n \xrightarrow{\tau_{cm}} x$ , and so  $x \in \widetilde{\mathcal{I}} \cap \mathcal{M}$ . Thus  $\widetilde{\mathcal{I}} \cap \mathcal{M}$  is norm-closed.

Certainly  $(\widetilde{\mathcal{I}} \cap \mathcal{M})^{-\tau_{cm}} \subset \widetilde{\mathcal{I}}$ . Suppose

$$x = v|x| = v \int_0^\infty \lambda \, de_\lambda(|x|) \in \widetilde{\mathcal{I}}$$

Put

$$x_n = v \int_0^n \lambda \, de_\lambda(|x|)$$

Then  $x_n = x e_n(|x|) \in \widetilde{\mathcal{I}} \cap \mathcal{M}$  and  $x_n \xrightarrow{\tau_{cm}} x$ . So  $\widetilde{\mathcal{I}} \subset (\widetilde{\mathcal{I}} \cap \mathcal{M})^{-\tau_{cm}}$ . ■

The result that  $(\widetilde{\mathcal{I}} \cap \mathcal{M})^{-\tau_{cm}} = \widetilde{\mathcal{I}}$  can be called a down-up theorem. We want to investigate under what conditions we have a so called up-down theorem i.e. under what conditions on  $\mathcal{I}$  do we have that  $\mathcal{I}^{-\tau_{cm}} \cap \mathcal{M} = \mathcal{I}$ .

We achieve this by examining the projections involved. Since  $\mathcal{I}^{-\tau_{cm}}$  is a measure-closed ideal,  $\mathcal{I}^{-\tau_{cm}} \cap \mathcal{M}$  is a norm-closed ideal by Theorem 6.1.1 (b). So by Theorem 1.1.9, the norm-closed ideals  $\mathcal{I}^{-\tau_{cm}} \cap \mathcal{M}$  and  $\mathcal{I}$  are equal iff the p-ideals

$$\mathcal{I}^{-\tau_{cm}} \cap \mathcal{M} \cap \mathcal{M}^p = \mathcal{I}^{-\tau_{cm}} \cap \mathcal{M}^p$$

and

$$\mathcal{I} \cap \mathcal{M}^p$$

are equal.

**Proposition 6.1.2** *Suppose  $\mathcal{I}$  is a norm-closed ideal in  $\mathcal{M}$ . Then*

$$\mathcal{I}^{-\tau_{cm}} \cap \mathcal{M}^p = (\mathcal{I} \cap \mathcal{M}^p)^{-\tau_{cm}} \quad (6.1)$$

**Proof:** We first show that  $\mathcal{M}^p$  is closed in the measure topology. Suppose

$$\mathcal{M}^p \supset p_n \xrightarrow{\tau_{cm}} x \in \widetilde{\mathcal{M}}$$

then by the measure continuity of multiplication and adjunction we have that  $x$  is idempotent and self-adjoint. Since  $1 \geq \mu_t(p_n) \rightarrow \mu_t(x)$  a.e., we have that  $\|x\| \leq 1$ . Thus  $x \in \mathcal{M}^p$ .

Since  $\mathcal{I} \subset \mathcal{I}^{-\tau_{cm}}$  we have that  $\mathcal{I} \cap \mathcal{M}^p \subset \mathcal{I}^{-\tau_{cm}} \cap \mathcal{M}^p$ . Thus

$$(\mathcal{I} \cap \mathcal{M}^p)^{-\tau_{cm}} \subset (\mathcal{I}^{-\tau_{cm}} \cap \mathcal{M}^p)^{-\tau_{cm}} = \mathcal{I}^{-\tau_{cm}} \cap \mathcal{M}^p$$

since the intersection of closed sets is closed.

Conversely, suppose  $p \in \mathcal{I}^{-\tau_{cm}} \cap \mathcal{M}^p$ . Since  $\mathcal{I} = ((\mathcal{I} \cap \mathcal{M}^p))^{-\|\cdot\|}$ , we have that

$$\mathcal{I}^{-\tau_{cm}} = ((\mathcal{I} \cap \mathcal{M}^p))^{-\|\cdot\|^{-\tau_{cm}}} = ((\mathcal{I} \cap \mathcal{M}^p))^{-\tau_{cm}}$$

Thus we can find  $((\mathcal{I} \cap \mathcal{M}^p)) \supset x_n \xrightarrow{\tau_{cm}} p$ . Then  $r(x_n p) \in \mathcal{I} \cap \mathcal{M}^p$  by Proposition 1.1.5, and  $r(x_n p) \xrightarrow{\tau_{cm}} p$  by Proposition 1.1.8. Thus  $p \in (\mathcal{I} \cap \mathcal{M}^p)^{-\tau_{cm}}$ . ■

It follows that the up-down theorem holds iff the p-ideal  $\mathcal{I} \cap \mathcal{M}^p$  is measure-closed.

We now introduce a condition on a p-ideal that is equivalent to it being measure-closed and which may be easier to verify in examples.

**Definition 6.1.3** *Suppose  $\mathcal{P}$  is a p-ideal. Then  $\mathcal{P}$  is said to be an mp-ideal if whenever we have  $(p_n) \subset \mathcal{P}$ ,  $p \in \mathcal{M}^p$ ,  $p_n \xrightarrow{\tau_{cm}} p$  and  $p_n \leq p$  for all  $n \in \mathbb{N}$ , then  $p \in \mathcal{P}$ .*

**Proposition 6.1.4** *A p-ideal  $\mathcal{P}$  is an mp-ideal iff it is closed in measure.*

**Proof:** It is clear that if  $\mathcal{P}$  is closed in measure then it is an mp-ideal.

Conversely, suppose  $\mathcal{P}$  is an mp-ideal and suppose  $\mathcal{P} \supset p_n \xrightarrow{\tau_{cm}} p$ . Then for all  $n \in \mathbb{N}$  we have that  $p_n p \in ((\mathcal{P}))$  and so  $r(p_n p) \in \mathcal{P}$ , by Proposition 1.1.5, and  $r(p_n p) \xrightarrow{\tau_{cm}} p$  by Proposition 1.1.8. Furthermore  $r(p_n p) \leq p$ , so  $p \in \mathcal{P}$  by hypothesis. Thus  $\mathcal{P}$  is closed in measure. ■

We summarise the discussion in the following theorem:

**Theorem 6.1.5** *Suppose  $\mathcal{I}$  is a norm-closed ideal. The following are equivalent:*

- (a) *the up-down theorem holds for  $\mathcal{I}$ , i.e.  $\mathcal{I}^{-\tau_{cm}} \cap \mathcal{M} = \mathcal{I}$ .*
- (b) *the  $p$ -ideal  $\mathcal{I} \cap \mathcal{M}^p$  is  $\tau_{cm}$ -closed.*
- (c) *the  $p$ -ideal  $\mathcal{I} \cap \mathcal{M}^p$  is an mp-ideal.*

**Proposition 6.1.6** *Suppose  $\mathcal{M}^p \supset p_n \xrightarrow{\tau_{cm}} p$  and  $p_n \leq p$  for all  $n \in \mathbb{N}$ . Then  $\tau(p - p_n) \rightarrow 0$  and  $p = \bigvee_{n=1}^{\infty} p_n$ .*

**Proof:** It is clear that  $p \geq \bigvee_{n=1}^{\infty} p_n$ . Now

$$\tau \left( p - \bigvee_{n=1}^{\infty} p_n \right) = \tau \left( \bigwedge_{n=1}^{\infty} (p - p_n) \right) \leq \tau(p - p_n)$$

for all  $n \in \mathbb{N}$ . Therefore by the faithfulness of  $\tau$  it suffices for both statements to show that  $\tau(p - p_n) \rightarrow 0$ . This is clear from the fact that

$$\chi_{(0, \tau(p - p_n))}(t) = \mu_t(p - p_n) \rightarrow 0$$

for  $t > 0$ . ■

**Corollary 6.1.7** *Any  $\sigma p$ -ideal is an mp-ideal.*

**Proof:** Suppose  $\mathcal{P}$  is a  $\sigma p$ -ideal and  $\mathcal{P} \supset p_n \xrightarrow{\tau_{cm}} p$ , with  $p_n \leq p$  for all  $n \in \mathbb{N}$ . Then by Proposition 6.1.6 we have that  $p = \bigvee_{n=1}^{\infty} p_n$ . Thus  $p \in \mathcal{P}$ , and  $\mathcal{P}$  is an mp-ideal. ■

In particular the up-down theorem holds for ideals whose  $p$ -ideal is a  $\sigma p$ -ideal.

**Theorem 6.1.8** *Suppose  $\mathcal{I}$  is a norm closed ideal with  $p$ -ideal  $\mathcal{P}$ . The mp-ideal generated by  $\mathcal{P}$  - i.e. the smallest mp-ideal containing  $\mathcal{P}$  - is the  $\tau_{cm}$ -closure of  $\mathcal{P}$ , and this coincides with the set*

$$\{p \in \mathcal{M}^p : \exists (p_n) \subset \mathcal{P} \text{ such that } p_n \uparrow p, p_n \xrightarrow{\tau_{cm}} p\} \quad (6.2)$$

*The  $p$ -ideal of the ideal  $\mathcal{I}^{-\tau_{cm}} \cap \mathcal{M}$  is this mp-ideal.*

**Proof:** We have that  $\mathcal{I}^{-\tau_{cm}} \cap \mathcal{M}$  is a norm closed ideal and by Proposition 6.1.2 that its p-ideal is the  $\tau_{cm}$ -closure of  $\mathcal{P}$ . In particular  $\mathcal{P}^{-\tau_{cm}}$  is indeed a p-ideal. Thus the generated mp-ideal is  $\mathcal{P}^{-\tau_{cm}}$ .

Clearly

$$\{p \in \mathcal{M}^p : \exists(p_n) \subset \mathcal{P} \text{ such that } p_n \uparrow p, p_n \xrightarrow{\tau_{cm}} p\} \subset \mathcal{P}^{-\tau_{cm}}$$

Conversely, suppose  $p \in \mathcal{P}^{-\tau_{cm}}$ . Suppose  $\mathcal{P} \supset p_n \xrightarrow{\tau_{cm}} p$ . By the same argument as in Proposition 6.1.4 we have that

$$\mathcal{P} \supset r(p_n p) \xrightarrow{\tau_{cm}} p$$

Now

$$p \geq \bigvee_{i=1}^n r(p_i p) \geq r(p_n p)$$

for all  $n \in \mathbb{N}$ , and so

$$\mu_t \left( p - \bigvee_{i=1}^n r(p_i p) \right) \leq \mu_t(p - r(p_n p)) \rightarrow 0$$

for all  $t > 0$ . It follows from Proposition 6.1.6 that

$$\bigvee_{i=1}^n r(p_i p) \uparrow p \text{ and } \bigvee_{i=1}^n r(p_i p) \xrightarrow{\tau_{cm}} p$$

Thus  $p \in \{p \in \mathcal{M}^p : \exists(p_n) \subset \mathcal{P} \text{ such that } p_n \uparrow p, p_n \xrightarrow{\tau_{cm}} p\}$ , which finishes the proof. ■

**Example 6.1.9** Not all p-ideals are mp-ideals. For example, consider the von Neumann algebra  $\ell^\infty \subset \mathcal{B}(\ell^2)$ , with the weighted trace determined by the values

$$\tau(\underline{e}_n) = \frac{1}{2^n}$$

Thus, if  $\underline{x} = (x_n) \in \ell^{\infty+}$ , then

$$\tau(\underline{x}) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$$

Of course  $\tau$  is a faithful semifinite normal trace on  $\ell^\infty$ , and  $\tau_{cm}$  is the topology of pointwise convergence.

Now both  $c_0$  and  $\ell^\infty$  itself are norm closed ideals in  $\ell^\infty$ . The p-ideal of  $c_0$  is

$$\{\chi_A : A \subset \mathbb{N} \text{ is finite}\}$$

It is clear that the mp-ideal generated by this p-ideal is

$$\{\chi_A : A \subset \mathbb{N}\}$$

which is the p-ideal for  $\ell^\infty$ . Thus the up-down theorem fails here; moreover, we immediately deduce that there is not a one-to-one correspondence between the norm-closed ideals in  $\mathcal{M}$  and the measure-closed ideals in  $\widehat{\mathcal{M}}$ .

**Example 6.1.10** Consider the ideal  $\mathcal{M}_0$ . Suppose

$$\mathcal{M}_0^p = \mathcal{P}_\tau \supset p_n \xrightarrow{\tau_{\text{cm}}} p$$

and  $p_n \leq p$  for all  $n \in \mathbb{N}$ . Then as shown in Proposition 6.1.6,  $\tau(p - p_n) \rightarrow 0$  and so

$$\tau(p) = \tau(p - p_n) + \tau(p_n) < \infty$$

i.e.  $p \in \mathcal{P}_\tau$ .

Thus  $\mathcal{P}_\tau$  is an mp-ideal and the up-down theorem holds for  $\mathcal{M}_0$ .

In fact, we show that if  $\mathcal{P}_\tau \subset \mathcal{P}$  then  $\mathcal{P}$  is an mp-ideal, and so the up-down theorem holds for  $[[\mathcal{P}]]$ .

**Proof:** Suppose  $\mathcal{P} \supset p_n \xrightarrow{\tau_{\text{cm}}} p$ , and  $p_n \leq p$  for every  $n \in \mathbb{N}$ . Then as argued in Proposition 6.1.6,  $p - p_n$  is eventually a member of  $\mathcal{P}_\tau$ , and thus a member of  $\mathcal{P}$ . Thus  $p = (p - p_n) + p_n \in \mathcal{P}$ . ■

## 6.2 Definition of $\widetilde{\mathcal{M}}_0$ and some examples

In this and the following sections we focus our attention on the  $\tau$ -compact operators.

Suppose  $x \in \widetilde{\mathcal{M}}$ . Put

$$\mu_\infty(x) = \lim_{t \rightarrow \infty} \mu_t(x) = \inf_{t > 0} \mu_t(x) \quad (6.3)$$

The above limit exists since the generalised singular function  $\mu_t(x)$  is decreasing (in  $t$ ).

**Proposition 6.2.1** *If  $x, x_1, x_2 \in \widetilde{\mathcal{M}}$  and  $\lambda \in \mathbb{C}$ , then*

$$\mu_\infty(x) \leq \|x\| \quad (6.4)$$

$$\mu_\infty(\lambda x) = |\lambda| \mu_\infty(x) \quad (6.5)$$

$$\mu_\infty(x_1 + x_2) \leq \mu_\infty(x_1) + \mu_\infty(x_2) \quad (6.6)$$

$$\mu_\infty(x_1 x_2) \leq \mu_\infty(x_1) \mu_\infty(x_2) \quad (6.7)$$

$$\mu_\infty(x) = \mu_\infty(x^*) \quad (6.8)$$

$$\mu_\infty(x) = \mu_\infty(|x|) \quad (6.9)$$

from which it follows (equations (6.5) to (6.8)) that  $\mu_\infty$  is a  $*$ -algebra semi-norm on  $\widetilde{\mathcal{M}}$ .

**Proof:** These follow from the following equations which are established in [FK86] Lemma 2.5 :

$$\mu_t(x) \leq \|x\|$$

$$\begin{aligned}
\mu_t(\lambda x) &= |\lambda| \mu_t(x) \\
\mu_{2t}(x_1 + x_2) &\leq \mu_t(x_1) + \mu_t(x_2) \\
\mu_{2t}(x_1 x_2) &\leq \mu_t(x_1) \mu_t(x_2) \\
\mu_t(x) &= \mu_t(x^*) \\
\mu_t(x) &= \mu_t(|x|)
\end{aligned}$$

for all  $t > 0$ . ■

Note that if  $x_i \xrightarrow{\tau_{cm}} 0$  then  $\mu_t(x_i) \rightarrow 0$  for all  $t > 0$ , and hence  $\mu_\infty(x_i) \rightarrow 0$ , since  $\mu_t(\cdot)$  is decreasing. It follows that  $\mu_\infty$  is continuous at 0, and hence continuous on  $\widetilde{\mathcal{M}}$ . In particular we have that its kernel is a  $\tau_{cm}$ -closed two-sided \*-ideal in  $\widetilde{\mathcal{M}}$ , namely, the set of operators whose generalised singular function decreases to 0.

Now it is well known that the compact operators are exactly those whose  $s$ -number sequence decreases to 0. For this reason we call the ideal under consideration the  $\tau$ -compact ideal, denoted  $\widetilde{\mathcal{M}}_0$ . This notation is consistent with that for the ideal  $\mathcal{M}_0$  discussed in Chapter 1 since it will be shown in Corollary 6.2.4 that  $\widetilde{\mathcal{M}}_0 \cap \mathcal{M} = \mathcal{M}_0$ .

**Note 6.2.2** By (0.22) we have that for  $p \in \mathcal{M}^p$ ,  $\mu_\infty(p)$  is either 0 or 1 and

$$p \in \widetilde{\mathcal{M}}_0^p \Leftrightarrow \tau(p) < \infty \Leftrightarrow p \in \mathcal{P}_\tau \quad (6.10)$$

By the definition of  $\tau$ -measurability, any  $x \in \widetilde{\mathcal{M}}$  admits a decomposition

$$xp + x(1 - p)$$

where  $p \in \mathcal{M}^p$ ,  $\tau(1 - p) < \infty$ , and  $xp \in \mathcal{M}$ . Then  $1 - p \in \widetilde{\mathcal{M}}_0$ , and so  $x(1 - p) \in \widetilde{\mathcal{M}}_0$ . It follows that

$$\widetilde{\mathcal{M}} = \mathcal{M} + \widetilde{\mathcal{M}}_0 \quad (6.11)$$

The equivalence of (a), (b) and (c) in the following theorem has already been established in [FK86] Proposition 3.2. We give a complete proof for the reader's convenience.

**Theorem 6.2.3** Suppose  $x \in \widetilde{\mathcal{M}}$ . The following are equivalent :-

- (a)  $x \in \widetilde{\mathcal{M}}_0$
- (b)  $d_t(x) < \infty$  for all  $t > 0$
- (c) There exists  $\{x_n\} \subset ((\mathcal{P}_\tau))$  such that  $x_n \xrightarrow{\tau_{cm}} x$
- (d) There exists  $\{x_n\} \subset H(\mathcal{M})$  such that  $x_n \xrightarrow{\tau_{cm}} x$

**Corollary 6.2.4**

$$\widetilde{\mathcal{M}}_0 \cap \mathcal{M} = \mathcal{M}_0 \quad (6.14)$$

i.e.

$$\mathcal{M}_0 = \{x \in \mathcal{M} : \mu_\infty(x) = 0\} \quad (6.15)$$

**Proof:** Suppose  $x \in \mathcal{M}_0$ , so there exists  $\{x_n\} \subseteq H(\mathcal{M})$  such that  $x_n \xrightarrow{\|\cdot\|} x$ . Then certainly  $x_n \xrightarrow{\tau_{cm}} x$ . Thus by Theorem 6.2.3 (d),  $x \in \widetilde{\mathcal{M}}_0$ , and certainly  $x \in \mathcal{M}$ .

Conversely, if  $x = v|x| \in \widetilde{\mathcal{M}}_0 \cap \mathcal{M}$  then the operators defined as

$$x_n = v \int_{\frac{1}{n}}^{\|x\|} \lambda \, d\mu_\lambda(|x|)$$

converge in norm to  $x$ , by the spectral theorem. Also by inspecting the argument (b)  $\Rightarrow$  (c)  $\Rightarrow$  (d) in Theorem 6.2.3 we see that  $\{x_n\} \subset H(\mathcal{M})$ . Thus  $x \in \mathcal{M}_0$ . ■

**Example 6.2.5** If  $\mathcal{M} = \mathcal{L}^\infty(X, \Sigma, \mu)$  and  $\tau = \int \cdot \, d\mu$  then it is well known that  $\widetilde{\mathcal{M}}$  is the space of all measurable functions bounded except on a set of finite measure, and

$$H(\mathcal{M}) = \mathcal{L}^\infty(X, \Sigma, \mu) \cap \mathcal{L}^1(X, \Sigma, \mu)$$

In this case

$$\widetilde{\mathcal{M}}_0 = \{f \in \widetilde{\mathcal{M}} : \mu\{x \in X : |f(x)| > t\} < \infty \text{ for all } t > 0\}$$

**Examples 6.2.6** (1) If  $\mathcal{M} = \mathcal{B}(\mathcal{H})$  with the canonical trace, then  $\widetilde{\mathcal{M}} = \mathcal{M}$  and

$$\widetilde{\mathcal{M}}_0 = \mathcal{M}_0 = \mathcal{K}(\mathcal{H})$$

(2) If  $\mathcal{M} = l^\infty$  with the canonical trace, then  $\widetilde{\mathcal{M}} = \mathcal{M}$  and  $\widetilde{\mathcal{M}}_0 = \mathcal{M}_0 = c_0$

In the above examples it is clear that  $\widetilde{\mathcal{M}} = \mathcal{M}$ ,  $\tau_{cm} = \|\cdot\|$ , and hence (via Theorem 6.2.3 (d)), that  $\widetilde{\mathcal{M}}_0 = \mathcal{M}_0$ . In fact, we have the following result:

**Proposition 6.2.7** *The following are equivalent:*

(a)  $\widetilde{\mathcal{M}} = \mathcal{M}$

(b)  $\inf_{0 \neq p \in \mathcal{M}^p} \tau(p) > 0$

(c)  $\tau_{cm} = \|\cdot\|$

(d)  $\widetilde{\mathcal{M}}_0 = \mathcal{M}_0$

**Proof:** It is clear that (b)  $\Rightarrow$  (c)  $\Rightarrow$  (d).

We have already seen that  $\widetilde{\mathcal{M}} = \mathcal{M} + \widetilde{\mathcal{M}}_0$ . This shows that (d)  $\Rightarrow$  (a).

We show that (a)  $\Rightarrow$  (b) by considering the contrapositive. Suppose

$$\inf_{0 \neq p \in \mathcal{M}^p} \tau(p) = 0$$

Choose a sequence  $\{p_n\} \subset \mathcal{M}^p$  such that  $0 < \tau(p_n) \leq \frac{1}{2^n}$ . Define

$$x_n = \sum_{i=1}^n i p_i$$

Since

$$\text{supp}(x_m - x_n) = \bigvee_{i=n+1}^m p_i \quad (\text{whenever } m > n)$$

it follows that

$$\tau(\text{supp}(x_m - x_n)) = \tau\left(\bigvee_{i=n+1}^m p_i\right) \leq \sum_{i=n+1}^m \tau(p_i) \leq \sum_{i=n+1}^m \frac{1}{2^i} = \frac{1}{2^n} \rightarrow 0$$

as  $n, m \rightarrow \infty$ . Hence  $\{x_n\}$  is Cauchy in measure, and so has a limit  $x \in \widetilde{\mathcal{M}}$ , by the completeness of  $\widetilde{\mathcal{M}}$ .

To conclude the argument it will suffice to show that this limit is not in  $\mathcal{M}$ : for *a.e.*  $t > 0$  we have that

$$\mu_t(x_n) \rightarrow \mu_t(x) \leq \|x\|$$

and so  $\limsup_n \|x_n\| \leq \|x\|$ . But  $\|x_n\| = n$  for all  $n \in \mathbb{N}$ , and so  $x \notin \mathcal{M}$ . ■

We now consider reduced von Neumann algebras. If  $Q \in \mathcal{M}^p$ , then  $\mathcal{M}_Q$  is a semifinite von Neumann algebra of operators. If  $\tau_Q$  is defined on  $\mathcal{M}_Q$  by putting

$$\tau_Q(x_Q) = \tau(QxQ) \tag{6.16}$$

then  $\tau_Q$  is a faithful semifinite normal trace on  $\mathcal{M}_Q$ , called the reduction of  $\tau$  by  $Q$ .

We now show that the  $\tau$ -measurable and  $\tau$ -compact operators are preserved under reductions.

It is important to note that for  $x \in \widetilde{\mathcal{M}}$ ,  $x_Q$  denotes the restriction of the *closure* of  $Qx$  to  $Q\mathcal{H}$ . We denote

$$\widetilde{\mathcal{M}}_Q = \{x_Q : x \in \widetilde{\mathcal{M}}\}$$

On the other hand,  $\widetilde{\mathcal{M}}_Q$  denotes the completion of the semifinite von Neumann algebra  $\mathcal{M}_Q$  in  $\tau_{Qcm}$ , the topology of convergence in measure determined by the reduced trace  $\tau_Q$ . The generalised singular function on  $\widetilde{\mathcal{M}}_Q$  determined by  $\tau_Q$  is also denoted  $\mu_t(\cdot)$  as there is no danger of confusion. This theme seems to have first been considered by Fack. In the following proposition, (6.17) improves [Fac82] Proposition 1.5 (i) and (iii).

**Theorem 6.2.8** Suppose  $Q \in \mathcal{M}^p$  and  $x \in \widetilde{\mathcal{M}}$ . Then

$$\mu_t(x_Q) = \mu_t(QxQ) \quad (6.17)$$

$$\widetilde{\mathcal{M}}_Q = \widetilde{\mathcal{M}}_Q \quad (6.18)$$

$$\widetilde{\mathcal{M}}_{0Q} = \widetilde{\mathcal{M}}_{Q_0} \quad (6.19)$$

**Proof:** Suppose  $x \in \widetilde{\mathcal{M}}$ . It is immediate that  $x_Q$  is a closed densely defined operator on  $Q\mathcal{H}$ .

We want to show that  $x_Q$  is affiliated to  $\mathcal{M}_Q$ . Since  $(\mathcal{M}')_Q = (\mathcal{M}_Q)'$ , it suffices to show that  $z_Q x_Q \subset x_Q z_Q$  for  $z \in \mathcal{M}'$ . If  $\xi \in D(z_Q x_Q) = Q\mathcal{H} \cap D(x)$  then

$$z_Q x_Q \xi = QzQxQ\xi = QzxQ\xi = QxzQ\xi = QxQz\xi = x_Q z_Q \xi$$

- using the fact that  $zx \subset xz$  - and so  $z_Q x_Q \subset x_Q z_Q$ , as required.

Suppose  $t > 0$ . Then there exists  $p \in \mathcal{M}^p$  such that  $p\mathcal{H} \subset D(x)$  and  $\tau(1-p) \leq t$ , by the definition of  $\tau$ -measurability. Put  $e = p \wedge Q$ , then

$$e_Q(Q\mathcal{H}) \subset Q\mathcal{H} \cap D(x) = D(x_Q)$$

and

$$\tau_Q(Q - e_Q) = \tau(Q - Q \wedge p) = \tau(Q \vee p - p) \leq \tau(1-p) \leq t$$

which shows that  $x_Q$  is  $\tau_Q$ -measurable. Thus  $\widetilde{\mathcal{M}}_Q \subset \widetilde{\mathcal{M}}_Q$ .

It also follows from this calculation that

$$\begin{aligned} \mu_t(QxQ) &= \inf_{\substack{p \in \mathcal{M}^p \\ \tau(1-p) \leq t}} \|QxQp\| \\ &\geq \inf_{\substack{p \in \mathcal{M}^p \\ \tau(1-p) \leq t}} \|QxQ(Q \wedge p)\| \\ &\geq \inf_{\substack{e_Q \in (\mathcal{M}_Q)^p \\ \tau_Q(Q - e_Q) \leq t}} \|x_Q e_Q\| \\ &= \mu_t(x_Q) \end{aligned}$$

On the other hand, if  $e_Q \in (\mathcal{M}_Q)^p$  and  $\tau_Q(Q - e_Q) \leq t$ , then by putting

$$p = (1 - Q) + Qe_Q$$

we have that  $\tau(1-p) \leq t$  and  $\|QxQp\| = \|x_Q e_Q\|$ . Hence

$$\mu_t(x_Q) = \inf_{\substack{e_Q \in (\mathcal{M}_Q)^p \\ \tau_Q(Q - e_Q) \leq t}} \|x_Q e_Q\| \geq \inf_{\substack{p \in \mathcal{M}^p \\ \tau(1-p) \leq t}} \|QxQp\| = \mu_t(QxQ)$$

Thus  $\mu_t(x_Q) = \mu_t(QxQ)$ .

Now suppose  $\Phi \in \widetilde{\mathcal{M}}_Q$ . Choose  $\{x_{nQ}\} \subset \mathcal{M}_Q$  such that  $x_{nQ} \xrightarrow{\tau_{Qcm}} \Phi$ . Then by (6.17)  $Qx_{nQ}Q$  is  $\tau_{cm}$ -Cauchy; say with limit  $x \in \widetilde{\mathcal{M}}$ . Clearly  $x = QxQ$ .

For  $t > 0$ ,

$$\begin{aligned} \mu_{2t}(\Phi - x_Q) &\leq \mu_t(\Phi - x_{nQ}) + \mu_t(x_{nQ} - x_Q) \\ &= \mu_t(\Phi - x_{nQ}) + \mu_t(Q(x_n - x)Q) \\ &= \mu_t(\Phi - x_{nQ}) + \mu_t(Qx_nQ - QxQ) \\ &\rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . Hence  $\Phi = x_Q$ , and so  $\widetilde{\mathcal{M}}_Q \subset \widetilde{\mathcal{M}}_Q$ .

It now follows that

$$\widetilde{\mathcal{M}}_{Q_0} = \{x_Q : x \in \widetilde{\mathcal{M}}, \mu_t(x_Q) \rightarrow 0\} = \{x_Q : x \in \widetilde{\mathcal{M}}, \mu_t(QxQ) \rightarrow 0\}$$

while

$$\widetilde{\mathcal{M}}_{0Q} = \{x_Q : x \in \widetilde{\mathcal{M}}, \mu_t(x) \rightarrow 0\}$$

Therefore the inclusion  $\widetilde{\mathcal{M}}_{0Q} \subset \widetilde{\mathcal{M}}_{Q_0}$  follows from the fact that if  $\mu_t(x) \rightarrow 0$  then  $\mu_t(QxQ) \rightarrow 0$ , while the reverse inclusion follows from the fact that

$$x_Q = (QxQ)_Q$$

for any  $x \in \widetilde{\mathcal{M}}$ . ■

**Corollary 6.2.9** *mp-ideals are preserved under reductions.*

**Proof:** Suppose  $\mathcal{P}$  is a mp-ideal and  $Q \in \mathcal{M}^p$ . Recall (Definition 1.2.11) that the p-ideal reduction is the set

$$\mathcal{P}_{\leq Q} = \{pQ : p \in \mathcal{P}, p \leq Q\}$$

Suppose  $(p_{nQ}) \subset \mathcal{P}_{\leq Q}$  and  $p_{nQ} \xrightarrow{\tau_{Qcm}} p_Q \in \mathcal{M}_Q^p$ . Then by definition  $p_n \leq Q$  and by Lemma 0.3.4 we may suppose  $p \leq Q$ . Then

$$\begin{aligned} \mu_t(p - p_n) &= \mu_t(QpQ - Qp_nQ) \\ &= \mu_t(Q(p - p_n)Q) \\ &= \mu_t((p - p_n)_Q) \\ &= \mu_t(pQ - p_nQ) \\ &\rightarrow 0 \end{aligned}$$

Thus  $p_n \xrightarrow{\tau_{cm}} p$ , and so  $p \in \mathcal{P}$  by hypothesis. Therefore  $pQ \in \mathcal{P}_{\leq Q}$ . ■

### 6.3 The $\tau$ -Calkin algebra

We have already seen that  $\mu_\infty$  is a  $*$ -algebra semi-norm on  $\widetilde{\mathcal{M}}$ , hence a  $*$ -algebra norm, also denoted by  $\mu_\infty$ , is canonically induced on the quotient  $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$ . We call  $(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0, \mu_\infty)$  the  $\tau$ -Calkin algebra.

It follows from (6.10) that  $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$  is non-trivial (and has unit  $1 + \widetilde{\mathcal{M}}_0$ ) iff  $\tau(1) = \infty$ . We henceforth assume that this is indeed the case.

Note that if  $f : [0, \infty) \rightarrow [0, \infty)$  is a continuous increasing function then by [FK86] Lemma 2.5 we have  $\mu_t(f(|x|)) = f(\mu_t(x))$  and so

$$\mu_\infty(f(|x|)) = f(\mu_\infty(x)) \quad (6.20)$$

Hence, by putting  $f(t) = t^2$ , we have that  $\mu_\infty$  satisfies the canonical  $C^*$ -algebra type property :

$$\mu_\infty(x^*x) = \mu_\infty(|x|^2) = \mu_\infty(x)^2 \quad (6.21)$$

We will show that  $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$  equipped with the norm  $\mu_\infty$  is indeed a  $C^*$ -algebra.

We have already noted that  $\widetilde{\mathcal{M}} = \mathcal{M} + \widetilde{\mathcal{M}}_0$ . Furthermore, note that for any such decomposition

$$x = x_\infty + x_0$$

of an operator  $x \in \widetilde{\mathcal{M}}$ , we have that

$$\|x_\infty\| \geq \mu_\infty(x_\infty) = \mu_\infty(x)$$

The following result, of which we make extensive use, shows that this lower bound on the norm can always be attained. This decomposition is suggested by the decomposition described in [GK69] II §7.2 and §7.3. In the case that  $\mathcal{M} = \mathcal{B}(\mathcal{H})$  the decomposition below reduces to that one.

**Lemma 6.3.1** *Suppose  $x \in \widetilde{\mathcal{M}}$ . Then  $x$  has a decomposition*

$$x = x_\infty + x_0$$

where  $x_\infty \in \mathcal{M}$ ,  $x_0 \in \widetilde{\mathcal{M}}_0$  such that

$$\|x_\infty\| = \mu_\infty(x)$$

**Proof:** We first suppose that  $x$  is positive. Let

$$x_\infty = x e_{\mu_\infty(x)}(x) + \mu_\infty(x) e_{(\mu_\infty(x), \infty)}(x)$$

so  $x_\infty \in \mathcal{M}$  and

$$\|x_\infty\| = \mu_\infty(x)$$

Then we put

$$\begin{aligned}
x_0 &= x - x_\infty \\
&= [x - \mu_\infty(x)] e_{(\mu_\infty(x), \infty)}(x) \\
&= \int_{\mu_\infty(x)}^{\infty} (\lambda - \mu_\infty(x)) de_\lambda(x) \\
&= \int_0^{\infty} \lambda de_{\lambda + \mu_\infty(x)}(x)
\end{aligned}$$

Then we have

$$e_t(x_0) = e_{t + \mu_\infty(x)}(x)$$

and so

$$d_t(x_0) = d_{t + \mu_\infty(x)}(x)$$

It follows by (6.13) that  $d_t(x_0) < \infty$  for all  $t > 0$ , and so  $x \in \widetilde{\mathcal{M}}_0$ .

If  $x$  is not positive, let  $x = v|x|$  be the polar decomposition, and consider the decomposition  $|x| = |x|_\infty + |x|_0$  as above. Then

$$x = v|x| = v|x|_\infty + v|x|_0$$

and  $v|x|_0 \in \widetilde{\mathcal{M}}_0$  and

$$\|v|x|_\infty\| \leq \| |x|_\infty \| = \mu_\infty(|x|) = \mu_\infty(x)$$

while as previously noted, the reverse inequality is guaranteed. The result follows. ■

**Remarks 6.3.2** (1) It follows from Lemma 6.3.1 that  $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$  is a  $C^*$ -algebra : it is clear that absolutely summable sequences in  $(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0, \mu_\infty)$  can be identified with absolutely summable sequences in  $(\mathcal{M}, \|\cdot\|)$ ; the result then follows by the completeness of  $(\mathcal{M}, \|\cdot\|)$  and the continuity of the canonical maps

$$(\mathcal{M}, \|\cdot\|) \rightarrow (\widetilde{\mathcal{M}}, \tau_{cm}) \rightarrow (\widetilde{\mathcal{M}}, \mu_\infty) \rightarrow (\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0, \mu_\infty)$$

(2) As already noted, the given decomposition is the best in the sense that it gives the smallest possible value for the norm of  $x_\infty$ . Moreover, the decomposition is the best in a stronger sense, namely that it gives the smallest possible generalised singular function. This follows from the fact that

$$\|x_\infty\| \geq \mu_t(x_\infty) \geq \mu_\infty(x_\infty) = \mu_\infty(x) = \|x_\infty\|$$

for all  $t > 0$ .

However, the decomposition is not unique under the given conditions :

Suppose  $\mathcal{M} = \ell^\infty$ , with  $\tau$  the canonical trace. Then  $\mathcal{M} = \widetilde{\mathcal{M}}$  and  $\widetilde{\mathcal{M}}_0 = c_0$ . We

show that  $\underline{1}$  has arbitrarily many decompositions satisfying the given conditions. Indeed, for any sequence  $\underline{k}$  where

$$1 \geq k_1 \geq k_2 \geq \dots \downarrow 0$$

one has that

$$\underline{1} = (\underline{1} - \underline{k}) + \underline{k}$$

with  $\underline{k} \in c_0$ , and

$$\|\underline{1} - \underline{k}\| = 1 = \mu_\infty(\underline{1})$$

(3) One should also note that since

$$d_t(|x|_0) = d_{t+\mu_\infty(x)}(|x|)$$

we have that

$$\mu_t(x_0) = \mu_t(x) - \mu_\infty(x) \tag{6.22}$$

for all  $t > 0$ .

**Proposition 6.3.3** For any  $x \in \widetilde{\mathcal{M}}$ ,

$$\mu_\infty(x) = \inf_{x_0 \in \widetilde{\mathcal{M}}_0} \|x - x_0\| \tag{6.23}$$

**Proof:** Since  $\mu_\infty$  is a seminorm and  $\widetilde{\mathcal{M}}_0$  is the kernel of  $\mu_\infty$ , it follows that

$$\mu_\infty(x - x_0) = \mu_\infty(x)$$

for all  $x_0 \in \widetilde{\mathcal{M}}_0$ . Hence by (6.4) we have that

$$\mu_\infty(x) = \inf_{x_0 \in \widetilde{\mathcal{M}}_0} \mu_\infty(x - x_0) \leq \inf_{x_0 \in \widetilde{\mathcal{M}}_0} \|x - x_0\|$$

Conversely, Lemma 6.3.1 constructs  $x_0 \in \widetilde{\mathcal{M}}_0$  for which

$$\mu_\infty(x) = \|x - x_0\|$$

Note then in particular that the given infimum is attained. ■

**Theorem 6.3.4**  $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$  is isometrically \*-isomorphic to  $\mathcal{M}/\mathcal{M}_0$

**Proof:** For any  $x \in \widetilde{\mathcal{M}}$  we can decompose  $x = x_\infty + x_0$  where  $x_\infty \in \mathcal{M}$ ,  $x_0 \in \widetilde{\mathcal{M}}_0$ . Define

$$\Phi : \widetilde{\mathcal{M}} \rightarrow \mathcal{M}/\mathcal{M}_0 : x \rightarrow x_\infty + \mathcal{M}_0 \tag{6.24}$$

$\Phi$  is well defined since if

$$x = x_{\infty 1} + x_{01} = x_{\infty 2} + x_{02}$$

where  $x_{\infty 1}, x_{\infty 2} \in \mathcal{M}$  and  $x_{01}, x_{02} \in \widetilde{\mathcal{M}}_0$  then

$$x_{\infty 1} - x_{\infty 2} = x_{02} - x_{01} \in \widetilde{\mathcal{M}}_0 \cap \mathcal{M} = \mathcal{M}_0$$

by Corollary 6.2.4.

It is then easy to verify that  $\Phi$  is a \*-algebra homomorphism onto  $\mathcal{M}/\mathcal{M}_0$ , with kernel  $\widetilde{\mathcal{M}}_0$ . Hence a \*-algebra isomorphism  $\tilde{\Phi} : \widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0 \rightarrow \mathcal{M}/\mathcal{M}_0$  is induced.

Finally

$$\begin{aligned} \mu_{\infty}(x) &= \mu_{\infty}(x_{\infty}) \\ &= \inf_{x_0 \in \widetilde{\mathcal{M}}_0} \|x_{\infty} - x_0\| \\ &= \inf_{x_0 \in \widetilde{\mathcal{M}}_0 \cap \mathcal{M}} \|x_{\infty} - x_0\| \\ &= \inf_{x_0 \in \mathcal{M}_0} \|x_{\infty} - x_0\| \end{aligned}$$

Thus  $\tilde{\Phi}$  is isometric. ■

## 6.4 The functions $q_t$ and $q_{\infty}$

In this section we generalise the work on the function  $q_T$  considered in Section 4.2. We move out of the setting considered in Section 4.2; instead of examining a norm closed ideal in  $\mathcal{M}$  we rather consider  $\widetilde{\mathcal{M}}_0$ , which is a measure closed ideal in  $\widetilde{\mathcal{M}}$ . That the operators in this setting are unbounded (that they are not everywhere defined) does cause certain technical difficulties; but these are overcome reasonably easily.

We obtain results quite similar to those in Section 4.2. However, our work here is not simply a isomorphic copy of the previous results, making use of Theorem 6.2.3 or Corollary 6.2.4 or Theorem 6.3.4. Firstly, we consider a parametrised version of the quantity  $q_{\mathcal{M}_0}$ , namely  $q_t$ , and consider  $\mu_t$  as a parametrised version of  $\alpha$  (where  $\alpha$  is relative to  $\mathcal{M}_0$ ). This is where one of the main obstacles to implementing Theorem 6.3.4 lies : if  $\tilde{\Phi}$  is the map considered there and it sends an operator  $x$  to  $x_{\infty}$ , then it is certainly a consequence of that theorem that

$$\mu_{\infty}(x) = \mu_{\infty}(x_{\infty}) = \alpha(x_{\infty})$$

where  $\alpha$  is relative to  $\mathcal{M}_0$ ; however, it is of course not the case that  $\mu_t(x) = \mu_t(x_{\infty})$  for all  $t > 0$ . So once we consider the parameters these results can play no role.

Secondly we replace the quantity  $q_{\mathcal{M}_0}$  by the quantity  $q_{\infty}$ , which takes on the same value as  $q_{\mathcal{M}_0}$  on bounded sets, but can also discriminate usefully between certain unbounded

sets, that is, take on finite values for certain unbounded sets. This will be discussed in greater detail later.

As we shall see, it is appropriate to consider  $\mathcal{H}$  to be equipped with the topology of convergence in measure as defined by Nelson - [Nel74]. Recall that this vector topology (which we also denote by  $\tau_{cm}$ , as there is no danger of confusion) has as basic neighbourhoods of 0 the sets

$$\mathcal{H}(\epsilon, t) = \{\xi \in \mathcal{H} : \exists p \in \mathcal{M}^p \text{ such that } \|p\xi\| \leq \epsilon, \tau(1-p) \leq t\} \quad (6.25)$$

where  $\epsilon, t > 0$  are allowed to vary.

We will also have occasion to consider such sets where  $\epsilon = 0$  or  $t = 0$ , and it is easy to see that

$$\mathcal{H}(0, t) = \bigcup \{p\mathcal{H} : p \in \mathcal{M}^p, \tau(p) \leq t\} \quad (6.26)$$

and

$$\mathcal{H}(\epsilon, 0) = \epsilon\mathcal{B}_{\mathcal{H}} \quad (6.27)$$

although we stress that these sets are not neighbourhoods of the topology.

Recall that Nelson represented the abstract completion  $\widetilde{\mathcal{M}}$  as an algebra of (everywhere defined) operators on the abstract completion of  $\mathcal{H}$  equipped with the topology of convergence in measure. Of course  $\widetilde{\mathcal{M}}$  has a concrete realisation (as shown by Marianne Terp in [Ter81], and in fact by Nelson - see the remarks at the end of Section 2 in [Nel74]), but this is not the case for the completion of  $\mathcal{H}$ , and so the approach of Nelson is somewhat unsatisfactory, as pointed out by Terp - [Ter81] page 24. In fact, Terp identifies  $\widetilde{\mathcal{M}}$  without any reference to  $\tau_{cm}$  on  $\mathcal{H}$ . For our purposes a middle road is most suitable : we make full use of  $\tau_{cm}$  on  $\mathcal{H}$  but do not consider the completion. The principal reason for the suitability of this approach is the fact, demonstrated in the next proposition by making use of the generalised singular function, that the  $\tau$ -measurable operators are continuous with respect to the restriction of this topology to the domain of the pertinent operator. The argument is inspired by one of Nelson's : see [Nel74] Theorem 1, the proof of (17'); or [Ter81], the proof of Proposition 5(ii). The reader is also in this regard referred back to the proof of (4.23).

**Proposition 6.4.1** *Suppose  $x \in \widetilde{\mathcal{M}}$ . Then*

$$x[D(x) \cap \mathcal{H}(\epsilon, t_2)] \subset \mathcal{H}(\mu_{t_1}(x), \epsilon, t_1 + t_2) \quad (6.28)$$

for all  $\epsilon, t_1, t_2 > 0$ . It follows that

$$x : (D(x), \tau_{cm|D(x)}) \rightarrow (\mathcal{H}, \tau_{cm})$$

is a continuous operator.

**Proof:** Suppose  $\epsilon, t_1, t_2 > 0$  are given. Let  $e = e_{\mu_{t_1}(x)}(|x^*|)$ ; then

$$\tau(1 - e) = d_{\mu_{t_1}(x)}(|x^*|) = d_{\mu_{t_1}(|x^*|)}(|x^*|) \leq t_1$$

and

$$\mu_{t_1}(x) = \mu_{t_1}(|x^*|) = \||x^*|e\| = \|x^*e\| = \|ex\|$$

Suppose  $\xi \in D(x) \cap \mathcal{H}(\epsilon, t_2)$ , so there exists  $f \in \mathcal{M}^p$  such that

$$\tau(1 - f) \leq t_2 \text{ and } \|f\xi\| \leq \epsilon$$

Let

$$q = N((1 - f)x^*) = 1 - R(x(1 - f))$$

Note that we have that

$$qx = qxf$$

Put

$$p = e \wedge q$$

Now we have that

$$1 - q = R(x(1 - f)) \sim R((1 - f)x^*) \leq 1 - f$$

and hence

$$\tau(1 - p) = \tau((1 - e) \vee (1 - q)) \leq \tau(1 - e) + \tau(1 - q) \leq \tau(1 - e) + \tau(1 - f) \leq t_1 + t_2$$

Furthermore

$$\|px\xi\| = \|pxf\xi\| = \|pexf\xi\| \leq \|p\| \|ex\| \|f\xi\| \leq \mu_{t_1}(x) \epsilon$$

Hence  $x\xi \in \mathcal{H}(\mu_{t_1}(x) \epsilon, t_1 + t_2)$ . ■

We now discuss the appropriate analogues of the function  $q_{\mathcal{I}}$  discussed in Section 4.2. There we saw that  $q_{\mathcal{I}}$  is essentially determined by boundedness and the members of  $\mathcal{I}^p$ . Here we make adjustments to both of these determining factors.

Firstly, it is clear that since the operators we consider are now unbounded, many of the sets of interest - for example the images of norm-bounded sets - will in general be unbounded. We have already noted that the members of  $\widetilde{\mathcal{M}}$  are continuous with respect to the measure topology when restricted to the domain of the operator. Hence we can anticipate that boundedness in this topology will be more appropriate than norm boundedness.

#### Definition 6.4.2 ([Nel74])

A set  $B \subset \mathcal{H}$  is said to be bounded in measure if it is bounded in the topology of convergence in measure.

(a)  $q_t(B) \leq \epsilon$  iff there exists  $p \in \mathcal{M}^p$  such that  $\tau(p) \leq t$  and  $B \subset p(B) + \epsilon\mathcal{B}_{\mathcal{H}}$ .

(b)  $q_{\infty}(B) \leq \epsilon$  iff there exists  $p \in \mathcal{M}^p$  such that  $\tau(p) < \infty$  and  $B \subset p(B) + \epsilon\mathcal{B}_{\mathcal{H}}$ .

Of course the proof of the above Proposition follows the lines of Proposition 4.2.3 - in this case we make an appeal to Proposition 6.4.1 to show that the set  $p(B)$  is bounded in measure. Therefore we omit the proof. It follows that for any  $B \subset \mathcal{H}$

$$q_t(B) = \begin{cases} \inf_{\substack{p \in \mathcal{M}^p \\ \tau(1-p) \leq t}} \sup_{\xi \in B} \|p\xi\| & \text{if } B \text{ is bounded in measure} \\ \infty & \text{otherwise} \end{cases} \quad (6.32)$$

$$q_{\infty}(B) = \begin{cases} \inf_{\substack{p \in \mathcal{M}^p \\ \tau(1-p) < \infty}} \sup_{\xi \in B} \|p\xi\| & \text{if } B \text{ is bounded in measure} \\ \infty & \text{otherwise} \end{cases} \quad (6.33)$$

But these formulas are not valid for sets that are not bounded in measure: consider the set  $p\mathcal{H}$ , where  $p \in \mathcal{M}^p, \tau(p) < \infty$ .

We should note at this point that besides being a parametrised version of  $q_{\mathcal{M}_0}$ , the function  $q_t$  does have another motivation. It is quite simply an analogue of the generalised singular function, this time in the space  $\mathcal{H}$  equipped with the topology of convergence in measure. Indeed, for  $x \in \widetilde{\mathcal{M}}$ , the generalised singular function can be described by means of the formula

$$\mu_t(x) = \inf_{\substack{p \in \mathcal{M}^p \\ \tau(1-p) \leq t}} \|px\|$$

while it is clear that for  $\xi \in \mathcal{H}$ ,

$$q_t(\{\xi\}) = \inf_{\substack{p \in \mathcal{M}^p \\ \tau(1-p) \leq t}} \|p\xi\|$$

We have used the notation  $q_{\infty}$  not only to remind us of (6.31), but because  $q_{\infty}$  is distinct from the quantity  $q_{\mathcal{M}_0}$ . Of course we have that

$$q_{\mathcal{M}_0}(B) = \begin{cases} \inf_{\substack{p \in \mathcal{M}^p \\ \tau(1-p) < \infty}} \sup_{\xi \in B} \|p\xi\| & \text{if } B \text{ is bounded in norm} \\ \infty & \text{otherwise} \end{cases}$$

and remembering that any norm bounded set is bounded in measure, it follows that  $q_{\mathcal{M}_0}$  and  $q_{\infty}$  agree on sets that are bounded in norm, and that  $q_{\infty}$  takes on finite values on (some) sets that are bounded in measure but not norm bounded. As has been previously indicated this is very desirable as many sets arising in applications are bounded in measure but not in norm.

However, everything is not so simple. Recall that for  $B \subset \mathcal{H}$  we have that  $q_{\mathcal{M}_0}(B) < \infty$  iff  $B$  is norm bounded. One might expect that  $q_{\infty}(B) < \infty$  iff  $B$  is bounded in measure. Unfortunately this is not the case, as shown by the following example. Thus the class of sets for which  $q_{\infty}(B) < \infty$  is in general properly included in the class of those which are bounded in measure.

We now establish the analogues of Propositions 4.2.2 and 4.2.4 for the function  $q_t$ .

**Proposition 6.4.7** Suppose  $B, B_1, B_2 \in \mathcal{H}$ ,  $\alpha \in \mathbb{C}$ .

$$B_1 \subset B_2 \Rightarrow q_t(B_1) \leq q_t(B_2) \quad (6.34)$$

$$q_t(\alpha B) = |\alpha| q_t(B) \quad (6.35)$$

$$q_t(\mathcal{B}_{\mathcal{H}}) = 1 \quad (6.36)$$

$$\max\{q_{t_1+t_2}(B_1); q_{t_1+t_2}(B_2)\} \leq q_{t_1+t_2}(B_1 \cup B_2) \leq \max\{q_{t_1}(B_1); q_{t_2}(B_2)\} \quad (6.37)$$

$$q_{t_1+t_2}(B_1 + B_2) \leq q_{t_1}(B_1) + q_{t_2}(B_2) \quad (6.38)$$

$$q_t(\overline{B}) = q_t(B) \quad (6.39)$$

**Proof:** These formulae are either obvious or follow in a similar way to those appearing in Proposition 4.2.4.

For example, to show (6.37) : it follows from (6.34) that

$$q_{t_1+t_2}(B_1 \cup B_2) \geq \max\{q_{t_1+t_2}(B_1); q_{t_1+t_2}(B_2)\}$$

Conversely, suppose that for  $i = 1, 2$  we have  $p_i \in \mathcal{M}^p$  such that  $\tau(1 - p_i) \leq t_i$  and a measure bounded  $B'_i \in \mathcal{H}$  such that  $B_i \subset B'_i + \epsilon_i \mathcal{B}_{\mathcal{H}}$ . Then the same argument as for (4.19) shows that

$$q_{t_1+t_2}(B_1 \cup B_2) \leq \max\{\epsilon_1, \epsilon_2\}$$

because  $\tau(p_1 \vee p_2) \leq t_1 + t_2$ .

The others follow similarly and so the proofs are omitted. ■

By taking limits in the equations above we obtain:

$$B_1 \subset B_2 \Rightarrow q_{\infty}(B_1) \leq q_{\infty}(B_2) \quad (6.40)$$

$$q_{\infty}(\alpha B) = |\alpha| q_{\infty}(B) \quad (6.41)$$

$$q_{\infty}(\mathcal{B}_{\mathcal{H}}) = 1 \quad (6.42)$$

$$q_{\infty}(B_1 \cup B_2) = \max\{q_{\infty}(B_1); q_{\infty}(B_2)\} \quad (6.43)$$

$$q_{\infty}(B_1 + B_2) \leq q_{\infty}(B_1) + q_{\infty}(B_2) \quad (6.44)$$

$$q_{\infty}(\overline{B}) = q_{\infty}(B) \quad (6.45)$$

for  $B, B_1, B_2 \in \mathcal{H}$ ,  $\alpha \in \mathbb{C}$ . It now follows that  $q_{\infty}$  is a  $q$ -function.

In (4.15) it was pointed out that if  $\mathcal{I}$  is norm-closed then  $q_{\mathcal{I}}$  is a convex  $q$ -function. It can be seen that this relies on the fact that the absolutely convex hull of a norm bounded set is norm bounded, which results from the fact that the norm topology is locally convex. Since the topology of convergence in measure is not necessarily locally

and the result follows.

(c) The proof of this is identical to that of (4.24). ■

By taking limits in (6.46) we obtain:

$$q_{\infty}(x[D_1(x)]) = \mu_{\infty}(x) \quad (6.49)$$

for  $x \in \widetilde{\mathcal{M}}$ .

By taking limits in (6.47) we obtain:

$$q_{\infty}(xB) \leq \mu_{\infty}(x) q_{\infty}(B) \quad (6.50)$$

for  $x \in \widetilde{\mathcal{M}}$ ,  $B \subset D(x)$  such that  $q_{\infty}(B) < \infty$ . Note that this equation also shows that the latter class of sets is invariant under the operator, as is the subclass of  $\tau$ -totally bounded sets (included in the domain of the operator).

By combining equations (6.49) and (6.50), we obtain the following corollary:

**Corollary 6.4.9** *The following are equivalent :*

(a)  $x \in \widetilde{\mathcal{M}}_0$

(b)  $xB$  is  $\tau$ -totally bounded for every  $B \subset D(x)$  such that  $q_{\infty}(B) < \infty$

(c)  $x[D_1(x)]$  is  $\tau$ -totally bounded

As before (6.50) fails for arbitrary  $B$  included in the domain of the operator. Consider any  $p \in \mathcal{M}^p$  such that  $\tau(p) < \infty$ . Then for any  $t_1 > \tau(p)$  and any  $t_2 > 0$  we have  $q_{t_1+t_2}(p\mathcal{H}) = \infty$ , but

$$\mu_{t_1}(p) q_{t_2}(\mathcal{H}) = 0 \cdot \infty = 0$$

This is not a very interesting example, as the set  $p\mathcal{H}$  is unbounded in every sense. The following example shows more : that (6.50) cannot be extended to the class of sets that are bounded in measure. In fact, we find a  $\tau$ -compact operator that sends the set previously discussed in Example 6.4.6 - bounded in measure but not having a finite  $q_{\infty}$  value - to a similar such set. Once again (6.50) fails by virtue of the agreement that  $0 \cdot \infty = 0$ .

**Example 6.4.10** Let  $\mathcal{H}, \mathcal{M}$  and  $B$  be as in Example 6.4.6. Let

$$g = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \chi_{[n, n+1)}$$

Then certainly  $g \in \widetilde{\mathcal{M}}_0$ .

To show that  $q_\infty(g(B)) = \infty$  it suffices to show that if we are given  $A \in \Sigma$  such that  $m(A) < \infty$  then for all  $\epsilon \geq 1$  there exists  $n, \alpha > 0$  such that

$$\|\chi_{A'} g f_{n,\alpha}\|_2^2 \geq \epsilon$$

Choose  $n \in \mathbb{N}$  such that

$$m(A \cap [n, n+1)) \leq \frac{1}{4\epsilon n}$$

This must be possible, for otherwise

$$\begin{aligned} \infty &> m(A) \\ &\geq \sum_{n=1}^{\infty} m(A \cap [n, n+1)) \\ &\geq \frac{1}{4\epsilon} \sum_{n=1}^{\infty} \frac{1}{n} \\ &= \infty \end{aligned}$$

Put

$$\alpha = \frac{1}{2\epsilon n}$$

Now

$$m(A \cap [n, n+\alpha]) \leq m(A \cap [n, n+1)) \leq \frac{1}{4\epsilon n} = \frac{\alpha}{2}$$

and so

$$m(A' \cap [n, n+\alpha]) \geq \frac{\alpha}{2}$$

Thus

$$\begin{aligned} \|\chi_{A'} g f_{n,\alpha}\|_2^2 &= \left(\frac{1}{\alpha}\right)^2 \frac{1}{n} m(A' \cap [n, n+\alpha]) \\ &\geq \left(\frac{1}{\alpha}\right)^2 \frac{1}{n} \frac{\alpha}{2} \\ &= \frac{1}{2\alpha n} \\ &= \epsilon \end{aligned}$$

as required. ■

The above also provides the example promised after Theorem 4.2.5 by considering the bounded function  $g$  as a member of the norm closed ideal  $\mathcal{M}_0$ .

## 6.5 Geometric characterisations of $\tau$ -compact operators

We now aim to generalise some results of Kaftal ([Kaf77] Lemma 1.2 and Theorem 1.3) and Ströh ([Str89] Proposition 2.4). We are faced with the problem that the operators

Let  $p = p_{\overline{M}} - N(x)$ . It is clear that  $x|_{p\mathcal{H}}$  is one-to-one.

We show that  $x|_{p\mathcal{H}}$  is onto  $q\mathcal{H}$ . From Lemma 6.5.1 we have that

$$x^{-1}(q\mathcal{H}) = \overline{x^{-1}(q\mathcal{H})} \cap D(x)$$

Hence

$$\begin{aligned} \text{range}(x|_{p\mathcal{H}}) &= \text{range}(xp) \\ &= \text{range}(xp_{\overline{M}}) \\ &= x[\overline{x^{-1}(q\mathcal{H})} \cap D(x)] \\ &= x[x^{-1}(q\mathcal{H})] \\ &= q\mathcal{H} \cap \text{range}(x) \\ &= q\mathcal{H} \end{aligned}$$

Finally we note that since  $N(xp) = 1 - p$  and  $R(xp) = q$ ,

$$q = R(xp) \sim R((xp)^*) = 1 - N(xp) = p$$

■

**Theorem 6.5.3** cf. [Kaf77] Theorem 1.3

The following are equivalent :-

- (a)  $x \in \overline{\mathcal{M}}_0$
- (b)  $x[D_1(x)]$  is  $\tau$ -totally bounded
- (c)  $xB$  is  $\tau$ -totally bounded, for every  $B \subset D(x)$  such that  $q_\infty(B) < \infty$
- (d) If  $q \in \mathcal{M}^p$  and  $q\mathcal{H} \subset \text{range}(x)$  then  $\tau(q) < \infty$  (generalised Calkin condition)
- (e) If  $p \in \mathcal{M}^p$  and  $x$  is bounded from below on  $p\mathcal{H}$  then  $\tau(p) < \infty$
- (f) For all  $\epsilon > 0$  there exists  $p \in \mathcal{M}^p$  such that  $\|xp\| \leq \epsilon, \tau(1-p) < \infty$  (generalised Rellich condition)

**Proof:** The equivalence of (a), (b) and (c) was established in Corollary 6.4.9.

(c)  $\Rightarrow$  (d) Suppose  $q \in \mathcal{M}^p, q\mathcal{H} \subset \text{range}(x)$ . By Lemma 6.5.2 we can choose  $p \in \mathcal{M}^p$  such that  $p \sim q$  and

$$x|_{p\mathcal{H}} : p\mathcal{H} \cap D(x) \rightarrow q\mathcal{H}$$

is bijective. It is clear that  $x|_{p\mathcal{H}}$  is a closed operator, since  $x$  is closed. Thus  $x|_{p\mathcal{H}}$  is one-to-one, has closed range, and is closed. Hence by the Banach Isomorphism Theorem  $x|_{p\mathcal{H}}$  has a bounded inverse, and so

$$(x|_{p\mathcal{H}})^{-1}[q\mathcal{H} \cap B_{\mathcal{H}}]$$

is bounded. Thus

$$x[(x|_{p\mathcal{H}})^{-1}[q\mathcal{H} \cap \mathcal{B}_{\mathcal{H}}]] = q\mathcal{H} \cap \mathcal{B}_{\mathcal{H}} = q\mathcal{B}_{\mathcal{H}}$$

is  $\tau$ -totally bounded, by hypothesis. Thus by Corollary 6.4.9,  $\tau(q) < \infty$ .

(d)  $\Rightarrow$  (e) Suppose  $p \in \mathcal{M}^p$  and  $x$  is bounded from below on  $p\mathcal{H}$ . It is then clear that  $\text{range}(xp)$  is closed, since  $x$  is closed. If we define  $q = R(xp)$  we have

$$q\mathcal{H} \subset \text{range}(xp) \subset \text{range}(x)$$

and so  $\tau(q) < \infty$  by hypothesis. Furthermore  $N(xp) = 1 - p$ . Thus  $p \sim q$ , and  $\tau(p) < \infty$ .

(e)  $\Rightarrow$  (f) Suppose  $\epsilon > 0$ . Put  $p = e_{\epsilon}(|x|)$ . Then  $\|xp\| \leq \epsilon$ , and  $x$  is bounded from below on  $1 - p$ . Thus  $\tau(1 - p) < \infty$  by hypothesis.

(f)  $\Rightarrow$  (a) Suppose  $\epsilon > 0$ , and  $p$  is as indicated. Then

$$p \wedge e_{(\epsilon, \infty)}(|x|) = 0$$

implies that

$$e_{(\epsilon, \infty)}(|x|) \lesssim 1 - p$$

and hence  $\tau(e_{(\epsilon, \infty)}(|x|)) < \infty$ . Therefore  $x \in \widetilde{\mathcal{M}}_0$ , by Theorem 6.2.3, since  $\epsilon$  was arbitrary. ■

**Corollary 6.5.4** Suppose  $y \in \widetilde{\mathcal{M}}$ ,  $x \in \widetilde{\mathcal{M}}_0$ , and  $\text{range}(y) \subset \text{range}(x)$ . Then  $y \in \widetilde{\mathcal{M}}_0$ .

**Proof:** Immediate from Theorem 6.5.3 (d). ■

It is well known that for a positive compact operator on a Hilbert space, the  $s$ -number sequence is made up of the eigenvalues of the operator, arranged in decreasing order and counted according to multiplicity. It then follows by the classical theorem of Riesz that the spectrum is exactly the closure of the  $s$ -number sequence. The following theorem generalises this observation to the  $\tau$ -compact operators.

**Theorem 6.5.5** Suppose  $0 \leq x \in \widetilde{\mathcal{M}}_0$ . Then

$$\sigma(x) = \overline{\{\mu_t(x) : t > 0\}} \quad (6.51)$$

**Proof:** Suppose  $t > 0$ . To show that  $\mu_t(x) \in \sigma(x)$ , it suffices to show that

$$\sup \sigma(x) \cap [0, \mu_t(x)] = \mu_t(x)$$

since  $\sigma(x)$  is closed.

Throughout this section we consider an operator  $x \in \widetilde{\mathcal{M}}$ , and suppose  $\phi(x) \in \mathcal{M}$  is any operator for which  $x - \phi(x) \in \widetilde{\mathcal{M}}_0$ . Thus  $\phi(x)$  is a bounded operator in  $x + \widetilde{\mathcal{M}}_0$ . Furthermore  $\pi : \widetilde{\mathcal{M}} \rightarrow \widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$  denotes the canonical quotient map.

It is clear from Theorem 6.3.4 that  $x$  is Fredholm relative to  $\widetilde{\mathcal{M}}_0$  iff  $\phi(x)$  is Fredholm relative to  $\mathcal{M}_0$ ; hence  $\sigma(\pi(x)) = \sigma(\pi(\phi(x)))$ , where of course the first essential spectrum is relative to  $\widetilde{\mathcal{M}}_0$  and the second relative to  $\mathcal{M}_0$ .

What is not evident is that the following characterisation of the  $\widetilde{\mathcal{M}}_0$ -essential spectrum of a self-adjoint operator in  $\widetilde{\mathcal{M}}$  can be deduced directly from its 'bounded' analogue - Theorem 2.1.4 - by use of Theorem 6.3.4. However, this characterisation is obviously derived in a manner quite similar to the bounded case, as the reader may easily verify. This characterisation then coincides with the (unmotivated) definition of the essential spectrum given on page 4 of [HN91].

**Proposition 6.6.1** *Suppose  $x \in \widetilde{\mathcal{M}}^{sa}$ . Then*

$$\sigma(\pi(x)) = \{\lambda \in \mathbf{R} : \tau(e_{(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}]}) = \infty \text{ for all } n \in \mathbf{N}\} \quad (6.52)$$

It is now clear that (2.23) generalises to  $\widetilde{\mathcal{M}}_0$ , while (2.24) is in fact generalised in the crucial (6.12). Furthermore, (2.25) also generalises. This relies crucially on the previously mentioned fact that (2.2) generalises in full, as seen in Proposition 6.3.3. For the same reason, we find that Theorem 2.1.10 generalises to the following result:

**Theorem 6.6.2**  $\mu_\infty$  is unique amongst the seminorms  $\delta$  on  $\widetilde{\mathcal{M}}$  satisfying

- (a)  $\delta$  has kernel  $\widetilde{\mathcal{M}}_0$
- (b)  $\delta(x) \leq \|x\|$  for  $x \in \mathcal{M}$
- (c)  $\delta(yx) \leq \delta(y)\delta(x)$  for  $y, x \in \widetilde{\mathcal{M}}$

Olsen ([Ols84]) has developed a complete Fredholm and index theory relative to any closed ideal  $\mathcal{I}$  in a von Neumann algebra. It is clear that the elementary characterisations of Fredholm operators given by Olsen (Theorem 2.2.5 for example) generalise to the  $\widetilde{\mathcal{M}}_0$  case.

We now discuss the index theory of Olsen. For  $x \in \mathcal{M}$  the index is defined as

$$i(x) = \dim(N(x)) - \dim(N(x^*)) \quad (6.53)$$

where  $\dim$  is the dimension function of Tomiyama, relative to  $\mathcal{I}$ , discussed in [Ols84] §5. Of course the function

$$i(x) = \dim(N(\phi(x))) - \dim(N(\phi(x^*))) \quad (6.54)$$

will define an index in  $\widetilde{\mathcal{M}}$  relative to the ideal  $\widetilde{\mathcal{M}}_0$ .

We now show that this index has a concrete formulation, namely that it coincides with the function (which in the meanwhile will bear a different name)

$$\text{index}(x) = \dim(N(x)) - \dim(N(x^*)) \quad (6.55)$$

For any  $x \in \widetilde{\mathcal{M}}$ , let  $x = v|x|$  be the polar decomposition of  $x$ , and let

$$|x| = |x|_\infty + |x|_0$$

be the decomposition for  $|x|$  constructed in the proof of Lemma 6.3.1. Then

$$x = v|x|_\infty + v|x|_0$$

It is clear from the construction given in the lemma that

$$N(|x|) = N(|x|_\infty)$$

and so

$$R(|x|) = R(|x|_\infty)$$

since both operators are positive. Thus

$$v^*v = R(|x|_\infty)$$

and so by the uniqueness clause in the polar decomposition theorem ([KR86] Theorem 6.1.11) the operator  $v|x|_\infty$  is in polar decomposition form. Thus

$$N(x) = N(|x|) = N(|x|_\infty) = N(v|x|_\infty)$$

and

$$N(x^*) = 1 - R(x) = 1 - vv^* = 1 - R(v|x|_\infty) = N((v|x|_\infty)^*)$$

Hence  $\text{index}(x) = i(v|x|_\infty)$ . But by standard reasoning, as for example seen in the proof of Theorem 6.3.4, we have that

$$v|x|_\infty - \phi(x) \in \mathcal{M}_0$$

Hence by [Ols84] Theorem 7.4,

$$\text{index}(x) = i(v|x|_\infty) = i(\phi(x))$$

It follows that the index theory of Olsen extends completely to an index theory for elements in  $\widetilde{\mathcal{M}}$  modulo  $\widetilde{\mathcal{M}}_0$ .

Remaining with the theme of Fredholm operators, we again have an Atkinson-type characterisation in the  $\widetilde{\mathcal{M}}_0$  case, generalising Theorem 2.2.7.

**Theorem 6.6.3** Suppose  $x \in \widetilde{\mathcal{M}}$ .

- (a) If  $x$  is left Fredholm with respect to  $\widetilde{\mathcal{M}}_0$  then there exists  $y \in \mathcal{M}$ ,  $e \in \mathcal{M}^p$  such that  $\tau(1 - e) < \infty$  and  $yx = e$
- (b) If  $x$  is right Fredholm with respect to  $\widetilde{\mathcal{M}}_0$  then there exists  $y \in \mathcal{M}$ ,  $e \in \mathcal{M}^p$  such that  $\tau(1 - e) < \infty$  and  $xy = e$
- (c) If  $x$  is Fredholm with respect to  $\widetilde{\mathcal{M}}_0$  then there exists  $y \in \mathcal{M}$ ,  $e_1, e_2 \in \mathcal{M}^p$  such that  $yx = e_1$ ,  $xy = e_2$ ,  $\tau(1 - e_1) = \tau(1 - e_2) < \infty$ , and  $e_1 \sim e_2$

Note that from the construction given in Theorem 2.2.7 we can ensure that the operator  $y$  is indeed bounded.

Theorem 2.2.8, Corollary 2.2.9 and Corollary 2.2.10 generalise easily.

Finally, we mention that the function  $q_\infty$  can be applied to operators that are left Fredholm with respect to  $\widetilde{\mathcal{M}}_0$  in a manner like that in Chapter 4, and we arrive at the following results:

**Theorem 6.6.4**  $x \in \widetilde{\mathcal{M}}$  is left Fredholm with respect to  $\widetilde{\mathcal{M}}_0$  iff there exists  $c > 0$  such that

$$q_\infty(xB) \geq c q_\infty(B)$$

for all sets  $B \subset D(x)$  such that  $q_\infty(B) < \infty$ . The largest possible value of  $c$  is  $m(x)$ , where  $m(x)$  is defined in the same way as for ideals of bounded operators.

**Corollary 6.6.5** For any  $x \in \widetilde{\mathcal{M}}$ ,

$$m(x) = \inf_{\substack{B \subset D(x) \\ 0 < q_\infty(B) < \infty}} \frac{q_\infty(xB)}{q_\infty(B)} \quad (6.56)$$

## 6.7 Commutativity of the trace

One of the most basic questions one can ask in non-commutative integration theory is the extent to which the trace  $\tau$  is commutative, that is, for which  $x, y \in \widetilde{\mathcal{M}}$  do we have  $\tau(xy) = \tau(yx)$ . In the elementary theory of the trace class ideal (as developed in [Tak79], for example) we quickly learn that if  $x \in H(\mathcal{M})$  and  $y \in \mathcal{M}$  then both  $xy$  and  $yx$  are of course members of  $H(\mathcal{M})$ , and  $\tau(xy) = \tau(yx)$ .

It is also shown that  $\tau$  is a continuous linear functional on  $H(\mathcal{M})$  when equipped with the 1-norm i.e. the norm

$$\|x\|_1 = \tau(|x|) = \int_0^\infty \mu_t(|x|) dt = \int_0^\infty \mu_t(x) dt \quad (6.57)$$

and so  $\tau$  extends to a continuous linear functional on  $\mathcal{L}^1(\mathcal{M})$ , which is the completion of  $H(\mathcal{M})$  in this norm. For this reason, the commutativity result mentioned above then holds for  $x \in \mathcal{L}^1(\mathcal{M})$ ,  $y \in \mathcal{M}$ .

As indicated, we are interested in how far this commutativity result can be extended. In this section we show that if  $x, y \in \widetilde{\mathcal{M}}$  and  $xy, yx \in \mathcal{L}^1(\mathcal{M})$  then  $\tau(xy) = \tau(yx)$ . This result has actually been shown in [BK90]; nevertheless we include it here since the techniques used are somewhat different to those of Brown and Kosaki. Our strategy is in fact an extension of that of Proposition 3.4 of [DDP90] where it is claimed that if one of the operators is in  $\widetilde{\mathcal{M}}_0$  and the other is in  $G(\mathcal{M})$  then the stated result holds. In fact, there is an error in the proof of this proposition, for there it is claimed that if  $x \in \widetilde{\mathcal{M}}_0$  then  $x$  commutes with  $e_{(\frac{1}{n}, n]}(|x|)$  for  $n \in \mathbb{N}$ . This is false, as the following example shows.

**Example 6.7.1** Let  $\mathcal{M} = \mathcal{B}(\ell^2)$  equipped with the canonical trace. We follow the convention of representing multiplication operators in  $\mathcal{B}(\ell^2)$  by the corresponding  $\ell^\infty$  sequence.

Let

$$x : (a_1, a_2, a_3, \dots) \rightarrow (a_2, a_3, 0, 0, \dots)$$

That is,  $x = pl$  where  $p = (1, 1, 0, 0, \dots)$ , and  $l$  is the left shift operator. Then  $x \in \widetilde{\mathcal{M}}_0$  since  $\tau(p) = 2$ .

Denote the right shift operator by  $r$ . Note that

$$|x| = (x^*x)^{\frac{1}{2}} = (l^*pl)^{\frac{1}{2}} = (rpl)^{\frac{1}{2}} = (0, 1, 1, 0, 0, \dots)^{\frac{1}{2}} = (0, 1, 1, 0, 0, \dots)$$

Therefore

$$e_t(|x|) = \begin{cases} 0 & \text{for } t < 0 \\ 1 - |x| & \text{for } 0 \leq t < 1 \\ 1 & \text{for } t \geq 1 \end{cases}$$

and so  $e_{(\frac{1}{2}, 2]}(|x|) = |x|$ .

Thus

$$x e_{(\frac{1}{2}, 2]}(|x|) : (a_1, a_2, a_3, \dots) \rightarrow (a_2, a_3, 0, 0, \dots)$$

while

$$e_{(\frac{1}{2}, 2]}(|x|) x : (a_1, a_2, a_3, \dots) \rightarrow (0, a_3, 0, 0, \dots)$$

We will show our result by building up a number of cases. The argument exploits a number of the facts about  $\widetilde{\mathcal{M}}_0$  that we have already seen. The proofs of each of the cases are quite similar, and in them we follow the essential idea of [DDP90], which is a clever use of the non-commutative Dominated Convergence Theorem. We now state this theorem, which was first proved by Fack and Kosaki in [FK86], in the form used in [DDP90].

**Theorem 6.7.2** Suppose  $(x_n) \subset \mathcal{L}^1(\mathcal{M})$ ,  $x_n \xrightarrow{\tau_{cm}} x \in \widetilde{\mathcal{M}}$  and  $\mu_t(x_n) \leq f(t) \in \mathcal{L}^1(0, \infty)$  for all  $n \in \mathbb{N}$ . Then  $x \in \mathcal{L}^1(\mathcal{M})$  and  $\|x_n - x\|_1 \rightarrow 0$ .

To begin with we isolate a simple result which we will use a number of times.

**Lemma 6.7.3** Suppose  $x, y \in \widetilde{\mathcal{M}}$ . Then

$$\mu_t(xy) = \mu_t(|x|y) \quad (6.58)$$

**Proof:** Suppose  $x = v|x|$  is the polar decomposition of  $x$ . Then  $|x| = v^*x$  and so

$$\mu_t(xy) = \mu_t(v|x|y) \leq \mu_t(|x|y) = \mu_t(v^*xy) \leq \mu_t(xy)$$

■

**Proposition 6.7.4** Suppose  $x \in \mathcal{M}$ ,  $y \in \widetilde{\mathcal{M}}$  and  $xy, yx \in \mathcal{L}^1(\mathcal{M})$ . Then

$$\tau(xy) = \tau(yx)$$

**Proof:** Let  $x = v|x|$  be the polar decomposition. For any fixed  $n \in \mathbb{N}$ , let

$$f_n(t) = \begin{cases} 0 & \text{for } t \leq \frac{1}{n} \\ \frac{1}{t} & \text{for } t > \frac{1}{n} \end{cases}$$

Then  $f_n(|x|) \in \mathcal{M}$ , since  $f_n$  is bounded, and

$$f_n(|x|) |x| = e_{(\frac{1}{n}, \infty)}(|x|)$$

For brevity we will denote  $e_{(\frac{1}{n}, \infty)}(|x|)$  by  $p_n$ .

Now  $xy \in \mathcal{L}^1(\mathcal{M})$ , thus

$$p_n y = f_n(|x|) |x| y = f_n(|x|) v^* xy \in \mathcal{L}^1(\mathcal{M})$$

since  $f_n(|x|) v^* \in \mathcal{M}$ ,  $xy \in \mathcal{L}^1(\mathcal{M})$ , and  $\mathcal{L}^1(\mathcal{M})$  is a  $\mathcal{M}$ -module.

Now  $|x|p_n \xrightarrow{\|\cdot\|} |x|$  as  $n \rightarrow \infty$  and so

$$xp_n = v|x|p_n \xrightarrow{\|\cdot\|} v|x| = x$$

as  $n \rightarrow \infty$ . Therefore  $xp_n \xrightarrow{\tau_{cm}} x$  since the measure topology is weaker than the norm topology, and so

$$xp_n y \xrightarrow{\tau_{cm}} xy \text{ and } yxp_n \xrightarrow{\tau_{cm}} yx$$

by the  $\tau_{cm}$  continuity of multiplication. Since  $|x|p_n = p_n|x|$  we have that

$$\mu_t(xp_n y) = \mu_t(|x|p_n y) = \mu_t(p_n|x|y) \leq \mu_t(|x|y) = \mu_t(xy)$$

and of course

$$\mu_t(yxp_n) \leq \mu_t(yx)$$

Thus by the Dominated Convergence Theorem we have that

$$xp_n y \xrightarrow{\|\cdot\|_1} xy$$

and

$$yxp_n \xrightarrow{\|\cdot\|_1} yx$$

Since  $\tau$  is  $\|\cdot\|_1$ -continuous, we have

$$\tau(xp_n y) \rightarrow \tau(xy) \text{ and } \tau(yxp_n) \rightarrow \tau(yx)$$

Since  $p_n y \in \mathcal{L}^1(\mathcal{M})$  and  $x \in \mathcal{M}$  we have that

$$\tau(xp_n y) = \tau(p_n yx)$$

Also, since  $yx \in \mathcal{L}^1(\mathcal{M})$  and  $p_n \in \mathcal{M}$  we have that

$$\tau(p_n yx) = \tau(yxp_n)$$

Therefore

$$\tau(xy) = \lim_n \tau(xp_n y) = \lim_n \tau(p_n yx) = \lim_n \tau(yxp_n) = \tau(yx)$$

as required. ■

**Proposition 6.7.5** *Suppose  $x \in \widetilde{\mathcal{M}}_0$ ,  $y \in \widetilde{\mathcal{M}}$  and  $xy, yx \in \mathcal{L}^1(\mathcal{M})$ . Then*

$$\tau(xy) = \tau(yx)$$

**Proof:** Suppose  $x = v|x|$  and  $y = w|y|$  are the polar decompositions. As before denote  $e_{(\frac{1}{n}, n]}(|x|)$  as  $p_n$ . Since  $x \in \widetilde{\mathcal{M}}_0$  we have that  $(xp_n) \subset H(\mathcal{M})$ , while of course  $(ye_n(|y|)) \subset \mathcal{M}$  and

$$xp_n \xrightarrow{\tau_{cm}} x \text{ and } ye_n(|y|) \xrightarrow{\tau_{cm}} y$$

Therefore

$$ye_n(|y|)xp_n \xrightarrow{\tau_{cm}} yx \text{ and } xp_n ye_n(|y|) \xrightarrow{\tau_{cm}} xy$$

by the continuity of multiplication in  $\widetilde{\mathcal{M}}$ .

Since

$$\mu_t(xp_nye_n(|y|)) \leq \mu_t(xp_ny) = \mu_t(|x|p_ny) = \mu_t(p_n|x|y) \leq \mu_t(|x|y) = \mu_t(xy)$$

and similarly

$$\mu_t(ye_n(|y|)xp_n) \leq \mu_t(yx)$$

we have by the Dominated Convergence Theorem that

$$xp_nye_n(|y|) \xrightarrow{\|\cdot\|_1} xy$$

and

$$ye_n(|y|)xp_n \xrightarrow{\|\cdot\|_1} yx$$

Since  $\tau$  is  $\|\cdot\|_1$ -continuous, we have that

$$\tau(xp_nye_n(|y|)) \rightarrow \tau(xy) \quad \text{and} \quad \tau(ye_n(|y|)xp_n) \rightarrow \tau(yx)$$

Since  $xp_n \in H(\mathcal{M})$ ,  $ye_n(|y|) \in \mathcal{M}$ , we have that

$$\tau(xp_nye_n(|y|)) = \tau(ye_n(|y|)xp_n)$$

Therefore

$$\tau(xy) = \lim_n \tau(xp_nye_n(|y|)) = \lim_n \tau(ye_n(|y|)xp_n) = \tau(yx)$$

as required. ■

**Theorem 6.7.6** *Suppose  $x, y \in \widetilde{\mathcal{M}}$  and  $xy, yx \in \mathcal{L}^1(\mathcal{M})$ . Then  $\tau(xy) = \tau(yx)$ .*

**Proof:** We decompose  $x$  into a member of  $\mathcal{M}$  and a member of  $\widetilde{\mathcal{M}}_0$ . Suppose  $x = v|x|$  is the polar decomposition.

Let  $x_\infty = xe_\alpha(|x|)$  and let  $x_0 = x(1 - e_\alpha(|x|))$ , where  $\alpha > \mu_\infty(x)$ . Since  $1 - e_\alpha(|x|) \in \mathcal{P}_\tau$  it follows that

$$x = x_\infty + x_0, \quad x_\infty \in \mathcal{M}, \quad x_0 \in \widetilde{\mathcal{M}}_0$$

Since  $xy \in \mathcal{L}^1(\mathcal{M})$ , we have from Lemma 6.7.3 that  $|x|y \in \mathcal{L}^1(\mathcal{M})$ , and therefore  $x_\infty y = ve_\alpha(|x|)|x|y \in \mathcal{L}^1(\mathcal{M})$ . Further  $yx_\infty = yxe_\alpha(|x|) \in \mathcal{L}^1(\mathcal{M})$ .

Likewise  $x_0y, yx_0 \in \mathcal{L}^1(\mathcal{M})$ .

Therefore we have that

$$\begin{aligned} \tau(xy) &= \tau(x_\infty y + x_0 y) \\ &= \tau(x_\infty y) + \tau(x_0 y) \\ &= \tau(yx_\infty) + \tau(yx_0) \\ &= \tau(yx_\infty + yx_0) \\ &= \tau(yx) \end{aligned}$$

by making use of Propositions 6.7.4 and 6.7.5. ■

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