University of Cape Town

Hamiltonian Cycles in Maximal Planar Graphs and Planar Triangulations

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Declaration

I Tshenolo Mofokeng confirm that this Master's thesis is my own work and that I have documented all sources and material used.

Acknowledgements

I would like to thank my supervisor Dr David Erwin for his guidance, and knowledge and his dedication to helping to complete my thesis. I would also like to thank my parents for everything that they have done for me. Lastly I would like to thank the National Research Foundation for funding my studies.

Abstract

In this thesis we study planar graphs, in particular, maximal planar graphs and general planar triangulations.

In Chapter 1 we present the terminology and notations that will be used throughout the thesis and review some elementary results on graphs that we shall need.

In Chapter 2 we study the fundamentals of planarity, since it is the cornerstone of this thesis. We begin with the famous Euler's Formula which will be used in many of our results. Then we discuss another famous theorem in graph theory, the Four Colour Theorem. Lastly, we discuss Kuratowski's Theorem, which gives a characterization of planar graphs.

In Chapter 3 we discuss general properties of a maximal planar graph, G particularly concerning connectivity. First we discuss maximal planar graphs with minimum degree i, for $i = 3, 4, 5$, and the subgraph induced by the vertices of G with the same degree. Finally we discuss the connectivity of G, a maximal planar graph with minimum degree i.

Chapter 4 will be devoted to Hamiltonian cycles in maximal planar graphs. We discuss the existence of Hamiltonian cycles in maximal planar graphs. Whitney[25] proved that any maximal planar graph without a separating triangle is Hamiltonian, where a separating triangle is a triangle such that its removal disconnects the graph. Chen[5] then extended Whitney's results and allowed for one separating triangle and showed that the graph is still Hamiltonian. Helden[14] also extended Chen's result and allowed for two separating triangles and showed that the graph is still Hamiltonian. G. Helden and O. Vieten[16] went further and allowed for three separating tri-

angles and showed that the graph is still Hamiltonian. In the second section we discuss the question by Hakimi and Schmeichel[11]: what is the number of cycles of length p that a maximal planar graph on n vertices could have in terms of n? Then in the last section we discuss the question by Hakimi, Schmeichel and Thomassen[12]: what is the minimum number of Hamiltonian cycles that a maximal planar graph on n vertices could have, in terms of n?

In Chapter 5, we look at general planar triangulations. Note that every maximal planar graph on $n \geq 3$ vertices is a planar triangulation. In the first section we discuss general properties of planar triangulations and then end with Hamiltonian cycles in planar triangulations.

Contents

Chapter 1

Introduction

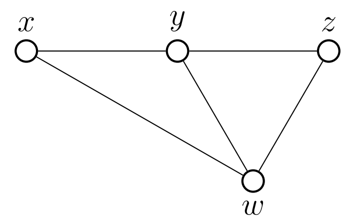
In this chapter we discuss the definitions and notations from graph theory that will be used throughout this dissertation. This material has been adapted from Chartrand and Lesniak [3], Harary [13] and Chartrand and Zhang [4]. Other definitions and notations will be defined as needed in the thesis.

1.1 Graph theoretic terminology

A *graph* G is an ordered pair (V, E) where V is a finite nonempty set of *vertices* and E is a set of 2-subsets of V called *edges*. We use $V(G)$ to denote the *vertex set* and $E(G)$ the *edge set* of G. The edge $\{x, y\} = \{y, x\}$ is denoted by xy. If $xy \in E(G)$, we say x and y are *adjacent* and we say x is a *neighbour* of y and y is a *neighbour* of x. The *neighbourhood* $N_G(x)$ of a vertex x in G is the set of all the neighbours of x. The *order* $n(G)$ of G is the number of vertices of G and the *size* $m(G)$ is the number of edges of G.

Example 1.1

G :



Example 1.1 shows a graph G with $V(G) = \{w, x, y, z\}$ and $E(G) = \{wx, wy, wz, xy, yz\}$. G has order $n = 4$ and size $m = 5$.

The Degree of a Vertex

The *degree* $d_G(x)$ of a vertex x in G is the number of vertices of G that are adjacent to x. A vertex of degree of 0 is called an *isolated vertex* and a vertex of degree 1 is an *end-vertex*. The *minimum degree* $\delta(G)$ and the *maximum degree* $\Delta(G)$ of G are the smallest and largest degrees of the vertices of G, respectively.

For the graph G in Example 1.1, we have

$$\begin{gathered} d_G(w) = 3 \\ d_G(x) = 2 \\ d_G(y) = 3 \\ d_G(z) = 2 \end{gathered}$$

and

$$\begin{gathered} \Delta(G) = 3 \\ \delta(G) = 2 \end{gathered}$$

Subgraphs

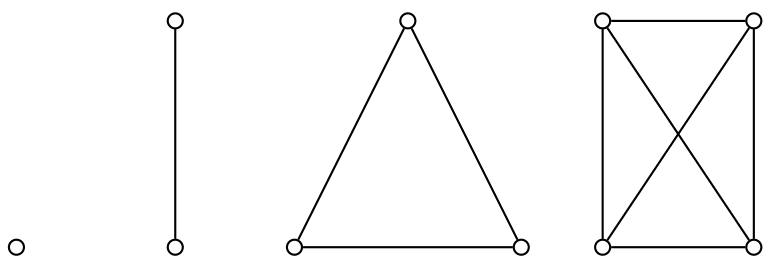
Let G and H be graphs. Then H is a subgraph of G (denoted $H \subseteq G$ and we say G contains H) if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $S \subseteq V(G)$. Then the subgraph of G *induced* by S (denoted as $G(S)$) is the graph with vertex set S and where vertices x and y of S are adjacent if and only if $xy \in E(G)$.

Isomorphism

A graph G is *isormorphic* to a graph H (written as $G \cong H$) if there exists a bijection β: $V(G) \to V(H)$ such that $xy \in E(G)$ if and only if $\beta(x)\beta(y) \in E(H)$. Such a bijection is called an *isomorphism*. An isomorphism from G to itself is called an *automorphism*.

Special Classes of Graphs

A *complete graph* with n vertices (denoted K_n) is a graph in which each pair of distinct vertices are adjacent.

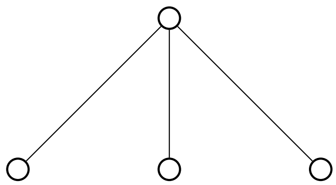


(a) K_1 (b) K_2 (c) K_3 (d) K_4

Figure 1.1: Some complete graphs

A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets such that every vertex of the graph is in one of the sets and no two vertices from the same set are adjacent. It is well known that a graph is bipartite if and only if it has no odd cycle. A *complete bipartite graph* is a bipartite graph such that every vertex of the first set is adjacent to every vertex of the second set. If there are m and n vertices in the two sets respectively, the complete bipartite graph is denoted as $K_{m,n}$.

$K_{1,3}$:



$K_{2,4}$:

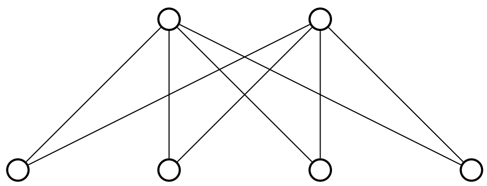


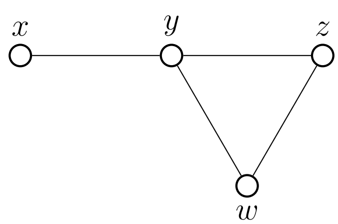
Figure 1.2: Some complete bipartite graphs

Walks, Trails, Paths and Cycles

A *walk* is an alternating sequence $v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k$ of vertices and edges where e_i joins v_{i-1} and v_i for all $i \in [1, k]$. The number of edges in a walk is called its *length*.

Example 1.2

G :



For the graph G in Example 1.2, the sequence W: $x, xy, y, yz, z, wz, w, wz, z$ is an $x - z$ walk of length 4.

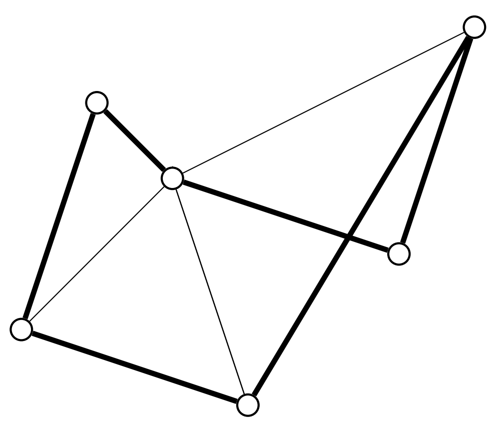
For brevity, we will write walks by leaving out the edges. For example, we rewrite the walk W in Example 1.2 as x, y, z, w, z. A walk in which no edge is repeated is called a *trail*. A walk in which no vertex (and hence no edge) is repeated is called a *path*. A walk is *closed* if it begins and ends at the same vertex, otherwise it is *open*. A closed walk with no repeated vertex (and hence no repeated edge) except the first and last vertices is called a *cycle*. A k-cycle is a cycle of length k.

Example 1.3 For the graph G in Example 1.2, the walk x, y, w, z, y is a trail. The walk x, y, z, w is a path. Lastly, the walk w, z, y, w is a cycle.

Hamiltonian Paths and Cycles

A path in G that contains every vertex of G is called a *hamiltonian path*. Similarly a cycle in G that contains every vertex of G is called a *hamiltonian cycle*. A graph G is said to be hamiltonian if it contains a hamiltonian cycle.

Example 1.4



The graph shown in Example 1.4 is hamiltonian since it contains a hamiltonian cycle, indicated in thicker edges.

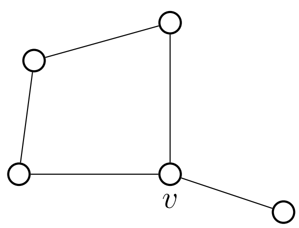
Connectivity, cut-vertices, bridges and blocks

A graph G is said to be *connected* if there is a path between any two vertices of G. A *component* of a graph G is a maximal connected subgraph of G.

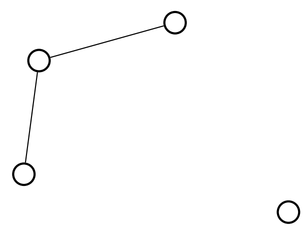
A vertex v of G is called a *cut vertex* of G if $G - v$ is disconnected. Let S be a set of vertices of G such that $G - S$ is disconnected. Then we call S a *vertex-cut* of G. Let e be an edge of G. Then e is called a *bridge* if $G - e$ is disconnected. A *separable graph* is a connected graph that contains at least one cut-vertex. A *non-separable* graph is a connected graph without a cut-vertex. A *block* is a maximal non-separable subgraph of a connected graph. A block that has only one cut-vertex is called an *end-block*. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal makes G either disconnected or reduces G to a trivial graph. A graph G is called *k-connected* if and only if it remains connected whenever fewer than k vertices are removed, i.e., if $\kappa(G) \geq k$.

Example 1.5

G :



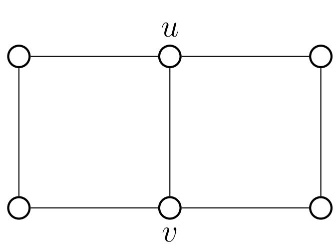
$G - v$:



In the example above v is a cut vertex since G is connected and $G - v$ is disconnected. The graph has $\kappa(G) = 1$ and is 1-connected.

Example 1.6

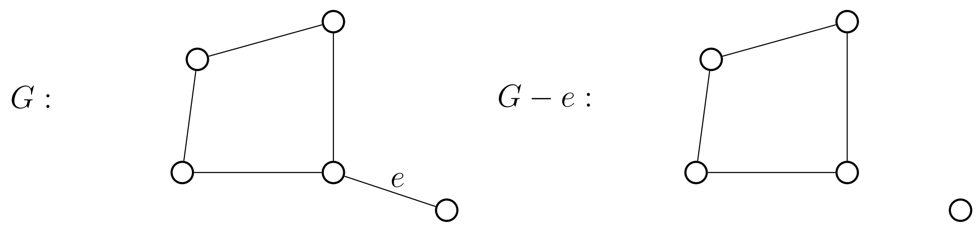
G :



$G - \{u, v\}$:

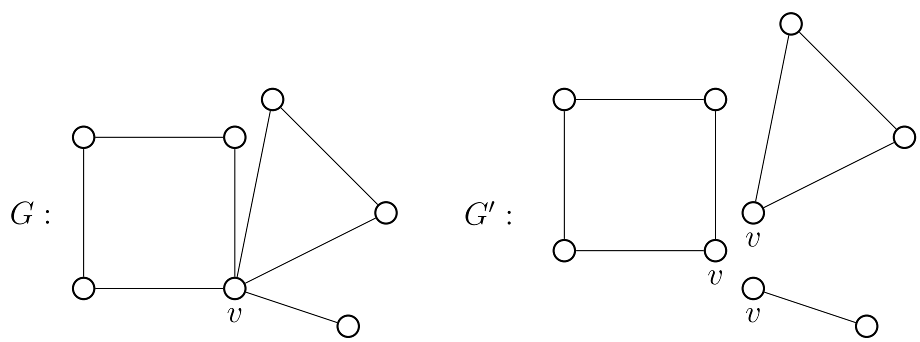
In the above example, u and v form a vertex-cut since G is connected and $G - \{u, v\}$ is disconnected. The graph has $\kappa(G) = 2$ and is 2-connected.

Example 1.7



In the example above G is connected and $G - e$ is disconnected. So e is a bridge.

Example 1.8

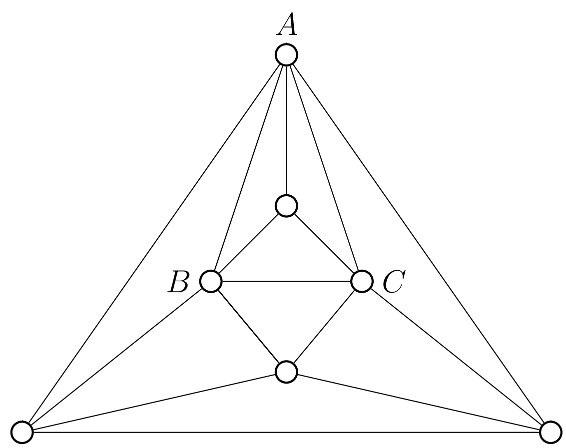


In the Example above, G' shows the three blocks of G.

Separating triangle

A *triangle* or a 3-cycle is a cycle of length 3. A *separating triangle* of a graph G is a triangle whose removal disconnects G.

Example 1.9



In the example above, the graph has a separating triangle ABC.

Chapter 2

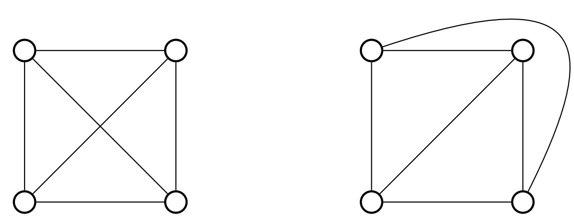
Planarity

In this chapter we discuss *planarity* which is the cornerstone of this thesis. The material has been adapted from Chartrand and Lesniak[3], Harary [13] and Chartrand and Zhang [4].

2.1 Planar graphs

Definition 2.1. *A graph G is planar if it can be embedded on the plane, that is, if it can be drawn on the plane (sheet of paper) in such a way that none of the edges cross each other. Such a drawing is called a plane graph.*

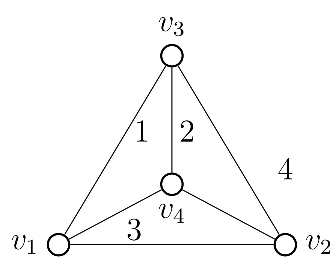
Example 2.1



Above we have a graph drawn with its edges crossing each other on the left and then redrawn with no edges crossing each other on the right. So this graph is embeddable on a plane and hence we say it is planar and we refer to the drawing on the right as a plane graph.

Definition 2.2. *When a graph G is embedded on the plane, the plane divides the graph into regions (also called faces). The (unique) unbounded region is called the exterior region and a bounded region is called an interior region. The boundary of a region R (denoted as ∂R) of the plane graph G is a subgraph of G consisting of all the edges and vertices incident with R. The degree of a region R in a plane graph is the number of edges bordering R.*

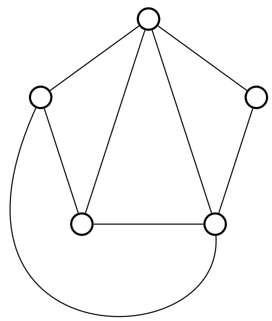
Example 2.2



In the plane graph shown above, region 4 is the unique exterior region and regions 1,2 and 3 are the interior regions. The boundary of region 1 is the cycle v_1, v_4, v_3, v_1, the boundary of region 2 is the cycle v_2, v_3, v_4, v_2, the boundary of the region 3 is the cycle v_1, v_2, v_4, v_1 and lastly the boundary of region 4 is the cycle v_1, v_2, v_3, v_1. All of the regions have degree 3.

Definition 2.3. *A planar graph is said to be maximal planar if the addition of any edge destroys its planarity.*

Example 2.3



The graph above is $K_5 - e$, which is clearly planar. The graph K_5, on the other hand, is non-planar (proved later). Therefore $K_5 - e$ is a maximal planar graph.

2.2 Fundamental results in planar graphs

We now discuss fundamental results on planar graphs.

Theorem 2.4. *(Chartrand and Lesniak [3]) If G is a connected graph and $e \in E(G)$. Then e is a bridge of G if and only if it does not lie on a cycle of G.*

Proof. ($\Longrightarrow$) Let $e = xy$ be a bridge of a graph G. Then $G - e$ is disconnected and x and y lie in different components of $G - e$. Therefore there is no $x - y$ path in $G - e$. Assume to the, to the contrary, that e lies on a cycle $C : x, y_1, y_2, \ldots, y_k, x$ in G. But clearly $y, y_1, y_2, \ldots, y_k, x$ is an $x - y$ path in $G - e$. This is a contradiction and so it follows that e lies on no cycle of G.
($\Longleftarrow$) Now let us assume that e is not a bridge of G. Then $G - e$ is connected and so it follows that there exists an $x - y$ path P in $G - e$. However the edge e together with the path P form a cycle in G that contains e. □

Theorem 2.5. ***(Euler's formula)*** *If G is a connected plane graph with n vertices, m edges and r regions, then $n - m + r = 2$.*

Proof. We prove by induction on the number of edges on the graph. The base case is when $m = 0$. Then since G is connected, it is the trivial graph that has a single vertex with a single region surrounding it. Thus we have $n - m + r = 1 - 0 + 1 = 2$. Suppose now that the formula works for all connected plane graphs of size less than m, where $m \geq 2$. Let G be a connected plane graph of size m. We consider two cases:

Case 1: G does not contain a cycle. So G is a tree. Then $m = n - 1$ and $r = 1$. Hence $n - m + r = n - (n - 1) + 1 = 2$.

Case 2: G contains a cycle. So G is not a tree. Let e be an edge that is on the cycle. Then e is on the boundary of two regions in G. When we remove the edge e we merge these two regions. So $G - e$ has one fewer region than G. By Theorem 2.4, the edge e is not a bridge. Therefore $G - e$ is a connected plane graph of order n and size $m - 1$. Therefore by the inductive hypothesis the formula works for $G - e$. That is $n - (m - 1) + (r - 1) = 2$, so $n - m + r = 2$. □

Corollary 2.6. *If G is a plane graph with m edges, n vertices, k components and r regions, then $n - m + r = k(G) + 1$.*

Lemma 2.7. *If G is a maximal plane graph on $n \geq 3$ vertices, then the boundary of every face is a triangle.*

Proof. Let R be a non-triangular region of G and $H \subseteq G$ its boundary. Then the order of H is at least 4. H is connected, otherwise it would consist of more than one component, and so we can add an edge between the components and thus contradicting maximality. We claim H contains a cycle. To prove this claim let us assume that H is acyclic. Then H is a tree. This implies that R is an exterior region of G and that G is actually a tree, contradicting maximality. Therefore the boundary of every region is a cycle. Now the non-triangular region R must have three consecutive vertices on its boundary, say, v_1, v_2 and v_3 where $v_1v_2 \in E(G)$, $v_2v_3 \in E(G)$ but $v_1v_3 \notin E(G)$ (Figure 2.1). Adding v_1v_3 within R in G preserve planarity but contradicts maximality. Therefore every region of a maximal planar graph with at least three vertices is a triangle.

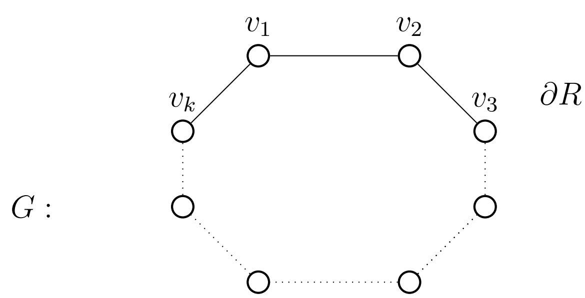


Figure 2.1: A region with more than four vertices in a maximal plane graph.

□

From Lemma 2.7 we obtain the following result;

Lemma 2.8. *Consider an embedding of a maximal planar graph G in the plane, and let $u \in V(G)$. If the neighbours of u, labelled sequentially clockwise around u in the plane, are $u_1, u_2, \ldots, u_k$, then $u_1, u_2, \ldots, u_k$, is a cycle. We call this cycle, the cycle induced by the neighbours of u and we denote it as C_u.*

Definition 2.9. *A planar triangulation is a planar graph in which the boundary of every region, except possibly the exterior region, is a 3-cycle. Thus every maximal planar graph with at least three vertices is a planar triangulation (but not conversely).*

Theorem 2.10. *If G is a maximal planar graph on $n \geq 3$ vertices and m edges, then $m = 3n - 6$.*

Proof. Consider a plane drawing of G with r regions. Let $\pi(R)$ be the degree of region R of G. By Lemma 2.7, the boundary of each region is a triangle. Therefore $\sum\limits_R \pi(R) = 3r$. Each edge is on the boundary of two regions. So $\sum\limits_R \pi(R) = 2m$. Therefore, $2m = 3r$, which implies $r = \frac{2}{3}m$. Substituting this into Euler's formula gives us $n - m + (\frac{2}{3}m) = 2$ and hence $m = 3n - 6$. □

Corollary 2.11. *If G is a planar graph with $n \geq 3$ vertices and n edges, then $m \leq 3n - 6$.*

Proof. Add edges to G until it is maximal planar. Then the resulting graph has m' edges $(m \leq m')$ edges and $n' = n$ vertices. By Corollary 2.10, $m' = 3n - 6$ and hence the result. □

Let us consider a planar graph with no triangles. We now prove a stronger result than Corollary 2.11.

Corollary 2.12. *If G is a planar graph with m edges, $n \geq 4$ vertices and no triangles, then $m \leq 2n - 4$.*

Proof. Consider a plane drawing of G with r regions. For each region R, let $\pi(R)$ be the number of edges on the boundary of R. Since G has at least four vertices and no triangles, each region of G is bounded by at least four edges and each edge is on the boundary of two regions. Therefore $4r \leq \sum_R \pi(R) = 2m$. Substituting Euler's formula, we get $4(m - n + 2) \leq 2m$ and hence $m \leq 2n - 4$. □

Corollary 2.13. *If G is a maximal planar graph of order 4 or more, then $\delta(G) \geq 3$.*

Proof. Let v be a vertex of G. Then $G - v$ is a graph on $n - 1$ vertices and $m - d_G(v)$ edges. Furthermore, $G - v$ is a planar graph on $n - 1 \geq 3$ vertices and so by Corollary 2.11, $m - d_G(v) \leq 3(n - 1) - 6$. Since $m = 3n - 6$, we obtain $3n - 6 - d_G(v) \leq 3n - 9$. Therefore the degree of v must be at least 3. □

Theorem 2.14. *K_5 and $K_{3,3}$ are not planar.*

Proof. Let us assume that K_5 is planar. K_5 has $n = 5$ and $m = 10$. Therefore by Corollary 2.11, we obtain $10 \leq 3(5) - 6 = 9$. A contradiction, so K_5 is non-planar.

$K_{3,3}$ is bipartite and so it has no odd cycles. In particular $K_{3,3}$ has no triangles. Let us assume that $K_{3,3}$ is planar. $K_{3,3}$ has $n = 6$ and $m = 9$. Therefore by Corollary, 2.12 we obtain $9 \leq 2(6) - 4 = 8$. A contradiction, so $K_{3,3}$ is non-planar. □

2.3 The Four Colour Theorem

One of the most famous theorems in graph theory is the *Four Colour Theorem*. It simply states that no more than four colours are required to colour regions in a plane in such a way that no two neighbouring regions receive the same colour.

The proof of the Four Colour Theorem is very long and complicated. Instead we will prove The *Five Colour Theorem*: That five colours are sufficient to colour regions in a plane in such a way that no two neighbouring regions receive the same colour. First let us look at some definitions so that we can restate this as a problem in graph theory. The Four Color Theorem can be expressed in terms of the dual graph as follows: the dual's vertices can be coloured with at most four colors such that no two vertices connected by an edge are coloured with the same color.

Definition 2.15. *Let be G be a graph, then a proper colouring of G is a colouring of the vertices of G such that no two adjacent vertices receive the same colour. A proper colouring that uses k colours is called a proper k-colouring. The chromatic number of G is the minimum k for which G has a proper k-colouring. If G has a proper colouring using at most k colours, then G is called k-colourable.*

Restated using this terminology, the Four Colour Theorem becomes the following:

Theorem 2.16. ***(The Four Colour Theorem)*** *Every planar graph is 4-colourable.*

To prove the Five Colour Theorem we need the following lemma.

Lemma 2.17. *If G is a planar graph, then G has a vertex of degree at most 5.*

Proof. We proceed by way of contradiction and assume that every vertex of G has degree at least 6. Then $2m = \sum_{v \in V(G)} deg(v) \geq 6n$. Hence $m \geq 3$, which contradicts Corollary 2.11. □

Theorem 2.18. ***(The Five Colour Theorem)*** *Every planar graph is 5-colourable.*

Proof. We proceed by induction on the order, n, of G. If G has order at most 5, then obviously the result holds. For some $n \geq 6$, let us assume that every planar graph on $n - 1$ vertices is 5-colourable. Let G be a planar graph on n vertices. By Lemma 2.17, G has a vertex v of degree at most 5. Consider the planar graph $G - v$. By the induction hypothesis, $G - v$ is 5-colourable. If fewer than five colours are used in the neighbourhood of v, then v may be coloured with a colour that does not appear in the neighbourhood of v, producing a proper 5-colouring of G. Now, let us assume $d_G(v) = 5$ and that all 5 colours are used in the neighbourhood of v. Consider an embedding of G and let $N_G(v) = \{v_1, v_2, v_3, v_4, v_5\}$ (v_i has colour i for $i = 1, 2, 3, 4, 5$) where $v_1 \ldots v_5$ are arranged cyclically in that order around v, (Figure 2.2).

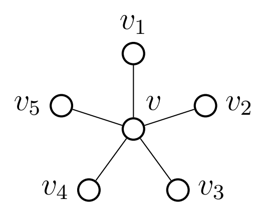


Figure 2.2: The five neighbours of the vertex v

Let $G_{1,3}$ be the subgraph of G induced by all the vertices that are coloured 1 or 3. If v_1 and v_3 are in different components of $G_{1,3}$, say $v_1 \in (D_1)$ and $v_3 \in (D_2)$, then by exchanging the colours 1 and 3 in D_1, we produce a colouring in which the colour 1 is not used at any neighbour of v. So we can colour v with 1 and get a proper 5-colouring of G. Now let us assume that v_1 and v_3 are in the same component of $G_{1,3}$. Then there is a path P from v_1 to v_3 that has only vertices coloured with 1 or 3. Then the cycle C starting at v, proceeding to v_1, then following P from v_1 to v_3, then back to v bounds the region containing the vertex v_2 (Figure 2.3).

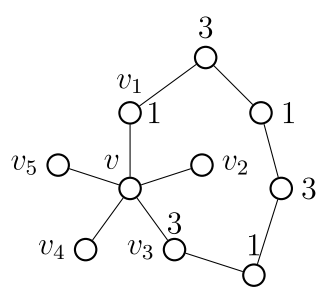


Figure 2.3: The cycle C containing v_2

Therefore there is no path from v_2 to v_4 having only vertices coloured 2 or 4.

Now let $G_{2,4}$ be the subgraph of G induced by the vertices that are coloured 2 or 4. By exchanging the colours of the vertices in the component of $G_{2,4}$ that contains v_2, we get a proper 5-colouring of $G - v$ in which the colour 2 is not used at any neighbour of v. Therefore by colouring v with the colour 2, we get a proper 5-colouring of G. □

2.4 Characterization of planar graphs

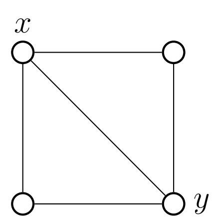
In Theorem 2.14 we proved that K_5 and $K_{3,3}$ are non-planar. We are now going to show that in fact these two graphs play an important role in the characterization of planar graphs. First we look at some definitions.

Definition 2.19. *An elementary subdivision of a non-empty graph G is a graph obtained from G by removing an edge $e = xy$ and adding a new vertex z and new edges xz and yz. A subdivision of a graph G is a graph obtained from G by a sequence of zero or more elementary subdivisions.*

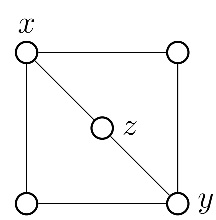
Note that if a graph G contains a subgraph that is a subdivision of a graph H, then we say G contains a subdivision of H, since a subdivision is a subgraph.

Example 2.4

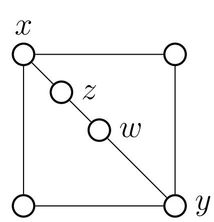
G :



G' :



G'' :



In this example, G' is an elementary subdivision of G and G'' is a subdivision of G. We also note that G is a subdivision of itself.

Lemma 2.20. *Every subdivision of a planar graph is planar and every subdivision of a non-planar graph is non-planar.*

Theorem 2.21. *Every graph that contains a subdivision of K_5 or $K_{3,3}$ is non-planar.*

Proof. This is due to Lemma 2.20. □

We now present a famous theorem that gives the characterization of planar graphs in terms of K_5 and $K_{3,3}$. This theorem is known as Kuratowski's theorem, named after the mathematician Kazimierz Kuratowski.

Theorem 2.22. ***(Kuratowski's Theorem)***
A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Proof. The necessity, ($\Longrightarrow$) is given by Theorem 2.21. To prove the sufficiency, ($\Longleftarrow$), we must first prove some results on blocks. □

Theorem 2.23. *(Chartrand and Lesniak [3]) Let G be a connected graph and $z \in V(G)$. Then z is a cut-vertex of G if and only if there exist vertices x and y ($x, y \neq z$) such that z is on every x-y path of G.*

Proof. ($\Longrightarrow$) Assume that z is cut-vertex of G. Then $G - z$ is disconnected. Suppose that x and y are in different components of $G - z$, then there does not exist an $x - y$ path in $G - z$. Given that G is connected, then there are $x - y$ paths in G and so it follows that every $x - y$ path in G contains z.
($\Longleftarrow$) Now assume that there exist $x, y \in V(G)$ such that z lies on every $x - y$ path in G. Then there does not exist an $x - y$ path in G. Therefore $G - z$ is disconnected and so it follows that z is a cut-vertex. □

Lemma 2.24. *Let $e = xy$ be a bridge of a connected graph G, then there are exactly two components in $G - e$, one containing x and the other containing y.*

Lemma 2.25. *If a graph G contains a bridge incident with a vertex x, then x is a cut-vertex of G if and only if $d_G(x) \geq 2$.*

Theorem 2.26. *If a graph G on $n \geq 3$ vertices contains a bridge, then G contains a cut-vertex.*

Proof. Let $e = xy$ be a bridge in G. Then by Lemma 2.24, the graph $G - e$ has two components, one containing x, which call G_x and one containing y, which we call G_y. Given that G has three or more vertices, at least one of G_x and G_y has more than one vertex. Assume without loss of generality that G_x has more than one vertex. Then $d_{G_x}(x) \geq 1$. Given that x is a neighbour of y in G but y is not in G_x, it follows that $d_G(x) \geq 2$. Therefore by Lemma 2.25, x is a cut-vertex of G. □

Theorem 2.27. *(Chartrand and Lesniak [3]) A graph G on $n \geq 3$ vertices is a block if and only if for every two vertices of G, there is a cycle containing both vertices.*

Proof. ($\Longleftarrow$) Let G be a graph such that each pair of vertices of G is on a common cycle. Suppose to the contrary that G has a cut-vertex, say, z. By Theorem 2.23 there exist vertices x and y such that z is on every $x - y$ path in G. Let C be a cycle of G that x and y lie on. But then, given that $C - z$ is connected, there exists an $x - y$ path in G that does not contain z. This is a contradiction and so it follows that G has no cut-vertex.
($\Longrightarrow$) Now suppose that G contains no cut-vertices. Also assume, to the contrary, that there exists a pair of vertices of G that do not lie on a common cycle of G. Of all such pairs choose x and y such that the distance $d(x, y)$ is minimum. By Theorem 2.4 each edge of G incident with x is on a cycle. So it follows that $d(x, y) \geq 2$. Let P be an $x-y$ geodesic, that is a path of minimum length between x and y. Let y' be the vertex of P that is a neighbour of y. It is clear that $x \neq y'$ since $d(x, y) \geq 2$. Given that $d(x, y') < d(x, y)$, the vertex y' is on a cycle, say, C that contains x. Moreover, given that y' is not a cut-vertex, there exists a $y - x$ path P' that does not contain y'. Let w be the first vertex of P' that lies on C. Let C' be the $w - y$ path in C that contains x. If we follow P' from y to w, then C' from w via x to y', then through the edge between y' and y, we get a cycle that contains y. This is a contradiction and so it follows that each pair of vertices of G lies on a common cycle. □

Theorem 2.28. *(Chartrand and Lesniak [3]) A graph is planar if and only if each of its blocks is planar.*

Proof. It is clear that a graph is planar if and only if each of its components is planar. So we assume that G is connected. It is also clear that if G is planar then each block of G is planar. All that remains is to show that if every block of G is planar, then G is planar. We proceed by induction on the number k of blocks. If $k = 1$, then clearly G is planar. Now suppose $k \geq 2$ and that if G is a graph with at most $k - 1$ blocks, all of which are planar, then G is planar. Let B be an end-block of G with cut-vertex v. Let G' be the graph obtained from G by deleting all vertices of B different from v. Then G' has $k - 1$ planar blocks and so by the inductive hypothesis G' is planar. Since both B and G' are planar, we may embed them in the plane so that in both plane graphs the vertex v lies on the boundary of the exterior region. If we identify the vertex v in B with the vertex v in G', the resulting graph is an embedding of G in the plane. Thus G is planar. □

Definition 2.29. *A block G is called a critical block if for all $v \in V(G)$, $G - v$ is not a block.*

Theorem 2.30. *(Chartrand and Lesniak [3]) Let G be a critical block of order $n \geq 4$, then G contains a vertex of degree 2.*

Proof. G is a critical block and so for each vertex x of G there exists a vertex y of $G - x$ such that $G - x - y$ is disconnected. Amongst all such pairs x, y of G, let u and v be a pair such that $G - u - v$ is disconnected and contains a component G_1 of minimum order k. If G_1 has only one vertex, that is, $k = 1$, then that vertex has degree 2, since it must be adjacent to both u and v. So G_1 has order at least 2. Denote the union of the components of $G - u - v$ different from G_1 as G_2. Let $H = \langle V(G_1) \cup \{u, v\}\rangle$. Also let $z_1 \in V(G_1)$. Since G is a critical block, there exists a vertex z_2 in $G - z_1$ such that $G - z_1 - z_2$ is disconnected. Let us consider two cases:

Case 1: Assume that $z_2 \in V(H)$. Given that both $\langle V(G_2) \cup \{u\}\rangle$ and $\langle V(G_2) \cup \{v\}\rangle$ are connected, some component of $G - z_1 - z_2$ has order less than k, producing a contradiction.

Case 2: Assume that $z_2 \in V(G_2)$.

We claim that $H - z_1$ is disconnected and u and v lie in different components of $H - z_1$. Assume to the contrary that this is not true. Then there exists a path P from u to v in $H - z_1$. Since z_2 is a cut-vertex in $G - z_1$, then $G - z_1 - z_2$ contains a pair of vertices a, b such that every $a - b$ path in $G - z_1$ includes the vertex z_2. Since every path that begins or ends in G_1 and includes z_2 must use at least one u and v, we may assume that $a, b \notin V(G_1)$. G is a block and so there is an

$a - b$ path P' in $G - z_2$ that must z_1, therefore it follows that it must use both u and v. If we use the paths P and P', we get an $a - b$ path in $G - z_1 - z_2$. This is a contradiction and thus the claim is proved.

Now let H_u be the component of $H - z_1$ that contains u and similarly H_v be the component of $H - z_1$ that contains v. Every vertex of $G - z_1$ is in H_u or H_v, and this is due to our choice of the component G_1. If H_u or H_v is trivial, then its only vertex has degree 2. So assume that H_u and H_v are non-trivial. Let us consider one of the components, say, H_u, then $G - u - z_1$ has a component smaller than G_1. This is a contradiction to the way u and v were chosen. □

Definition 2.31. *A block G is called a minimal block if for all $e \in E(G)$ $G - e$, is not a block.*

Definition 2.32. *An edge is called a pendant edge if it is incident with a vertex of degree 1.*

Theorem 2.33. *(Chartrand and Lesniak [3]) Let G be a minimal block of order $n \geq 4$, then G contains a vertex of degree 2.*

Proof. Let us assume that G is a minimal block on $n \geq 4$ vertices and with no vertex of degree 2. Then by Theorem 2.30, G is not a critical block and so G has a vertex z such that $G - z$ is a block. Let us consider an edge e of G incident with z. $G - e$ is not a block because G is a minimal block and so $G - e$ has a cut-vertex u, distinct from z. Therefore $G - e - u$ is disconnected and so e is a bridge of $G - u$. Let us consider $G - u - z$ which is connected. We know that e is a pendant edge of $G - u$ that z is an end-vertex of $G - u$. Therefore z is of degree 1 in $G - u$ and has degree 2 in G. This is a contradiction and hence the result. □

Finally we prove the sufficiency in Kuratowski's theorem. By Theorem 2.28 it suffices to prove the following theorem:

Theorem 2.34. *(Chartrand and Lesniak [3]) A block is planar if it contains no subdivision of K_5 or $K_{3,3}$.*

Proof. Let us assume that this is not true. Of all non-planar blocks containing no subdivision of K_5 or $K_{3,3}$, pick G to be one of minimum size.
Claim: Minimum degree of G is at least three.
Proof: Given that G is a block, it has no vertices of degree one. Let us

assume, to the contrary that there exists a vertex $v \in V(G)$ such that $d_G(v) = 2$. Let u and w be the two vertices adjacent to v. Then we have the following two cases:

Case 1: $uw \in E(G)$

By Theorem 2.27, $G - v$ is a block, and since G has no subdivision of K_5 or $K_{3,3}$, it follows that $G - v$ also has no subdivision of K_5 or $K_{3,3}$. Since $G - v$ is a non-planar block with no subdivision of K_5 or $K_{3,3}$, then it follows that $G - v$ is planar. Since $G - v$ is embeddable on the plane, G is also embeddable on a plane, this is because in any plane embedding of $G - v$, we can add the vertex v and the edges vw and uv so that the graph we obtain is planar. But this is a contradiction.

Case 2: $uw \notin E(G)$

Again by Theorem 2.27, $G' = G - v + uw$ is a block and has smaller size than G. Given that G does not contain a subdivision of K_5 or $K_{3,3}$ and that G is a subdivision of G', the graph G' does not contain a subdivision of K_5 or $K_{3,3}$. Therefore by our choice of G, the graph G' is planar, again a contradiction.

In both cases we obtain a contradiction and so it follows that G has no vertex of degree 2, that is, the minimum degree of G is at least three.

By Theorem 2.33, G is not a minimal block and so it follows that there is an edge $e = uv$ such that $H = G - e$ is also a block. Given that H contains no subdivision of K_5 or $K_{3,3}$ and H has smaller size than G, we see that H is planar. By Theorem 2.27, we know that H has a cycle that contains both u and v. Now consider a plane embedding of H, and let C be a cycle of H that contains u and v and has maximum number of interior regions. Let

$$C : u = v_0, v_1, \ldots, v_i = v, \ldots, v_n = u, \text{ where } 1 < i < n - 1.$$

We make some observations about the plane graph H. But first let us define some special subgraphs of H. Let the *exterior subgraph (interior subgraph)* of H be the subgraph of G induced by the edges in the exterior (interior) of the cycle C.

Observation 2.35. *G is non-planar and so both the interior and exterior subgraphs are non-empty or else, the edge $e = uv$ could be added into H in either the interior or exterior of C so that G is embeddable on a plane.*

Observation 2.36. *No two vertices of the set* $\{v_0, v_1, \ldots, v_i\}$ *or* $\{v_i, v_{i+1}, \ldots, v_n\}$ *are connected by a path in the exterior subgraph of* H *or else this contradicts our choice of* C *having maximum number of interior regions.*

The two observations together with the fact that $H+e$ is non-planar means that there exists a $v_a - v_b$ path P, $0 < a < i < b < n$ in the exterior subgraph of H such that there is no vertex of P different from v_a and v_b that belongs to C (Figure 2.4). Also no vertex of P different from v_a and v_b is a neighbour to any vertex of C other than v_a or v_b. Furthermore, if there is a path that connects a vertex of P with a vertex of C, then that path must contain at least one of v_a and v_b.

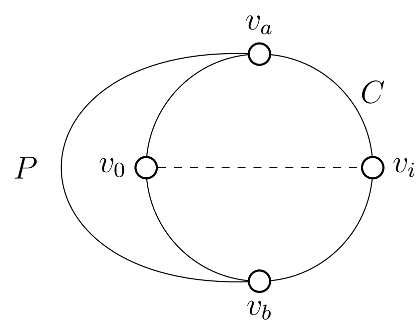


Figure 2.4: The graph showing the path P

Now consider H_1 to be the component of $H - \{v_l | 0 \leq l < n, l \neq a, b\}$ that contains P. H_1 cannot be added in the interior of C in a plane manner, this due to the choice of C. This fact, together with the assumption that G is non-planar, means that the interior subgraph of H must contain one of the following:

1. A $v_x - v_y$ path R, $0 < x < a, i < y < b$ (or equivalently , $a < x < i$ and $b < y < n$), such that no vertices of that R apart from v_x and v_y belong to C. In this case, we observe that G contains a subdivision of $K_{3,3}$ with partite sets $\{v_a, v_y, v_i\}$ and $\{v_b, v_0, v_x\}$ (Figure 2.5), which is a contradiction.

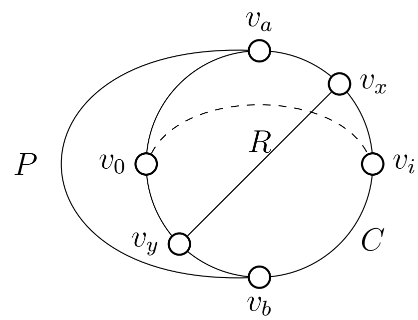


Figure 2.5: The graph for case 1

2. A vertex $z \notin V(C)$ that is connected to C by three internally disjoint paths in such a way that the end-vertex of one such path R_1 is one of v_0, v_a, v_i and v_b. Suppose R_1 ends at v_0, then the end-vertices of the other paths are v_x and v_y, $a \leq x < i$ and $i < y \leq b$ but not both $x = a$ and $y = b$ hold. We obtain three analogous results if R_1 ends at any of v_a, v_i or v_b. Again we observe that G contains a subdivision of $K_{3,3}$ with partite sets $\{z, v_b, v_i\}$ and $\{v_0, v_x, v_y\}$ (Figure 2.6), which is a contradiction.

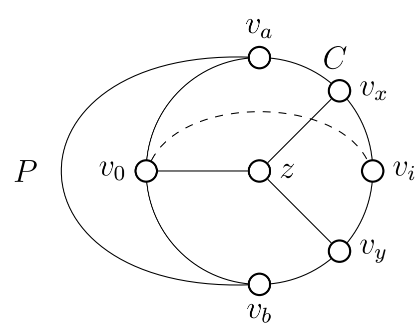


Figure 2.6: The graph for case 2

3. A vertex $z \notin V(C)$ that is connected to C by three internally disjoint paths R_1, R_2, R_3 such that the end-vertices of the paths are three of the four vertices v_0, v_a, v_i, v_b. Suppose the that the three vertices are v_0, v_a and v_i, and R_1 is a path from v_0 to z and R_2 is a path from

v_i to v_z. Then there exists a vertex $s \neq v_0, z, v_i$ on either R_1 or R_2 and a path R_4 from s to v_b. The other choices for R_1, R_2 and R_3 gives us three analogous results. Again we observe that G contains a subdivision of $K_{3,3}$ with sets $\{v_0, s, v_a\}$ and $\{z, v_b, v_1\}$, (Figure 2.7), which is a contradiction.

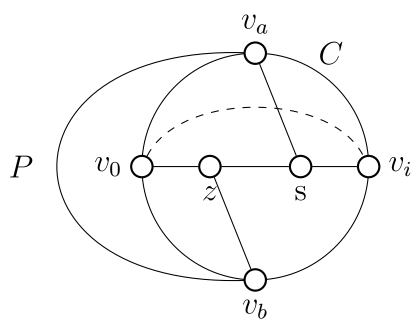


Figure 2.7: The graph for case 3

4. A vertex $z \notin V(C)$ that is connected to the vertices v_0, v_a, v_i, v_b by four internally disjoint paths. In this case observe that G contains a subdivision of K_5 (Figure 2.8), which is a contradiction.

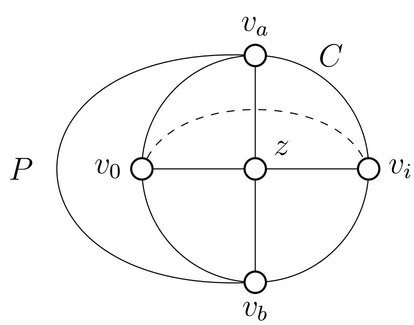


Figure 2.8: The graph for case 4

All these four cases are the only possible ones. Since all of these four cases contradicts our assumption, it follows that no such graph G exists. This concludes the proof of the theorem. □

Chapter 3

General Properties of Maximal Planar Graphs

In this chapter we discuss the general properties, including the connectivity of maximal planar graphs.

3.1 Maximal planar graphs with minimum degree i

In this section we discuss maximal planar graphs with minimum degree i. We denote by S_i the set of vertices of G of degree i. $G(S_i)$ is then the subgraph of G induced by vertices of degree i.

Theorem 3.1. *(Helden [15]) Let G be a maximal planar graph. If the minimum degree of G is 3, then G has K_4 as a subgraph.*

Proof. Let u be a vertex of G with minimum degree 3. Let u_1, u_2 and u_3 be the neighbours of u. By Lemma 2.7, every face of G is a triangle and so it follows that u_1, u_2 and u_3 are mutually adjacent. The resulting graph is K_4 (Figure 3.1).

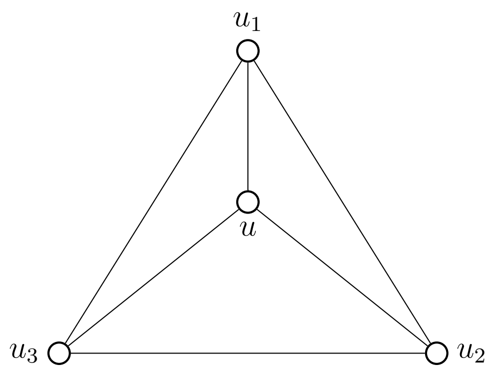


Figure 3.1: The graph K_4.

□

Observation 3.2. *Let G be a maximal planar graph with order at least 5 and minimum degree 3. Then G has one or more separating triangles.*

Theorem 3.3. *(Helden [15]) Let G be a maximal planar graph of order n. If the minimum degree of G is 3, then all components of $G(S_3)$ are either K_4 if $n = 4$ or isolated vertices if $n \geq 5$.*

Proof. Suppose that G has minimum degree 3 and that G contains two adjacent vertices u and v of degree 3. Let u_1 and u_2 be adjacent to u and let v_1 and v_2 be adjacent to v (Figure 3.2).

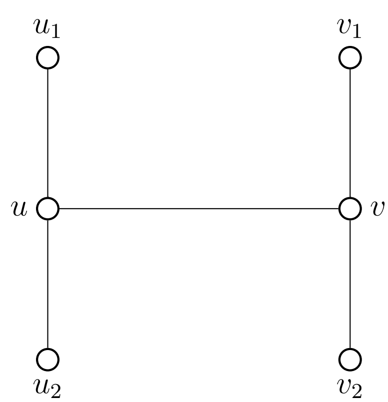


Figure 3.2: The vertices u and v and their neighbours.

Now suppose $u_1 \neq v_1$. Since every face of G is a triangle, then without loss of generality, $uv_1 \in E(G)$ and $uv_2 \in E(G)$ and therefore u has degree greater than 3 (Figure 3.3), a contradiction.

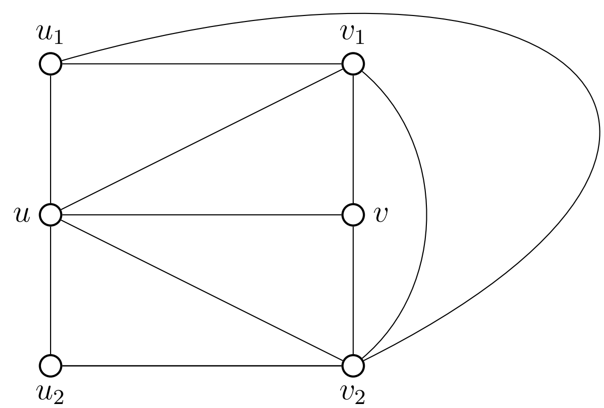


Figure 3.3: Graph showing the triangular faces of G.

Therefore u_1 and v_1 are the same vertex. Let us call this vertex w. Similarly u_2 and v_2 are the same vertex and we call it z. We consider two cases:

Case1: $n = 4$.

Since G is a maximal planar graph, w and z must be neighbours. Therefore the resulting graph is the complete graph on four vertices, that is, $G(S_3) = K_4$ (Figure 3.4).

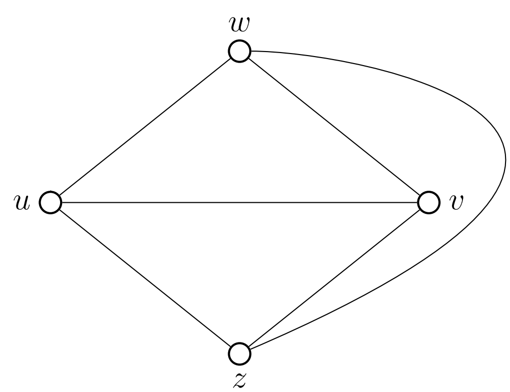


Figure 3.4: The subgraph $G(S_3)$.

Case2: $n \geq 5$.

Since $n \geq 5$ and G a maximal planar, then without loss of generality, there exists a vertex x that is adjacent to u and so this means that the degree of u is greater than 3, a contradiction. Therefore, all components of $G(S_3)$ are isolated vertices. □

Theorem 3.4. *(Helden [15]) Let G be a maximal planar graph. If the minimum degree of G is 4, then there is no vertex u of degree 4 with $d_{G(S_4)}(u) = 3$.*

Proof. Assume, to the contrary, that there exists a vertex u in G with $d_G(u) = 4$ and $d_{G(S_4)}(u) = 3$. Then u has three neighbours, say, u_1, u_2 and u_3 each with degree 4 and one neighbour, say, u_4 with $d_G(u_4) \geq 5$. Since G is a maximal planar graph, every face of G is a triangle. So u_1 is adjacent to u_2 and u_4. Similarly u_3 is adjacent to u_2 and u_4 (Figure 3.5(a)). Since, without loss of generality, $d_G(u_3) = 4$, u_3 has a neighbour, say, u_5 with $d_G(u_5) \geq 4$ (since minimum degree of G is 4) (Figure 3.5(b)). Since G is a maximal planar graph and $d_G(u_5) \geq 4$, u_5 is also adjacent to u_1, u_2 and u_4 (Figure 3.5(c)). Since $d_G(u_4) \geq 5$, there exists a vertex u_6 which is adjacent to u_4. Also u_6 is adjacent to u_1, since G is a maximal planar graph (Figure 3.5(d)). Therefore, we see that $d_G(u_1) = 5$, which is a contradiction. Hence there is no vertex of degree three in the subgraph $G(S_4)$.

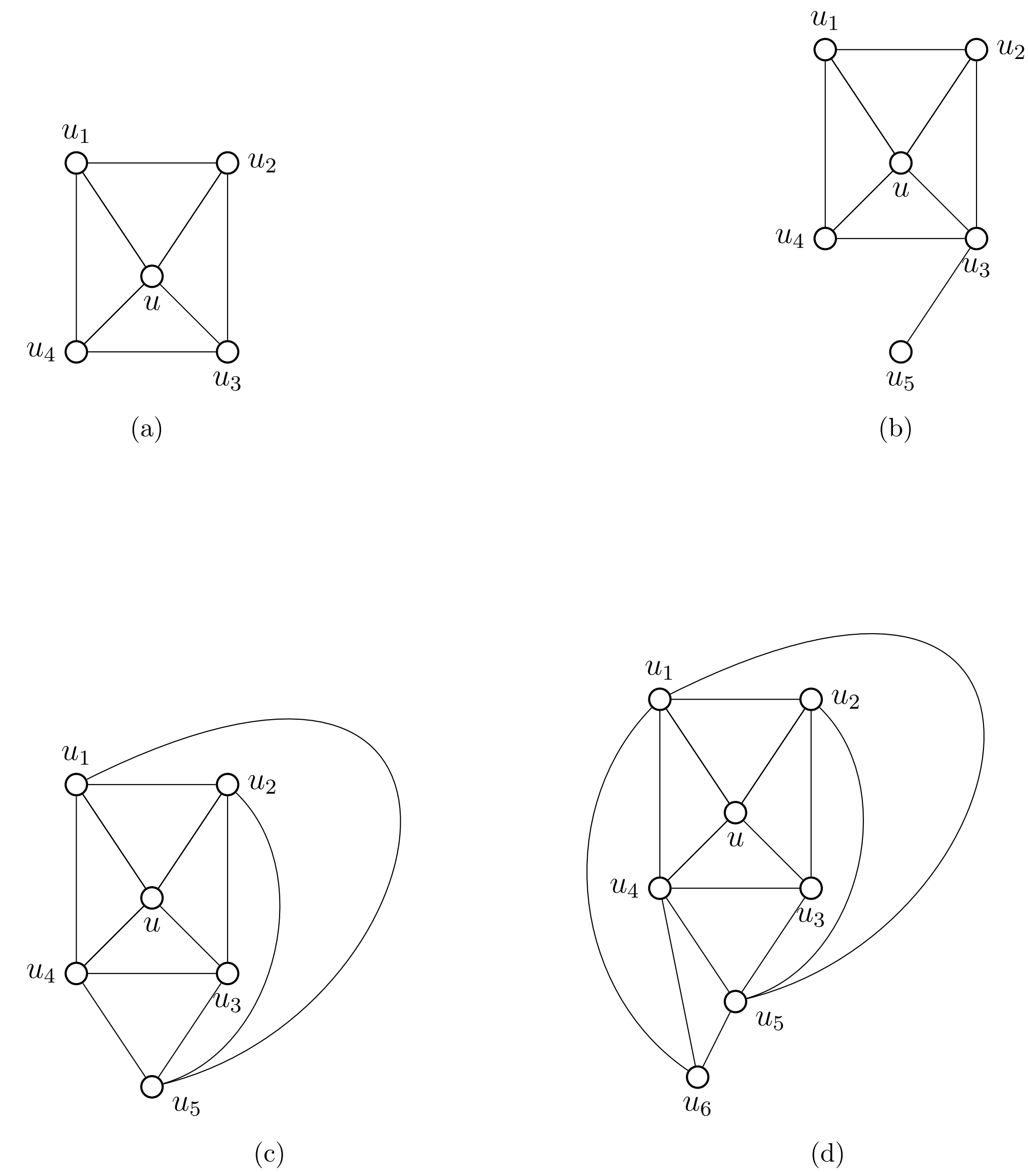


Figure 3.5: Steps in the proof of Theorem 3.4

□

Theorem 3.5. *(Helden [15]) Let G be a maximal planar graph of minimum degree 4. If there exists a vertex u in $G(S_4)$ with $d_{G(S_4)}(u) = 4$, then $\Delta(G(S_4)) = 4$ and $n(G(S_4)) = 6$.*

Proof. Let u be a vertex of G such that $d_{G(S_4)}(u) = 4$. Then u has four neighbours of degree 4 and we call them u_1, u_2, u_3 and u_4. Since G is a maximal planar graph, every face of G is a triangle. So u_1 is adjacent to u_2 and u_4. Similarly u_3 is adjacent to u_2 and u_4 (Figure 3.6). If u_1 is adjacent to u_3, then $d_{G(S_4)}(u_2) = 3$ or $d_{G(S_4)}(u_4) = 3$. Similarly If u_2 is adjacent to u_4, then $d_{G(S_4)}(u_1) = 3$ or $d_{G(S_4)}(u_3) = 3$. But this contradicts Theorem 3.4, which states that there is no vertex of degree 3 in the subgraph $G(S_4)$.

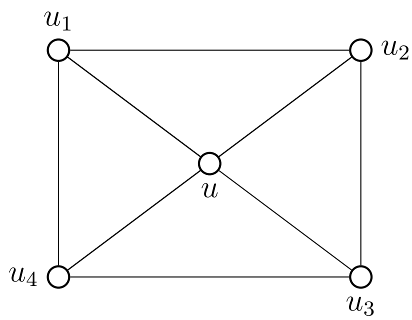


Figure 3.6: A step in the proof of Theorem 3.5

Therefore, it follows that every vertex of $G(S_4)$ must have degree 4. This means that there exists a vertex u_5 with degree 4. Since G is a maximal planar graph, u_5 is adjacent to u_1, u_2, u_3 and u_4. Therefore the resulting graph $G(S_4)$ has order 6, with every vertex having degree four (Figure 3.7). In fact this graph is $C_4 + \overline{K}_2$.

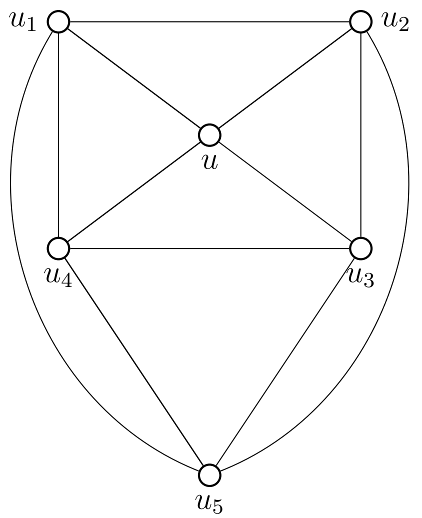


Figure 3.7: The graph $C_4 + \overline{K}_2$.

□

Theorem 3.6. *(Helden [15]) Let G be a maximal planar graph. If the minimum degree of G is 4, then all components of $G(S_4)$ belong to one of the following graphs:*

- $C_4 + \overline{K}_2$.
- *A cycle.*
- *A path.*
- K_1.

Proof. By Theorem 3.5, $G(S_4)$ has maximum degree four. By Theorem 3.4 $G(S_4)$ has no vertex of degree 3. So the vertices in $G(S_4)$ have degree 0,1,2 or 4. Let us consider a vertex u. If $d_{G(S_4)}(u) = 4$, then by Theorem 3.5 we obtain $C_4 + \overline{K}_2$ as a component of $G(S_4)$. If $d_{G(S_4)}(u) = 2$, then a cycle occurs as a component of $G(S_4)$. If $d_{G(S_4)}(u) = 1$, then we obtain a path as a component of $G(S_4)$. Lastly, if $d_{G(S_4)}(u) = 0$, we obtain K_1 as the component of $G(S_4)$. □

We now state two theorems that describe graphs of the form MPG_n-5, that is a maximal planar graph of order n with minimum degree 5.

Theorem 3.7. *(Batagelj[1]) Let G be a maximal planar graph with minimum degree 5 and maximal degree 5, then G is unique up to isomorphism. We call G an icosahedron or MPG_{12}-5, a maximal planar graph on twelve vertices and with minimum degree* 5*.*

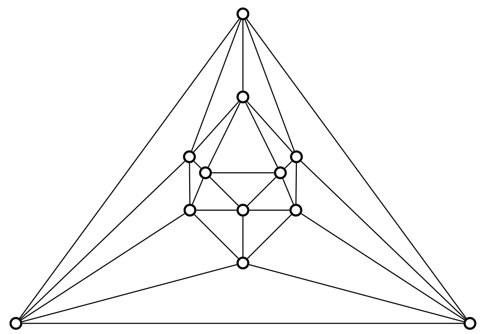


Figure 3.8: The Icosahedron graph or MPG_{12}-5.

Theorem 3.8. *(Rolland [20]) Let G be a maximal planar graph with order* 14 *and minimum degree* 5*. If the maximum degree of G is* 6*, then G is unique up isomorphism. We call G, MPG_{14}-5, a maximal planar graph on fourteen vertices and with minimum degree* 5 *(Figure 3.9).*

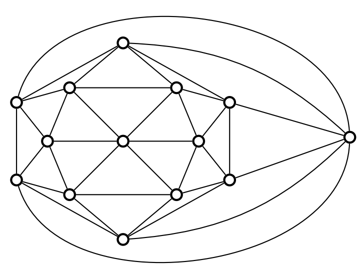


Figure 3.9: The graph MPG_{14}-5.

3.2 Connectivity of maximal planar graphs

In this section we discuss the connectivity of maximal planar graphs. If G is a maximal planar graph of order at most 3, then G is K_1, K_2 or K_3. So we consider only maximal planar graphs with order at least four.

Theorem 3.9. *(Whitney [26]) If G is a maximal planar graph with at least four vertices, then G is 3-connected.*

Proof. No vertex of G is a cut-vertex. Hence G is 2-connected. We claim that G is 3-connected. Let us assume to the contrary that G is not 3-connected. Then there exist $u, v \in V(G)$ such that $G - \{u, v\}$ is disconnected. Let $x, y \in V(G)$. Since G is connected, there is an $x - y$ path P in G. We now prove that there is an $x - y$ path P' in $G - \{u, v\}$. If neither u nor v lie on P, then let $P' = P$. We assume, then, that at least one of u and v lies on P. We consider two cases:

Case 1: $uv \in E(G)$

Suppose first that v does not lie on P. By Lemma 2.8, C_u is the cycle induced by the neighbours of u. Let u' be the first vertex of P that is in C_u and let u'' be the last vertex of P that is in C_u (Figure 3.10). Let Q be a $u' - u''$ path in C_u. Note that the path Q contains the vertex v. Then let P' be the path that follows P from x to u', then follows Q from u' to u'', and then follows P from u'' to y. We see that u and v do not lie on P'. Therefore P' is an $x - y$ path in $G - \{u, v\}$.

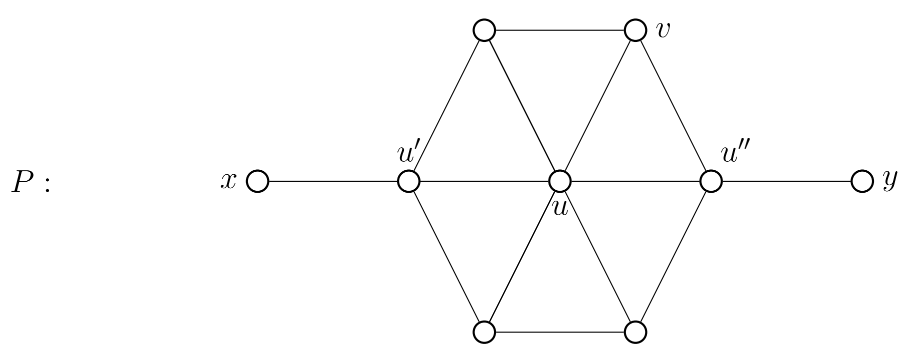


Figure 3.10: When $uv \in E(G)$ and only one of u and v, say, u lies on P.

Now suppose that both u and v lie on P. By Lemma 2.8, C_u is the cycle induced by the neighbours of u. Similarly C_v is the cycle induced by the neighbours of v. Let u' be the first vertex of P that is in C_u and let v' be the last vertex of P that is in C_v (Figure 3.11). Since G is a maximal planar graph, the edge uv lies on the boundary of a triangle, so there is a vertex $z \in C_u \cap C_v$. Let Q' be a $u' - z$ path in C_u and Q'' be a $z - v'$ path in C_v. Let P' be the path that follows the path P from x to u',then follows the path Q' from u' to z,then follows the path Q'' from z to v', and then the path P from v' to y. We see that u and v do not lie on P'. Therefore P' is an $x - y$ path in $G - \{u, v\}$.

P :

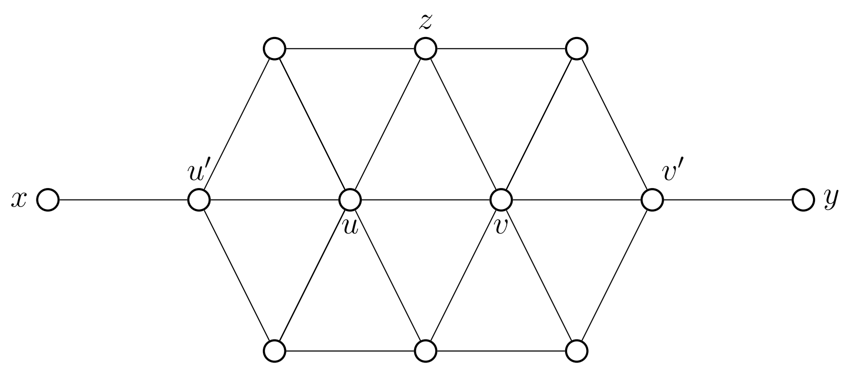


Figure 3.11: When $uv \in E(G)$ and both u and v lie on P.

For this case, P' is an $x - y$ path in $G - \{u, v\}$, that is, $G - \{u, v\}$ is connected, a contradiction. Thus G is 3-connected.

Case 2 : $uv \notin E(G)$

Suppose first that v does not lie on P. By Lemma 2.8, C_u is the cycle induced by the neighbours of u. Let u' be the first vertex of P that is in C_u and let u'' be the last vertex of P that is in C_u (Figure 3.12). Let Q be a $u' - u''$ path in C_u. Then let P' be the path that follows P from x to u', then follows Q from u' to u'',and then follows P from u'' to y. We see that u and v do not lie on P'. Therefore P' is an $x - y$ path in $G - \{u, v\}$.

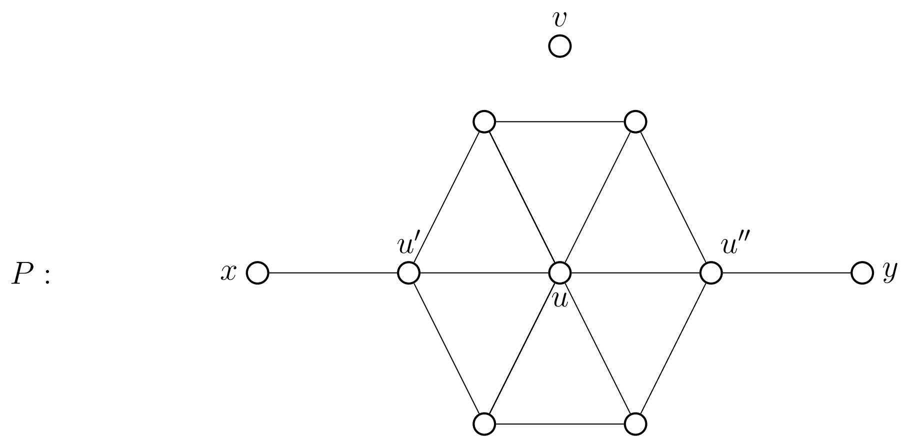


Figure 3.12: When $uv \notin E(G)$ and only one of u and v, say, u lies on P

Suppose that both u and v lie in P. By Lemma 2.8, C_u is the cycle induced by the neighbours of u. Similarly C_v is the cycle induced by the neighbours of v. Let u' be the first vertex of P that is in C_u and let u'' be the last vertex of that is in C_u. Let v' be the first vertex of P that is in C_v and let v'' be the last vertex of that is in C_v (Figure 3.13). Let Q' be a $u' - u''$ path in C_u and Q'' be a $v' - v''$ path in C_v. Let P' be the path that follows P from x to u', then follows the path Q' from u' to u'', then follows the path P from u'' to v', then follows the path Q'' from v' to v'' and then the path P from v'' to y. We see that u and v do not lie on P'. Therefore P' is an $x - y$ path in $G - \{u, v\}$.

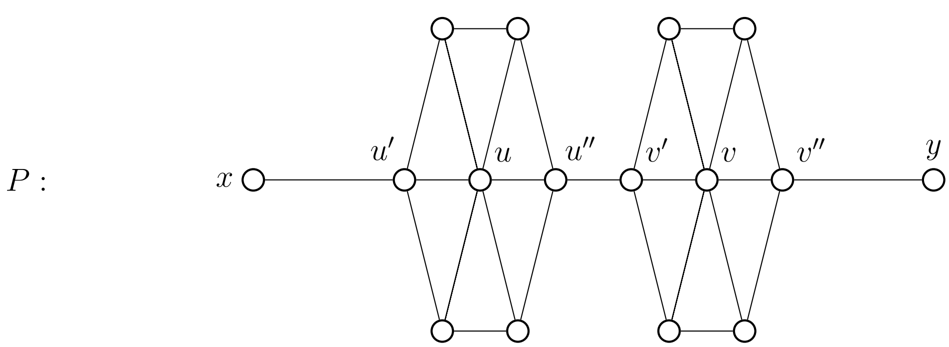


Figure 3.13: When $uv \notin E(G)$ and both u and v lie on P

For this case P' is also an $x - y$ path in $G - \{u, v\}$, that is $G - \{u, v\}$ is connected, a contradiction. Thus G is 3-connected. □

Lemma 3.10. *(Baybars [2]) If $S \subset V(G)$ is a minimal vertex-cut of a maximal planar graph G with $|S| = \kappa(G) = k$, then the subgraph induced by S is a cycle of length k.*

Theorem 3.11. *(Chen [5]) If G is a maximal planar graph with at least five vertices and without a separating triangle, then G is 4-connected.*

Proof. Let S be a minimal vertex-cut of G. Since G is a maximal planar graph on at least four vertices, G is 3-connected (By Lemma 3.9). So $\kappa(G) \geq 3$. If $\kappa(G) = 3$, then by Lemma 3.10, the subgraph of G induced by S is triangle, but we made the assumption that G has no separating triangles. So this is a contradiction. Therefore, $\kappa(G) \geq 4$, that is, G is 4-connected. □

Theorem 3.12. *(Helden [15]) Let G be a maximal planar graph. If the minimum degree of G is five, then G is either 5-connected or G has a separating triangle or separating 4-cycle.*

Proof. Since $\delta(G) = 5$, we have by Whitney's Theorem [26] $\kappa(G) \leq 5$. Since G is a maximal planar graph, $\kappa(G) \geq 3$. So $\kappa(G) \in \{3, 4, 5\}$. If $\kappa(G) = 5$, then G is 5-connected. If $\kappa(G) = 4$, then by Lemma 3.10, G has a separating 4-cycle. Similarly if $\kappa(G) = 3$, then by Lemma 3.10, G has a separating 3-cycle. □

Now we focus on a theorem proved by Hakimi and Schmeichel in 1978. It gives sufficient conditions in terms of vertex degrees for a maximal planar graph to be 4 or 5 connected. We need the following two lemmas in order to prove the results by Hakimi and Schmeichel.

Lemma 3.13. *(Hakimi and Schmeichel [10]) Let G be a maximal planar graph with $d_s \geq 4$. Suppose G has a separating triangle induced by the vertex set $T = \{u_1, u_2, u_3\}$ and let the number of vertices of degree k in the interior of T be denoted by $\lambda_k(T)$.*

i. *If $\lambda_4(T) = 0$, then $\lambda_5(T) \geq 7$.*

ii. *If $\lambda_4(T) = 1$, then $\lambda_5(T) \geq 5$.*

iii. *If $\lambda_4(T) = 2$, then $\lambda_5(T) \geq 3$.*

Proof. Let the set of vertices inside triangle T be denoted by $I(T)$. If the lemma holds for a separating triangle induced by the set of vertices $T' \subseteq T \cup I(T)$, then the lemma holds for T itself. Suppose that there does not exist a separating triangle $T' \neq T$ with $T' \subseteq T \cup I(T)$. Let $d_{I(T)}(u_i)$, $i = 1, 2, 3$ be the number of vertices in $I(T)$ to which u_i is adjacent. Given that G is a maximal planar graph and $I(T) \neq \emptyset$, we have that $d_{I(T)}(u_i) > 0$ for every i. We consider the following cases;

Case 1: If $d_{I(T)}(u_i) = 1$, for some $i = 1, 2, 3$, then minimum degree of G is three. So we consider $d_{I(T)}(u_i) = 1$, for some i, say, $i = 1$. Let $w \in I(T)$ be the neighbour of u_i, for $i = 1, 2, 3$. Since the minimum degree of G is at least four, $d(w) > 3$ and hence the set of vertices $\{w, u_1, u_2\} \subset T \cup I(T)$ induce a separating triangle. This is a contradiction and so it follows that $d_{I(T)}(u_i) \geq 2$ for each i.

Case 2: $d_{I(T)}(u_i) \geq 2$, for each $i = 1, 2, 3$. First assume that $d_{I(T)}(u_i) = 2$ for each i. For this case the situation is shown in Figure 3.14. We observe that $d(w_i) = 4$, for $i = 1, 2, 3$ and hence $\lambda_4(T) = 3$. However, in $((i) - (iii))$ we made the assumption that $\lambda_4(T) \leq 2$ and so we must also assume that $d_{I(T)}(u_i) \geq 3$ for some i. That is,

$$\sum_{i=1}^{3} d_{I(T)}(u_i) \geq 3 + 2 + 2 = 7. \tag{3.1}$$

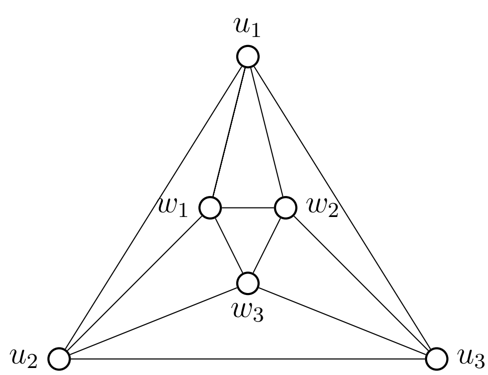


Figure 3.14: An illustration for when $d(u_i) = 2$ for each i.

Note that for a maximal planar graph G on $n \geq 4$. Euler's formula gives us $|E(G)| = 3(n-2)$. Hence,

$$\sum_{i=1}^{n} d_i = 6(n-2). \tag{3.2}$$

Now let the maximal planar graph induced by $T \cup I(T)$ be denoted by H. Using Equations 3.1 and 3.2, we obtain,

$$\begin{aligned}
6(|V(H)|-2) &= \sum_{w \in H} d(w) \\
&= \sum_{i=1}^{3}[d_{I(T)}(u_i)+2] + \sum_{w \in I(T)} d(w) \geq 7+6+\sum_{w \in I(T)} d(w) \\
So, 6(|I(T)|-2) &\geq -5 + \sum_{w \in I(T)} d(w) \\
\Rightarrow \sum_{w \in I(T)} d(w) &\leq 6(|I(T)|-7. \qquad (3.3)
\end{aligned}$$

If $\lambda_4(T) = 0$, then $d(w) \geq 5$ for all $w \in I(T)$. Hence if $\lambda_5(T) < 7$, that is, $\lambda_5(T) \leq 6$, then we have

$$\begin{aligned}
6 \times 5 + (|I(T)|-6)6 &\leq \sum_{w \in I(T)} d(w) \\
\Rightarrow 6|I(T)| - 6 &\leq \sum_{w \in I(T)} d(w)
\end{aligned}$$

So Equation 3.3 does not hold. Therefore, if $\lambda_4(T) = 0$, then $\lambda_5(T) \geq 7$. Figure 3.15 shows the graph for which we have equality in (i).

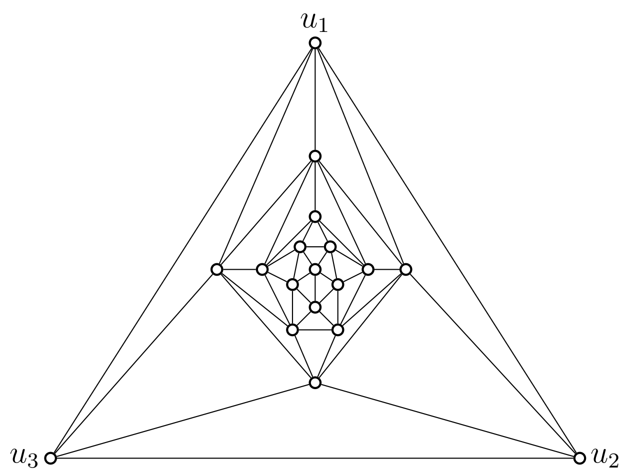


Figure 3.15: An illustration for the equality in (i)

If $\lambda_4(T) = 1$, then $d(w) \geq 4$ for all $w \in I(T)$. Hence if $\lambda_5(T) < 5$, that is,$\lambda_5(T) \leq 4$, then we have

$$\begin{aligned} 1 \times 4 + 4 \times 5 + (|I(T)| - 1 - 4)6 &\leq \sum_{w \in I(T)} d(w) \\ \Rightarrow 6|I(T)| - 6 &\leq \sum_{w \in I(T)} d(w) \end{aligned}$$

So Equation 3.3 does not hold. Therefore, if $\lambda_4(T) = 1$, then $\lambda_5(T) \geq 5$. Figure 3.16 shows the graph for which we have equality in (ii).

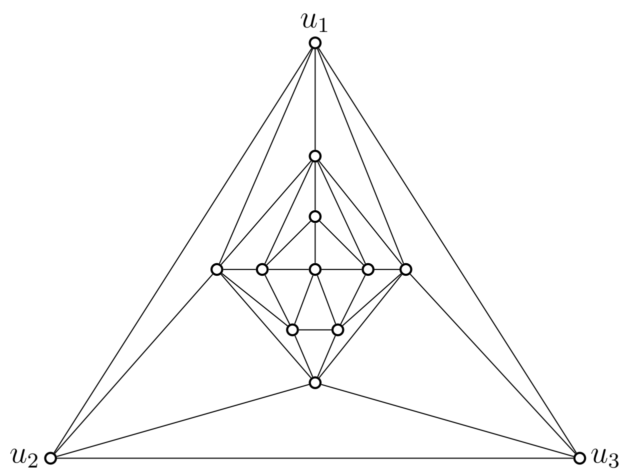


Figure 3.16: An illustration for the equality in (ii).

If $\lambda_4(T) = 2$, then $d(w) \geq 4$ for all $w \in I(T)$. Hence if $\lambda_5(T) < 3$, that is,$\lambda_5(T) \leq 2$, the we have

$$\begin{aligned} 2 \times 4 + 2 \times 5 + (|I(T)| - 2 - 2)6 &\leq \sum_{w \in I(T)} d(w) \\ \Rightarrow 6|I(T)| - 6 &\leq \sum_{w \in I(T)} d(w) \end{aligned}$$

So Equation 3.3 does not hold. Therefore, if $\lambda_4(T) = 2$, then $\lambda_5(T) \geq 3$. Figure 3.17 .

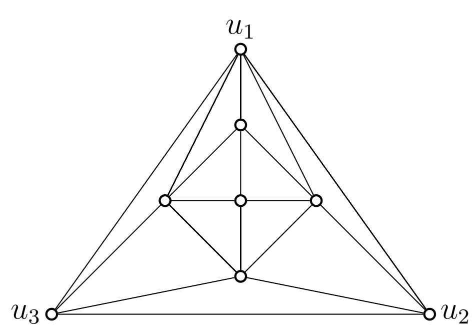


Figure 3.17: An illustration for the equality in (iii).

□

Lemma 3.14. *(Hakimi and Schmeichel [10]) Let G be a maximal planar graph with minimum degree 5. If G contains a separating 4-cycle induced by the set of vertices $T = \{u_1, u_2, u_3, u_4\}$. Then in any planar embedding of G, the interior and exterior of T must each contain at least seven vertices of degree 5.*

Proof. Let the set of vertices inside of the 4-cycle T be denoted by $I(T)$. Let $d_{I(T)}(u_i)$, $i = 1, 2, 3, 4$ denote the vertices in $I(T)$ to which u_i is adjacent. Given that G is a maximal planar graph and $I(T) \neq \emptyset$, we have that $d_{I(T)}(u_i) > 0$ for every i. By the proof of Lemma 3.13, it follows that $d_{I(G)}(u_i) \geq 2$, for each i and $d_{I(T)}(u_i) \geq 3$ for some i. That is,

$$\sum_{i=1}^{4} d_{I(T)}(u_i) \geq 3 + 2 + 2 + 2 = 9. \tag{3.4}$$

Let H denote the planar graph induced by the set of vertices $T \cup I(T)$. Since adding one more edge (any edge) to H results in a maximal planar graph, it follows from Equations 3.1 and 3.4 that

$$\begin{aligned}
6(|V(H)| - 2) - 2 &= \sum_{w \in H} d(w) \\
&= \sum_{i=1}^{4}[d(u_i) + 2] + \sum_{w \in I(T)} d(w) \geq 9 + 8 + \sum_{w \in I(T)} d(w) \\
So, 6(|I(T)| - 2) &\geq 17 + \sum_{w \in I(T)} d(w) \\
\Rightarrow \sum_{w \in I(T)} d(w) &\leq 6(|I(T)| - 7. \qquad (3.5)
\end{aligned}$$

First let us consider the interior of T. If $\lambda_5(T) < 7$, that is, $\lambda_5(T) \leq 6$, then we have

$$\begin{aligned}
6 \times 5 + (|I(T)| - 6)6 &\leq \sum_{w \in I(T)} d(w) \\
\Rightarrow 6|I(T)| - 6 &\leq \sum_{w \in I(T)} d(w)
\end{aligned}$$

So Equation 3.5 does not hold. Hence $\lambda_5(T) \geq 7$. The inequality is shown in Figure 3.18. Since T is the vertex set of an arbitrary separating 4-cycle, the result also hold for the exterior of T.

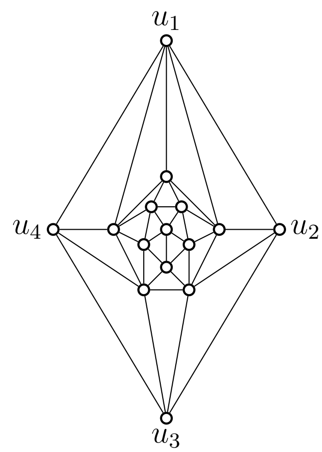


Figure 3.18: An illustration for the equality in Lemma 3.14

□

Now we prove the main results by Hakimi and Schmeichel.

Theorem 3.15. *(Hakimi and Schmeichel [10]) Let G be a maximal planar graphs with n vertices and vertex degrees $d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_n$ and with $d_n \geq 4$. If*

$$\frac{7}{3}\lambda_4(T) + \lambda_5(T) < 14, \tag{3.6}$$

then G is d_n-connected.

Proof. Let us assume that Equation 3.6 holds. If $d_n = 5$, then it implies that $\lambda_4(T) = 0$ and so from Equation 3.6, we have that

$$\Rightarrow \lambda_5(T) < 14,$$

That is G has at most thirteen vertices of degree 5. Suppose that G is not 5-connected, then G has a separating triangle or 4-cycle with at most six vertices of degree 5 in, say, its interior. But this is a contradiction to both Lemma 3.13(i) and Lemma 3.14. Therefore if Equation 3.6 holds and $\lambda_4(T) = 0$, then G is 5-connected. Now suppose that $d_n = 4$. Then $1 \leq \lambda_4(T) \leq 5$. Let us consider all the cases:

Case 1: $\lambda_4(T) = 1$. Then Equation 3.6 gives us that $\lambda_5(T) < \frac{35}{3}$, that is G contains at most eleven vertices of degree 5.

Case 2: $\lambda_4(T) = 2$. Then Equation 3.6 gives us that $\lambda_5(T) < \frac{28}{3}$, that is G contains at most nine vertices of degree 5.

Case 3: $\lambda_4(T) = 3$. Then Equation 3.6 gives us that $\lambda_5(T) < 7$, that is G contains at most six vertices of degree 5.

Case 4: $\lambda_4(T) = 4$. Then Equation 3.6 gives us that $\lambda_5(T) < \frac{14}{3}$, that is G contains at most four vertices of degree 5.

Case 5: $\lambda_4(T) = 5$. Then Equation 3.6 gives us that $\lambda_5(T) < \frac{7}{3}$, that is G contains at most two vertices of degree 5.

Now if G is not 4-connected then G has a separating 3-cycle. Without loss of generality let us assume that there is a vertex of degree 4 on the interior of the 3-cycle. Then by Lemma 3.13, there must be at least five vertices of degree 5 in the interior of the 3-cycle and and at least seven vertices of degree 5 in the exterior of the 3-cycle. But this contradicts cases $(1 - 5)$. Therefore if Equation 3.6 holds and $1 \leq \lambda_4(T) \leq 5$, then G is 4-connected. This concludes the proof of the theorem. □

Corollary 3.16. *(Hakimi and Schmeichel [10]) Let G be a maximal planar graph with vertex degrees* $d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_n$ *and with* $d_1 - d_n \leq 1$*. Then G is* d_n*-connected.*

Proof. Suppose $d_1 - d_n = 0$, the G has one of the following vertex degrees;

1. Four vertices of degree 3. We have $d_1 = 3 = d_n$. The unique graph with these vertex degrees is the 1-skeletons of the tetrahedron (Figure 3.19). By Theorem 3.9, the graph is d_n-connected.

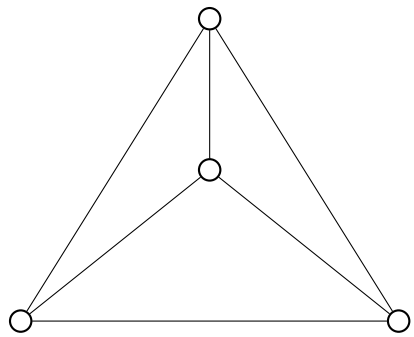


Figure 3.19: 1-skeletons of the tetrahedron

2. Six vertices of degree 4. We have $d_1 = 4 = d_n$. The unique graph with this vertex degrees is the octahedron (Figure 3.20). By Theorem 3.11, the graph is d_n-connected.

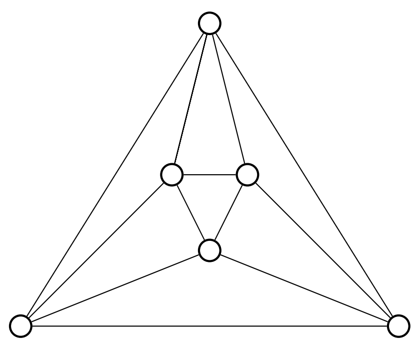


Figure 3.20: The octahedron graph.

3. Twelve vertices of degree 5. We have $d_1 = 5 = d_n$. The unique graph with this vertex degrees is the icosahedron (Figure 3.21). By Theorem 3.12, the graph is d_n-connected.

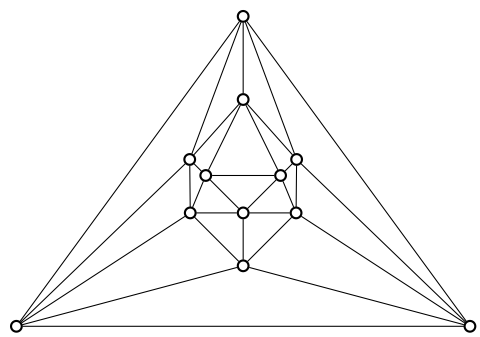


Figure 3.21: The icosahedron graph

Now suppose that $d_1 - d_n = 1$. We consider three cases;

Case 1: $d_n = 3$. The order of G is at least 4, so by Theorem 3.9 G is d_n-connected.

Case 2: $d_n = 4$. Then Euler's formula give us that

$$\sum_{i=1}^{n} d_i = 4\lambda_4(T) + 5\lambda_5(T) = 6(\lambda_4(T) + \lambda_5(T) - 2)$$

$$\Rightarrow \lambda_5(T) + 2\lambda_4(T) = 12. \tag{3.7}$$

Given that $\lambda_5(T) \geq 1$ (because $d_1 = 5$), then by Equation 3.7, $\lambda_4(T) \leq 5$. Clearly Equation 3.6 holds for $\lambda_5(T) \geq 1$ and $\lambda_4(T) \leq 5$. Therefore G is d_s-connected.

Case 3: $d_n = 5$. Then Euler's formula give us that

$$\sum_{i=1}^{n} d_i = 5\lambda_5(T) + 6\lambda_6(T) = 6(\lambda_5(T) + \lambda_6(T) - 2)$$

$$\Rightarrow \lambda_5(T) = 12. \tag{3.8}$$

$d_n = 5 \Rightarrow \lambda_4(T) = 0$. Clearly Equation 3.6 holds for $\lambda_4(T) = 0$ and $\lambda_5(T) = 12$. Therefore G is d_n-connected. The proof is complete. □

Chapter 4

Hamiltonicity Properties Of Maximal Planar Graphs

4.1 Hamiltonicity of maximal planar graphs with k separating triangles

It is well known that the problem of establishing whether a graph is Hamiltonian is NP-complete. In 1976 Garey, Johnson and Tarjan [9] proved that the problem of establishing whether a 3-connected planar graph is Hamiltonian is NP-complete. In 1985, Chvátal [6] showed that the problem of establishing whether a maximal planar graph is Hamiltonian is NP-complete. In 1931, Whitney [25] proved that any maximal planar graphs with no separating triangles is Hamiltonian. In this section we discuss the Hamiltonicity of maximal planar graphs with k separating triangles, for $k = 1, 2, 3$. We will begin our discussions by formally stating the result by Whitney. But before we state this result we need the following two definitions and lemma.

Definition 4.1. *An edge that joins two non-consecutive vertices of a cycle is called a chord.*

Definition 4.2. *Let G be a planar triangulation, let F be the exterior face of G, and let x and y be the two vertices on F. We say that (G, F, x, y) satisfies Whitney's Condition (Condition W for short) if (G, F, x, y) satisfies Conditions $(W1)$ and $(W2)$ described below;*

1. *We say that (G, F, x, y) satisfies Condition $(W1)$ if G has no separating 3-cycles.*

2. *We say that* (G,F,x,y) *satisfies Condition* $(W2)$ *if* (G,F,x,y) *satisfies Condition* $W(2a)$ *F is divided into two paths,* $x_0x_1\ldots x_m$ *is the path from* x *to* y *and* $y_0y_1\ldots y_n$ *is the path from* y *to* x $(x_0 = y_n = x, y_0 = x_m = y)$ *and there is no chord of the form* x_px_q *or* y_py_q *(Figure 4.1 (a)), or*

   Condition $(W2b)$ *F is divided into three paths,* $x_0x_1\ldots x_m$ *is the path from* x *to* y*,* $y_0y_1\ldots y_n$ *is the path from* y *to* z *and* $z_0z_1\ldots z_l$ *is the path from* z *to* x *for some vertex* z *on* F $(x_0 = z_l = x, y_0 = x_m = y, z_0 = y_n = z)$ *and there is no chord of the form* x_px_q*,* y_py_q *or* z_pz_q *(Figure 4.1 (b))*

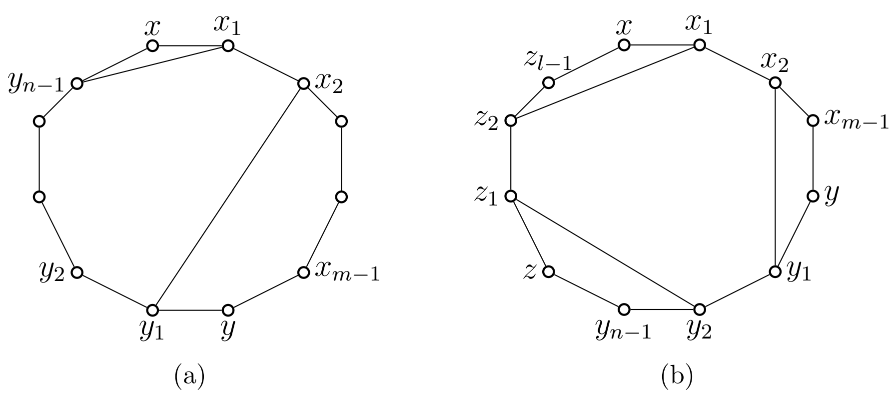


Figure 4.1: Graphs illustrating Conditions $(W2a)$ and $(W2b)$

Lemma 4.3. *(Whitney [25]) Let G be a planar triangulation with the exterior face F. Let x and y be two vertices of F. If (G,F,x,y) satisfies Condition (W), then G has a Hamiltonian path from x and y.*

We now state Whitney's theorem.

Theorem 4.4. *(Whitney [25]) Consider a maximal planar graph G with no separating triangles. Then G is Hamiltonian.*

Now we are ready to discuss the Hamiltonicity of maximal planar graphs with k separating triangles, for $k = 1, 2, 3$. In 2003 Chen [5] extended Whitney's Theorem [25] by proving that any maximal planar graphs with one separating triangle is Hamiltonian. To prove this theorem by Chen we discuss some results and definition first.

Lemma 4.5. *(Chen [5]) Consider a maximal planar graph G without separating triangles and with the exterior faces F. Let x and y be the two vertices of F. Then (G, F, x, y) satisfies Condition (W).*

Proof. Given that G contains no separating triangles, (G, F, x, y) satisfies Condition $(W1)$. Given that F is a triangle, it has no chords, and therefore (G, F, x, y) satisfies Condition $(W2)$. So it follows that G satisfies Condition (W). □

The following lemma is a variation of Lemma 4.3 by Whitney.

Lemma 4.6. *(Chen [5]) Consider a maximal planar graph G without a separating triangle and with the exterior face F. Let x, y and z be the three vertices of F. Then G has a Hamiltonian path from x to y passing through the edge xz.*

Proof. Obviously the lemma is true if the order of G is three. Now assume that G has order at least four. Let $a_0, a_1, \ldots, a_s$ $(a_0 = y, a_s = z)$ be the sequence of vertices that are neighbours of x such that each xa_i is the immediate clockwise edge of xa_{i-1} around x (Figure 4.2) Note that G has an edge of the form a_ia_{i+1} for all i, where $0 \leq i \leq s - 1$.

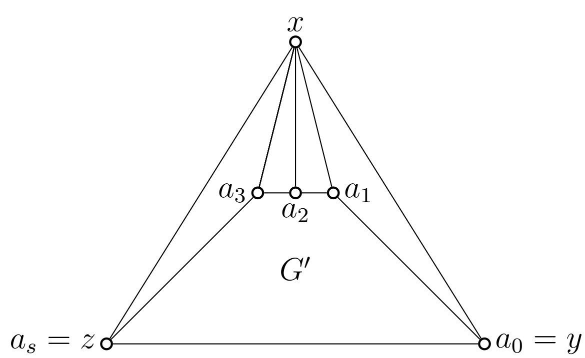


Figure 4.2: Graph illustrating the proof of Lemma 4.6

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Given that G contains no separating triangles we observe that the following three properties hold:

a. There is no edge of the form a_ia_j $(0 \leq i < i + 2 \leq j \leq s)$ in G, or else there would exist a separating triangle of the form a_ia_jx.

b. There is no edge of the form ya_i $(2 \leq i \leq s-1)$ in G, or else there would exist a separating triangle of the form ya_ix.

c. There is no edge of the form za_i $(1 \leq i \leq s-2)$ in G, or else there would exist a separating triangle of the form za_ix.

Let us consider the subgraph G' obtained by deleting x from G. Then G is a triangulation. Let $F' = y, a_1, \ldots, a_{s-1}, z$ be the exterior face of G' (Figure 4.2). Given that G has no separating triangle, it follows that G' also has no separating triangles. Therefore (G', F', y, z) satisfies Condition $(W1)$. The properties $a - c$ mean that F' has no chord and so (G', F', y, z) satisfies Condition $(W2)$. Therefore (G', F', y, z) satisfies Condition (W). By Lemma 4.3 there is a Hamiltonian path P from z to y in G. This path P together with the edge xz form a Hamiltonian path from x to y passing through the edge xz. □

Definition 4.7. *Consider a maximal planar graph G. We define a boundary edge of G as an edge that lies on the exterior face of G. We say that G is Hamiltonian for any two boundary edges if for any two boundary edges, G has a Hamiltonian cycle passing through them. Similarly we say that G is non-Hamiltonian for any two boundary edges if for any two boundary edges, G does not have a Hamiltonian cycle passing through them.*

Theorem 4.8. *(Chen [5]) Consider a maximal planar graph G with no separating triangles. Then G is Hamiltonian for any two boundary edges.*

Proof. Consider the exterior face F of G. Let x, y and z be the three vertices on F. By Lemma 4.6, G has a Hamiltonian path P from x and y passing through the edge xz. The path P together with the edge yx form a Hamiltonian cycle in G that passes through the edges xy and xz (Figure 4.2). By similar arguments, there is Hamiltonian cycle in G that passes through the edges yx and zy and another Hamiltonian cycle that passes through the edges zy and yx. Therefore we conclude that G is Hamiltonian for any two boundary edges. □

We are now ready to prove the main result By Chen.

Theorem 4.9. *(Chen [5]) Consider a maximal planar graph G with only one separating triangle. Then G is Hamiltonian.*

Proof. Let T be the unique separating triangle of G with $x, y, z \in V(T)$. Let G_{in} be the subgraph of G obtained by deleting all the vertices outside T. Similarly, let G_{out} be the subgraph obtained by deleting all the vertices inside T. Then G_{in} and G_{out} are maximal planar graphs without separating triangles. First let us consider G_{in}. The vertices x, y and z form the exterior face of G_{in}. By Theorem 4.8, there is a Hamiltonian cycle $C_{in} = y, x, z, P_{in}(z, y), y$ in G_{in} that passes through the edges yx and xz and with $P_{in}(z, y)$ being the subpath of C_{in} between z and y. Now let us consider G_{out}. The vertices x, y and z form the interior face, say F', of G_{out}. Since a planar graph can be embedded in the plane so that a given face of a plane graph becomes the exterior face, we can embed G_{out} in the plane such that F' becomes the exterior face of G_{out}. By Theorem 4.8, there is a Hamiltonian cycle $C_{out} = x, z, y, P_{out}(y, x), x$ in G_{in} that passes through the edges xz and xy and with $P_{out}(z, y)$ being the subpath of C_{out} between y and x. Then we observe that $C_H = x, z, P_{in}(z, y), y, P_{out}(y, x), x$, is a Hamiltonian cycle in G. □

We now extend Whitney's Theorem [25] to two separating triangles by first discussing the result by L. Li, R. Li and S. Li [18] that shows that any maximal planar graph with two separating triangles has a Hamiltonian path and then we discuss the result by Helden [14] that proves that any maximal planar graph with two separating triangles is Hamiltonian.

Lemma 4.10. *(Li, Li and Li [18]) Consider a maximal planar graph G of order at least four and without separating triangles. Let F be the exterior face of G and x, y and z be the vertices on F, then G has a path which contains all the vertices except x,y and z.*

Proof. Let $a_0 = z, a_1, a_2, \ldots, a_{p-1}, a_p = x(b_0 = y, b_1, b_2, \ldots, b_{q-1}, b_q = z; c_0 = x, c_1, c_2, \ldots, c_{r-1}, c_r = y$,respectively) be the sequence of vertices adjacent to $y(x; z$, respectively) so that every $ya_i(xb_j; zc_k$, respectively) is the immediate clock-wise edge of $ya_{i-1}(xb_{j-1}; zc_{k-1}$, respectively) around y,($x; z$,respectively) (Figure 4.3). Note that G has edges of the form $a_ia_{i+1}(b_jb_{j+1}; c_kc_{k+1}$, respectively) where $0 \le i \le p-1$ ($0 \le j \le q-1$;$0 \le k \le r-1$, respectively).

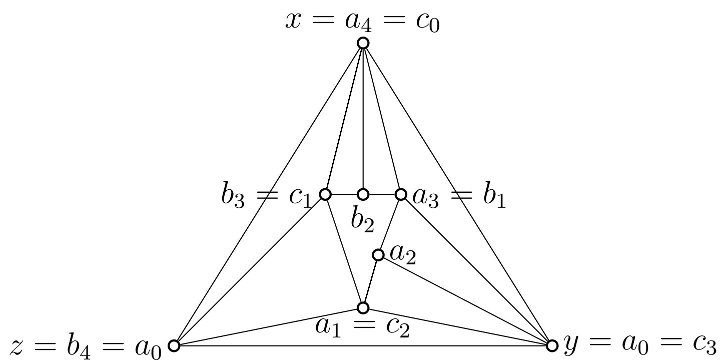

Figure 4.3: Graph illustrating the proof of Lemma 4.10

Given that G contains no separating triangles we observe that the following three properties hold:

a. There is no edge of the form a_ia_j $(0 \leq i \leq i+2 \leq j \leq p)$ in G, or else there would exists a separating triangle of the form a_ia_jy.

b. There is no edge of the form b_ib_j $(0 \leq i+2 \leq j \leq q)$ in G, or else there would exists a separating triangle of the form b_ib_jx.

c. There is no edge of the form c_ic_j $(0 \leq i+2 \leq j \leq r)$ in G, or else there would exists a separating triangle of the form c_ic_jz.

Let us consider the subgraph G' obtained by deleting x, y and z from G. Then G is a triangulation. Let F' be the exterior face of G'). Given that G has no separating cycle, it follows that G' also has no separating triangles. Therefore (G', F', y, z) satisfies Condition $(W1)$. Note that $a_1 = c_{k-1}$, or else a_1z and c_{k-1} are edges in G, contradicting planarity. Similarly $a_{p-1} = b_1$ and $b_{q-1} = c_1$. The properties $a - c$ mean that F' has no chord and so (G', F', a_1, b_1) satisfies Condition $(W2)$. Therefore (G', F', a_1, b_1) satisfies Condition (W). By lemma 4.3 there is a Hamiltonian path P from a_1 to b_1 in G. □

We now present the result by L. Li, R. Li and S. Li.

Theorem 4.11. *(Li, Li and Li [18]) Consider a maximal planar graph G with two separating triangles. Then G has a Hamiltonian path.*

Proof. Consider a maximal planar graph G with two separating triangles T_1 and T_2, where $x, y, z \in V(T_1)$ and $x', y', z' \in (T_2)$. Let G_{out} be the subgraph of G obtained by deleting all the vertices inside T_2. Then G_{out} is a maximal planar graph with only one separating triangle. By Theorem 4.9, G_{out} has Hamiltonian cycle $C_{out} = x', y', \ldots, x'$. Now let G_{in} be the subgraph of G obtained by deleting all the vertices outside T_2. Then G_{in} is a maximal planar graph without a separating triangle. By Lemma 4.10 G_{in} has a path P, which contains all the vertices of G_{in} not including x', y' and z'. Note that all the end-vertices of P belong to the neighbours of x', y' and z'. So we can join C_{out} and P together and obtain a Hamiltonian path in G. □

To prove the main result by Chen discuss some important results first.

Definition 4.12. *Consider a graph G. Let P be a subgraph of a graph G. A P-bridge of G is either a single edge of $G - E(P)$ with both ends on P (which is trivial), or a component of $G - V(P)$ together with the edges joining the component to P (and all incident vertices). For any P-bridge B of G, the set of attachments of B on P is $V(B) \cap V(P)$. We say that P is a Tutte subgraph of G, if every P-bridge of G has at most three attachments on P. Given a subgraph C of G, we say that P is a C-Tutte subgraph of G, if P is a Tutte subgraph of G, and every P-bridge of G containing an edge of C has at most two attachments. A Tutte path (or Tutte cycle) is a path (or cycle), which is a Tutte subgraph.*

Theorem 4.13. *(Sanders [22]) Let G be a 2-connected plane graph, then exterior cycle is the cycle that bounds the exterior face of G and is denoted as X_G. Let β be an edge of X_G, and let a and b be arbitrary distinct vertices of G. Then G has a Tutte path P from a to b containing β.*

Lemma 4.14. *(Helden [14]) Consider a maximal planar graph G with only one separating triangle. Then G is Hamiltonian for any two boundary edges.*

Proof. Let T be the separating triangle of G with $a, b, c \in V(T)$ and such that ab is not a boundary edge. Let G_{in} be the subgraph of G obtained by deleting all the vertices outside T. Then G_{in} is a maximal planar graph without separating triangles. By Theorem 4.8, G_{in} is Hamiltonian for any two boundary edges. Now let G_{out} be a subgraph of G obtained by deleting all the vertices inside T. Let x, y and z be the vertices that form the exterior face of G_{out} and let x be different from a and b. Moreover, let G'_{out} be the subgraph of G_{out} obtain by deleting x. By Theorem 4.13, G'_{out} has a Tutte path from a to b that contains the edge yz. Then it follows that G_{out} has a Tutte path from a to b and contains the edges yx and xz. It is well known that a Tutte path in a 4-connected planar graph is also a Hamiltonian path (Since in a 4-connected planar graph G a Tutte path contains every vertex of G). So since G_{out} is a maximal planar graph without separating triangles, it is 4-connected. Hence G_{out} has a Hamiltonian path. Given that G_{in} is Hamiltonian for any two boundary edges, there is a path $P_{in}(a, b)$ in the cycle between a and b. Therefore we have that $x, z, y, \ldots, a, P_{in}(a, b), b, \ldots, x$ is a Hamiltonian cycle of G passing through two boundary edges. Now since x is an arbitrary vertex of G, it follows therefore that G is Hamiltonian for any two boundary edges. □

Definition 4.15. *A nested triangle graph with n vertices is a planar graph formed from a sequence of $n/3$ triangles, by connecting pairs of corresponding vertices on consecutive triangles in the sequence.*

We now present the main result by Helden.

Theorem 4.16. *(Helden [14]) If G is a maximal planar graph with exactly two separating triangles, then G is Hamiltonian.*

Proof. Let x, y and z be the vertices that form one separating triangle T of G. Let G_{in} be the subgraph obtained by deleting all the vertices outside T. Similarly, let G_{out} be the subgraph obtained by deleting all the vertices inside T. Note that both G_{in} and G_{out} are separating triangles. We consider the two cases where both G_{in} and G_{out} are either nested or not.

Case 1: G_{in} and G_{out} are not nested. G_{in} and G_{out} could be disjoint or not and so in both of this cases G_{in} is a maximal planar graph without separating triangles and G_{out} is a maximal planar graph with only one separating triangle. Then by Theorem 4.8, G_{in} is Hamiltonian for any two boundary edges. Since a planar graph can be embedded in the plane so that a given face of the graph becomes the exterior face, we can embed G_{out} in the plane so that T becomes the exterior face of G_{out}. Then by Theorem 4.14, G_{out} is Hamiltonian for any two boundary edges. Therefore we observe that $C_H = x, z, P_{in}(z, y), y, P_{out}(y, x), x$ is Hamiltonian cycle in G. Note that there is a common edge of P_{in} and P_{out} because both graphs are Hamiltonian for any two boundary edges.

Case 2: G_{in} and G_{out} are nested. Again G_{in} and G_{out} could be disjoint or not and so in both cases we have the following,

Case 2.1 G_{in} is a maximal planar graph without separating triangles and G_{out} is a maximal planar graph with only one separating triangle.

Case 2.2 G_{in} is a maximal planar graph with only one separating triangle and G_{out} is a maximal planar graph without separating triangles. In both of these subcases we have that that G_{in} is Hamiltonian any two boundary edges and G_{out} is Hamiltonian for any two boundary edges.

Therefore, we observe that $C_H = x, z, P_{in}(z, y), y, P_{out}(y, x), x$ is a Hamiltonian cycle in G. □

In 2006 Helden and Vieten [16] extended Theorem 4.16 to three separating triangles and showed that G is still Hamiltonian. Before we discuss this theorem we first look at some definitions and results needed for the theorem.

Definition 4.17. *Consider a maximal planar graph G with k separating triangles, say, $T_1, T_2, \ldots, T_k$. Let $G_{in,T_1}(G_{in,T_2}, \ldots, G_{in,T_k}$, respectively) be the subgraph of G obtained by deleting all the vertices outside the separating triangle $T_1, (T_2, \ldots, T_k$, respectively). Similarly, let $G_{out,T_1}(G_{out,T_2}, \ldots, G_{out,T_k}$, respectively) be the subgraph of G obtained by deleting all the vertices inside the separating triangle $T_1, (T_2, \ldots, T_k$, respectively).*

Note that these subgraphs $G_{out,T_1}, G_{out,T_2}, \ldots, G_{out,T_k}$ and $G_{in,T_1}, G_{in,T_2}, \ldots, G_{in,T_k}$ are actually maximal planar graphs with fewer separating triangles than G. We now discuss a lemma that tells us how to get a Hamiltonian cycle in G from two known Hamiltonians cycles in $G_{out,T_1}, G_{out,T_2}, \ldots, G_{out,T_k}$ and $G_{in,T_1}, G_{in,T_2}, \ldots, G_{in,T_k}$.

Definition 4.18. *A facial cycle is a cycle that bounds a face in a graph.*

Lemma 4.19. *(Helden and Vieten [16]) Consider a maximal planar graph G with at the minimum one separating triangle T. Suppose $G_{in,T}(G_{out,T}$, respectively) is Hamiltonian for any two boundary edges of $G_{in,T}(G_{out,T}$, respectively), then G is Hamiltonian.*

Proof. Let $x, y, z \in V(T)$. First we consider $G_{in,T}$. The vertices x, y and z form the exterior cycle of $G_{in,T}$. Without loss of generality, $G_{in,T}$ contains a Hamiltonian cycle $C_{in} = x, y, z, P_{in}(z, x), x$ with $P_{in}(z, x)$ being a Hamiltonian path of $G_{in,T} - y$ between z and x. Now consider $G_{out,T}$. The vertices

x, y and z are the form the facial cycle of $G_{out,T}$. Since a planar graph can be embedded in the plane so that a given face of the graph becomes the exterior face, we can embed $G_{out,T}$ in the plane so that this facial cycle of $G_{out,T}$ becomes the exterior face of $G_{out,T}$. Without loss of generality, $G_{out,T}$ contains a Hamiltonian cycle $C_{out} = y, z, a, P_{out}(x, y), y$ with $P_{out}(x, y)$ being a Hamiltonian path of $G_{out,T} - z$ between x and y. So it follows that, $C_H = x, P_{out}(x, y)y, z, P_{in}(z, x), x$ is a Hamiltonian cycle in G. □

In 2002 Jackson and Yu [17] showed that there is a Hamilton cycle through edges in specified triangles in a plane triangulation which has no separating triangles. To prove this Theorem by Jackson and Yu, we first need the following definition and Lemma.

Definition 4.20. *An ordered pair (G, C_G) consisting of a connected plane graph and a facial cycle C_G of G in such a way that for each 2-cut S of G, each component of $G - S$ contains a vertex of C_G is called a circuit graph.*

Lemma 4.21. *(Jackson and Yu [17]) If (G, C_G) is a circuit graph with $x, y \in V(G)$ and $e \in E(G)$, then G has a C_G-Tutte Cycle C_T through e, x and y.*

Theorem 4.22. *(Jackson and Yu [17]) Consider a planar triangulation G without a separating triangle and with three three distinct triangles T, T_1 and T_2. Let $x, y, z \in V(T)$. Then there is a Hamiltonian cycle C_G in G and edges $e_1 \in E(T_1)$ and $e_2 \in E(T_2)$ in such a way that xy, xz, e_1, e_2 are distinct and contained in $E(C_G)$.*

Proof. Let $G' = G - x$ and let R be the face of G', which contained x. We assume that R is the infinite face of G'. Let $H_k = T_k \cap G'$ for $1 \leq k \leq 2$. Then we have that $H_k = T_k$ or H_k is an edge incident with R. Consider the graph G'' derived from G' in the following way; suppose $H_k \in E(G')$, then we subdivide H_k with a new vertex a_i then join a_i to each vertex of T_k. Let R' to be the outercycle of G''. Then (G'', R') is a circuit graph, since for each 2-cut S of G, each component of $G - S$ contains a vertex of R'. By Lemma 4.21, G'' has an R'-Tutte cycle C'_T passing through yz, a_1 and a_2. Since the only possible separating triangles are T_2 and T_3 and that C'_T is an R'-Tutte cycle in G'', we have that C_T is a Hamiltonian cycle in G''. Let C_H be the cycle derived from $C_T - \{yz, a_1, a_2\}$ by adding xy, xz and the edges e_i between the neighbours of a_i on C_T for $1 \leq i \leq 2$. Then we observe that C_H is the desired Hamiltonian cycle of G. □

The following two corollaries follow from Theorem 4.22

Corollary 4.23. *(Helden and Vieten [16]) Consider a maximal planar graph G without a separating triangle and with two distinct triangles T' and T''. Let $x, y, z \in V(X_G)$. Then there is a Hamiltonian cycle C_H in G and edges $e_1 \in E(T')$ and $e_2 \in E(T'')$ in such a way that xy, xz, e_1, e_2 are distinct and contained in $E(C_G)$.*

Corollary 4.24. *(Helden and Vieten [16]) Consider a maximal planar graph G with no separating triangles and with two distinct, arbitrary and facial triangles T' and T''. Then there are edges $e_1 \in E(T')$ and $e_2 \in E(T'')$ such that G is Hamiltonian for any two boundary edges and the Hamiltonian cycles contains the edges e_1 and e_2.*

Definition 4.25. *Let T be a rooted tree. Suppose that a vertex v of T adjacent to u lies below v, we say u is a child of v.*

The following Corollary by Helden and Vieten [16] is a stronger version of Theorem 4.8.

Corollary 4.26. *(Helden and Vieten [16]) Consider a maximal planar graph G with no separating triangles and with an arbitrary facial triangle T. Then there is an edge $e \in E(T)$ such that G is Hamiltonian for any two boundary edges of G and the Hamiltonian cycle contains e.*

Let G be a maximal planar graph. Suppose that T is a separating triangle in G. Then we may separate G into two graphs $G_{in,T}$ and $G_{out,T}$. Then $G_{in,T}$ and $G_{out,T}$ are maximal planar graphs and T is a facial cycle of $G_{in,T}$ and $G_{out,T}$. We shall refer to T as a marker triangle in $G_{in,T}$ and $G_{out,T}$. We now iterate this procedure for both $G_{in,T}$ and $G_{out,T}$. We continue until we obtain a collection S of maximal planar graphs each of which has no separating triangles. We shall refer to S as pieces of G. Note that each separating triangle will occur as a marker triangle in exactly two pieces of G. We define a new graph B whose vertices are the pieces in S, and in which two pieces are joined by an edge if they have a marker triangle in common. It follows from the decomposition theory developed by W.H Cunningham and J. Edmonds [27] that B is a tree and also that the set S and the tree B are uniquely defined by G. We shall refer to B as the decomposition tree of G. For a given embedding of G with k separating triangles $T_1, T_2, \ldots, T_k$ we define the piece $G_{out,T_1,T_2,\ldots,T_k}$ as the root of the decomposition tree B. Thus we get a rooted decomposition tree B.

Lemma 4.27. *(Helden and Vieten [16]) Consider a maximal planar graph G with at the minimum one separating triangle T_1. Let B be the rooted decomposition tree of the given embedding of G. Let every vertex of the the tree B have not more than two children and the root $G_{out,T_1,T_2,\ldots,T_k}$ has exactly one child G_{in,T_1}. Suppose G_{in,T_1} is Hamiltonian for any two boundary edges of G_{in,T_1}, then G is Hamiltonian for any two boundary edges of G for the given plane graph of G.*

Proof. Let $x, y, z \in V(T_1)$. First we consider G_{in,T_1} and let x, y and z be the vertices of the exterior cycle of G_{in,T_1}. Without loss of generality, there is a Hamiltonian cycle $C_{in} = x, y, z, P_{in}(z, x), x$ in G_{in,T_1} with $P_{in}(z, x)$ being a Hamiltonian path of $G_{in,T_1} - y$ between z and x. Now consider G_{out,T_1}. G_{out,T_1} is a maximal planar graph with no separating triangles. The vertices x, y and z form the facial cycle of G_{out,T_1}. By Corollary 4.26, G_{out,T_1} is Hamiltonian for any two boundary edges of G_{out,T_1} and has without loss of generality the edge xz. If we substitute $P_{in}(z, x)$ in the place of the edge xz, then we have that G is Hamiltonian for any two boundary edges of G for the given plane graph of G. □

Lemma 4.28. *(Helden and Vieten [16]) Consider a maximal planar graph G with at the minimum two separating triangles T_1 and T_2. Let B be the rooted decomposition tree of the given embedding of G. Let every vertex of the tree B have not more than two children and the root $G_{out,T_1,T_2,\ldots,T_k}$ has exactly two children G_{in,T_1} and G_{in,T_2}. Suppose G_{in,T_1} is Hamiltonian for any two boundary edges of G_{in,T_1} and G_{in,T_2} is Hamiltonian for any two boundary edges of G_{in,T_2}, then G is Hamiltonian for any two boundary edges of G for the given plane graph of G.*

Proof. Let $x', y', z' \in V(T_1)$. First we consider $G_{in,T'}$ and let x', y' and z' be the vertices of the exterior cycle of G_{in,T_1}. Without loss of generality, there is a Hamiltonian cycle $C_{in} = x', y', z', P_{in}(z', x'), x'$ in G_{in,T_1} with $P_{in}(z', x')$ being a Hamiltonian path of $G_{in,T_1} - y'$ between z' and x'. Let $x'', y'', z'' \in V(T_2)$. First we consider G_{in,T_2} and let x'', y'' and z'' be the vertices of the exterior cycle of G_{in,T_2}. Without loss of generality, there is a Hamiltonian cycle $C_{in} = x'', y'', z'', P_{in}(z'', x''), x''$ in $G_{in,T''}$ with $P_{in}(z'', x'')$ being a Hamiltonian path of $G_{in,T_2} - y''$ between z'' and x''. Now consider G_{out,T_1,T_2}. G_{out,T_1,T_2} is a maximal planar graph with no separating triangles. The vertices x', y', z' and x'', y'', z'' form the facial cycle of G_{out,T_1,T_2}. By Corollary 4.26, G_{out,T_1,T_2} is Hamiltonian for any two boundary edges of G_{out,T_1,T_2} and has without loss of generality the edges $x'z'$ and $x''z''$. If we substitute $P_{in}(z', x')$ in the place of the edge $x'z'$ and $P_{in}(z'', x'')$ in the place of the edge $x''z''$, then we have that G is Hamiltonian for any two boundary edges of G for the given plane graph of G. □

Theorem 4.29. *(Helden and Vieten [16]) Consider a maximal planar G with only two separating triangles. Then G is Hamiltonian for any two boundary edges of G.*

Proof. Consider the two separating triangles of G to be T_1 and T_2. Let B be the rooted decomposition tree of an arbitrary embedding of G. Every vertex of the tree B has no more than two children and the root G_{out,T_1,T_2} has at the minimum on child. We consider two cases.

Case 1 The root G_{out,T_1,T_2} has only one child G_{in,T_1}. Then we have that the piece G_{in,T_2} is a child of the piece G_{in,T_1}. G_{in,T_1} has only one separating triangle T_2. By Theorem 4.14, G_{in,T_1} is Hamiltonian for any two boundary edges of G_{in,T_1}. By Lemma 4.27, G is Hamiltonian for any two boundary edges of G.

Case 2 The root G_{out,T_1,T_2} has only two children G_{in,T_1} and G_{in,T_2}. G_{in,T_1} has no separating triangles and similarly G_{in,T_2} has no separating triangles. By Theorem 4.8, G_{in,T_1} is Hamiltonian for any two boundary edges of G_{in,T_1} and similarly G_{in,T_2} is Hamiltonian for any two boundary edges of G_{in,T_2}. Then by Lemma 4.28, G is Hamiltonian for any two boundary edges of G. □

Theorem 4.30. *(Helden and Vieten [16]) Consider a maximal planar graph G and let B be the rooted decomposition tree of the given embedding of G. Suppose every vertex of the tree B has no more than two children, then G is Hamiltonian for any two boundary edges of G for the given plane graph of G.*

Proof. We proceed by induction on the number k of separating triangles. If $k = 0, 1, 2$, then by Theorem 4.8, Theorem 4.14 and Theorem 4.29 this theorem is true. Now assume that G contains $k + 1$ separating triangles. We have that the root $G_{out,T_1,T_2,\ldots,T_{k+1}}$ has at the minimum one child. We consider two cases.

Case 1 $G_{out,T_1,T_2,...,T_{k+1}}$ has only one child G_{in,T_1}. G_{in,T_1} has k separating triangles. Given that the rooted decomposition tree B' of G_{in,T_1} is a subtree of B, it follows that every vertex of B' has no more than two children. Then by induction, G_{in,T_1} is Hamiltonian for any two boundary edges of G_{in,T_1}. Therefore, by Lemma 4.27, G is Hamiltonian for any two boundary edges of G for the given plane graph of G.

Case 2 $G_{out,T_1,T_2,...,T_{k+1}}$ has only two children G_{in,T_1} and G_{in,T_2}. G_{in,T_1} has p separating triangles and G_{in,T_2} has q separating triangles. Given that $p+q=k-1$, it must be that $p \leq k$ and $q \leq k$. Given that the rooted decomposition tree B' of G_{in,T_1} and the rooted decomposition tree B'' of G_{in,T_2} are subtrees of B, every vertex of both of this subtrees has no more than two children. Then by induction, G_{in,T_1} is Hamiltonian for any two boundary edges of G_{in,T_1} and G_{in,T_2} is Hamiltonian for any two boundary edges of G_{in,T_2}. Therefore, by Lemma 4.28, G is Hamiltonian for any two boundary edges of G for the given plane graph of G. □

We now present the main results by Helden and Vieten [16]

Theorem 4.31. *(Helden and Vieten [16] Let G be a maximal planar graph with only three separating triangles. Then G is Hamiltonian.*

Proof. Consider the the three separating triangles of G to be T_1, T_2 and T_3. Let B be the rooted decomposition tree of an arbitrary embedding of G. Every vertex of B has no more than three children and the root G_{out,T_1,T_2,T_3} has at the minimum one child. We consider three cases.

Case 1 G_{out,T_1,T_2,T_3} has only one child G_{in,T_1}. Then, G_{in,T_1} has two separating triangles T_2 and T_3. By Theorem 4.29, G_{in,T_1} is Hamiltonian for any two boundary edges of G_{in,T_1}. G_{out,T_1} has no separating triangles. Then by Theorem 4.8, G_{out,T_1} is Hamiltonian for any two boundary edges of G_{out,T_1}. Therefore, by Lemma 4.19, G is Hamiltonian.

Case 2 G_{out,T_1,T_2,T_3} has only two children G_{in,T_1} and G_{in,T_1}. Without loss of generality, G_{in,T_1} contains only separating triangle T_3. Then G_{out,T_1} also contains only one separating triangle T_2. Therefore, by Theorem 4.14, G_{in,T_1} is Hamiltonian for any two boundary edges of G_{in,T_1} and similarly G_{out,T_1} is Hamiltonian for any two boundary edges of G_{out,T_1}. Therefore, by Lemma 4.19, G is Hamiltonian.

Case 3 G_{out,T_1,T_2,T_3} has only three children G_{in,T_1}, G_{in,T_2} and G_{in,T_3}. G_{in,T_1} contains no separating triangles and G_{out,T_1} contains two separating triangles T_2 and T_3. Then by Theorem 4.8, G_{in,T_1} is Hamiltonian for any two boundary edges of G_{in,T_1} and by Theorem 4.29, G_{out,T_1} is Hamiltonian for any two boundary edges of G_{out,T_1}. Therefore, by Lemma 4.19, G is Hamiltonian. □

Note if $k \geq 4$, then there is a counter-example which shows that G is not immediately Hamiltonian.

4.2 Cycles of length p in maximal planar graphs

In this section we discuss the number of cycles of length p that a maximal planar graph of order n could have, in terms of n.

4.2.1 Short cycles in maximal planar graphs

We regard short cycles as cycles with length 3 and 4. Let G be a maximal planar graph of order n, and let $C_p(G)$ denote the number of cycles of length p in G. We discuss the tight bounds for $C_3(G)$ and $C_4(G)$ in terms of n.

Theorem 4.32. *(Hakimi and Schmeichel [11]) If G is a maximal planar graph of order $n \geq 6$, then $2n-4 \leq C_3(G) \leq 3n-8$.*

Note that we obtain the lower bound if and only if G is 4-connected, and we obtain an upper bound if and only if G is obtained from K_3 by recursively putting a vertex of degree 3 inside a face and then add this vertex to three vertices that are incident to that face. Furthermore, for each integer $s \neq 3n-9$ such that $2n-4 \leq s \leq 3n-8$, there is a maximal planar graph with n vertices and with $C_3(G) = s$. Note that there is no maximal planar graph with n vertices and with $C_3(G) = 3n-9$.

Proof. A maximal planar graph of order n has size $3n-6$ and $2n-4$ faces. Given that the boundary of every face of G is a triangle and that all such triangles are distinct if $G \neq K_3$, then $C_3(G) \geq 2n-4$ when $n \geq 6$. By Theorem 3.11, there is a 4-connected maximal planar graph on $n \geq 6$ vertices. Given that a maximal planar graph G is 4-connected precisely if it has no separating triangles, that is, the only triangles are those that bounds the faces, then $C_3(G) = 2n - 4$ if and only if G is 4-connected. If $n = 6$, then upper bound is true. So we proceed by induction on n. Let us suppose that G is not 4-connected, then by embedding G in the plane, G will have a triangle T with some of the vertices of G lying in both the interior and exterior of T. As a consequence, we have that $G = G_1 \cup G_2$, where G_i is a maximal planar graph of order n_i, for $i = 1, 2$ and both G_1 and G_2 have exactly T in common. By the induction hypothesis, if $n_i \geq 6$, then, $C_3(G) \leq 3n_i - 8$ for $i = 1, 2$. Actually, if $n_i = 4$ or 5, then $C_3(G) = 3n_i - 8$. Therefore we have that

$$\begin{aligned} C_3(G) &= C_3(G_1) + C_3(G_2) - 1 \\ &\leq 3(n_1 + n_2) - 17 \\ &= 3(n + 3) - 17 \\ &= 3n - 8, \end{aligned}$$

as required. If $C_3(G) = 3n - 8$, it follows that $C_3(G) = 3n_i - 8$ as well, for $i = 1, 2$ and so by induction, G_1 and G_2 have the special structure defined in Theorem 4.32. Then it follows that G also has this special structure. Furthermore, $C_3(G) \neq 3n - 9$, or else $C_3(G) = 3n_i - 9$ for $i = 1, 2$, which cannot be because of the induction hypothesis. Now all we need to do is to show that there is a maximal planar graph of order n with $C_3(G) = s$ for any integer s such that $2n - 4 \leq s \leq 3n - 10$. We proceed by constructing a 4-connected maximal planar graph G' of order $n' = 3n - s - 4 \geq 6$. G' has exactly $(2n' - 4)$ triangles. Then, we recursively add $n - n'$ vertices to G' as described in the statement of the theorem, and by doing so we create three new triangles when we add each new vertex. We call the maximal planar graph obtained from this process G. Then we have that G has exactly $(2n' - 4) + 3(n - n') = s$ triangles. This concludes the proof. □

Theorem 4.33. *(Hakimi and Schmeichel [11]) If G is a maximal planar graph of order $n \geq 5$, then $3n - 6 \leq C_4(G) \leq \frac{1}{2}(n^2 + 3n - 22)$.*

Note that we obtain the lower bound if and only if $n = 5$ or G is 5-connected and we obtain the upper bound if G is the graph in Figure 4.5.

Proof. If we remove any edge of G, then we form a face that is bounded by a 4-cycle. G would have at least $(3n - 6)$ 4-cycles if all the resulting faces were distinct. If $G = K_4$, then the removal of any two non-incident edges result in different faces being bounded by the same 4-cycle. If $n = 5$, then $C_4(G) = 3n - 6$ and so if $n \geq 5$, we have that $C_4(G) \geq 3n - 6$. By Havel and Schmeichel [10], there is a 5-connected maximal planar graph of order $n \geq 12, n \neq 13$. But a maximal planar graph is 5-connected if and only if there exists no 4-cycles apart from the ones described above. As a consequences, $C_4(G) = 3n - 6$ if and only if $n = 5$ or G is 5-connected. Suppose $5 < n < 12$ or $n = 13$, then there exists no maximal planar graph such that $C_4(G) = 3n - 6$, because every one of those graphs has a vertex of degree at most 4. Now to determine the upper bound $C_4(G) \leq \frac{1}{2}(n^2+3n-22)$. We must first prove the following lemma.

Lemma 4.34. *(Hakimi and Schmeichel [11]) If G is a 4-connected maximal planar graph of order $n \geq 7$, then $C_4(G) \leq \frac{1}{2}(n^2 - n - 2)$.*

Proof. First there is only one 4-connected maximal planar graph G on $n = 7$ vertices with $C_4(G) = 20$. So we proceed by induction on n. We assume that G is not 5-connected, or else $C_4 = 3n - 6 < \frac{1}{2}(n^2 - n - 2)$ if $n \geq 7$. For this reason, if we embed G in the plane, G will contain a 4-cycle C having vertices in both its interior and exterior. (We call such 4-cycle separating). Thus we can write $G = G_1 \cup G_2$, where G_i will be a planar graph on n_i vertices for $i = 1, 2$, and G_1 and G_2 will have exactly the 4-cycle C in common. Assume first that $n_i \geq 7$ for $i = 1, 2$. Then we can add one of the two possible edges, say e_1, to G_i to form a maximal planar graph G_i' which is 4-connected. Then by the induction hypothesis $C_4(G_i') \leq \frac{1}{2}(n_i^2 - n_i - 2)$, for $i = 1, 2$. There will be exactly four non-separating 4-cycles in G containing vertices on both sides of C, while the number of separating 4-cycles containing vertices on both sides of C will be at most $(n_1 - 4)(n_2 - 4)$, given that $n \geq 7$ and G is 4-connected. In addition, the edge e_i appears in at least four cycles of G_i' and so we have

$$\begin{aligned} C_4(G_i) &\leq \frac{1}{2}(n_i^2 - n_i - 2) - 4 \\ &= \frac{1}{2}(n_i^2 - n_i - 10) \end{aligned}$$

considering both these observations, we obtain,

$$\begin{aligned}
C_4(G) &\le C_4(G_1) + C_4(G_2) - 1 + 4 + (n_1-4)(n_2-4)\\
&\le \frac{1}{2}(n_1^2 - n_1 - 10) + \frac{1}{2}(n_2^2 - n_2 - 10) - 1 + 4 + (n_1-4)(n_2-4)\\
&= \frac{1}{2}[(n_1+n_2-4)^2 - (n_1+n_2-4) - 2]\\
&= \frac{1}{2}(n^2 - n - 2),
\end{aligned}$$

as required. If $n_i = 5$, then we obtain $C_4(G_i) = 5$. If $n_i = 6$ and G_i' is 4-connected, then we obtain $C_4(G_i) = 10$. Therefore, $C_4(G) \le \frac{1}{2}(n^2 - n - 2)$ for these cases as well. This concludes the proof of the lemma. □

Note that the equality in Lemma 4.34 holds if and only if G is the graph represented in Figure(4.4).

Now we complete the proof of Theorem 4.33 by showing that $C_4(G) \le \frac{1}{2}(n^2 + 3n - 22)$ with equality holding precisely when G is the graph in Figure 4.5. By Lemma 4.34, we may assume that G is not 4-connected, and hence we can choose a triangle T, G_1 and G_2 as in the proof of Theorem 4.32, but with the additional assumption that n_1 is minimum. Assume first that $n_1 \ge 7$. Then G_1 is 4-connected by the minimality of n_1, and it follows therefore by Lemma 4.34 that $C_4(G_1) \le \frac{1}{2}(n_1^2 - n_1 - 2)$. Additionally by the induction hypothesis, $C_4(G) \le \frac{1}{2}(n_2^2 + 3n_2 - 22)$. Finally we observe that there is at most one vertex in G_2 adjacent to every vertex in T, and G_1 has no separating triangles by the minimality of n_1. As a consequences the number of 4-cycles in G containing vertices on both sides of C is at most $n_2 - 1$. Merging these observations together, we obtain

$$\begin{aligned}
C_4(G) &\le C_4(G_1) + C_4(G_2) + (n_2 - 1)\\
&\le \frac{1}{2}(n_1^2 - n_1 - 2) + \frac{1}{2}(n_2^2 + 3n_2 - 22) + (n_2 - 1)\\
&< \frac{1}{2}[(n_1+n_2-3)^2 + 3(n_1+n_2-3) - 22]\\
&= \frac{1}{2}(n^2 + 3n - 22),
\end{aligned}$$

as required, if given $n_2 \geq n_1 > 6$. Suppose $n_1 = 6$, then G_1 is 4-connected by the minimality of n_1, and therefore G_1 is the graph represented in Figure (4.4). So we have that $C_4(G_1) = 15$ and therefore, we obtain

$$\begin{aligned}
C_4(G) &\leq C_4(G_1) + C_4(G_2) + (n_2 - 1) \\
&\leq 15 + \frac{1}{2}(n_2^2 + 3n_2 - 22) + (n_2 - 1) \\
&< \frac{1}{2}[(n_2 + 3)^2 + 3(n_2 + 3) - 22] \\
&= \frac{1}{2}(n^2 + 3n - 22),
\end{aligned}$$

as required, given that $n_2 \geq n_1 = 6$. Lastly, suppose $n_1 \leq 5$, then we have that $p_1 = 4$, (or else G_1 has a separating triangle), and hence $C_4(G_1) = 3$. As a consequences, we have that

$$\begin{aligned}
C_4(G) &\leq C_4(G_1) + C_4(G_2) + (n_2 - 1) \\
&\leq 3 + \frac{1}{2}(n_2^2 + 3n_2 - 22) + (n_2 - 1) \\
&< \frac{1}{2}[(n_2 + 1)^2 + 3(n_2 + 1) - 22] \\
&= \frac{1}{2}(n^2 + 3n - 22),
\end{aligned}$$

as required. Lastly, we discuss what happens when the equality holds. We observe that the equality holds if and only if $C_4(G_2) = \frac{1}{2}(n_2^2 + 3n_2 - 22)$ and that G_2 is actually the graph represented in Figure 4.5 by the induction hypothesis and there are exactly $(n_2 - 1)$ 4-cycles in G with vertices on both vertices of T. It follows therefore that G must be the graph represented in Figure 4.5. The proof is complete.

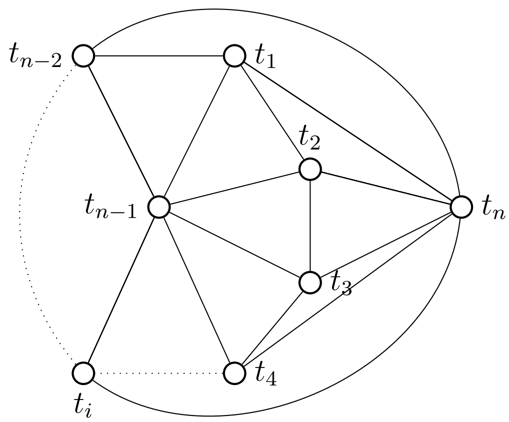

Figure 4.4: A 4-connected maximal planar graph.

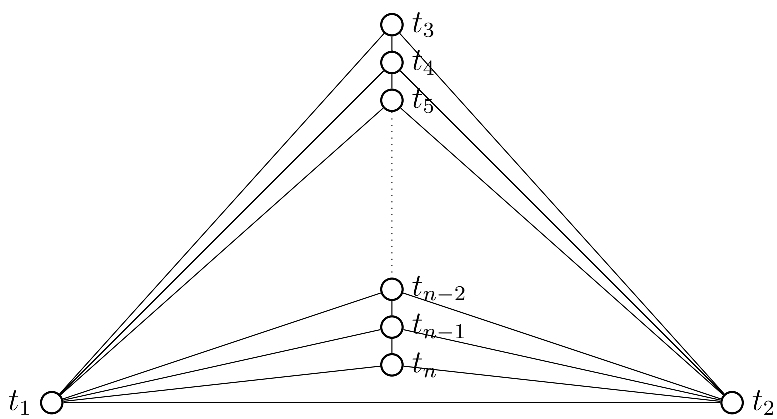

Figure 4.5: Maximum number of cycles on a planar graph.

□

4.2.2 Long cycles in maximal planar graphs

We regard long cycles as cycles with length at least 5. In this section we discuss the bounds for $C_5(G)$ in terms of n.

Lemma 4.35. *(Etourneau [8]) For every $n \geq 14$, there exist 5-connected maximal planar graphs with $(n-12)$ of degree 5 and twelve vertices of 5.*

Lemma 4.36. *(Hakimi and Schmeichel [11]) If G is a maximal planar graph of order $n \geq 8$, then $C_5(G) \geq 6n$.*

There is a maximal planar graph G for which $C_5(G) = 6n$ for every $n \geq 14$.

Proof. Let $t \in V(G)$ with $d_G(t) > 3$. Consider e_1t and e_2t to be any two edges incident at t that share a face. It follows therefore that if we remove e_1t and e_2t from G, then we form a face that is bounded by a 5-cycle. Note that all 5-cycles that occur in this way are distinct if $n \geq 6$, and as a result we have $d_G(t)$ 5-cycles that are associated with each vertex t of degree greater than 3. Then it follows that,

$$C_5(G) \geq \sum_t d(t) - 3\phi(3)$$

where $\phi(k)$ defined as the number of vertices of degree k in G. Consider $t \in V(G)$ with $d_G(t) = 5$. Note that in $G - t$ there is a face that is bounded by a 5-cycle. Furthermore, if $n \geq 8$, then this 5-cycle is distinct from all 5-cycles counted previously. Therefore we have that

$$C_5(G) \geq \sum_t d(t) - 3\phi(3) + \phi(5)$$

Consider $t \in V(G)$ with $d_G(t) = 4$. If $n \geq 6$, then t is a neighbour to no more than one vertex of degree 3. Note $G - t$ has a face which is bounded by a 4-cycle,say, t_1, t_2, t_3, t_4, t_1. Since no more than one vertex among t_1, t_2, t_3, t_4, t_1 is of degree 2 in $G - t$, there are at the minimum two edges in this cycle such that the removal of either edge from $G - t$ forms a 5-cycle if $n \geq 8$. Therefore, if $n \geq 8$, we obtain

$$C_5(G) \geq \sum_t d(t) - 3\phi(3) + \phi(5) + 2\phi(4) \tag{4.1}$$

but, we have

$$
\begin{aligned}
\sum_{t} d(t) &= \sum_{k\geq 3} k\phi(k) \\
&= 3\phi(3) + 4\phi(4) + 5\phi(5) + \sum_{k\geq 3} k\phi(k) \\
&\geq 3\phi(3) + 4\phi(4) + 5\phi(5) + 6(n - \sum_{k=3}^{5} \phi(k)) \\
&\geq 6n - 3\phi(3) - 2\phi(4) - \phi(5). \qquad (4.2)
\end{aligned}
$$

Now we substitute Equation 4.2 in Equation 4.1 and we have $C_5(G) \geq 6n - 6\phi(3)$, when $n \geq 8$. If $\phi(3) = 0$, then Lemma 4.36 is proved. So, we continue by way of induction on $\phi(3)$. Suppose that Lemma 4.36 is correct when $\phi(3) < k$. Assume that $\phi(3) = k$, and momentarily suppose $n \geq 9$. Let $t \in V(t)$. Then we observe that $G - t$ is a maximal planar graph , and has $n \geq 8$ vertices. By the induction hypothesis, $C_5(G - t) \geq 6(n - 1)$. Given that the vertices adjacent to t are of degree ≥ 4, there are at least 12 5-cycles in G which pass through t and therefore

$$C_5(G) \geq C_5(G - t) + 12 \geq 6(n - 1) + 12 > 6n.$$

The case when $n = 8$ can be handled by considering all maximal planar graphs on 8 vertices.
By Lemma 4.35 there exist 5-connected maximal planar graphs with $(n-12)$ of degree 5 and twelve vertices of 5 for every $n \geq 14$. Moreover by Etourneau's construction all the resulting graphs have exactly 12 separating 5-cycles. Now given that Etourneau's graphs contains exactly $6n - 12$ separating 5-cycles, we conclude that Etourneau's graphs have $6n$ 5-cycles. The proof is complete. □

Lemma 4.37. *(Hakimi and Schmeichel [11]) If G is a maximal planar graph of order at least six, then $C_5(G) \leq 5n^2 - 26n$*

Proof. If $n = 6$, then the inequality holds. So we proceed by induction on n. Let t be the a vertex of minimum degree in G. Then $3 \leq d(t) \leq 5$. Since the number of non-separating 5-cycles in G that contains t is at most 25, we have that $C_5(G) \leq C_5(G - t) + 25 + T_5(t)$, where $T_5(t)$ is the number of separating 5-cycles in G that contains t. Now by the induction hypothesis,

$C_5(G-t) \leq 5(n-1)^2-26(n-1)$. Therefore our inequality becomes $C_5(G) \leq 5(n-1)^2-26(n-1)+25+T_5(t)$. Suppose $T_5(t) \leq 10n-56$, then we have that $C_5(G) \leq 5n^2-26n$ as required. Now assume that $T_5(t) > 10n-56 > 10(n-6)$. This means that for some pair of vertices $r, s \in N_G(t)$, there is more than $(n-6)$ paths of length 3 in $G-t$ joining r and s so that every one of these paths together with the path r, t, s forms a separating 5-cycle in G. Note that the contraction of a pair of vertices x_i and x_j of a graph produces a graph in which the two vertices x_1 and x_2 are replaced with a single vertex x such that x is adjacent to the union of the vertices to which x_1 and x_2 were originally adjacent. So if we contract r and s in $G-t$, we form a $(n-2)$-vertex planar graph G'. We observe that in G' there are more than $(n-6)$ separating triangles containing the vertex $r = s$. Let G'' be a $(n-2)$-vertex maximal planar graph derived from G' by adding an edge to G'. Given that G'' has only $[2(n-2)-4]$ non-separating triangles triangles, we have that $C_3(G) > [2(n-2)-4]+(n-6) = 3(n-2)-8$. But this contradicts Theorem 4.32. □

From Lemma 4.36 and 4.37, we obtain the following Theorem:

Theorem 4.38. *(Hakimi and Schmeichel [11]) If G is a maximal planar graphs on $n \geq 8$ vertices, then $6n \leq C_5(G) \leq 5n^2-26n$.*

Now we discuss an upper bound for $C_p(G)$ for an arbitrary p.

Definition 4.39. *We define $C_p(n) = max_G C_p(G)$ to be the maximum being taken over all maximal planar graphs with n vertices.*

Theorem 4.40. *(Hakimi and Schmeichel [11]) Consider a maximal planar graph with n vertices. Then, $C_p(G)$ and $C_p(n) \in O(n^{[\frac{p}{2}]})$, for $p = 3, 4, \ldots, n$.*

Proof. All we have to do to prove the theorem is to show that $C_p(n) \in O(n^{[\frac{p}{2}]})$. We proceed by induction on n. Obviously the result is true for $n = 3, 4, 5$. We assume that it is true for all integers less than n. Now to prove that it also true for all n, we have to show that $C_p(n)-C_p(n-1) \in O(n^{[\frac{p}{2}]-1})$ since $C_p(n) = \sum_{i=p}^{n} C_p(i)-C_p(i-1)$, and thus we would have $C_p(n) \in O(n^{[\frac{p}{2}]})$. Otherwise for any constant C_0, $C_p(i)-C_p(i-1) > C_0 n^{[\frac{p}{2}]-1}$, fr some integer n. Consider G to be a maximal planar graph with n vertices and with $C_p(n)$ p-cycles. Let t be a vertex of minimum degree with $d_G(t) \leq 5$. Then,

$$C_p(G)-C_p(G-t) \geq C_p(n)-C_p(n-1) > C_0 n^{[\frac{p}{2}]-1},$$

so it follows that t must occur in more than $C_0 n^{[\frac{p}{2}]-1}$ p-cycles in G. Therefore, for some vertices $t_1, t_2 \in N_G(t)$, there exists more than $C_0 n^{[\frac{p}{2}]-1}/\binom{5}{2}$ paths between t_1 and t_2 of length $p-2$ in $G-t$. Now we construct a maximal planar graph G' with $n-2$ vertices by contracting t_1 and t_1 in $G-t$. Then

$$C_{p-2}(G') > (C_0/\binom{5}{2})n^{[\frac{p}{2}]-1}.$$

Since C_0 is any constant, then we have $C_{p-2}(G') \notin O(n^{[\frac{p}{2}]-1})$. But this contradicts the inductive hypothesis, hence have the desired result that $C_p(n) \in O(n^{[\frac{p}{2}]})$. □

4.3 Number of Hamiltonian cycles in maximal planar graphs

In this section we discuss the minimum number of Hamiltonian cycles that a maximal planar graph on n vertices could have, in terms of n. But first we construct a maximal planar graph on $n \geq 12$ vertices, and with exactly four Hamiltonian cycles.

Lemma 4.41. *(Hakimi, Schmeichel and Thomassen [12] If G' is the maximal planar graph shown in Figure 4.6, then*

(a) There exists no Hamiltonian path in $G - x_p$ from x_q to x_r, for $p = 1, 2, 3$ and $q, r \neq p$.

(b) There exists only two Hamiltonian paths in G_1 from x_1 to x_2.

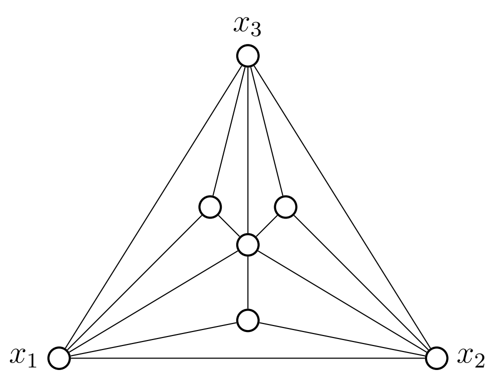

Figure 4.6: The maximal planar graph G'.

Lemma 4.42. *(Hakimi, Schmeichel and Thomassen [12]) If G'' is the maximal planar graph shown in Figure 4.7, then*

(a) There exists no Hamiltonian path from x_1 to x_3(x_2 to x_3, respectively) in $G'' - x_2$($G'' - x_1$, respectively).

(b) There exists only two Hamiltonian paths in $G'' - x_3$ from x_1 to x_2.

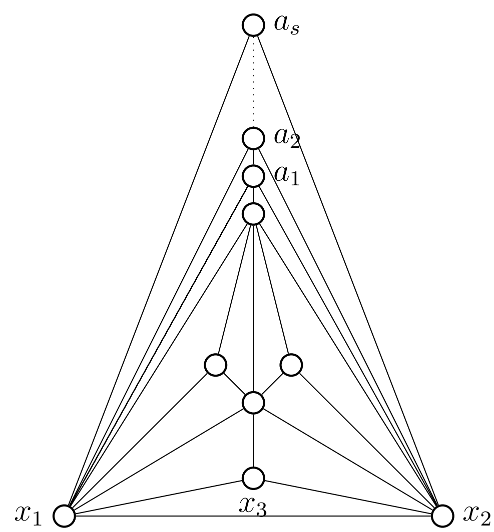

Figure 4.7: The maximal planar graph G''.

Theorem 4.43. *(Hakimi, Schmeichel and Thomassen [12]) Construct a maximal planar graph G on $n \geq 12$ vertices as follows: Start with the graph G'' as shown in Figure 4.7 with $(s = n-1)$ and in the interior of the triangle $T = \{x_1, x_2, x_3\}$ place the graph of G' as shown in Figure 4.6 in such a way that both G' and G'' have exactly T in common. Then G has exactly four Hamiltonian cycles.*

Proof. First assume that G is Hamiltonian. Consider a Hamiltonian cycle C_H of G. Given that C_H could enter and exit the interior of T exactly once, then the edges of C_H that also belong to G' must form a Hamiltonian path in G' or $G' - x_r$, for some r. Actually, according to Lemma 4.41(a) these edges must form a Hamiltonian path in G' itself from x_p to x_q, for some p, q.

If either x_p or x_q is x_3, then the edges of C_H that also belong to G'' form a Hamiltonian path x_p to x_q in $G - x_r$, contradicting Lemma 4.42(a). For this reason, any Hamiltonian cycle of G must contain a Hamiltonian path in G' from x_1 to x_2, and a Hamiltonian path in $G'' - x_3$ from x_1 to x_2. By Lemma 4.41(b) and Lemma 4.42(b), there are precisely two Hamiltonian paths of each of these types. Therefore G is Hamiltonian as assumed and has precisely four Hamiltonian cycles. □

Since the maximal planar graphs formed in Theorem 4.43 have connectivity three, the question arises, what is the minimum number of Hamiltonian cycles in a 4-connected maximal planar graph with order n? Theorem 4.4 by Whitney tells us that such graph has a at least one Hamiltonian cycles. To answer this question we first state the following lemma by Whitney.

Lemma 4.44. *(Whitney [25]) Consider a cycle C in a 4-connected maximal planar graph together with the vertices and edges on one side of C, which we shall call the outside. Let x and y be two distinct vertices of C, dividing C into two paths P_1 and P_2 and each path contains both x and y. Suppose*

1. *No two vertices of P_1 touch each other outside C, that is, they are not joined by an edge which lies outside of C; and*
2. *Either no pair of vertices of P_2 are joined by an edge which lies outside C, otherwise there exists a vertex z in P_2 distinct from x and y, and dividing P_2 into paths P_3 and P_4 each of which contains z such that no two vertices of P_3 and no two vertices of P_4 are joined by an edge that edge which lies outside C.*

Then there is a path from x to y using only edges on and outside of C which passes through every vertex on and outside of C exactly once.

Theorem 4.45. *(Hakimi, Schmeichel and Thomassen [12]) If G is a 4-connected maximal planar graph of order n, then G has at least $\frac{n}{\log_2 n}$ Hamiltonian cycles.*

Proof. Consider any edge in G, say, ab. Then ab will be incident to two facial triangles in G, say, $T_1 = a, b, c, a$ and $T_2 = a, b, d, a$. First we want to show that there exists a Hamiltonian cycle in G that contains the path $P = c, a, b, d$. Let a be adjacent to the vertices $b, c, a_1, a_1, \dots, a_p, d$ and let b be adjacent to the vertices $a, d, b_1, b_2, \dots, b_q, c$ (Figure 4.8).

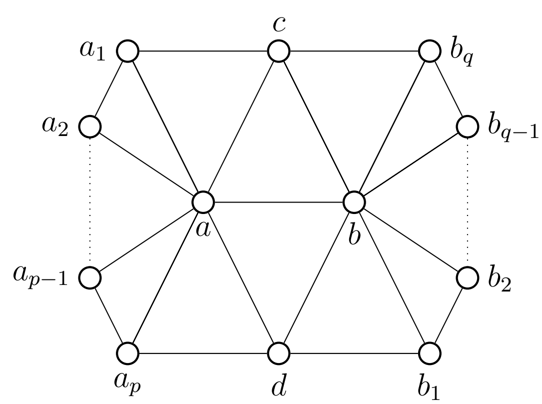

Figure 4.8: An illustration of the proof of Theorem 4.45.

Let $C = c, a_1, a_2, \ldots, a_p, d, b_1, b_1, \ldots, b_q, c$ be the cycle in G. We observe that c and d divide C into two paths that satisfy the condition in Lemma 4.44, because if there was an edge of the form $a_i a_j$ outside C, then this would mean that there exists a separating triangle $T_3 = a_i, a_j, a, a_i$, contradicting the fact that G is a 4-connected maximal planar graph. So the path P together with the path from c to d outside C as described in Lemma 4.44 is the required Hamiltonian cycle in G. Now for every edge ab of G, choose a Hamiltonian cycle that contains P. So we have $3n - 6$ (not necessarily distinct) Hamiltonian cycles in G. Denote as β, the largest number of times a Hamiltonian cycle of G occurs as described above. Denote the number of distinct Hamiltonian cycles in G as $\omega(G)$, then $\omega(G) \geq \frac{3n-6}{\beta}$. Let C_H be a Hamiltonian cycle of G counted β times as described above. Then at the minimum, $\frac{\beta}{3}$ of the corresponding 4-cycles a, c, b, d, a will bound regions which will pairwise meet at up to one vertex (Figure 4.9). For every one of these $\frac{\beta}{3}$ 4-cycles a, c, b, d, a, if we replace the path $P = c, a, b, d$ with the path $P' = c, b, a, d$, then we obtain a new Hamiltonian cycle. Therefore, we have that $\omega(G) \geq 2^{\frac{\beta}{3}}$. It follows therefore that,

$$\log_2 \omega(G) \geq \frac{\beta}{3} \geq \frac{n-2}{\omega(G)}$$

or

$$\omega(G) \log_2 \omega(G) \geq n - 2$$

$$\Rightarrow \omega(G) > \frac{n}{\log_2 n}$$

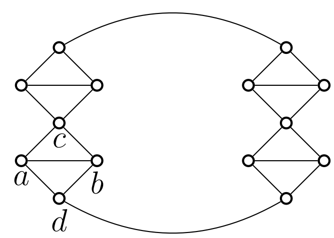

Figure 4.9: The Hamiltonian cycle C_H

□

Since the lower bound from Theorem 4.45 appears far from tight, Hakimi, Schmeichel and Thomassen [12] gave the following conjecture,

Conjecture (Hakimi, Schmeichel and Thomassen [12]) Let G be a 4-connected maximal planar graph of order n. Then G has at the minimum $2(n-2)(n-4)$ Hamiltonian cycles, with the equality occurring if and only if G is the graph $\overline{K}_2 + C_{n-2}$ shown in (Figure 4.10)

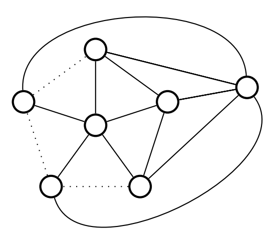

Figure 4.10: The graph $\overline{K}_2 + C_{n-2}$

Chapter 5

Planar Triangulations

5.1 General properties of planar triangulations

In this section we discuss general properties of planar triangulations, including connectivity of planar triangulations.

Theorem 5.1. *(Helden [15]) Consider a planar triangulation T. Then T is at least 2-connected.*

Proof. Let X_T be the exterior cycle of T. Now given that X_T is a cycle and that cycles are 2-connected, it follows that T is at least 2-connected. □

Given that maximal planar graphs with $n \geq 4$ vertices have minimum degree at least two or minimum degree at most five. It is known that when we delete a vertex from a maximal planar graph we obtain a planar triangulation. So we make the following observation

Observation 5.2. *(Helden [15]) Consider a planar triangulation T. Then $\delta(G) \in \{2, 3, 4, 5\}$. This follows from the previous statement.*

From the proceeding observation we obtain the following corollary.

Corollary 5.3. *(Helden [15]) Consider a planar triangulation T. Then T is at most 5-connected.*

Lemma 5.4. *(Helden [15]) Consider a planar triangulation T. Let u be a vertex of T with degree 2. Then u lies on the exterior cycle X_T of T.*

Proof. Suppose to the contrary that u does not lie on X_T. Then u lies on a two facial cycle F. Given that T is a planar triangulation, every facial cycle is a 3-cycle. Given that u has degree 2, it follows that F cannot be a 3-cycle, a contradiction and hence the result. □

Definition 5.5. *We define an NST-triangulation as a triangulation without separating triangles.*

Definition 5.6. *We call a vertex of a planar triangulation T, a boundary vertex if it is incident with the exterior face of the triangulation. The subgraph induced by the boundary vertices of T is called the boundary graph of T and is denoted as $BG(T)$. We call an edge of $BG(T)$ a chordal edge of $BG(T)$ if it is a chord of T.*

Definition 5.7. *Consider a 2-connected planar triangulation T and let $Ch(T)$ be the set of vertices of chords of G. Let $H = T - Ch(T)$. Then H decomposes into several components. Now we add the chords to each corresponding component in the following sense. A component K of $T - Ch(T)$ together with all vertices of H adjacent to vertices of K and all edges with one end in H and the other in K. Now we add the chords to each component. We call the subgraphs obtained from this decomposition the chordal-subgraphs of T.*

Lemma 5.8. *(Helden [15]) Consider a 2-connected planar triangulation T. The chordal-subgraphs of T have one of the following structures.*

1. *The chordal-subgraph is 3-connected.*
2. *The chordal-subgraph is a triangle whose vertices are boundary vertices. Then we have three possibilities.*
   (a) *Only one edge is a chord.*
   (b) *Only two edges are chords.*
   (c) *All three edges are chords.*

Definition 5.9. *Let T be a planar triangulation and let a subset $S \subset V(G)$ with $|S| = 3$ be not an induced cycle. If $T - S$ is not connected, then S is called chordal 3-cut.*

Lemma 5.10. *(Helden [15]) Consider a planar triangulation T. Let S be a chordal 3-cut. Then only two vertices of S are boundary vertices of T.*

Proof. The exterior cycle X_T of a planar triangulation is at least 2-connected. So we need at least two non-adjacent boundary vertices whose removal causes X_T to become disconnected. □

5.2 Hamiltonicity properties of planar triangulations

In this section we study the hamiltonicity of planar triangulations.

Theorem 5.11. *(Dillencourt [7]) If T is an NST-triangulation such that the boundary of each face of the boundary graph $BG(T)$ of T has at most three chordal edges, then T is hamiltonian.*

Proof. The faces F_i of $BG(T)$ decompose T into a collection of smaller triangulations T_i, where each T_i is the subgraph induced the vertices of T lying either inside or on the boundary of the face F_i. Suppose every T_i contains a Hamiltonian cycle that contains all its chordal edges, then these cycles can be joined together to form a Hamiltonian cycle through T. □

Corollary 5.12. *(Dillencourt [7]) Consider an NST-triangulation T with no more than three chords. Then T is Hamiltonian.*

Proof. Every chordal edge of any face of the boundary graph $BG(T)$ is a chord of T. Then by Theorem 5.11, T is Hamiltonian. □

Corollary 5.13. *(Dillencourt [7]) Consider an NST-triangulation T with no more than seven boundary vertices. Then T is Hamiltonian.*

Proof. Assume that T is a non-Hamiltonian NST-triangulation. By Theorem 5.11, there is some face F' of $BG(T)$ with at the minimum four chordal edges. Every one of these edges is a chord of T, and therefore separates F' from some boundary vertex, a different vertex for each chord. All four of these vertices together with at the minimum four endpoints of the chords, constitute at the minimum eight distinct boundary vertices. A contradiction. □

Theorem 5.14. *(Helden [15]) Consider a 2-connected NST-triangulation T such that every chordal-subgraph of T has no more than three chordal edges. Then T is Hamiltonian.*

Proof. We know by Theorem 5.11 that a NST-triangulation T is Hamiltonian if the boundary of each face of the boundary graph $BG(T)$ of T has at most three chordal edges. We observe that vertices of one face of the boundary graph $BG(T)$ correspond to the vertices of the exterior cycle X_H of one chordal-subgraph H. Given that the chordal-subgraphs are formed with the aid of the chords, we can extend the claim to chordal-subgraphs. □

Lemma 5.15. *(Helden [15]) Consider a 3-connected NST-triangulation T and any two edges of X_T. Then, T has a hamiltonian cycle containing those edges. Consider a 3-connected NST-triangulation T and $|V(X_T)| \geq 4$. Given any three edges of X_T, T contains a Hamiltonian cycle containing those edges.*

Definition 5.16. *Consider a planar triangulation T with k separating triangles, say, $S_1, S_2, \ldots, S_k$. Let $T_{in,S_1}(T_{in,S_2}, \ldots, S_{in,S_k}$, respectively) be the subgraph of T obtained by deleting all the vertices outside the separating triangle $S_1, (S_2, \ldots, S_k$, respectively). Similarly, let $G_{out,S_1}(G_{out,S_2}, \ldots, T_{out,S_k}$, respectively) be the subgraph of T obtained by deleting all the vertices inside the separating triangle $S_1, (S_2, \ldots, S_k$, respectively).*

Theorem 5.17. *(Helden [15]) Consider a 3-connected planar triangulation T with $|X_T| \geq 4$ and let B be the rooted decomposition tree of the given plane graph of T. Suppose the root of the tree B has no more than three children whose related disjoint separating triangles have one edge in common with X_T and if all other vertex of the tree B has no more than two children, then T is Hamiltonian.*

Proof. Let $S_1 = x_1, y_1, z_1, x_1, S_2 = x_2, y_2, z_2, x_2$ and $S_3 = x_3, y_3, z_3, x_3$ be the three separating triangles of the root. Without loss of generality, let $a_i = x_i y_i$, for $1 \leq i \leq 3$ be the edges which are in common with X_T. T_{out,S_1,S_2,S_3} is an NST-triangulation with $|X_{T_{out,S_1,S_2,S_3}}| > 3$. Then by Lemma 5.15, there is a Hamiltonian cycle C_H in T_{out,S_1,S_2,S_3} which contains the edges $a_i = x_i y_i$, for $1 \leq i \leq 3$. By Theorem 4.30 each T_{in,S_i} is Hamiltonian for any two boundary edges of T_{in,S_i}, for $1 \leq i \leq 3$. Therefore without loss of generality, there is a Hamiltonian cycle $C_{H1} = x_i, z_i, y_i, Pin(y_i, x_i), x_i$ in T_{in,S_i} with $P_{in}(y_i, x_i)$ being the Hamiltonian path of $T_{in,S_i} - z_i$ from between y_i and x_i. Now substitute $P_{in}(y_i, x_i)$ in place of the edges $x_i y_i$ in the graph T. Then T is Hamiltonian. □

Theorem 5.18. *(Helden [15]) Consider a a 3-connected planar triangulation T and let B be the rooted decomposition tree of the given plane graph of T. If every vertex of the tree B has no more than two children and if the two separating triangles S_1 and S_2 of the root have each one edge which are both located on the same face, then T is Hamiltonian.*

Proof. Let Let $S_1 = x_1, y_1, z_1, x_1, S_2 = x_2, y_2, z_2, x_2$ be the two separating triangles of T. Without loss of generality let $a_1 = x_1, y_1$ and $a_2 = x_2, y_2$ with $x_1 = x_2$ be the edges which are both located on the same face F. Since a planar graph can always be embedded in the plane such that a given face of the graph becomes the exterior face. We can embed T in the plane such that F becomes the exterior face of T. Let this graph be T'. T'_{out,S_1,S_3} is a 3-connected planar graph without interior component 3-cuts. Therefore T'_{out,S_1,S_3} has a Hamiltonian cycle C_H which contains the edges a_1 and a_2. It follows from Theorem 4.30 that T_{in,S_i} are Hamiltonian for any two boundary edges of T_{in,S_i}, for $1 \leq i \leq 2$. Without loss of generality, there is a Hamiltonian cycle $C_{in,S_1} = x_1, z_1, y_1, P_{in,S_1}(y_1, x_1), x_1$ with $P_{in,S_1}(y_1, x_1)$ being the path of $P_{in,S_1} - z_1$ between y_1 and x_1. Now we substitute in P_{in,S_1} in the place of the edge $a_1 = x_1y_1$ in T'. Also without loss of generality, there is a Hamiltonian cycle $C_{in,S_2} = x_2, y_2, z_2, P_{in,S_2}(z_2, x_2), x_2$ with $P_{in,S_2}(z_2, x_2)$ being the path of $P_{in,S_2} - y_2$ between z_2 and x_2. Now we substitute in P_{in,S_2} in the place of the edge $a_2 = x_2z_2$ in T'. Then T' contains a Hamiltonian cycle and therefore we conclude that T is Hamiltonian. □

Lemma 5.19. *(Helden [15]) Consider a maximal planar graph G with only one separating triangle and let u be a vertex of G. Then $G-u$ is Hamiltonian. We consider two cases.*

Theorem 5.20. *(Helden [15]) Consider a 3-connected planar triangulation T with no more than two separating triangles. Then T is Hamiltonian.*

Proof. We begin by inserting a new vertex p in the exterior face which is bounded by X_T , and connect this vertex with every vertex of X_T .Note that by insertion from the vertex p, we get no new separating triangles. Suppose we get a new separating triangle, then p must be one of three vertices. Therefore, this implies that the two remaining vertices would be a chord in T, but this is a contradiction. A chordal 3-cut could only be extended to a separating triangle if both vertices on X_T were connected. As, however, after construction only the boundary vertices are connected with the new vertex p, no new separating triangle can arise from this. This implies that graph $T+p$ is a maximal planar graph with no more than 2 separating triangles. Then by Lemma 5.19 that the graph $T + p - p = T$ has a Hamiltonian cycle. □

Lemma 5.21. *(Helden [15]) Consider a planar a planar triangulation T without chords, separating triangles and chordal 3-cuts. Then T is 4-connected.*

Proof. By Theorem 5.1, every planar triangulation is at least 2-connected. Since there is no chord there is no 2-cut whose removal causes the graph to become disconnected. Furthermore, since there are no separating triangles and chordal 3-cuts there is no 3-cut whose removal causes the graph to become disconnected. Therefore, T is 4-connected. □

The following corollary follows from the preceding theorem.

Corollary 5.22. *(Helden [15]) Consider a 4-connected planar triangulation T. Then T is Hamiltonian.*

5.3 Conclusion

We showed that any maximal planar graph with $k = 1, 2, 3$ is Hamiltonian and that if $k \geq 4$, there is counter-example to show that G is not immediately Hamiltonian. For $4 \leq k \leq 6$, it is possible to develop a special structure of the position of the separating triangles to each other, which can ultimately generate Hamiltonicity, but this is beyond the scope of this thesis.

Bibliography

[1] Batagelj, V. (1983). An inductive definition of the class of all triangulations with no vertex of degree smaller than 5. In *Proceedings of the Fourth Yugoslav Seminar on Graph Theory, Novi Sad*, pages 15–24.

[2] Baybars, I. (1982). On k-path hamiltonian maximal planar graphs. *Discrete Mathematics*, 40(1):119–121.

[3] Chartrand, G. and Lesniak, L. (1996). *Graphs and Digraphs*. Chapman & Hall, 3rd edition.

[4] Chartrand, G. and Zhang, P. (2008). *Chromatic Graph Theory*. CRC press.

[5] Chen, C. (2003). Any maximal planar graph with only one separating triangle is hamiltonian. *Journal of Combinatorial Optimization*, 7(1):79–86.

[6] Chvátal, V. (1985). *Hamiltonian Cycles. Chapter 11 in Lawler, EL, Lenstra, JK, Rinnooy Kan, AHG, Shmoys, DB (Eds.): The Traveling Salesman Problem. A Guided Tour of Combinatorial Optimization*. John Wiley & Sons, Chichester.

[7] Dillencourt, M. B. (1990). Hamiltonian cycles in planar triangulations with no separating triangles. *Journal of Graph Theory*, 14(1):31–49.

[8] Etourneau, E. (1975). Existence and connectivity of planar graphs having 12 vertices of degree 5 and n- 12 vertices of degree 6. In *A. Hajnal et al. Infinite and finite sets, Colloq. Math. Soc. J. Bolyai. Amsterdam-Oxford-New York: North-Holland*.

[9] Garey, M. R., Johnson, D. S., and Tarjan, R. E. (1976). The planar hamiltonian circuit problem is np-complete. *SIAM Journal on Computing*, 5(4):704–714.

[10] Hakimi, S. L. and Schmeichel, E. F. (1978). On the connectivity of maximal planar graphs. *Journal of Graph Theory*, 2:307–314.

[11] Hakimi, S. L. and Schmeichel, E. F. (1979). On the number of cycles of length k in a maximal planar graph. *Journal of Graph Theory*, 3(1):69–86.

[12] Hakimi, S. L., Schmeichel, E. F., and Thomassen, C. (1979). On the number of hamiltonian cycles in a maximal planar graph. *Journal of Graph Theory*, 3(4):365–370.

[13] Harary, F. (1969). *Graph Theory.* Addison-Wesley, Reading, MA.

[14] Helden, G. (2007a). Each maximal planar graph with exactly two separating triangles is hamiltonian. *Discrete Applied Mathematics*, 155(14):1833–1836.

[15] Helden, G. (2007b). Hamiltonicity of maximal planar graphs and planar triangulations. *These de doctorat, RWTH Aachen University.*

[16] Helden, G. and Vieten, O. (2006). Hamilton cycles in maximal planar graphs. *Electronic Notes in Discrete Mathematics*, 25:71.

[17] Jackson, B. and Yu, X. (2002). Hamilton cycles in plane triangulations. *Journal of Graph Theory*, 41(2):138–150.

[18] Li, L., Li, R., and Li, S. (2005). Any maximal planar graph with two separating triangles has a hamilton path. *Journal-Shanxi University Natural Science Edition*, 28(2):133.

[19] Nishizeki, T. and Chiba, N. (1988). *Planar graphs: Theory and algorithms*, volume 32. Elsevier.

[20] Rolland, P. New approach to construct planar triangulations with minimum degree 5. `http://prolland.free.fr/works/research/MPG5/doc/genMPG5_2e.pdf/`.

[21] Sanders, D. P. (1996). On hamilton cycles in certain planar graphs. *Journal of Graph Theory*, 21(1):43–50.

[22] Sanders, D. P. (1997). On paths in planar graphs. *Journal of Graph Theory*, 24(4):341–345.

[23] Thomassen, C. (1983). A theorem on paths in planar graphs. *Journal of Graph Theory*, 7(2):169–176.

[24] Tutte, W. T. (1956). A theorem on planar graphs. *Transactions of the American Mathematical Society*, 82(1):99–116.

[25] Whitney, H. (1931). A theorem on graphs. *Annals of Mathematics*, pages 378–390.

[26] Whitney, H. (1932). Congruent graphs and the connectivity of graphs. *American Journal of Mathematics*, 54(1):150–168.

[27] William, H. C. and Edmonds, J. (1980). A combinatorial decomposition theory. *Canadian Journal of Mathematics*, 32(3):734–765.