UNIVERSITY OF CAPE TOWN

DEPARTMENT OF MATHEMATICS

REALCOMPACT ALEXANDROFF SPACES

AND

REGULAR σ-FRAMES

Christopher R. A. Gilmour

A thesis prepared under the supervision of Associate Professor G.C.L. Brümmer for the degree of Doctor of Philosophy in Mathematics.

To

Ingrid

John and Betty

...things which are too wonderful
for me....
The way of an eagle in the air...

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CONTENTS

SYNOPSIS

In the early 1940's, A.D. Alexandroff [1940], [1941] and [1943] introduced a concept of space, more general than topological space, in order to obtain a simple connection between a space and the system of real-valued functions defined on it. Such a connection aided the investigation of the relationships between the linear functionals on these systems of functions and the additive set functions defined on the space. The Alexandroff spaces of this thesis are what Alexandroff himself called the completely normal spaces and what H. Gordon [1971] called the zero-set spaces.

An Alexandroff space may be viewed as a Tychonoff space with a distinguished base for the closed sets, here called an Alexandroff base, the model being the zero-sets of a subalgebra of the algebra of continuous functions. The Alexandroff bases are, in greater or lesser generality, closely related to the bases investigated by Wallman [1938], Banaschewski [1963], Frink [1964], Steiner and Steiner [1970], Alò and Shapiro [1974], Sultan [1978], Blasco [1979] and others.

A.W. Hager [1974] introduced, in the setting of uniform spaces, the separable $\mathcal{M}$-fine spaces and showed that they correspond, via a categorical isomorphism, to

the Alexandroff spaces. Hager's construction led to the more general concepts of A - c uniform spaces, for different classes A and coreflectors c of uniform spaces. The latter spaces have been extensively studied, particularly by the Seminar Uniform Spaces led by Z. Frolík in Prague (SUS [1973-74]).

One of the most useful and interesting facts about the Alexandroff spaces is that they form a category *Alex*, wider than the category of Tychonoff spaces, in which pseudocompactness is productive and in which the realcompactness reflector distributes over arbitrary products (Gordon [1971]).

We outline our thesis.

Chapter 1 This chapter gives some basic results on Alexandroff spaces that we shall need later on and briefly outlines the connection between the Alexandroff spaces and the structures mentioned above.

Chapter 2 The theory of realcompactifications, in particular Wallman realcompactifications and their relationship to Hewitt's universal realcompactification is unified by means of the Alexandroff bases.

The Tychonoff spaces are located in the category *Alex* as the fine (or topological) Alexandroff spaces.

The Alexandroff bases that are complete (in the sense of Blasco [1979]) correspond to the realcompact-fine Alexandroff spaces. Our terminology conforms with that of the A - c construction of Hager.

After characterising the realcompact-fine spaces we show how they and their associated coreflector may be applied to the following four problems.

I When does $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ (for Tychonoff spaces X and Y)?

II When is the *Alex*-product of topological spaces topological?

III When does a Wallman realcompactification of a Tychonoff space coincide with the Hewitt realcompactification?

IV Which continuous functions on a Tychonoff space extend to a given Wallman realcompactification?

The first of these problems has been considered by many authors. For references to this question and a discussion of it see Walker ([1974], 8.39). Our contribution to this and the second problem is closely related to that of Blair and Hager [1977].

The last section of this chapter is devoted to those Alexandroff spaces with realcompact topology, the corresponding epireflector υ_t in *Alex* and its properties.

Chapter 3 The realcompact spaces arise naturally when a description corresponding to 'topology without points' is sought for the Alexandroff spaces. The setting for this description is that of σ-frames (Banaschewski [1980a,b,c]) which are generalizations of frames (i.e. locales or local lattices, in the sense of Bénabou [1957-58], Papert [1964] and Isbell [1972a]).

My thesis advisor conjectured that the Alexandroff spaces which could be described solely by their cozero-sets (i.e. in terms of their cozero-set σ-frames) would be the Alexandroff spaces which are complete in the uniformity induced by the correspondence of Hager [1974]. These latter spaces are precisely the realcompact Alexandroff spaces (ibid.). The conjecture was proved with the generous help of Professor Bernhard Banaschewski and appears as the first half of theorem 3.2.7.

We begin chapter 3 with a short introductory section on regular σ-frames (distilled from Banaschewski [1980a]). We then give the dual adjunction between the Alexandroff spaces and the regular σ-frames. The largest duality contained in this adjunction is that between the realcompact Alexandroff spaces and the Alexandroff σ-frames. There is a corresponding dual adjunction between the Tychonoff spaces and regular σ-frames and the largest duality therein is that between the realcompact

Tychonoff spaces and the topological σ-frames. This last duality extends that of Banaschewski [1980a,b] between the compact Hausdorff spaces and the compact regular σ-frames.

As a consequence of the above we can give alternate descriptions for the analogues of υ and β, the realcompact epireflector and compact epireflector, in *Alex*. The unit corresponding to υ, unlike its counterpart for Tychonoff spaces, is an essential embedding.

The Alexandroff spaces provide a natural setting for the theory of z-embedding in topology (e.g. see Blair and Hager [1977]). We illustrate this observation by means of some applications in section 4.

This chapter concludes with two short sections. The first of these is a note on complete objects (in the sense of Brümmer [1979]) in *Alex*. In the last section we establish, directly from results of Banaschewski [1980a], a way of generating the cozero-sets of the fine coreflection of an Alexandroff space via its open sets.

INDEX OF SELECTED DEFINITIONS

CHAPTER 1

Alexandroff Spaces

1. Alexandroff bases and the category *Alex*

Further details and proofs of the material in this and the following section may be found in Alexandroff [1940], Gordon [1971], Hager [1974] and Gilmour [1975].

1.1 Definition An *Alexandroff space* is a pair $(X,\mathcal{Z})$ where X is a set and $\mathcal{Z}$, the *Alexandroff structure* on X, is a collection of subsets of X satisfying Z1 - Z4 below. The sets in $\mathcal{Z}$ are called *zero-sets* and their complements with respect to X are *cozero-sets.*

Z1 $\mathcal{Z}$ is closed under finite unions and countable intersections; $\emptyset$ and X are in $\mathcal{Z}$.

Z2 If $A,B \in \mathcal{Z}$ and $A \cap B = \emptyset$ then there are $C,D \in \mathcal{Z}$ such that $A \cap C = \emptyset = B \cap D$ and $C \cup D = X$.

Z3 If $A \in \mathcal{Z}$ then there is a sequence $\{A_n\}$ in $\mathcal{Z}$ such that $X \setminus A = \cup A_n$.

Z4 For each pair of distinct points in X there is an $A \in \mathcal{Z}$ containing just one of them.

Where no confusion will arise, $(X,\mathcal{Z})$ is often abbreviated to simply X .

1.2 Examples (1) The zero-sets of any topological space satisfy Z1 - Z3 .

(2) Any σ-algebra $\mathcal{B}$ of subsets of a set X is an Alexandroff structure on X .

(3) The set $\mathcal{P}X$ of all subsets of a set X is the *discrete* Alexandroff structure on X .

(4) The closed sets of the real line $\mathbb{R}$ with its usual topology satisfy Z1 - Z4. We denote the associated Alexandroff space by $\mathbb{R}_z$.

1.3 The zero-sets of an Alexandroff space $(X,\mathcal{Z})$ form a base for the closed sets of a Tychonoff topology on X ; such a base will be called an *Alexandroff base* for the underlying topology. Other authors use the terms : *strong delta normal base* (Alò and Shapiro [1974]); *separating nest generated intersection ring* (Steiner and Steiner [1970]); *strongly normal complement generated delta lattice* (Bachman and Sultan [1976]).

In particular a family $\mathcal{Z}$ of closed sets in X is (i) *normal* if $\mathcal{Z}$ satisfies Z2 (ii) *complement generated if* Z3 is satisfied (iii) *nest generated* if for each $A \in \mathcal{Z}$ there exists a sequence $\{A_n, B_n\}$ in $\mathcal{Z}$ such that $X \setminus B_{n+1} \subset A_{n+1} \subset X \setminus B_n \subset A_n$, $n = 1,2,\ldots.$ and $A = \cap A_n$.

Steiner [1966] showed that each family of closed sets $\mathcal{Z}$ which is nest generated and satisfies Z1 , is normal (a special case of 3.1.5). It is easy to see that such a $\mathcal{Z}$ is complement generated. Conversely it is not difficult to verify that an Alexandroff base is nest generated.

A base $\mathcal{N}$ for the closed sets of a topological space X , is a *normal base* if it is (i) closed under finite unions and finite intersections (ii) satisfies Z2 (iii) such that for each closed set F of X and each $x \notin F$, there is an $N \in \mathcal{N}$ with $x \in N$ and $F \cap N = \emptyset$.

Each Alexandroff base is a normal base.

1.4 The axioms Z1 - Z4 for the zero-sets of an Alexandroff space X translate (via complements) to axioms which characterise the cozero-sets of X . We shall find it appropriate in chapter 3 to refer to the *cozero-set structure* of X , and we denote it by $\mathfrak{U}X$.

1.5 Functions between Alexandroff spaces for which preimages of cozero-sets (zero-sets) are cozero-sets (respectively, zero-sets) are called *coz-maps*. The Alexandroff spaces and coz-maps are the objects and morphisms of the category *Alex*. The collection of all coz-maps between Alexandroff spaces X and Y is denoted by $\mathcal{Alex}(X,Y)$. In particular we abbreviate $\mathcal{Alex}(X,\mathbb{R}_z)$ to $A(X)$. We note that $A(X)$ is an algebra of functions (in the sense of Hager [1969]). An Alexandroff space X is *pseudocompact* if each coz-map of $A(X)$ is bounded.

1.6. Theorem (Gordon [1971]) *If* $(X,\mathfrak{Z})$ *is an Alexandroff space, then*

$$\mathfrak{Z} = \{Zf : f \in A(X)\} .$$

We note that this result is contained in the duality of Steiner and Steiner ([1970], 4.3).

1.7 Dropping the separation axiom Z4 we obtain a larger category $\mathcal{Zero}$ that is topological over $\mathcal{Ens}$.

Thus for any class of pairs (f_α, Y_α) of functions f_α on a set S to $Y_\alpha \in \mathcal{Zero}$, there is an $X \in \mathcal{Zero}$ with underlying set S satisfying :

(1) Each $f_\alpha: X \to Y_\alpha$ is a coz-map.

(2) Whenever $W \in \mathcal{Zero}$ and coz-maps $g_\alpha: W \to Y_\alpha$ and a function k are given such that $f_\alpha k = g_\alpha$ for each α, then k is a coz-map.

Such an X is called *initial* for the given class of pairs (f_α, Y_α). The zero-sets of X are precisely those sets which are countable intersections of finite unions of preimages of zero-sets of the Y_α under the f_α .

We call an object $(X,\mathcal{Z})$ *coarser* than $(X,\mathcal{W})$ (or $(X,\mathcal{W})$ *finer* than $(X,\mathcal{Z})$) if $\mathcal{Z} \subset \mathcal{W}$. The initial structure on X above is the coarsest for which (1) is satisfied.

The initial *Zero*-object X above will be an Alexandroff space if and only if the class $\{f_\alpha\}$ distinguishes points.

1.8 It is well known that every topological category is cotopological. Thus *Zero* admits coinitial structures.

Theorem *Let* (Y_α, g_α) *be a class of pairs of functions* g_α *from* $Y_\alpha \in$ *Zero* *to a given set* S . *Let* $\mathcal{F}$ *be the family of all functions* f *on* S *to* $\mathbb{R}$ *such that* $fg_\alpha \in Zero(Y_\alpha, \mathbb{R}_z)$ *for each* α . *The following are equivalent for* $X \in$ *Zero* *with underlying set* S :

(1) X *is coinitial for the* (Y_α, g_α)

(2) X *is initial for the* $(f, \mathbb{R}_z)$ *with* $f \in \mathcal{F}$

(3) $Zero(X, \mathbb{R}_z) = \mathcal{F}$.

Zero, being topological over *Ens*, is complete, and cocomplete. A *subspace* X of an Alexandroff space

Y is a subset of Y with the initial structure for (i,Y) where i is the inclusion function (i.e. restriction of zero-sets). An *embedding* f is an initial map that distinguishes points.

1.9 *Alex* is epi-reflective in *Zero* (being productive and hereditary) and is thus also complete. A coz-map is an epimorphism if and only if it is dense (for the underlying topology). The closed embeddings in *Alex* coincide with the extreme monomorphisms. *Alex* is well - and cowell-powered and it follows that an epireflective subcategory of *Alex* is one which is productive and closed-hereditary.

1.10 Theorem *Every Alexandroff space* X *is initial for the set of all pairs* $(f,\mathbb{R}_z)$ *with* $f \in Alex(X,\mathbb{R}_z)$.

1.11 Corollary *Every Alexandroff space* X *is embedded in a product of copies of* $\mathbb{R}_z$ (viz. $\mathbb{R}_z^{A(X)}$).

2. Some functorial relationships between *Tych*, *Alex* and *Unif*.

The symbols F, R, G, H and $\mathcal{U}$ will have a fixed meaning throughout the sequel.

2.1 The underlying topology of an Alexandroff space, taking zero-sets as a base for the closed sets, is delivered by a forgetful functor F: *Alex* → *Tych* . When FX = Y we say the Tychonoff space Y *admits* the Alexandroff space X . The zero-sets of the continuous functions on a Tychonoff space Y satisfy Z1 - Z4 in 1.1. The associated Alexandroff space is denoted RY and defines a functor R: *Tych* → *Alex* which is the unique section (i.e. right inverse) to F . R delivers the finest admissible structure on Y and is left adjoint to **F** . RY is initial for the set of all pairs $(f, \mathbb{R}_z)$ with f ∈ C(X) (Gilmour [1974]). Thus *Tych* is embedded as a full bicoreflective subcategory of *Alex*. The Alexandroff spaces X for which X = RFX will be called ***fine***.

2.2 A result that we use frequently is :

Theorem (Hager [1974]) *A Tychonoff space* X *admits a unique Alexandroff structure if and only if* X *is either Lindelöf or almost compact.*

In particular $\mathbb{R}$, the real line with its usual topology, admits only $\mathbb{R}_z$.

An amusing consequence of the previous theorem is the following :

Corollary *Let* X *be a Tychonoff space. The following are equivalent :*

(1) X *is Lindelöf*

(2) $X \times Y$ *admits a unique Alexandroff structure for each compact Hausdorff* Y

(3) $X \times \beta X$ *admits a unique Alexandroff structure.*

Proof The implications (1) ⇒ (2) ⇒ (3) are trivial.
(3) ⇒ (1) If $X \times \beta X$ is Lindelöf, so is X .
Otherwise $X \times \beta X$ is almost compact, hence pseudocompact so that $\beta(X \times \beta X) = \beta X \times \beta X$ and
$1 \geqslant \text{card}\,((\beta X \times \beta X) \setminus (X \times \beta X)) = \text{card}\,(\beta X \setminus X).\text{card}(\beta X)$
so that $\text{card}(\beta X \setminus X) = 0$ and X is compact.

2.3 The functor F preserves initiality and consequently preserves products and closed subspaces (extreme monomorphisms). Thus as a special case of a result of Brümmer ([1971], 1.9.2) we have :

Proposition *Let* S *be an epireflective subcategory of Tych. The full subcategory of* **Alex** *consisting of those Alexandroff spaces* X *for which* $FX \in S$ *, is epireflective in* **Alex**.

Examples (1) The compact Tychonoff spaces lift to the *compact* Alexandroff spaces. We denote the associated epireflector in **Alex** by β . No confusion should arise with the Stone-Čech epireflector β in *Tych*. The following properties are easily verified :

(i) $R\beta \cong \beta R$ (: *Tych* → **Alex**) ,

(ii) $F\beta X \cong \beta FX$ if and only if X is a fine Alexandroff space,

(iii) R and F restrict to an equivalence between the categories *Comp* **Alex** of compact Alexandroff spaces and *Comp Tych*, of compact Tychonoff spaces.

(2) The Alexandroff spaces with realcompact topology form an epireflective subcategory of **Alex**. The associated epireflector is denoted by υ_t , and is studied in section 4 of chapter 2.

There is another class of Alexandroff spaces, called realcompact and introduced in chapter 2 section 1, that is epireflective in **Alex**. *The corresponding epireflector is denoted by* υ . *The context will ensure that this notation is not confused with that for the Hewitt realcompact epireflector in Tych.*

2.4 There is a one-to-one correspondence between the cozero-sets of X and its Tychonoff reflection which will be an isomorphism of the corresponding lattices of cozero-sets (in the sense of chapter 3).

2.5 Let $\mathbb{R}_{su}$ denote the real line with the standard (metric) uniformity. The functor G from the category of separated uniform spaces and uniformly continuous maps *Unif* to *Alex*, assigns to each uniform space Y the Alexandroff space with the zero-sets of the uniformly continuous (bounded) functions on Y to $\mathbb{R}_{su}$ for zero-sets.

There are two right inverses of G with which we shall be concerned :

(1) $\mathcal{U}$: *Alex* → *Unif* , where $\mathcal{U}X$ is initial for the set of all pairs $(f,\mathbb{R}_{su})$ with $f \in Alex(X,\mathbb{R}_z)$,

(2) H: *Alex* → *Unif* , where the countable coz-covers (i.e. cozero-set covers) of an Alexandroff space X are a basis for the uniform covers of HX (Hager [1974]).

2.6 Let αY denote the (topologically) fine coreflection, and γY the completion of a uniform space Y . Let A be a class of uniform spaces.

Definition (Hager [1974]) A uniform space X is $\mathcal{A}$-*fine* if for every uniform space Y in $\mathcal{A}$, each uniformly continuous map $f: X \to Y$ factors through αY.

A uniform space is *separable* if it has a basis of countable covers. The category of separable $\mathcal{M}$-fine spaces (where $\mathcal{M}$ is the class of metric uniform spaces) is denoted by $S\mathcal{M}F$. For every Alexandroff space X, $HX \in S\mathcal{M}F$. Indeed :

Theorem (Hager [1974]) *The functors* H *and* G *restrict to an isomorphism of the categories* **Alex** *and* $S\mathcal{M}F$.

2.7 Two technical results :

Theorem (Gordon [1971]) $\mathcal{U}\upsilon \cong \gamma\mathcal{U}$.

Lemma (Hager [1974]) *If* X *is a separable uniform space then* X *has a basis of countable zero-set covers (the zero-sets being zero-sets of functions in* $\mathcal{U}nif(X,\mathbb{R}_{su})$).

2.8 Those $\mathcal{A}$-fine spaces which are not necessarily separable have been studied extensively by Frolík [1974 a,b], [1976]; Rice [1975], [1976]; Vilímovský [1973] and Hager [1979]. See especially the survey article of Frolík [1977] and the seminar notes SUS [1973-4].

In greater generality than SℳF we have the coz-fine spaces: A uniform space X is *coz-fine* if for each uniform space Y , *Unif*(X,Y) = *Alex*(GX,GY) .

In *Unif*, each coz-fine space is ℳ-fine, but not conversely. In separable uniform spaces the concepts are equivalent.

We note that the functor G: *Unif* → *Alex* does *not* have a left adjoint (Hager [1977]). We have the analogue of A-fine in *Alex* (see chapter 2), the fine functor being RF: *Alex* → *Alex*. This can of course be expressed in SℳF . However the "fine" functor in this situation is HRFG (resticted to SℳF) which is not a restriction of the (topologically) fine functor α .

CHAPTER 2

Realcompact Alexandroff Spaces

1. Realcompact Alexandroff Spaces and Wallman Realcompactifications

Realcompactness for an Alexandroff space $(X,\mathcal{Z})$ is defined (firstly by Gordon [1971] in analogy with the concept in topology) in terms of the convergence of certain ultrafilters on $\mathcal{Z}$ - the *real $\mathcal{Z}$-ultrafilters* (that is those $\mathcal{Z}$-ultrafilters which are closed under countable intersection - equivalently, having the countable intersection property).

1.1 Definitions An Alexandroff space $(X,\mathcal{Z})$ is *realcompact* if each real $\mathcal{Z}$-ultrafilter F is *fixed* (i.e. $\cap F \neq \emptyset$).

If Y is realcompact and X is densely embedded in Y , then we call Y a *realcompactification* of X .

1.2 Examples (a) RX is realcompact for each realcompact Tychonoff space X : RX and X have the same zero-sets.

(b) (Alò and Shapiro [1969], A. W. Hager) An Alexandroff space may have realcompact topology without being realcompact. Let A be the σ-algebra generated by the countable subsets of $\mathbb{R}$. Then $(\mathbb{R},A)$ is an Alexandroff space which is not realcompact : the real A-ultrafilter comprising all the co-countable subsets is not fixed. However $F(\mathbb{R},A)$ is discrete.

1.3 The full subcategory *Realcompact Alex* of realcompact Alexandroff spaces is reflective in *Alex*, (Gilmour [1974]). We shall denote the epireflection of the Alexandroff space $(X,\mathcal{Z})$ on *Realcompact Alex* by $\upsilon(X,\mathcal{Z})$ and, where no confusion should arise, by υX. In the construction of υX given below, we shall use a notation similar to that of Alò and Shapiro [1974]. The relationship between $\upsilon(X,\mathcal{Z})$ and the Wallman realcompactification $\rho(X,\mathcal{Z})$ of a Tychonoff space X (more precisely $F(X,\mathcal{Z})$) with Alexandroff base $\mathcal{Z}$, then becomes evident.

1.4 For an Alexandroff space $(X,\mathcal{Z})$ let $\mathcal{Z}^\upsilon$ denote the collection of all sets of the form

$Z^\upsilon = \{F: \ F$ a real $\mathcal{Z}$-ultrafilter containing $Z\}$

where $Z \in \mathcal{Z}$. Then $\mathcal{Z}^\upsilon$ satisfies the axioms Z1 - Z4 for the zero-sets of an Alexandroff space (ibid. 5.21 (10)). Denote the set of all real $\mathcal{Z}$-ultrafilters on X, with the Z^υ for zero-sets, by $\upsilon(X,\mathcal{Z})$.

1.5 Lemma *Whenever* $\mathfrak{S} = \{Z_\alpha^\upsilon\}$ *is a real* $\mathcal{Z}^\upsilon$*-ultrafilter on* $\upsilon(X,\mathcal{Z})$, *then* $S = \{Z_\alpha\}$ *is a real* $\mathcal{Z}$*-ultrafilter on* X.

Proof Let $Z_1, Z_2 \in S$ then $S \in Z_1^\upsilon, Z_2^\upsilon$ and these in turn are both in $\mathfrak{S}$.

Hence $\mathcal{S} \in Z_1^{\upsilon} \cap Z_2^{\upsilon} = (Z_1 \cap Z_2)^{\upsilon} \in \mathfrak{S}$ and $Z_1 \cap Z_2 \in \mathcal{S}$. Whenever $A \in \mathcal{Z}$ with $Z \in \mathcal{S}$ and $A \supset Z$ then $Z^{\upsilon} \subset A^{\upsilon}$ and $A^{\upsilon} \in \mathfrak{S}$. Thus $A \in \mathcal{S}$. Since $\emptyset \notin \mathfrak{S}$, $\emptyset \notin \mathcal{S}$. $\mathcal{S}$ is thus a filter. $\mathcal{S}$ is maximal :

Suppose $\mathcal{H} \supset \mathcal{S}$ and $A \in \mathcal{H}$, $A \notin \mathcal{S}$. Then A meets each $Z \in \mathcal{S}$ and A^{υ} meets each Z^{υ} $(Z \in \mathcal{S})$: For if $A^{\upsilon} \cap Z^{\upsilon} = \emptyset$ then $(A \cap Z)^{\upsilon} = \emptyset$, that is, no real $\mathcal{Z}$-ultrafilter contains $A \cap Z$. If $A \cap Z \neq \emptyset$, with $x \in A \cap Z$, then the point $\mathcal{Z}$-ultrafilter F_x generated by x is in $(A \cap Z)^{\upsilon}$, thus $A \cap Z = \emptyset$.

By maximality of $\mathfrak{S}$, $A^{\upsilon} \in \mathfrak{S}$ and $A \in \mathcal{S}$ giving the necessary contradiction. $\mathcal{S}$ is real: If $\{Z_{\alpha_n}\}$ is a countable sequence in $\mathcal{S}$ then $(\cap Z_{\alpha_n})^{\upsilon} = \cap (Z_{\alpha_n})^{\upsilon} \neq \emptyset$ since $\mathfrak{S}$ is real; thus $\cap Z_{\alpha_n} \neq \emptyset$.

1.6 As a result of the above, we conclude :

Proposition $\upsilon(X,\mathcal{Z})$ *is a realcompactification of* $(X,\mathcal{Z})$.

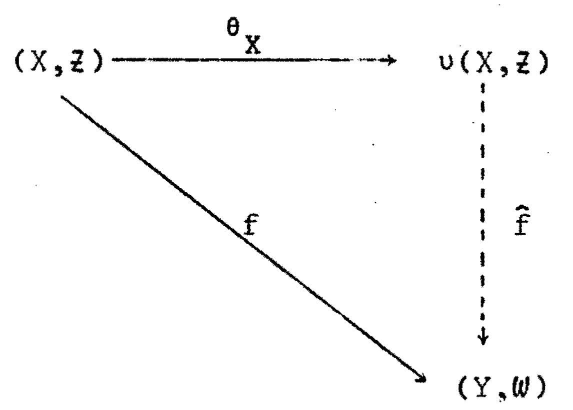
Proof Whenever $\mathfrak{S} = \{Z_\alpha^{\upsilon}\}$ is any real $\mathcal{Z}^{\upsilon}$-ultrafilter on $\upsilon(X,\mathcal{Z})$, then $\mathcal{S} = \{Z_\alpha\}$ is a real $\mathcal{Z}$-ultrafilter on X. Moreover $\mathcal{S} \in Z_\alpha^{\upsilon}$, for all α. Thus $\mathcal{S} \in \cap \mathfrak{S}$ and $\mathfrak{S}$ is fixed. It follows that $(X,\mathcal{Z})$ is realcompact.

It is straightforward to show that the mapping $\theta_X\colon (X,\mathcal{Z}) \to \upsilon(X,\mathcal{Z})$ given by $\theta_X(x) = F_x$ - the point

$\mathcal{Z}$-ultrafilter generated by x - is a dense embedding in *Alex*.

1.7 Theorem *The embedding* $\theta_X: (X,\mathcal{Z}) \to \upsilon(X,\mathcal{Z})$ *defines an epi-reflection of Alex onto the full subcategory of realcompact Alexandroff spaces.*

Proof



Given $(Y,\mathcal{W})$ realcompact and $f \in Alex(X,Y)$. If $\mathcal{F}$ is a real $\mathcal{Z}$-ultrafilter on X , then $f^{\#}(\mathcal{F}) = \{Z \in \mathcal{W}: f^{-1}(Z) \in \mathcal{F}\}$ is a real $\mathcal{W}$-ultrafilter on Y . (Note: $f^{\#}$ does not preserve $\mathcal{Z}$-ultrafilters in general (Gillman and Jerison [1960] 4H(2)), but $f^{\#}$ does preserve prime $\mathcal{Z}$-filters and every prime $\mathcal{Z}$-filter that is closed under countable intersections is a real $\mathcal{Z}$-ultrafilter (cf. Blair [1976], 2.3).)

Since $(Y,\mathcal{W})$ is realcompact we may define $\hat{f}: \upsilon(X,\mathcal{Z}) \to (Y,\mathcal{W})$ by :

$$\hat{f}(\mathcal{F}) = \cap f^{\#}(\mathcal{F}) \quad .$$

Property Z4 ensures that $\hat{f}$ is well-defined and that $(\hat{f} \circ \theta_X)(x) = \hat{f}(F_x) = \cap f^{\#}(F_x) = f(x)$.

If $W \in \mathcal{W}$ then

$$\begin{aligned}\hat{f}^{-1}(W) &= \{F : F \text{ a real } \mathcal{Z}\text{-ultrafilter on } X \text{ with } W \in f^{\#}(F)\} \\ &= \{F : F \text{ a real } \mathcal{Z}\text{-ultrafilter on } X \text{ with } f^{-1}(W) \in F\} \\ &= (f^{-1}(W))^{\upsilon} \in \mathcal{Z}^{\upsilon} .\end{aligned}$$

Then $\hat{f}$ is a coz-map and moreover is the unique extension of f , since θ_X is epi.

1.8 (1) As a consequence of 1.7, $\upsilon(X,\mathcal{Z})$ may be identified with the corresponding construct of Gordon [1971]. (See Gilmour [1974]).

(2) An Alexandroff space $(X,\mathcal{Z})$ is compact if every covering by cozero-sets admits a finite subcover. Equivalently the underlying topology is compact. By considering *all* $\mathcal{Z}$-ultrafilters in the above construction and making the necessary changes we obtain the compact epi-reflection $\beta(X,\mathcal{Z})$ of $(X,\mathcal{Z})$ in ***Alex*** (see Alexandroff [1941], Gordon [1971], Gilmour [1974]).

1.9 As in the case for Tychonoff spaces (Gillman and Jerison [1960], 8.4) the points of υX , for an Alexandroff space X , may be identified with those points x of βX for which $f^*(x)$ is finite for every coz-map $f: X \to \mathbb{R}_z$, where f^* is the Stone extension of f from βX into $\mathbb{R}_z^*$, the one point compactification of $\mathbb{R}$. One approach is via the observation that the zero-sets of βX are countable intersections of closures in βX of zero-sets of X (ibid. 6E(3)). For a nice exposition for the topological case see Walker [1974], 1.53.

1.10 Theorem *The realcompact Alexandroff spaces comprise the epi-reflective hull of* $\mathbb{R}_z$ *in* *Alex*.

Proof As *Realcompact Alex* is an epi-reflective subcategory of *Alex*, each closed subspace of a product of copies of the realcompact space $\mathbb{R}$, is realcompact (1.1.9).

Conversely, let X be realcompact. The continuous functions $g: F\beta X \to \mathbb{R}$ distinguish points and closed sets and hence also distinguish points. Also *Alex*$(\beta X, \mathbb{R}_z)$ = *Tych*$(F\beta X, \mathbb{R})$. Moreover each continuous $f^*: F\beta X \to \mathbb{R}_z^*$ is the Stone extension of some bounded coz-map $f: \beta X \to \mathbb{R}_z$. Thus the set of all coz-maps $f^*: \beta X \to \mathbb{R}_z^*$ distinguishes points and closed sets and distinguishes points. Then the evaluation map $e: \beta X \to (\mathbb{R}_z^*)^{A(X)}$ is an embedding in *Tych* and, with compactness of domain and codomain, an embedding in *Alex*. $e(\beta X)$ is closed in $(\mathbb{R}_z^*)^{A(X)}$. From the description of

υX in 1.9, X then is embedded via βX as a closed subspace of $\mathbb{R}_z^{A(X)}$.

1.11 It is not surprising that the compact Alexandroff spaces comprise the epi-reflective hull in ***Alex*** of the unit interval [0,1] with the subspace structure induced by $\mathbb{R}_z$. This follows immediately from the corresponding and well-known result in ***Tych***.

1.12 The following result has been proved, in equivalent situations, by Isbell [1958], Hager [1969], [1974], Steiner and Steiner [1970] and Alò and Shapiro [1974]. A very simple and direct proof was given by Salbany [1974].

Proposition *The underlying topology of each realcompact Alexandroff space is realcompact.*

1.13 The above result may also be quickly obtained as a corollary to 1.10 which states that each realcompact Alexandroff space is a closed subspace of a product of copies of $\mathbb{R}_z$. F preserves products and subspaces and as is well-known the epi-reflective hull of $\mathbb{R}$ in ***Tych*** is the full subcategory of realcompact spaces. Hence Proposition 1.12.

1.14 In their construction of the Wallman realcompactifiction $\rho(X,\mathcal{Z})$ of a topological space X with a strong delta normal (= Alexandroff) base $\mathcal{Z}$, Alò and Shapiro [1974] showed that (our notation) $\mathcal{Z}^{\upsilon}$ forms a base for the closed sets of $\rho(X,\mathcal{Z})$. Hence the identity :

$$F\upsilon = \rho \quad .$$

In the case of the Wallman compactification $\omega(X,\mathcal{Z})$, it is well-known that this is a Hausdorff compactification when $\mathcal{Z}$ is a normal base for X (Frink [1964]) - in this circumstance $\omega(X,\mathcal{Z})$ is called the Wallman-Frink compactification of X for the base $\mathcal{Z}$. Alexandroff [1941] had shown that a Hausdorff compactification is obtained when $\mathcal{Z}$ is an Alexandroff base for X. In this more restrictive setting (see 1.15(2) below) we obtain from 1.8(2) the identity :

$$F\beta = \omega \quad .$$

We may thus regard ρ and ω as functors : *Alex* $\to$ *Tych* .

1.15 (1) If $X = (\mathbb{R},A)$, A the σ-algebra generated by the countable subsets of $\mathbb{R}$ (see 1.2(b)), then the *only* free real A-ultrafilter is that consisting of all cocountable subsets of $\mathbb{R}$: We firstly show that no free real A-ultrafilter F on X can contain a countable set. Suppose A is countable and a member of $\mathcal{F}$. For each $a \in A$, there exists $B_a \in \mathcal{F}$ such that $a \notin B_a$ (otherwise $\mathcal{F}$ is fixed). Then $B = \bigcap_{a \in A} B_a$ is a zero-set of X and $A \cap B = \emptyset$. But if $\mathcal{F}$ is real then $B \in \mathcal{F}$ giving the necessary contradiction.

Thus every real free $\mathcal{A}$-ultrafilter has only co-countable sets as members. Every co-countable set meets every other co-countable set so maximality ensures uniqueness.

FX is discrete of Ulam non-measurable cardinality and hence a realcompact Tychonoff space. υX is not discrete and $\upsilon X \setminus X$ consists of one point.

This then is an example of a space X whose Wallman-Frink realcompactification $\rho X \cong F\upsilon X$ is distinct from its Hewitt realcompactification υFX .

(2) It was a long standing question of Frink [1964], whether every compactification of a Tychonoff space X is a Wallman-Frink compactification (for some normal base on X). This question has been answered in the negative by Ul'janov [1977].

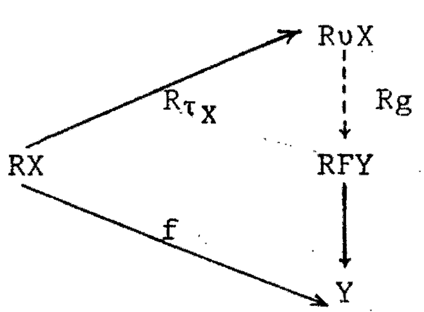
Here we give an example (in the more restricted setting) of a compactification Y of an Alexandroff space X which is not $\beta(X,\mathcal{Z})$ for any admissible Alexandroff structure $\mathcal{Z}$ on FX , yet $FY = \omega(FX,\mathcal{N})$ for some normal base $\mathcal{N}$ of FX .

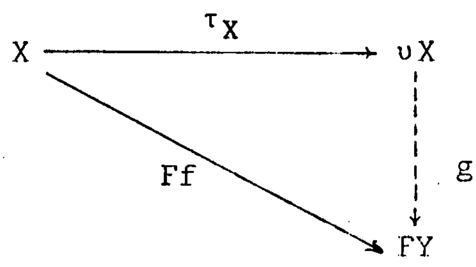
The one point compactification $\mathbb{R}^*$ of $\mathbb{R}$ admits a unique Alexandroff space Y. Then $Y|_{\mathbb{R}} = \mathbb{R}_z$, for $\mathbb{R}$ admits a unique Alexandroff structure (1.2.2) and F preserves subspaces (1.2.3). Clearly Y is not isomorphic to $\beta\mathbb{R}_z$ (for $\mathbb{R}^*$ is not isomorphic to $\beta\mathbb{R}$). Brooks [1967] has shown that the collection N of all zero-sets of continuous functions constant on the complement of some compact subset of $\mathbb{R}$ forms a normal base on $\mathbb{R}$, and $\omega(\mathbb{R},N)$ is $\mathbb{R}^*$.

$\mathbb{R}^*$ is of course a realcompactification of $\mathbb{R}$ distinct from $\rho(\mathbb{R}_z) \cong \mathbb{R}$. In general (Steiner and Steiner [1970]) a realcompact Tychonoff space X which admits a unique Alexandroff base will be homeomorphic to its only Wallman realcompactification, υX.

1.16 Proposition $R\upsilon \cong \upsilon R$.

Proof $R\upsilon X$ is realcompact for each space X. We show that $R\upsilon X$ has the universal property reserved for υRX. Let Y be realcompact and $f \in \mathcal{Alex}(RX,Y)$.





Then FY is realcompact (1.12) and there exists a unique g with $g.\tau_X = Ff$ where τ_X is the natural embedding $X \hookrightarrow \upsilon X$. Then $Rg: R\upsilon X \to RFY$ and as RFY is finer than Y we have the required extension of f which is unique as $R\tau_X$ is epi.

1.17 Corollary $\rho R \cong \upsilon$.

1.18 The following remarkable results of Gordon [1971] will be used repeatedly in the sequel. A proof of (1) for the case of a product with a finite number of factors is given in 3.4.14 and for the unrestricted case in 3.4.23 which follows quickly from (2) which is proved in 3.4.22.

(1) The $\mathcal{A}lex$-product of any number of pseudocompact Alexandroff spaces is again pseudocompact.

(2) υ distributes over arbitrary $\mathcal{A}lex$-products.

Since F preserves products, we may conclude from (2) and 1.14:

Theorem *The functor* ρ: *Alex* $\to$ *Tych preserves arbitrary products.*

1.19 An Alexandroff space $(X,\mathcal{Z})$ is pseudocompact if and only if $\upsilon(X,\mathcal{Z}) = \beta(X,\mathcal{Z})$ (Gordon [1971]). Compact spaces admit unique Alexandroff structure. Thus $(X,\mathcal{Z})$ is pseudocompact if and only if $\rho(X,\mathcal{Z}) = \omega(X,\mathcal{Z})$. Each pseudocompact Tychonoff space X admits only pseudocompact Alexandroff structures. Hence, if X is a pseudocompact Tychonoff space then $\rho(X,\mathcal{Z}) = \omega(X,\mathcal{Z})$ for every Alexandroff base $\mathcal{Z}$. The space G of Gillman and Jerison ([1960], 9.15) is pseudocompact with $G \times G$ not pseudocompact. Then $RG \times RG$ is a pseudocompact Alexandroff space (by 1.18(1)) and $\rho(RG \times RG) = \omega(RG \times RG)$. Note that $R(G \times G)$ is not pseudocompact so that this example shows that R does not preserve products.

We shall prove in 3.4.19 the result of Hager [1969] that every compactification of a pseudocompact Tychonoff space X is a Wallman-Frink compactification (in fact for an Alexandroff base on X).

2. Realcompact-fine Alexandroff Spaces

Our terminology is motivated by that of Hager [1974].

2.1 Definition An Alexandroff space X is *realcompact-fine* if for each realcompact Alexandroff space Y, each coz-map $f: X \to Y$ factors through RFY.

2.2 It can be easily proved that in the above definition we need only consider those coz-maps which are *onto* realcompact Y (Heldermann [1980], Hager [1974], Vilímovský [1973]).

2.3 Let $r(X,\mathcal{Z})$ (usually abbreviated rX) denote the restriction of $RF\upsilon(X,\mathcal{Z})$ to X. From the definition of R and 1.14, the zero-sets of rX are precisely the zero-sets of ρX restricted to X. Blasco [1979] calls (in our notation) the Alexandroff base $\mathcal{Z}$ *complete* if $(X,\mathcal{Z}) = r(X,\mathcal{Z})$. As we shall see in 2.6 $(X,\mathcal{Z})$ is realcompact-fine if and only if $\mathcal{Z}$ is complete.
We also note that FrX = FX for every Alexandroff space X.

2.4 Theorem $\upsilon(rX) = RF\upsilon X$ *for each Alexandroff space* X.

Proof $RF\upsilon X$ is a realcompactification of rX. It will suffice (Gordon [1971]; see 3.4.21) to show that each non-empty zero-set of $RF\upsilon X$ meets rX. This follows immediately form the result of Alò and Shapiro ([1974], 5.16) which shows that for any X , FX is G_δ-dense in ρX .

2.5 Corollary (Blasco [1979]) $\rho(rX) = \rho X$ *for each Alexandroff space* X .

2.6 Theorem *The following are equivalent for an Alexandroff space* X :

(1) X *is realcompact-fine*

(2) $X \cong rX$

(3) υX *is fine*

(4) RFY *is a realcompactification of* X *for each realcompactification* Y *of* X

(5) $\beta F \upsilon X \cong F\beta X$ *(i.e.* $\beta\rho X \cong \omega X$ *).*

Proof (1) ⇒ (4) If the realcompact-fine X is embedded in the realcompactification Y , with embedding $q: X \hookrightarrow Y$ say, then $m: X \to RFY$ is a coz-map, where $im = q$ and $i: RFY \to Y$ is the identity. Clearly m distinguishes points and m is initial because $im = q$ is. (4) ⇒ (3) If (4) holds then $RF\upsilon X$ is a realcompactification of X which has the universal property reserved for υX . The implication "(3) ⇒ (2)" is trivial. (2) ⇒ (1) Each coz-map f on X to realcompact Y extends to $\hat{f}: \upsilon X \to Y$. The restriction of $RF\hat{f}$ to rX gives the required factor $X \to RFY$ of f , since $X \cong rX$.

The equivalence of (2) and (5) for their particular settings was given first by Hager ([1969], 1.2) and later by Blasco [1979]. A simple proof uses the commutativity of R over β. We delay the proof however until 3.4.6.

2.7 Examples (1) Each realcompact realcompact-fine Alexandroff space is fine (e.g. use 2.6(3)). Each fine Alexandroff space is realcompact-fine; the converse is false (e.g. see (3) below).

(2) If E denotes the Sorgenfrey line then $RE \times RE$ is realcompact, but not fine: the set $A = \{(x,y) \in E \times E: y \geq x\}$ is a cozero-set of $E \times E$ (i.e. of $R(E \times E)$) but is not a cozero-set of $RE \times RE$. Thus the product of realcompact-fine (even fine) Alexandroff spaces need not be realcompact-fine.

We show below that if a product of Alexandroff spaces is realcompact-fine then the factor spaces are realcompact-fine (2.9).

(3) The members of a σ-algebra $\mathcal{B}$ on a set X may be taken as the zero-sets for an Alexandroff space $(X,\mathcal{B})$. If $\mathcal{B}$ consists of the Borel sets of $\mathbb{R}$ or of the Lebesgue measurable sets of $\mathbb{R}$, then $(\mathbb{R},\mathcal{B})$ is not realcompact-fine (Blasco [1979]). If $\mathcal{A}$ is the σ-algebra generated by the countable subsets of $\mathbb{R}$, then $(\mathbb{R},\mathcal{A})$ is realcompact-fine: $F(\mathbb{R},\mathcal{A})$ is Lindelöf.

$(\mathbb{R}, A)$ is neither realcompact nor fine. The underlying topology of each of the above three spaces is the discrete topology on $\mathbb{R}$.

(4) If X is a pseudocompact Alexandroff space then $\upsilon X = \beta X$ and hence X is realcompact-fine. Indeed every pseudocompact Tychonoff space admits only realcompact-fine Alexandroff structures. The converses to both the above statements are false. Consider $\mathbb{R}$ with the usual structures in both instances.

(5) Trivially, a discrete Alexandroff space is realcompact-fine. Thus the image of a realcompact-fine space need not be realcompact-fine.

2.8 Proposition *If* $X \times Y$ *is a fine Alexandroff space then* X *and* Y *are fine.*

Proof Since $X \times Y$ is fine and F preserves products, it is clear that $X \times Y = RFX \times RFY$. Restriction of cozero-sets then shows that $X = RFX$ and $Y = RFY$.

2.9 Corollary *If* $X \times Y$ *is a realcompact-fine Alexandroff space then* X *and* Y *are realcompact-fine.*

Proof If $X \times Y$ is realcompact-fine then $\upsilon(X \times Y)$ is fine (2.6). Thus $\upsilon X \times \upsilon Y$ is fine (1.18). From 2.8, υX and υY are fine, and reapplying 2.6, X and Y are realcompact-fine.

2.10 Proposition *The Alexandroff space* rX *is initial for the class of all pairs* (f, RFY) *with* Y *realcompact and* $f \in \mathcal{A}lex(X,Y)$.

Proof Immediate from definition 2.3.

2.11 Proposition *The following diagram is a pull-back in* $\mathcal{A}lex$.

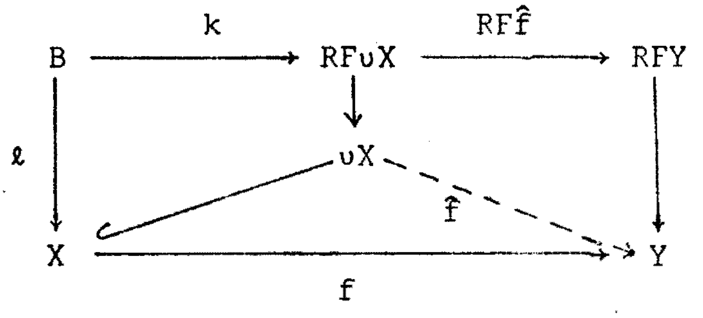
$$\begin{array}{ccc} rX & \hookrightarrow & RF\upsilon X \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \upsilon X \end{array} \qquad (*)$$

The horizontal arrows are the respective embeddings. The vertical arrows are the respective identity functions.

Proof Clearly the diagram (*) commutes.

Suppose
$$\begin{array}{ccc} B & \xrightarrow{\;k\;} & RF\upsilon X \\ {\scriptstyle \ell}\downarrow & & \downarrow \\ X & \hookrightarrow & \upsilon X \end{array}$$
commutes.

Then, for each $f: X \to Y$ with Y realcompact, the diagram



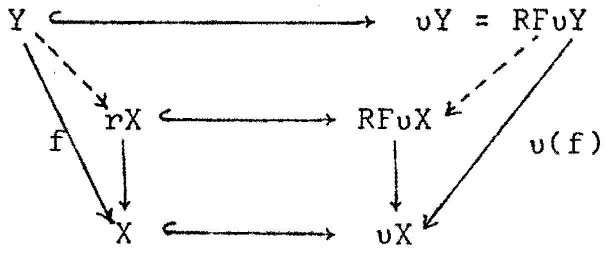
commutes, where $\hat{f}$ is the Hewitt extension for f and the arrow $RFY \to Y$ is the identity function.

In particular $RF\hat{f} \cdot k: B \to RFY$ is a coz-map, and coincides as function with the function $f \cdot \ell$. By the initiality of rX (see 2.10 above), ℓ and k factor (uniquely) through rX as required.

2.12 As is to be expected (Hager [1974], Vilímovský [1973] and Rice [1973]) we have :

Corollary *The correspondence* $X \mapsto rX$ *defines a coreflection of* **Alex** *onto the full subcategory of realcompact-fine Alexandroff spaces.*

Proof From 2.4, rX is realcompact-fine. If Y is realcompact-fine and $f: Y \to X$ is a coz-map then the diagram



commutes, where υ(f) (the Hewitt extension of f from υY to υX) factors through RFυX , as υY is fine. Apply 2.11 to obtain the unique factorization of f through rX ∘

3. Some Applications

3.1 Theorem *Let* X *and* Y *be Tychonoff spaces. Then,* $\upsilon(X \times Y) \cong \upsilon X \times \upsilon Y$ *if and only if* $r(RX \times RY) \cong R(X \times Y)$.

Proof Necessity. Applying 1.16 and 1.18(2) :

$$
\begin{aligned}
RF\upsilon(RX \times RY) &= R(F\upsilon RX \times F\upsilon RY) = R(\upsilon X \times \upsilon Y) \\
&= R\upsilon(X \times Y) \quad \text{by assumption} \\
&= \upsilon R(X \times Y) .
\end{aligned}
$$

Restrict to $X \times Y$.

Sufficiency.

$$
\begin{aligned}
\upsilon(X \times Y) &= F\upsilon R(X \times Y) \\
&= F\upsilon r(RX \times RY) \quad \text{by assumption} \\
&= F\upsilon(RX \times RY) \quad \text{by 2.4} \\
&= \upsilon X \times \upsilon Y \quad \text{applying 1.16 and 1.18(2).}
\end{aligned}
$$

The condition $r(RX \times RY) \cong R(X \times Y)$ is quickly seen to be equivalent to the condition of Blair that $X \times Y$ be z-embedded in $\upsilon X \times \upsilon Y$.

3.2 We have already noted that $RX \times RY$ and $R(X \times Y)$ may differ (1.19 or 2.7(2)). Another example: If X is uncountably infinite and discrete, then $RX \times RX$ is not discrete. The diagonal $\{(x,x): x \in X\}$ is not a cozero-set.

A sufficient condition for RX × RY to be fine (and thus equal R(X × Y)) is that βX × βY be C*-embedded in β(X × Y) ; informally - β(X × Y) = βX × βY (Blair and Hager [1977] and independently Gilmour [1978]). The condition is however not necessary: Consider $\mathbb{R}$ × Y , for any infinite compact space Y . β does not distribute over $\mathbb{R}$ × Y as neither factor space is finite nor is the product pseudocompact (Glicksberg [1959]). Being Lindelöf $\mathbb{R}$ × Y admits a unique Alexandroff structure.

3.3 Theorem *Let* X *and* Y *be Tychonoff spaces. Then* R(X × Y) = RX × RY *if and only if* RX × RY *is realcompact-fine and* υ(X × Y) ≅ υX × υY .

Proof Sufficiency follows immediately from 3.1. Necessity. Under the hypothesis RX × RY is fine and hence realcompact-fine. That the hypothesis implies that υ(X × Y) ≅ υX × υY is known (Blair and Hager [1977], Gilmour [1978]). A simple proof follows :

$$\begin{aligned} \upsilon(X \times Y) &= FR\upsilon(X \times Y) = F\upsilon R(X \times Y) \\ &= F\upsilon(RX \times RY) \quad \text{(by hypothesis)} \\ &= \upsilon X \times \upsilon Y . \end{aligned}$$

Blair and Hager [1977] proved : R(X × Y) = RX × RY if and only if X × Y is z-embedded in βX × βY (cf. the comment after 3.1). This follows quickly since one always has (i) $R(\beta X \times \beta Y) = \beta RX \times \beta RY$ (ii) R commutes with β.

We remark that the proofs of both 3.1 *and* 3.3 *hold for arbitrary products.*

3.4 We draw special attention to the paper of Ohta [1978] from which the following result is taken.

Theorem *The following are equivalent for a Tychonoff space* X.

(1) X *is locally compact with a countable base*

(2) $X \times Y$ *is* z-*embedded in* $\beta X \times \beta Y$ *for each space* Y

(3) $R(X \times Y) = RX \times RY$ *for each space* Y.

3.5 The next result characterises those Alexandroff bases $\mathcal{Z}$ for which $\rho(X,\mathcal{Z}) = \upsilon X$.

Theorem *Let* X *be an Alexandroff space.* $F\upsilon X \cong \upsilon FX$ *if and only if* rX *is fine.*

Proof If rX is fine, then $rX = RFrX = RFX$ (see 2.3) and thus $F\upsilon X = F\upsilon(rX)$ (applying 2.4)

$= F\upsilon(RFX) = \upsilon FX$ (applying 1.16).

Conversely, $\upsilon(rX) = RF\upsilon X$ (applying 2.4)

$= R\upsilon FX$ by assumption

$= \upsilon RFX$ (applying 1.16).

Restrict to X.

3.6 Proposition *An Alexandroff space* Y *is realcompact if and only if* FY *is realcompact and* rY *is fine.*

Proof Straightforward.

The Alexandroff space $Y = (\mathbb{R}, A)$ of 2.7(3) has FY realcompact. rY = Y is not fine.

Problem Characterise those Tychonoff spaces X for which all admissible Alexandroff structures $\mathcal{Z}$ have $r(X,\mathcal{Z})$ fine. This class contains all Lindelöf spaces and all almost compact spaces; for Hager [1974] has characterised these as the spaces which admit a unique Alexandroff structure (which is then fine).

3.7 Necessary and sufficient conditions for the extension of continuous functions on a Tychonoff space X to $\rho(X,\mathcal{Z})$, for a given Alexandroff base $\mathcal{Z}$, have been obtained by Bentley and Naimpally [1974] and Blasco [1979]. Formulated in *Alex* we have the result, with simple proof:

Proposition *Given an Alexandroff base* $\mathcal{Z}$ *on the Tychonoff space* X *, a real-valued continuous function* f *on* X *can be extended to* $\rho(X,\mathcal{Z})$ *if and only if* $f: r(X,\mathcal{Z}) \to \mathbb{R}_z$ *is a coz-map.*

Proof Sufficiency. If $f: r(X,\mathcal{Z}) \to \mathbb{R}_z$ then there exists a unique extension g of f with $g: \upsilon r(X,\mathcal{Z}) \to \mathbb{R}_z$. Then $Fg: F\upsilon r(X,\mathcal{Z}) = F\upsilon X \to \mathbb{R}$ extends Ff uniquely.
Necessity. The hypothesis implies that f extends to a coz-map : $R\rho(X,\mathcal{Z}) \to \mathbb{R}_z$. Restrict to X .

A comparison of the above proposition with the result of Bentley and Naimpally ([1974], Theorem 6) leads to the following characterization of their concept of an ω-map (in our setting) :

A continuous map $f: (X,\mathcal{Z}) \to (Y,\mathcal{W})$ is an ω-map if and only if $f: r(X,\mathcal{Z}) \to r(Y,\mathcal{W})$ is a coz-map.

3.8 Consider the three related problems :

(I) *Find a constructive method for obtaining* rX *from* X .

This problem was raised by Blasco [1979].

(II) *Find an internal, set theoretic method for obtaining* RFX *from* X .

(III) *Find a constructive method for obtaining the set* C(FX) *of all continuous functions on* X *from the algebra* A(X) *of all coz-functions on* X .

The latter problem was discussed by Hager [1969] (see also other papers cited there).

The solution of (II) will (via 2.11 say) give a solution for (I). In section 6 of chapter 3 we give a solution, in a certain sense, for (II).

4. Topologically realcompact Alexandroff spaces

It was observed in 2.7(3) that an Alexandroff space need not be realcompact in order to have realcompact topology. We introduce :

4.1 Definition An Alexandroff space X is *topologically realcompact* if it has realcompact topology (i.e. FX is realcompact). Trivially, X is topologically realcompact if and only if RFX is realcompact.

The full subcategory of all topologically realcompact Alexandroff spaces is epi-reflective in *Alex* (1.2.3). Denote the corresponding epi-reflector by υ_t and the unit of the adjunction by λ . We indicate that certain of the nice properties of υ in *Alex* are held by υ_t .

4.2 Proposition *Every Alexandroff space* X *is embedded as a dense subspace of* $\upsilon_t X$.

Proof Let $\theta_X: X \hookrightarrow \upsilon X$ and $\lambda_X: X \to \upsilon_t X$ be the units of the respective reflections. Since υX is topologically realcompact (1.12), θ_X extends (uniquely) to a coz-map $\hat{\theta}_X$ so that $\hat{\theta}_X . \lambda_X = \theta_X$. Since θ_X is an embedding, this implies that λ_X is an embedding.

Since each realcompact Alexandroff space is topologically realcompact, $\upsilon(\upsilon_t X)$ has the universal property of υX and $\upsilon_t X$ is naturally embedded in υX .

4.3 The following theorem is due to Comfort and Herrlich [1976] who state it and give an elegant proof for it in the setting of Hausdorff topological spaces. Their proof carries over with only the necessary small changes to $\mathcal{Alex}$. Let $\mathcal{RD}$ denote, for a subcategory $\mathcal{D}$ of $\mathcal{Alex}$, all those Alexandroff spaces X for which there exists $Y \in \mathcal{D}$ with X a subspace of Y .

Theorem *Let* $\mathcal{D}$ *and* $\mathcal{E}$ *be epi-reflective subcategories of* $\mathcal{Alex}$ *with reflectors* d *and* e *respectively. If* $\mathcal{D} \subset \mathcal{E} \subset \mathcal{RD}$ *and* $X, Y \in \mathcal{RD}$ *with* $d(X \times Y) = dX \times dY$, *then* $e(X \times Y) = eX \times eY$.

4.4 Corollary *For any Alexandroff spaces* X *and* Y $\upsilon_t(X \times Y) = \upsilon_t X \times \upsilon_t Y$.

Proof We need only observe the following facts and apply 4.3.

(i) Every realcompact Alexandroff space is topologically realcompact.

(ii) Every Alexandroff space X is a subspace of realcompact υX .

(iii) υ distributes over products.

4.5 With a similar proof to that of 1.16, $R\upsilon \cong \upsilon_t R$ and consequently $\upsilon_t R \cong \upsilon R$, i.e. υ_t and υ coincide on "topological spaces". Indeed υ and υ_t coincide on the larger class consisting of the (topologically realcompact) - fine Alexandroff spaces.

We call an Alexandroff space X *(topologically realcompact)-fine* if for each topologically realcompact Alexandroff space Y, each coz-map $f: X \to Y$ factors through RFY.

Clearly, each (topologically realcompact)-fine space is realcompact-fine. The converse is false - take $(\mathbb{R}, A)$ of 2.7(3). Denote by $r_t X$ the restriction of $RF\upsilon_t X$ to X. We omit the proofs of the following results - they are identical to their analogues in section 2.

Theorem *The following are equivalent for an Alexandroff space* X.

(1) X *is (topologically realcompact)-fine*

(2) $X \cong r_t X$

(3) $\upsilon_t X$ *is fine*

(4) RFY *is a (topologically realcompact)-ification of* X *for each (topologically realcompact)-ification* Y *of* X.

(5) $\beta F \upsilon_t X \cong F\beta X$.

4.6 Proposition *The Alexandroff space* r_tX *is initial for the class of all pairs* (f,Y) *with* Y *topologically realcompact and* $f \in \mathcal{A}lex(X,Y)$.

4.7 Proposition *The following diagram is a pull-back in* $\mathcal{A}lex$

$$\begin{array}{ccc} r_tX & \hookrightarrow & RF\upsilon_tX \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \upsilon_tX \end{array}$$

The horizontal arrows are the respective embeddings. The vertical arrows are the respective identity functions.

4.8 Corollary *The correspondence* $X \mapsto r_tX$ *defines a coreflection of* $\mathcal{A}lex$ *onto the full subcategory of (topologically realcompact)-fine Alexandroff spaces.*

4.9 If X is (topologically realcompact)-fine then υ_tX is fine (4.5) and υ_tX is realcompact. Consequently if $X = r_tX$ then $\upsilon_tX = \upsilon X$. The converse is false: $\upsilon_t(RE \times RE) = \upsilon(RE \times RE)$ but $RE \times RE$ is not (topologically realcompact)-fine (2.7(2)).

1 Preliminaries

1.1 Definition A *σ-frame* is a lattice L which has all countable joins and finite meets, possesses a greatest element 1 and a least element 0 and satisfies the distribution law $x \wedge \bigvee x_n = \bigvee x \wedge x_n$ ($n \in I$, countable). A *homomorphism* between σ-frames is a map preserving countable joins, finite meets, 0 and 1 .

These are the objects and maps of a category, σ *Frm* .

1.2 Examples (a) The family $\mathcal{P}X$ of all subsets of any set X forms a σ-frame with intersection for meet and union for join. The greatest and least elements are X and ∅ .

(b) A *sub-σ-frame* A of a σ-frame L is a subset A of L which is a σ-frame with the operations and distinguished elements of L .

The lattice $\mathfrak{U}X$ of cozero-sets of an Alexandroff space X is a σ-frame (regarded as a sub-σ-frame of $\mathcal{P}X$). In particular the cozero-sets of any topological space form a σ-frame.

(c) Each Heyting σ-algebra is a σ-frame :
A *Heyting σ-algebra* is a lattice L with $\wedge$, $\bigvee_n$, 0, 1

and such that for each pair a,b in L the set of all $x \in L$ with $a \wedge x \leq b$ contains a greatest element $b:a$ (cf. Birkhoff's [1979] definition for a Brouwerian lattice).

(d) An uncountable set X with all its countable subsets is an example of a σ-frame that is not a Heyting σ-algebra.

1.3 The σ-frames that we shall chiefly be concerned with satisfy a further property. For elements a and b of a σ-frame L, a *is rather below* b, written $a \ll b$, if there exists an element c of L with $a \wedge c = 0$, $c \vee b = 1$.

Definition A σ-frame is *regular* if each element a of L is a countable join of elements rather below it, i.e. for each $a \in L$ there exist $x_n \in L$ with $a = \bigvee x_n$, $x_n \ll a$.

The relation $\ll$ satisfies the following properties:

(1) If $a \leq x \ll y \leq b$ then $a \ll b$

(2) If $a \ll x$ and $b \ll y$ then $a \wedge b \ll x \wedge y$ and $a \vee b \ll x \vee y$.

Moreover it is easily verified that a homomorphism of σ-frames preserves $\ll$.

1.4 Definition A σ-frame L is *normal* if for each pair a,b of elements of L with $a \vee b = 1$, there exist u,v in L with $a \vee u = 1 = v \vee b$ and $u \wedge v = 0$.

Apparently the following result, due to Banaschewski [1980a] is not widely known.

1.5 Proposition *Every regular σ-frame is normal.*

Proof This follows directly from the following Lemma, taking $a \vee b = 1$.

Lemma *For any σ-frame* L, *if* $a = \bigvee a_n$ $(a_n \ll a)$, $b = \bigvee b_n$ $(b_n \ll b)$ *then there exist* u, v *in* L *with* $a \vee u = a \vee b = v \vee b$, $u \wedge v = 0$.

Proof By 1.3(2) we may assume that $a_n \leq a_{n+1}$ and $b_n \leq b_{n+1}$. Choose u_n and v_n with
$a_n \wedge u_n = 0$, $a \vee u_n = 1$ and $b_n \wedge v_n = 0$, $b \vee v_n = 1$.
Let $u = \bigvee u_n \wedge b_n$, $v = \bigvee a_n \wedge v_n$, then

$$a \vee u = a \vee \bigvee u_n \wedge b_n = \bigvee (a \vee a_n) \wedge (a \vee b_n) = \bigvee a \vee b_n$$
$$= a \vee \bigvee b_n = a \vee b .$$

Similarly $v \vee b = a \vee b$.

$$u \wedge v = (\bigvee u_n \wedge b_n) \wedge (\bigvee a_n \wedge v_n)$$
$$= \bigvee_n \bigvee_k u_n \wedge b_n \wedge a_k \wedge v_k .$$

If $k \leqslant n$ then $a_k \wedge u_n \leqslant a_n \wedge u_n = 0$.
If $k \geqslant n$ then $b_n \wedge v_k \leqslant b_k \wedge v_k = 0$.
Thus $u \wedge v = 0$.

The regular σ-frames then are precisely the *Alexandroff algebras* of Reynolds [1979].

1.6 Corollary *In any regular σ-frame L the relation $\ll$ interpolates.*

Proof If $a \ll b$ then there exists x with $a \wedge x = 0$, $b \vee x = 1$. By normality of L , there exist u, v with $b \vee u = 1 = v \vee x$ and $u \wedge v = 0$. Then $a \ll v \ll b$.

1.7 Example 1.2(d) is of a *non*-regular σ-frame (here $A \ll B$ if and only if $A = \emptyset$, $B = X$).

Cozero-set structures on a set X are the regular σ-frames of subsets of X : If X is an (Alexandroff) space and if $A , B \in \mathfrak{U}X$ then $A \ll B$ if and only if there exists $C \in \mathfrak{U}X$ with $A \subset X \setminus C \subset B$.

$\mathfrak{U}X$ is regular as a σ-frame as it is nest generated (1.1.3) .

Any regular σ-frame of subsets of a set X , is normal, and regularity ensures that it is complement generated.

2 Adjoint functors and dualities

Reg σ*Frm* is the full subcategory of σ*Frm* whose objects are the regular σ-frames. If X and Y are Alexandroff spaces and $f: X \to Y$ is a coz-map, then $\mathfrak{U}f = f^{-1}: \mathfrak{U}Y \to \mathfrak{U}X$ is a homomorphism in *Reg* σ*Frm*. This defines a contravariant functor $\mathfrak{U}$: *Alex* $\to$ *Reg* σ*Frm* .

2.1 Definition A *filter* P of a σ-frame L is a subset of L satisfying :

(a) $E \subset P$, E finite $\Rightarrow \wedge E \in P$

(b) $a \in P$, $b \geq a \Rightarrow b \in P$

If P further satisfies

(c) $\vee S \in P$ ($S \subset L$ countable) $\Rightarrow S \cap P \neq \emptyset$

then P is a σ-*prime filter*.

Note that regarding $\emptyset$ as finite, (a) implies $1 \in P$ and (c) implies $0 \notin P$.

The σ-prime filter of $\mathfrak{U}X$ whose members are the cozero-sets which contain a given point x , is denoted by $\mathfrak{U}(x)$.

2.2 We define a (contravariant) functor Ψ: *Reg* σ*Frm* $\to$ *Alex* as follows : The points of ΨL - the *spectrum* of $L \in$ *Reg* σ*Frm* - are the σ-prime filters on L . The cozero-sets of ΨL are the sets of form

$$\Psi_a = \{P: a \in P \in \Psi L\} \quad , \quad a \in L .$$

Then $\mathfrak{U}\Psi L = \{\Psi_a\}_{a \in L}$:

As may be quickly verified

$$\Psi_a \cap \Psi_b = \Psi_{a \wedge b}$$

$$\cup \Psi_{a_n} = \Psi_{\vee a_n}$$

$$\Psi_1 = \Psi L \quad , \quad \Psi_0 = \emptyset$$

and distributivity is immediate.

Thus $\mathfrak{U}\Psi L$ is a σ-frame; in fact these identities also verify that the map $a \longmapsto \Psi_a$ is a homomorphism which then preserves regularity. We conclude from 1.7 that ΨL is an Alexandroff space - ΨL being separated : $P \neq Q$, $P,Q \in \Psi L$ then there exists $a \in P$, $a \notin Q$ say. Then $P \in \Psi_a$, $Q \notin \Psi_a$.

If $f: L \to M$ is a homomorphism then $\Psi f: \Psi M \to \Psi L$ is given by $\Psi f(P) = f^{-1}(P)$ for each $P \in \Psi M$. It is easily checked that $f^{-1}(P) \in \Psi L$.

2.3 Proposition Ψ , $\mathfrak{U}$ *are adjoint on the right.*

Proof Let $\eta_X: X \to \Psi\mathfrak{U}X$ be the map defined by $\eta_X(x) = \mathfrak{U}(x)$ - where $\mathfrak{U}(x)$ is the neighbourhood base of the point x comprising all cozero-sets containing x . η_X is a coz-map : If $U \in \mathfrak{U}X$ then

$$\begin{aligned} \eta_X^{-1}(\Psi_u) &= \{x: \eta_X(x) \in \Psi_u\} \\ &= \{x: \mathfrak{U}(x) \in \Psi_u\} \\ &= \{x: U \in \mathfrak{U}(x)\} = U . \end{aligned}$$

Naturality of η : If $g: X \to Y$ is any coz-map and $x \in X$

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ \Psi\mathfrak{U}X & \xrightarrow[\Psi\mathfrak{U}g]{} & \Psi\mathfrak{U}Y \end{array} \qquad \begin{array}{ccc} x & \longmapsto & g(x) \\ \downarrow & & \downarrow \\ \mathfrak{U}(x) & & \mathfrak{U}(g(x)) \end{array}$$

then
$$\begin{aligned} \Psi\mathfrak{U}g(\mathfrak{U}(x)) &= (\mathfrak{U}g)^{-1}\mathfrak{U}(x) \\ &= \{U \in \mathfrak{U}Y : \mathfrak{U}g(U) \in \mathfrak{U}(x)\} \\ &= \{U \in \mathfrak{U}Y : g^{-1}(U) \in \mathfrak{U}(x)\} \\ &= \{U \in \mathfrak{U}Y : x \in g^{-1}(U)\} \\ &= \mathfrak{U}(g(x)) . \end{aligned}$$

We have already observed in the previous paragraph that the map $\varepsilon_L: L \to \mathfrak{U}\Psi L$ given by $\varepsilon_L(a) = \Psi_a$ is a homomorphism. Naturality of ε: If $f: L \to M$ is any homomorphism and $a \in L$

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle \varepsilon_L}\downarrow & & \downarrow{\scriptstyle \varepsilon_M} \\ \mathfrak{U}\Psi L & \xrightarrow[\mathfrak{U}\Psi f]{} & \mathfrak{U}\Psi M \end{array} \qquad \begin{array}{ccc} a & \longmapsto & f(a) \\ \downarrow & & \downarrow \\ \Psi_a & & \Psi_{f(a)} \end{array}$$

then
$$\begin{aligned} \mathfrak{U}\Psi f(\Psi_a) &= (\Psi f)^{-1}\Psi_a \\ &= \{P \in \Psi M : \Psi f(P) \in \Psi_a\} \\ &= \{P \in \Psi M : f^{-1}(P) \in \Psi_a\} \\ &= \{P \in \Psi M : a \in f^{-1}(P)\} \\ &= \Psi_{f(a)} . \end{aligned}$$

comprising those objects for which η (respectively ε) is pointwise an isomorphism.

In the case of η and ε introduced above, the restriction is proper as the following examples illustrate.

(1) Let $X = (\mathbb{R}, A)$ where A is the σ-algebra generated by the countable subsets of $\mathbb{R}$. Then X is an Alexandroff space. If P is the collection of all cocountable subsets of $\mathbb{R}$, then P is a σ-prime filter on A which is not of the form $\mathfrak{U}(x)$ for any $x \in \mathbb{R}$. Thus $\eta_X : X \to \Psi\mathfrak{U}X$ is not onto (see 2.1.2(b)).

(2) (Banaschewski) If L is the complete Boolean algebra of regular open subsets of the unit interval, then L is a regular σ-frame without atoms. It follows that L has no σ-prime filters :

Every σ-prime filter P on L is *completely prime* (i.e. $\bigvee U_\alpha \in P \Rightarrow U_\beta \in P$ for some β , over an arbitrary index set). Now a completely prime filter Q on any complete Boolean algebra B gives rise to an atom as follows :

Let $c = \bigvee x \; (x \notin Q)$. Suppose there exists a , $c < a < 1$. Then $c = a \wedge (c \vee \sim a)$ and $a \in Q$. If $c \vee \sim a > c$, then $c \in Q$, which is not possible. Thus $c \vee \sim a = c$, and $\sim a \leqslant c$ and then $\sim a \leqslant a$

and $1 = a \vee (\sim a) \leqslant a$. Thus $a = 1$ and $\sim c$ is an atom.

Hence $\Psi L = \emptyset$ and certainly $\mathfrak{U}\Psi L$ is not isomorphic to L .

2.5 Properties of η and ε For $U \in \mathfrak{U}X$, $\eta_X^{-1}(\Psi_U) = \{x: \mathfrak{U}(x) \in \Psi_U\} = \{x: U \in \mathfrak{U}(x)\} = U$. Thus η_X is an initial coz-map. Moreover, as cozero-sets of X separate points, η_X is $1-1$ and hence an embedding. In fact η_X is an *essential embedding* i.e. each f with $f\eta_X$ an embedding is itself an embedding:

If $f\eta_X$ is an embedding then $\mathfrak{U}\eta_X\mathfrak{U}f$ is onto and thus $\mathfrak{U}\eta_X$ is onto. We show that $\mathfrak{U}\eta_X$ is an isomorphism (anticipating a result (2.12) this says simply that $\mathfrak{U}X \cong \mathfrak{U}\upsilon X)$. Now $\mathfrak{U}\eta_X: \mathfrak{U}\Psi\mathfrak{U}X \to \mathfrak{U}X$ and a typical element of $\mathfrak{U}\Psi\mathfrak{U}X$ is Ψ_U for $U \in \mathfrak{U}X$. Then, as above, $\mathfrak{U}\eta_X(\Psi_U) = \eta_X^{-1}(\Psi_U) = U$, so that $\mathfrak{U}\eta_X$ is clearly $1-1$ and hence an isomorphism. It follows that $\mathfrak{U}f$ is onto and consequently f is initial. All spaces being separated we conclude that f is an embedding.

For $U \neq \emptyset$, $U \in \mathfrak{U}X$, $\mathfrak{U}(x) \in \Psi_U$ for any $x \in U$ so that each non-empty cozero set Ψ_U of $\Psi\mathfrak{U}X$ meets $\eta_X X$, thus η_X is an epi-morphism.

$\varepsilon_L: L \to \mathfrak{U}\Psi L$ with $\varepsilon_L(a) = \Psi_a$ is clearly onto. In the next paragraph we characterise Fix ε .

2.6 If $\mathbf{2}$ denotes the σ-frame with two distinct elements, then we say the σ-frame L *has enough homomorphisms* if the homomorphisms : $L \to \mathbf{2}$ separate the elements of L . Similarly, L *has enough σ-prime filters* if they separate the elements of L .

Proposition *For* L *a regular σ-frame the following statements are equivalent :*

(1) $L \cong \mathfrak{U}X$ *, for some Alexandroff space* X

(2) $\varepsilon_L : L \to \mathfrak{U}\Psi L$ *is an isomorphism*

(3) L *has enough σ-prime filters*

(4) L *has enough homomorphisms.*

Proof (1) ⇒ (3) For $U, V \in \mathfrak{U}X$, with $U \neq V$, there exists $x \in U$, $x \notin V$ say. Then $U \in \mathfrak{U}(x)$, $V \notin \mathfrak{U}(x)$.

(3) ⇒ (2) We've seen that ε_L is onto.
$\varepsilon_L(a) = \varepsilon_L(b) \iff \Psi_a = \Psi_b \Rightarrow a = b$ by hypothesis.
That (2) ⇒ (1) is trivial.

(3) ⇒ (4) If a and b are distinct elements of L take $P \in \Psi L$ with $a \in P$, $b \notin P$ say. Define $h: L \to \mathbf{2}$ by

$$h(x) = 1 , \; x \in P$$
$$= 0 , \; x \notin P .$$

It is easily verified that h is a homomorphism which then separates a and b .

(4) ⇒ (3) a and b distinct elements of L with $h(a) = 1$, $h(b) = 0$ for a homomorphism h , then $h^{-1}\{1\}$ is a σ-prime filter on L separating a and b .

2.7 The following theorem characterises Fix η . The equivalence (2) ⟺ (3) is due to Hager [1974] and it is his proof of (2) ⇒ (3) that we give here.

Following Bourbaki [1966] we shall identify the points of the completion γY of a uniform space Y with the minimal Cauchy filters on Y . Recall also (1.2.6) that the functor $H: \mathcal{Alex} \to \mathcal{SMF}$ is an isomorphism and that HX has the countable coz-covers as basic uniform covers.

We need the concept of a *regular filter* on a σ-frame L : $\mathcal{F}$ is a regular filter on L if for each $a \in \mathcal{F}$ there exists $b \in \mathcal{F}$ with $b \ll a$.

Theorem *The following are equivalent for an Alexandroff space* X :

(1) $X \cong \Psi\mathfrak{A}X$

(2) HX *is complete*

(3) X *is realcompact.*

$\mathcal{F}$ is Cauchy $U_m \in \mathcal{F}$ for some m .

Finally since $\mathcal{F}$ is σ-prime, $P = \{a \in L : \Psi_a \in \mathcal{F}\}$ is σ-prime and $\mathcal{F}$ is a base for the neighbourhood filter of P .

2.9 Proof of the Theorem The implication (1) ⇒ (2) follows immediately from the above Proposition.

(2) ⇒ (1) We show that each point of the spectrum $\Psi\mathfrak{U}X$ is a point (of the completion) of HX (as a minimal Cauchy filter).

If $\mathcal{F} \in \Psi\mathfrak{U}X$ then $\mathcal{F}$ is Cauchy : If U is a basic entourage of HX , $U = \bigcup U_n \times U_n$ where $X = \bigcup U_n$, $U_n \in \mathfrak{U}X$. But $X \in \mathcal{F}$ and $\mathcal{F}$ is σ-prime, so there exists some $U_n \in \mathcal{F}$ and $U_n \times U_n \subset U$. $\mathcal{F}$ is regular: If $U \in \mathcal{F}$ then by regularity of $\mathfrak{U}X$, $U = \bigcup V_n$ where $V_n \ll V_{n+1}$. But $\mathcal{F}$ is σ-prime thus there exists $V_n \in \mathcal{F}$ and $V_n \ll U$.

Finally $\mathcal{F}$ is minimal Cauchy : If $U \in \mathcal{F}$ then there exists a $W \in \mathcal{F}$ with $W \ll U$ (since $\mathcal{F}$ regular). Take $V \in \mathfrak{U}X$, $V \cap W = \emptyset$, $V \cup U = X$. Then $\{V,U\}$ is a coz-cover of X and thus $E = (U \times U) \cup (V \times V)$ is a basic entourage in HX with $E(V) \subset U$.

(3) ⇒ (2) If X is realcompact then $\mathcal{U}X$ is complete (1.2.7) , where $\mathcal{U}X$ is initial for the set of all pairs $(f,\mathbb{R}_{su})$ with $f \in \mathrm{Alex}(X,\mathbb{R}_z)$. For each $f \in \mathrm{Alex}(X,\mathbb{R}_z)$ and each

countable coz-cover $\mathcal{B}$ of $\mathbb{R}_z$, $f^{-1}(\mathcal{B})$ is a countable coz-cover of X and thus $f \in \mathrm{Unif}(HX, \mathbb{R}_{su})$ (recall, $\mathbb{R}_{su}$ has as basis, the countable open covers of $\mathbb{R}$). Hence HX is finer than $\mathcal{U}X$ and HX is complete.

(2) ⇒ (3) Let $\mathcal{F}$ be a real z-ultrafilter on X. Suppose that $\mathcal{F}$ does not have the Cauchy property on HX. Now HX has a basis of countable zero-set covers (1.2.7), thus there exists a countable cover $\{Z_n\}$ of zero-sets of X so that no Z_n is in $\mathcal{F}$. $\mathcal{F}$ is an ultrafilter, thus there exists $F_n \in \mathcal{F}$ with $F_n \cap Z_n = \emptyset$, for each n. Then $(\cup Z_n) \cap (\cap F_n) = \emptyset$ and hence $\cap F_n = \emptyset$ giving the necessary contradiction.

Thus $\mathcal{F}$ has the Cauchy property on HX and the filter of subsets of X generated by $\mathcal{F}$ converges. Then $\mathcal{F}$ has a unique cluster point p which since each member of $\mathcal{F}$ is closed says that $\cap \mathcal{F} = \{p\}$.

2.10 Proposition $\gamma H \cong H\Psi\mathfrak{U}$.

Proof From the theorem (2.7) $H\Psi\mathfrak{U}X$ is complete for each Alexandroff space X. Moreover $\eta_X: X \to \Psi\mathfrak{U}X$ is a dense embedding (2.5). $H\eta_X: HX \to H\Psi\mathfrak{U}X$ is initial:

Let $U = \cup U_n \times U_n$, where $U_n' \in \mathfrak{U}X$ for each n and $\cup U_n = X$, be a basic entourage of HX. Observe that if $A \in \mathfrak{U}X$ then $x \in A$ if and only if $\eta_X(x) \in \Psi_A$.

Thus $(x,y) \in U$ if and only if

$(\eta_X(x), \eta_X(y)) \in \bigcup \Psi_{U_n} \times \Psi_{U_n}$, where each Ψ_{U_n} is a cozero-set of $\Psi\mathfrak{U}X$ and $\bigcup \Psi_{U_n} = \Psi\mathfrak{U}X$: If $P \in \Psi\mathfrak{U}X$, then $X \in P$. But $X = \bigcup U_n$ and P is σ-prime, hence there exists $U_m \in P$ and then $P \in \Psi_{U_m}$.

It follows that $\bigcup \Psi_{U_n} \times \Psi_{U_n}$ is a basic entourage of $H\Psi\mathfrak{U}X$ and $H\eta_X$ is initial; being one-one and an epimorphism, $H\eta_X$ is a uniform embedding of HX into the complete uniform space $H\Psi\mathfrak{U}X$. Then $\gamma HX \cong H\Psi\mathfrak{U}X$, for every X. Naturality follows by uniqueness of the relevant extensions.

2.11 The first equivalence in the following Proposition is the expected result (Isbell [1972], Lambek and Rattray [1973]) Fix $\eta = \text{Im}\ \Psi$, where $\text{Im}\ \Psi$ is the full subcategory of *Alex* whose objects are isomorphic to some ΨL.

Proposition *For an Alexandroff space* X *the following are equivalent* :

(1) $X \cong \Psi L$, *for some regular* σ*-frame* L

(2) $\eta_X : X \to \Psi\mathfrak{U}X$ *is an isomorphism*

(3) *Every* σ*-prime filter of* $\mathfrak{U}X$ *is of the form* $\mathfrak{U}(x)$, *for some* $x \in X$

(4) X *is realcompact.*

Proof Clearly (2) ⇒ (1). The implication (1) ⇒ (4) and the equivalence (2) ⟺ (4) follow directly from the Theorem 2.7, Proposition 2.8 and Proof 2.9.

(2) ⇒ (3) If P is a σ-prime filter of $\mathfrak{U}X$ then by (2) there exists $x \in X$ with $\mathfrak{U}(x) = \eta_X(x) = P$.

(3) ⇒ (2) η_X is an embedding (2.5). By (3) it is onto.

2.12 We have here an alternate construction of the Hewitt realcompactification υX of an Alexandroff space X .

Corollary $\Psi\mathfrak{U} \cong \upsilon$

Proof For each Alexandroff space X and coz-map $f: X \to Y$ with Y realcompact the diagram

$$\begin{array}{ccc} X & \xhookrightarrow{\eta_X} & \Psi\mathfrak{U}X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Psi\mathfrak{U}f} \\ Y & \xhookrightarrow{\eta_Y} & \Psi\mathfrak{U}Y \end{array}$$

commutes

with η_Y an isomorphism. Then $\Psi\mathfrak{U}f \cdot \eta_Y^{-1}$ extends f , and uniquely so as η_X is an epimorphism.

Thus $\Psi\mathfrak{U}X \cong \upsilon X$, for each X . Naturality follows quickly.

2.13 As in the last paragraph it is easy to verify that Fix ε is a reflective sub-category of *Reg σFrm* with reflection map $\varepsilon_L : L \to \mathfrak{A}\Psi L$. Motivated by Proposition 2.6 we denote Fix ε by *Alex σFrm* , the full sub-category of *Reg σFrm* whose objects we call the Alexandroff σ-frames. (Not to be confused with the Alexandroff algebras of Reynolds [1979] which are precisely the regular σ-frames.) Prepared by 2.4 we may now observe :

Proposition *$\mathfrak{A}$ and Ψ induce a dual equivalence*

$$\begin{array}{ccc} \textit{Realcompact Alex} & \hookrightarrow & \textit{Alex} \\ \wr\!\wr & & \mathfrak{A}\downarrow\;\uparrow\Psi \\ \textit{Alex } \sigma\textit{Frm} & \hookrightarrow & \textit{Reg } \sigma\textit{Frm} \end{array}$$

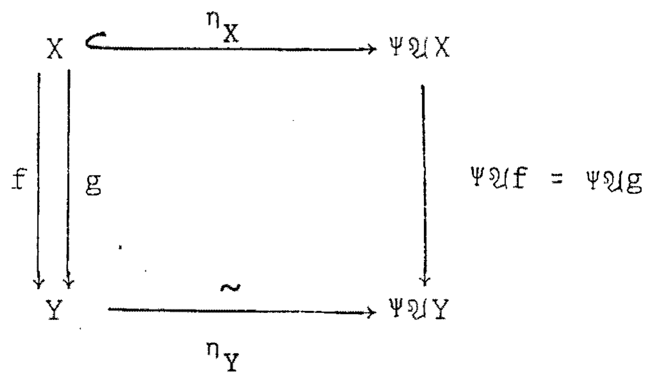
on the category of realcompact Alexandroff spaces and the category of Alexandroff σ-frames.

2.14 The following two Propositions may be compared with the corresponding results, of Broverman [1978b], for topological spaces which we shall recover in section 4 of this chapter.

Proposition *The following are equivalent for a realcompact Alexandroff space* Y :

(1) $h: \mathfrak{A}Y \to \mathfrak{A}X$ *is a homomorphism*

(2) $h = \mathfrak{A}f$ *for a unique coz-map* $f: X \to Y$.



commutes, by naturality of η , for both f and g . As η_Y is an isomorphism, $f = g$.

2.15 It was observed in 2.5 that $\mathfrak{U}\eta_X : \mathfrak{U}X \to \mathfrak{U}\Psi\mathfrak{U}X$ is an isomorphism. Applying 2.12 we obtain the expected isomorphism (as σ-frames) : $\mathfrak{U}X \cong \mathfrak{U}\upsilon X$. Further :

Proposition *If* X *and* Y *are Alexandroff spaces, then* $\mathfrak{U}X \cong \mathfrak{U}Y$ *if and only if* $\upsilon X \cong \upsilon Y$.

Proof If $\upsilon X \cong \upsilon Y$ then $\mathfrak{U}\upsilon X \cong \mathfrak{U}\upsilon Y$ hence as observed above, $\mathfrak{U}X \cong \mathfrak{U}Y$. Conversely, if $\mathfrak{U}X \cong \mathfrak{U}Y$ then $\Psi\mathfrak{U}X \cong \Psi\mathfrak{U}Y$.

3 Compact regular σ-frames - the Stone-Čech compactification in *Alex*

In [1980a] Banaschewski developes the theory of compact regular σ-frames.

The definitions, lemma and its corollary with which we initiate our study are due to him.

3.1 Definitions A σ-frame L is *compact* if given $1 = \bigvee x_n$ then there exist a finite number of the x_n $x_{n_1}, x_{n_2}, \ldots x_{n_k}$ say, such that $1 = x_{n_1} \vee \ldots \vee x_{n_k}$.

A filter P of a σ-frame is a σ-*open filter* if given $\bigvee x_n \in P$ then there exist $x_{n_1}, \ldots, x_{n_k}$ with $x_{n_1} \vee \ldots \vee x_{n_k} \in P$.

3.2 Lemma *If* F *is a* σ-*open filter on a* σ-*frame* L *and* $a \notin F$ *then there exists a* σ-*prime filter* P *on* L *such that* $F \subset P$ *and* $a \notin P$.

Proof By Zorn's Lemma there is a maximal σ-open filter P with $a \notin P$ and $F \subset P$. We show P is prime and thus (immediately) σ-prime.

Suppose $b \vee c \in P$, with neither b nor c in P. Let $Q = \{x: x \vee c \in P\}$. Then Q is a σ-open filter. Now $P \subset Q$, but $b \in Q$ with $b \notin P$, so $a \in Q$ and $a \vee c \in P$. Repeat with a replacing c . Then $a \vee a = a \in P$ giving the required contradiction. Thus either b or c is in P .

3.3 Corollary *A compact regular σ-frame has enough σ-prime filters.*

Proof Suppose $a < b$. Now $b = \bigvee b_n$, $b_n \ll b$. If $b_n \leqslant a$ for every n, then $b \leqslant a$ - contradicting our hypothesis, so there exists c, with $c \ll b$ and $c \not\leqslant a$.

Let $F = \{x: c \ll x\}$. Applying the properties of $\ll$ (1.3(1) and (2)); F is a filter. Furthermore F is σ-open: If $\bigvee x_n \in F$ then there exists z with $z \wedge c = 0$, $z \vee \bigvee x_n = 1$. By compactness, there exist $x_{n_1}, \ldots, x_{n_k}$ with $z \vee x_{n_1} \vee \ldots \vee x_{n_k} = 1$ and thus $c \ll x_{n_1} \vee \ldots \vee x_{n_k} \in F$. Moreover $b \in F$ and $a \notin F$. Apply the above Lemma to F.

3.4 If L is a compact regular σ-frame the Corollary confirms $\varepsilon_L: L \to \mathfrak{A}\Psi L$ is an isomorphism (2.6 Proposition) Consequently each countable coz-cover of ΨL has a finite subcover - a property of Alexandroff spaces that we shall characterise later (4.13). In this situation however more can be said :

Proposition *If L is a compact regular σ-frame then ΨL is a compact Alexandroff space.*

Proof Every uniform cover $\mathcal{B}$ of $H\Psi L$ has a refinement of form $\{\Psi_{a_n}\}$ with $\bigcup \Psi_{a_n} = \Psi L$. Then $\Psi_{\bigvee a_n} = \Psi_1$ and as L has enough σ-prime filters, $\bigvee a_n = 1$. There exist $n_1, \dots, n_k$ with $a_{n_1} \vee \dots \vee a_{n_k} = 1$. Thus $\{\Psi_{a_{n_i}}\}$, $i = 1, \dots, k$ is a finite uniform cover refining $\mathcal{B}$ and $H\Psi L$ is totally bounded and hence (2.8) compact.

3.5 From the previous two paragraphs Fix ε contains the compact regular σ-frames. Of course Fix η contains the compact Alexandroff spaces and we obtain a sub-duality to that duality presented in 2.13.

Proposition *Ψ and $\mathfrak{U}$ induce a dual equivalence*

$$\begin{array}{ccccc}
\mathit{Comp\ Alex} & \hookrightarrow & \mathit{Realcompact\ Alex} & \hookrightarrow & \mathit{Alex} \\
\wr\!\! \| & & \wr\!\! \| & & \mathfrak{U} \downarrow \; \uparrow \Psi \\
\mathit{K\,Reg\ \sigma Frm} & \hookrightarrow & \mathit{Alex\ \sigma Frm} & \hookrightarrow & \mathit{Reg\ \sigma Frm}
\end{array}$$

on the category of compact Alexandroff spaces and the category of compact regular σ-frames.

3.6 **Definitions** An *ideal* I of a σ-frame L is a subset of L satisfying :

(a) $E \subset I$, E finite $\Rightarrow \bigvee E \in I$

(b) $a \in I$, $b \leq a \Rightarrow b \in I$.

An ideal J is *countably generated* if there is an at most countable subset A of J such that if $b \in J$ then there exist $a_1, \ldots, a_n$ in A with $b \leq a_1 \vee \ldots \vee a_n$.

3.7 Let $\mathcal{I}L$ denote the set of all countably generated ideals of the σ-frame L. Then $\mathcal{I}L$ with intersection for meet and with $\bigvee J_n$ ($J_n \in \mathcal{I}L$) generated by $\bigcup A_n$, where A_n countably generates J_n, is a σ-frame.

Moreover $\mathcal{I}L$ is compact: the greatest element of $\mathcal{I}L$ is L and is generated by $\{1\} \subset L$. If $L = \bigvee J_n$, and A_n countably generates J_n, for each n, then $\bigcup A_n$ countably generates L. Thus $1 \leq b_1 \vee \ldots \vee b_m$, where $b_i \in A_{n_i}$. Then $L = \bigvee J_{n_i}$.

If $L_\alpha \subset L$ (arbitrary index set) are regular sub-σ-frames of the σ-frame L, then the sub-σ-frame generated by the L_α (whose elements are countable joins of finite meets of elements of the L_α) is regular.

The largest regular sub-σ-frame of L will be denoted by $\mathcal{R}L$. The compact regular coreflection of a σ-frame L in σFrm, due to Banaschewski, is simply $KL = \mathcal{R}\mathcal{I}L$.

The details below 3.8 - 3.11 are adapted in part from those presented in Banaschewski [1980a] for the analogous construction for frames.

3.8 Lemma *If* M *is a compact* σ*-frame and* $x \ll \bigvee m_n$ *in* M *, then* $x \leqslant m_{n_1} \vee \ldots \vee m_{n_\ell}$.

Proof $x \ll m$ implies there exists p with $x \wedge p = 0$, $p \vee m = 1$. By compactness there exist $m_{n_1}, \ldots, m_{n_\ell}$ with $p \vee m_{n_1} \vee \ldots \vee m_{n_\ell} = 1$. Then $x \wedge (m_{n_1} \vee \ldots \vee m_{n_\ell}) = x$.

3.9 Let $\downarrow m = \{x : x \ll m\}$.

Lemma *If* M *is a compact regular* σ*-frame then* $\downarrow : M \to \mathcal{I}M$ *is a homomorphism.*

Proof That $\downarrow m$ is an ideal for each $m \in M$ follows immediately from 1.3(1) and (2). Now for $m \in M$, $m = \bigvee m_n$ $(m_n \ll m)$ by regularity. Thus $x \in \downarrow m$ implies $x \leqslant m_{n_1} \vee \ldots \vee m_{n_\ell}$ (by 3.8), hence $\{m_n\}$ countably generates $\downarrow m$.

Clearly $\downarrow 1 = M$ and $\downarrow 0 = \{0\}$ and applying 1.3(2), $\downarrow(x \wedge y) = \downarrow x \cap \downarrow y$. To show $\downarrow$ a homomorphism it only remains to verify $\downarrow \bigvee x_n = \bigvee \downarrow x_n$.

Clearly $\bigvee \downarrow x_n \subseteq \downarrow \bigvee x_n$. For the converse we first consider the finite case. If $z \in \downarrow(a \vee b)$ then $z \ll \bigvee_{n,m} a_n \vee b_m$ where $a = \bigvee a_n$, $b = \bigvee b_m$. Hence by 3.8, $z \leqslant a_{n_1} \vee \ldots \vee a_{n_\ell} \vee b_{m_1} \vee \ldots \vee b_{m_k}$ and hence $z \in \downarrow a \vee \downarrow b$. Thus $\downarrow(a \vee b) = \downarrow a \vee \downarrow b$.

If $w \in \downarrow \bigvee x_n$, then there exists p with $w \ll p \ll \bigvee x_n$ (by 1.6). Then $p \leq x_{n_1} \vee \ldots \vee x_{n_\ell}$ (by 3.8) and hence $w \ll x_{n_1} \vee \ldots \vee x_{n_\ell}$; and $w \in \downarrow x_{n_1} \vee \ldots \vee \downarrow x_{n_\ell}$. Consequently, $w \in \bigvee \downarrow x_n$.

3.10 Corollary *If* M *is a compact regular* σ*-frame* $\downarrow : M \to KM$ *is an isomorphism.*

Proof Define $k_L : KL \to L$ as follows : If $J \in KL$ is countably generated by A , then $k_L J = \bigvee A$. Clearly k_L is a homomorphism.

In this case we have compact M , $\downarrow$ is a homomorphism and for each $m \in M$, $k_M \downarrow m = \bigvee \downarrow m = m$, thus $k_M \downarrow = id_M$.

It remains to prove that $\downarrow k_M J = J$ for each $J \in KM$. Let A countably generate J .

If $x \in \downarrow k_M J$ then $x \ll \bigvee A$ and hence $x \leq a_1 \vee \ldots \vee a_n$, with each $a_i \in A \subset J$ (3.8). Thus $x \in J$ and $J \subset \downarrow k_M J$.

Now $J = \bigvee K_n$ where $K_n \ll J$. Take $x \in J = \bigvee K_n$. Then $x \leq \ell_1 \vee \ell_2 \vee \ldots \vee \ell_m$ where the ℓ_i are generating elements for the K_n . We show that $\ell_i \ll \bigvee A$ for each i .

If $\ell_i \in K_p$ say, then there exists H_p with

$$K_p \cap H_p = \{0\} \quad , \quad H_p \vee J = M \quad .$$

Then $1 = (a_1 \vee \ldots \vee a_q) \vee h$ for some $h \in H_p$ and $a_i \in A$. But $\ell_i \wedge h = 0$, thus

$$\ell_i \leqslant a_1 \vee \ldots \vee a_q \leqslant \bigvee A \ .$$

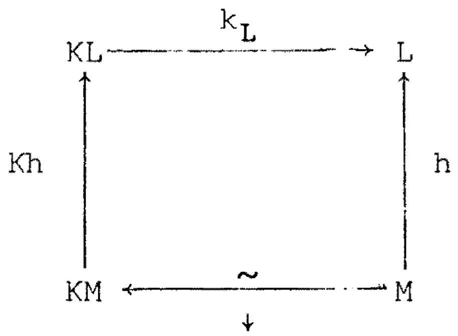
Thus $x \leqslant \ell_1 \vee \ldots \vee \ell_m \leqslant \bigvee A$ and $J \subset \downarrow \bigvee A$.

•

3.11 The correspondence induced by K is functorial: $\sigma Frm \to K\,Reg\ \sigma Frm$. If $h: M \to L$ is a homomorphism and $J \in KM$ is countably generated by A , then $Kh(J)$ is the ideal countably generated by $h(A)$ in L . The map k_L discussed below was defined in 3.10 above.

Theorem *$K\,Reg\ \sigma Frm$ is a full coreflective subcategory of σFrm with coreflection map $k_L: KL \to L$.*

Proof Let $h: M \to L$, with M compact regular, be a given homomorphism.



If $m \in M$, then $m = \bigvee m_n$, $m_n \leqslant m$ and $A = \{m_n\}$ generates $\downarrow m$. Then $k_L \circ Kh(\downarrow m) = \bigvee h(A) = h(\bigvee A) = h(m)$.

Uniqueness : Let $f, g : M \to KL$ with $k_L f = k_L g = h$.
Then $f(m) = \bigvee f(m_n)$ with $f(m_n) \ll f(m)$ for each n
(f preserves $\ll$).
There exists $H_n \in KL$ with $f(m_n) \cap H_n = \{0\}$,
$H_n \vee f(m) = KL$. Then there exists $h \in H_n$ and
$a_i \in A$, where A countably generates $f(m)$ such that
$(a_1 \vee \ldots \vee a_n) \vee h = 1$. Set $a = a_1 \vee \ldots \vee a_n$.
Then for each $z \in f(m_n)$, $z \leqslant a$. Thus

$$\bigvee f(m_n) \leqslant a \in f(m) \quad \text{i.e.} \quad k_L f(m_n) \in f(m) .$$

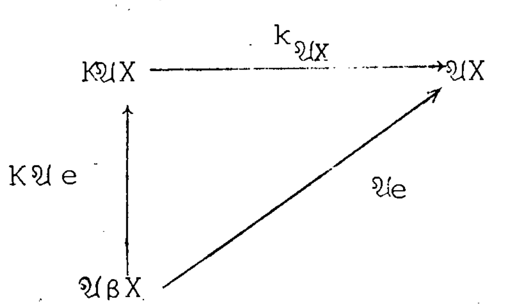
But $k_L f(m_n) = k_L g(m_n)$, and $\bigvee g(m_n) \in f(m)$ for each n.
It follows that $g(m_n) \subset f(m)$, for each n , hence
$\bigvee_n g(m_n) \subset f(m)$ i.e. $g(m) \subset f(m)$.
Similarly, $f(m) \subset g(m)$.-

3.12 Theorem $\Psi K \mathfrak{U} \cong \beta$.

Proof By 3.4, $\Psi K \mathfrak{U} X$ is compact for each Alexandroff space X . We show that it has the required universal property.

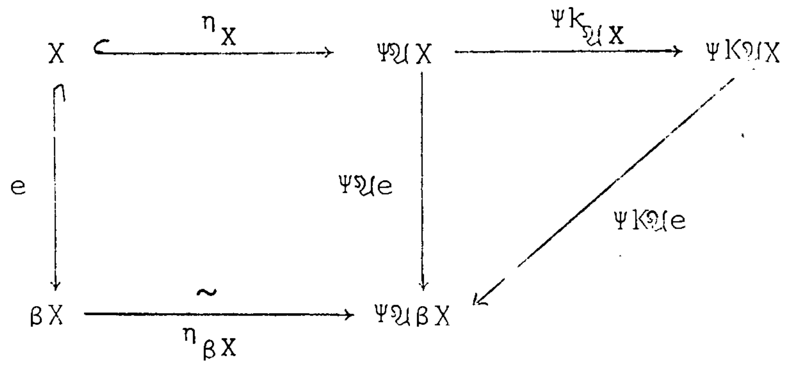
Firstly, if $e : X \hookrightarrow \beta X$ is the Stone-Čech reflection map, then $\mathfrak{U} \beta X$ is compact and the diagram

(1)



commutes.

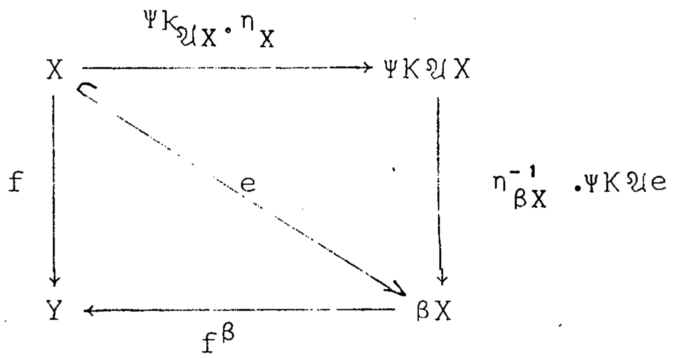
Consider the diagram



The left-hand rectangle commutes by naturality of η and $\eta_{\beta X}$ is an isomorphism.

The right-hand triangle commutes - it is obtained by applying Ψ to the triangle in (1).

If Y is a compact Alexandroff space and $f: X \to Y$ is a coz-map then the outer rectangle of

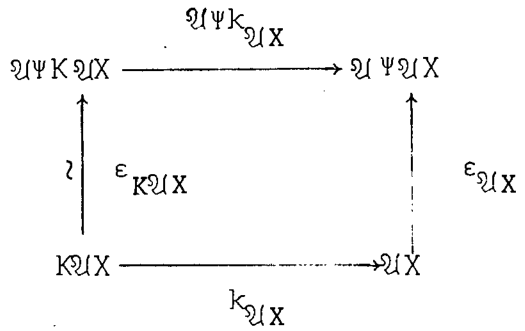


commutes, where $f^{\beta}e = f$ is the Stone-Čech extension of f.

Then $\hat{f} = f^{\beta}\eta^{-1}_{\beta X}\,\Psi K\mathfrak{U}e$ extends f to $\Psi K\mathfrak{U}X$. We show this extension is unique.

Let $g \cdot \Psi k_{\mathfrak{U}X} \cdot \eta_X = f$ where $g: \Psi K\mathfrak{U}X \to Y$. Then
$\mathfrak{U}\eta_X \cdot \mathfrak{U}\Psi k_{\mathfrak{U}X} \cdot \mathfrak{U}g = \mathfrak{U}f : \mathfrak{U}Y \to \mathfrak{U}X$.

Now



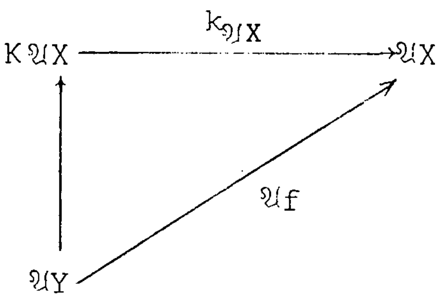
commutes by naturality of ε and $\mathfrak{U}\eta_X \cdot \varepsilon_{\mathfrak{U}X} = id_{\mathfrak{U}X}$ (2.3(2)), thus

$$\mathfrak{U}\eta_X \cdot \mathfrak{U}\Psi k_{\mathfrak{U}X} = k_{\mathfrak{U}X} \cdot \varepsilon^{-1}_{K\mathfrak{U}X} \cdot$$

It follows then that

$$\mathfrak{U}f = k_{\mathfrak{U}X} \cdot (\varepsilon^{-1}_{K\mathfrak{U}X} \cdot \mathfrak{U}g) \quad \text{and}$$

$$\mathfrak{U}f = k_{\mathfrak{U}X} \cdot (\varepsilon^{-1}_{K\mathfrak{U}X} \cdot \mathfrak{U}\hat{f})$$



But such extensions are unique, thus $\varepsilon^{-1}_{K\mathfrak{U}X} \cdot \mathfrak{U}g = \varepsilon^{-1}_{K\mathfrak{U}X}\mathfrak{U}\hat{f}$ and $\mathfrak{U}g = \mathfrak{U}\hat{f}$.

Applying 2.14, $g = \hat{f}$. Naturality follows easily.

4. Applications to realcompact and pseudocompact spaces.

Recall (1.2.1) $R: \mathit{Tych} \to \mathit{Alex}$ is the unique right inverse and a left adjoint to $F: \mathit{Alex} \to \mathit{Tych}$.
Let $\Sigma = F\Psi: \mathit{Reg}\ \sigma\mathit{Frm} \to \mathit{Tych}$ and $T = \mathfrak{U}R: \mathit{Tych} \to \mathit{Reg}\ \sigma\mathit{Frm}$.
We may conclude :

4.1 Proposition *Σ and T are adjoint on the right.*

4.2 Let $\nu: 1_{\mathit{Tych}} \to FR$ denote the unit for the adjoint pair F,R which is, as observed, the identity natural transformation.

The counit is $\delta: RF \to 1_{\mathit{Alex}}$ where $\delta_X(x) = x$, for all $x \in X \in \mathit{Alex}$.

The adjunctions for Σ and T then are given by :

$$\zeta: 1_{\mathit{Tych}} \to \Sigma T \text{ where}$$

$\zeta = (F*\eta*R).\nu$, and

$$\varphi: 1_{\mathit{Reg}\ \sigma\mathit{Frm}} \to T\Sigma \text{ with}$$

$\varphi = (\mathfrak{U}*\delta*\Psi).\varepsilon$.

ζ_X is, for each Tychonoff space X, a dense embedding since ν is an identity, η a dense embedding in Alex and F preserves initiality. Unlike η_Y in Alex, ζ_X is not an essential embedding for every X, for reasons that will become obvious once we have identified ζ_X (4.6).

In contrast to ε_L for a regular σ-frame L, φ_L need not be onto.

$\mathfrak{A}\delta_{\Psi L}: \mathfrak{A}\Psi L \to \mathfrak{A}RF\Psi L$ is described by :
$\mathfrak{A}\delta_{\Psi L}(\Psi_a) = \Psi_a$ for each $a \in L$. Thus $\mathfrak{A}\delta_{\Psi L}$ is onto if and only if ΨL is fine.

4.3 Mimicking 2.4 and 2.13, Σ and T induce a dual equivalence between Fix ζ and Fix φ. We now characterise these subcategories :

Proposition *For a Tychonoff space X the following are equivalent*

(1) $X \cong \Sigma L$ *, for some regular σ-frame* L .

(2) $\zeta_X: X \to \Sigma TX$ *is an isomorphism.*

(3) The σ-prime filters of cozero-sets of X are precisely those whose members contain a given point $x \in X$.

(4) X *is realcompact.*

Proof The implication (2) $\Rightarrow$ (1) is trivial.
(1) $\Rightarrow$ (4) ΨL is realcompact for each L (2.11) hence $\Sigma L = F\Psi L$ is realcompact (2.1.12).
(4) $\Rightarrow$ (2) If X is realcompact then RX is realcompact and by 2.11 $\eta_{RX}: RX \to \Psi\mathfrak{A}RX$ is an isomorphism. Then $\zeta_X = F\eta_{RX}.\nu_X$ is an isomorphism as ν_X is always the identity map (4.2).

(3) $\Leftrightarrow$ (4) RX and X have the same cozero-sets hence this equivalence follows directly from the like in 2.11.

We remark that we have not seen the characterization of realcompactness given by (3) $\Leftrightarrow$ (4) in the above Proposition in the literature.

4.4 Blair and Hager [1974] have shown that each cozero-set of a Tychonoff space is z-embedded. With this observation the simple proof of the following result carries over to a proof for the analogue for Tychonoff X (Gillman and Jerison [1960], 8.14).

Proposition *Each cozero-set* A *of a realcompact Alexandroff space* X *is realcompact as a subspace of* X.

Proof Let F be a σ-prime filter of cozero-sets of A . Set $G = \{G \in \mathfrak{U}X : G \cap A \in F\}$. G is a σ-prime filter on $\mathfrak{U}X$. Thus $G = \mathfrak{U}(x)$ for some $x \in X$ (2.11). As $A \in G$, $x \in A$ and $x \in F$ for each $F \in F$. Finally, each cozero-set of A containing x is the restriction of a cozero-set in G to A .

4.5 The domain for the definition of T: *Tych* → *Reg σFrm* can be extended to *Top* - the category of all topological spaces - in the obvious way ("take cozero-sets"). It is in this spirit that we have abused notation in (2) below.

Proposition *For a regular σ-frame* L *the following are equivalent* :

(1) $L \cong TY$ *for some Tychonoff space* Y .

(2) $L \cong TY$ *for some topological space* Y .

(3) $\varphi_L : L \to T\Sigma L$ *is an isomorphism.*

(4) $L \cong \mathfrak{A}X$ *for some realcompact-fine Alexandroff space* X .

(5) L *has enough σ-prime filters and* ΨL *is fine*

(6) L *has enough σ-prime filters and* $\beta\Sigma L \cong \Sigma KL$.

Proof Since the cozero-set lattices of a topological space Y and its Tychonoff reflection are isomorphic as σ-frames (1.2.4) , $(1) \Longleftrightarrow (2)$.

Clearly $(3) \Rightarrow (1)$.

$(1) \Rightarrow (4)$ $L \cong TY \cong \mathfrak{A}\upsilon RY = \mathfrak{A} R\upsilon Y$

(2.15) and $R\upsilon Y$ is of course realcompact-fine.

$(4) \Rightarrow (5)$ By (2.2.6) if X is realcompact-fine then $\Psi L \cong \Psi\mathfrak{A}X \cong \upsilon X$ is fine. If $L \cong \mathfrak{A}X$ then L has enough σ-prime filters (2.6).

$(5) \Longleftrightarrow (6)$ Recall that an Alexandroff space X is fine if and only if $F\beta X \cong \beta FX$ (1.2.3). Thus

ΨL is fine if and only if $\beta\Sigma L \cong F\beta\Psi L$. When L has enough σ-prime filters, $\mathfrak{A}\Psi L \cong L$ and applying 3.12 ,

$F\beta\Psi L \cong F\Psi K\mathfrak{A}\Psi L = \Sigma KL$.

$(5) \Rightarrow (3)$ If L has enough σ-prime filters then ε_L is an isomorphism (2.6). Now $\mathfrak{A}\delta_{\Psi L}$ is one-one, and (4.2) since ΨL is fine, onto. Thus $\varphi_L = \mathfrak{A}\delta_{\Psi L} . \varepsilon_L$ is an isomorphism.

4.6 Corollary *The Alexandroff space* X *is realcompact-fine if and only if* $\beta\Sigma\mathfrak{U}X \cong \Sigma K\mathfrak{U}X$.

4.7 Fix φ and Fix ζ are reflective subcategories of *Reg* σ*Frm* and *Tych* respectively. It is clear from 4.5 that Fix $\varphi \subset$ Fix ε and motivated by 4.5(2) we call the objects of Fix φ the *topological* σ*-frames* and denote the corresponding full subcategory of *Reg* σ*Frm* by *Top* σ*Frm*.

ζ is the unit of the Hewitt realcompactification in *Tych* and we have here the alternative construction for υX given by $\upsilon X \cong \Sigma TX$. Note also that $F\eta R \cong \zeta$.

The first duality 4.8(1) given below is a consequence of the characterizations of Fix φ and Fix ζ given above. The second 4.8(2), due to Banaschewski [1980a,b], may be deduced from the corresponding duality given in 3.5: the counit $\delta: RF \to 1_{\mathcal{A}\ell ex}$ restricts on *Comp Alex* to an isomorphism and the category of compact Tychonoff spaces, denoted *Comp Tych*, is isomorphic to *Comp Alex*.

4.8 Proposition T *and* Σ *induce dual equivalences.*

$$\begin{array}{ccccc}
\textit{Comp Tych} & \hookrightarrow & \textit{Realcompact Tych} & \hookrightarrow & \textit{Tych} \\
\| \wr & & \| \wr & & T \downarrow \; \uparrow \Sigma \\
K\ \textit{Reg}\ \sigma\textit{Frm} & \hookrightarrow & \textit{Top}\ \sigma\textit{Frm} & \hookrightarrow & \textit{Reg}\ \sigma\textit{Frm}
\end{array}$$

(1) on the category of realcompact Tychonoff spaces and the category of topological σ*-frames,*

(2) on the category of compact Tychonoff spaces and the category of compact règular σ*-frames.*

4.9 Theorem $\Sigma KT \cong \beta$

Proof $\Sigma KT = F\Psi K\mathfrak{A} R \cong F\beta R$ (3.12) and $F\beta R \cong \beta$ (1.2.3).

4.10 Broverman [1978b] obtained the zero-set lattice analogues for the following two results.

Proposition *If* Y *is a realcompact Tychonoff space then the following are equivalent*

(1) $h: TY \to TX$ *is a homomorphism*

(2) $h = Tf$ *for a unique continuous map* $f: X \to Y$.

Proof The implication (2) ⇒ (1) is trivial

(1) ⇒ (2) $h = \mathfrak{A}\, g$ for a unique coz-map $g: RX \to RY$ (from 2.14). Set $f = Fg$. Uniqueness follows by faithfulness of R and uniqueness of g .

4.11 Proposition *If* X *and* Y *are Tychonoff spaces then* $TX \cong TY$ *if and only if* $\upsilon X \cong \upsilon Y$.

Proof This result follows immediately on observing firstly that $TX \cong T\upsilon X$ (well-known) and secondly that $\Sigma T \cong \upsilon$.

4.12 The following theorem may be contrasted with similar though essentially different results obtained by Broverman [1978a] and Speed [1973].

Theorem *The following are equivalent for a regular* σ*-frame* L :

(1) L *is compact*

(2) ΨL *is compact and* L *has enough* σ*-prime filters*

(3) $L \cong \mathfrak{U}X$ *for some compact Alexandroff space* X

(4) $L \cong TY$ *for some compact Tychonoff space* Y

(5) $L \cong \mathfrak{U}X$ *for some pseudocompact Alexandroff space* X

(6) $L \cong TY$ *for some pseudocompact Tychonoff space* Y .

Proof The implication (1) ⇒ (2) is the substance of 3.3 and 3.4.

(2) ⇒ (3) Take $X = \Psi L$

(3) ⇒ (4) Take $Y = FX$

(4) ⇒ (6) Trivial

(6) ⇒ (5) Take $X = RY$

(5) ⇒ (3) If $L \cong \mathfrak{U}X$ with X pseudocompact, then $\Psi L \cong \upsilon X \cong \beta X$ and $L \cong \mathfrak{U}\beta X$.

The implication (3) ⇒ (1) is trivial.

4.13 Corollary *(1) An Alexandroff space* X *is pseudocompact if and only if* $\mathfrak{A}X$ *is compact.*

(2) A Tychonoff space X *is pseudocompact if and only if* TX *is compact.*

Proof (1) If $\mathfrak{A}X$ is compact, $\psi\mathfrak{A}X \cong \upsilon X$ is compact, hence X is pseudocompact. The reverse implication follows directly from 4.12 above.

(2) Similar.

The Alexandroff spaces satisfying the condition in (1) above have been called *countably compact* (Alexandroff [1940], Hager [1974]) and *semicompact* (Hager [1979a]).

4.14 The foregoing corollary yields the following simple proof that pseudocompactness is finitely productive in *Alex*, which we give although we shall also prove Gordon's [1971] result on arbitrary productiveness in 4.23.

Proposition *If* X *and* Y *are pseudocompact Alexandroff spaces then* $X \times Y$ *is pseudocompact.*

Proof As each cozero-set of $X \times Y$ is a countable union of sets of the form $A \times B$, with $A \in \mathfrak{A}X$, $B \in \mathfrak{A}Y$ we choose, without loss of generality, any countable coz-cover of $X \times Y$ of form $\{A_n \times B_n\}_{n \in \mathbb{N}}$

where $A_n \in \mathfrak{U}X$ and $B_n \in \mathfrak{U}Y$ for each n .

For each $x \in X$, there exists $K_x \subset \mathbb{N}$ such that $\{B_k\}_{k\in K_x}$ covers Y and $x \in A_k$, for each $k \in K_x$.

Since Y is pseudocompact there exists by 4.13 a finite set $J_x \subseteq K_x$ such that $\mathcal{B}_x = \{B_j\}_{j\in J_x}$ already covers Y .

Let $D_x = \bigcap_{j\in J_x} A_j$, then D_x is a cozero-set containing x . Moreover there are only countably many distinct D_x's as x runs through X . As X is pseudocompact the countable cover $\{D_x\}_{x\in X}$ has a finite subcover $\{D_{x_1}, \ldots, D_{x_n}\}$ say.

Then $\{A_j \times B_j\}_{j\in J_{x_i}}$, where $i = 1, \ldots, n$, is a cover of $X \times Y$.

4.15 As for topological spaces, any Alexandroff space X that admits a unique realcompactification is pseudocompact - the realcompactification being $\upsilon X \cong \beta X$. We showed in 2.5 that η_X is an essential embedding of each Alexandroff space X into $\Psi\mathfrak{U}X \cong \upsilon X$ (by 2.12). This generalises a theorem of Gordon ([1971], 7.9). Thus υX is densely embedded in every realcompactification Y of X . In particular, if X is pseudocompact, $\upsilon X \cong \beta X$ is densely embedded in Y and thus $Y \cong \upsilon X$. We have proved:

Proposition *An Alexandroff space* X *is pseudocompact if and only if* X *admits a unique realcompactification.*

4.16 Corollary *A Tychonoff space* X *is pseudocompact if and only if* υX *is the only realcompactification in which* X *is* z*-embedded.*

Proof X is z-embedded in Y if and only if RX is embedded in RY . If X is pseudocompact then RX is pseudocompact and by 4.15 υRX is the unique real-compactification of RX . Conversely, X is z-embedded in both υX and βX , always.

4.17 The analogue of Proposition 4.15 for topological spaces fails. A pseudocompact space need not admit a unique realcompactification. This is because it may not be z-embedded in some compactification :

Corollary *The following are equivalent for a Tychonoff space* X

(1) X *is pseudocompact and* z*-embedded in every compactification*

(2) X *admits a unique realcompactification*

(3) X *admits a unique compactification.*

Proof (1) ⇒ (2) Each realcompactification Y of pseudocompact X is compact, thus by assumption X is z-embedded in Y and by 4.16, $Y \cong \upsilon X$.

(2) ⇒ (3) is trivial

(3) ⇒ (1) It is well-known (Gillman and Jerison [1960]) that if X admits a unique compactification then X is pseudocompact. The compactification is βX, in which X is z-embedded.

A further equivalent condition on X is that it be almost compact (Gillman and Jerison [1960]). If X is Lindelöf then every embedding of X is a z-embedding (attributed to Henriksen and Johnson, Alò and Shapiro [1974]). Note however that if X admits a unique compactification it need not be Lindelöf (e.g. the ("deleted") Tychonoff plank).

The corollaries 4.16 and 4.17 may also be quickly deduced using the result of Blair and Hager [1974], that the notions of z-embedding, C*- and C-embedding are equivalent on pseudocompact subspaces. The latter result is also easily obtained as follows :

If A is z-embedded in X then RA is a subspace of RX. By essentiality of the embedding η_{RA} (2.5) υRA is embedded in υRX. For pseudocompact A, RA is pseudocompact and then $\upsilon RA \cong \beta RA$ which is embedded in υRX. Since R commutes with both υ and β and F preserves embeddings it follows that βA is embedded in υX . Each compact subset of a Tychonoff space is C-embedded. A is C*-embedded in βA which as we have seen is C-embedded in υX . Consequently (by restricting), A is C-embedded in X .

4.18 As a consequence of 4.15 and thus a consequence of the fact that η_X is an essential embedding, we have that a pseudocompact Alexandroff space admits a unique compactification. The converse holds for Tychonoff spaces and may be proved for Alexandroff spaces in like manner (e.g. Gillman and Jerison [1960], Gilmour [1974]). Gordon [1971] first stated and proved this result which may also be recovered in the setting of SMF from the work of Hager [1974]. We state then :

Theorem (Gordon) *An Alexandroff space* X *is pseudo-compact if and only if* X *admits a unique compactification.*

Other equivalent conditions for an Alexandroff space to be pseudocompact may be deduced as in Gillman and Jerison [1960]15Q,R) , (see Gilmour [1974] and especially Hager [1979a]).

4.19 The Corollary following is a result of Hager [1969]. His proof is different to that given here.

Corollary *A Tychonoff space* X *is pseudocompact if and only if every compactification* Y *of* X *is* ωA *, where* A *is the restriction of* RY *to* X *in* *Alex*.

Proof If X is pseudocompact then A is a pseudocompact Alexandroff space (FA = X) . Thus (4.18) A admits a unique compactification $\beta A \cong RY$.
Then $Y \cong F\beta A$.
If RX is not pseudocompact then RX admits at least two distinct compactifications βRX and W say. Then FW is a compactification of X distinct from $\omega RX \cong F\beta RX \cong \beta X$.

4.20 **Corollary** *A Tychonoff space* X *is pseudocompact if and only if* βX *is the only compactification in which* X *is* z-*embedded.* (cf. 4.16 Corollary)

Proof If X is pseudocompact then by 4.19 each compactification Y of X is ωA where A is the restriction of RY to X . If A = RX then $Y = \beta X$

The converse follows by assuming the contrapositive as in 4.19.

4.21 We shall now prove the two results (4.22 and 4.23) quoted and used in Chapter 2. These results are due to Gordon and the proofs we shall give are his. The proofs rely on the following theorem which may be found in Gordon [1971]. The proof presented here is our own.

Theorem υX *is the only realcompactification of the Alexandroff space* X *with the property that each of its non-empty zero-sets meets* X .

Proof Firstly, using the construction of υX given in 2.12, each cozero-set of $\upsilon X \cong \psi \mathfrak{U} X$ is of the form Ψ_U where $U \in \mathfrak{U} X$ and

$$\Psi_U = \{P \in \psi \mathfrak{U} X : U \in P\} .$$

Suppose there is an Ψ_U such that $\mathfrak{U}(x) \in \Psi_U$ for each $x \in X$. Then $U = X$ and $\Psi_U = \psi \mathfrak{U} X$. Thus the only cozero-set of υX completely containing "X" is υX itself and each zero-set of υX meets X.

Suppose now that $(Y, \mathcal{Z})$ is a realcompactification of $(X, \mathcal{W})$, distinct from υX , and such that each zero-set of Y meets X . As η_X is an essential embedding we may regard $X \subset \upsilon X \subset Y$. Take $y \in Y$ so that $y \notin \upsilon X$. Let F_y be the point $\mathcal{Z}$-ultrafilter on $(Y, \mathcal{Z})$.

Then $G = \{A \cap X : A \in F_y\}$ is a real $\mathcal{W}$-ultrafilter on $(X, \mathcal{W})$. From the construction of υX given in chapter 2, G converges (as a filter base on Y) to some point of υX as well as to y -contradiction.

4.22 Theorem (Gordon [1971]) υ *distributes over arbitrary Alex-products.*

Proof Let $\{X_\alpha\}$ be a family of Alexandroff spaces. $\Pi\upsilon X_\alpha$ is a realcompactification of ΠX_α. By 4.21 we need only show that each non-empty zero-set of $\Pi\upsilon X_\alpha$ meets ΠX_α for the isomorphism of $\Pi\upsilon X_\alpha$ and $\upsilon\Pi X_\alpha$.

Each zero-set A of $\Pi\upsilon X_\alpha$ is a countable intersection of finite unions of sets of the form $\pi_\alpha^{-1} A_\alpha$ with A_α zero-sets of υX_α. If A is non-empty we may choose countably many distinct A_{α_n} zero-sets of υX_{α_n} such that $B = \cap \pi_{\alpha_n}^{-1} A_{\alpha_n}$ is non-empty. Then B is a zero-set and $B \subset A$. Now for each n, there exists $x_n \in A_{\alpha_n} \cap X_{\alpha_n}$ (by 4.21). Take any $x \in \Pi X_\alpha$ with $\pi_{\alpha_n}(x) = x_n$ for all n. Then $x \in B \subset A$ and hence A meets ΠX_α.

4.23 Corollary (Gordon [1971]) *The Alex-product of pseudocompact Alexandroff spaces is again pseudocompact.*

Proof If $\{X_\alpha\}$ is a family of pseudocompact Alexandroff spaces then $\upsilon(\Pi X_\alpha) = \Pi\upsilon X_\alpha = \Pi\beta X_\alpha$ is compact. Thus ΠX_α is pseudocompact.

5 A note on complete objects

The category of sober spaces is dual to the category of spatial frames (see 6.2). A T_0-topological space has a unique (up to homeomorphism) sobrification. Similarly a separated uniform space has (up to uniform isomorphism) a unique completion. This characteristic of the sober spaces in *Top* and the complete spaces in *Unif* has been abstracted by Brümmer [1979] in defining the concepts of rigidity and of complete objects for concrete categories. The definitions of Brümmer for the categories with which we are concerned reduce to :

5.1 Definition Let $\mathcal{D}$ be any subcategory of *Zero* (*Top*, *Unif*). An epireflective subcategory $\mathcal{S}$ (with epireflector S and unit π) is a *rigid epireflective subcategory* of $\mathcal{D}_0$ (the full subcategory of separated $\mathcal{D}$-objects) if:

(1) For each $X \in \mathcal{D}_0$, $\pi_X\colon X \to SX$ is a dense embedding,

(2) If $A \in \mathcal{S}$ and $f\colon X \to A$ is a dense embedding, then there exists a unique isomorphism h such that $h.f = \pi_X$.

It turns out (Brümmer - Herrlich : private communication) that $\mathcal{D}_0$ can have at most one rigid epireflective subcategory $\mathcal{S}$. If there is such an $\mathcal{S}$, then its objects are called the *complete objects* of $\mathcal{D}$.

5.2 Examples (1) The complete separated uniform spaces are the complete objects of *Unif*.

(2) The sober spaces are the complete objects of *Top*.

(3) The compact spaces are the complete objects of *Psc Alex* (the full subcategory of *Alex* consisting of all pseudocompact spaces) : the epireflector is β - here restricted to *Psc Alex*. Rigidity follows immediately from 4.18.

5.3 If $\mathcal{D}$ is a subcategory of *Alex*, $\mathcal{D}$ is said to be *closed under compactification* if each compactification of an object of $\mathcal{D}$ is again in $\mathcal{D}$.

Proposition *If a subcategory $\mathcal{D}$ of Alex is closed under compactification then : $\mathcal{D}$ admits a rigid epireflective subcategory if and only if each* $D \in \mathcal{D}$ *is pseudocompact.*

Proof If $\mathcal{D}$ has $\mathcal{S}$ as a rigid epireflective subcategory, then $\mathcal{S}$ contains all compact $\mathcal{D}$-objects : each compact X is densely embedded in its $\mathcal{S}$ reflection.

If $X \in \mathcal{D}$ is not pseudocompact, then X admits at least two distinct compactifications (4.18) which are then both in $\mathcal{S}$ - contradicting the rigidity of $\mathcal{S}$.

5.4 Corollary *The category Alex has no complete objects.*

6. Generating cozero-sets from open sets

In this short section we outline certain results from the theory of frames in order to obtain an internal set-theoretical construction of the topologically fine coreflection RFX in *Alex* (6.7) as promised in 2.3.8. All the material (with the exception of 6.7) is drawn from Banaschewski [1980a]. Related material can be found in Reynolds [1979].

6.1 Definitions A *frame* L is a complete lattice that satisfies the distribution law $x \wedge \bigvee x_\alpha = \bigvee x \wedge x_\alpha$ ($\alpha \in I$, arbitrary). A *homomorphism* between frames is a map preserving finite meets, arbitrary joins, 0 and 1 .

These are the objects and maps of the category of frames, denoted by *Frm*.

6.2 The functors Ω: *Top* → *Frm* , where ΩX is the lattice of open sets of X , and Λ: *Frm* → *Top* , with ΛL the spectrum of L (i.e. all completely prime filters on L) with open sets $\Lambda_a = \{P: a \in P \in \Lambda L\}$, are adjoint on the right. Λ and Ω restrict to a dual equivalence between the category of sober spaces (a topological space X is *sober* if and only if each union irreducible closed set is the closure of some unique point of X) and the category of *spatial frames* (i.e. those frames L such that $L \cong \Omega X$ for some $X \in$ *Top*.)

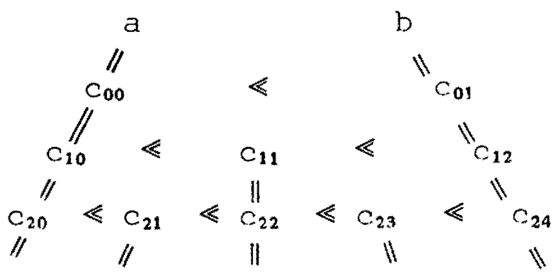
6.3 Definition Let L be a frame. Coz L is the subset of L whose elements a are of the form $a = h(\mathbb{R} \setminus \{0\})$ where $h \in$ *Frm* $(\Omega\mathbb{R}, L)$.

Coz L is the largest regular sub-σ-frame of L i.e. Coz $L = \mathcal{R}L_\sigma$ where $\mathcal{R}$ is described in 3.7 and L_σ is the underlying σ-frame of L .

6.4 Definitions The *rather below* relation $\ll$ described in 1.3 is defined in precisely the same way for *Frm*.

An element a of a frame L is *completely below* b , written $a \triangleleft b$ if there exists $\{c_{nk}\}$ with $n = 0,1,2,\ldots$ and $k = 0, \ldots, 2^n$ such that $c_{00} = a$, $c_{01} = b$, $c_{nk} = c_{n+1,2k}$ and $c_{nk} \ll c_{n,k+1}$

Pictorially :



Then $\triangleleft$ satisfies the corresponding properties of 1.3(1) and (2) for $\ll$. Moreover $\triangleleft$ interpolates.

If L is a *compact regular frame* (defined analogously as for σ*Frm*) then $\ll$ interpolates and $\lhd$ coincides with $\ll$.

In ΩX, $U \ll V$ if and only if $\overline{U} \subseteq V$ ($\overline{U}$ is the closure of U).

In ΩX, $U \lhd V$ if and only if there is a continuous function $f: X \to [0,1]$ with $f(x) = 0$ if $x \in U$ and $f(x) = 1$ if $x \notin V$.

6.5 If L is spatial then the paragraph 6.2 tells us that $\mathit{Frm}(\Omega\mathbb{R},L) \cong \mathit{Top}(\Lambda L,\mathbb{R})$. If further $T: \mathit{Top} \to \mathit{Frm}$ is the functor "take cozero-sets" then (from 6.3) $\mathrm{Coz}\ L \cong T\Lambda L$.

In particular, if X is an Alexandroff space, then

$$\mathrm{Coz}\ \Omega FX \cong T\Lambda\Omega FX \cong TFX = \mathfrak{U}RFX \quad .$$

6.6 Proposition (Banaschewski [1980a]) *Let* L *be a frame. Then* $a \in \mathrm{Coz}\ L$ *if and only if* $a = \bigvee a_n$ *where* $a_0 \lhd a_1 \lhd a_2 \lhd \ldots$, *and* $a_n \in L$.

6.7 Corollary *Let* X *be an Alexandroff space.* U *is a cozero-set of* RFX *if and only if* $U = \bigcup U_n$ *where* $U_0 \lhd U_1 \lhd \ldots$ *in* ΩFX *and the* U_n *are open in* FX.

6.8 Corollary (Banaschewski [1980a]) *Let* Y *be a compact Tychonoff space.* U *is a cozero-set of* Y *if and only if* $U = \bigcup U_n$ *where each* U_n *is open and* $\overline{U}_n \subset U_{n+1}$ $(n = 0,1,2,\ldots)$.

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