Topics in Modified Gravity

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Declaration

The work presented in this thesis is partly based on collaboration with Peter K S Dunsby, Rituparno Goswami, Amare Abebe and Alvaro de la Cruz-Dombriz (Department of Mathematics and Applied Mathematics, University of Cape Town), Sante Carloni (ESA-Advanced Concept Team, European Space Research Technology Center (ESTEC)), and Kishore N. Ananda. The list below identifies sections or paragraphs which are partially based on the listed publications and preprints:

- Chapter 6
  Cosmological dynamics of exponential gravity.
  Mohamed Abdelwahab, Sante Carloni, Peter K.S. Dunsby.
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- Chapter 7
  Cosmological dynamics of fourth order gravity: A compact view.
  Mohamed Abdelwahab, Rituparno Goswami, Peter K. S. Dunsby.
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- Chapter 8
  Covariant gauge-invariant perturbations in multifluid $f(R)$ gravity.
  Amare Abebe, Mohamed Abdelwahab, Alvaro de la Cruz-Dombriz, Peter K.S. Dunsby.
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- Chapter 9
  CMB tensor anistropies in $f(R)$-gravity.
  Mohamed Abdelwahab, Alvaro de la Cruz-Dombriz, Peter K. S. Dunsby.

- Chapter 10
  Unifying the study of background dynamics and perturbations in $f(R)$-gravity.
  Sante Carloni, Kishore N. Ananda, Peter K. S. Dunsby, Mohamed Abdelwahab.
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I hereby declare that this thesis has not been submitted, either in the same or different form, to this or any other university for a degree and that it represents my own work.

Mohamed Elshazli Abdelwahab

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Chapter 1

INTRODUCTION

1.1 The standard model of cosmology

The key element of modern cosmology is the assumption that the General Theory of Relativity (GR) is the correct theory of gravitation. It replaced the Newtonian theory of gravity which was presented in the Principia in 1687. The fundamental idea in GR is that gravity is a manifestation of the curvature of the spacetime, while in Newton's theory gravity acts directly as a force between bodies. Many of the predictions of GR, such as the bending of star light by gravity and a tiny shift in the orbit of the planet Mercury, have been quantitatively confirmed by experiment (for a review, see [38]).

In a homogeneous and isotropic spacetime the Einstein field equations give rise to the Friedmann equations, which provide us with the most elegant mathematical framework for understanding the dynamics of the background space-time and hence of the Universe as a whole. In fact the Big Bang model (BB) based on the assumptions that the large-scale geometry and dynamics of the Universe can be described by an exact Friedmann-Lemaitre-Robertson-Walker (FLRW) model and the existence of the radiation and matter-dominated epochs, is one of the greatest successes of GR, where it reproduces beautifully all the main observational results, for example Background Radiation (CMB) anisotropies [34], large-scale structure formation [35], baryon oscillations [40] and weak lensing [42]). Until recently the BB model is a broadly accepted as the standard model of cosmology and hence of the evolution of our Universe. Unfortunately, this model is affected by significant fine-tuning problems. The Universe appears to be homogeneous, isotropic and (almost) flat, which requires very special initial conditions [158]. Also, the formation of a large-scale structure is believed to be due to initial density fluctuations. These fluctuations must have a very special spectrum in order to produce the structure we observe today.

The introduction of an inflationary stage in the early Universe offers an elegant solution to these problems [158]. Inflation is a general term for scenarios of the very early Universe, all of which share the common feature that the Universe experiences a finite period of accelerated expansion, while it is in a vacuum-like state containing only homogeneous classical fields (for a review see [17]). The most popular candidate for inflation in the framework of GR is a scalar field ϕ with a slowly varying potential.

The inflationary paradigm not only provides an excellent way in solving flatness and homogeneity problems but also explains the origin of the large scale structure of the cosmos as resulting from quantum fluctuations in the microscopic inflationary region [16]. The theory states that the spontaneous fluctuations in the pre-inflationary epoch were greatly magnified by inflation. In the post-inflationary cosmos, these fluctuations produced regions just slightly denser than their surroundings. The differences in density were in turn amplified by gravity, which pulls matter into the denser regions. Thus inflation provides a mechanism to generate the seeds of density perturbations observed in the CMB anisotropies [21]. Currently no other theory can explain both why the Universe is so uniform overall, and yet contains exactly the kind of ripples represented by the distribution of galaxies through space and by the variations in the cosmic microwave background radiation [21]. However, inflation is believed to be caused by a massless scalar-field, the inflaton, which has not yet been observed .

In 1998, observations of type Ia supernova by the Supernova Cosmology Project at the Lawrence Berkeley National Laboratory and the High-z Supernova Search Team, surprisingly indicated that the Universe is undergoing another phase of cosmic acceleration [31], which started after the matter domination. In order to explain these observations in the framework of GR, a new component with a negative pressure has to be added to the energy budget of the Universe, this new component is called dark energy. Furthermore, these cosmological observations indicate that the around 70% of the present Universe energy content is dark energy and is homogeneously distributed in the Universe, 30% dust matter (cold dark matter plus baryons), and negligible radiation. Although, it has become quite clear how dark energy works, its nature remains an unsolved problem.

Recently there have been several explanations of the cosmic acceleration. The simplest one is the introduction of small positive cosmological constant originates from a vacuum energy with a constant equation of state parameter $\omega = -1$ in the framework of GR, the so-called ΛCDM Model [32]. Unfortunately, this model is affected by significant fine-tuning problems related to the vacuum energy scale [10], and also it has no explicit physical interpretation for the origin or the physical nature of dark energy. Another problem with the cosmological constant is referred to as the cosmic coincidence problem ([12]-[15]), that is, the near equivalence of the matter and Λ contribution to the total energy density at the present time. The simplest extension of the ΛCDM model is to replace the cosmological constant by a dynamical, spatially inhomogeneous, negative pressure cosmic scalar field This assumption has led to the so-called quintessence model [30]. In this case, the equation of state of dark energy is allowed to differ from -1. Although the success of quintessence in a scalar field explaining the cosmic acceleration and the coincidence problem is evident, it is still not clear where the scalar field arises and how to choose the self-interacting potential [27]. These facts have led to a considerable number of efforts to develop a more viable theoretical schemes.

1.1.1 Alternative theories of gravity

Cosmological observations indicate that the standard model of cosmology appears unable to offer a simple explanation of the phenomena observed without assuming the existence of (1) inflation, (2) Dark Matter, (3) Dark Energy. These assumptions are needed to explain the early and the late expansion history of the Universe as well as to understand the formation of structures such as galaxies and clusters of galaxies. It has been shown in literature [25], [26], [27], [23] that, all these assumptions are questionable and suffer from serious problems. Instead of modifying the energy-momentum tensor in Einstein equations by introducing new scalar fields, it is possible to solve these problems by assuming that the gravitational interaction changes behavior on large scales. GR might not be the only classical theory capable of successfully explaining the gravitational interaction. Currently serious attention is being paid to the classical formulation of gravity and scientists investigating the possibilities of formulating a viable alternative theory, which could explain the low-energy Universe without recourse to new unobserved particles. Several theoretical models have been proposed the most popular ones are:

1. Scalar-tensor theories [104].
2. Higher order theories of gravity (HOTG) [171],[172], [173] ,[175] ,[177].
3. Braneworld models [56].
4. Backreaction of matter inhomogeneities [57], [58].

There is currently active research into many aspects of these theories. Other reasons to modify GR are provided by modern theories such as String, M-theory and Branes. These theories indicate that modification of the standard gravitational action is necessary [159], where it has been shown in [160], that higher-order gravity theories naturally appear when quantization is performed on a curved spacetime and the renormalization problem is addressed. In 1962, Utiyama and DeWitt proved that renormalization at one loop requires adding higher-order curvature terms to Hilbert-Einstein action [160], [159]. On the other hand, from the conceptual point of view, there is no prior reason to restrict the gravitational Lagrangian to the simple Hilbert-Einstein action when a more general formulation is allowed. Also the conformal equivalence between the HOTG and the scalar-tensor theories might suggest the existence of a more general framework, which might incorporate both of them as special cases.

Recently, serious attention has been paid to the alternative formulations of gravity, which are based on Lagrangians that include higher-order corrections with respect to the linear term in the scalar curvature (Hilbert-Einstein action), either in the form of higher order curvature invariants such as $R^2, R^{\nu\mu}\, R_{\nu\mu}, R^{\nu\mu ab}R_{\nu\mu ab}$, or $R\,\Box R, R\,\Box^k R$. In order to explain the fact that the Universe has started to accelerate only recently, these modifications must become important only at late times. The most popular modified gravity theories are those which are based on gravitational actions that are non-linear in the Ricci curvature R and/or contain terms involving combinations of derivatives of R, the so called $f(R)$ gravity [18, 44, 46, 48, 50]. Such theories were first proposed in the 1980's because it was shown that they can be derived from fundamental physical theories (Like M-theory) and naturally lead to a phase of accelerated expansion, which could explain the early Universe inflationary phase [59]. The phenomenology of Dark Energy also requires the presence of a similar phase. The possibility of the late-time cosmic acceleration in $f(R)$ gravity was first suggested by Capozziello [133] in 2002. The addition of a nonlinear function of the Ricci scalar R to the Hilbert-Einstein action has been demonstrated to produce acceleration for a wide variety of $f(R)$ functions [165], [175]. Also $f(R)$ theories possess a number of interesting features with regards to cosmological and astrophysical scales. In fact, they are known to admit natural inflation phases [82] [67] and to explain the flattening of the galactic rotation curves [84]. Another very interesting feature of these models is that the higher order corrections to the Hilbert-Einstein action can be viewed as an effective fluid which could mimic the properties of Dark Energy. Thus Dark Energy may have a geometrical origin, i.e., that there is a connection between Dark Energy and a non-standard behavior of gravitation on cosmological scales (for more details see [62, 223, 68, 69, 71, 68, 73]).

Efforts to obtain an understanding of the physics of these theories are constrained by the complexity of the field equations, making it difficult to obtain both exact and numerical solutions that can be tested against observations. In this thesis we address some of these issues using a number of useful techniques, such as the 1+3 covariant and gauge invariant approach which was developed by Ellis and Bruni [75] and the dynamical system technique [119] .

1.1.2 Thesis outline

The structure of this thesis is as follows:

In Chapter 2 we present a review of the 1+3 formalism. The main aspects of the 1+3 formalism, is to project tensors and tensorial equations parallel and orthogonal to some preferred vector field, decompose tensor into a sum of algebraically simpler parts, and finally derive the evolution and constraint equations for the basic kinematical and dynamical variables by using the Ricci identities and Bianchi identities in conjunction with the Einstein field equations.

In Chapter 3 we present a review of cosmological perturbation theory. According to the perturbation theory, fluctuations generated during the inflationary epoch produced regions slightly denser than their surroundings. The differences in density were in turn amplified by gravity, which pulls matter into the denser regions. The imprint of these primordial density perturbations is recorded in the CMB anisotropies, and the evolution of these perturbations results in the large-scale structure that we observe today.

In Chapter 4 we give the actions and relevant field equations for general $f(R)$-theories of gravity and perform a detailed discussion of some cosmological solutions and the conditions that are required for this class of theories to be cosmologically viable. We also discuss the conformal relationships between the $f(R)$ and the scalar-tensor. We conclude by commenting on some of the outstanding problems of all infrared-modified $f(R)$-theories of gravity.

In Chapters 5 and 6 we argue that, owing to the complexity of the field equations, and the lack of an existence of an exact solutions, dynamical systems analysis is an extremely powerful mathematical method in cosmology. It is very useful in understanding the qualitative behavior of the cosmological dynamics and obtaining special exact solutions of the cosmological equations. This chapter contains an overview of this approach. We start by introducing some of the basic dynamical systems concepts and discuss the stability of 2-D linear dynamical systems. We then present the Hartman-Grobman and the center manifold theorems, which completely solve the problem of determining the stability and qualitative behavior in a neighborhood of a hyperbolic as well as nonhyperbolic critical point of a nonlinear system. Finally we give a detailed investigation of the cosmological dynamics based on $exp(-R/\Lambda)$ gravity. The study of this model shows that cosmic histories exist which admit a double de Sitter phase which could be useful for describing the early and the late-time accelerating Universe .

Chapter 7 is based on a work completed in collaboration with Peter K S Dunsby and Rituparno Goswami. In this paper we construct a compact phase space for flat FLRW spacetimes with standard matter described by a perfect fluid with a barotropic equation of state for a general $f(R)$ theories of gravity, subject to certain conditions on the function f. We then use this framework to study the background evolution of Universe for $R + \alpha R^n$ gravity model. We find a number of interesting cosmological evolutions which include the possibility of an initial unstable power-law inflationary phase, followed by a curvature fluid dominated phase mimicking standard radiation, then passing through a standard matter (CDM) era and ultimately evolving asymptotically towards a de-Sitter-like late-time accelerated phase. Also Our compact analysis shows that there are more equilibrium points than in the corresponding non-compact analysis in [123]. Furthermore, we find that for $n > 1/2(1+\sqrt{3})$ the phase space of $R + \alpha R^n$, contains two accelerated fixed points, together with two other saddle points (one represent a radiation phase and the other represent a matter-like phase). The presence of all these phases in the state space of $R + \alpha R^n$ makes a more detailed investigation worth pursuing .

In Chapter 8 we study the evolution of scalar cosmological perturbations by extending the 1+3 covariant gauge-invariant formalism to generic $f(R)$ theory of gravity. We derived a complete set of equations describing the evolution of matter and curvature fluctuations for a single and multi-fluid cosmological Universe with a general equation of state parameter. We also study the evolution of the density perturbations for the R^n gravity model using the general 1+3 covariant gauge-invariant formalism. Since the radiation-dust era represents the major period in the history of the Universe, we focus on the radiation and dust fluids described by barotropic equations of state. We finally study the exact solutions for scales much smaller and much larger than the Hubble radius. Finally we give a new covariant characterization of the quasi-static approximation and used this to show that on small scales this approximation is valid for values of n in the neighbourhood of 1, i.e., it is in good agreement with a numerical integration of the full set of equations. This work has been completed in collaboration with Peter K S Dunsby, Amare Abebe, and Alvaro de la Cruz-Dombriz .

In Chapter 9 we present for the first time a complete analysis of the imprint of tensor anisotropies on the Cosmic Microwave Background for a class of $f(R)$ gravity theories within the parameterized post-Friedmann (ppf-CAMB) framework. We derive equations, both for the cosmological background and gravitational wave perturbations, required to obtain the standard temperature and polarization power spectra, taking care to include all effects which arise from $f(R)$ modifications of both the background and the perturbation equations. Our analysis makes it possible to distinguish these models form GR and demonstrates the importance of considering the correct background when alternative theories of gravity are considered. This work has been completed in collaboration with Peter K S Dunsby, Amare Abebe, and Alvaro de la Cruz-Dombriz.

Chapter 10 is based on a paper completed in collaboration with Peter K S Dunsby, Sante Carloni and Kishore N Ananda. In this paper we show how the covariant gauge invariant equations for the evolution of scalar, vector and tensor perturbations for a generic $f(R)$ gravity theory can be recast in order to exploit the power of dynamical system methodology. In this way, recent results describing the dynamics of the background FRW model can be easily combined with these equations to reveal important details pertaining to the evolution of cosmological models in fourth order gravity.

Finally in Chapter 11 we conclude this thesis with a brief discussion of the main results obtained, open problems and an outlook for further work.

Unless otherwise specified, natural units ($\hbar = c = k_B = 8\pi G = 1$) will be used throughout this paper, Latin indices run from 0 to 3. The symbol ∇ represents the usual covariant derivative and ∂ corresponds to partial differentiation. We use the $-,+,+,+$ signature, and the Riemann tensor is defined by

$$R^a{}_{bcd} = W^a{}_{bd,c} - W^a{}_{bc,d} + W^e{}_{bd}W^a{}_{ce} - W^f{}_{bc}W^a{}_{df} \ , \tag{1.1}$$

where the $W^a{}_{bd}$ are the Christoffel symbols (i.e. symmetric in the lower indices), defined by

$$W^a_{bd} = \frac{1}{2}g^{ae}\left(g_{be,d} + g_{ed,b} - g_{bd,e}\right) \ . \tag{1.2}$$

The Ricci tensor is obtained by contracting the *second* and the *fourth* indices

$$R_{ab} = g^{cd}R_{acbd} \ . \tag{1.3}$$

Finally the Hilbert–Einstein action in the presence of matter is given by

$$\mathcal{A} = \int d^4x\sqrt{-g}\left[\frac{1}{2}R + L_m\right] \ . \tag{1.4}$$

Chapter 2

THE COVARIANT APPROACH TO COSMOLOGY

2.1 Introduction

The goal of cosmology is to explain the large-scale structure, and the evolution of the Universe as a single dynamical system on a particular averaging scale. The essential elements of any cosmological model are:

1. A space-time manifold M.
2. A metric structure, which describes the geometry of the space-time manifold.
3. A matter field.
4. A preferred four-velocity field (usually associated with the preferred matter motion at each point in the space-time on a suitable averaging scale).
5. A theory of gravity.

The space-time manifold M is assumed to be four-dimensional (Hausdorff), connected, time-oriented and C^{∞}, and the metric g is globally defined, C^{∞}, nondegenerate and Lorentzaian. The pair (M,g) define the space-time of GR, the simplest space-time is Minkowski space-time (R^4, η_{ab}), where η_{ab} is Minkowski metric.

The basic elements of the 1+3 formalism, is to project tensors and tensorial equations parallel and orthogonal to some preferred four-velocity vector u, and then use Ricci identities and Bianchi identities in conjunction with the Einstein field equations to derive the evolution and constraint equations for the kinematical and dynamical quantities. In the subsequent sections we will introduce the basic relations and the concepts of this formalism.

2.1.1 Orthogonal decomposition

Associated with the preferred four-velocity 2.73 there is a uniquely defined rest-frame at each point $p \in M$. Consequently the space-time manifold M can be locally split into space plus time through the orthogonal decomposition of the tangent space $T_p(M)$ to M at p,

$$T_p(M) = T_p(\Sigma) \oplus N_p(u)\,, \tag{2.1}$$

where $N_p(u)$ represent a 1-dimensional subspace of $T_p(M)$ generated by the vector u, and $T_p(\Sigma)$ is a 3-dimensional subspace orthogonal to u. These are the local hypersurfaces of simultaneity for the fundamental observers. The existence of these hypersurfaces allows us to define a time function, t, such that t =constant corresponds "at least locally" to a unique hypersurface$\Sigma_{\,t}$.
The local decomposition of the space-time manifold M with respect to the vector field u^a is accomplished by the following projection operators,

$$h : T(M) \longrightarrow T(\Sigma)\,, \qquad N : T(M) \longrightarrow N(u)\,, \tag{2.2}$$

$$v \longrightarrow v + g(u,v)\,u \qquad v \longrightarrow -u\,g(u,v)\,, \tag{2.3}$$

where N projects parallel to u and h projects on the rest space orthogonal to u. In component form,

$$h(u) = \delta + u \otimes u\,, \qquad h^a{}_b = g^a{}_b + u^a u_b = \delta^a{}_b + u^a u_b\,, \tag{2.4}$$

$$N(u) = -u \otimes u\,, \qquad N^a{}_b = -u^a u_b = \delta^a{}_b - h^a{}_b\,. \tag{2.5}$$

Accordingly,

$$h^a{}_b n^b = 0\,, \qquad N^a{}_b u^b = u^a\,, \qquad , h^a{}_a = 3\,, \tag{2.6}$$

$$h^a{}_c h^c{}_b = h^a{}_b\,, \qquad N^a{}_c N^c{}_b = N^a{}_b\,, \qquad N^a{}_c h^c{}_b = 0\,. \tag{2.7}$$

To project an arbitrary tensor T_{ab} into Σ each free index has to be contracted with the projection operator $h(u)$, that is,

$$T_{ab} = h^c{}_b h^d{}_a T_{cd}\,. \tag{2.8}$$

in particular the induced metric on the hypersurface Σ can be obtained by projecting the four dimensional metric g,

$$g_{ab\perp} = h_{ab} = h^c{}_a h^d{}_b g_{cd}\,, \tag{2.9}$$

$$= g_{ab} + u_a u_b\,. \tag{2.10}$$

For any space-time observer the local spatial and temporal splitting of any general tensor into its irreducible parts is the set of all tensors which arise from the spatial and temporal projection on each of its indices. For example,

1-*For a vector field* T^a:

$$T^a = \delta^a{}_b T^b = \Big[N^a{}_b + h^a{}_b\Big] T^b \tag{2.11}$$

$$= -(u_b S^b) u^a + h^a{}_b S^b \,. \tag{2.12}$$

The irreducible parts of T^a are $-(n_b S^b)$ and $h^a{}_b S^b$.

2-*For a 2-form* T_{ab}:

$$T_{ab} = \delta^c{}_a \delta^d{}_b T_{cd} = \Big[N^c{}_a + h^c{}_a\Big]\Big[N^d{}_b + h^d{}_b\Big] T_{cd} \,, \tag{2.13}$$

$$= N^c{}_a N^d{}_b T_{cd} + N^c{}_a h^d{}_b T_{cd} + h^c{}_a N^d{}_b T_{cd} + h^c{}_a h^d{}_b T_{ab} \,, \tag{2.14}$$

$$= n^c u_a n^d u_b T_{cd} - u^c u_a h^d{}_b T_{cd} - u^d u_b h^c{}_a T_{cd} + h^c{}_a h^d{}_b T_{cd} \,. \tag{2.15}$$

The irreducible parts of T_{ab} are the invariant $u^a u^b T_{ab}$, the two vectors $-u^a h^b{}_d T_{ab}$, $u^b h^a{}_c T_{ab}$ that are orthogonal to u^c and the spatial tensor $h^a{}_c h^b{}_d T_{ab}$ which is also orthogonal to u^c.

3-*For a mixed tensor* $T^a{}_b$:

$$T^a{}_b = \Big[N^a{}_c + h^a{}_c\Big]\Big[N^d{}_b + h^d{}_b\Big] T^c{}_d \,, \tag{2.16}$$

$$= \Big(u^d u_c T^c{}_d\Big) u^a n_b - u^a u_c h^d{}_b T^c{}_d \,, \tag{2.17}$$

$$= -u^d u_b h^a{}_c T^c{}_d + h^a{}_c h^d{}_b T^c{}_d \,. \tag{2.18}$$

4-*For a general tensor* $T^{a_1 \dots a_r}_{b_1 \dots b_s}$:

$$T^{a_1 \dots a_r}{}_{b_1 \dots b_s} = \Big[N^{a_1}{}_{c_1} + h^{a_1}{}_{c_1}\Big] \dots \Big[N^{a_r}{}_{c_r} + h^{a_r}{}_{c_r}\Big]\Big[N^{d_1}{}_{b_1} + h^{d_1}{}_{b_1}\Big] \dots \tag{2.19}$$

$$\Big[N^{d_s}{}_{b_s} + h^{d_s}{}_{b_s}\Big] T^{c_1 \dots c_r}{}_{d_1 \dots d_s} \,, \tag{2.20}$$

$$= h^{a_1}{}_{c_1} \dots h^{a_r}{}_{c_r} h^{d_1}{}_{b_1} \dots h^{d_s}{}_{b_s} T^{c_1 \dots c_r}{}_{d_1 \dots d_s} \,. \tag{2.21}$$

Since all these are tensor equations, the orthogonal decomposition is the same irrespective of what coordinates we use.

The projected volume element in the three hypersurfaces Σ is given by contracting the space-time volume element (η_{abcd}) along the time direction,

$$\eta_{abc} = \eta_{dabc} \; u^d \,. \tag{2.22}$$

Accordingly,

$$\eta_{abc} = \eta_{[abc]}\,, \quad \eta_{abc}\,u^c = 0\,, \tag{2.23}$$

$$\eta^{abc}\eta_{def} = 3!\,h^{[a}{}_d\,h^b{}_e\,h^{c]}{}_f\,, \quad \eta^{abc}\,\eta_{cef} = 2!\,h^{[a}{}_e\,h^{b]}{}_f\,, \tag{2.24}$$

$$\eta^{abc}\,\eta_{bcf} = 2!\,h^a{}_f\,, \quad \eta^{abc}\,\eta_{abc} = 3!\,, \tag{2.25}$$

where η_{abcd} is the natural canonical four-form representing the future pointing, right-handed unit volume element in four dimensions, such that $(\eta_{abc}d = \eta_{[abcd]} = 2\,\eta_{ab[c}u_{d]} - 2\,u_{[a}\eta_{b]cd},\ \eta_{0123} = \sqrt{|det\,g_{ab}|},\ \eta^{0123} = [-det(g_{ab})]^{-1/2})$. Also the projected permutation tensor is then defined by,

$$\epsilon_{abc} = \eta_{abcd}\,u^d\,. \tag{2.26}$$

Consequently we can define the spatial curl of a vector and a rank 2 tensor by

$$\mathrm{curl}\,X_a = \epsilon_{abc}\,\tilde{\nabla}^b\,X^c\,, \qquad \mathrm{curl}\,T_{ab} = \epsilon_{cd(a}\tilde{\nabla}^c\,T^d_{b)}\,. \tag{2.27}$$

We can always go back and forth between M and Σ, using the transformation equation,

$$A^{ab} = A^{\alpha\beta}\,e^a{}_\alpha e^b{}_\beta\,, \qquad A_{\alpha\beta} = A_{ab}\,e^a{}_\alpha e^b{}_\beta\,, \tag{2.28}$$

where e^a_α are the basis vectors defined on Σ.

2.1.2 Spatial covariant derivative

When u^a has a zero vorticity, the spatial metric $h_{ab} = g_{ab} + u_a\,u_b$, induces a unique covariant derivative operator $\tilde{\nabla}_b$ that is intrinsic to the hypersurfaces Σ and is obtained by totally projecting the four-dimensional covariant derivative ∇_a into Σ.
Given a spatial vector field X_a, the spatial covariant derivative is defined as,

$$\tilde{\nabla}_b\,X_a = (\nabla\,X_a)_\perp = h^c{}_b\,h^d{}_a\,\nabla_d\,X_c\,. \tag{2.29}$$

The spatial covariant derivative is compatible with the spatial metric,

$$\tilde{\nabla}_c\,h_{ab} = h^d{}_c\,h^f{}_a\,h^e{}_b\,\nabla_d\,h_{fe} = 0\,. \tag{2.30}$$

Note that when we apply $\tilde{\nabla}_a$ to space-time tensors we cannot use its compatibility with the spatial metric, which by construction applies only when we deal with a purely spatial tensors. Also it is crucial to regard spatial tensors as tensors over M and not overΣ , otherwise the covariant and Lie derivatives in the direction of Σ would not make sense. The action of $\tilde{\nabla}_a$ on a general space-time tensor and scalar is given by,

$$[\tilde{\nabla}_c\,T]^{a\cdots}{}_{b\ldots} = h^a a_1 \ldots h^{b_1}{}_b \ldots h^f{}_c\,\nabla_f\,T^{a_1\cdots}{}_{b_1\ldots}\,, \qquad \tilde{\nabla} f = h^b{}_a\,\nabla_b\,f\,. \tag{2.31}$$

The differentiation along the fundamental congruence, defined for scalars and tensors as,

$$\dot{f} = f_{,a}\, u^a\,, \qquad \dot{T}_{abc..c} = u^d\, \nabla_d\, T_{ab...c}\,. \tag{2.32}$$

Thus, the covariant derivative of a scalar can then be decomposed into a derivative along the fundamental congruence u and a spatial covariant derivative,

$$\nabla_a\, f = \tilde{\nabla}_a - u_a\, \dot{f}\,. \tag{2.33}$$

Following [76] we use angle brackets to denote the orthogonal projections of vectors, the orthogonally projected symmetric trace-free part of tensors, and their time derivatives,

$$v^{\langle a\rangle} = h^a{}_b v^b, \quad T^{\langle ab\rangle} = \left[h^{(a}{}_c h^{b)}{}_d - \frac{1}{3}h^{ab}h_{cd}\right]T^{cd}, \tag{2.34}$$

$$\dot{v}^{\langle a\rangle} = h^a{}_b \dot{v}^b, \quad \dot{T}^{\langle ab\rangle} = \left[h^{(a}{}_c h^{b)}{}_d - \frac{1}{3}h^{ab}h_{cd}\right]\dot{T}^{cd}. \tag{2.35}$$

Other useful spatial derivatives are:
The spatially projected Lie derivative along a vector field X,

$$[\tilde{L}_X\, T]^{a\cdots}{}_{b\ldots} = h^a{}_f \ldots h e_b \ldots [L_X\, T]^{f\cdots}{}_{e\ldots}\,, \tag{2.36}$$

and the spatially projected absolute derivative along a curve with unit tangent vector X,

$$[h\, \nabla_X\, T]^{a\cdots}{}_{b\ldots} = h^a{}_{a_1} \ldots h^{b_1}{}_b \ldots [\nabla_X\, T]^{a_1\cdots}{}_{b_1\ldots}\,. \tag{2.37}$$

2.2 Kinematics of timelike congruence

In this section we introduce the quantities that characterize the kinematical features of time-like congruence, all the equations that we will to encounter in this section are purely geometrical and essentially independent of the field equations of GR. Moreover, all the results of this section actually apply to any general fluid flow.

The comoving coordinates: To describe the geometry we need to set up a coordinate system. Although GR allows one to formulate the laws of physics using arbitrary coordinates, some coordinates are more natural and easier to work with. The coordinates that are comoving with the average motion of matter at each space-time point is an example of such a natural choice.
If a fixed spatial coordinate y^α is assigned to each fundamental world-line, we obtain the comoving coordinates $\{t, y^\alpha\}$, where t is the cosmic time function measuring proper time along every fundamental world-line. The available coordinate freedoms which preserve this form are,

$$t' = t'(t, y^\alpha)\,, \qquad y^{\alpha'} = y^\alpha\,, \tag{2.38}$$

$$y^{\alpha'} = y^{\alpha'}(y^\alpha)\,, \qquad t' = t\,. \tag{2.39}$$

The first transformation corresponds to a choice of new time surfaces, while the second one is nothing other than a relabelling of the world- lines in the initial surface. With respect to the comoving coordinates $\{t, y^\alpha\}$ the 4-velocity vector becomes,

$$u^a = (\frac{dt}{d\tau}, 0, 0, 0)\,. \tag{2.40}$$

It is always possible to choose t such that $dt/d\tau = 1$. General coordinates x^a are related to the comoving coordinates by the transformation $x^a = x^a(t, y^\alpha)$, thus in general coordinates,

$$u^a = \left(\frac{\partial x^a}{\partial t}\right)_{y^\alpha=\mathrm{const}}\,. \tag{2.41}$$

Note that the fundamental observers are hypothetical observers, where in more realistic models, matter acquires peculiar velocities with respect to the smooth Hubble flow. The dipolar anisotropy of the cosmic microwave background (CMB) is actually due to our peculiar motion relative to the cosmic rest-frame, the frame that redshifts with the expansion and in which the dipole vanishes.

The evolution of time-like world lines To determine how a congruence of time-like world-lines evolves with time we need to study the behavior of the connecting vector ξ^a between neighboring world lines under small displacement along the congruence.

Consider the normalized comoving coordinates $\{s, y^\alpha(v)\}$. Let ξ^a be a vector connecting two adjacent world-lines separated by an infinitesimal distance $\delta\, y^\alpha$ =const at some initial surface Σ_{s0} define the connecting vector

$$\xi^a = \frac{d\,x^a}{d\,v} = (0, \delta\, y^\alpha)\,, \qquad \text{where} \qquad \delta\, y^\alpha = \frac{d\,y^\alpha}{dv}\,dv\,. \tag{2.42}$$

With respect to a general coordinates $z^a = x^a(s, y^\alpha)$, ξ is given by,

$$\xi^a = \frac{\partial x^a}{\partial y^\alpha}|_{\mathrm{t=const}}\ \delta y\,. \tag{2.43}$$

The relative velocity associated with ξ is,

$$\frac{d\xi^a}{d\tau} = \xi^a{}_{;b}\, u^b\,. \tag{2.44}$$

During a time dt the connecting vector ξ^a is dragged along the world lines by the four-velocity $u^a = \partial x^a/\partial s|_{\mathrm{y=const}}$. From the properties of Lie derivative we have,

$$\mathcal{L}_u \xi^a = u^a{}_{;b}\xi^b - \xi^a{}_{;b}u^b = 0\,. \tag{2.45}$$

It follows from equation 2.44 that,

$$\frac{d\xi^a}{d\tau} = u^a{}_{;b}\ \xi^b\,. \tag{2.46}$$

According to this equation the relative velocity of nearby world lines is related to their relative position vector by a linear transformation. For a short time interval$\Delta\ t = t_1 - t_0$,

$$\xi^a(t_1) = \xi^a(t_0) + \Delta\xi^a(t_0)\,, \qquad \text{where}\Delta \qquad \xi^a = u^a b(t_0)\,\xi^b(t_0)\,\Delta t + O(\Delta t^2)\,. \tag{2.47}$$

To describe the action of the tensor $u^a{}_{;b}$ on ξ first we need to decompose it into its components parallel and orthogonal to u^a. Doing this we obtain ,

$$u_{a;b} = h^c{}_a h^d{}_b u_{c;d} - \dot{u}_a u_b\,, \tag{2.48}$$

where the vector $\dot{u}_a = u^b u_{a;b}$ is the acceleration due to non-gravitational forces, and vanishes if and only if there is no external force other than gravity and inertia acting on the fluid. In this case the fluid is said to be in a free fall or a geodesics motion. The first term in 2.48 can be decomposed into symmetric and anti-symmetric parts as follows ,

$$h^c{}_a\, h^d{}_b\, u_{c;d} = \Theta_{ab} + \omega_{ab}\,, \tag{2.49}$$

where ,

$$\Theta_{ab} = \frac{1}{2}(u_{cd} + u_{cd})\, h^c{}_a\, h^d{}_b\,, \qquad \omega_{ab} = \frac{1}{2}(u_{cd} - u_{cd})\, h^c{}_a\, h^d{}_b\,. \tag{2.50}$$

The tensor$\Theta\ _{ab}$ can be further decomposed into trace and trace free parts ,

$$\Theta_{ab} = \frac{1}{3}u^d{}_{;d} h_{ab} + \sigma_{ab}\,, \tag{2.51}$$

By substituting back into 2.48 we obtain ,

$$u_{a;b} = \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab} - \dot{u}_a u_b\,, \tag{2.52}$$

where $\Theta =\ u^d{}_{;d}$ is the expansion scalar, σ_{ab} is the shear tensor and ω_{ab} is the rotation tensor. By construction we have $\sigma_{ab}u^a = 0 = \omega_{ab}u^a = \dot{u}_a u^a$. σ_{ab} and ω_{ab} can be written in term of u as ,

$$\sigma_{ab} \;=\; u_{(a;b)} + \dot{u}_{(a}u_{b)} - \frac{1}{3}\Theta h_{ab}\,, \tag{2.53}$$

$$\omega_{ab} \;=\; u_{[a;b]} + \dot{u}_{[a}u_{b]}\,, \tag{2.54}$$

where we have used the definition $h^c{}_a = \delta^c{}_a + u^c u_a$ and the relation $u_{a;b}u^a = 0$. The magnitude of the kinematical quantities that we have introduced above is given by ,

$$\omega^2 \;=\; \omega^a\omega_a = \frac{1}{2}\omega_{ab}\omega^{ab} \geq 0\,, \;\; \text{where} \;\; \omega^a = \frac{1}{2}\eta^{abc\delta}u_b\omega_{c\delta} \tag{2.55}$$

$$\sigma^2 \;=\; \frac{1}{2}\sigma_{ab}\sigma^{ab} \geq 0\,, \tag{2.56}$$

where ,

$$\omega^2 \;=\; 0 \Leftrightarrow \omega^a = 0 \Leftrightarrow \omega_{ab} = 0\,, \qquad u^a\,\omega_a = 0 \;\; \omega_{ab}\,\omega^b = 0\,, \tag{2.57}$$

$$\sigma^2 \;=\; 0 \Leftrightarrow \sigma_{ab} = 0\,. \tag{2.58}$$

An irrotational vector field u^a is the one for which the vorticity is zero. The tensor $u_{;ab}$ has 12 independent components 5 independent components of σ_{ab}, 3 of ω_{ab}, 1 of Θ and 3 of $\dot{u}_a$. To understand the physical significance of Θ, σ_{ab} and ω_{ab}, consider the 2-dimensional matrix ,

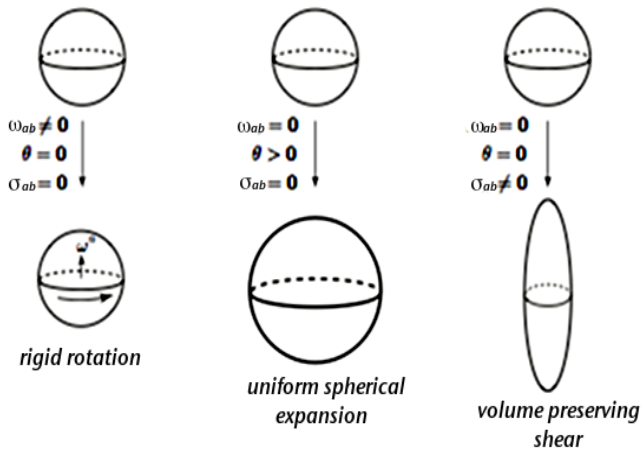

Figure 2.1: Rotation, expansion and shear.

$$u_{a;b} = \begin{pmatrix} \Theta/2 & 0 \\ 0 & \Theta/2 \end{pmatrix} + \begin{pmatrix} \sigma_+ & \sigma_x \\ \sigma_x & -\sigma_+ \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \tag{M1}$$

where $\Theta = \;\; u^a a$ is the trace of $u_{a;b}$, while σ_{ab} and ω_{ab} are the symmetric-tracefree and the antisymmetric parts of $u_{a;b}$ respectively . Let $\xi^b_\perp = (\cos\phi, \sin\phi)$ on $\Sigma_{t=t_0}$, then the action of the first term of (M1) on $\xi^b_\perp$ result in a new circle $(1/2)\Theta(\cos\phi, \sin\phi)$ of radius $r = 1 + (1/2)\Theta$. The corresponding change in area of the circle is then,$\Delta\ A \equiv A - A_0 = \pi\Theta$, and therefore $\Theta = \Delta A/A_0\,\Delta t$. It is clear that Θ measures the fractional change of area per unit time. In three dimensions, $\Theta =\!\Delta\ V/V_0\,\Delta t$, where in this case it measures the fractional change of volume V per unit time. It is convenient to define a representative length scale a(t) determining the average distance behavior of the fluid flow, by the relation ,

$$\frac{\dot{a}}{a} = \frac{1}{3}\Theta\,. \tag{2.59}$$

Thus the change change of volume along the fluid flow will be characterized by$\Delta\ V \propto a(t)^3$. The quantity $H = \dot{a}/a$ is called the Hubble parameter, it is present day value $H_0 = 72$ Km $s^{-1} Mpc^{-1}$ [137] being the Hubble constant.

By acting on the same original unit circle by the second part of $u_{a;b}$ we obtain,

$$\Delta\xi^a = \Delta\tau(\sigma_+ \cos\phi + \sigma_x \sin\sigma\phi, -\sigma_+ \sin\phi + \sigma_x \cos\sigma\phi)\,. \tag{2.60}$$

This is an equation of an ellipse with a major axis at angle ϕ with respect to x-axis. In parametric form equation 2.60 becomes $r(\phi) = 1 + \sigma_+\Delta\tau\cos 2\phi + \sigma_x\Delta\tau\sin 2\phi$. It is easy to check that the area of this ellipse is the same as the area of the original unit circle. Thus the action of the second part of (M1) is nothing else than a shearing of the original figure. σ_+ and σ_x are known by the shear parameters.

Finally by acting on the unit circle by the last part of (M1) we obtain,$\Delta\ \xi^a = \omega\Delta\tau(\sin\phi, -\cos\phi)$, and hence the new deviation vector is $\xi^a(\tau_1) = (\ \phi', \sin\phi')$, where $\phi' = \omega\Delta\tau$. This is a pure rotation of the original figure. The rotation tensor ω_{ab} determines both the (instantaneous) axis of rotation, and the (instantaneous) speed of rotation through the vector $\omega^a = \frac{1}{2}\epsilon^{abcd}\,\xi_b\,\omega_{cd}$. The acceleration, expansion, shear, and vorticity are the basic kinematical quantities, which can be used to characterize simple cosmological models.

2.2.1 The generalized Hubble's law

By decomposing the vector ξ^a into its components parallel and orthogonal to u^a, we obtain,

$$\xi^a = \xi^a_\perp + n^a n_b\,\xi^b\,, \tag{2.61}$$

where $\xi^a_\perp = h^a{}_b\xi^b$ is the instantaneous relative position vector between two adjacent world-lines in some initial hypersurface$\Sigma\ _{\tau_0}$. This position vector can be decomposed into a relative distance a, which represent the length of the connecting vector measured in the congruence rest fame, and directions e^a, which represents the direction cosines of a in the spatial triad,

$$\xi^a_\perp = e^a a\,, \qquad a^2 = \xi^a\,\xi_a = h_{ab}\,\xi^a\,\xi^b\,, \qquad e^a e_a = 1\,, \qquad e^a u_a = 0\,. \tag{2.62}$$

In term of $\xi^a_\perp$ equation 2.46 takes the form,

$$\frac{d\xi^a_\perp}{d\tau} = u^a{}_{;b}\,\xi^b_\perp\,. \tag{2.63}$$

By projecting this equation orthogonal to u^a we obtain a vector in the rest frame of u^a,

$$v^a = V^a{}_b\,\xi_\perp\,, \qquad \text{where} \qquad V_{ab} = h^c{}_a\,h^d{}_b\,u_{c;d}\,. \tag{2.64}$$

Using equation 2.63, 2.62 we obtain ,

$$\dot{a} = \Theta_{cb} e^c e^b a = (\frac{1}{3}\Theta + \sigma_{cb} e^c e^b)\, a\,. \tag{2.65}$$

This equation is a generalization of the Hubble relation, where it is clear that the rate of change of distance between world lines is proportional to the distance between them. Equation 2.65 also shows that the change of distance between neighboring world lines is due to two factors: an isotropic variation (expansion) measured by Θ and anisotropic expansion, measured by σ_{ab} .
The directions cosines e^a of l also change along the congruence. In fact from 2.62 and 2.65 we obtain ,

$$h^a{}_b\, \dot{e}^b = \omega^a{}_b\, e^b + \sigma^a{}_b - d^a{}_b\, (\sigma_{cd}\, e^c\, e^d) e^b\,. \tag{2.66}$$

According to this equation the change in the direction of the connecting vector along the congruence consist of two parts: a uniform rotation due to the vorticity of the congruence and a non-uniform rotation due to the shear .

2.2.2 Frobenius theorem

Let $\Phi(x^a) = c$ where $c \in R$ be a family of hypersurfaces foliating the space-time. A congruence of curves (timelike, spacelike, or null) is said to be hypersurface orthogonal if ,

$$n_a \propto \Phi_{,a} \qquad \Longrightarrow \qquad u_a = -r(x^a)\, \Phi_{,a}\,. \tag{2.67}$$

For some positive scalar function $r(x^a)$, this function can be determined from the normalization condition $u^a\, u_a = -1$.

Proposition: *A congruence of curves (timelike, spacelike, or null) is hypersurface orthogonal if and only if* $u_{[a;b}u_{c]} = 0$*, where* u^a *is tangent to the curves .*

Proof: Consider the completely antisymmetric tensor ,

$$u_{[a;b}u_{c]} \;\; \equiv \;\; \frac{1}{3!}(u_{ab}u_c + u_{c;a}u_b + u_{b;c}u_a - u_{b;a}u_c - u_{a;c}u_b - u_{c;b}u_a)\,. \tag{2.68}$$

By using equation 2.67, we obtain $u_{[a;b}u_{c]} = 0$, and consequently $\omega_a = 0$ by equation 2.55. Thus if a congruence is hypersurface orthogonal then $u_{[a;b}u_{c]} = 0$, the converse is also true. According to equation 2.68 we don't need to know any information about the scalar field Φ to decide whether or not u^a is hypersurface orthogonal .
If we can further impose the condition of zero acceleration $\dot{u}_a = 0$, we obtain the following result .

Proposition: $u_{[a;b}u_{c]} = 0, \dot{u}_a = 0 \iff \omega_{ab} = 0$

Proof: By using equation 2.52 and the relation 2.68, the tensor $u_{[a;b}u_{c]}$ can be written as,

$$u_{[a;b}u_{c]} = \frac{2}{3!}(u_{[a;b]}u_c + u_{[c;a]}u_b + u_{[b;c]}u_a)\,, \tag{2.69}$$

$$= \frac{2}{3!}(\omega_{ab}u_c + \omega_{ca}u_b + \omega_{bc}u_a)\,, \tag{2.70}$$

$$= 0\,. \tag{2.71}$$

By multiplying this equation by u^c we obtain $\omega_{ab} = 0$, the converse is non-trivial (Wald [p. 436]). In this case we can also prove that $u_a = -\Phi_{,a}$.

Thus for an irrotational flow the set of rest-spaces of all the fundamental observers define what is known by the hypersurfaces of simultaneity. In the presence of vorticity, however, Frobenius theorem forbids the existence of such integrable hypersurfaces. In this case the observers rest-spaces no longer mesh together smoothly.

2.3 Matter description

There are several different descriptions of the observed matter in the Universe, such as: The kinetic theory approach [39], and the continuum fluid approximations, has proven to be quite successful in describing the history of the Universe.

In the fluid approximation the basic thermodynamical quantities used to characterize the physical state of all matter fluids are the stress-energy tensor T^{ab}, which is a symmetric tensor field of type $(0,2)$, that describes the energy and momentum content of space-time, the particle flux vector N^a and the entropy flux vector S^a. These quantities satisfy,

$$T^{ab}{}_{;b} = 0\,, \qquad N^a{}_{;a} = 0\,, \qquad S^a{}_{;a} \geq 0\,. \tag{2.72}$$

The first equation expresses the conservation of energy and momentum, which is guaranteed by the twice-contracted Bianchi identities. The second one represents the conservation of particle number, which is true only under certain circumstances, and the last inequality relates that the entropy flux of the matter fluid obeys the second law of thermodynamics.

2.3.1 Average velocity

At late times and on a suitable averaging scale the average motion of the standard matter results in a preferred four-velocity, and hence a preferred world-line (the fundamental world-lines) in the space-time. If the fundamental world lines are given in terms of the local coordinates $x^{\alpha}(\tau)$, then ,

$$u^a \equiv \frac{dx^a}{d\tau}\,, \qquad \text{where} \quad g_{ab}\, u^a u^b = -1\,, \tag{2.73}$$

where τ is the proper time. This four-velocity is the key ingredient of the 1+3 approach, and it is defined by the vanishing of the dipole of the cosmic microwave background radiation (CBR). The coordinates adapted to this preferred four-velocity are known by the comoving coordinates. Consider a set of particles contained in some initial hypersurface$\Sigma_{\ t0}$. If the total mass within this hypersurface is not conserved, then one cannot identify the same average velocity at the beginning and the end of the time period used to measure the four-velocities. Therefore the average velocity can only be defined in terms of a conserved quantity. For example, if the particle flux $N^a = nu^a + q^a$ is conserved, there is a well-defined hydrodynamical average velocity $u_N = N^a/\sqrt{-N_b N^b}$ associated with it. The frame corresponding to this velocity is called the particle frame; this is the frame in which the particle drift vector $j^a = h^a{}_b N^b = 0$ vanishes. On the other hand, if the strong energy condition $T_{ab}V^aV^b \geq 0$ is satisfied for all time-like vectors V^a, then T^{ab} has a unique time-like eigenvector u^a_E defined by the vanishing of the energy flux and the stress tensor $q^a = 0 = \pi$, and satisfying the eigenvalue equation ,

$$T^a{}_b\, u^E v_a = -\rho\, u^E{}_b\,. \tag{2.74}$$

The kinematical choice $u^a{}_E$ is referred to as the energy frame or rest-frame of the fluid. The four-velocity of the observer in the energy frame is always parallel to the energy flux.

2.3.2 Imperfect fluid

For the imperfect fluids the hydroynamical time-like vectors $u^a{}_N, u^a{}_E$ point in different directions. The irreducible decomposition of T^{ab}, N^a and S^a with respect to the fundamental observers moving with four-velocity u^a is given by ,

$$\begin{aligned} T^{ab} &= d^a{}_c d^b{}_d T^{cd} = \Big[N^a{}_c + h^a{}_c\Big]\Big[N^b{}_d + h^b{}_d\Big]T^{cd}\,, \\ &= \mu n^a n^b + n^a q^b + n^b q^a + S^{ab}\,, \\ N^a &= n\, u^a + k^a\,, \qquad k^a\, u_a = 0\,, \\ S^a &= s\, u^a + r^a\,, \qquad r^a\, u_a = 0\,. \end{aligned} \tag{2.75}$$

where $\mu = n_c n_d T^{cd}$ is the relativistic energy density (the rest mass density plus the total internal energy due to heat, chemical energy, etc) measured by an observer comoving with the fluid;

$q^a = -h^a{}_c n_d T^{cd} = -n_d T^{ad} - n^a \rho$ is the relativistic momentum density (due to process such as diffusion and heat conduction); and $S^{ab} = h^a{}_c h^b{}_d T^{cd}$ is the relativistic stress tensor, $n = -N^a u_a$ and $s = -S^a n_a$ are the particle and entropy densities, respectively, k^a is the number flux (particle drift vector), and r^a the entropy flux. The quantities q^a and S^{ab} satisfy the orthogonality conditions ,

$$S^{ab} n_a = 0 \,, \qquad S^{ab} n_b = 0 \,, \qquad q^a n_a = 0 \,. \tag{2.76}$$

If T^{ab} is symmetric then $q^a = q^b$, and if T^{ab} is antisymmetric $\rho = 0$; $q^a = -q^b$, with S^{ab} now antisymmetric. S^{ab} can be decomposed into isotropic and trace-free parts as follows ,

$$S^{ab} = \pi^{ab} + p h^{ab} \,, \quad p = \frac{1}{3} S^a_a \,, \tag{2.77}$$

$$\pi^{ab} = \pi^{ba} \,, \ \ \pi^a_a = 0 \,, \ \ \pi_{ab} u^b = 0 \,, \tag{2.78}$$

where $p = S^{ab} \, h_{ab}/3 = T^{ab} \, h_{ab}/3$ is the effective isotropic pressure of the fluid, namely the sum of the equilibrium pressure and the associated bulk viscosity and $\pi^{ab} = h^c_{<a} \, h^d{}_{b>} \, S_{cd} = h^c_{<a} \, h^d{}_{b>} \, T_{cd}$ is the symmetric and trace-free anisotropic stress tensor, due to viscosity and/or elasticity and it describes the anisotropic pressure of the fluid. Thus the stress energy tensor in 2.75 takes the form ,

$$T^{ab} = \mu n^a n^b + p h^{ab} + q^a u^b + q^b u^a + \pi^{ab} \,. \tag{2.79}$$

The components $u_a T^{ab}{}_{;b} = 0$ and $h^c{}_a T^{ab}{}_{;b} = 0$ of the conservation equations $T^{ab}{}_{;b} = 0$ are given by ,

$$\dot{\mu} + (\mu + p)\Theta + \ \pi^{ab}\sigma_{ab} + q^a \dot{u}_a + q^a{}_{;a} = 0 \,, \tag{2.80}$$

$$(\mu + p)\dot{u}_a + h^c{}_a (p_{,c} + \pi^b{}_{c;b} + \dot{q}_c) + (\omega^b{}_a + \sigma^b{}_a + \frac{4}{3}\Theta h^b{}_a) q_b = 0 \,, \tag{2.81}$$

where the dot denotes the covariant derivative with respect to the proper time. The first equation is the energy conservation equation, where it shows how matter density varies along the fluid flow, the second equation is the momentum conservation equation and it determines the evolution equation for the acceleration. According to these equations we can see that the inertial mass density of matter is $(\mu + p)$, therefore any form of internal energy (e.g. heat or chemical energy) contributes to the effective inertial mass density in a direct way by contributing to μ or in an indirect way by contributing to p .

2.3.3 A non-tilted perfect fluid

When the fluid is perfect or in equilibrium state, all three the vectors S^a, $u^a{}_E$ and $u^a{}_N$ point in the same direction, thus in this case there are only two naturally defined timelike vector fields: The unit vector field, n^a, orthogonal to the surfaces of homogeneity (the geometric congruence), and the four-velocity, u^a, of the fluid (the matter congruence). If u^a is not aligned with n^a, we obtain what is called tilted cosmology. The geometric congruence is necessarily geodesic, vorticity-free and acceleration-free. The matter congruence, on the other hand, is not necessarily geodesic and can have both vorticity and acceleration. For a perfect fluid described by its fundamental four velocity u^a, the stress energy tensor, particle and entropy flux vectors are given by ,

$$T^{ab} = \mu u^a u^b + ph^{ab} = (\mu + p)u^a u^b + pg^{ab} \qquad \Longrightarrow \qquad q_a = 0 = \pi_{ab} \,. \tag{2.82}$$

$$N^a = n\, u^a \,, \qquad S^a = s\, u^a \,. \tag{2.83}$$

By combining the last two equations we obtain ,

$$S^a = m\, N^a \,, \tag{2.84}$$

where $m = s/n$ represents the specific entropy (the entropy per particle) of the fluid. For a perfect fluid T^{ab} and S^a are conserved, that is ,

$$T^{ab}{}_{;a} = 0 \,, \tag{2.85}$$

$$S^a{}_{;a} = 0 \,, \tag{2.86}$$

from which we obtain the following set of equations,

$$\dot{\mu} + (\mu + p)\Theta = 0 \,, \tag{2.87}$$

$$(\mu + p)u^a u_{a;b} + p_{,a}(g^{ab} + u^a u^b) = 0 \,, \tag{2.88}$$

$$\dot{m} = 0 \,. \tag{2.89}$$

The first equation determines the evolution of the energy density μ, once the a relation between p and μ is specified. The second equation is a generalization of Newtons second law $\mu \dot{u}_a + p_{,a} = 0$, the only difference being the use of $(\mu + p)$ as the inertial mass instead of μ in the Newtonian case. If we assume that $p = 0$, we have the simplest case of pressure-free matter, namely dust, which includes baryonic matter (after decoupling) and cold dark matter. In this case the conservation of momentum implies the geodesics equation ,

$$u^a u_{a;b} = 0 \,, \tag{2.90}$$

which means that the dust particles do not interact with anything other than the gravitational field and hence they are freely falling. Also we can have special combinations of the pressure gradient and the metric tensor such that one has $p_{,a}(g^{ab} + u^a u^b) = 0$, in this case the vector field u^a is also geodesic even though $p \neq 0$. The third equation in 2.87 means that the specific entropy of the system is constant along the fluid world-lines; a flow to satisfy this condition is called adiabatic. If we further assume the conservation of particle numbers $N^a{}_{;a} = 0$, we obtain one additional equation ,

$$\dot{n} \;=\; 0\,. \tag{2.91}$$

This equation ensures that the particle numbers do not change during the fluid motion .

2.3.4 A tilted perfect fluid

An observer O moving with a velocity $\widetilde{u}^a$ with respect to a perfect fluid will not determine it to have a perfect fluid form. If ψ is the hyperbolic angle between $\widetilde{u}^a$ and u^a, then $\widetilde{u}^a\, u_a = -\cosh\psi$ and $\widetilde{u}^a = \cosh\psi\, u^a + \sinh\psi\, e^a$, $e^a\, e_a = 1$, $e^a\, u_a = 0$. The projection tensor into the rest space of the observer O is $\widetilde{h}_{ab} = g_{ab} + \widetilde{u}_a\widetilde{u}_b$. Thus for the observer O the stress energy tensor will take the form ,

$$\widetilde{T}_{ab} = \widetilde{\mu}\widetilde{u}_a\widetilde{u}_b + \widetilde{p}\widetilde{h}_{ab} + 2\widetilde{q}_{(a}n_{b)} + \widetilde{\pi}_{ab}\,, \tag{2.92}$$

where ,

$$\widetilde{\mu} \;=\; \mu + (\mu + p)\sinh^2\psi\,, \tag{2.93}$$

$$\widetilde{p} \;=\; p + \frac{1}{3}(\mu + p)\sinh^2\psi\,, \tag{2.94}$$

$$\widetilde{q}_a \;=\; -(\mu + p)\cosh^2\psi\,\widetilde{u}_a\,, \tag{2.95}$$

$$\widetilde{\pi}_{ab} \;=\; \widetilde{h}^c{}_a\,\widetilde{h}^d{}_b\,T_{cd} = (\mu + p)[\cosh^2\psi\,\widetilde{u}_a\,\widetilde{u}_b - \frac{1}{3}\sinh^2\psi\,\widetilde{h}_{ab}]\,. \tag{2.96}$$

2.4 Multi-fluid cosmology

The thermodynamical description of a relativistic fluid is dictated by the energy momentum tensor T_{ab}, the particle flux N^a and the entropy flux S^a of the system. If there are N fluid components then the total stress energy tensor will be the sum of the contribution from each component, that is,

$$\begin{aligned} T^{\text{Total}}{}_{ab} \;&=\; \Sigma_i T^{(i)}{}_{ab}\,, \\ &=\; \Sigma_i\mu_i u^i{}_a u^i{}_b + \Sigma_i p_i h^i{}_{ab} + \Sigma_i q^i{}_a u^i{}_b + \Sigma_i q^i{}_b u^i{}_a + \Sigma_i\,\pi^i{}_{ab}\,. \end{aligned} \tag{2.97}$$

We can also define the total particle flux $N^{\text{Total}}{}_a$ and the total entropy flux $S^i{}_a$ as follow ,

$$N^a_{\text{total}} = \Sigma_i n_i u^a{}_i + f^a{}_i\,, \qquad S^a{}_{\text{total}} = s_i u^a{}_i\,, \tag{2.98}$$

where $u^a{}_i$ can be fixed by either choosing the energy frame $u^a{}_i = u^a{}_{E(i)}$ or the particle frame $u^a{}_i = u^a{}_{N(i)}$ of any matter component i. Following King, Ellis (1973) and Dunsby (1991), we are going to look at the situation in a frame define by the four-velocity u^α. Let ψ_i be the hyperbolic angle between u^a and $u^a{}_i$, then $u_a\, u^a{}_i = -\cosh\psi$, and the projection tensors into the instantaneous rest spaces orthogonal to $u^a{}_i$ is $h^i{}_{ab} = g_{ab} + u^i{}_a u^i{}_b$. see figure 2.4 . With respect

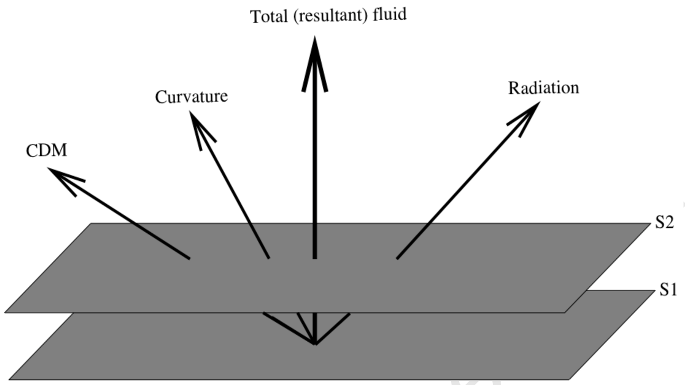

Figure 2.2: The Multi-fluid diagram: The different arrows show the unit time-like four-velocity vectors at different hyper-surfaces S1 and S2. The vectors do not coincide at the perturbative level.

to the observer O the tensors 2.97, 2.98 takes the form

$$\widetilde{T}_{ab} = \widetilde{\mu}\widetilde{u}_a\widetilde{u}_b + \widetilde{p}\widetilde{h}_{ab} + 2\widetilde{q}_{(a}u_{b)} + \widetilde{\pi}_{ab}\,, \tag{2.99}$$

$$\widetilde{N}^a = \widetilde{n}\,\widetilde{u}^a\,, \qquad \widetilde{S}^a = \widetilde{s}\,\widetilde{u}^a + \widetilde{r}^a\,, \tag{2.100}$$

where ,

$$\widetilde{n} = n\cosh\psi - k_a\,\widetilde{u}^a\,,$$

$$\widetilde{k}_a = k_a + (k_b\,\widetilde{u}^b - n\cosh\psi)\,\widetilde{u}_a + n\,u_a\,,$$

$$\widetilde{s} = s\,\cosh\psi - r_a\,\widetilde{u}^a\,,$$

$$\widetilde{r}_a = r_a + (r_b\,u^b - s\cosh\psi)\widetilde{u}_a + s\,u_a\,,$$

$$\widetilde{\mu} = \mu + (\mu + p)\sinh^2\psi + \pi_{ab}\widetilde{u}^a\widetilde{u}^b - 2q_a\widetilde{u}^a\cosh\psi$$

$$
\begin{aligned}
\widetilde{p} &= p + \frac{1}{3}(\mu + p)\sinh^2\psi + \frac{1}{3}\pi_{ab}\widetilde{u}^a\widetilde{u}^b - \frac{2}{3}q_a\widetilde{u}^a\cosh\psi\,, \\
\widetilde{q}_a &= q_a\cosh\psi - \pi_{ab}\widetilde{u}^b + [2q_b\widetilde{u}^b\cosh\psi - (\mu+p)\cosh^2\psi - \pi_{bc}\widetilde{u}^b\widetilde{u}^c]\widetilde{u}_a \\
&+ [(\mu+p)\cosh\psi - q_b\widetilde{u}^b]u_a\,, \\
\widetilde{\pi}_{ab} &= \pi_{ab} + 2\widetilde{u}_{(a}\pi_{b)c}\widetilde{u}^c + \frac{1}{3}[2q_c\widetilde{u}^c\cosh\psi - (\mu+p)\sinh^2\psi - \pi_{cd}\widetilde{u}^c\widetilde{u}^d]\widetilde{h}_{ab} \\
&+ [(\mu+p)\cosh^2\psi + \pi_{cd}\widetilde{u}^c\widetilde{u}^d - 2q_c\widetilde{u}^c\cosh\psi]\widetilde{u}_a\widetilde{u}_b + 2[q_c\widetilde{u}^c - (\mu+p)\cosh\psi]\widetilde{u}_{(a}u_{b)} \\
&+ (\mu+p)u_a u_b + 2q_{(a}u_{b)} - 2q_{(a}\widetilde{u}_{b)}\cosh\psi\,.
\end{aligned}
$$

If there is an interaction between the matter components then $T^{i\,\alpha\beta}{}_{;\beta}$ does not vanish for each component i, but the total stress energy tensor is conserved,

$$
T^{Total\ ab}{}_{;b} \equiv \Sigma_i T^{i\,ab}{}_{;b} = 0\,. \tag{2.101}
$$

If the matter components are non-interacting then $T^{i\,ab}{}_{;b} = 0$ for each i. when $\psi_i \ll 1$ the energy momentum tensor 2.99 reduces to the simple form 2.92. The stress energy tensor of this fluid relative to the fundamental four-velocity u^a takes the form,

2.4.1 Scalar fields

Consider a massless scalar filed ϕ with a potential $V(\phi)$. The equation of motion of this scalar field is Klein-Gordon equation,

$$
\nabla^a\nabla_a\phi - V'(\phi) = 0\,, \tag{2.102}
$$

and its stress energy tensor is given by,

$$
T_{ab} = \nabla_a\phi\,\nabla_b\phi - \left[\frac{1}{2}\nabla_c\phi\,\nabla^c\phi + V(\phi)\right]g_{ab}\,. \tag{2.103}
$$

If the field is spatially homogeneous $\phi = \phi(t)$, then it is possible to define a unit normal vector u^a orthogonal to the hypersurfaces $\phi =$cont, that is,

$$
u^a = \frac{\nabla^a\phi}{(-\nabla_c\phi\,\nabla^c\phi)^{\frac{1}{2}}}\,, \tag{2.104}
$$

with respect to this unit vector the stress-energy tensor 2.103 takes the perfect fluid form,

$$
\mu = -\frac{1}{2}\nabla_a\phi\,\nabla^a\phi + V(\phi)\,, \qquad p = -\frac{1}{2}\nabla_a\phi\,\nabla^a\phi - V(\phi)\,, \qquad q_a = 0\,, \qquad \pi_{ab} = 0\,. \tag{2.105}
$$

Scalar fields are needed in the early Universe to source inflation.

2.4.2 Equation of state

The pressure, which appears in the energy-momentum tensor, is determined by the equation of state of the medium. For a perfect fluid 2.82 the equation of state is given by ,

$$p = w\,\mu\,, \qquad 0 \leq w \leq 1\,. \tag{2.106}$$

This equation is a good approximation provided that no significant interactions between different components of the cosmic fluid take place and when the bulk viscosity is negligible. The general case, where the interaction between the different components of the cosmic fluid cannot be neglected, the equation of state takes the form ,

$$p = p(\mu, T, s)\,, \tag{2.107}$$

where s and T are the entropy per particle and the temperature respectively.

2.5 General relativity

The central equations of modern cosmology are the field equations of GR, which were first published in 1915. The basic ingredients of GR are:

1. Asymmetric metric tensor (Lorentzian four-metric) .
2. A torsion free connection,$\Gamma^{\;a}_{bc}$, which determines the covariant derivatives and is related to the metric through the relation $g_{ab;c} = 0$.
3. A stress-energy tensor describing the energy, momentum, and stress of all matter content of the space-time, and is related to the space-time metric through Einstein's field equation .
4. A four-dimensional manifold equipped with an arbitrary coordinates x^a .

The action $S[g]$ that gives rise to Einstein field equations is given by the following integral ,

$$S[g] = \int [R + L_M]\,\sqrt{-g}\,d^4x\,, \tag{2.108}$$

where L_M is the matter Lagrangian, R the scalar curvature (the trace of the Ricci tensor R_{ab}), and g is the determinant of the metric tensor g_{ab}. This action is known as the Einstein-Hilbert action. The variation of $S[g]$, with respect to the metric yields the field equations of GR ,

$$G_{ab} = R_{ab} - \frac{1}{2} R\, g_{ab} = -T_{ab}\,, \tag{2.109}$$

where T_{ab} is the stress-energy tensor of the various matter fields appearing in the action, and is defined in terms of the matter term L_M through ,

$$T_{ab} = -\Big(\frac{\partial\, L_M}{\partial\, g^{ab}} - g_{ab}\, L_M\Big)\,. \tag{2.110}$$

The expression on the left side of the equation 2.109 describes the curvature of space-time, while the expression on the right side represents the matter/energy content of Universe. The field equations 2.109 can then be interpreted as a set of equations dictating how the curvature of space-time is related to the matter/energy content of the Universe.

It is clear by definition that the conservation of energy comes as an automatic consequence of the field equations rather than a separate constraint,

$$G^{ab}{}_{;b} = 0 \qquad \Rightarrow \qquad T^{ab}{}_{;b} = 0 \, . \tag{2.111}$$

Raising a and contracting it with b in 2.109 yields, $R = T$, thus the field equations 2.109 can be rewritten in the form,

$$R_{ab} = -(T_{ab} - \frac{1}{2} g_{ab} T) \, . \tag{2.112}$$

According to this field equations, the only explicit source of the gravitational field g_{ab} is the stress energy tensor T_{ab}. When $T_{ab} = 0$, 2.112 reduce to the vacuum equations $R_{ab} = 0$.

Before 1915, Einstein believed that the Universe was static, but the solutions of the field equations 2.109, suggest that the Universe should be expanding. Einstein therefore proposed a modification of his equations, by introducing a new term Λ,

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = -T_{ab} \, , \tag{2.113}$$

where Λ is the cosmological constant. Λ, like R, has the dimensions of a curvature, namely $(\text{length})^2$. After the discovery of the Hubble redshift and the introduction of the expanding space paradigm, Einstein abandoned this concept. In 1998 measurements by two different groups of researchers of the apparent brightness of supernovae with redshifts near $z = 1$ showed that the expansion of the Universe is accelerating. These results have generated a new interest in the old idea of the cosmological constant. Analogously to 2.112, we find from 2.113 that,

$$R_{ab} = \Lambda \, g_{ab} - (T_{ab} - \frac{1}{2} \, g_{ab} \, T) \, . \tag{2.114}$$

Clearly in the absence of a matter sources, 2.114 reduces to equations $R_{ab} = \Lambda g_{ab}$, which implies nonzero curvature rather than Minkowski flat space-time.

Non-Local gravitational field

In GR, the gravitational field is described by means of the Riemann curvature tensor. This tensor can be split into a symmetric massless (non-local) traceless part C_{abcd} and the trace part

R_{ab} ,

$$R_{abcd} = C_{abcd} + \frac{1}{2}(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{bc}R_{ad} - g_{ad}R_{bc}) - \frac{1}{6}(g_{ac}g_{bd} - g_{ad}g_{bc})\,, \qquad (2.115)$$

and satisfies ,

$$R_{abcd} = R_{[ab][cd]} = R_{cdab}\,, \qquad R_{a[bcd]} = 0\,, \qquad (2.116)$$

where C_{abcd} is the Weyl tensor (conformal curvature), which represents the part of the space-time curvature that is not determined by the local matter but rather determined by conditions elsewhere. R_{ab} is Ricci tensor. This Ricci field describes the local gravitational field at each event due to matter via the Einstein field equations 2.113.
The Weyl tensor C_{abcd} contains all the information about the non-local, free gravitational field, mediated via gravitational waves and tidal forces. This tensor has the symmetries ,

$$C_{abcd} = C_{[ab][cd]} = C_{cdab}, \qquad C_{a[bcd]} = 0\,, \qquad C^{c}_{acb} = 0\,. \qquad (2.117)$$

It is clear that the Weyl tensor shares all the symmetries of the Riemann tensor and is also trace-free. Relative to the fundamental observers, the conformal curvature tensor decomposes further into its irreducible "electric" and "magnetic" parts E_{ab} , H_{ab} , where ,

$$C_{abcd} = (g_{abqp}\, g_{cdsr} + \eta_{abqp}\, \eta_{cdsr})\, u^q\, u^s\, E^{pr} + (\eta_{abqp}\, g_{cdsr} + g_{abqp}\, \eta_{cdsr})\, u^q\, u^s\, H^{pr}\,, \qquad (2.118)$$

where η_{abqp} is defined in chapter 2 section 2.1.1, $g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}$, and ,

$$E_{ab} = C_{acbd}\, u^c\, u^d \quad \Rightarrow \quad E^a_a = 0\,, \quad E_{ab} = E_{(ab)}\,, \quad E_{ab}\, u^b = 0\,, \qquad (2.119)$$

$$H_{ab} = \frac{1}{2}\eta_{ade}\, C^{de}{}_{bc}\, u^c \quad \Rightarrow \quad H^a{}_a = 0\,, \quad H_{ab} = E_{(ab)}\,, \quad H_{ab}\, u^b = 0\,. \qquad (2.120)$$

It is clear from the latter relations that E_{ab} and H_{ab} are trace-free, symmetric and orthogonal to u^a. The Newtonian analogue of E_{ab} is $E_{\mu\nu} = \varphi_{,\mu\nu} - (1/3)(\varphi^{,\kappa}{}_{,\kappa})h_{\mu\nu}$, where φ is the Newtonian gravitational potential. There is no Newtonian analogue of H_{ab} . From the once-contracted Bianchi identities we obtain ,

$$\nabla^d\, C_{abcd} = \nabla_{[b}R_{a]c} + \frac{1}{6}g_{c[b}\nabla_{a]}R\,. \qquad (2.121)$$

This equation can be decomposed into a set of two propagation and two constraint equations, which govern the dynamics of the electric and magnetic Weyl components .

2.5.1 Energy and causality condition

The stress energy tensor 2.82 is assumed to satisfy a series of conditions (see Table 2.5.1). These conditions are independent of the equation of state and are required to guarantee the stability and

Table 2.1: Energy conditions, where u^b and k^a are arbitrary timelike and null-future-directed vectors respectively

.

Name	Condition	For a perfect fluid
Weak	$T_{ab}u^a\, u^b \geq 0$	$\rho \geq 0, \quad \rho + p \geq 0$
Null	$T_{ab}k^a k^b \geq 0$	$\rho + p \geq 0$
Strong	$T_{ab}u^a\, u^b + 1/2T \geq 0$	$\rho + 3p \geq 0, \quad \rho + p \geq 0$
Dominant	$-T^a_b u^b$ future directed	$\rho \geq 0, \quad \rho \geq \lvert p_i \rvert$

having a proper gravitational behavior . For example, if the inertial mass density of matter $\mu + p$ is negative, then it follows from 2.87, that on compressing the fluid its energy density decreases. This result in an instability of the fluid. On the other hand, if the the active gravitational mass density $\mu + 3p$ is negative, then the gravitational effect of matter will be negative. According to the field equations of GR the strong energy condition is equivalent to $R_{ab}\, u^a\, u^b \geq 0$. Also the null-energy condition is equivalent to $R_{ab}\, k^a\, k^b \geq 0$. If the matter distribution is not of a form of an ideal fluid, then it is not possible to interpret the spatial components of $T^i{}_q$ as pressure. Therefore, we do not expect that the equation of state for matter at high energies obey the condition $(\mu + 3p) > 0$.

2.5.2 1+3 exact covariant propagation and constraint equations

By using Ricci identities and Bianchi identities, Einstein field equations 2.113 can be written as a set of evolution and constraint equations for the kinematic quantities and Weyl tensor components .

Ricci identities

For a general vector field u^a Ricci identities are given by ,

$$2\nabla_{[a}\nabla_{b]}\, u^c = R^c{}_{abd}\, u^d\,. \tag{2.122}$$

By projecting this equation along u^a, we obtain the general propagation equation ,

$$\frac{du_{a;d}}{dt} - \dot{u}_{a;d} + u^c{}_{;d} u_{a;c} = R_{abcd} u^b u^c\,. \tag{2.123}$$

By separating out this equation into a trace, symmetric trace-free, and skew symmetric parts, we obtain .

1- *The Raychaudhuri equation*

By taking the trace of 2.123 and using equations 2.52, we obtain,

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 2(\sigma^2 - \omega^2) - R_{ab}\, u^a\, u^b\,. \tag{2.124}$$

Finally by using 2.113 we obtain Raychaudhuri equation,

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) + \dot{u}^a{}_{;a} + \frac{1}{2}(\mu + 3p) - \Lambda = 0\,. \tag{2.125}$$

This equation is the fundamental equation for the gravitational attraction, telling us how the expansion scalar Θ varies along the fluid flow lines. Since the shear and rotation tensors are purely spatial $\omega^2 \geq 0$ and $\sigma^2 \geq 0$. The Raychaudhuri equation is the key equation for the relativistic gravitational attraction, showing that $(\mu + 3p)$ is the active gravitational mass density of the fluid. It is also clear that energy density and pressure, as well as the shear tend to decrease the expansion $\dot{\Theta} \leq 0$, while the vorticity and a positive cosmological constant tends to accelerate the expansion $\dot{\Theta} \geq 0$. The conventional (non-phantom) matter is always attractive unless $p < -\rho/3$.

2- *The vorticity propagation equation*

From the symmetric trace-free part of 2.123 we obtain,

$$\dot{\omega}_{<a>} = -\frac{2}{3}\Theta\,\omega_a - \frac{1}{2}\text{curl}\,A_a + \sigma_{ab}\,\omega^b\,. \tag{2.126}$$

3- *The shear propagation equation*

From the skew symmetric part of 2.123 we obtain,

$$\dot{\sigma}_{<ab>} = -\frac{2}{3}\Theta\,\sigma_{ab} - \sigma_{c<a}\sigma^c_{b>} - \omega_{<a}\omega_{b>} + A_{<a}A_{b>} - E_{ab} + \frac{1}{2}\pi_{ac}\,. \tag{2.127}$$

In these equations the acceleration provides a source for both the shear and the vorticity.

The constraint equations

By projecting 2.122 into the rest space of u^a we obtain in an analogous way the corresponding three kinematical constraints,

1- *The* $(0a)$*-equation*

$$\tilde{\nabla}^b\,\sigma_{ab} = \frac{2}{3}\tilde{\nabla}_a\,\Theta + \text{curl}\omega_a + 2\varepsilon_{abc}\,A_b\,\omega^c - q_a\,, \tag{2.128}$$

2- *The vorticity divergence identity*

$$\tilde{\nabla}^a\,\omega_a = A^a\,\omega_a\,, \tag{2.129}$$

3- *The H_{ab}-equation*

$$H_{ab} = \operatorname{curl}\sigma_{ab} + \tilde{\nabla}_{<a}\omega_{b>} + 2A_{<a}\omega_{b>}\,, \tag{2.130}$$

where $\operatorname{curl} v_a \equiv \varepsilon_{abc}\,\tilde{\nabla}^b\, v^c$ for any orthogonally projected vector v_a and $\operatorname{curl} T_{ab} \equiv \varepsilon_{cd<a}\tilde{\nabla}^c T^d{}_{b>}$ for any orthogonally projected symmetric tensor T_{ab}. Equation 2.129 represents the divergence of vorticity, equation 2.128 links the divergence of the shear to the rotation of the vorticity and equation 2.130 characterizes the gravitomagnetic field as the distortion of vorticity and the rotation of shear.

Twice contracted Bianchi identities

The Bianchi identities read,

$$\nabla_{[a}R^e{}_{bc]d} = 0\,. \tag{2.131}$$

By contracting this identity twice we obtain,

$$\nabla_a\, R^a{}_c + \nabla_b\, R^b{}_c - \nabla_c R^c{}_a = 0 \quad \Leftrightarrow \quad \nabla^a\, G_{ab} = 0 \Rightarrow\; \nabla^a\, G_{ab} = 0\,. \tag{2.132}$$

The last equality comes from the field equations 2.113. Projecting parallel and perpendicular to u^a we obtain the propagation equations,

$$\dot{\mu} + (\mu+p)\Theta + \;\pi^{ab}\sigma_{ab} + q^a\dot{u}_a + q^a{}_{;a} \;=\; 0\,, \tag{2.133}$$

$$(\mu+p)\dot{u}_a + h^c{}_a(p_{,c} + \pi^b{}_{c;b} + \dot{q}_c) + (\omega^b{}_a + \sigma^b{}_a + \frac{4}{3}\Theta h^b{}_a)\, q_b = 0\,. \tag{2.134}$$

Once contracted Bianchi identities

The Bianchi identities read,

$$R_{ab[cd;e]} = 0\,. \tag{2.135}$$

By splitting R_{abcd} into R_{ab} and C_{abcd} the identities 2.135 can be written as,

$$C^{abcd}{}_{;d} = R^{c[a;b]} - \frac{1}{6}\, g^{c[a}R^{;b]} \equiv q^{abc}\,, \tag{2.136}$$

where q^{abc} is the current of the gravitational charge. Consequently the contracted Bianchi identities take the form,

$$q^{abc}{}_{;c} = 0\,. \tag{2.137}$$

This equation implies the conservation of current. Making a $1+3$ split of these equations, using the definitions 2.5 and the field equations 2.113, the once contracted Bianchi identities 2.137

give a set of two propagation equations that are similar to Maxwell equations in an expanding Universe together with two constraint equations.

1- *The $\dot{E}$ equation*

$$
\begin{aligned}
&(\dot{E}^{\langle ab\rangle} + \frac{1}{2}\dot{\pi}^{\langle ab\rangle}) - \mathrm{curl}H^{ab} + \frac{1}{2}\tilde{\nabla}^{\langle a}q^{b\rangle} \\
&= -\frac{1}{2}(\mu+p)\sigma^{ab} - \Theta(E^{ab} + \frac{1}{6}\pi^{ab}) + 3\sigma^{\langle a}{}_{c}(E^{b\rangle c} - \frac{1}{6}\pi^{b\rangle c}) - \dot{u}^{\langle a}q^{b\rangle} \\
&\quad + \eta^{cd\langle}\left[2\dot{u}_c H^{b\rangle}{}_d + \omega_c(E^{b\rangle}{}_d + \frac{1}{2}\pi^{b\rangle}{}_d)\right].
\end{aligned}
\tag{2.138}
$$

2- *The $\dot{H}$ equation*

$$
\begin{aligned}
\dot{H}^{\langle ab\rangle} + curlE^{ab} - \frac{1}{2}curl\pi^{ab} &= -\Theta H^{ab} + 3\sigma^{\langle a}{}_{c}H^{b\rangle c} + \tfrac{3}{2}\omega^{\langle a}q^{b\rangle} \\
&- \eta^{cd\langle a}\left[2\dot{u}_c E^{b\rangle}{}_d - \frac{1}{2}\sigma^{b\rangle}{}_c q_d - \omega_c H^{b\rangle}{}_d\right].
\end{aligned}
\tag{2.139}
$$

The corresponding constraints are,

1- *The divergence of E equation*

$$
\tilde{\nabla}_a(E^{ab} + \frac{1}{2}\pi^{ab}) - \frac{1}{3}\tilde{\nabla}^a\mu + \frac{1}{3}\Theta q^a - \frac{1}{2}\sigma^a{}_b q^b - 3\omega H^{ab} - \eta^{abc}[\sigma_{bd}H^d{}_c - \frac{3}{2}\omega_b q_c] = 0. \tag{2.140}
$$

2- *The divergence of H equation*

$$
\tilde{\nabla}_b H^{ab} + (\mu+p)\omega^a + 3\omega_b(E^{ab} - \frac{1}{6}\pi^{ab}) + \eta^{abc}\left[\frac{1}{2}\tilde{\nabla}_b q_c + \sigma_{bd}(E^d{}_c + \tfrac{1}{2}\pi^d{}_c)\right]. \tag{2.141}
$$

Using equations 2.138; 2.139, we can obtain wave equations for E^{ab} and H^{ab} that describe the propagation of the gravitational radiation. The constraint equations 2.140; 2.141 show that $\tilde{\nabla}_a E^{ab}$; $\tilde{\nabla}_a H^{ab}$ are sourced by the spatial gradient of the energy density and the vorticity, respectively.

2.6 The Friedmann model

The high symmetry of the Robertson-Walker (RW) line-element puts a huge constraint on the kinematics of the fundamental congruence, and on the geometrical quantities that characterize the space-time manifold.

2.6.1 The kinematic characterization

The RW metric imposes specific conditions on the kinematic quantities of the fundamental congruence. The covariant derivative associated with RW metric,

$$
ds^2 = -dt^2 + a(t)^2\, h_{\alpha\beta}(x^a)\, dx^\alpha\, dx^\beta\,, \tag{2.142}
$$

is given by ,

$$\nabla_b \, u_a = \frac{1}{2} \, g_{ab,0} \, . \tag{2.143}$$

It follows from the metric 2.142 that ,

$$\nabla_b \, u_a = \frac{\dot{a}}{a} \, h_{ab} \, . \tag{2.144}$$

By comparing with 2.52 we obtain ,

$$\sigma_{ab} = 0 \, , \qquad \omega_a = 0 \, , \qquad \dot{u}_a = 0 \, , \qquad \tilde{\nabla}_a \, \Theta = 0 \, . \tag{2.145}$$

2.6.2 The geometric characterization

For the RW metric 2.142 Weyl tensor, and Ricci tensor satisfies ,

$$C_{abcd} = 0 \, , \Rightarrow E_{ab} = 0 \, , \qquad H_{ab} = 0 \, , \tag{2.146}$$

$$R_{ab} \, u^a \, h^b{}_c = 0 \, , \qquad R_{<ab>} = 0 \, . \tag{2.147}$$

These results follow immediately from Ricci identities, together with 2.130, and 2.145.

2.6.3 The dynamics of Friedmann-Lemaitre-Robertson-Walker (FLRW) models

The kinematics and dynamics of the cosmological model (M, g, u^a) satisfying the cosmological principle are quite simple. Since the line element 2.142 depends only on a single function of time, the scale factor $a(t)$, it follows that the non-trivial components of Einstein tensor are:

$$G^{00} = \frac{3}{a^2} \left(\dot{a}^2 + K \right) \, , \tag{2.148}$$

$$G^{11} = G^{22} = G^{33} = -\frac{1}{a^2} \left(2a\ddot{a} + \dot{a}^2 + K \right) \, . \tag{2.149}$$

As required by the cosmological principle the stress-energy tensor takes the perfect fluid form, which is given by ,

$$T_{ab} = (\mu + p) u_a \, u_b - p \, g_{ab} \, . \tag{2.150}$$

The functions μ and p are interpreted as the local energy density of the matter (including the rest mass energy and radiation in the Universe), and the local pressure at a given time respectively. According to the cosmological principle the gradients of p and μ should not select a particular direction in the surface of homogeneity Σ, thus they can only be functions of the cosmic time t .

Given the stress-energy tensor T_{ab}, Einstein equations 2.109 reduce to the following two differential equations:

$$\frac{\dot{a}^2}{a^2} = \frac{\mu}{3} - \frac{K}{a^2}\,, \tag{2.151}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\left(\mu + 3\,p\right)\,. \tag{2.152}$$

Equations 2.151 and 2.152 are termed the Friedmann and the Raychaudhuri equations, respectively. The Raychaudhuri equation determines how the rate of the expansion of the Universe changes, i.e., whether it is slowing down or speeding up. while the Friedmann equation is nothing else than a constraint equation. From 2.151 and 2.152 we can obtain the following equation,

$$\dot{\mu} + 3H\left(\mu + p\right) = 0\,. \tag{2.153}$$

Equation 2.153 is in fact the energy conservation law. The first term $\dot{\mu}$ tells us how fast density changes, while the second term is the loss of kinetic energy from the fluid, into gravitational potential energy. Any two of 2.151, 2.152 and 2.153 imply the third. When they are satisfied, all the 10 Einstein equations 2.109 are satisfied.

Before discussing the solutions Friedmann equation 2.151 and 2.152, we shall first mention some general features of these equations. To clarify the meaning of equation 2.153 we write it in the following form,

$$\frac{d}{dt}(\mu\, a^3) + p\frac{d}{dt}(a^3) = 0\,. \tag{2.154}$$

This means that in any local volume comoving with the expansion $a(t)$, the change of proper energy (i.e., in the frame of a local observer) is just the work done on its surroundings by the adiabatic change of that volume. This is, of course, conservation of energy in the form of the first law of thermodynamics, $dE = -p\,dV$. If $p = 0$, then $\mu\, a^3$ is the total mass in the comoving volume and it stays constant.

For normal matter, $(\mu + 3p) > 0$, thus according to equation 2.152 the Universe will have a decelerating expansion $(\ddot{a} < 0)$.

2.6.4 The singularity theorem

In cosmology a physical singularity is assumed to exist if some physical quantity become infinite or discontinuous. The existence of singular solutions is one of the generic features of GR. In terms of the scale factor $a(t)$ Raychaudhuri equation 2.125, can be written as,

$$3\frac{\ddot{a}(t)}{a(t)} = -2(\sigma^2 - \omega^2) + \dot{u}^{\alpha}{}_{;\alpha} - \frac{1}{2}k(\mu + 3p) + \Lambda\,. \tag{2.155}$$

If$\Lambda \leq 0$, $\mu + 3p \geq 0$ and $\mu + p > 0$ in a fluid flow for which $\dot{u} = 0$, $\omega = 0$, and $H_0 > 0$ at some time t_0, then a space-time singularity $a(t) \to 0$ at a finite proper time $t_s \leq 1/H_0$ occurs. Consequently equation 2.153 implies that the energy density and temperature diverge. This shows that the Universe in the standard model originated from a hot and dense state. According to Raychaudhuri equation pressure increases the active gravitational mass, thus in the standard model of cosmology an increase in pressure does not avoid the occurrence of the singularity.

In the framework of GR the singularity can be avoided if there are spatial pressure gradients, vorticity, a positive cosmological constant or the strong energy condition $\mu + 3p \geq 0$ is violated. If we abandon the FRW geometry a singularity will still occur for $a(\tau) < 0$ due to the divergence of the shear. Physically the existence of the BB singularity means that GR which is the classical theory of gravity has enters a physical domain in which it is not applicable.

2.6.5 Hubble-normalized parameters

In the standard model of cosmology the global dynamics of the Universe is described by a set of parameters. These are the Hubble parameter H, the matter density parameter Ω, which tells us how much matter the Universe contains, the dark energy density parameterΩ_{DE}, which controls the present expansion of the Universe and the spatial curvature density parameterΩ_K, which determines the geometry of the spatial sections of the space-time. The cosmological evolution is determined by these parameters through the gravitational field equations 2.151 and 2.152 (or 2.153) and a suitable equation of state. The density parameters are defined as,

$$\Omega_m = \frac{\mu_m}{3\,H^2}\,, \qquad \Omega_{DE} = \frac{\mu_{DE}}{3\,H^2}\,, \qquad \Omega_K = \frac{K}{a^2\,H^2}\,. \tag{2.156}$$

It follows from the Friedmann equation 2.151 that,

$$\Omega_m + \Omega_{DE} + \Omega_K = 1\,. \tag{2.157}$$

Another important cosmological parameter is the deceleration parameter q, which provides another mean of quantifying the rate of expansion of the Universe,

$$q \equiv -\frac{\ddot{a}}{a\,H^2}\,, \tag{2.158}$$

where $q > 0$ and $q < 0$ corresponds to deceleration and acceleration of the Universe respectively. Friedmann models can also be classified in terms of the signs of q and H as follows:

1. $H > 0; q > 0 \Rightarrow$ expanding and decelerating.

2. $H > 0; q < 0 \Rightarrow$ expanding and accelerating.

3. $H < 0; q > 0 \Rightarrow$ contracting and decelerating.

4. $H < 0; q < 0 \Rightarrow$ contracting and accelerating.

5. $H > 0; q = 0 \Rightarrow$ expanding with zero deceleration.

6. $H < 0; q = 0 \Rightarrow$ contracting with zero deceleration.

7. $H = 0; q = 0 \Rightarrow$ static.

2.6.6 Radiation, dust and dark energy-dominated Universes

To be able to solve the Friedmann Equation it is necessary to assume a suitable equation of state, which is an equation that relates p with μ. Generally, barotropic perfect fluids obey,

$$p = w\,\mu\,, \tag{2.159}$$

where the parameter w obeys,

$$0 \leq w \leq 1 \qquad \Leftrightarrow 0 \leq c_s^2 \leq 1\,, \tag{2.160}$$

where $c_s^2 = (\partial\, p/\partial\, \mu)_{\rm s=const}$ is the adiabatic speed of sound. The lower bound in 2.160 is required for local mechanical stability of matter. Any two of the three equations 2.151 and 2.152 2.153, combined with the equation of state 2.159, completely determine the three functions a(t), $\mu(t)$ and p(t). By solving 2.151 and 2.153 using , 2.159 we obtain,

$$\mu \propto a^{-3(w+1)}\,, \qquad a(t) \propto t^{\frac{2}{3(w+1)}}\,. \tag{2.161}$$

We know that for radiation, $w = 1/3$. It follows from Friedmann equation 2.151 that the cosmic evolution during the radiation-dominated epoch is given by $a \propto t^{1/2}$, and for non-relativistic matter $w = 0$, thus the cosmic evolution during the matter dominated era is given by $a \propto t^{2/3}$.

The Universe has passed through two phases of cosmic acceleration. The first one is called inflation, which is believed to have occurred prior to radiation domination. The second accelerating phase started after the matter domination, and is known as the dark energy phase. The existence these phases has been confirmed by a number of observations (for a review see [51],[52], [55],[53].[54]). The accelerated expansion of the Universe can be achieved if the condition,

$$\mu + 3p < 0 \Longrightarrow w < -1/3\,, \tag{2.162}$$

is satisfied. The dark energy component with energy density μ_{DE} and pressure p_{DE} satisfies the continuity equation,

$$\mu_{DE} + 3H(\dot{\mu}_{DE} + p_{DE})\,. \tag{2.163}$$

Integrating this equation we obtain ,

$$\mu_{DE} = \mu_{DE}^{(0)}\, exp\Big[\int_{a_0}^{a} -\frac{3(1+w_{DE})}{a}da\Big] \,, \tag{2.164}$$

where "0" stands for the present value and $w_{DE} = p_{DE}/\mu_{DE}$. Equation 2.151 can be written as ,

$$H^2(a) = H_0^2\Big[\Omega_m^{(0)}a^{-3} + \Omega_r^{(0)}a^{-4} + \Omega_K^{(0)}a^{-2} + \Omega_{DE}^{(0)}\, exp\Big[\int_0^{a} -\frac{3(1+w_{DE})}{a}da\Big] \,. \tag{2.165}$$

In the flat Universe dominated by dark energy with constant w_{DE}, equation 2.6.6 reduced to $H^2 \propto a^{-3(1+w_{DE})}$, thus the scale factor evolves as ,

$$a \propto t^{\frac{2}{3(1+w_{DE})}} \,, \quad \text{for } w_{DE} > -1 \,, \qquad\qquad a \propto e^{Ht} \,, \quad \text{for } w_{DE} = -1 \,.$$

The cosmic acceleration occurs for $-1 \leq w_{DE} < -1/3$. It is clear that the accelerated expansion generated by the cosmological constant $w_{DE} = -1$ does not end, therefore the cosmological constant can only be responsible for dark energy era in the late-time Universe. Modified gravity is another possible alternatives to derive the late-time cosmic acceleration and in some cases for inflation as well (see [205, 206, 207, 208, 209] for more details).

Chapter 3

COSMOLOGICAL PERTURBATION THEORY

3.0.7 Introduction

Newtonian theory can provide a good approximation for analyzing the gravitational instability on scales well below the Hubble length. However, on larger scales a relativistic treatment is needed. Hawking [19] developed a fully covariant formalism to study the perturbation of the curvature tensor. This work was developed further by Oslon [20], but due to their use of the gauge-dependent quantity density contrast $\delta\mu/\mu$ to study density perturbations, nonphysical gauge modes appears in their results. The first successful covariant and gauge-invariant approach to perturbations developed by Ellis and Bruni [75], where they make use of Stewart & Walker lemma to define gauge invariant variables. This work has been extended by Dunsby [149] to study perturbations in a multifluid cosmology. Based on this work the evolution of density perturbation in multifluid $f(R)$ gravity model can be developed [36].

3.0.8 The background Universe

Our Universe is well described in terms of a FLRW model at large scales but deviates from it at small scales due to the inhomogeneities. A possible way to take these inhomogeneities into account is by perturbing the FLRW model. Thus in cosmological perturbation theory one deals with two different space-times or manifolds, one being the unperturbed FRW background space-time, while the other is the perturbed, physical space-time. Using the 3+1 [37] decomposition the line element ds has the general form,

$$ds = [-\alpha^2 + \beta^\alpha\,\beta_\alpha]dt^2 + 2\beta_\alpha\,dt\,dx^\alpha + h_{\alpha\beta}dx^\alpha\,dx^\beta\,, \tag{3.1}$$

where α and $+\beta^{\alpha}$ are the lapse function and the shift vector respectively. In FRW Universe the above metric reduces to ,

$$d\overline{s}^2 = \overline{g}_{ab}\, dx^a\, dx^b = a^2(\eta)[-d\eta^2 + h_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}]\,, \tag{3.2}$$

where η is the conformal time which is related to cosmic time via $d\eta = dt/a(t)$, and $h_{\alpha\beta}$ is the metric on the hypersurfaces of homogeneity of constant curvature. The cosmological background evolution equations are ,

$$H^2 = (\frac{a'}{a})^2 = \frac{8\pi G}{3}\,\overline{\rho}\, a^2\,, \tag{3.3}$$

$$H' = -\frac{4\pi\, G}{3}(\overline{\rho} + 3\overline{p})a^2\,, \tag{3.4}$$

$$\overline{\rho}' = -3H(\overline{\rho} + \overline{p})\,\overline{\rho}\,, \tag{3.5}$$

where the prime represent the derivative with respect to η. The homogeneity of the background Universe implies that implies that $\overline{\rho} = \overline{\rho}(\eta); \overline{p} = \overline{p}(\eta)$,

The background energy tensor is of a perfect fluid form, that is ,

$$\overline{T}^{ab} = (\overline{\rho} + \overline{p})\overline{u}^a\,\overline{u}^b + \overline{p}\,\overline{g}^{ab}\,. \tag{3.6}$$

The isotropy of the background Universe implies that the spatial components of the four-velocity vector of matter fluid has to vanish, $u^i = 0$. To obtain the zeroth component u^0 we use the constraint $\overline{g}_{ab}\overline{u}^a\overline{u}^b = -1$,

$$\overline{g}_{ab}\overline{u}^a\overline{u}^b = a^2\,\eta_{ab}\overline{u}^a\overline{u}^b = -a^2(\overline{u}^0)^2 = -1\,, \Longrightarrow\ \overline{u}^0 = a^{-1}\,. \tag{3.7}$$

The background four-velocity is thus ,

$$\overline{u}^a = a^{-1}(\delta^{\alpha}0,\, \overrightarrow{0}) \qquad \text{and} \qquad \overline{u}_a = a(-\delta^0{}_b,\, \overrightarrow{0})\,. \tag{3.8}$$

where $\delta^{\alpha}{}_0$ is Krönecker delta .

3.0.9 Decomposition of vectors and tensors

The metric degrees of freedom in linear perturbation theory were classified first by Lifshitz [1] in 1946 into scalar vector and tensor parts. The study of the first-order perturbation theory shows that the scalar, vector, and tensor parts of the perturbed Einsteinian equations evolve independently, and the total evolution of the full perturbation is nothing else than a linear superposition of the independent evolution of the scalar, vector, and tensor part of the perturbation .

With the help of the orthogonal decomposition technique presented in chapter 2 section 2.1.1., any three-vector V can be split into a divergence-free (transverse or vector) part $V^{\perp}$ and a non-rotational (longitudinal or scalar) part $V^{\parallel}$,

$$V = V^{\parallel} + V^{\perp}\,, \qquad \text{where} \quad \bigtriangledown \times V^{\parallel} = 0 = \bigtriangledown \cdot V^{\perp}\,. \tag{3.9}$$

The curl and divergence are defined using the spatial covariant derivative, e.g $\overrightarrow{\nabla} \cdot \overrightarrow{V} = h^{\alpha\beta}\nabla_{\alpha} V_{\beta}$. Since $V^{\parallel}$has a vanishing curl, we can write it as the divergence of some scalar field, ϕ, thus

$$V = \nabla\phi + V^{\perp}\,. \tag{3.10}$$

Similarly a symmetric tensor S_{ab} can be decomposed into three parts correspond to: both indices are longitudinal $S^{\parallel}_{ab}$, one is transverse $S^{\perp}_{ab}$, and two are transverse S^{T}_{ab}.,

$$S_{ab} = S^{\parallel}_{ab} + S^{\perp}_{ab} + S^{T}_{ab}\,, \tag{3.11}$$

where,

$$h^{bc}\nabla_k\, S_{ab} = h^{bc}\nabla_k\, S^{\parallel}_{ab} + h^{bc}\nabla_k\, S^{\perp}_{ab}\,, \qquad h^{bc}\nabla_k S^{T}_{ab} = 0\,. \tag{3.12}$$

h^{a}_{b} is the projection operator defined in chapter 2, section 2.1.1. The two terms in this equation represent the longitudinal and the transverse vectors respectively. If S_{ab} is a traceless symmetric tensor then, $S^{\parallel}_{ab}$ and $S^{\perp}_{ab}$ can be obtained from the gradients of a scalar and a transverse vector, respectively,

$$S^{\parallel}_{ab} \;=\; (\nabla_a \nabla_b - \frac{1}{3}\, h_{ab}\nabla^2)\phi\,, \tag{3.13}$$

$$S^{\perp}_{ab} \;=\; \nabla_i\, S^{\perp}_{b} + \nabla_b\, S^{\perp}_{a}\,, \text{where} \quad h^{ab}\, S_{a,b} = \partial_a\, S_a = 0\,, \tag{3.14}$$

and they satisfy the following constraints,

$$\epsilon_{abc}\, \partial_b\, \partial_d\, S^{\parallel}_{dc} = 0\,, \qquad \partial_a \partial_b\, S^{\perp}_{ab} = 0\,, \qquad \partial_a\, S^{T}_{ab} = 0\,. \tag{3.15}$$

S^{T}_{ab} cannot be obtained from a gradient of a scalar or a vector, therefore it represents the tensor part, while $S^{\parallel}_{ab}$ and $S^{\perp}_{ab}$ represent the scalar and vector parts respectively. It is clear that $S^{\parallel}_{ab}$ is symmetric and traceless by definition, and $S^{\perp}_{ab}$ is also symmetric by definition and the condition $h^{ab}\, S_{a,b} = 0$ make it traceless. On the other hand the symmetric tensor S^{T}_{ab} satisfy the constraints,

$$h^{ab} S^{T}_{ab} = 0\,, \qquad h^{bc}\nabla_c\, S^{T}_{ab} = 0\,. \tag{3.16}$$

The first constraints make the tensor S^{T}_{ab} transverse and tracless, while the second constraint reduces the number of degrees of freedom of S^{T}_{ab} from five to two; these two degrees of freedom correspond to the two polarization states of gravitational waves.

3.0.10 Gauge transformations

To be able to compare tensors defined in two different space-times M; N, we must first define how points in these two space-times relate to each other, any such choice define what is known as gauge choice. Formally a gauge is nothing other than a diffeomorphism $\varphi: M \to N$, between the two space-times. Given a tensor quantity Q defined in the physical space-time N, and the corresponding background quantity $\widehat{Q}$, defined in the background space-time M, then the perturbation δQ of Q at the point $p \in M$ is defined as the difference between its value at some event in the physical space-time and its value at the corresponding event (associated via the gauge) in the background. ,

$$\delta Q(p) = Q(p) - \overline{Q}(\varphi^{-1}(p)) \, . \tag{3.17}$$

$\overline{Q}(\varphi^{-1}(p))$ is the image in M of the perturbed quantity (the pullback). We cannot assign a unique background quantity $\overline{Q}$ to a point in the perturbed space-time, because in different gauges this point is associated with different points in the background, with different values of $\overline{Q}$. Therefore there is also no unique perturbation $\delta\, Q$,. The value of $\delta\, Q$ is entirely gauge-dependent and therefore arbitrary.

Since the composition of two diffeomorphisms is itself a diffeomorphism, therefore if we fix the coordinates in the background space-time, gauge transformation can be viewed as a diffeomorphism inside the physical manifold see figure 3.0.10. Thus, if we want to study gauge transformations we need simply to look at diffeomorphisms in the physical space-time N, instead of dealing with diffeomorphisms between different space-times . Let ξ^b be an arbitrary vector field on N, the flow φ^ξ generated by this vector field is a map from N to N ,

$$\varphi^\xi_\lambda : \quad N \longrightarrow \quad N \tag{3.18}$$

$$q \quad \longrightarrow \quad \varphi^\xi_\lambda(q) \, , \tag{3.19}$$

where λ is an affine parameter. φ^ξ_λ is a translation of the manifold N a parameter distance λ along the integral curves of ξ. For a given λ, φ^ξ_λ is therefore a diffeomorphism on N, identifying the point $\varphi^\xi(\lambda, x)$ with x, this diffeomorphism define gauge transformation. For a given point $q \in N$ the diffeomorphism φ^ξ_λ depends continuously on λ, by Taylor expand Q at q we obtain ,

$$\widehat{Q}(q) = \Sigma^\infty_{n=0} \frac{(\lambda)^n}{n!} \, L^n_\xi \, Q(q) \, , \tag{3.20}$$

where $\widehat{Q}(q) = \varphi^\xi_\lambda(Q(q))$ is the pullback of Q from $\varphi^\xi_\lambda(q)$ to q. The infinitesimal form of the gauge transformations ,

$$\widehat{Q}(q) = Q(q) + L_\xi \, Q(q) + \frac{1}{2} \, L_\xi \, L_\xi \, Q(q) + O(\xi^3) \, . \tag{3.21}$$

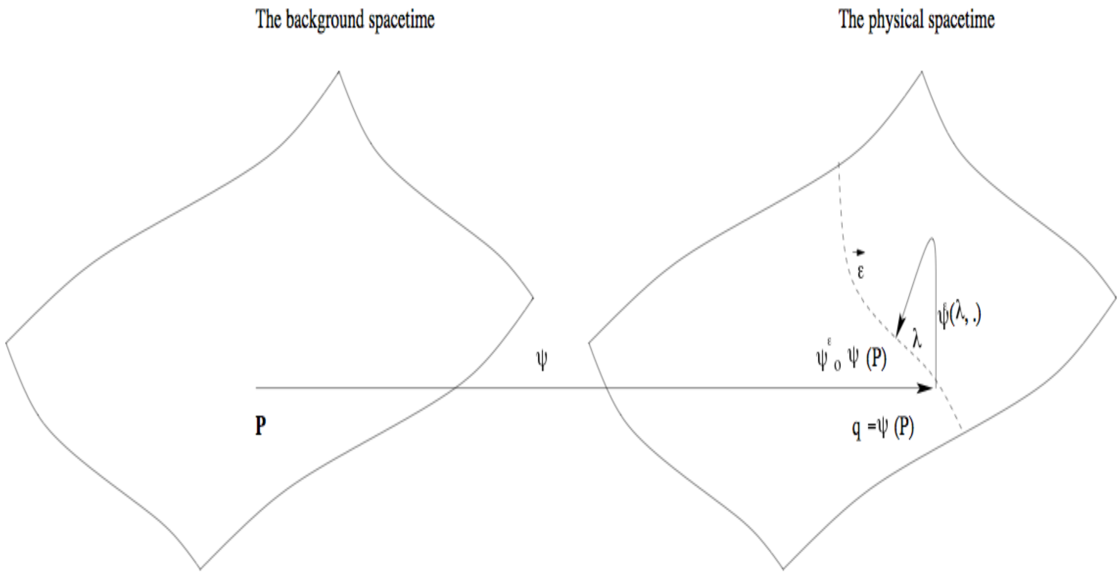

Figure 3.1: Both Ψ anΨ $^{\xi} \circ \Psi$ are valid diffeomorphisms corresponding to two different gauges. Ψ^{ξ}, being an embedding, represents a gauge transformation.

Note that λ has been absorbed in the rescaling of ξ^b. equation 3.21 describes how the quantity Q transform under the gauge transformation $\varphi^{\xi}(\lambda, q)$.

Thus a general tensor quantity is gauge invariant if and only if it has a vanishing Lie derivative along every infinitesimal vector field on the background space-time . Another way to state this condition is via the following lemma .

Stewart & ***Walker lemma*** [203] The linear perturbation δT of a quantity T_0 on the background space-time is gauge invariant if and only if one of the following holds:

1. T_0 vanishes ,
2. T_0 is a constant scalar ,
3. T_0 is a constant linear combination of products of Kronecker deltas .

3.0.11 Scalar, vector and tensor perturbations

Scalar perturbations couple to density and pressure perturbations; they are responsible for the gravitational collapse and for seeding the structure formation in the Universe. Vector perturbations, on the other hand, couple to the vorticity perturbations and they always decay in an expanding Universe, therefore they are not important in cosmology. Tensor perturbations are gauge-invariant and they are responsible for gravitational waves. Scalar, vector and tensor perturbations behave like a spin 0 spin 1 and spin 1/2 field under spatial rotations.

Any gauge transformation (a change in the correspondence between the background and the physical Universe) induces a coordinate transformation in the perturbed space-time. Thus the gauge freedom is usually expressed as a freedom of coordinate choice in the perturbed space-time. If x^b is a set of coordinates defined on N, then the gauge transformation φ^{ξ}_{λ}, generated by an infinitesimal vector field $\xi = (\xi^0, \xi^\alpha)$, induces the first-order coordinate transformation ,

$$x^b = (\eta, \overrightarrow{x}) \longrightarrow^{\xi} x^{b'} = \varphi^{\xi}_{\lambda}(x^b) = (\widetilde{\eta}, \widetilde{x})\,, \tag{3.22}$$

where ,

$$\widetilde{\eta} = \eta + \xi^0(\eta, \overrightarrow{x})\,, \qquad \widetilde{x}^\alpha = x^\alpha + \xi^\alpha(\eta, \overrightarrow{x})\,. \tag{3.23}$$

Scalar quantities that are homogeneous in the hypersurfaces$\Sigma_{\ \eta}$, such as the density $\overline{\rho}$ and the the 3-curvature K depend only on ξ^0, while the 3-vectors and 3-tensors on$\Sigma_{\ \eta}$ are functions of ξ. By splitting ξ^α into a divergence-free (transverse) part $\xi^\perp$ and a non-rotational (longitudinal) part $\xi^\parallel$, expressible as a gradient of some function ζ, we obtain,

$$\xi^i = \xi^\perp + \xi^\parallel = \xi^\parallel + \nabla\zeta\,, \qquad \text{where} \qquad \nabla \cdot \zeta^\perp = 0\,. \tag{3.24}$$

The two infinitesimal scalar functions ξ^0 and $\zeta_{,i}$ are responsible for the scalar perturbation, whereas the longitudinal part $\xi^\parallel$ introduces vector perturbation.

3.0.12 Perturbations of the metric

In this thesis we focus only on first-order perturbation. For any tensor quantity $\overline{T}$ defined in the background Universe we can define a corresponding first-order perturbed quantity ,

$$T = \overline{T} + \delta T\,, \tag{3.25}$$

in the perturbed Universe. In particular the first-order perturbed metric can be written as ,

$$g_{ab} = \overline{g}_{ab} + \delta g_{ab}\,. \tag{3.26}$$

We write the perturbation δg_{ab} as ,

$$\delta\, g_{ab} = \begin{pmatrix} -2\phi & -\varrho_\alpha \\ -\varrho_\alpha & \frac{1}{3}D\delta_{\alpha\beta} + M_{\alpha\beta} \end{pmatrix}, \qquad M2$$

where D and and $M_{\alpha\beta}$ are the trace and the trace-free parts of the spatial metric perturbation $\delta\, g_{\alpha\beta}$,

$$D = \delta\, g^{\alpha}_{\alpha}\,, \qquad \text{and} \quad \delta^{\alpha\beta}\, M_{\alpha\beta} = 0\,, \tag{3.27}$$

and, $\phi(\eta, x^\alpha); \varrho_i(\eta, x^\alpha)$ are the lapse function and the shift vector respectively. The physical, inhomogeneous line element is thus,

$$ds^2 = (\overline{g}_{ab} + \delta\, g_{ab})dx^a\, dx^b = a^2(\eta)[(1+2\phi)d\eta^2 - 2\varrho_\alpha\, d\eta dx^{\,\alpha} + [(1-\frac{1}{3}D)\delta_{\alpha\beta} + M_{\alpha\beta}]dx^\alpha\, dx^\beta]\,.$$

Since $\delta\, g_{ab}$ is a symmetric 4×4 metric it has in total 10 independent component.

3.0.13 Decomposition of the perturbed metric

Now we can apply the results of the previous section to the metric (M2). The time-time component of the metric is the scalar function ϕ, the space-time components are the three-vector ϱ_α, which can be splitted into,

$$\varrho = \varrho^{\parallel} + \varrho^{\perp} \implies \varrho_\alpha = B_{,\alpha} + S_\alpha\,, \tag{3.28}$$

for some scalar function B and a divergence-free vector field S_a. The comma represent the covariant derivative with respect to $h_{\alpha\beta}$. The space-space components are given by the traceless tensor $M_{\alpha\beta}$ and can also be split into,

$$M_{\alpha\beta} \;=\; (\partial_\alpha\partial_\beta - \frac{1}{3}h_{\alpha\beta}\nabla^2)\,\mu + \partial_\alpha\, F_\beta + \partial_\beta\, F_\alpha + M^T_{\alpha\beta}\,, \tag{3.29}$$

where μ is some scalar function and F_α is some divergence-free vector field and $\partial_\alpha\, M^T_{\alpha\beta} = 0$. Equation 3.29 can be written in a compact form as follows,

$$M_{\alpha\beta} = 2\,\psi\,\gamma_{\alpha\beta} - 2E_{,\alpha\beta} + \partial_\alpha\, F_\beta + \partial_\beta\, F_\alpha + M^T_{\alpha\beta}\,, \tag{3.30}$$

where,

$$\psi \equiv \frac{1}{6}(D - \nabla^2 b)\,, \qquad E \equiv -\frac{1}{2}\,\mu\,. \tag{3.31}$$

Scalar part Scalar perturbations are constructed from a scalar quantity or its derivatives, and any background quantities such as the 3-metric $h_{\alpha\beta}$. Thus the scalar parts of the metric (M2) are,

$$\delta\, g_{00} = -2a^2\phi(\eta, \overrightarrow{x})\,,\ \delta\ g(0\alpha) = a^2 B_{,\alpha}(\eta, \overrightarrow{x})\,,\ \delta\ g_{\alpha\beta} = -2a^2\,(\psi(\eta, \overrightarrow{x})h_{\alpha\beta} + 2E_{,\alpha\beta}(\eta, \overrightarrow{x})) \tag{3.32}$$

The scalar function $\psi(\eta, \overrightarrow{x})$ determines the perturbation of the three-curvature of the surfaces of homogeneity$\Sigma_{\,t}$. and $\phi(\eta, \overrightarrow{x})$ plays the same role as the Newtonian potential (since it determines particle acceleration in this metric).

Vector part Vector perturbations are constructed from a pure rotational vectors that has no scalar parts, which means that vector quantities that are constructed from scalars, which are irrotational should be excluded. The vector part of the metric (M2) are ,

$$\delta\, g_{0\alpha} = -a^2\, S_\alpha(\eta, \overrightarrow{x})\,, \qquad \delta\, g_{\alpha\beta} = 2a^2 F_{\alpha,\beta}(\eta, \overrightarrow{x})\,. \tag{3.33}$$

Tensors part Tensor perturbations are constructed from symmetric $M_{[\alpha\beta]} = 0$, trace-free $\gamma^{\alpha\beta}\, M_{\alpha\beta} = 0$, and divergence-free $\gamma^{\beta\gamma} M_{\alpha\beta,\gamma} = 0$ three-tensors. The Tensor part of the metric (M2) are ,

$$\delta\, g_{\alpha\beta} = a^2 M_{\alpha\beta} \tag{3.34}$$

Note that the names scalar, vector, and tensor refer to their transformation properties under rotations of the three-dimensional hypersurfaces$\Sigma_{\,\eta}$ in the background space-time. The most general linear perturbed metric can now be written as ,

$$\delta\, g_{ab} = a^2(\eta) \begin{pmatrix} 2\phi & -B_{,\alpha} \\ -B_{,\alpha} & 2(\psi\, \delta_{\alpha\beta} - E_{,\alpha\beta}) \end{pmatrix} - a^2(\eta) \begin{pmatrix} 0 & S_\alpha \\ S_i & F_{\alpha,\beta} + F_{\beta,\alpha} \end{pmatrix} + a^2(\eta) \begin{pmatrix} 0 & 0 \\ 0 & M_{\alpha\beta} \end{pmatrix} \quad (M3)$$

and consequently the perturbed line element is ,

$$ds^2 = a^2(\eta) {-}(1 + 2\phi)d\eta^2 + 2(B_{,\alpha} - S_\alpha)d\eta dx^{\,\alpha} + [(1 - 2\psi)\gamma_{\alpha\beta} + 2E_{,\alpha\beta} + 2F_{\alpha,\beta} + M_{\alpha\beta}]dx^\alpha\, dx^\beta\,.$$

The three terms in (M3) represent the scalar, vector and tensor metric perturbations respectively. In total the metric perturbations have ten degrees of freedom. Where the scalar perturbations contribute by four degrees of freedoms through the scalar functions ϕ, B, ψ and E, the condition for the three-vectors B_α and F_α to be transverse reduces their independent components from three to two thus the vector perturbations have in total four degrees of freedom, and finally in the tensor perturbations there are only two degrees of freedom, since they are made up of a symmetric, transverse and trace-free three-tensor $M_{\alpha\beta}$. This coincides with the number of degrees of freedom in the original metric perturbation $\delta\, g_{ab}$. Because of the ability to make continuous deformations of the coordinates, four of these degrees of freedom are gauge coordinate dependent, leaving six physical degrees of freedom . Under the first order gauge transformation

3.22, the new perturbed metric is defined as,

$$\delta\widetilde{g}_{ab} = \delta g_{ab} - L_\xi g_{ab}, \tag{3.35}$$

$$= \delta g_{ab} - g_{b\lambda}\,\xi^\lambda_{,a} + g_{\lambda a}\,\xi^\lambda_{,b} + g_{ba,\lambda}\xi^\lambda\,. \tag{3.36}$$

By substituting these components into (M3) we obtain,

$$2\,a^2\widetilde{\phi} = 2\,a^2\phi - 2a^2\xi^{,0} - 2\,a\,a'\,\xi^0\,, \tag{3.37}$$

$$-a^2\,\widetilde{B}_{,\alpha} = -a^2\,B_{,\alpha} - a^2\,(\xi^0 - \xi^{'})_{,\alpha}\,, \tag{3.38}$$

$$2\,a^2\,(\widetilde{\psi}\delta_{\alpha\beta} - \widetilde{E}_{\alpha\beta}) = 2a^2\,(\psi\,\delta_{\alpha\beta} - E_{,\alpha\beta}) + 2\,a^2\,(\xi_{,\alpha\beta} + \frac{a'}{a}\xi^0\,\delta_{\alpha\beta})\,. \tag{3.39}$$

We can now read off the transformation equations for the scalar metric perturbations ,

$$\widetilde{\phi} = \phi - \xi^{'0} - \frac{a'}{a}\xi^0\,, \tag{3.40}$$

$$\widetilde{B} = B + \xi^0 - \xi^{'}, \tag{3.41}$$

$$\widetilde{\psi} = \psi + \frac{a'}{a}\xi^0\,, \tag{3.42}$$

$$\widetilde{E} = E - \xi\,. \tag{3.43}$$

Since all physical, measurable quantities must be gauge independent, the metric perturbations do not constitute some physical property. But by arranging the metric perturbations appropriately into quantities that are gauge independent we get quantities that can be interpreted physically, Bardeen potentials are the most common examples of such gauge independent quantities.

Similarly the two spatial vectors S_α and F_α transform as,

$$\widetilde{F}_\alpha = F_\alpha - \overline{\xi}_\alpha \qquad \widetilde{S}_\alpha = S_\alpha + \overline{\xi}'\,. \tag{3.44}$$

The tensor part of the perturbation, $M_{\alpha\beta}$, is gauge invariant.

Bardeen potentials

From the gauge dependent quantities ϕ, ψ, B and E we can construct two-gauge invariant scalar functions Φ and Ψ known by Bardeen potentials,

$$\Phi := \phi + \frac{1}{a}[a(B - \dot{E})]^{\cdot}\,, \qquad \Psi = \psi - \frac{\dot{a}}{a}(B - \dot{E})\,. \tag{3.45}$$

These two potentials are analogous to in electromagnetism the potentials ϕ; A, which are gauge-dependent, while the physical measurable quantities B; E "magnetic and electric fields" which are derived from these potentials, are gauge-independent.

3.0.14 Perturbed kinematic quantities

The time-like four-vector field orthogonal to hypersurfaces$\Sigma_{\ \eta}$ of the perturbed metric is ,

$$n^a = a^{-1}(1-\phi, S^\alpha - B_{,\alpha})\,, \qquad \text{and} \qquad n_a = a(-1-\phi, \overrightarrow{0})\,. \tag{3.46}$$

The covariant derivative of this vector field can be decomposed into the following irreducible kinematical quantities ,

$$n_{a;b} = \sigma_{ab} + \frac{1}{3}\Theta\, h_{ab} - a_a\, n_b\,, \tag{3.47}$$

where

$$\sigma_{ab} = \frac{1}{2}h_a^c\, h_b^d(n_{c;d} + n_{d;c}) - \frac{1}{3}\Theta\, h_{ab}\,, \tag{3.48}$$

$$\Theta = n^a_{;a}\,, \tag{3.49}$$

$$a_a = n_{a;b}\, n^a\,. \tag{3.50}$$

In the perturbed FRW models these quantities reduce to ,

$$\Theta = \frac{3}{a}[\mathcal{H}(1-\phi) - \psi' - \frac{1}{3}\nabla^2(B-E')]\,, \tag{3.51}$$

$$\sigma_{ab} = a(-(B-E')_{,ab} + \frac{1}{3}\gamma_{ab}\nabla^2(B-E'))\,, \tag{3.52}$$

$$a_a = \phi_{,a}\,. \tag{3.53}$$

The temporal components $a_0 = 0$ and $\sigma_{0a} = 0$. In the case of vector perturbations $\overline{a}_a = 0, \overline{\Theta} = 0$ and $\overline{\sigma_0 0} = \overline{\sigma}_0\alpha = 0$, while the shear is given by ,

$$\Sigma_{\alpha\beta} \equiv \overline{\sigma}_{\alpha\beta} = \frac{1}{2}a[(S_{\alpha,\beta} + S_{\beta,\alpha}) + (F_{\alpha,\beta} + F_{\beta,\alpha})']\,. \tag{3.54}$$

And finally for the tensor perturbations the only nonzero quantity is ,

$$\sigma^T_{\alpha\beta} = \frac{1}{2}a\, M_{\alpha\beta}\,. \tag{3.55}$$

3.0.15 Perturbation of the energy-momentum tensor

To be able to perturb the energy-momentum tensor we need first to perturb the four-velocity . In the background ,

$$\overline{u}^a = a^{-1}(\delta^b{}_0, \overrightarrow{0}) \qquad \text{and} \qquad \overline{u}_a = a(-\delta^0{}_b, \overrightarrow{0})\,, \tag{3.56}$$

where $\overline{u}^a$ is orthogonal to the hypersurfaces of constant proper time η comoving with the fluid. Up to first order in the metric perturbations the perturbed four-velocity is ,

$$u^a \equiv \frac{1}{a}\frac{dx^a}{d\eta} = \overline{u}^a + \delta\, u^a = (a^{-1} + \delta\, u^0, a^{-1}(v_{,i} + v_i))\,, \tag{3.57}$$

where we split the spatial three-velocity into its scalar part v, the velocity potential, and its vector part, v_α, the rotational velocity field. The temporal component δu^0 can be found with the help of the constraint equation $g_{ba} u^a u^b = -1$,

$$g_{ba} u^a u^b = g_{ab}(a^{-1}\delta^b{}_0 + \delta u^b)(a^{-1}\delta^b{}_0 + \delta u^b), \tag{3.58}$$

$$= g_{ab}(a^{-2}\delta^b{}_0\,\delta^b{}_0 + a^{-1}\delta^b{}_0\,\delta u^a + a^{-1}\delta^a{}_0\,\delta u^b), \tag{3.59}$$

$$= a^{-2}g_{00} + 2a^{-1}g_{00}\delta u^0 + 2a^{-1}g_{0\alpha}\delta u^\alpha, \tag{3.60}$$

$$= 1 + 2\Phi + 2a\,\delta u^0 = -1. \tag{3.61}$$

Thus,

$$\delta u^0 = a^{-1}(1 - \Phi). \tag{3.62}$$

It is clear that there is no constraint over the spatial components of the four-velocity, since the spatial components of the background four-velocity is zero, then $\delta u_\alpha = u_\alpha$ and also,

$$\delta u_\alpha = u_\alpha = g_{\alpha\,a} u^a = a[(v + B)_{,\alpha} + v_\alpha - S_\alpha]. \tag{3.63}$$

The perturbed total four-velocity will take the form,

$$u^a = a^{-1}(1 - \Phi, v_{,\alpha} + v_\alpha) \quad \text{and} \quad u_a = a(-1 - \Phi, (v + B)_{,\alpha} + v_\alpha - S_\alpha). \tag{3.64}$$

The energy tensor of the perturbed Universe is,

$$T^b{}_a = \overline{T}^b{}_a + \delta T^b{}_a. \tag{3.65}$$

The perturbation $\delta T^b{}_a$ can be divided into perfect fluid plus non-perfect fluid contributions, where the perfect fluid degrees of freedom are those which keep T^b_a in the perfect fluid form,

$$T^b a = (\rho + p) u^b\, u_a + p\,\delta^b{}_a, \tag{3.66}$$

where,

$$\rho = \overline{\rho} + \delta\rho, \qquad p = \overline{p} + \delta p. \tag{3.67}$$

For a single perfect fluid the pressure perturbation is coupled to the density perturbation via,

$$\delta p = c_s^2\,\delta\rho. \tag{3.68}$$

When considering a system of interacting multiple fluids, the perturbations in p are also coupled to the perturbations in the entropy S, $\delta p = c_s^2 \delta p + \partial p / \partial S$.

By using this expression for the four-velocity we can obtain the components of the energy-momentum tensor,

$$T^0{}_0 = (\rho+p)u^0\,u_0 + p = (\rho+p)u^0\,g_{0b}u^b + p\,, \tag{3.69}$$

$$= (\rho+p)a^{-1}(1-\Phi)g_{00}u^0 + p \tag{3.70}$$

$$= -(\rho+p)a^{-1}(1-\Phi)a^2(1+2\Phi)a^{-1}(1-\Phi) + p = -\rho = -(\overline{\rho}+\delta\rho)\,, \tag{3.71}$$

$$T^0{}_i = (\rho+p)u^0\,u_i = a^{-1}(1-\Phi)g_{\alpha\,b}u^b\,, \tag{3.72}$$

$$= (\rho+p)a^{-1}(1-\Phi)g_{\alpha\beta}u^\beta\,, \tag{3.73}$$

$$= (\rho+p)a^{-1}(1-\Phi)[-a^2[(1-2\Phi)\delta_{\alpha\beta} + 2E_{,\alpha\beta}]\delta\,u^\alpha] = -(\rho+p)a\,\delta\,u^\alpha\,, \tag{3.74}$$

$$= -(\overline{\rho}+\overline{p})a\,\delta\,u^\alpha = -(\overline{\rho}+\overline{p})(v_{,\alpha} + B_{,\alpha} + v_\alpha - S_\alpha)\,, \tag{3.75}$$

$$T^\alpha{}_0 = (\overline{\rho}+\overline{p})(v_{,\alpha} + v_\alpha)\,, \tag{3.76}$$

$$T^{\alpha\beta} = (\rho+p)u^\alpha\,u_\beta + p\,\gamma^\alpha{}_\beta = (\rho+p)\delta\,u^\alpha\,\delta\,u_\beta + p\,\gamma^\alpha{}_\beta\,, \tag{3.77}$$

$$= p\,\delta^\alpha_\beta = (\overline{p}+\delta\,p)\,\gamma^\alpha{}_\beta\,. \tag{3.78}$$

$$T^a{}_b = \overline{T}^a{}_b + \delta\,T^a{}_b = \begin{pmatrix} -\overline{\rho} & \\ & \overline{p}\,\delta^\alpha{}_\beta \end{pmatrix} + \begin{pmatrix} -\delta\rho & (\overline{\rho}+\overline{p})(v_{,\alpha} + B_{,\alpha} + v_\alpha - S_\alpha) \\ (\overline{\rho}+\overline{p})(v_{,\alpha} + v_\alpha) & \delta\,p\,\delta^\alpha{}_\beta \end{pmatrix}\,, \tag{M4}$$

The non-perfect fluid contributions to the perturbed energy momentum tensor are contained in the space part $\delta\,T^\alpha{}_\beta$, where,

$$\delta\,T^\alpha{}_\beta = \delta\,p\,\delta^\alpha{}_\beta + \pi^\alpha{}_\beta\,, \tag{3.79}$$

where we have introduced the anisotropic stress tensor $\pi^\alpha{}_\beta$ which decomposes into a trace-free scalar part, Π, a vector part, π^α, and a tensor part, $\Pi^\alpha{}_\beta$, according to,

$$\pi^\alpha{}_\beta = \Pi^\alpha{}_{,\beta} - \frac{1}{3}\nabla^2\,\Pi\,\delta^\alpha{}_\beta + \frac{1}{2}(\pi^\alpha{}_{,\beta} + \pi^\alpha{}_{\beta,}) + \Pi^\alpha{}_\beta\,. \tag{3.80}$$

3.0.16 Gauge transformation of the energy tensor perturbations

$$\begin{aligned}
\widetilde{\delta T}^{0}{}_{0} &= -\widetilde{\delta\rho} = \delta T^{0}{}_{0} - L_{\xi}\, T^{0}{}_{0}\,, & (3.81)\\
&= -\delta\rho + \overline{\rho}'\,\xi^{0}\,, & (3.82)\\
\widetilde{\delta T}^{\alpha}{}_{0} &= -(\overline{\rho}+\overline{p})\widetilde{v}_{\alpha} = \delta T^{\alpha}{}_{0} - L_{\xi}\, T^{\alpha}{}_{0}\,, & (3.83)\\
&= -(\overline{\rho}+\overline{p})\widetilde{v}_{\alpha} - \xi^{\alpha}{}_{,0}(\overline{\rho}+\overline{p})\,, & (3.84)\\
\frac{1}{3}\widetilde{\delta T}^{\gamma}{}_{\gamma} &= \widetilde{\delta p} = \frac{1}{3}\delta T^{\gamma}{}_{\gamma} - \frac{1}{3}L_{\xi}\, T^{k}{}_{\gamma}\,, & (3.85)\\
&= \delta p - \overline{p}'\,\xi^{0}\,, & (3.86)\\
\widetilde{\delta T}^{i}{}_{\beta} - \frac{1}{3}\delta^{i}{}_{\beta}\,\widetilde{\delta T}^{k}{}_{k} &= \overline{p}\widetilde{\Pi}_{\alpha\beta} = \delta T^{i}{}_{\beta} - \frac{1}{3}\delta^{i}_{\beta}\,\widetilde{\delta T}^{k}{}_{k} = \overline{p}\,\Pi_{\alpha\beta}\,. & (3.87)
\end{aligned}$$

Thus,

$$\begin{aligned}
\widetilde{\delta\,\rho} &= \delta\rho - \overline{\rho}'\,\xi^{0}\,, & (3.88)\\
\widetilde{\delta\,p} &= \delta\,p - \overline{p}'\,\xi^{0}\,, & (3.89)\\
\widetilde{v}_{\alpha} &= v_{\alpha} + \xi^{\alpha}_{,0}\,, & (3.90)\\
\widetilde{\delta\,\Pi}_{\alpha\beta} &= \Pi_{\alpha\beta}\,. & (3.91)
\end{aligned}$$

It is clear that the anisotropic stress tensor$\Pi_{\alpha\beta}$ is gauge-invariant. Just as for the metric perturbations, we can extract a scalar perturbation out of v_{α}, ξ^{α} and$\Pi_{\alpha\beta}$,

$$v_i = -v_{,\alpha}\,, \qquad \xi^{\alpha} = -\xi_{,\alpha}\,, \qquad \Pi_{\alpha\beta} = (\partial_{\alpha}\,\partial_{\beta} - \frac{1}{3}\,\delta_{\alpha\beta}\,\nabla^{2})\Pi\,. \tag{3.92}$$

Therefore,

$$\widetilde{v} = v + \xi'\,, \qquad \widetilde{\Pi} = \Pi\,. \tag{3.93}$$

3.0.17 The longitudinal gauge

This gauge is defined by the two constraints,

$$B = E = 0\,. \tag{3.94}$$

According to the definition 3.45 the two remaining metric perturbations, ϕ and ψ, are equal to the gauge invariant quantities Φ and Ψ in this gauge.

$$\Phi = \phi_{\text{longitudinal}}\,, \qquad \Phi = \phi_{\text{longitudinal}}\,. \tag{3.95}$$

The longitudinal coordinates, as well as the metric perturbations are unique. The line element of longitudinal gauge takes the form,

$$ds^2 = a^2(\eta)(1 + 2\Psi)d\eta^2 - (1 - 2\Phi)\delta_{\alpha\beta}dx^\alpha\, dx^\beta\,. \tag{3.96}$$

We can also see from 3.45 that the condition $B = E = 0$ implies,

$$\xi = -E\,, \xi \qquad ^0 = -B + E'\,. \tag{3.97}$$

In the longitudinal gauge the energy momentum tensor perturbation (M4) take the form,

$$\delta\, T^a{}_b = \begin{pmatrix} -\delta\rho & (\overline{\rho} + \overline{p})v_{,\alpha} \\ (\overline{\rho} + \overline{p})v_{,\alpha}\,\delta\, u^\alpha & \delta\,\rho\,\delta^\alpha{}_\beta + \overline{p}(\Pi_{,\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta}\nabla^2\Pi) \end{pmatrix}. \tag{M5}$$

Also the perturbation of Einstein tensor in this gauge is given by,

$$\begin{aligned} \delta\, G^t{}_t &= \frac{2}{a^2}\Big[3\,H\,(H\,\Phi + \dot{\Psi}) - (3\,K + \Delta\,)\Big], & (3.98)\\ \delta\, G^t{}_\alpha &= -\frac{2}{a^2}\,(H\,\Phi + \dot{\Psi})\,, & (3.99)\\ \delta\, G^\alpha{}_\beta &= -\frac{2\,K\,\Psi}{a^2}\,\delta^\alpha{}_\beta - (\Delta\,\Psi - \Delta\,\Phi\,)\frac{\delta^\alpha{}_\beta}{a^2} + \frac{\nabla^\alpha\nabla_\beta}{a^2}\,(\Psi - \Phi) \\ &\quad + \frac{2}{a^2}\Big[H(\dot{\Phi} + 2\,\dot{\Psi}) + (2\,\dot{H} + H^2)\,\Phi + \ddot{\Psi}\Big]\,\delta^\alpha{}_\beta\,, & (3.100) \end{aligned}$$

where Δ is Laplace operator. Define the gauge invariant potentials,

$$\begin{aligned} V &\equiv (v - B) + (\dot{E} + B)\,, & (3.101)\\ \Delta_s &\equiv \frac{1}{\rho}\,(\delta\,\rho - \dot{\rho}\,(\dot{E} + B))\,, & (3.102)\\ \Gamma &\equiv \frac{1}{p}(\delta\,p = \dot{p}\,(\dot{E} + B))\,, & (3.103) \end{aligned}$$

where V and$\Delta\,_s$ are the gauge invariant velocity and density contrast respectively. Using these gauge invariant together with 3.98 and (M5) Einstein equations take the following gauge invariant form,

$$\begin{aligned} 3\,H\,(H\,\Phi + \dot{\Psi}) - (3\,K + \Delta)\,\Psi = &\; -\frac{1}{2}\,\rho\,a^2\,\Delta_s\,, & (3.104)\\ H\,\Phi + \dot{\Psi} &= -\frac{1}{2}\rho\,a^2\,(1 + \omega)\,V\,, & (3.105)\\ (\nabla_\alpha\nabla_\beta - \frac{1}{3}\,h_{\alpha\beta}\Delta)(\Phi + \Psi) &= -\rho\,a^2\,\omega\,\Pi_{\alpha\beta}\,, & (3.106)\\ \frac{1}{3}\,\Delta\,(\,\Phi - \Psi) + (H^2 + 2\,\dot{H})\,\Phi - K\,\Psi + (\ddot{\Psi} + H\,\dot{\Phi} + 2\,H\,\dot{\Psi}) &= \frac{1}{2}\,\rho\,a^2\,(\omega\,\Gamma + c_s^2\,\Delta_s) & (3.107) \end{aligned}$$

The first two equations correspond to the time-time and space-time components of $\delta\, G^a_b$, while the last two equations correspond to the traceless part and the trace of $\delta\, G^\alpha_\beta$. From the field equations 3.104, Bardeen potentials can now be written in term of the matter quantities as,

$$\Psi \;=\; \frac{1}{2}\,\rho\, a^2\, \Delta(\Delta + 3\,K)^{-1}\,, \tag{3.108}$$

$$\Phi \;=\; \frac{1}{2}\,\rho\, a^2\, [\Delta(\Delta + 3\,K)^{-1} - 2\,\omega\,\Pi]\,. \tag{3.109}$$

Evolution of density contrast and velocity From the conservation equation of the stress-energy tensor,

$$\delta T^a{}_{b;a} = 0\,, \tag{3.110}$$

one can obtain the following evolution equations of the of density contrast,

$$\dot{\delta} - 3\,H\,\omega\,\delta = (\delta + 3\,K)[3\,\omega\,H\,\Pi - (1+\omega)\,V]\,. \tag{3.111}$$

$$(1+\omega)[\dot{V} + H\,V] = \Big[\frac{1}{2}\,\rho\,a^2 - \frac{2}{1}(\delta + 3\,K)\Big]\,\omega\,\Pi - \omega\,\Gamma\,, \tag{3.112}$$

where $\delta = \Delta_s - 3\,(1+\omega)\Psi + 3\,(1+\omega)[\Psi - H\,V]$. By expanding the gauge invariant quantities in harmonic modes, the system 3.111 is equivalent to the second order equation,

$$\ddot{\delta} + [\delta + 3\,(c_s^2 - 2\,\omega)]\,H\,\dot{\delta} + 3\,A\,\delta - [2\,B\,\Pi - 2\,H\,\omega\,\dot{\Pi} - \omega\,k^2\,\Gamma](1 - \frac{3\,K}{k^2})\,, \tag{3.113}$$

where,

$$A \;=\; H^2\,[\frac{3}{2}\,\omega^2 - 4\,\omega + 3\,c_s^2 - \frac{1}{2}] + \frac{1}{2}(3\,\omega^2 - 1)\,K + \frac{1}{3}(k^2 - 3\,K)\,c_s^2\,. \tag{3.114}$$

$$B \;=\; H^2[3\,\omega^2 - 2\,\omega + 3\,c_s^2] + \omega(3\,\omega + 2)\,K + (k^2 - 3\,K)\,c_s^2\,, \tag{3.115}$$

where k is the wave number. It is clear from 3.113 that the sources of density perturbations are entropy perturbations Γ and anisotropic pressure perturbations Π.

3.1 Covariant and gauge-invariant approach

In gauge invariant perturbation theory one works only with quantities that are gauge invariant. This guarantees that they are physical and that the results are unique. The first step in doing gauge invariant perturbation theory, is to find a set of gauge invariant quantities and then reformulate the equations using only these quantities. According to Stewart and Walker lemma [111] the simplest gauge invariant variable are scalar that is constant in the background, a tensor that vanishes in the background, and a tensor that is a constant linear combination of Kronecker deltas.

The FLRW model is characterized by,

$$\sigma_{ab} = 0\,, \quad \omega_{ab} = 0\,, \quad \dot{u}^a = 0\,, \quad E_{ab} = 0\,, \quad H_{ab} = 0\,, \tag{3.116}$$

$$\Longrightarrow \quad \mu = \mu(t)\,, \quad p = p(t)\,, \quad \Theta = \Theta\,(t)\,. \tag{3.117}$$

The simplest covariantly defined quantities in a homogeneous and isotropic space-times are:

$$\{\omega_{ab}, \sigma_{ab}, E_{ab}, H_{ab}, J_a, \pi_{ab}\}\,. \tag{3.118}$$

Of these only ω_{ab} and $\dot{u}_a$ are, in general, gauge invariant for non-tilted Bianchi models. To describe the density inhomogeneities we need to define gauge invariant variables that characterize the variation of the zero-order quantities $\{\mu, p, \Theta\}$ which are in general nonzero in expanding FLRW models, and so are not gauge invariant, The most simple gauge invariant quantities that we can define out of these variables are their orthogonal spatial gradients,

$$X_a \equiv\, 3\nabla_b\mu\,, \qquad Y_a \equiv\, 3\nabla_b\, p \qquad Z_a \equiv\, 3\nabla_b\,\Theta\,. \tag{3.119}$$

Two other important gauge invariant quantities are the divergence of the acceleration and its spatial gradient,

$$A \equiv \nabla_a \dot{u}^a\,, \qquad A_a \equiv\, 3\nabla_b\, A\,. \tag{3.120}$$

The quantity X_a is in principle observable (Ellis & Bruni 1989). To characterize the significance of this variable we divide it by the density itself $\dfrac{X_a}{\mu}$. In the context of considering the growth of protogalaxy fluctuations we want to consider density variations at a fixed comoving scale. Thus we define the comoving fractional density gradient and the comoving gradient of the expansion as,

$$D_a \equiv\, a\,\frac{X_a}{\mu}\,, \qquad \mathcal{Z}_a = a\, Z_a\,. \tag{3.121}$$

3.1.1 Exact non-linear evolution equations

Imperfect fluid

The propagation equations for the zero-order quantities μ, p and Θ along the fluid flow are,

$$\dot{\mu} + (\mu+p)\Theta + \pi^{ab}\sigma_{ab} + J^a\dot{u}_a + J^a;a \;=\; 0\,, \tag{3.122}$$

$$(\mu+p)\dot{u}_a + h^{\gamma}{}_a(p_{,\gamma} + \pi^b{}_{\gamma;b} + \dot{J}_\gamma) + (\omega^b{}_a + \sigma^b{}_a + \frac{4}{3}\Theta h^b{}_a)J_b \;=\; 0\,, \tag{3.123}$$

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) + \dot{u}^a{}_{;a} + \frac{1}{2}(\mu+3p) - \Lambda \;=\; 0\,. \tag{3.124}$$

Taking the covariant derivative of 3.121 and using the equations above we obtain,

$$\begin{aligned} h^a{}_c(\dot{D}_a) &= \frac{p}{\mu}\,\Theta\, D_c - (1+\frac{p}{\mu})\mathcal{Z}_c - (\omega^a{}_c + \sigma^a{}_c)\, D_a - \frac{a}{\mu}\, h^d{}_c\, (\pi_{ab}\,\sigma^{ab})_{,d} \\ &- \frac{a}{\mu} h^d{}_c\,(q^a{}_{;a})_{,d} - \frac{a}{\mu} h^d{}_c\,(\dot{u}_a\; q^a)_{,d} + \frac{a}{\mu}\,\Theta[h^d{}_c\,(\pi^b{}_{d;b} + \dot{q}_d) \\ &+ (\omega^b{}_c + \sigma^b{}_c + \frac{4}{3}\,\Theta\, h^b{}_c) q_b] + \frac{1}{\mu}\,(\pi_{ab}\sigma^{ab} + q^a{}_{;a} + \dot{u}_a\, q^a)(D_c - a\,\dot{u}_c)\,, \quad (3.125) \\ h^a{}_c(\dot{\mathcal{Z}}_a) &= a\,[\dot{u}_c\,\mathcal{R} + A_c - 2\, h^a{}_c\,(\sigma^2)_{,a} + 2\, h^a{}_c\,(\omega^2)_{,a} + \frac{3}{2} h^a{}_c\,(\pi^b{}_{a;b} + \dot{q}_a) \\ &+ \frac{3}{2}\,(\omega^b{}_c + \sigma^b{}_c + \frac{4}{3}\,\Theta\, h^b{}_c)\, q_b] - \frac{1}{2}\mu\, D_c - (\sigma^b{}_c + \omega^b{}_c)\,\mathcal{Z}_b - \frac{2}{3}\,\Theta\,\mathcal{Z}_c\,, \quad (3.126) \end{aligned}$$

where,

$$\mathcal{R} = -\frac{1}{3}\,\Theta^2 - 2\,\sigma^2 + 2\,\omega^2 + A + \mu + \Lambda\,. \qquad (3.127)$$

The 3-curvature scalar in the tangent space is given by,

$$3R = 2\,(-\frac{1}{3}\,\Theta^2 + \sigma^2 - \omega^2 + \mu + \Lambda\,), \qquad \Rightarrow \qquad \mathcal{R} = \frac{1}{2}\,3R - 3\,\sigma^2 + 3\,\omega^2 + A\,. \qquad (3.128)$$

When $\omega = 0$, R is the Ricci scalar in the hypersurface orthogonal to the fluid flow lines.

Perfect fluids

When dealing with perfect fluids, a choice of frame can be made in which the fluid appears isotropic, thus the propagation equations for the zero-order quantities μ, p and Θ along the fluid flow are,

$$\begin{aligned} \dot{\mu} &+ (\mu + p)\Theta = 0\,, & (3.129) \\ (\mu &+ p)\,\dot{u}_a + 3\nabla_a\, p = 0\,, & (3.130) \\ \dot{\Theta} &+ \frac{1}{3}\Theta^2 + 2\,(\sigma^2 - \omega^2) - A + \frac{1}{2}(\mu + 3\,p) - \Lambda = 0\,. & (3.131) \end{aligned}$$

If we assume a barotropic equation of state $\omega = p/\mu$ then it follows from the first equation in 3.129,

$$\dot{\omega} = -\Theta(1+\omega)(c_s^2 - \omega)\,, \qquad \text{where } c_s \text{ is the speed of sound } \quad c_s^2 = \frac{d\,p}{d\,\mu}\,. \qquad (3.132)$$

The propagation equation for the comoving fractional density gradient D is given by,

$$h^b{}_a(\dot{D}_b) = \omega\,\Theta\, D_a - (\sigma^b{}_a + \omega^b{}_a)\, D_b - (1+\omega)\, Z_a\,. \qquad (3.133)$$

In deriving this equation we have used the identity,

$$h^b{}_a(a\, 3\,\dot{\nabla}_b\, f) = a\,(3\nabla_a + \dot{u}_a)\,\dot{f} - a\,(\sigma^b{}_a + \omega^b{}_a)\, 3\,\nabla\, f\,. \qquad (3.134)$$

The propagation equation for Z_a is ,

$$h^b{}_a(\dot{Z}_b) = -\frac{2}{3}\Theta\, Z_a - (\sigma^b{}_a + \omega^b{}_a)\, Z_b - \frac{1}{2}\,\mu\, D_a + a\, R\, \dot{u}_a + a\, 3\nabla_a\, (A - 2\,\sigma^2 + 2\,\omega^2)\,. \quad (3.135)$$

3.1.2 Linearization about a FLRW Universe

We now linearize the previous equations about FLRW background by taking the quantities $\{\mu, p, \Theta\}$ to be zero order and $\{D_a, Z_a, \sigma_{ab}, \omega_{ab}, \dot{u}_a, \pi_{ab}, J_a\}$ and their derivatives as first order. The linearization is performed by dropping all the products of first-order quantities. The linearized Raychaudhuri equation ,

$$\dot{\Theta} + \frac{1}{3}\Theta^2 - A + \frac{1}{2}(\mu + 3\,p) - \Lambda = 0\,. \quad (3.136)$$

The energy and momentum conservation equations are unaffected by the linearization procedure. The linearized equations for propagation of D_a and Z_a ,

$$\dot{D}_{\perp\, a} = \omega\,\Theta\, D_a - (1+\omega)\, Z_a\,, \quad (3.137)$$

$$\dot{Z}_{\perp\, a} = -\frac{2}{3}\,\Theta\, Z_a - [\frac{1}{2}\,\mu + c_s^2\,\frac{(\mu - \frac{1}{3}\Theta^2)}{1+\omega}]\, D_a + a\, 3\nabla_a\, A \quad (3.138)$$

where $\dot{D}_{\perp\, a} = h^b{}_a\,(\dot{D}_b)$; $\dot{Z}_{\perp\, a} = h^b{}_a\,(\dot{Z}_b)$.

3.1.3 The scalar variables

The variables defined above 3.121 contain information that are not relevant to the growth of matter clumping, thus it is convenient to introduce a local decomposition by by taking the spatial derivative of the inhomogeneous variables, for example, the spatial derivative of D_a reads ,

$$\tilde{\nabla}_a\, D_b = W_{ab} + \Sigma_{ab} + \frac{1}{3}\,\Delta\, h_{ab}\,.$$

where $W_{ab} = W_{[ab]}$ contains information about vorticity,$\Sigma\,_{ab}$ describes the evolution of anisotropies in the Universe, and $\Delta = \tilde{\nabla}^a\, D_a$ characterizes the clumping of matter, thus we will only consider this part of the density evolution. By taking the spatial derivative of the inhomogenous variables we obtain the following scalar variables ,

$$\Delta = a\,\tilde{\nabla}^a\, D_a\,, \qquad Z = a\,\tilde{\nabla}^a\, Z_a\,. \quad (3.139)$$

The scalar dynamical equations are given by ,

$$\dot{\Delta} = \omega\,\Theta\,\Delta - (1+\omega)\, Z\,, \quad (3.140)$$

$$\dot{Z} = -\frac{2}{3}\,\Theta\, Z - [\frac{1}{2}\,\mu + c_s^2\,\frac{(\mu - \frac{1}{3}\Theta^2)}{1+\omega}]\,\Delta + a\, 3\nabla_a\, A\,. \quad (3.141)$$

3.1.4 The harmonic decomposition

In order to separate the time and space variations of matter inhomogeneities we expand every scalar X using the harmonics decomposition ,

$$X = \sum_k X^k(t) Q_k(\vec{x})\,, \tag{3.142}$$

where $Q_k(x)$ are the eigenfunctions of the covariantly defined Laplace-Beltrami operator on an almost FLRW space-time:

$$\tilde{\nabla}^2 Q = -\frac{K^2}{a^2} Q\,. \tag{3.143}$$

Here $k = \frac{2\pi a}{\lambda}$ is the order of the harmonic and $\dot{Q}_k(\vec{x}) = 0$ (Q is covariantly constant). Using these harmonics the evolution equations 3.140, 3.141 can be converted into a system of ordinary differential equations.

$$\dot{\Delta}^k = \omega\,\Theta\,\Delta^k - (1+\omega)\,Z^k\,, \tag{3.144}$$

$$\dot{Z}^k = -\frac{2}{3}\,\Theta\,Z^k - [\frac{1}{2}\,\mu + c_s^2\,\frac{(\mu - \frac{1}{3}\Theta^2)}{1+\omega}]\,\Delta^k - \frac{c_s^2}{(1+\omega)}(3\nabla^2 - \frac{2\,K}{a^2})\,, \tag{3.145}$$

which can be reduced to a single second-order differential equation ,

$$\ddot{\Delta}^k + A(t)\,H\,\dot{\Delta}^k + (B(t) + \frac{c_s^2\,n^2}{H^2\,a^2})\,H^2\,\Delta^k = 0\,, \tag{3.146}$$

where ,

$$A(t) = 2 - 3\,\omega - 3\,(\omega - c_s^2)\,, \tag{3.147}$$

$$B(t) = -\frac{3}{2}\,[(1-\omega)(1+3\,\omega) + 6\,(\omega - c_s^2)]\,\Omega + \frac{12\,k}{H^2\,a^2}\,(\omega - c_s^2)\,. \tag{3.148}$$

This differential equation is the basic linearized equation for structure growth in a FLRW background. It describes the evolution of the gauge invariant density perturbation along the fluid flow lines and it depends on the background and the equation of state through the variables H, Ω, a, ω and c_s^2, that appears in the coefficients $A(t)$; $B(t)$.

3.1.5 Perturbations of flat FLRW

We now consider the flat FLRW background ,

$$\Omega = 1\,, \qquad H = \frac{2}{3\,\gamma}\,t^{-1}\,, \qquad a = t^{\frac{2}{3\,\gamma}}\,, \tag{3.149}$$

together with the equation of state $p = (\gamma - 1)\mu$. It follows from 3.132 that ,

$$\omega = \gamma - 1\,, \qquad c_s^2 = \gamma - 1\,. \tag{3.150}$$

Accordingly Equation 3.145 becomes,

$$\ddot{\Delta}^k + (5-3\,\gamma)\,\frac{2}{3\,\gamma}\,t^{-1}\,\dot{\Delta}^k + [-\frac{3}{2}\,(2-\gamma)(3\,\gamma-2)\,+\,\frac{9\,\gamma^2(\gamma-1)\,n^2}{4\,t^{\frac{4}{3\,\gamma}-2}}]\,\frac{4}{9\,\gamma^2}\,t^{-2}\,\Delta^k = 0\,. \quad (3.151)$$

Dust case For the dust case $\gamma = 1$, the general solution of 3.151 is,

$$\Delta^k = a_+\,t^{\frac{2}{3}} + c_-\,t^{-1}\,, \quad (3.152)$$

where t is proper time along the flow lines and the $+;\,-$ corresponds to the growing and decaying modes respectively. It is clear the growing modes lead to structure formation, and on the other hand the decaying modes dissipate inhomogeneities.

Radiation Case For simplicity we will consider the solutions for the 'long-wavelength limit' $H^{-1}/\lambda \ll 1$, where $\lambda = 2\,\pi\,a/n$, in this limit equation 3.151 reduces to,

$$\ddot{\Delta}^k + (5-3\,\gamma)\,H\,\dot{\Delta}^k - \frac{3}{2}\,(2-\gamma)(3\,\gamma-2)\,H^2\,\Delta^k = 0\,. \quad (3.153)$$

The background variable H can be absorbed into the time derivative by introducing the dimensionless time variable $\tau = log\,(a/a_0)$, equation 3.153 becomes,

$$\Delta''^k + \frac{1}{2}\,(10-9\,\gamma)\,\Delta'^k - \frac{3}{2}\,(2-\gamma)(3\,\gamma-2)\Delta^{\,k} = 0\,. \quad (3.154)$$

The general solution of this equation in term of t, is,

$$\Delta^k = c_+\,t^{\frac{2(3\,\gamma-2)}{3\,\gamma}} + c_-\,t^{-\frac{(2-\gamma)}{\gamma}}\,. \quad (3.155)$$

By substituting ($\gamma = \frac{4}{3}$) we obtain,

$$\Delta^k = c_+\,t + c_-\,t^{-\frac{1}{2}}\,. \quad (3.156)$$

In the dust case we obtained the standard growing and decaying modes without any extra gauge-dependent fictitious modes. On the other hand in the radiation case the result that we obtained is different from the one obtained using the synchronous and comoving gauges [115]. But it agrees with the results obtained by Bardeen [116] and Sakai [117], when using the comoving time orthogonal gauge [116]. Since the covariant and gauge invariant approach does not result in any fictitious modes, we expect that the comoving time orthogonal gauge to be more appropriate in dealing with this physical situation. In chapter 8 we apply this covariant and gauge invariant formalism to a single fluid and multi-fluid cases. In the next chapter we introduce $f(R)$ gravity which is one of the most extensively studied modified gravity theories.

Chapter 4

$f(R)$-GRAVITY

4.1 Introduction

As already mentioned, cosmological observations indicate that the standard theory of gravity (GR) is unable to provide a simple explanation of the gravitational dynamics of the low-energy Universe. The observational evidence for the accelerated expansion rate of the Universe, and the introduction of the concept of dark energy have put theoretical cosmology into crisis. Despite the increasing amount and quality of data, no model has been proposed so far that is able to give a completely satisfactory theoretical explanation of these observations.

It is, however, quite possible that the late time acceleration of the Universe is the result of large-scale modification of gravity. Amongst all the schemes of modification of gravity in the infrared regime, the higher order gravity theories have recently gained much attention [171, 222, 173, 175, 177]. The reason for this popularity is the fact that these models provide a somewhat more natural explanation of the cosmic acceleration in the high-curvature regime as well as inflation in the low-curvature regime. This effect is due to corrections to Einstein gravity which are directly related to the characteristic properties of the gravitational interaction. Among the possible modifications of the gravitational action, those consisting of non-linear functions of the scalar curvature (the so-called $f(R)$-gravity) (see [164, 206, 207, 208, 209] for more details) are among the most widely studied.

The motivation for considering this subclass of modified theories of gravity is due to the fact that any quadratic Lagrangian leading to an isotropic, homogeneous cosmological model can be written as a quadratic function of Ricci scalar [85], where all contributions to field equation due to the other curvature invariant can be expressed in terms of R and R^2. This is possible because of the identities ,

$$(\delta/\delta g_{ab}) \quad \int \quad d^4x\sqrt{-g}(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2) = 0\,,$$

$$(\delta/\delta g_{ab}) \quad \int \quad d^4x\sqrt{-g}\epsilon^{iklm}R_{ikst}R^{st}{}_{lm} = 0\,,$$

$$(\delta/\delta g_{ab}) \quad \int \quad d^4x\sqrt{-g}(3R_{ab}R^{ab} - R^2) = 0\,.$$

Therefore, without loss of generality we can limit our studies to cosmological models result from a gravitational Lagrangian that is an arbitrary function of the scalar curvature R. The first papers on cosmological models in modified $f(R)$ gravity appeared in 1969-1970 [77] .

It has been shown in literature [176] that $f(R)$ gravity can be recast as GR with an additional scalar degree of freedom (the scalaron) conformally coupled to all matter. This new degree of freedom gives rise to a large coupling of the order of unity in Planck units. Thus any change of the standard Einstein-Hilbert of GR will induce new effects in cosmological as well as in the solar system scales. We know that GR works well in local regions whose densities are much larger than the homogeneous cosmological density, thus we need a screening mechanism in order to suppress these local effects while bringing it back to existence in cosmological scales. There are four screening mechanisms that have been proposed so far:

- The Chameleon [220] .
- The Symmetron [193], [220] .
- The Vainshtein [194] .
- The Least Coupling Principle [192] .

All these mechanisms rely on changing behavior depending on the environment. The chameleon mechanism is the most common one, it was first discovered in quintessence models of dark energy [72]. In this mechanism, the scalar field associated with the gravity modification changes its behavior with respect of the density of medium (like Chameleons). A number of viable f (R) models that can satisfy both cosmological and local gravity constraints have been proposed in [64], [83]. The large-scale modification of the gravitational interaction resulting from these models led to several interesting observational signatures such as the modification to the spectra of galaxy clustering [118], [41] CMB [43],[41],[47], and weak lensing [49],[74] .

Discrimination between different $f(R)$ models requires highly precise cosmic tests which are not available at present. Although the available data from the solar system is highly precise, the fact that the gravitational dynamics of this system is governed only by the ordinary matter, places a major limitation on its viability with regards to $f(R)$ cosmology. However, in general $f(R)$ models must mimic ΛCDM in the high-redshift regime where it is well tested by the CMB. Second, it should accelerate the expansion at low redshift with an expansion history that is close to ΛCDM, but without a true cosmological constant. Also the theory should give rise to cosmological perturbations compatible with the data from the cosmic microwave background and large-scale structure surveys. Based on these conditions the function $f(R)$ must satisfy :

1. $f''(R) > 0$ for $R \geq R_0 (> 0)$: The theory must have post-Newtonian limits compatible with the available Solar System tests. In [45] it has been shown that only $f(R)$ theories with $f'' > 0$ exhibit stable high-curvature limits and well-behaved cosmological solutions with a proper era of matter domination. In general, any $f(R)$ models with terms that become dominant at low cosmic curvatures are not viable at solar system scales .

2. $f'(R) > 0$ for $R \geq R_0 (> 0)$: This condition ensures that the effective Newton's constant $G_{eff} = G/f'$ to be positive at all times and the graviton energy to be positive [101] .

3. $f \rightarrow R - 2\Lambda$ for $R \gg R_0$: This is required for the consistency with the available Solar System tests, and for the presence of the matter-dominated epoch .

4. $0 < \frac{Rf''}{f'}(r = -2) < 1$ at $r = -\frac{Rf'}{f} = -2$: This is required for the stability of the late-time de Sitter point [66] .

Here R_0 is the Ricci scalar today. Models that satisfy $f''(R) \neq 0$ apart from the massless spin-2 graviton, necessarily contain scalaron. Stability of these models requires that the scalaron is not tachyon and the graviton is not a ghost which can be ensured by demanding the positivity of the first and the second derivatives of $f(R)$. The violation of the condition $f''(R) > 0$ gives rise to a negative mass squared for the scalaron filed. Hence the condition $f''(R) > 0$ avoids tachyonic instabilities. The condition $f'(R) > 0$ is also required to avoid the appearance of ghosts. Most of the $f(R)$ models are either not cosmologically viable or simply reduce to ΛCDM [78]. A number of viable cosmological models satisfying all these requirements has been proposed in [165], [70],

[175] .

$$
\begin{aligned}
(A)\;\; f(R) &= R - \mu R_0 \frac{(R/R_0)^{2n}}{(R/R_0)^{2n}+1} \qquad \text{with} \;\; n, \mu, R_0 > 0\,, \\
(B)\;\; f(R) &= R - \mu R_0 \Big[1 - (1 + \frac{R^2}{R_0^2})^{-n}\Big] \qquad \text{with} \;\; n, \mu, R_0 > 0\,, \\
(C)\;\; f(R) &= R - \mu R_0 \tanh(\frac{R}{R_0}) \qquad \text{with} \;\; \mu, R_0 > 0\,, \\
(D)\;\; f(R) &= R - \mu R_c (R/R_0)^p \qquad \text{with} \;\; 0 < p < 1;\; \mu, R_0 > 0\,,
\end{aligned}
$$

where μ, are constants and R_0 is the present cosmological Ricci scalar. In the regions where the density is high $R \gg R_0$, these models reduced to ΛCDM model ($f(R) \simeq R - \mu\, R_0$), where in this limit the models A; B have following asymptotic behavior ,

$$
f(R) \simeq R - \mu\, R_0 \Big[1 - (\frac{R}{R_0})^{-2n}\Big]\,. \tag{4.1}
$$

Thus GR is recovered locally even for $n = \mathcal{O}(1)$, the model C approaches ΛCDM even faster. The model D approaches ΛCDM asymptotically only if the power p is close to 0 .

Although viable $f(R)$ models are indistinguishable from the cosmological constant at the background level [167], but as already shown in [165], [41] and as we also demonstrate in this thesis, the study of the dynamics of perturbations shows of $f(R)$ models can lead to interesting signatures that might be seen in future observations.

Recently there have been many attempts to construct a new modified gravity model that leads to a natural unification of dark energy and dark matter. In [28], [29] a model which consists of two scalar fields where one of scalars represents the Lagrange multiplier was proposed. This multiplier imposes specific constraints on the second scalar field. As a result, the whole system contains the single dynamical degree of freedom [28], [29] .

Recently, new modified gravity theories, namely the f($\mathcal{T}$)-theory and the f(G)-theory, where $\mathcal{T}$ and G are the torsion scalar and the Gauss-Bonnet scalar respectively ,

$$
\mathcal{T} = S^{ab}{}_{\rho} T^{\rho}{}_{ab}\,, \qquad S^{ab}{}_{\rho} \equiv \frac{1}{2}(K^{ab}{}_{\rho} + \delta^{a}{}_{\rho} T^{\theta b}{}_{\theta} - \delta^{b}{}_{\rho} T^{\theta a}{}_{\theta})\,. \tag{4.2}
$$

$$
G = R^2 - 4R^{ab}R_{ab} + R^{ab\rho\sigma}R_{ab\rho\sigma}\,. \tag{4.3}
$$

Here K^{ab} is the extrinsic curvature tensor. f($\mathcal{T}$) theory is a generalization of the teleparallel gravity theory [185]. In this theory, torsion, instead of curvature, is responsible of the gravitational interaction and the Weitzenbock connection replaces the Levi-Civita connection, which results in null curvature but a non-vanishing torsion [105]. f(G) theories on the other hand have a property that the field equations are second order [107]. The common feature of all infrared-modified gravity theories is that the generalized gravitational action is assumed to contain some additional terms which start to grow with decreasing curvature and thus lead a late time-acceleration epoch. It is known that these two classes of modified gravity models can also explain the accelerated expansion of the Universe with no need of dark energy, and even the inflationary epoch [106], [108] .

4.2 The field equations

There are two approaches in deriving field equations of $f(R)$ gravity. The first is the standard metric approach in which the field equations are derived by the variation of the action with respect to the metric tensor g_{ab}. The second approach is the Palatini formalism in which the metric g_{ab} and the connection$\Gamma^{\ a}{}_{bc}$ are treated as independent variables. Therefore, in this formalism the field equations are derived by the variation of the action with respect to the metric tensor g_{ab} as well as the affine connection$\Gamma^{\ a}{}_{bc}$ [1] . In this thesis we consider only the metric approach. The general $f(R)$ gravity action is ,

$$S[g] = \int f(R)\sqrt{-g}\, d^4x + S^{\rm matter}\,, \tag{4.4}$$

where $f(R)$ is non-linear function of its argument [102] [103], and $S^{\rm matter}$ describes all non-gravitational kinds of matter including non-relativistic (cold) dark matter. The energy-momentum tensor of matter is defined in section 2.5 . In the metric-$f(R)$ gravity varying the action 4.4 yields the field equations ,

$$f'R_{ab} - \frac{1}{2}fg_{ab} - \nabla_a\nabla_b f' + g_{ab}\Box f' = T^m{}_{ab}\,, \tag{4.5}$$

where the prime denotes a derivative with respect to R. and T^m_{ab} represents the stress energy tensor of standard matter. Equations 4.5 are obviously fourth-order partial differential equations in the metric since R already includes second derivatives of the latter. By splitting 4.5 orthogonal and parallel to u^a we obtain the following set of relations ,

$$R = f'^{-1}\Big[3\,p - \mu + 2f - 3S\Big]\,, \tag{4.6}$$

$$R_{ab}\,u^a u^b = f'^{-1}\Big[\mu - \frac{1}{2}f + h^{ab}S_{ab}\Big]\,, \tag{4.7}$$

$$R_{ab}u^a\,h^b{}_c = f'^{-1}\Big[-q_c + S_{ab}u^a\,h^b{}_c\Big]\,, \tag{4.8}$$

$$R_{ab}\,h^a{}_b h^b{}_d = f'^{-1}\Big[\pi_{cd} - (p + \frac{1}{2}f + S)h_{cd} + S_{ab}\,h^a{}_c h^b{}_d\Big]\,. \tag{4.9}$$

[1] the distinction between metric and Palatini formalisms is irrelevant for GR, but they give rise to different field equations for non-linear $f(R)$ functions .

We assume that the standard matter to be a perfect fluid. In GR the trace equation 4.6 reduce the algebraic relation $R = -T = \mu_m - p_m$, where μ_m and p_m are the energy density and the pressure of the matter, respectively. This means that the Ricci scalar R is directly determined by the matter. In $f(R)$ the Ricci scalar R is determined by the differential equation 4.6 which governs the propagation of the scalaron field f'.

4.2.1 Decomposition of the energy-momentum tensor

In a general frame the irreducible decomposition of the stress energy momentum tensor is,

$$T_{ab} = \mu\, u_a\, u_b + q_a\, u_b + u_a\, q_b + p\, h_{ab} + \pi_{ab}\,. \tag{4.10}$$

In the frame u^a that is comoving with standard matter represented by baryonic matter such as galaxies and clusters of galaxies, the decomposition of the effective energy momentum tensors yields,

$$\mu^{\rm tot} = T^{\rm tot}_{ab}u^a u^b = \tilde{\mu}^{\,m} + \mu^{\,R}\,, \qquad p^{\rm tot} = \frac{1}{3}T^{\rm tot}_{ab}h^{ab} = \tilde{p}^{\,m} + p^{\,R}\,, \tag{4.11}$$

$$q^{\rm tot}_a = -T^{\rm tot}_{bc}h^b{}_a u^c = \tilde{q}^{\,m}_a + q^{\,R}_a\,, \qquad \pi^{\rm tot}_{ab} = T^{\rm tot}_{cd}h^c_{<a}h^d_{b>} = \tilde{\pi}^{\,m}_{ab} + \pi^{\,R}_{ab}\,, \tag{4.12}$$

with

$$\tilde{\mu}^{\,m} = \frac{\mu^{\,m}}{f'}\,, \qquad \tilde{p}^{\,m} = \frac{p^{\,m}}{f'}\,, \qquad \tilde{q}^{\,m}_a = \frac{q^{\,m}_a}{f'}\,, \qquad \tilde{\pi}^{\,m}_{ab} = \frac{\pi^{\,m}_{ab}}{f'}\,. \tag{4.13}$$

The effective thermodynamical quantities for the curvature fluid are

$$\mu^R = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R + f'''\,\widetilde{\nabla}^b R\widetilde{\nabla}_b R\right]\,, \tag{4.14}$$

$$p^R = \frac{1}{f'}\left[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\Theta f''\dot{R} - \frac{2}{3}f''\tilde{\nabla}^2 R + \right.$$
$$\left. -\frac{2}{3}f'''\widetilde{\nabla}^a R\widetilde{\nabla}_a R + f''\, a_b\widetilde{\nabla}^b R\right]\,, \tag{4.15}$$

$$q^R_a = -\frac{1}{f'}\left[f'''\dot{R}\widetilde{\nabla}_a R + f''\widetilde{\nabla}_a\dot{R} - \frac{1}{3}\Theta f''\widetilde{\nabla}_a R\right]\,, \tag{4.16}$$

$$\pi^R_{ab} = \frac{1}{f'}\left[f''\widetilde{\nabla}_{\langle a}\widetilde{\nabla}_{b\rangle}R + f'''\widetilde{\nabla}_{\langle a}R\widetilde{\nabla}_{b\rangle}R - \sigma_{ab}f''\dot{R}\right]\,. \tag{4.17}$$

4.2.2 Energy conservation

It is possible to treat fourth-order gravity as standard Einstein gravity in the presence of two effective fluids, by rewriting the field equation 4.5 in the following form,

$$G_{ab} = \tilde{T}^m_{ab} + T^R_{ab} = T^{\rm tot}_{ab}\,, \tag{4.18}$$

where $\tilde{T}^m_{ab} = \frac{T^m_{ab}}{f'}$, and ,

$$T^R_{ab} = \frac{1}{f'}\Big[\frac{1}{2}\, g_{ab}\,(R - R\, f') + \nabla_b\, \nabla_a\, f - g_{ab}\, \nabla_\sigma\, \nabla^\sigma\, f\Big]\,. \tag{4.19}$$

These two sources can be interpreted as an effective fluids, the curvature fluid (associated with T^R_{ab}) and the effective matter fluid (associated with $\tilde{T}^m_{ab}$). The conservation of the total effective fluid $T^{\rm tot}_{ab}$ follows directly from the Bianchi identities. Note that the individual effective fluids are not conserved but exchange energy and momentum,

$$\tilde{T}^{m;b}{}_{ab} = -\frac{f''}{f'^2}\, T^m_{ab}\, R^{;b}\,, \qquad T^{R;b}{}_{ab} = \frac{f''}{f'^2}\, T^m_{ab}\, R^{;b}\,. \tag{4.20}$$

In this way dark energy can be thought of as have a geometrical origin, rather than be due to the vacuum energy or additional scalar fields which are added by hand to the energy momentum tensor. It is clear from 4.18 that the term $\frac{1}{f'(R)}$ couples matter non-minimally to geometry in a way analogous to what is done in scalar-tensor theories. When $f(R) = R$, the curvature stress-energy tensor $T^R_{\mu\nu}$ identically vanishes and equation (4.18) reduces to the standard second-order Einstein field equations.

4.2.3 Exact non-linear evolution equations

We are now ready to derive the exact nonlinear equations that govern the exact gravitational dynamics of fourth-order gravity relative to observers comoving with standard matter. These equations are fully covariant and hold for any space-time. The twice-contracted Bianchi identities lead to evolution equations for μ^m, p^m, μ^R, q^R_a,

$$\dot{\mu}^m \; = \; -\,\Theta\,(\mu^m + p^m)\;,$$

$$\widetilde{\nabla}^a p^m = -(\mu^m + p^m)\,\dot{u}^a\,.$$

$$\dot{\mu^R} + \widetilde{\nabla}^a q^R_a = -\,\Theta\,(\mu^R + p^R) - 2\,(\dot{u}^a q^R_a) - (\sigma^{ab}\pi^R_{ab}) + \mu^m\frac{f''\,\dot{R}}{f'^2}\;,$$

$$\dot{q}^R_{\langle a\rangle} + \widetilde{\nabla}_a p^R + \widetilde{\nabla}^b \pi^R_{ab} = -\,\tfrac{4}{3}\,\Theta\, q^R_a - \sigma^b a\, q^R_b - (\mu^R + p^R)\,\dot{u}_a - \dot{u}^b\,\pi^R_{ab} - \eta^{bc}{}_a\,\omega_b\, q^R_c + \mu^m\frac{f''\,\widetilde{\nabla}_a R}{f'^2}$$

The Raychaudhuri equation:

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \sigma_{ab}\sigma^{ab} - 2\omega_a\omega^a - \widetilde{\nabla}^a\dot{u}_a + \dot{u}_a\dot{u}^a + \tfrac{1}{2}(\tilde{\mu}^m + 3\tilde{p}^m) = -\tfrac{1}{2}(\mu^R + 3p^R)\,. \tag{4.21}$$

Vorticity propagation:

$$\dot{\omega}_{\langle a\rangle} + \tfrac{2}{3}\Theta\omega_a + \tfrac{1}{2}{\rm curl}\,\dot{u}_a - \sigma_{ab}\omega^b = 0\,.$$

Shear propagation:

$$\dot{\sigma}_{\langle ab\rangle} + \tfrac{2}{3}\Theta\sigma_{ab} + E_{ab} - \widetilde{\nabla}_{\langle a}\dot{u}_{b\rangle} + \sigma_{c\langle a}\sigma_{b\rangle}{}^{c} + \omega_{\langle a}\omega_{b\rangle} - \dot{u}_{\langle a}\dot{u}_{b\rangle} \;=\; \tfrac{1}{2}\pi^{R}_{ab} \;.$$

Gravito-electric propagation:

$$\dot{E}_{\langle ab\rangle} + \Theta E_{ab} - \mathrm{curl}\, H_{ab} + \tfrac{1}{2}(\tilde{\mu}^{m} + \tilde{p}^{m})\sigma_{ab} - 2\dot{u}^{c}\eta_{cd(a}H^{d}_{b)} - 3\sigma_{c\langle a}E^{c}_{b\rangle} + \omega^{c}\eta_{cd(a}E^{d}_{b)}$$

$$= -\tfrac{1}{2}(\mu^{R} + p^{R})\sigma_{ab} - \tfrac{1}{2}\dot{\pi}^{R}_{\langle ab\rangle} - \tfrac{1}{2}\widetilde{\nabla}_{\langle a}q^{R}_{b\rangle} - \tfrac{1}{6}\Theta\pi^{R}_{ab} - \tfrac{1}{2}\sigma^{c}{}_{\langle a}\pi^{R}_{b\rangle c} - \tfrac{1}{2}\omega^{c}\eta^{d}_{c(a}\pi^{R}_{b)d} \;.$$

Gravito-magnetic propagation:

$$\dot{H}_{\langle ab\rangle} + \Theta H_{ab} + \mathrm{curl}\, E_{ab} - 3\sigma_{c\langle a}H^{c}_{b\rangle} + \omega^{c}\eta_{cd(a}H^{d}_{b)} + 2\dot{u}^{c}\eta_{cd(a}E^{d}_{b)}$$

$$= \tfrac{1}{2}\mathrm{curl}\, \pi^{R}_{ab} - \tfrac{3}{2}\omega_{\langle a}q^{R}_{b\rangle} + \tfrac{1}{2}\sigma^{c}{}_{(a}\eta^{d}_{b)c}q^{R}_{d} \;.$$

Vorticity constraint:

$$\widetilde{\nabla}^{a}\omega_{a} - \dot{u}^{a}\omega_{a} = 0 \;.$$

Shear constraint:

$$\widetilde{\nabla}^{b}\sigma_{ab} - \mathrm{curl}\, \omega_{a} - \tfrac{2}{3}\widetilde{\nabla}_{a}\Theta + 2[\omega, \dot{u}]_{a} = -q^{R}_{a} \;.$$

Gravito-magnetic constraint:

$$\mathrm{curl}\, \sigma_{ab} + \widetilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} + 2\dot{u}_{\langle a}\omega_{b\rangle} = 0 \;.$$

Gravito-electric divergence:

$$\widetilde{\nabla}^{b}E_{ab} - \tfrac{1}{3}\widetilde{\nabla}_{a}\tilde{\mu}^{m} - [\sigma, H]_{a} + 3H_{ab}\omega^{b} = \tfrac{1}{2}\sigma^{b}_{a}q^{R}_{b} - \tfrac{3}{2}[\omega, q^{R}]_{a} - \tfrac{1}{2}\widetilde{\nabla}^{b}\pi^{R}_{ab} + \tfrac{1}{3}\widetilde{\nabla}_{a}\mu^{R} - \tfrac{1}{3}\Theta q^{R}_{a} \;.$$

Gravito-magnetic divergence:

$$\widetilde{\nabla}^{b}H_{ab} - (\tilde{\mu}^{m} + \tilde{p}^{m})\omega_{a} + [\sigma, E]_{a} - 3E_{ab}\omega^{b} = -\tfrac{1}{2}\mathrm{curl}\, q^{R}_{a} + (\mu^{R} + p^{R})\omega_{a} - \tfrac{1}{2}[\sigma,\pi^{\,R}]_{a} - \tfrac{1}{2}\pi^{R}_{ab}\omega^{b} \;.$$

In the equations above the spatial curl of a vector and a tensor is

$$(\mathrm{curl}\ X)^{a} = \eta^{abc}\,\widetilde{\nabla}_{b}X_{c}\,, \qquad\qquad (\mathrm{curl}\ X)^{ab} = \eta^{cd\langle a}\,\widetilde{\nabla}_{c}X^{b\rangle}{}_{d}\,,$$

respectively. Finally, $\omega_{a} = \frac{1}{2}\eta^{bc}{}_{a}\omega_{bc}$ and the covariant commutators are

$$[X, Y]_{a} = \eta_{acd}X^{c}Y^{d}\;, \qquad [W, Z]_{a} = \eta_{acd}W^{c}{}_{e}Z^{de}\,.$$

Note that for $f(R) = R$, one has $f'(R) = 1$ and $\mu^{R}, p^{R}, q^{R}_{a}, \pi^{R}_{ab} = 0$ and therefore the above equations reduce to the ones for GR. Using 4.21-4.21 one can analyze any type of $f(R)$ cosmology. Such analysis can be performed either directly or with the use of alternative techniques, like the Dynamical System Approach (see e.g. [113, 112, 114, 186]) .

4.3 $f(R)$ theories and the cosmic acceleration

The main motivation for $f(R)$ gravity is to explain the accelerated expansion without the need for dark energy. For the spatially homogeneous and isotropic space-times with vanishing 3-curvature and barotropic perfect fluid as the standard matter source with equation of state $p = \omega\,\mu_m$, the independent field equations of $f(R)$ gravity read ,

$$H^2 = \frac{1}{3f'}\Big[\mu_m + \frac{Rf'-f}{2} - 3H\dot{R}f''\Big]\,, \tag{4.22}$$

$$2\dot{H} + 3H^2 = -\frac{1}{f'}\Big[w\mu_m + \dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{f-Rf'}{2}\Big]\,. \tag{4.23}$$

By combining the Raychaudhuri and Friedman equations we obtain ,

$$R = 6\dot{H} + 12H^2\,. \tag{4.24}$$

In vacuum equations these equations can take the form of the standard Friedmann equation ,

$$H^2 = \frac{1}{3}\mu^R\,. \tag{4.25}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{2}[\mu^R + 3p^R]\,, \tag{4.26}$$

where ,

$$\mu^R = \frac{1}{f'}\Big[\frac{Rf'-f}{2} - 3H\dot{R}f''\Big]\,, \tag{4.27}$$

$$p^R = \frac{1}{f'}\Big[\frac{f-Rf'}{2} + \ddot{R}f'' + 2H\dot{R}f'' + \dot{R}^2 f'''\Big]\,. \tag{4.28}$$

The cosmic acceleration is achieved when the right-hand side of the acceleration equation remains positive. The conservation equations are

$$\dot{\mu}_m + \Theta\mu_m(1+w) = 0\,, \qquad \dot{\mu}_{eff} + \Theta\mu_{eff}(1+w_{eff}) = 0\,, \tag{4.29}$$

where w_{eff} is the effective equation of state parameter of modified gravity ,

$$w_{eff} \equiv \frac{p^R}{\mu^R} - \frac{\dot{R}f''\mu^m}{f'^2}\,. \tag{4.30}$$

In vacuum the effective equation of state parameter of modified gravity w_{eff} can be expressed as ,

$$w_{eff} \equiv \frac{p^R}{\mu^R} = \frac{\dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + (f-Rf')/2}{(Rf'-f)/2 - 3H\dot{R}f''}\,. \tag{4.31}$$

We assume that $f' > 0$ and $f'' > 0$. It is clear form equation 4.27 that μ^R is non-negative. Since the denominator is stricly positive, the sign of w^R is determined by its numerator. For

$w_{eff} = -1$, the condition,

$$\frac{f'''}{f''} = \frac{\dot{R}H - \ddot{R}}{\dot{R}^2}\,, \tag{4.32}$$

must be satisfied. To show that the accelerated behavior can be obtained in a simple way by extending GR without exotic dark energy, let us consider the following example $f(R) \propto R^n$. In [113] it has been shown that there exist a power-law solution,

$$a(t) = (t/t_0)^c\,, \tag{4.33}$$

By substituting in 4.31 we obtain,

$$w_{eff} = -\frac{6n^2 - 7n - 1}{6n^2 - 9n + 3} \qquad n = 2 \Rightarrow w_{eff} = -1\,. \tag{4.34}$$

A general scale factor a(t) would lead to a time-dependent w^R. If we define the quantity $\phi \equiv f'$, then equation 4.30 takes the form,

$$w_{eff} = -1 + \frac{\ddot{\phi} - H\dot{\phi}}{3\phi H^2}\,. \tag{4.35}$$

Accordingly the deviation from de Sitters equation of stat $w = -1$ can be parametrized by,

$$\mu_{eff} + p_{eff} = \frac{\ddot{\phi} - H\dot{\phi}}{3\phi H^2} = \frac{\dot{\phi}}{\phi}\frac{d}{dt}\Big[\ln\Big(\frac{\dot{\phi}}{a}\Big)\Big]\,. \tag{4.36}$$

The only acceptable exact de Sitter solution corresponds to $\dot{\phi} = f''\dot{R} = 0$.

4.4 Cosmological solutions

The phase space structure of $f(R)$ gravity is quite rich, where a variety of interesting cosmological behaviors at both early and late-times were found. In particular non-singular and accelerating behavior in the early Universe, as well as late-time accelerating expansion has been shown to exist for a number of interesting $f(R)$ gravity theories [166], [168],[169],[170]. For $R = const$ and $T_{ab} = 0$ equation 4.6 reduces for a given $f(R)$ to an algebraic equation in R,

$$f'\,R - 2\,f = 0\,. \tag{4.37}$$

- If $R = 0$, equation 4.5 implies $R_{ab} = 0 \Rightarrow$ The maximally symmetric solution is Minkowski space-time.
- If $R = const$, equation 4.5 implies $R_{ab} = \frac{const}{4} g_{ab} \Rightarrow$ The maximally symmetric solution is de Sitters or anti de Sitters space depending on the sign of constant.

The model $f(R) = R + \alpha R^2$ which has been proposed by Starobinsky as an inflation model, satisfies the condition 4.37, thus it gives rise to an exact de Sitter solution. It has also been shown in literature that any $f(R)$ gravity for which equation 4.37 is satisfied, there exists a de Sitter solution with $R = 4\,\Lambda$. Equation 4.37 holds for both the metric and the Palatini formalisms.

Another important class of solutions that have existed for some $f(R)$ are the power-law solutions. For a spatially flat FLRW background in the presence of a perfect barotropic fluid with equation of state $p = \omega\,\rho$, theories of the type $f(R) = R + R^n$ has power-law solutions of the form,

$$a(t) = t^{\frac{2(1+n)}{3(1+\omega)}}\,. \tag{4.38}$$

These are the only power-law perfect fluid FLRW solutions that exist for any $f(R)$ gravity theory [9]. The stability of these solutions, have been investigated in [8].

Exact Bianchi cosmological solutions for $f(R) = R^n$ have been studied in [3][4][2]. Bianchi type I, V and V IIA solutions have been studied in [5], [6] and [7] respectively.

It has been shown in [174] that, $f(R)$ only admit an Einstein Static solution for the very special form of gravitational Lagrangian.

$$f(R) = \alpha + \beta\,R^n\,, \tag{4.39}$$

the constants α,β and n are fixed functions of the equation of state parameter w and cosmological constant$\Lambda \neq 0$. If $\Lambda = 0$, $\alpha = 0$ and β is arbitrary. Also It has been shown in [174] that, the stability analysis of this $f(R)$ model with respect to generic linear inhomogeneous and anisotropic perturbations shows that the Einstein static model is stable against vector and tensor perturbations for arbitrary w on all scales, while Scalar perturbations are only stable on all scales only if the matter fluid equation of state satisfies $c_s^2 > (\sqrt{5}-1)/6$, which is quite similar to the GR result where Einstein Static model is stable for $c_s^2 > 1/5$.

4.4.1 Reconstruction methods in $f(R)$ gravity

Due to the complexity of the field equations of $f(R)$ gravity, it is extremely difficult to obtain both exact and numerical solutions. The reconstruction methods in $f(R)$ gravity has proven to be very successful in obtaining exact solutions. In this technique one assumes that the expansion history of the Universe is known exactly and one inverts the field equations to deduce the class of $f(R)$ gravity models that give rise to this solution. In this subsection we discuss some of the main reconstruction methods,

- **Reconstruction from the condition a=a(t)**

This method is called the classical reconstruction method. By substituting $a = a(t)$ in equation 4.24 and solving for t in term of Ricci scalar R, we obtain the functional relation $t = m(R)$. By changing the variable t to R using this transformation, Friedmann equation 4.22 becomes,

$$3H[m(R)]\dot{R}[m(R)]f'' + f'\Big[3H[m(R)]^2 - \frac{R}{2}\Big] + \frac{f}{2} = \mu_0\, m(R)^{-3(1+w)}\,. \tag{4.40}$$

a second order differential equation for the function $f(R)$, the solution of which gives the class of theories of gravity for which the given function $a = a(t)$. For $a = a_0\, t^{\alpha}$ then from 4.24 we obtain the transformation $R(t) = \varepsilon t^{-2}$. By solving Friedmann equation 4.22 after applying this transformation we obtain,

$$f(R) = A\Big(\frac{R}{\varepsilon}\Big) + B\, R^{\frac{3}{4}-\frac{\alpha}{4}+\frac{\sqrt{d}}{4}} + \frac{2}{\sqrt{c}}C\, R^{\frac{3}{4}-\frac{\alpha}{4}-\frac{\sqrt{d}}{4}}\,. \tag{4.41}$$

where A and d are functions of α, μ_0 and w, while B, C are constants of integration.

- **Reconstruction from the condition $\dot{a} = h(a)$**

By substituting $\dot{a} = h(a)$ in 4.24 we obtain,

$$R(a) = 6\Big(\frac{h_{,a}^2}{2a} + \frac{h^2}{a^2}\Big)\,. \tag{4.42}$$

Using this transformation the Friedmann equation 4.22 becomes,

$$3\frac{h[m(R)]}{m(R)}R_{,a}[m(R)]h[m(R)]f'' + f'\Big[3\frac{h[m(R)]^2}{m(R)^2} + \frac{Rf'}{2}\Big] + \frac{f}{2} = \mu_0\, m(R)^{-3(1+w)}\,. \tag{4.43}$$

For a dust-like matter (w=0) we assume $\dot{a} = 2\,\Omega/(\sqrt{\Lambda})(\sqrt{a - \Lambda\, a^2})$. Accordingly the scale factor evolves with time as,

$$a(t) = \frac{1}{\sqrt{\Lambda}}\sin^2(\Omega\, t)\,, \qquad \text{and} \qquad R(a) = \frac{12\Omega^{\,2}(3 - 4\Lambda\, a)}{\Lambda\, a}\,, \tag{4.44}$$

the corresponding reconstructed class of $f(R)$ models is,

$$f(R) = A\, R + B\, R^2 + C\, R^3 + D\,, \tag{4.45}$$

where A,B,C and D are functions ofΛ , μ_0 and Ω.

- **Reconstruction from the condition $\dot{H} = h(H)$**

In this case equation 4.24 takes the form,

$$R(H) = 6h(H) + 12H^2\,. \tag{4.46}$$

If $H = m(R)$ be the required transformation equation, then Friedmann equation 4.22 becomes,

$$3m(R)R_{,H}[m(R)]h[g(R)]f'' + f'\Big[3m(R)^2 - \frac{Rf'}{2}\Big] + \frac{f}{2} = \frac{\mu_0}{a(R)^{3(1+w)}}\,. \tag{4.47}$$

In the vacuum case we assume $\dot{H} = \alpha$ where α is a constant. The corresponding reconstructed class of $f(R)$ models is ,

$$f(R) = A\,R + B\,R^2 + C\,, \tag{4.48}$$

where A,B and C are functions of α.

For any given dynamical variable X, the reconstruction of the class of $f(R)$ models that corresponds to specific evolution history, is possible if and only if the function R(X), as obtained by substituting the solution in equation 4.24, is analytically invertible .

4.5 Conformal transformations

Conformal transformations allows one to obtain a different representation of the same gravity theory. Conformal transformations are defined by a conformal rescaling of the space-time metric ,

$$g_{\nu\mu} \longrightarrow \tilde{g}_{\nu\mu} = \Omega^2(x^a)\, g_{\nu\mu}\,. \tag{4.49}$$

Where Ω is a continuous regular function, and the tilde represents quantities in the Einstein frame. The issue of the physical equivalence of these conformal frames is one of the open problems in theoretical physics [93] . By using conformal transformations, it is possible to show that the higher order and non-minimally coupled terms always correspond to Einstein gravity plus one (or more) scalar field(s) minimally coupled to the curvature [80], [86]; [81].

4.5.1 Conformal transformation of the geometric and matter quantities

Consider an n-dimensional space-time $(M, g_{\nu\mu})$, under the conformal transformation 4.49 we obtain a new space-time $(M, \tilde{g}_{\nu\mu})$ such that ,

1- The inverse of the metric

$$\tilde{g}^{\nu\mu} = \Omega^{-2}\, g^{\nu\mu}\,, \qquad\qquad \tilde{g} = \Omega^{2n}\, g\,. \tag{4.50}$$

2- Christoffel symbols

$$\tilde{\Gamma}^a{}_{b\gamma} = \Gamma^a{}_{b\gamma} + \Omega^{-1}(\delta^a{}_b\,\Omega_{;\gamma} + \delta^a{}_\gamma\,\Omega_{;b} - g_{b\gamma}\,\Omega^{;a})\,. \tag{4.51}$$

3- Riemann tensor

$$\begin{aligned}\tilde{R}^\delta{}_{ab\gamma} = R^\delta{}_{ab\gamma} &+ 2\,\delta^\delta{}_{[a}\nabla_{b]}\nabla_\gamma\,(\ln\Omega) - 2g^{\delta\sigma}g_{\gamma[a}\nabla_{b]}\nabla_\sigma\,(\ln\Omega) + 2\,\nabla_{[a}(\ln\Omega)\delta^\delta{}_{b]}\nabla_\gamma\,(\ln\Omega)\\ &- 2\,\nabla_{[a}(\ln\Omega)g_{b]\gamma}\,g^{\delta\sigma}\nabla_\sigma(\ln\Omega) - 2\,g_{\gamma[a}\delta^\delta{}_{b]}g^{\sigma\rho}\nabla_\sigma\,(\ln\Omega)\nabla_\sigma\,(\ln\Omega)\,.\end{aligned} \tag{4.52}$$

4- Ricci tensor

$$\tilde{R} = \Omega^{-2}\,[R - 2(n-1)\,\Box\,(\ln\Omega) - (n-1)(n-2)\frac{g^{ab}\nabla_a\Omega\,\nabla_b\Omega}{\Omega^2}]\,. \tag{4.53}$$

5-Weyltensor

$$\tilde{C}^{\delta}{}_{ab\gamma} = C^{\delta}{}_{ab\gamma}\,. \tag{4.54}$$

6-Stress-energy tensor

$$\tilde{T}_{ab} = \Omega^{-2}T_{ab}\,, \qquad \tilde{T}^{b}{}_{a} = \Omega^{-4}T^{b}{}_{a}\,, \qquad \tilde{T}^{ab} = \Omega^{-6}T^{ab}\,. \qquad \tilde{T} = \Omega^{-4}T\,. \tag{4.55}$$

where $\Box$ is the d'Alembert operators with respect to the metric g_{ab}. Although vectors, tensorial quantities and consequently the laws of physics are not invariant under conformal transformations, but the causal structure of space-time $(M, g_{\nu\mu})$ remains the same. For example the the covariant conservation equation ,

$$\nabla^{b}\,T_{ab} = 0\,, \tag{4.56}$$

is transformed under the conformal transformation 4.49 to ,

$$\tilde{\nabla}^{b}\,\tilde{T}_{ab} = -\tilde{T}\,\tilde{\nabla}_{a}\,(\ln\Omega)\,. \tag{4.57}$$

It is clear from this equation that the geometric factor Ω is directly coupled to matter in the Einstein frame. It also follows that timelike geodesics in $(M, g_{\nu\mu})$ are not the same as those in $(M, \tilde{g}_{\nu\mu})$. The conservation equation (2.11) is conformally invariant only if the trace T of the energy-momentum tensor T_{ab} vanishes.

4.5.2 $f(R)$ and scalar-tensor theories

In this section we discuss the conformal relationship between scalar-tensor and $f(R)$ theories of gravity. It has bee shown in literature [109] that if $f''(R) \neq 0$, the metric $f(R)$ gravity is equivalent to scalar-tensor theories with Brans-Dicke parameter $\omega = 0$. In $f(R)$ theories, the additional degree of freedom can be interpreted as a new scalar field $\phi \equiv f'(R)$. In Jordan-frame the $f(R)$ gravity action 4.4 reads ,

$$S = \int d^4x\sqrt{-g}\,(\phi\,R - U(\phi)) + S^{matter}\,, \tag{4.58}$$

where ,

$$U(\phi) = \phi\,R(\phi) - f(R(\phi))\,. \tag{4.59}$$

To see how the action looks in the Einstein frame, consider the conformal transformation ,

$$\tilde{g}_{ab} = \Omega^2 \, g_{ab} \, , \qquad \psi \sqrt{3} \ln \phi \, , \tag{4.60}$$

where a tilde represents quantities in the Einstein frame and$\Omega^{\,2}$ is the conformal factor . In term of $\tilde{g}$ and ψ, the action 4.58 becomes ,

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\tilde{R} + \frac{1}{2} \, \tilde{g}^{ab} \, \partial_a \psi \, \partial_b \psi + U(\psi) \right] + S^{matter} \, , \tag{4.61}$$

where ,

$$V = \frac{U(\phi(\psi))}{\phi(\psi)^2} \, . \tag{4.62}$$

Meanwhile the action in the Brans-Dicke theory with potential $V(\phi)$ is given by ,

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\tilde{R} + \frac{\omega_{BD}}{2} \, \tilde{g}^{ab} \, \partial_a \psi \, \partial_b \psi + U(\psi) \right] + S^{matter} \, , \tag{4.63}$$

where ω_{BD} is the Brans-Dicke parameter. Comparing this action with the action 4.58 it follows that $f(R)$ theory in the metric formalism is equivalent to Brans-Dicke theory with the parameter $\omega_{BD} = 0$, whereas solar system and binary pulsar data currently require $\omega_{BD} > 40000$. This problem can be evaded through the screening mechanisms that we mentioned earlier .

4.6 Ostrogradski instability

It has been shown by Ostrogradski [190], that any dynamical theory whose Lagrangian depends non-degenerately upon more than one time derivative there exist a linear instability. To clarify this point let us consider a one-dimensional point particle whose position as a function of time is $q(t)$. Let us assume that the Lagrangian depends nondegenerately upon $\ddot{q}$, that is $L = L(q, \dot{q}, \ddot{q})$. The corresponding Euler-Lagrange equation is,

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} = 0 \, . \tag{4.64}$$

The phase space of this theory is four-dimensional and corresponds to the canonical coordinates (q_1, p_1) and $(q_2, p2)$, given by .

$$q_1 \equiv q \, , \qquad p_1 \equiv \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} \, , \tag{4.65}$$

$$q_2 \equiv \dot{q} \, , \qquad p_2 \equiv \frac{\partial L}{\partial \ddot{q}} \, . \tag{4.66}$$

The Hamiltonian of the system can be obtained by using Legendre transformation ,

$$H(q_1, q_2, p_1, p_2) \equiv p_1 q_2 + p_2 f(q_1, q_2, p_2) - L(q_1, q_2, f(q_1, q_2, p_2)) \, . \tag{4.67}$$

It is clear from the Ostrogradski Hamiltonian 4.67 it is linear in the canonical momentum p_1, since p_1 can take any arbitrary value, there is nothing to prevent the Hamiltonian from being infinitely negative. This result is generic for all theories whose Lagrangian depend nondegenerately [2]. upon more than one time derivative. It has been shown in [191] that $f(R)$ theories are the only modified gravity theories that do not suffer from Ostrogradski instability, because they violate Ostrogradskis assumption of non-degeneracy.

4.6.1 The stability of the de Sitter Universe

The stability of de Sitter solutions can be studied either with respect to the homogeneous perturbations, which depend only on time, or with respect to the more general inhomogeneous perturbations, which depend on both space and time. In [60] it is demonstrated that the stability conditions for these two cases coincide, thus for simplicity we are going to focus only on the stability condition of de Sitter space with respect to homogeneous perturbations in modified gravity. To obtain the evolution equation for the homogeneous perturbation δH, we use,

$$H(t) = H_0 + \delta H(t)\,, \tag{4.68}$$

$$R = R_0 + \delta R\,, \delta R = 6(\delta\dot{H} + 4H_0\delta H)\,, \tag{4.69}$$

$$f' = f_0' + f''\delta R\,, \qquad f = f_0 + f_0'\delta R\,, \tag{4.70}$$

together with the field equations 4.22 and the relation 4.37,

$$\delta\ddot{H} + \Big(4H_0 - \frac{f_0}{6H_0 f_0'}\Big)\delta\dot{H} + \frac{1}{3}\Big(\frac{f_0'}{f''} - \frac{2f_0}{f_0'}\Big)\delta H = 0\,. \tag{4.71}$$

By assuming the anstaz $\delta H = \epsilon exp(st)$, we obtain,

$$s_\pm = \frac{1}{2}\Big[-\frac{f_0}{2H_0 f_0'} \pm \sqrt{\Big(\frac{f_0}{2H_0 f_0'}\Big)^2 - \frac{4}{3}\Big(\frac{f_0'}{f''} - \frac{2f_0}{f_0'}\Big)}\Big]\,. \tag{4.72}$$

If $f_0, H_0, f_0' > 0$, then the root s_- is real and negative, corresponding to stable mode decaying exponentially in time if and only if,

$$\frac{f_0}{f''} - \frac{2f_0}{f_0'} \geq 0\,. \tag{4.73}$$

This stability condition helps to rule out models from a theoretical point of view. As an example of the application of the stability condition see [61]. For $f(R)$ a theory, characterized by,

- $f(R) = R - \frac{\mu^4}{R}$

[2]nondegeneracy simply means that one cannot eliminate $\ddot{q}$ by partial integration[191]

The stability condition 4.73 reduces to

$$1+\frac{6\mu^2}{R_o^2}-\frac{3\mu^8}{R_o^4}\leq 0\,. \tag{4.74}$$

The condition for the existence of a nontrivial de Sitter solution is $R_0 = \sqrt{3}\,\mu^2$; therefore the stability condition 4.74 can never be satisfied. This situation can be compensated by a quadratic correction in the theory with

- $f(R) = R - \frac{\mu^4}{R} + aR^2$

The condition for the existence of a nontrivial de Sitter space is again $R_0 = \sqrt{3}\,\mu^2$, the stability condition 4.73 reduces to

$$\frac{1}{3\sqrt{3}\,a\,\mu^2-1}\geq 0\,, \tag{4.75}$$

and therefore the de Sitter space is stable only if

$$a>\frac{1}{3\sqrt{3}\,\mu^2}\,, \tag{4.76}$$

otherwise it is unstable.

- $f(R) = R^n$

One of the fixed points in the phase space of this model corresponds to powerlaw inflation $a \propto t^\alpha$ with,

$$\alpha=\frac{-2n^2+3n-1}{n-2}\,. \tag{4.77}$$

The condition 4.37 for the existence of a nontrivial de Sitter solutions becomes,

$$nR_0 = 2R_0^2\,, \tag{4.78}$$

which is only satisfied for $n = 2$ or $R_0 = 0$. The stability condition 4.73 yields,

$$R_0\frac{2-n}{n(n-1)}\geq 0\,. \tag{4.79}$$

According to this condition the de Sitter solutions is stable for $1 < n \leq 2$ and for $n < 0$.

- $f(R) = R + \alpha\, R^n$

The condition for the existence of a nontrivial de Sitter space is,

$$R_0^{n-1}=\frac{1}{\alpha\,(n-2)}\,,\quad n\neq 2\,. \tag{4.80}$$

The stability condition 4.73 yields,

$$\frac{-n^2+2n-2}{\alpha\,n(n-1)}\geq 0\,, \tag{4.81}$$

Thus de Sitter space is stable only if $\alpha\, n < 0$

4.6.2 Curvature singularity

Although of the great success of modified $f(R)$ gravity models, there are, however, several problems. It has been shown in literature that [162], all infrared-modified $f(R)$ gravity models suffer from an unprotected curvature singularity that is a finite distance away both in field and energy values from the place we are supposed to live in .

Infrared-modified gravity theories introduces a new scalar degree of freedom that is not there in GR. The equation of motion of this new scalar degree of freedom is given by the trace of 4.5 ,

$$\Box f' = \frac{1}{3}(2f - f' R) + \frac{8\pi G}{3} T \,. \tag{4.82}$$

If we define the scalar filed $\phi = f' - 1$, the above equation can be rewritten as ,

$$\Box\phi = V'(\phi) - F \,, \tag{4.83}$$

where $F = (8\pi G/3)(\rho - 3p)$ and $V(\phi)$ is effective scalar field potential, that depends on the $f(R)$ gravity model and is determined by ,

$$V'(\phi) = \frac{1}{3}(2f - f' R) \,. \tag{4.84}$$

For a FRW background space-time scalar gravitational degree of freedom ϕ obeys a usual scalar field equation,

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = F \,. \tag{4.85}$$

The expansion rate of the Universe starts feeling the effect of the gravity modification only when the local curvature drops below the infrared modification scale R_0. in this domain the space-time curvature is controlled by the potential $V(\phi)$ and the deriving term F. Figure 4.1 shows the effective potential for Starobinsky's model with $\lambda = 2$ and $n = 1$. The minimum of the potential $V(\phi)$, is very near to the field configuration corresponding to infinitely large values of R for solar physics constraints to be evaded. It has been shown in [162] that the existence of this curvature singularity is a generic feature of all infrared-modified $f(R)$ gravity models and any proposal to solve this problem requires fine tuning more than the one encountered in ΛCDM model ϕ [161]. It has been shown in [64] The problem can alleviated by invoking an R^2 correction, but the scenario becomes problematic if extended to compact objects like neutron stars [65] .

One of the main problems occurring in the study of higher-order theories of gravity is that finding exact cosmological solutions is extremely difficult owing to the high degree of nonlinearity exhibited by these theories and the fact that the field equations arising from $f(R)$-action are

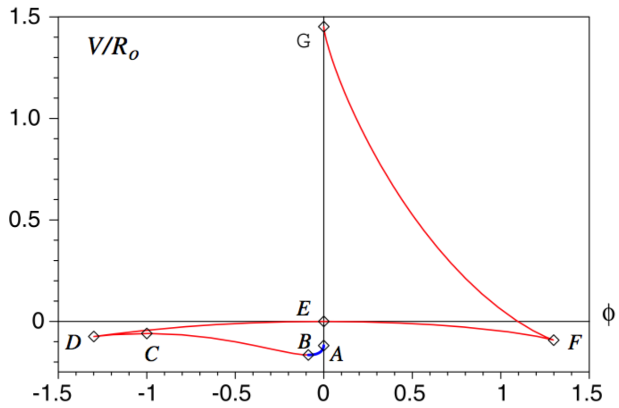

Figure 4.1: Point A is a positive curvature singularity $R = +\infty$. Point B is the stable de Sitter minimum in this model, and point C is the unstable de Sitter maximum. Point E corresponds to a flat space-time. Points D and F are critical points with $f'' = 0$ that occur at $R = \pm 1/\sqrt{3}$; potential branches there. Finally, point G is a negative curvature singularity $R = -\infty$ [162].

typically of the fourth order. This problem can be partially addressed by employing a suitable choice of generalized coordinates. In this case, the field equations can be written as a system of first-order autonomous differential equations together with a constraint equation. In this way the methods of dynamical systems theory can be exploited in order to both understand the qualitative behavior of the cosmological dynamics and obtain special exact solutions of the cosmological equations.

Chapter 5

DYNAMICAL SYSTEMS IN COSMOLOGY

5.1 Introduction

This chapter provides an elementary introduction to the theory of dynamical systems [180], and its application to cosmology [63],[181], [210]. A dynamical system is a set of states that are normally described by a set of variables $x_1, ..., x_n$, which in general are a function of time, along with a rule that describes how the present state is determined by the previous state and the initial conditions. This rule is given in terms of differential equations of the form:

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1......, x_n)\,, \\ \dot{x}_2 &= f_2(t, x_1......, x_n)\,, \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1......, x_n)\,, \end{aligned} \tag{5.1}$$

where $\mathbf{x}_i \in$ the set of real numbers $\mathbf{R}$ and $f_i : \mathbf{R} \to \mathbf{R}$, $i = 1, ..., n$. By defining $\mathbf{x} = (x_1, ..., x_n)^T$ and $\mathbf{F}(t, \mathbf{x}) = (f_1, ...f_n)^T$, in the matrix form, the system 5.1 can be written as,

$$\dot{\mathbf{x}} = \mathbf{F}(t, \mathbf{x}(t))\,. \tag{5.2}$$

One may use various methods to study the system 5.2; generally they are classified as (1) qualitative, (2) analytical, (3) numerical. This chapter focuses only on qualitative methods. We begin by defining some fundamental concepts closely related to the qualitative approach to dynamical systems.

Autonomous System In the system 5.1, if none of the functions, f_j, depend on time, the system is termed autonomous. Autonomous systems can be written in the matrix form:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})\,. \tag{5.3}$$

In the following sections we will focus only on autonomous systems.

Phase Space The phase space is the set of all possible states $(x_1, ..., x_n)$ of the system; generally the phase Space is a subspace of $\mathbf{R}^n$. For example, in classical mechanics the state of a single particle is determined by the its coordinates and velocity $(x^1, x^2, x^3, v^1, v^2, v^3)$.

Trajectory The set of points $(t, \mathbf{x}(t)); t \in \mathbf{R}$ which solve the system 5.3, and for which $\mathbf{x}(t_0) = x_{i0}$, is called the trajectory or solution curve of the dynamical system passing through $\mathbf{x}_0$.

Phase flow The collection of all trajectories obtained by varying t_0 and $\mathbf{x}_0$ throughout all allowed values is called the flow of the dynamical system.

Orbit/Phase Curve The set of points $(\mathbf{x}(t)); t \in \mathbf{R}$ which solve the system 5.3, and for which $\mathbf{x}(t_0) = \mathbf{x}_0$ is called the orbit or phase curve of the said system passing through $\mathbf{x}_0$.

5.2 Linear dynamical systems

Theorem 1.2 ***Let*** A ***be any*** $n \times n$ ***matrix. Then for a given*** $\mathbf{x}_0 \in \mathbf{R}^n$***, the initial value problem***

$$\dot{\mathbf{x}} = A\mathbf{x}\,, \qquad \mathbf{x}(0) = \mathbf{x}_0\,, \tag{5.4}$$

has a unique solution given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0\,, \tag{5.5}$$

where x_0 ***is determined by the initial conditions*** [180].

The set of vectors $\mathbf{x} \in \mathbf{R}^n$ spans the phase space of the system 5.4, and the set of all solution curves $\mathbf{x}(t) \in \mathbf{R}^n$, defines the phase portrait of the system 5.4. The function,

$$f(x) = e^{At}\,. \tag{5.6}$$

On the right-hand side of 5.5 a set of linear mappings defines $f : \mathbf{R}^n \to \mathbf{R}^n$. This set of mappings may be regarded as describing the motion of the points $\mathbf{x}_0 \in \mathbf{R}^n$, along trajectories of 5.4, where the point $\mathbf{x}_0 \in \mathbf{R}^n$, moves to the point $\mathbf{x}(t) \in \mathbf{R}^n$, given by 5.5 after time t. The set of all the mappings 5.6 is called the flow of the linear system 5.4.

Fixed/Critical point A point x_0 is called a fixed or critical point of the autonomous dynamic system 5.3 if,

$$\dot{\mathbf{x}} = 0 \Longleftrightarrow A\mathbf{x}_0 = 0\,, \tag{5.7}$$

which means that if a system is at an equilibrium point x_0 it will remain there forever.
An equilibrium point x_0 is called a hyperbolic equilibrium point of 5.4 if none of the eigenvalues of the matrix A contain a zero real part. An important property of an equilibrium point is its stability. The following describes the basic cases:

- If all solutions that start close to an equilibrium converge to the equilibrium asymptotically as $t \to \infty$, the equilibrium is described as asymptotically stable.
- If all solutions in a sufficiently small neighborhood of the equilibrium remain close to the equilibrium point, the equilibrium point is likewise regarded as stable. Note that asymptotic stability implies stability.
- If every neighborhood of the equilibrium contains solutions arbitrarily close to the equilibrium point and leaves the neighborhood, we say the equilibrium point is unstable.

5.2.1 Diagonalization

If the matrix A is diagonal then the linear system 5.4 is termed an uncoupled linear system. For uncoupled systems it is easy to compute the matrix e^{At} and hence obtain an explicit form for the solution 5.5. Since the matrix elements depend on the bases that are used to represent them, if A is not diagonal, we can look for the transformation that reduces the matrix elements of A to a diagonal form. This section presents the various cases for which this transformation exists.

Theorem 2.2 ***If the eigenvalues*** $\lambda_1, ..., \lambda_n$ ***of an*** $n \times n$ ***matrix*** A ***are real and distinct, and if the linear transformation*** $\boldsymbol{T}: \boldsymbol{R}^n \to \boldsymbol{R}^n$ ***is represented by the matrix*** A ***with respect to the standard basis*** $[e_1, ..., e_n]$***, then, with respect to any basis of eigenvectors*** $[v_1, ..., v_n]$ ***of*** A***,*** $\boldsymbol{T}$ ***is represented by***

$$\text{diag}[\lambda_1, ..., \lambda_n] = P^{-1}AP\,, \tag{5.8}$$

where the transformation matrix $P = [v_1 ... v_n]$ [180].
With respect to the basis $[v_1, ..., v_n]$, the system 5.4 takes the form,

$$P^{-1}\dot{\mathbf{x}} = P^{-1}APP^{-1}\mathbf{x}_0 \qquad \Longleftrightarrow \qquad \dot{\mathbf{y}} = \text{diag}[\lambda_1, ..., \lambda_n]\mathbf{y}\,, \tag{5.9}$$

where $\mathbf{y} = P^{-1}\mathbf{x}$. This uncoupled linear system has the solution ,

$$\mathbf{y}(t) = \text{diag}[e^{\lambda_1 t}, ..., e^{\lambda_n t}]\, \mathbf{y}(0)\,. \tag{5.10}$$

It follows that the linear system 5.4 has the solution ,

$$\mathbf{x}(t) = P^{-1}[e^{\lambda_1 t}, ..., e^{\lambda_n t}]P\, \mathbf{x}(0)\,. \tag{5.11}$$

An equilibrium point x_0 of 5.4 is called a sink if all of the eigenvalues of the matrix A contain a negative real part; it is called a source if all of the eigenvalues of the matrix A have a positive real part. and it is called a saddle if it is a hyperbolic point and A has at least one eigenvalue with a positive real part and at least one with a negative real part. The phase flows of these different cases are shown in fig 5.2.1 .

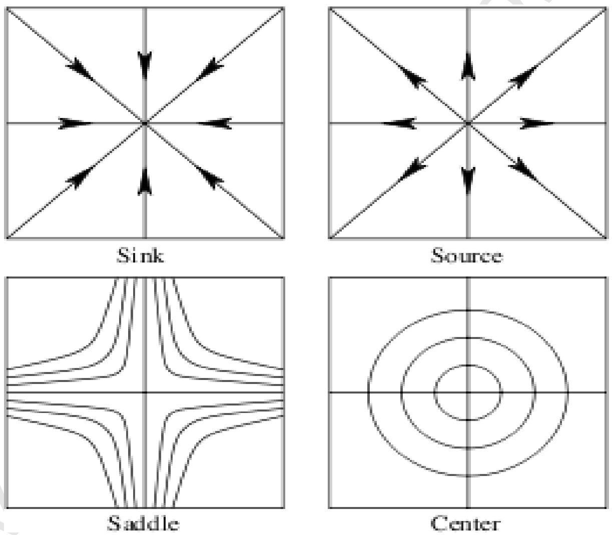

Figure 5.1: The phase portrait of a sink, source saddle and center respectively

Theorem 3.2 ***If the*** $2n \times 2n$ ***real matrix*** A ***has*** $2n$ ***distinct complex eigenvalues*** $\lambda_j = a_j + ib_j$ ***and*** $\lambda_j = a_j - ib_j$ ***and corresponding complex eigenvectors*** $w_j = u_j + iv_j$ ***and*** $w_j = u_j - iv_j$***,*** $j = 1, ..., n$***, then*** $[u_1, v_1, ..., u_n, v_n]$ ***are the basis of*** R^n***. The system 5.4 can be written in the diagonal form***

$$P^{-1}\dot{\mathbf{x}} = P^{-1}APP^{-1}\mathbf{x}_0 \qquad \Longleftrightarrow \qquad \dot{\mathbf{y}} = \text{diag}[\lambda_1, ..., \lambda_n]\mathbf{y}\,, \tag{5.12}$$

where $P = [u_1 v_1 ... u_n v_n]$ ***and*** $P^{-1}AP$ ***is a*** $2n \times 2n$ ***with*** 2×2 ***block diagonal*** [180] .
The solution of 5.12 is given by

$$\mathbf{y}(t) = \text{diag}\; e^{a_j t} \begin{bmatrix} \cos b_j t & -\sin b_j t \\ \sin b_j t & \cos b_j t \end{bmatrix} \mathbf{y}_0\,.$$

It then follows that 5.4 has the solution

$$\mathbf{x}(t) = P \text{ diag } e^{a_j t} \begin{bmatrix} \cos b_j t & -\sin b_j t \\ \sin b_j t & \cos b_j t \end{bmatrix} P^{-1}\mathbf{x}_0 .$$

Definition Let λ be an eigenvalue of the $n \times n$ matrix A of multiplicity $m \leq n$. Then for $k = 1, .., m$ any nonzero solution of $(A - \lambda I)^k v = 0$ is called a generalized vector of A.

Definition An $n \times n$ matrix N is said to be nilpotent of order k if $N^{k-1} \neq 0$ and $N^k = 0$.

Theorem 4.2 ***Let*** A ***be a real*** $n \times n$ ***matrix with real eigenvalues*** $\lambda_1, ..., \lambda_n$***, repeated according to their multiplicity. Also, a basis of generalized eigenvectors for*** R^n ***exists. If*** $[v_1, ..., v_n]$ ***forms any basis of generalized eigenvectors for*** R^n***, the matrix*** P ***is invertable***,

$$A = S + N , \tag{5.13}$$

where $N = A - S$ ***is nilpotent of order*** $k \leq n$ [180].

The solution of 5.4 is given by,

$$\mathbf{x}(t) = P \text{ diag}[e^{\lambda_j t}]\, P^{-1} \left[I + Nt + ... + \frac{N^{k-1}t^{k-1}}{(k-1)!} \right] . \tag{5.14}$$

Finally let us consider the case of multiple complex eigenvalues. If the matrix A in 5.4 is $2n \times 2n$ with multiple complex eigenvalues, then the solution of 5.4 is given by,

$$\mathbf{x}(t) = P \text{ diag } e^{a_j t} \begin{bmatrix} \cos b_j t & -\sin b_j t \\ \sin b_j t & \cos b_j t \end{bmatrix} P^{-1} \left[I + Nt + ... + \frac{N^k t^k}{(k)!} \right] \mathbf{x}_0 ,$$

where $P = [v_1 u_1 ... v_n u_n]$ and $N = A - S$ is nilpotent of order $k \leq 2n$.

5.2.2 Invariant subspaces

Theorem 5.2 ***Let*** A ***be a real*** $n \times n$ ***matrix. Then*** $\boldsymbol{R}^n$ ***can be written as a direct sum of three subspaces denoted by*** $\boldsymbol{E}^s, \boldsymbol{E}^u$ ***and*** $\boldsymbol{E}^c$ [180],

$$\mathbf{R}^n = \mathbf{E}^s + \mathbf{E}^u + \mathbf{E}^c , \tag{5.15}$$

where

- E^s *=span* $e_1, ..., e_s$ *, the stable subspace.*
- E^u *=span* $e_{s+1}, ..., e_{s+u}$ *, the unstable subspace.*
- E^c *=span* $e_{s+u+1}, ..., e_{s+u+c}$ *, the center subspace.*

$[e_1, ..., e_s]$ are the generalized eigenvectors of A corresponding to the eigenvalues of A having a negative real part, $[e_{s+1}, ..., e_{s+u}]$ are the generalized eigenvectors of A corresponding to the eigenvalues of A having a positive real part, and $[e_{s+u+1}, ..., e_{s+u+c}]$ are the generalized eigenvectors of A corresponding to the eigenvalues of A having a zero real part (see fig 5.2.2). These

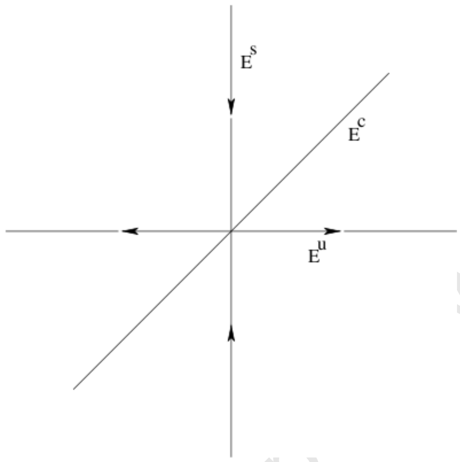

Figure 5.2: The stable, unstable and center subspaces.

three subspaces are invariant with respect to the flow e^{At} of 5.4, where the solutions of 5.4 with initial conditions entirely contained in either $\mathbf{E}^s, \mathbf{E}^u$, or $\mathbf{E}^c$ must forever remain in that particular subspace.

5.3 Non-linear systems

In general it is not always possible to find an analytical solution for a given non-linear system, and in such cases one is forced to use approximation methods. The following sections present some basic approximation techniques that enable one to deal with non-linear systems.

5.3.1 Linearization

Let us consider the n dimensional non-linear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}). \tag{5.16}$$

If $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$ is an equilibrium point of 5.16, then by Taylor's theorem

$$\mathbf{f}(\mathbf{x}) = D\mathbf{f}(\mathbf{0})\mathbf{x} + \frac{1}{2}D^2\mathbf{f}(\mathbf{0})(\mathbf{x}, \mathbf{x}) + ... \tag{5.17}$$

it follows that the linear term $D\mathbf{f}(\mathbf{0})\mathbf{x}$ is a good first approximation to the non-linear function $\mathbf{f}(\mathbf{x})$ near $\mathbf{x} = \mathbf{0}$. The non-linear system 5.16 can now be approximated by the linear system.

$$\dot{\mathbf{x}} = A\mathbf{x}\,, \tag{5.18}$$

where

$$A = D\mathbf{f}(\mathbf{0}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

This is called the Jacobian matrix.

5.3.2 Hartman-Grobman theorem

This theorem expresses one of the most important results in the local qualitative theory of the ordinary differential equations; according to it, near the hyperbolic point $\mathbf{x}_0$, the non-linear system,

$$\dot{\mathbf{x}} = f(\mathbf{x})\,, \tag{5.19}$$

is topologically equivalent to the linear system [180],

$$\dot{\mathbf{x}} = A\mathbf{x}\,, \tag{5.20}$$

which means that the two autonomous systems 5.19 and 5.20 have the same qualitative structure near the equilibrium point $\mathbf{x}_0$.

5.3.3 Lyapunov theorem

If $A = D\mathbf{f}(\mathbf{0})$ has no eigenvalues with zero real part; then there exists stable and unstable manifolds of the non-linear system, $\mathbf{W}^s$ and $\mathbf{W}^u$, which are tangential to $\mathbf{E}^s$ and $\mathbf{E}^u$ respectively at the fixed point (see fig 5.3.3). All we know about these manifolds is the fact that they are tangential to the invariant subspaces in the neighborhood of the fixed point.

5.4 The center manifold theorem

The previous section presented the Hartman-Grobman Theorem, which completely solves the problem of determining the stability and qualitative behavior in the neighborhood of a hyperbolic critical point of a non-linear system. In this section we will discuss the problem of determining the stability and qualitative behavior of the flow near a non-hyperbolic critical point of 5.19 using

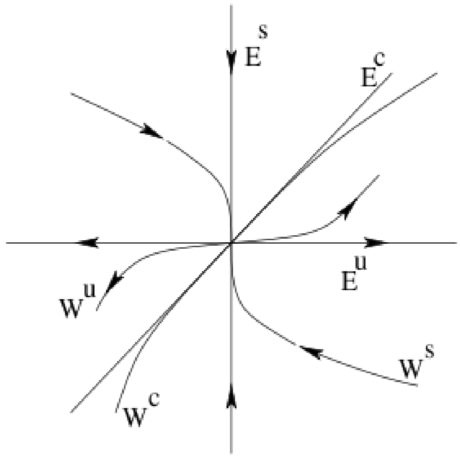

Figure 5.3: The stable, unstable and center manifolds.

the center manifold approach. The center manifold theorem is a generalization of the Lyapunov theorem to the case when there exist a center manifold $\mathbf{W}^c$ tangential to the center subspace $\mathbf{E}^c$ at the fixed point. For simplicity we shall assume that the origin is a non-hyperbolic fixed point as regards this system (this assumption does not affect the generality of our treatment because it is always possible to change the coordinates to make the fixed point the origin of the new coordinate system). If $\mathbf{f} \in C^1(E)$ and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, then the system 5.19 can be written in the diagonal form [180] ,

$$\dot{\mathbf{w}} = A\mathbf{w} + \mathbf{F}(\mathbf{w})\,, \tag{5.21}$$

where $A = \mathbf{Df}(\mathbf{0})$, $\mathbf{F}(\mathbf{w}) = \mathbf{w} - A\mathbf{w}$ and $\mathbf{F}(\mathbf{0}) = D\mathbf{F}(\mathbf{0}) = \mathbf{0}$. Now let us try to decouple the non-linear system 5.21. We can find a linear transformation $\mathbf{T}$ which transforms 5.19 into the form ,

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{y}} \\ \dot{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} A_s & 0 & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_c \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} + \begin{pmatrix} \mathbf{F}_s(\mathbf{x},\mathbf{y},\mathbf{z}) \\ \mathbf{F}_u(\mathbf{x},\mathbf{y},\mathbf{z}) \\ \mathbf{F}_c(\mathbf{x},\mathbf{y},\mathbf{z}) \end{pmatrix}\,, \tag{4.22}$$

where $(\mathbf{x},\mathbf{y},\mathbf{z}) \in R^s \times R^u \times R^c$, $s+u+c = n$, A_s is a diagonal $s \times s$ matrix having eigenvalues with a negative real part, A_u is a diagonal $u \times u$ matrix having eigenvalues with a positive real part, A_c is a diagonal $c \times c$ matrix having eigenvalues with a zero real part and $\mathbf{F}_s(\mathbf{x},\mathbf{y},\mathbf{z}), \mathbf{F}_s u(\mathbf{x},\mathbf{y},\mathbf{z})$, and $\mathbf{F}_c(\mathbf{x},\mathbf{y},\mathbf{z})$ are the s, u and c components, respectively, of the vector $\mathbf{T}^{-1}F(\mathbf{T}(\mathbf{x},\mathbf{y},\mathbf{z}))$.
We did not decouple the stable and unstable states associated with the negative and positive eigenvalues respectively, from the states associated with the zero eigenvalue, because of the term $(\mathbf{F}_s(\mathbf{x},\mathbf{y},\mathbf{z}), \mathbf{F}_s u(\mathbf{x},\mathbf{y},\mathbf{z}), \mathbf{F}_c(\mathbf{x},\mathbf{y},\mathbf{z}))^T$. In what follows, we will find two functions $\mathbf{h}^c_x$ relating $\mathbf{x}$

and $\mathbf{z}$ and $\mathbf{h}_y^c$ relating $\mathbf{y}$ and $\mathbf{z}$, i.e, $\mathbf{x} = \mathbf{h}_x^c(\mathbf{z})$ and $\mathbf{y} = \mathbf{h}_y^c(\mathbf{z})$. The center manifold theorem guarantees the existence of such mapping.

5.4.1 The local center manifold theorem

Suppose 4.22 is $\mathbf{C}^r$, $r \geq 2$, then these exist as a $\mathbf{C}^r$ s-dimensional local stable manifold $\mathbf{W}_{loc}^s(\mathbf{0})$ tangent to the stable subspace $\mathbf{E}^s$ at the origin, a $\mathbf{C}^r$ u-dimensional local stable manifold $\mathbf{W}_{loc}^{su}(\mathbf{0})$ tangential to the unstable subspace $\mathbf{E}^u$ at the origin and $\mathbf{C}^r$ c-dimensional local center manifold $\mathbf{W}_{loc}^c(\mathbf{0})$ tangential to the center subspace $\mathbf{E}^c$ at the origin. In particular, we have [180],

$$\begin{aligned}
\mathbf{W}_{loc}^s(\mathbf{0}) &= \left\{(\mathbf{x},\mathbf{y},\mathbf{z}) \in R^s \times R^u \times R^c | \mathbf{y} = \mathbf{h}_y^s(\mathbf{x}), \mathbf{z} = \mathbf{h}_z^s(\mathbf{x}); D\mathbf{h}_y^s = \mathbf{0}, D\mathbf{h}_z^s = \mathbf{0};\ |x| < \delta\right\}, \\
\mathbf{W}_{loc}^u(\mathbf{0}) &= \{(\mathbf{x},\mathbf{y},\mathbf{z}) \in R^s \times R^u \times R^c | \mathbf{x} = \mathbf{h}_x^u(\mathbf{y}), \mathbf{z} = \mathbf{h}_z^u(\mathbf{y}); D\mathbf{h}_x^u = \mathbf{0}, D\mathbf{h}_z^u = \mathbf{0};\ |y| < \delta\}, \\
\mathbf{W}_{loc}^c(\mathbf{0}) &= \left\{(\mathbf{x},\mathbf{y},\mathbf{z}) \in R^s \times R^u \times R^c | \mathbf{x} = \mathbf{h}_x^c(\mathbf{z}), \mathbf{y} = \mathbf{h}_y^c(\mathbf{z}); D\mathbf{h}_x^c = \mathbf{0}, D\mathbf{h}_y^c = \mathbf{0};\ |z| < \delta\right\},
\end{aligned}$$

where $\mathbf{h}_y^s(\mathbf{x}), \mathbf{h}_z^s(\mathbf{x}), \mathbf{h}_x^u(\mathbf{y}), \mathbf{h}_z^u(\mathbf{y}), \mathbf{h}_x^c(\mathbf{z})$ and $\mathbf{h}_y^c(\mathbf{z})$ are $\mathbf{C}^r$. In order to find the center manifold we need to solve

$$\mathbf{D}\mathbf{h}_x^c(\mathbf{z})\left[A_c\mathbf{z} + F_c(\mathbf{h}_x^c(\mathbf{z}), \mathbf{h}_y^c(\mathbf{z}), \mathbf{z})\right] - A_s\mathbf{h}_x^c(\mathbf{z}) - F_s\left(\mathbf{h}_x^c(\mathbf{z}), \mathbf{h}_y^c(\mathbf{z}), \mathbf{z}\right), \tag{4.23}$$

$$\mathbf{D}\mathbf{h}_y^c(\mathbf{z})\left[A_c\mathbf{z} + F_c\left(\mathbf{h}_x^c(\mathbf{z}), \mathbf{h}_y^c(\mathbf{z}), \mathbf{z}\right)\right] - A_u\mathbf{h}_y^c(\mathbf{z}) - F_u(\mathbf{h}_x^c(\mathbf{z}), \mathbf{h}_y^c(\mathbf{z}), \mathbf{z}), \tag{4.24}$$

with the boundary conditions,

$$\mathbf{h}_x^c(\mathbf{0}) = \mathbf{D}\mathbf{h}_x^c(\mathbf{0}) = \mathbf{0},$$

$$\mathbf{h}_y^c(\mathbf{0}) = \mathbf{D}\mathbf{h}_y^c(\mathbf{0}) = \mathbf{0}.$$

In general, these equations are difficult to solve analytically, so we resort to approximations. There are various approximations to the previous boundary value problem and we restrict ourselves to only the polynomial approximation method. In this method we approximate, $\mathbf{h}_x^c(\mathbf{0})$ and $\mathbf{h}_y^c(\mathbf{0})$ as

$$\mathbf{h}_x^c(\mathbf{z}) = a_0 + a_1\mathbf{z} + a_2\mathbf{z}^2 + a_3\mathbf{z}^3 + \ldots, \tag{4.25}$$

$$\mathbf{h}_y^c(\mathbf{z}) = b_0 + b_1\mathbf{z} + b_2\mathbf{z}^2 + b_3\mathbf{z}^3 + \ldots. \tag{4.26}$$

form the boundary conditions $a_0 = b_0 = a_1 = b_1 = 0$. To find the other coefficient, substitute these polynomials into 4.23 and 4.24, and match the coefficients. Now the flow on the center manifold $\mathbf{W}^c(\mathbf{0})$ in the neighborhood of the origin is defined by the system

$$\dot{\mathbf{z}} = A_c\mathbf{z} + \mathbf{F}_c\left(\mathbf{z}, \mathbf{h}_x^c(\mathbf{z}), \mathbf{h}_y^c(\mathbf{z})\right).$$

for all $\mathbf{z} \in \mathbf{R}^c$. Since the dimension of the center manifold is typically less than n, this simplifies the problem of determining the qualitative behavior of the system 5.19 near a non-hyperbolic critical point. In general, the flow on the center manifold near the fixed point takes the form ,

$$\dot{\mathbf{z}} = a\,\mathbf{z}^r + \dots .$$

If $r \geq 2$ and $a_r \neq 0$, then for r we have a saddle-node at the fixed point, for r odd and $a_r > 0$ we have an unstable node and for r odd and $a_r < 0$ we have a topological saddle.

Now let us consider the case when we have a double zero eigenvalue and one eigenvalue with a negative or a positive real part. Assume that we have the system,

$$\begin{pmatrix} \dot{x} \\ \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_s & 0 & 0 \\ 0 & A_{c1} & 0 \\ 0 & 0 & A_{c2} \end{pmatrix} \begin{pmatrix} x \\ z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \mathbf{F}_s(x, z_1, z_2) \\ \mathbf{F}_{c1}(x, z_1, z_2) \\ \mathbf{F}_{c2}(x, z_1, z_2) \end{pmatrix},$$

where z_1 and z_2 are the states that correspond to the zero eigenvalues and x is the state that corresponds to the eigenvalue with a negative real part. In this case it follows from the local center manifold theory, that a 2-dimensional invariant center manifold $\mathbf{W}^c_{local}(\mathbf{0})$ exists, defined by,

$$\mathbf{W}^c_{local}(\mathbf{0}) = \{(x, z_1, z_2) \in R^s \times R^c |\; x = \mathbf{h}(z_1, z_2) \text{ for} |z_1|, |z_2| < \delta\} \;,$$

for some $\delta > 0$, where $\mathbf{h} \in C^r(N_\delta(\mathbf{0}))$, $\mathbf{h}(\mathbf{0})) = D\mathbf{h}(\mathbf{0}) = 0$. In order to find the center manifold in this case, we need to solve

$$\begin{bmatrix} \frac{\partial \mathbf{h}(z_1,z_2)}{\partial z_1} & \frac{\partial \mathbf{h}(z_1,z_2)}{\partial z_2} \end{bmatrix} \begin{bmatrix} A_{c1} z_1 \\ A_{c2} z_2 \end{bmatrix} - A_s x - \mathbf{F}_s(x, \mathbf{h}(z_1, z_2)) \;,$$

with the boundary conditions,

$$\mathbf{h}(z_1, z_2) = \mathbf{Dh}(z_1, z_2) = 0 .$$

By considering these boundary conditions the function $\mathbf{h}(z_1, z_2)$ can be approximated by,

$$\mathbf{h}(z_1, z_2) = a z_1^2 + b z_1 z_2 + d z_2^2 .$$

Now, the flow on the center manifold $\mathbf{W}^c(\mathbf{0})$ in the neighborhood of the origin is defined by the reduced system,

$$\begin{aligned} \dot{z}_1 &= A_{c1} z_1 + \mathbf{F}_s(x, \mathbf{h}(z_1, z_2) , \\ \dot{z}_2 &= A_{c1} z_2 + \mathbf{F}_s(x, \mathbf{h}(z_1, z_2) . \end{aligned}$$

Example 4.1

Consider the following non-linear system [180] ,

$$\dot{z} = z^2 x - z^5 \, , \qquad (4.27)$$

$$\dot{x} = -x + z^2 \, . \qquad (4.28)$$

In this case we have $A_c = [0], A_s = [-1], F_c = z^2 x - z^5$, and $F_s = z^2$. By substitution of the expansion ,

$$h^c_x(z) = a_2 z^2 + a_3 z^3 + \dots . \qquad Dh^c_x(z) = 2a_2 z + 3a_3 z + \dots ,$$

into the equation

$$Dh^c_x(z)\left[A_c z + F_c(h^c_x(z), h^c_y(z), z)\right] - A_s \, , h^c_x(z) - F_s\left(h^c_x(z), h^c_y(z), z\right) \, ,$$

we obtain

$$(2az + 3bz^2 + \dots)(az^4 + bz^5 + \dots - z^5) + az^2 + bz^3 + \dots - z^2 = 0 \, .$$

Setting the coefficients of like powers of z equal to zero yields $a = 1, b = c = 0, \dots$, it follows that ,

$$\mathbf{h}^c_x(z) = z^2 + 0(z^5) \, .$$

Substituting $x = h^c_x(z)$ into 4.26 yields ,

$$\dot{z} = z^4 + 0(z^5) \, .$$

This equation describes the flow of the original non-linear system in the neighborhood of the origin (see fig 4.4). It is clear that the origin is a saddle-node . In the next section, we will investigate Friedmann-Lemaitre models with a cosmological constant using the dynamical systems approach .

5.5 Dynamical systems in cosmology

As an example we study homogeneous and isotropic cosmologies using the dynamical system techniques that we have developed so far [179]. We start by investigating the simplest case of a single fluid. The dynamical variables that characterize the dynamic of a single fluid Friedmann cosmology are Hubble parameter H and the density parameter Ω. The field equations 2.151 and 2.152 (or 2.153) are equivalent to the following dynamical system ,

$$\frac{dH}{d\tau} = -(1+q)H \, , \qquad (5.22)$$

$$\frac{d\Omega}{d\tau} = -2q(1-\Omega) \, , \qquad (5.23)$$

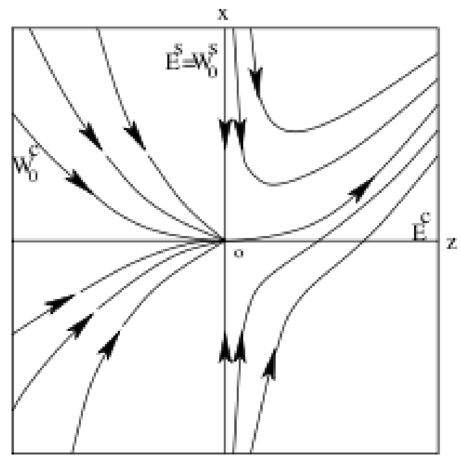

Figure 5.4: The phase portrait in the neighborhood of the origin of the system in example 4.1 .

where $\tau = \ln \dfrac{a}{a_0}$, a_0 is the scale factor at some arbitrary time t_0. Using equation 2.151 and 2.159 the deceleration parameter q takes the form ,

$$q = \frac{1}{2}(3\gamma - 2)\Omega\,. \tag{5.24}$$

Substituting this expression for q back into 5.22 we obtain an autonomous system that is equivalent to the field equations 2.151 and 2.152 (or 2.153) ,

$$\frac{dH}{d\tau} = -(1 + (\frac{3}{2}\gamma - 1)\Omega)\, H\,, \tag{5.25}$$

$$\frac{d\Omega}{d\tau} = -(3\gamma - 2)(1 - \Omega)\,. \tag{5.26}$$

It is clear that the evolution equation for Ω decouples, thus we can consider it separately as a one-dimensional dynamical system . For $\gamma > 2/3$ and $\Omega \geq 0$ the phase space contains two fixed points $\Omega = (0, 1)$. The stability of these points is shown in figure 5.5 . It is clear that near the

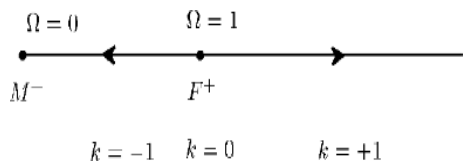

Figure 5.5: The phase space of a single-fluid FL model.

initial singularity the flat FRW model $\Omega = 1$ is unstable, as we shall see later .
In the next chapter we apply the dynamical system technology to study the cosmological dynamic of some $f(R)$ gravity models .

Chapter 6

COSMOLOGICAL DYNAMICS OF EXPONENTIAL GRAVITY

6.1 Introduction

In this chapter a detailed investigation of the cosmological dynamics based on $\exp(-R/\Lambda)$ [114] will be presented. The dynamical system technology will be used to investigate the cosmological dynamic of these models in FLRW Universes using Einstein frame.
Consider the $f(R)$ gravity action,

$$A = \int d^4x\mathcal{L} = \int d^4x\sqrt{-g}\left[e^{-\frac{R}{\Lambda}} + L_M\right] \,, \tag{6.1}$$

where Λ is the cosmological constant, R is the Ricci scalar and L_M is the Lagrangian of standard matter. This Lagrangian possesses some interesting features. Firstly, its treatment is equivalent to one involving a combination of powers of the Ricci scalar and, secondly, it reduces to,

$$\exp\left(-\frac{R}{\Lambda}\right) = 1 - \frac{R}{\Lambda} + O[R^2] \,, \tag{6.2}$$

in the small curvature limit, which is equivalent to the Hilbert-Einstein action. It has been shown in [88] that, in the presence of a barotropic perfect fluid, these solutions are unstable against either homogeneous or inhomogeneous perturbations. Before investigating the cosmological dynamics of $\exp(-R/\Lambda)$ gravity we present a brief review of the dynamics of R^n gravity model.

6.2 Cosmological dynamics of R^n gravity

In this section we focus on the finite analysis of the R^n-gravity model [8], whose action for the gravitational interaction read,

$$S = \int d^4x\sqrt{-g}\Big[\mathcal{X}\,(n)R^n\Big] + S^{matter} \,. \tag{6.3}$$

where $\mathcal{X}(n) > 0$ for all n and reduces to 1 when $n = 1$. Using the metric formalism the field equations for this theory read,

$$nR^{n-1}G_{ab} = \mathcal{X}(n)^{-1}\tilde{T}^{M}_{ab} + \frac{1}{2}g_{ab}(1-n)R^{n} + \Big[n(n-1)R^{n-2}R^{;cd} + n(n-1)(n-2)R^{n-3}R^{;c}R^{;d}\Big](g_{ca}g_{db} - g_{cd}g_{ab})\,. \tag{6.4}$$

In the FLRW metric, the above equations can be written as,

$$H^2 + H\frac{\dot{R}}{R}(n-1) - \frac{R}{6n}(1-n) + \frac{K}{a^2} - \frac{\mu}{3n\mathcal{X}(n)R^{n-1}} = 0\,, \tag{6.5}$$

$$2n\frac{\ddot{a}}{a} + n(n-1)H\frac{\dot{R}}{R} + n(n-1)\frac{\dot{R}}{R} + n(n-1)(n-2)\frac{\dot{R}^2}{R^2} - (1-n)\frac{R}{3} + \frac{\mu}{3n\mathcal{X}R^{n-1}}(1+3w) = 0\,, \tag{6.6}$$

together with the conservation equation for matter

$$\dot{\mu} + 3H\mu(1+w) = 0\,. \tag{6.7}$$

Before discussing the general case in which matter is present, we begin with an analysis of the vacuum case.

6.2.1 The vacuum case

The form of equations 6.5, 6.6 suggests the following choice of expansion normalized variables::

$$\begin{aligned} x &= \frac{\dot{R}}{RH}(n-1) \\ y &= \frac{R}{6nH^2}(1-n) \\ \chi &= \frac{K}{a^2H^2}\,. \end{aligned} \tag{6.8}$$

The variable x is associated with the rate of variation of the Ricci curvature, y represents a measure of the expansion normalized Ricci curvature and χ the spatial curvature of the Friedmann model. If we consider the Ricci curvature as a scalar field, we can think of x as the kinetic term for this field and y as its potential. However, this analogy does not work completely, because x appears only linearly in the Friedmann equation. The cosmological equations 6.5, 6.6 are equivalent to the autonomous system,

$$x' = 2 + x - x^2 + \frac{y}{1-n}(2+nx) + (2+x)\chi\,,$$

$$y' = \frac{xy}{n-1} + 2y\Big(\frac{ny}{1-n} + 2 + \chi\Big)\,, \tag{6.9}$$

$$\chi' = 2z(1 + \frac{ny}{1-n} + \chi)\,. \tag{6.10}$$

Table 6.1: Coordinates of the fixed points, eigenvalues, and solutions for R^n-gravity in vacuum.

Point	Coordinates(x,y)	Eigenvalues	Solution
$\mathcal{A}$	$[0,0]$	$[-2,2]$	$a = a_o\, t$
$\mathcal{B}$	$[-1,0]$	$[2, \frac{4n-5}{n-1}]$	$a = a_o(t-t_o)^{1/2}$ for n=3/2
$\mathcal{C}$	$[\frac{2(n-2)}{2n-1}, \frac{4n-5}{2n-1}]$	$[\frac{5-4n}{n-1}, \frac{4n+2-4n^2}{2n^2-3n+1}]$	$a = a_o t^{(1-n)(2n-1)/(n-2)}$
$\mathcal{D}$	$[2(1-), 2(n-1)^2]$	$[n-2+\sqrt{3n(3n-4)}, n-2-\sqrt{3n(3n-4)}]$	$a = a_o\, t$ for $K = 0$
			$a = K\, t/(2n^2-2n-1)$ for $K \neq 0$

and the constraint equation,

$$1 + x - y + \chi = 0\,. \tag{6.11}$$

The prime represents the derivative with respect to the time variable $N = ln\, a$. The constraint equation can be used to reduce the dimensionality of the dynamical system 6.9. It is clear that $y = 0$ is an invariant sub-manifold in the phase space of the system, which means that a general orbit starting at $y \neq 0$ can approach $y = 0$ only asymptotically. The fixed points of this system and their stability can be obtained by a local stability analysis and this is summarized in Table 6.2.1. The fixed point $\mathcal{A}$ is always a saddle. The point $\mathcal{B}$ is an unstable node for $n < 1$ and $n > 5/4$ and a saddle otherwise. The point $\mathcal{C}$ is stable node for $n < (1-\sqrt{3})/2\,, 1/2 < n < 1\,, n > (1+\sqrt{3})/2$, an unstable node for $1 < n < 5/4$ and a saddle otherwise. The fixed point $\mathcal{D}$ is stable node for $(1-\sqrt{3})/2 < n < 0$ and $4/3 < n < (1-\sqrt{3})/2$, saddle for $n < (1-\sqrt{3})/2$ and $n > (1+\sqrt{3})/2$ and an attractor otherwise.
It is useful to define the deceleration parameter q in terms of the dynamical variables,

$$q = -\frac{H'}{H^2} - 1 = -x - \frac{y}{n-1}\,, \tag{6.12}$$

which means, for $q > 0$,

$$n > 1\,, \qquad x(n-1) + y < 0\,.$$

$$n < 1\,, \qquad x(n-1) + y > 0\,.$$

These conditions are never satisfied for the fixed points, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$. For the point $\mathcal{C}$ the constraint on n are ,

$$n > (1+\sqrt{3})/2 \qquad \text{Expansion}\,, \tag{6.13}$$

$$1/2 < n < 1 \qquad \text{Expansion}\,, \tag{6.14}$$

$$n < (1-\sqrt{3})/2 \qquad \text{Contraction}\,. \tag{6.15}$$

6.2.2 The matter case

When matter is present (as a perfect fluid), a new dynamical variable need to be defined to account for this extra component ,

$$\Omega = \frac{\mu}{3n\mathcal{X}H^2R^{n-1}}\,, \tag{6.16}$$

Differentiating 6.8 and 6.16 with respect to the logarithmic time, we obtain the autonomous system ,

$$\begin{aligned} x' &= x - 2x^2 + \frac{y(2+nx)}{1-n} - (2+x)\chi + 2 - \chi(1+3w)\,. \\ y' &= \frac{xy}{n-1} + 2y\Big(\frac{ny}{1-n} + \chi\Big) \\ \Omega' &= -3\Omega(1+w) + 2\,\Omega\Big(2 + \frac{ny}{n-1} + \chi\Big) - x\Omega\,, \\ \chi' &= 2\chi\Big(1 + \frac{ny}{n-1} + \chi\Big)\,, \end{aligned} \tag{6.17}$$

with the constraint ,

$$1 + x - y + \chi - \Omega = 0\,. \tag{6.18}$$

The invariant plane $y = 0, \Omega = 0$ divide the phase space of the system into two disconnected sectors. The coordinates of the fixed points, their stability and the solutions at these points are shown in Tables 6.2.2 and 6.3. The fixed point $\mathcal{A}$ is always a saddle-node .

The fixed point $\mathcal{B}$ is an unstable node for $n < 1$ and $n > 5/4$ if the standard matter is dust or radiation and a saddle otherwise .

The fixed point $\mathcal{C}$ is an unstable node for $1 < n < 5/4$, $w = 0, 1/3$ and for $1 < n < (11+\sqrt{37})/14$ in case of stiff matter, and represent a stable node for $n < (1-\sqrt{3})/2$, $1/2 < n < 1$ and $n > (1+\sqrt{3})/2$, whatever the value of the parameter w. For all the other values of n this point is a saddle-node.

The fixed point $\mathcal{D}$ is a pure attractive node for $(1-\sqrt{3})/2 < n \leq 0$ and $4/3 \leq n < (1+\sqrt{3})/2$,

Table 6.2: Coordinates of the fixed points, eigenvalues, and solutions for R^n-gravity with matter.

Point	Coordinates(x,y,Ω)	Solution
$\mathcal{A}$	$[0,0,0]$	$a = a_o\,(t - t_0)$
$\mathcal{B}$	$[-1,0,0]$	$a = a_o(t - t_o)^{1/2}$ for n=3/2
$\mathcal{C}$	$[\frac{2(n-2)}{2n-1}, \frac{4n-5}{2n-1}, 0]$	$a = a_o t^{(1-n)(2n-1)/(n-2)}$
$\mathcal{D}$	$[2(1-n), 2(n-1)^2, 0]$	$a = a_o\,t$ for $K = 0$
		$a = \frac{kt}{2n^2-2n-1}$ for $K \neq 0$
$\mathcal{E}$	$[-1-3w, 0, -1-3w]$	$a = a_o\,(t - t_0)$
$\mathcal{F}$	$[1-3w, 0, 2-3w]$	$a = a_o\,(t - t_o)^{1/2}$ for $n = 3/2$
$\mathcal{G}$	$[-\frac{3(n-1)(1+w)}{n}, \frac{(n-1)(4n-3(w+3))}{2n^2}, \frac{n(13+9w)-2n^2(4+3w)-3(1+w)}{2n^2}]$	$a = a_o\,t^2 n/(3(1+w))$

for $0 < n < 4/3$ the eigenvalues are complex and an analysis of the real parts show that the focus-nodes are always attractive, and a saddle-node otherwise.

The fixed point $\mathcal{E}$ is a saddle-node whatever the values of n and w.

The fixed point $\mathcal{F}$ is a saddle-node for every value of n if the matter is dust or radiation. In the stiff matter case, this fixed point is an unstable node if $n > 3$.

For the fixed point $\mathcal{G}$ numerical calculations show that there are complex eigenvalues for $0.33 \leq n \leq 0.71$ and $1 \leq n \leq 1.33$ if $w = 0, 0.35 \leq n \leq 1.28$ if $w = 1/3, 0.37 \leq n \leq 1$ and $1.224 \leq n \leq 1.47$ if $w = 1$. For all these values of n, $\mathcal{G}$ behaveslike a saddle focus. For the other values of n this point is always a saddle-node apart in case of stiff matter, for which $1.220 \leq n \leq 1.223$ and $1.47 \leq n \leq 1.5$. In this case we have a pure repulsive node.

Again it is useful to define the deceleration parameter q in terms of the dynamical variables,

$$q = -\frac{H'}{H^2} - 1 = \Omega - x - \frac{y}{n-1}\,, \tag{6.19}$$

which means, for $q > 0$,

$$n > 1\,, \qquad (\Omega - x)(n-1) - y < 0\,.$$

$$n < 1\,, \qquad (\Omega - x)(n-1) - y < 0\,.$$

These conditions are never satisfied for the points $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$. For the point $\mathcal{C}$ the constraint on n are the same as in the vacuum case, and for the point $\mathcal{G}$ we have an accelerated evolution only if $n > 3(1+w)$ (expansion) and $2n < 0$ (contraction).

Table 6.3: Stability of the fixed points for R^n-gravity with matter. We consider here only dust, radiation. The term "spiral^{+}" has been used for pure attractive focus-nodes and the term "saddle-focus" for unstable focus-nodes.

	$n < \frac{1}{2}(1-\sqrt{3})$	$\frac{1}{2}(1-\sqrt{3}) < n < 0$	$0 < n < 1/2$	$1/2 < n < 1$
$\mathcal{A}$	saddle	saddle	saddle	saddle
$\mathcal{B}$	repellor	repellor	repellor	repellor
$\mathcal{C}$	saddle	saddle	saddle	saddle
$\mathcal{D}$	saddle	saddle	saddle	saddle
$\mathcal{E}$	saddle	attractor	spiral	spiral
$\mathcal{F}$	attractor	saddle	saddle	attractor

	$1 < n < 5/4$	$5/4 < n < 4/3$	$4/3 < n < \frac{1}{2}(1+\sqrt{3})$	$n > \frac{1}{2}(1+\sqrt{3})$
$\mathcal{A}$	saddle	saddle	saddle	saddle
$\mathcal{B}$	saddle	repellor	repellor	repellor
$\mathcal{C}$	saddle	saddle	saddle	saddle
$\mathcal{D}$	saddle	saddle	saddle	saddle
$\mathcal{E}$	spiral	spiral	attractor	saddle
$\mathcal{F}$	repellor	saddle	saddle	attractor

$\mathcal{G}$	$n \lesssim 0.33$	$0.33 \lesssim n \lesssim 0.35$	$0.35 \lesssim n \lesssim 0.37$	$0.37 \lesssim n \lesssim 0.71$	$0.71 \lesssim n \lesssim 1$
$w = 0$	saddle	saddle-focus	saddle-focus	saddle-focus	saddle
$w = 1/3$	saddle	saddle	saddle-focus	saddle-focus	saddle-focus

	$1 \lesssim n \lesssim 1.220$	$1.220 \lesssim n \lesssim 1.223$	$1.223 \lesssim n \lesssim 1.224$	$1.224 \lesssim n \lesssim 1.28$
$w = 0$	saddle-focus	saddle-focus	saddle-focus	saddle-focus
$w = 1/3$	saddle-focus	saddle-focus	saddle-focus	saddle-focus

	$1.28 \lesssim n \lesssim 1.32$	$1.32 \lesssim n \lesssim 1.47$	$1.47 \lesssim n \lesssim 1.50$	$n \gtrsim 1.50$
$w = 0$	saddle-focus	saddle	saddle	saddle
$w = 1/3$	saddle	saddle	saddle	saddle

6.3 cosmological dynamics of $\exp(-R/\Lambda)$ gravity

By assuming $f(R) = \exp(-R/\Lambda)$ we obtain the field equations:

$$\begin{aligned} G_{\mu\nu} &= -\Lambda e^{R/\Lambda}\tilde{T}^{M}_{\mu\nu} + \\ &- e^{-R/\Lambda}\left[\frac{1}{2}g_{\mu\nu}(\Lambda + R) + \frac{1}{\Lambda^2}(R^{;\alpha}R^{;\beta} + \Lambda R^{;\alpha\beta})\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}\right)\right]. \end{aligned} \tag{6.20}$$

In the case of the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, the above equations reduce to:

$$\begin{aligned} H^2 + \frac{K}{a^2} - \frac{H}{\Lambda}\dot{R} + \frac{R}{6} + \frac{\Lambda}{6} + \frac{\Lambda\rho}{3e^{-R/\Lambda}} &= 0\,, \\ 2\frac{\ddot{a}}{a} + \frac{R}{3} - \frac{H}{\Lambda}\dot{R} + \frac{1}{\Lambda^2}\dot{R}^2 - \frac{1}{\Lambda}\ddot{R} - \frac{\Lambda\rho}{3e^{-R/\Lambda}}(1+3w) + \frac{\Lambda}{3} &= 0\,, \end{aligned}$$

with,

$$R = -6\left(\frac{\ddot{a}}{a} + H^2 + \frac{K}{a^2}\right)\,, \tag{6.21}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter, a is the usual scale factor, k is the spatial curvature, ρ is the energy density of standard matter and w its barotropic factor. The Bianchi identities applied to the total stress-energy tensor $T^{TOT}_{\mu\nu}$ lead to the energy conservation equation for standard matter:

$$\dot{\rho} + 3H\rho(1+w) = 0\,. \tag{6.22}$$

6.4 The vacuum case

In the case of a vacuum, ($\rho = 0$) equation 6.21 and 6.21 can be written as a closed system of first order differential equations using the dimensionless variables:

$$x = \frac{\dot{R}}{\Lambda H}, \quad y = \frac{R}{6H^2}, \quad z = \frac{\Lambda}{6H^2}, \quad \chi = \frac{K}{a^2H^2}\,. \tag{6.23}$$

Here the variables y and z are a measure of the expansion normalized Ricci curvature and the cosmological constant respectively, K is the spatial curvature parameter of the Friedmann model, while x is a measure of the time rate of the change of the Ricci curvature. The evolution equations for the variables 6.23 are given by

$$\begin{aligned} \frac{dx}{dN} &= \varepsilon(2z + 2\chi - 2) + x\varepsilon(1 + x + y + \chi), \\ \frac{dy}{dN} &= x\varepsilon z + 2y\varepsilon(2 + y + \chi)\,, \\ \frac{dz}{dN} &= 2z\varepsilon(2 + y + \chi)\,, \\ \frac{d\chi}{dN} &= 2\chi\varepsilon(y + 1 + \chi)\,, \end{aligned} \tag{6.24}$$

where $N = |\ln a|$ and $\varepsilon = |H|/H$. This system is completed with the Friedmann constraint,

$$1 + \chi + y + z - x = 0\,, \tag{6.25}$$

which defines a hyperplane in the total phase space of the system. Consequently, all solutions of the dynamical system will be located in a non-compact sub-manifold of the phase space associated with 6.24. The time derivative of 6.25 is nothing other than the Raychaudhuri equation.

6.4.1 Finite analysis

The dimensionality of the state space of the system 6.24 can be reduced by eliminating any one of the four variables using the constraint equation 6.25. If we choose to eliminate x^1, the dynamic equations become:

$$\begin{aligned} \frac{dy}{dN} &= y\varepsilon(4+2\chi+2y+z)+z\varepsilon(1+\chi+z)\,, \\ \frac{dz}{dN} &= 2z\varepsilon(2+\chi+y), \\ \frac{d\chi}{dN} &= 2\chi\varepsilon(1+\chi+y)\,. \end{aligned} \tag{6.26}$$

The system 6.26 admits two invariant sub-manifolds: $z = 0$ and $\chi = 0$. This means that the points contained in these sub-manifolds form an invariant set under the transformation defined by the system 6.26, that is, 6.26 sends points on these sets only to points of the same set. As a consequence, if we choose $z = 0$ ($\chi = 0$) as an initial condition for an orbit, this orbit will never leave the plane $z = 0$ ($\chi = 0$), while for orbits with an initial condition $z \neq 0$ ($\chi \neq 0$), the only way to smoothly approach $z = 0$ ($\chi = 0$) is to approach it asymptotically. This implies that no orbit crosses the $z = 0$ plane and consequently no global attractor can exist, because the phase space is divided into independent sectors which contain complete cosmological histories.
Setting $\chi' = 0$, $y' = 0$, $z' = 0$, we obtain four fixed points that we label with capital letters and the subscript v to indicate those which are found for the vacuum case (see Table 6.4).
We can obtain exact cosmological solutions at these points by using the Raychaudhuri equation,

$$\dot{H} = -(y+\chi+2)H^2\,. \tag{6.27}$$

In fact, at any fixed point with $y + \chi + 2 \neq 0$, the equation 6.27 reduces to

$$\dot{H} = -\frac{1}{\alpha}H^2\,, \qquad \alpha = (y_* + \chi_* + 2)^{-1}\,, \tag{6.28}$$

where the subscript "$*$" indicates that a quantity has been calculated at the fixed point. Equation 6.27 applies to both the matter and vacuum cases and describes a general power law evolution

[1] Of course we could have chosen to eliminate any other variable of the system; our choice is motivated by the fact that the equation for x is by far the most complicated one to solve and that, with this choice, the number of invariant sub-manifolds is maximized. As we will see, this will assist in the investigation of the properties of the cosmology, particularly in the matter case

of the scale factor. In addition, integrating with respect to time we obtain

$$a = a_o(t - t_o)^{\alpha} \ . \quad \alpha \neq 0 \ . \tag{6.29}$$

This means that by finding the value of α at a given fixed point, we can obtain the solutions associated with it using equation 6.27 and this solution will be given by 6.29 for $\alpha \neq 0$.

In this way, points $\mathcal{A}_v$ and $\mathcal{B}_v$ are found to represent Milne and power-law evolutions respectively (see Table 6.4). However, by direct substitution into the cosmological equations it can be shown that these fixed points cannot be considered physical because, in order to satisfy 6.21, one needs to violate the weak energy condition $\rho \geq 0$. This does not constitute a problem because, as we will see below, these points are always unstable, which means that we can choose initial conditions as close to these points as we wish.

For the points $\mathcal{C}_v$ and $\mathcal{D}_v$ we have $\alpha = 0$ so that 6.27 reduces to $\dot{H} = 0$ and the scale factor is given by

$$a = a_o e^{\gamma(t - t_o)}. \tag{6.30}$$

The value of the constant γ can be obtained by direct substitution into equations 6.21. For both $\mathcal{C}_v$ and $\mathcal{D}_v$ we obtain

$$\gamma = \pm\sqrt{\frac{\Lambda}{6}} \ , \tag{6.31}$$

so that they represent an exponential evolution. The contracting or expanding nature of this solution depends on the direction of approach of the orbits with respect to the hypersurface $y + \chi + 2 = 0$. This hypersurface divides the phase space into two hypervolumes characterized by a contracting or expanding evolution. In particular, for $y < -\chi - 2$ the orbits describe a contracting Universe, while for $y > -\chi - 2$ they represent an expanding one.

The stability of the hyperbolic fixed points $\mathcal{A}_v$, $\mathcal{B}_v$ and $\mathcal{D}_v$ is obtained by using the Hartman-Grobman theorem. The point $\mathcal{C}_v$, instead, is non-hyperbolic and must use the local center manifold theorem in order to find its stability. In our case, using the transformation

$$\begin{aligned} y &= u_1 - 2u_2 - m \ , \\ z &= 4m \ , \\ k &= u_2 \end{aligned} \tag{6.32}$$

the system 6.26 can be written in the form

$$\dot{u_1} \;=\; -4u_1\varepsilon + \varepsilon(12m^2 + 2mu_1 + 2u_1^2 - 4mu_2 - 2u_1)\,, \tag{6.33}$$

$$\dot{u_2} \;=\; -2u_2\varepsilon + \varepsilon(-2mu_2 + 2u_1u_2 - 2u_2^2)\,, \tag{6.34}$$

$$\dot{m} \;=\; \varepsilon(-2m^2 + 2mu_1 - 2mu_2)\,, \tag{6.35}$$

where

$$F_{s1}(u_1, u_2, m) = \varepsilon(12m^2 + 2mu_1 + 2u_1^2 - 4mu_2 - 2u_1)\,, \tag{6.36}$$

$$F_{s2}(u_1, u_2, m) = \varepsilon(-2mu_2 + 2u_1u_2 - 2u_2^2)\,, \tag{6.37}$$

$$F_m(u_1, u_2, m) = \varepsilon(-2m^2 + 2mu_1 - 2mu_2)\,. \tag{6.38}$$

represent the non-linearterms. By substituting the expansions

$$h1(m) \;= am^2 + bm^3 + O(m^4)\,, \tag{6.39}$$

$$h2(m) \;= cm^2 + dm^3 + O(m^4) \tag{6.40}$$

into equations 2.23 and 2.24 and then solving for the coefficients a, b, c and d, we obtain

$$h1(m) = 3m^2 + \frac{9}{2}m^3 + O(m^4), \qquad h2(m) = O(m^4)\,. \tag{6.41}$$

Substituting this result into equation 6.35 then yields

$$\dot{m} = -2m^2 + O(m^3) \tag{6.42}$$

on the center manifold $W^c(\mathbf{0})$, around the point $\mathcal{C}_v$. This implies that the point $\mathcal{C}_v$ is a saddle-node, that is, it behaves like a saddle or an attractor depending on the direction from which the orbit approaches. The local phase portrait in the neighborhood of $\mathcal{C}_v$ is depicted in figure 6.1 .. If one considers the transformation (6.32) now, one realizes that $m \propto z$, so that $\mathcal{C}_v$ is an attractor for $z > 0$ and a saddle for $z < 0$. This is also clear from Figure 6.2 in which the invariant sub-manifold $K = 0$ is depicted .

Finally, it is useful to derive an expression for the deceleration parameter q in terms of the dynamical variables:

$$q = -\frac{\dot{H}}{H^2} - 1 = -(y + \chi + 1)\,. \tag{6.43}$$

This equation holds for both the vacuum and the matter case. Note that that $q > 0$ is realized only when $(y + \chi + 1) < 0$. This condition is satisfied only for the point $\mathcal{C}_v$ as is expected by looking at the solution associated with this fixed point (see Table 6.4). In Figure 6.3 we give the location of the $q = 0$ plane relative to the fixed points $\mathcal{A}_v$, $\mathcal{C}_v$ and $\mathcal{B}_v$.

Table 6.4: Coordinates of the fixed points, eigenvalues, stability and solutions for $\exp(-\frac{R}{\Lambda})$-gravity in vacuum.

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Point	Coordinates(y,z,χ)	Eigenvalues	Stability	Solution
$\mathcal{A}_v$	$[0, 0, 0]$	$[2, 4, 4]$	repeller	$a = a_o(t - t_o)^{\frac{1}{2}}$
$\mathcal{B}_v$	$[0, 0, -1]$	$[-2, 2, 2]$	Saddle	$a = a_o(t - t_o)$
$\mathcal{C}_v$	$[-2, 0, 0]$	$[-4, -2, 0]$	Saddle-node	$a = a_o e^{\gamma(t-t_o)}$
$\mathcal{D}_v$	$[-2, 1, 0]$	$[-\frac{(3+\sqrt{17})}{2}, -2, \frac{(3+\sqrt{17})}{2}]$	Saddle	$a = a_o e^{\gamma(t-t_o)}$

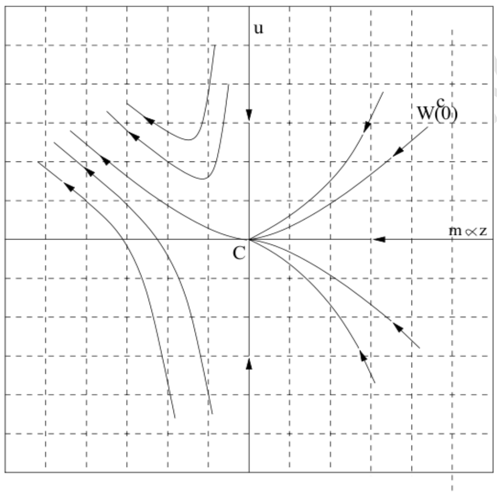

Figure 6.1: The phase portrait for the system 6.26 in the neighborhood of the fixed point $\mathcal{C}$ for $\exp(-R/\Lambda)$-gravity in vacuum.

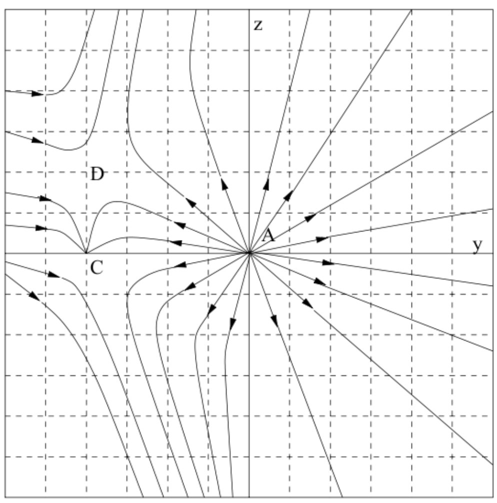

Figure 6.2: The invariant sub-manifold $K = 0$ for $\exp(-R/\Lambda)$-gravity in vacuum.

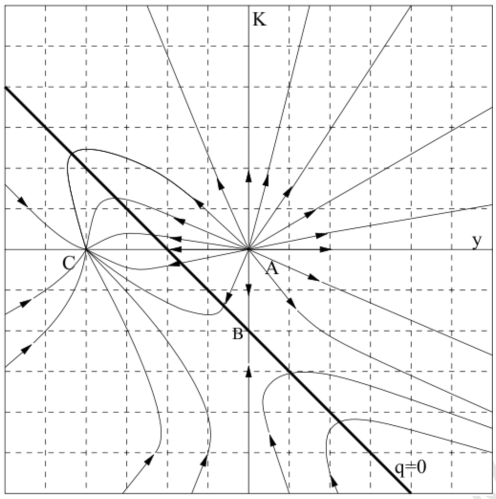

Figure 6.3: The invariant sub-manifold $z = 0$. We explicitly indicate the location of the $q = 0$ plane relative to the fixed points $\mathcal{A}_v$, $\mathcal{C}_v$ and $\mathcal{B}_v$ for $\exp(-R/\Lambda)$-gravity in vacuum.

6.4.2 Asymptotic analysis

In this section we will determine the fixed points at infinity and study their stability. In order to simplify the asymptotic analysis we will compactify the phase space using the Poincaré method. Transforming to polar coordinates (r,θ,ϕ):

$$z \to r\ \cos\theta,\ \ K \to r\ \sin\theta\ \cos\phi,\ \ y \to r\ \sin\theta\ \sin\phi$$

and substituting $r \to \frac{\mathcal{R}}{1-\mathcal{R}}$, the regime $r \to \infty$ corresponds to $\mathcal{R} \to 1$. Using this coordinate transformation and taking the limit $\mathcal{R} \to 1$, the system 6.26 can be written as

$$\begin{aligned} \mathcal{R}' \to \quad & \frac{1}{4}\left\{8\cos\phi\,\sin^3\theta\ - \sin\phi\,[-7\sin\theta\ + \sin 3\theta]\right. \\ + \quad & \left. 8A\cos^2\theta\,\sin\theta + 4A\cos\theta\,\sin^2\theta\,\sin\phi\right\}\ , \end{aligned} \tag{6.44}$$

$$\mathcal{R}\theta' \to -\frac{\varepsilon\cos^2\theta\,\sin\phi\,\sin\theta}{\mathcal{R}-1}\,(A + \cot\theta)\ , \tag{6.45}$$

$$\mathcal{R}\phi' \to \frac{\varepsilon\cos\phi\,\cot\theta}{\mathcal{R}-1}\,(A + \cot\theta)\ , \tag{6.46}$$

where $A = \cos\phi + \sin\phi$. Since equation 6.44 does not depend on the coordinate $\mathcal{R}$, we can find the fixed points of the above system using equations 6.45 and 6.46 only. From equations (6.45) and (6.46) if,

$$A + \cot\theta = 0\,, \tag{6.47}$$

then $\theta' = \phi' = 0$, which means that $\mathcal{O}_v^\infty = [-\text{arccot}\, A, \phi]$ represent a fixed sub-space. The other fixed sub-space is given by $\mathcal{I}_v^\infty = [\pi/2, \phi]$, and there are no single fixed points (see Table 6.5).

Let us now derive the solution for $\mathcal{I}_v^\infty$. In this sub-space the first equation of the system 6.26 reduces to

$$\frac{dy}{dN} = 2\epsilon y^2(\cot\phi_0 + 1)\,, \quad \text{where} \quad \cot\phi_0 = K/y\,, \tag{6.48}$$

and equation 6.27 becomes

$$\dot{H} = -y(\cot\phi_0 + 1)H^2\,. \tag{6.49}$$

Integrating equation 6.48 we obtain

$$y = \frac{-1}{2\epsilon(1 + \cot\phi_0)(N - N_\infty)}\,, \tag{6.50}$$

where N_∞ is an integration constant. Substituting y back into equation 6.49 and solving for N, we obtain

$$N - N_\infty = [\frac{1}{2\epsilon}(c_1 \pm c_o(t - t_o))]^2\,. \tag{6.51}$$

The same procedure can be employed to obtain solutions for $\mathcal{O}_v^\infty$ (see Table 6.5 for the result). Using the 6.44 to take into account the radial behaviour of the orbits, the stability of $\mathcal{I}_v^\infty$ is:

$-\pi/4 < \phi < \pi/2$	Stable,
$\pi/2 < \phi < 3\pi/4$	Unstable,
$3\pi/4 < \phi < 3\pi/2$	Stable,
$3\pi/2 < \phi < 7\pi/4$	Unstable.

We obtain the stability of $\mathcal{O}_v^\infty$ in the same way from which it turns out that these points are never stable. For the values of ϕ for which $L_1(\phi)$ and $L_2(\phi)$ are both positive, the points in $\mathcal{O}_v^\infty$ are repellers, while for the values of ϕ for which these functions have opposite signs, they are saddles.

In the next section we will observe how the introduction of matter modifies the picture we obtained in the vacuum case.

6.5 The matter case

In this case we can utilise the same dynamical variables we used for the vacuum case together with one additional variable D, related to the matter energy density:

$$x = \frac{\dot{R}}{\Lambda H}, \quad y = \frac{R}{6H^2}, \quad z = \frac{\Lambda}{6H^2}, \quad \chi = \frac{K}{a^2H^2}, \quad \Omega = \frac{\Lambda\rho}{3H^2 e^{R/\Lambda}}\,. \tag{6.52}$$

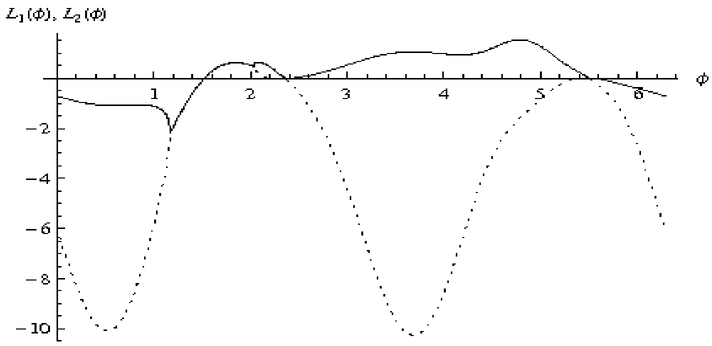

Figure 6.4: The graph of the eigenvalues of the fixed space as a function of ϕ. Here $L_1(\phi)$ is represented by a solid curve and $L_2(\phi)$ is the dashed one.

Table 6.5: Coordinates, eigenvalues and the stability of the fixed points in the asymptotic regime for $\exp(-\frac{R}{\Lambda})$ gravity in vacuum. L_1 and L_2 are functions of ϕ which are too complicated to be recorded here (see Figure 6.4 for their plots), $E = [2 + 2\cot\phi_0 + \cot\theta \text{cosec}\, \phi_0 + A_0 \,\text{cosec}\, \phi_0^2(1 + A_0^{-1}\cos\phi_0)]$ and A_0 is the value of A in ϕ_0

Point (θ,ϕ)	Eigenvalues	Solution
$\mathcal{I}_v^\infty$ $[\frac{\pi}{2},\phi]$	$[0, -A\cos\phi]$	$(N - N_\infty) = [\frac{1}{2\epsilon}(c_1 \pm c_0(t_1 - t_0))]^2$
$\mathcal{O}_v^\infty$ $[-\text{arccot}\, A,\phi]$	$[L_1(\phi) < 0\ \ \forall\phi,\ \ L_2(\phi) < 0\ \ \forall\phi]$	$(N - N_\infty) = [\frac{E-1}{\epsilon E}(c_1 \pm c_0(t_1 - t_0))]^{\frac{E}{E-1}}$
	$[L_1(\phi) > 0\ \ \forall\phi,\ \ L_2(\phi) > 0\ \ \forall\phi]$	
	$[L_1(\phi) > 0\ \ \forall\phi,\ \ L_2(\phi) < 0\ \ \forall\phi]$	

The definition of the variables reveals that not all of the phase space corresponds to physical situations. This becomes clear if we divide D by z. We obtain

$$\frac{D}{z} = 2\,\rho\,\exp\left(-\frac{R}{\Lambda}\right)\ , \tag{6.53}$$

which has the same sign as ρ. This means that the sectors in the phase space for which the sign of D is different from the sign of z contain orbits in which standard matter violates the weak energy condition $\rho > 0$, and have to be discarded as not physical. As we will observe, this implies restrictions on the cosmic evolution scenarios allowed in this model. Following the same procedure we used in the vacuum case, we obtain the autonomous system:

$$\begin{aligned}
\frac{dx}{dN} &= \varepsilon(2+2z+2\chi) + x\,\varepsilon(1-x+y+\chi) - \Omega\,\varepsilon(1+3w)\;, \\
\frac{dy}{dN} &= xz\varepsilon \; + 2y\,\varepsilon(2+y+\chi)\;, \\
\frac{dz}{dN} &= 2z\,\varepsilon(2+y+\chi)\;, \\
\frac{d\chi}{dN} &= 2\chi\,\varepsilon(y+1+\chi)\;, \\
\frac{d\Omega}{dN} &= \Omega\,\varepsilon(1-3w+2y+2\chi-x)\;,
\end{aligned} \tag{6.54}$$

together with the constraint equation

$$1+\chi+x+y-z-\Omega = 0\;, \tag{6.55}$$

where $N = |\ln a|$.

6.5.1 Finite analysis

The system 6.54 can be further simplified using the constraint equation 6.55 to eliminate x. We obtain:

$$\begin{aligned}
\frac{dy}{dN} &= y\varepsilon(4+2\chi+2y+z) + z\varepsilon(1+\chi+\Omega+z)\,, \\
\frac{d\chi}{dN} &= 2\chi\varepsilon(1+\chi+y)\;, \\
\frac{dz}{dN} &= 2z\varepsilon(2+y+\chi)\;, \\
\frac{d\Omega}{dN} &= \Omega\varepsilon(2-3w+3\chi+\Omega+3y+z)\,.
\end{aligned} \tag{6.56}$$

The structure of 6.56 reveals that in this case we have three invariant sub-manifolds: $\Omega = 0$, $z = 0$ and $\Omega = 0$; hence also in this case no global attractor can exist. Setting $\chi' = 0$, $y' = 0$, $z' = 0$ and $\Omega' = 0$ we obtain seven fixed points (see Table 6.6).

As in the vacuum case, we can use the coordinates of these fixed points and equation 6.27 to find the behaviour of the scale factor at these points. In addition, the behaviour of the energy density ρ can be obtained from equation 6.22, which at a fixed point reads

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\alpha}{t}\;, \tag{6.57}$$

where α is defined by 6.28. However, direct substitution in the cosmological equations reveals that all the fixed points correspond to vacuum states.

Table 6.6: Coordinates of the fixed points, the eigenvalues, and solutions for $\exp(-R/\Lambda)$-gravity in the matter case.

Point	Coordinates(y,z,χ,Ω)	Eigenvalues	Solution
$\mathcal{A}_m$	$[0,0,0,0]$	$[2-3w,2,4,4]$	$a=a_o(t-t_o)^{\frac{1}{2}}$
$\mathcal{B}_m$	$[0,0,-1,0]$	$[2,2,-2,-(1+3w)]$	$a=a_o(t-t_o)$
$\mathcal{C}_m$	$[0,0,0,3w-2]$	$[3w-2,2,4,4]$	$a=a_o(t-t_o)^{\frac{1}{2}}$
$\mathcal{D}_m$	$[0,0,-1,3w+1]$	$[2,2,-2,(1+3w)]$	$a=a_o(t-t_o)$
$\mathcal{E}_m$	$[-2,1,0,0]$	$[-\frac{\sqrt{17}+3}{2},\frac{\sqrt{17}-3}{2},-2,-3-3w]$	$a=a_oe^{\gamma(t-t_o)}$
$\mathcal{F}_m$	$[-2,0,0,0]$	$[-2,-4,-3w-4,0]$	$a=a_oe^{\gamma(t-t_o)}$
$\mathcal{G}_m$	$[-2,0,0,3w+4]$	$[3w+4,-2,-4,0]$	$a=a_oe^{\gamma(t-t_o)}$

The exact solutions at the fixed points are summarized in Table 6.6. As in the vacuum case, we use the Hartman-Grobman theorem together with the center manifold theorem to analyze the stability of all the fixed points. The results appear in Table 6.7 .

6.5.2 Asymptotic analysis

We complete the analysis for the matter case by investigating the asymptotic behaviour of the system 6.56. In order to achieve this we compactify the phase space by transforming it to 4-D polar coordinates. The transformation equations are ,

$$\Omega\to r\ \cos\delta\,,\quad z\to r\ \sin\delta\ \ \cos\theta,\chi\quad\to r\ \sin\delta\ \ \sin\theta\ \ \cos\phi,$$
$$y\to r\ \sin\theta\ \ \sin\delta\ \ \sin\phi\ ,$$

where $r\in[0,\infty[$, $\delta\in[0,\pi]$, $\theta\in[0,\pi]$, and $\phi\in[0,2\pi]$. We then transform the radial coordinate $r\to\frac{\mathcal{R}}{1-\mathcal{R}}$ and in the limit $\mathcal{R}\to 1$, the system 6.56 reduces to

$$\begin{aligned}\mathcal{R}'\to\cos^3\delta\quad &+\quad \cos\delta\,\cos\theta\,\sin^2\delta\,\sin\theta\,\sin\phi\,+\cos^2\delta\,\sin\delta\,(\cos\theta\\ &+\quad 3A\sin\theta)+\sin^3\delta\,\sin\theta\left[\cos\phi\,(2+B)\right.\\ &+\quad \left.\sin\phi\,(3\cos^2\theta\,+2\sin^2\theta\,+B)\right]\,,\end{aligned}\tag{6.58}$$

$$\begin{aligned}\mathcal{R}\delta'\to\frac{\sin\delta\,\cos\delta}{8(\mathcal{R}-1)}\quad &\{8\cos\delta\,(B-1)-\sin\delta\,[\cos 3\theta+8\cos\phi\,\sin\theta\\ &+8\sin^3\theta\,\sin\phi\,+7\cos\theta\,\,+4\cos\theta\,\sin^2\theta\,(\cos 2\phi\\ &-\sin 2\phi)]\}\,\,,\end{aligned}\tag{6.59}$$

Table 6.7: Stability of the fixed points for $\exp(-R\Lambda)$-gravity in the matter case.

Point	$w=0$	$0<w<\frac{1}{3}$	$w=\frac{1}{3}$
$\mathcal{A}_m$	Repeller	Repeller	Repeller
$\mathcal{B}_m$	Saddle	Saddle	Saddle
$\mathcal{C}_m$	Saddle	Saddle	Saddle
$\mathcal{D}_m$	Saddle	Saddle	Saddle
$\mathcal{E}_m$	Saddle	Saddle	Saddle
$\mathcal{F}_m$	Saddle-node	Saddle-node	Saddle-node
$\mathcal{G}_m$	Saddle-node	Saddle-node	Saddle-node
Point	$\frac{1}{3}<w<\frac{2}{3}$	$w=\frac{2}{3}$	$\frac{2}{3}<w<1$
$\mathcal{A}_m$	Repeller	Saddle-node	Saddle
$\mathcal{B}_m$	Saddle	Saddle	Saddle
$\mathcal{C}_m$	Saddle	Saddle-node	Repeller
$\mathcal{D}_m$	Saddle	Saddle	Saddle
$\mathcal{E}_m$	Saddle	Saddle	Saddle
$\mathcal{F}_m$	Saddle-node	Saddle-node	Saddle-node
$\mathcal{G}_m$	Saddle-node	Saddle-node	Saddle-node

$$\mathcal{R}\theta' \to \frac{\cos^2\theta}{2(\mathcal{R}-1)} \{2\cos\delta\,\sin\phi + \sin\delta\,[2\cos\theta\,\sin\phi + \sin\theta\,(1-\cos 2\phi + \sin 2\phi)]\}\ , \tag{6.60}$$

$$\mathcal{R}\phi' \to \frac{\cos\phi\,\{\cos\delta\,\cot\theta + \cos\theta\,\sin\delta\,[A+\cot\theta]\}}{\mathcal{R}-1}\ , \tag{6.61}$$

where $B = \cos\theta\,\sin\theta\,\sin\phi$. Notice that the first equation of the previous system does not depend on $\mathcal{R}$, which means that the fixed points of this system can be determined by the angular equations alone. As in the vacuum case, there are no isolated fixed points (see Table 6.8).

The solutions at the fixed points can be obtained by following the same procedure we used in the vacuum case.

Taking into account the radial behaviour of the orbits we can deduce the stability of the first two fixed sub-spaces. We have that $\mathcal{A}_m^\infty$, and $\mathcal{B}_m^\infty$ attractors for $0<\theta<3\pi/4$ and $0<\theta<\pi/4$ respectively. The fixed sub-spaces $\mathcal{C}_m^\infty$ and $\mathcal{D}_m^\infty$ are unstable for all ϕ (see Figure 6.7). The stability analysis of the sub-space $\mathcal{E}_m^\infty$ is complicated by the fact that the eigenvalues are all zero. This means that the system for these points is not structurally stable or, in other words, their stability is determined mainly by the non-linear terms. One can gain an idea of the stability of $\mathcal{E}_m^\infty$ by studying second order small perturbations around this fixed sub-space. To the second order the evolution equations for the perturbations $(\bar{\delta}, \bar{\theta}, \bar{\phi}, \bar{\mathcal{R}})$ around $\mathcal{E}_m^\infty$ are,

Table 6.8: Coordinates, eigenvalues, and the solutions for fixed points in the asymptotic regime for the $\exp(-R/\Lambda)$ gravity in matter case. Here f_1 and f_2 are functions of θ while g is a function of ϕ (see Figures 6.5 and 6.6), $S = [2 + \cot\theta + \cot^2\theta(1 + \cot\delta\sec\theta)]$ and $\tilde{S} = [-2 + \cot\theta - \cot^2\theta(1 + \cot\delta\sec\theta)]$

Point (δ, θ, ϕ)	Eigenvalues	Solution
$\mathcal{A}_m^\infty$ $[\,\mathrm{arccot}(-\sin\theta - \cos\theta), \theta, \frac{\pi}{2}]$	$[0, 0, f_1(\theta)]$	$(N - N_\infty) = [\frac{S-1}{\epsilon S}(c_1 \pm c_0(t_1 - t_0))]^{\frac{s}{s-1}}$
$\mathcal{B}_m^\infty$ $[\,\mathrm{arccot}(\sin\theta - \cos\theta), \theta, \frac{3\pi}{2}]$	$[0, 0, f_2(\theta)]$	$(N - N_\infty) = [\frac{\tilde{S}-1}{\epsilon\tilde{S}}(c_1 \pm c_0(t_1 - t_0))]^{\frac{\tilde{s}}{\tilde{s}-1}}$
$\mathcal{C}_m^\infty$ $[\,\mathrm{arccot}(-A), \frac{\pi}{2}, \phi]$	$[0, 0, g(\phi) > 0 \;\; \forall\phi]$	$(N - N_\infty) = [\frac{1}{2\epsilon}(c_1 \pm c_o(t - t_o))]^2$
$\mathcal{D}_m^\infty$ $[\,\mathrm{arccot}(A), \frac{3\pi}{2}, \phi]$	$[0, 0, g(\phi) > 0 \;\; \forall\phi]$	$(N - N_\infty) = [\frac{1}{2\epsilon}(c_1 \pm c_o(t - t_o))]^2$
$\mathcal{E}_m^\infty$ $[\,\mathrm{arccot}(-\cos\theta), \theta, \frac{3\pi}{4}]$	$[0, 0, 0]$	$a = ce^{\gamma(t-t_0)}$

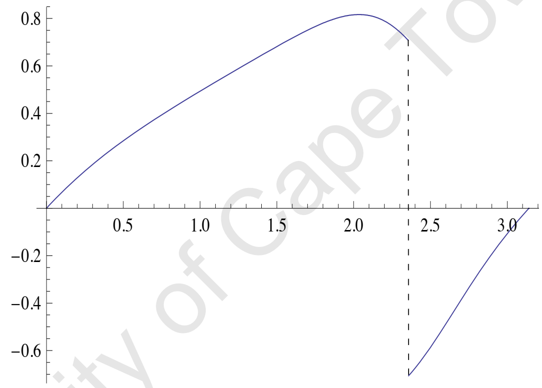

Figure 6.5: The graph of the function $f_1(\theta)$.

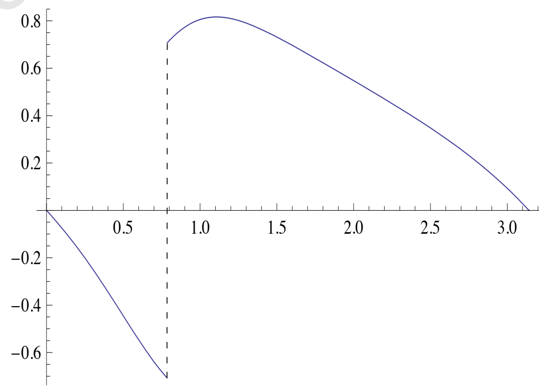

Figure 6.6: The graph of the function $f_2(\theta)$.

$$
\begin{aligned}
\bar{\delta}' &= \varepsilon(a_1\,\bar{\delta} + a_2\,\bar{\delta}^2 + a_3\,\bar{\delta}\,\bar{\theta} + a_4\,\bar{\delta}\,\bar{\phi} + a_5\,\bar{\theta} + a_6\,\bar{\theta}^2 + a_7\,\bar{\phi}^2 + a_8\,\bar{\phi} + a_9\,\bar{\phi}\bar{\theta})\,, \\
\bar{\theta}' &= \varepsilon(b_1\,\bar{\theta} + b_2\,\bar{\theta}/,\bar{\phi} + b_3\,\bar{\delta} + b_4\,\bar{\delta}\,\bar{\phi} + b_5\,\bar{\delta}\,\bar{\theta} + b_6\,\bar{\phi} + b_7\,\bar{\theta}^2 + b_8)\,\bar{\phi}^2\,, \qquad (6.62) \\
\bar{\phi}' &= \varepsilon(c_1\,\bar{\delta} + c_2\,\bar{\delta}\,\bar{\phi} + c_3\,\bar{\delta}\,\bar{\theta} + c_4\,\bar{\theta}\,\bar{\phi} + c_5\,\bar{\theta} + c_6\,\bar{\phi}^2 + c_7\,\bar{\phi} + c_8\,\bar{\theta}^2)\,, \\
\bar{\mathcal{R}}' &= \varepsilon(d_1\bar{\delta}^2 + \bar{\theta}d_3\bar{\delta} + \bar{\phi}d_7\bar{\delta} + \bar{\theta}^2 d_5 + \bar{\phi}d_6 + \bar{\theta}\bar{\phi}d_8 + \bar{\phi}^2 d_9 + \bar{\delta}d_2 + \bar{\theta}d_4)\,,
\end{aligned}
$$

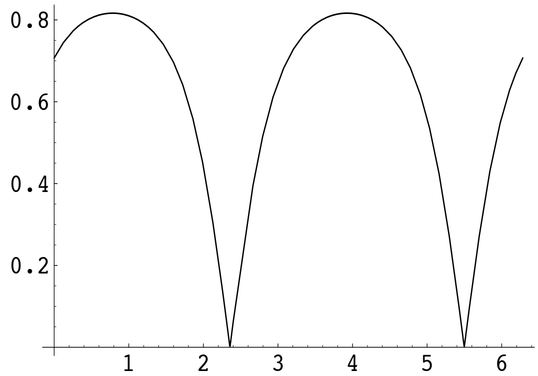

Figure 6.7: The graph of the function $g(\phi)$.

where

$$a_1 = -\frac{(3+\cos 2\theta_0)\sec\theta_0^2(-4+\sqrt{2}\sin 2\theta_0)}{8(1+\sec\theta_0^2)^{\frac{3}{2}}},$$

$$a_2 = -(\frac{(3+\cos 2\theta_0)\sec\theta_0(-4+\sqrt{2}\sin 2\theta_0)\tan\theta_0^2}{8(1+\sec\theta_0^2)^{\frac{3}{2}}}),$$

$$a_3 = \frac{-\sqrt{2}\cos 2\theta_0 + (-2+\sec\theta_0^2)\tan\theta_0}{(1+\sec\theta_0^2)^{\frac{3}{2}}},$$

$$a_4 = \frac{(4\cos 2\theta_0 - \sqrt{2}(\sin 2\theta_0 - 4\tan\theta_0))\tan\theta_0^2}{4(1+\sec\theta_0^2)^{\frac{3}{2}}},$$

$$a_5 = \frac{(2\sec\theta_0 - \sqrt{2}\sin\theta_0)\tan\theta_0}{2(1+\sec\theta_0^2)^{\frac{3}{2}}},$$

$$a_6 = \frac{-5\sqrt{2}\sin\theta_0 + 2\sec\theta_0(1+\sqrt{2}\tan\theta_0)}{4(1+\sec\theta_0^2)^{\frac{3}{2}}},$$

$$a_7 = \frac{\sin\theta_0\tan\theta_0}{(1+\sec\theta_0^2)^{\frac{3}{2}}}, \qquad a_8 = \frac{\sqrt{2}\sec\theta_0 - \sin\theta_0\tan\theta_0}{(1+\sec\theta_0^2)^{\frac{3}{2}}},$$

$$a_9 = -\frac{\sqrt{2}\cos\theta_0 - 6\sin\theta_0 + \sec\theta_0(3\sqrt{2}2\tan\theta_0)}{2(1+\sec\theta_0^2)^{\frac{3}{2}}},$$

$$b_1 = \frac{\sin\theta_0\cos\theta_0}{\sqrt{2}(\sqrt{1+\sec\theta_0^2})}, \qquad b_2 = -\frac{2-6\cos 2\theta_0 + \sqrt{2}\sin 2\theta_0}{4(\sqrt{1+\sec\theta_0^2})},$$

$$b_3 = \frac{\cos\theta_0 + \cos\theta_0^3}{\sqrt{2}(\sqrt{1+\sec\theta_0^2})}, \qquad b_4 = -\frac{\cos\theta_0(\sqrt{2}+\sqrt{2}\cos\theta_0^2+\sin 2\theta_0)}{2(\sqrt{1+\sec\theta_0^2})},$$

$$b_5 = -\frac{(2+3\cos\theta_0^2)\sin\theta_0}{\sqrt{2}(\sqrt{1+\sec\theta_0^2})}, \qquad b_6 = \frac{\cos\theta_0\sin\theta_0}{\sqrt{1+\sec\theta_0^2}},$$

$$b_7 = \frac{-3+5\cos 2\theta_0}{4\sqrt{2}(\sqrt{1+\sec\theta_0^2})}, \qquad b_8 = -\frac{\cos\theta_0\sin\theta_0}{\sqrt{1+\sec\theta_0^2}},$$

$$
\begin{aligned}
c_1 &= \frac{-2\sec\theta_0+\sin\theta_0}{\sqrt{2}(\sqrt{1+\sec\theta_0^2})}\,, \qquad c_2 = \frac{2\cos\theta_0+\sqrt{2}(-2\csc\theta_0+\sin\theta_0)}{2(\sqrt{1+\sec\theta_0^2})}\,,\\
c_3 &= \frac{\cos\theta_0+2\cot\theta_0\csc\theta_0+\sec\theta_0}{\sqrt{2}(\sqrt{1+\sec\theta_0^2})}\,,\\
c_4 &= \frac{2(\frac{\cos\theta_0+\sec\theta_0}{\sqrt{2}}+\sqrt{2}\cot\theta_0\csc\theta_0-\sin\theta_0)-\sqrt{2}-2\tan\theta_0}{\sqrt{1+\sec\theta_0^2)}}\,,\\
c_5 &= -\frac{1}{\sqrt{2}(\sqrt{1+\sec\theta_0^2})}\,, \qquad c_6 = -\frac{1}{\sqrt{1+\sec\theta_0^2}}\,,\\
c_7 &= -\frac{1}{\sqrt{1+\sec\theta_0^2}}\,, \qquad c_8 = \frac{\cot\theta_0+2\tan\theta_0}{2\sqrt{2}\sqrt{1+\sec\theta_0^2}}\,,\\
d_1 &= -\frac{(\cos(2\theta_0)+3)\sec^2(\theta_0)\left(\sqrt{2}\sin(2\theta_0)-4\right)}{4\left(\sec^2(\theta_0)+1\right)^{3/2}}\,,\\
d_2 &= \frac{(\cos(2\theta_0)+3)\sec(\theta_0)\left(\sqrt{2}\tan(\theta_0)+2\right)}{4\left(\sec^2(\theta_0)+1\right)^{3/2}}\,,\\
d_3 &= -\frac{\sqrt{2}\sec^3(\theta_0)+2\left(\sqrt{2}-2\tan(\theta_0)\right)\sec(\theta_0)-5\sqrt{2}\cos(\theta_0)+2\sin(\theta_0)}{2\left(\sec^2(\theta_0)+1\right)^{3/2}}\,,\\
d_4 &= \frac{\tan(\theta_0)\left(\sqrt{2}\tan(\theta_0)+2\right)}{2\left(\sec^2(\theta_0)+1\right)^{3/2}}\,,\\
d_5 &= \frac{\sqrt{2}\left(5-2\sec^2(\theta_0)\right)\tan(\theta_0)+2}{4\left(\sec^2(\theta_0)+1\right)^{3/2}}\,,\\
d_6 &= \frac{\sec^2(\theta_0)\left(3\sqrt{2}\cos(2\theta_0)+\sin(2\theta_0)+7\sqrt{2}\right)\tan(\theta_0)}{2\left(\sec^2(\theta_0)+1\right)^{3/2}}\,,\\
d_7 &= -\frac{\sec(\theta_0)\left(7\sqrt{2}\cos(2\theta_0)+6\sin(2\theta_0)+9\sqrt{2}\right)\tan(\theta_0)}{4\left(\sec^2(\theta_0)+1\right)^{3/2}}\,,\\
d_8 &= \frac{\left(3\sqrt{2}-2\tan(\theta_0)\right)\sec^2(\theta_0)+6\tan(\theta_0)+7\sqrt{2}}{2\left(\sec^2(\theta_0)+1\right)^{3/2}}\,,\\
d_9 &= -\frac{\tan^2(\theta_0)}{\left(\sec^2(\theta_0)+1\right)^{3/2}}\,.
\end{aligned}
$$

The system above is very difficult to solve exactly, so we will limit ourselves to plotting the solutions for specific values of θ and to deduce the stability from them. Figure 6.8 shows the solutions of the perturbed system 6.62 for specific values of θ (i.e. $\theta = \pi/6, \pi/4, \pi/3, 3\pi/4$ and $5\pi/6$); it is clear from the graphs that all the solutions are unstable.

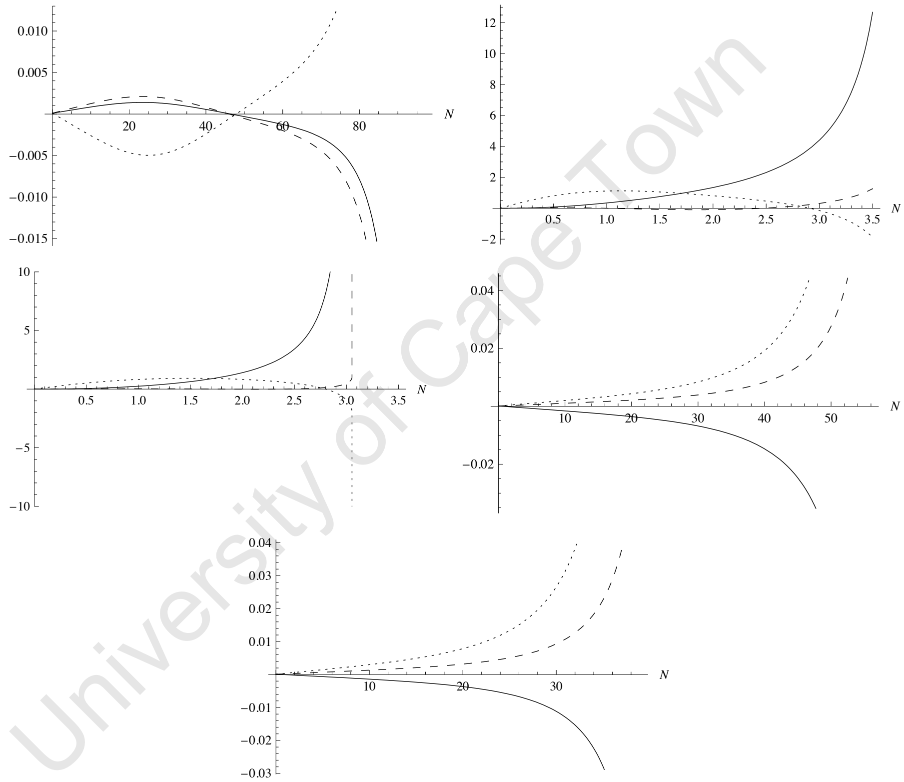

Figure 6.8: The solutions for $\bar{\delta}$, $\bar{\theta}$ and $\bar{\phi}$ of the system 6.62 for $\theta = \pi/56, \pi/4, \pi/3, 3\pi/4$ and $5\pi/6$ respectively. The solid, dashed and dotted lines represent $\bar{\delta}$, $\bar{\theta}$ and $\bar{\phi}$ respectively.

6.6 Discussion

We have applied the dynamical systems approach to the exponential gravity cosmological model, and found exact solutions together with their stability for both the vacuum and matter cases.
In the vacuum case, we identified four finite critical points $\mathcal{A}_v$, $\mathcal{B}_v$, $\mathcal{C}_v$ and $\mathcal{D}_v$, of which only two $\mathcal{C}_v$ and $\mathcal{D}_v$ are found to be physical. These last points were established to represent a solution whose nature depends on the parameter $\gamma(\Lambda)$; for $\Lambda > 0$ we can have either exponential expansion ($\gamma > 0$) or exponential contraction ($\gamma < 0$), while for $\Lambda < 0$ the solution oscillates.
From the stability point of view, the point $\mathcal{C}_v$, which resides in the invariant sub-manifold $z = 0$, is of particular interest because, since it is non-hyperbolic, it represents an attractor for $z > 0$ and a saddle for $z < 0$, while the other physical point $\mathcal{D}_v$ is found to be a saddle.
On the other hand, the solution connected with the non-physical points $\mathcal{A}_v$ and $\mathcal{B}_v$ is found to correspond to power law evolution and is also interesting because the orbits can approach them arbitrarily closely.
The invariant sub-manifold $z = 0$ divides the phase space into two regions, $z > 0$ and $z < 0$ which correspond to $\Lambda > 0$ and $\Lambda < 0$ respectively. The fact that no orbit can cross the plane $z = 0$ is then consistent with the fact that Λ must always have the same sign during a cosmic history.
In the vacuum case, we established that the region $z < 0$ does not contain any finite critical point. Thus, the only attractors in the region $z < 0$ are asymptotic, which means that all the models that begin their evolution in this region will re-collapse. However, in the plane $z = 0$, the physical point $\mathcal{C}_v$ represents a saddle and the non-physical points $\mathcal{A}_v$ and $\mathcal{B}_v$ are a repeller and saddle point respectively. This means that during the evolution towards the asymptotic attractors a transient de-Sitter or a power law phase(s) might be present, depending on the initial conditions.
From a physical point of view, the region $z > 0$ appears to be more interesting because in this region the point $\mathcal{C}_v$ represents a de-Sitter attractor which might be associated with a dark energy/inflation era. The same region also contains the point $\mathcal{D}_v$, which represents an unstable de Sitter phase (see Figure 6.1). This implies that the subset of the orbits which converge to $\mathcal{C}_v$ can also contain cosmic histories that present a second, unstable, de Sitter phase. In addition, orbits that evolve near the non-physical point $\mathcal{B}_v$ can also present an intermediate power law phase.
Finally, it is apparent from Figure 6.3 that the de Sitter phases $\mathcal{C}_v$ and $\mathcal{D}_v$ are separated from the past attractor $\mathcal{A}_v$ by the plane $q = 0$; therefore any model with initial conditions near the past attractor $\mathcal{A}_v$ and evolving toward the future de-Sitter attractor $\mathcal{C}_v$ will cross the plane $q = 0$,

indicating a transition from an accelerating evolution to a decelerating one.

In the asymptotic regime, no isolated fixed point was found, but only two fixed sub-spaces: $\mathcal{I}_v^\infty$ and $\mathcal{O}_v^\infty$. The stability analysis reveals that the only asymptotic attractors are in $\mathcal{I}_v^\infty$. This means that all the models that evolve towards an asymptotic attractor are bound to re-collapse.

The introduction of matter into this model increases the dimensionality of the phase space, making it more difficult to visualize. By a direct substitution into the field equations we found that all the fixed points in the matter case do not represent physical solutions. This means that the series expansion of the action function $exp(-R/\Lambda)$ should be truncated at some point before matter can be treated within the higher order gravity scheme.

In the finite regime (see Table 6.7), we determined that the four points $\mathcal{A}_m$, $\mathcal{B}_m$,$\mathcal{F}_m$ and $\mathcal{E}_m$ are generalizations of the vacuum fixed points $\mathcal{A}_v$, $\mathcal{B}_v$, $\mathcal{C}_v$ and $\mathcal{D}_v$ respectively and present the same solutions.

The points $\mathcal{A}_m$ and $\mathcal{B}_m$ are found to represent Milne solutions while $\mathcal{C}_m$ and $\mathcal{D}_m$ represent a power law evolution. For points $\mathcal{E}_m$, $\mathcal{F}_m$ and $\mathcal{G}_m$ we established that $\dot{H} = 0$, which means that these points represent Einstein-de Sitter solutions.

In the asymptotic regime we identified three different classes of solutions (see Table 6.8). The first class contains the fixed points $\mathcal{A}_m^\infty$ and $\mathcal{B}_m^\infty$, and the solutions at these points depends on the value of $S(\bar{\delta},\theta)$ and $\tilde{S}(\bar{\delta},\theta)$. The solutions at these points represent an expansion if $S, \tilde{S} > 1$, a contraction if $S, \tilde{S} < 1$. These fixed sub-spaces contain attractive parts (specifically $0 < \theta < 3\pi/4$ for $\mathcal{A}_m^\infty$ and $0 < \theta < \pi/4$ for $\mathcal{B}_m^\infty$).

The second class of solutions contains the two fixed points $\mathcal{C}_m^\infty$ and $\mathcal{D}_m^\infty$. These sub-spaces represent a recollapsing evolution, but are always unstable.

The third class contains a single fixed sub-space $\mathcal{E}_m^\infty$ and the solution at this point depends on the parameter γ. By substituting this solution into the field equations 6.21, and by ignoring the subdominant terms we obtain

$$\gamma^2 = 0\,, \tag{6.63}$$

$$\gamma^2\left[-1+\frac{144}{\Lambda}\left(\frac{k}{a^2}\right)^2+24\left(\frac{k}{a^2}\right)\right]+3\left(\frac{k}{a^2}\right)=0\,. \tag{6.64}$$

It is clear that the only consistent solution is $\gamma = 0$, when the spatial part of the spacetime is flat $\chi = 0$. The stability of the points in this sub-space cannot be performed in general because of their non-hyperbolic character. We limited ourselves to investigate a few specific cases analyzing the evolution of the second order perturbations around a general point of $\mathcal{E}_m^\infty$. These points seem to be always unstable. Such a behaviour is interesting because it points towards the presence of bounce solutions for this cosmological model.

In conclusion, $\exp(-\frac{R}{\Lambda})$ gravity possesses a very rich structure that includes a series of diverse and interesting cosmological histories. Particularly important are the ones including multiple de Sitter phases because they could provide us with natural models describing the early and late time acceleration of the Universe. Unfortunately, as is clear from Figure 6.2, this scenario does not include a decelerated expansion phase between these two de Sitter phases. This implies that these cosmic histories will not, in general, admit a standard structure formation scenario (as it was claimed using a different argument in [182]). On the other hand, the evolution of scalar perturbations in fourth order gravity does not necessarily need the presence of a decelerated expansion phase. Only a detailed numerical analysis of these specific orbits (and on the perturbation evolution along them) will be able to clarify this matter.

Using Hubble normalized variables we can not study bouncing, recollapsing or static behaviours, that is because the dynamical variables diverge when $H = 0$. In the next chapter we develop an alternative scheme to deal with this issue. We then use this framework to study the behavior of $R + \alpha R^n$ gravity model.

Chapter 7

COSMOLOGICAL DYNAMICS OF FOURTH ORDER GRAVITY: A COMPACT VIEW

7.1 Introduction

In this chapter we develop an alternative scheme which involves compactifying the phase space for general $f(R)$ [124], [125], [126] theories of gravity, subject to certain conditions on the function f. We then use this framework to study the behavior of the phase space of Universes with a non-negative Ricci scalar in $R + \alpha R^n$ gravity. We find a number of interesting cosmological evolutions, which include the possibility, at least in principle, where the Universe begins close to at an unstable power-law inflationary equilibrium point, then evolves towards a curvature fluid-dominated phase where the effective equation of state mimics standard radiation with $w \sim 1/3$ (we will refer to such phases as radiation-like), then passes through a standard matter CDM era and ultimately evolves asymptotically towards a de-Sitter-like late-time accelerated phase.

We also show that as $n \to 0$, all the fixed points that approach the ΛCDM subspace of the complete state space of $R + \alpha R^n$ gravity, are unstable. This implies that the behavior of the solutions of a fourth order theory which is close to ΛCDM may be completely different from those of ΛCDM and one needs to do a careful analysis of the solutions rather than a prior assuming any global behavior of the trajectories.

7.2 Compactification of Friedmann-Lemaitre models

This section is a continuation of the analysis presented in section 4.5. We study the evolution of spatially homogeneous cosmological models with a positive cosmological constant [210]. Equations 7.10 and 7.11 (14-15) are the starting point of a dynamical analysis of this cosmological model. Let us define the dimensionless parameters

$$\Omega_k = \frac{K}{3H^2} \qquad \Omega = \frac{\rho}{3H^2} \qquad \Omega_\Lambda = \frac{\Lambda}{3H^2} \,. \tag{7.1}$$

In what follows, $H \neq 0$ is assumed. Using these variables, equation 7.11 can be written as a system of first order differential equations

$$\begin{aligned} \Omega_k' &= ([3\gamma-2]\Omega - 2\Omega_\Lambda)\Omega_k \,, \\ \Omega_\Lambda' &= 2(1 + \frac{3\gamma-2}{2}\Omega - \Omega_\Lambda)\Omega_\Lambda \,, \\ \Omega' &= ([3\gamma-2](\Omega-1) - 2\Omega_\Lambda)\Omega \,, \end{aligned} \tag{7.2}$$

where the prime represents the derivative with respect to the time variable $' = H^{-1}d/dt$. This system is completed by the Friedmann constraint

$$\Omega + \Omega_\Lambda - \Omega_k = 1 \,. \tag{7.3}$$

By considering a weak energy condition and assuming$\Lambda \geq 0$, the dynamical variables proposed in 7.1 can be compactified to the range [0,1]. Using the Friedmann constraint the system 7.2 can be reduced to a two-dimensional system,

$$K' = ([3\gamma-2]\Omega - 2\Omega_\Lambda)\Omega_k \,, \tag{7.4}$$

$$\Omega_\Lambda' = 2(1 + \frac{3\gamma-2}{2}\Omega - \Omega_\Lambda)\Omega_\Lambda \,. \tag{7.5}$$

Equilibrium point solutions, and the stability analysis of this system, are shown in Table 7.1. In

Table 7.1: Coordinates of the fixed points, eigenvalues, solutions and the stability.

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Fixed points	Eigenvalues	Solution	Stability	
$(\Omega_k, \Omega_\Lambda)$			$0 < \gamma < 2/3$	$2/3 < \gamma < 2$
$[0, 0]$	$[3\gamma - 2, 3\gamma]$	F	Saddle	Reppler
$[-1, 0]$	$[-(3\gamma - 2), 2]$	M	Reppler	Saddle
$[0, 1]$	$[-2, -3\gamma]$	ds	Attractor	Attractor

order to compactify the region$\Omega_{\ k} > 0$ of the phase space we will begin by writing 7.10 as

$$\rho = 3H^2 + \frac{3K}{a^2} - \Lambda\,, \tag{7.6}$$

Let us define the compact variables

$$Q = \frac{H}{D} \qquad \tilde{\Omega_\Lambda} = \frac{\Lambda}{3D^2}\,, \tag{7.7}$$

where $D = \sqrt{H^2 + k/a^2}$. Then by using the new time variable $' = D^{-1}d/dt$, the evolution equations for Q and $\tilde{\Omega_\Lambda}$ are given by

$$Q' = (1 - \frac{3\gamma}{2}[1 - \tilde{\Omega_\Lambda}])(1 - Q^2)\,, \tag{7.8}$$

$$\tilde{\Omega_\Lambda} = 3\gamma(1 - \tilde{\Omega_\Lambda})Q\,\tilde{\Omega_\Lambda}\,. \tag{7.9}$$

The equilibrium points, and the solutions of this system, are depicted in Table 7.2 . The compact space for all Friedmann models with $2/3 < \gamma < 2$ is illustrated in figure 7.2 . The left and right

Table 7.2: Coordinates of the fixed points, solutions and the stability.

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Coordinates$(Q, \tilde{\Omega_\Lambda})$	Solution	Stability
$[1, 0]$	+F	Reppler
$[-1, 0]$	-F	Attractor
$[1, 1]$	+ds	Attractor
$[-1, 1]$	- ds	Reppler
$[0, \frac{3\gamma-2}{3\gamma}]$	E	Saddle

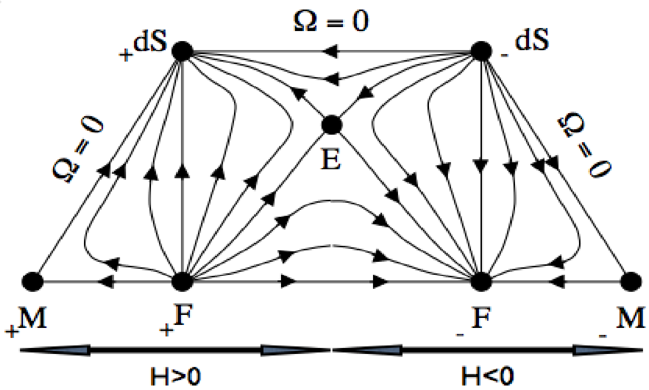


Figure 7.1: The phase space for Friedmann-Lemaitre models with $-1/3 < w < 1$.

sides of the phase space correspond to expanding and contracting models respectively. In the

$\Omega_k < 0$ and $H > 0$ region orbits with $\Omega > 0$ and $\Lambda > 0$ evolve from F to ds. For $\Omega_k > 0$ and $H > 0$ there are three possibilities:

- If Λ is sufficiently small, we obtain Friedmann-Lemaitre models,
- If Λ is large, we obtain Lemaitre models,
- For $\Lambda = \frac{3\gamma-2}{\gamma}\frac{K}{a_0^2}$, we obtain the Einstein static Universe.

There are also models which start with $H < 0$ and cross to $H > 0$. The models which passed asymptotic to the Einstein static Universe are called Eddington-Lemaitre models. The straight line $\Omega = 0$ from $-ds$ to $+ds$ corresponds to a de-Sitter Universes.

7.2.1 The Einstein static Universe

For the FRW metric background, the Einstein field equation 2.109 takes the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{7.10}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3P) + \frac{\Lambda}{3}. \tag{7.11}$$

Equation 7.10 can be written in a more compact form as

$$\dot{a}^2 + V(a) = -k, \tag{7.12}$$

where

$$V(a) = \frac{\rho_c}{3}a^{-(1+3\omega)} - \frac{\Lambda}{3}a^2, \tag{7.13}$$

This equation is very useful in studying the dynamic stability of the Universe. For the Universe to be static $\ddot{a} = \dot{a} = 0$, this condition is satisfied when

$$\frac{dV(a)}{da} = 0. \tag{7.14}$$

Thus the radius of the Einstein Static Universe is consequently given by

$$a_0^{3\gamma} = \frac{1}{2}(3\gamma - 2)\frac{\rho_c}{\Lambda}, \tag{7.15}$$

where $\gamma = \omega + 1$. Figure 7.2 shows the graph of the V(a) a function of a, It is clear from figure 7.2 that the Einstein Static Universe, which take place at (v_0, a_0) is not stable, so that a small change in the energy density of the Universe will result in a fast collapse or infinite expansion. If the initial conditions are chosen in such a manner that the Universe starts in region I, then there will be three possible evolutions.

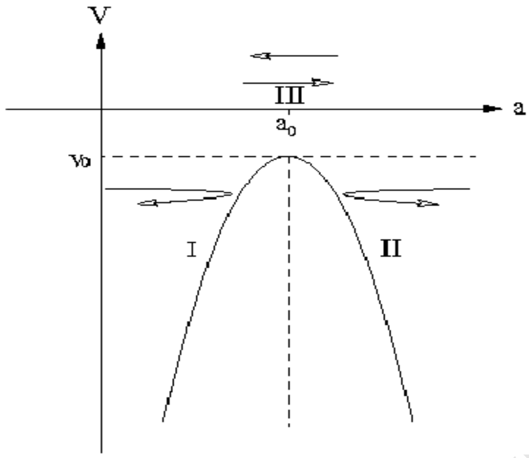


Figure 7.2: The graph of $V(a)$ a function of a.

- $a(t)$ expands up to a maximum radius a_0 and then re-collapses,
- $a(t)$ expands up to a maximum radius a_0 with $q < 0$ and then expands with $q > 0$,
- $a(t)$ expands to a_0 in an infinite time.

If the initial conditions are chosen in such a manner that the Universe starts in region II then,

- $a(t)$ contracts to a minimum radius a_0 and then expands,
- $a(t)$ contracts to a minimum radius a_0 with $q < 0$ and then contracts to zero with $q > 0$,
- $a(t)$ contracts to a_0 in an infinite time.

Figure 7.3 shows schematic diagrams of all these cases.

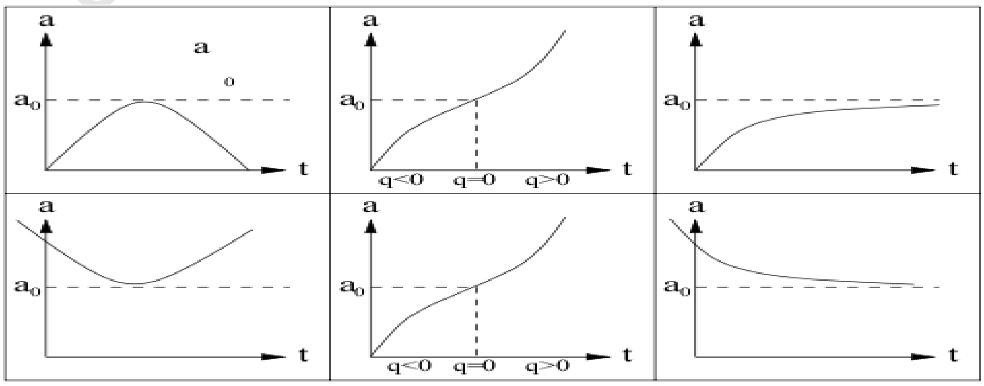


Figure 7.3: The evolution of the scale factor for Friedmann-Lemaitre models.

In region III, depending on the initial conditions, the Universe can either expand to infinity or contract to zero.

7.3 Compact phase space for positive Ricci scalar Universe

In this section we will study the dynamics of Friedmann - Lemaître - Robertson - Walker (FLRW) models only in the sector $R \geq 0$. This is because the sector $R < 0$ is not of much physical interest and also, as we shall see later, the sectors $R > 0$ and $R < 0$ are connected by the invariant sub-manifold $R = 0$, making the physically interesting dynamics completely confined to the sector $R > 0$. Also, we consider the 3-curvature to be vanishing, which is an invariant sub-manifold by itself. As required by the *no-ghost condition* we also assume $f' > 0$. To compactify the phase space we rewrite the Friedmann equation (8.2) in the following form:

$$D^2 = \frac{3\rho}{f'} + \frac{3}{2}R + \frac{9}{4}(\frac{(\dot{f'})}{f'})^2 \,, \tag{7.16}$$

where

$$D = \sqrt{\left(\Theta + \frac{3}{2}\frac{(\dot{f'})}{f'}\right)^2 + \frac{3}{2}\frac{f}{f'}} \,. \tag{7.17}$$

Where the variable Θ is defined in chapter 2 section 2.2. We can now define the following set of normalized variables:

$$x = \frac{3}{2}\frac{(\dot{f'})}{f'D} \qquad y = \frac{3}{2}\frac{f}{f'D^2} \qquad \Omega_m = \frac{3\rho}{f'D^2} \qquad z = \frac{3}{2}\frac{R}{D^2} \qquad Q = \frac{\Theta}{D} \,. \tag{7.18}$$

To guarantee that the propagation equations for these compact variables will result in a dimensionless dynamical system, we need to define a new time variable τ, such that,

$$\frac{d}{d\tau} \equiv' = \frac{1}{D}\frac{d}{dt} \,. \tag{7.19}$$

For τ to be a monotonously increasing time variable, a normalization D is chosen such that it is strictly positive at all times. It is clear by construction that when $\Theta = 0$ the normalized dynamical variables as well as the time variable are well defined, thus this normalization allows the study of general static, re-collapsing and bouncing solutions. From the Friedmann equation we obtain the following constraints,

$$\begin{aligned} \Omega_m + z + x^2 &= 1 \,, \\ (Q + x)^2 + y &= 1. \end{aligned} \tag{7.20}$$

The first constraint comes directly from Friedmann equation, while the second one arises from the definition of the normalization parameter D. According to these constraints and considering

$R>0$, $\rho>0$ and $f'>0$, we see that the above dynamical variables have to be defined in the following ranges,

$$0\leq\Omega_m\leq 1\,,\qquad 0\leq z\leq 1\,,\qquad -1\leq x\leq 1\qquad -2\leq Q\leq 2;\qquad 0\leq y\leq 1\,,\tag{7.21}$$

making the complete phase space compact. Also since the variable Q is a normalized Hubble parameter, the cosmological solutions will naturally include both expanding and collapsing as well as static solutions and these two sets of solutions are connected via the non-invariant subset $Q=0$.

7.3.1 The propagation equations

An autonomous system, which is equivalent to cosmological equations (8.2-8.4) can be derived by differentiating the compact variables (7.18), with respect to τ and using (8.2-8.4). The dimensionality of the resultant system can then be reduced by using the two constraints (7.20). By eliminating the dynamical variables$\Omega_{\,m}$ and y, we obtain the following 3-dimensional effective autonomous system:

$$\begin{aligned}
x' &= \frac{1}{6}\Big(-3(1+\omega)-(1+3\omega)x^4-4Q^2(-1+x^2)+(1+3\omega)z\\
&- Qx\Big[(5+3\omega)(-1+x^2)+3(1+\omega)z\Big]+x^2\Big[4+6\omega+z(-3(1+\omega)-2\Gamma)\Big]\Big)\\
z' &= \frac{z}{3}\Big(-4Q^2x-Q((5+3\omega)x^2+3(1+\omega)(-1+z))\\
&+ x(5+3\omega-(1+3\omega)x^2-\Gamma\Big[-2-3n(1+\omega)+2z+3n(1+\omega)(z+(Q+x)^2)\Big]\Big)\\
Q' &= \frac{1}{6}\Big(-4Q^2x-Q\Big[(5+3\omega)x^2+3(1+\omega)(-1+z)\Big]\\
&+ x(5+3\omega-(1+3\omega)x^2-\Gamma\Big[-2-3n(1+\omega)+2z+3n(1+\omega)((Q+x)^2+z)\Big]\Big)\,,
\end{aligned}\tag{7.22}$$

where$\Gamma\;\equiv f'/Rf''$. In general, the system is not closed unless Γ is expressed in terms of the dynamical variables (7.18). For example, in the case of $R+\alpha R^n$, we have,

$$\Gamma\equiv-\frac{z}{n(y-z)}=\frac{z}{n[(q+x)^2+z-1]}\,.\tag{7.23}$$

Thus, the above system defines the dynamics of all well defined $f(R)$ theories for which f'/Rf'' is invertible in terms of the dynamical variables. From equations (7.22) we can see that $z=0$ is an invariant sub-manifold. and in the $z=0$ 2-surface the line $Q=0$ is an invariant subset. Since $z=0$ corresponds to $R=0$, we obtain an important result: *For all well-defined functions $f(R)$, with $f'>0$ and f'/Rf'' invertible in terms of the dynamical variables defined by (7.18), a FLRW Universes with non-negative Ricci Scalar, continues to be so, both in the future and in* *the past. Also an $R = 0$ Universe can never undergo a bounce in the future or the past.* In the next section we will fix the function f to be the class of theories $f(R) = R + \alpha R^n$ and study the dynamics of the flat FLRW Universes and their stability for those theories. In order to study the stability of the fixed points of the dynamical systems (7.22), we will use the very well-known techniques, which involve linearizing the dynamical equations around the equilibrium points and then finding the eigenvalues of the linearization matrix (the Jacobian) at the equilibrium points. If the Jacobian is well-defined, then they can be classified according to the sign of the real part of eigenvalues as attractors, repellers and saddle points .

7.3.2 The fixed points of $R + \alpha R^n$-gravity

As we have seen from the equation (7.23), f'/Rf'' is invertible in terms of the dynamical variables for $f(R) = R + \alpha R^n$. It is interesting to note that the constant 'n' couples to the dynamical equations (7.22) only via the quantity Γ and the constant α does not couple to the equations at all. Hence all the fixed point of the system are necessarily independent of α. The coordinates of the fixed points are shown in Table 7.3. Note that each fixed point has an expanding ($Q > 0$) and a collapsing ($Q < 0$) version as indicated by the subscripts $(+; -)$ respectively. Also some points only occur in the compact state space defined by (7.21) for certain ranges of n. The occurrence of the fixed points outside the compact region for specific n and ω means that the constraints (7.20) are not satisfied and consequently these fixed points are not physical for these values of n and ω). Fixed points that are not physical for these values of n and ω have been excluded from the analysis . By looking at the coordinates of the fixed points in Table 7.3, we can distinguish two classes; the first corresponds to points with coordinates that are independent of n, which means that these points are common to all $f(R)$ theories. This class contains the fixed points $\mathcal{A}_{\pm}, \mathcal{B}, \mathcal{C}_{\pm}, \mathcal{D}_{\pm}, \mathcal{E}_{\pm}$ and $\mathcal{N}_{\pm}$ and they all lie on the boundary of the compact region except for the point $\mathcal{N}$. In the non - compact analysis developed in [123], non of these boundary points appear. Furthermore, even though $\mathcal{N}_{\pm}$ is not a boundary point, it does not appear in [123], because of its special location in the phase space -: it lies exactly on the intersection of the plane $x = 0$ and the surface $z = y = 1 - (Q + x)^2$. In this case one has to take the limit of Γ carefully as one approaches this point and the standard techniques of finding fixed points breaks down for this case. The other class contains fixed points with coordinates that depend on n and ω; this class contains the three points $\mathcal{L}_{\pm}, \mathcal{I}_{\pm}$ and $\mathcal{F}_{\pm}$. $\mathcal{F}_{\pm}$ is the only boundary point and it lies in the invariant sub-manifold $z = 0$. The expanding versions of the points $\mathcal{L}_{\pm}$ and $\mathcal{I}_{\pm}$ correspond to the equally labeled finite points in [123]. The point H in [123] enters the compact sector, which we consider in this work only when $n = (1 + \sqrt{3})/2$ and for this value of n it merges with the

Table 7.3: Coordinates of the equilibrium points for $R + \alpha R^n$-gravity. We will not explicitly state the expressions for s,g1,...,g4 and f1,...,f4, which are rational functions of n and ω, however we give them at the following link [120].

Fixed points	Coordinates (x, Ω, z, Q)	Solution $a(t)$
$\mathcal{A}_{\pm}$	$(1, 0, 0, \pm 2)$	$a_0\sqrt{t - t_0}$
$\mathcal{B}$	$(\pm 1, 0, 0, 0)$	a_0
$\mathcal{C}$	$(-\frac{\sqrt{3 + 12\omega + 9\omega^2}}{1 + 3\omega}, -\frac{2}{1 + 3\omega}, 0, 0)$	a_0
$\mathcal{D}_{\pm}$	$(\frac{1 - 3\omega}{3(\omega - 1)}, -\frac{4(3\omega - 2)}{9(\omega - 1)^2}, 0, \pm\frac{2}{3(\omega - 1)})$	$a_0\sqrt{t - t_0}$
$\mathcal{E}_{\pm}$	$(0, 0, 1, \pm\frac{1}{\sqrt{2}})$	$a_0 e^{Ct}$
$\mathcal{F}_{\pm}$	$(f_1(n,\omega), g_1(n,\omega), l_1(n,\omega), n_1(n,\omega))$	$a_0\sqrt{t - t_0}$
$\mathcal{G}_{\pm}$	$(f_2(n,\omega), g_2(n,\omega), l_2(n,\omega), n_2(n,\omega))$	$a_0(t - t_0)^{s(n,\omega)}$
$\mathcal{I}_{\pm}$	$(f_3(n), g_3(n), l_3(n), n_3(n))$	$a_0((n-2)t - t_0)^{\frac{-1+3n-2n^2}{-2+n}}$
$\mathcal{L}_{\pm}$	$(f_4(n,\omega), g_4(n,\omega), l_4(n,\omega), n_4(n,\omega))$	$a_0(3t(1+\omega) - t_0)^{\frac{2n}{3(1+\omega)}}$
$\mathcal{N}_{\pm}$	$(0, \frac{2}{3}, \frac{1}{3}, \pm\frac{\sqrt{6}}{3})$	$a_0(2t - t_0)^{2/3}$

point I. All the other points that appear in the above-mentioned reference do not appear in the sector we are studying in this work.

7.4 The exact solutions

The exact solutions at the fixed points are also summarized in Table 7.3 and the stability analysis for the dust and radiation cases are summarized in Table 7.4. First we discuss the static solutions. From definition (7.22), $Q = 0 \Rightarrow \Theta = 0$, so any fixed point that lies on the surface $Q = 0$ represents a static Universe. By looking at the coordinates of the fixed points in Table 7.3, we can see that the point $\mathcal{B}$ is static for all values of n and ω. The point $\mathcal{I}_{\pm}$ is static only for $n = 1/2$ and $n = 1$ and we find that for these values of n the point $\mathcal{I}_{\pm}$ represent an unstable saddle point. We now proceed to find the exact solutions for the scale factor at the non-static fixed points. The expansion rate and the deceleration parameter $q = -\frac{\ddot{a}a}{\dot{a}^2}$ are related by the Raychaudhuri equation,

$$\dot{\Theta} = -\frac{1}{3}(1 + q)\Theta^2 \, . \tag{7.24}$$

If we know the value of the deceleration parameter q_i at some fixed point i, we can use the above equation to obtain the behavior of the scale factor at that point. When $q_i = -1$ we have de-Sitter solutions ($\Theta =$ *constant*) or static solutions ($\Theta = 0$). For $q_i = 0$ we have a

Table 7.4: The stability of the fixed points for $\omega = 0; 1/3$

points	Physical range		Stability	
	$\omega = 0$	$\omega = \frac{1}{3}$	$\omega = 0$	$\omega = \frac{1}{3}$
$\mathcal{A}_-$	$\forall$n	$\forall$n	Attractor	Attractor
$\mathcal{A}_+$	$\forall$n	$\forall$n	Repeller	Repeller
$\mathcal{B}$	$\forall$n	$\forall$n	Attractor	Attractor
$\mathcal{D}_\pm$	$\forall$n	$\forall$n	Saddle	Saddle
$\mathcal{E}_-$	$\forall$n	$\forall$n	Repeller	Repeller
$\mathcal{E}_+$	$\forall$n	$\forall$n	Attractor for $n \in (0, 2)$	Attractor for $n \in (0, 2)$
$\mathcal{F}_\pm$	$n \in (0, \frac{1}{3} + \frac{\sqrt{57}}{9})$	$n \in (0, \frac{1}{8} + \frac{\sqrt{17}}{8})$	Saddle	Saddle
$\mathcal{I}_-$	$n \in (\frac{1}{2}, 1)$ and $n > 5/4$	$n \in (\frac{1}{2}, 1)$ and $n > 5/4$	Saddle for $n \in (1/2, 1)$	Saddle for $n \in (1/2, 1)$
			Attractor for $n = 5/4$	Attractor for $n = 5/4$
			Saddle for $n \in (5/4, 2)$	Saddle for $n \in (5/4, 2)$
			Attractor for $n > 2$	Attractor for $n > 2$
$\mathcal{I}_+$	$n \in (\frac{1}{2}, 1)$ and $n > 5/4$	$n \in (\frac{1}{2}, 1)$ and $n > 5/4$	Saddle for $n \in (1/2, 1)$	Saddle for $n \in (1/2, 1)$
			Repeller for $n = 5/4$	Repeller for $n = 5/4$
			Saddle for $n \in (5/4, 2)$	Saddle for $n \in (5/4, 2)$
			Repeller for $n > 2$	Repeller for $n > 2$
$\mathcal{L}_\pm$	$n \in (\frac{3}{4}, \frac{4}{7} + \frac{\sqrt{37}}{7})$	$n \in (1, \sqrt{2})$	Saddle	Saddle
$\mathcal{N}_-$	$\forall$n	$\forall$n	Spiral$^+$	Spiral$^+$
$\mathcal{N}_+$	$\forall$n	$\forall$n	Spiral$^-$	Spiral$^-$

Milne evolution and when $-1 < q_0 < 0$; $q_0 > 0$ we have accelerated and decelerated power law behaviors respectively. To obtain the exact solutions for the scale factor $a(t)$ associated with the non - static$\Theta \neq 0$ equilibrium points we need to have an expression for q in term of the compact variables. From the definition q we obtain:

$$q_i = 1 - \frac{z_i}{Q_i^2} \,. \tag{7.25}$$

The non-invariant surface $z_i = Q_i^2$ is the transition surface between accelerated and decelerated expansions phases. By substituting 7.25 in 7.24 we obtain:

$$\dot{\Theta} = -\frac{1}{3}(2 - \frac{z_i}{Q_i^2})\Theta^2 \,, \tag{7.26}$$

where $Q \neq 0$. The evolution of the scale factor can now be given directly by integrating equation 7.26:

$$a(t) = a_0(t - t_0)^{\beta_i} \,, \qquad \text{where} \;\; \beta_i = (2 - \frac{z_i}{Q_i^2}) \,. \tag{7.27}$$

The constants of integration can be obtained by substituting the solutions into the original equations. As explained in [123], these solutions must satisfy all the cosmological equations in order to be considered physical. By looking at Table 7.3 we can distinguish two classes of non-static solutions. The first class $\{\mathcal{A}_\pm, \mathcal{D}_\pm, \mathcal{E}_\pm, \mathcal{F}_\pm$ and $\mathcal{N}_\pm\}$ contain solutions that are independent of n and ω. The fixed point $\mathcal{B}$ represents a static phase as mentioned earlier, the points $\mathcal{A}, \mathcal{D}$ and $\mathcal{F}$ are radiation-like phases. The expanding version of the point $\mathcal{A}$ is a saddle for $z > 0$ and repeller for $z = 0$ and the other two points $\mathcal{D}$ and $\mathcal{F}$ are saddles. The fixed point $\mathcal{N}$ represent a matter phase and the expanding version of this point is a spiral$^-$. The evolution of the scale factor a(t) for the fixed point $\mathcal{L}$ is a function of (n, ω, t) and for fixed point $\mathcal{I}$ is function of (n, t). The dependence of these solutions on n and/or ω provide us with additional degrees of freedom that can lead to interesting cosmological scenarios. When $\omega = 0$, the fixed point $\mathcal{L}$ merges with $\mathcal{N}$ for $n = 1$, and it merges with the point $\mathcal{D}_\pm$ for $n = 3/4$. When $\omega = 1/3$ it merges with $D_\pm$ for $n = 1$, and for $n = 4/3$ it corresponds to the matter point $\mathcal{N}$. In the case $\omega = 0$ or $\omega = 1/3$, we find that for all values n for which this point is physical, the expansion (contraction) is never accelerating. As mentioned earlier, the evolution of the scale factor for the point $\mathcal{I}$ is independent of the equation of state parameter ω. For $n = 5/4$ the fixed point $\mathcal{I}$ merges with $\mathcal{A}$, for $n = 2$ it merges with point $\mathcal{E}$ and for $n = 7/12 + \sqrt{73}/12$ it is a matter point. We also find that for this fixed point the expansion (contraction) is accelerating for $n > 1/2(1 + \sqrt{3})$. The existence of this accelerated phase together with the fact that for $n > 1/2(1 + \sqrt{3})$ the point $\mathcal{L}$ is a matter-like point, leads to the possibility of an extremely interesting cosmological scenario, where it is possible in principle to find an orbit that starts close to the unstable accelerating phase $\mathcal{I}_+$, evolves past the unstable radiation-like point $\mathcal{D}_+$, followed by the unstable matter point $\mathcal{L}_+$ and finally ends up at the de-Sitter attractor $\mathcal{E}_+$. In figures 7.4 and 7.4 we have plotted two interesting orbits. The orbit in figure 7.4 is for $\omega = 0$ and $n = 5/4$. It begins near the radiation-like points $\mathcal{A}_+/\mathcal{I}_+$, passes nearby the radiation-like point $\mathcal{D}_+$, followed by the standard matter point $\mathcal{L}_+$ and ends up at the de-Sitter attractor E_+. The orbit in figure 7.4 is for $\omega = -11/18 + \sqrt{73}/18$ and $n = 7/12 + \sqrt{73}/12$. It begins near the radiation-like point $\mathcal{A}_+$, passes nearby the matter point $\mathcal{I}_+$, then close to the matter point $\mathcal{L}_+$ and ends up at the de Sitter attractor $\mathcal{E}_+$. It is interesting to see that for all fixed points whose x-coordinate goes to zero as $n \to 0$, are unstable. As the ΛCDM subspace lies on the $x = 0$ surface and this surface is not an invariant sub-manifold, this implies that the behavior of the solutions of a fourth order theory which is close to ΛCDM, may be completely different from those of ΛCDM. Furthermore, this suggests that the best fit model to the current observational data within the complete state space of $R + \alpha R^n$, may be given by a non-infinitesimal value of n.

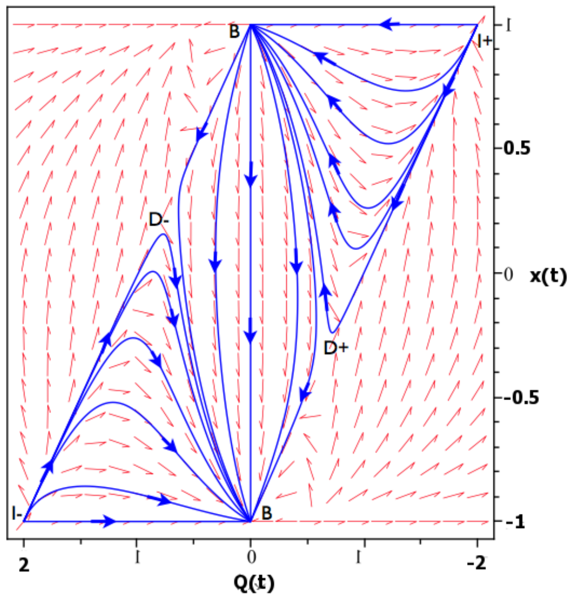


Figure 7.4: plot of the invariant subspace $z = 0$ for $\omega = 0$ and $n = 5/4$. The left half of the state space corresponds to collapsing models, while the right half contain expanding models. This is indicated by the subscripts of the various equilibrium points.

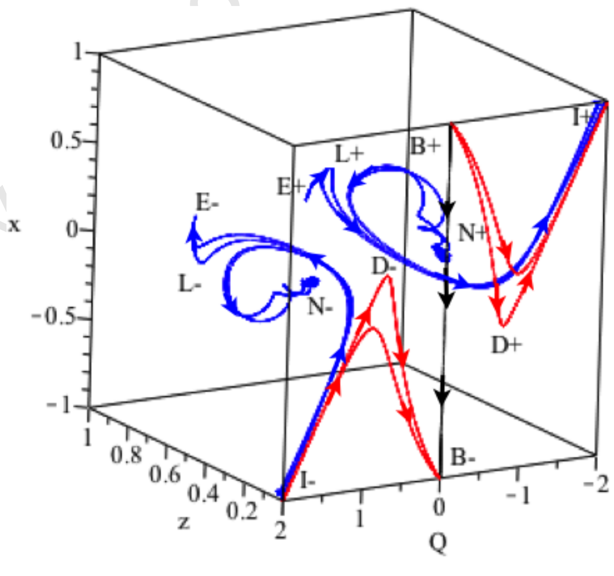


Figure 7.5: For this orbit $\omega = 0$ and $n = 5/4$.

7.5 Conclusion

In this work, we have presented a careful analysis of the state space of the class of $R + \alpha R^n$ theories of gravity, focusing on the $R > 0$ sector with $K = 0$, together with the no-ghost

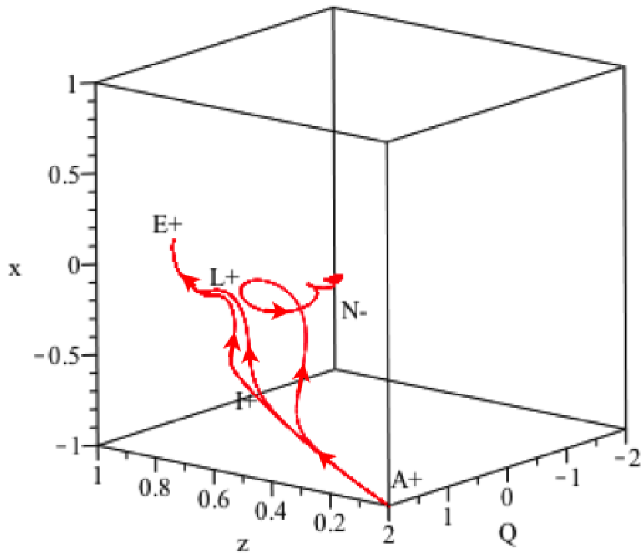


Figure 7.6: For this orbit $\omega = -11/18 + \sqrt{73}/18$ and $n = 7/12 + \sqrt{73}/12$.

condition $f(R); f'(R) > 0$. Due to the complexity of this class of gravity theories, the standard Hubble normalization does not lead to a compact dynamical variables. In order to construct variables defining a compact dynamical system one has to use an appropriate normalization. In this project we used the same formalism used in [124], where we absorbed all the negative contributions of the Friedmann equation into the normalization. First of all we obtained the following important result: For all well defined function $f(R)$, with $f' > 0$ and f'/Rf'' invertible in terms of the dynamical variables defined by 7.18, the FLRW Universes with non-negative Ricci Scalar, continues to be so both in the future and in the past. Also an R = 0 Universe can never undergo a bounce in the future or past. Our compact analysis shows that there are more equilibrium points than in the corresponding non-compact analysis in [123]. In particular we find a new finite fixed point $\mathcal{N}_{\pm}$. Because of its very special location in the phase space, it is quite difficult to obtain this point using the standard techniques. This point is found to represent a matter phase and the expanding version of this point is $spiral^{-}$. Furthermore, we find that for $n > 1/2(1+\sqrt{3})$ the phase space of $R + \alpha R^n$, contains two accelerated fixed points $\mathcal{E}_+; \mathcal{I}_+$, together with two other saddle points (one represent a radiation phase $\mathcal{D}_+$ and the other represent a matter-like phase $\mathcal{L}_+$). Although we have obtained all the desired fixed points and desired stability, this does not necessarily imply that there is an orbit connecting them. Due to the fact that for $n > 1/2(1+\sqrt{3})$, the two accelerated points and the matter-like point are quite close to each other in the phase space, which makes it difficult to prove the existence of an orbit connecting these points together with the radiation-like point. But the presence of all these phases in the state space of $R + \alpha R^n$ makes a more detailed investigation worth pursuing. By a suitable choice of the $f(R)$ gravity action it is possible to construct any background evolution we desire, and at the same time the theory pass solar system test. In order to pursue this program, one needs to compute not only the linear cosmological perturbations and their signature in the matter power spectrum and the CMB anisotropies. In the next chapter we will review the standard cosmological perturbation theory as well as the covariant and gauge invariant approach.

Chapter 8

COVARIANT GAUGE-INVARIANT PERTURBATIONS IN SINGLE AND MULTI-FLUIDS

8.1 Introduction

The 1+3 covariant approach was developed originally to study the evolution of linear perturbations of FRW models in GR with great success [189, 75, 196, 195, 148, 197, 198, 199]. For our purposes this approach has two major advantages. Firstly, a specific recasting of the field equations allows one to easily make extensions to a wide range of modified gravity theories including Braneworlds [200] and FOG [201, 202]. Secondly, the structure of the formalism is such that, unlike other approaches, it is possible to keep track of the physical meaning of the equations at any stage of the calculations, which can be crucial when one modifies the theory of gravity. In particular, in a number of recent papers [201, 202, 204] the evolution equations for scalar and tensor perturbations of a subclass of fourth order theories of gravity characterized by an action which is a general analytic function of the Ricci scalar was presented. The results obtained in [201, 202, 204] clearly demonstrate that the evolution of scalar perturbations is determined by a fourth order differential equation rather than a second order one. This implies that the evolution of the density fluctuations contains, in general, four modes rather than two and can give rise to a more complex evolution than what is obtained in GR. It was also found that the perturbations depend on the scale for any value of the equation of state parameter of standard matter (while in GR the evolution of the dust perturbations are not scale-dependent) and that there is a characteristic scale-dependent signature in the matter power spectrum [204]. This means that in FOG the evolution of super-horizon and sub-horizon perturbations are different. It also turns out that the growth of large density fluctuations can occur also in backgrounds in which the expansion rate is increasing in time. This surprising result is strikingly different with what one finds in GR and could provide a strong constraint on some FOG theories. In addition to that the analysis of scalar perturbations shows a characteristic signature of FOG in the matter power spectrum [202] that could be tested against observations.

8.1.1 The linearized equations

The exact non-linear equations 3.125-3.126 derived in the chapter 3 can be linearized around any chosen background. In what follows we will choose a background that is homogeneous and isotropic,i.e., a FLRW metric model. In the linearization procedure all the inhomogeneous and anisotropic quantities that vanish in this background, e.g. q_a^R and π_{ab}^R, are taken to be of linear order, the Stuart & Walker lemma ensures that since these quantities vanish in the background, they are automatically gauge-invariant. Homogeneity and isotropy imply:

$$\sigma = \omega = 0\,, \qquad \tilde{\nabla}_a \mu = \tilde{\nabla}_a p = 0 \quad \Rightarrow \dot{u} = 0\,. \tag{8.1}$$

In this background the cosmological equations for a generic $f(R)$ read:

$$\Theta^2 = 3\tilde{\mu}^m + 3\mu^R - \frac{3\tilde{R}}{2}\,, \tag{8.2}$$

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \tfrac{1}{2}(\tilde{\mu}^m + 3\tilde{p}^m) + \tfrac{1}{2}(\mu^R + 3p^R) = 0\,, \tag{8.3}$$

$$\dot{\mu}^m + \Theta\,(\mu^m + p^m) = 0\,, \tag{8.4}$$

where $\tilde{R} = 6\,K/a^2$ is the 3-Ricci scalar. The structure of these equations shows clearly that, higher order gravity behaves like an additional fluid in the model, and also it influences the way in which standard matter interacts gravitationally.

Making use of the background cosmological equations (8.2-8.4), we can write the linearized equations of propagation and constraint in the following form:

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 - \tilde{\nabla}^a \dot{u}_a + \tfrac{1}{2}(\tilde{\mu}^m + 3\tilde{p}^m) = -\tfrac{1}{2}(\mu^R + 3p^R), \tag{8.5}$$

$$\dot{\omega}_a + \tfrac{2}{3}\Theta\omega_a + \tfrac{1}{2}curl\ \dot{u}_a = 0, \tag{8.6}$$

$$\dot{\sigma}_{ab} + \tfrac{2}{3}\Theta\sigma_{ab} + E_{ab} - \tilde{\nabla}_{\langle a}\dot{u}_{b\rangle} = -q_a^R, \tag{8.7}$$

$$\dot{E}_{\langle ab\rangle} + \Theta E_{ab} - curl\ H_{ab} + \tfrac{1}{2}(\tilde{\mu}^m + \tilde{p}^m)\sigma_{ab}$$

$$= -\tfrac{1}{2}(\mu^R + p^R)\sigma_{ab} - \tfrac{1}{2}\dot{\pi}^R_{\langle ab\rangle} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q^R_{b\rangle} - \tfrac{1}{6}\Theta\pi^R_{ab}, \tag{8.8}$$

$$\dot{H}_{ab} + \Theta H_{ab} + curl E_{ab} = \tfrac{1}{2} curl \pi\ ^R_{ab}\,, \tag{8.9}$$

$$\tilde{\nabla}^a\omega_a = 0, \tag{8.10}$$

$$\tilde{\nabla}^a\sigma_{ab} - curl\omega\ _a - \tfrac{2}{3}\tilde{\nabla}_a\Theta = -q^R_a, \tag{8.11}$$

$$curl\sigma\ _{ab} + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} = 0, \tag{8.12}$$

$$\tilde{\nabla}^b E_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\tilde{\mu}^m = -\tfrac{1}{2}\tilde{\nabla}^b\pi^R_{ab} + \tfrac{1}{3}\tilde{\nabla}_a\mu^R - \tfrac{1}{3}\Theta q^R_a, \tag{8.13}$$

$$\tilde{\nabla}^b H_{ab} - (\tilde{\mu}^m + \tilde{p}^m)\omega_a = -\tfrac{1}{2}curl\ q^R_a + (\mu^R + p^R)\omega_a. \tag{8.14}$$

together with the linearized conservation equations

$$\dot{\mu}^m = -\Theta(\mu^m + p^m), \tag{8.15}$$

$$\dot{\mu}^R + \tilde{\nabla}^a q^R_a = -\Theta(\mu^R + p^R) + \mu^m\frac{f''\dot{R}}{f'^2}, \tag{8.16}$$

$$\dot{q}^R_{\langle a\rangle} + \tilde{\nabla}^a p^R + \tilde{\nabla}^b\pi^R_{ab} = -\tfrac{4}{3}\Theta q^R_a - (\mu^R + p^R)\dot{u}_a + \mu^m\frac{f''\tilde{\nabla}_a R}{f'^2}, \tag{8.17}$$

$$\tilde{\nabla}^a p^m = -(\mu^m + p^m)\dot{u}^a. \tag{8.18}$$

These equations provide the basis for a covariant and gauge-invariant description of perturbations of $f(R)$ theories of gravity.

8.2 Perturbation equations

We are now ready to analyze the evolution of the density perturbations on a FLRW background. The scalar perturbations are characterized by the variables.

$$D^m_a \equiv \frac{a}{\mu^m}\tilde{\nabla}_a\,\mu\,, \qquad Z_a \equiv a\,\tilde{\nabla}_a\,\theta\,, \qquad C_a \equiv a^3\,\tilde{\nabla}_a\,3R\,, \tag{8.19}$$

$$\mathcal{R}_a \equiv a\,\tilde{\nabla}_a\,R\,, \qquad \Re_a \equiv a^2\,\tilde{\nabla}_a\,\dot{R} \tag{8.20}$$

which represent the spatial gradient of the energy density, the spatial gradient of the expansion, the spatial gradient of the 3-curvature, the spatial gradient of Ricci scalar and the spatial gradient of the $\dot{R}$.

By assuming the matter to be a barotropic perfect fluid with barotropic factor $w = p^m/\mu^m$ and that the vorticity is zero [36], we obtain the following system of evolution equations for the above variables:

$$
\begin{aligned}
&\dot{D}^m_a - w\Theta D^m_a + (1+w)Z_a = 0\,.\\
&\dot{Z}_a = (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z_a + \left[\frac{w-1}{1+w}\frac{\mu_m}{f'} + \frac{w}{1+w}\left(\frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'} - \frac{K}{a^2}\right)\right] D^m_a\\
&\quad + \Theta\frac{f''}{f'}\Re_a + \left[\frac{1}{2} - \frac{1}{2}\frac{f f''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'} + \frac{2K}{a^2}\frac{f''}{f'}\right]\mathcal{R}_a\\
&\quad - \frac{f''}{f'}\tilde{\nabla}^2\mathcal{R}_a - \frac{w}{1+w}\tilde{\nabla}^2 D^m_a\,.\\
&\dot{\mathcal{R}}_a = \Re_a - \frac{w}{1+w}\dot{R}D^m_a\,.\\
&\dot{\Re}_a = -\left(\Theta + 2\dot{R}\frac{f'''}{f''}\right)\Re_a - \dot{R}Z_a + \left(\frac{(1-3w)\mu_m}{3f''} - \frac{w}{1+w}\ddot{R}\right)D^m_a\\
&- \left[\ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \Theta\dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3} + \frac{2K}{a^2}\right]\mathcal{R}_a + \tilde{\nabla}^2\mathcal{R}_a.
\end{aligned}
\tag{8.21}
$$

together with the constraint ,

$$
\begin{aligned}
&\frac{C_a}{a^2} + \left[\tfrac{4}{3}\Theta + 2\frac{\dot{R}f''}{f'}\right] Z_a - 2\frac{\mu_m}{f'}D^m_a + \left[2\Theta\dot{R}\frac{f'''}{f'} - \frac{f''}{f'}(\frac{f}{f'} - 2\frac{\mu_m}{f'} + 2\dot{R}\Theta\frac{f''}{f'} + \frac{4K}{a^2})\right]\mathcal{R}_a\\
&+ 2\,\Theta\frac{f''}{f'}\Re_a - 2\frac{f''}{f'}\tilde{\nabla}^2\mathcal{R}_a = 0.
\end{aligned}
\tag{8.22}
$$

8.3 The scalar variables

The variables defined above 8.19 contain information that are not relevant to the growth of matter clumping, thus it is convenient to introduce a local decomposition by considering the spatial derivatives of these variables, for example the spatial derivative of D_a reads ,

$$
\tilde{\nabla}_a\,D_b = W_{ab} + \Sigma_{ab} + \frac{1}{3}\,\Delta\,h_{ab}\,.
$$

where $W_{ab} = W_{[ab]}$ contains information about vorticity,$\Sigma_{\ ab}$ describes the evolution of anisotropies in the Universe, and $\Delta = \tilde{\nabla}^a\,D_a$ characterizes the clumping of matter, thus we will only consider this part of the density evolution. By taking the spatial derivative of the inhomogenous variables we obtain the following scalar variables ,

$$
\Delta_m \equiv a\,\tilde{\nabla}^a\,D^m_a\,,\qquad Z \equiv a\,\tilde{\nabla}^a\,Z_a\,,\qquad C \equiv a\,\tilde{\nabla}^a\,C_a\,,\qquad \mathcal{R} \equiv a\,\tilde{\nabla}^a\,\mathcal{R}_a\,,\qquad \Re \equiv a\,\tilde{\nabla}^a\,\Re_a\,.
$$

The scalar dynamical equations are given by:

$$\dot{\Delta}_m = w\Theta\Delta_m - (1+w)Z, \tag{8.23}$$

$$\dot{Z} = (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z + \left[\frac{w-1}{1+w}\frac{\mu_m}{f'} + \frac{w}{1+w}\left(\frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'} - \frac{3K}{a^2}\right)\right]\Delta_m,$$

$$+\Theta\frac{f''}{f'}\Re + \left[\frac{1}{2} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R} - \frac{f''}{f'}\tilde{\nabla}^2\mathcal{R} - \frac{w}{1+w}\tilde{\nabla}^2\Delta_m, \tag{8.24}$$

$$\dot{\mathcal{R}} = \Re - \frac{w}{1+w}\dot{R}\Delta_m, \tag{8.25}$$

$$\dot{\Re} = -\left(\Theta + 2\dot{R}\frac{f'''}{f''}\right)\Re - \dot{R}Z + \left(\frac{(1-3w)\mu_m}{3f''} - \frac{w}{1+w}\ddot{R}\right)\Delta_m$$

$$-\left[\ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \Theta\dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3}\right]\mathcal{R} + \tilde{\nabla}^2\mathcal{R}, \tag{8.26}$$

$$\frac{C}{a^2} + \left[\frac{4}{3}\Theta + 2\frac{\dot{R}f''}{f'}\right]Z - 2\frac{\mu_m}{f'}\Delta_m + \left[2\Theta\dot{R}\frac{f'''}{f'} - \frac{f''}{f'}(\frac{f}{f'} - 2\frac{\mu_m}{f'} + 2\dot{R}\Theta\frac{f''}{f'})\right]\mathcal{R}$$

$$+2\,\Theta\frac{f''}{f'}\Re - 2\frac{f''}{f'}\tilde{\nabla}^2\mathcal{R} = 0. \tag{8.27}$$

In addition, from the Friedmann constraint (B-15), we can obtain an equation connecting Ca with other gradient variables,

$$\frac{C}{a^2} + \left[\tfrac{4}{3}\Theta + 2\frac{\dot{R}f''}{f'}\right]Z - 2\frac{\mu_m}{f'}\dot{\Delta}_m + \left[2\Theta\dot{R}\frac{f'''}{f'} - \frac{f''}{f'}(\frac{f}{f'} - 2\frac{\mu_m}{f'} + 2\dot{R}\Theta\frac{f''}{f'} + \frac{4K}{a^2})\right]\mathcal{R}$$

$$+2\,\Theta\frac{f''}{f'}\Re - 2\frac{f''}{f'}\tilde{\nabla}^2\mathcal{R} = 0. \tag{8.28}$$

The evolution of the scalar constrain is given by

$$\dot{C} = 12K^2\left[\frac{f'^2 + 2a^2(\Theta f'' - 3\ddot{R}f''')}{a^2 f'^2}\frac{f''}{2\Theta f' + 3\dot{R}f''}\right]\mathcal{R} + K\left\{\frac{6f'C}{a^2(2\Theta f' + 3\dot{R}f'')}\right.$$

$$+\Delta_m\left(\frac{16w\Theta}{3(1+w)} - \frac{12\mu_m}{2\Theta f' + 3\dot{R}f''}\right) - \frac{12f''}{2\Theta f' + 3\dot{R}f''}\tilde{\nabla}^2\mathcal{R}$$

$$-\left[\frac{2a^2(\Theta f'' - 3\dot{R}f''')}{3f'}\frac{12\dot{R}\Theta f'f''' - 6f''\left(f - 2\mu_m + 2\dot{R}\Theta f''\right)}{(2\Theta f' + 3\dot{R}f'')f'}\right]\mathcal{R}$$

$$\left.+\left(\frac{12\Theta f''}{2\Theta f' + 3\dot{R}f''} + \frac{2f''}{f'}\right)\Re\right\} + \tilde{\nabla}^2\left[\frac{4wa^2\Theta}{3(1+w)}\Delta_m + \frac{2a^2f''}{f'}\Re - \frac{2a^2(\Theta f'' - 3\dot{R}f'')}{3f'}\mathcal{R}\right]. \tag{8.29}$$

8.3.1 Harmonic analysis

The above evolution equations 8.23 can be thought of as a coupled system of harmonic oscillators of the form,

$$\ddot{X} + A\dot{X} + BX = C(Y, \dot{Y}) \,, \tag{8.30}$$

which can be reduced to a system of ordinary differential equations by using the harmonic decomposition. In equation 8.30 the second term from the left represents the friction (damping) term, the third one, the restoring force term while C represents the source-forcing term. A key assumption in the analysis of the equation here is that we can apply the separation of variables technique,

$$X(x,t) = X(\vec{x})X(t) \,, \quad Y(x,t) = Y(\vec{x})Y(t) \,, \tag{8.31}$$

and write,

$$X = \sum_k X^k(t)Q_k(\vec{x}) \,, \qquad Y = \sum_k Y^k(t)Q_k(\vec{x}) \,, \tag{8.32}$$

where $Q_k(x)$ are the eigenfunctions of the covariantly defined Laplace-Beltrami operator on an almost FLRW space-time:

$$\tilde{\nabla}^2 Q = -\frac{k^2}{a^2}Q \,. \tag{8.33}$$

Here $k = \frac{2\pi a}{\lambda}$ is the order of the harmonic and $\dot{Q}_k(\vec{x}) = 0$ (Q is covariantly constant). Using these harmonics the evolution and the constraint equations can be converted into a system of ordinary differential equations. In this way, one obtains the equations describing the k^{th} mode for scalar perturbations in $f(R)$ gravity. By applying the harmonic decomposition to (8.23-8.29) we obtain,

$$\dot{\Delta}^k_m = w\Theta\Delta^k_m - (1+w)Z^k, \tag{8.34}$$

$$\dot{Z}^k = (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z^k + \left[\frac{w-1}{1+w}\frac{\mu_m}{f'} + \frac{w}{1+w}\left(\frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'} - \frac{3K}{a^2}\right) + \frac{w}{1+w}\frac{k^2}{a^2}\right]\Delta^k_m$$
$$+ \Theta\frac{f''}{f'}\Re^k + \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k, \tag{8.35}$$

$$\dot{\mathcal{R}}^k = \Re^k - \frac{w}{1+w}\dot{R}\Delta^k_m, \tag{8.36}$$

$$\dot{\Re}^k = -\left(\Theta + 2\dot{R}\frac{f'''}{f''}\right)\Re^k - \dot{R}Z^k + \left(\frac{(1-3w)\mu_m}{3f''} - \frac{w}{1+w}\ddot{R}\right)\Delta^k_m$$
$$- \left[\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \Theta\dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3}\right]\mathcal{R}^k, \tag{8.37}$$

$$\frac{C^k}{a^2}+\left[\frac{4\Theta}{3}+\frac{2\dot{R}f''}{f'}\right]Z^k-2\frac{\mu_m}{f'}\Delta^k_m+\left[2\Theta\dot{R}\frac{f'''}{f'}-\frac{f''}{f'}(\frac{f}{f'}-\frac{2\mu_m}{f'}+2\dot{R}\Theta\frac{f''}{f'}-2\frac{k^2}{a^2})\right]\mathcal{R}$$
$$+\frac{2\Theta f''}{f'}\Re^k=0. \tag{8.38}$$

along with

$$\dot{C}^k=12K^2\left[\frac{f'^2+2a^2(\Theta f''-3\ddot{R}f''')}{a^2f'^2}\frac{f''}{2\Theta f'+3\dot{R}f''}\right]\mathcal{R}^k+K\left\{\frac{6f'C^k}{a^2(2\Theta f'+3\dot{R}f'')}\right.$$
$$+\Delta^k_m\left(\frac{16w\Theta}{3(1+w)}-\frac{12\mu_m}{2\Theta f'+3\dot{R}f''}\right)+\frac{12f''}{2\Theta f'+3\dot{R}f''}\frac{k^2}{a^2}\mathcal{R}^k$$
$$-\left[\frac{2a^2(\Theta f''-3\dot{R}f''')}{3f'}\frac{12\dot{R}\Theta f'f'''-6f''\left(f-2\mu_m+2\dot{R}\Theta f''\right)}{(2\Theta f'+3\dot{R}f'')f'}\right]\mathcal{R}^k$$
$$\left.+\left(\frac{12\Theta f''}{2\Theta f'+3\dot{R}f''}+\frac{2f''}{f'}\right)\Re\right\}-\frac{k^2}{a^2}\left[\frac{4wa^2\Theta}{3(1+w)}\Delta_m+\frac{2a^2f''}{f'}\Re^k-\frac{2a^2(\Theta f''-3\dot{R}f'')}{3f'}\mathcal{R}^k\right]. \tag{8.39}$$

The first four first-order equations can be reduced to a coupled system of second-order differential equations:

$$\ddot{\Delta}^k_m-\left[(w-\frac{2}{3})\Theta+\frac{\dot{R}f''}{f'}\right]\dot{\Delta}^k_m+\left[w\frac{k^2}{a^2}+(w-1)\frac{\mu_m}{f'}-w\frac{f}{f'}\right]\Delta^k_m$$
$$=\frac{1+w}{2}\left[-1-\frac{2k^2}{a^2}\frac{f''}{f'}+(f-2\mu_m+2\dot{R}\Theta f'')\frac{f''}{f'^2}-2\dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}-\frac{1+w}{f'}\Theta f''\dot{\mathcal{R}}^k, \tag{8.40}$$
$$\ddot{\mathcal{R}}^k+(2\dot{R}\frac{f'''}{f''}+\Theta)\dot{\mathcal{R}}^k+\left[\frac{k^2}{a^2}+\ddot{R}\frac{f'''}{f''}+\dot{R}^2\frac{f^{(4)}}{f''}+\Theta\dot{R}\frac{f'''}{f''}+\frac{f'}{3f''}-\frac{R}{3}\right]\mathcal{R}^k$$
$$=-\left[\frac{1}{3}(3w-1)\frac{\mu_m}{f''}+\frac{w}{1+w}\left(2\ddot{R}+2\dot{R}^2\frac{f'''}{f''}+2\dot{R}\Theta\right)\right]\Delta^k_m+\frac{1-w}{1+w}\dot{R}\dot{\Delta}^k_m. \tag{8.41}$$

The corresponding equations in GR can be retrieved by setting $f(R)=R$. This yields

$$\ddot{\Delta}^k_m-(w-\frac{2}{3})\Theta\dot{\Delta}^k_m+\left[w\frac{k^2}{a^2}+(\frac{1}{2}+w-\frac{3}{2}w^2)\mu_m\right]\Delta^k_m=0, \tag{8.42}$$
$$\mathcal{R}^k=(1-3w)\mu_m\Delta^k_m. \tag{8.43}$$

The evolution of density perturbations in $f(R)$ gravity is determined by a fourth-order differential equation. Also we notice that the system 8.23 contains friction terms and source terms due to the interaction and the gravitation of the two effective fluids. Thus the perturbations in the fourth-order theories of gravity have a much richer structure than the GR ones.

8.4 Example R^n-gravity

In [113] it has been shown that for $1.37 \leq n \leq 2$ there is a set of initial conditions for which the Universe passes through the transient decelerated phase $a = a_0\, t^{2n/3(1+3w)}$ which evolves towards an accelerated expansion one $a = a_0\, t^{2n/3(1+3w)}$. This first phase is argued to be adequate for the structure formation to take place. In this section we will analyze the evolution of the scalar perturbations during the transient phase. The expansion, the Ricci scalar, the curvature fluid pressure, the curvature fluid-energy density and the effective matter-energy density are given by:

$$\Theta = \frac{2n}{(1+w)t}, \tag{8.44}$$

$$R = \frac{4n\,[4n-3(1+w)]}{3(1+w)^2t^2}, \tag{8.45}$$

$$\mu_R = \frac{2(n-1)\,[2n(3w+5)-3(1+w)]}{3(1+w)^2t^2}, \tag{8.46}$$

$$p_R = \frac{2(n-1)\left[n(6w^2+8w-2)-3w(1+w)\right]}{3(1+w)^2t^2}, \tag{8.47}$$

$$\mu_m = \left(\frac{3}{4}\right)^{1-n} n\chi\left(\frac{n(4n-3(1+w))}{(1+w)^2t^2}\right)^{n-1}\frac{4n^2-2(n-1)\,[2n(3w+5)-3(1+w)]}{3(1+w)^2t^2}. \tag{8.48}$$

For spatially flat (K=0) background and in the long wavelength equation 8.29 reduces to $\dot{C} = 0$ i.e. the variable C is conserved. By substituting the expressions above into the first order equations 8.23-8.27 we obtain ,

$$\begin{aligned}\dot{\Delta}_m = {} & \left[\frac{1+w-2n}{1+w} - \frac{6(n-1)n}{n+3(n-1)w-3}\right]\frac{\Delta_m}{t} - \frac{3(1+w)^2}{4a_0^2\,[n+3(n-1)w-3]\,[4n-3(1+w)]}\times \\ & t^{1-\frac{4n}{3(1+w)}}C_0 - \frac{9(n-1)(1+w)^3t^2}{4\,[n+3(n-1)w-3]\,[4n-3(1+w)]}t^2\Re \\ & + \left[\frac{3(n-1)(1+w)^2\,[n(6w+8)-15(1+w)]}{4\,[n+3(n-1)w-3]\,[4n-3(1+w)]}\right]t\mathcal{R},\end{aligned} \tag{8.49}$$

$$\dot{\mathcal{R}} = \Re + \frac{8nw\,[4n-3(1+w)]}{3(1+w)^3}\frac{\Delta_m}{t^3}, \tag{8.50}$$

$$\dot{\Re} = -2\left[\frac{(n-4)+2(n-2)w}{(1+w)t} - \frac{3n(n-1)}{n+3w(n-1)-3}\right]\Re + \frac{2n(4n-3w-3)}{(1+w)\left[n+3(n-1)w-3\right]}\frac{C_0}{a_0^2}\times$$
$$t^{-\frac{4n}{3(1+w)}-2}$$
$$-2\left[\frac{9n(n-2)(n-1)}{n+3(n-1)w-3} + 2n^2 - 7n - \frac{3n^2(9n-26)+57n}{9(1+w)(n-1)} - \frac{8n^2(n-2)}{9(1+w)^2(n-1)} + 6\right]\frac{\mathcal{R}}{t^2} +$$
$$\frac{16n\left[4n-3(1+w)\right]\left[4n+3(n-1)w-3\right]}{27(n-1)(1+w)^4\left[n+3(n-1)w-3\right]}\times$$
$$\left[(9w(1+w)+8)\,n^2 - (3w(9w+8)+13)\,n + 3(1+w)(1+6w)\right]\frac{\Delta_m}{t^4}, \tag{8.51}$$

where C_0 is a constant representing the value of the conserved quantity C. By decoupling the evolution of density perturbations we obtain the following third-order perturbation equation,

$$(n-1)\dddot{\Delta}_m - (n-1)\left(\frac{4nw}{1+w} - 5\right)\frac{\ddot{\Delta}_m}{t} + \mathcal{D}_1(n,w)\frac{\dot{\Delta}_m}{t^2} + \mathcal{D}_2(n,w)\frac{\Delta_m}{t^3} + \mathcal{D}_3(n,w)C_0\times$$
$$t^{-\left(\frac{4n}{3(1+w)}\right)-1} = 0, \tag{8.52}$$

where the $\mathcal{D}$s are constant coefficients for prescribed n and w values defined as

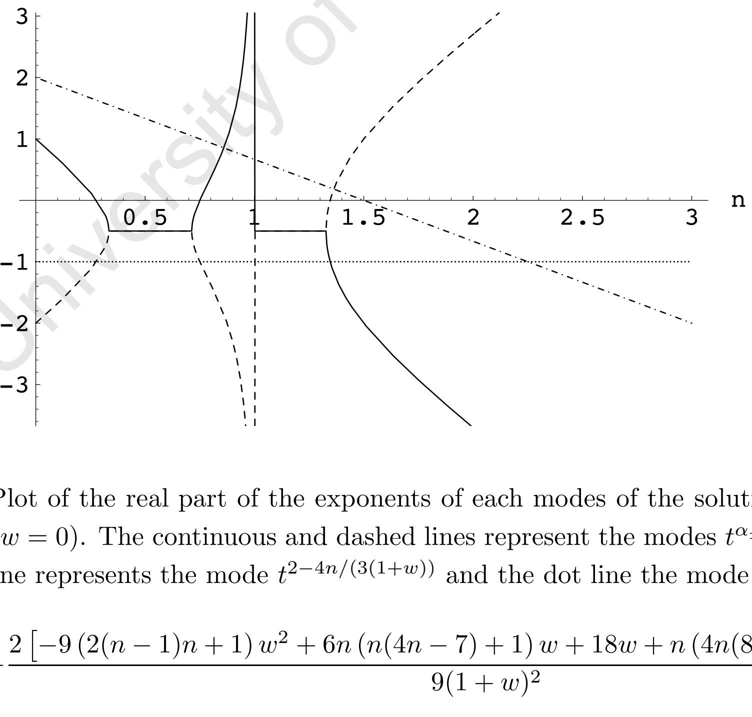


Figure 8.1: Plot of the real part of the exponents of each modes of the solution 8.54 against n for the case ($w = 0$). The continuous and dashed lines represent the modes $t^{\alpha_\pm}$ respectively, the dashed-dot line represents the mode $t^{2-4n/(3(1+w))}$ and the dot line the mode t^{-1}.

$$\mathcal{D}_1(n,w) = -\frac{2\left[-9\left(2(n-1)n+1\right)w^2 + 6n\left(n(4n-7)+1\right)w + 18w + n\left(4n(8n-19)+33\right)+9\right]}{9(1+w)^2},$$

$$\mathcal{D}_2(n,w) = \frac{\left[(2n-1)w-1\right]\left[4n-3(1+w)\right]\left[3(1+w)+n\left(n(6w+8)-9w-13\right)\right]}{9(1+w)^3},$$

$$\mathcal{D}_3(n,w) = -\frac{n\left[21w-6n(2+w)+31\right]-18(1+w)}{6a_0^2}. \tag{8.53}$$

The general solution of this equation is ,

$$\Delta_m(t) = K_1 t^{(\frac{2nw}{1+w})-1} + K_2 t^{\alpha} + K_3 t^{\beta} - K_4 \frac{C_0}{a_0^2} t^{2-(\frac{4n}{3(1+w)})}, \tag{8.54}$$

A_1,A_2 and A_3 being arbitrary integration constants to be determined from initial conditions and with

$$\alpha_{\pm} = -\frac{1}{2} + \frac{nw}{1+w} \pm \frac{\sqrt{(n-1)\left[4(3w+8)^2 n^3 - 4\left(3w(18w+55)+152\right)n^2 + 3(1+w)(87w+139)n - 81(1+w)\right]}}{6(n-1)(1+w)^2}$$

$$K_4 = \frac{9(1+w)^3\left[18(1+w) + (6n(2+w) - 21w - 31)\right]}{8\left[n(6w+4) - 9(1+w)\right]\left[6(2+w)n^3 - (19+9w)n^2 - 3(1+w)(1+3w)n + 9(1+w)^2\right]}. \tag{8.55}$$

In the limit $n \to 1$ the two GR modes $t^{2/3}$ and t^{-1} are reproduced. The other two extra modes characterize the R^n-gravity modification, for $0.33 < n < 0.7$ and $1 < n < 1.32$: these two extra modes describe a damped oscillatory solutions. The most interesting feature of the solutions 8.54 is that, the long wavelength perturbations grow for every value of n, even if the Universe is in a state of accelerated expansion (see figure 8.4), which means that in Rn gravity large- scale structures can in principle also be formed in accelerating backgrounds .

In the next section we study the more general multi-fluid case .

8.5 Perturbations in multi-fluids

The real Universe consist of a number of interacting matter fields such as: Scalar fields, Yang-Mills fields, Spinors, Electromagnetic field, dark energy, etc. We believe that these fields played a significant role in the evolution of the Universe during different epochs. In [148] a full treatment of the covariant and gauge invariant perturbation of an interacting multifluid cosmological medium in GR has been presented. In this section we extend this formalism study multicomponent fluid system in $f(R)$ gravity, we start first by studying the the total fluctuation dynamics of the entire fluid and then we extend the single fluid $f(R)$ perturbations into multi-component ones, we conclude by applying this formalism to a Universe filled with a mixture of radiation and pressureless matter or dust .

If we fix our frame to be the energy frame of the standard matter, then all decompositions will be with respect to the four-velocity vector $u^a_i = u^a_m$. For multi-component matter fluids, the decomposition of the EMT yields:

$$T^m_{ab} = \mu_a u^m_a u^m_b + p_m h^m_{ab} + q^m_a u^m_b + q^m_b u^m_a + \pi^m_{ab}\,, \tag{8.56}$$

where ,

$$\mu^m = T^m_{ab} u^a_m u^b_m = \sum_{i=1}^{N} \mu_i, \tag{8.57}$$

$$p^m = \frac{1}{3} T^m_{ab} h^{ab}_m = \sum_{i=1}^{N} p_i, \tag{8.58}$$

$$q^m_a = -T^m_{bc} h^b{}_{a(m)} u^c = \sum_{i=1}^{N} (\mu_i + p_i) V^i_a, \tag{8.59}$$

$$\pi^m_{ab} = T^m_{cd} h^c{}_{\langle a(m)} h^d{}_{b(m)\rangle} = 0 \quad \text{(to} \quad \text{linear order)} \tag{8.60}$$

V^a_i is the velocity of the i^{th} fluid component relative to this frame and is given by ,

$$V^a_i = u^a_i - u^a_m \tag{8.61}$$

It is clear that for a titled fluid $V^a_i \neq 0$. Also in this frame the exact FLRW background models will be characterized by ,

$$X_a = \tilde{\nabla}_a \mu_m = 0, \quad Y_a = \tilde{\nabla}_a p_m = 0, \quad Z_a = \tilde{\nabla}_a \Theta = 0 \tag{8.62}$$

Consequently $\mu^m = \mu^m(t)$, $p^m = p^m(t)$ and $\Theta = \Theta\,(t)$. Therefore the EMT will have the perfect fluid form.

8.5.1 The background Universe

We will consider a background that is homogeneous and isotropic,i.e., a FLRW model. In this background the cosmological equations for a generic $f(R)$ are given by ,

$$\Theta^2 = 3(\tilde{\mu}^m + \mu^R) - \tfrac{3}{2}\tilde{R}, \tag{8.63}$$

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \tfrac{1}{2}(\tilde{\mu}^m + 3\tilde{p}^m) + \tfrac{1}{2}(\mu^R + 3p^R) = 0, \tag{8.64}$$

$$\dot{\mu}^m + \Theta\,(\mu^m + p^m) = 0\,, \tag{8.65}$$

where $\tilde{R} = 6K/a^2$ is the 3-Ricci scalar, K=0,$\pm$ 1 abd a is the scale factor.

8.5.2 The inhomogeneity variables for the total matter

The key variables characterizing the inhomogeneities of matter are ,

$$D^m_a = a\frac{\tilde{\nabla}_a \mu_m}{\mu_m}\,, \qquad Z_a = a\tilde{\nabla}_a \Theta, \qquad C_a = a\tilde{\nabla}_a \tilde{R},$$

$$\varepsilon_a = \frac{a}{p_m}\left(\frac{\partial p}{\partial s}\right)\tilde{\nabla}_a s\,. \tag{8.66}$$

where $a \equiv a(t)$ here is the usual FLRW cosmological scale factor. D^m_a and Z_a define the comoving fractional density gradient and comoving gradient of the expansion respectively and can in principle be measured observationally [195]. The relation ,

$$p\varepsilon_a = \sum_i p_i \varepsilon^i_a + \frac{1}{2}\sum_{i,j} \frac{h_i h_j}{h}(c^2_{si} - c^2_{sj}) S^{ij}_a \tag{8.67}$$

defines the dimensionless variable ε_a that quantifies entropy perturbations in the total fluid. We have defined the shorthand $h \equiv \mu_m + p_m$ for the total matter fluid and $h_i \equiv \mu_i + p_i$ for the component matter fluids. w and c_s^2 denote the effective barotropic equation of state and speed of sound of the total matter fluid, respectively, and are defined by ,

$$w \equiv \frac{p_m}{\mu_m}, \qquad c_s^2 \equiv \frac{\partial p_m}{\partial \mu_m} \,, \tag{8.68}$$

These two quantities are related to linear order by:

$$\dot{w} = (1+w)(w - c_s^2)\Theta \,. \tag{8.69}$$

8.5.3 Total fluid equations

The equations characterize the temporal fluctuations of inhomogeneities of a generic perfect cosmological fluid with an equation of state given by 8.69, are ,

$$\dot{D}_a + (1+w)Z_a - w\Theta D_a = 0 \,, \tag{8.70}$$

$$\begin{aligned}
\dot{Z}_a &= (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z_a + \left[\frac{(2c_s^2 - w - 1)}{(1+w)}\frac{\mu_m}{f'} + \frac{c_s^2}{(1+w)}\left(\frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'} - \frac{K}{a^2}\right)\right] D_a^m \\
&+ \frac{w}{(1+w)}\left[2\frac{\mu_m}{f'} + \frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'} - \frac{K}{a^2}\right]\varepsilon_a + \Theta\frac{f''}{f'}\Re_a \\
&+ \left[\frac{1}{2} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'} + \frac{2K}{a^2}\frac{f''}{f'}\right]\mathcal{R}_a - \frac{f''}{f'}\tilde{\nabla}^2\mathcal{R}_a - \frac{c_s^2\tilde{\nabla}^2 D_a^m}{1+w} - \frac{w\tilde{\nabla}^2\varepsilon_a}{1+w},
\end{aligned} \tag{8.71}$$

whereas the equations for the curvature inhomogeneities and the constraint are as follows:

$$\dot{\mathcal{R}}_a = \Re_a - \dot{R}\left[\frac{c_s^2}{1+w}D_a^m + \frac{w}{1+w}\varepsilon_a\right], \tag{8.72}$$

$$\begin{aligned}
\dot{\Re}_a &= -\left(2\dot{R}\frac{f'''}{f''} + \Theta\right)\Re_a - \dot{R}Z_a + \left[\frac{(1-3c_s^2)\mu_m}{3f''} - \frac{c_s^2}{1+w}\ddot{R}\right]D_a^m - \left[\frac{w\mu_m}{f''} + \frac{w}{1+w}\ddot{R}\right]\varepsilon_a \\
&- \left(\frac{2K}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}_a + \tilde{\nabla}^2\mathcal{R}_a,
\end{aligned} \tag{8.73}$$

$$\begin{aligned}
&\frac{C_a}{a^2} + \left(\frac{4}{3}\Theta + 2\frac{\dot{R}f''}{f'}\right)Z_a - 2\frac{\mu_m}{f'}D_a^m + \left[2\dot{R}\Theta\frac{f'''}{f'} - \frac{f''}{f'}\left(\frac{f}{f'} - 2\frac{\mu_m}{f'} - \frac{4K}{a^2}\right)\right]\mathcal{R}_a \\
&+ 2\,\Theta\frac{f''}{f'}\Re_a - 2\frac{f''}{f'}\tilde{\nabla}^2\mathcal{R}_a = 0.
\end{aligned} \tag{8.74}$$

The constraint evolves as,

$$
\begin{aligned}
\dot{C}_a = 12K^2 & \left[\frac{3f'^2 + 2a^2(\Theta f'' - 3\ddot{R}f''')}{a^2 f'^2}\frac{f''}{2\Theta f' + 3\dot{R}f''}\right]\mathcal{R}_a + K\Bigg\{\frac{6f'C_a}{a^2(2\Theta f' + 3\dot{R}f'')} \\
& + D^m_a\left(\frac{16w\Theta}{3(1+w)} - \frac{12\mu_m}{2\Theta f' + 3\dot{R}f''}\right) - \frac{12f''}{2\Theta f' + 3\dot{R}f''}\tilde{\nabla}^2\mathcal{R}_a \\
& - \left[\frac{2a^2(\Theta f'' - 3\dot{R}f''')}{3f'}\frac{12\dot{R}\Theta f' f''' - 6f''\left(f - 2\mu_m + 2\dot{R}\Theta f''\right)}{(2\Theta f' + 3\dot{R}f'')f'}\right]\mathcal{R}_a \\
& + \left(\frac{12\Theta f''}{2\Theta f' + 3\dot{R}f''} + \frac{2f''}{f'}\right)\Re_a\Bigg\} + \tilde{\nabla}^2\left[\frac{4wa^2\Theta}{3(1+w)}D^m_a + \frac{2a^2 f''}{f'}\Re_a - \frac{2a^2(\Theta f'' - 3\dot{R}f'')}{3f'}\mathcal{R}_a\right].
\end{aligned}
\tag{8.75}
$$

The linearized three-curvature scalar of the projected metric h_{ab} orthogonal to the four-velocity vector u^a to linear order [148] is,

$$
\tilde{R} = 2\left(-\frac{1}{3}\Theta^2 + \mu\right) \tag{8.76}
$$

reduces to the Ricci scalar in the hypersurfaces orthogonal to u^a when $\omega = 0$. The covariant, GI gradient C_a gives, to linear order,

$$
\begin{aligned}
\frac{C_a}{a^2} & + \left(\tfrac{4}{3}\Theta + 2\tfrac{\dot{R}f''}{f'}\right)Z_a - 2\tfrac{\mu_m}{f'}D^m_a + 2\,\Theta\tfrac{f''}{f'}\Re_a - 2\tfrac{f''}{f'}\tilde{\nabla}^2\mathcal{R}_a \\
& + \left[2\Theta\dot{R}\tfrac{f'''}{f'} - \tfrac{f''}{f'}\left(\tfrac{f}{f'} - 2\tfrac{\mu_m}{f'} + 2\dot{R}\Theta\tfrac{f''}{f'} + \tfrac{4K}{a^2}\right)\right]\mathcal{R}_a = 0\,.
\end{aligned}
\tag{8.77}
$$

This variable quantifies the spatial variation in the three-curvature and is a geometrically natural quantity useful in the long wavelength analysis of our perturbation equations. The time evolution of this quantity is given by,

$$
\begin{aligned}
\dot{C}_a = 2K & \Bigg\{\frac{3f'C_a}{a^2(2\Theta f' + 3\dot{R}f'')} + D^m_a\left[\frac{8w\Theta}{3(1+w)} - \frac{2f'\Theta^2 - 6f'\mu_R}{2\Theta f' + 3\dot{R}f''}\right] - \frac{6f''\tilde{\nabla}^2\mathcal{R}_a}{2\Theta f' + 3\dot{R}f''} \\
& - \left[\frac{a^2(\Theta f'' - 3\dot{R}f''')}{3f'}\frac{12\dot{R}\Theta f' f''' - 2f''\left(3f - 2\Theta^2 f' + 6\,\Theta^2\mu_R + 6\dot{R}\Theta f''\right)}{2\Theta f' + 3\dot{R}f''}\right]\mathcal{R}_a \\
& + \left[\frac{6\Theta f''}{2\Theta f' + 3\dot{R}f''} + \frac{f''}{f'}\right]\Re_a\Bigg\} + K^2\left[\frac{36f''\mathcal{R}_a}{a^2(2\Theta f' + 3\dot{R}f'')} - \frac{36f'D^m_a}{a^2(2\Theta f' + 3\dot{R}f'')}\right] \\
& + \frac{2}{3}\tilde{\nabla}^2\left\{\frac{2wa^2\Theta}{(1+w)}D^m_a + \frac{a^2}{f'}\left[3f''\Re_a - (\Theta f'' - 3\dot{R}f'')\mathcal{R}_a\right]\right\}.
\end{aligned}
\tag{8.78}
$$

8.5.4 Scalar equations

Define the scalar variables,

$$
\Delta_m = a\tilde{\nabla}^a D^m_a\,, \qquad Z = a\tilde{\nabla}^a Z_a\,, \qquad C = a\tilde{\nabla}^a C_a\,, \qquad \mathcal{R} = a\tilde{\nabla}^a\mathcal{R}_a\,, \qquad \Re = a\tilde{\nabla}^a\Re_a\,,
$$
$$
\varepsilon = a\tilde{\nabla}^a\varepsilon_a\,, \tag{8.79}
$$

the corresponding scalar evolution and constraint equations are,

$$\dot{\Delta}_m = w\Theta\Delta_m - (1+w)Z, \tag{8.80}$$

$$\begin{aligned}\dot{Z} &= (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z + \left[\frac{(2c_s^2 - w - 1)}{(1+w)}\frac{\mu_m}{f'} + \frac{c_s^2}{(1+w)}\left(\frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'} - \frac{3K}{a^2}\right)\right]\Delta_m \\ &+ \frac{w}{(1+w)}\left[2\frac{\mu_m}{f'} + \frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'} - \frac{3K}{a^2}\right]\varepsilon + \Theta\frac{f''}{f'}\Re \\ &+ \left[\frac{1}{2} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R} - \frac{f''}{f'}\tilde{\nabla}^2\mathcal{R} - \frac{c_s^2\tilde{\nabla}^2\Delta_m}{1+w} - \frac{w\tilde{\nabla}^2\varepsilon}{1+w}\end{aligned} \tag{8.81}$$

$$\dot{\mathcal{R}} = \Re - \dot{R}\left[\frac{c_s^2}{1+w}\Delta_m + \frac{w}{1+w}\varepsilon\right], \tag{8.82}$$

$$\begin{aligned}\dot{\Re} &= -(2\dot{R}\frac{f'''}{f''} + \Theta)\Re - \dot{R}Z - \left[\ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right]\mathcal{R} \\ &+ \left(\frac{(1-3c_s^2)\mu_m}{3f''} - \frac{c_s^2}{1+w}\ddot{R}\right)\Delta_m - \left(\frac{w\mu_m}{f''} + \frac{w}{1+w}\ddot{R}\right)\varepsilon + \tilde{\nabla}^2\mathcal{R},\end{aligned} \tag{8.83}$$

$$\begin{aligned}&\frac{C}{a^2} + \left[\frac{4}{3}\Theta + 2\frac{\dot{R}f''}{f'}\right]Z - 2\frac{\mu_m}{f'}\Delta_m + \left[2\Theta\dot{R}\frac{f'''}{f'} - \frac{f''}{f'}(\frac{f}{f'} - 2\frac{\mu_m}{f'} + 2\dot{R}\Theta\frac{f''}{f'})\right]\mathcal{R} + 2\,\Theta\frac{f''}{f'}\Re \\ &- 2\frac{f''}{f'}\tilde{\nabla}^2\mathcal{R} = 0,\end{aligned} \tag{8.84}$$

$$\begin{aligned}\dot{C} &= 12K^2\left[\frac{f'^2 + 2a^2(\Theta f'' - 3\ddot{R}f''')}{a^2 f'^2}\frac{f''}{2\Theta f' + 3\dot{R}f''}\right]\mathcal{R} + K\left\{\frac{6f'C}{a^2(2\Theta f' + 3\dot{R}f'')}\right. \\ &+ \Delta_m\left(\frac{16w\Theta}{3(1+w)} - \frac{12\mu_m}{2\Theta f' + 3\dot{R}f''}\right) - \frac{12f''}{2\Theta f' + 3\dot{R}f''}\tilde{\nabla}^2\mathcal{R} \\ &- \left[\frac{2a^2(\Theta f'' - 3\dot{R}f''')}{3f'}\frac{12\dot{R}\Theta f' f''' - 6f''\left(f - 2\mu_m + 2\dot{R}\Theta f''\right)}{(2\Theta f' + 3\dot{R}f'')f'}\right]\mathcal{R} \\ &+ \left.\left(\frac{12\Theta f''}{2\Theta f' + 3\dot{R}f''} + \frac{2f''}{f'}\right)\Re\right\} + \tilde{\nabla}^2\left[\frac{4wa^2\Theta}{3(1+w)}\Delta_m + \frac{2a^2f''}{f'}\Re - \frac{2a^2(\Theta f'' - 3\dot{R}f'')}{3f'}\mathcal{R}\right].\end{aligned} \tag{8.85}$$

8.5.5 Harmonic analysis

Following the same harmonic decomposition procedure presented in Section 8.3.1, the above scalar equations become,

$$\dot{\Delta}^k_m + (1+w)Z^k - w\Theta\Delta^k_m = 0, \tag{8.86}$$

$$\dot{Z}^k = (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z^k + \left[\frac{(1+3w)c_s^2 - 2(1+w)}{2(1+w)}\frac{\mu_m}{f'} + \frac{2c_s^2\Theta^2 + 3c_s^2(\mu_R + 3p_R)}{6(1+w)} + \frac{c_s^2}{1+w}\frac{k^2}{a^2}\right]\Delta^k_m$$
$$+ \left[\frac{2f'\Theta^2 + 3(1+3w)\mu_m + 3f'(\mu_R + 3p_R)}{6f'(1+w)} + \frac{1}{1+w}\frac{k^2}{a^2}\right]w\varepsilon^k + \Theta\frac{f''}{f'}\Re^k$$
$$+ \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k, \tag{8.87}$$

$$\dot{\mathcal{R}}^k = \Re^k - \dot{R}\left[\frac{c_s^2}{1+w}\Delta^k_m + \frac{w}{1+w}\varepsilon^k\right], \tag{8.88}$$

$$\dot{\Re}^k = -\left(2\dot{R}\frac{f'''}{f''} + \Theta\right)\Re^k - \dot{R}Z^k + \left[\frac{(1-3c_s^2)\mu_m}{3f''} - \frac{c_s^2}{1+w}\ddot{R}\right]\Delta^k_m - \left[\frac{w\mu_m}{f''} + \frac{w}{1+w}\ddot{R}\right]\varepsilon^k$$
$$- \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}^k, \tag{8.89}$$

$$\frac{C^k}{a^2} + \left[\frac{4}{3}\Theta + 2\frac{\dot{R}f''}{f'}\right]Z^k - 2\frac{\mu_m}{f'}\Delta^k_m + \left[2\Theta\dot{R}\frac{f'''}{f'} - \frac{f''}{f'}(\frac{f}{f'} - 2\frac{\mu_m}{f'} + 2\dot{R}\Theta\frac{f''}{f'} - 2\frac{k^2}{a^2})\right]\mathcal{R}$$
$$+ 2\,\Theta\frac{f''}{f'}\Re^k = 0. \tag{8.90}$$

The evolution of the constraint equation is given by,

$$\dot{C}^k = 12K^2\left[\frac{f'^2 + 2a^2(\Theta f'' - 3\dot{R}f''')}{a^2 f'^2}\frac{f''}{2\Theta f' + 3\dot{R}f''}\right]\mathcal{R}^k + K\left\{\frac{6f'C^k}{a^2(2\Theta f' + 3\dot{R}f'')}\right.$$
$$+ \Delta^k_m\left(\frac{16w\Theta}{3(1+w)} - \frac{12\mu_m}{2\Theta f' + 3\dot{R}f''}\right) + \frac{12f''}{2\Theta f' + 3\dot{R}f''}\frac{k^2}{a^2}\mathcal{R}^k$$
$$- \left[\frac{2a^2(\Theta f'' - 3\dot{R}f''')}{3f'}\frac{12\dot{R}\Theta f'f''' - 6f''\left(f - 2\mu_m + 2\dot{R}\Theta f''\right)}{(2\Theta f' + 3\dot{R}f'')f'}\right]\mathcal{R}^k$$
$$\left. + \left(\frac{12\Theta f''}{2\Theta f' + 3\dot{R}f''} + \frac{2f''}{f'}\right)\Re\right\} - \frac{k^2}{a^2}\left[\frac{4wa^2\Theta}{3(1+w)}\Delta_m + \frac{2a^2f''}{f'}\Re^k - \frac{2a^2(\Theta f'' - 3\dot{R}f'')}{3f'}\mathcal{R}^k\right]. \tag{8.91}$$

8.5.6 Second-order equations

From the first-order equations (8.86, 8.87, 8.88, 8.89) we can obtain the following second-order pair of equations,

$$\ddot{\Delta}^k_m + \left[(c_s^2 + \frac{2}{3} - 2w)\Theta - \dot{R}\frac{f''}{f'}\right]\dot{\Delta}^k_m + \left[\left(\frac{3}{2}w^2 + 5c_s^2 - 4w - 1\right)\frac{\mu_m}{f'} + \frac{1}{2}(3w - 5c_s^2)\frac{f}{f'}\right.$$
$$\left. +(c_s^2 - w)\left(2R - 4\dot{R}\Theta\frac{f''}{f'} - \frac{12K}{a^2}\right) + c_s^2\frac{k^2}{a^2}\right]\Delta^k_m + \left[2\frac{\mu_m}{f'} + \frac{R}{2} - \frac{f}{f'} - \dot{R}\Theta\frac{f''}{f'} - \frac{3K}{a^2} + \frac{k^2}{a^2}\right]w\varepsilon^k$$
$$= \frac{1+w}{2}\left[-1 - \frac{2k^2}{a^2}\frac{f''}{f'} + (f - 2\mu_m + 2\dot{R}\Theta f'')\frac{f''}{f'^2} - 2\dot{R}\Theta(\frac{f''}{f'})^2 - 2\dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k$$
$$- (1+w)\Theta\frac{f''}{f'}\dot{\mathcal{R}}^k, \tag{8.92}$$

$$\ddot{\mathcal{R}}^k + (2\dot{R}\frac{f'''}{f''} + \Theta)\dot{\mathcal{R}}^k + \left[\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right]\mathcal{R}^k + \frac{c_s^2 - 1}{1+w}\dot{R}\dot{\Delta}^k_m$$
$$+ \left[\frac{(3c_s^2 - 1)\mu_m}{3f''} + \frac{w + c_s^2}{1+w}\dot{R}\Theta + \frac{c_s^2}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right) + \frac{\dot{R}}{1+w}\left(\dot{c_s^2} + c_s^2(c_s^2 - w)\Theta\right)\right]\Delta^k_m$$
$$+ \frac{w}{1+w}\dot{R}\dot{\varepsilon}^k + \left[\frac{w\mu_m}{f''} + \frac{2w - c_s^2}{1+w}\dot{R}\Theta + \frac{w}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right)\right]\varepsilon^k = 0. \tag{8.93}$$

In the general relativistic limit, the corresponding equations are [148],

$$\mathcal{R}^k = (1 - 3c_s^2)\mu_m\Delta^k_m - 3w\mu_m\varepsilon^k, \tag{8.94}$$

$$\ddot{\Delta}^k_m + (c_s^2 + \frac{2}{3} - 2w)\Theta\dot{\Delta}^k_m + \left[\frac{3w(w + c_s^2) + (c_s^2 - w) - 2}{2}\mu_m + c_s^2\frac{k^2}{a^2} - \frac{4}{3}(w - c_s^2)\Theta^2\right]\Delta^k_m$$
$$+ \left[\frac{2\Theta^2 + 3(1+3w)\mu_m}{6} + \frac{k^2}{a^2}\right]w\varepsilon^k + \frac{1+w}{2}\left((1 - 3c_s^2)\mu_m\Delta^k_m - 3w\mu_m\varepsilon\right) = 0, \tag{8.95}$$

$$\Rightarrow \ddot{\Delta}^k_m + (c_s^2 + \frac{2}{3} - 2w)\Theta\dot{\Delta}^k_m + \left[(\frac{3}{2}w^2 + 3c_s^2 - 4w - \frac{1}{2})\mu_m + (w - c_s^2)\frac{12K}{a^2} + c_s^2\frac{k^2}{a^2}\right]\Delta^k_m$$
$$+ w\left(\frac{k^2}{a^2} - \frac{3K}{a^2}\right)\varepsilon^k = 0. \tag{8.96}$$

8.5.7 Matter inhomogeneity variables for the components

The variables characterizing inhomogeneities of matter for the i^{th}- component fluid are defined as,

$$D^i_a = a\frac{\tilde{\nabla}_a\mu_i}{\mu_i}, \qquad Y^i_a = \tilde{\nabla}_a p_i, \qquad \varepsilon^i_a = \frac{a}{p^i}\left(\frac{\partial p^i}{\partial s_i}\right)\tilde{\nabla}_a s_i. \tag{8.97}$$

In near-perfect fluid analyses such as the present one, ε^i_a is often taken to be negligible. Thus in subsequent discussions all terms containing this quantity are dropped. The information about our deviation from standard GR is carried by the following dimensionless gradient quantities,

$$\mathcal{R}_a = a\tilde{\nabla}_a R \,, \qquad \Re_a = a\tilde{\nabla}_a \dot{R} \,. \tag{8.98}$$

These variables describe the inhomogeneities in the Ricci scalar. Finally, the velocity of the curvature fluid is defined by,

$$V^R_a = -\frac{\tilde{\nabla}_a R}{\dot{R}} \,. \tag{8.99}$$

The component background equations in the FLRW spacetime are given by,

$$\Theta^2 = 3(\tilde{\mu}^m + \mu^R) - \tfrac{3}{2}\tilde{R}, \tag{8.100}$$

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \tfrac{1}{2}(\tilde{\mu}^m + 3\tilde{p}^m) + \tfrac{1}{2}(\mu^R + p^R) = 0, \tag{8.101}$$

$$\dot{\mu}_i + \Theta\,(\mu_i + p_i) = 0, \tag{8.102}$$

The speed of sound $c^2_{si} = \frac{\dot{p}_i}{\dot{\mu}_i}$ and the barotropic equation of state $w_i = \frac{p_i}{\mu_i}$ of the i^{th}-component fluid are related by the familiar expression,

$$\dot{w}_i = -(1 + w_i)(c^2_{si} - w_i)\Theta. \tag{8.103}$$

8.5.8 Component equations

These are the equations that describe the evolution dynamics of the individual fluid component fluctuations. For the component matter and velocity fluctuations, these are given by,

$$\dot{D}^i_a - (w_i - c^2_{si})\Theta D^i_a + (1 + w_i)Z_a = \frac{1}{\mu_i h} h_i \Theta(c^2_s \mu D_a + p\varepsilon_a) - a(1 + w_i)\tilde{\nabla}_a\tilde{\nabla}^b V^i_b \,, \tag{8.104}$$

$$\dot{V}^i_a - \left(c^2_{si} - \frac{1}{3}\right)\Theta V^i_a = \frac{1}{ahh_i}\left(h_i c^2_s \mu D_a + h_i p\varepsilon_a - hc^2_{si}\mu^i D^i_a\right) \,. \tag{8.105}$$

We note that the equations involving the gradients of the inhomogeneities in the expansion and curvature variables $(Z_a, \mathcal{R}_a, \Re_a, C_a)$ remain the same as in the total fluid equations. This is to be expected since these quantities are global intrinsic properties of the spacetime itself rather than of the individual components of matter in the fluid.

8.5.9 Relative equations

Let us now define the variables that relate features of pairs of the different components of the fluid, and derive their governing evolution equations. These relative variables depend only on the choice of the individual velocities, not on the choice of the overall frame.

$$S^{ij}_a \equiv \frac{\mu_i D^i_a}{h_i} - \frac{\mu_j D^j_a}{h_j} \,, \qquad V^{ij}_a \equiv V^i_a - V^j_a \,. \tag{8.106}$$

These are the quantities that allow us to distinguish between adiabatic and isothermal perturbations [138, 148]. The derivation of the evolution equations for the above quantities is straightforward and yields,

$$\dot{V}^{ij}_a - (c^2_{sj} - \tfrac{1}{3})\Theta V^{ij}_a - (c^2_{si} - c^2_{sj})\Theta V^i_a = -\tfrac{1}{ah_i}(c^2_{si} - c^2_{sj})\mu_i D^m_i - \tfrac{1}{a}c^2_{sj}S^{ij}_a \,, \tag{8.107}$$

$$\dot{S}^{ij}_a + a\tilde{\nabla}_a\tilde{\nabla}^b V^{ij}_b = 0 \,. \tag{8.108}$$

8.6 Scalar equations

On the basis of the decomposition scheme 8.3, our scalar variables are:

$$\Delta^i_m = a\tilde{\nabla}^a D^i_a\,, \qquad V_i = a\tilde{\nabla}^a V^i_a\,, \qquad S_{ij} = a\tilde{\nabla}^a S^{ij}_a\,, \qquad V_{ij} = a\tilde{\nabla}^a V^{ij}_a\,. \tag{8.109}$$

For the scalar variables describing component inhomogeneities and interactions in the fluid, the evolution equations are given by ,

$$\dot{\Delta}^i_m - \Theta(w_i - c^2_{si})\Delta^i_m + (1+w_i)Z = \frac{1+w_i}{1+w}\left(c^2_s\Theta\Delta_m + w\Theta\varepsilon\right) - a(1+w_i)\tilde{\nabla}^2 V_i\,, \tag{8.110}$$

$$\dot{V}_i - \left(c^2_{si} - \frac{1}{3}\right)\Theta V_i = \frac{1}{ahh_i}\left(h_i c^2_s \mu\Delta_m + h_i p\varepsilon - hc^2_{si}\mu^i\Delta^i_m\right)\,, \tag{8.111}$$

$$\dot{V}_{ij} - \left(c^2_{sj} - \frac{1}{3}\right)\Theta V_{ij} - (c^2_{si} - c^2_{sj})\Theta V_i = -\frac{1}{ah_i}(c^2_{si} - c^2_{sj})\mu_i\Delta^i_m - \frac{1}{a}c^2_{sj}S_{ij}\,, \tag{8.112}$$

$$\dot{S}_{ij} + a\tilde{\nabla}^2 V_{ij} = 0\,. \tag{8.113}$$

8.6.1 Second-order equations

The above first-order equations 8.110-8.113 can be reduced to a set of linearly independent second-order equations. This has the advantage of simplifying the equations and making comparisons to GR more transparent [154]:

$$\begin{aligned}
&\ddot{\mathcal{R}} + \left(2\dot{R}\frac{f'''}{f''} + \Theta\right)\dot{\mathcal{R}} + \left(\ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(iv)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R} \\
&+ \frac{c^2_s - 1}{1+w}\dot{R}\dot{\Delta}_m + \left\{\frac{(3c^2_s - 1)\mu_m}{3f''} + \frac{w + c^2_s}{1+w}\dot{R}\Theta + \frac{2c^2_s}{1+w}\left(\ddot{R} + \dot{R}^2\frac{f'''}{f''}\right)\right. \\
&\left.+ \frac{\dot{R}}{1+w}\left[\dot{c^2_s} + c^2_s(c^2_s - w)\Theta\right]\right\}\Delta_m - \tilde{\nabla}^2\mathcal{R} + \frac{w}{1+w}\dot{R}\dot{\varepsilon} \\
&+ \left[\frac{w\mu_m}{f''} + \frac{2w - c^2_s}{1+w}\dot{R}\Theta + \frac{2w}{1+w}\left(\ddot{R} + \dot{R}^2\frac{f'''}{f''}\right)\right]\varepsilon = 0\,,
\end{aligned} \tag{8.114}$$

$$\ddot{\Delta}_i + \left(\frac{2}{3} - w_i\right)\Theta\dot{\Delta}_i - \frac{1+w_i}{1+w}\left[\dot{R}\frac{f''}{f'} + (c_s^2 - c_{si}^2)\Theta\right]\dot{\Delta}_m - (1+w_i)\frac{f''}{f'}\tilde{\nabla}^2\mathcal{R}$$

$$- \frac{1+w_i}{1+w}\left[(1+w)\frac{\mu_m}{f'} - \left(2\frac{\mu_m}{f'} - \frac{f}{f'} - 2\Theta\dot{R}\frac{f''}{f'}\right)c_s^2 + \dot{c_s^2}\Theta\right.$$

$$\left. + (c_s^2 - c_{si}^2)(c_s^2 - w)\Theta^2 - (c_s^2 + w)\dot{R}\Theta\frac{f''}{f'}\right]\Delta_m - \frac{1+w_i}{1+w}w\Theta\dot{\varepsilon} - c_{si}^2\tilde{\nabla}^2\Delta_i$$

$$+ (1+w_i)\Theta\frac{f''}{f'}\dot{\mathcal{R}} + (1+w_i)\left[\frac{1}{2} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta\left(\frac{f''}{f'}\right)^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}$$

$$- \frac{1+w_i}{1+w}\left[(w - c_s^2 - c_{si}^2 w)\Theta^2 - w\left(2\frac{\mu_m}{f'} - \frac{f}{f'} - \dot{R}\Theta\frac{f''}{f'}\right)\right]\varepsilon = 0\,. \quad (8.115)$$

$$\ddot{S}_{ij}^k = \frac{k^2}{a}\dot{V}_{ij} - \frac{k^2}{3a}\Theta V_{ij}, \quad (8.116)$$

$$\ddot{V}_{ij}^k = (c_z^2 - \frac{1}{3})\Theta\dot{V}_{ij} + \left[\dot{c}_z^2\Theta - (c_z^2 - \frac{1}{3})\left(\frac{1}{3}\Theta^2 + \frac{1}{2}(1+3w)\frac{\mu_m}{f'} + \frac{1}{2}(\mu_R + 3p_R)\right)\right]V_{ij}$$

$$- \frac{c_{si}^2 - c_{sj}^2}{a(1+w)}\dot{\Delta}_m + \frac{c_{si}^2 - c_{sj}^2}{a(1+w)}(\frac{1}{3} + w - c_s^2)\Theta\Delta_m - \frac{c_z^2}{a}\dot{S}_{ij} + \frac{c_z^2\Theta - 3\dot{c}_z^2}{3a}S_{ij}. \quad (8.117)$$

The last two equations 8.116 and 8.117 are linearly dependent .

8.7 Harmonic analysis

After harmonic decomposition the first order total and component fluid equations 8.110-8.113 can be rewritten in the following form:

$$\dot{\Delta}_i^k - (w_i - c_{si}^2)\Theta\Delta_i^k + (1+w_i)Z^k = \frac{1+w_i}{1+w}\left(c_s^2\Delta_m^k + w\varepsilon^k\right)\Theta + (1+w_i)\frac{k^2}{a}V_i\,, \quad (8.118)$$

$$\dot{V}_i^k - \left(c_{si}^2 - \frac{1}{3}\right)\Theta V_i^k = \frac{1}{ahh_i}\left(h_i c_s^2\mu\Delta_m + h_i p\varepsilon^k - hc_{si}^2\mu^i\Delta_i^k\right), \quad (8.119)$$

$$\dot{V}_{ij}^k - \left(c_{sj}^2 - \frac{1}{3}\right)\Theta V_{ij}^k - (c_{si}^2 - c_{sj}^2)\Theta V_i^k = -\frac{1}{ah_i}(c_{si}^2 - c_{sj}^2)\mu_i\Delta_i^k - \frac{1}{a}c_{sj}^2 S_{ij}^k\,, \quad (8.120)$$

$$\dot{S}_{ij}^k - \frac{k^2}{a}V_{ij}^k = 0\,. \quad (8.121)$$

The harmonically decomposed second-order equations 8.114-8.117 will become,

$$\ddot{\Delta}^k_i + (\frac{2}{3} - w_i)\Theta\dot{\Delta}^k_i - \frac{1+w_i}{1+w}\left[\dot{R}\frac{f''}{f'} + \left(c_s^2 - c_{si}^2\right)\Theta\right]\dot{\Delta}^k + (1+w_i)\Theta\frac{f''}{f'}\dot{\mathcal{R}}^k$$
$$- \frac{1+w_i}{1+w}\left[(1+w)\frac{\mu_m}{f'} - \left(2\frac{\mu_m}{f'} - \frac{f}{f'} - 2\Theta\dot{R}\frac{f''}{f'}\right)c_s^2 + \dot{c}_s^2\Theta\right.$$
$$\left.+(c_s^2 - c_{si}^2)(c_s^2 - w)\Theta^2 - (c_s^2 + w)\dot{R}\Theta\frac{f''}{f'}\right]\Delta^k - \frac{1+w_i}{1+w}w\Theta\dot{\varepsilon}^k$$
$$- \frac{1+w_i}{1+w}\left[(w - c_s^2 - c_{si}^2 w)\Theta^2 - w\left(2\frac{\mu_m}{f'} - \frac{f}{f'} - \dot{R}\Theta\frac{f''}{f'}\right)\right]\varepsilon^k + c_{si}^2\frac{k^2}{a^2}\Delta^k_i$$
$$+ (1+w_i)\left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta\left(\frac{f''}{f'}\right)^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k = 0\,. \quad (8.122)$$

$$\ddot{\mathcal{R}}^k + \left(2\dot{R}\frac{f'''}{f''} + \Theta\right)\dot{\mathcal{R}}^k + \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(iv)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}^k$$
$$+ \frac{c_s^2 - 1}{1+w}\dot{R}\dot{\Delta}^k_m + \left[\frac{w\mu_m}{f''} + \frac{2w - c_s^2}{1+w}\dot{R}\Theta + \frac{w}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right)\right]\varepsilon^k$$
$$+ \left\{\frac{(3c_s^2 - 1)\mu_m}{3f''} + \frac{w + c_s^2}{1+w}\dot{R}\Theta + \frac{c_s^2}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right)\right.$$
$$\left.+\frac{\dot{R}}{1+w}\left[\dot{c}_s^2 + c_s^2(c_s^2 - w)\Theta\right]\right\}\Delta^k_m + \frac{w}{1+w}\dot{R}\dot{\varepsilon}^k = 0\,, \quad (8.123)$$

$$\ddot{S}^k_{ij} = \frac{k^2}{a}\dot{V}_{ij} - \frac{k^2}{3a}\Theta V_{ij}\,, \quad (8.124)$$

$$\ddot{V}^k_{ij} = \left(c_z^2 - \frac{1}{3}\right)\Theta\dot{V}_{ij} + \frac{c_{si}^2 - c_{sj}^2}{a(1+w)}\left(\frac{1}{3} + w - c_s^2\right)\Theta\Delta_m - \frac{c_z^2}{a}\dot{S}_{ij} + \frac{c_z^2\Theta - 3\dot{c}_z^2}{3a}S_{ij}$$
$$+ \left\{\dot{c}_z^2\Theta - \left(c_z^2 - \frac{1}{3}\right)\left[\frac{1}{3}\Theta^2 + \frac{1}{2}(1+3w)\frac{\mu_m}{f'} + \frac{1}{2}(\mu_R + 3p_R)\right]\right\}V_{ij}$$
$$- \frac{c_{si}^2 - c_{sj}^2}{a(1+w)}\dot{\Delta}_m\,. \quad (8.125)$$

Since Equations (8.124) and (8.125) are not linearly independent equations, we can choose either one of them to close our system. The form and use of the equations above will be more transparent when we discuss the long wavelength limits of our perturbations for radiation and dust backgrounds in the next section.

8.8 Perturbations in a radiation-dust Universe

Now that we have derived the equations for perturbations of a general multi-fluid system in the previous section, we consider an application of the equations for a cosmological medium containing a non-interacting radiation-dust mixture and described by a flat ($K = 0$) FLRW space-time. Since our component fluids satisfy the conservation equations separately, we write,

$$\dot{\mu}_d + \Theta\mu_d = 0 \ , \tag{8.126}$$

$$\dot{\mu}_r + \tfrac{4}{3}\Theta\mu_r = 0 \ . \tag{8.127}$$

where d and r subindices hold for dust and radiation respectively. The general equation of state w for such a radiation-dust mixture is given by,

$$w = \frac{p_m}{\mu_m} = \frac{p_d + p_r}{\mu_d + \mu_r} = \frac{1}{3}\frac{\mu_r}{\mu_d + \mu_r} \tag{8.128}$$

and the speed of sound in the mixture is,

$$c_s^2 = \frac{\dot{p}_m}{\dot{\mu}_m} = \frac{4\mu_r}{3(3\mu_d + 4\mu_r)} \ . \tag{8.129}$$

Wherever necessary, we will use the shorthand,

$$c_z^2 \equiv \frac{1}{h}\left(h_r c_{sd}^2 + h_d c_{sr}^2\right) \ . \tag{8.130}$$

In general, since we do not have an explicit expression of the Hubble parameter H and the curvature scalar R as functions of the scale factor a in generic $f(R)$ gravity theories, an exact multi-fluid background solution is not available and numerical solutions need to be obtained. This important issue will be investigated in a future work. In this thesis, we confine our discussion to R^n models [154, 156] and look for solutions in the short wavelength and long wavelength approximations for perturbations deep in the radiation- and dust- dominated epochs. During these epochs, since one fluid is negligible with respect to the other,we can use the exact single fluid background transient solution for R^n models given by $a = a_{eq}(t/t_{eq})^{\frac{2n}{3(1+w)}}$ where a_{eq} is the scale factor at the time of radiation-dust equality t_{eq} and will henceforth be normalized to unity. In R^n models the expressions for the expansion, the Ricci scalar, the curvature fluid-energy density, the curvature-fluid pressure and the effective matter-energy density are given by 8.44,

8.8.1 Total fluid equations

Upon expanding Eqn. (8.67) for a mixture of dust and radiation, we obtain,

$$p_m\varepsilon = -\frac{4\mu_d\mu_r}{3(3\mu_d + 4\mu_r)}S_{dr} \ , \tag{8.131}$$

and hence

$$\varepsilon = -\frac{4\mu_d}{3\mu_d + 4\mu_r}S_{dr} \ . \tag{8.132}$$

We can thus readily derive the evolution equation for ε as follows,

$$\dot{\varepsilon} = -\frac{16\mu_d\mu_r\Theta}{3(3\mu_d + 4\mu_r)^2}S_{dr} - \frac{4\mu_d}{3\mu_d + 4\mu_r}\dot{S}_{dr} = -4c_z^2c_s^2\Theta S_{dr} - 4c_z^2\dot{S}_{dr} \ . \tag{8.133}$$

Using these relations and applying the general total fluid second order equations to the radiation-dust mixture yields,

$$\begin{aligned}\ddot{\Delta}^k_m \quad & + \left[\left(c_s^2 + \tfrac{2}{3} - 2w\right)\Theta - \dot{R}\tfrac{f''}{f'}\right]\dot{\Delta}^k_m - 4wc_z^2\left[2\tfrac{\mu_m}{f'} + \tfrac{R}{2} - \tfrac{f}{f'} - \dot{R}\Theta\tfrac{f''}{f'} + \tfrac{k^2}{a^2}\right]S^k_{dr} \\ & + \left[\left(\tfrac{3}{2}w^2 + 5c_s^2 - 4w - 1\right)\tfrac{\mu_m}{f'} + \tfrac{1}{2}(3w - 5c_s^2)\tfrac{f}{f'} + (c_s^2 - w)\left(2R - 4\dot{R}\Theta\tfrac{f''}{f'}\right)\right. \\ & \left. + c_s^2\tfrac{k^2}{a^2}\right]\Delta^k_m - \tfrac{1+w}{2}\left[-1 - \tfrac{2k^2}{a^2}\tfrac{f''}{f'} + (f - 2\mu_m + 2\dot{R}\Theta f'')\tfrac{f''}{f'^2} - 2\dot{R}\Theta\left(\tfrac{f''}{f'}\right)^2\right. \\ & \left. -2\dot{R}\Theta\tfrac{f'''}{f'}\right]\mathcal{R}^k + (1+w)\Theta\tfrac{f''}{f'}\dot{\mathcal{R}}^k = 0\,, \end{aligned} \qquad (8.134)$$

$$\begin{aligned}\ddot{\mathcal{R}}^k \quad & + \left(2\dot{R}\tfrac{f'''}{f''} + \Theta\right)\dot{\mathcal{R}}^k + \left[\tfrac{k^2}{a^2} + \ddot{R}\tfrac{f'''}{f''} + \dot{R}^2\tfrac{f^{(iv)}}{f''} + \dot{R}\Theta\tfrac{f'''}{f''} + \tfrac{1}{3}\tfrac{f'}{f''} - \tfrac{R}{3}\right]\mathcal{R}^k \\ & + \tfrac{c_s^2 - 1}{1+w}\dot{R}\dot{\Delta}^k_m + \left\{\tfrac{(3c_s^2 - 1)\mu_m}{3f''} + \tfrac{w + c_s^2}{1+w}\dot{R}\Theta + \tfrac{c_s^2}{1+w}\left(2\ddot{R} + 2\dot{R}^2\tfrac{f'''}{f''}\right)\right. \\ & \left. + \tfrac{\dot{R}}{1+w}\left[\dot{c_s^2} + c_s^2(c_s^2 - w)\Theta\right]\right\}\Delta^k_m - 4wc_z^2\left[\tfrac{2}{1+w}\left(\ddot{R} + \dot{R}\Theta + \dot{R}^2\tfrac{f'''}{f''}\right)\right. \\ & \left. + \tfrac{\mu_m}{f''}\right]S^k_{dr} - \tfrac{4w}{1+w}c_z^2\dot{R}\dot{S}^k_{dr} = 0\,, \end{aligned} \qquad (8.135)$$

$$\ddot{S}^k_{dr} + \left(c_s^2 + \tfrac{1}{3}\right)\Theta\dot{S}^k_{dr} + \tfrac{k^2}{a^2}c_z^2 S^k_{dr} - \tfrac{k^2}{a^2}\left(c_z^2 + \tfrac{3}{4}c_s^2\right)\Delta^k_m = 0\,, \qquad (8.136)$$

where$\Delta_{\ m}$ and S_{dr} are given by$\Delta_{\ m} = \frac{\mu_d\Delta_d + \mu_r\Delta_r}{\mu_d + \mu_r}$, $S_{dr} = \Delta_d - \frac{3}{4}\Delta_r$.

8.8.2 Component equations

The perturbations of the density gradients of the radiation component of the fluid are described by the propagation equation,

$$
\begin{aligned}
&\ddot{\Delta}^k_r + \left\{\left[\frac{1}{3} - \frac{4}{3(1+w)}\left(\frac{3w\mu_d}{3\mu_d+4\mu_r} + \frac{(c_s^2-\frac{1}{3})\mu_r}{\mu_d+\mu_r}\right)\right]\Theta - \frac{4\mu_r\dot{R}f''/f'}{3(1+w)(\mu_d+\mu_r)}\right\}\dot{\Delta}^k_r \\
&+ \frac{4\mu_d}{3(1+w)}\left[\left(\frac{4w}{3\mu_d+4\mu_r} - \frac{c_s^2-\frac{1}{3}}{\mu_d+\mu_r}\right)\Theta - \frac{\dot{R}f''}{(\mu_d+\mu_r)f'}\right]\dot{\Delta}^k_d \\
&+ \frac{4}{3(1+w)}\Bigg[\frac{k^2}{3a^2} + \left(\frac{(w-c_s^2)\mu_r}{\mu_d+\mu_r} - \frac{3w\mu_d}{3\mu_d+4\mu_r} + \frac{\mu_d\mu_r}{3(\mu_d+\mu_r)^2}\right)\dot{R}\Theta\frac{f''}{f'} \\
&- \Bigg(\frac{4\mu_d\mu_r}{(3\mu_d+4\mu_r)^2}\frac{3w\mu_d+(3w-1)\mu_r}{3(\mu_d+\mu_r)} - \frac{3(c_s^2-\frac{2w}{3})\mu_d}{3\mu_d+4\mu_r} + \frac{(c_s^2-\frac{1}{3})(c_s^2-w)\mu_r}{\mu_d+\mu_r} \\
&- \frac{(c_s^2-\frac{1}{3})\mu_d\mu_r}{3(\mu_d+\mu_r)^2}\Bigg)\Theta^2 - \left(\frac{\left(1+w-2c_s^2\right)\mu_r}{\mu_d+\mu_r} - \frac{6w\mu_d}{3\mu_d+4\mu_r}\right)\frac{\mu_d+\mu_r}{f'} \\
&- \left(\frac{3w\mu_d}{3\mu_d+4\mu_r} + \frac{c_s^2\mu_r}{\mu_d+\mu_r}\right)\frac{f}{f'}\Bigg]\Delta^k_r + \frac{4}{3(1+w)}\Bigg[\Bigg(\frac{(w-c_s^2)\mu_d}{\mu_d+\mu_r} + \frac{4w\mu_d}{3\mu_d+4\mu_r} \\
&- \frac{\mu_d\mu_r}{3(\mu_d+\mu_r)^2}\Bigg)\dot{R}\Theta\frac{f''}{f'} + \Bigg(\frac{4\mu_d\mu_r}{3(3\mu_d+4\mu_r)^2}\frac{4w\mu_r+(4w+1)\mu_d}{\mu_d+\mu_r} - \frac{(c_s^2-\frac{1}{3})\mu_d\mu_r}{3(\mu_d+\mu_r)^2} \\
&- \frac{4(c_s^2-\frac{2w}{3})\mu_d}{3\mu_d+4\mu_r} - \frac{(c_s^2-\frac{1}{3})(c_s^2-w)\mu_d}{\mu_d+\mu_r}\Bigg)\Theta^2 + \left(\frac{4w\mu_d}{3\mu_d+4\mu_r} - \frac{c_s^2\mu_d}{\mu_d+\mu_r}\right)\frac{f}{f'} \\
&- \left(\frac{\left(1+w-2c_s^2\right)\mu_d}{\mu_d+\mu_r} + \frac{8w\mu_d}{3\mu_d+4\mu_r}\right)\frac{\mu_d+\mu_r}{f'}\Bigg]\Delta^k_d + \frac{4}{3}\Theta\frac{f''}{f'}\dot{\mathcal{R}}^k \\
&+ \frac{4}{3}\left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''(\mu_r+\mu_d)}{f'^2} - \dot{R}\Theta\left(\frac{f''}{f'}\right)^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k = 0\,.
\end{aligned}
\tag{8.137}
$$

Similarly the propagation equation of the dust-density gradient is given by,

$$
\begin{aligned}
&\ddot{\Delta}^k_d + \left[\left(\frac{2}{3} + \frac{\mu_d}{1+w}(\frac{4w}{3\mu_d+4\mu_r} - \frac{c_s^2}{\mu_d+\mu_r})\right)\Theta - \frac{\mu_d}{(1+w)(\mu_d+\mu_r)}\dot{R}\frac{f''}{f'}\right]\dot{\Delta}^k_d \\
&-\frac{1}{1+w}\left[\left(\frac{3w\mu_d}{3\mu_d+4\mu_r} + \frac{c_s^2\mu_r}{\mu_d+\mu_r}\right)\Theta + \frac{\mu_r}{\mu_d+\mu_r}\dot{R}\frac{f''}{f'}\right]\dot{\Delta}^k_r \\
&+\frac{\mu_d}{1+w}\left[\left(\frac{w-c_s^2}{\mu_d+\mu_r} + \frac{4w}{3\mu_d+4\mu_r} - \frac{\mu_r}{3(\mu_d+\mu_r)^2}\right)\dot{R}\Theta\frac{f''}{f'} + \left(\frac{4\mu_r}{3(3\mu_d+4\mu_r)^2}\times\right.\right. \\
&\left.\frac{4w\mu_r+(4w+1)\mu_d}{\mu_d+\mu_r} - \frac{4(c_s^2-w)}{3\mu_d+4\mu_r} - \frac{(c_s^2-w)c_s^2}{\mu_d+\mu_r} - \frac{c_s^2\mu_r}{3(\mu_d+\mu_r)^2}\right)\Theta^2 \\
&\left.-\left(\frac{1+w-2c_s^2}{\mu_d+\mu_r} + \frac{8w}{3\mu_d+4\mu_r}\right)\frac{\mu_d+\mu_r}{f'} + \left(\frac{4w}{3\mu_d+4\mu_r} - \frac{c_s^2}{\mu_d+\mu_r}\right)\frac{f}{f'}\right]\Delta^k_d \\
&+\frac{1}{1+w}\left[\left(\frac{(w-c_s^2)\mu_r}{\mu_d+\mu_r} - \frac{3w\mu_d}{3\mu_d+4\mu_r} - \frac{\mu_d\mu_r}{3(\mu_d+\mu_r)^2}\right)\dot{R}\Theta\frac{f''}{f'} + \left(\frac{c_s^2\mu_d\mu_r}{3(\mu_d+\mu_r)^2}\right.\right. \\
&\left.+\frac{3(c_s^2-w)\mu_d}{3\mu_d+4\mu_r} - \frac{(c_s^2-w)c_s^2\mu_r}{\mu_d+\mu_r} + \frac{4\mu_d\mu_r}{(3\mu_d+4\mu_r)^2}\frac{(1-3w)\mu_r-3w\mu_d}{3(\mu_d+\mu_r)}\right)\Theta^2 \\
&-\left(\frac{3w\mu_d}{3\mu_d+4\mu_r} + \frac{c_s^2\mu_r}{\mu_d+\mu_r}\right)\frac{f}{f'} - \left(\frac{\left(1+w-2c_s^2\right)\mu_r}{\mu_d+\mu_r} - \frac{6w\mu_d}{3\mu_d+4\mu_r}\right)\times \\
&\left.\frac{\mu_d+\mu_r}{f'}\right]\Delta^k_r + \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''(\mu_r+\mu_d)}{f'^2} - \dot{R}\Theta\left(\frac{f''}{f'}\right)^2\right. \\
&\left.+\dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k + \Theta\frac{f''}{f'}\dot{\mathcal{R}}^k = 0\,. \qquad (8.138)
\end{aligned}
$$

In terms of the component perturbation variables we can rewrite the propagation equation for the curvature fluid gradient as,

$$
\begin{aligned}
&\ddot{\mathcal{R}}^k + \left(2\dot{R}\frac{f'''}{f''} + \Theta\right)\dot{\mathcal{R}}^k + \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(iv)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}^k \\
&\quad + \left(\frac{c_s^2 - 1}{1+w}\frac{\mu_d}{\mu_d + \mu_r} - \frac{4wc_z^2}{1+w}\right)\dot{R}\dot{\Delta}_d^k + \left(\frac{c_s^2 - 1}{1+w}\frac{\mu_r}{\mu_d + \mu_r} + \frac{3wc_z^2}{1+w}\right)\dot{R}\dot{\Delta}_r^k \\
&\quad + \left\{\left[(3c_s^2 - 1)\frac{\mu_d + \mu_r}{3f''} + \frac{w + c_s^2}{1+w}\dot{R}\Theta + \frac{c_s^2}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right)\right.\right. \\
&\quad \left. + \frac{\dot{R}}{1+w}\left(\dot{c_s^2} + c_s^2(c_s^2 - w)\Theta\right)\right]\frac{\mu_d}{\mu_d + \mu_r} - 4wc_z^2\left[\frac{\mu_d + \mu_r}{f''} + \frac{2}{1+w}\left(\ddot{R} + \dot{R}\Theta\right.\right. \\
&\quad \left.\left.\left. + \dot{R}^2\frac{f'''}{f''}\right)\right] + \frac{c_s^2 - 1}{3(1+w)}\frac{\mu_d\mu_r}{(\mu_d + \mu_r)^2}\dot{R}\Theta\right\}\Delta_d^k + \left\{\left[(3c_s^2 - 1)\frac{\mu_d + \mu_r}{3f''}\right.\right. \\
&\quad \left. + \frac{w + c_s^2}{1+w}\dot{R}\Theta + \frac{c_s^2}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right) + \frac{\dot{R}}{1+w}\left(\dot{c_s^2} + c_s^2(c_s^2 - w)\Theta\right)\right]\frac{\mu_r}{\mu_d + \mu_r} \\
&\quad + 3wc_z^2\left[\frac{\mu_d + \mu_r}{f''} + \frac{2}{1+w}\left(\ddot{R} + \dot{R}\Theta + \dot{R}^2\frac{f'''}{f''}\right)\right] \\
&\quad \left. - \frac{c_s^2 - 1}{3(1+w)}\frac{\mu_d\mu_r}{(\mu_d + \mu_r)^2}\dot{R}\Theta\right\}\Delta_r^k = 0\,. \qquad (8.139)
\end{aligned}
$$

8.9 Short wavelength solutions

In this section, we study the evolution of the short-wavelength modes, i.e., large values of the wave number k, by using the equations presented in Section 6 valid for a radiation-and-dust mixture. The general results will then be considered for R^n models and a proposal for a *quasi-static* approximation for the matter perturbations will be introduced for both radiation- and dust-dominated epochs.

8.9.1 Perturbations in the radiation-dominated epoch

Let us now look at the case where the characteristic size of the fluid inhomogeneities is much less than the Jeans length for the radiation fluid but is still larger than the mean free path of the photon, i.e., $\lambda \ll \lambda_H \ll \lambda_J$. Similar investigation has been made by [149] for the case of GR. Here we assume that we can neglect the interaction between the component fluids and that the radiation energy density can be taken as *almost* homogeneous, meaning$\Delta_r \approx 0$. This amounts to studying dust and curvature fluctuations on a homogeneous radiation background, whereby radiation affects the growth of the dust fluctuations by speeding up the cosmic expansion [148]. We consider the flat ($K = 0$) background, and hence the equations for such a background are given by,

$$\dot{\Delta}_d^k + Z^k = \frac{\Theta}{h}(c_s^2\mu_m\Delta_m^k + p_m\varepsilon^k) + a(\frac{k^2}{a^2})V_d^k, \tag{8.140}$$

$$\dot{Z}^k - (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z^k - \left[\frac{(2c_s^2 - w - 1)}{(1+w)}\frac{\mu_m}{f'} - \frac{c_s^2}{(1+w)}\left(\frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'}\right)\right]\Delta_m^k$$
$$- \frac{w}{(1+w)}\left[2\frac{\mu_m}{f'} + \frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'}\right]\varepsilon^k - \frac{1}{h}(\frac{k^2}{a^2})\left[c_s^2\mu_m\Delta_m^k + p_m\varepsilon^k\right] + \Theta\frac{f''}{f'}\Re^k$$
$$- \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k = 0, \tag{8.141}$$

$$\dot{V}_d^k + \frac{1}{3}\Theta V_d^k = \frac{1}{ah}\left[c_s^2\mu_m\Delta_m^k + p_m\varepsilon^k\right], \tag{8.142}$$

$$\dot{V}_{dr}^k - (c_z^2 - \frac{1}{3})\Theta V_{dr}^k = \frac{1}{3ah}\mu_m\Delta_m^k - \frac{1}{a}c_z^2 S_{dr}^k, \tag{8.143}$$

$$\dot{\mathcal{R}}^k = \Re^k - \dot{R}\frac{1}{h}\left[c_s^2\mu_m\Delta_m^k + p_m\varepsilon^k\right], \tag{8.144}$$

$$\dot{\Re}^k = -\left(2\dot{R}\frac{f'''}{f''} + \Theta\right)\Re^k - \dot{R}Z^k + \frac{\mu_m}{3f''}\Delta_m^k - \frac{1}{f''}\left[c_s^2\mu_m\Delta_m^k + p_m\varepsilon^k\right]$$
$$- \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}^k. \tag{8.145}$$

Since$\Delta_r \ll \Delta_d$ we have,

$$c_s^2\mu\Delta_m^k + p\varepsilon^k = \frac{1}{3}\mu_r\Delta_r^k \approx 0, \tag{8.146}$$

and so,

$$S_{dr}^k \approx \Delta_d^k. \tag{8.147}$$

In these limits the above set of equations (8.140-8.145) can be rewritten as,

$$\dot{\Delta}_d^k + Z^k - a(\frac{k^2}{a^2})V_d^k = 0, \tag{8.148}$$

$$\dot{Z}^k - (\dot{R}\frac{f''}{f'} - 2H)Z^k + \frac{\mu_d}{f'}\Delta_d^k - \Theta\frac{f''}{f'}\Re^k$$
$$- \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_r}{f'^2} - 3H\dot{R}(\frac{f''}{f'})^2 + 3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k = 0, \tag{8.149}$$

$$\dot{V}_d^k + HV_d^k = 0, \tag{8.150}$$

$$\dot{V}_{dr}^k + \frac{4}{3}\frac{\mu_r}{h}HV_{dr}^k = 0, \tag{8.151}$$

$$\dot{\mathcal{R}}^k = \Re^k, \tag{8.152}$$

$$\dot{\Re}^k = -\left(2\dot{R}\frac{f'''}{f''} + 3H\right)\Re^k - \dot{R}Z^k + \frac{\mu_d}{3f''}\Delta_d^k$$
$$- \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + 3H\dot{R}\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}^k. \tag{8.153}$$

From Eqns. (8.148)-(8.153) we obtain the following two second order differential equations:

$$\ddot{\Delta}_d^k + \left(2H - \frac{3\dot{R}f''}{4f'}\frac{\mu_d}{\mu_r}\right)\dot{\Delta}_d^k - \frac{\mu_d}{f'}\Delta_d^k + 3H\frac{f''}{f'}\dot{\mathcal{R}}^k$$

$$+ \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_r}{f'^2} - 3H\dot{R}(\frac{f''}{f'})^2 + 3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k = 0, \tag{8.154}$$

$$\ddot{\mathcal{R}}^k + \left(2\dot{R}\frac{f'''}{f''} + 3H\right)\dot{\mathcal{R}}^k - \frac{3\dot{R}\mu_d}{4\mu_r}\dot{\Delta}_d^k - \frac{\mu_d}{3f''}\Delta_d^k$$

$$+ \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + 3H\dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3}\right)\mathcal{R}^k = 0. \tag{8.155}$$

It can be shown that H and $\dot{R}f''/f'$ are of the same order for R^n models, whereas $\mu_d \ll \mu_r$, implying that curvature and radiation fluids effectively dominate the fluctuation dynamics. In effect, terms like $\mu_d\Delta_d^k$ merely sub-dominate in the curvature-radiation-dust mixture. Hence we can safely approximate the above equations by ,

$$\ddot{\Delta}_d^k + 2H\dot{\Delta}_d^k + 3H\frac{f''}{f'}\dot{\mathcal{R}}^k + \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_r}{f'^2} - 3H\dot{R}(\frac{f''}{f'})^2 + 3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k = 0, \tag{8.156}$$

$$\ddot{\mathcal{R}}^k + \left(2\dot{R}\frac{f'''}{f''} + 3H\right)\dot{\mathcal{R}}^k + \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + 3H\dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3}\right)\mathcal{R}^k = 0. \tag{8.157}$$

These two equations tell us that, deep in the radiation-dominated era, the matter and curvature fluctuations are decoupled in the second order equations. GR is a specific example of the generalized R^n models where $n = 1$. In this limit, Eqns. (8.156) and (8.157) reduce to ,

$$\ddot{\Delta}_d^k + 2H\dot{\Delta}_d^k + \tfrac{1}{2}\mathcal{R}^k = 0\,, \tag{8.158}$$

$$\mathcal{R}^k = 0\,, \tag{8.159}$$

thus yielding the standard GR equation for the density contrast in a radiation background ,

$$\ddot{\Delta}_d^k + \frac{1}{t}\dot{\Delta}_d^k = 0\,, \tag{8.160}$$

whose general solution is given by ,

$$\Delta_d^k(t) = C_1 + C_2\ln t\,. \tag{8.161}$$

with $C_{1,2}$ arbitrary constants. For $n \neq 1$, with $w = \frac{1}{3}$ in the radiation-dominated epoch, Eqns. (8.156) and (8.157) take the following forms:

$$\ddot{\Delta}_d^k + \tfrac{n}{t}\dot{\Delta}_d^k + \tfrac{t}{2}\dot{\mathcal{R}}^k + \left[\tfrac{12-5n}{4} + \tfrac{n}{12}\left(\tfrac{\lambda_H}{\lambda}\right)^2_{eq} t^{2-n}\right]\mathcal{R}^k = 0\,, \tag{8.162}$$

$$\ddot{\mathcal{R}}^k - \left(\tfrac{5n-16}{2t}\right)\dot{\mathcal{R}}^k + \left[\tfrac{n^2}{4}\left(\tfrac{\lambda_H}{\lambda}\right)^2_{eq} t^{-n} - \tfrac{6(n-2)}{t^2}\right]\mathcal{R}^k = 0\,, \tag{8.163}$$

where we have used the fact that $\frac{k^2}{a^2} = \frac{n^2}{4}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{-n}$ with normalized time $t_{eq} = 1$ at the time of *radiation-matter equality* .

Quasi-static analysis

In general the system of equations (8.162)-(8.163) yields Bessel hypergeometric-type analytic solutions. However, since we are dealing with small scales we can take a *quasi-static approximation* where the time variations in $\mathcal{R}^k$ are treated as negligible, i.e., $\ddot{\mathcal{R}}^k \simeq 0$ and $\dot{\mathcal{R}}^k \simeq 0$. In this scheme the overall dynamics of the density perturbations lead to the simplified, k- independent, equation ,

$$\ddot{\Delta}^k_d + \frac{n}{t}\dot{\Delta}^k_d = 0 \; . \tag{8.164}$$

This equation admits the general solution

$$\Delta^k_d(t) = C_1 + C_2 t^{1-n} \; . \tag{8.165}$$

On small scales, radiation suppresses the growth of fluctuations as they enter the horizon before radiation-dust equality, and dust (baryon) self-gravitation is not yet strong enough to offset the cosmic expansion. This is because the expansion scale factor grows faster than the perturbation amplitudes do. The phenomenon is known in the literature as the *Mészáros effect* [157]. It is clear from the above analysis that the Mészáros effect puts a constraint on the value of n in R^n gravity. To do so, all we need do is determine the allowed values of n for which the perturbation amplitudes grow slower than the expansion in the radiation dominated era, i.e. ,

$$\frac{d}{dt}\left[\frac{\Delta^k_d(t)}{a(t)}\right] \propto \frac{d}{dt}\left[\frac{t^{1-n}}{t^{\frac{n}{2}}}\right] < 0 \Rightarrow 1 - \frac{3n}{2} < 0 \; . \tag{8.166}$$

This means that only values of $n > \frac{2}{3}$ give a growth rate compatible with the Mészáros effect. In figure 8.9.1, we plot the normalized dust-density contrast $\delta(t) \equiv \Delta_m(t)/\Delta_{eq}$ in the radiation-dominated epoch . The figure on the right shows the relative error of the full solutions and the quasi-static approximation.

8.9.2 Perturbations in the Dust-dominated epoch

During this epoch of the Universe the dust energy density is dominating in the two-fluid dynamics and all order-of-magnitude approximations go in line with the assumption that $\mu_d >> \mu_r$. The equations (8.140)-(8.145) will, upon imposing the short-wavelength assumptions (8.146,8.147),

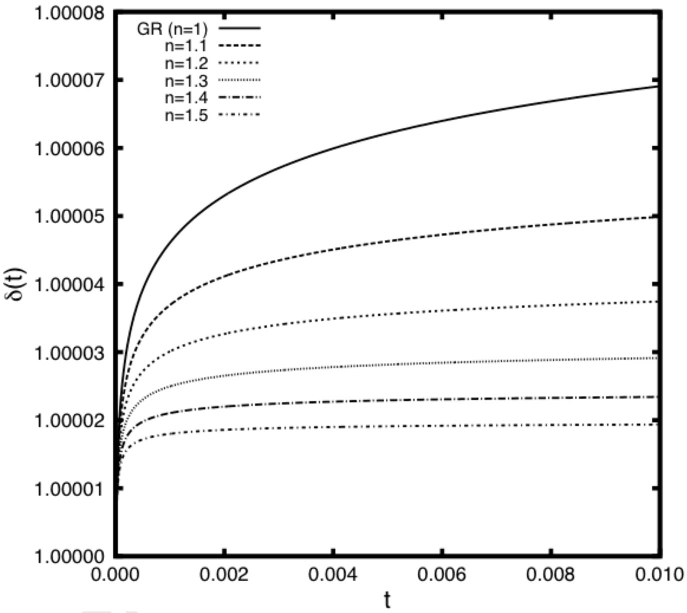


Figure 8.2: Dust growth factor in the radiation-dominated epoch for R^n models: The plots show the growth factor obtained by solving numerically the full system of equations (8.162) and (8.163) for scale $\lambda_H = 100\lambda$ and the quasi-static solution (8.164) for $n = 1.5, 1.4, 1.3, 1.2, 1.1$ from bottom to top. The topmost plot corresponds to GR ($n = 1$). It can be seen that quasi-static results are quite close to those of the full system for the stated values of n, only slightly (but invisibly) lower in the plots. For values of $\lambda_H > 100\lambda$ the growth factor appears to be insensitive to scale showing the convenience of introducing the *quasi-static* approximation.

become ,

$$\dot{\Delta}_d^k + Z^k + a\tilde{\nabla}^2 V_d^k = 0, \tag{8.167}$$

$$\dot{Z}^k - (\dot{R}\frac{f''}{f'} - 2H)Z^k + \frac{\mu_d}{f'}\Delta_d^k - \Theta\frac{f''}{f'}\Re^k$$
$$-\left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_d}{f'^2} - 3H\dot{R}(\frac{f''}{f'})^2 + 3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k = 0, \tag{8.168}$$

$$\dot{V}_d^k + HV_d^k = 0, \tag{8.169}$$

$$\dot{V}_{dr}^k + \frac{4}{3}\frac{\mu_r}{h}HV_{dr}^k = 0 \Rightarrow \dot{V}_{dr}^k + \frac{4\mu_r}{3\mu_d}HV_{dr}^k = 0, \tag{8.170}$$

$$\dot{\mathcal{R}}^k = \Re^k, \tag{8.171}$$

$$\dot{\Re}^k = -\left(2\dot{R}\frac{f'''}{f''} + 3H\right)\Re^k - \dot{R}Z^k + \frac{\mu_d}{3f''}\Delta_d^k$$
$$-\left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + 3H\dot{R}\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}^k. \tag{8.172}$$

The resulting second-order equation is therefore ,

$$\ddot{\Delta}_d^k + \left(2H - \dot{R}\frac{f''}{f'}\right)\dot{\Delta}_d^k - \frac{\mu_d}{f'}\Delta_d^k + 3H\frac{f''}{f'}\dot{\mathcal{R}}^k$$
$$+\left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_d}{f'^2} - 3H\dot{R}(\frac{f''}{f'})^2 + 3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k = 0, \tag{8.173}$$

$$\ddot{\mathcal{R}}^k + \left(2\dot{R}\frac{f'''}{f''} + 3H\right)\dot{\mathcal{R}}^k - \dot{R}\dot{\Delta}_d^k - \frac{\mu_d}{3f''}\Delta_d^k$$
$$+\left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + 3H\dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3}\right)\mathcal{R}^k = 0. \tag{8.174}$$

As can be observed, these two equations differ from their counterparts in the radiation-dominated epoch in that they form a *coupled* system of equations. The limiting GR perturbation equations for (8.173) and (8.174) in this epoch are given by ,

$$\ddot{\Delta}_d^k + 2H\dot{\Delta}_d^k - \mu_d\Delta_d^k + \tfrac{1}{2}\mathcal{R}^k = 0\ , \tag{8.175}$$

$$-\tfrac{\mu_d}{3}\Delta_d^k + \tfrac{1}{3}\mathcal{R}^k = 0\ , \tag{8.176}$$

and combine to give the equation

$$\ddot{\Delta}_d^k + \frac{4}{3t}\dot{\Delta}_d^k - \frac{2}{3t^2}\Delta_d^k = 0\ . \tag{8.177}$$

This equation admits the well known solution

$$\Delta_d^k(t) = C_1 t^{-1} + C_2 t^{\frac{2}{3}}\ . \tag{8.178}$$

For R^n models, equations (8.173,8.174) take the form,

$$\ddot{\Delta}_d^k + \left(\frac{10n-6}{3t}\right)\dot{\Delta}_d^k + \frac{2(8n^2-13n+3)}{3t^2}\Delta_d^k + \frac{3(n-1)}{2(4n-3)}t\dot{\mathcal{R}}^k + \left[\frac{n(n-1)}{3(4n-3)}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{2-\frac{4n}{3}} + \frac{27n^2-8n^3-18n}{2n(4n-3)}\right]\mathcal{R}^k = 0\ , \tag{8.179}$$

$$\ddot{\mathcal{R}}^k + \left\{\frac{8n\left[n(8n-13)+3\right](4n-3)}{27(n-1)t^4}\right\}\Delta_d^k + \frac{8n(4n-3)}{3t^3}\dot{\Delta}_d^k + \frac{8-2n}{t}\dot{\mathcal{R}}^k + \left\{\frac{4n^2}{9}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{-\frac{4n}{3}} - \frac{2\left[n(8n+5)-69\right]+54}{9(n-1)t^2}\right\}\mathcal{R}^k = 0\ , \tag{8.180}$$

where $\frac{k^2}{a^2} = \frac{4n^2}{9}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{-\frac{4n}{3}}$ during this epoch.

Quasi-static analysis

In the quasi-static limit with $\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} >> 1$ we get a single second order k-scale independent equation,

$$\ddot{\Delta}_d^k + \frac{4n}{3t}\dot{\Delta}_d^k + \left[\frac{4(8n^2-13n+3)}{9t^2}\right]\Delta_d^k = 0\ , \tag{8.181}$$

the solution of which is given by,

$$\Delta_d^k(t) = C_1 t^{\alpha_+} + C_2 t^{\alpha_-}\ , \tag{8.182}$$

where $\alpha_\pm = -\frac{2n}{3} + \frac{1}{2} \pm \frac{\sqrt{-112n^2+184n-39}}{6}$. The coefficients $C_{1,2}$ can be determined by imposing initial conditions. At $t = t_{eq} = 1$ we have,

$$\Delta^k_{(d)\,eq} \equiv \Delta^k_{(d)}(t_{eq}) = C_1 + C_2\ , \tag{8.183}$$

and differentiating (8.182) gives,

$$\dot{\Delta}_d^k(t) = \quad \alpha_+ C_1 t^{\alpha_+ -1} + \alpha_- C_2 t^{\alpha_- -1}\ , \tag{8.184}$$

which, at equality, will give,

$$\dot{\Delta}^k_{(d)\,eq} \equiv \dot{\Delta}^k_{(d)}(t_{eq}) = \alpha_+ C_1 + \alpha_- C_2\ . \tag{8.185}$$

Solving (8.183) and (8.185) simultaneously we obtain,

$$C_{1,2} = \frac{\pm\dot{\Delta}^k_{(d)\,eq} \mp \alpha_\mp \Delta^k_{(d)\,eq}}{\alpha_+ - \alpha_-}\ .\ ;\quad C_2 = \frac{-\dot{\Delta}^k_{(d)\,eq} + \alpha_+\Delta^k_{(d)\,eq}}{\alpha_+ - \alpha_-}. \tag{8.186}$$

The following plots show the evolution of the density perturbations $\delta(t) \equiv \Delta^k_{(d)}(t)/\Delta^k_{(d)eq}$ in time (t from 1 onwards, where $t = t_{eq} = 1$ is the normalized time at equality) for the above linearly independent solutions, $C_{1,2}$ having been obtained by setting$\Delta\ ^k_{(d)eq} = 10^{-5}$ and, $\dot{\Delta}^k_{(d)eq} = 10^{-5}$.

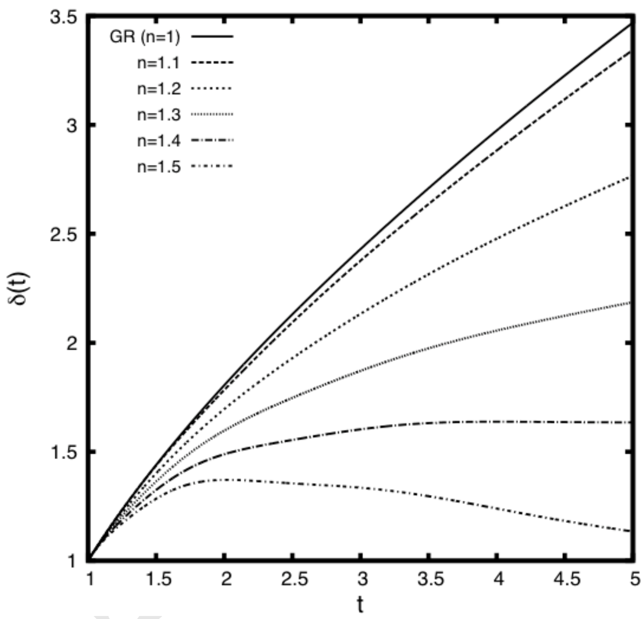


Figure 8.3: Dust growth factor in the dust-dominated epoch for R^n models: The plots show the growth factor obtained by solving numerically the full system equations (8.179) and (8.180) for scale $\lambda_H = 100\lambda$ and the quasi-static analytic solution (8.181) . It can be seen that quasi-static results are indistinguishable from the full results for $n = 1.5, 1.4, 1.3, 1.2, 1.1$ from bottom to top, with the full system solutions slightly higher than those of the quasi-static approximation. It can also be seen that for higher values of n the growth factor increases more slowly till a critical value of n somewhere between 1.4 & 1.5 where the growth factor becomes a decreasing function of time. Note the $n = 1$ case (GR) is presented on the topmost plot.

8.10 Long wavelength solutions

For specific intervals of n, a set of initial conditions gives rise to cosmic histories which include a transient decelerated phase which evolves towards an accelerated phase. Structure formation takes place during the transient regime [154]. In this section we analyze the evolution of scalar perturbations during this phase, in the *long wavelength limit.* In this limit the wavenumber k is so small that $\lambda = \frac{2\pi a}{k} \gg \lambda_H$, i.e., $\frac{k^2}{a^2H^2} \ll 1$. All Laplacian terms can therefore be neglected and spatially flat ($K = 0$) backgrounds guarantee the conservation of C, i.e., $\dot{C}^k = 0$. In this thesis we are considering only adiabatic perturbations, i.e. $S_{ij} = 0$ and hence, for a radiation-dust mixture, the equation for the evolution of entropy perturbations,

$$\dot{S}_{dr} + a\tilde{\nabla}^2 V_{dr} = 0 \; . \tag{8.187}$$

implies that

$$V_{dr} = 0 \; . \tag{8.188}$$

And from this and the equation,

$$\dot{V}_{dr} - \left(c_z^2 - \frac{1}{3}\right)\Theta V_{dr} = -\frac{1}{ah}(c_{sd}^2 - c_{sr}^2)\mu\Delta_m - \frac{1}{a}c_z^2 S_{dr} \; . \tag{8.189}$$

follows,

$$(c_{sd}^2 - c_{sr}^2)\mu\Delta_m = 0 \; . \tag{8.190}$$

We therefore have the following system of equations:

$$\dot{\Delta}_m + (1+w)Z - w\Theta\Delta_m = 0, \tag{8.191}$$

$$\dot{Z} = (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z + \left[\frac{(2c_s^2 - w - 1)}{(1+w)}\frac{\mu_m}{f'} + \frac{c_s^2}{(1+w)}\left(\frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'}\right)\right]\Delta_m$$
$$+ \Theta\frac{f''}{f'}\Re + \left[\frac{1}{2} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}, \tag{8.192}$$

$$\dot{\mathcal{R}} = \Re - \frac{c_s^2}{1+w}\dot{R}\Delta_m, \tag{8.193}$$

$$\dot{\Re} = -\left(2\dot{R}\frac{f'''}{f''} + \Theta\right)\Re - \dot{R}Z - \frac{(3c_s^2 - 1)\mu_m}{3f''}\Delta_m$$
$$- \left(\ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}, \tag{8.194}$$

$$\frac{C_0}{a^2} + \left(\frac{4}{3}\Theta + 2\frac{\dot{R}f''}{f'}\right)Z - 2\frac{\mu_m}{f'}\Delta_m + \left[2\dot{R}\Theta\frac{f'''}{f'} - \frac{f''}{f'}\left(\frac{f}{f'} - 2\frac{\mu_m}{f'} + 2\dot{R}\Theta\frac{f''}{f'}\right)\right]\mathcal{R}$$
$$+ 2\,\Theta\frac{f''}{f'}\Re = 0. \tag{8.195}$$

C_0 being the conserved value for the quantity C. In terms of the background R^n solutions and making use of the conservation of C the above equations can be rewritten as,

$$\dot{\Delta}_m = \left[\frac{1+w-2n}{1+w} - \frac{6(n-1)n}{n+3(n-1)w-3}\right]\frac{\Delta_m}{t} - \frac{3(1+w)^2}{4a_0^2\left[n+3(n-1)w-3\right]\left[4n-3(1+w)\right]}\times$$
$$t^{1-\frac{4n}{3(1+w)}}C_0 - \frac{9(n-1)(1+w)^3t^2}{4\left[n+3(n-1)w-3\right]\left[4n-3(1+w)\right]}t^2\Re$$
$$+\left[\frac{3(n-1)(1+w)^2\left[n(6w+8)-15(1+w)\right]}{4\left[n+3(n-1)w-3\right]\left[4n-3(1+w)\right]}\right]t\mathcal{R}, \tag{8.196}$$

$$\dot{\mathcal{R}} = \Re + \frac{8nc_s^2\left[4n-3(1+w)\right]}{3(1+w)^3}\frac{\Delta_m}{t^3}, \tag{8.197}$$

$$\dot{\Re} = -2\left[\frac{(n-4)+2(n-2)w}{(1+w)t} - \frac{3n(n-1)}{n+3w(n-1)-3}\right]\Re + \frac{2n(4n-3w-3)}{(1+w)\left[n+3(n-1)w-3\right]}$$
$$\frac{C_0}{a_0^2}t^{-\frac{4n}{3(1+w)}-2} - 2\left[\frac{9n(n-2)(n-1)}{n+3(n-1)w-3} + 2n^2 - 7n - \frac{3n^2(9n-26)+57n}{9(1+w)(n-1)} - \frac{8n^2(n-2)}{9(1+w)^2(n-1)} + 6\right]$$
$$\frac{\mathcal{R}}{t^2} + \frac{16n\left[4n-3(1+w)\right]\left[4n+3(n-1)w-3\right]}{27(n-1)(1+w)^4\left[n+3(n-1)w-3\right]}$$
$$\frac{\left[(9w(1+w)+8)\,n^2-(3w(9w+8)+13)\,n+3(1+w)(1+6w)\right]}{27(n-1)(1+w)^4\left[n+3(n-1)w-3\right]}\frac{\Delta_m}{t^4}. \tag{8.198}$$

8.10.1 Perturbations in the Radiation-dominated epoch

The second-order set of equations governing the dynamics of density perturbations in this epoch is given by,

$$\ddot{\Delta}^k_m + \frac{n(9n-14)+4}{2(n-2)t}\dot{\Delta}^k_m + \frac{n\left(n(n(19n-54)+58)-32\right)+8}{2(n-2)^2t^2}\Delta^k_m$$
$$+\frac{2\left(n(3n-4)+2\right)}{3(n-2)^2}t\dot{\mathcal{R}}^k - \frac{(n(15n-22)+14)}{3(n-2)}\mathcal{R}^k + \frac{4(n^2-1)}{3(n-2)^2a_0^2}t^{-n}C_0 = 0, \tag{8.199}$$
$$\ddot{\mathcal{R}}^k - \frac{n(11n-32)+32}{2(n-2)t}\dot{\mathcal{R}}^k + \frac{3\left(n(5n-9)+8\right)}{2t^2}\mathcal{R}^k - \frac{3n\left(n(n-3)+2\right)}{2(n-2)t^3}\dot{\Delta}^k_m$$
$$-\frac{3n(n-1)\left(n(19n-28)+4\right)}{4(n-2)t^4}\Delta^k_m - \frac{3n(n-1)}{(n-2)a_0^2}t^{-(n+2)}C_0 = 0. \tag{8.200}$$

Making use of the conservation of C, we can eliminate $\dot{\mathcal{R}}^k$ and $\mathcal{R}^k$ quantities in favor of$\Delta^{\ k}_{r}$ (and its derivatives) and C_o. This way we can get a decoupled third order k-scale independent equation for$\Delta^{\ k}_{r}$:

$$\dddot{\Delta}^k_r - \tfrac{n-5}{t}\ddot{\Delta}^k_r + \tfrac{24n-19n^2+8}{4t^2}\dot{\Delta}^k_r + \tfrac{(n-2)\left[5n^2-8n+2\right]}{2t^3}\Delta^k_r - \tfrac{(12-7n)C_0}{3t^{(n+1)}} = 0\,. \tag{8.201}$$

This equation admits the general solution,

$$\Delta^k_r(t) = C_1t^{\frac{n}{2}-1} + C_2t^{\beta_+} + C_3t^{\beta_-} + C_4t^{2-n}\,. \tag{8.202}$$

where $C_{1,2,3}$ are arbitrary integration constants to be evaluated from initial conditions with ,

$$C_4 \equiv \frac{2\,(24-14n)\,C_0}{9(7n^3-18n^2+16)} \tag{8.203}$$

and

$$\beta_{\pm} \equiv -\frac{1}{2}+\frac{n}{4}\pm\frac{\sqrt{3(81n^2-44n+12)}}{4}\,. \tag{8.204}$$

Provided that the initial values of$\Delta\,^k_r$, $\dot{\Delta}^k_r$, $\ddot{\Delta}^k_r$ and C_0 are known at $t_{eq}=1$, the integration constants can be determined since ,

$$\begin{aligned}
\Delta^k_{(r)eq} &= C_1+C_2+C_3+C_4,\\
\dot{\Delta}^k_{(r)eq} &= \left(\tfrac{n-2}{2}\right)C_1+C_2\beta_+ + C_3\beta_- + (2-n)C_4\,,\\
\ddot{\Delta}^k_{(r)eq} &= \left[\tfrac{(n-2)(n-4)}{4}\right]C_1+C_2\beta_+(\beta_+-1)\\
&\quad +C_3\beta_-(\beta_--1)+(2-n)(1-n)C_4\,.
\end{aligned} \tag{8.205}$$

We do not present $C_{1,2,3}$ explicitly for the sake of simplicity .

8.10.2 Perturbations in the Dust-dominated epoch

Proceeding in a similar fashion for the dust-dominated, long-wavelength regime gives the second-order evolution equations given by ,

$$\begin{aligned}
&\ddot{\Delta}^k_m+\frac{n(8n-13)+3}{(n-3)t}\dot{\Delta}^k_m+\frac{(n(8n-13)+3)\,(n(16n-15)+9)}{3(n-3)^2t^2}\Delta^k_m\\
&+\frac{3(n-1)\,(n(16n-15)+9)}{4(n-3)^2(4n-3)}t\dot{\mathcal{R}}^k-\frac{n\left[(n(16n(8n-31)+711)-540\right]+189}{4(n-3)^2(4n-3)}\mathcal{R}^k\\
&-\frac{n\left(27+54n-56n^2\right)-27)}{4(n-3)^2(4n-3)a_0^2}t^{-\frac{4n}{3}}C_0=0,
\end{aligned} \tag{8.206}$$

$$\begin{aligned}
&\ddot{\mathcal{R}}^k-\frac{4(n-1)\,(n(2n-5)+6)}{(n(n-4)+3)\,t}\dot{\mathcal{R}}^k+\frac{4\left[n\left(n(2n(16n-65)+213)-198\right)+81\right]}{9\,(n(n-4)+3)\,t^2}\mathcal{R}^k\\
&-\frac{16n(3-4n)^2\,(n(8n-13)+3)}{27\,((n-4)n+3)\,t^4}\Delta^k_m-\frac{2n\,(n(4n-7)+3)}{(n(n-4)+3)\,a_0^2}t^{-(n+2)}C_0=0.
\end{aligned} \tag{8.207}$$

which reduce to a single third-order evolution equation for the density perturbations given as ,

$$\begin{aligned}
&\dddot{\Delta}^k_m+\frac{5}{t}\ddot{\Delta}^k_m-\frac{2\left[n\left(4n(8n-19)+33\right)+9\right]}{9(n-1)t^2}\dot{\Delta}^k_m\\
&-\frac{2(4n-3)\,(n(8n-13)+3)}{9(n-1)t^3}\Delta^k_m-\frac{(n(12n-31)+18)}{6(n-1)a_0^2}t^{-(n+1)}C_0=0,
\end{aligned} \tag{8.208}$$

which is a third-order decoupled k-scale independent equation. The general solution of (8.208) is given by ,

$$\Delta^k_d(t)=C_1t^{-1}+C_2t^{\gamma_+}+C_3t^{\gamma_-}+C_4t^{2-\frac{4n}{3}}\,, \tag{8.209}$$

where $C_{1,2,3}$ are arbitrary integration constants to be evaluated from initial conditions and,

$$C_4 \equiv \frac{9\left(12n^2-31n+18\right)C_0}{8(48n^4-184n^3+159n^2+63n-81)} \tag{8.210}$$

together with,

$$\gamma_{\pm} \equiv -\frac{1}{2} \mp \frac{1}{6}\sqrt{\frac{256n^3-608n^2+417n-81}{n-1}}\,. \tag{8.211}$$

As in the radiation epoch, the integration constants $C_{1,2,3}$ can be determined from the initial values of$\Delta^{\ k}_{d}$, $\dot{\Delta}^k_d$, $\ddot{\Delta}^k_d$ and C_0 known at $t_{eq} = 1$ as follows:

$$\begin{aligned}\Delta^k_{(d)eq} &= C_1 + C_2 + C_3 + C_4,\\ \dot{\Delta}^k_{(d)eq} &= -C_1 + C_2\gamma_+ + C_3\gamma_- + \left(\tfrac{6-4n}{3}\right)C_4,\\ \ddot{\Delta}^k_{(d)eq} &= 2C_1 + C_2\gamma_+(\gamma_+ - 1)\\ &+ C_3\gamma_-(\gamma_- - 1) + \tfrac{(6-4n)(3-4n)}{9}C_4\,. \end{aligned} \tag{8.212}$$

Once again, for the sake of simplicity, we do not present them here explicitly. It turns out that in the adiabatic limit, the long-wavelength solutions of the growth factor both in the radiation and dust epochs are exactly the same as those found in [154].

8.11 Conclusions

In this work we have extended the $1+3$ covariant and gauge-invariant cosmological perturbations formalism to a multi-component fluid Universe with a general equation of state parameter for an arbitrary $f(R)$ theory of gravity. The linearized evolution equations of the density and curvature perturbations of such a Universe have been derived in the energy frame of the total matter. We then have taken the background transient solutions of R^n gravity for a two-fluid system dominated respectively by radiation and dust and obtained solutions in both the short- and long-wavelength approximations. We found that for R^n gravity to be consistent with the Mészáros effect, the parameter n needs to satisfy $n > 2/3$.

In the short-wavelength limit, the quasi-static approximation turns out to be a good approximation for values of n in the vicinity of 1. This is the first time such a quasi-static analysis has been presented in a covariant way both for radiation and dust Universes and a comparison of this analysis with what is found using the metric formalism, together with a full computation of the power spectra will be presented in a future work.

In the next chapter we present for the first time a complete analysis of the imprint of tensor anisotropies on the CMB for a class of f (R) gravity.

Chapter 9

CMB TENSOR ANISTROPIES IN $f(R)$ GRAVITY

9.1 Introduction

The study of the CMB tensor perturbations in modified gravity theories has not received much attention in comparison with the scalar counterpart devoted to the study the density contrast of large-scale structure in these theories [226]. This fact which is related to the difficulty of obtaining the required tensor-perturbed equations which are in general of higher order. An alternative route in order to circumvent this difficulty consists of tackling the problem by using the simulations performed by several codes available such as CAMB [231], which is based on CMBFast [232].

Different attempts have been made for several modified gravity scenarios. For instance, the contribution made by cosmic strings in an Abelian Higgs model to the temperature power spectrum of the CMB was studied in [233]. The strings and their decay products source metric perturbations via their energy-momentum tensor, the unequal-time correlation functions of which were used as input into the CMB calculations. These calculations were performed in a modified version of CMB-Easy and were able to constrain the string tension when normalized with available WMAP data. Some predictions in the CMB spectrum for this model were presented in [234] and its effects on the integrated Sachs-Wolfe effect in [235]. Most of the attention devoted to the study of the tensor perturbations evolution in the last years has focused on brane-world theories. For instance, in [236] the evolution of cosmological tensor perturbations in the Randall-Sundrum II model was discussed. In the near-brane limit, the separation of the wave equations becomes possible and make the study of the evolution of perturbations feasible. Massive excitations were proved to decay outside the horizon leading to some novel cosmological signatures.

A complementary study on the evolution of tensor perturbations in a brane cosmology embedded in a five-dimensional anti-de Sitter bulk was given in [237]. In a Randall-Sundrum brane-world, the zero mode of the 5-dimensional graviton is generated during slow-roll inflation. When this zero mode of the 5-dimensional graviton re-enters the Hubble radius, massive modes are produced. It was shown that massive modes decouple in the low-energy/near-brane limit and the

mode-mixing at finite energy is then calculated.
Other attempts made in [238] involving the CAMB code [231] were able to compute the tensor anisotropies in the CMB, as generated for a generalized $1+4$ Randall-Sundrum II brane-world model. Corrections to the power spectra for standard temperature and polarization anisotropies were seen to depend upon a single dimensionless parameter.

Finally, with regard to $f(R)$ fourth-order gravity theories, the only attempt to encapsulate the main features of tensor perturbation was made in [240]. The authors of this investigation analyzed the tensor perturbations of flat thick domain wall branes in $f(R)$ gravity. They showed that under the transverse and traceless gauge, the metric perturbations decouple from the perturbation of the background scalar field which generates the brane. In addition, they also found that only when the bulk curvature is a constant or when $f(R) = R$, the perturbed equation reduces to the standard Klein-Gordon equation for massless spin-2 particles.
The purpose of this investigation is therefore to address for the first time in the literature a complete calculations of tensor perturbations for a class of f(R) gravity theories. This analysis can shed light on the viability of cosmological evolution provided by this kind of modified gravity, and constrains the possible candidates for the underlying gravitational action.

9.2 The background dynamics and tensor perturbations for $f(R)$ theories

For homogeneous and isotropic, i.e. Robertson-Walker, space-times with vanishing 3-curvature and barotropic perfect fluid - with equation of state $p = w\mu$ - as the standard matter source, the independent field equations for general $f(R)$ gravity can be written as

$$\Theta^2 = 3\,\frac{\mu^m}{f'} + 3\,\mu^R\,, \tag{9.1}$$

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \frac{1}{2\,f'}(\mu^m + 3p^m) + \frac{1}{2\,f'}(\mu^R + 3\,p^R) = 0\,, \tag{9.2}$$

$$\dot{\mu}^m + \Theta\,(\mu^m + p^m) = 0\,, \tag{9.3}$$

The linearization of the exact propagation and constraint equations around this background for a pure tensor perturbations then leads to the system [244]:

$$\dot{\sigma}_{ab} + \frac{2}{3}\,\Theta\,\sigma_{ab} + E_{ab} - \frac{1}{2}\pi_{ab} = 0\,, \tag{9.4}$$

$$\dot{H}_{ab} + H_{ab}\,\Theta + (\mathrm{curl}\,E)_{ab} - \frac{1}{2}(\mathrm{curl}\,\pi)_{ab} = 0\,, \tag{9.5}$$

$$\dot{E}_{ab} + E_{ab}\,\Theta - (\mathrm{curl}\,H)_{ab} + \frac{1}{2}\,(\mu + p)\;\sigma_{ab} + \frac{1}{6}\Theta\,\pi_{ab} + \frac{1}{2}\dot{\pi}_{ab} = 0\,, \tag{9.6}$$

$$\tilde{\nabla}_b H^{ab} = 0\,, \quad \tilde{\nabla}_b E^{ab} = 0\,, \quad H_{ab} = (\mathrm{curl}\,\sigma)_{ab}\,, \tag{9.7}$$

together with the linearized conservation equations,

$$\dot{\mu}^m = -\Theta\,\mu^m\,(1+\omega) \quad , \quad \dot{\mu}^R = -\Theta\,\mu^R(1+\omega_{eff})\,, \tag{9.8}$$

that are obtained from the twice contracted Bianchi identities, where

$$\omega_{eff} \equiv \frac{p^R}{\mu^R} - \frac{\dot{R}\, f''\, \mu^m}{f'^2}\,. \tag{9.9}$$

Taking the time derivative of equations (9.4)-(9.6) we obtain,

$$\ddot{\sigma}_{ab} - \tilde{\nabla}^2\sigma_{ab} + \frac{5}{3}\,\Theta\,\dot{\sigma}_{ab} + \left(\frac{1}{2}\mu - \frac{3}{2}p\right)\,\sigma_{ab} = \dot{\pi}_{ab} + \frac{2}{3}\,\Theta\,\pi_{ab}\,, \tag{9.10}$$

To simplify the above equations we use the standard procedure we perform an harmonic decomposition,

$$\sigma_{ab} = \sum_k \frac{k}{a}\left[\sigma_k Q^{(k)}_{ab} + \bar{\sigma}_k \bar{Q}^{(k)}_{ab}\right], \tag{9.11}$$

$$\pi_{ab} = \rho\sum_k \left[\pi_k Q^{(k)}_{ab} + \bar{\pi}_k \bar{Q}^{(k)}_{ab}\right]. \tag{9.12}$$

Then, equation (9.10) reduces to ,

$$\ddot{\sigma}_k + \Theta\,\dot{\sigma}_k + \left[\frac{k^2}{a^2} - \frac{1}{3}(\mu + 3\,p)\right]\sigma_k = \frac{a}{k}\left[\mu\,\dot{\pi}_k - \frac{1}{3}(\mu + 3\,p)\,\Theta\,\pi_k\right]. \tag{9.13}$$

Once the form of the anisotropic pressure has been determined, the above equations can be solved to give the evolution of tensor perturbations. From equation (4.17) and for pure tensor modes, one gets

$$\pi^R_k = -\frac{k}{a\,\mu}\frac{f''\,\dot{R}}{f'}\,\sigma_k\,. \tag{9.14}$$

From the equation above, it is clear that since π_k is proportional to σ_k. This fact guarantees that the tensor perturbations equations will be second-order differential equations, contrarily to the well-known fact in their scalar counterparts for $f(R)$ theories [226] where the involved equations are usually fourth-order.

9.2.1 The initial conditions

In the radiation dominated era, the anisotropic stress π is dominated by the radiation fluid contribution. Therefore, in this scenario $\pi = \pi^\gamma$ and consequently, equation (9.13) reduces to

$$\ddot{\sigma}_k + A\dot{\sigma}_k + B\sigma_k = \frac{a}{k}\frac{\mu}{f'}\left[\dot{\pi}^\gamma_k - \left(H + 3\,H\omega + \frac{f''}{f'}\right)\pi^\gamma_k\right], \tag{9.15}$$

where the quantities A and B are defined as

$$A \equiv 3H + \frac{f''}{f'}\dot{R}\,, \qquad B \equiv \frac{k^2}{a^2} + 2\frac{\ddot{a}}{a} + \dot{R}^2\left[\frac{f'''}{f'} - \left(\frac{f''}{f'}\right)^2\right] + \frac{f''}{f'}\ddot{R} + H\frac{f''}{f'}\dot{R}. \tag{9.16}$$

Thanks to the homogeneity of the early Universe, equation (9.15) can be further simplified by assuming that the radiation anisotropic stress vanishes. It follow that,

$$\frac{\mathrm{d}^2\sigma_k}{\mathrm{d}\tau^2} + (aA - \mathcal{H})\,\frac{\mathrm{d}\sigma_k}{\mathrm{d}\tau} + a^2 B\sigma_k = 0\,, \tag{9.17}$$

where τ is the conformal time and $\mathcal{H} \equiv aH$. In order to remove the damping term in the previous equation, we perform the variable change $u_k = a^m \sigma_k$ and choose $m = \frac{Aa-\mathcal{H}}{2\mathcal{H}}$. After this redefinition, equation (9.17) reads

$$\frac{\mathrm{d}^2 u_k}{\mathrm{d}\tau^2} + \left(-\frac{1}{2} m\mathcal{H}\frac{f''}{f'} - \frac{m}{a}\frac{\mathrm{d}^2 a}{\mathrm{d}\tau^2} + a^2 B\right) u_k = 0\,. \tag{9.18}$$

Note that in the derivation of the previous equation, the exponent m has been assumed to be constant. This in fact the case for R^n models which will be studied in the following section.

9.3 Dynamics of R^n models, background and tensor perturbations evolution

In order to illustrate the formalism described in the previous sections, we considered a one-dependent parameter kind of $f(R)$ models, the $f(R) = R^n$ models. Let us then study the background and tensor perturbations evolution for these models.

9.3.1 Background setup and the evolution equations

Let us define the following set of dynamical variables 6 :

$$x = \frac{\dot{R}(n-1)}{HR},\;\; y = \frac{R(1-n)}{6nH^2},\;\; \Omega_d = \frac{\mu_d}{3H^2 nR^{n-1}}\;\; \Omega_r = \frac{\mu_r}{3H^2 nR^{n-1}}\,. \tag{9.19}$$

where μ_d and μ_r are the dust and radiation densities respectively. In terms of these variables, the Friedmann equation (9.1) takes the simple form ,

$$1 + x + y - \Omega_d - \Omega_r = 0\,. \tag{9.20}$$

At this stage, an autonomous system, which is equivalent to cosmological equations (9.1)-(9.3) can be derived by differentiating the dynamical variables defined in (9.19). Thus,

$$\begin{aligned} a\,\frac{\mathrm{d}x}{\mathrm{d}a} &= -x - x^2 + \frac{(4-2n+nx)y}{n-1} + \Omega_d\,,\\ a\,\frac{\mathrm{d}y}{\mathrm{d}a} &= \left[4 + \frac{(x+2ny)}{n-1}\right] y\,,\\ a\,\frac{\mathrm{d}\Omega_d}{\mathrm{d}a} &= \left[1 - x + \frac{2ny}{n-1}\right]\Omega_d\,,\\ a\,\frac{\mathrm{d}\Omega_r}{\mathrm{d}a} &= \left[-x + \frac{2ny}{n-1}\right]\Omega_r\,, \end{aligned} \tag{9.21}$$

The constraint equation (9.20) can be used to reduce the dimensionality of the system above. The evolution equation of Hubble parameter H is given in terms of the dynamical variables (9.19) by ,

$$a\,\frac{\mathrm{d}H}{\mathrm{d}a} = -H\left(2 + \frac{ny}{n-1}\right)\,. \tag{9.22}$$

The fixed points of the system (9.21) are shown in Table 6.2.2. In order to study the stability of these fixed points we use the well-known techniques, which involve linearizing the dynamical equations around the equilibrium points and then finding the eigenvalues of the linearization matrix the Jacobian at the equilibrium points. There are two interesting points in the phase space of the R^n-gravity models: the point J which is a transient decelerated power law expansion phase and the point B which represents an accelerated expansion phase. In fact, a large number of orbits connecting these two points can be found. Since we are interested in a background evolution that is similar to ΛCDM, we used a numerical procedure to single out the orbit which gives the best fit to ΛCDM evolution. In order to illustrate this procedure, we chose the value $n \simeq 1.28$ which provides the best fit to SN-Ia data (reference required about the SN catalogue we are using). Figure 9.1 shows the Hubble parameter and distance modulus for this value of n.

9.3.2 Perturbations setup

When $f(R) = R^n$ and for the case $n \neq 2$ equation (9.18) reduces to [246],

$$\frac{\mathrm{d}^2 u_k}{\mathrm{d}\tau^2} + \left(k^2 - 2\,\tau^{-2}\right) u_k = 0\,, \tag{9.23}$$

where $m = \frac{2-n}{n}$ due to the fact that for R^n models, the scale factor in the radiation dominated era satisfies $a(\tau) = \tau^{\frac{n}{2+n}}$ [248] and therefore the parameter m is constant as it was assumed in order to obtain (9.18). The result in (9.23) is exactly the same as the one for tensor perturbations in GR obtained in [249]. On the other hand,equation (9.14) becomes,

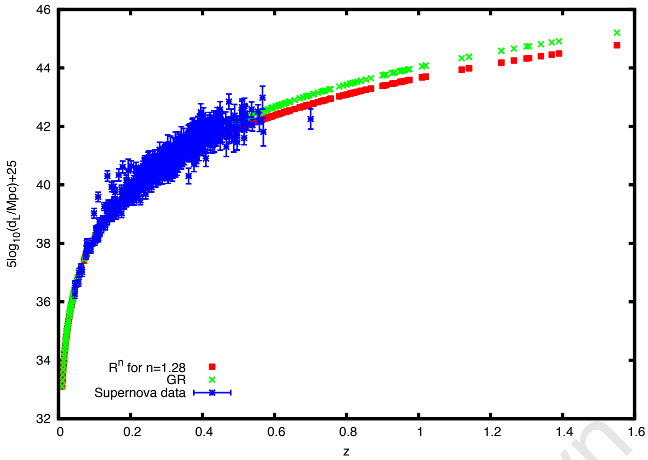
$$\pi^R_k = -\frac{k}{a^2}\frac{(n-1)}{\mu\,R}\frac{\mathrm{d}R}{\mathrm{d}\tau}\,\sigma_k. \tag{9.24}$$

whose importance was stressed after (9.14).

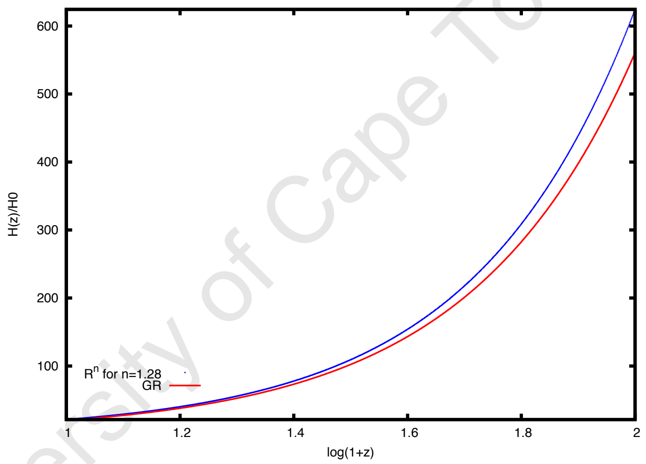
9.4 CMB tensor spectra for R^n models

In the latest version of CAMB, the so-called parameterized post-Friedmann (ppf-CAMB) [239], there exists the possibility of inputting the equation of state parameter of the dark energy contribution via a data file. Since the curvature fluid plays the role of dark energy in the $f(R)$ theories, equation (9.9) can be used to generate the required equation of state parameter data file for the curvature fluid. By supplying the ppf-CAMB code with this data file and using the effective matter density $\mu_{eff} \equiv \mu/f'$, together with equation (9.24), we are able to implement the correct background evolution in CAMB. This procedure is usually missing in previous investigations which for the sake of simplicity assumed a GR background when studying the tensor perturbations of modified gravity theories.

We considered different values of n to illustrate the general procedure. First, we consider values of n very close to unity to test that our method converges to the usual GR calculations ($n = 1$). Then, the studied values of n are taken to be $n = 1.22, 1.23, 1.24, 1.25, 1.26, 1.27, 1.28$. This choice is motivated by the fact that the best fit for R^n models to SN-Ia data was obtained for $n \simeq 1.28$. Once the correct background and perturbations evolution are implemented in the



(a)



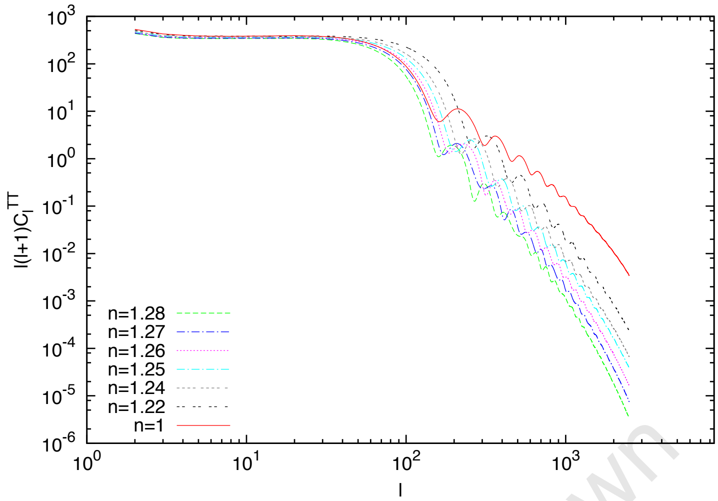
(b)

Figure 9.1: The distance modulus (left panel) and Hubble parameter evolution (right panel) for $n = 1.28$. The ΛCDM evolution is also plotted in both figures with GR assumed and density parameters$\Omega_{\ b} = 0.3$,$\Omega_{\ \Lambda} = 0.7$, no massive neutrinos, and the Hubble constant obtained for the R^n background evolution that best fit ΛCDM is $H_0 = 57$ km s^{-1} Mpc^{-1}. SN-Ia catalogue data [247] are used.

CAMB simulation, distinct features can be found depending on the value of n. There are notable effects that the modifications in the background and tensor perturbations produce in both the $c_l^{\rm TT}$ and $c_l^{\rm EE}$ coefficients with respect to the usual results obtained from GR plus cosmological constant. Let us summarize these results as follows:

9.4.1 $c_l^{\rm TT}$ features

In all the studied cases we find that the amplitude for fully modified $c_l^{\rm TT}$ coefficients is suppressed for large l's with respect to the usual GR simulations. The suppression increases with increasing values of parameter n. For small l's, a small reduction is also found. All these features can be seen in figure 9.2. For $n = 1.24, 1.25, 1.26$ the maximum amplitude suppression is two orders of



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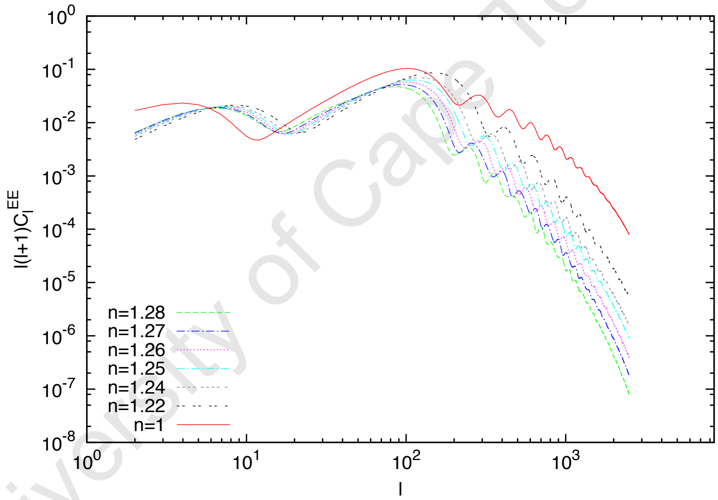
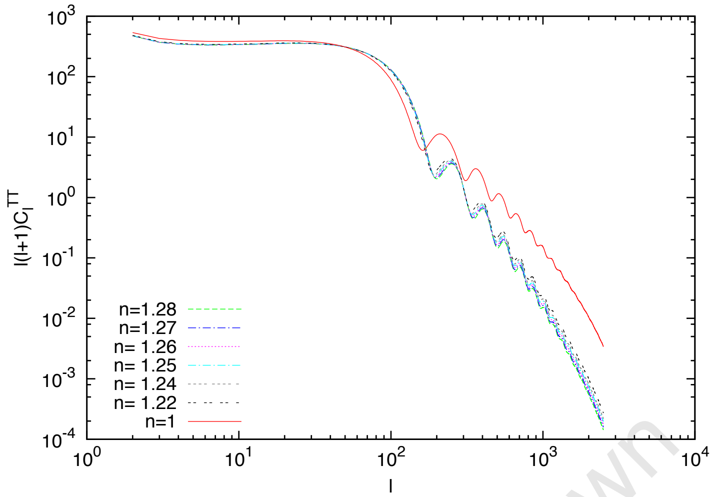


(b)

Figure 9.2: The temperature (left panel) and EE (right panel) power spectra for tensor perturbations using the correct background and perturbation equations. R^n models are shown for $n = 1.22$, 1.23, 1.24, 1.25, 1.26, 1.27 and 1.28. We also plot $n = 1$, i.e., GR for comparison. The initial tensor power spectra are scale-invariant and we have adopted an absolute normalisation to the power in the primordial gravity wave background. The GR ($n = 1$) cosmology is the spatially flat ΛCDM (concordance) model with density parameters $\Omega_b = 0.3$,$\Omega_{\Lambda} = 0.7$, no massive neutrinos. The Hubble constant $H_0 = 57\,\mathrm{km}\,s^{-1}\mathrm{Mpc}^{-1}$.

magnitude at largest ($l \approx 2000$) whereas for $n = 1.27$ and 1.28 this suppression attained three orders of magnitude as can be seen in figure 9.2. Thus, the amplitudes at high l's moves from a numerical values of $2 \cdot 10^{-3}$ compared with GR to $2 \cdot 10^{-6}$ ($n = 1.28$).
For $n = 1.22$ we observe a horizontal shift to the right for the modified c_l^{TT} with respect to GR at intermediate scales ($l \approx 100 - 200$). For the rest of n values considered, this shift moves towards the left as can be seen in figure 9.2. For $n \simeq 1.27$ the horizontal shift is cancelled out. Finally for $n = 1.28$ the shift is now towards the right with respect to GR results. The number of relative maxima and minima remain invariant, although their location is shifted to the right



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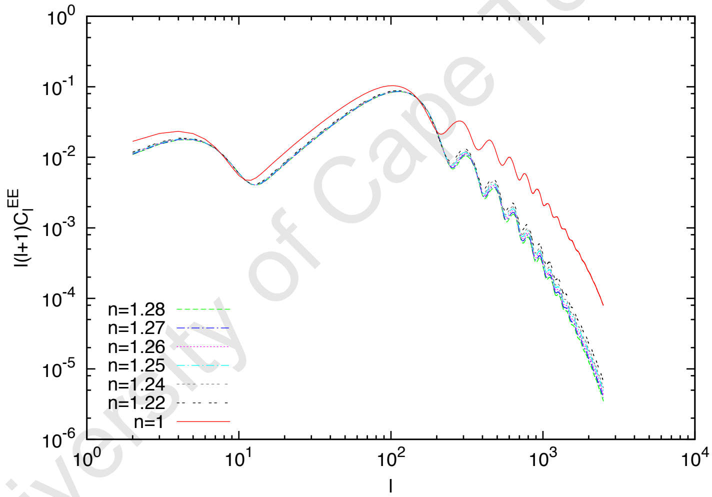
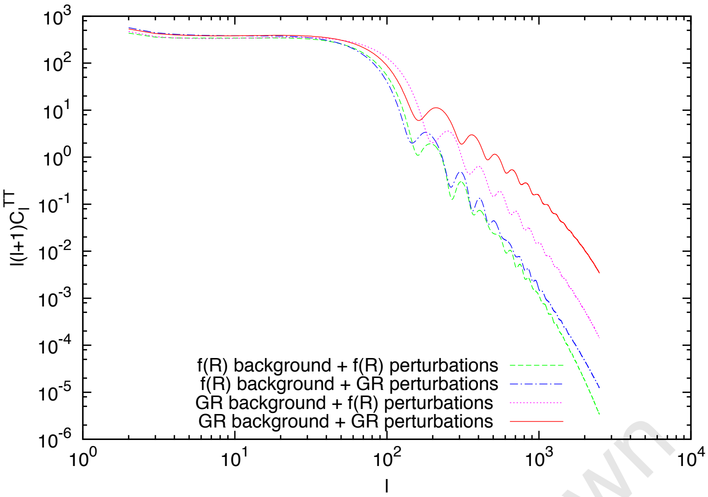


(b)

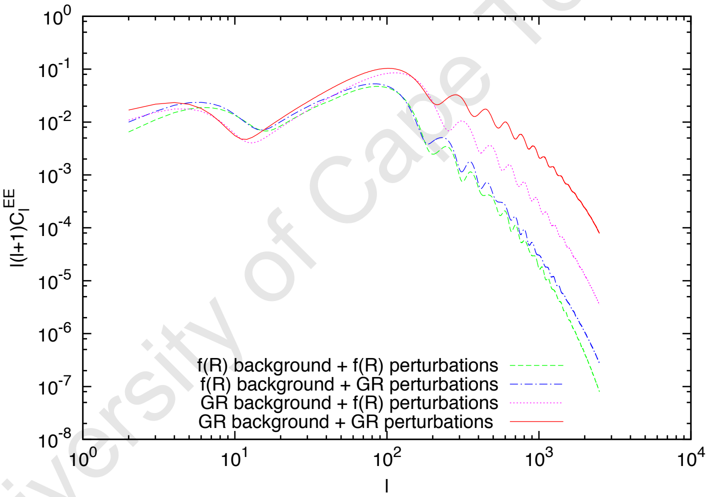
Figure 9.3: The temperature (left panel) and EE (right panel) power spectra for tensor perturbations for GR background and $f(R)$ perturbations, for $n = 1.22$, 1.23, 1.24, 1.25, 1.26, 1.27 and 1.28. We also plot $n = 1$, i.e., GR for comparison. The initial tensor power spectra are scale-invariant and we have adopted an absolute normalisation to the power in the primordial gravity wave background. The GR ($n = 1$) cosmology is the spatially flat ΛCDM (concordance) model with density parameters$\Omega_{\ b} = 0.3$,$\Omega_{\ \Lambda} = 0.7$, no massive neutrinos. The Hubble constant $H_0 = 65\,\mathrm{km}\,s^{-1}\mathrm{Mpc}^{-1}$.

for $n = 1.22 - 1.26$ and then slightly shifted to the left for $n > 1.26$.
Provided that the correct $f(R)$ background evolution is considered, the suppression is more severe than when a GR background is assumed, as can be seen by comparing figures 9.2 and 9.3.

Finally, by studying separately the simulations involving either only modifications in the background or in the perturbations, we noticed that the reduction can be attributed mainly to the modification of the background evolution. These features were observed for all the studied n values and are presented in figure 9.4 for $n = 1.28$. It can be seen, both partial modifications reduce significantly the amplitude at large $l's$. Consequently the total observed pattern (when



(a)



(b)

Figure 9.4: The temperature (left panel) and electrical (right panel) power spectra for tensor perturbations in all the possible background and perturbations scenarios. R^n model for $n = 1.28$: Power spectra for GR background and GR perturbations are depicted in red continuous, with no dependence on the R^n model and shown just for comparison; GR background and $f(R)$ perturbations pink dotted line; $f(R)$ background and GR perturbations in dotted-dashed blue line; $f(R)$ both background and perturbations in dashed green line.

both background and perturbations are obtained by considering the correct $f(R)$ modifications) can be regarded as a combined effect coming from the two partial modifications.

9.4.2 $c_l^{\rm EE}$ features

For the full calculations presented in figure 9.2, one sees that for bigger values of n, the suppression with respect to GR spectra at high l's is bigger. This suppression ranges from two ($n = 1.24$) to three ($n = 1.28$) orders of magnitude, from a numerical value of 10^{-4} (GR) to 10^{-7} ($n = 1.28$). Unlike the c_l^{TT} patterns, The number of relative maxima-minima remains invariant, although their location is shifted to the right for $n = 1.22$-1.26 and then back to the left for $n > 1.26$. This horizontal shift affects mainly low $l's$,being almost insignificant for high $l's$. The initial amplitudes for $l = 2$ remain below the GR case for all values of n under consideration. At intermediate scales (l between 6 and 20) the amplitudes for $f(R)$ are bigger than GR. From that scale onwards, the c_l^{EE} remain smaller in amplitude than GR for all values of n, except $n = 1.22$. By studying separately the simulations involving only modifications in the perturbations and keeping the background as GR, we argue that the reduction can be attributed mainly to the modification of the background evolution introduced by the $f(R)$ models. This fact can be seen by straightforward comparison of right panels on figures 9.2 and 9.3. The strongest suppression demonstrates the $f(R)$ background consideration, as can be seen in figure 9.4. This fact shows by the importance of considering the correct background when performing these calculations.

9.4.3 General comments

First, according to (9.24) the relation between shear and momentum flux is proportional to the factor $n - 1$ and therefore, as n departures from unity, the presence of this extra shear contribution affects the evolution of the tensor perturbations. This statement is independent of the background consideration and is present even when a GR background is considered. In figures 9.3 we can see how this term (assuming GR background) shifts and modifies the amplitude of the c_l^{TT} and c_l^{EE} coefficients .
Secondly, the consideration of the correct background evolution proved the necessity of determining accurately the cosmological background evolution for every $f(R)$ model under consideration. In fact, figure 9.2 in comparison with 9.3 and Figure 9.4 (for the paradigmatic case of $n = 1.28$), shows that the background modification are 9.2 and 9.3.

9.5 Conclusions

In this work we have presented for the first time a detailed analysis of the ppf-CMB features for $f(R)$ modified gravity theories by using a CAMB implementation. These simulations considered the correct cosmological background evolution as provided by these fourth-order gravity theories as well as the required tensor perturbations equations .
We applied our general results to R^n models for different values of n verifying the convergence to GR result when n approaches unity and determining the features that may distinguish those models from Concordance model predictions. Our implementation makes it possible to distinguish these models form GR and demonstrates the importance of considering the correct background when alternative theories of gravity are subjected to this kind of analyses .
According to our results, the sole consideration of perturbations for the c_l^{TT}, having assumed the usual GR background, would lead for instance to not see any appreciable difference between pure GR and the $n = 1.22$ case. Moreover, values in the interval $n = 1.26 - 1.28$ are also indistinguishable from each other .
With regards to the c_l^{EE} coefficients, it can be seen for instance how for $n = 1.22$ the sole consideration of perturbations (keeping the GR background) is hardly detectable for c_l^{EE} as well as the interval $n = 1.26 - 1.28$ provide the same c_l^{EE} pattern as GR. Once the correct background

is implemented, this degeneracy is broken and therefore observable effects may be detected.
Our code provides then a powerful tool able to show the key features of the effect that fourth-order gravity theories may have in the CMB tensor perturbations.
In the next chapter we show how the covariant-gauge invariant equations for the evolution of scalar, vector and tensor perturbations for a generic f(R) gravity theory can be recast in order to exploit the power of dynamical systems approach.

Chapter 10

UNIFYING THE STUDY OF BACKGROUND DYNAMICS AND PERTURBATIONS IN $f(R)$-GRAVITY

10.1 Introduction

Because the field equations resulting from fourth-order gravity are extremely complicated, difficult conceptual and technical issues arise which need to be resolved in order to uncover the detailed physics of these models. Consequently it is important to develop new methods which are able to assist in resolving these problems. Two such approaches, the *dynamical systems approach to cosmology* and the *covariant approach to cosmological perturbations*, have proved particularly useful in this respect . The dynamical system approach, first developed by Collins [178] and extensively reviewed in the book by Ellis and Wainwright, [181] has proved to be an important tool in the understanding of cosmology of $f(R)$-gravity models. In fact, studying cosmological models using these techniques has the advantage of providing a relatively simple method for obtaining exact solutions, which appear as fixed points of the system, and for obtaining a global picture of the dynamics of these models. Consequently, such an analysis allows for an efficient preliminary investigation of these theories, allowing one to identify specific models which merit further investigation. In a series of recent papers, a wide range of features of $f(R)$ cosmology have been presented, ranging from an analysis of the standard Friedmann-Lemaître-Robertson-Walker (FLRW) models [182, 183, 184] to a discussion of the properties of the Einstein Universe and the isotropization of Bianchi models [186, 187, 188] .

The aim of this chapter is to combine dynamical systems methods with linear perturbation theory in such a manner that one is able to apply directly the results coming from the former to the latter and is able to gain a semi-qualitative idea of both the behavior of the background and that of the first order perturbations in a general $f(R)$-gravity theory. In order to achieve this we will express the coefficients of the perturbation equations in terms of the dynamical system variables in such a way that at any fixed point it will correspond a set of perturbation equations and as a consequence an evolution law for the linear fluctuations.

10.2 The dynamical system

The DS approach [181] has been used with great success in the analysis of the background evolution of cosmological models with fourth-order gravity. In [182, 183] it was shown that using the dimensionless variables:

$$x = \frac{3\dot{f}'}{f'\Theta}, \qquad y = \frac{3R}{2\Theta^2}, \qquad z = \frac{3f}{2f'\Theta^2}, \qquad \Omega = \frac{3\mu_m}{f'\Theta^2}, \qquad \chi = \frac{9K}{S^2\Theta^2}\,, \tag{10.1}$$

together with the characteristic variable

$$\mathcal{Q} \equiv \left(\frac{d\log f'}{d\log R}\right)^{-1} = \frac{f'}{Rf''} \tag{10.2}$$

and logarithmic time $N = |\ln S|$, the cosmological equations are equivalent to the autonomous system:

$$\begin{aligned}
\frac{dy}{dN} &= \varepsilon y[2\chi - 2y + \mathcal{Q}(-\chi + y - z + \Omega - 1) + 4]\,,\\
\frac{dz}{dN} &= \varepsilon z(3\chi - 3y + z - \Omega + 5) + \mathcal{Q}\epsilon y(-\chi + y - z + \Omega - 1)\,,\\
\frac{d\Omega}{dN} &= -\varepsilon\Omega(3w - 3\chi + 3y - z + \Omega - 2)\,,\\
\frac{d\chi}{dN} &= 2\varepsilon\chi(\chi - y + 1)\,,\\
1 &= y + \Omega - x - z - \chi\,.
\end{aligned} \tag{10.3}$$

where $\varepsilon = |\Theta|/\Theta$ [1]. This system allows one to analyze many interesting fourth-order gravity cosmological models and leads to the result that some of them present cosmic histories which posses a transient Friedmann phase and evolve naturally towards an accelerated (DE-like) expansion phase. A detailed analysis of the properties and caveats of this method can be found in [182, 183]; here it is important only to remind the reader that the solutions associated with the fixed points can be found by substituting the coordinates of the fixed points into the system,

$$\dot{\Theta} = \gamma\Theta^2\,, \qquad \gamma = \frac{-2 + y_i - \chi_i}{3}\,, \tag{10.4}$$

$$\dot{\mu}_m = -\frac{(1+w)}{\gamma\, t}\mu_m\,, \tag{10.5}$$

[1] We have chosen here to use the constraint to eliminate the variable x. This is different to what has been done in the other works on this subject. The reason for this choice is due to the fact that, since we will eventually express the coefficients of the perturbation equations in terms of the dynamical system variables, it is more useful to retain the variables which have a more direct physical meaning.

where the subscript "i" stands for the value of a generic quantity at a fixed point. This means that for $\gamma \neq 0$, the general solutions can be written as,

$$a = a_0(t-t_0)^{1/3\gamma}\,, \tag{10.6}$$

$$\mu_m = a_0(t-t_0)^{-\frac{(1+w)}{3\gamma}}\,. \tag{10.7}$$

The expression above gives the solution for the scale factor and the evolution of the energy density for every fixed point in which $\gamma \neq 0$. When $\gamma = 0$ the (10.4) reduces to $\dot{\Theta} = 0$, which corresponds to either a static or a de Sitter solution. It is important to stress at this point that sometimes the solutions associated with the fixed points are "non-physical" i.e., they do not satisfy the cosmological equations. Then one might ask how is it possible that, although derived from the cosmological equations themselves, our dynamical system equations give non-physical solutions. The reason for this apparent contradiction needs to be looked for in the very structure of the dynamical system approach. As we have seen, the condition to obtain the fixed points of the system (10.3)- as well as every dynamical system - is to set the first derivative of the dynamical variables to zero (i.e., to set the left-hand side of (10.3) to zero) and solve the system obtained. Such a step in the standard dynamical system theory is usually trivial, however, in our specific formulation this step becomes much more subtle. In fact, the requirement of the existence of a fixed point also imposes the requirement that all the variables acquire a constant value (or equivalently that their first derivative with respect to the time coordinate is zero). This is equivalent to an additional system of equations that is not necessarily satisfied by the solutions of the system. For example, in,

$$\mathbf{x}' = \mathbf{F}(\mathbf{x})\,, \tag{10.8}$$

the condition to obtain the fixed points would be

$$\mathbf{F}(\mathbf{x}) = 0 \qquad \text{but also} \qquad \mathbf{x} = const \quad (\mathbf{x}' = 0)\,, \tag{10.9}$$

as mentioned earlier, the second system is usually trivially satisfied. In the formalism above, however, these variables are a combination of quantities appearing in the cosmological equations. This means that $\mathbf{x} = const$ becomes a set of conditions to be satisfied by all the physical fixed points of the system. In GR, because of the structure of the variables [181, 210], these constraints are automatically satisfied. In fact, in this case one has$\Omega_{GR} = 3\mu^m/\Theta^2$ and $\chi_{GR} = 9K/S^2\Theta^2$ which means,

$$\frac{d\Omega_{GR}}{dN} = 0 \qquad \Rightarrow \qquad \Omega_{GR} = const. \qquad \Rightarrow \qquad \mu^m \propto \Theta^2\,, \tag{10.10}$$

$$\frac{d\chi_{GR}}{dN} = 0 \qquad \Rightarrow \qquad \chi_{GR} = const. \qquad \Rightarrow \qquad S^2 \propto K\Theta^2\,, \tag{10.11}$$

so the physical points can either have $K = 0$ and $\mu^m \propto \Theta^2$ or $K \neq 0$, $S \propto \Theta$. Both the fixed points that one obtains (corresponding to Milne and Friedmann solutions) satisfy these criteria. This then raises the following question: how do we consider the non- physical fixed points? The answer depends on the stability. If the fixed point is unstable then, although the solution associated with the fixed point does not satisfy the cosmological equations, it can be used to approximate the behavior in the neighborhood of the fixed point. Orbits nearby this point will bounce on or run away from it, but they will never reach it. Instead in the case in which the point is stable, the set of orbits approaching the point will not correspond to any physical evolution for the system and the dynamical system approach fails to give an appropriate description of the cosmological evolution. This imposes an intrinsic limitation on the predictive power of this approach and has to be taken into account to avoid incorrect conclusions.

Another consequence of this limitation is that the structure of the variables characterizes the type of solution associated with the fixed points and, consequently, the fixed points one derives with a dynamical system formalism are not necessarily the complete set of the elementary solutions of the system. This is also important because it implies that the absence of a specific cosmic history in a dynamical systems formalism does not necessarily indicate that this cosmic history cannot be realized, but only that the specific formalism used is not able to show its presence.

10.3 Perturbation equations and dynamical system analysis

In this section we will rewrite the evolution equations for scalar, vector and tensor perturbations in terms of the dynamical system variables. As we will see, the requirement to obtain a closed form for the coefficients will require a redefinition of some of the variables in the equations. Once this has been done the behavior of the perturbations at a fixed point of the system (10.3) can be inferred in all generality. This can be extremely useful in determining the set of cosmic histories which are compatible with observations for a given $f(R)$ model.

10.3.1 Scalar perturbations

The evolution equations for the scalar perturbations for a generic $f(R)$ theory of gravity in the covariant approach were derived in [211, 201] and successively analyzed in detail in [202]. These types of perturbations are characterized by the quantities,

$$\Delta^m = \frac{S^2}{\mu^m}\tilde{\nabla}^2\mu^m\,, \qquad Z = S^2\tilde{\nabla}^2\Theta\,, \qquad C = S^4\tilde{\nabla}^2\tilde{R}\,, \qquad \mathcal{R} = S^2\tilde{\nabla}^2 R\,, \qquad \Re = S^2\tilde{\nabla}\dot{R}\,, \tag{10.12}$$

and their evolution equations are given in chapter 8. In order to express these equations in terms of the dynamical system variables we first have to convert them into equations in which the independent variable is N. In addition to that one can only obtain a closed form for the coefficients if the curvature variable is redefined as,

$$\mathcal{R} = S^2\tilde{\nabla}^2\ln[f'(R)]\,. \tag{10.13}$$

Substituting for the new curvature variable and using the definitions (10.1), the perturbation equations developed in harmonics can be written as,

$$\frac{d^2\,\Delta^{(\ell)}}{dN^2} + \mathcal{A}_1\,\frac{d\Delta^{(\ell)}}{dN} + \mathcal{B}_1\,\Delta^{(\ell)} + \mathcal{C}_1\,\mathcal{R}^{(\ell)} + \mathcal{D}_1\,\frac{d\mathcal{R}^{(\ell)}}{dN} = 0\,, \tag{10.14}$$

$$\frac{d^2\mathcal{R}^{(\ell)}}{dN^2} + \mathcal{E}_1\,\frac{d\mathcal{R}^{(\ell)}}{dN} + \mathcal{F}_1\,\mathcal{R}^{(\ell)} + \mathcal{G}_1\,\Delta^{(\ell)} + \mathcal{H}_1\,\frac{d\Delta^{(\ell)}}{dN} = 0\,. \tag{10.15}$$

where ,

$$\mathcal{A}_1 = \varepsilon(1 - 3w + z - \Omega)\,, \tag{10.16}$$

$$\mathcal{B}_1 = -3(2wz - 3w\mathcal{K} + (1-w)\Omega)\,, \tag{10.17}$$

$$\mathcal{C}_1 = -3(w+1)(z - \mathcal{Q}y - 3\mathcal{K} - \Omega)\,, \tag{10.18}$$

$$\mathcal{D}_1 = 3\varepsilon(w+1)\,, \tag{10.19}$$

$$\mathcal{E}_1 = -\varepsilon(3\chi - 3y + 2z - 2\Omega + 1)\,, \tag{10.20}$$

$$\mathcal{F}_1 = 4z - 2\mathcal{Q}y - 9\mathcal{K} + (3w-1)\Omega\,, \tag{10.21}$$

$$\mathcal{G}_1 = \frac{w(4y - 8z) - \left(3w^2 - 4w + 1\right)\Omega}{w+1}\,, \tag{10.22}$$

$$\mathcal{H}_1 = -\frac{\varepsilon(w-1)}{w+1}(\chi - y + z - \Omega + 1)\,, \tag{10.23}$$

and the form of $\mathcal{K}(N)$ is given by ,

$$\frac{d\mathcal{K}}{dN} = 2\varepsilon\mathcal{K}(\chi - y + 1)\,. \tag{10.24}$$

Note that in this form the above equations are such that two different forms of the Lagrangian have the same evolution of the scalar perturbations if they both have a fixed point with the same coordinates *and* and in this fixed point $\mathcal{Q}$ is the same. As we will see this will allow us to attach a fixed point to a certain evolution law of the perturbations .

10.3.2 Vector perturbations

In order to analyze the evolution of the vector perturbations one has to extract the vector parts of the variables in the 1+3 equations (4.21-4.21) and the propagation equations for $D_a, Z_a, \mathcal{R}_a$ and $\Re_a$ in chapter 8. In our specific case some important facts have to be noted. First, looking at (4.14) the heat flux q_a and the anisotropic pressure π_{ab} are not independent, i.e., they can be written as functions of the other variables, specifically σ_{ab}, $\mathcal{R}_a$ and $\Re_a$. Secondly, the variables $(D_a, Z_a\mathcal{R}_a, \Re_a)$ are one index objects, which means that their purely vector part is obtained by taking their *curl* multiplied by the scale factor. In addition, looking at (10.12) above, one notices that these variables are in fact gradients of scalars and we know that $\text{curl}\tilde{\nabla}_a X = 2\omega_a \dot{X}$. This means that at first order one can write:

$$(D_a)^V = -2S^2\Theta(1+w)\,\omega_a\,, \tag{10.25}$$

$$(Z_a)^V = 2S^2\dot{\Theta}\,\omega_a\,, \tag{10.26}$$

$$(\mathcal{R}_a)^V = 2S^2\dot{R}\,\omega_a = 4S^2\left(\frac{4}{3}\dot{\Theta}\Theta - \frac{2}{3}\frac{K}{S^2}\Theta + \ddot{\Theta}\right)\omega_a\,, \tag{10.27}$$

$$(\Re_a)^V = 2S^2\ddot{R}\,\omega_a = 4S^2\left(\frac{8}{9}\frac{K}{S^2}\Theta^2 - \frac{4}{3}\frac{K}{S^2}\dot{\Theta} + \frac{8}{3}\dot{\Theta}^2 + \frac{8}{3}\Theta\ddot{\Theta} + 2\,\Theta^{(3)}\right)\omega_a\,, \tag{10.28}$$

i.e., one can express all these quantities in terms of the vorticity vector. Also one can use the constraints(4.2.3- 4.22) to express the remaining quantities $\sigma_{ab}, E_{ab}, H_{ab}$ in terms of the vorticity vector. Thus, in the end the only relevant equations for the dynamics of the vector perturbations is the vorticity equation (4.2.3):

$$\epsilon\frac{d\omega_a}{dN}+(2-3w)\omega_a=0\,, \tag{10.29}$$

which does not depend on ℓ. This means that the vorticity evolution is independent of the scale. Specifically one has,

$$\omega^\ell=\omega_0^\ell\exp\left[-\varepsilon N\;(2-3w)\right]=\omega_0^\ell S^{-\epsilon(2-3w)}\,, \tag{10.30}$$

which implies that the vorticity is always decreasing regardless of the features of the action. This is the same result that one obtains in GR [196] and shows that $f(R)$ gravity does not affect the evolution of this quantity. However, the quantities (10.25-10.28) will change behavior according to changes in the background and the form of the action.

10.3.3 Tensor perturbations

As already noticed in [212] the only independent equation in the evolution of the tensor perturbation is the shear. This happens because the second order equation for σ_{ab} is closed and from the constraint (4.2.3) one obtains $H_{ab}=(curl\sigma)_{ab}$ where $(\mathrm{curl}\,X)^{ab}=\epsilon^{cd\langle a}\,\tilde{\nabla}_c X^{b\rangle}{}_d$. In addition to that the first order equation for σ_{ab} (Eq. (8.7)) can be used to derive E_{ab}, because [212] in $f(R)$ gravity (as in the scalar tensor case [215]) the tensor component of the anisotropic pressure π_{ab} can be proven to be proportional to the shear. Thus we are left only with the equation for σ_{ab}. This equation developed in harmonics and written in terms of the dynamical systems variables reads,

$$\frac{d^2\sigma^{(\ell)}}{dN^2}-\mathcal{A}_2\frac{d\sigma^{(\ell)}}{dN}-\mathcal{B}_2\sigma^{(\ell)}=0 \tag{10.31}$$

$$\mathcal{A}_2=\varepsilon(2\chi-2y+z-\Omega-2)\,, \tag{10.32}$$

$$\mathcal{B}_2=\chi^2-2(y-z+\Omega-1)\chi+(y-z+\Omega)^2-6y+5z-9\mathcal{K}+(3w-2)\Omega\,. \tag{10.33}$$

$$\tag{10.34}$$

For (10.31) the same remark given for scalars holds: since the coefficients depend only on the coordinate of the fixed points two theories with the same fixed point will have, in the fixed point, the same evolution law for the tensor perturbations, as in the case of the scalars. However, since the coefficients (10.32) and (10.33) do not contain $\mathcal{Q}$ such occurrences are even more common.

10.4 Examples

In this section we will apply the equations defined above to some specific forms of $f(R)$ to illustrate the utility of the above approach.

10.4.1 R^n-gravity

Consider the action 6.3. In this case the characteristic function $\mathcal{Q}$ is always constant. In particular, we have,

$$\mathcal{Q}=\frac{1}{n-1}\,, \tag{10.35}$$

which implies that the variables z and y are not independent, i.e., the phase space of R^n-gravity is contained in the subspace $y = nz$ of the general phase space described by (10.3). This can easily be seen if one substitutes $y = nz$ into (10.3). Then the equations for y and z turn out to be exactly the same and (10.3) reduces to:

$$\frac{dy}{dN} = ny\varepsilon\left(\frac{y-2n(y+2)+5}{(n-1)^2} + \frac{(3-2n)\chi}{(n-1)^2} - \frac{\Omega}{(n-1)^2}\right) , \tag{10.36}$$

$$\frac{d\Omega}{dN} = -\varepsilon\Omega\left(-3w + \left(3 + \frac{2}{n-1}\right)y + 3\chi - \Omega + 2\right) , \tag{10.37}$$

$$\frac{d\chi}{dN} = 2\varepsilon\chi\left(\chi + 1 - \frac{ny}{1-n}\right) , \tag{10.38}$$

with the constraint

$$1 + x + y + \chi - \Omega = 0 , \tag{10.39}$$

which is equivalent to the one given in [183]. The fixed point with their stability and the associated solutions are given in Tables 6.2.2 and 6.3. In Table 10.1, the long- wavelength modes of the solutions for the matter scalar perturbations and the tensor perturbations are given. As expected, for the background $t^{2n/3(1+w)}$ corresponding to the point $\mathcal{G}$, the results are the same as the ones already found in [201, 212].For $\ell \neq 0$, however, one has to use numerical methods to obtain the solution of the equations.

It is interesting to observe the behavior of the matter perturbation modes of the point $\mathcal{F}$. Here the matter perturbations possess a constant mode and the other modes can be growing or decaying, depending on the value of the parameter n.

As shown in [184] for some specific values of the parameter n ($1.37 \lesssim n \lesssim 2$) this model has a set of cosmic histories characterized by the presence of a transient, decelerated power law expansion that evolves towards an accelerated expansion. Using (10.14) and (10.31) we are now able to see directly the evolution of the scalar and tensor perturbations along these orbits.

In particular, as the Universe approaches the point $\mathcal{F}$, for $1.37 \lesssim n \lesssim 2$ and $w = 0$, the large scale scalar perturbations, which nearby $\mathcal{G}$ have a growing mode (see Figure 10.1), start dissipating, which is consistent with what one would expect in a late-time acceleration scenario. The large-scale tensor perturbations instead do not change their behavior and keep being dissipated, but at a much faster rate.

In order to analyze the behavior of the perturbations for smaller scales ($\ell \neq 0$) one needs to integrate the equations numerically. This can be done in a relatively easy manner and an example of the results obtained in the case of dust and $\ell = 100$ are shown in Figures 10.3 and 10.4. It is clear that in the point $\mathcal{F}$ the matter scalar perturbations approach a constant value which depends on the initial conditions, while in the point $\mathcal{G}$ the perturbations first have a phase of growth and then start to decay which is consistent with what was found in [202]. The same can be done with the tensor perturbations, but we find that, as for the $\ell = 0$, case these types of perturbations are dissipated on small scales.

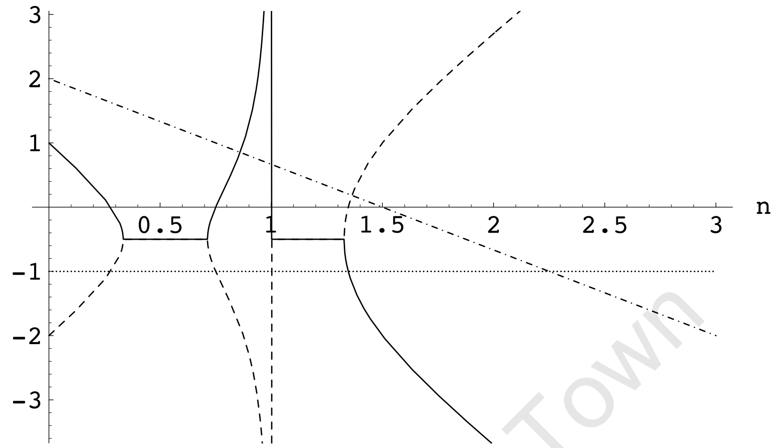


Figure 10.1: Plot against n of the real part of the exponents of the long- wavelength modes for R^n-gravity in the point $\mathcal{G}$ and in the dust case ($w = 0$).

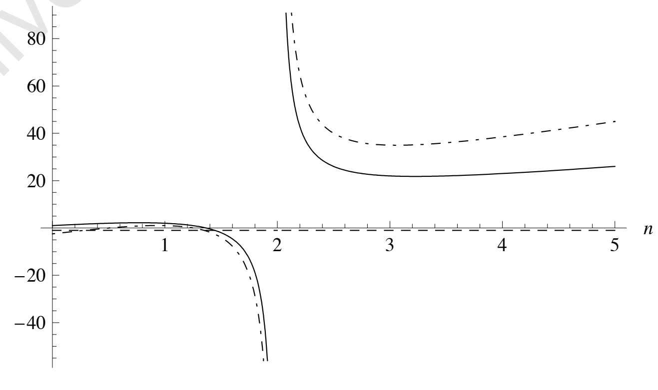
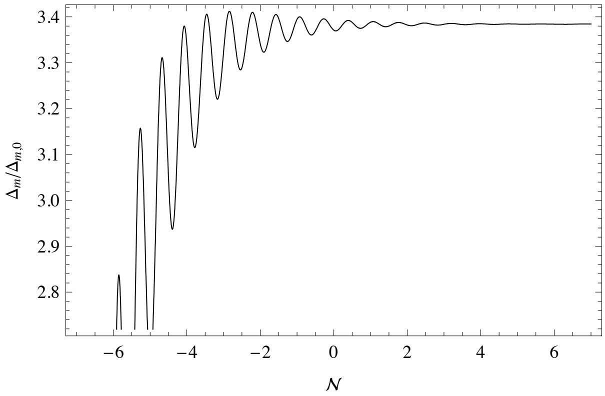
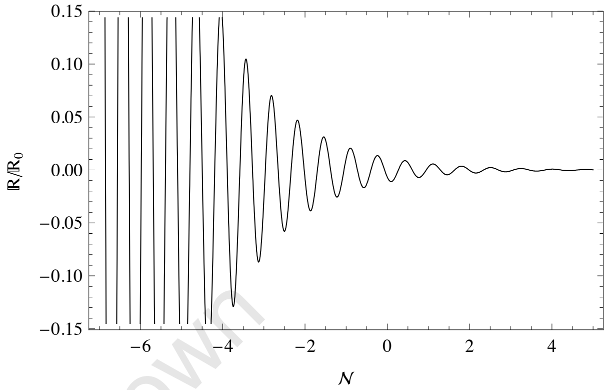


Figure 10.2: Plots against n of the real part of the exponents of long wavelength matter perturbation modes for R^n-gravity in the point $\mathcal{F}$ in the case of dust ($w = 0$).

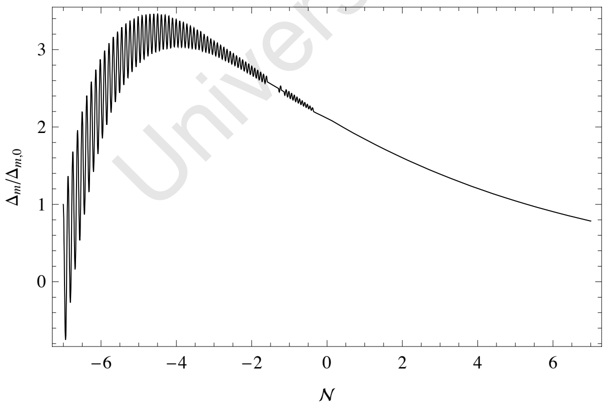


(a) Normalized plot of the evolution ofΔ $_m$

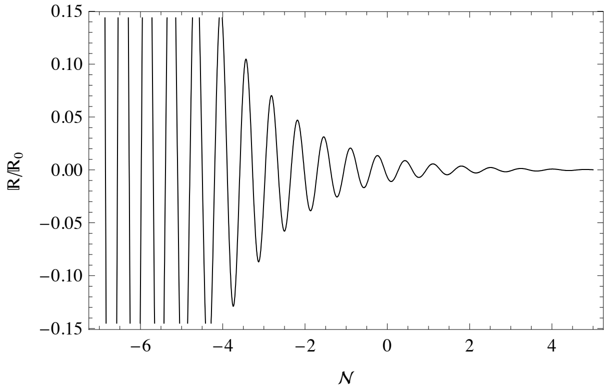


(b) Normalized plot of the evolution of $\mathcal{R}$

Figure 10.3: Plots of the solutions of the equations (10.14) in the fixed point $\mathcal{F}$ in the case $n \approx 1.37$, $w = 0$ and $\ell = 100$.



(a) Normalized plot of the evolution ofΔ $_m$



(b) Normalized plot of the evolution of $\mathcal{R}$

Figure 10.4: Plots of the solutions of the equations (10.14) in the fixed-point $\mathcal{G}$ in the case $n \approx 1.37$, $w = 0$ and $\ell = 100$

Table 10.1: Exponent of the modes for scalar and tensor in the fixed points of the system (10.36).

Point	Scalar Modes Exponents	Tensor Modes Exponents
$\mathcal{A}$	$\{\ \ 2\varepsilon, 0\}$	$\{-\varepsilon(2+\sqrt{3}), -\varepsilon(2-\sqrt{3})\}$
$\mathcal{B}$	$\{0, \varepsilon\, , \varepsilon\left(\frac{3w}{2}-1\right)\}$	$\{-\varepsilon, 0\}$
$\mathcal{C}$	$\{\ \ 2\varepsilon, 0, \varepsilon(3w-1), \varepsilon(3w+1)\}$	$\{\frac{\varepsilon}{2}\left(3w-\sqrt{9w^2-6w+9}-3\right),$ $\frac{\varepsilon}{2}\left(3w+\sqrt{9w^2-6w+9}-3\right)\}$
$\mathcal{D}$	$\{0, \varepsilon\, , \varepsilon\left(\frac{3w}{2}-1\right))\}$	$\{0, \varepsilon\left(\frac{3w}{2}-2\right)\}$
$\mathcal{E}$	$\Big\{0, \varepsilon(3w-1), \varepsilon\left[n-2-\sqrt{3}\sqrt{n(3n-4)}\right],$ $, \varepsilon\left[n-2+\sqrt{3}\sqrt{n(3n-4)}\right]\Big\}$	$\{\varepsilon\left(n-3-\sqrt{6-3n^2}\right), \varepsilon\left(n-3+\sqrt{6-3n^2}\right)\}$
$\mathcal{F}$	$\Big\{0, \varepsilon\left(8n+2+\frac{9}{n-2}\right), \varepsilon\left[4(n+1)+\frac{6}{n-2}\right],$ $-\varepsilon\left[\frac{3(n(2n-3)+1)w}{n-2}+1\right]\Big\}$	$\Big\{\varepsilon n\left(4+\frac{3}{n-2}\right), 8\varepsilon\left(n+1+\frac{9}{n-2}\right)\Big\}$
$\mathcal{G}$	$\Big\{\varepsilon\left(2-\frac{4n}{3(w+1)}\right), \varepsilon\left(\frac{2nw}{w+1}-1\right),$ $\varepsilon\left[\frac{3(w+1)+3n((2n-3)w-1)+\sqrt{A}}{6(n-1)(w+1)}\right],$ $\varepsilon\left[\frac{3(w+1)+3n((2n-3)w-1)-\sqrt{A}}{6(n-1)(w+1)}\right]\Big\}$	$\Big\{\varepsilon\left(1-\frac{4n}{3(w+1)}\right), \varepsilon\frac{2(n-1)w-2}{w+1}\Big\}$
$A = (n-1)(4(3w+8)^2n^3 - 4(3w(18w+55)+152)n^2 + 3(w+1)(87w+139)n - 81(w+1)^2)$		

10.4.2 $R + \alpha R^n$-gravity

In this case the action reads,

$$L = \int d^4x\sqrt{-g}\left[R + \alpha R^n +\right] + S^{matter}\ . \tag{10.40}$$

This theory has gained much popularity as a fourth order gravity model within the context of both inflation and dark energy [213, 59, 214, 173]. The characteristic function $\mathcal{Q}$ is :

$$\mathcal{Q}\ =\ \frac{y}{n(z-y)}\,, \tag{10.41}$$

and substituting this relation into the system of equations (10.3) we obtain,

$$\frac{dy}{dN} = \varepsilon y\left[2\chi - 2y + \frac{y(-\chi+y-z+\Omega-1)}{ny-nz} + 4\right]\ , \tag{10.42}$$

$$\frac{dz}{dN} = \varepsilon\left[\frac{(-\chi+y-z+\Omega-1)y^2}{ny-nz} + z(3\chi-3y+z-\Omega+5)\right]\ , \tag{10.43}$$

$$\frac{d\Omega}{dN} = -\varepsilon\Omega(3w-3\chi+3y-z+\Omega-2)\ , \tag{10.44}$$

$$\frac{d\chi}{dN} = 2\varepsilon\chi(\chi-y+1)\ , \tag{10.45}$$

with the constraint,

$$1 + x - y + z + \chi = 0\ , \tag{10.46}$$

The fixed points and their stability for the phase space of (10.42) are shown in Tables 7.3 and 7.4. In Table 10.2 we show the exponent of the long wavelength scalar and tensor perturbation modes.

The first point to note is that there exists a fixed point that corresponds to a transient Friedmann-type behavior just as in the $f(R) = R^n$ case for exactly the same values of the dynamical system variables. It is therefore not surprising that the long-wavelength scalar perturbations modes for these fixed points have the same solutions in both theories. At first glance this may seem to contradict the results in [202], where it was found that the scalar perturbations at the fixed point $\mathcal{L}$ depend on the value of α. However, this discrepancy can be resolved when one considers the structure of our dynamical system formalism and in particular the conditions (10.9). These additional equations impose further constraints on our system. This can be seen more clearly if we consider the fixed point $\mathcal{L}$ in the case of dust ($w = 0$) in more detail. Using the definition of the dynamical systems variable, x and our choice of $f(R)$ we find that the following constraint must be obeyed at $\mathcal{L}$,

$$x = -\frac{3(n-1)}{n} \qquad \Rightarrow \qquad \frac{a}{3\dot{a}} \frac{\alpha n^2 \dot{R} R^{(n-1)}}{R + \alpha n R^n} = -1 \,. \tag{10.47}$$

If we then use the fact that

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right), \qquad a(t) = a_0\,(t - t_0)^{\frac{2n}{3}} \,, \tag{10.48}$$

and that we require that the constraint above is satisfied for all time, it is trivial to show that (10.47) is only satisfied if one requires $\alpha \to \infty$. This means that supposing that the point $\mathcal{L}$ is associated with a solution of the cosmological equations, means that the function $f(R)$ we are dealing with is very close to R^n. As a consequence, when we insert constant values of the dynamical system variables in the coefficient of the perturbation equations we are in fact imposing that we are dealing with a theory which is essentially R^n-gravity. Thus, naturally, we recover exactly the same results of the previous section. This can be also seen from the fact that if we substitute the coordinates of the fixed point in the definition of $\mathcal{Q}$ we obtain,

$$\mathcal{Q} = \frac{1}{1-n} \,, \tag{10.49}$$

which is exactly the $\mathcal{Q}$ of R^n-gravity. Since $\mathcal{Q}$ is the only parameter that differentiates the theory of gravity in the coefficients of the perturbation equations (as well as the dynamical system) we clearly expect that after substituting the coordinates of the fixed point into the coefficients we obtain the same results of R^n-gravity.

This seems to suggest that in some sense the fixed points carry information about the theory of gravity other than the background evolution, so that a fixed point represents a physical solution only of a specific form of the Lagrangian. Hence, if the phase space of a generic theory of gravity possesses that fixed point, it means that there can be regimes in which this theory can be approximated by a Lagrangian for which that background is a “physical” solution.

Furthermore, from what was said above, one can also conclude that somehow the evolution of the perturbations is attached to the fixed point in such a way that regardless of the theory, the evolution of scalar perturbations is determined only by the fixed point. In other words we are somehow obtaining the “fixed point” of the perturbation theory which corresponds to the fixed point of the dynamical systems approach. Such a fixed point is an approximation of the real behavior of the equations. This means that we expect the scalar perturbations around this fixed point to be approximated by the results we found, which is consistent with what we obtained in [202] .

Table 10.2: Exponent of the modes for scalar and tensor in the fixed points of the system (10.42).

Point	Scalar Modes Exponents	Tensor Modes Exponents
$\mathcal{A}$	$\{\ 2\varepsilon, 0\}$	$\left\{-\varepsilon(2+\sqrt{3}), -\varepsilon(2-\sqrt{3})\right\}$
$\mathcal{B}$	$\left\{0, \varepsilon\,, \varepsilon\left(\frac{3w}{2}-1\right)\right\}$	$\{-\varepsilon, 0\}$
$\mathcal{C}$	$\{\ 2\varepsilon, 0, \varepsilon(3w-1), \varepsilon(3w+1)\}$	$\left\{\frac{\varepsilon}{2}\left(3w-\sqrt{9w^2-6w+9}-3\right),\right.$
		$\left.\frac{\varepsilon}{2}\left(3w+\sqrt{9w^2-6w+9}-3\right)\right\}$
$\mathcal{D}$	$\left\{0, \varepsilon\,, \varepsilon\left(\frac{3w}{2}-1\right)\right)\}$	$\left\{0, \varepsilon\left(\frac{3w}{2}-2\right)\right\}$
$\mathcal{E}$	$\{\varepsilon\frac{-3n-\sqrt{25n-32}\sqrt{n}}{2n}, \varepsilon\frac{-3n+\sqrt{25n-32}\sqrt{n}}{2n}\}$	$\{0, 0\}$
$\mathcal{F}$	$\{\ 2\varepsilon, -2\varepsilon, 0, -\varepsilon(1-3w)\}$	$\left\{\varepsilon(-3-\sqrt{6}), \varepsilon(-3+\sqrt{6})\right\}$
$\mathcal{G}$	$\{\ 5\varepsilon, 0, 2\varepsilon, \varepsilon\,(3w-2)\}$	$\left\{-\frac{7}{2}\varepsilon, 0\right\}$
$\mathcal{H}$	$\left\{0, \varepsilon(3w-1), \varepsilon\left[n-2-\sqrt{3}\sqrt{n(3n-4)}\right],\right.$	
	$\left., \varepsilon\left[n-2+\sqrt{3}\sqrt{n(3n-4)}\right]\right\}$	$\left\{\varepsilon\left(n-3-\sqrt{6-3n^2}\right), \varepsilon\left(n-3+\sqrt{6-3n^2}\right)\right\}$
$\mathcal{I}$	$\left\{0, \varepsilon\left(8n+2+\frac{9}{n-2}\right), \varepsilon\left[4(n+1)+\frac{6}{n-2}\right],\right.$	
	$\left.-\varepsilon\left[\frac{3(n(2n-3)+1)w}{n-2}+1\right]\right\}$	$\left\{\varepsilon n\left(4+\frac{3}{n-2}\right), 8\varepsilon\left(n+1+\frac{9}{n-2}\right)\right\}$
$\mathcal{L}$	$\left\{\varepsilon\left(2-\frac{4n}{3(w+1)}\right), \varepsilon\left(\frac{2nw}{w+1}-1\right),\right.$	
	$\varepsilon\left[\frac{3(w+1)+3n((2n-3)w-1)+\sqrt{A}}{6(n-1)(w+1)}\right],$	
	$\left.\varepsilon\left[\frac{3(w+1)+3n((2n-3)w-1)-\sqrt{A}}{6(n-1)(w+1)}\right]\right\}$	$\left\{\varepsilon\left(1-\frac{4n}{3(w+1)}\right), \varepsilon\frac{2(n-1)w-2}{w+1}\right\}$
$A=(n-1)(4(3w+8)^2n^3-4(3w(18w+55)+152)n^2+3(w+1)(87w+139)n-81(w+1)^2)$		

10.5 Conclusions

In this project we have discussed the connection between the dynamical system approach and the covariant-gauge invariant theory of perturbations, presenting a method to calculate directly the evolution of the scalar, vector and tensor perturbations at a fixed point of the phase space of a generic $f(R)$-gravity theory. Within the limitations of the dynamical system approach one is then able to obtain an idea of the evolution of the perturbations in any $f(R)$ model .

Because of the non-linearity and the peculiar structure of the dynamical system formalism, the concept of fixed points in $f(R)$-gravity is more subtle than the one of GR. In particular, one can have fixed points which do not correspond to solutions which satisfy the cosmological equations. This is due to the fact that the fixed-point conditions (10.9) constitute additional constraints that can be incompatible with the cosmological equations and that the exact solution associated to the fixed points seems to depend more on the definition of the variables than the model itself . This has profound implications on the interpretation of the results of the dynamical system approach. Specifically it suggests that this kind of approach offers only partial information on the actual evolution of the cosmology in this framework and that this information depends on the formalism used. In particular (i) one might not be able to see all the fixed points in the phase space because of the form of the variables and (ii) some of these fixed points might not correspond to actual solutions of the system with obvious problems in the interpretations of the orbits which have these points as attractors. As a consequence one has to be very careful in using these tools to derive general conclusions about the dynamics for these cosmological models.

These peculiarities also have consequences on the results of the perturbations equations. In particular, we found that the evolution laws obtained using this form of the equations do not in general coincide with those that one would obtain by simply subsittuting in the background corresponding to the fixed point in the dynamical system equations. This can be explained when one considers that the additional conditions associated with the fixed points, further constrain the perturbation equations and therefore lead to different results. On the other hand, such a fact also implies that one can associate an evolution law for first-order perturbations to a specific fixed point, which in some sense may be considered as a "fixed point" for the perturbation theory. Thus one can use the results of the equations given above in the same way in which one used the solution at the fixed points: gaining qualitative information about the behavior of the perturbations along an orbit. Such a feature, confirmed by the direct calculations presented in [202], can help with the understanding of the behavior of the perturbations in models for which a direct numerical integration is too complex or too resource-intensive to be performed. It is also worth noticing that since the peculiarities in the predictions of the equations presented above derive ultimately from the additional condition on the fixed points, they apply only when one deals with fixed points and will not be present when one considers the dynamical system variables as functions of time.

In conclusion, in spite of these difficulties the dynamical system approach, when combined with the covariant gauge invariant formalism, represents an extremely powerful tool for the study of $f(R)$-gravity cosmological models. We believe that a careful use of these methods could be invaluable in determining the relevance of these models on cosmological scales and to constrain them using current and future observations.

Chapter 11

DISCUSSION AND CONCLUSION

As mentioned before, the existence of dark energy has been inferred from the standard model of cosmology. The bounds for the value of the equation of state parameters of dark energy w_{DE} is given by the Wilkinson Microwave Anisotropy Probe (WMAP) data [250], in the range $-1.11 < w_{DE} < -0.86$. When the effective equation of state parameter of dark energy w_{DE} is equal to -1 the Universe passes through the ΛCDM epoch, if w_{DE} is is slightly less than -1 then we live in a phantom-dominated Universe, and if w_{DE} is slightly more than -1 the quintessence dark epoch occurs. Although it is believed that the Universe has passed through a de Sitter-type accelerating-expansion 'inflation' in the early Universe, the possibility that the current (or future) acceleration could be of quintessence or phantom type should not be completely excluded and hence the Universe might be evolving toward a finite-time future singularity. Therefore, until more accurate observational data are available we cannot judge the viability of $f(R)$ gravity models.

We believe that any modified theory of gravity must account for the early and late-time cosmologies which are well established by observations, and at the same time must be consistent with solar system and laboratory tests of gravity. In order for the gravity model to produce a late-time acceleration, however, one must introduce a new degree of freedom, such as scalar filed. The effective mass of this scalar field should depend on the local matter density. In particular, the scalar field should be very light for the cosmological density and heavy for the solar system density. Hu-Sawicki and Starobinsky models are the most common models that satisfy these requirements. The study of the strong gravity aspects shows that all $f(R)$ models generically suffer from a curvature singularity problem [221], these results raise doubts on the viability of $f(R)$ as an alternative theory of gravity. However $f(R)$ theories make excellent candidates for toy-theories from which one gains some insight in gravity modifications..

Several extensions of $f(R)$ gravity have been proposed in order to address some of the viability issues, for example consider the action,

$$S = \int d^4x\sqrt{-g}\Big[\frac{f_1(R)}{2} + (1 + \lambda\, f_2(R)L_m)\Big]\,, \tag{11.1}$$

where L_m is the matter Lagrangian and f_1, f_2 are functions of the Ricci curvature R. This action has been motivated by Bertolami [218] as a substitute for dark matter. Bertolami action gives rise to extra forces that might account for the modified Newtonian dynamics (MOND) gravity, but at the cost of violation of energy conservation and of the equivalence principle as well. Another extension of $f(R)$ gravity is what is known by extended quintessence [219] in which scalar field couple to different curvature invariants. Also it has been shown in [222] that, additional modification of any $f(R)$ gravity by the terms relevant at the early Universe is necessary to avoid future singularities. To conclude, when all of the above concerns are taken into account, $f(R)$ gravity is still a toy theory that might help a lot in understand the principles and limitations of modified gravity as well as GR itself.

11.0.1 Some of the results obtained

The study of the exponential gravity model shows that (see chapter 5) cosmic histories exist which admit a double de Sitter phase which could be useful for describing the early and the late-time accelerating Universe. In the vacuum case we find two finite physical critical points c_v; D_v, these two points represent a solution whose nature depends on the parameter $\gamma(\Lambda)$; for$\Lambda\ > 0$ we can have either exponential expansion ($\gamma > 0$) or exponential contraction ($\gamma < 0$), while for $\Lambda < 0$ the solution oscillates. From the stability point of view, the point $\mathcal{C}_v$, which resides in the invariant submanifold $z = 0$, is of particular interest because, since it is non-hyperbolic, it represents an attractor for $z > 0$ and a saddle for $z < 0$, while the other physical point $\mathcal{D}_v$ is found to be a saddle. From a physical point of view, the region $z > 0$ appears to be more interesting because in this region the point $\mathcal{C}_v$ represents a de-Sitter attractor which might be associated with a Dark Energy/inflation era. The same region also contains the point $\mathcal{D}_v$, which represents an unstable de Sitter phase (see Figure 5.1). This implies that the subset of the orbits which converge to $\mathcal{C}_v$ can also contain cosmic histories that present a second, unstable, de Sitter phase. In addition, orbits that evolve near the non-physical point $\mathcal{B}_v$ can also present an intermediate power law phase. Finally, it is apparent from Figure 5.3 that the de Sitter phases $\mathcal{C}_v$ and $\mathcal{D}_v$ are separated from the past attractor $\mathcal{A}_v$ by the plane $q = 0$; therefore any model with initial conditions near the past attractor $\mathcal{A}_v$ and evolving toward the future de-Sitter attractor $\mathcal{C}_v$ will cross the plane $q = 0$, indicating a transition from an accelerating evolution to a decelerating one. By a direct substitution into the field equations we found that all the fixed points in the matter case do not represent physical solutions.

By applying the compactification scheme [126] to $R + \alpha R^n$ (see chapter 6), we find a number of interesting cosmological evolutions which include the possibility of an initial unstable power-law inflationary point, followed by a curvature-fluid-dominated phase mimicking standard radiation, then passing through a standard matter era and ultimately evolving asymptotically towards a de Sitter-like late-time accelerated phase. Our compact analysis shows that there are more equilibrium points than in the corresponding non-compact analysis in [123]. In particular we find a new finite fixed point $\mathcal{N}_{\pm}$. Because of its very special location in the phase space, it is quite difficult to obtain this point using the standard techniques. This point is found to represent a matter phase and the expanding version of this point is *spiral*$^{-}$. Furthermore, we find that for $n > 1/2(1+\sqrt{3})$ the phase space of $R+\alpha R^n$, contains two accelerated fixed points $\mathcal{E}_+;\mathcal{I}_+$, together with two other saddle points (one represent a radiation phase $\mathcal{D}_+$ and the other represent a matter-like phase $\mathcal{L}_+$). Although we have obtained all the desired fixed points and desired stability, this does not necessarily imply that there is an orbit connecting them. Due to the fact that for $n > 1/2(1+\sqrt{3})$, the two accelerated points and the matter-like point are quite close to each other in the phase space, which makes it difficult to prove the existence of an orbit connecting these points together with the radiation-like point. But the presence of all these phases in the state space of $R+\alpha R^n$ makes a more detailed investigation worth pursuing.

In chapter 9 we study the evolution of scalar cosmological perturbations in the 1+3 covariant gauge-invariant formalism for generic $f(R)$ theories of gravity. Extending previous works, we give a complete set of equations describing the evolution of matter and curvature fluctuations for a multi-fluid cosmological medium and we showed that they reduce to the corresponding GR evolution equations in the limit when $f(R) = R$. We then specialize to a radiation-dust fluid described by barotropic equations of state and solve the perturbation equations around a background solution of R^n gravity. In particular we study exact solutions for scales much smaller and much larger than the Hubble radius and show that $n > 2$ in order to have a growth rate compatible with the Meszaros effect. We also gave a new covariant characterisation of the quasi-static approximation and used this to show that on small scales this approximation is valid for values of n in the neighbourhood of 1, i.e., it is in good agreement with a numerical integration of the full set of equations for the given set of initial conditions. This is the first time such a quasi-static analysis has been presented in a covariant way both for radiation and dust universes and provided the foundations for detailed comparison with what is found using the metric formalisms, together with a full computation of the power spectra.

The connection between the dynamical system approach and the covariant-gauge invariant theory of perturbations has been investigated in details in chapter 11, where we present a method to calculate directly the evolution of the scalar, vector and tensor perturbations at a fixed point of the phase space of a generic $f(R)$ gravity theory. This method represents an extremely powerful tool for the study of perturbations in $f(R)$ gravity cosmological models. It is interesting to observe the behavior of the matter perturbation modes of the point $\mathcal{F}$. Here the matter perturbations possess a constant mode and the other modes can be growing or decaying, depending on the value of the parameter n. As shown in [184] for some specific values of the parameter n $(1.37 \lesssim n \lesssim 2)$ this model has a set of cosmic histories characterized by the presence of a transient, decelerated power law expansion that evolves towards an accelerated expansion. Using (10.14) and (10.31) we are now able to see directly the evolution of the scalar and tensor perturbations along these orbits.
In particular, as the Universe approaches the point $\mathcal{F}$, for $1.37 \lesssim n \lesssim 2$ and $w = 0$, the large scale scalar perturbations, which nearby $\mathcal{G}$ have a growing mode (see Figure 10.1), start dissipating, which is consistent with what one would expect in a late-time acceleration scenario.

From the study of the CMB anisotropies generated by tensor perturbations for R^n gravity model ($n \neq 2$) (see chapter 10), we find that the initial conditions for R^n gravity model ($n \neq 2$) are exactly the same as the GR initial conditions. Thus by modifying the perturbation and the background equations in CAMB package, we manage to generate the temperature and the E-mode polarization power spectra for this $f(R)$ gravity model. We applied our general results to R^n models for different values of n verifying the convergence to GR result when n approaches unity and determining the features that may distinguish those models from Concordance model predictions. Our implementation makes it possible to distinguish these models form GR and demonstrates the importance of considering the correct background when alternative theories of gravity are subjected to this kind of analyses. According to our results, the sole consideration of perturbations for the $c_l^{\rm TT}$, having assumed the usual GR background, would lead for instance to not see any appreciable difference between pure GR and the $n = 1.22$ case. Moreover, values in the interval $n = 1.26 - 1.28$ are also indistinguishable from each other. With regards to the $c_l^{\rm EE}$ coefficients, it can be seen for instance how for $n = 1.22$ the sole consideration of perturbations (keeping the GR background) is hardly detectable for c_l^{EE} as well as the interval $n = 1.26 - 1.28$ provide the same c_l^{EE} pattern as GR. Once the correct background is implemented, this degeneracy is broken and therefore observable effects may be detected.
Our code provides then a powerful tool able to show the key features of the effect that fourth-order gravity theories may have in the CMB tensor perturbations. This work is expected to be generalized to more realistic $f(R)$ gravity models in the future.

APPENDIX

A Covariant identities

For any scalar f,vector V^a and second-rank tensor W^{ab},the following commutation relations are valid identities in a FLRW geometry:

$$(\tilde{\nabla}_a f)^{\cdot} = \tilde{\nabla}_a \dot{f} - \tfrac{1}{3}\Theta\tilde{\nabla}_a f + \dot{f}\dot{u}_a \tag{A-1}$$

$$\tilde{\nabla}^2(\nabla_a f) = \tilde{\nabla}_a(\nabla^2 f) - \frac{2K}{a^2}\tilde{\nabla}_a f - 2\dot{f}\omega_a \tag{A-2}$$

$$(\tilde{\nabla}^2 f)^{\cdot} = \tilde{\nabla}^2 \dot{f} - \frac{2}{3}\Theta\tilde{\nabla}^2 f + \dot{f}\tilde{\nabla}^a\dot{u}_a \tag{A-3}$$

$$(\tilde{\nabla}_a V_b)^{\cdot} = \tilde{\nabla}_a \dot{V}_b - \tfrac{1}{3}\Theta\tilde{\nabla}_a V_b \tag{A-4}$$

$$\tilde{\nabla}_{[a}\tilde{\nabla}_{b]}V_c = -\frac{K}{a^2}V_{[a}h_{b]c} \tag{A-5}$$

$$\tilde{\nabla}^b(\tilde{\nabla}_{\langle a}V_{b\rangle}) = \tfrac{1}{2}\tilde{\nabla}^2 V_a + \tfrac{1}{6}\tilde{\nabla}_a(\tilde{\nabla}^b V_b) + \frac{K}{a^2}V_a \tag{A-6}$$

$$(\tilde{\nabla}_a W_{cd})^{\cdot} = \tilde{\nabla}_a \dot{W}_{cd} - \tfrac{1}{3}\Theta\tilde{\nabla}_a W_{cd} \tag{A-7}$$

where $V_a = V_{\langle a\rangle}$ and $W_{ab} = W_{\langle ab\rangle}$ are first order quantities.

B Some useful relations in $f(R)$

The following $f(R)$ relations are generally used throughout this thesis:

$$\mu = \frac{\mu_m}{f'} + \mu_R \tag{B-8}$$

$$\mu_R = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R\right] \tag{B-9}$$

$$p = \frac{p_m}{f'} + p_R \tag{B-10}$$

$$p_R = \frac{1}{f'}\left[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\Theta f''\dot{R} - \frac{2}{3}f''\tilde{\nabla}^2 R\right] \tag{B-11}$$

$$R = \mu - 3p = \tilde{\mu}_m + \mu_R - 3\tilde{p}_m - 3p_R \tag{B-12}$$

$$3\ddot{R}f'' + 3\dot{R}^2 f''' + 3\,\Theta\dot{R}f'' - 3\tilde{\nabla}^2 f' = \mu_m + Rf' - 2f - 3p_m \tag{B-13}$$

$$\mu_R + 3p_R = (1 - 3w)\frac{\mu_m}{f'} - \frac{f}{f'} - 2\Theta\dot{R}\frac{f''}{f'} + 2\frac{f''}{f'}\tilde{\nabla}^2 R \tag{B-14}$$

$$\Theta^2 = 3\frac{\mu_m}{f'} + \frac{3R}{2} - \frac{3f}{2f'} - 3\dot{R}\Theta\frac{f''}{f'} + 3\frac{f''}{f'}\tilde{\nabla}^2 R - \frac{9K}{a^2} \tag{B-15}$$

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