Multivalued semi-Fredholm Operators in Normed Linear Spaces

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A thesis submitted
in fulfilment of the requirements
for the degree of
PhD in Mathematics

JULY 2002

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

UNIVERSITY OF CAPE TOWN

Abstract

Semi-Fredhom linear relations between normed linear spaces X and Y are defined as follows : A linear relation (or multivalued linear operator) $T : X \to Y$ is said to be upper semi-Fredholm, denoted $T \in \mathcal{F}_+$, if there exists a finite codimensional subspace M of the domain $D(T)$ such that the restriction $T|_M$ is open and injective, i.e. if T has a single-valued continuous inverse on some finite codimensional subspace. When X and Y are complete and T is closed, i.e. when its graph $G(T) := \{(x, y) \mid y \in Tx\}$ is a closed subspace of $X \times Y$, then $T \in \mathcal{F}_+$ if and only if its range $R(T)$ is closed and $\alpha(T)$, the dimension of the nullspace $N(T) := \{x \mid Tx = 0\}$, is finite. Hence the theory of upper semi-Fredholm relations is a theory of continuous relations in the sense that the inverse relation T^{-1} of $T \in \mathcal{F}_+$ is continuous. T is said to be lower semi-Fredholm, denoted $T \in \mathcal{F}_-$ if is adjoint relation, denoted T' and also sometimes referred to as conjugate, dual or transpose, is upper semi-Fredholm. T is called a Fredholm relation if it is both upper and lower semi-Fredholm. Certain properties associated with these classes are stable under small perturbation, i.e. stable under additive perturbation by continuous operators whose norms are less than the minimum modulus of the relation being perturbed, and are also stable under perturbation by compact, strictly singular or strictly cosingular operators. In this work we continue the study of these classes and introduce the classes of α–Atkinson and β–Atkinson relations. These are subclasses of upper and lower semi-Fredholm relations respectively, having generalised inverses and defined in terms of the existence of continuous projections onto their ranges and nullspaces. We show that the existence of generalised inverses is stable under perturbation and extend known stability theorems for the index of an operator. The classification of Fredholm operators is connected to studies of the spectra of operators. We extend investigations in the multivalued case by introducing essential spectra for multivalued operators, and briefly consider the invariant subspace problem in the multivalued context.

Acknowledgments

Sincere and heartfelt thanks go to Prof. R.W. Cross for introducing me to multivalued operators, and for nurturing my interest and understanding of the mathematics of operators and normed linear spaces; I thank him for the enlightening conversations through the duration of this project, and for his patience and support, even when other commitments slowed its progress.

I am grateful to my parents, Toni and Robert Wilcox, for all their support, particularly for their assistance in securing accommodation while I have been a student. I also deeply appreciate their various contributions to the foundations of my education.

I thank the many educators and researchers who have offered advice, encouragement or training. Particularly, I thank Teresa Alvarez for her collaboration on Atkinson relations and thank Louis Labuschagne for reading the entire manuscript in detail and making numerous observations and suggestions.

Warm thanks to Pauline M. Groves for supporting mathematics education for all by providing much-needed sponsorship in the form of the Chisnall endowed Tutorship. This work was also partially supported by research grants from the NRF and former FRD.

This project was undertaken within the Department of Mathematics and Applied Mathematics at the University of Cape Town, where I have also worked as a lecturer, tutor and admin. assistant. I thank members of the department , including its students, and the greater university community for endeavours to create an environment for education and fundamental research. Particularly, I thank the mathematics postgraduate students group for enjoyable and enriching discussions and seminars on various topics of research and life in mathematics.

In South Africa and further away, many thanks to friends and family for the lessons shared, questions asked, the challenges and the encouragement and companionship. To those dead and gone, thank you for your endeavors to see into the nature of things.

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Chapter 0

Introductory Remarks

Fredholm Operators in Mathematical Analysis

Origins

The general theory of Fredholm type linear operators arose out of investigations to expand the scope of general methods for solving integral equations and out of the development of spectral theory in functional analysis.

In 1900 Ivar Fredholm [53] presented a technique for solving what are now referred to as Fredholm integral equations of the second kind :

$$f(s) = g(s) + \lambda \int_a^b K(s,t)f(t)dt, \tag{1}$$

where K is a bounded and piece-wise continuous function on $[a,b] \times [a,b]$, g is continuous and f is unknown. Before him, Volterra had given representations of solutions to equation (1) for the case when the function $K(s,t) = 0$ for $t > s$. Such equations are referred to as Volterra equations of the 2nd kind; equations of the 1st kind :

$$\lambda \int_a^b K(s,t)f(t)dt = g(s), \tag{2}$$

can be reduced to equations of 2nd kind. Prior to to these works, integral equations had only been investigated in isolated problems (cf. Dieudonné [46] or Kline [77]) - general methods for finding solutions were not known till the pioneering work of Volterra.

Volterra had noted that such equations resembled the limiting case of a system of n linear equations of n unknowns, with n tending to infinity. It was this idea which Fredholm advanced in order to construct solutions for equations of the form (1). By doing so, he was able to give a proof of the existence of solutions to the *Dirichlet problem* (cf. Kellog [76]). In its simplest form, this concerns the existence of a harmonic function u, i.e. a solution to Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

within a 2-dimensional domain D such that the values of u are given on the boundary S of D. Fredholm's solution involves reducing the problem into two integral equations of 2nd kind (cf. Tricomi, [139]). This approach had already been investigated for specific examples of Dirichlet problems by Beer and Neumann (cf. Dieudonné [46]), and by Poincaré. However, until Fredholm's paper, the existence of a solution could only be proved for a restricted class of domains.

It is likely that Fredholm proceeded with three central ideas:

[I] Equation (1) was replaced by the Riemann sum

$$f(y_j) \; + \; \tfrac{\lambda(b-a)}{n} \sum_{k=1}^{n} K(y_j, y_k) f(y_k) \; = \; g(y_j).$$

[II] Based on a formula for infinite determinants due to von Koch, a series expansion was derived for the so-called *determinant* of the system.

[III] Using techniques attributed to earlier work by Hadamard, the series was shown to be uniformly convergent.

In 1903, Fredholm completed some of the results which were lacking details - in particular, he concluded by giving necessary and sufficient conditions on the function g for the existence of a solution to equation (1). The latter is now referred to as the *Fredholm alternative* or the *alternative theorem*, and may be stated more generally as follows:

If the homogeneous Fredholm integral equation

$$f(s) \; - \; \lambda \int_a^b K(s,t) f(t) dt \; = \; 0, \tag{3}$$

has only the trivial solution, then there exists a unique solution for the corresponding non-homogeneous equation. If the homogeneous equation has some nontrivial solution, then the non-homogeneous equation has either no solution or infinitely many solutions, depending on the given function.

Fredholm also noted that :

the number of solutions of the "transposed" equation,

$$f(s) \; = \; g(s) \; + \; \lambda \int_a^b K(t,s) f(t) dt, \tag{4}$$

the equation in which $K(s,t)$ is replaced with $K(t,s)$ in equation (1), is equal to the number of solutions to (1).

Fredholm's work both advanced the general Dirichlet problem, and introduced a fundamental method for solving integral equations, and hence, sparked widespread interest in a general theory for integral equations. Hilbert recognised that the development of the new subject would be important for the theory of definite integrals, for the development of arbitrary functions in series, for the theory of linear differential equations, for potential theory and for the calculus of variations (cf. Kline [77]). In a series of papers, he (and other mathematicians) began by improving upon

Fredholm's work with a more rigourous treatment of the passage to the limit. Considering the case in which the kernel K is continuous, real valued and symmetric, i.e. $K(s,t) = K(t,s)$, Hilbert considered the roots of the Fredholm determinant which he referred to as *eigenvalues*. For a function defined by :

$$f(s) \;=\; \lambda \int_a^b K(s,t)g(t)dt,$$

he then showed that the corresponding eigenfunctions yielded a Fourier type expansion

$$f(s) = \sum_n \left[\int_a^b f(t)x_n(t)dt\right] x_n(s)$$

which converged absolutely and uniformly, where the values of $\int_a^b f(t)x_n(t)dt$ are the "Fourier coefficients".

The most fundamental of Hilbert's contributions, however, was his paper of 1906. In this work he considered equation (1) as an infinite system of infinitely many linear equations in infinitely many unknowns. Thus, he obtained the basic results of *infinite bilinear* and *quadratic forms*, and introduced techniques which are now well-known for the sequence space l_2. He also verified the phenomenon of the *continuous spectrum*. Fourier, and later Wirtinger, had considered periodic solutions to equations of the form

$$y'' + \lambda q(x)y \;=\; 0. \tag{5}$$

Fourier had noted that the eigenvalues for the problem with boundary conditions $y(-a) = y(a) = 0$ and satisfying $q(x) = 0$, tended to fill the real line as the value of a tended to infinity. Wirtinger observed similar behaviour when investigating complex solutions of period n as n tended to infinity.

The Spectral Theory of Compact Operators

The early years of the 20th century saw the crystallisation of many of the fundamental concepts and techniques of functional analysis. In 1913 Riesz presented a derivation of Hilbert's reduction in terms the analysis of *continuous operators* on l_2. Using the property that an endomorphism T is uniquely associated with the bilinear form

$$(x,y) \;\rightarrow\; < Tx, y >,$$

where $< .,. >$ denotes the inner product on l_2, he considered functions of operators, which are also continuous endomorphisms, and obtained a spectral decomposition for symmetric operators as Hilbert had done.

In his analysis of infinite systems of linear equations, Riesz maintained Fredholm's approach of considering continuous mappings (rather than bilinear forms as Hilbert had done). Thus, in his work of 1918 he introduced *compact operators* as the maps which transformed bounded sets to relatively compact ones. While Hilbert had formulated such maps as those which transformed weakly convergent sequences to strongly convergent sequences, Riesz's definition applied the general concept of compactness which was introduced by Frèchet , and was not restricted to sequence spaces.

It is worthwhile noting that Frèchet's thesis of 1906 on metric spaces was the first comprehensive treatment of an abstract theory of function spaces and of linear functionals (cf. Kline [77]).

Riesz showed that a Fredholm operator with continuous kernel is compact. He then presented what is now called *Riesz's Lemma* and the important *characterisation of finite-dimensional normed spaces* in terms of compactness of the unit ball. Next he considered compact endomorphisms of a normed space (though normed spaces were not yet formally defined). Considering the map

$$T = I - K$$

where K is a compact operator defined on a normed space X, and I is the identity, he proved that

(i) the nullspace $N(T)$ is finite dimensional, and

(ii) the range $R(T)$ is closed and finite codimensional.

Furthermore, by considering iterates T^n of T, he deduced that the space X could be decomposed into the topological direct sum of invariant subspaces. Riesz was then able to furnish the general theory for the eigenvalues of compact linear operators, including the famous properties that :

> *The spectrum $\sigma(K)$ of a compact operator K is a countable set which has no non-zero points of accumulation and if λ is an eigenvalue, then the dimension of the subspace spanned by the corresponding eigenvectors, i.e. the* multiplicity *of λ, is finite.*

(cf. Dieudonné [46] for further details).

One may regard the Fredholm alternative theorem as an infinite-dimensional analogue of the *fundamental theorem of linear algebra* (cf. Section 2.7). Furthermore, one may note that the Riesz's use of iterates of a compact operator is analogous to techniques for identifying the properties of and deriving the Jordan normal form of a matrix on a finite dimensional vector space. His formulation of the general spectral theory for compact operators remains essentially unchanged as part of the classic core of spectral theory.

The Emergence of the General Theory of Fredholm Linear Operators

The properties of the range and kernel of the perturbed compact opertor $T = I - K$, which were elucidated by Riesz, were in fact the properties which would come to be used to define *Fredholm operators.* For a long time, the notion of a Fredholm operator referred to the operator associated with a Fredholm integral equation. Eventually, the study of integral equations proceeded to tackle equations to which the alternative theorem (or one of its variants) did not apply. These included nonlinear and singular equations, where the latter refers to equations in which the limits are unbounded or in which the integrand takes on infinite values on the domain of definition. Fredholm had considered the case where the kernel was a piecewise continuous function after which his results were extended to include L_2 kernels. For the case when the kernel was not an L_2 function, one of the lines of investigation was to consider the manner in which the behaviour of solutions deviated from the alternative theorem.

In 1921, F. Noether discovered that for a particular class of equations, the number of solutions to the homogeneous equation differed from the number of solutions of the homogeneous "transposed equation" (equation (4) above) . Nevertheless, it was observed that the conditions on the solvability of the corresponding non-homogeneous equation could correspond to those given by Fredholm. This motivated the definition of the quantity :

$$index(\lambda - T) = \alpha(\lambda - T) - \alpha(\lambda - T^*).$$

Here T is an integral operator, T^* is its transpose in the sense of the "transposed" kernel, and $\alpha(\lambda - T)$ and $\alpha(\lambda - T^*)$ denote the dimensions of the nullspaces of the operators, $\lambda - T$ and $\lambda - T^*$, respectively. In the case when λ is an eigenvalue, these quantities are the dimensions of the corresponding eigenspaces.

Similar quantities were investigated by Carleman and, a short while thereafter, by von Neumann, in their studies of Hermitian matrices. Von Neumann considered the general case of Hermitian operators and considered the *defects* which were defined as the dimensions of subspaces associated with the Cayley transform of the matrix (cf. Dieudonné). The latter quantity may be compared with that of the codimension of the range of an operator, which was referred to as the *deficiency* in later literature.

In their contributions to the development of spectral theory, Weyl (in work dating back to 1909) and Courant (in 1920) , formulated a theorem for the comparison of the eigenvalues of compact operators. They showed that, for compact operators A_1 and A_2, the eigenvalues values of the sum, $A := A_1 + A_2$, are related to those of A_1 and A_2 via a collection of inequalities (cf. Riesz and Sz-Nagy [124], section 95). Weyl was able to show that

> *the eigenvalues of a bounded symmetric operator A were invariant under additive perturbation of A by a compact symmetric operator .*

Von Neumann gave a simpler proof of this result in his paper on the spectral theory of integral operators in 1935 (cf. Riesz and Sz-Nagy [124], section 134).

By the late 1940's, it was shown by Gahov (and the Tiflis school of mathematics) that the index of a bounded operator (where the index quantity is defined in the sense given below) on a Hilbert space was stable under perturbation by a compact operator. This theory was then extended to bounded operators on a Banach space by Atkinson and Gohberg, and to unbounded linear operators by Krein, Krasnosel'skii, Sz-Nagy and Gohberg (see Gohberg and Krein [124]).

Gradually it was realised that the properties identified by Riesz could serve to define abstract classes of operators on Banach spaces whose general theory unified results obtained in the areas of singular integral equations, Hermitian matrices and spectral theory. In literature stemming from the Russian school, these would become denoted as Φ operators, and were formally defined as follows :

$$
\begin{aligned}
\Phi(X,Y) &:= \{T \in C(X,Y) \mid R(T) \text{ is closed, } \alpha(T) < \infty \text{ and } \beta(T) < \infty\} \\
\Phi_+(X,Y) &:= \{T \in C(X,Y) \mid R(T) \text{ is closed and } \alpha(T) < \infty\} \\
\Phi_-(X,Y) &:= \{T \in C(X,Y) \mid R(T) \text{ is closed and } \beta(T) < \infty\},
\end{aligned}
$$

where $C(X,Y)$ denotes the class of closed bounded linear operators, $\alpha(T) := dim\, N(T)$, $\beta(T) := codim\, R(T)$. It is worth noting that $codim\, R(T) = \alpha(T')$, where T' denotes the *adjoint* of the operator T (in the contemporary sense). The *index* of the operator is defined to be the quantity

$$\kappa(T) := \alpha(T) - \beta(T).$$

An exposition of the above classes, and the quantities $\alpha(T)$, $\beta(T)$ and $\kappa(T)$, is given in a paper Gohberg and Krein [57]. The theory of linear Fredholm operators thus came to incorporate fundamental connections with the theory of compact operators (and more generally strictly singular operators), perturbation and stability theorems of the index, and theorems on the essential spectra of a linear operator. The latter refer to subsets of the spectrum which remain invariant under compact and small perturbation.

Contributions of this work

This work continues the generalisation of the theory of Fredholm operators in normed linear spaces to the multivalued case (cf. Cross [35]). Chapter 1 summarises a selection of known properties of normed linear spaces and set-valued maps relevant to the sequel and includes further historical information. Chapter 2 contains the basic theorems and properties of linear relations. In Chapter 3, some important theorems for linear relations are discussed, namely, the Baire property of linear relations, the Closed Graph and Closed Range theorems, and State Diagram, the Small Perturbation theorem, and in the last section, properties are given for a new class of relations, Multivalued Linear Projections. Results from this chapter are applied in the succeeding chapters. Chapter 3 includes original work - further details are given both at the start of the chapter and at the end, in Section 3.7. In Chapter 4, properties of the operator quantities are reviewed for application in the perturbation theorems of Chapters 5, 6 and 8. In Chapter 5, Fredholm type linear relations are discussed (the definitions used are those of Cross [35]). The connections between Fredholm relations and compact and strictly-singular relations, and the perturbation theorems are given here. In Chapter 6, the classes of $\alpha-$Atkinson and $\beta-$Atkinson relations are introduced, and their characterising properties are described. The following sections of the chapter are concerned with the development of perturbation theory for Atkinson relations. Chapter 7 begins with a review of known results in the spectral theory of linear relations. A theorem on the Domains of Iterates of a linear relations is given in this chapter. This result is followed by a section on the Invariant Subspace Problem for multivalued linear operators. In Chapter 8, various Essential Spectra of a linear relation are investigated. These subsets of the spectrum are defined in terms of Fredholm properties of the operator $\lambda - T$. Perturbation theorems from previous chapters are applied to show the stability of essential spectra under additive perturbation.

Applications of Fredholm operators

We conclude this general introduction by mentioning some examples of where Fredholm operators arise in other areas of mathematics.

The theory of Fredholm operators has various applications in *nonlinear analysis* . For example, in the study of classes of nonlinear and singular integral equations which cannot be solved directly via alternative theorems, it is known that three new phenomena may occur (cf. Tricomi [139]). In one case, the number of solutions to the equation may change as the parameter λ in the Fredholm equation of the 2nd kind varies, i.e. there may be points of bifurcation. It is sometimes possible to determine properties of bifurcations and solutions by means of linear Fredholm projections and alternative theorems. In other problems, the Frèchet (or Gâteaux) derivative of a nonlinear map may be a linear Fredholm operator. In such cases, Fredholm theory is sometimes applied at a local level (cf. Krasnosel'skii and Zabreiko [78] or Georgescu and Oprea [54]).

In the theory of *smooth manifolds*, if X is a smooth compact manifold, and E and F are smooth vector bundles on X, then the differential operator $d : C^{\infty}(E) \to C^{\infty}(F)$, which is given locally by a matrix of differential operators, is a linear Fredolm operator with finite index. Gelfand had asked whether the invariance of the index could be expressed in terms of topological data. Atiyah and Singer succeeded in showing this by introducing a rational cohomology class to define a *topological index* i_t. In 1963 they proved that $\kappa(d) = i_t(d)$ (cf. Palais [117]). Fredholm operators continue to feature in the investigation of properties of manifolds and in generalised cohomology theories.

In the development of *Banach space theory*, Gowers and Maurey [66] gave a counterexample for the famous unconditional basic sequence problem in 1993. They also showed that their example is a hereditarily indecomposable space, i.e. neither the space itself, nor any of its infinite dimensional subspaces, can be decomposed into the topological sum of infinite dimensional subspaces. They then showed that the only bounded operators on the space are of the form $\lambda - S$, where S is a strictly singular operator. Operators of this form are Fredholm operators for non-zero λ (cf. Chapter 5, Section 5.3). Further comments on the Gowers and Maurey space are made at the ends of Chapters 1 and 5.

Chapter 1

Preliminaries

1.1 Normed Linear Spaces

Before abstract normed linear spaces were formally considered, the spaces L_p and l_p were investigated by Riesz in 1910. Earlier, Schmidt had already applied a norm function (using notation $||x||$) to address general systems of linear equations on l_2 and, in his *Geometry of Numbers* , 1896, Minkowski [108] had considered norm functions on $I\!R^n$ defined in terms of closed symmetric bounded convex sets centred at the origin. In 1912 and after the war in 1921, Helly extended these ideas to sequence spaces by defining a norm on $\mathbb{C}^n$ which satisfied the standard axioms for a norm function, and gave a special case of what is now called the Hahn-Banach extension theorem. Banach, in his thesis in 1920, and Hahn, in 1922, independently generalised these ideas to consider norm functions on abstract vector spaces over $I\!R$ or $\mathbb{C}$, and considered the continuous linear operators between such spaces. In his famous book published in 1932, Banach [15] gave a comprehensive presentation of the theory known at that time, including the Closed Graph and Banach-Steinhaus theorems (these are discussed in the context of linear relations in Chapter 3).

Definitions 1.1.1 *A* **linear topological vector space** *is a linear vector space over a field $I\!K = I\!R$ or $\mathbb{C}$ with a topology such the map $(x, y) \to x + y$ is continuous from $X \times X \to X$ and the map $(\alpha, y) \to \alpha x$ is continuous from $I\!K \times X \to X$.*

A **normed linear space** *X (over a field $I\!K = I\!R$ or $\mathbb{C}$) is a linear topological vector space with topology given by a real valued function, referred to as* **norm**, *and denoted $||\_|| : X \to I\!R$, which satisfies:*

$$\begin{aligned} ||x+y|| &\leq ||x|| + ||y||, \\ ||\alpha x|| &= |\alpha|\ ||x||, \\ ||x|| \geq 0, \quad &and \quad ||x|| = 0 \Leftrightarrow x = 0 \end{aligned}$$

for $x,\ y \in X$ and $\alpha \in I\!K$. The norm defines a natural metric $d(x, y) := ||x - y||$. If X is complete under this topology, then it is called a Banach space.

Examples 1.1.2

(1) The spaces $I\!R^n$ *and* $\mathbb{C}^n$ *with norms defined by:*

$$||(x_1, x_2, \ldots, x_n)|| := (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}.$$

(2) The spaces c_o, c *and* l_∞ *, where the latter is also denoted by* m *, consisting of null, convergent and bounded scalar sequences, respectively, with norms defined by:*

$$||x|| := \sup |x_i|.$$

(3) The spaces l_p, $1 \le p < +\infty$ *of p-summable scalar sequences with norms defined by:*

$$||x||_p := (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}.$$

(4) The spaces $C(K)$ *and* $B(K)$ *consisting of continuous and bounded functions, respectively, defined on a compact set* K*, with norms defined by :*

$$||f|| := \sup_{x \in K} |f(x)|.$$

(5) The function space $L_\infty(X, \mu)$ *which is defined as follows. Let* (X, μ) *be a measure space, and let* $\mathcal{L}_\infty$ *denote the set of measurable functions which satisfy*

$$||f||_\infty := \text{ess sup}\,|f(x)| < \infty.$$

The space $L_\infty(X, \mu)$ *consists of the equivalence classes of elements of* $\mathcal{L}_\infty$, *where a function* f *is equivalent to* g *if and only if* $f(x) = g(x)$, $\mu-$*almost everywhere.*

(6) The function spaces $L_p(X, \mu)$, $1 \le p < +\infty$ *which are defined as follows. Let* (X, μ) *be a measure space, and let* $\mathcal{L}_p$ *denote the set of measurable functions which satisfy*

$$||f||_p := (\int_X |f(x)|^p d\mu(x))^{\frac{1}{p}} < \infty.$$

The space $L_p(X, \mu)$, $1 \le p < +\infty$, *consists of the equivalence classes of elements of* $\mathcal{L}_p$, *where a function* f *is equivalent to* g *if and only if* $f(x) = g(x)$, $\mu-$*almost everywhere.*

For the spaces L_∞ *and* L_p, $1 \le p < +\infty$, *it is customary to let* f *denote the class of elements which are equivalent to* f *under the equivalence relation described.*

Definition 1.1.3 *A set of vectors* $\{e_i\}_{i \in I}$ *is called a* **Hamel** *or an* **algebraic basis** *if* $\forall x \in X$ *there is a unique decomposition* $x = \sum_i a_i e_i$ *as a finite linear combination of* $e_i's$.

Theorem 1.1.4 *Any linear space* X *has a Hamel basis. Furthermore, if* X *is a Banach space, then the cardinality of the basis is either finite or uncountable.*

In finite dimensional normed spaces the algebraic bases are essential tools. While Zorn's lemma ensures that an algebraic basis always exists for an infinite-dimensional Banach space, in general such a basis is not necessarily connected to the topology of the space, i.e. if $\{e_i\}_{i\in I}$ is an algebraic basis for X, and $(x_n)_{n\in\mathbb{N}}$ is a sequence converging in the norm topology to x, then each of the $x_n's$ and x have finite decompositions $x_n = \sum_i (a_n)_i e_i$ and $x = \sum_i a_i e_i$ while it is not the case, for each i, that $(a_n)_i$ converges to a_i. This leads to the definition of the *Schauder basis.* First we clarify the notion of convergence of series in a Banach space:

Definitions 1.1.5 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points on a Banach space X. Then*

(a) the series $\sum_{n\in\mathbb{N}} x_n$ **converges** *to a point x, written $x = \sum_{n\in\mathbb{N}} x_n$ if*

$$\lim_{p\to\infty} ||x - \sum_{n\leq p} x_n|| = 0.$$

(b) the series $\sum_{n\in\mathbb{N}} x_n$ **converges unconditionally** *to a point x if for all permutations π of $\mathbb{N}$, the series $\sum_{n\in\mathbb{N}} x_{\pi(n)}$ converges to x.*

(c) the series $\sum_{n\in\mathbb{N}} x_n$ **converges absolutely** *or* **normally** *to a point x if it converges to x and the series of positive numbers $\sum_{n\in\mathbb{N}} ||x_n||$ is convergent.*

(d) a sequence $(x_n)_{n\in\mathbb{N}}$ is a called a **basic sequence** *if for all $x \in \overline{span}\{x_n\}_{n\in\mathbb{N}}$ there exists a unique sequence $(a_n)_{n\in\mathbb{N}}$ of scalars such that $\sum_{n\in\mathbb{N}} a_n x_n$ converges to x.*

(e) a sequence $(x_n)_{n\in\mathbb{N}}$ is a called a **Schauder basis** *of X if it is a basic sequence and if $\overline{span}\{x_n\}_{n\in\mathbb{N}} = X$.*

It is easy to see that unconditional convergence implies convergence.

Proposition 1.1.6 (cf. Beauzamy, [17], II.1) *If a sequence converges absolutely, then it converges unconditionally.*

Theorem 1.1.7 (cf. Goldberg, [60], I.2.5.) *If X is a normed linear space, then X is complete if and only if every series which converges absolutely also converges in X.*

If a normed space X has a Schauder basis, $\{x_n\}_{n\in\mathbb{N}}$, then every point has a unique decomposition $x = \sum_n a_n x_n$. The scalars $(a_n)_{n\in\mathbb{N}}$ are called the **co-ordinates** of x on the basis, which is usually assumed to be normalized. If the sequence $(x_n)_{n\in\mathbb{N}}$ is basic and normalized, we can define **co-ordinate functionals** $(f_k)_{k\in\mathbb{N}}$ by

$$f_k(x) = a_k \quad \text{when} \quad x = \sum_n a_n x_n.$$

There exist spaces without Schauder bases (see Section 1.8 below), and furthermore, there is no property which characterises the existence of a Schauder basis in a normed linear space. Nevertheless, the following does hold:

Theorem 1.1.8 (Banach, [15]) *Every infinite-dimensional Banach space contains a basic sequence.*

Examples 1.1.9

(1) c_0 *and* l_p, $(1 \leq p < +\infty)$ *have the natural basis for the sequences spaces as Schauder bases.*

(2) $L_p([0,1])$, $(1 \leq p < +\infty)$ *have the Haar system as Schauder bases.*

Definition 1.1.10 *A normed space is said to be* **separable** *if it contains a countable dense subset.*

Examples 1.1.11

(1) The finite sequences with rational coefficients form dense countable subsets of l_p, $1 \leq p < +\infty$.

(2) $C([0,1])$ *is separable.*

(3) l_∞ *is not separable (consider sequences made up of* $+1$ *and* -1*).*

Theorem 1.1.12 (cf. Beauzamy [17], III.2.) *If* X *is separable and infinite dimensional, then there is a dense linearly independent sequence* $(x_n)_{n \in \mathbb{N}} \subset X$.

Theorem 1.1.13 *Let* X *be a normed linear space. The following are equivalent:*

(i) X *is separable.*

(ii) The unit ball B_X *is separable.*

(iii) The unit sphere $S_X = \{x \in X \mid ||x|| = 1\}$ *is separable.*

Theorem 1.1.14 (cf. Beauzamy [17], III.2.) *If* X *is separable, then so are all its subspaces.*

1.2 Linear Operators and Linear Relations

Definitions 1.2.1 *Let* X *and* Y *be arbitrary non-empty sets. A* **relation** T *from* X *to* Y *is a mapping defined on a non-empty subset* $D(T)$ *of* X*, called the* **domain** *of* T*, which takes on values in* $P(Y) \setminus \emptyset$. *We denote the class of relations from* X *to* Y *by* $R(X,Y)$. *For* $T \in R(X,Y)$ *we formally define its* **graph** $G(T)$, *a subset of* $X \times Y$, *as follows :*

$$G(T) := \{(x,y) \in X \times Y \mid x \in D(T),\ y \in T(x)\}.$$

If T *maps the points in its domain to singletons, then* T *is said to be a single-valued relation or a* **function**.

The **range** $R(T)$ *of* T *is defined*

$$R(T) \ := \bigcup_{x \in D(T)} Tx.$$

If $R(T) = Y$, *then we say* T *is* **surjective**, *and if* $A \subset X$ *then the image of* A *under* T *is defined to be the set*

$$T(A) \quad := \bigcup_{a \in A \cap D(T)} Ta\,.$$

The **inverse** *of a relation* T *is the relation* T^{-1} *given by the graph*

$$G(T^{-1}) := \{(y,x) \in Y \times X \mid (x,y) \in G(T)\}.$$

A relation is said to be **injective** *if* T^{-1} *is single-valued.*

If $B \subset Y$, *then the* **inverse image** *of* B *under* T *is defined to be the set*

$$T^{-1}(B) := \{x \in D(T) \mid Tx \cap B \neq \emptyset\},$$

and the **core** *of* B *under* T *is defined to be the set*

$$T^{+1}(B) := \{x \in D(T) \mid T(x) \subset B\}.$$

Let $S \in R(Y,Z)$. *The* **composition** *or* **product** $ST \in R(X,Z)$ *of* T *and* S *is defined by*

$$ST(x) := S(Tx), \quad x \in X.$$

If $A \subset X$, *then the* **restriction** *of* T *to* A, *denoted by* $T|_A$ *is defined by*

$$G(T|_A) \ := \ \{(x,y) \in G(T) \mid x \in A\} \ = \ G(T) \cap (A \times Y).$$

Suppose $S \in R(X,Y)$. *Then* R *is said to be an* **extension** *of* T *if*

$$S|_{D(T)} = T.$$

The following properties follow easily from the definitions:

Proposition 1.2.2 *Let* $T \in R(X,Y)$. *Then*

(a) $T^{-1}y = \{x \in D(T) \mid y \in Tx\}$ *for* $y \in R(T)$, *and hence,* $D(T^{-1}) = R(T)$ *and* $R(T^{-1}) = D(T)$.

(b) If T *is injective then* $Tx_1 = Tx_2$ *implies* $x_1 = x_2$ *for* $x_1, x_2 \in D(T)$.

(c) If T *is single-valued, then* $T^{-1}(B) = T^{+1}(B)$ *for* $B \subset Y$.

(d) For $S \in R(Y,Z)$, *the domain and graph of* ST *are given by*

$$\begin{aligned} D(ST) &= \{x \in X \mid STx \neq \emptyset\} = \{x \in X \mid Tx \cap D(S) \neq \emptyset\} = T^{-1}(D(S)), \ \text{and} \\ G(ST) &= \{(x,z) \in X \times Z \mid \exists y \in Y \ \text{such that} \ (x,y) \in G(T) \ \text{and} \ (y,z) \in G(S)\}. \end{aligned}$$

(e) For non-empty subsets A_1 *and* A_2 *of* X *we have*

$$\begin{aligned} T(A_1 \cup A_2) &= T(A_1) \cup T(A_2), \\ T(A_1 \cap A_2) &= T(A_1) \cap T(A_2), \\ T(X \setminus A_1) &\supset R(T) \setminus T(A_1), \quad \text{and} \\ A_1 \subset A_2 &\Rightarrow T(A_1) \subset T(A_2). \end{aligned}$$

Definitions 1.2.3 *Let X and Y be vector spaces over a field $I\!K$, and let $x_1, x_2 \in X$ and $\alpha \in I\!K$. A* **linear operator** *$T : X \to Y$ is a single-valued map from X into Y such that*

$$\begin{aligned} T(x_1 + x_2) &= Tx_1 + Tx_2, \quad \text{and} \\ \alpha Tx_1 &= T(\alpha x_1). \end{aligned}$$

We denote the class of linear operators from a space X into a space Y by $L(X, Y)$.

A **multivalued linear operator** *or* **linear relation** *$T : X \to Y$ is a relation whose graph is a linear subspace of $X \times Y$. We let $LR(X, Y)$ denote the class of linear relations from a space X into a space Y.*

Clearly, T is a linear relation if and only if T^{-1} is a linear relation.

Proposition 1.2.4 (cf. Cross [35], I.2.3) *Let $T \in R(X, Y)$. The following are equivalent.*

(i) T is a multivalued linear operator.

(ii) For $x_1, x_2 \in D(T)$ and each non-zero scalar $\alpha \in I\!K$ we have

$$Tx_1 + Tx_2 = T(x_1 + x_2), \text{ and}$$
$$\alpha Tx_1 = T(\alpha x_1).$$

Corollary 1.2.5 *Let $T \in R(X, Y)$. Then T is a linear relation if and only if the equality*

$$\alpha Tx_1 + \beta Tx_2 = T(\alpha x_1 + \beta x_2)$$

holds for all $x_1, x_2 \in D(T)$ and non-zero scalars $\alpha, \beta \in I\!K$.

Corollary 1.2.6 *Let $T \in LR(X, Y)$. Then $T(0)$ and $T^{-1}(0)$ are linear subspaces of Y and X, respectively.*

Corollary 1.2.7 *Let $T \in LR(X, Y)$ and let M be a linear subspace of X. Then $T|_M \in LR(X, Y)$.*

Definition 1.2.8 *The subspace $T^{-1}(0)$ is called the* **null-space** *or* **kernel** *of T and is denoted $N(T)$.*

The following property and its corollaries are used extensively and without specific reference in the sequel.

Proposition 1.2.9 *[cf. Cross [35], I.2.8] Let $T \in LR(X, Y)$*

(a) Let $x \in D(T)$. We have the following equivalence:

$$y \in Tx \Leftrightarrow Tx = y + T(0).$$

In particular,

$$0 \in Tx \Leftrightarrow Tx = T(0).$$

(b) For $x_1, x_2 \in D(T)$ we have the following equivalence :

$$Tx_1 \cap Tx_2 \neq \emptyset \Leftrightarrow Tx_1 = Tx_2.$$

Corollary 1.2.10 *Let $T \in LR(X, Y)$.*

(a) $TT^{-1}(0) = T(0)$.

(b) $T^{-1}T(0) = T^{-1}(0)$.

Corollary 1.2.11 *Let $T \in LR(X, Y)$.*

(a) If $y \in R(T)$ then $TT^{-1}y = y + T(0)$.

(b) If $x \in D(T)$ then $T^{-1}Tx = x + T^{-1}(0)$.

Corollary 1.2.12 *Suppose T, $S \in LR(X, Y)$ and $G(S) \subset G(T)$. Then T is an extension of S if and only if $S(0) = T(0)$.*

The following rules are easy to verify.

Proposition 1.2.13 *Let $\alpha \in I\!K$, $\alpha \neq 0$, and let $A, B \subset X$, $C \subset Y$.*

(a) $T(\alpha A) = \alpha T(A)$.

(b) $T(A) + T(B) \subset T(A + B)$.

(c) *If $A \subset D(T)$ or $B \subset D(T)$, then $T(A + B) = T(A) + T(B)$.*

(d) *If $A \subset D(T)$ or $B \subset D(T)$ and $A \cap B = \{0\}$,*
then $T(A + B) = T(A) + T(B)$ and $T(A) \cap T(B) = T(0)$.

(e) $TT^{-1}C = C \cap R(T) + T(0)$.

(f) $T^{-1}T(A) = A \cap D(T) + T^{-1}(0)$.

(g) $T^{-1}(0) \times \{0\} = G(T) \cap (X \times \{0\})$.

(h) $\{0\} \times T(0) = G(T) \cap (\{0\} \times Y)$.

(i) $X \times R(T) = G(T) + (X \times \{0\})$.

(j) $D(T) \times Y = G(T) + (\{0\} \times Y)$.

Equality does not necessarily hold in (b) - one may consider the case $A = \{a\}$, $B = \{b\}$ such that $a + b \in D(T)$ while $a \notin D(T)$ and $b \notin D(T)$.

1.3 Semi-Continuous Relations, Continuity and the Norm of Linear Operators

Throughout this section, X and Y will denote normed linear spaces.

Definitions 1.3.1 *Let $\epsilon > 0$, and $M \subset X$. Then the sets $B(M,\epsilon)$, B_X, $U(M,\epsilon)$, U_X, and S_X are defined by :*

$$\begin{aligned} B(M,\epsilon) &:= \{x \in X \mid d(x,M) \leq \epsilon \}, \\ B_X &:= \{x \in X \mid d(x,0) \leq 1 \}, \\ U(M,\epsilon) &:= \{x \in X \mid d(x,M) < \epsilon \}, \\ U_X &:= \{x \in X \mid d(x,0) < 1 \}, \\ S_X &:= \{x \in X \mid d(x,0) = 1 \}. \end{aligned}$$

Definitions 1.3.2 *A subset U of X is set to be a* **neighbourhood** *of a point $x \in X$ if U contains an open set containing x.*

A relation $T \in R(X,Y)$ is said to be **upper semicontinuous (u.s.c.) at a point** *$x \in D(T)$ if for any neighbourhood U of $T(x)$ there exists $\epsilon > 0$ such that for any $z \in B(x,\epsilon)$ we have $T(z) \subset U$. T is said to be* **upper semicontinuous** *if it is upper semicontinuous at every x in its domain $D(T)$.*

It follows from the definition that T is u.s.c. at $x \in D(T)$ if and only if the core of any neighbourhood of $T(x)$ is a neighbourhood of x. Thus T is u.s.c. if and only if the core of any open set is open.

Definitions 1.3.3 *A set-valued map or relation $T \in R(X,Y)$ is said to be* **lower semicontinuous (l.s.c.) at a point** *$x \in D(T)$ if for any $y \in T(x)$ and for any sequence $\{x_n\} \subset D(T)$ such that $x_n \to x$ there exists $y_n \in T(x_n)$ such that $y_n \to y$. T is said to be* **lower semicontinuous** *if it is lower semicontinuous at every x in its domain $D(T)$.*

It follows that T is l.s.c. at $x \in D(T)$ if and only if the inverse image of any open set which intersects $T(x)$ is a neighbourhood of x. Thus T is l.s.c. if and only if the inverse image of any open set is open.

Examples 1.3.4

(a) The set-valued map $T_1 \in R(I\!R, I\!R)$ defined by:

$$T_1(x) = \begin{cases} [-1,1] & \text{if } x \neq 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

is l.s.c. at zero but not u.s.c. at zero.

(b) The set-valued map $T_2 \in R(I\!R, I\!R)$ defined by:

$$T_2(x) = \begin{cases} \{0\} & \text{if } x \neq 0 \\ [-1,1] & \text{if } x = 0 \end{cases}$$

is u.s.c. at zero but not l.s.c. at zero.

The definitions of upper and lower semicontinuity are equivalent for single valued maps. Furthermore, it is well known that the continuity of a (single-valued) linear operator can be characterised in terms of the operator norm.

Definitions 1.3.5 *Let X and Y be normed linear spaces, and $T \in L(X,Y)$. The* **norm** *of T is defined as follows:*

$$||T|| := \sup_{||x||=1} ||Tx|| = \sup_{x \neq 0} \frac{||Tx||}{||x||}.$$

Theorem 1.3.6 (cf. Goldberg, [60], I.3.2.) *Let $T \in L(X,Y)$. Then the following are equivalent:*

(i) T is continuous at a point,

(ii) T is uniformly continuous on its domain,

(iii) There exists $M \in I\!R$ such that $||Tx|| \leq M||x||$ for every x in the domain of T,

(iv) $||T|| < \infty$.

Remarks 1.3.7

The definition of the operator norm quantity can be extended to linear relations. Furthermore, it can be shown that the property of lower semicontinuity of a multivalued operator is equivalent to the property of having a finite norm. For this reason we choose the notion of lower semicontinuity to serve as the definition for continuity of a multivalued linear operator. We provide formal definitions in the next chapter. Of course, the term *continuous* is also more convenient to use frequently, than the more precise expression *lower semicontinuous.* We note that in the literature of convex analysis, a map is said to be continuous if and only if it is both upper and lower semicontinuous.

Notation 1.3.8 *We let $B(X,Y)$ denote the class of continuous single-valued linear operators from a normed linear space X into a normed linear space Y, and $B(X)$ denotes this class for the case $X = Y$.*

Definitions 1.3.9 *Let $T \in LR(X,Y)$. If T and its inverse map T^{-1} are single-valued, continuous and everywhere-defined, then T is said to be an* **isomorphism** . *T is said to be an* **isometry** *if $||Tx|| = ||x||$.*

Theorem 1.3.10 (cf. Goldberg, [60], I.3.7.) *Let $T \in L(X,Y)$. Then T^{-1} is continuous and single-valued if and only if there exists $m > 0$ such that*

$$||Tx|| \geq m||x||, \quad x \in D(T).$$

1.4 Classification of Normed Linear Spaces

This section serves to introduce properties which are used in the sequel.

Definitions 1.4.1 *A pair of normed linear spaces are said to be isormorphic (isometric) if there exists an isomorphism (isometry) which maps the one into the other.*

A normed linear space may be classified in terms of isomorphisms and isometries from the space itself, or from its subspaces, into subspaces of the classic spaces (or the lack thereof). Characterisation of normed space properties may be *topological*, e.g. weak compactness of the unit ball, or *geometric*, e.g. structure of a sequence of points, or of two sequences of points, or of sequences of points and of linear functionals. Spaces are also studied via their local properties, i.e. how they are built up from finite-dimensional subspaces. Most generally we have the following identifications:

Theorem 1.4.2 (Banach, [15]) *If K and H are compact metric spaces, then K is homeomorphic to H if and only if $C(K)$ is isometric to $C(H)$.*

Theorem 1.4.3 (Milutin, [107]) *If K and H are uncountable compact metric spaces, then $C(K)$ is isomorphic to $C(H)$.*

Theorem 1.4.2 was first extended by M.H. Stone to compact Hausdorff spaces. However, $C(K)$ may be *isomorphic* to $C(H)$ without K being homeomorphic to H. By Theorem 1.4.3, if K is an uncountable compact metric space, we need only consider $K = [0,1]$ (other references sometimes use the Cantor set for this purpose). On the other hand as K varies over *countable* compact metric spaces, there are uncountably many isomorphism classes for $C(K)$ (see Bessaga and Pełczyński [21]).

Theorem 1.4.4 (cf. Goldberg, [60], I.4.2.) *If X is an n-dimensional normed linear space over $I\!R$ (or over $\mathbb{C}$), then X is isomorphic to $I\!R^n$ (respectively, $\mathbb{C}^n$).*

Lemma 1.4.5 (cf. Goldberg, [60], I.4.8.) *If X is isomorphic to a Banach space, then X is also a Banach space.*

Corollary 1.4.6 *If X is a finite-dimensional normed linear space, then X is complete.*

Corollary 1.4.7 *If B is a closed bounded set in a finite-dimensional normed linear space, X, then B is compact.*

The converse is also true, i.e. the property of the the unit ball given in Corollary 1.4.7 characterises finite-dimensional normed linear spaces. Riesz's Lemma is usually used to prove this.

Theorem 1.4.8 (Riesz's Lemma) *Let M be a subspace of a normed linear space X such that M is not dense. Then there exists a sequence $\{x_n\} \subset S_X$ such that $d(x_n, M) \to 1$.*

Theorem 1.4.9 (cf. Goldberg, [60], I.4.6) *If X is a normed linear space such that B_X is totally bounded, then X is finite-dimensional.*

1.5 Linear Functionals and Conjugate spaces

Definition 1.5.1 *Let X be a topological vector space. The* **algebraic conjugate** *of X, denoted $X^{\#}$, is the set of all linear functionals defined on X, i.e.*

$$X^{\#} := \{f : X \to \mathbb{R} \quad | \quad f(x+y) = f(x) + f(y) \ \text{ and } \ f(\alpha x) = \alpha f(x) \\ \text{ for any } \ x, y \in X \ \text{ and } \ \alpha \in \mathbb{K}\}.$$

If X is a normed linear space, then the **topological conjugate** *of X, denoted X', is the subset of $X^{\#}$ consisting of the linear functionals which satisfy*

$$||f|| := \sup_{x \in X} |f(x)| < \infty.$$

We usually let x' denote an arbitrary element in X', and refer to X' simply as the *conjugate* or *adjoint* of X, when there is no ambiguity. We note that X' is a Banach space with the norm given above.

Definition 1.5.2 *Let $x_0' \in X'$. The sets*

$$U_{\epsilon, x_1, \ldots x_n}(x_0') := \{ x' \in X' \mid |x'(x_i) - x_0'(x_i)| < \epsilon, \ 1 \leq i \leq n \},$$

$\epsilon > 0$, $\{x_1, \ldots x_n\} \subset X$, $n \geq 1$ *form a neighbourhood basis for $x_0' \in X'$. The topology given by these sets is referred to as the* **weak*-topology** *on X', and is denoted by $\sigma(X', X)$.*

Theorem 1.5.3 (Alaoglu) *The unit ball $B_{X'}$ of X' is $\sigma(X', X)-$compact.*

Definition 1.5.4 *The elements of a normed space X determine linear functionals on X' by the formula $x''(x') := x'(x)$ for $x \in X$. We say X is* **reflexive** *if all the continuous linear functionals on X' are determined in this way, i.e. if $X'' = X$ under this identification. The $\sigma(X'', X')$ topology on X, where X is considered as a subspace of X'', is referred to as the* **weak topology** *on X.*

Properties of the various topologies on X and X' are discussed in Wilansky [143], for example. The geometry of reflexive spaces is discussed in detail in Beauzamy [17] and Wojtaszczyk [145]. We mention some basic properties.

Theorem 1.5.5 (cf. Goldberg [60], I.6.11.) *If X is a Banach space, then X is reflexive if and only if X' is reflexive.*

Theorem 1.5.6 (cf. Goldberg [60], I.6.12.) *If X is reflexive, then so are all its closed subspaces.*

Theorem 1.5.7 (cf. Wojtaszczyk [145], II.A.14.) *X is reflexive if and only if its unit ball B_X is $\sigma(X, X')-$compact.*

Examples 1.5.8

(1) c_0 *is not reflexive since* $c_0'' = l_\infty$.

(2) l_p, $1 < p < +\infty$ *and* L_p, $1 < p < +\infty$ *are reflexive.*

(3) l_1 *is not reflexive, and hence it follows that the spaces* L_1, $C([0,1])$, *and* L_∞ *are also not reflexive since they have subspaces isomorphic to* l_1.

Theorem 1.5.9 (cf. Beauzamy [17], III.2.) *If* X' *is separable, then* X *is separable.*

Theorem 1.5.10 (cf. Wojtaszczyk [145], II.A.15.) *If* X *is separable, then the* $\sigma(X', X)$ *topology of* X' *is metrisable.*

In general, for separable topological spaces, we may use sequences to test continuity and convergence (instead of nets or filters). Sequences also suffice when the space is metrisable (and, hence, if the space is normable). However, when considering the weak topologies on (infinite dimensional spaces) X and X', it it not generally the case that these topologies are metrizable.

1.6 The Hahn-Banach Extension and Separation Theorems

Properties of a space may be inferred from the study of the continuous linear functionals on the space (cf. Theorems 1.4.2 and 1.4.3). The Hahn-Banach theorem and its various corollaries establish that there are a sufficient number of linear functionals for useful observation. It is also the case that every infinite-dimensional Banach space contains a basic sequence (cf. Theorem 1.1.8). Suppose $(x_n)_{n\in\mathbb{N}}$ is a basic sequence. Then the Hahn-Banach Theorem establishes that the associated co-ordinate functionals can be extended to the whole space, and thus, they may serve as generalised co-ordinates of the space. Furthermore, if $(x_n)_{n\in\mathbb{N}}$ is a Schauder basis and if $(z_n)_{n\in\mathbb{N}}$ is a sequence of points converging to z, then for each k we have $\lim_{n\to\infty} f_k(z_n) = f_k(z)$. Such a sequence of co-ordinate functionals is sometimes called the **biorthogonal sequence** associated with $(x_n)_{n\in\mathbb{N}}$ since $f_k(x_n) = 0$ if $n \neq k$, $= 1$ if $n = k$.

Theorem 1.6.1 (The Hahn-Banach Extension Theorem) *Let* X *be a vector space over* $\mathbb{R}$ *or* $\mathbb{C}$. *Suppose* p *is a real-valued function defined on* X *satisfying*

$$\begin{aligned} p(x+y) &\leq p(x) + p(y), \quad \textit{and} \\ p(\alpha x) &= |\alpha| p(x). \end{aligned}$$

If M *is a subspace of* X *and* f *is a linear functional defined on* M *which satisfies* $|f(m)| \leq p(m)$ *for* $m \in M$, *then there exists a linear functional* F *which extends* f *to all of* X *and satisfies* $|F(x)| \leq p(x)$ *for* $x \in X$.

Definition 1.6.2 *A subset* K *of a vector space* X *is said to be* **convex** *if* $\lambda x + (1-\lambda)y \in K$ *whenever* $x, y, \in K$ *and* $\lambda \in [0,1]$.

Theorem 1.6.3 (A separation theorem) *Let K be closed, convex subset of a normed linear space X. If $x \in X$ and $x \notin K$ then there exists $x' \neq 0,\ x' \in X'$ such that*

$$Re\, x'x \geq Re\, x'k, \quad k \in K.$$

1.7 Quotients, Subspaces and Projections

Definition 1.7.1 *Let M be a subspace of a linear space X, and denote by $[x]$ the set of all elements equivalent to x under the equivalence relation*

$$yRx \quad \Leftrightarrow \quad y - x \in M.$$

The **quotient space** *X/M is defined by :*

$$X/M := \{\ [x] \mid x \in X\}.$$

If M is closed subspace of a normed linear space $(X, ||\_||_X)$, then X/M is a normed space with the norm $||\_||$ defined by :

$$||[x]|| := d(x, M) = \inf_{m \in M} ||x - m||.$$

Remarks 1.7.2

The fact that the norm on X/M is well-defined follows from the observation that if yRx then

$$\begin{aligned} ||[y]|| &= \inf_{m \in M} ||y - m|| \\ &= \inf_{m \in M} ||x - ((y - x) + m)|| \\ &= \inf_{m \in M} ||x - m|| \\ &= ||[x]||. \end{aligned}$$

It thus follows that

$$||[x]|| = \inf_{y \in [x]} ||y||.$$

Definition 1.7.3 *The operator $Q_M^X : X \to X/M$ defined by $Q_M^X x = [x]$ is called the* **natural quotient map** *with domain X and nullspace M.*

Theorem 1.7.4 (cf. Goldberg, [60], I.2.8.) *Let X be a Banach space. If M is a closed subspace of X, then X/M is Banach space.*

◇

Proposition 1.7.5 *Let M be a closed subspace of X, and let $N \subset X$ be a subspace such that $M \subset N$. Then N is closed if and only if N/M is closed in X/M.*

PROOF

Suppose N/M is closed, and $\{x_n\}_{n\in\mathbb{N}} \subset N$ is a sequence converging to $x \in X$. Then $\{\ [x_n]\ \}_{n\in\mathbb{N}} \subset N/M$, and

$$\begin{aligned} & \lim_{n\in\mathbb{N}} \inf_{m\in M} \|(x_n - x) - m\| = 0 \\ \Rightarrow\quad & \lim_{n\in\mathbb{N}} \|[x_n - x]\| = 0 \\ \Rightarrow\quad & \lim_{n\in\mathbb{N}} \|[x_n] - [x]\| = 0. \end{aligned}$$

Since N/M is closed, $[x] \in N/M$ and $\exists y \in N$ such that $[y] = [x]$. Since $y - x \in M$, it follows that $x \in N$.

The reverse implication is similar.

$\diamond$

A proof of the next proposition can be found in Cross [35], I.6.1.

Proposition 1.7.6 *Let $T \in LR(X,Y)$, and let $M \subset X$. Then*

$$dim\, R(T)/T(M) \ \le\ dim\, D(T)/D(T)\cap M \ \le\ dim\, X/M.$$

In particular, if M is finite codimensional in $D(T)$, then $T(M)$ is a finite codimensional subspace of $R(T)$.

Definitions 1.7.7 *Let X be a linear space, and let $M \subset X$. Then we define what is sometimes referred as the* **annihilator** *$M^\perp$ of M by:*

$$M^\perp := \{x' \in X' \mid x'x = 0\ \forall x \in M\}$$

Similarly, if $N \subset X'$ then $N^\top$ is defined by:

$$N^\top := \{x \in X \mid x'x = 0\ \forall x' \in N\}$$

Remarks 1.7.8 *$M^\perp$ and $N^\top$ are closed subspaces of X' and X, respectively. Moreover, $M^{\perp\top} = \overline{M}$, and $N^{\top\perp}$ is the weak*-closure of N (by the Bipolar Theorem - see, for example, Wilansky [143]).*

Proposition 1.7.9 (cf. Goldberg, [60], I.6.4.) *Let M be a subspace of a normed linear space X. Then*

(a) $X'/M^\perp$ is isometrically isomorphic to M' under the map U defined by:

$$U[x'] := x'_M$$

where $[x'] \in X/M^\perp$ and x'_M is the restriction of x' to M.

(b) If M is closed, then $(X/M)'$ is isometrically isomorphic to $M^\perp$ under the map V defined by:

$$(Vz')x := z'[x],$$

where $z' \in (X/M)'$.

Definitions 1.7.10 *If M is a subspace of a linear space X, then a (single-valued)* **linear projection** *from X onto M is a (single-valued) linear operator which satisfies the condition $P^2 = P$.*

If M and N are subsets of a linear space X, then the sum $M + N$ denotes the set $\{m + n \mid m \in M \text{ and } n \in N\}$.

If M and N are linear subspaces of X which satisfy $X = M + N$ and $M \cap N = \{0\}$, then N is called a **complement** *of M. If furthermore, there exists a continuous linear projection from X onto M, then N is called a* **topological complement** *of M. In this case we write $X = M \oplus N$.*

Multivalued Linear Projections are defined and discussed in Chapter 3.

Proposition 1.7.11 (cf. Wilansky, [143], 2.8.) *Let M and N be linear subspaces of X. If N is a complement of M, then N is isomorphic to the quotient space X/M.*

The existence of a continuous projection onto a subspace is connected to the existence of a continuous extension of a continuous operator defined on that subspace to an operator defined on the whole space. Banach and Mazur [16] gave an uncomplemented subspace of $C([0,1])$ as the first example illustrating that such extensions are not always possible. Properties of projections and extensions may be quantified (results relating these quantities have been developed) and the structure or geometry of normed spaces can be characterised via the properties of projections and topologically complemented subspaces. In general the finite dimensional and closed finite codimensional subspaces in a normed subspace are topologically complemented. However, the associated projections need not be of norm 1, for example, in $L_p([0,1], dt)$, $(1 \leq p < +\infty)$, $p \neq 2$, none of the closed hyperplanes admit norm 1 projections. Lindenstrauss and Tzafriri [98] showed that if every closed subspace of a normed linear space is topologically complemented, then it is isomorphic to a Hilbert space.

Theorem 1.7.12 (Goldberg, [60], II.1.14.) *Let M be a closed subspace of a Banach space X. There exists a continuous linear projection from X onto M if and only if there exists a closed subspace N such that $X = M + N$ and $M \cap N = \{0\}$.*

Theorem 1.7.13 (Goldberg, [60], II.1.16.) *If M is a finite-dimensional subspace of a normed linear space, then M is topologically complemented.*

Definitions 1.7.14 *A closed subspace M of of a Banach space X is said to be* **quasicomplemented** *if there exists closed subspace N such that $M \cap N = \{0\}$ and $M + N$ is dense in X.*

Theorem 1.7.15 (cf. Lindenstrauss [93], see also [110] and [100])
Every closed subspace in a separable or reflexive Banach space is quasicomplemented.

Examples of subspaces without quasicomplements are discussed in Lindenstrauss [94].

1.8 Further Notes and Remarks

The definitions, propositions and results which are summarised above are already well-known, widely used or introduced elsewhere, and hence, references rather than proofs are provided. Historical data has been gleaned from various sources.

Remarks on Normed Linear Spaces

The spaces given in Section 1.1 above are amongst those now referred to as *classic spaces* in Banach space theory. The first truly non-classical Banach space was given by Tsirelson [141] (cf. Tomczak-Jaegermann [138], and also Casazza and Shura [27]). In Tsirelson's space, the unit ball is defined implicitly so that the space and all of its infinite dimensional subspaces have a particular geometric property (the space is said to be saturated with the property) which prevents them from containing l_p, $1 < p < \infty$ or c_0.

It is clear that a space which has a Schauder basis must be separable (linear combinations with rational coefficients of the basis elements form a dense countable set). The converse, however, does not necessarily hold. In 1973 P. Enflo [50] provided an example of a separable Banach space without a Schauder basis. In fact, for $p \neq 2$, the spaces $L_p([0,1], dt)$, $1 \leq p < +\infty$ have subspaces without bases.

Definition 1.8.1 *A Banach space X is said to have the* **approximation property** *(A.P.) if, for every compact subset $M \subset X$, and $\forall \epsilon > 0$, there is a finite rank operator T such that $\|Tx - x\| < \epsilon$ for all $x \in X$.*

Equivalently, X has A.P. if for every compact operator T from a Banach space Y into X there exists a sequence of finite rank operators converging in norm to T (see Lindenstrauss [95], or Lindenstrauss and Tzafriri [97]). Enflo's example, mentioned above, of a Banach space without a Schauder basis was in fact one which did not have A.P. By considering the projections on to the first n co-ordinates, if X has a Schauder basis, then it has A.P.

Banach showed that every space contains a basic sequence (Theorem 1.1.8 above). The existence of an *unconditional* basis indicates some additional structure. While it was known that the spaces $C([0,1])$ and L_1, for example, do not have unconditional bases, it was an open question as to whether every space contains an unconditional basic sequence (or, equivalently, whether every space has a subspace with an unconditional basis). In 1994 a counterexample was given by Gowers and Maurey [66]. The space which was constructed, now known as the *Gowers and Maurey space*, is a Tsirelson type space.

Whether every infinite dimensional Banach space admitted a non-trivial decomposition into topologically complemented subspaces was also an open problem till recently, and the Gowers and Maurey space provided a counterexample to this question as well. Furthermore, it was established that the Gowers and Maurey space contains no decomposable subspaces. In this sense it is referred to as being *hereditarily indecomposable* or H.I.. Though this was not the line of argument adopted initially in the presentation by Gowers and Maurey, it is easy to see that an H.I space cannot contain an unconditional basic sequence. The H.I property was also applied to solve the *hyperplane problem* of Banach - their space is not isomorphic to any subspace of codimension 1. Consideration of the Gowers and Maurey space has revealed an open question in the Invariant Subspace Problem (cf. Androulakis and Schlumprecht [6]) - further comments on this space are made at the ends of Chapters 5 and 7.

Contributions on general normed and Banach spaces include Beauzamy [17], Diestel [45], Lindenstrauss [95], Lindenstrauss and Pełczyński [96], Lindenstrauss and Tzafriri [97], Wojtaszczyk [145] and, more recently, the collection of papers by several leading contributers [71].

Remarks on Linear Operators and Linear Relations

Single-valued maps were favoured as the natural morphisms in the rigorous development of topology at the start of the 20th century. Nevertheless, limits of sequences of sets were considered by Painlevé in 1909 (cf. Aubin and Frankowska [14]) and later by Kuratowski [82], for example, in 1958. Furthermore, extension problems in topology led to the study of selections or single-valued parts of upper and lower semi-continuous set-valued maps (cf. Michael [105]). Multivalued maps, of course, occur quite naturally, but the earnest development of mathematical methods for set-valued or multivalued problems came in the 1960's.

In the study of properties of linear operators, it was recognised by von Neumann that the adjoint (defined in Chapter 2) of a non-densely defined single-valued operator is a multivalued linear operator (cf. Cross [35]). In general the closure of an operator may also be multivalued. In 1961, Arens [7] assembled some basic properties of multivalued operators, referring to these maps as linear relations. At about the same time, non-densely defined symmetric differential operators arising in differential equations were considered (see Coddington [29], [30], [31], [32], Coddington and De Snoo [33] and Coddington and Dijksma [34]). This work has continued into the 1990's by H.S.V. De Snoo, A. Diksma, H. Langer and A.V. Strauss, among others. Other contributions involving linear relations include those by Lee [88], and Lee and Nashed [89], [90]. The basic algebraic and dual properties of multivalued linear operators were established in these studies as generalisations of the single-valued operator.

Problems in optimisation and control also led to the study of set-valued maps and differential inclusions (cf. Aubin and Cellina [13], Clarke [28] and Rockafellar [125]). Studies of convex processes, tangent cones, subgradients and epiderivatives, etc., form part of the theory of convex analysis developed to deal with nonsmooth problems in viability and control theory, for example. Some of the basic topological properties from this area coincide with the core of the topological results for multivalued linear operators. It should be noted that the contents of Section 1.3 extend naturally to more general linear topological spaces. In Aubin and Frankowska [14], the authors regard *closed convex processes*, i.e. maps whose graphs are closed convex cones in the product space $X \times Y$, as the natural generalisations of single-valued linear operators. In the language of convex processes, the subclass of linear processes is equivalent to the class of closed multivalued linear operators (in Chapter 2 we discuss the properties of *closed* multivalued operators, i.e. operators whose graphs are closed). The important Closed Graph and Open Mapping theorems can be extended to the class of convex processes (cf. Aubin and Cellina [13] - we discuss the linear case in Chapter 3). Conjugates (or adjoints or transposes) of convex processes have also been investigated (cf. Aubin and Frankowska [14] or Rockerfellar [125]). Nevertheless, certain useful algebraic techniques can be exploited in the case of linear relations which fail with the alternative generalisation (subtraction is not permitted in cones).

Other works on multivalued mappings include the treatise on *Partial Differential Relations* by Gromov [67], the application of multivalued methods to solving differential equations by Favini and Yagi [52] and the development of *fixed point theory* for multivalued maps (cf. Gorniewicz [65], Saveliev [127] and Agarwal, Meehan and O'Regan [8]).

General references on functional analysis and linear operators include the books by Akhiezer and Glazman [2], Dunford and Schwartz [48], Gohberg and Goldberg [55], Gohberg, Goldberg and Kaashoek [56], Goldberg [60], Kato [75], Nikol'skij [114], Taylor and Lay [136], Wilansky [143], and Yosida [147]. The monograph by Cross [35] served as the main source for the definitions and notation adopted for linear relations. Definitions for upper and and lower semicontinuity of set-valued maps can be found in Aubin and Cellina [13] and Aubin and Frankowska [14], for example.

Chapter 2

Linear Relations in Normed Linear Spaces

2.1 The Algebra of Linear Relations

In this section we discuss the operations of addition and scalar muliplication in $LR(X, Y)$. We first verify that the composition of two multivalued linear operators is also a multivalued linear operator:

Proposition 2.1.1 *Let $T \in LR(X,Y)$ and let $S \in LR(Y,Z)$. Then $ST \in LR(X,Z)$.*

PROOF

Let $x_1, x_2 \in D(ST)$ and let α, β be non-zero scalars. Then the following equalities follow from the Proposition 1.2.13:

$$\begin{aligned}
\alpha(STx_1) + \beta(STx_2) &= \alpha S(Tx_1) + \beta S(Tx_2) \\
&= S(\alpha Tx_1) + S(\beta Tx_2) \\
&= S(T(\alpha x_1)) + S(T(\beta x_2)) \\
&= S(T(\alpha x_1)) + T(\beta x_2)) \\
&= ST(\alpha x_1 + \beta x_2).
\end{aligned}$$

◇

Definitions 2.1.2 *Let S, $T \in LR(X,Y)$, and let $\alpha \in I\!K$.* **Addition** *and* **scalar multiplication** *of linear relations are defined, respectively, in the obvious way:*

$$(S+T)x := Sx + Tx \quad x \in X \quad \textit{and}$$
$$(\alpha T)x := \alpha(Tx) \quad x \in X.$$

Remarks 2.1.3

The following properties for $R,\ S,\ T\ \in LR(X,Y)$ and $\alpha,\ \beta \in I\!K$ follow easily from the definitions :

$$\begin{aligned}
D(S+T) &= D(S)\cap D(T)\\
G(S+T) &= \{\ (x,y)\in X\times Y \mid y=s+t,\ s\in Tx,\ t\in Sx\ \}\\
S+T &= T+S\\
R+(S+T) &= (R+S)+T
\end{aligned}$$

$$\begin{aligned}
D(\alpha T) &= D(T)\\
G(\alpha T) &= \{\ (x,\alpha y)\in X\times Y \mid (x,y)\in G(T)\ \}\\
&= \{\ (x,y)\in X\times Y \mid (x,\alpha^{-1}y)\in G(T)\ \}\\
&= \{\ (\alpha^{-1}x,y)\in X\times Y \mid (x,y)\in G(T)\ \}\\
\alpha(\beta T) &= (\alpha\beta)T
\end{aligned}$$

It follows that $S+T$ and αT are linear relations, i.e. $LR(X,Y)$ is closed under addition and scalar multiplication.

Proposition 2.1.4 *Let $T, T_2 \in LR(X,Y)$ and $R,\ S \in LR(Y,Z)$. Then*

(a) $TT^{-1} = I_{R(T)} + (TT^{-1} - TT^{-1})$.

(b) $\alpha(ST) = (\alpha S)T = S(\alpha T), \quad 0 \neq \alpha \in I\!K$.

(c) If $G(S) \subset G(R)$, then $G(ST) \subset G(RT)$.

(d) $G((R+S)T) \subset G(RT) + G(ST)$ with equality if T is single-valued.

(e) $S(T+T_2)$ is an extension of $ST + ST_2$,
with equality if $D(S)$ contains both $R(T)$ and $R(T_2)$.

PROOF

(a) Let $y \in R(T)$. Then $TT^{-1}y = y + T(0)$ and $(TT^{-1} - TT^{-1})y = T(0)$. Thus

$$TT^{-1} = I_{R(T)} + TT^{-1} - TT^{-1} = I + TT^{-1} - TT^{-1}$$

where the subscript $R(T)$ can be dropped since the restriction is implied by the definition of addition.

(b) Let $0 \neq \alpha \in I\!K$, and let $x \in D(\alpha(ST)) = D((\alpha S)T) = D(S(\alpha T))$. It follows from the definitions of scalar multiplication and the composition of relations that

$$\alpha(ST)x \;=\; \alpha(S(Tx)) \;=\; \alpha S(Tx) \;=\; S(\alpha(Tx)) \;=\; S(\alpha Tx).$$

(d)

$$
\begin{aligned}
G((R+S)T) &= \{\ (x,z) \mid \exists y \in Y \ \ s.t.\ \ (x,y) \in G(T) \ \ and \ \ (y,z) \in G(R+S)\ \} \\
&= \{\ (x,z) \mid \exists y \in Y \ \ s.t.\ \ (x,y) \in G(T),\ (y,z_1) \in G(R) \\
&\qquad\qquad and \ \ (y,z_2) \in G(S),\ z_1+z_2=z\ \} \\
&\subset \{\ (x,z) \mid \exists y_1 \in Y \ \ s.t.\ \ (x,y_1) \in G(T),\ \ (y_1,z_1) \in G(R), \\
&\qquad\qquad and \ \ \exists y_2 \in Y \ \ s.t.\ \ (x,y_2) \in G(T),\ \ (y_2,z_2) \in G(S), \\
&\qquad\qquad z_1+z_2=z\ \} \\
&= \{\ (x,z) \mid (x,z_1) \in G(RT),\ (x,z_2) \in G(ST),\ \ z_1+z_2=z\ \} \\
&= G(RT+ST).
\end{aligned}
$$

If T is single-valued then it follows that the inclusion above is in fact an equality.

(e) We first show that

$$G(ST+ST_2) \subset G(S(T+T_2)) \tag{2.1}$$

$$
\begin{aligned}
G(ST+ST_2) &= \{\ (x,z) \mid (x,z_1) \in G(ST),\ \ (x,z_2) \in G(ST_2),\ \ z_1+z_2=z\ \} \\
&= \{\ (x,z) \mid \exists y_1 \in Y \ \ s.t.\ \ (x,y_1) \in G(T),\ \ (y_1,z_1) \in G(S), \\
&\qquad\qquad \exists y_2 \in Y \ \ s.t.\ \ (x,y_2) \in G(T_2),\ \ (y_2,z_2) \in G(S), \\
&\qquad\qquad z_1+z_2=z\ \} \\
&\subset \{\ (x,z) \mid \exists y_1 \in Y \ \ s.t.\ \ (x,y_1) \in G(T), \\
&\qquad\qquad \exists y_2 \in Y \ \ s.t.\ \ (x,y_2) \in G(T_2),\ \ y_1+y_2 \in D(S), \\
&\qquad\qquad (y_1+y_2,z) \in G(S)\ \} \\
&= \{\ (x,z) \mid \exists y \in Y \ \ s.t.\ \ (x,y) \in G(T+T_2),\ \ (y,z) \in G(S)\ \} \\
&= G(S(T+T_2)).
\end{aligned}
$$

Choose $x \in D(ST+ST_2) \subset D(S(T+T_2))$, and let $z \in S(T+T_2)x$. Then $z \in Sy$ for $y = y_1+y_2 \in (T+T_2)x$, $y_1 \in Tx$, $y_2 \in T_2x$. It follows that

$$z \in S(y_1+y_2) = Sy_1 + Sy_2 \subset STx + ST_2x.$$

Thus $S(T+T_2)x \subset STx + ST_2x$. From (2.1) it follows that $S(T+T_2)x = STx + ST_2x$, i.e. $S(T+T_2)$ is an extension of $ST+ST_2$.

If the $R(T)$ and $R(T_2)$ are both contained in $D(S)$, then

$$
\begin{aligned}
D(S(T+T_2)) &= D(T+T_2) = D(T) \cap D(T_2) = D(ST) \cap D(ST_2) \\
&= D(ST+ST_2).
\end{aligned}
$$

Thus the inclusion in the proof of (2.1) is in fact equality.

$\diamond$

The following examples show that equality may not hold in 2.1.4 (d) and (e).

Examples 2.1.5

(1) Let $R \in LR(Y, Z)$ be single-valued and non-zero, let $S = -R$, let $G(T) = X \times Y$ where $Y \neq \{0\}$. Then

$$(R+S)T(0) = (R-R)Y = \bigcup_{y \in D(R)} (R-R)y = \{0\}$$

while

$$(RT+ST)(0) = (RT-RT)(0) = RT(0) - RT(0) = R(Y) - R(Y) = R(Y) \neq \{0\}$$

Thus, applying Corollary 1.2.12, $RT + ST$ is not always an extension of $(R+S)T$.

(2) We construct an example in which $S(T+T_2)$ is a proper extension of $ST + ST_2$. Let $S, T, T_2 \in LR(\mathbb{R}^2)$ be defined as follows:

$$\begin{aligned} T &= I \\ G(T_2) &= \left\{ \left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} w \\ x \end{pmatrix} \right) \mid x, w \in \mathbb{R} \right\} \\ S &= I_M \qquad \textit{where } M = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \mid y \in \mathbb{R} \right\} \end{aligned}$$

It follows that $D(ST) = D(S) = M$ while $D(ST_2) = D(T_2)$. Thus $D(ST_2 + ST) = \{0\}$. On the other hand

$$D(S(T+T_2)) = (T+T_2)^{-1}(D(S)) = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \right\},$$

showing that equality does not hold.

2.2 Continuity and the Norm function for Linear Relations

Notation 2.2.1 *Let $T \in LR(X, Y)$. We let Q_T, or simply Q, when T is understood, denote the natural quotient map, $Q^Y_{\overline{T(0)}}$, of Y onto $Y/\overline{T(0)}$.*

The approach of factoring out the set-valued part $T(0)$ of a linear relation T, by using an associated quotient map, is central to many of the proofs in the theory of linear relations. In particular, in this section the quotient map is used to extend the definition of the operator norm to linear relations. Some elementary properties are reviewed, we provide a geometric characterisation of the norm and show that the continuity of a linear relation is equivalent to the finiteness of its norm.

Proposition 2.2.2 *QT is single-valued*

PROOF

Let $x \in D(T)$, and let $y_1, y_2 \in QTx$. Then $y_1 - y_2 \in QTx - QTx = QT(0) \subset Q\overline{T(0)} = 0$.

◇

Proposition 2.2.3 *Let $T \in LR(X,Y)$. Then*

$$N(T) \subset N(QT)$$

with equality if $T(0)$ is relatively closed in $R(T)$.

PROOF

We apply Proposition 1.2.9:

$$\begin{aligned} N(T) &= \{x \in X \mid Tx = T(0)\} \subset \{x \in X \mid Tx \subset \overline{T(0)}\} \\ &= \{x \in X \mid QTx = 0\} = N(QT). \end{aligned}$$

If $T(0)$ is relatively closed in $R(T)$, then equality holds.

◇

The next example shows that equality does not hold generally in Proposition 2.2.3.

Example 2.2.4 *Let X be an infinite dimensional normed space and let $f \in LR(X, I\!K)$ be an everywhere-defined discontinuous linear functional. If $T = f^{-1}$, then $T(0)$ is a dense hyperplane in X and $Q_T = 0$. Thus*

$$N(T) = 0 \neq I\!K = N(QT) .$$

Proposition 2.2.5 *Let $T \in LR(X,Y)$. If $R(T)$ is closed, then $R(QT)$ is closed. Conversely, if $R(QT)$ is closed and $\overline{T(0)} \subset R(T)$, then $R(QT)$ is closed.*

PROOF

This is just a special case of Proposition 1.7.5.

◇

Definitions 2.2.6 *Let $T \in LR(X,Y)$. Then T is said to be* **continuous at a point** *$x \in D(T)$ if the inverse image of any neighbourhood of Tx is a neighbourhood of x. T is said to be* **continuous** *if it is continuous at every point in its domain.*

For $x \in D(T)$ we define $||Tx||$ by

$$||Tx|| := ||QTx||$$

and the quantity $||T||$, which is called the **norm** *of T, is defined by*

$$||T|| := ||QT||.$$

We note that $||T||$ is not a true "norm" function since $||T|| = 0$ does not imply that $T = 0$.

Proposition 2.2.7 *Let $T \in LR(X,Y)$.*

(a) For $x \in D(T)$

$$\begin{aligned} ||Tx|| &= d(y,T(0)) \quad \text{for all } y \in Tx \\ &= \inf_{y\in Tx} ||y|| \\ &= d(y+T(0),0) \quad \text{for all } y \in Tx \\ &= d(Tx,0) = d(Tx,T(0)). \end{aligned}$$

(b) $||T|| = \sup_{x\in B_{D(T)}} ||Tx||$

PROOF

(a) The first equality follows from definition of $||Tx||$ and Proposition 1.2.9 and the second equality follows from the definition and properties of the norm on $X/\overline{T(0)}$. The rest are obvious.

(b) We have $||T|| = ||QT|| = \sup_{x\in B_{D(T)}} ||QTx|| = \sup_{x\in B_{D(T)}} ||Tx||$.

◇

We will show that $||T|| < \infty$ if and only if T is continuous. But first we illustrate the extension of some well-known results about the norm of the sum and scalar multiples of linear operators.

Proposition 2.2.8 *(a) For $S,T \in LR(X,Y)$ and $x \in D(S+T)$ we have*

$$||Sx+Tx|| \leq ||Sx|| + ||Tx||.$$

If additionally $S(0) \subset \overline{T(0)}$ then

$$||Tx|| - ||Sx|| \leq ||Tx - Sx||.$$

(b) For $\alpha \in I\!K$ and $x \in D(T)$ we have

$$||\alpha Tx|| = |\alpha|\,||Tx||.$$

PROOF

(a) Let $s \in Sx$ and $t \in Tx$. Then $s+t \in Sx+Tx = (S+T)x$. Thus :

$$\begin{aligned} ||Sx+Tx|| &= d(s+t,(S+T)(0)) \\ &\leq d(s,S(0)+T(0)) + d(t,S(0)+T(0)) \\ &\leq d(s,S(0)) + d(t,T(0)) \\ &= ||Sx|| + ||Tx||. \end{aligned}$$

If $S(0) \subset \overline{T(0)}$, then by what we have just shown,

$$||Tx|| = ||Tx+Sx-Sx|| \leq ||Tx-Sx|| + ||Sx||.$$

(b) We have $||\alpha Tx|| = ||Q(\alpha T)(x)|| = ||\alpha QTx|| = |\alpha|\,||QTx|| = |\alpha|\,||Tx||$.

◇

The next example shows that it is NOT true in general that $||Tx|| - ||Sx|| \leq ||Tx - Sx||$ for linear relations.

Example 2.2.9 *Let X be a nonzero normed space, let $G(S) = X \times X$ and let $T = I_X$. Then for $x \neq 0$ we have*

$$||Tx - Sx|| = 0$$

while

$$||Tx|| - ||Sx|| = ||x|| \neq 0.$$

Thus $||Tx|| - ||Sx|| \not\leq ||Tx - Sx||$.

Proposition 2.2.10 *Let $S, T \in LR(X,Y)$ and let $\alpha \in I\!K$. Then*

(a) $||S + T|| \leq ||S|| + ||T||$.

(b) If $S(0) \subset \overline{T(0)}$ then $||T|| - ||S|| \leq ||T - S||$.

(c) $||\alpha T|| = |\alpha|\,||T||$.

PROOF

(a) Applying Proposition 2.2.8:

$$\begin{aligned}
||S+T|| &= \sup\{\ ||Sx + Tx||\ \mid x \in B_X \cap D(S) \cap D(T)\ \} \\
&\leq \sup\{\ ||Sx|| + ||Tx||\ \mid x \in B_X \cap D(S) \cap D(T)\ \} \\
&\leq \sup\{\ ||Sx||\ \mid x \in B_X \cap D(S) \cap D(T)\ \} \\
&\quad + \sup\{\ ||Tx||\ \mid x \in B_X \cap D(S) \cap D(T)\ \} \\
&\leq \sup\{\ ||Sx||\ \mid x \in B_{D(S)}\ \} + \sup\{\ ||Tx||\ \mid x \in B_{D(T)}\ \} \\
&= ||S|| + ||T||.
\end{aligned}$$

(b) follows from Proposition 2.2.8 and (a), and (c) follows from Propositions 2.2.7 and 2.2.8.

◇

The next result gives a geometric characterisation of the norm.

Proposition 2.2.11 *Let $T \in LR(X,Y)$. Then*

(a) $||T|| < \infty \Leftrightarrow$ *there exists $\lambda > 0$ such that*

$$TB_{D(T)} \subset \lambda B_{R(T)} + T(0). \tag{2.2}$$

(b) If $||T|| < \infty$ then

$$||T|| = \inf_{\lambda > 0}\{\lambda \mid TB_{D(T)} \subset \lambda B_{R(T)} + T(0)\}. \tag{2.3}$$

PROOF

(a) Suppose $||T|| < \infty$. We apply Proposition 2.2.7: for $x \in B_{D(T)}$ and $y \in Tx$ there exists $k \in T(0)$ such that given $\epsilon > 0$,

$$||y - k|| < ||T|| + \epsilon,$$

i.e. $y - k \in (||T|| + \epsilon)B_{R(T)}$. Thus

$$y \in (||T|| + \epsilon)B_{R(T)} + T(0) \tag{2.4}$$

as required.

Conversely, suppose (2.2) holds. Let $x \in B_{D(T)}$ and choose $y \in Tx$. Then $y = \lambda y_1 + k$ where $||y_1|| \leq 1$ and $k \in T(0)$. Thus $||y - k|| \leq \lambda$, in particular, $d(y, T(0)) \leq \lambda$. It follows from Proposition 2.2.7 that $||T|| \leq \lambda < \infty$.

(b) Suppose $||T|| < \infty$. If $||T|| = 0$, then $TB_{D(T)} \subset \overline{T(0)}$ and (2.3) holds. Suppose $||T|| > 0$. Then it follows from (2.4) that

$$||T|| \geq \inf_{\lambda > 0} \{\lambda \mid TB_{D(T)} \subset \lambda B_{R(T)} + T(0)\}$$

Let α be such that $0 < \alpha < ||T||$, and choose $x \in B_{D(T)}$, $y \in Tx$ such that

$$\alpha < d(y, T(0)) \tag{2.5}$$

If $TB_{D(T)} \subset \alpha B_{R(T)} + T(0)$ then there exists $y_1 \in B_{R(T)}$, and $k \in T(0)$ such that $y = \alpha y_1 + k$. But then

$$||y - k|| \leq \alpha,$$

which contradicts (2.5). Thus,

$$\alpha < \inf_{\lambda > 0} \{\lambda \mid TB_{D(T)} \subset \lambda B_{R(T)} + T(0)\}$$

and the result follows.

◇

Proposition 2.2.12 *Let $T \in LR(X, Y)$*

(a) T is continuous if and only if $||T|| < \infty$.

(b) If $\dim D(T) < \infty$, then T is continuous.

PROOF

(a) Suppose T is continuous. Since $T(0) + B_Y$ is a neighbourhood of $T(0)$, it follows that $T^{-1}(T(0) + B_Y) = T^{-1}B_{R(T)}$ is a neighbourhood of 0. Thus $\exists\ \lambda > 0$ s.t.

$$\lambda B_{D(T)} \subset T^{-1}B_{R(T)}$$

and, therefore,

$$\lambda T B_{D(T)} \subset B_{R(T)} + T(0) = TT^{-1}B_{R(T)}.$$

By Proposition 2.2.11, this implies that $||T|| < \infty$.

Conversely, suppose $||T|| < \infty$. Let $x \in D(T)$, and let V be a nontrivial closed ball in $R(T)$ with center y where $y \in Tx$. Then $V_0 = V - \{y\} = \alpha B_{R(T)}$ for some $\alpha > 0$. Applying Proposition 2.2.11, there exists $\lambda > 0$ such that

$$TB_{D(T)} \subset \lambda B_{R(T)} + T(0).$$

It follows that

$$B_{D(T)} + T^{-1}(0) \subset \lambda T^{-1}B_{R(T)} = \alpha^{-1}\lambda T^{-1}V_0$$

or, equivalently,

$$\lambda^{-1}\alpha B_{D(T)} + T^{-1}(0) \subset T^{-1}V_0 = T^{-1}(V - y).$$

Hence,

$$\lambda^{-1}\alpha B_{D(T)} + T^{-1}y \subset T^{-1}V - T^{-1}y + T^{-1}y = T^{-1}V,$$

and $\lambda^{-1}\alpha B_{D(T)} + T^{-1}y$ is a neighbourhood of x in $D(T)$. Now suppose W is a neighbourhood of Tx, let $U \subset W$ be an open set containing $y \in Tx$, and let $V \subset U$ be a non-trivial closed ball with centre y. From what has already been shown, it follows that $T^{-1}W$ is neighbourhood of x. Hence T is continuous.

(b) If $dim\, D(T) < \infty$, then QT is a continuous single-valued operator, i.e. $||QT|| < \infty$. Since $||T|| = ||QT||$, the result follows from (a).

$\diamond$

2.3 Open Relations and the Minimum Modulus

In this section we define what it means for a linear relation to be open. Next we introduce the minimum modulus $\gamma(T)$ of a linear relation T, and give equivalent characterisations of this quantity (Propositions 2.3.2 and 2.3.3). We then show that $\gamma(T) = ||T^{-1}||^{-1}$, and deduce that a linear relation T is open if and only if $\gamma(T) > 0$. In Proposition 2.3.6 we consider the relationship between $\gamma(T)$ and $\gamma(QT)$, and in Proposition 2.3.8 we give criteria for $T(0)$ to be closed in $R(T)$, and for the equality $N(T) = N(QT)$. We conclude this section by giving inequalities for the minimum modulus and the norm of the composition of linear relations.

Definitions 2.3.1 *A linear relation $T \in LR(X,Y)$ is said to be* **open** *if its inverse T^{-1} is a continuous linear relation.*

The **minimum modulus** *of T is the quantity*

$$\gamma(T) := sup\{\lambda \mid ||Tx|| \geq \lambda d(x, N(T)) \ for\ x \in D(T)\}.$$

The following proposition provides an equivalent definition for $\gamma(T)$.

Proposition 2.3.2 *Let $T \in LR(X,Y)$.*

$$\gamma(T) = \begin{cases} \infty & if\ \ D(T) \subset \overline{N(T)} \\ \inf\{ \frac{\|Tx\|}{d(x,N(T))} \mid x \in D(T) \setminus \overline{N(T)}\} & otherwise \end{cases} \tag{2.6}$$

PROOF

Let $\gamma_1(T)$ denote the expression in (2.6), and for all $x \in D(T)$ let $\lambda \geq 0$ satisfy

$$\|Tx\| \geq \lambda d(x, N(T)). \tag{2.7}$$

If $D(T) \subset \overline{N(T)}$, then $\lambda < \gamma_1(T) = \infty$. If $D(T) \not\subset \overline{N(T)}$, then by (2.7) we have $\frac{\|Tx\|}{d(x,N(T))} \geq \lambda$ for $x \in D(T) \setminus \overline{N(T)}$. Thus $\lambda \leq \gamma_1(T)$. Taking the supremum over λ in (2.7) yields $\gamma(T) \leq \gamma_1(T)$.

Suppose $\gamma(T) < \gamma_1(T)$. Then $\gamma(T) < \infty$, and thus from the definition of $\gamma(T)$ it follows that $D(T) \not\subset \overline{N(T)}$. It also follows that there exists $x \in D(T)$ such that $\|Tx\| < \gamma_1(T)d(x, N(T)$, which contradicts (2.6). Thus $\gamma(T) \geq \gamma_1(T)$.

◇

We now give a geometric characterisation of the minimum modulus.

Proposition 2.3.3

$$\gamma(T) = \sup\{\lambda \mid TB_{D(T)} \supset \lambda B_{R(T)}\} \tag{2.8}$$

PROOF

Let γ_1 denote the expression in (2.8). Suppose $\gamma_1 \neq \infty$ and $\gamma_1 > \gamma(T)$, and choose $\epsilon > 0$ sufficiently small so that $\gamma_1 > (1+2\epsilon)\gamma(T)$. Since $\|Tx\| = \inf_{y \in Tx} \|y\|$ and $\gamma(T) = \inf_{x \in D(T)\setminus\overline{N(T)}} \frac{\|Tx\|}{d(x,N(T))}$, we may choose $(x,y) \in G(T)$ such that

$$\gamma(T) \ \leq \ \frac{\|y\|}{d(x, N(T))} \ < \ \gamma_1(1+2\epsilon)^{-1}.$$

Now, if $\eta > 0$, then $\frac{\|\eta y\|}{d(\eta x, N(T))} = \frac{\|y\|}{d(x,N(T))}$. Thus, letting $x_1 = \eta x$, $y_1 = \eta y$ and $\eta = \frac{1}{\|y\|}$, we have

$$1 = \|y_1\| < \gamma_1 d(x_1, N(T)).(1+2\epsilon)^{-1} \tag{2.9}$$

From (2.8), $TB_{D(T)} \supset \gamma_1(1+\epsilon)^{-1}B_{R(T)}$. Thus, $y_1 \in \gamma_1^{-1}(1+\epsilon)TB_{D(T)}$ and $x_2 \in \gamma_1^{-1}(1+\epsilon)B_{D(T)}$ may be chosen so that $y_1 \in Tx_2$. Since $x_2 - x_1 \in N(T)$, it follows from (2.9) that

$$1+2\epsilon \ < \ \gamma_1\, d(x_1, N(T)) \ = \ \gamma_1\, d(x_2, N(T)) \ \leq \ \gamma_1\|x_2\| \ \leq \ 1+\epsilon$$

a contradiction. Thus $\gamma_1 \leq \gamma(T)$.

For the opposite inequality we may clearly assume that $\gamma_1 < \infty$. Suppose $\gamma_1 < \alpha < \gamma(T)$. Then $TB_{D(T)} \not\supset \alpha B_{R(T)}$ and there exists $y \in R(T)$ such that $\|y\| = 1$ and $y \notin \alpha^{-1}TB_{D(T)}$ and, hence,

$$\|x\| \leq \alpha^{-1} \Rightarrow y \notin Tx. \tag{2.10}$$

Let $x_0 \in D(T)$ satisfy $y \in Tx_0$ and fix β such that $\alpha < \beta < \gamma(T)$. Then

$$1 = ||y|| \geq ||Tx_0|| \geq \beta d(x_0, N(T)).$$

Choose $x_1 \in N(T)$ so that $||Tx_0|| \geq \alpha||x_0 + x_1||$. Letting $x := x_0 + x_1$, we have that $Tx_0 = Tx$ and, therefore, $y \in Tx$ where $1 = ||y|| \geq ||Tx|| \geq \alpha||x||$, which contradicts (2.10). Thus $\gamma_1 \geq \gamma(T)$.

For the case $\gamma_1 = \infty$ we have

$$TB_{D(T)} \supset \bigcup_{n \in I\!N} \{nB_{R(T)}\} = R(T).$$

Taking inverses yields

$$B_{D(T)} + T^{-1}(0) \supset D(T).$$

Hence, for each $\epsilon > 0$ we have

$$\epsilon B_{D(T)} + T^{-1}(0) \supset D(T).$$

Taking the intersection over $\epsilon > 0$ yields $\overline{N(T)} \supset D(T)$ which by Proposition 2.3.2 implies $\gamma(T) = \infty$.

$\diamond$

The next result yields the relationship between the minimum modulus and the norm of a linear relation.

Proposition 2.3.4 *Let $T \in LR(X,Y)$. Then*

$$\gamma(T) = ||T^{-1}||^{-1}. \tag{2.11}$$

PROOF

Let $\lambda \geq 0$. Then

$$\begin{aligned} T^{-1}B_{R(T)} \supset \lambda B_{D(T)} \quad &\Rightarrow \quad B_{R(T)} + T(0) \supset \lambda TB_{D(T)} \\ &\Rightarrow \quad T^{-1}B_{R(T)} + T^{-1}(0) \supset \lambda B_{D(T)} + T^{-1}(0) \\ &\Rightarrow \quad T^{-1}B_{R(T)} \supset \lambda B_{D(T)}. \end{aligned}$$

In particular

$$\lambda TB_{D(T)} \subset B_{R(T)} + T(0) \;\Leftrightarrow\; \lambda B_{D(T)} \subset T^{-1}B_{R(T)}. \tag{2.12}$$

We consider two cases

Case 1: $||T|| < \infty$

From Propositions 2.2.11 and 2.3.3, the geometric characterisations of the norm and minimum modulus respectively, and applying the equivalence (2.12) we have

$$\begin{aligned} ||T|| &= \inf\{\lambda > 0 \mid \lambda^{-1}TB_{D(T)} \subset B_{R(T)} + T(0)\,\} \\ &= (\,\sup\{\lambda > 0 \mid \lambda TB_{D(T)} \subset B_{R(T)} + T(0)\,\}\,)^{-1} \\ &= (\,\sup\{\lambda > 0 \mid T^{-1}B_{R(T)} \supset \lambda B_{D(T)}\,\}\,)^{-1} \\ &= \gamma(T^{-1})^{-1}. \end{aligned}$$

Case 2: $||T|| = \infty$

Again we apply Proposition 2.2.11. In this case we have:

$$\forall \lambda > 0, \quad TB_{D(T)} \not\subset \lambda B_{R(T)} + T(0).$$

Thus,

$$\forall \lambda > 0, \quad \lambda^{-1} TB_{D(T)} \not\subset B_{R(T)} + T(0).$$

Hence, applying the equivalence (2.12)

$$\forall \lambda > 0, \quad \lambda B_{D(T)} \not\subset T^{-1} B_{R(T)}.$$

Applying Proposition 2.3.3, $\gamma(T^{-1}) = 0$, and the desired equality holds.

$\diamond$

Proposition 2.3.5 *Let $T \in LR(X,Y)$.*

(a) T is open if and only if $\gamma(T) > 0$.

(b) If $\dim R(T) < \infty$, then T is open.

PROOF

(a) Apply Proposition 2.3.4 to Proposition 2.2.12 (a).

(b) This follows on substituting T^{-1} for T in Proposition 2.2.12 (b).

$\diamond$

We examine the relationship between $\gamma(T)$ and $\gamma(QT)$.

Proposition 2.3.6 *Let $T \in LR(X,Y)$. Then*

$$\gamma(T) \leq \gamma(QT)$$

with equality if $T(0)$ is relatively closed in $R(T)$.

PROOF

$$\begin{aligned}
\gamma(T) &= \sup\{\lambda \mid ||Tx|| \geq \lambda d(x, N(T)) \ \ \forall x \in D(T)\} \\
&\leq \sup\{\lambda \mid ||Tx|| \geq \lambda d(x, N(QT)) \ \ \forall x \in D(T)\} \\
&= \sup\{\lambda \mid ||QTx|| \geq \lambda d(x, N(T)) \ \ \forall x \in D(T)\} \\
&= \gamma(QT).
\end{aligned}$$

By Proposition 2.2.3, equality holds when $T(0)$ is closed in $R(T)$.

$\diamond$

Proposition 2.3.6 proves that T is open, then so is QT. The converse is false and equality need not hold. This is illustrated in the following example:

Example 2.3.7 *Let X be an infinite-dimensional normed linear space, let f be a discontinuous linear functional with domain $D(T) = X$ and let $T := f^{-1}$. Then $T(0)$ is a dense hyperplane and $N(T) = \{0\}$, while $N(QT) = I\!K$ and $\gamma(QT) = \infty$. However, since T is not open, $\gamma(T) = 0$.*

The next result provides criteria for $T(0)$ to be closed in $R(T)$. This may also be applied to Proposition 1.7.5.

Proposition 2.3.8 *Let $T \in LR(X,Y)$ be open. Then*

$$R(T) \cap \overline{T(0)} \subset T\overline{T^{-1}(0)},$$

with equality if $N(T)$ is relatively closed in $D(T)$.

PROOF

We first establish the result for the case when T is injective. Suppose $\{y_n\}$ is a sequence in $T(0)$ such that $y_n \to y \in R(T)$. Then

$$\lim_{n\to\infty} ||T^{-1}(y_n - y)|| \;\leq\; ||T^{-1}|| \lim_n ||y_n - y|| = 0$$

since T^{-1} is continuous. Thus $||T^{-1}y|| = 0$. Thus $T^{-1}y = 0$ and, since T^{-1} is single-valued, $y \in N(T^{-1}) = T(0) = TT^{-1}(0) \subset T\overline{T^{-1}(0)}$. If $N(T)$ is closed in $D(T)$, then $TT^{-1}(0) = T\overline{T^{-1}(0)}$.

Now let $Q := Q_{T^{-1}}$. Then

$$TQ^{-1}(B_{X/\overline{T^{-1}(0)}}) = T(B_X + \overline{T^{-1}(0)}) \supset TB_X \supset \lambda B_{R(T)}$$

for some $\lambda > 0$ since T is open. Thus, TQ^{-1} is open. Applying the first part of the proof, we have that $D(QT^{-1}) \cap \overline{N(QT^{-1})} = R(TQ^{-1}) \cap \overline{TQ^{-1}(0)} \subset TQ^{-1}(\overline{QT^{-1}(0)})$. Hence

$$\begin{aligned} R(T) \cap \overline{T(0)} &= D(T^{-1}) \cap \overline{N(T^{-1})} \\ &\subset D(QT^{-1}) \cap \overline{N(QT^{-1})} \\ &\subset TQ^{-1}(\overline{QT^{-1}(0)}) = TQ^{-1}(0) = T\overline{T^{-1}(0)}. \end{aligned}$$

If $N(T)$ is relatively closed in $D(T)$, then

$$R(T) \cap \overline{T(0)} \supset T(0) = TT^{-1}(0) = T\overline{T^{-1}(0)},$$

and equality holds.

◇

Corollary 2.3.9 *If $T \in LR(X,Y)$ is open and $N(T)$ is closed then*

(a) $N(T) = N(QT)$.

(b) $\gamma(T) = \gamma(QT)$.

PROOF

(a) Since $T^{-1}(0)$ is closed, $R(T) \cap \overline{T(0)} = T(0)$. The result follows from Proposition 2.2.3.

(b) As in (a), the result follows from Proposition 2.3.6

◇

The last results in this section will be concerned with the behaviour of the minimum modulus and norm of the composition of linear relations. We will need the following property:

Proposition 2.3.10 *Let M be a non-empty subset of $R(T)$, and let $\gamma(T) < \infty$. Then for $N \subset D(T)$ we have*

$$d(TN, M) \geq \gamma(T) d(N, T^{-1}M).$$

PROOF

If $TN \cap M \neq \emptyset$, then $\emptyset \neq T^{-1}(TN \cap M) = (N + N(T)) \cap (T^{-1}M)$ and, thus, $d(N, T^{-1}M) = 0$. Suppose $TN \cap M = \emptyset$, let $\epsilon > 0$, and choose $m \in M$ and $n \in N$ such that

$$d(TN, M) \; > \; d(Tn - m, 0) - \epsilon. \tag{2.13}$$

Now

$$\begin{aligned} d(Tn - m, 0) &= d(Tn - m - T(0), 0) \; = \; d(Tn, m + T(0)) \; = \; d(Tn, TT^{-1}m) \\ &= \inf_{h \in T^{-1}m} d(Tn, Th) \; = \; \inf_{h \in T^{-1}m} d(T(n-h), 0) \; = \; \inf_{h \in T^{-1}m} \|T(n-h)\| \\ &\geq \gamma(T) \inf_{h \in T^{-1}m} d(n - h, T^{-1}(0)) \; = \; \gamma(T) \inf_{h \in T^{-1}m} d(n, h + T^{-1}(0)) \\ &= \gamma(T) d(n, T^{-1}m) \; \geq \; \gamma(T) d(n, T^{-1}M) \; \geq \; \gamma(T) d(N, T^{-1}M). \end{aligned}$$

Since ϵ was chosen arbitrarily, it follows from (2.13) that

$$d(TN, M) \geq \gamma(T) d(N, T^{-1}M).$$

◇

Proposition 2.3.11 *Let $T \in LR(X, Y)$ and $S \in LR(Y, Z)$. Then*

$$\gamma(ST) \geq \gamma(S|_{R(T)})\gamma(T) \quad (\infty.0 \textit{ excluded}), \tag{2.14}$$

with $\gamma(ST) = \infty$ when $\gamma(T) = \infty$ (even if $\gamma(S|_{R(T)}) = 0$). Furthermore

$$S^{-1}(0) \subset R(T) \Rightarrow \gamma(ST) \geq \gamma(S)\gamma(T). \tag{2.15}$$

PROOF

Let $x \in D(ST)$. We first consider the case when $\gamma(S), \gamma(T) < \infty$. Since $ST = (S|_{R(T)})T$, we suppose that $S = S|_{R(T)}$. Then $S^{-1}ST(0) = T(0) + S^{-1}(0) \subset R(T)$ and $T^{-1}(0) \subset T^{-1}S^{-1}(0)$. It follows from Proposition 2.3.10 that

$$\begin{aligned} ||STx|| = d(STx, ST(0)) &\geq \gamma(S)d(Tx, S^{-1}ST(0)) \\ &\geq \gamma(S)\gamma(T)(x, T^{-1}S^{-1}ST(0)) \\ &= \gamma(S)\gamma(T)d(x, T^{-1}S^{-1}(0)). \end{aligned}$$

Thus, applying Proposition 2.3.2, inequality (2.14) holds. Now suppose $S^{-1}(0) \subset R(T)$ and that it is not necessarily the case that $S = S|_{R(T)}$. Then

$$\begin{aligned} d(Tx, S^{-1}ST(0)) &\geq \gamma(T)d(x, T^{-1}S^{-1}ST(0)) \\ &= \gamma(T)d(x, T^{-1}(\,(T(0) \cap D(S)\,) + S^{-1}(0)\,)\,) \\ &= \gamma(T)d(x, T^{-1}S^{-1}(0)). \end{aligned}$$

Thus, as before,

$$||STx|| \geq \gamma(S)\gamma(T)d(x, T^{-1}S^{-1}(0)),$$

and implication (2.15) follows.

Next we consider the case $\gamma(T) = \infty$, and suppose $S = S|_{R(T)}$. By Proposition 2.3.2, $\overline{N(T)} = D(T)$. Since $N(T) \subset N(ST)$ and $D(T) \supset D(ST)$, it follows that $N(ST)$ is dense in $D(ST)$. Thus $\gamma(ST) = \infty$, and the desired inequality in (2.14) holds. If $S^{-1}(0) \subset R(T)$, then the proof of inequality (2.15) is similar.

Lastly suppose that $\gamma(S) = \infty$ and $0 < \gamma(T) < \infty$, and suppose $S = S|_{R(T)}$. Since $S^{-1}(0)$ is dense in $D(S)$ it follows that $d(Tx, S^{-1}(0)) = 0$. Therefore, $\gamma(T)d(x, T^{-1}S^{-1}(0)) = 0$ (applying Proposition 2.3.10), i.e. $N(ST)$ is dense in $D(ST)$. Thus $\gamma(ST) = \infty$, and again and the desired inequality in (2.14) follows. If $S^{-1}(0) \subset R(T)$, then the proof of inequality (2.15) is similar.

◇

Remarks 2.3.12 *If $\gamma(S) = \infty$ and $\gamma(T) = 0$, then inequality (2.14) may fail to hold: consider $S := f$ and $T := f^{-1}$ where f is a discontinuous linear functional on an infinite dimensional space. Then $\gamma(ST) = \gamma(ff^{-1}) = \gamma(I) = 1$, while $\gamma(S) = \infty$ (since $\overline{N(f)} = X$), and $\gamma(T) = ||f||^{-1} = 0$.*

Corollary 2.3.13 *Let $T \in LR(X,Y)$ and $S \in LR(Y,Z)$. Then*

$$||ST|| \leq ||S||\ ||I_{D(S)}T|| \quad (\infty.0 \text{ excluded}). \tag{2.16}$$

with $||ST|| = 0$ when $||S|| = 0$ (even if $I_{D(T)} = \infty$). Furthermore,

$$T(0) \subset D(S) \Rightarrow ||ST|| \leq ||S||\ ||T||. \tag{2.17}$$

PROOF

Applying Proposition 2.3.4, inequality (2.16) follows from (2.14) of Theorem 2.3.11.

If $T(0) \subset D(S)$, then for $x \in D(ST)$ and $y \in Tx$ we have that $Tx \subset D(S) + T(0) = D(S)$ (since $x \in T^{-1}(D(S))$). Thus $I_{D(S)}Tx = Tx \cap D(S) = Tx$, and

$$||ST|| \leq ||S|| \; ||T|_{D(ST)}||,$$

from which (2.17) follows.

◇

Remarks 2.3.14

The case when $\infty.0$ appears on the right hand side of (2.16) corresponds to the analogous case in Theorem 2.3.11 (applying Proposition 2.3.4). In this case no conclusion can be made from the hypothesis (see the remarks following Theorem 2.3.11).

In Cross [35], V.1.13, the author constructs a surprising example where the composition of a pair of continuous linear relations is defined but is not continuous .

The following proposition serves as a useful tool for working with inequalities involving the norm of the composition of two relations (see for example Corollaries 5.3.4 and 5.3.8).

Proposition 2.3.15 *Suppose* $T \in LR(X,Y)$, *and* $S \in LR(Y,Z)$ *is continuous with* $D(S) \supset \overline{T(0)}$. *Then*

$$Q_{ST}ST = Q_{ST}SQ_T^{-1}Q_TT.$$

PROOF

Let Q denote Q_T, and let $y \in Tx$. Then $SQ^{-1}Qy = S(y + \overline{T(0)})$, and since S is continuous with $D(S) \supset \overline{T(0)}$,

$$S\overline{T(0)} \;\subset\; \overline{ST(0)},$$

and, hence,

$$SQ^{-1}Qy \;\subset\; Sy + \overline{ST(0)}.$$

It follows that

$$Q_{ST}SQ^{-1}Qy \;\subset\; Q_{ST}Sy \;\subset\; Q_{ST}(SQ^{-1}Qy),$$

and, hence,

$$Q_{ST}SQ^{-1}Qy \;=\; Q_{ST}Sy.$$

Thus

$$Q_{ST}SQ^{-1}QTx \;=\; Q_{ST}STx,$$

and the result follows.

◇

2.4 Linear Selections

Selections or the single-valued parts of set-valued maps were considered in extension problems in topology (see for example Michael [105]), and are still investigated today. Selections play an important role in convex analysis, and there is a well-developed theory on selections of set-valued maps which satisfy various properties (see for example Aubin and Cellina [13] or Aubin and Frankowska [14]). In this section we give a brief review of linear selections of linear relations and consider some conditions for continuity.

Definition 2.4.1 *A single-valued linear operator A is called a* **linear selection** *(or* **single-valued part***) of a linear relation T if*

$$T = A + T - T. \tag{2.18}$$

If A is a selection of T then for all $x \in D(T)$ we have

$$Tx = Ax + T(0). \tag{2.19}$$

It follows from (2.19) that $R(T) = R(A) + T(0)$. However, this sum may not always be direct. The following result provides a method for constructing selections.

Proposition 2.4.2 *If P is a single-valued linear projection with domain $R(T)$ and kernel $T(0)$, then PT is a selection of T. Conversely, if A is a selection of T and $R(A) \cap T(0) = \{0\}$, then the single-valued projection defined on $R(T)$ with range $R(A)$ and kernel $T(0)$ satisfies $A = PT$.*

PROOF

Let P be as described. Then $PT(0) = \{0\}$, and for $y \in Tx$ we have

$$Tx = y + T(0) = Py + (I - P)y + T(0) = PTx + T(0).$$

Conversely, let A be a selection of T and let P be a linear projection defined on $R(T)$ with range $R(A)$ and kernel $T(0)$. Then for $x \in D(T)$ we have $Tx = Ax + T(0)$, whence $PTx = PAx = Ax - (I - P)Ax = Ax$.

◇

In Cross [35] the author gives another method for obtaining selections of a linear relation T by considering projections on $G(T)$.

Proposition 2.4.3 *Let $T \in LR(X, Y)$.*

(a) If T has a continuous selection A, then T is continuous and

$$||T|| \leq ||A|| \ .$$

(b) If $T(0)$ is topologically complemented in $R(T)$, then T is continuous if and only if T has a continuous selection.

PROOF

(a) Suppose A is a continuous selection of T. Then for $x \in D(T)$, we have $Tx = Ax + T(0)$, i.e. $Ax \in Tx$. Since

$$||Tx|| = \inf_{y \in Tx} ||y|| \leq ||Ax|| \leq ||A||\,||x||,$$

the result follows.

(b) Let P be a continuous projection defined on $R(T)$ with kernel $T(0)$. If T is continuous, then PT is a continuous selection of T. The reverse implication is contained in (a).

$\diamond$

2.5 Closed and Closable Linear Relations

In this section we give the basic properties of closed relations and we consider the relationship between a linear relation T and its closure. We also consider the connection between the properties of continuity and closedness.

Definitions 2.5.1 *The* **closure** *of a relation $T \in LR(X,Y)$ is the relation $\overline{T}$ defined*

$$G(\overline{T}) := \overline{G(T)}. \tag{2.20}$$

A relation is called **closed** *if its graph $G(T)$ is closed in $X \times Y$ or, equivalently, if $T = \overline{T}$.*

Proposition 2.5.2 *Let $T \in LR(X,Y)$. Then*

(a) $\overline{T} \in LR(X,Y)$.

(b) T is closed if and only if T^{-1} is closed. Furthermore, $(\overline{T})^{-1} = \overline{T^{-1}}$.

(c) If T is closed then $T(0)$ is closed.

(d) If T is continuous and $D(T)$ and $T(0)$ are closed, then T is closed.

PROOF

(a) and (b) follow from the definition of $\overline{T}$ and the fact that $\overline{G(T)}$ is a linear subspace of $X \times Y$.

(c) If $\{y_n\}$ is a sequence in $T(0)$ such that $y_n \to y$, then $(0,y) \in G(T)$ since T is closed.

(d) Suppose $\{(x_n, y_n)\}$ is a sequence in $G(T)$ such that $(x_n, y_n) \to (x,y)$. Then $x \in D(T)$ since $D(T)$ is closed. Letting $z \in Tx$, it follows from the continuity of T that $\exists z_n \in Tx_n$ such that $z_n \to z$. Since $z_n - y_n \in T(0)$ and $z_n - y_n \to z - y \in T(0)$, it follows that $y \in z + T(0) = Tx$.

$\diamond$

In the next chapter we will show that the converse of Proposition 2.5.2(d) holds, i.e. if T is closed with closed domain, then T is continuous - this is the multivalued version of the classic Closed Graph Theorem. Thus, continuity and closedness do agree in some cases, however, the two notions are quite different.

Definition 2.5.3 *A linear relation is said to be* **closable** *if* $\overline{T}$ *is an extension of* T.

We give an example of a continuous relation which is neither closed nor closable, and give an example of a closed relation which is not continuous. We note that $\overline{T}$ need not be an extension of T. We only have that $G(T) \subset G(\overline{T})$ and $Tx \subset \overline{T}x$ for all $x \in D(T)$. Furthermore, $\overline{T(0)} \subset \overline{T}(0)$, and equality need not hold, as is seen in the third of the examples below.

Examples 2.5.4

(1) Let T *be a linear relation such that* $T(0) \neq \overline{T(0)}$. *Let* F *be any finite-dimensional subspace of* $D(T)$. *Then* $T|_F$ *is continuous. However,* $\overline{T|_F(0)} = \overline{T(0)} \neq T(0)$. *Thus* $T|_F(0)$ *is not closed, and* $T|_F$ *is not closable.*

(2) Let $X = C([0,1])$, *and let* $C'([0,1]) := \{f \in X \mid f' \text{ is continuous}\}$. *Define* $T \in LR(C'([0,1]), X)$ *by* $(Tf)(t) := f'(t),\ t \in [0,1]$. *Then* T *is closed: if* $x_n \to x$ *and* $Tx_n \to y$, *then* $\{x_n\}$ *converges uniformly to* x *and* $\{x'_n\}$ *converges uniformly to* y *on* $[0,1]$. *Taking antiderivatives of* x'_n *and* y, *it follows that* $x \in D(T)$ *and* $Tx = y$ *on* $[0,1]$. *However,* T *is not continuous: consider the sequence* $\{x_n(t)\} = \{t^n\}$. *Then* $||Tx_n|| = n$ *while* $||x_n|| = 1$.

(3) Let $\{x_\alpha \mid \alpha \in S\}$ *be a normalised Hamel basis of* $l_2(I\!N)$. *Recall that* $I\!K^S := \{f : S \to I\!K\}$ *and* $l_1(S)$ *is the collection of sequences indexed by* S *such that* $\sup\limits_{\zeta(S)\subset S} \sum\limits_{\alpha\in\zeta(S)} |x_\alpha| < \infty$ *where* $\zeta(S)$ *denotes some finite subset of* S. *Let* Z_1 *be* $l_2(I\!N)$ *renormed by*

$$|| \sum_{\alpha\in S} c_\alpha x_\alpha ||_{z_1} := \sum_{\alpha\in S} |c_\alpha|,$$

and let $Z := \{f \in I\!K^S : f(\alpha) \neq 0 \text{ for at most finitely many } \alpha\}$ *with* $||f||_z := \sum\limits_{\alpha\in S} |f(\alpha)|$.

Then Z_1 *and* Z *are isometric to each other under the correspondence* $\sum c_\alpha x_\alpha \leftrightarrow \{c_\alpha\}$. *We now show that* Z *is dense in* $l_1(S)$. *Let* $f \in l_1(S)$. *Then* $\{\alpha \mid f(\alpha) \neq 0\}$ *is countable. Let* $\{\alpha_1, \alpha_2, \ldots\}$ *be the set of* α*'s for which* $f(\alpha) \neq 0$. *Define the sequence* f_n *as follows:*

$$\begin{aligned} f_n(\alpha_k) &:= f(\alpha_k) \text{ if } 1 \le k \le n \\ f_n(\alpha_k) &:= 0 \text{ otherwise} \end{aligned}$$

Then $||f_n - f|| = \sum\limits_{k=n+1}^{\infty} |f(\alpha_k)| \to 0$ *as* $n \to \infty$.

Let $T \in LR(l_1(S), l_2(I\!N))$ *be defined:* $T\{c_\alpha\} = \sum c_\alpha x_\alpha$. *Since* $l_2(I\!N)$ *is embeddable as a set in* $l_1(S)$, $\overline{T}$ *is onto. Thus* $l_1(S)/N(\overline{T})$ *is isomorphic to* $l_2(I\!N)$. *Since* $l_2(I\!N)$ *is separable, and* $l_1(S)$ *is not,* $N(\overline{T})$ *must be infinite dimensional. Clearly* $N(T) = \{0\}$.

Remarks 2.5.5 *Clearly* T *is closable if and only if* $T(0) = \overline{T}(0)$, *and* T^{-1} *is closable if and only if* $N(T) = N(\overline{T})$. *However, there is an unfortunate lack of symmetry in that* T *closable does not imply that* T^{-1} *is closable. The following example illustrates this.*

Example 2.5.6 *Let M be a non-closed subspace of a Banach space X. Let $G(T) = X \times M$. Then $G(\overline{T}) = \overline{G(T)} = X \times \overline{M}$. Since $T(0) = M \neq \overline{M} = \overline{T}(0)$, T is not closable. However, $N(T) = X = N(\overline{T})$, i.e. T^{-1} is closable.*

Proposition 2.5.7 *Let $T \in LR(X,Y)$. Then*

$$Q_T\overline{T} = \overline{Q_T T}.$$

PROOF

Let $Q := Q_T = Q^{Y}_{\overline{T(0)}}$, and let $(x,z) \in G(Q\overline{T})$. Then $\exists y \in R(\overline{T})$ such that $(x,y) \in G(\overline{T})$ and $Qy = z$. Let (x_n, y_n) be a sequence in $G(T)$ converging to (x,y). Now Qy_n converges to $Qy = z$. Hence $(x_n, QTx_n) = (x_n, Qy_n) \to (x,z)$, i.e. $(x,z) \in G(\overline{QT})$.

Conversely, let $(x,z) \in G(\overline{QT})$. Then $\exists (x_n, z_n) \in G(QT)$ such that $(x_n, z_n) \to (x,z)$. Thus, $\exists y_n \in R(T)$ such that $(x_n, y_n) \in G(T)$ and $Qy_n = z_n \to z \in Qy \in Y/\overline{T(0)}$ for some $y \in Y$. Thus $\exists k \in \overline{T(0)}$ such that $y_n \to y - k$. Since $\overline{T(0)} \subset \overline{T}(0)$, it follows that $(x, y-k) \in G(\overline{T})$. Since $Q(y-k) = Qy = z$ we have that $(x,z) \in G(Q\overline{T})$.

◇

Proposition 2.5.8 *Let $T \in LR(X,Y)$. The following properties are equivalent:*

(i) T is closed.

(ii) QT is closed and $T(0)$ is closed.

PROOF

(i) $\Rightarrow$ (ii) This follows from Proposition 2.5.7 and Proposition 2.5.2 (c).

(ii) $\Rightarrow$ (i) Applying Proposition 2.5.7,

$$D(T) = D(QT) = D(\overline{QT}) = D(Q\overline{T}) = D(\overline{T}).$$

Furthermore, for $x \in D(T)$ we have $Q\overline{T}x = QTx$. Thus, $\overline{T}x + T(0) = Tx + T(0)$, i.e. $\overline{T}x = Tx$. It follows that $T = \overline{T}$.

◇

Proposition 2.5.9 *Let $T \in LR(X,Y)$.*

$$\|\overline{T}\| \leq \|T\|$$

with equality holding if $\overline{T}(0) = \overline{T(0)}$.

PROOF

Clearly we may assume that $||T|| < \infty$. We prove by contradiction. Suppose $||\overline{T}|| > ||T||$. Choose $\epsilon > 0$ and $(x_0, y_0) \in G(\overline{T})$ such that $||x_0|| \leq 1$ and

$$d(y_0, \overline{T}(0)) > ||T|| + 2\epsilon. \tag{2.21}$$

Now choose $(x,y) \in G(T)$ such that $||x|| \leq 1$ and $||x - x_0|| + ||y - y_0|| < \epsilon$. Since $|d(y_0, T(0)) - d(y, T(0))| \leq ||y_0 - y|| < \epsilon$, it follows from (2.21) that

$$||T|| + 2\epsilon \ < \ d(y_0, \overline{T}(0)) \ \leq \ d(y_0, T(0)) \ \leq \ d(y, T(0)) + \epsilon \ \leq \ ||T|| + \epsilon,$$

which is a contradiction. Thus $||\overline{T}|| \leq ||T||$.

Suppose now that $\overline{T}(0) = \overline{T(0)}$. Then for $x \in D(T)$ and $y \in Tx \subset \overline{T}x$ we have $||Tx|| = d(Tx, \overline{T}(0)) = d(\overline{T}x, \overline{T}(0)) = ||\overline{T}x||$, and the desired equality follows.

◇

Corollary 2.5.10 *Let $T \in LR(X,Y)$, then*

$$\gamma(\overline{T}) \geq \gamma(T)$$

with equality holding if $N(\overline{T}) = \overline{N(T)}$.

Proposition 2.5.11 *Let $T \in LR(X,Y)$ be closed and let Y be complete. If $S \in LR(X,Y)$ is continuous with $S(0) \subset \overline{T(0)}$ and $D(S) \supset \overline{D(T)}$, then $T + S \in LR(X,Y)$ is closed.*

PROOF

Suppose T and S are single-valued, and let (x_n) be a sequence in $D(T)$ such that $(T+S)x_n \to y \in Y$ and $x_n \to x$. Then

$$||T(x_n - x_m)|| \leq ||(T+S)(x_n - x_m)|| + ||S||\,||x_n - x_m||, \tag{2.22}$$

and, since S is continuous, the right-hand side of (2.22) converges to zero as $m, n \to \infty$. Thus, $\{Tx_n\}$ is a Cauchy sequence, and $\exists z \in Y$ such that $Tx_n \to z$. Since T is closed, it follows that $x \in D(T) = D(T+S) \subset D(S)$ and $Tx = z$. Since S is continuous, $Sx_n \to Sx$. Thus $(T+S)x_n \to (T+S)x = y$, i.e. $T+S$ is closed.

Passing to the general case, it follows from Proposition 2.5.8 that $(S+T)(0) = T(0)$ is closed, and $Q_T T$ is closed. Furthermore, $Q_T S$ is single-valued and continuous (by Proposition 2.3.13). By what has already been shown, $Q_{T+S}(T+S) = Q_T T + Q_T S$ is closed. Applying Proposition 2.5.8, $T+S$ is closed.

◇

2.6 The Adjoint of a Linear Relation

Definition 2.6.1 *The* **adjoint** *or* **conjugate** T' *of a linear relation* $T \in LR(X,Y)$ *is defined*

$$G(T') := G(-T^{-1})^{\perp} \subset Y' \times X'.$$

where

$$[(y,x),(y',x')] := [x,x'] + [y,y'] = x'x + y'y.$$

Remarks 2.6.2

We note that the terms *adjoint* and *conjugate* are used interchangeably throughout.

If $(y',x') \in G(T')$ then $y'y = x'x$ for all $y \in Tx$, $x \in D(T)$, i.e. $x' \in T'y' \Leftrightarrow x'x = y'Tx$ for all $x \in D(T)$, i.e.

$$x'|_{D(T)} = y'T.$$

If T is densely defined, then $y'T$, which is single-valued, has a unique extension to X, making T' single-valued. Thus, we may make the following assertions:

Proposition 2.6.3 $T' \in LR(Y',X')$ *is a closed relation with*

$$D(T') = \{y' \in Y' \mid y'T \textit{ is continuous and single-valued } \}$$

and $T'y'x = y'Tx \in I\!K$ *for* $x \in D(T)$ *and* $y' \in D(T')$.

Proposition 2.6.4 *Let* $T \in LR(X,Y)$. *Then*

(a) $(\overline{T})' = T'$

(b) $(T')^{-1} = (T^{-1})'$

(c) $(\lambda T)' = \lambda T'$

PROOF

We need only verify (c): Let $\lambda \in I\!K$, $\lambda \neq 0$. Then

$$\begin{aligned} G((\lambda T)') &= \{(y',x') \mid y'y = x'x \textit{ for } (x,y) \in G(\lambda T)\} \\ &= \{(y',\lambda x') \mid y'(\lambda y) = (\lambda x')x \textit{ for } (x,y) \in G(T)\} \\ &= G(\lambda T'). \end{aligned}$$

◇

The following equivalences are generalisations of well-known properties for closed single-valued operators (cf. Goldberg [60], IV.1.2). The proofs are significantly simplified by the multivalued concept unifying them.

Proposition 2.6.5 *Let $T \in LR(X,Y)$. Then*

(a) $N(T') = R(T)^{\perp}$

(b) $T'(0) = D(T)^{\perp}$

(c) $N(\overline{T}) = R(T')^{\top}$

(d) $\overline{T}(0) = D(T')^{\top}$

PROOF

Statements in (b) and (d) follow from (a) and (c), respectively, by substituting T with T^{-1}. Thus we only show that the latter two hold.

(a)

$$\begin{aligned} y' \in N(T') &\Leftrightarrow (y',0) \in G(T') \\ &\Leftrightarrow y'y = 0 \ \ \forall y \in R(T) \\ &\Leftrightarrow y' \in R(T)^{\perp}. \end{aligned}$$

(c)

$$\begin{aligned} x \in N(\overline{T}) &\Leftrightarrow (x,0) \in G(\overline{T}) = G(T)^{\perp\top} = G((-T^{-1})')^{\top} = G(-(T')^{-1})^{\top} \\ &\Leftrightarrow x'x = 0 \ \ \forall x' \in R(T') \\ &\Leftrightarrow x \in R(T')^{\top}. \end{aligned}$$

$\diamond$

Proposition 2.6.6 *Let $S, T \in LR(X,Y)$. Then*

(a) $G(S' + T') \subset G(\,(S+T)'\,)$.

(b) $(S+T)'$ *is an extension of* $S' + T'$ *if and only if* $(\,D(S) \cap D(T)\,)^{\perp} = D(S)^{\perp} + D(T)^{\perp}$.

(c) If $D(T) \subset D(S)$ *and* S *is continuous, then* $S' + T' = (S+T)'$.

PROOF

(a) Let $(y', x') \in G(S' + T')$. Then $(y', x_1') \in G(S')$ and $(y', x_2') \in G(T')$ where $x_1' \in S'y'$ and $x_2' \in T'y'$, $x' = x_1' + x_2'$. Let $(x, s+t) \in G(S+T)$, $s \in Sx$, $t \in Tx$. Then

$$y'(s+t) - x'x = y's + y't - x_1'x - x_2'x = 0,$$

i.e $(y', x') \in G(\,(S+T)'\,)$.

(b) Since $(S+T)'$ is an extension of $S' + T'$ if and only if $(S+T)'(0) = S'(0) + T'(0)$, the result follows from Proposition 2.6.5 (b).

(c) We first show that the domains are equal. Suppose $y' \in D(\,(S+T)'\,)$. Then $y'S(0) + y'T(0) = y'(S+T)(0) = 0$ and thus, $y'S(0) = y'T(0) = 0$. Since S is continuous, so is $y'S$, and also $y'T$

since $y'Tx = y'(S+T)x - y'Sx$ for all $x \in D(T) \subset D(S)$. Thus, $y' \in D(T') \cap D(S') = D(S'+T')$. Since it follows from (a) that $D(S'+T') \subset D((S+T)')$, the domains are equal. Now $D(S) \supset D(T)$ implies $(D(S) \cap D(T))^{\perp} = D(S)^{\perp} + D(T)^{\perp}$, and hence (b) holds. It follows that $S'+T' = (S+T)'$.

◇

The following collection of results deal with the composition of relations. The next result is due to Kascic [73] (see Cross [35]).

Proposition 2.6.7 *Let $T \in LR(X,Y), S \in LR(Y,Z)$. Then*

(a) $G(T'S') \subset G((ST)')$.

(b) If either

$$\begin{array}{ll} & (1) \quad R(T') = X' \text{ and } D(S) \subset R(T) \\ or & (2) \quad D(S') = Z' \text{ and } R(T) \subset D(S) \end{array}$$

then $(ST)' = T'S'$.

PROOF

(a) Suppose $(z', x') \in G(T'S')$. Then there exists $y' \in Y'$ such that $(z', y') \in G(S')$ and $(y', x') \in G(T')$. Hence $z'z = y'y = x'x$ for all $(x,z) \in G(ST)$, i.e. $(z',x') \in G((ST)')$.

(b) Suppose (1) holds, and let $(z', x') \in G((ST)')$. Since $x' \in X' = R(T')$, there exists $y' \in Y'$ such that $(y', x') \in G(T')$. Let $(y, z) \in G(S)$. Then $y \in D(S) \subset R(T)$, and there exists $x \in D(T)$ such that $(x,y) \in G(T)$. Thus $y'y = x'x$, and since $(x,z) \in G(ST)$, $z'z = x'x = y'y$, i.e. $(z', y') \in G(S')$. From this it follows that $(z',x') \in G(T'S')$, and equality follows from (a).

Now suppose that (2) holds. By Proposition 2.6.4 (b), we have that $R((S^{-1})') = Z'$, and $D(T^{-1}) \subset R(S^{-1})$. Thus, by what has been shown,

$$(T^{-1}S^{-1})' = (S^{-1})'(T^{-1})'.$$

Another application of Proposition 2.6.4 (b) yields the desired result.

◇

An example showing that equality need not hold is given in Cross (III.1.7).

Notation 2.6.8 *Let E be a subspace of a normed linear space X. We let J_E^X denote the* **natural injection** *map from E into X, i.e. for $x \in E$, $J_E^X x = x \in X$.*

Proposition 2.6.9 *Let E be a subspace of X. Then*

(a) $(J_E^X)' = Q_{E^{\perp}}^{X'}$.

(b) If E is closed, then $(Q_E^X)' = J_{E^{\perp}}^{X'}$.

PROOF

(a) Applying Proposition 1.7.9, $Q^{X'}_{E^\perp} : X' \to E'$ with

$$(Q^{X'}_{E^\perp} x')\,(e) = x'e \tag{2.23}$$

for $x' \in X'$ and $e \in E$.

Similarly $(J^X_E)' : X' \to E'$ and

$$(\,(J^X_E)'x'),(e) = x'(J^X_E)e = x'e \tag{2.24}$$

for $x' \in X'$ and $e \in E$. Equality follows on combining (2.23) and (2.24).

(b) Applying Proposition 1.7.9 again, $(Q^X_E)' : E^\perp \to X'$ with

$$(\,(Q^X_E)'e')\,(x) = e'(Q^X_E)x = e'x \tag{2.25}$$

for $x \in X$ and $e' \in E^\perp$.

Similarly $J^{X'}_{E^\perp} : E^\perp \to X'$ with

$$(J^{X'}_{E^\perp} e')\,(x) = e'x \tag{2.26}$$

for $x \in X$ and $e' \in E^\perp$. Equality follows on combining (2.25) and (2.26).

◇

Proposition 2.6.10 *Let $T \in LR(X,Y)$. Then*

(a) $(Q_T T)' = T' J^{Y'}_{T(0)^\perp}$.

(b) $(T J_{D(T)})' = Q_{T'} T'$.

(c) $(Q_T T J_{D(T)})' = Q_{T'} T' J_{T(0)^\perp}$.

PROOF

These equalities follow from direct applications of Propositions 2.6.7 and 2.6.9.

◇

Corollary 2.6.11 *Let $T \in LR(X,Y)$. Then $D(T') = D((QT)')$. Furthermore, $T'y' = (QT)'y'$ for $y \in D(T')$.*

Proposition 2.6.12 *Let $T \in LR(X,Y)$. Then*

$$\|T'\| \le \|T\|.$$

PROOF

We may clearly assume that $\|T\| < \infty$. Letting $J := J_{D(T)}$ we have from Proposition 2.6.10 that

$$(QTJ)' = Q_{T'} T' J_{T(0)^\perp} . \tag{2.27}$$

By Proposition 2.6.5, the domain of the above relation contains $D(T')$, and hence for $y' \in D(T')$ we have $(QTJ)'y' = Q_{T'}T'J_{T(0)^\perp}y'$. Furthermore, $(QTJ)'$ is single-valued since QTJ is everywhere defined. Thus

$$\begin{aligned}
||T'|| = ||Q_{T'}T'J_{T(0)^\perp}|| &= \sup_{y'\in B_{D(T')}} ||(QTJ)'y'|| \\
&\leq \sup_{y'\in B_{D(T')}} \sup_{x\in B_{D(T)}} ||y'(QTJ)x|| \\
&\leq \sup_{y'\in B_{D(T')}} \sup_{x\in B_{D(T)}} ||y'||\ ||QTJ||\ ||x|| \\
&\leq ||QTJ|| = ||T||.
\end{aligned}$$

◇

Proposition 2.6.13 *Let $T \in LR(X,Y)$. Then*

$$\gamma(T') \geq \gamma(T).$$

PROOF

This follows from Proposition 2.6.12 combined with Proposition 2.3.4.

◇

In the Chapter 3, we show that the converses of (a) and (b) of the next proposition also hold; the converses of (c) and (d) are contained in Propositions 2.2.12 and 2.3.5.

Proposition 2.6.14 *Let $T \in LR(X,Y)$. Then*

(a) If T is continuous, then $D(T') = T(0)^\perp$.

(b) If T is open, then $R(T') = N(T)^\perp$.

(c) If T is continuous, then $||T'|| = ||T|| < \infty$.

(d) If T is open, then $\gamma(T') = \gamma(T) > 0$.

PROOF

We need only show that (a) and (c) hold.

(a) Suppose T is continuous. Then by Proposition 2.6.12, $(QTJ)'$ is continuous, and, by Proposition 2.6.3, its domain is the whole space, i.e. $T(0)^\perp$. Thus, by Proposition 2.6.10, the desired equality holds.

(c)

$$\begin{aligned}
\|T'\| &= \sup_{y' \in B_{D(T')}} \|(QTJ)'y'\| \\
&= \sup_{y' \in B_{D(T')}} \sup_{x \in B_{D(T)}} \|y'(QTJ)x\| \\
&= \sup_{y' \in B_{T(0)^\perp}} \sup_{x \in B_{D(T)}} \|y'(QTJ)x\| \\
&= \sup_{x \in B_{D(T)}} \|(QTJ)x\| \\
&= \|QTJ\| = \|T\|.
\end{aligned}$$

◇

2.7 Dimension Theorems and The Nullity, Deficiency and Index

In this section we show that the *fundamental theorem of linear algebra* holds for multivalued operators (Proposition 2.7.2). We then give an algebraic proof of *an index theorem for the composition of multivalued operators* in Proposition 2.7.3.

We give the duality relations between the dimensions of the kernels and the codimensions of the ranges of T and T' in Proposition 2.7.6. These quantities also satisfy important inequalities when an open relation T is perturbed by another relation of suitably small norm. This is given in a multivalued generalisation of the classic *small perturbation theorem for linear operators* and its corollaries in Chapter 3. These perturbation results are central to the proofs of the various stability results for Fredholm type operators which are considered later on. We begin with some definitions.

Definitions 2.7.1 *The* **nullity** *and the* **deficiency** *of a linear relation $T \in LR(X,Y)$ are defined respectively as follows:*

$$\begin{aligned}
\alpha(T) &:= \mathit{dim}\, N(T), \quad \mathit{and} \\
\beta(T) &:= \mathit{codim}\, R(T) := \mathit{dim}\, Y/R(T).
\end{aligned}$$

The quantity $\bar{\beta}(T)$ is defined as follows :

$$\beta(T) := \mathit{codim}\, \overline{R(T)} := \mathit{dim}\, Y/\overline{R(T)}.$$

If either $\alpha(T) < \infty$ or $\beta(T) < \infty$, then we define the **index** *of a linear relation as follows :*

$$\kappa(T) := \alpha(T) - \beta(T),$$

where the value of the differences is taken to be $\kappa(T) := \infty$ if $\alpha(T)$ is infinite and $\beta(T) < \infty$ and $\kappa(T) := -\infty$ if $\beta(T)$ is infinite and $\alpha(T) < \infty$.

The **reduced index** $\bar{\kappa}(T)$ *of a linear relation is defined analogously :*

$$\bar{\kappa}(T) \ := \ \alpha(T) - \bar{\beta}(T),$$

provided $\alpha(T) < \infty$ *or* $\bar{\beta}(T) < \infty$, *and where* $\bar{\kappa}(T) := \infty$ *if* $\alpha(T)$ *is infinite and* $\bar{\beta}(T) < \infty$, $\bar{\kappa}(T) := -\infty$ *if* $\bar{\beta}(T)$ *is infinite and* $\alpha(T) < \infty$.

Proposition 2.7.2 *Let* $T \in LR(X,Y)$. *Then*

$$dim\, D(T) + dim\, T(0) = dim\, R(T) + dim\, N(T).$$

PROOF

For single-valued operators we have the equality

$$dim\, D(T) = dim\, R(T) + dim\, N(T). \tag{2.28}$$

Let $q : Y \to Y/T(0)$ be the quotient map from Y onto $Y/T(0)$. Then qT is single-valued with $D(qT) = D(T)$ and $N(qT) = N(T)$. Furthermore, qT satisfies the equality (2.28). Now

$$dim\, R(T) = dim\, R(qT) + dim\, T(0). \tag{2.29}$$

Thus, combining (2.28) and (2.29), we have

$$\begin{aligned} dim\, D(T) + dim\, T(0) &= dim\, D(qT) + dim\, T(0) \\ &= dim\, R(qT) + dim\, N(T) + dim\, T(0) \\ &= dim\, R(T) + dim\, N(T). \end{aligned}$$

◇

Proposition 2.7.3 *Let* $T \in LR(X,Y)$ *and* $S \in LR(Y,Z)$. *Suppose* $D(S) = Y$. *Then*

$$\alpha(ST) + \beta(T) + \beta(S) + dim\,(\,T(0) \cap N(S)\,) = \beta(ST) + \alpha(T) + \alpha(S). \tag{2.30}$$

PROOF

We first suppose that S is single-valued. The map

$$\begin{aligned} \eta \quad &: \quad N(ST)/N(T) \to R(T) \cap N(S) \\ \eta[x] \quad &:= \quad Tx \cap N(S) \end{aligned}$$

is onto and has a single-valued inverse. Thus, by Proposition 2.7.2, $dim\, N(ST)/N(T) + dim\, \eta[0] = dim\,(\,R(T) \cap N(S)\,)$, and hence

$$dim\, N(ST) + dim\, \eta[0] \ = \ dim\,(R(T) \cap N(S)\,) + dim\, N(T). \tag{2.31}$$

Let $A := R(T) \cap N(S)$, and choose a subspace B such that $N(S) = A + B$, $A \cap B = \{0\}$. Thus we have that

$$dim\, N(S) \ = \ dim\, A + \dim\, B. \tag{2.32}$$

Furthermore $R(T) \cap B = \{0\}$, and we may choose a subspace C such that $Y = R(T) + B + C$ and $(R(T) + B) \cap C = \{0\}$. Then

$$codim\, R(T) \;=\; dim\, B + dim\, C. \tag{2.33}$$

Now S is a one-one map on C since $N(S) = A + B \subset R(T) + B$. Thus $dim\, S(C) = dim\, C$ and $R(S) = S(R(T) + C) = R(ST) + S(C)$ with $R(ST) \cap S(C) = S(0) = \{0\}$. It follows that $dim\, Y/R(S) = dim\, Y/(R(ST) + S(C))$ and thus,

$$codim\, R(S) + dim\, C = dim\, Y/R(ST). \tag{2.34}$$

By equalities (2.31), (2.32), (2.33) and (2.34), we have

$$\begin{aligned} & \alpha(ST) + dim\,(T(0) \cap N(S)) + \beta(T) + \beta(S) + dim\, C \\ = \;& dim\, A + \alpha(T) + dim\, B + dim\, C + \beta(ST) \\ = \;& \alpha(S) + \alpha(T) + \beta(ST) + dim\, C \end{aligned} \tag{2.35}$$

Now if $dim\, C = \infty$, then $\beta(T) = \infty$ by (2.33) and (2.34) implies that $\beta(ST) = \infty$ and equality, hence, holds in equation (2.30). If $dim\, C < \infty$, then subtracting $dim\, C$ from equation (2.35) yields the desired equality.

For the case when S is multivalued, we consider the single-valued operator $q_S S : Y \to Z/S(0)$, where $q_S : Z \to Z/S(0)$ is the natural quotient map. Let D be a subspace of Z such that $Z = R(ST) + D$ and $R(ST) \cap D = \{0\}$. Then $\beta(ST) = dim\, D$, and $q_S(Z) = q_S(R(ST)) + q_S(D)$ with $q_S(R(ST)) \cap q_S(D) = \{0\}$. Now q_S is a one-one map on D since $S(0) \subset R(ST)$. Thus $dim\, q_S(D) = dim\, D$, and

$$\beta(q_S ST) = \beta(ST).$$

Since $N(ST) \subset N(q_S ST) \subset N(q_{ST} ST) = N(ST)$, where $q_{ST} : Z \to Z/ST(0)$ is the natural quotient map, it follows that

$$\alpha(q_S ST) = \alpha(ST).$$

It follows similarly that

$$\alpha(q_S S) = \alpha(S)$$

and

$$\beta(q_S S) = \beta(S).$$

Thus, substituting $q_S S$ for S, the result follows from the case when S is assumed to be single-valued.

$\diamond$

Corollary 2.7.4 *Let $T \in LR(X,Y)$ and $S \in LR(Y,Z)$. Suppose $D(S) = Y$ and suppose T and S have finite indices. Then*

$$\kappa(ST) = \kappa(T) + \kappa(S) - dim\,(T(0) \cap N(S)).$$

Lemma 2.7.5 *Let $S, T \in LR(X,Y)$ and let S be an extension of T such that $\dim D(S)/D(T) = n < \infty$.*

(a) If T is closed , then so is S.

(b) If $T(0)$ is closed and $R(T)$ is closed, then $R(S)$ is closed.

(c) If T has an index, then $\kappa(S) = n + \kappa(T)$.

PROOF

(a) By the hypothesis, $D(S) = D(T) \oplus N$ where $\dim N = n$. Let $x \in D(T)$, $s \in N$. If $(x+s, y) \in G(S)$, then, since $S(0) = T(0)$, is follows that $y = y_1 + y_2$ where $y_1 \in Sx = Tx$ and $y_2 \in Ss$. Thus $(x, y_1) \in G(T)$, $(s, y_2) \in G(S|_N)$ and $(x+s, y) = (x, y_1) + (s, y_2)$. Thus, $G(S) = G(T) + G(S|_N)$. Furthermore, since $S(0) = T(0)$, it follows that

$$G(QS) = G(QT) + G(QS|_N) \tag{2.36}$$

Since QS is single-valued, $\dim G(QS|_N) \leq \dim N < \infty$. Hence, since $G(QT)$ is a closed subspace, $G(QS)$ is closed.

(b) If $R(T)$ and $T(0)$ are closed, then $R(QT)$ is closed and by (2.36)

$$R(QS) = R(QT) + QS(N) \tag{2.37}$$

Thus, since $QS(N)$ is finite-dimensional, $R(QS)$ is closed. It follows that $R(S)$ is closed.

(c) If q denotes the quotient map defined on Y with kernel $T(0)$, then $\kappa(S) = \kappa(qS)$ and $\kappa(T) = \kappa(qT)$. It suffices then to prove the statement for the case when T, and so S, are single-valued, and $n = 1$.

Suppose $N = sp\{x\}$ for some $x \in D(S)$, $x \neq 0$. If $Sx \notin R(T)$, then $R(S) = R(T) + sp\{w\}$, where $w = Sx$, $N(S) = N(T)$ and, hence, $\beta(T) = \beta(T) + 1$ and $\alpha(T) = \alpha(S)$. Thus $\kappa(S) = \kappa(T) + 1$.

If $Sx \in R(T)$, then $R(S) = R(T)$. Thus, there exists a non-zero $z \in D(T)$ such that $Sx = Tz = Sz$. Since $S(z - x) = 0$ and $z - x \notin D(T)$, it follows that $N(S) = N(T) + sp\{z - x\}$, i.e. $\alpha(S) = \alpha(T) + 1$. Thus $\kappa(S) = \kappa(T) + 1$.

$\diamond$

Proposition 2.7.6 *Let $T \in LR(X,Y)$. Then*

(a) $\alpha(T') = \bar{\beta}(T)$.

(b) $\alpha(T) \leq \bar{\beta}(T')$.

PROOF

We apply Propositions 2.6.5 and 1.7.9.

(a)

$$\begin{aligned} \alpha(T') &= \dim N(T') = \dim R(T)^{\perp} \\ &= \dim(Y/\overline{R(T)})' = \dim Y/\overline{R(T)} \\ &= \bar{\beta}(T)\,. \end{aligned}$$

(b)

$$\begin{aligned}\alpha(T) &= \dim N(T) = \dim N(T)' \\ &= \dim X'/N(T)^{\perp} \\ &\leq \dim X'/\overline{R(T')} \\ &= \bar{\beta}(T')\end{aligned}$$

◇

2.8 The Graph Operator and Relative Boundedness

An arbirary linear relation $T \in LR(X,Y)$ may be considered as a bounded relation on $D(T)$ as follows:

Definitions 2.8.1 *Let $T \in LR(X,Y)$. Then $(X_T, ||\_||_T)$ denotes the vector space $D(T)$ endowed with the norm $||x||_T := ||x|| + ||Tx||$ for $x \in D(T)$. The* **graph operator,** *$G_T \in LR(X_T, X)$ is defined*

$$D(G_T) := X_T, \quad G_T x := x \ \ for \ \ x \in X_T.$$

We let G denote the graph operator G_T when T is understood. Clearly $X_T = X_{QT}$ and, thus, $G_{QT} = G_T$. We note also that X_T is norm isomorphic to $G(T)$ when T is single-valued.

Proposition 2.8.2 *Let X and Y be complete and let $T \in LR(X,Y)$ be closed. Then X_T is complete.*

PROOF
The space $X_T = X_{QT}$ is norm isomorphic to $G(QT)$ which is closed in the Banach space $X \times Y/T(0)$.

◇

Proposition 2.8.3 *Let $T \in LR(X,Y)$. Then TG is bounded with*

$$||TG|| = \tfrac{||T||}{1+||T||}$$

where $\frac{\infty}{\infty} := 1$.

PROOF

$$\begin{aligned}||TG|| &= \sup_{x \in X_T} \frac{||TGx||}{||x||_T} \\ &= \sup_{x \in D(T)} \frac{||Tx||}{||x|| + ||Tx||} \\ &= \sup_{x \in D(T)} \frac{||Tx||}{1 + ||Tx||\,||x||^{-1}} . ||x||^{-1} \\ &= \frac{||T||}{1+||T||}.\end{aligned}$$

◇

Proposition 2.8.4 *Let $T \in LR(X,Y)$. Then TG is open if and only if T is open and*

$$\gamma(TG) = \begin{cases} \infty & if \ \ T=0 \\ \frac{\gamma(T)}{1+\gamma(T)} & \quad otherwise \end{cases} \tag{2.38}$$

where $\frac{\infty}{\infty} := 1$.

PROOF

Clearly the expression is true for $T = 0$. For the case $T \neq 0$, $N(TG) = N(T)$ as a subspace of the vector space $D(T)$. Now

$$\begin{aligned} d(x, N(TG)) &= \inf_{z \in N(TG)} \|x - z\|_T \\ &= \inf_{z \in N(TG)} (\|x - z\| + \|T(x - z)\|) \\ &= \inf_{z \in N(T)} \|x - z\| + \|Tx\| \\ &= d(x, N(T)) + \|Tx\|. \end{aligned}$$

Thus,

$$\begin{aligned} \gamma(TG) &= \inf_{x \in X_T \setminus N(TG)} \frac{\|TGx\|}{d(x, N(TG))} \\ &= \inf_{x \in D(T) \setminus N(T)} \frac{\|Tx\|}{d(x, N(T)) + \|Tx\|} \\ &= [\sup_{x \in D(T) \setminus N(T)} \frac{d(x, N(T)) + \|Tx\|}{\|Tx\|}]^{-1} \\ &= [\gamma(T)^{-1} + 1]^{-1} \\ &= \frac{\gamma(T)}{1 + \gamma(T)}. \end{aligned}$$

◇

The renorming of $D(T)$ is also useful when one wishes to consider a perturbation $T + S$ of the relation T by some $S \in LR(X,Y)$. For example, to show that some property holds, it may be sufficient for the relation $G_T S \in LR(X_T, Y)$ to be continuous.

Definitions 2.8.5 *A relation S is said to be* **T-bounded** *if $D(S) \subset D(T)$ and $\overline{T(0)} \supset S(0)$, and there exist $a, b \in I\!R$ such that*

$$\|Sx\| \leq a\|x\| + b\|Tx\| \quad for \ \ x \in D(T). \tag{2.39}$$

If S is $T-$bounded, then the infimum of all b such that (2.39) holds is called the $T-$bound of S.

The next proposition generalises a result due to B. Sz. Nagy (see [124]) :

Proposition 2.8.6 *Let $T \in LR(X,Y)$ and suppose $S \in LR(X,Y)$ satisfies $S(0) \subset T(0)$ and $D(S) \supset \overline{D(T)}$, and is $T-$bounded with $a, b > 0, b < 1$ such that for $x \in D(T)$*

$$||Sx|| \leq a||x|| + b||Tx||.$$

(a) The norms $||\_||_T$ and $||\_||_{T+S}$ are equivalent.

(b) If X, Y are complete and T is closed, then $T + S$ is closed.

PROOF

We prove (a) and (b) together and note that the argument for (a) does not require that T be closed or that the spaces be complete. We first assume that T and S are single-valued. Let $x \in D(T)$. Then

$$||(T+S)x|| \leq ||Tx|| + ||Sx|| \leq a||x|| + (1+b)||Tx|| \tag{2.40}$$

and ,

$$||(T+S)x|| \geq ||Tx|| - ||Sx|| \geq -a||x|| + (1-b)||Tx||$$

or equivalently

$$\begin{aligned} ||Tx|| &\leq (1-b)^{-1}(||(T+S)x|| + a||x||) \\ &= \frac{a}{(1-b)}||x|| + \frac{1}{(1-b)}||(T+S)x||. \end{aligned} \tag{2.41}$$

Thus the norms are equivalent. Now suppose T is closed, the spaces are complete and suppose $\exists \{x_n\} \subset D(T)$ such that $x_n \to x$ and $(T+S)x_n \to y \in Y$. Then, by (2.41), $\{Tx_n\}$ is Cauchy and converges in Y. Furthermore, since T is closed, $x \in D(T) = D(T+S)$ and $Tx_n \to Tx$. Thus, applying (2.40),

$$||(T+S)(x_n - x)|| \leq a||x_n - x|| + (1+b)||Tx_n - Tx|| \ \to \ 0.$$

It follows that $(T+S)x_n \to (T+S)x$.

Passing to the general case, we note that $(T+S)(0) = T(0)$ is closed, and that $Q_T T$ is closed. We also have that

$$||Q_T Sx|| \leq ||Q_S Sx|| = ||Sx|| \leq a||x|| + b||Tx|| = a||x|| + b||Q_T Tx||.$$

Thus, $Q_T S$ is single-valued and $T-$*bounded*, and, by what has already been shown, $Q_{T+S}(T+S) = Q_T T + Q_T S$ is closed. Applying Proposition 2.5.8, it follows that $T + S$ is closed.

◇

Corollary 2.8.7 *The norms $||\_||_T$ and $||\_||_{\lambda - T}$ are equivalent.*

PROOF

Since $||\lambda x|| \leq |\lambda|\,||x|| + b||-Tx||$ for any $b < 1$, the property follows by substituting λI for S in Proposition 2.8.6.

◇

2.9 Canonical Factorisation

The canonical factorisation of $T \in LR(X, Y)$ provides a single-valued inverse for T similar to the way that the product $Q_T T$ is a single-valued operator corresponding to T.

Definitions 2.9.1 *Let $T \in LR(X, Y)$. Then the* **injective component** *$\hat{T}$ of T is the map*

$$\hat{T} := T(Q^{X}_{N(T)})^{-1} \in LR(X/N(T), Y).$$

The represenatation $T = \hat{T}Q^{X}_{N(T)}$ is referred to as the **canonical factorisation** *of T.*

Clearly $\hat{T}[x] = Tx$ for $x \in D(T)$, where $[x] = x + N(T) \in X/N(T)$. We note that if $N(T)$ is not closed then $X/N(T)$ is not a normed linear space.

Proposition 2.9.2 *Let $T \in LR(X, Y)$. If $N(T)$ is closed, then*

(a) T is closed if and only if $\hat{T}$ is closed,

(b) $\gamma(T) = \gamma(\hat{T})$.

PROOF

We first note that

$$\hat{T}^{-1} = Q^{X}_{N(T)}T^{-1} = Q_{T^{-1}(0)}T^{-1}. \tag{2.42}$$

(a) Applying Proposition 2.5.8 to T^{-1}, it follows that from (2.42) above that T^{-1} is closed if and only if $\hat{T}^{-1}$ is closed.

(b) By (2.42) above and the definition of the norm quantity, we have

$$||\hat{T}^{-1}|| = ||Q_{T^{-1}(0)}T^{-1}|| = ||T^{-1}||. \tag{2.43}$$

Thus by (2.43) and Proposition 2.3.4, the desired equality holds.

◇

Corollary 2.9.3 *Let $T \in LR(X, Y)$. If $T(0)$ is closed, then $||T|| = ||\hat{T}||$.*

Substituting T with T^{-1} in Proposition 2.9.2, the desired equality follows from (2.43).

◇

2.10 Notes and Remarks

The material of this chapter is derived from Cross [35], Chapters I, II and III (references to original sources and research papers can be found there). For reference and completeness, we give proofs here as well - most of the proofs given are essentially the same as those provided by Cross, though minor differences occur in some arguments. We note, however, that the proof given for Proposition 2.8.4 (which is simpler than the arguments given in Cross [35], IV.3.10 and IV.3.11) and the generalisation given in Proposition 2.8.6 are due to the author. The definitions and proofs given in Section 2.9 are also due to the author - the definitions given here are equivalent to those in Cross [35], V.13; the proofs of the statements in Proposition 2.9.2 are simpler than the arguements given in [35], V.13.

The contents of Sections 2.1 to 2.6 form part of the basic toolbox for investigating linear relations, and are applied variously in the sequel. Properties on the nullity and deficiency introduced in Section 2.7 form a fundamental part of the theory of Fredholm relations. Results on the index are applied in Chapter 6 (remarks are also made in the introduction to Section 2.7). Properties in Section 2.8 on the graph operator and relative boundedness are also applied in Chapter 6 where some of the stabilty theorems are extended to classes of relatively bounded and relatively compact relations. Properties of the canonical factorisation of a linear relation are given in Section 2.9 for application in Chapters 6.

Chapter 3

Theorems for Multivalued Linear Operators

Three famous theorems are said to form the basis of functional analysis, namely the *Hahn-Banach theorem* (stated in Chapter 1), the *Uniform Boundedness principle* (also referred as the Banach-Steinhaus theorem), and the *Closed Graph / Open Mapping theorem.* In this chapter we give an operator version of the Baire property, and we illustrate this with a proof for multivalued linear operators. Both the Uniform Boundedness Principle and the Closed Graph/Open Mapping theorem are consequences of the Baire property. Furthermore, these results are closely related to one another. The former (or one of its corollaries) is sometimes given as a consequence(s) of the latter (see for example Goldberg [60], Wilansky [143] or Willard [144]). In our treatment, we give the Uniform Boundedness Principle as a simple consequence of the Baire property. In Section 3 we discuss the Closed Graph, Open Mapping and Closed Range theorems. In Section 4 we derive the *State Diagram* for linear relations from results contained in the earlier sections in this chapter. This construction summarises some of the relationships between properties of a linear relation and those of its adjoint. The *Small Perturbation* theorem, given in Section 5, serves as the basis for the perturbation and stability results of the index which are discussed Chapter 6. The basic theorem in this section only requires that the operator be open. The final section introduces a new class of linear relations, *Multivalued Linear Projections.* We derive conditions for the sums of closed subspaces in a Banach space to be closed by applying properties of continuous projections, the characterisation of multivalued projections in terms of pairs of subspaces combined with the closed graph and closed range theorems.

3.1 The Baire Category Theorem for Linear Relations

Definitions 3.1.1 *Let M be a subset of a topological space X. We let M° denote the* interior *of the set M, i.e.*

$$M^\circ := X \setminus (\overline{X \setminus M}).$$

If M is dense in some non-trivial open set, then M is said to be **somewhere dense,** *i.e.*

$$\overline{M}^\circ \neq \emptyset,$$

otherwise M is called **nowhere dense.**

A set is said to be of **first category** *if it is the union of a countable family of nowhere dense sets. Otherwise it is said to be of* **second category.**

Example 3.1.2 *$\mathbb{Q}$ is dense in $\mathbb{R}$, $\mathbb{R}$ is not dense in $\mathbb{R}^2$, $\mathbb{Q}$ is first category in $\mathbb{R}$, $\mathbb{Z}$ is first category in $\mathbb{R}$, but second category in itself.*

Letting $A = X \setminus M$, it follows that M is nowhere dense if and only if $\overline{A^\circ} = X$, i.e. a closed set is nowhere dense if and only if it's complement is open dense in X. We note equivalent characterisations of the Baire property:

Let X be a topological space, let $\{X_n\}$ be an arbitrary sequence of closed subsets of X, and let $A_n := X \setminus X_n$. The following are equivalent:

(i) X *is second category in itself.*

(ii) $X = \bigcup_{n \in \mathbb{N}} X_n \;\Rightarrow\; \exists\, n \in \mathbb{N}$ *such that* $X_n^\circ \neq \emptyset$.

(iii) $\forall\, n \in \mathbb{N},\; X_n^\circ = \emptyset \;\Rightarrow\; (\bigcup_{n \in \mathbb{N}} X_n)^\circ = \emptyset$.

(iv) $\forall\, n \in \mathbb{N},\; \overline{A_n} = X \;\Rightarrow\; \overline{(\bigcap_{n \in \mathbb{N}} A_n)} = X$.

The Baire category theorem states that a complete metric space is of second category in itself. Instead of considering an indexed family of dense subsets of a complete metric space, we consider a sequence of spaces linked by a sequence of continuous maps, each having dense range. The classic result, which proves that the intersection of a family of dense subsets is dense, then follows as a special case. The Baire theorem for linear relations, Theorem 3.1.6, is applied in Theorem 7.6.2 on the domain of iterates of a linear relation in Chapter 7.

Definitions 3.1.3 *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of non-empty sets, and let $(T_n)_{n \in \mathbb{N}}$ be a sequence of maps such that $T_n : X_n \to X_{n-1}$. Then the* **inverse limit** $\varprojlim(X_n, T_n)_{n \in \mathbb{N}}$ *is defined by:*

$$\varprojlim(X_n, T_n) := \{\ (x_n)_{n \in \mathbb{N}} \in \prod X_n \mid x_{n-1} \in T_n(x_n) \quad for \quad all \quad n \in \mathbb{N}\}.$$

For $k \in I\!N$, let π_k denote the k^{th} co-ordinate projection from $\prod X_n$ onto X_k. Then

$$\pi_k\left(\varprojlim(X_n,T_n)\right) \;:=\; \{\, x \in X_k \mid \exists (x_n)_{n\geq k} \in \prod_{n\geq k} X_n \;\; such \;\; that \;\; x_k = x \;\; and \;\; x_n \in T_{n+1}(x_{n+1}) \;\; \forall n \geq k \,\}.$$

Example 3.1.4 *If $(X_n)_{n\in I\!N}$ is a sequence of non-empty subsets of a set X such that $X_1 \supset X_2 \supset \dots$, and if $T_n : X_n \to X_{n-1}$ is the injection mapping for each n, then $\varprojlim(X_n,T_n)$ is isomorphic to $\bigcap_{n=1}^{\infty} X_n$. This identification follows easily from the followng equalities:*

$$\varprojlim(X_n,T_n) \;=\; \{(x_1,x_2,\dots) \mid x_n = x_{n-1} \;\forall\, n\,\} \;=\; \{(x_1,x_2,\dots) \mid x_1 \in \bigcap_{n=1}^{\infty} X_n\}.$$

The notion of the inverse limit of a sequence of sets arose out of questions about the topological properties of infinite products $\prod X_n$. Particularly, if each set X_n has the discrete topology, then the same does not necessarily hold for $\prod X_n$ (cf. Willard [144]).

Questions about the existence of **invariant subspaces** of linear operators led to the investigation of properties of sets of the form

$$F_x := \{\, x, Tx, T^2x, \dots \,\},$$

where T is a linear operator on a normed linear space X. The set F_x is referred to as the **orbit** of the point $x \in X$. The relevance of *invariant subspace problems* is discussed briefly in the concluding section of Chapter 7 on spectral theory.

Example 3.1.5 If $T \in L(X)$, $X_n := X$ and $T_n := T^n$, then set F_x may be identified with the point

$$(x, Tx, T^2x, \dots) \in \varprojlim(X_n,T_n) = \{\, (x_n)_{n\in I\!N} \mid x_n \in T^n(x_{n-1}) \;\; for \;\; all \;\; n \in I\!N\}.$$

Theorem 3.1.6 (The Baire property for Linear Relations) *Let $\{(X_n,\, \|\text{-}\|_n)\}_{n=0}^{\infty}$ be a sequence of Banach spaces and let $\{T_n\}_{n=1}^{\infty}$ be a sequence of continuous linear relations such that each T_n maps X_n into X_{n-1} with dense range and $T_n(0)$ is closed for each n. Then*

$$\bigcap_{n=1}^{\infty} T_1T_2\dots T_nX_n \quad is \;\; dense \;\; in \;\; (X_0,\, \|\text{-}\|_0) \tag{3.1}$$

PROOF

We first suppose $T_n(0) = X_{n-1}$ for some $n \in I\!N$. Let T_k be the first map in the sequence such that $T_k(0) = X_{k-1}$. Since $0 \in T_{k+1}T_{k+2}\dots T_{k+n}X_{k+n}$ for any $n \in I\!N$, it follows that for any $m \in I\!N$ such that $m \geq k$ we have :

$$T_1T_1\dots T_mX_m \;=\; T_1T_2\dots T_{k-1}X_{k-1}.$$

Hence,

$$\bigcap_{n=1}^{\infty} T_1T_2\dots T_nX_n \;=\; \bigcap_{n=1}^{k-1} T_1T_2\dots T_nX_n$$

which is clearly dense in $(X_0,\ ||\text{-}||_0)$.

Now suppose $T_n(0) \neq X_{n-1}$ for any $n \in I\!N$. For $m \leq n$ we abbreviate the composition of relations, $T_{m+1}T_{m+2}\dots T_n \in LR(X_n, X_m)$ as follows: define $f_{m,n} \in LR(X_n, X_m)$ by

$$f_{m,n} := \begin{cases} T_{m+1}T_{m+2}\dots T_n & if\ \ m < n \\ I_{X_m} & if\ \ m = n \end{cases} \tag{3.2}$$

We show that $\lim\limits_{\leftarrow}(X_n, T_n) \neq \emptyset$, and that given any $x_0 \in X_0$, we may construct a sequence $z_n \in X_n$ such that $z_0 \in \pi_0(\lim\limits_{\leftarrow}(X_n, T_n))$ is arbitrarily close to x_0.

Let $x_0 \in X_0$, let $\epsilon > 0$, and for $n \geq 1$, select $x_n \in X_n$ such that

$$d(\,x_{n-1}\ ,\ f_{n-1,n}(x_n)\,) \ <\ \frac{\epsilon.2^{-(n-1)}}{||T_1||\,||T_2||\dots||T_{n-1}||}$$

Fixing m, it follows that for $n \geq m$,

$$\begin{aligned} d(\,f_{m,n}(x_n)\ ,\ f_{m,n+1}(x_{n+1})\,) &= d(\,f_{m,n}(x_n)\ ,\ f_{m,n}(f_{n,n+1}(x_{n+1}))\,) \\ &\leq ||f_{m,n}||\,d(\,x_n\ ,\ f_{n,n+1}(x_{n+1})\,) \\ &< \frac{\epsilon.2^{-n}}{||T_1||\,||T_2||\dots||T_m||}. \end{aligned}$$

Furthermore, if $p \geq m$ and $q > 0$, then

$$\begin{aligned} & d(\,f_{m,p}(x_p)\ ,\ f_{m,p+q}(x_{p+q})\,) \\ \leq\ & ||f_{m,p}||\ d(\,x_p\ ,\ f_{p,p+q}(x_{p+q})\,) \\ \leq\ & ||f_{m,p}||\ [\ d(\,x_p\ ,\ f_{p,p+1}(x_{p+1})\,) \ +\ d(\,f_{p,p+1}(x_{p+1})\ ,\ f_{p,p+2}(x_{p+2})\,) \\ & \qquad +\ \dots\ +\ d(\,f_{p,p+q-1}(x_{p+q-1})\ ,\ f_{p,p+q}(x_{p+q})\,)\ ] \\ \leq\ & ||f_{m,p}||\ [\ d(\,x_p\ ,\ f_{p,p+1}(x_{p+1})\,) \ +\ ||f_{p,p+1}||\ d(\,x_{p+1}\ ,\ f_{p+1,p+2}(x_{p+2})\,) \\ & \qquad +\ \dots\ +\ ||f_{p,p+q-1}||\ d(\,x_{p+q-1}\ ,\ f_{p+q-1,p+q}(x_{p+q})\,)\ ] \\ <\ & ||f_{m,p}||\ [\ \frac{\epsilon.2^{-p}}{||T_1||||T_2||\dots||T_p||} \ +\ ||f_{p,p+1}||\,\frac{\epsilon.2^{-(p+1)}}{||T_1||\,||T_2||\dots||T_{p+1}||} \\ & \qquad +\ \dots\ +\ ||f_{p,p+q-1}||\ \frac{\epsilon.2^{-(p+q-1)}}{||T_1||\,||T_2||\dots||T_{p+q-1}||}\ ] \\ \leq\ & ||f_{m,p}||\ [\ \frac{\epsilon.2^{-p}}{||T_1||\,||T_2||\dots||T_p||} \ +\ \frac{\epsilon.2^{-(p+1)}}{||T_1||\,||T_2||\dots||T_p||} \\ & \qquad +\ \dots\ +\ \frac{\epsilon.2^{-(p+q-1)}}{||T_1||\,||T_2||\dots||T_p||}\ ] \\ \leq\ & \epsilon\,\frac{\sum_{i=p}^{p+q-1} 2^{-i}}{||T_1||\dots||T_m||}. \end{aligned}$$

In particular, for $m = p$ and $n > m$,

$$d(\, x_m \;,\; f_{m,n}(x_n)\,) \;<\; \epsilon \frac{\sum_{i=m}^{n} 2^{-i}}{||T_1|| \ldots ||T_m||}.$$

Let $\delta_{m,n} := \epsilon \frac{\sum_{i=m}^{n} 2^{-i}}{||T_1||\ldots||T_m||}$, and for each $n > m$, choose $z_{m,n} \in f_{m,n}(x_n) \subset X_m$ such that

$$||x_m - z_{m,n}|| < \delta_{m,n} + \frac{\epsilon}{||T_1||\ldots||T_m||}.$$

Then it follows from the above that $\{z_{m,n}\}_{n>m}$ is a Cauchy sequence, and since X_m is complete, $\{z_{m,n}\}_{n>m}$ converges to some $z_m \in X_m$. Furthermore, $||x_m - z_m|| < \epsilon.2.(||T_1|| \ldots ||T_m||)^{-1}$.

Now for $n \geq m+1$ we have

$$\begin{aligned} z_{m,n} \in f_{m,n}(x_n) &= f_{m,m+1}(\, f_{m+1,n}(x_n)\,) \\ &= f_{m,m+1}(\, z_{m+1,n} + f_{m+1,n}(0)\,) \\ &= f_{m,m+1}(z_{m+1,n}). \end{aligned}$$

Moreover, since $z_{m,n} \to z_m$, it follows that

$$d(\, z_m, f_{m,m+1}(z_{m+1,n})\,) \to 0.$$

Similarly, $z_{m+1,n} \to z_{m+1}$, and since $f_{m,m+1}$ is continuous, it follows that

$$d(\, z_m, f_{m,m+1}(z_{m+1})\,) = 0,$$

i.e.

$$z_m \in \overline{f_{m,m+1}(z_{m+1})}.$$

Since $f_{m,m+1}(0)$ is closed, it follows that

$$z_m \in f_{m,m+1}(z_{m+1})$$

Since m was fixed arbitrarily, it follows that

$$(z_0, z_1, \ldots) \;\in\; \varprojlim(X_n, f_{n-1,n}) \;=\; \varprojlim(X_n, T_n).$$

Thus, $z_0 \in \bigcap_{n=1}^{\infty} T_1T_2\ldots T_nX_n$, and, since ϵ was arbitrary,

$$\bigcap_{n=1}^{\infty} T_1T_2\ldots T_nX_n \text{ is dense in } X_0.$$

$\diamond$

Corollary 3.1.7 *Let* $\{(X_n,\ ||\text{-}||_n)\}_{n=0}^{\infty}$ *be a sequence of Banach spaces, and* $\{T_n\}_{n=1}^{\infty}$ *be a sequence of continuous everywhere-defined linear relations such that each* T_n *maps* X_n *into* X_{n-1} *with dense range, and* $T_n(0)$ *is closed for each* n*. Let* G_n *be a sequence of subspaces such that each* G_n *is dense in* X_n*. Then*

$$\Gamma := G_0 \cap \bigcap_{n=1}^{\infty} T_1T_2\ldots T_nG_n \quad \textit{is dense in} \quad (X_0,\ ||\text{-}||) \tag{3.3}$$

PROOF

First, let $A_0 := G_0$ and define $A_n := G_n \cap T_n^{-1}(A_{n-1})$. Then $T_n(A_n) \subset A_{n-1}$ for all n. Furthermore, since T_n is continuous for each $n \in I\!N$, it follows that T_nA_n is dense in $\overline{A_{n-1}}$ for each $n \in I\!N$. Thus, by Theorem 3.1.6,

$$\bigcap_{n=1}^{\infty} T_1T_2\ldots T_n\overline{A_n} \text{ is dense in } A_0.$$

Furthermore, since $A_n \subset G_n$ for all $n \in I\!N$, and A_0 is dense in X_0, it follows that Γ is dense in X_0.

◇

Corollary 3.1.8 *Let $\{(X_n,\ d_n)\}_{n=0}^{\infty}$ be a sequence of Banach spaces, and $\{T_n\}_{n=1}^{\infty}$ be a sequence of continuous linear relations such that T_n maps X_n into X_{n-1} with dense range, and $T_n(0)$ is closed for each $n \in I\!N$. Then $\forall k \in I\!N$*

$$\pi_k(\underleftarrow{\lim}(X_n, T_n)) \text{ is dense in } X_k.$$

◇

Theorem 3.1.6 is a topological result and can be proved in terms of open balls rather than the derivation given above which used properties of the norm. In Lennard [91], this method was used to prove the same theorem for a sequence of single-valued operators acting on complete metric spaces. The requirement that the operators be linear is not necessary in that approach, and hence, considering dense subsets rather than dense subspaces, Theorem 3.1.9 below, the Baire Category Theorem for Banach spaces, which is stated here without proof, could be deduced as a corollary.

Theorem 3.1.9 (The Baire Category theorem for normed linear spaces) *A Banach space X is second category in itself.*

3.2 The Uniform Boundedness Principle

Theorem 3.2.1 (The Uniform Boundedness Principle) *Let $\{T_\lambda \in LR(X, Y_\lambda) \mid \lambda \in \Lambda\}$ be an indexed family of everywhere defined continuous linear relations from a Banach space X into normed spaces Y_λ, $\lambda \in \Lambda$. If for each $x \in X$ we have*

$$\sup_{\lambda\in\Lambda} ||T_\lambda x|| < \infty,$$

then

$$\sup_{\lambda\in\Lambda} ||T_\lambda|| < \infty.$$

PROOF

Let $A_n := \{x \in X \mid ||T_\lambda x|| \leq n,\ \lambda \in \Lambda\}$. Now $X = \bigcup_{n \in I\!N} A_n$. Thus, by the Baire Category theorem, there exists $k \in I\!N$ such that A_k, which is a closed set, contains a non-empty open ball. Let $B(a, r) \subset A_k$ be a ball with radius $r > 0$ and centred at a. Then for $x \in X$, $||x|| < r$, we have $x + a \in B(a, r)$. Thus, for each $\lambda \in \Lambda$,

$$||T_\lambda x|| = ||T_\lambda(x + a - a)|| \leq ||T_\lambda(x + a)|| + ||T_\lambda a|| \leq 2k.$$

It follows that $||T_\lambda|| \leq 2k/r$ for all $\lambda \in \Lambda$.

◇

The following example illustrates that the assumption of completeness cannot be omitted from Theorem 3.2.1.

Example 3.2.2 *Let c_{00} denote the collection of real-valued sequences $x = \{x_n\}$ such that $x_n \neq 0$ for at most a finite number of $n \in I\!N$, and $||x|| := \sup_n |x_n|$. Let $\{t_n\}$ be a sequence of linear functionals defined by $t_n(x) := nx_n$. Then $\{t_n\}$ is pointwise bounded and $\lim_n t_n(x) \to 0$ for each $x \in X$. However, $||t_n|| = n$ for each $n \in I\!N$.*

The next corollary, also called the Banach-Steinhaus (closure) theorem, is sometimes given as an alternative form of the Uniform Boundedness principle (the latter may be derived from the former).

Corollary 3.2.3 *Let X be complete and let $\{T_n\} \subset LR(X, Y)$ be a sequence of everywhere defined continuous linear relations. If $T \in LR(X, Y)$ is an everywhere defined relation such that $T_n(0) \subset \overline{T(0)}$ for each $n \in I\!N$ and $\lim_{n \to \infty} ||T_n x - Tx|| = 0$ for all $x \in X$, then T is continuous.*

The next example shows that the condition, $T_n(0) \subset \overline{T(0)}$ for each $n \in I\!N$, is necessary in Corollary 3.2.3.

Example 3.2.4 *Let X be a nonzero normed space, let $T \in LR(X)$ be an unbounded linear relation and suppose $R(T_n) = X$ for all n. Then for $x \neq 0$,*

$$||Tx - T_n x|| = 0.$$

Completeness is not required in the next corollary.

Corollary 3.2.5 *Let X be a normed linear space. Suppose that for each $x' \in X'$, $W \subset X$ satisfies*

$$\sup_{w \in W} |x'w| < \infty.$$

Then W is bounded.

PROOF

The result follows from Theorem 3.2.1 with W considered as a subset of X''.

◇

3.3 The Closed Graph, Open Mapping and Closed Range Theorems

In Proposition 2.5.2 we showed that a continuous linear relation T, with closed domain and $T(0)$ closed, is closed. The Closed Graph theorem (Theorem 3.3.7 below) gives a partial converse of this property. The Closed Range Theorem is closely related to this result and, in the setting of linear relations, the Open Mapping and Closed Domain theorems are trivially equivalent to the Closed Graph and Closed Range theorems, respectively. These theorems apply to closed relations defined on complete spaces. Proposition 3.3.2 and its Corollary 3.3.3 are consequences of the Uniform Boundedness principle, but hold more generally, i.e. the relation need not be closed, nor is it necessary that the spaces be complete.

We note that the Closed Graph and Closed Range theorems are well-known for single-valued operators on general topological linear spaces. We prove results for multivalued linear relations in the setting of normed spaces. In [13], the authors prove the Closed Graph theorem for convex processes.

Notation 3.3.1 *We let $\tilde{X}$ denote the completion of a normed linear space X, and if $T \in LR(X,Y)$, then $\tilde{T}$, defined by $G(\tilde{T}) := (G(T))^{\sim}$, denotes the completion of T.*

In the Closed Graph / Open Mapping and Closed Range / Closed Domain Theorems, it is assumed that the spaces are complete and that the operator is closed. Analogous properties follow by passing to the completion of the spaces and of the operator. Thus, more generally, we show that :

$$R(\tilde{T}) \text{ is closed } \Leftrightarrow R(\tilde{T}') \text{ is closed } \Leftrightarrow R(\tilde{T}') \text{ is weak*-closed,}$$
$$D(\tilde{T}) \text{ is closed } \Leftrightarrow D(\tilde{T}') \text{ is closed } \Leftrightarrow D(\tilde{T}') \text{ is weak*-closed,}$$

and

$$\tilde{T} \text{ is open } \Leftrightarrow R(\tilde{T}) \text{ is closed,}$$
$$\tilde{T} \text{ is continuous } \Leftrightarrow D(\tilde{T}) \text{ is closed.}$$

Proposition 3.3.2 *Let $T \in LR(X,Y)$. Then*

(a) T is continuous if and only if $D(T') = T(0)^{\perp}$.

(b) T is open if and only if $R(T') = N(T)^{\perp}$.

PROOF
(a) By Proposition 2.6.14, we only need to prove the reverse implication. We first assume that T is single-valued. Then $D(T') = T(0)^{\perp} = Y'$, and hence, for any $y' \in Y'$ we have

$$\begin{aligned} \sup_{y \in T(U_{D(T)})} |y'y| &= \sup_{x \in U_{D(T)}} |y'Tx| \\ &\leq \|y'T\| \\ &< \infty. \end{aligned}$$

By Corollary 3.2.5, $T(U_{D(T)})$ is bounded, and hence, $||T|| < \infty$. More generally, if T is multivalued, then the result follows from the above and the equivalence $||QT|| = ||T||$.

Clearly (b) is equivalent to (a).

$\diamond$

Corollary 3.3.3 *Let $T \in LR(X,Y)$ be closed. Then*

(a) T is continuous if and only if $D(T')$ is weak-closed.*

(b) T is open if and only if $R(T')$ is weak-closed.*

PROOF

We need only verify (a). By Proposition 3.3.2, $D(T') = T(0)^{\perp} = D(T')^{\top\perp} = \overline{D(T')}^{*}$.

$\diamond$

Lemma 3.3.4 *Let $T \in LR(X,Y)$.*

(a) If T' is open, then $\overline{TU_{D(T)}} \supset \gamma(T')U_{R(T)}$.

(b) If T' is continuous, then $||T'||\overline{T^{-1}U_{R(T)}} \supset U_{D(T)}$.

PROOF

We note that (b) follows from (a) by replacing T with T^{-1}. To prove (a), let $\gamma := \gamma(T')$. We first assume that T' is injective. Suppose $y \in U_{R(T)}(0,\gamma)$, $y \notin \overline{TU_{D(T)}}$. Since $\overline{TU_{D(T)}}$ is closed and convex, by Theorem 1.6.3, there exists $y' \neq 0$, $y' \in Y'$ such that

$$Re\, y'y \;\geq\; Re\, y'k \;\; \forall\, k \in \overline{TU_{D(T)}}.$$

Now if $x \in U_{D(T)}$ then, using polar form, $y'Tx = |y'Tx|e^{i\theta}$. Since $e^{-i\theta}x \in U_{D(T)}$, it follows that

$$Re\, y'y \;\geq Re\, y'T(e^{i\theta}x) = |y'Tx|$$

for all $x \in U_{D(T)}$. In particular, if $x = 0$, then $|y'T(0)| \leq Re\, y'y$, and hence, it must be the case that $y'T(0) = 0$. Thus $y' \in D(T')$, and, since $(T')^{-1}$ is single-valued,

$$||y'||\,||y|| \;\geq\; |y'y| \;\geq\; \sup_{x \in U_{D(T)}} |y'Tx| \;=\; ||T'y'|| \;\geq\; \gamma||y'||.$$

It follows that $||y|| \geq \gamma$, which is a contradiction.

Suppose T' is not injective. Let $S \in LR(X, \overline{R(T)})$ be defined $Sx := Tx$. Then S' is injective since $\{0\}_{\overline{R(T)}'} = \overline{R(T)}^{\perp} = N(S')$, where $\overline{R(T)}^{\perp}$ is considered as a subset of $\overline{R(T)}'$. Furthermore, $T = J_{\overline{R(T)}}S$ and identifying spaces up to isomorphism, $Y'/\overline{R(T)}^{\perp} = \overline{R(T)}'$ and $T' = S'Q^{Y'}_{\overline{R(T)}^{\perp}} = S'Q^{Y'}_{N(T')}$ (Proposition 1.7.9). Thus, by Proposition 2.9.2, $S' = T'(Q^{Y'}_{N(T')})^{-1}$ is open and $\gamma(S') = \gamma(T')$. Thus, by what has already been proved,

$$\overline{SU_{D(T)}} \;\supset\; \gamma(S')U_{R(S)},$$

and the result follows.

◇

Lemma 3.3.5 *Let X be a normed linear space. If $K \subset X$ is a convex set satisfying $K = -K$, and the interior of K is nonempty, then the point 0 (zero) is also an interior point in K.*

PROOF

By the hypothesis, there exists $u \in K$ and $\lambda > 0$ such that $\lambda U_X + u \subset K$. Choose $v \in X$ such that $||v|| < 2\lambda$. Then

$$\begin{aligned} v &= (u + \frac{v}{2}) - (u - \frac{v}{2}) \\ &\in U_X(u, \lambda) - U_X(u, \lambda) \\ &\subset K + K = 2K \end{aligned}$$

since K is convex. Thus $U_X(0, 2\lambda) \subset 2K$, from which it follows that $U_X(0, \lambda) \subset K$.

◇

Lemma 3.3.6 *Let $T \in LR(X, Y)$ be closed, and single-valued.*

(a) If Y is a Banach space then

$$\lambda U_{D(T)} \subset \overline{T^{-1}B_{R(T)}} \quad \Rightarrow \quad \lambda U_{D(T)} \subset T^{-1}B_{R(T)}. \tag{3.4}$$

(b) If X is a Banach space then

$$\lambda U_{R(T)} \subset \overline{TB_{D(T)}} \quad \Rightarrow \quad \lambda U_{R(T)} \subset TB_{D(T)}. \tag{3.5}$$

PROOF

(a) Let $V := T^{-1}B_{R(T)} = \{x \in X \mid ||Tx|| \leq 1\}$, and assume $U_{D(T)}(0, \lambda) \subset \overline{V}$. Then

$$U(0, a\lambda) \subset \overline{aV}, \quad a > 0.$$

Let $0 < \epsilon < 1$. If $x \in X$ and $||x|| < \lambda$, then $x \in \overline{V}$ and there exists $x_1 \in V$ such that $||x - x_1|| < \epsilon\lambda$ and

$$x - x_1 \in U(0, \epsilon\lambda) \subset \overline{\epsilon V}.$$

Thus there exists $x_2 \in \epsilon V$ such that $||x - x_1 - x_2|| < \epsilon^2\lambda$. By induction, we may construct a sequence $(x_n)_{n \in \mathbb{N}}$ such that

$$||x - \sum_{i=1}^{n} x_i|| < \epsilon^n\lambda, \quad x_i \in \epsilon^{i-1}V. \tag{3.6}$$

Let $s_n := \sum_{i=1}^{n} x_i$. Then by (3.6) and the definition of V, it follows that $s_n \to x$ and

$$||Tx_n|| < \epsilon^{n-1}.$$

For $n > m$, we have

$$||Ts_n - Ts_m|| \leq \sum_{i=m+1}^{\infty} ||Tx_i|| \leq \sum_{i=m+1}^{\infty} \epsilon^{i-1} = \frac{\epsilon^m}{1-\epsilon} \to 0$$

as $m \to \infty$. Thus $(Ts_n)_{n \in I\!N}$ is a Cauchy sequence, and $Ts_n \to y$ for some $y \in Y$. Since T is closed, it follows that $Tx = y$ and

$$||Tx|| = ||y|| \leq \sum_{i=1}^{\infty} ||Tx_i|| \leq \frac{1}{1-\epsilon},$$

provided $||x|| < \lambda$. Since $\epsilon > 0$ is arbitrary, it follows that $x \in V$.

(b) The implication, (3.5) follows from (3.4) of (a) by replacing T with T^{-1}. $\diamond$

By Propositions 2.2.11 and 2.3.3, the right hand sides of (3.4) and (3.5) are geometric characterisations of continuity and openess, respectively.

Theorem 3.3.7 (The Closed Graph and Open Mapping theorems for Linear Relations) *Let X and Y be Banach spaces, and let $T \in LR(X,Y)$ be closed.*

(a) T is continuous if and only if $D(T)$ is closed.

(b) T is open if and only if $R(T)$ is closed.

PROOF

Suppose T is continuous. Let $\{x_n\} \subset D(T)$ be a sequence such that $x_n \to x,\ \ x \in X$. Since QT is continuous, QTx_n is a Cauchy sequence and hence, converges to some $y \in Y/T(0)$. Since QT is closed, $(x, y) \in G(QT)$, and hence, $x \in D(T)$.

To see that the converse holds, we first suppose T is single-valued. Without loss of generality, we may assume that $D(T) = X$. Let $V := T^{-1}B_{R(T)}$. Since T is linear, $X = \bigcup_{n=1}^{\infty} nV$ and, by the Baire Category theorem, there exists $k \in I\!N$ such that $k\overline{V} = \overline{kV}$ has an interior point. It follows that $\overline{V}$ has a interior point. Furthermore, $\overline{V}$ is convex and satisfies $\overline{V} = -\overline{V}$. Thus, by Lemma 3.3.5, there exists $\lambda > 0$ such that $U(0,\lambda) \subset \overline{V}$. Thus, it follows from Lemma 3.3.6 that for any point $x \in D(T)$, the inverse image of a neighbourhood Tx is a neighbourhood of x.

Passing to the the general case, we have that QT is closed and $D(QT) = D(T)$. By what has already been proved, the claim holds for QT. Thus, $||T|| = ||QT|| < \infty$ and T is continuous.

Clearly the statement in (b) is equivalent to the one in (a).

$\diamond$

Theorem 3.3.8 (Closed Range theorem for multivalued linear operators) *Let X and Y be Banach spaces and let $T \in LR(X,Y)$ be closed. The following are equivalent:*

(i) $R(T)$ is closed.

(ii) $R(T')$ is closed.

(iii) $R(T')$ is weak-closed.*

PROOF

Clearly (iii) $\Rightarrow$ (ii).

(i)$\Rightarrow$ (ii)

By Theorem 3.3.7, if $R(T)$ is closed then T is open. By Proposition 2.6.13, this implies T' is open. The desired implication follows from Theorem 3.3.7 applied to T'.

(i)$\Leftrightarrow$ (iii)

By Theorem 3.3.7, $R(T)$ is closed if and only if T is open. By Proposition 3.3.2, T is open if and only if $R(T') = N(T)^{\perp}$. Since $R(T')^{\top} = N(T)$ (Proposition 2.6.5), it follows that T is open if and only if $R(T') = R(T')^{\top\perp} = \overline{R(T')}^{*}$.

(ii)$\Rightarrow$ (i)

By Theorem 3.3.7, T' is open. By Lemma 3.3.4, $\overline{TB_{D(T)}} \supset \gamma(T')U_{R(T)}$. Since T is closed, it follows from Lemma 3.3.6 and Propositions 2.3.3 and 2.3.5 that T is open. Hence, it follows from another application of Theorem 3.3.7 that $R(T)$ is closed.

$\diamond$

Theorem 3.3.9

(a) If X is complete, then $\gamma(\tilde{T}) = \gamma(T')$. Furthermore, if $\gamma(\tilde{T}) > 0$, then $R(\tilde{T})$ is closed.

(b) If Y is complete, then $||\tilde{T}|| = ||T'||$. Furthermore, if $||\tilde{T}|| < \infty$, then $D(\tilde{T})$ is closed.

PROOF

(a) If T' is not open, then, by Proposition 2.6.13, $\gamma(T) \leq \gamma(T') = 0$. If T' is open, then, by the Open Mapping and Closed Range Theorems, 3.3.7 and 3.3.8, respectively, $R(\tilde{T})$ is closed, and furthermore, by Proposition 2.3.3, there exists $\lambda > 0$ such that

$$\tilde{T}B_X \supset \lambda B_{R(\tilde{T})}. \tag{3.7}$$

Without loss of generality, assume that T is closed. Now if $x \in B_X$ and $(x, y) \in G(\tilde{T})$ such that $y \in Y$, then there is a sequence $(x_n, y_n) \in G(T)$ such that $(x_n, y_n) \to (x, y)$, and hence $(x, y) \in X \times Y$ since X is complete. We have shown that

$$\tilde{T}B_X \cap Y \subset TB_{D(T)}.$$

Furthermore, since T is closed, it follows that $x \in B_{D(T)}$ and $y \in TB_{D(T)}$. Thus,

$$R(\tilde{T}) \cap Y \supset R(T) = \bigcup_{n=1}^{\infty} nTB_{D(T)} \supset (\bigcup_{n=1}^{\infty} n\tilde{T}B_X) \cap Y = R(\tilde{T}) \cap Y,$$

and hence, $R(T)$ is closed. By taking intersections of both sides of (3.7) with $R(T)$, it follows that

$$TB_{D(T)} \supset \lambda B_{R(T)}. \tag{3.8}$$

Applying Proposition 2.3.3 again, it follows that T is open, and hence, by Proposition 2.6.14, $\gamma(T) = \gamma(T')$.

(b) follows from (a) by substituting T^{-1} for T and applying Proposition 2.3.4.

$\diamond$

3.4 The State Diagram of Linear Relations

State diagrams for unbounded and/or closed linear operators were compiled for single-valued operators by Goldberg [60] as summaries of some of the relationships which exist between a linear relation T and its adjoint T'. M. Möller extended the diagrams to linear relations(M.Sc. dissertation, 1976, cf. Cross [35]), and Cross showed that similar diagrams hold for the *essential states* of linear relations (see Theorem 5.8.4).

Definition 3.4.1 *A linear relation $T \in LR(X,Y)$ is classified according to the following states:*

$$\begin{aligned} I &: \; R(T) = Y \\ II &: \; R(T) \neq Y \text{ but } \overline{R(T)} = Y \\ III &: \; \overline{R(T)} \neq Y \end{aligned}$$

$$\begin{aligned} 1 &: \; T^{-1} \text{ is single-valued and continuous} \\ 2 &: \; T^{-1} \text{ is single-valued but is not continuous} \\ 3 &: \; T^{-1} \text{ is not single-valued} \end{aligned}$$

As examples, if $R(T) = Y$ then T is said to be in state I, written $T \in I$. Similarly, $T \in 3$ means that T is in state 3, i.e. T is not injective. If for example $T \in I$ and $T \in 3$, then we write $T \in I_3$. The same classification and corresponding notation is applied to T'. If for example we have that $T \in I_3$ and $T' \in III_1$ then we write $(T,T') \in (I_3, III_1)$. The implications and equivalences of Proposition 3.4.2 are summarised in a table of Theorem 3.4.3.

Proposition 3.4.2 *Let $T \in LR(X,Y)$. Then*

(a) $T \in 1 \Leftrightarrow T' \in I$.

(b) $T' \in 1 \Rightarrow T' \notin II$.

(c) $T \in I_1 \cup II_1 \Leftrightarrow T' \in I_1$.

(d) $T \in III \Leftrightarrow T' \in 3$.

(e) $T \in 3 \Rightarrow T' \in III$.

(f) If Y is complete and $T \in I$, then $T' \in 1$.

PROOF

(a)
$$\begin{aligned} T \in 1 &\Leftrightarrow R(T') = N(T)^{\perp} = X' \quad (\textit{ Proposition } 3.3.2 \text{ }) \\ &\Leftrightarrow T' \in I \end{aligned}$$

(b)
$$\begin{aligned} T' \in 1 &\Rightarrow R(T') \textit{ is closed} \quad (\textit{ Theorem } 3.3.7 \text{ }) \\ &\Rightarrow T' \notin II. \end{aligned}$$

(c)

$$
\begin{aligned}
T \in I_1 \cup II_1 \;&\Leftrightarrow\; T' \in I \;\; and \;\; R(T)^{\perp} = \{0\} \;\; (\; applying \;\; part(a) \;) \\
&\Leftrightarrow\; T' \in I \;\; and \;\; N(T') = \{0\} \;\; (\; Proposition \; 2.6.5 \;) \\
&\Leftrightarrow\; T' \in I_1 \;\; (\; Theorem \; 3.3.7 \;)
\end{aligned}
$$

(d)

$$
\begin{aligned}
T \in III \;&\Leftrightarrow\; R(T)^{\perp} = N(T') \neq \{0\} \;\; (\; Proposition \; 2.6.5 \;) \\
&\Leftrightarrow\; T' \in 3
\end{aligned}
$$

(e)

$$
\begin{aligned}
T \in 3 \;&\Rightarrow\; \{0\} \neq N(T) \subset R(T')^{\top} \;\; (\; Proposition \; 2.6.5 \;) \\
&\Rightarrow\; T' \in III
\end{aligned}
$$

(f) Suppose Y is complete. Then

$$
\begin{aligned}
T \in I \;&\Rightarrow\; R(\tilde{T}) = Y \\
&\Rightarrow\; N(T') = \{0\} \;\; and \;\; R(T') \;\; is \;\; closed \\
&\qquad (\; Proposition \; 2.6.5 \;\; and \;\; Theorem \; 3.3.8, \;\; respectively \;) \\
&\Rightarrow\; T' \in 1 \;\; (\; Theorem \; 3.3.7 \;)
\end{aligned}
$$

◇

Theorem 3.4.3 The State Diagram for Linear Relations

<table>
<tr><td></td><td>III_3</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td></td><td></td></tr>
<tr><td></td><td>III_2</td><td>||||||</td><td>Y</td><td>Y</td><td>||||||</td><td></td><td></td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td></td><td>III_1</td><td>||||||</td><td></td><td></td><td>||||||</td><td></td><td></td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td></td><td>II_3</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td></td><td>||||||</td></tr>
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<tr><td></td><td>II_1</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td></td><td>I_3</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td></td><td>||||||</td><td>||||||</td></tr>
<tr><td>↑</td><td>I_2</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td>T'</td><td>I_1</td><td></td><td>||||||</td><td>||||||</td><td></td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td></td><td></td><td>I_1</td><td>I_2</td><td>I_3</td><td>II_1</td><td>II_2</td><td>II_3</td><td>III_1</td><td>III_2</td><td>III_3</td></tr>
<tr><td></td><td></td><td>$T \rightarrow$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

Y : this state cannot occur if Y is complete

Proposition 3.4.4 *Let X be complete, and let $T \in LR(X,Y)$ be closed.*

(a) $T' \in I_1 \Rightarrow T \in I_1$.

(b) $T' \in III_1 \Rightarrow T \in I_3$.

(c) If X is reflexive and T is injective, then $R(T')$ is dense in X'.

PROOF

(a) Suppose $T' \in I_1$. By the State Diagram for linear relations Theorem 3.4.3, $T' \in I_1 \Leftrightarrow T \in I_1 \cup II_1$. By Theorem 3.3.9, $\gamma(T) = \gamma(T') > 0$ and $R(T)$ is closed. Hence, $T \in I_1$.

(b) $T' \in III_1 \Leftrightarrow N(T') = \{0\}$, T' is open and $R(T') \neq X'$. Thus, if $T' \in III_1$, then it follows from Theorem 3.3.9 that $\gamma(T) = \gamma(T') > 0$ and $R(T)$ is closed. Hence, by Proposition 2.6.5, $R(T) = R(T)^{\perp\top} = N(T')^{\top} = Y$ and $N(T) = R(T')^{\top} \neq \{0\}$.

(c)
$$
\begin{aligned}
N(T) = \{0\} \quad &\Rightarrow \quad \{0\}^{\perp} = N(T)^{\perp} = \overline{R(T')}^{*} \quad (\textit{ Proposition } 2.6.5 \text{)} \\
&\Rightarrow \quad \overline{R(T')} = X \quad \textit{since } X \textit{ is reflexive}
\end{aligned}
$$

◇

Theorem 3.4.5 The State Diagram for Closed Linear Relations

<table>
<tr><td></td><td>III_3</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>Xrc</td><td></td></tr>
<tr><td></td><td>III_2</td><td>||||||
					</td><td>Y Xrc</td><td>Y Y</td><td>					
					</td><td>Xrc</td><td></td><td>					
					</td><td>					
					</td><td>					
					</td></tr>
<tr><td></td><td>III_1</td><td>||||||</td><td>Xc</td><td></td><td>||||||</td><td>Xc</td><td>Xc</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td></td><td>II_3</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td></td><td>||||||</td></tr>
<tr><td></td><td>II_2</td><td>||||||</td><td>Y</td><td>||||||</td><td>||||||</td><td></td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td></td><td>II_1</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td></td><td>I_3</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td></td><td>||||||</td><td>||||||</td></tr>
<tr><td>$\uparrow$</td><td>I_2</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td>T'</td><td>I_1</td><td></td><td>||||||</td><td>||||||</td><td>Xc</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td></td><td></td><td>I_1</td><td>I_2</td><td>I_3</td><td>II_1</td><td>II_2</td><td>II_3</td><td>III_1</td><td>III_2</td><td>III_3</td></tr>
<tr><td></td><td></td><td>$T \rightarrow$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

Y : this state cannot occur if Y is complete

Xc : this state cannot occur if X is complete and T is closed

Xrc : this state cannot occur if X is complete and reflexive and T is closed

In Goldberg [60], examples are given to illustrate the states (T, T') which can occur for single-valued linear operators. These can be extended to linear relations. Noting that $(T^{-1})' = (T')^{-1}$, one may also derive the *Inverse State Diagram* from the above, by replacing the operator T with it's inverse T^{-1}. In this case the states for a linear relation $T \in LR(X, Y)$ may be given

$$\begin{aligned} I &: \ D(T) = X \\ II &: \ D(T) \neq X \ \textit{but} \ \overline{D(T)} = X \\ III &: \ \overline{D(T)} \neq X \end{aligned}$$

$$\begin{aligned} 1 &: \ T \ \textit{is single-valued and continuous} \\ 2 &: \ T \ \textit{is single-valued but is not continuous} \\ 3 &: \ T \ \textit{is not single-valued} \end{aligned}$$

Theorem 3.4.6 The Inverse State Diagram for Linear Relations

<table>
<tr><td></td><td>III_3</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>Yrc</td><td></td></tr>
<tr><td></td><td>III_2</td><td>||||||
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					</td><td>Yrc</td><td></td><td>					
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					</td><td>					
					</td></tr>
<tr><td></td><td>III_1</td><td>||||||</td><td>Yc</td><td></td><td>||||||</td><td>Yc</td><td>Yc</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td></td><td>II_3</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td></td><td>||||||</td></tr>
<tr><td></td><td>II_2</td><td>||||||</td><td>X</td><td>||||||</td><td>||||||</td><td></td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td></td><td>II_1</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td></td><td>I_3</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td></td><td>||||||</td><td>||||||</td></tr>
<tr><td>$\uparrow$</td><td>I_2</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td>T'</td><td>I_1</td><td></td><td>||||||</td><td>||||||</td><td>Yc</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td></td><td></td><td>I_1</td><td>I_2</td><td>I_3</td><td>II_1</td><td>II_2</td><td>II_3</td><td>III_1</td><td>III_2</td><td>III_3</td></tr>
<tr><td></td><td></td><td>$T \rightarrow$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

X : this state cannot occur if X is complete

Yc : this state cannot occur if Y is complete and T is closed

Yrc : this state cannot occur if Y is complete and reflexive and T is closed

3.5 The Small Perturbation Theorem

We use the following fundamental lemma, due to Krein, Krasnosel'skii and Milman, which gives a geometric property of normed linear spaces. It applies Borsuk's antipodal mapping theorem.

Lemma 3.5.1 *Let M, N be subspaces of X with $dimM > dimN$. Then $\exists m \in M$, $m \neq 0$ such that $||m|| = d(m, N)$.*

If we take X to be $I\!R^2$ and M and N to be non-perpendicular lines through the origin, we see that the lemma need not apply when $dimM = dimN$.

Theorem 3.5.2 *Let $T \in LR(X,Y)$ with $\gamma(T) > 0$. Suppose $S \in LR(X,Y)$ satisfies $D(S) \supset D(T)$, $S(0) \subset \overline{T(0)}$ and $||S|| < \gamma(T)$. Then*

(a) $\alpha(T+S) \leq \alpha(T)$

(b) $\bar{\beta}(T+S) \leq \bar{\beta}(T)$

PROOF

(a) We may clearly assume that $\alpha(T+S) > 0$ and choose $x \in N(T+S)$, $x \neq 0$. Then

$$\begin{aligned}
\gamma(T)d(x, N(T)) &\leq ||Tx|| = ||QTx|| \quad \textit{where } Q := Q_T \\
&= ||QSx|| \qquad \textit{since } Q(Tx+Sx) = 0 \\
&\leq ||S||\,||x|| \\
&< \gamma(T)||x||.
\end{aligned}$$

Since the choice of x was arbitrary, we have shown that

$$d(x, N(T)) < ||x|| \quad \forall x \in N(T+S), x \neq 0.$$

Thus, by Lemma 3.5.1, $\alpha(T+S) \leq \alpha(T)$.

(b) Now, from the properties of adjoints, we have $\gamma(T') = \gamma(T)$, $||S'|| = ||S||$, and $T'+S' = (T+S)'$. By Proposition 2.6.5, $T'(0) \supset S'(0)$. Replacing X by $D(S)$ if necessary, we may assume that S' is single-valued. Applying Proposition 2.7.6 and (a) yields the desired result, i.e.

$$\bar{\beta}(T+S) = \alpha(T'+S') \leq \alpha(T') \leq \bar{\beta}(T).$$

◇

Theorem 3.5.3 *Let $T \in LR(X,Y)$ be open and injective. If $S \in LR(X,Y)$ satisfies $D(S) \supset D(T)$, $S(0) \subset \overline{T(0)}$ and $||S|| < \gamma(T)$, then $T+S$ is open and injective, and $\bar{\beta}(T+S) = \bar{\beta}(T)$.*

PROOF

Let $\gamma := \gamma(T)$, and choose $n \in I\!N$ such that

$$\tfrac{||S||}{n} \leq \gamma - ||S||.$$

Let $x \in D(T+S)$, then for $0 \leq k \leq n$, we have

$$\begin{aligned} ||(T+S-\frac{k}{n}S)x|| &\geq ||Tx|| - ||(1-\frac{k}{n})Sx|| \qquad since \;\; S(0) \subset \overline{T(0)} \\ &\geq \gamma||x|| - (1-\frac{k}{n})||S||\,||x|| \\ &\geq (\gamma - ||S||)\,||x||. \end{aligned}$$

Thus for $0 \leq k \leq n$ we have

$$\gamma(T+S-\tfrac{k}{n}S) \geq \gamma - ||S||.$$

Particularly $\gamma(T+S) > 0$. Replacing T by $T+S-\frac{k}{n}S$ and S by $-\frac{1}{n}S$ in Theorem 3.5.2, we see that

$$\bar{\beta}(T+S-\tfrac{k}{n}S-\tfrac{1}{n}S) \leq \bar{\beta}(T+S-\tfrac{k}{n}S).$$

Letting $k = n-1, n-2, \ldots, 1, 0$ successively yields

$$\bar{\beta}(T) \leq \bar{\beta}(T+S).$$

Equality follows from Theorem 3.5.2.

◇

Corollary 3.5.4 *Let $T \in LR(X,Y)$ be open, injective and have dense range. If $S \in LR(X,Y)$ satisfies $D(S) \supset D(T)$, $S(0) \subset \overline{T(0)}$ and $||S|| < \gamma(T)$. Then $T+S$ is open, injective and has dense range.*

◇

3.6 Multivalued Linear Projections

A **multivalued linear projection operator** P defined on linear space X is a multivalued linear operator which is idempotent and has invariant domain. We investigate the properties of such relations in normed linear spaces. After giving a formal definition, we show that a multivalued projection may be characterised in terms of a pair of linear subspaces. Descriptions of adjoints and closures (or completions) of linear projections follow naturally in terms of the adjoints and closures (completions), respectively, of subspaces.

The continuity of a projection is related to the properties of the subspaces associated with it. Criteria for continuity are summarised, and a well-known theorem on the sums of closed subspaces in Banach spaces is deduced as a corollary of Theorem 3.6.7. We note, however, that a continuous multivalued projection does not necessarily decompose the space into topologically complemented subspaces (examples are given at the end of the section).

Definition 3.6.1 *Let $P \in LR(X)$.*
Then P is said to be a **multivalued linear projection** *if it satisfies the conditions*

(1) $P^2 = P$, and

(2) $R(P) \subset D(P)$.

Projections can be characterised in terms of subspace pairs, i.e. any pair of subspaces of a normed linear space determines a projection and vice-versa. We have:

Proposition 3.6.2 (Characterisation of Multivalued Linear Projections) *Let M and N be linear subspaces of a normed linear space X. Define*

$$G(P) := \{ (m+n, m) \mid m \in M, \ n \in N \}.$$

Then P is a multivalued linear projection satisfying $D(P) = M + N$, $R(P) = M$, $N(P) = N$ and $P(0) = M \cap N$.

Conversely, if P is a multivalued linear projection, then P determines a pair of subspaces M and N such that $G(P) = \{ (m+n, m) \mid m \in M, \ n \in N\}$ with $D(P) = M + N$, $R(P) = M$, $N(P) = N$ and $P(0) = M \cap N$.

◇

From the above it follows that the relation $I - P$ is a projection whenever P is, and $R(I-P) = N(P)$, $N(I-P) = R(P)$ and $G(I-P) = \{ (m+n, n) \mid m \in R(P), \ n \in N(P)\}$.

Note that not all idempotents are projections. The equivalence

$$G(P) = G(P^2) \ \Leftrightarrow \ G(P^{-1}) = G((P^{-1})^2) \tag{3.9}$$

shows that P is an idempotent if and only if its inverse P^{-1} is an idempotent. However, the inverse of a projection P is generally not a projection. Part (a) of Proposition 3.6.3 below follows from (3.9), while (b) is easy to verify.

Proposition 3.6.3 *Let $P \in LR(X,Y)$. Then*
(a) P is an idempotent if and only if P^{-1} is an idempotent,
(b) If P is projection then the following are equivalent:
(i) P^{-1} is a projection.
(ii) $D(P) = R(P)$.

◇

Proposition 3.6.4 *If P is a projection with subspace pair $R(P) = M$ and $N(P) = N$, then P' is a projection with subspace pair satisfying $R(P') = N^{\perp}$ $N(P') = M^{\perp}$ and $P'(0) = M^{\perp} \cap N^{\perp}$.*

PROOF
We have

$$G((P')^2) \subset G((P^2)') = G(P'). \tag{3.10}$$

(Proposition 2.6.7). It follows from (3.10) that $(P')^2(0) \subset P'(0)$ and, thus, $(P')^2(0) = P'(0)$. To see $R(P') \subset D(P')$, let $x' \in D(P')$ and suppose $y' \in P'x'$. Then, for $x \in D(P)$,

$$y'(Px) \subset P'x'(Px) = x'P(Px) = x'Px$$

since $R(P) \subset D(P)$. Since $x'P$ is single-valued, we have $y'Px = x'Px$, and, since $x \in D(P)$ was arbitrarily chosen, it follows that $y'P$ is continuous and single-valued, i.e. $y' \in D(P')$.

To see $G((P')^2) \supset G((P^2)') = G(P')$, let $(y', x') \in G(P')$. Since $R(P') \subset D(P')$, it follows that $y' \in D((P')^2)$ and for $(x, Px) \in G(P)$ we have

$$x'x \in (P'y')x = y'(Px) = y'(P^2)x = P'y'(Px) = P'(P'y')x.$$

Thus $(y', x') \in G((P')^2)$, and the desired inclusion holds.

By Proposition 2.6.5, $N(P') = R(P)^\perp = M^\perp$.
Similarly, $R(P') = N(I_{X'} - P') = R(I - P)^\perp = N(P)^\perp = N^\perp$.

$\diamond$

Proposition 3.6.5 *If P is a projection with $R(P) = M$ and $N(P) = N$, then $\overline{P}$ is a projection with $R(\overline{P}) = \overline{M}$, $N(\overline{P}) = \overline{N}$, and $\overline{P}(0) = \overline{M} \cap \overline{N}$.*

PROOF
We first show that $R(\overline{P}) \subset D(\overline{P})$ and $G(\overline{P}) \subset G(\overline{P}^2)$. Let $(x, y) \in G(\overline{P})$. Thus, there exists a sequence $\{(x_n, y_n)\} \in G(P)$ such that $(x_n, y_n) \to (x, y)$. It follows that $\{(y_n, y_n)\} \in G(P)$ and $(y_n, y_n) \to (y, y) \in G(\overline{P})$, i.e. $y \in D(\overline{P})$ and $(x, y) \in G(\overline{P}^2)$.

For the reverse inclusion we have:

$$G(-(\overline{P}^2)^{-1})^\perp = G((\overline{P}^2)') \supset G(\overline{P}'\overline{P}') = G(P'P') = G(P') = G(\overline{P}') = G(-\overline{P}^{-1})^\perp. \quad (3.11)$$

Hence,

$$G(\overline{P}^2) \subset \overline{G(\overline{P}^2)} = G(\overline{P}^2)^{\perp\top} \subset G(\overline{P})^{\perp\top} = \overline{G(\overline{P})} = G(\overline{P}),$$

and thus, $G(\overline{P}^2) = G(\overline{P})$.

To verify $\overline{M} = \overline{R(P)} \supset R(\overline{P})$, we have that $y \in R(\overline{P})$ if and only if $(y, y) \in G(\overline{P})$ if and only if there exists $\{(x_n, y_n)\} \subset G(P)$ such that $(x_n, y_n) \to (y, y)$. Now $\{(y_n, y_n)\} \subset G(P)$ since $R(P) \subset D(P)$, and thus $(y_n, y_n) \to (y, y)$. Since $\{y_n\} \subset M$, it follows that $y \in \overline{M}$. Therefore $R(\overline{P}) \subset \overline{R(P)}$. To see that the reverse inclusion holds, we note that each step in the argument just given is reversible, and, hence $R(\overline{P}) = \overline{R(P)} = \overline{M}$. The case $\overline{N} = N(\overline{P}) = R(I - \overline{P})$ follows similarly.

$\diamond$

Corollary 3.6.6 *If M and N are closed subspaces of a normed linear space X, with associated projection $P \in LR(X)$, then P is closed.*

$\diamond$

The familiar duality properties about subspace pairs are immediate consequences of Proposition 3.6.5:

$$\begin{aligned} M^{\perp} \cap N^{\perp} &= (M+N)^{\perp} \\ (M^{\perp}+N^{\perp})^{\top} &= \overline{M} \cap \overline{N}. \end{aligned}$$

Combining the Closed Graph and Closed Domain theorems for multivalued linear operators with Propositions 2.6.5, 3.6.4 and 3.6.5 yields the following theorem:

Theorem 3.6.7 *Let X be a normed linear space, and let $P \in LR(X)$ be a projection with $R(P) = M$ and $N(P) = N$. The following are equivalent:*

(i) $\tilde{P}$ *is continuous*

(ii) $D(\tilde{P})$ *is closed*

(iii) $D(\tilde{P}')$ *is weak*-closed*

(iv) $D(P') = D(\tilde{P}')$ *is closed*

(v) $P' = \tilde{P}'$ *is continuous*

(vi) $\tilde{M} + \tilde{N}$ *is closed*

(vii) $\tilde{M} + \tilde{N} = (\tilde{M}^{\perp} \cap \tilde{N}^{\perp})^{\top}$

(viii) $\tilde{M}^{\perp} + \tilde{N}^{\perp}$ *is weak*-closed*

(ix) $\tilde{M}^{\perp} + \tilde{N}^{\perp} = (\tilde{M} \cap \tilde{N})^{\perp}$

(x) $\tilde{M}^{\perp} + \tilde{N}^{\perp}$ *is closed*

$\diamond$

Corollary 3.6.8 *Let M and N be closed subspaces of a Banach space X. Then*

$$\begin{aligned} M+N \text{ is closed} \quad &\Leftrightarrow \quad M+N = (M^{\perp} \cap N^{\perp})^{\top} \\ &\Leftrightarrow \quad M^{\perp}+N^{\perp} \text{ is weak}*\text{-closed} \\ &\Leftrightarrow \quad M^{\perp}+N^{\perp} \text{ is closed} \\ &\Leftrightarrow \quad M^{\perp}+N^{\perp} = (M \cap N)^{\perp}. \end{aligned}$$

$\diamond$

Corollary 3.6.8 may be proved via techniques involving quantities referred to as the *geometric opening, opening* or *gap* between subspaces of a Banach space (cf. Kato [75], Mennicken and Sagraloff [103] and [104], and also Cross [35] (III.4)). Proposition 3.6.9 below restates Propositions 3.3.2 and 2.6.14 for the particular case when the linear relation is a projection, and gives a necessary and sufficient condition for the equality $M^{\perp}+N^{\perp} = (M \cap N)^{\perp}$ to hold. The latter does not require that the subspaces M and N be closed subspaces of a Banach space (cf. Corollary 3.6.8).

Proposition 3.6.9 *Let $P \in LR(X)$ be a projection with $R(P) = M$ and $N(P) = N$.*

(a) P is continuous if and only if $D(P') = M^{\perp} + N^{\perp} = (M \cap N)^{\perp} = P(0)^{\perp}$.

(b) If P is continuous, then $||P'|| = ||P||$.

$\diamond$

We recall a well-known proposition on subspaces and the existence of continuous projections. It is extended to a multivalued analogue in Proposition 3.6.11 (cf. Theorems 1.7.13 and 1.7.13).

Proposition 3.6.10 *Let X be a normed linear space, and let M and N be closed subspaces with $\dim N < \infty$, and $M \cap N = \{0\}$. Then the single-valued projection P with domain $M + N$, range, M, and kernel, N, is continuous.*

◇

Proposition 3.6.11 *Let P be a multivalued projection in a normed linear space.*
(a) If $R(P)$ is closed and $\dim N(P) < \infty$ then P is continuous.
(b) If $N(P)$ is closed and $\dim R(P) < \infty$ *then P is continuous.*

PROOF
(a) Let A denote a single-valued projection defined on $R(P)$ with kernel $P(0) = R(P) \cap N(P)$. Then A is continuous, and AP is a selection of P. To see that P is continuous, it is sufficient to show that AP is continuous (Proposition 2.4.3). Now AP is a single-valued projection defined on $D(P)$ with finite-dimensional kernel $N(AP) = P^{-1}A^{-1}(0) = N(P)$ and range $R(AP) = A(R(P))$. Since A is continuous, so is $I_{R(P)} - A$, and hence, $N(I - A) = R(A) = R(AP)$ is closed in $D(A)$. Thus, by Proposition 3.6.10, AP is continuous.
(b) follows from (a) by replacing P with $I - P$.

◇

Proposition 3.6.12 *If $P \in LR(X)$ is a multivalued projection, then P is open.*

PROOF
Let M and N be the subspaces associated with P where $R(P) = M$ and $N(P) = N$, and let U be an open set in $M + N$. Then $P(U) = P(U \cap (M + N)) \supset P(U \cap M) = U \cap M$ which is relatively open in M.

◇

Idempotents are generally not open: if P is an unbounded projection, then its inverse P^{-1} is idempotent but is not open. We conclude with some examples:

Examples 3.6.13
(1) Let N be a dense proper subspace of an infinite dimensional normed space X. Let $P \in LR(X)$ be a projection with kernel N and range $M = R(P)$ satisfying $X = M + N$ and $M \cap N = \{0\}$. Then P is not continuous, while $\tilde{P}$ and P' are.

(2) A separable Banach space X admits a pair of quasi-complements, i.e. for a given closed subspace M of X, there exists closed subspace N such $M \cap N = \{0\}$, and $M + N$ is dense in X

(see for example Lindenstrauss [93]; see also Murray [110] and Mackey [100]]). In this case, the associated projection P is closed but not continuous.

(3) Let X be a Banach space and let M and N be closed infinite-dimensional subspaces such that $X = M + N$. Suppose M is topologically complemented in X, but $M \cap N$ is not topologically complemented in M. Then the multivalued projection P with range $R(P) = M$ and kernel $N(P) = N$ is continuous. However, if A is a single-valued projection defined on M with kernel $M \cap N = P(0)$, then A is not continuous. Furthermore, the selection AP of P with $N(AP) = M \cap N$ is not continuous.

Remarks 3.6.14

Example (1) illustrates that continuity of P does not necessarily follow from the continuity of $\tilde{P}$ or P', while Example (2) shows that a closed projection need not be continuous. In Example (3) it is shown that continuous multivalued projections do not in general have the same decomposition properties with regard to topological complementation as their single-valued counterparts. If however, a projection, with range $R(P) = M$ and kernel $N(P) = N$, is continuous and $P(0)$ is topologically complemented (for example if $P(0)$ is finite dimensional or closed and finite codimensional), then M and N are topologically complemented in $M + N$. An illustrative example is also given in the context of Atkinson relations - see Example 6.2.11.

3.7 Notes and Remarks

The Baire property for (single-valued) linear operators is due to Beauzamy [17] (see remarks in Section 3.1 above). The proof of the generalisation to linear relations, Theorem 3.1.6, is due to the author, and is based on the Mittag-Leffler theorem on inverse limits for single-valued operators (cf. Bourbaki [22]). Lennard ([91] and [92]) proved Corollary 3.1.7 directly for single-valued operators, and then deduced the theorem of Beauzamy as a special case. In the second paper he deduced Corollary 3.1.7 from the Mittag-Leffler theorem. The same approach is used here. However, the proof given for Corollary 3.1.7 is simpler than the argument given by Lennard in [92]. Theorem 3.1.6 is applied in Theorem 7.6.2 of Chapter 7.

The proofs given here for the Uniform Boundedness Principle for linear relations, Theorem 3.2.1, and it's corollaries, are given in Cross [35].

In Section 3.3 on the Closed Graph and Closed Range theorems, Proposition 3.3.2 is due to P. Pillay (cf. Cross, [35]). Lemma 3.3.4 is based on Lemma II.4.1. given in Goldberg [60]. The more general result given here is due to the author; in Goldberg, the lemma is proved for the case when T is a densely-defined single-valued linear relation and its adjoint T' has a bounded inverse. Lemmas 3.3.5 and 3.3.6 can be found in Gohberg and Goldberg [55]. Lemma 3.3.6 is presented here as an independent result, whilst in Gohberg and Goldberg it is contained in the proof of the Closed Graph theorem. The proof of the Closed Graph theorem (and, hence, the Open Mapping theorem) for multivalued linear operators, given here as Theorem 3.3.7, is based on the proof given in Gohberg and Goldberg [55]. The proof of the Closed Range theorem, given here as Theorem 3.3.8, is due to the author.

As already mentioned, the State Diagrams for unbounded and for closed linear (single-valued) operators were compiled by Goldberg [60]; M. Möller extended the diagrams to linear relations (M.Sc. dissertation, 1976, cf. Cross [35]). Further comments on the Essential State Diagram, due to Cross, are made at the end of Chapter 5.

The Small Perturbation theorems for open relations are given in this chapter for reference in Chapters 5 and 6 ; the generalisation of these theorems for multivalued linear operators were given in Cross [35].

Multivalued Linear Projections were first considered by R.W. Cross (cf. Cross [39]), who gave proofs of Properties 3.6.2, 3.6.3, 3.6.4 and 3.6.5. The proofs given here are based on the ones given in [39]. The statements comprising Theorem 3.6.7 and Corollary 3.6.8 were summarised by the author. Further communication with Professor Cross led to the formulation given here. In particular, the proof of Corollary 3.6.8, on the sums of subspaces, is deduced from the Closed Graph and Closed Range theorems and from properties of subspace pairs. The proof does not apply techniques based on *gap* quantities. On the other hand, the properties for pairs of closed subspaces in a Banach space may be proved via *gap* quantities, and without use of the Closed Graph and Closed Range theorems. The Closed Graph and Closed Range theorems may then be deduced from a theorem analogous to Corollary 3.6.8 (cf. Cross [35]; further comments are given in [35], and are also given immediately after Corollary 3.6.8 above). Propositions 3.6.9 and 3.6.11, and the concluding examples, are also due to the author. The examples, to further illustrate the properties of multivalued projections, were assembled by the author. Properties of Multivalued Projections are applied in the proofs of the duality properties of Atkinson relations in Section 6.2.

Chapter 4

Operator Quantities

4.1 Quantities for Linear Relations

Definitions 4.1.1 *Let* $\mathcal{I}(X)$, $\mathcal{C}(X)$ *and* $\mathcal{P}(X)$ *denote the infinite dimensional, finite codimensional and closed finite codimensional subspaces, respectively, of a normed linear space* X. *Let* $A_{XY} := \{\Gamma, \Gamma_0\,\bar{\Gamma}_0, \Delta, \tau, \tau_0, \bar{\tau}_0\}$ *where* $f : LR(X,Y) \to [0,\infty] \in A_{XY}$ *is defined as follows:*

If $\dim D(T) < \infty$ *then* $f(T) := 0$ *for all* $f \in A_{XY}$.
Otherwise,

$$\begin{aligned}
\Gamma(T) &:= \inf_{M\in\mathcal{I}(D(T))} \|T|_M\| \\
\Gamma_0(T) &:= \inf_{M\in\mathcal{C}(D(T))} \|T|_M\| \\
\bar{\Gamma}_0(T) &:= \inf_{M\in\mathcal{P}(D(T))} \|T|_M\| \\
\Delta(T) &:= \sup_{M\in\mathcal{I}(D(T))} \Gamma(T|_M) \\
\tau(T) &:= \sup_{M\in\mathcal{I}(D(T))} \inf_{m\in S_M} \|Tm\| \\
\tau_0(T) &:= \sup_{M\in\mathcal{C}(D(T))} \inf_{m\in S_M} \|Tm\| \\
\bar{\tau}_0(T) &:= \sup_{M\in\mathcal{P}(D(T))} \inf_{m\in S_M} \|Tm\|
\end{aligned}$$

In this chapter, we will restrict our attention to the quantities $\Gamma(T)$, $\Gamma_0(T)$, $\bar{\Gamma}_0(T)$ and $\Delta(T)$ which will be applied directly in the sequel. Properties of operator quantities are applied in perturbation theorems, and show the stability properties of Fredholm operators in greater generality (cf. Gohberg and Krein [57] and Goldberg [60]). Further properties of operator quantities are given in Cross [35] for linear relations and in Labuschagne [84] for single-valued operators.

Proposition 4.1.2 *Let* X *be a normed linear space.*
(a) Let $M \in \mathcal{C}(X)$, $N \in \mathcal{I}(X)$. *Then* $M \cap N \in \mathcal{C}(N)$.
(b) Let $M \in \mathcal{P}(X)$, $N \in \mathcal{I}(X)$. *Then* $M \cap N \in \mathcal{P}(N)$.

(c) Let $M \in \mathcal{I}(X)$. Then

$$\begin{aligned} \mathcal{C}(M) &= \{M \cap N | N \in \mathcal{C}(X)\} \\ \mathcal{P}(M) &= \{M \cap N | N \in \mathcal{P}(X)\} \end{aligned}$$

PROOF

(a) The space X/M is finite-dimensional. Define $A : N/M \cap N \to X/M$ by $A(n + M \cap N) = n + M,\ n \in N$. Since A is injective, $dim\, N/M \cap N < \infty$.

(b) This follows from (a)

(c) Suppose $N \in \mathcal{P}(X)$. Since $M \cap N \in \mathcal{P}(M)$ (from (b) above), we have $\{M \cap N \mid N \in \mathcal{P}(X)\} \subset \mathcal{P}(M)$. If $L \in \mathcal{P}(M)$, then there exists a subspace $F \subset M$ such that $dim\, F < \infty,\ L + F = M$ and $L \cap F = \{0\}$. Let $x_1, \ldots x_k$ be a basis for F. Since L is closed, it follows from the Hahn-Banach theorem that there exists $x'_1, \ldots x'_k \in X'$ such that $x'_i x_j = \delta_{ij}$ and $x'_i(x) = 0$ for all $x \in L$, where $i, j \leq k$. The subspace $N := \bigcap_{i \leq k} x_i^{-1}(0) \subset X$ is closed and finite-codimensional. Furthermore $L = M \cap N$, and the reverse inclusion follows.

◇

Proposition 4.1.3 *Let $T \in LR(X, Y)$.*
If $M \in \mathcal{C}(X)$ and $f \in \{\Gamma,\ \Gamma_0,\ \Delta\}$ then $f(T|_M) = f(T)$, and if $M \in \mathcal{P}(X),\ f = \bar{\Gamma}_0$ then $f(T|_M) = f(T)$.

PROOF

The result follows trivially if $D(T)$ is finite-dimensional. Suppose $dim\, D(T) = \infty$, and let $M \in \mathcal{C}(X)$.

For the case $f = \Gamma$, clearly $\Gamma(T) \leq \Gamma(T|_M)$. For the reverse inequality, let $\epsilon > 0$. Then there exists $E \in \mathcal{I}(X)$ such that

$$\Gamma(T) + \epsilon \geq ||T|_E|| \geq \inf_{N \in \mathcal{I}(M \cap E)} ||T|_N|| \geq \inf_{N \in \mathcal{I}(M \cap D(T))} ||T|_N|| = \Gamma(T|_M).$$

The case $f = \Gamma_0,\ \bar{\Gamma}_0$ follow similarly, noting that if $E \in \mathcal{C}(X)$, ($E \in \mathcal{P}(X)$), then $M \cap E \in \mathcal{C}(M)$, ($M \cap E \in \mathcal{P}(M)$) by Proposition 4.1.2.

Now suppose $f = \Delta$. For $N \in \mathcal{I}(D(T))$, it follows from Proposition 4.1.2, and what has just been shown, that $\Gamma(T|_N) = \Gamma(T|_{M \cap N})$. Thus

$$\Delta(T|_M) = \sup_{N \in \mathcal{I}(M)} \Gamma(T|_N) = \sup_{N \in \mathcal{I}(D(T))} \Gamma(T|_{N \cap M}) = \sup_{N \in \mathcal{I}(D(T))} \Gamma(T|_N) = \Delta(T)$$

◇

Corollary 4.1.4 *Let $F, T \in LR(X, Y)$, and suppose F satisfies $D(F) \supset D(T)$ and $dim\, R(F) < \infty$.*

(a) If $f \in \{\Gamma,\ \Gamma_0,\ \Delta\}$, then $f(T + F) \leq f(T)$ with equality if $F(0) \subset \overline{T(0)}$.

(b) If F is continuous and $f = \bar{\Gamma}_0$, then $f(T + F) \leq f(T)$ with equality if $F(0) \subset \overline{T(0)}$.

PROOF

(a) Now $dim\, D(F)/N(F) = dim\, D(F)/N(QF) < \infty$ since QF is single-valued and $R(F)$ is finite-dimensional. If $x \in N(F)$ then

$$||(T+F)x|| = ||Q_{\overline{(T+F)(0)}}(T+F)x|| \leq ||Q_{\overline{T(0)}}Tx|| = ||Tx||. \tag{4.1}$$

Thus, by Proposition 4.1.3,

$$f(T+F) = f((T+F)|_{N(F)}) \leq f(T|_{N(F)}) = f(T). \tag{4.2}$$

If $F(0) \subset \overline{T(0)}$, then equality holds in (4.1), and hence also in (4.2).

(b) If F is continuous then $N(F) \in \mathcal{P}(X)$. Thus applying Proposition 4.1.3 again,

$$\bar{\Gamma}_0(T+F) = \bar{\Gamma}_0((T+F)|_{N(F)}) \leq \bar{\Gamma}_0(T|_{N(F)}) = \bar{\Gamma}_0(T).$$

with equality if $F(0) \subset \overline{T(0)}$.

◇

Proposition 4.1.5 *Let $T \in LR(X,Y)$. Then*

$$\Gamma(T) \leq \Delta(T).$$

PROOF

By Proposition 4.1.3 we have

$$\begin{aligned} \Delta(T) &= \sup_{M \in \mathcal{I}(D(T))} \Gamma(T|_M) \\ &\geq \sup_{M \in \mathcal{C}(D(T))} \Gamma(T|_M) \\ &= \Gamma(T). \end{aligned}$$

◇

Proposition 4.1.6 *Let $T \in LR(X,Y)$. Then*

$$\Delta(T) \leq \Gamma_0(T).$$

PROOF

We assume without loss of generality that $D(T) = X$ and $dim\, X = \infty$.

Suppose $N \in \mathcal{I}(D(T))$. Then

$$\begin{aligned} \Gamma(T|_N) &\leq \Gamma_0(T|_N) \\ &= \inf_{M \in \mathcal{C}(N)} ||T|_M|| = \inf_{M \in \mathcal{C}(X)} ||T|_{M \cap N}|| \\ &\leq \inf_{M \in \mathcal{C}(X)} ||T|_M|| = \Gamma_0(T). \end{aligned}$$

Since N was arbitrary, the desired inequality follows by taking taking the supremum over all $N \in \mathcal{I}(D(T))$.

◇

Proposition 4.1.7 *Let $T, S \in LR(X,Y)$. Then*

$$\Gamma(S+T) \;\leq\; \Delta(S) + \Gamma(T|_{D(S)}).$$

PROOF

The result follows trivially if $D(T+S)$ is finite-dimensional. Suppose $dim\, D(T+S) = \infty$, and without loss of generality, let $X = D(T)$.

Let $M \in \mathcal{I}(D(S))$, and for $\epsilon > 0$, choose $N \in \mathcal{I}(M)$ such that $||S|_N|| \leq \Gamma(S|_M) + \epsilon$. Then

$$\begin{aligned}
\Gamma(T+S) \leq ||(S+T)|_N|| \;&\leq\; ||T|_N|| + ||S|_N|| \\
&\leq\; ||T|_N|| + \Gamma(S|_M) + \epsilon \\
&\leq\; ||T|_N|| + \Delta(S) + \epsilon \\
&\leq\; |||T|_M|| + \Delta(S) + \epsilon.
\end{aligned}$$

Since M and $\epsilon > 0$ are arbitrary, the desired inequality holds.

◇

Corollary 4.1.8 *Let $T, S \in LR(X,Y)$. Then*

$$\Delta(S+T) \;\leq\; \Delta(S) + \Delta(T).$$

PROOF

For any $M \in \mathcal{I}(D(S+T))$ we have

$$\begin{aligned}
\Gamma((S+T)|_M) \;&\leq\; \Delta(S|_M) + \Gamma(T|_M) \\
&\leq\; \Delta(S) + \Delta(T).
\end{aligned}$$

Thus

$$\Delta(S+T) \;=\; \sup_{M \in \mathcal{I}(D(T+S))} \Gamma((S+T)|_M) \;\leq\; \Delta(S) + \Delta(T).$$

◇

Proposition 4.1.9 *Let $T, S \in LR(X,Y)$. Then*

$$\Gamma_0(S+T) \;\leq\; \Gamma_0(S) + \Gamma_0(T), \quad \textit{and} \tag{4.3}$$

$$\bar{\Gamma}_0(S+T) \;\leq\; \bar{\Gamma}_0(S) + \bar{\Gamma}_0(T). \tag{4.4}$$

PROOF

We assume without loss of generality that $D(T) = X$ and $dim\, X = \infty$.

To see that inequality (4.3) holds, let $\epsilon > 0$, choose $M_1, M_2 \in \mathcal{C}(X)$ such that $\Gamma_0(T) > ||T|_{M_1}|| - \frac{\epsilon}{2}$ and $\Gamma_0(S) > ||S|_{M_2}|| - \frac{\epsilon}{2}$, and let $M := M_1 \cap M_2$. It follows that $M \in \mathcal{C}(X)$, and

$$\Gamma_0(T) + \Gamma_0(S) \; > \; ||T|_M|| + ||S|_M|| - \epsilon \; > \; ||(T+S)|_M|| - \epsilon \; \geq \; \Gamma_0(T+S) - \epsilon.$$

Since ϵ was arbitrary, the desired inequality follows.

Inequality (4.3) follows similarly.

$\diamond$

Lemma 4.1.10 *Let $T \in LR(X,Y)$ and $S \in LR(Y,Z)$. If T is single-valued with $R(T) \subset D(S)$ and, if M is a subspace of X, then*

$$\Gamma(ST|_M) \leq \Gamma(T|_M)\Delta(S|_{T(M)})$$

whenever the righthand side is defined.

PROOF

The inequality is not defined for the cases $\Gamma(T|_M) = 0$, $\Delta(S|_{T(M)}) = \infty$, and $\Gamma(T|_M) = \infty$, $\Delta(S|_{T(M)}) = 0$. It is enough to show that the statement holds for $0 < \Gamma(T|_M) < \infty$ and $\Delta(S|_{T(M)}) < \infty$. Furthermore, we need only consider the case $dim\, D(ST) \cap M = \infty$.

$D(T) = D(ST)$ since $R(T) \subset D(S)$. Let $\epsilon > 0$, and choose $N \in \mathcal{I}(M \cap D(T))$ such that

$$||T|_N|| \leq \Gamma(T|_M) + \epsilon. \tag{4.5}$$

By applying Corollary 2.3.13,

$$\begin{aligned}
\Gamma(ST|_N) &= \inf_{L \in \mathcal{I}(N)} ||ST|_L|| \\
&\leq \inf_{L \in \mathcal{I}(N)} ||S|_{T(L)}||\, ||T|_L|| \\
&\leq ||T|_N|| \inf_{L \in \mathcal{I}(N)} ||S|_{T(L)}||
\end{aligned}$$

If $dim\, T(N) = \infty$, let $V := N \cap T^{-1}(W)$ where $W \in \mathcal{I}(T(N))$. Then $T(V) = W$ and $dim\, V = \infty$ since $dim\, T(V) = \infty$ and T is single-valued. Thus,

$$\begin{aligned}
\inf_{L \in \mathcal{I}(N)} ||S|_{T(L)}|| &\leq \inf_{W \in \mathcal{I}(T(N))} ||S|_W|| \\
&\leq \Gamma(S|_{T(N)}) < \infty
\end{aligned}$$

since $\Delta(S|_{T(M)}) < \infty$. Thus

$$\Gamma(ST|_N) \leq \Gamma(S|_{T(N)})||T|_N||. \tag{4.6}$$

If $dim\, T(N) < \infty$, let $K := N(T|_N) \in \mathcal{I}(N)$. Then $\Gamma(ST|_N) \leq ||ST|_K|| = 0$, and inequality (4.6) holds. Applying (4.5),

$$\Gamma(ST|_M) \leq \Delta(S|_{T(M)})\,(\Gamma(T|_M) + \epsilon), \tag{4.7}$$

and, since M and $\epsilon > 0$ are arbitrary, the desired inequality holds.

$\diamond$

Proposition 4.1.11 *Let $T \in LR(X,Y)$ and $S \in LR(Y,Z)$. If T is single-valued then*

$$\Delta(ST) \leq \Delta(S)\Delta(T)$$

whenever the righthand side is defined.

PROOF

Let $S_1 := S|_{R(T)}$. By Lemma 4.1.10 we have

$$\Gamma(S_1 T|_M) \leq \Gamma(T|_M)\Delta(S_1)$$

for $M \in I(D(T))$. Taking the supremum over such M, we have

$$\Delta(S_1 T) \leq \Delta(T)\Delta(S_1).$$

Since $S_1 T = ST$ and $\Delta(S_1) \leq \Delta(S)$, the desired inequality follows.

◇

Proposition 4.1.12 *Let $T \in LR(X,Y)$ and $M \in \mathcal{P}(X)$. Then*

(a) If $T|_M$ is continuous then so is T.

(b) T is continuous if and only if $\bar{\Gamma}_0(T) < \infty$.

PROOF

(a) Without loss of generality, we may assume that $D(T) = X$. Suppose $M \in \mathcal{P}(X)$ and $||T|_M|| < \infty$. Then there exists a subspace $N \subset X$ such that $dim\, N < \infty$, $M + N = X$ and $M \cap N = \{0\}$, and a continuous projection P such that $R(P) = M$ and $N(P) = N$. Since $dim\, N < \infty$, it follows that $T|_N$ is continuous. Thus for $x \in X$ we have

$$||Tx|| \leq ||TPx|| + ||T(I-P)x|| \leq ||T|_M||\, ||P||\, ||x|| + ||T|_N||\, ||I-P||\, ||x||$$

(b)The forward implication is clear from the defintion of $\bar{\Gamma}_0$, and if $\bar{\Gamma}_0(T) < \infty$ then there exists $M \in \mathcal{P}(X)$ such that $||T|_M|| < \infty$. The reverse implication then follows from (a).

◇

Proposition 4.1.13 *Let $T \in LR(X,Y)$ be single-valued, and $S \in LR(Y,Z)$. Then*

(a) $\bar{\Gamma}_0(ST) \leq \bar{\Gamma}_0(T)\bar{\Gamma}_0(S)$, and

(b) $\Gamma_0(ST) \leq \Gamma_0(T)\Gamma_0(S)$,

whenever the righthand side of the inequalities are defined.

PROOF

(a) The inequality does not apply when either $\bar{\Gamma}_0(T) = 0$ and $\bar{\Gamma}_0(S) = \infty$, or $\bar{\Gamma}_0(T) = \infty$ and $\bar{\Gamma}_0(S) = 0$. For the cases when the inequality is defined, it is enough to show that the statement holds for $0 < \bar{\Gamma}_0(T) < \infty$ and $\bar{\Gamma}_0(S) < \infty$.

By Proposition 4.1.12, S and T are continuous, and since $T(0) \subset D(S)$, $\|ST\| < \infty$ (Corollary 2.3.13). Let $\epsilon > 0$, and choose $M \in \mathcal{P}(X)$ and $N \in \mathcal{P}(Y)$ such that

$$\begin{aligned} \|T|_M\| &< \bar{\Gamma}_0(T) + \epsilon, \quad \textit{and} \\ \|S|_N\| &< \bar{\Gamma}_0(S) + \epsilon. \end{aligned}$$

Let $W := T^{-1}(N \cap D(S)) \cap M$. By Proposition 1.7.6, $T^{-1}(N \cap D(S)) \in \mathcal{C}(D(ST))$, and hence $W \in \mathcal{C}(D(ST))$. Thus we have

$$\begin{aligned} \bar{\Gamma}_0(ST) = \inf_{E \in \mathcal{P}(X)} \|ST|_E\| &\leq \sup_{x \in \overline{W} \cap D(ST)} \frac{\|STx\|}{\|x\|} \\ &= \sup_{x \in W \cap D(ST)} \frac{\|STx\|}{\|x\|} \qquad (4.8) \\ &\leq \|S|_N\| \sup_{x \in W \cap D(ST)} \frac{\|Tx\|}{\|x\|} \\ &\leq (\bar{\Gamma}_0(S) + \epsilon)(\bar{\Gamma}_0(T) + \epsilon) \end{aligned}$$

where (4.8) follows from the continuity of ST.

(b) The proof is similar to the one given for (a). Continuity of ST is not needed since the infimum for $\Gamma_0(ST)$ is taken over sets in $\mathcal{C}(D(ST))$.

$\diamond$

Proposition 4.1.14 *Let $T \in LR(X,Y)$ and suppose dim $D(T) = \infty$. Then $\alpha(T) < \infty$ implies $\gamma(T) \leq \Gamma(T)$.*

PROOF
Clearly we need only consider the case $\gamma(T) > 0$. Letting $M \in \mathcal{I}(D(T))$, it follows from Lemma 3.5.1 that there exists $m \in M$, $m \neq 0$ such that $\|m\| = d(m, N(T))$. Thus,

$$\frac{\|Tm\|}{\|m\|} = \frac{\|Tm\|}{d(m,N(T))} \geq \gamma(T),$$

and therefore, $\|T|_M\| \geq \gamma(T)$. Since $M \in \mathcal{I}(D(T))$ was arbitrary,

$$\Gamma(T) = \inf_{M \in \mathcal{I}(D(T))} \|T|_M\| \geq \gamma(T).$$

$\diamond$

4.2 Conjugate Quantities for Linear Relations

Definition 4.2.1 *Let $\mathcal{F}(Y)$ and $\mathcal{E}(Y)$ denote the classes of finite dimensional and closed subspaces of infinite codimension, respectively, of a normed linear space Y. Let $A'_{XY} := \{\Gamma'_0, \Gamma', \Delta'\}$ and let $f : LR(X,Y) \to [0,\infty] \in A'_{XY}$ be defined as follows:*

If $\dim Y < \infty$ then $f(T) := 0$ for all $f \in A'_{XY}$.
Otherwise,

$$\begin{array}{rcl} \Gamma_0'(T) & := & \inf\{\ \|Q_M T\| \mid M \in \mathcal{F}(Y)\ \} \\ \Gamma'(T) & := & \inf\{\ \|Q_M J_Y T\| \mid M \in \mathcal{E}(\tilde{Y})\ \} \\ \Delta'(T) & := & \sup\{\ \Gamma'(Q_M T) \mid M \in \mathcal{E}(Y)\} \end{array}$$

where J_Y denotes the natural injection of Y into $\tilde{Y}$.

Lemma 4.2.2 *Suppose $M \subset X$ is a closed subspace.*

(a) If $N \subset X$ is a closed subspace such that $M \subset N$, then

$$\begin{array}{rcl} X/M & = & (X/M)/(N/M), \quad \textit{with} \\ Q^X_N & = & Q^{X/M}_{N/M} Q^X_M, \end{array}$$

where equality of spaces is given up to isometry.

(b) If E is a closed subspace of X/M, then $N := (Q_M)^{-1}(E)$ is closed, $M \subset N$, and $(X/M)/E = X/N$.

PROOF

(a) Elements in X/M are the form $\{x + m \mid m \in M\}$ while elements in N/M are the form $\{n + m \mid m \in M\}$. Thus, if $[x] \in (X/M)/(N/M)$, then,

$$[x] \;=\; \{x+m \mid m \in M\} + \{\ \{n+m \mid m \in M\} \mid n \in N\} \;=\; \{x+n \mid n \in N\} \;\in\; X/N$$

since $N \supset M$ i.e. $(X/M)/(N/M) \subset X/N$. The reverse inclusion follows similarly. Furthermore,

$$\begin{array}{rcl} \|Q^{X/M}_{N/M} Q^X_M\| & = & \inf\limits_{m\in M} \|Q^{X/M}_{N/M}(x-m)\| \\ & = & \inf\limits_{n\in N} (\inf\limits_{m\in M} \|(x-m)-n\|) \\ & = & \inf\limits_{n\in N} \|x-n\| \;=\; \|Q^X_N x\|. \end{array}$$

(b) Clearly $(Q_M)^{-1}(E)$ is closed, and $M = (Q_M)^{-1}(0) \subset N$. Furthermore, $(Q_M)(N) = E$ implies

$$(X/M)/E = (X/M)/(N/M) = X/N.$$

◇

Lemma 4.2.3 *Let Y be infinite dimensional and suppose $M \in \mathcal{E}(Y)$. Then*

(a) $\Gamma_0'(Q_M T) = \inf\{\|Q_{M+F}T\| \mid F \in \mathcal{F}(Y)\}$

(b) $\Gamma'(Q_M T) = \inf\{\|Q_N J_Y T\| \mid N \in \mathcal{E}(\tilde{Y}),\ N \supset M\}$

(c) $\Delta'(Q_M T) = \sup\{\Gamma'(Q_N T) \mid N \in \mathcal{E}(Y),\ N \supset M\}$

PROOF

(a) By definition and Lemma 4.2.2

$$\begin{array}{rcl} \Gamma_0'(Q_M T) & = & \inf\{\|Q^{Y/M}_H (Q_M T)\| \mid H \in \mathcal{F}(Y/M)\} \\ & = & \inf\{\|Q^Y_N T\| \mid N = M + H,\ H \in \mathcal{F}(Y)\} \end{array}$$

(b) Letting $\overline{M}$ denote the closure of M in $\tilde{Y}$, we note that $\tilde{Y}/\overline{M} = (Y/M)\tilde{}$, and $J_{Y/M}Q_M = Q^{\tilde{Y}}_{\overline{M}}J_Y$. Thus, by the definition and Lemma 4.2.2,

$$\begin{aligned}\Gamma'(Q_M T) &= \inf\{\,||Q^{\tilde{Y}/\tilde{M}}_H J_{Y/M}(Q_M T)|| \mid H \in \mathcal{E}((Y/M)\tilde{})\,\} \\ &= \inf\{\,||Q^{\tilde{Y}}_N J_Y T|| \mid N \in \mathcal{E}(\tilde{Y}),\ N \supset \overline{M}\,\}\end{aligned}$$

(c) As before, it follows from the definition and Lemma 4.2.2 that

$$\begin{aligned}\Delta'(Q_M T) &= \sup\{\,\Gamma'(Q^{Y/M}_H(Q_M T)) \mid H \in \mathcal{E}(Y/M)\,\} \\ &= \sup\{\,\Gamma'(Q_N T) \mid N \in \mathcal{E}(Y),\ N \supset M\,\}\end{aligned}$$

◇

Corollary 4.2.4 *Let Y be infinite dimensional and suppose $M \in \mathcal{E}(Y)$. Then*

(a) $\Gamma'_0(Q_M T) \leq \Gamma'_0(T)$,

(b) $\Gamma'(Q_M T) \geq \Gamma'(T)$,

(c) $\Delta'(Q_M T) \leq \Delta'(T)$.

PROOF

(a) If $F \in \mathcal{F}(Y)$, then $||Q_{M+F}T|| \leq ||Q_F T||$, since $F \subset M + F$. The desired inequality follows from Lemma 4.2.3 (a) and the definition of Γ'_0.

(b) and (c) follow from Lemma 4.2.3 (b) and the definition of Γ', and Lemma 4.2.3 (c) and the definition of Δ', respectively.

◇

Proposition 4.2.5 *Let Y be infinite dimensional and suppose $F \in \mathcal{F}(Y)$. Then*

(a) $\Gamma'_0(Q_F T) = \Gamma'_0(T)$,

(b) $\Gamma'(Q_F T) = \Gamma'(T)$,

(c) $\Delta'(Q_F T) = \Delta'(T)$.

PROOF

(a) By Lemma 4.2.3

$$\begin{aligned}\Gamma'_0(Q_F T) &= \inf\{\,||Q^Y_N T|| \mid N = F + H,\ H \in \mathcal{F}(Y)\,\} \\ &\geq \inf\{\,||Q^Y_N T|| \mid N \in \mathcal{F}(Y)\,\} \\ &= \Gamma'_0(T).\end{aligned}$$

The reverse inequality follows from Corollary 4.2.4 (a).

(b) By Corollary 4.2.4 (b), we need only consider the case $\Gamma'(T) < \infty$. Let $\epsilon > 0$, and choose $E \in \mathcal{E}(\tilde{Y})$ such that $||Q_E J_Y T|| - \epsilon < \Gamma'(T)$. Then $E + J_Y F \in \mathcal{E}(\tilde{Y})$. Applying Lemma 4.2.3 (b),

$$\begin{aligned} \Gamma'(Q_F T) &= \inf\{\, ||Q^{\tilde{Y}}_N J_Y T|| \mid N \in \mathcal{E}(\tilde{Y}),\ N \supset J_Y F \,\} \\ &\leq ||Q^{\tilde{Y}}_{E+J_Y F} J_Y T|| \\ &\leq ||Q^{\tilde{Y}}_E J_Y T|| \\ &< \Gamma'(T) + \epsilon. \end{aligned}$$

Since ϵ was arbitrary, $\Gamma'(Q_F T) \leq \Gamma'(T)$. The reverse inequality follows from Corollary 4.2.4 (b).

(c) We have

$$\begin{aligned} \Delta'(Q_F T) &= \sup\{\, \Gamma'(Q_N T) \mid N \in \mathcal{E}(Y),\ N \supset F \,\} &(4.9)\\ &= \sup\{\, \Gamma'(Q_{M+F} T) \mid M \in \mathcal{E}(Y)\} \\ &= \sup\{\, \Gamma'(Q^{Y/M}_{(M+F)/M} Q^{Y}_{(M} T) \mid M \in \mathcal{E}(Y)\} &(4.10)\\ &= \sup\{\, \Gamma'(Q^{Y}_{M} T) \mid M \in \mathcal{E}(Y)\} &(4.11)\\ &= \Delta'(T), \end{aligned}$$

where equality (4.9) follows from Lemma 4.2.3, equality (4.10) follows from Lemma 4.2.2, and equality (4.11) is a consequence of (b) above since $(M+F)/M \in \mathcal{F}(Y/M)$.

◇

Proposition 4.2.6 *Let $T \in LR(X,Y)$. Then*

$$\Gamma'(T) \ \leq\ \Delta'(T) \ \leq \Gamma'_0(T).$$

PROOF
It suffices to prove the result for the case $dim\, Y = \infty$. Let $E \in \mathcal{E}(Y)$. Then we have

$$\begin{aligned} \Gamma'(T) &\leq \Gamma'(Q_E T) &(4.12)\\ &\leq \Gamma'_0(Q_E T) &(4.13)\\ &\leq \Gamma'_0(T), &(4.14) \end{aligned}$$

where inequalities (4.12) and (4.14) follow from Corollary 4.2.4, and inequality (4.13) follows directly from the definitions. Taking the supremum over $E \in \mathcal{E}(Y)$ yields the desired result.

◇

Proposition 4.2.7 *Let $T \in LR(X,Y)$ and suppose $S \in LR(Z,X)$ satisfies $S(0) \subset D(T)$. Then for $f \in \{\Gamma'_0,\ \Gamma',\ \Delta'\ \}$*

$$f(TS) \ \leq\ ||S||\, f(T) \qquad (\infty.0 \ \ excluded).$$

PROOF

By Corollary 2.3.13, we have

$$\begin{aligned} \|Q_M TS\| &\leq \|Q_M T\|\,\|S\|, \quad M \subset Y, \ \textit{and} \\ \|Q_M J_Y TS\| &\leq \|Q_M J_Y T\|\,\|S\|, \quad M \subset \tilde{Y} \end{aligned}$$

(since $S(0) \subset D(T)$). Thus, by the definitions, inequality holds for $f = \Gamma'_0$ and $f = \Gamma'$. In particular, we have

$$\Gamma'(Q_M TS) \leq \Gamma'(Q_M T)\|S\|,$$

and hence the inequality for $f = \Delta'$ follows as well.

◇

Proposition 4.2.8 *Let $T, S \in LR(X, Y)$. Then*

(a) $\Gamma'_0(T + S) \leq \Gamma'_0(T) + \Gamma'_0(S)$,

(b) $\Gamma'(T + S) \leq \Delta'(J_Y T) + \Gamma'(S)$,

(c) $\Delta'(T + S) \leq \Delta'(J_Y T) + \Delta'(S)$,

PROOF

(a) Let $\mathcal{F} \in \mathcal{F}(Y)$. Then

$$\Gamma'_0(T + S) \leq \|Q_F(T + S)\| \leq \|Q_F T\| + \|Q_F S\| \leq \|Q_F T\| + \|S\|$$

Thus, by taking the infimum over $F \in \mathcal{F}(Y)$, $\Gamma'_0(T + S) \leq \Gamma'_0(T) + \|S\|$. Applying Proposition 4.2.5, we have for $\mathcal{F} \in \mathcal{F}(Y)$.

$$\begin{aligned} \Gamma'_0(T + S) &= \Gamma'_0(Q_F(T + S)) \\ &\leq \Gamma'_0(Q_F T) + \|Q_F S\| \\ &= \Gamma'_0(T) + \|Q_F S\| \end{aligned}$$

Again, by taking the infimum over $F \in \mathcal{F}(Y)$, the desired inequality follows.

(b) Let $\epsilon > 0$ and choose $E \in \mathcal{E}(\tilde{Y})$ and $M \in \mathcal{E}(\tilde{Y}/E)$ such that

$$\begin{aligned} \|Q_E J_Y S\| - \epsilon/2 &\leq \Gamma'(S), \ \textit{and} \\ \|Q_M^{\tilde{Y}/E} Q_E^{\tilde{Y}} J_Y T\| - \epsilon/2 &\leq \Gamma'(Q_E J_Y T), \end{aligned}$$

respectively. We have

$$\begin{aligned} \|Q_M^{\tilde{Y}/E} Q_E^{\tilde{Y}} J_Y (T + S)\| &\leq \|Q_M^{\tilde{Y}/E} Q_E^{\tilde{Y}} J_Y T\| + \|Q_M^{\tilde{Y}/E} Q_E^{\tilde{Y}} J_Y S\| \\ &\leq \Gamma'(Q_E J_Y T) + \epsilon/2 \ + \ \|Q_M\|\,\|Q_E J_Y S\|, \end{aligned}$$

where the last inequality follows from Lemma 4.2.3. Thus, by Corollory 4.2.4 (b), and the initial choice E, we have

$$\Gamma'(T+S) \;\leq\; \Gamma'(Q_{E}J_{Y}T) \;+\; \Gamma'(S) \;+\; \epsilon.$$

Taking the supremum over $E \in \mathcal{E}(\tilde{Y})$, it follows that

$$\Gamma'(T+S) \;\leq\; \Delta'(J_{Y}T) \;+\; \Gamma'(S) \;+\; \epsilon.$$

Since ϵ was arbitrary, the result follows.

(c) Let $E \in \mathcal{E}(Y)$, and let $\overline{E}$ denote the closure of E in $\tilde{Y}$. As in Lemma 4.2.3 (b), we identify $J_{Y/E}Q_{E} = Q^{\tilde{Y}}_{\overline{E}}J_{Y}$. By Corollary 4.2.4 we have

$$\Delta'(J_{Y/E}Q_{E}T) \;=\; \Delta'(Q_{\overline{E}}J_{Y}T) \;\leq\; \Delta'(J_{Y}T). \tag{4.15}$$

Thus, by (a) above and (4.15), we have

$$\begin{aligned}\Gamma'(Q_{E}(T+S)) \;&\leq\; \Delta'(J_{Y/E}Q_{E}T) \;+\; \Gamma'(Q_{E}S)\\ &\leq\; \Delta'(J_{Y}T) \;+\; \Gamma'(Q_{E}S).\end{aligned}$$

The desired inequality follows by taking the supremum over $E \in \mathcal{E}(Y)$.

◇

Corollary 4.2.9 *Let $T \in LR(X,Y)$. Then*

(a) $\Gamma_0'(J_YT) \;\leq\; \Gamma_0'(T)$.

(b) $\Gamma'(J_YT) \;=\; \Gamma'(T)$.

(c) $\Delta'(J_YT) \;\geq\; \Delta'(T)$.

PROOF

(a) Since $Y \subset \tilde{Y}$, the infimum for lefthand side of the inequality is evaluated over $\mathcal{F}(\tilde{Y}) \supset \mathcal{F}(Y)$.

(b) This follows directly from the definition.

(c) This follows from Proposition 4.2.8 (c) with $S = 0$.

◇

Proposition 4.2.10 *Let $T \in LR(X,Y)$ and suppose $S \in LR(Y,Z)$ satisfies $\dim S(0) < \infty$ and $D(S) = Y$. Then*

$$\Gamma_0'(ST) \;\leq \Gamma_0'(S)\Gamma_0'(T).$$

PROOF

Without loss of generality we assume that $\Gamma_0'(S) < \infty$ and $\Gamma_0'(T) < \infty$.

Letting $\epsilon > 0$, we may select, $F \in \mathcal{F}(Y)$ such that $||Q_{F}T|| \leq \Gamma_0'(T) + \epsilon$ and, since $S(0)$ is finite dimensional, $S(F) \in \mathcal{F}(Z)$. By Proposition 4.2.5, the latter implies that $\Gamma_0'(S) = \Gamma_0'(Q_{S(F)}S)$. Thus, by Lemma 4.2.3, we may select $G \in \mathcal{F}(Z)$ such that

$$||Q_{S(F)+G}S|| \leq \Gamma_0'(Q_{S(F)}S) + \epsilon = \Gamma_0'(S) + \epsilon.$$

Now let $N := N(Q_{S(F)+G}S) \supset F$. Representing $Q_{S(F)+G}S = (Q_{S(F)+G}S)\hat{}Q_N$ (it's canonical factorisation), we have

$$\begin{aligned}
\Gamma_0'(T) = \inf_{F \in \mathcal{F}(Z)} ||Q_F ST|| &\leq ||Q_{S(F)+G}ST|| = ||(Q_{S(F)+G}S)\hat{}Q_N T|| \\
&\leq ||(Q_{S(F)+G}S)\hat{}||\,||Q_N T|| \\
&\leq ||Q_{S(F)+G}S||\,||Q_N T|| \\
&\leq ||Q_{S(F)+G}S||\,||Q_F T|| \\
&\leq (\Gamma_0'(S) + \epsilon)\,(\Gamma_0'(T) + \epsilon).
\end{aligned}$$

Since ϵ was arbitrary, the result follows.

$\diamond$

4.3 Notes and Remarks

The material of this chapter is based on Cross [35], Chapter IV. Labuschagne presented properties of conjugate quantities for single-valued operators in [85]. References to original sources and research papers can be found in both these works. Proofs of theorems, which are applied in the sequel, are given in this chapter for reference and completeness - most of the proofs are essentially the same as those presented in [35], though minor differences occur in some arguments.

Chapter 5

Fredholm Type Linear Relations

5.1 Multivalued Semi-Fredholm Operators

Definitions 5.1.1 *We denote the classes of* **upper semi-Fredholm, lower semi-Fredholm** *and* **Fredholm linear relations** *from X into Y by $\mathcal{F}_+(X,Y)$, $\mathcal{F}_-(X,Y)$ and $\mathcal{F}(X,Y)$, respectively, and use the corresponding abbreviations $\mathcal{F}_+$, $\mathcal{F}_-$ and $\mathcal{F}$ when X and Y are understood. The sets are defined as follows:*

$$\begin{aligned} \mathcal{F}_+(X,Y) &:= \{T \in LR(X,Y) \mid \exists M \in \mathcal{C}(D(T)) \text{ s.t. } T|_M \textit{ is injective and open}\} \\ \mathcal{F}_-(X,Y) &:= \{T \in LR(X,Y) \mid T' \in \mathcal{F}_+(Y',X')\} \\ \mathcal{F}(X,Y) &:= \mathcal{F}_+(X,Y) \cap \mathcal{F}_-(X,Y). \end{aligned}$$

We say T is **partially continuous** *or* **partially open** *if there exists a finite codimensional subspace M of $D(T)$ such that $T|_M$ is continuous or open, respectively.*

Remarks 5.1.2 – The generalised definitions for Fredholm Relations

Many of the properties and theorems for semi-Fredholm type relations considered below are proved by verifying the single-valued case, and inferring the properties of T from the quotient QT. The following equivalences are used extensively, and without further reference:

$$T \in \mathcal{F}_- \quad \Leftrightarrow \quad T' \in \mathcal{F}_+ \tag{5.1}$$

$$T \in \mathcal{F}_+ \quad \Leftrightarrow \quad QT \in \mathcal{F}_+ \tag{5.2}$$

$$T \in \mathcal{F}_- \quad \Leftrightarrow \quad QT \in \mathcal{F}_-, \tag{5.3}$$

where (5.1) is an immediate consequence of the definitions, (5.2) follows from Corollary 2.3.9 and (5.3) follows from (5.1) and (5.2).

The definitions given for the classes $\mathcal{F}_+(X,Y)$, $\mathcal{F}_-(X,Y)$ and $\mathcal{F}(X,Y)$ explicitly include properties of openess which are implicit in the classes Φ_+, Φ_- and Φ defined below, while not specifying that the relations be closed or defined on Banach spaces. Moreover, the condition of *partial openess* is sufficient to ensure stability of Fredholm properties under additive perturbation. The notation for the classes of Φ_+, Φ_- and Φ linear relations stems from the classic definitions and notation for closed (single-valued) semi-Fredholm operators on Banach spaces (see for example Gohberg and Krein [57] or Goldberg [60]). We later show that the converse implications of Proposition 5.1.4 below also hold (see Theorem 5.2.10).

Notation 5.1.3
We write $T \in \Phi_+(X,Y)$ if $\alpha(T) < \infty$ and $R(T)$ is closed, and write $T \in \Phi_-(X,Y)$ if $\beta(T) < \infty$ and $R(T)$ is closed. The set $\Phi(X,Y)$ is defined $\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$. When X and Y are understood, the abbreviations Φ_+, Φ_- and Φ, respectively, are used.

Proposition 5.1.4 *Let $T \in LR(X,Y)$ be closed and let X and Y be complete.*

(a) $T \in \Phi_+(X,Y) \Rightarrow T \in \mathcal{F}_+$, and

(b) $T \in \Phi_-(X,Y) \Rightarrow T \in \mathcal{F}_-$.

PROOF
By the Open Mapping Theorem 3.3.7, it follows that a linear relation $T \in \Phi_+(X,Y) \cup \Phi_-(X,Y)$ is an open map.

(a) Since $\alpha(T) < \infty$, $D(T) = M \oplus N(T)$ where $M \in \mathcal{P}(D(T))$. Thus $R(T|_M) = R(T)$ and $T|_M$ is closed, and $T|_M$ is open and injective, i.e. $T \in \mathcal{F}_+$.

(b) By the Closed Range Theorem 3.3.8, $R(T')$ is closed. Thus, $\alpha(T') = \bar{\beta}(T) = \beta(T) < \infty$, (Proposition 2.7.6), and $T' \in \mathcal{F}_+(Y', X')$, i.e. $T \in \mathcal{F}_-$.

◇

Proposition 5.1.5 *Let $T \in LR(X,Y)$. The following properties are equivalent.*

(i) $T \in \mathcal{F}_+$

(ii) $\exists M \in \mathcal{C}(D(T))$ and $c > 0$ such that $||Tm|| \geq c||m||$ for all $m \in M$.

PROOF
(i) $\Rightarrow$ (ii): Suppose $T \in \mathcal{F}_+$. Then $\exists M \in \mathcal{C}(D(T))$ such that $T|_M$ is bounded below, i.e $\exists\ M \in \mathcal{C}(D(T))$ such that $\gamma(T|_M) > 0$ and $N(T|_M) = \{0\}$. Thus for $m \in M$,

$$||Tm|| \geq \gamma(T|_M)\, d(m, N(T|_M)) = \gamma(T|_M)\, ||m||.$$

(ii) $\Rightarrow$ (i): Clearly the stated inequality implies injectivity. Furthermore, $0 < c \leq \gamma(T|_M)$ implies that $T|_M$ is open.

◇

Proposition 5.1.6 *Let $T \in LR(X,Y)$. The following are equivalent:*

(i) $T \in \mathcal{F}_-$

(ii) $TJ_{D(T)} \in \mathcal{F}_-$.

(iii) $Q_F T \in \mathcal{F}_-$ *for* $F \subset Y$ *such that* $\dim F < \infty$.

PROOF

Applying Proposition 2.6.10, the equivalence (i) $\Leftrightarrow$ (ii) follows from the equivalences

$$T \in \mathcal{F}_- \Leftrightarrow T' \in \mathcal{F}_+ \Leftrightarrow Q_{T'}T' \in \mathcal{F}_+ \Leftrightarrow (TJ_{D(T)})' \in \mathcal{F}_+ \Leftrightarrow TJ_{D(T)} \in \mathcal{F}_-.$$

(i) $\Leftrightarrow$ (iii): By Propositions 2.6.7 and 2.6.9, $(Q_F T)' = T'J_{F^\perp}$. Thus, the desired equivalence follows noting that $F^\perp$ is finite codimensional, and $T' \in \mathcal{F}_+ \Leftrightarrow T'J_{F^\perp} \in \mathcal{F}_+$.

$\diamond$

The next example illustrates that the class of Fredholm relations may contain non-closed single-valued relations even when X and Y are complete.

Example 5.1.7 *Let X and Y be arbitrary Banach spaces, and let $S \in \mathcal{F} = \mathcal{F}_+ \cap \mathcal{F}_-$ be single-valued. Let P denote the natural bounded projection of $G(S)$ onto $D(S)$, let M be a dense subspace of $G(S)$ of codimension one, and let $J_{P(M)}$ denote the identity embedding of $P(M)$ in X. Choose $y_0 \in Y$, $y_0 \neq 0$ arbitrarily and let $(x_0, Sx_0) \in G(S) \setminus M$. Let $T \in LR(X,Y)$ be defined by :*

$$TJ_{P(M)} := SJ_{P(M)} \quad \text{with} \quad \begin{cases} Tx_0 = y_0 & if \quad Sx_0 = 0 \\ Tx_0 = 0 & if \quad Sx_0 \neq 0. \end{cases}$$

Then $D(T) = D(S)$, and we may choose $(x_n, Tx_n) = (x_n, Sx_n) \in M$ such that $(x_n, Sx_n) \to (x_0, Sx_0) \neq (x_0, Tx_0)$. Thus T is not closed. Since $P(M)$ is finite codimensional and $T|_{P(M)} = S|_{P(M)}$, it follows that $T \in \mathcal{F}_+$. To see that $T \in \mathcal{F}_-$, we let $F = sp\{Tx_0, Sx_0\}$. Since $S \in \mathcal{F}_-$, the desired property follows from the equality $Q_F T = Q_F S$ and Proposition 5.1.6

The next proposition gives equivalent conditions for an upper semi-Fredholm relation to be open:

Proposition 5.1.8 *Suppose $T \in LR(X,Y)$ and $\alpha(T) < \infty$. Then the following are equivalent:*

(i) *T is open,*

(ii) *For every $M \in \mathcal{P}(D(T))$, $T(M)$ is closed in $R(T)$, and if $N(T) \cap M = \{0\}$ then $T|_M$ is open and injective,*

(iii) *There exists $M \in \mathcal{P}(D(T))$ such that $T(M)$ is closed in $R(T)$ and $T|_M$ is open and injective.*

PROOF

Clearly we may assume that $D(T)$ is infinite dimensional.

(i) $\Rightarrow$ (ii) : Let $M \in \mathcal{P}(D(T))$. Then $M + N(T)$ is closed since $\alpha(T) < \infty$. Now suppose $QTx_k \longrightarrow QTx$ for $\{x_k\} \subset M$. Since T is open, we have $d(x - x_k, N(T)) \longrightarrow 0$. Thus, there exists $\{n_k\} \subset N(T)$ such that $x_k + n_k \longrightarrow x$. Since $M + N(T)$ is closed, $x \in M + N(T)$ and hence, $Tx \in T(M)$, i.e. $R(T|_M)$ is closed in $R(T)$.

Since $\alpha(T) < \infty$, there exists $M \in \mathcal{P}(D(T))$ such that $M \cap N(T) = \{0\}$ and $T|_M$ is injective. Hence, suppose $M \cap N(T) = \{0\}$, let P be a continuous single-valued projection defined on $M + N(T)$ with kernel $N(T)$ and let $\{x_k\} \subset M$ be as before. We have

$$Px_k = P(x_k + n_k) \longrightarrow Px.$$

Furthermore, for $x \in D(T)$ we have $TPx = Tx$ since $(I-P)x \in N(T)$, and hence, $(T|_M{}^{-1})Tx_k = x_k \longrightarrow x$. Since $\{Tx_k\} \subset T(M)$ was arbitrary, it follows that $T|_M$ is open.

The implication (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$(i) : Suppose $M \in \mathcal{P}(D(T))$, $T|_M$ is open and injective and $T(M)$ is closed in $R(T)$. Since $\alpha(T) < \infty$, there exists a finite dimensional subspace $F \subset D(T)$ such that $M+F+N(T) = D(T)$, $(M + F) \cap N(T) = \{0\}$, and $M \cap F = F \cap N(T) = M \cap N(T) = \{0\}$. Furthermore, we have

$$dim\, R(T)/T(M) \leq dim\, D(T)/M < \infty.$$

Thus, $R(T) = T(M) + F_2$, where F_2 is finite dimensional, and $(T|_{M+F})^{-1}$ is single-valued with domain $T(M) + F_2$. Now $((T|_{M+F})^{-1})|_{F_2}$ is continuous since $dim\, F_2 < \infty$, and $((T|_{M+F})^{-1})|_{T(M)} = T|_M{}^{-1}$ is continuous. Thus, $(T|_{M+F})^{-1}$ is continuous, i.e. $T|_{M+F}$ is open. Thus

$$\begin{aligned}
0 &< \gamma(T|_{M+F}) \\
&= \inf_{x \in M+F} \frac{||Tx||}{||x||} \\
&\leq \inf_{x \in M+F} \frac{||Tx||}{d(x, N(T))} \\
&= \inf_{x \in D(T)\setminus N(T)} \frac{||Tx||}{d(x, N(T))} \\
&= \gamma(T).
\end{aligned}$$

$\diamond$

5.2 Compact, Strictly Singular and Upper Semi-Fredholm Relations

The theory of compact (single-valued) operators is attributed mainly to F. Riesz, and forms part of the classic core of functional analysis and operator theory. Riesz showed that if K is a compact operator defined on a Banach space then the operator $T = \lambda I - K$ has finite dimensional kernel and closed finite codimensional range (see the Chapter 0, the Introduction). Bounded strictly singular operators were introduced by Kato [74] (see also Goldberg [60]), while unbounded strictly singular operators are discussed in Cross [37]. It is clear from the definitions that if T is a strictly singular relation then $T \notin \mathcal{F}_+$. The proof that precompact linear relations form a subclass of the class of strictly singular relations, is given in Theorem 5.2.4 below. While the adjoint of a compact operator is also compact (see Theorem 5.2.2 below), the same does not hold for strictly singular operators. This motivated the consideration of the the class of strictly cosingular relations which are discussed in Section 5.5. For convenience, we introduce the term **Singular Type Relation** to refer to any linear relation which is precompact, compact, strictly singular or strictly cosingular. This section recalls some of the basic properties of singular type relations, and the relationships between Fredholm type and singular type properties. Theorem 5.2.9 is fundamental to the latter, and is used to establish Theorem 5.2.10 which completes Proposition 5.1.4.

Definitions 5.2.1 *A relation $T \in LR(X,Y)$ is* **precompact** *if $Q_T T B_X$ is totally bounded. If $\overline{Q_T T B_X}$ is compact in Y, then T is called* **compact**. *T is said to be* **strictly-singular** *if there does not exist $M \in \mathcal{I}(D(T))$ such that $T|_M$ is injective and open.*

These definitions coincide with the standard definitions for precompact, compact and strictly singular single-valued operators. Indeed, if T is single-valued, then $Q_T T B_X = T B_X$.

Theorem 5.2.2 (Schauder) *Let $T \in LR(X,Y)$ be continuous and single-valued. Then T is precompact if and only if T' is compact.*

Corollary 5.2.3 *Let $T \in LR(X,Y)$ be continuous. Then T is precompact if and only if T' is compact.*

PROOF

Since T is continuous and precompact if and only if $QTJ_{D(T)}$ is continuous and precompact, and $(QTJ_{D(T)})' = Q_{T'}T'J_{T(0)^\perp}$ is single-valued, the result follows from the theorem for single-valued relations.

◇

Theorem 5.2.4 *If $T \in LR(X,Y)$ is precompact, then T is continuous and strictly singular.*

PROOF

Continuity follows from the fact that totally bounded sets are bounded, i.e. $QT(B_{D(T)})$ is a bounded set, and hence QT is continuous. To see that T is strictly singular, suppose $QT|_M$ is open and injective for some subspace M of $D(T)$. Then it follows that B_M is totally bounded since $QT(B_M)$ is totally bounded. Thus, by Theorem 1.4.9, M must be finite dimensional.

◇

Proposition 5.2.5 *Let $T \in LR(X,Y)$ be continuous with finite dimensional range. Then T is compact.*

PROOF
By Proposition 2.2.11,

$$QTB_X \subset ||T||QB_{R(T)} \subset ||T||B_{R(QT)}.$$

Since $dim\, R(T) < \infty$, $B_{R(QT)}$ is a compact subset of QY. Thus, T is compact.

$\diamond$

The next example illustrates that the property may fail if T is not assumed to be continuous.

Example 5.2.6 *Let M be a nonclosed, dense subspace of a normed linear space X and let P denote a projection of X onto M with kernel N satisfying $M + N = X$ and $M \cap N = \{0\}$. Then $R(I - P) = N$ is finite-dimensional, while $I - P$ is not continuous.*

Lemma 5.2.7 *Let B be a bounded subset of a normed linear space X, and suppose $\{x'_1, x'_2, \ldots, x'_n\}$ is a finite subset of X'. Then for any $\epsilon > 0$ there exists a finite set $\{x_1, x_2, \ldots, x_m\} \subset B$ such that for any $x \in B$, there exists $x_j \in \{x_1, x_2, \ldots, x_m\}$ satisfying*

$$\sum_{i=1}^{n} |x'_i x - x'_i x_j| \leq \epsilon, \qquad and$$

$$|x'_i x - x'_i x_j| \leq \epsilon, \quad 1 \leq i \leq n.$$

PROOF
Define $K : X \to I\!K^n$ by $Kx := (x'_1 x,\ x'_2 x,\ \ldots,\ x'_n x)$. Then K is continuous with finite dimensional range, and therefore, by Proposition 5.2.5, K is compact. Let $\epsilon > 0$. Since $K(B)$ is totally bounded, it follows that there exists a finite set $\{x_1, x_2, \ldots, x_m\} \subset B$ such that for $x \in B$, there exists $x_j \in \{x_1, x_2, \ldots, x_m\}$ such that

$$\sum_{i=1}^{n} |x'_i x - x'_i x_j| \ \leq \ ||Kx - Kx_j|| \ \leq \ \epsilon.$$

Hence, the inequalities $|x'_i x - x'_i x_j| \leq \epsilon,\ 1 \leq i \leq n,$ also hold.

$\diamond$

Proposition 5.2.8 *Let $T,\ T_n \in LR(X,Y)$ be everywhere-defined relations with each T_n precompact, $\bigcup_{n=1}^{\infty} T_n(0) \subset \overline{T(0)}$ and $\lim_{n\to\infty} ||T_n - T|| = 0$. Then T is precompact.*

PROOF
Let $\epsilon > 0$, and choose $N \in I\!N$ such that

$$||T - T_N|| < \frac{\epsilon}{3} \tag{5.4}$$

Since T_N is precompact, $Q_{T_N} T_N B_X$ is totally bounded. Thus there exists a subset $\{x_1, x_2, \ldots x_m\} \subset B_X$ such that for each $x \in B_X$, $\exists\, x_i \in \{x_1, x_2, \ldots x_m\}$ for which

$$||T_N x - T_N x_i|| < \frac{\epsilon}{3}. \tag{5.5}$$

Since $T_N(0) \subset \overline{T(0)}$, it follows from (5.4) and (5.5) that

$$\begin{aligned} ||Tx - Tx_i|| &= ||Tx - T_N x + T_N x - T_N x_i + T_N x_i - Tx_i|| \\ &\leq ||Tx - T_N x|| + ||T_N x - T_N x_i|| + ||T_N x_i - Tx_i|| \\ &< \epsilon. \end{aligned}$$

◇

Theorem 5.2.9 *Let $T \in LR(X,Y)$. Then the following properties are equivalent.*

(i) $T \notin \mathcal{F}_+$.

(ii) *There does not exist $M \in \mathcal{P}(X)$ such that $T|_M$ is injective and open.*

(iii) *Given $\epsilon > 0$, $\exists\; M \in \mathcal{I}(X)$ such that $T|_M$ is precompact with $||T|_M|| \leq \epsilon$.*
In this case, if T is closed, then we may assume M is closed.

PROOF
Since $T(0) = T|_M(0)$, it suffices to prove the result for QT. Therefore, we will assume that T is single-valued.

Clearly (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii): Suppose (ii) holds. Let $\epsilon > 0$. By Proposition 5.1.5 and the Hahn-Banach Theorem, we may select $x_1 \in D(T)$ and $x_1' \in X'$ such that

$$||x_1|| = 1 \quad \textit{and} \quad ||Tx_1|| < 3^{-1}\epsilon;$$
$$||x_1'|| = 1 \quad \textit{and} \quad x_1' x_1 = ||x_1|| = 1.$$

Since $N(x_1')$ is closed and finite codimensional, we may apply Proposition 5.1.5 and the Hahn-Banach Theorem again, and select $x_2 \in N(x_1')$ and $x_2' \in X'$ such that

$$||x_2|| = 1 \quad \textit{and} \quad ||Tx_2|| < 3^{-2}\epsilon;$$
$$||x_2'|| = 1 \quad \textit{and} \quad x_2' x_2 = ||x_2|| = 1.$$

Similarly $N(x_1') \cap N(x_2')$ is closed and finite codimensional, and there exists $x_3 \in N(x_1') \cap N(x_2')$ and $x_3' \in X'$ such that $||x_3|| = 1$, $||Tx_3|| < 3^{-3}\epsilon$, $||x_2'|| = 1$, and $x_2' x_2 = ||x_2|| = 1$. Continuing in this way we obtain:

$$\begin{aligned} ||x_k|| &= ||x_k'|| = x_k' x_k = 1 \\ ||Tx_k|| &< 3^{-k}\epsilon \quad \textit{for} \quad 1 \leq k < \infty \end{aligned} \tag{5.6}$$

$$\begin{aligned} x_k &\in \bigcap_{i=1}^{k-1} N(x_i') \\ i.e., \ \ x_i' x_k &= 0 \ \ for \ \ 1 \le i < k \end{aligned} \tag{5.7}$$

Clearly the set $\{x_k \mid 1 \le k < \infty\}$ is linearly independent. Let $M = sp\{x_k \mid 1 \le k < \infty\}$. Then M is an infinite dimensional subspace of $D(T)$. For $x \in M$, we have $x = \sum_{i=1}^{m} a_i x_i$, and, applying (5.6) and (5.7),

$$|a_1| = |x_1' x| \le ||x_1'|| \, ||x|| = ||x||.$$

Now by (5.6) and (5.7) again, for $j \le m$,

$$x_j' x = \sum_{i=1}^{j-1} a_i x_j'(x_i) + a_j.$$

Hence, the inequality $|a_k| \le 2^{k-1}||x||$ for $k < j \le m$, implies

$$\begin{aligned} |a_j| &\le |x_j' x| + \sum_{i=1}^{j-1} |a_i| \, |x_j'(x_i)| \\ &\le ||x|| + \sum_{i=1}^{j-1} 2^{i-2}||x|| \\ &= 2^{j-1}||x||, \end{aligned}$$

and thus, by induction, $|a_k| \le 2^{k-1}||x||$ for $1 \le k \le m$.

Let P_n denote the projection of M onto $sp\{x_1, x_2, \ldots, x_n\}$ with kernel $sp\{x_{n+1}, x_{n+2}, \ldots\}$. Then for $x = \sum_{i=1}^{m} a_i x_i \in M$, we have

$$\begin{aligned} ||Tx - TP_n x|| &= ||\sum_{i=n+1}^{\infty} a_i T x_i|| \\ &\le \sum_{i=n+1}^{\infty} |a_i| \, ||Tx_i|| \\ &< \sum_{i=n+1}^{\infty} 2^{i-1} 3^{-i} \epsilon ||x|| \\ &= \epsilon \frac{1}{2} ||x|| \sum_{i=n+1}^{\infty} \frac{2}{3}^{i} \\ &\to 0 \ \ as \ \ n \to \infty. \end{aligned}$$

Since

$$||TP_n x|| = ||\sum_{i=1}^{n} a_i T x_i|| < \epsilon(\tfrac{1}{2} \sum_{i=1}^{n} \tfrac{2}{3}^{i}) \, ||x||,$$

it follows that TP_n is continuous. Thus, since $dim\, R(TP_n) < \infty$, it follows from Proposition 5.2.5 that TP_n is precompact. Hence $T|_M$ is precompact (Proposition 5.2.8). Furthermore, for $x \in B_M$ we have

$$||Tx|| \le \sum_{i=1}^{m} |a_i|\,||Tx_i|| \le \sum_{i=1}^{m} 2^{i-1}3^{-i}\epsilon||x|| \le \epsilon||x||$$

Thus $||T|_M|| \le \epsilon$ as required.

(iii) $\Rightarrow$ (i): Suppose (iii) holds while $T \in \mathcal{F}_+$. Let $M \in \mathcal{I}(X)$ be a subspace such that $T|_M$ is precompact. Choose $E \in \mathcal{C}(X)$ such that $T|_E$ is injective and open. Hence $\exists\ \lambda > 0$ such that $TB_{E\cap M} \supset \lambda B_{T(E\cap M)}$. Since $TB_{E\cap M}$ is totally bounded, the unit ball of $R(T|_{E\cap M})$ is also totally bounded and $R(T|_{E\cap M})$ is finite-dimensional. Since $T|_{E\cap M}$ is injective, $E \cap M$ must be finite-dimensional as well, a contradiction.

◇

Theorem 5.2.10 *Let X and Y be complete, and T be closed. The following are equivalent:*

(i) $T \in \mathcal{F}_+$

(ii) $T \in \Phi_+$

PROOF
The implication (ii) $\Rightarrow$ (i) follows from Proposition 5.1.4.

We prove (i) $\Rightarrow$ (ii) for the case when T is single-valued first.

By Proposition 5.2.9, $\exists\ M \in \mathcal{C}(D(T))$ such that $\gamma(T|_M) > 0$ and $(T|_M)^{-1}(0) = 0$ and since T is closed, we may assume that M is closed as well. From this it follows that $D(T) = M \oplus N$, where N is finite dimensional and $N(T) \subset N$. Thus $\alpha(T) < \infty$ and $N = F \oplus N(T)$ for some $F \subset D(T)$, $dim F < \infty$. Since $T|_M$ is closed and $\gamma(T|_M) > 0$, $R(T|_M)$ is closed. Thus, since $dim TF < \infty$, it follows that $R(T) = T(M + F) = TM + TF$ is closed.

Passing to the general case, since $T(0)$ is closed it follows that $N(T) = N(QT)$. Furthermore, since $R(QT) = R(T)/T(0)$ is closed, we have that R(T) is closed.

◇

Corollary 5.2.11 $T \in \mathcal{F}_- \Leftrightarrow \tilde{T} \in \Phi_-$

PROOF
Since T' is closed, the result follows from Proposition 2.7.6 and the Closed Range Theorem 3.3.8.

◇

In Corollary 5.8.5, we show that the equivalence $T \in \mathcal{F}_+ \Leftrightarrow \tilde{T} \in \Phi_+$ also holds.

5.3 Operator Quantities and the Classification of Linear Relations I

Certain classes of linear relations may be characterised as sets for which a particular operator quantity is zero. In particular, compact relations and, more generally, linear relations which are not upper semi-Fredholm, occur as the zero sets of quantities given in Section 4.1 of Chapter 4. The classification of relations in terms of operator quantities is applied in Section 5.6 where it is shown that semi-Fredholm properties are stable under various additive perturbations. Analogous theorems for lower semi-Fredholm relations are given Section 5.5 and Section 5.7. of this chapter.

Theorem 5.3.1 *Let $T \in LR(X,Y)$ and $\dim D(T) = \infty$. Then*

$$T \in \mathcal{F}_+ \textit{ if and only if } \Gamma(T) > 0.$$

PROOF

It suffices to prove the result for QT, and hence we may assume that T is single-valued.

Suppose $T \in \mathcal{F}_+$. By Proposition 5.1.5, there exists $M \in \mathcal{C}(D(T))$ and $c > 0$ such that $\frac{||Tm||}{||m||} \geq c$ for any $m \in M$, $m \neq 0$. Thus $||T|_N|| \geq c$ for any $N \in \mathcal{I}(M)$, and hence

$$\Gamma(T|_M) = \inf_{N \in \mathcal{I}(M)} ||T|_N|| \geq c.$$

By Proposition 4.1.3, it follows that $\Gamma(T) \geq c > 0$.

Conversely, suppose $T \notin \mathcal{F}_+$. Given $\epsilon > 0$, it follows from Theorem 5.2.9 that there exists $M \in \mathcal{I}(D(T))$ such that $||T|_M|| \leq \epsilon$. Thus

$$\Gamma(T) = \inf_{M \in \mathcal{I}(D(T))} ||T|_M|| \leq \epsilon.$$

Since ϵ is arbitrary, $\Gamma(T) = 0$.

◇

Theorem 5.3.2 *Let $T \in LR(X,Y)$. Then T is strictly singular if and only if $\Delta(T) = 0$.*

PROOF

As before, it suffices to prove the result for QT, and hence we may assume that T is single-valued.

Suppose T is strictly singular. Then clearly $T|_M$ is strictly singular for any $M \in \mathcal{I}(D(T))$, and hence, $T|_M \notin \mathcal{F}_+$ for any $M \in \mathcal{I}(D(T))$. By Theorem 5.3.1, this implies that

$$\Delta(T) = \sup_{M \in \mathcal{I}(D(T))} \Gamma(T|_M) = 0.$$

Conversely, if $\Delta(T) = 0$, then $\Gamma(T|_M) = 0$ for every $M \in \mathcal{I}(D(T))$. By Theorem 5.3.1 again, $T|_M \notin \mathcal{F}_+$ for any $M \in \mathcal{I}(D(T))$. In particular, $T|_M$ does not have a continuous single-valued inverse for any $M \in \mathcal{I}(D(T))$, and hence, T is strictly singular.

◇

Corollary 5.3.3 *If T, $S \in LR(X,Y)$ are strictly singular then $T + S$ is strictly singular.*

PROOF

This follows from Theorem 5.3.2 and Corollary 4.1.8.

◇

Corollary 5.3.4

(a) If $T \in LR(X,Y)$ is strictly singular and $S \in LR(Y,Z)$ is continuous, and if $\overline{T(0)} \subset D(S)$, then ST is strictly singular.

(b) If $S \in LR(Y,Z)$ is continuous and strictly singular and $T \in LR(X,Y)$, and if $\overline{T(0)} \subset D(S)$ and $\Delta(T) < \infty$, then ST is strictly singular.

PROOF

(a) Suppose first that T is single-valued. If S is continuous then, by Proposition 4.1.12, $\Delta(S) \leq \bar{\Gamma}_0(S) < \infty$, and hence the inequality

$$\Delta(ST) \leq \Delta(S)\Delta(T)$$

of Proposition 4.1.11 is defined. Thus the result follows from Theorem 5.3.2.

More generally, let Q denote Q_T. Now QT is single-valued and strictly singular, and since S is continuous, so is SQ^{-1}. Thus, since $\overline{T(0)} \subset D(S)$, it follows from Proposition 2.3.15 and what has already been proved that $Q_{ST}ST = Q_{ST}SQ^{-1}QT$ is strictly singular. The result follows.

(b) The argument for this case is the similar to (a).

◇

Theorem 5.3.5 *Let $T \in LR(X,Y)$. Then*

(a) T is precompact if and only if $\bar{\Gamma}_0(T) = 0$.

(b) T is partially precompact if and only if $\Gamma_0(T) = 0$.

PROOF

(a) It suffices to prove the result for QT, and hence we may assume that T is single-valued. Without loss of generality, we assume further that $D(T) = X$.

Suppose T is precompact, and $\epsilon > 0$. There exists $\{x_1, x_2, \ldots, x_n\} \subset B_X$ such that for $x \in B_X$ there exists $x_k \in \{x_1, x_2, \ldots, x_n\}$ such that

$$||Tx - Tx_k|| < \tfrac{\epsilon}{2}.$$

By the Hahn-Banach theorem, we may choose $\{y'_1, y'_2, \ldots, y'_n\} \subset B_{Y'}$ such that $y'_iTx_i = ||Tx_i||$, $1 \leq i \leq n$, and $|y'_iTx| \leq ||Tx||$. Let $N := \bigcap_{i=1}^{n} N(y'_iT)$. Then for $x \in N \cap B_X$ we have

$$\begin{aligned}
||Tx|| &\leq ||Tx_k|| + ||Tx - Tx_k|| \\
&< y'_kTx_k + \frac{\epsilon}{2} \\
&= |y'_k(Tx_k - Tx)| + \frac{\epsilon}{2} \\
&\leq ||Tx_k - Tx|| + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2}.
\end{aligned}$$

Since $x \in N \cap B_X$ was arbitrary, $||T|_N|| < \epsilon$. Since N is closed and finite codimensional in X and ϵ was arbitrary, it follows that $\bar{\Gamma}_0(T) = \inf_{N \in \mathcal{P}(X)} ||T|_N|| = 0$.

Now suppose $\bar{\Gamma}_0(T) = 0$, and let $\epsilon > 0$. Choose $M \in \mathcal{P}(X)$ such that $||T|_M|| < \epsilon$. There exists a finite dimensional subspace F of X such that $X = M + F$ and $M \cap F = \{0\}$, and hence we may fix some basis $\{b_1,\ b_2,\ \ldots,\ b_n\}$ for F. By the Hahn-Banach theorem there exists $\{x'_1,\ x'_2,\ \ldots,\ x'_n\} \subset M^{\perp}$ such that $x'_i b_j = \delta_{ij}$. Thus, for $x \in X$ we may write

$$x \;=\; m + \sum_{i=1}^{n} x'_i(x)\, b_i,$$

where $m \in M$, and hence,

$$||m|| \;\leq\; ||x|| + \sum_{i=1}^{n} |x'_i(x)|\, ||b_i||.$$

Since $||T|_M|| < \epsilon$, it follows that

$$\begin{aligned}
||Tx|| \;&<\; \epsilon||m|| + \sum_{i=1}^{n} |x'_i(x)|\, ||Tb_i|| \\
&\leq\; \epsilon||x|| + \sum_{i=1}^{n} |x'_i(x)|\, (\, \epsilon\, ||b_i|| + ||Tb_i||\,) \\
&\leq\; \epsilon||x|| + c\sum_{i=1}^{n} |x'_i(x)| \qquad (5.8)
\end{aligned}$$

where $c := max\{\ \epsilon\, ||b_i|| \ + \ ||Tb_i|| \ \mid 1 \leq i \leq n\}$. By Lemma 5.2.7, there exists $\{x_1,\ x_2,\ \ldots,\ x_k\} \subset B_X$ such that for $z \in B$, we have $\sum_{i=1}^{n} |x'_i z - x'_i x_k| < \epsilon$ for some $x_k \in \{x_1,\ x_2,\ \ldots,\ x_k\}$. From (5.8) it follows that

$$||Tz - Tx_k|| \;<\; \epsilon||z - x_k|| + c\epsilon \;\leq\; (2+c)\epsilon.$$

Since ϵ is arbitrary, TB_X is totally bounded.

(b) We may assume that $D(T)$ is infinite dimensional. If T is partially precompact then there exists $M \in \mathcal{C}(X)$ such that $T|_M$ is precompact and hence, by (a), $\bar{\Gamma}_0(T|_M) = 0$. Since $M \in \mathcal{C}(X)$,

$$\Gamma_0(T) \;=\; \Gamma_0(T|_M) \;\leq\; \bar{\Gamma}_0(T|_M) \;=\; 0.$$

Conversely, if $\Gamma_0(T) = 0$ then for any $\epsilon > 0$ there exists $M \in \mathcal{C}(X)$ such that $||T|_M|| < \epsilon$, and hence, $||T|_M||$ is continuous. It follows that

$$0 \;=\; \Gamma_0(T) \;=\; \Gamma_0(T|_M) \;=\; \inf_{N \in \mathcal{C}(M)} ||T|_N|| \;=\; \inf_{\overline{N} \in \mathcal{P}(M)} ||T|_N|| \;=\; \bar{\Gamma}_0(T|_M).$$

Thus, by (a), $T|_M$ is precompact.

$\diamond$

Corollary 5.3.6 *Let $T \in LR(X,Y)$. Then*

(a) T is precompact if and only if for any $\epsilon > 0$ there exists $M \in \mathcal{P}(D(T))$ such that $||T|_M|| < \epsilon$.

(b) T is partially precompact if and only if for any $\epsilon > 0$ there exists $M \in \mathcal{C}(D(T))$ such that $||T|_M|| < \epsilon$.

PROOF

(a) and (b) follow from the definitions of $\bar{\Gamma}_0(T) = 0$ and $\Gamma_0(T) = 0$, respectively.

◇

The arguments for the following two corollaries are similar to those given for Corollaries 5.3.2 and 5.3.4.

Corollary 5.3.7 *If $T,\ S \in LR(X,Y)$ are precompact (partially precompact) then $T + S$ is precompact (partially precompact).*

PROOF

This follows from Theorem 5.3.5 and Proposition 4.1.9.

◇

Corollary 5.3.8

(a) If $T \in LR(X,Y)$ is precompact (partially precompact) and $S \in LR(Y,Z)$ is continuous, and if $\overline{T(0)} \subset D(S)$ then ST is precompact (partially precompact).

(b) If $S \in LR(Y,Z)$ is precompact (partially precompact) and $T \in LR(X,Y)$, and if $\overline{T(0)} \subset D(S)$ and $\bar{\Gamma}_0(T) < \infty$, (or $\Gamma_0(T) < \infty$,) then ST is precompact (partially precompact).

PROOF

(a) Suppose first that T is single-valued. If S is continuous then, by Proposition 4.1.12, $\Gamma_0(S) \leq \bar{\Gamma}_0(S) < \infty$, and hence the inequalities

$$\begin{aligned} \bar{\Gamma}_0(ST) &\leq \bar{\Gamma}_0(S)\bar{\Gamma}_0(T), \text{ and} \\ \Gamma_0(ST) &\leq \Gamma_0(S)\Gamma_0(T) \end{aligned}$$

of Proposition 4.1.13 are defined. Thus the result follows from Theorem 5.3.5.

More generally, let Q denote Q_T. Now QT is single-valued and precompact (partially precompact) and, since S is continuous, so is SQ^{-1}. Thus, since $\overline{T(0)} \subset D(S)$, it follows from Proposition 2.3.15 and what has already been proved that $Q_{ST}ST = Q_{ST}SQ^{-1}QT$ is precompact (partially precompact). The result follows.

(b) follows by the same argument applied in (a).

◇

5.4 Perturbation Theorems for Upper Semi-Fredholm Linear Relations

Proposition 5.4.1 *Let $T \in LR(X,Y)$ and $S \in LR(Y,Z)$. Then*

(a) If $||S|| < \infty$ and $T(0) \subset D(S)$, then $ST \in \mathcal{F}_+ \Rightarrow T|_{D(ST)} \in \mathcal{F}_+$.
In addition, if $D(ST) \in \mathcal{C}(D(T))$, for example if $R(T) \subset D(S)$, then $T \in \mathcal{F}_+$.

(b) If $S, T \in \mathcal{F}_+$ then $ST \in \mathcal{F}_+$.

PROOF

(a) We apply Corollary 2.3.13 and Proposition 5.1.5. Since $T(0) \subset D(S)$ and $ST \in \mathcal{F}_+$, there exists $M \in \mathcal{C}(D(ST))$ and $c > 0$ such that

$$||S||\,||Tm|| \;\geq\; ||STm|| \;\geq\; c||m|| \quad \forall\, m \in M.$$

Thus, $T|_M$ is bounded below, and by Proposition 5.1.5 again, $T|_{D(ST)} \in \mathcal{F}_+$. If $D(ST) \in \mathcal{C}(D(T))$, then $M \in \mathcal{C}(D(T))$, and $T \in \mathcal{F}_+$.

(b) Clearly it is enough to consider the case $dim\, D(ST) = \infty$. It follows that $dim\, D(T) = \infty$, $\Gamma(T) > 0$ and, since $\alpha(T) < \infty$, we also have $dim\, R(T) = \infty$ and $dim\, D(S) = \infty$.

Choose $M \in \mathcal{C}(X)$ and $V \in \mathcal{C}(Y)$ such that $T|_M$ and $S|_V$ are injective and open. Then $T(M) \in \mathcal{C}(R(T))$, and hence $dim\, T(M) = \infty$. Since $V \cap T(M) \in \mathcal{C}(T(M))$, it follows that $T^{-1}(V \cap T(M)) \in \mathcal{C}(M + N(T))$. Thus, since $\alpha(T) < \infty$ and M is finite codimensional, $T^{-1}(V \cap T(M)) \in \mathcal{C}(D(T))$. Let $L := T^{-1}(V \cap T(M)) \cap M \in \mathcal{C}(D(T))$ and let $S_1 := S|_{V \cap T(M)}$. Then

$$D(S_1T) \;=\; T^{-1}(V \cap T(M) \cap D(S)) \;=\; T^{-1}(V) \cap M \cap D(ST) \;\in\; \mathcal{C}(D(ST)).$$

Now $T|_L$ is bounded below, $||S_1^{-1}|| < \infty$ and $S_1T(0) \subset D(S_1^{-1})$. Thus, for $x \in L$ we have, by Corollary 2.3.13,

$$||S_1^{-1}||\,||S_1Tx|| \;\geq\; ||S_1^{-1}S_1Tx|| \;=\; ||Tx|| \;\geq\; c||x||,$$

for some $c > 0$. Thus S_1T is bounded below on $L \in \mathcal{C}(D(T))$, and hence $\Gamma(ST) \;=\; \Gamma(S_1T) \;>\; 0$.

◇

Proposition 5.4.2 *Let $S, T \in LR(X,Y)$. If $S(0) \subset \overline{T(0)}$ then*

$$\Delta(S) \;<\; \Gamma(T) \;\Rightarrow\; T + S \in \mathcal{F}_+.$$

PROOF

Assume $\Gamma(T) - \Delta(S) > 0$. Then we may choose ϵ such that $\Gamma(T) - \Delta(S) > \epsilon > 0$. Suppose $T + S \notin \mathcal{F}_+$. Then $\Gamma(T+S) = 0$, and there exists $M \in \mathcal{I}(D(T+S))$ such that $||(T+S)|_M|| < \epsilon$. Since $S(0) \subset \overline{T(0)}$, we have for $x \in M$

$$||Tx|| \;\leq\; ||Tx + Sx|| + ||Sx|| \;<\; \epsilon||x|| + ||Sx||.$$

Thus for $x \in B_M$, $||Tx|| - ||Sx|| < \epsilon$, and hence, $||T|_N|| < ||S|_N|| + \epsilon$ for $N \in \mathcal{I}(M)$. Taking the infimum over $N \in \mathcal{I}(M)$, it follows that $\Gamma(T|_M) \leq \Delta(S) + \epsilon$ which is a contradiction.

$\diamond$

Proposition 5.4.3 *Suppose $T \in \mathcal{F}_+(X,Y)$ with $G(S) \subset G(T)$ and $\dim D(S) = \infty$. Then $S \in \mathcal{F}_+$.*

PROOF
Since $G(S) \subset G(T)$, we have for $x \in D(S)$

$$\begin{aligned} ||T|_{D(S)}x|| &= d(y, T(0)) \ \ for \ \ y \in Tx \\ &\leq d(y, T(0)) \ \ for \ \ y \in Sx \\ &\leq d(y, S(0)) \ \ since \ \ S(0) \subset T(0) \\ &= ||Sx||, \end{aligned}$$

i.e. $||T_{D(S)}|| \leq ||S||$. Thus

$$\begin{aligned} 0 < \Gamma(T) &= \inf_{M \in \mathcal{I}(D(T))} ||T|_M|| \leq \inf_{M \in \mathcal{I}(D(S))} ||T|_M|| \\ &\leq \inf_{M \in \mathcal{I}(D(S))} ||S|_M|| = \Gamma(S) \end{aligned}$$

and the result follows.

$\diamond$

5.5 Lower Semi-Fredholm Relations

We begin by summarising some properties.

Proposition 5.5.1 *Let $T \in LR(X,Y)$. The following are equivalent:*

(i) $T \in \mathcal{F}_-$,

(ii) $\tilde{\beta}(T) < \infty$ *and* $\gamma(T') > 0$,

(iii) $\tilde{T} \in \Phi_-$,

(iv) $TJ_{D(T)} \in \mathcal{F}_-$,

(v) $QT \in \mathcal{F}_-$,

(vi) $Q_F T \in \mathcal{F}_-$ *for* $F \in F(Y)$.

PROOF
(i) $\Leftrightarrow$ (v) was observed from the definitions (see equivalence (5.3) in Section 5.1),
(i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vi) are contained in Proposition 5.1.6, and
(i) $\Leftrightarrow$ (iii) is a restatement of Corollary 5.2.11.

(i) $\Leftrightarrow$ (ii): Since $T' \in \mathcal{F}_+$, we have $\tilde{\beta}(T) = \alpha(T') < \infty$ (Proposition 2.7.6). Since $T' = \tilde{T}' \in \Phi_+$ (Theorem 5.2.10), $R(T')$ is closed. By the Open Mapping Theorem 3.3.7, this implies that T'

is open, i.e. $\gamma(T') > 0$. Conversely, if $\tilde{\beta}(T) = \alpha(T') < \infty$, and $\gamma(T') > 0$, then $R(T')$ is closed (another application of the Open Mapping Theorem), and hence applying Theorem 5.2.10 again, $T' \in \mathcal{F}_+$.

◇

Theorem 5.5.2 *Let $T \in LR(X,Y)$. The following properties are equivalent.*

(i) $T \notin \mathcal{F}_-$,

(ii) There does not exist $F \in \mathcal{F}(Y)$ such that $T'|_{F^\perp}$ is bounded below.

(iii) Given $\epsilon > 0$, there exists a compact operator $K \in LR(X, \tilde{Y})$ such that $||K|| < \epsilon$, $D(K) \supset D(T)$ and $\tilde{\beta}(J_Y T - K) = \infty$.

PROOF

(i) $\Rightarrow$ (ii): Suppose $T \notin \mathcal{F}_-$ and let $F \subset Y$ be finite dimensional. Then $F^\perp \in \mathcal{P}(Y')$. By Theorem 5.1.5, since $T' \notin \mathcal{F}_+$, there does not exist $M \in \mathcal{C}(Y')$ such that $T'|_M$ is bounded below. In particular, $T'|_{F^\perp}$ is not bounded below.

(ii) $\Rightarrow$ (iii): Let $\epsilon > 0$ and $\delta > 0$. Since T' is not bounded below, we may select $y_1' \in D(T')$ and $y_1 \in Y$ such that

$$\begin{aligned} ||y_1'|| = 1 \quad &\textit{and} \quad ||T'y_1'|| \ < \ 2^{-1}(1+\delta)^{-1}.\epsilon; \\ y_1' y_1 = 1 \quad &\textit{and} \quad ||y_1|| \ \le \ (1+\delta). \end{aligned}$$

Suppose $\{y_1',\ y_2',\ \dots, y_{n-1}'\} \subset D(T')$ and $\{y_1,\ y_2,\ \dots, y_{n-1}\} \subset Y$ have been selected such that

$$\begin{aligned} ||y_k'|| = 1 \quad &\textit{and} \quad ||T'y_k'|| \ < \ 2^{-k}(1+\delta)^{-1}(2+\delta)^{1-k}.\epsilon; \\ y_k' y_j = \delta_{kj} \quad &\textit{and} \quad ||y_k|| \ \le \ (1+\delta)(2+\delta)^{k-1} \end{aligned} \tag{5.9}$$

for $1 \le k,\ j \le n-1$ and $n \ge 2$. By (ii), the restriction of T' to the subspace $\{y_1,\ y_2,\ \dots, y_{n-1}\}^\perp \subset Y'$ is not bounded below and hence there exists $y_n' \in \{y_1,\ y_2,\ \dots, y_{n-1}\}^\perp \cap D(T')$ and $y \in Y$ such that

$$\begin{aligned} ||y_n'|| = 1 \quad &\textit{and} \quad ||T'y_n'|| \ < \ 2^{-n}(1+\delta)^{-1}(2+\delta)^{1-n}.\epsilon; \\ y_n' y = 1 \quad &\textit{and} \quad ||y|| \ \le \ (1+\delta). \end{aligned}$$

Then $y_n := y - \sum\limits_{k=1}^{n-1} y_k'(y) y_k$ satisfies

$$\begin{aligned} y_n'(y_k) \quad &= \quad \begin{cases} 0 & if \ \ 1 \le k < n, \\ y_n'(y) \ = \ 1, & if \ \ k = n \end{cases} \\ y_k'(y_n) \quad &= \quad y_k'(y) - y_k'(y) \ = \ 0 \quad if \ \ 1 \le k < n, \end{aligned}$$

i.e. $y'_k(y_j) = \delta_{kj}$ for $1 \leq k,\ j \leq n$, and

$$\begin{aligned} ||y_n|| &\leq ||y||\,(1 + \sum_{k=1}^{n-1} ||y_k||\,) \\ &\leq (1+\delta)(1 + \sum_{k=1}^{n-1}(1+\delta)(2+\delta)^{k-1}) \\ &= (1+\delta)(2+\delta)^{n-1}. \end{aligned}$$

Thus, there exists sequences $\{y'_n\} \subset Y'$ and $\{y_n\} \subset Y$ such that the conditions (5.9) hold.

Define the single-valued operators $K_n,\ K \in LR(X, \tilde{Y})$ as follows:

$$\begin{aligned} K_n x &:= \sum_{k=1}^{n} T'y'_k(x) y_k, \quad \textit{and} \\ Kx &:= \sum_{k=1}^{\infty} T'y'_k(x) y_k. \end{aligned}$$

Clearly $D(K) \supset D(T)$. Furthermore,

$$\begin{aligned} ||K_n x|| &\leq \sum_{k=1}^{n} ||T'y'_k||\,||x||\,||y_k|| \;<\; \epsilon(\sum_{k=1}^{n} 2^{-k})\,||x|| \\ ||Kx|| &\leq \sum_{k=1}^{\infty} ||T'y'_k||\,||x||\,||y_k|| \;<\; \epsilon(\sum_{k=1}^{\infty} 2^{-k})\,||x|| \;=\; \epsilon||x||. \end{aligned}$$

Thus K exists since $\tilde{Y}$ is complete, and $K_n \to K$. Now K_n is continuous and has finite-dimensional range for each $n \in I\!N$. Thus, by Proposition 5.2.5, K_n is compact for each $n \in I\!N$. By Proposition 5.2.8, it follows that K is compact. Furthermore, if $x \in D(T)$ then (5.9) implies that $y'_k(Kx) = T'y'_k(x) = y'_kT(x)$, i.e. $y'_k \in R(J_YT - K)^{\perp}$ for any $k \in I\!N$. Since $\{y'_n\}$ is an infinite sequence of linearly independent elements, $\bar{\beta}(J_YT - K) = \infty$.

(iii) $\Rightarrow$ (i): Let K be a compact operator satisfying (iii), and suppose $T \in \mathcal{F}_-$. Since $J_YT \in \mathcal{F}_-$, it suffices to consider the case when Y is complete.

Now $T' \in \mathcal{F}_+$, and K' is compact (Corollary 5.2.3) with $\Delta(K') = 0$ (Theorem 5.3.2) and $D(K') = K(0)^{\perp} = Y'$. The latter implies that the equality $(T-K)' = T' - K'$ holds (Proposition 2.6.6). Since $K'(0) = D(K)^{\perp} \subset D(T)^{\perp} = T'(0)$, it follows from Theorem 5.4.2 that $T' - K' =\in \mathcal{F}_+$. Thus $\bar{\beta}(T-K) = \alpha(T'-K') < \infty$, which contradicts (iii).

◇

Corollary 5.5.3 *Let $T \in LR(X,Y)$. The following are equivalent*

(i) $T \in \mathcal{F}_-$,

(ii) $\bar{\beta}(J_YT + K) < \infty$ *for every single-valued compact operator* $K \in LR(X, \tilde{Y})$ *such that* $D(K) \supset D(T)$.

PROOF

(ii) $\Rightarrow$ (i): The arguments of (i) $\Rightarrow$ (iii) in Theorem 5.5.2 apply.

(i) $\Rightarrow$ (ii): Suppose (i) holds and $K \in LR(X, \tilde{Y})$ is a single-valued compact operator such that $D(K) \supset D(T)$. Then K' is compact (Corollary 5.2.3), and hence $(T+K)' = T'+K'$ (Proposition 2.6.6). By Theorem 5.4.2 $T' + K' \in \mathcal{F}_+$, and thus

$$\bar{\beta}(T+K) = \alpha(T'+K') < \infty.$$

$\diamond$

Proposition 5.5.4 *Let T be closed and let $F \in \mathcal{F}(Y)$. Then $Q_F T$ is closed.*

PROOF

Suppose T is single-valued. Let $Q := Q_F^Y$, and let (x_n, Qy_n), where $y_n \in Tx_n$, be a sequence in $G(QT)$ such that $(x_n, Qy_n) \to (x, [z]) \in X \times Y/F$. Then there exists a sequence $\{f_n\} \subset F$ such that $y_n + f_n \to y$, $y \in [z]$.

Suppose $\{f_n\}$ is unbounded. Then there exists a subsequence $\{f_{n'}\}$ of $\{f_n\}$ such that $||f_{n'}|| \to \infty$ and $\frac{||y_{n'}+f_{n'}||}{||f_{n'}||} \to 0$. Now $\frac{f_{n'}}{||f_{n'}||}$ is bounded and hence, since F is finite dimensional, there exists a subsequence $\{f_{n''}\} \subset \{f_{n'}\}$ such that $\frac{f_{n''}}{||f_{n''}||} \to f$ for some $f \in F$, $||f|| = 1$. It follows that $\frac{y_{n''}}{||f_{n''}||} \to -f$ and $\frac{x_{n''}}{||f_{n''}||} \to 0$. Since T is closed, $(0, -f) \in G(T)$ and $-f = T(0)$, which contradicts $||f|| = 1$. It follows that $\{f_n\}$ is bounded.

Since $\{f_n\}$ is bounded and F is finite dimensional, there exists a convergent subsequence $\{f_{n'}\}$ of $\{f_n\}$ such that $f_{n'} \to f$ for some $f \in F$. It follows that $y_n \to y - f$ and since F is closed, $(x, y-f) \in G(T)$. Thus $(x, y) \in G(Q_F T)$.

Passing to the general case, we have that $Q_F(T(0)) = T(0) + F$ which is closed, since T is closed and F is finite dimensional. Furthermore, we have

$$Q^{Y/F}_{T(0)/F} Q_F T \;=\; Q^{Y/T(0)}_{F/T(0)} Q_{T(0)} T$$

Since $Q_{T(0)}T$ is closed and single-valued and $F/T(0)$ is a finite dimensional subspace of $Y/T(0)$, it follows from what has already been proved that $Q^{Y/F}_{T(0)/F} Q_F T$ is closed, and hence, $Q_F T$ is closed.

$\diamond$

Proposition 5.5.5 *Let X be complete and let $T \in LR(X, Y)$ be closed. Then the following are equivalent:*

(i) $T \in \mathcal{F}_-$,

(ii) $\bar{\beta}(T) < \infty$ *and* $\gamma(T) > 0$,

(iii) $\bar{\beta}(T) < \infty$ *and* $R(T)$ *is closed,*

(iv) $Q_F T$ *is open and surjective for some* $F \in \mathcal{F}(Y)$,

(v) $\bar{\beta}(Q_F T) < \infty$ *and* $\gamma(Q_F T) > 0$ *for some* $F \in \mathcal{F}(Y)$.

PROOF
(i) $\Rightarrow$ (ii) : Since $T' \in \mathcal{F}_+$, we have $\bar{\beta}(T) = \alpha(T') < \infty$. By Theorem 3.3.9 $\gamma(T) = \gamma(T') > 0$.

The equivalence (ii) $\Leftrightarrow$ (iii) follows from Theorem 3.3.9.

(ii) $\Rightarrow$ (iv) : Since $\gamma(T') \geq \gamma(T) > 0$, and $\alpha(T') = \bar{\beta}(T) < \infty$, we have $T' \in \Phi_+$, or equivalently, $T' \in \mathcal{F}_+$. By Theorem 5.5.2, there exists $F \in \mathcal{F}(Y)$ such that $T'|_{F^\perp} = (Q_F T)'$ is bounded below. By Proposition 5.5.4, $Q_F T$ is closed. Hence, by the State Diagram for closed linear relations, Theorem 3.4.3, $Q_F T$ is surjective. Since X is complete, it follows from Theorem 3.3.9 that $\gamma(Q_F T) = \gamma((Q_F T)') > 0$.

The implication (iv) $\Rightarrow$ (v) is obvious. If (v) holds, then $(Q_F T)' \in \mathcal{F}_+$, and hence $Q_F T \in \mathcal{F}_-$. Since $F \in \mathcal{F}(Y)$, it follows that $T \in \mathcal{F}_-$ (Proposition 5.5.1), and hence, (i) holds.

$\diamond$

5.6 Operator Quantities and the Classification of Linear Relations II

Recall that a single-valued operator $T \in LR(X,Y)$ is said to be *strictly cosingular* if *there is no closed infinite codimensional subspace M of Y such that $Q_M T$ is surjective* (see Pelczynski [118] or Whitely [142]). This property is generalised in the definition given below.

Definition 5.6.1 *A linear relation $T \in LR(X,Y)$ is said to be* **strictly cosingular** *if $\Delta'(T) = 0$.*

Proposition 5.6.2 *Let $T \in LR(X,Y)$. Then the following are equivalent:*

(i) $\Gamma_0'(T) = 0$,

(ii) $\exists\ F \in \mathcal{F}(Y)$ *such that* $Q_F T$ *is precompact.*

PROOF
(i) $\Rightarrow$ (ii): Let $\epsilon > 0$. Since $\Gamma_0'(T) = 0$, we may choose $F \in \mathcal{F}(Y)$ such that $\|Q_F T\| < \epsilon$. Then $\|(Q_F T)'\| = \|T' J_{F^\perp}\| < \epsilon$, and hence $\bar{\Gamma}_0(T') = 0$. By Theorem 5.3.5, T' is compact. Thus $Q_F T$ is precompact, since $(Q_F T)'$ is compact.

(ii) $\Rightarrow$ (i): Suppose (ii) holds. Since $\Gamma_0'(Q_F T) = \Gamma_0'(T)$, it suffices to consider the case when T is precompact.

Given $\epsilon > 0$ there exists $\{x_1, x_2, \ldots, x_n\} \subset B_{D(T)}$ such that for any $x \in B_{D(T)}$ there exists $x_k \in \{x_1, x_2, \ldots, x_n\}$ such that $\|Tx - Tx_k\| < \epsilon$. Let $F := sp\{y_1, y_2, \ldots, y_n\}$ where $y_i \in Tx_i$, $1 \leq i \leq n$. Then

$$\begin{aligned} \|Q_F Tx\| &= \|Q_F(Tx - y_k)\| = \|Q_F(Tx - Tx_k)\| \\ &\leq \|Q_F\|\,\|Tx - Tx_k\| \\ &< \epsilon \end{aligned}$$

$\diamond$

Proposition 5.6.3 *Let $T \in LR(X,Y)$ and $F \in \mathcal{F}(Y)$. If $Q_F T$ is precompact, then T is strictly cosingular. In particular, T is strictly cosingular if T is precompact or $dim\, R(T) < \infty$.*

PROOF

By Proposition 5.6.2, $\Gamma_0'(Q_F T) = 0$. Thus, by Proposition 4.2.6, $\Delta'(T) \leq \Gamma_0'(Q_F T) = 0$.

$\diamond$

Theorem 5.6.4 *Let $T \in LR(X,Y)$ and $\overline{T(0)} \in \mathcal{E}(Y)$. Then the following are equivalent:*

(i) $T \in \mathcal{F}_-$,

(ii) $\Gamma'(QT) > 0$.

PROOF

Since $T \in \mathcal{F}_-$ if and only if $QT \in \mathcal{F}_-$, it suffices to consider the case when T is single-valued and $dim\, Y = \infty$.

(ii) $\Rightarrow$ (i): Suppose $T \notin \mathcal{F}_-$, and let $\epsilon > 0$. Then there exists a compact operator $K \in LR(X, \tilde{Y})$ such that $||K|| < \epsilon$ and $M := \overline{R(K - J_Y T)} \in \mathcal{E}(\tilde{Y})$ (Theorem 5.5.2). Thus, $Q_M J_Y T = Q_M K$ and furthermore, $Q_M K$ is compact. It follows that

$$\Gamma'(T) \leq \Gamma'(Q_M J_Y T) = \Gamma'(Q_M K) \leq \Gamma_0'(Q_M K) = 0$$

(applying Corollary 4.2.4 and Proposition 5.6.2).

(i) $\Rightarrow$ (ii): Suppose $\Gamma'(T) = 0$, and let $\epsilon > 0$. Then there exists $M \in \mathcal{E}(\tilde{Y})$ such that $||Q_M J_Y T|| < \epsilon$. Thus $||(Q_M J_Y T)'|| = ||T' J_{M^\perp}|| < \epsilon$. Since T is single-valued, $Y' \supset D(T') \supset M^\perp$, and $M^\perp \in \mathcal{I}(Y')$, it follows that $\Gamma(T') = 0$. Hence, by Theorem 5.3.1, $T' \notin \mathcal{F}_+$, i.e. $T \notin \mathcal{F}_-$.

$\diamond$

Corollary 5.6.5 *Let $dim\, Y = \infty$ and let $T \in LR(X,Y)$ satisfy $T(0) \in \mathcal{F}(Y)$. Then the following are equivalent:*

(i) $T \in \mathcal{F}_-$,

(ii) $\Gamma'(T) > 0$.

$\diamond$

The condition $\overline{T(0)} \in \mathcal{E}(Y)$ is included to ensure that $dim\, Y/\overline{T(0)} = \infty$ and, hence, $\Gamma'(QT)$ is not strictly zero.

5.7 Perturbation Theorems for Lower Semi-Fredholm Linear Relations

Proposition 5.7.1 *Let $T, S \in LR(X,Y)$, $D(S) \supset D(T)$, and let $T \in \mathcal{F}_-$. Then*

(a) If $\dim R(S) < \infty$, then $T + S \in \mathcal{F}_-$.

(b) If S is precompact, then $T + S \in \mathcal{F}_-$.

(c) If $||S|| < \gamma(T')$, then $T + S \in \mathcal{F}_-$.

PROOF

(a) By Proposition 5.5.1, $Q_{R(S)}(T+S) = Q_{R(S)}T \in \mathcal{F}_-$, and hence, $T + S \in \mathcal{F}_-$.

(b) $\Delta(S') = 0$ since S' is compact, and since $D(S) \supset D(T)$, $(T+S)' = T' + S'$ (Proposition 2.6.6), and $S'(0) \subset T'(0)$. Thus by Proposition 5.4.2, $(T+S)' \in \mathcal{F}_+$.

(c) $T' \in \mathcal{F}_+$ and thus $\alpha(T') < \infty$ and $\gamma(T') \leq \Gamma(T')$ (Proposition 4.1.14). We have

$$\Delta(S') \;\leq\; ||S'|| \;\leq\; ||S|| \;\leq\; \gamma(T').$$

Thus, as in (b), $(T+S)' \in \mathcal{F}_+$ by Proposition 5.4.2.

$\diamond$

Proposition 5.7.2 *Let $T \in \mathcal{F}_-$ and suppose $J_Y S \in LR(X,Y)$ is strictly cosingular, $D(S) \supset D(T)$ and $\dim Q_T S(0) < \infty$. Then $T + S \in \mathcal{F}_-$.*

PROOF

Now $J_Y T \in \mathcal{F}_-$ if and only if $T \in \mathcal{F}_-$, and thus, we may assume that Y is complete. We first consider the case when S is single-valued. Then $Q_{T+S}(T+S-S) = Q_T T$, and hence,

$$\begin{aligned} \Gamma'(Q_T T) &= \Gamma'(\,Q_{T+S}(T+S) - Q_{T+S}S\,) \\ &\leq \Gamma'(\,Q_{T+S}(T+S)\,) + \Delta'(\,Q_{T+S}S\,) \end{aligned} \tag{5.10}$$

Now if $\overline{T(0)} \in \mathcal{E}(Y)$, then by Corollary 4.2.4, $\Delta'(Q_{T+S}S) \leq \Delta'(S) = 0$. Thus, since $Q_T T \in \mathcal{F}_-$, (5.10) implies that $\Gamma'(\,Q_{T+S}(T+S)\,) > 0$ and $T + S \in \mathcal{F}_-$ (Theorem 5.6.4). If $\overline{T(0)} \in \mathcal{C}(Y)$, then $Q_{T+S}S$ has finite dimensional range and, hence, $Q_{T+S}(T+S) = Q_T T + Q_T S \in \mathcal{F}_-$ (Proposition 5.7.1).

More generally, if $\dim Q_T S(0) < \infty$, let $F := Q_T S(0) \in \mathcal{F}(Q_T(Y))$. Then

$$Q_F Q_T \;=\; Q_{S(0)+\overline{T(0)}} \;=\; Q_{\overline{S(0)+T(0)}}. \tag{5.11}$$

Since $Q_T T \in \mathcal{F}_-$, it follows from Proposition 5.5.1 that $Q_F Q_T T \in \mathcal{F}_-$, and thus, by (5.11) and what has already been shown, $Q_{\overline{(S+T)(0)}}(T+S) = Q_F Q_T(T+S) \in \mathcal{F}_-$, i.e. $T + S \in \mathcal{F}_-$.

$\diamond$

5.8 The Essential State Diagram

We present an analogue of the State Diagram given in Section 3.4. The Essential State Diagram, Theorem 5.8.3 below, was developed by Cross (see [35], V.7) and has been generalised to include the dual properties of closed relations ([35], V.8). We begin by defining the *Essential States.*

Definitions 5.8.1 *The* **essential states** *of a linear relation $T \in LR(X,Y)$ are defined as follows:*

$$\begin{aligned} I_e &: \ R(T) \text{ is closed and } \beta(T) < \infty \\ II_e &: \ R(T) \text{ is not closed and } \bar{\beta}(T) < \infty \\ III_e &: \ \bar{\beta}(T) = \infty \end{aligned}$$

$$\begin{aligned} 1_e &: \ T \in \mathcal{F}_+ \\ 2_e &: \ \alpha(T) < \infty \text{ and } T \notin \mathcal{F}_+ \\ 3_e &: \ \alpha(T) = \infty \end{aligned}$$

We use notation analogous to the notation used for the State Diagram, i.e. if $T \in I_e$ and $T \in 2_e$ then we write $T \in I_{2e}$, and similarly for the adjoint. The results of Proposition 5.8.2 yields the Essential State Diagram for linear relations, the configuration of which coincides with the State Diagram for linear relations, Theorem 3.4.3.

The proof of Proposition 5.8.2 below uses properties of *operator ranges* (see, for example, [35], V.6). Recall that a subspace M of a normed linear space X is said to be an operator range if there exists a continuous single-valued and injective operator $T : Y \to X$ defined everywhere on a Banach space Y with range $R(T) = M$. Hence, the domain and range of a completely closed relation are operator ranges. Furthermore, it can be shown that if $R(\tilde{T})$ has a complementary subspace, then it is complete (see [35], V.6.5).

Proposition 5.8.2 *Let $T \in LR(X,Y)$. Then*

(a) $T \in 1_e \Leftrightarrow T' \in I_e$

(b) $T \in I_e \cup II_e \Leftrightarrow T' \in 1_e \cup 2_e$

(c) $T \in 3_e \Rightarrow T' \in III_e$

(d) $T \in I_{1e} \cup II_{1e} \Rightarrow T' \in I_{1e}$

(e) $T' \notin II_{1e}$

(f) If Y is complete and $T \in I_{2e} \cup I_{3e}$ then $T' \in III_{1e}$

PROOF

(a) Suppose $T \in 1_e$. Choose $M \in \mathcal{P}(X)$ such $T|_{M \cap D(T)}$ is injective and open. Applying Proposition 3.3.2, $R(Q_{M^\perp}T') = R((TJ_M)') = N(TJ_M)^\perp = M'$. In particular, $R(Q_{M^\perp}T')$ is closed. Since $M' = X'/M^\perp$ and $Q_{M^\perp}$ is continuous,

$$X' = (Q_{M^\perp})^{-1}M' = (Q_{M^\perp})^{-1}(R(Q_{M^\perp}T')) = R(T') + M^\perp.$$

Since $dim\, M^{\perp} < \infty$, it follows that $dim\,(R(T') \cap M^{\perp}) < \infty$ and hence $M^{\perp} = F \oplus (M^{\perp} \cap R(T'))$, $R(T') + M^{\perp} = R(T') + F$, and $R(T') \cap F = \{0\}$ for some finite dimensional subspace $F \subset X'$. Thus, $\beta(T') < \infty$, and, since $R(T')$ is an operator range and is complemented in a Banach space, it follows that $R(T')$ is closed (see remarks preceding Proposition 5.8.2).

Now suppose $T' \in I_e$, i.e. $R(T')$ is closed and $\beta(T') < \infty$. By Proposition 2.7.6,

$$\alpha(T) \;\le\; \alpha(\tilde{T}) \;\le\; \bar{\beta}(\tilde{T}') \;=\; \bar{\beta}(T') = \beta(T').$$

By the Closed Range Theorem 3.3.8, $R(\tilde{T})$ is closed. Thus, $\tilde{T} \in \Phi_+$, and by Theorem 5.2.10, $\tilde{T} \in \mathcal{F}_+$. By Proposition 5.1.5, there exists $M \in \mathcal{C}(D(\tilde{T}))$ and $c > 0$ such that $\|\tilde{T}m\| \ge c\|m\|$ for $m \in M$. Hence, $\|Tm\| \ge c\|m\|$ for $m \in M \cap D(T)$, where $M \cap D(T) \in \mathcal{C}(D(T))$. Applying Proposition 5.1.5 again, it follows that $T \in \mathcal{F}_+$.

(b) By Proposition 2.7.6 we have $\alpha(T') = \bar{\beta}(T) \le \beta(T)$ from which the result follows.

(c) Suppose $T \in 3_e$. Then $dim\, N(T) = dim\, N(\tilde{T}) = \infty$. By Proposition 2.7.6, we have

$$\infty \;=\; \alpha(T) \;=\; \alpha(\tilde{T}) \;\le\; \bar{\beta}(\tilde{T}') \;=\; \bar{\beta}(T').$$

(d) Suppose $T \in I_{1e} \cup II_{1e}$. Then by (a), $T' \in I_e$, and by (b), $T' \in 1_e \cup 2_e$. Now $T' \in I_e$ implies $T' \in \Phi_+$, and hence, by Proposition 5.1.4, $T' \in \mathcal{F}_+$.

(e) If $T' \in 1_e$, i.e. if $T' \in \mathcal{F}_+$, then by Theorem 5.2.10, $T' \in \Phi_+$, and hence, $T' \notin II_e$.

(f) Suppose $T \in I_e$. Since $\beta(T) < \infty$ we have $Y = R(T) + M$, where $dim\, M < \infty$ and $R(T) \cap M = \{0\}$. Thus $R(T)/M = Y/M$, and $Q_M T$ is in state I. Since Y/M is complete, it follows from the the State Diagram for linear relations (Section 3.4) that $T' J_{M^{\perp}} \in 1$. In particular, $R(T' J_{M^{\perp}})$ is closed and $N(T' J_{M^{\perp}})$ is finite dimensional. Thus $T' J_{M^{\perp}} \in \mathcal{F}_+$, and hence, since $M^{\perp} \in \mathcal{C}(Y')$, it follows that $T' \in \mathcal{F}_+$, i.e. $T' \in 1_e$. Now if $T \in I_{2e} \cup I_{3e}$, then from (a) above, $T \notin 1_e \Rightarrow T' \notin I_e$, i.e. $T' \in II_e \cup III_e$. By (e) above, $T' \notin II_{1e}$. Hence, $T' \in III_{1e}$.

◇

Theorem 5.8.3 The Essential State Diagram for Linear Relations

<table>
<tr><td></td><td>III_{3e}</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td></td><td></td></tr>
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<tr><td></td><td>II_{1e}</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td></td><td>I_{3e}</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td></td><td>||||||</td><td>||||||</td></tr>
<tr><td>↑</td><td>I_{2e}</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td>T'</td><td>I_{1e}</td><td></td><td>||||||</td><td>||||||</td><td></td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td><td>||||||</td></tr>
<tr><td></td><td></td><td>I_{1e}</td><td>I_{2e}</td><td>I_{3e}</td><td>II_{1e}</td><td>II_{2e}</td><td>II_{3e}</td><td>III_{1e}</td><td>III_{2e}</td><td>III_{3e}</td></tr>
<tr><td></td><td></td><td>$T \rightarrow$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

Y : this state cannot occur if Y is complete

◇

Corollary 5.8.4 *Let $T \in LR(X,Y)$. Then*

$$T \in \mathcal{F}_+ \Leftrightarrow T' \in \mathcal{F}_-$$

PROOF

By Theorem 5.8.3 and Proposition 5.5.5, $T \in \mathcal{F}_+$ if and only if $T' \in I_e$ if and only if $T' \in \mathcal{F}_-$.

◇

Corollary 5.8.5 *Let $T \in LR(X,Y)$. Then*

$$T \in \mathcal{F}_+ \Leftrightarrow \tilde{T} \in \Phi_+$$

PROOF

By Proposition 5.5.1 and Corollary 5.8.4, we have $T \in \mathcal{F}_+$ if and only if $\tilde{T}' \in \Phi_-$. The latter condition holds if and only if $R(\tilde{T})$ is closed (the Closed Range Theorem, 3.3.8) and $\alpha(\tilde{T}) = \beta(T') < \infty$ (Proposition 2.7.6 together with Proposition 3.3.2; in this case equality holds in Propositions 2.7.6 (b)).

◇

5.9 Singular Sequences

Definition 5.9.1 *Let $T \in LR(X,Y)$. A sequence $\{x_n\}_{n\in I\!N}$ in $D(T)$ is said to be a* **singular sequence** *if $||Tx_n|| \to 0$ but $||x_n|| = 1$ for each $n \in I\!N$, and $\{x_n\}_{n\in I\!N}$ does not have a Cauchy subsequence.*

Proposition 5.9.2 *Let $T \in \mathcal{F}_+$. Then any bounded sequence $\{x_n\}_{n\in I\!N}$ in $D(T)$ such that $\{QTx_n\}$ is Cauchy has a Cauchy subsequence.*

PROOF

Let $\{x_n\}_{n\in I\!N}$ be a bounded sequence in $D(T)$ for which $\{QTx_n\}$ is Cauchy. By Corollary 5.8.5, $\tilde{T} \in \Phi_+$. We first show that the result holds for the case when $\tilde{T}$ is single-valued. By Theorem 5.2.9 and Proposition 3.6.11, there exists a continuous single-valued projection P with $D(P) = D(\tilde{T})$, $R(P)$ is closed and finite codimensional, $dimN(P) < \infty$, and $\tilde{T}|_{R(P)}$ is injective and open. Since $N(P) = R(I-P)$ is finite dimensional, $\tilde{T}|_{R(I-P)}$ is continuous, and the sequence $\{\tilde{T}(I-P)x_n\}$ is bounded in the finite dimensional space $R(\tilde{T}(I-P))$. Choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{\tilde{T}(I-P)x_{n_k}\}$ is Cauchy. Then letting $k, l \to \infty$ it follows that

$$||\tilde{T}P(x_{n_k} - x_{n_l})|| \leq ||\tilde{T}(x_{n_k} - x_{n_l})|| + ||\tilde{T}(I-P)(x_{n_k} - x_{n_l})|| \to 0.$$

Thus, $\{\tilde{T}Px_{n_k}\}$ is Cauchy and, since $\tilde{T}|_{R(P)}$ is injective and open, we have

$$||\tilde{T}(P(x_{n_k} - x_{n_l}))|| \geq \gamma(\tilde{T}|_{R(P)})||P(x_{n_k} - x_{n_l})||.$$

Hence $\{Px_{n_k}\}$ is Cauchy. Choosing a subsequence $\{w_n\}$ of $\{x_{n_k}\}$ such that $\{(I-P)w_n\}$ is Cauchy, it follows that $\{w_n\} = \{Pw_n + (I-P)w_n\}$ is Cauchy.

Since $||Q_T Tx|| \geq ||Q_{\tilde{T}}\tilde{T}x||$, $x \in D(T)$, it follows that if $\{Q_T Tx_n\}$ is a Cauchy sequence, then so is $\{Q_{\tilde{T}}\tilde{T}x_n\}$. Hence, replacing $\tilde{T}$ by $Q_{\tilde{T}}\tilde{T}$, it follows from the above that the result holds for the multivalued case.

$\diamond$

Theorem 5.9.3 *The following properties are equivalent:*

(i) $T \notin \mathcal{F}_+$

(ii) There exists are non-precompact bounded subset W of $D(T)$ such that $QT(W)$ is precompact.

(iii) T has a singular sequence.

PROOF

Clearly (iii) $\Rightarrow$ (ii), while (ii) $\Rightarrow$ (i) follows from Proposition 5.9.2.

(i) $\Rightarrow$ (iii) : It suffices to show that the result holds for the case when T is single-valued. Assume $T \notin \mathcal{F}_+$. We first consider the case when T is bounded, and show that $\tilde{T}$ has a singular sequence. Since $\tilde{T} \notin \Phi_+$, by Theorem 5.2.9, there exists a closed subspace $M \in \mathcal{I}(X)$ such that $\tilde{T}|_M$ is compact. If $dim(M \cap N(\tilde{T})) = \infty$, , then clearly $\tilde{T}$ does have a singular sequence. If $M \cap N(\tilde{T})$ is finite dimensional, there exists a closed infinite dimensional subspace N of M such that $N + (M \cap N(\tilde{T})) = M$ and $N \cap N(\tilde{T}) = \{0\}$. By Proposition 5.1.5, there exists a norm one sequence $\{x_n\} \subset N$ such that $||\tilde{T}x_n|| \to 0$. Then $\{x_n\}$ cannot have a Cauchy subsequence for then $x_n \to y$ for some $y \in N$ with $||y|| = 1$ and, since $\tilde{T}$ is continuous, $\tilde{T}y = 0$, a contradiction. Thus, since $\tilde{T}$ has a singular sequence $\{x_n\}$, it follows that a sequence $\{z_n\} \subset D(T)$ such that $||z_n|| = 1$ and $||z_n - x_n|| < \frac{1}{n}$ for each n is a singular sequence for a bounded relation T.

Since $N(T) = N(TG)$ as subsets of $D(T)$, it follows from Proposition 2.8.4, $T \notin \mathcal{F}_+$ if and only if $TG \notin \mathcal{F}_+$. Thus, if T is unbounded, then, we may pass to TG and it follows from the above that TG has a singular sequence $\{z_n\} \subset X_T$. Now, $\{z_n\}$ does not have a Cauchy subsequence with respect to $||\text{\_}||_T$, and since $||z_n||_T = 1$ and $||TGz_n|| \to 0$, it follows that $\{Gz_n\} \subset D(T)$ does not have a Cauchy subsequence and $||Gz_n|| \to 1$. Since $(||Gz_n||^{-1})TGz_n \to 0$, it follows that $\{||Gz_n||^{-1}Gz_n\}$ is a singular sequence for T.

$\diamond$

5.10 Notes and Remarks

The material of this chapter is based on Cross [35], Chapter V. Proofs of the theorems given here are applied in the sequel, and are given in this chapter for reference in Chapters 6 and 8, and for completeness. Most of the proofs are essentially the same as those provided by Cross, where further properties of upper and lower semi-Fredholm relations are given. Discussion of the zero sets of operator quantities, given in Section 5.3 above, differs from that given in Cross, though. In Cross [35], V.2, it is first established that T is precompact (partially precompact) if and only if $\bar{\Gamma}_0(T) = 0$ ($\Gamma_0(T) = 0$). This property is then used to show that $T \in \mathcal{F}_+$ if and only if $\Gamma(T) > 0$, and the latter is used, in turn, to show that T is strictly singular if and only if $\Delta(T) = 0$. In Theorem 5.3.1 above, we first show that $T \in \mathcal{F}_+$ if and only if $\Gamma(T) > 0$; next, in Theorem 5.3.2, we show that T is strictly singular if and only if $\Delta(T) = 0$, and lastly, in Theorem 5.3.5 we show that T is precompact (partially precompact) if and only if $\bar{\Gamma}_0(T) = 0$ ($\Gamma_0(T) = 0$).

In Section 5.4, the proof of Proposition 5.4.1 (b) is also due to the author. This proof does not require that the relation T be single-valued (cf. Cross [35]). Proposition 5.4.3 is due to the author, and is applied in the proof of the Characterisation Theorem for $\beta-$Atkinson relations 6.2.5.

We have already noted that the characterisations of classes of linear relations by means of the operator quantities provide more generally applicable techniques for discussing the stability of Fredholm properties of linear relations, and hence of quantities of relations which are defined in terms of Fredholm properties. These ideas are applied in the perturbation theorems of Chapters 6 and 8.

We conclude with the comment that a notable property of the *Gowers and Maurey space* is that the continuous operators on that space are of the form $\lambda I + S$, where S is strictly singular. This property received some attention for its connection with the invariant subspace problem (see the notes for Chapter 7.).

Chapter 6

The Index and Generalised Inverses of Fredholm Type Linear Relations

In this chapter we discuss further properties of Fredholm type linear relations. In particular, we consider unbounded multivalued Atkinson operators with generalised inverses, i.e multivalued semi-Fredholm operators for which there are continuous projections onto the kernels and onto the closures of the ranges.

Our first series of stability theorems for the index are presented in Section 6.1. In Section 6.2 we introduce the classes of $\alpha-$Atkinson and $\beta-$Atkinson relations. In Sections 6.3, 6.4 and 6.5, the stability of Atkinson type properties, as well as further stability theorems for the index of linear relations, are investigated.

6.1 The Index of Fredholm Type Linear Relations

The *small perturbation* theorem (and its variants) of Section 3.5 and general properties of the index, the nullity and the deficiency given in Section 2.7 are applied in this section to give multivalued analogues of the well-known theorems that the index of a closed semi-Fredholm operator is stable under small perturbation and under strictly singular perturbation. The perturbation theorems are effectively given for the case when the relation T is closed and the spaces are complete. More general cases are discussed in Sections 6.3, 6.4 and 6.5.

Theorem 6.1.1 *Let $T \in \mathcal{F}_+(X,Y) \cup \mathcal{F}_-(X,Y)$, and suppose $S \in LR(X,Y)$ satisfies $D(S) \supset \overline{D(T)}$, $S(0) \subset \overline{T(0)}$, and $||S|| < \gamma(\tilde{T})$, then $\kappa(\tilde{T}+\tilde{S}) = \kappa(\tilde{T})$.*

PROOF

By Corollaries 5.2.11 and 5.8.5, $\tilde{T} \in \Phi_+ \cup \Phi_-$, and, by the Open Mapping theorem, $\gamma(\tilde{T}) > 0$. Furthermore, $||\tilde{S}|| \leq ||S|| < \gamma(\tilde{T})$ (Proposition 2.5.9). Now let $x \in D(\tilde{T})$. Then there exists $(x_n, y_n) \in G(T)$ such that $(x_n, y_n) \to (x, y) \in G(\tilde{T})$ where $x_n \in D(T) \subset D(S)$. Since $\tilde{S}$ is continuous, $(QS)\tilde{\ } x_n$ is Cauchy and converges to some $[z] \in Q_{\tilde{S}}\tilde{Y}$. Since $(x,[z]) \in G((QS)\tilde{\ })$, $x \in D((QS)\tilde{\ }) = D(Q\tilde{S}) = D(\tilde{S})$. Thus, since $D(\tilde{S})$ is closed ($\tilde{S}$ is continuous), $\overline{D(\tilde{T})} \subset D(\tilde{S})$. To see $\tilde{S}(0) \subset \tilde{T}(0)$, let $y \in \tilde{S}(0)$. Then there exists $y_n \in S(0)$ such that $(0, y_n) \to (0, y) \in G(\tilde{S})$. Since $S(0) \subset T(0)$, it follows that $(0, y) \in G(\tilde{T})$. Thus, we may assume that X and Y are complete and T and S are closed, and show that the stability holds for $\kappa(T)$. Since S is continuous and $D(S) \supset \overline{D(T)}$, $T + S$ is closed (Proposition 2.5.11).

Suppose $T \in \Phi_+$. By Proposition 5.4.2, $T + S \in \mathcal{F}_+$, and since $T + S$ is closed, $T + S \in \Phi_+$. Now there exists $M \in \mathcal{P}(D(T))$ such that $T|_M$ is injective and open. Since M is closed, $T|_M$ is closed, and it follows from the Open Mapping theorem, that $R(T|_M)$ is closed, i.e. $T|_M \in \Phi_+$. We first deduce the conclusion for the case $||S|| < \gamma(T|_M)$. Applying Proposition 5.4.2 again, $(T + S)|_M \in \mathcal{F}_+$, and since $T + S$ is closed, $(T + S)|_M \in \Phi_+$ and $R((T + S)|_M)$ is closed. Thus by Theorem 3.5.3, $\beta((T + S)|_M) = \bar{\beta}((T + S)|_M) = \bar{\beta}(T|_M) = \beta(T|_M)$. Applying the finite dimensional extension lemma, 2.7.5, provided $||S|| < \gamma(T|_M)$, we have:

$$\begin{aligned}
\kappa(T+S) &= \kappa((T+S)|_M) + \ codim\, M \\
&= \kappa(T|_M) + \ codim\, M \\
&= \kappa(T).
\end{aligned}$$

Passing to the case $||S|| < \gamma(T)$, let I denote the closed interval $[0,1]$, and let $\mathcal{Z} := \mathbb{Z} \cup \{-\infty, \infty\}$ (where $\mathbb{Z}$ denotes the integers). Let I be endowed with the usual topology, and $\mathcal{Z}$ with the discrete topology, and define $\psi : I \to \mathcal{Z}$ by $\psi(\lambda) := \kappa(T + \lambda S)$. It follows from the above that, provided λ_0 is sufficiently close to λ,

$$\begin{aligned}
\psi(\lambda) &= \kappa(T + \lambda_0 S + (\lambda - \lambda_0)S) \\
&= \kappa(T + \lambda_0 S) \\
&= \psi(\lambda_0).
\end{aligned}$$

Hence, ψ is continuous and $\psi(I)$ is connected and consists of only one point. It follows that $\kappa(T) = \psi(0) = \psi(1) = \kappa(T + S)$.

If $T \in \Phi_-$, then $T' \in \Phi_+$ and $R(T')$ is closed. Since $||S|| < \infty$ and $D(S) \supset D(T)$, we have that $T' + S' = (T + S)'$ (Proposition 2.6.6). Furthermore, $||S'|| \leq ||S|| < \gamma(T) \leq \gamma(T')$ (Propositions 2.6.12 and 2.6.13). Now $D(\tilde{S}) \supset \overline{D(\tilde{T})}$ implies $\tilde{S}'(0) \subset \tilde{T}'(0)$ (Proposition 2.6.5) and, since $\tilde{S}$ is closed and continuous, $\tilde{S}(0) \subset \tilde{T}(0)$ implies $D(\tilde{S}') = \overline{D(\tilde{S}')}^* \supset \overline{D(\tilde{T}')}^* \supset D(\tilde{T}')$. Thus, $D(\tilde{S}') \supset \overline{D(\tilde{T}')}$. Hence, as before, if X and Y are complete and T and S are closed, then $T' + S'$ is closed (Proposition 2.5.11). As for the case $T \in \Phi_+$, it follows that $R(T' + S')$ is closed, and

$$\kappa(T) = -\kappa(T') = -\kappa(T' + S') = -\kappa((T + S)') = \kappa(T + S).$$

Proposition 6.1.2

(a) Suppose $T \in \mathcal{F}_+(X,Y)$, *and let* $S \in LR(X,Y)$ *be strictly singular.*
If $\|S\| < \infty$, $S(0) \subset \overline{T(0)}$ *and* $D(S) \supset \overline{D(T)}$, *then* $\kappa(\tilde{T}+\tilde{S}) = \kappa(\tilde{T})$.

(b) Suppose $T \in \mathcal{F}_-(X,Y)$, *and let* $S \in LR(X,Y)$ *be such that* S' *is strictly singular.*
If $\|S'\| < \infty$, $S(0) \subset \overline{T(0)}$ *and* $D(S) \supset \overline{D(T)}$, *then* $\kappa(\tilde{T}+\tilde{S}) = \kappa(\tilde{T})$.

PROOF
(a) By Corollary 5.8.5, $\tilde{T} \in \Phi_+$, and hence, as in Proposition 6.1.1, we may assume that X and Y are complete, that T is closed, and show that the stability holds for $\kappa(T)$. Since $\overline{D(T)} \subset D(S)$ and S is continuous, $T+S$ is closed.

By Proposition 5.4.2, $T+\lambda S \in \mathcal{F}_+$, and since $T+S$ is closed, $T+\lambda S \in \Phi_+$ for all λ. Let $I := [0,1]$ with its usual topology and let $\mathcal{Z} = \mathbb{Z} \cup \{-\infty\}$ with the discrete topology. Since λS is continuous, by substituting $T+\lambda S \in \Phi_+$ for T in Proposition 6.1.1, it follows that $\psi(\lambda) := \kappa(T+\lambda S) : I \longrightarrow \mathcal{Z}$ is continuous and, is therefore, constant. Thus, $\kappa(T) = \psi(0) = \psi(1) = \kappa(T+S)$.

(b) By Corollary 5.2.11, $\tilde{T} \in \Phi_-$, and hence, as in Proposition 6.1.1, we may assume that X and Y are complete, that T is closed, and show that the stability holds for $\kappa(T)$. Since $\overline{D(T)} \subset D(S)$ and S is continuous, $T+S$ is closed. By Proposition 2.6.6, $T'+S' = (T+S)'$. Now $\|S'\| < \|S\| < \infty$, and as in Proposition 6.1.1, $S'(0) \subset T'(0)$, and $D(S') \supset \overline{D(T')}$. Thus, applying part (a) to $T' \in \Phi_+$ it follows that :

$$\kappa(T) = -\kappa(T') = -\kappa(T'+S') = -\kappa((T+S)') = \kappa(T+S)$$

◇

6.2 Generalised Inverses and Atkinson Relations

In consideration of the existence of continuous generalised inverses for Fredholm type relations, we define and characterise the classes of $\alpha-$Atkinson and $\beta-$Atkinson linear relations.

The concept of a *generalised inverse* may be traced back to early contributions by Fredholm, Hurwitz, Hilbert, and others, in the study of integral equations, which evolved into the rich theory on Fredholm type operators (cf. Atkinson [11], Gohberg and Krein [57], Goldberg [60], Kato [74] and [75], Krein, Krasnosel'skii, and Milman [80] and Yood [146]). In 1920 Moore also introduced the notion of an inverse A for a singular or rectangular matrix M which satisfied

$$MAM = A \text{ and } AMA = M$$

(cf. [112] edited by Nashed; applications of generalised inverses are also given in [26] edited by Campbell).

If $T \in \Phi(X,Y)$, then the equalities

$$\begin{aligned} AT &= I - P_N \\ TA &= I - P_{R^c} \end{aligned}$$

hold, where P_N denotes a continuous projection onto $N(T)$, P_{R^c} denotes a continuous projection onto a complement of $R(T)$, and A is the associated continuous generalised inverse satisfying $TAT = T$ and $ATA = A$ (cf. Schechter [130] and Taylor and Lay [136]).

Atkinson considered the class of *relatively regular* operators, where a bounded linear operator T is said to be relatively regular if there exists a bounded operator A such that $TAT = T$. This condition is equivalent to the existence of a continuous generalised inverse. However, we note that the class of relatively regular operators includes operators for which $\alpha(T)$ and $\beta(T)$ may both be infinite, and is not closed under compact additive perturbation (see Goldman [61] and also Goldberg [60]).

Continuous left and right generalised inverses can be constructed for closed upper and lower semi-Fredholm operators which have topologically complemented ranges and kernels in Banach spaces. Muller-Horrig [111] showed that such operators may be characterised in terms of these inverses (cf. Moore and Nashed [109] and Nashed [112]). Applying perturbation theorems for semi-Fredholm operators, it follows that these subclasses of relatively regular operators are stable under small and compact additive perturbation.

We extend these results to multivalued operators in arbitrary normed linear spaces. The characterisation theorems presented, Theorems 6.2.6 and 6.2.7, improve the characterisation theorem given in [111] by extending it to non-closed multivalued operators (cf. Gonzalez and Onieva [62]). We also give properties of adjoints of Atkinson relations.

As in the classic case, the existence of generalised inverses is associated with the existence of continuous projections onto the kernels and ranges.

Definitions 6.2.1 *The classes of $\alpha-$Atkinson and $\beta-$Atkinson linear relations, denoted $\mathcal{A}_\alpha(X,Y)$ and $\mathcal{A}_\beta(X,Y)$ respectively, are defined as follows:*

$$\begin{aligned} \mathcal{A}_\alpha(X,Y) &:= \{\, T \in \mathcal{F}_+ \mid \overline{R(T)} \textit{ is topologically complemented in } Y \,\} \\ \mathcal{A}_\beta(X,Y) &:= \{\, T \in \mathcal{F}_- \mid N(T) \textit{ is topologically complemented in } D(T) \,\} \end{aligned}$$

Characterisations of upper semi-Fredholm relations are given in Cross [35], V.10. We recall two of these results, and note that the generalised inverses for this class are not necessarily continuous, but are *partially continuous*, i.e. continuous on a finite codimensional subspace of Y.

Theorem 6.2.2 ([35], V.10.2) *Let $T \in LR(X,Y)$. The following are equivalent:*

(i) $T \in \mathcal{F}_+$

(ii) $\exists\ A \in L(Y,X)$ *and* $K \in B(X)$ *such that A is partially continuous, $D(A) = R(T)$, and K has finite rank, and* $AT = I_{D(T)} - K$.

Theorem 6.2.3 ([35], V.10.3) *Let $T \in LR(X,Y)$. If X is complete, then the following are equivalent:*

(i) $T \in \mathcal{F}_+$

(ii) $\exists\ A \in B(Y,X)$ *and a finite rank projection $K \in B(X)$ such that* $AT = I_{D(T)} - K$.

We require the following properties to verify intermediate claims in Theorems 6.2.6 and 6.2.7 :

Proposition 6.2.4 (Labuschagne [84], Proposition 10) *Let T be a relatively open operator, i.e. $\gamma(T) > 0$, with closed range and $\alpha(T) < \infty$. For any compact operator K with $D(K) \supset D(T)$, $T + K$ is relatively open, $R(T+K)$ is closed and $\bar{\kappa}(T) = \bar{\kappa}(T+K)$.*

Proposition 6.2.5 (Cross [35], Proposition V.13.3) *If $N(T)$ is closed then under the usual canonical identification $G(T') = G(\hat{T}')$.*

Theorem 6.2.6 (Characterisation of $\alpha-$Atkinson Relations) *Let $T \in LR(X,Y)$. If $\overline{T(0)}$ is topologically complemented in Y, then the following are equivalent:*

(i) $T \in \mathcal{A}_\alpha(X,Y)$

(ii) $\exists\ A \in B(Y,X)$ *and a finite rank projection* $K \in B(X)$ *such that* $R(K) \subset D(T)$, *A is densely defined,* $P\overline{R(T)} \subset \overline{R(T)}$, *where P is a continuous projection from Y onto $N(A)$,* $R(A) \subset D(T)$, $R(T) \cap D(A)$ *is relatively closed in* $D(A)$, $Y = D(A) + \overline{R(T)}$, *and* $AT = I_{D(T)} - K$.

PROOF

(i) $\Rightarrow$ (ii) Since $T \in \mathcal{A}_\alpha$ there exists a closed finite-codimensional subspace $M \subset D(T)$ such that $T|_M$ is open and injective. Letting $T_M := T|_M$, it follows that $R(T|_M)$ is finite codimensional in $R(T)$, and hence, there exists a finite-dimensional subspace $J \subset R(T)$ such that $\overline{R(T_M)} \oplus J = \overline{R(T)}$. Since $\overline{R(T)}$ is topologically complemented in Y, there exists a closed subspace $L \supset J$ such that $Y = \overline{R(T_M)} \oplus L$. Let P_R denote a continuous projection of Y onto $\overline{R(T_M)}$ with kernel L, and let $A := T_M^{-1}P_R$. We show that $D(AT) = D(T)$ and that $R(AT) = M$. We have

$$D(A) = D(T_M^{-1}P_R) = P_R^{-1}(R(T_M)) = R(T_M) + L, \tag{6.1}$$

and $D(AT) = T^{-1}D(A) = D(T)$. Since $R(T_M) \subset D(A)$ and $R(T_M) \in \mathcal{C}(R(T))$, it follows that $R(T) \cap D(A) = R(T_M) + J_2$ where $dim\, J_2 < \infty$ and $R(T_M) \cap J_2 = \{0\}$. Since $R(T_M)$ is relatively closed in $D(A)$, so is $R(T) \cap D(A)$. For $x \in D(T)$ and $y \in Tx$ write $y = r + l$ where $r \in R(T_M)$ and $l \in L$. Then

$$\begin{aligned} ATx &= A(y + T(0)) = T_M^{-1}P_R(y + T(0)) \\ &= T_M^{-1}(r + T(0)) \ = \ m \in M \end{aligned}$$

for some $m \in M$. Thus $R(AT) = M$. Now $D(T) = M + N(AT) = M + N(T) + F$ where $dim\, F < \infty$, $N(AT) = N(T) + F$ and $M \cap F = F \cap N(T) = N(T) \cap M = \{0\}$. Letting K denote a continuous finite rank projection of X onto $F + N(T)$ with complement M, it follows that $R(K) \subset D(T)$ and $AT = I_{D(T)} - K$.

(ii) $\Rightarrow$ (i) We first show that the result holds for the case when T is single-valued. Since K is precompact (Proposition 5.2.5), $\Delta(K) = 0$. By Proposition 5.4.2 we have that $AT = (I - K)|_{D(T)} \in \mathcal{F}_+(X,Y)$. Since T and A satisfy the conditions of Proposition 5.4.1, it follows that $T \in \mathcal{F}_+$. Thus, $\alpha(T) < \infty$. We need to show that $\overline{R(T)}$ is complemented. Now since $R(K) \subset D(T)$, we have

$$R(I - K) \cap D(T) \subset R(AT) \subset D(T). \tag{6.2}$$

Since $R(I-K)\cap D(T)$ is finite codimensional in $D(T)$, it follows that $R(AT)$ is finite codimensional in $R(A)$. Hence there exists a finite dimensional subspace $N \subset R(A)$ such that $\overline{R(A)} = \overline{R(AT)} \oplus N$. Let $\{Ay_1, Ay_2, \ldots Ay_n\}$ be a basis for N. Clearly $y_i \notin N(A)$, $i = 1 \ldots n$ and, letting $W := span\{y_1, \ldots, y_n\}$, we have $W \cap N(A) = \{0\}$. Furthermore, $W \cap \overline{R(T)} = \{0\}$. Since K is compact, $R(K) \subset D(T)$ and $AT = I_{D(T)} - K$, it follows from Proposition 6.2.4 that $R(AT)$ is relatively closed in $D(T)$. Thus, $R(AT)$ is relatively closed in $R(A)$ and hence, $\overline{R(A)} = \overline{R(AT)} + N$ implies that $R(A) = R(AT) + N$ and $D(A) = A^{-1}(R(A)) = R(T) + W + N(A)$.

We now show that $N(A) \cap \overline{R(T)}$ is finite-dimensional. The map

$$\eta : N(AT)/N(T) \to R(T) \cap N(A)$$

defined by

$$\eta[x] := Tx$$

is onto, single-valued, and has a single-valued inverse. Thus we have

$$dim\, \overline{R(T)} \cap N(A) = dim\, R(T) \cap N(A) = dim\, N(AT)/N(T) < \infty,$$

since $R(T) \cap D(A)$ is relatively closed in $D(A)$ so that $\overline{R(T)} \cap N(A) = R(T) \cap N(A)$. Let P denote a continuous projection from Y onto $N(A)$ and let P_2 denote a continuous projection from $N(A)$ onto $\overline{R(T)} \cap N(A)$. Then $P_1 := P_2 P$ is a continuous projection from Y onto $\overline{R(T)} \cap N(A)$ such that $N(P_1) \supset N(P)$. Since $\overline{R(T)} \cap N(A) \subset \overline{R(T)}$, we have

$$\begin{aligned} \overline{R(T)} &= (I-P_1)\overline{(R(T))} \oplus R(P_1) \;\; and \\ N(A) &= (I-P_1)N(A) \oplus R(P_1). \end{aligned}$$

Since $Y = \overline{R(T)} + D(A)$ and $D(A) = R(T) + W + N(A)$ and we have

$$\begin{aligned} N(P_1) = (I-P_1)Y &= (I-P_1)\overline{R(T)} + (I-P_1)D(A) \\ &= (I-P_1)\overline{R(T)} + (I-P_1)W + (I-P_1)N(A). \end{aligned}$$

Since $Y = N(P) \oplus N(A) = N(P) \oplus [(I-P_1)N(A) \oplus R(P_1)]$ and $N(P) \subset N(P_1)$ we have

$$N(P_1) = N(P) \oplus (I-P_1)N(A).$$

Furthermore, since $(I-P)(I-P_1) = I-P$ and $P(I-P_1) = (I-P_1)P$, $P_{N(P_1)}$ is a projection such that

$$\begin{aligned} P(N(P_1)) &= P(I-P_1)Y = (I-P_1)P(Y) = (I-P_1)N(A) \;\; and \\ (I-P)(N(P_1)) &= (I-P)(I-P_1)Y = (I-P)(Y) = N(P). \end{aligned}$$

We show that $(I-P_1)\overline{R(T)} \subset N(P)$: First note that $P(I-P_1)\overline{R(T)} \subset P(I-P_1)Y = (I-P_1)N(A)$. Suppose $x \in P(I-P_1)\overline{R(T)}$. Then $x \in (I-P_1)P\overline{R(T)} \subset (I-P_1)\overline{R(T)} \subset \overline{R(T)}$ since $P\overline{R(T)} \subset \overline{R(T)}$. On the other hand, $x \in (I-P_1)N(A) \subset N(A)$. Thus $x \in \overline{R(T)} \cap N(A) = R(P_1)$ and hence, $x = 0$. It follows that

$$(I-P_1)\overline{R(T)} \subset P(I-P_1)\overline{R(T)} + (I-P)(I-P_1)\overline{R(T)} = (I-P)\overline{R(T)} \subset N(P).$$

Now $(I-P_1)\overline{(R(T))} \in \mathcal{P}(N(P))$ since $(I-P_1)\overline{(R(T))}$ is closed ($I-P_1$ is open, $R(I-P_1)$ and $N(I-P_1)$ are closed and $dim\, N(I-P_1) < \infty$) and $dim\,(I-P_1)W < \infty$. Thus, $(I-P_1)\overline{(R(T))}$ is topologically complemented in $N(P_1)$. Hence, $(I-P_1)\overline{(R(T))}$ is topologically complemented in Y. Since $dim\, R(P_1) < \infty$, $\overline{R(T)}$ is topologically complemented in Y.

For the case when T is multivalued, let $Q := Q_T$ and $A_Q := AQ^{-1} \in LR(Y/\overline{T(0)},\ X)$. For $[y] \in Y/\overline{T(0)}$, we have $A_Q[y] = A(y+\overline{T(0)}) \subset Ay + \overline{AT(0)} = Ay$. Thus A_Q is single-valued and continuous. Similarly we may verify that $A_Q QT = AT = I_{D(T)} - K$. Since $A_Q Q, K$ and QT satisfy the hypotheses, it follows from what has been shown that $QT \in \mathcal{A}_\alpha(X,\ Y/\overline{T(0)})$. Thus $T \in \mathcal{F}_+(X,Y)$ since $QT \in \mathcal{F}_+$. Since $\overline{R(QT)}$ is topologically complemented in $Y/\overline{T(0)}$ and $\overline{T(0)}$ is topologically complemented in Y, it follows that $\overline{R(T)}$ is topologically complemented in Y.

◇

Theorem 6.2.7 (Characterisation of $\beta-$Atkinson Relations)

Let $T \in LR(X,Y)$. If $N(T)$ and $T(0)$ are relatively closed in $D(T)$ and $R(T)$ respectively, then the following are equivalent:

(i) $T \in \mathcal{A}_\beta(X,Y)$

(ii) $\exists\ B \in L(Y,X)$ *and a finite rank projection* $K \in B(Y)$ *such that* $B\overline{T(0)} = 0$,
$D(B)$ is dense, $R(B)$ is topologically complemented in $D(T)$, $R(T) \subset D(B)$, $R(K) \subset D(B)$,
B' is continuous, either $R(T') \subset D(B')$ *or* $D(B'T') \in \mathcal{C}(D(T'))$, *and*
$(I-K)|_{D(B)}$ *is a linear selection of* TB.

PROOF

(i) $\Rightarrow$ (ii) Suppose (i) holds. Let P_N be a necessarily continuous projection of $D(T)$ onto $N(T)$ with kernel M. Let $I - P_{R^c}$ be a projection of Y onto $\overline{R(T)}$. Let $B := T_M^{-1}(I-P_{R^c})$ and let $K := P_{R^c}$. Then B is densely-defined with $R(B) = M \subset D(T)$ which is topologically complemented in $D(T)$, and K is finite rank and continuous. We have $B\overline{T(0)} = T_M^{-1}(I-P_{R^c})\overline{T(0)} = T_M^{-1}\overline{T(0)} = 0$ since $T(0)$ is relatively closed in $R(T)$. Now $G(B') \supset G\,((I-K)'(T_M^{-1})')$ (Proposition 2.6.7). We show that the graphs are equal by composing B' and $(I-K)'(T_M^{-1})'$ with $((I-K)^{-1})' = ((I-K)')^{-1}$:

$$\begin{aligned} & G(\ ((I-K)')^{-1}\,(I-K)'(T_M^{-1})'\ ) \\ \subset\ & G(\ ((I-K)^{-1})'B'\ ) \subset G(\ (B(I-K)^{-1})'\ ) \\ =\ & G(\ (T_M^{-1}(I-K)(I-K)^{-1})'\ ) = G(\ (T_M^{-1})'\ ), \end{aligned}$$

where the second inclusion follows from Proposition 2.6.7 and the last equality holds since $D(T_M^{-1}) \subset R(I-K)$. It follows that $((I-K)')^{-1}\,(I-K)'(T_M^{-1})' = (T_M^{-1})'$. Thus, since

$$D(\ ((I-K)')^{-1}) = R((I-K)') = N(I-K)^\perp \supset N(B)^\perp \supset N(\overline{B})^\perp \supset R(B'),$$

we have

$$D(B') = D(\ ((I-K)')^{-1}B'\ ) = D((T_M^{-1})').$$

Since B is densely defined, B' is single-valued (Proposition 2.6.5) and hence, since $D((T_M^{-1})') = D(\ (I-K)'(T_M^{-1})'\ )$, we have $B' = (I-K)'(T_M^{-1})'$. Since $T \in \mathcal{F}_-$, M is isomorphic to $D(T)/N(T)$ and $N(T)$ is closed in $D(T)$, T'_M may be identified with $\hat{T}' \in \mathcal{F}_+$ (Proposition 6.2.5). Thus T'_M is open. Since $(I-K)'$ is continuous, it follows that B' is continuous.

Next, $R(T') \subset R(T'_M) = D((T_M^{-1})')$ $(R(T')^\top = N(\overline{T}) \supset N(\overline{T}_M) = R(T'_M)^\top$) and, hence, $D(B'T') = (T')^{-1}D(B') = (T')^{-1}R(T') = D(T')$, in particular, $D(B'T') \in \mathcal{C}(D(T'))$. Lastly,

$$TB = T(T_M^{-1}(I - P_{R^c})) = (I - P_{R^c}) + (TB - TB).$$

Thus $I - K$ is a linear selection of TB.

(ii) $\Rightarrow$ (i) Suppose (ii) holds. We first show that the result holds for the case when T is single-valued. Since K is precompact, K' is compact (Corollary 5.2.3). Therefore, applying Proposition 5.4.2, $I - K \in \mathcal{F}_+(Y)$ and $I - K' \in \mathcal{F}_+(Y')$, i.e. $TB = I - K|_{D(B)} \in \mathcal{F}(Y)$. Since $B'T' \subset (TB)'$, it follows from Proposition 5.4.3 that $B'T' \in \mathcal{F}_+$ as well. If $R(T') \subset D(B')$, then $D(B'T') \in \mathcal{C}(D(T'))$ ($D(B'T') = (T')^{-1}D(B') = (T')^{-1}R(T') = D(T')$) . Therefore, since T' and B' satisfy conditions of Proposition 5.4.1, $T' \in \mathcal{F}_+$ and $T \in \mathcal{F}_-$.

We need to show that $N(T)$ is topologically complemented. Since $TB = I - K|_{D(B)}$ it follows that $\alpha(B) \leq \alpha(TB) < \infty$. Now $N(TB) = B^{-1}(N(T))$, and, since B is single-valued and $\alpha(TB) < \infty$, we have that $B(N(TB)) = N(T) \cap R(B)$ is finite dimensional. Let $N \subset R(B)$ denote a topological complement of $N(T) \cap R(B)$ in $R(B)$. We have $\overline{R(TB)} = \overline{T(R(B))} = \overline{T(N)}$ which is finite codimensional in Y since $TB \in \mathcal{F}(Y)$ and, hence, $\overline{R(TB)}$ is finite codimensional in $\overline{R(T)}$. Thus, $\overline{R(T)} = \overline{T(N)} + M$ for some finite-dimensional subspace $M \subset R(T)$, $M \cap \overline{T(N)} = \{0\}$. Let L be a topological complement of $R(B)$ in $D(T)$. Then

$$D(T) = R(B) \oplus L = R(B) \cap N(T) \oplus N \oplus L.$$

Now let D be a complement of $N(T) \cap L$ in L, i.e. $L = (L \cap N(T)) + D$ and $N(T) \cap D = \{0\}$. We show that D must be finite-dimensional. We have:

$$D(T) = [\, R(B) \cap N(T) \oplus N \,] + L \cap N(T) + D,$$

and

$$R(T) = T(N) + T(D), \quad T(D) \cap T(N) = \{0\}.$$

Let $L_2 := T^{-1}M$. Then $N(T) \subset L_2$ ($0 \in M$) and if D_2 denotes a complement of $N(T)$ in L_2, then $L_2 = D_2 + N(T)$. Now $N \cap L_2 \subset N(T)$ (since $x \in N \cap L_2$ implies $Tx \in T(N) \cap M$), and hence, $D_2 \cap N = \{0\}$. Furthermore, since K is compact, $R(K) \subset D(B)$ and $TB = I_{D(B)} - K$, it follows from Proposition 6.2.4 that $T(N) = R(TB) = R(I_{D(B)} - K)$ is relatively closed in $D(B)$. Hence, since $R(T) \subset D(B)$ and $M \subset R(T)$ is finite dimensional, $\overline{T(N)} + M = \overline{R(T)}$ implies that that $T(N) + M = R(T)$. Thus

$$D(T) = R(B) + L_2 = [\, R(B) \cap N(T) \oplus N \,] + D_2 + N(T)$$

and

$$R(T) = T(N) + T(D_2), \quad T(D_2) \cap T(N) = \{0\}.$$

Thus

$$dim\, T(D) = dim\, T(D_2) = dim\, T(L_2) = dim\, M,$$

and, since T is injective on D and on D_2, it follows that $dim\, D = dim\, D_2 < \infty$. Since L and $N(T)$ are closed, $L \cap N(T)$ is topologically complemented in L by D. Furthermore, $D \cap N = \{0\}$ since $L \cap N = \{0\}$. Thus, we have

$$
\begin{aligned}
D(T) &= R(B) \oplus L = R(B) \cap N(T) \oplus N \oplus L \\
&= R(B) \cap N(T) \oplus N \oplus L \cap N(T) \oplus D.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
N(T) &= [R(B) \cap N(T) \oplus N \oplus L \cap N(T) \oplus D] \cap N(T) \\
&= R(B) \cap N(T) \oplus [(L \cap N(T) \oplus N \oplus D) \cap N(T)] \\
&= R(B) \cap N(T) \oplus L \cap N(T) \oplus [(N \oplus D) \cap N(T)].
\end{aligned}
$$

Now, since $dim\, N \cap N(T) = \{0\}$ and $dim\, D < \infty$, it follows that have $dim\, [(D+N) \cap N(T)] < \infty$. Letting V denote the topological complement of $(D+N) \cap N(T)$ in $D+N$, it follows that

$$
\begin{aligned}
D(T) &= R(B) \cap N(T) \oplus L \cap N(T) \oplus D \oplus N \\
&= R(B) \cap N(T) \oplus L \cap N(T) \oplus V \oplus (D+N) \cap N(T).
\end{aligned}
$$

Let P_B denote a projection of $D(T)$ onto $R(B) \cap N(T)$ with kernel $L \cap N(T) \oplus V \oplus (D+N) \cap N(T)$; let P_L denote a projection of $D(T)$ onto $L \cap N(T)$ with kernel $R(B) \cap N(T) \oplus V \oplus (D+N) \cap N(T)$; let P_D denote a projection of $D(T)$ onto $(D+N) \cap N(T)$ with kernel $R(B) \cap N(T) \oplus L \cap N(T) \oplus V$. Then $P_B + P_L + P_D$ is a projection of $D(T)$ onto $N(T)$.

For the case when T is multivalued let $Q := Q_T$. Since B is single-valued, $TB(0) = T(0)$. Let $B_Q := BQ^{-1} \in LR(Y/\overline{T(0)},\ X)$. Thus, for $[y] \in Y/\overline{T(0)}$, we have $B_Q[y] = B(y + \overline{T(0)}) = By$. Hence B_Q is single-valued. Furthermore, $B_Q{}' = (BQ^{-1})' = (Q^{-1})'B'$, where $(Q^{-1})'$ is continuous. Define $K_Q \in LR(Y/\overline{T(0)})$ as follows: for $[y] \in Y/\overline{T(0)}$, let $K_Q[y] := [Ky]$. Clearly K_Q is compact. Thus $QTB_Q = I_{D(B)/\overline{T(0)})} - K_Q$, and QT, B_Q and K_Q satisfy the hypotheses. From what has already been shown, it follows that $QT \in \mathcal{F}_-$ and $N(QT)$ is topologically complemented. Thus $T \in \mathcal{F}_-$ and, since $T(0)$ is closed in $R(T)$, we have that that $N(QT) = N(T)$. The result follows.

◇

Remarks 6.2.8

In Theorem 6.2.6 above, it suffices to consider the proof for the case $I_{D(T)} - K \in \mathcal{F}_+(X)$ instead of $K \in B(X)$ such that K is of finite rank. Similarly, in the proof of Theorem 6.2.7 it suffices to consider the case $I - K|_{D(B)} \in \mathcal{F}(Y)$ instead of $K \in B(Y)$ such that K is of finite rank.

Definitions 6.2.9 *An operator $A \in B(Y, X)$ satisfying Theorem 6.2.6 (ii) above is referred to as a* **left regulariser** *or* **left generalised inverse** *of $T \in \mathcal{A}_\alpha$, and an operator $B \in L(Y, X)$ satisfying Theorem 6.2.7 (ii) above is referred to as a* **right regulariser** *or* **right generalised inverse** *of $T \in \mathcal{A}_\beta$.*

The characterisation theorem for $\alpha-$Atkinson relations establishes the existence of a continuous left regulariser. In the characterisation of $\beta-$Atkinson relations it is not known whether a right regulariser is necessarily continuous.

Corollaries 6.2.10 and 6.2.11 below consider the characterisations for the case when the spaces X and Y are complete and the relation T is closed. In this case, $R(T)$ is closed in Theorem 6.2.6, and hence, a densely-defined left regulariser A can be continuously extended to the whole of Y. For the case of $\beta-$Atkinson relations, the characterisation becomes stronger. In particular, the right regulariser B is continuous and everywhere-defined when the spaces are complete and T is closed. Furthermore, we may characterise $\beta-$Atkinson relations by appealing to the duality property that $T \in \mathcal{A}_\beta$ if and only if $T' \in \mathcal{A}_\alpha$, given in Proposition 6.2.14 below, when the spaces are complete and T is closed. In particular, if $I - K$ is a selection of TB , and B is continuous and everywhere-defined, then $B'T' = (TB)' \in \Phi_+$ and B' and T' satisfy the conditions of Theorem 6.2.6.

Corollary 6.2.10 (to Theorem 6.2.6) *Let X and Y be complete and let $T \in LR(X,Y)$ be closed. If $T(0)$ is topologically complemented, then the following are equivalent.*

(i) $T \in \mathcal{A}_\alpha(X,Y)$

(ii) $\exists A \in B(Y,X)$ *and a finite rank projection* $K \in B(X)$ *such that*

$D(A) = Y$, $P(R(T)) \subset R(T)$, *where P is a continuous projection from Y onto $N(A)$,*

$R(A) \subset D(T)$, $R(K) \subset D(T)$, *and*

$AT = (I-K)|_{D(T)}$.

Corollary 6.2.11 (to Theorem 6.2.7) *Let X and Y be complete and let $T \in LR(X,Y)$ be closed. If $N(T)$ is closed in $D(T)$, then the following are equivalent.*

(i) $T \in \mathcal{A}_\beta(X,Y)$

(ii) $\exists B \in B(Y,X)$ *and a finite rank projection* $K \in B(Y)$ *such that B is everywhere-defined,*

$R(B)$ *topologically complemented in* $D(T)$, $R(T) \subset D(B)$, $R(K) \subset D(B)$,

B' *is continuous, either* $R(T') \subset D(B')$ *or* $D(B'T') \in \mathcal{C}(D(T'))$, *and*

$(I-K)|_{D(B)}$ *is a selection of* TB.

We conclude this section with Propositions 6.2.12 and 6.2.14 which give the duality relationships between $\alpha-$Atkinson and $\beta-$Atkinson properties. The properties of *multivalued linear projections* are applicable here. In the definition for $\beta-$Atkinson relations, we note that $N(T)$ is topologically complemented in $D(T)$, and not necessarily in the whole space X. Thus, if P is a projection from $D(T)$ onto $N(T)$, P' is not necessarily single-valued.

Proposition 6.2.12 *Let $T \in LR(X,Y)$. Then*

(a) $T \in \mathcal{A}_\alpha(X,Y) \;\Rightarrow\; T' \in \mathcal{A}_\beta(Y',X')$.

(b) If T is open and $D(T)$ is topologically complemented in X,

then $T \in \mathcal{A}_\beta(X,Y) \;\Rightarrow\; T' \in \mathcal{A}_\alpha(Y',X')$.

PROOF

(a) $T \in \mathcal{F}_+ \Leftrightarrow T' \in \Phi_-$, in particular, $R(T')$ is closed and $\beta(T') < \infty$. Thus, $R(T')$ is topologically complemented in X'.

If P_R is the continuous projection from Y onto $\overline{R(T)}$. Then P_R' is a projection defined on Y' with $R(P_R') = N(P_R)^\perp$ and $N(P_R') = R(P_R)^\perp = \overline{R(T)}^\perp = N(T')$. Hence $N(T') = R(I_{Y'} - P_R')$.

(b) Since $T \in \mathcal{F}_- \Leftrightarrow T' \in \mathcal{F}_+ \Leftrightarrow T' \in \Phi_+$ we need only verify that topological complementation is preserved. Without loss of generality we assume $D(T) = X$. Let P_N denote a continuous projection defined on X with range $N(T)$. Thus P'_N is open and $R(P'_N) = N(P_N)^\perp$. Applying Propositions 2.6.5 and 3.6.4 again, we have

$$N(P'_N) = R(P_N)^\perp = N(T)^\perp = R(T')$$

Hence, $R(T')$ is the range of the continuous projection $I_{X'} - P'_N$. Since $\alpha(T') < \infty$, $N(T')$ is the range of the continuous projection.

◇

The condition that T is open for Proposition 6.2.12 (b) is included to ensure that $R(T') = N(T)^\perp$. More generally $R(T') = N(\overline{T})^\perp \subset N(T)^\perp$, and if $T \in \mathcal{A}_\beta$ is closed and X and Y are complete then $T \in \Phi_-$ is open. It is not known whether $R(T')$ is topologically complemented without the additional assumption.

In general, the reverse implications of Proposition 6.2.12 do not hold (see also Example 3.6.13 (3)):

Example 6.2.13

$T' \in \mathcal{A}_\alpha \not\Rightarrow T \in \mathcal{A}_\beta$: Let X be an infinite-dimensional Banach space, and let P be an everywhere-defined linear projection with $N(P)$ a dense, non-closed hyperplane. Then $N(P)$ is not topologically complemented, and $P \notin \mathcal{A}_\beta$. However, P' is a projection with $N(P') = R(P)^\perp$ and $R(P') \subset \overline{R(P')}^* = N(\overline{P})^\perp = \{0\}$. It follows that $N(P')$ and $R(P')$ are topologically complemented, and thus, $P' \in \mathcal{A}_\alpha$. Similarly we may show that $T' \in \mathcal{A}_\beta \not\Rightarrow T \in \mathcal{A}_\alpha$.

Proposition 6.2.14 *Let X and Y be complete and let $T \in LR(X,Y)$ be closed. Then*
(a) $T \in \mathcal{A}_\alpha(X,Y) \Leftrightarrow T' \in \mathcal{A}_\beta(Y',X')$.
(b) If $D(T)$ is topologically complemented in X then, $T \in \mathcal{A}_\beta(X,Y) \Leftrightarrow T' \in \mathcal{A}_\alpha(Y',X')$.

PROOF
By Proposition 6.2.12, we need only establish the reverse implications.

(a) By Theorem 5.8.5, $T \in \Phi_+$, and $R(T)$ is closed. Let M denote some topological complement of $N(T')$ in Y'. Now $R(T) = \overline{R(T)} = N(T')^\top$,

$$\overline{R(T) \cap M^\top} = (N(T') + M)^\top = (Y')^\top = \{0\},$$

and

$$R(T) + M^\top = (N(T') \cap M)^\top = \{0\}^\top = Y.$$

It follows that the projection associated with the pair $R(T)$ and $M^{\top}$ is closed. By the Closed Graph theorem, it is continuous and hence, $M^{\top}$ is a topological complement of $R(T)$.

(b) By Theorem 5.2.11, $T \in \Phi_{-}$. Let N denote a topological complement of $R(T')$ in X'. Since T is closed, $N(T) = R(T')^{\top}$, and,

$$\overline{N(T) \cap N^{\top}} = (R(T') + N)^{\top} = (X')^{\top} = \{0\},$$

and

$$N(T) + N^{\top} = (R(T') \cap N)^{\top} = \{0\}^{\top} = X.$$

As in (a), it follows that $N^{\top}$ is a topological complement of $N(T)$.

◇

6.3 Small Perturbation of Atkinson Relations

In this section we investigate the stability of Atkinson relations under small perturbation, as well as the behaviour of the index under perturbation. If X and Y are not complete or if T is not closed, then $R(T)$ may not be closed. In particular, the index may fail to be stable under bounded perturbation. Furthermore, the reduced index $\bar{\kappa}(T)$ may also not be stable. Important counterexamples are given in Labuschagne [84]. Nevertheless, it is possible to consider stability of $\kappa(\tilde{T})$ and of $\bar{\kappa}(T)$ under suitable restrictions. In particular, Proposition 6.3.5 shows that the index $\kappa(T)$ is stable when the space containing the domain is complete and the relation T is closed. For the quantity $\kappa(\tilde{T})$, Proposition 6.3.1 (c) is given as a special case of Proposition 6.1.1.

The proofs given below of stability of Atkinson properties under small perturbation depend on the existence of continuous regularisers. With the characterisation theorems, we noted that it is not known whether a right regulariser of a β–Atkinson relation is necessarily continuous, and hence, this condition is stated as an additional assumption when required.

Proposition 6.3.1
Let $T \in LR(X,Y)$, and suppose $S \in LR(X,Y)$ satisfies $D(S) \supset \overline{D(T)}$ and $S(0) \subset \overline{T(0)}$.

(a) If $T \in \mathcal{A}_{\alpha}$, $\overline{T(0)}$ is topologically complemented in Y, $R(T+S) \cap D(A)$ is relatively closed in $D(A)$, $Y = R(T+S) + D(A)$ and $\overline{PR(T+S)} \subset \overline{R(T+S)}$, where P is a continuous projection from Y onto $N(A)$, then $\exists\ \epsilon > 0$ such that $||S|| < \epsilon \Rightarrow T+S \in \mathcal{A}_{\alpha}(X,Y)$.

(b) If $T \in \mathcal{A}_{\beta}$ has a continuous right regulariser B, $R(S') \subset D(B')$, $R(T+S) \subset D(B)$, and $(T+S)(0)$ and $N(T+S)$ are closed in $R(T+S)$ and $D(T+S)$, respectively, then $\exists\ \epsilon > 0$ such that $||S|| < \epsilon \Rightarrow T+S \in \mathcal{A}_{\beta}(X,Y)$.

(c) If (a) or (b) holds, and $||S|| < \gamma(\tilde{T})$, then $\kappa(\tilde{T}+\tilde{S}) = \kappa(\tilde{T})$.

PROOF
(a) Let $\epsilon := \frac{\gamma(AT)}{||A||} > 0$, where A denotes a left regulariser of T. If $||S|| < \epsilon$, then by Proposition 2.3.13,

$$||AS|| \leq ||A|| \, ||S|| < \gamma(AT).$$

Thus, by Theorem 6.2.6,

$$A(T+S) = AT + AS = (I - K_{AT} + AS)|_{D(T)},$$

where K_{AT} is the precompact operator associated with $T \in \mathcal{A}_\alpha$ and the given regulariser A. Hence, on perturbing $I - K_{AT}$ by AS, it follows from Proposition 5.4.2, that $A(T+S) \in \mathcal{F}_+(X,Y)$. Now

$$R(A) \subset D(T+S),$$

and

$$R(K_{AT} - AS) \subset D(T+S)$$

since $R(A) \subset D(T) \subset D(S)$ and $R(K_{AT}) \subset D(T) \subset D(S)$, respectively. Furthermore $\overline{(S+T)(0)}$ is topologically complemented in Y since $\overline{(S+T)(0)} = \overline{S(0)+T(0)} = \overline{T(0)}$. Thus, applying Theorem 6.2.6 to $T+S$ with A serving as a left regulariser, and substituting $K_{AT} - AS$ for the operator K in the same theorem, it follows from Remarks 6.2.8 that $T + S \in \mathcal{A}_\alpha(X,Y)$.

(b) For the case $T \in \mathcal{A}_\beta$, let $\epsilon := \frac{\gamma(TB)}{||B||} > 0$ where B denotes a continuous right regulariser of T. Then, as in the proof of (a), $||S|| < \epsilon$ implies $||SB|| < \gamma(TB)$. Now

$$(T+S)B = TB + SB = I - K_{TB} + SB,$$

where K_{TB} is the precompact operator of Theorem 6.2.7 associated with $T \in \mathcal{A}_\beta$ and the given regulariser B. Hence, on perturbing $I - K_{TB} \in \mathcal{F}_+ \cap \mathcal{F}_-$ by SB, it follows from Propositions 5.4.2 and 5.7.1 that $(T+S)B \in \mathcal{F}_+ \cap \mathcal{F}_-$. By Theorem 6.2.7 , $R(B)$ is topologically complemented in $D(T) = D(T+S)$. Last, we have $R((S+T)') = R(S' + T') \subset D(B')$ since $R(S') \subset D(B')$ and $R(T') \subset D(B')$. Thus, applying Theorem 6.2.7 to $T+S$ with B serving as a right regulariser, and substituting $K_{TB} - SB$ for the operator K in the same theorem, it follows from Remarks 6.2.8 that $T + S \in \mathcal{A}_\beta(X,Y)$.

(c) This follows immediately from Theorem 6.1.1.

◇

Proposition 6.3.2
Let $T \in LR(X,Y)$, and let $S \in LR(X,Y)$ satisfy $D(S) \supset \overline{D(T)}$ and $S(0) \subset \overline{T(0)}$.

(a) If $T \in \mathcal{A}_\alpha$ and $I + AS \in \mathcal{F}_+(X)$, and if $\overline{T(0)}$ is topologically complemented in Y , $R(T+S) \cap D(A)$ is relatively closed in $D(A)$, $Y = R(T+S) + D(A)$, and $P\overline{R(T+S)} \subset \overline{R(T+S)}$, where P is a continuous projection from Y onto $N(A)$, then $T + S \in \mathcal{A}_\alpha(X,Y)$.

(b) If $T \in \mathcal{A}_\beta$ and $I + SB \in \mathcal{F}_+ \cap \mathcal{F}_-(X)$, where B denotes a right regulariser of T, if $(T+S)(0)$ and $N(T+S)$ are closed in $R(T+S)$ and $D(T+S)$, respectively, and if $R(T+S) \subset D(B)$ and $R(S') \subset D(B')$, then $T + S \in \mathcal{A}_\beta(X,Y)$.

PROOF

(a) Let K_{AT} denote precompact operator of Theorem 6.2.6 associated with $T \in \mathcal{A}_\alpha$ and the given regulariser A. Then we have

$$A(T+S) = AT + AS \subset I - K_{AT} + AS.$$

Thus, since $I + AS \in \mathcal{F}_+$, it follows from Proposition 5.4.2 that $I - K_{AT} + AS \in \mathcal{F}_+(X)$. The conclusion now follows by the arguments given for Proposition 6.3.1 (a).

(b) Let K_{TB} denote precompact operator of Theorem 6.2.7 associated with $T \in \mathcal{A}_\beta$ and the given regulariser B. Then we have

$$(T+S)B = TB + SB = I - K_{TB} + SB.$$

Thus, since $I+SB \in \mathcal{F}_+ \cap \mathcal{F}_-$, it follows from Propositions 5.4.2 and 5.7.1 that $I - K_{TB} + SB \in \mathcal{F}_+ \cap \mathcal{F}_-$. The conclusion now follows by the arguments given for Proposition 6.3.1 (b).

◇

Corollary 6.3.3 *Let $T \in \mathcal{A}_\alpha(X,Y)$ with $\overline{T(0)}$ topologically complemented in Y, and suppose $S \in L(X,Y)$ is everywhere-defined and single-valued and $||S|| < ||A||^{-1}$. If $R(T+S) \cap D(A)$ is relatively closed in $D(A)$, $Y = R(T+S) + D(A)$ and $P\overline{R(T+S)} \subset \overline{R(T+S)}$, where P is a continuous projection from Y onto $N(A)$, , then $T+S \in \mathcal{A}_\alpha(X,Y)$.*

PROOF If $||S|| < ||A||^{-1}$ then $||AS|| < 1$. Hence $I + AS$ is invertible in the operator algebra. In particular, $I + AS \in \mathcal{F}_+$. The conclusion follows from Proposition 6.3.2 (a).

◇

Corollary 6.3.4 *Let $T \in \mathcal{A}_\beta(X,Y)$, and suppose $S \in L(X,Y)$ is everywhere-defined and single-valued and $||S|| < ||B||^{-1}$, where B denotes a continuous right regulariser of T. If $R(T+S) \subset D(B)$, $R(S') \subset D(B')$ and if $(T+S)(0)$ and $N(T+S)$ are closed in $R(T+S)$ and $D(T+S)$, respectively, then $T+S \in \mathcal{A}_\beta(X,Y)$.*

PROOF If $||S|| < ||B||^{-1}$ then $||SB|| < 1$. Hence $I + SB$ is invertible in the operator algebra. In particular, $I + SB \in \mathcal{F}_+ \cap \mathcal{F}_-$. The conclusion follows from Proposition 6.3.2 (b).

◇

Proposition 6.3.5 *Let X be complete, let $T \in LR(X,Y)$ be closed and suppose $S \in LR(X,Y)$ satisfies $D(S) \supset \overline{D(T)}$ and $S(0) \subset \overline{T(0)}$.*

(a) If $T \in \mathcal{A}_\alpha$, $\overline{T(0)}$ is topologically complemented in Y, $Y = R(T+S) + D(A)$, $R(T+S) \cap D(A)$ is relatively closed in $D(A)$ and $P\overline{R(T+S)} \subset \overline{R(T+S)}$, where P is a continuous projection from Y onto $N(A)$, then $\exists\ \epsilon > 0$ such that $||S|| < \epsilon$ implies $T+S \in \mathcal{A}_\alpha(X,Y)$ is open and has closed range.

(b) If $T \in \mathcal{A}_\beta$ has a continuous right regulariser B, $R(T+S) \subset D(B)$, $R(S') \subset D(B')$ and if $(T+S)(0)$ and $N(T+S)$ are closed in $R(T+S)$ and $D(T+S)$, respectively, then $\exists\ \epsilon > 0$ such that $||S|| < \epsilon$ implies $T+S \in \mathcal{A}_\beta(X,Y)$ is open and has closed range.

(c) If (a) or (b) holds and $||S|| < \gamma(T)$, then $\kappa(T+S) = \kappa(T)$.

PROOF

(a) Since $T' \in \mathcal{A}_\beta$ is open (Proposition 6.2.12) it follows from Theorem 3.3.9, $T = \overline{T}$ is open and $R(T)$ is closed. Let A be a left regulariser of T, and let $\epsilon := \min\{\ \gamma(T),\ \frac{\gamma(AT)}{||A||}\ \}$. Then $\epsilon > 0$. As in Proposition 6.3.1 (a), if $||S|| < \epsilon$, then $T + S \in \mathcal{A}_\alpha$. In this case, $(T+S)' \in \mathcal{A}_\beta$ is open. Furthermore, since S is continuous and $D(S) \supset \overline{D(T)}$, $T + S$ is closed and $T' + S' = (T+S)'$. Thus, applying Theorem 3.3.9 again, it follows that $T + S$ is open and has closed range.

(b) Applying Proposition 6.3.1 (b) and Theorem 3.3.9, the proof is similar to that given in (a).

(c) The relations T and S satisfy the hypotheses of Proposition 6.1.1. Since $T \in \mathcal{A}_\alpha \cup \mathcal{A}_\beta$ implies $T' \in \mathcal{A}_\alpha \cup \mathcal{A}_\beta$ it follows that $R(T')$ is closed. As in the proof of Proposition 6.1.1, $||S'|| \leq ||S|| < \gamma(T) \leq \gamma(T')$ (Propositions 2.6.12 and 2.6.13). Hence,

$$\kappa((T+S)') = \kappa(T' + S') = \kappa(T')$$

Thus, since $R(T)$ and $R(T+S)$ are closed (from (a) or (b) above), and since T is closed,

$$\kappa(T+S) = \bar{\kappa}(T+S) = -\kappa((T+S)') = -\kappa(T'+S') = -\kappa(T') = \bar{\kappa}(T) = \kappa(T).$$

◇

6.4 Directed Perturbations

Certain Fredholm type properties of linear operators and the reduced index $\bar{\kappa}(T)$ are known to be stable under "directed" perturbation (cf. Labuschagne [84]). In this section we consider the stability of Atkinson properties and of the quantity $\bar{\kappa}(T)$ of linear relations under directed perturbation. Propositions 6.4.1 and 6.4.2 below apply the characterisation theorems to give a simple proof for the stability of Atkinson properties (cf. Cross [36], V.16.1). Propositions 6.4.4 and 6.4.5 are based on results in Labuschagne [84] (Theorems 12 and 14, respectively), and require that T be open and that the kernel $N(S)$ of the perturbing operator be directed with respect $N(T)$. Before proving these propositions, we give Proposition 6.4.3 on the stability of the index under finite rank perturbation (cf. [84], Proposition 9).

Proposition 6.4.1

Suppose $T \in \mathcal{A}_\alpha$ with $\overline{T(0)}$ topologically complemented in Y and $A := T|_M^{-1} P_R$, where $M \in \mathcal{P}(D(T))$ such that $T|_M$ is injective and open and P_R denotes a continuous projection of Y onto $\overline{T(M)}$. If S satisfies $D(S) \supset D(T)$, $S(0) \subset \overline{T(0)}$, $R(T+S) \cap D(A) = \overline{R(T+S)} \cap D(A)$, $Y = R(T+S) + D(A)$, $P\overline{R(T+S)} \subset \overline{R(T+S)}$, where P is a continuous projection from Y onto $N(A)$ and $R(S) \subset R(I - P_R)$, then $T + S \in \mathcal{A}_\alpha(X,Y)$.

PROOF

Since $P_R S = 0$, we have $A(T+S) = AT + AS = AT + T_M^{-1} P_R S = AT$.

Let K_{AT} denote the precompact operator associated with $T \in \mathcal{A}_\alpha$ and the given regulariser A. Applying Theorem 6.2.6 to $T + S$ with A serving as a left regulariser, and K_{AT} serving as the operator K in the same theorem, it follows that $T + S \in \mathcal{A}_\alpha(X,Y)$.

◇

Proposition 6.4.2

Suppose $T \in \mathcal{A}_\beta(X,Y)$ and B is a right regulariser given by $B := T|_N^{-1}(I - P_{R^c})$, where N is a topological complement of $N(T)$ and P_{R^c} is a continuous projection of Y onto a complement of $\overline{R(T)}$. If S satisfies $D(S) \supset D(T)$, $S(0) \subset \overline{T(0)}$ and $N(S) \supset N$, if $(T+S)(0)$ and $N(T+S)$ are closed in $R(T+S)$ and $D(T+S)$, respectively and $R(T+S) \subset D(B)$, then $T+S \in \mathcal{A}_\beta(X,Y)$.

PROOF

Since $R(T_N^{-1}(I - P_{R^c})) = N \subset N(S)$, we have $(T+S)B = TB + SB = TB + ST_N^{-1}(I - P_{R^c}) = TB$. Furthermore, $N \oplus N(T) = D(T)$, and hence, $(T+S)|_N = T|_N$. Thus, $R((T+S)') \subset R(((T+S)|_N)') = R((T|_N)') = D(B')$ and hence, $D(B'(T+S)') = D((T+S)')$.

Let K_{TB} denote the precompact operator associated with $T \in \mathcal{A}_\beta$ and the given regulariser B. Since $D(B'(T+S)') \in \mathcal{C}(D((T+S)')$, we may apply Theorem 6.2.7 to $T+S$ with B serving as a left regulariser, and K_{TB} serving as the operator K in the same theorem. It follows that $T+S \in \mathcal{A}_\beta(X,Y)$.

◇

Proposition 6.4.3

Let T be open with closed range, and suppose S is continuous with finite rank, $D(S) \supset D(T)$ and $S(0) \subset T(0)$. Then $R(T+S)$ is closed and $\kappa(T+S) = \kappa(T)$ if $\kappa(T)$ exists.

PROOF

Without loss of generality we assume that $D(T) = X$. Let $M := N(S)$. Since $S(0)$ is finite-dimensional, and hence closed, M is closed and finite codimensional in X (Proposition 2.2.3). Thus, since $M \subset M + N(T)$, so is $M + N(T)$. Furthermore, since $x \in N((T+S)|_M)$ if and only if $(T+S)x = (T+S)(0)$ for $x \in M$ if only if $x \in N(T|_M)$, we have

$$\alpha(T|_M) = \alpha((T+S)|_M). \tag{6.3}$$

To see that $(T+S)|_M$ has closed range, suppose $Q_T T x_k \to y$ for $\{x_k\} \subset M$. Since $R(T)$ is closed, $y \in Tx$ for $x \in X$. Since T is open, $d(x - x_k,\ N(T)) \to 0$. Thus $\exists \{n_k\} \subset N(T)$ such that $x_k + n_k \to x$. Since $M + N(T)$ is closed, $x \in M + N(T)$, $y \in TM$, $TM = (T+S)M$ is closed and

$$\beta(T|_M) = \beta((T+S)|_M). \tag{6.4}$$

Furthermore, $R(T+S)$ is closed since $\mathit{dim}\ R(T+S)/R((T+S)|_M) \leq \mathit{dim}\ X/M < \infty$. Thus, if $\kappa(T|_M)$ exists, then, combining (6.3) and (6.4), we have

$$\kappa(T|_M) = \kappa((T+S)|_M). \tag{6.5}$$

Letting $\eta := \mathit{codim}\ M$, it follows from the Finite Dimensional Extension Lemma, 2.7.5, that

$$\kappa(T) = \kappa(T|_M) + \eta = \kappa((T+S)|_M) + \eta = \kappa(T+S).$$

◇

Proposition 6.4.4

Let T be open, and let S be continuous with $D(S) \supset \overline{D(T)}$, $S(0) \subset \overline{T(0)}$ and $S(0)$ closed. If $||S|| < \gamma(T)$ and $N(S) \supset N(T)$, then $N(T+S) = N(T)$ and $\bar{\kappa}(T+S) = \bar{\kappa}(T)$.

PROOF

We show that $N(T+S) = N(T)$. Let $\hat{S} \in LR(\ X/N(S)\ ,\ Y)$ denote the injective component of S. Since $S(0)$ is closed, it follows from Corollary 2.9.3 that $||\hat{S}|| = ||S||$, and, therefore,

$$||Sx|| \leq ||S||(d(x, N(S)) \leq ||S||(d(x, N(T)).$$

Thus,

$$\begin{aligned} ||(T+S)x|| &\geq ||Tx|| - ||Sx|| \\ &\geq \gamma(T)d(x, N(T)) - ||S||d(x, N(T)) \\ &= (\gamma(T) - ||S||)d(x, N(T)). \end{aligned} \tag{6.6}$$

Hence,

$$x \in N(T+S) \Rightarrow x \in \overline{N(T)} \Rightarrow x \in \overline{N(S)} = N(S),$$

and

$$N(T) = N(T) \cap N(S) \subset N(T+S) = N(T+S) \cap N(S) \subset N(T),$$

i.e. $N(T) = N(T+S)$. Furthermore, applying (6.6), it follows that $\gamma(T+S) \geq \gamma(T) - ||S|| > 0$.

By Theorem 3.5.2, $\bar{\beta}(T+S) \leq \bar{\beta}(T)$. For the reverse inequality, choose N such $\frac{||S||}{N} < \gamma(T) - ||S||$. Then, from what has just been shown,

$$\gamma(T + S - \tfrac{k}{N}S) \geq \gamma(T) - (1 - \tfrac{k}{N})||S|| \geq \gamma(T) - ||S||$$

for $0 \leq k \leq N$. Since $\frac{||S||}{N} < \gamma(T + S - \frac{k}{N}S)$, it follows from Theorem 3.5.2 that

$$\bar{\beta}(T + S - \frac{k+1}{N}S) \leq \bar{\beta}(T + S - \frac{k}{N}S). \tag{6.7}$$

Substituting $N-1, N-2, \ldots, 1, 0$ into (6.7) it follows that $\bar{\beta}(T) \leq \bar{\beta}(T+S)$. Thus, if $\bar{\kappa}(T)$ exists, then

$$\bar{\kappa}(T+S) = \bar{\kappa}(T).$$

◇

Proposition 6.4.5

Let T be open with $R(T)$ topologically complemented in Y and $N(T)$ topologically complemented in $D(T)$, and let S be continuous with $D(S) \supset \overline{D(T)}$ and $S(0) \subset T(0)$. If P is a continuous projection onto $N(T)$ and $N(S) \supset N(P)$, then $R(T+S)$ is closed and $\kappa(T+S) = \kappa(T)$.

PROOF

Without loss of generality, we assume that $D(T) = X$.

Let $M := N(P)$. Define $L := P - (\widehat{I-P})\ q_nT^{-1}S$ where $(\widehat{I-P}) \in LR(X/N(T), X)$ is the injective component of $I - P$, and $q_n := q_{N(T)}$ is the natural quotient map from X into $X/N(T)$. Then L is a continuous projection onto $N(T + S)$ with kernel M. To see this, we first note

$$\begin{aligned} L^2 &= (P - (\widehat{I-P})\ q_nT^{-1}S)(P - (\widehat{I-P})\ q_nT^{-1}S) \\ &= P^2 - (\widehat{I-P})\ q_nT^{-1}SP + ((\widehat{I-P})\ q_nT^{-1}S)((\widehat{I-P})\ q_nT^{-1}S) \\ &= P - (\widehat{I-P})\ q_nT^{-1}, S \end{aligned}$$

since $R(I - P) = M \subset N(S)$ and $S = S(P + (I - P)) = SP$. Choosing $x \in N(T + S)$ we have

$$\begin{aligned} x &= (P + (I - P))x \\ &= (P + (\widehat{I-P})\ q_n)x \\ &= (P + (\widehat{I-P})\ q_nT^{-1}T)x \\ &= (P - (\widehat{I-P})\ q_nT^{-1}S)x, \end{aligned}$$

i.e. $N(T + S) \subset R(L)$. For the reverse inclusion,

$$\begin{aligned} (T + S)(P - (\widehat{I-P})\ q_nT^{-1}S)x &= SPx - T(\widehat{I-P})\ q_nT^{-1}Sx \\ &= Sx - TT^{-1}Sx \\ &= Sx - (Sx + T(0)) \\ &= S(0) + T(0) \\ &= (T + S)(0). \end{aligned}$$

Thus $R(L) \subset N(T+S)$, and hence $R(L) = N(T+S)$, i.e. $N(T+S)$ is topologically complemented.

Now suppose $R(S) \subset R(T)$. Then $R(T + S) \subset R(T)$. Furthermore, since $X = N(T) \oplus M$ and $M \subset N(S)$, we have that $R(T) = TM = (T + S)M \subset R(T + S)$. Thus,

$$R(T) = R(T + S). \tag{6.8}$$

Furthermore,

$$\begin{aligned} (I - L)X &= ((I - P) + (\widehat{I-P})\ q_nT^{-1}S)X \\ &= (\widehat{I-P})(q_n +\ q_nT^{-1}S)X \\ &= (\widehat{I-P})q_nT^{-1}(T + S)X \\ &= (\widehat{I-P})q_nT^{-1}TX \\ &= (\widehat{I-P})q_nX \\ &= (I - P)X\ =\ M. \end{aligned}$$

Since L is a continuous projection, it follows that $X = N(T + S) \oplus M$. In particular,

$$\alpha(T + S) = \alpha(T). \tag{6.9}$$

Thus, if $R(S) \subset R(T)$, it follows from (6.8) and (6.9) that

$$\kappa(T+S) = \kappa(T).$$

More generally, suppose $R(T)$ is topologically complemented in Y and let P_R denote a continuous projection of Y onto $R(T)$. Then $R(P_R S) \subset R(T)$ and $N(P_R S) \supset N(S) \supset N(T)$. From what has already been shown, it follows that $R(T+P_R S)$ is closed and $\kappa(T+P_R S) = \kappa(T)$. Now $(I-P_R)S$ is a finite rank operator. Thus, from Proposition 6.4.3, we have that $R(T+S)$ is closed and

$$\kappa(T+S) = \kappa(T+P_R S) = \kappa(T).$$

◇

6.5 Strictly Singular and Strictly Cosingular Perturbation of Atkinson Relations

Propositions 6.5.1 and 6.5.2 are strictly singular and strictly cosingular analogues of Proposition 6.3.1 parts (a) and (b), respectively. As in previous sections, some arguments depend on the existence of continuous regularisers - this assumption is added for $\beta-$Atkinson relations when required.

Propositions 6.5.1 and 6.5.2 give conditions for the stability of the index of the completion $\kappa(\tilde{T})$ of an Atkinson relation T. We next consider stability of the index $\kappa(T)$ and the reduced index $\bar{\kappa}(T)$. Proposition 6.5.3 (a) and (b) are multivalued analogues of Labuschagne [84], Propositions 10 and 17, respectively (see also Pietsch [120]). These results are concerned with the case when perturbation is by a compact operator. The property fails if $R(T)$ is not closed (counterexamples can be found in [84]). In Propositions 6.5.4 and 6.5.5, it is shown that if the space containing $D(T)$ is complete and T is closed, then the conclusions of Proposition 6.5.3 (a) and (b) hold for strictly singular and strictly cosingular perturbation as well.

Proposition 6.5.1
If $T \in \mathcal{A}_\alpha(X,Y)$ with $\overline{T(0)}$ topologically complemented in Y and S is strictly singular and satisfies $D(S) \supset \overline{D(T)}$ and $S(0) \subset \overline{T(0)}$, $R(T+S) \cap D(A)$ is relatively closed in $D(A)$, $Y = D(A) + R(T+S)$ and $P\overline{R(T+S)} \subset \overline{R(T+S)}$, where P is a continuous projection from Y onto $N(A)$. Then

(a) $T+S \in \mathcal{A}_\alpha(X,Y)$,

(b) If $||S|| < \infty$, then $\kappa(\tilde{T}+\tilde{S}) = \kappa(\tilde{T})$.

PROOF
(a) Since AS is strictly singular (Corollary 5.3.4), by Theorem 6.2.6 we have

$$A(T+S) = AT + AS = (I - K_{AT} + AS)|_{D(T)},$$

where K_{AT} is the precompact operator associated with $T \in \mathcal{A}_\alpha$ and the given regulariser A. Hence, on perturbing $I - K_{AT}$ by AS, it follows from Proposition 5.4.2, that $A(T+S) \in \mathcal{F}_+(X,Y)$. As in Proposition 6.3.1 (a) we have

$$R(A) \subset D(T+S),$$

and

$$R(K_{AT} - AS) \subset D(T+S)$$

since $R(A) \subset D(T) \subset D(S)$ and $R(K_{AT}) \subset D(T) \subset D(S)$, respectively. Furthermore $\overline{(S+T)(0)}$ is topologically complemented in Y since $\overline{(S+T)(0)} = \overline{S(0)+T(0)} = \overline{T(0)}$. Thus, applying Theorem 6.2.6 to $T+S$ with A serving as a left regulariser, and substituting $K_{AT} - AS$ for the operator K in the same theorem, it follows from Remarks 6.2.8 that $T+S \in \mathcal{A}_\alpha(X,Y)$.

(b) This follows immediately from Proposition 6.1.2 (a) .

◇

Proposition 6.5.2 *Suppose $T \in \mathcal{A}_\beta(X,Y)$ has a continuous right regulariser B, and suppose $S \in LR(X,Y)$ satisfies $D(S) \supset \overline{D(T)}$, $S(0) \subset \overline{T(0)}$ and $R(S') \subset D(B')$. If $(T+S)(0)$ and $N(T+S)$ are closed in $R(T+S)$ and $D(T+S)$, respectively, $R(T+S) \subset D(B)$ and if S is strictly singular, $J_Y S$ is strictly cosingular and $\dim Q_T S(0) < \infty$, then*

(a) $T+S \in \mathcal{A}_\beta(X,Y)$,

(b) If $||S'|| < \infty$ and S' is strictly singular, then $\kappa(\tilde{T}+\tilde{S}) = \kappa(\tilde{T})$.

PROOF

(a) Let B denote a continuous right regulariser of T. Then, since $||B|| < \infty$, SB is strictly singular (Corollary 5.3.4(b)) and $J_Y SB$, is strictly cosingular (Proposition 4.2.7). As in Proposition 6.3.1 (b),

$$(T+S)B = TB + SB = I - K_{TB} + SB,$$

where K_{TB} is the precompact operator of Theorem 6.2.7 associated with $T \in \mathcal{A}_\beta$ and the given regulariser B. Hence, on perturbing $I - K_{TB} \in \mathcal{F}_+ \cap \mathcal{F}_-$ by SB, it follows from Propositions 5.4.2 and 5.7.2 that $(T+S)B \in \mathcal{F}_+ \cap \mathcal{F}_-$. By Theorem 6.2.7 , $R(B)$ is topologically complemented in $D(T) = D(T+S)$. Thus, applying Theorem 6.2.7 to $T+S$ with B serving as a right regulariser, and substituting $K_{TB} - SB$ for the operator K in the same theorem, it follows from 6.2.8 that $T+S \in \mathcal{A}_\beta(X,Y)$.

(b) This follows immediately from Proposition 6.1.2 (b) . ◇

Proposition 6.5.3

Let $T \in LR(X,Y)$ be open with closed range, and suppose S is compact with $D(S) \supset \overline{D(T)}$ and $S(0) \subset \overline{T(0)}$.

(a) If $T \in \mathcal{A}_\alpha$ then $T+S$ is open with closed range, and $\kappa(T) = \kappa(T+S)$, and if $\overline{T(0)}$ is topologically complemented in Y and $P\overline{R(T+S)} \subset \overline{R(T+S)}$, where P is a continuous projection from Y onto $N(A)$, then $T+S \in \mathcal{A}_\alpha$.

(b) If $T \in \mathcal{A}_\beta$ then $T+S$ is open with closed range and $\kappa(T) = \kappa(T+S)$, and if T has a continuous right regulariser B, $R(S') \subset D(B')$ and $(T+S)(0)$ and $N(T+S)$ are closed in $R(T+S)$ and $D(T+S)$, respectively, then $T+S \in \mathcal{A}_\beta$.

PROOF

(a) In this case $D(A) = Y$ and hence, by Proposition 6.5.1, if $T \in \mathcal{A}_\alpha$ and $\overline{T(0)}$ is topologically complemented in Y, it follows that $T + S \in \mathcal{A}_\alpha$. For the index, we assume, without loss of generality, that $D(T) = X$. Since T is open with $\alpha(T) < \infty$, it follows from Proposition 5.1.8 that there exists $M \in \mathcal{P}(D(T))$ such that $R(T|_M)$ is relatively closed in $R(T)$ and $T|_M$ is injective and open. Since S is compact, $\exists\ N \in \mathcal{P}(M)$ such that $||S|_N|| < \gamma(T|_M)$ (Corollary 5.3.6). However,

$$\begin{aligned}\gamma(T|_M) &= \inf\{\frac{||Tx||}{||x||} \mid x \in M\} \\ &\leq \inf\{\frac{||Tx||}{||x||} \mid x \in N\} \\ &\leq \gamma(T|_N).\end{aligned}$$

Thus $||S|_N|| < \gamma(T|_N)$. Thus, by Proposition 5.1.8 again, $R(T|_N)$ is closed in $R(T|_M)$ and $T|_N$ has a continuous single-valued inverse. Thus, by Theorem 3.5.3, $(T + S)|_N$ is injective and open.

Since $S(0) \subset \overline{T(0)}$ we have $Q_{T+S}(T + S) = Q_T(T + S)$. Letting Q denote Q_T, it follows that $\gamma(Q(T+S)|_N) \geq \gamma((T+S)|_N) > 0$. We therefore assume that $T+S$ is single-valued to show that $T+S$ is open and $R(T+S)$ is closed. Suppose $(T+S)x_k \longrightarrow y$ for $\{x_k\} \subset N$. Since $(T+S)|_N$ is open, $\{x_k\}$ is a Cauchy sequence. Since S is compact, $\{Sx_k\}$ has a convergent subsequence, which we assume to be $\{Sx_k\}$ itself, which converges to z. Then $Tx_k \longrightarrow y - z$, and since TN is closed, $y - z \in TN$. Thus $\exists\ x \in N$ such that $Tx = y - z$. Since $T|_N$ is open, it follows that $Tx_k \longrightarrow Tx$ implies that $x_k \longrightarrow x$. Thus $Sx_k \longrightarrow Sx = z$ and $y = (y - z) + z = Tx + Sx \in (T + S)N$. Thus, $(T + S)N$ is closed. Since $(T + S)|_N$ has a continuous inverse, it follows from another application of Proposition 5.1.8 that $T + S$ is open. Furthermore, $codim(T + S)N < \infty$ in $R(T + S)$. Thus $R(T + S)$ is closed .

Since S' is compact and $(T + S)' = T' + S'$ it follows Proposition 6.1.2 that $\kappa(T') = \kappa(T' + S')$. Thus, since $R(T)$ and $R(T + S)$ are closed, and T and $T + S$ are open,

$$\kappa(T + S) = \bar{\kappa}(T + S) = -\kappa((T + S)') = -\kappa(T' + S') = -\kappa(T') = \bar{\kappa}(T) = \kappa(T).$$

(b) By Proposition 6.5.2, if $T \in \mathcal{A}_\beta$ and if $(T + S)(0)$ and $N(T + S)$ are closed in $R(T + S)$ and $D(T + S)$, respectively, then $T + S \in \mathcal{A}_\beta$. For stability of the index, let P denote a projection of X onto $N(T)$, and let M denote the complement of $N(T)$ under P. Then $T|_M$ has a continuous single-valued inverse with $R(T|_M) = R(T)$. Since $\alpha(T|_M) = 0$ and $S(I - P))|_M$ is compact, we may apply (a):

$$\begin{aligned}\kappa((T + S(I - P))|_M) &= \kappa(T|_M) \\ &= -\beta(T) \\ &\leq 0.\end{aligned}$$

Furthermore, $R((T + S(I - P))|_M)$ is closed and $\alpha((T + S(I - P))|_M) < \infty$ so that $N((T + S(I - P))|_M)$ is topologically complemented in M by W, say. Then, since $N(T) \subset N(S(I - P))$ we have $N(T) \subset N(T + S(I - P))$ and $N(T + S(I - P)) = N(T) \oplus N((T + S(I - P))|_M)$.

Thus, since $X = M \oplus N(T)$ and $M = W \oplus N((T + S(I-P))|_M)$, we have $N(T + S(I-P))$ complemented in X by W.

We now show that $\kappa(T+S(I-P)) = \kappa(T)$. Let $q_n \in LR(X,\ X/N(T))$ denote the natural quotient from X onto $X/N(T)$. Then Tq_n^{-1} is injective and, since $N(S(I-P)) \supset N(T)$, $S(I-P)q_n^{-1}$ is well-defined and $N(S(I-P)q_n^{-1}) \supset N(Tq_n^{-1})$. It follows that $S(I-P)q_n^{-1}$ is compact and since Tq_n^{-1} is open, injective and has closed range, it follows from the arguments given in (a) that $R(T + S(I-P)) = R(Tq_n^{-1} + S(I-P)q_n^{-1})$ is closed and

$$\kappa((T + S(I-P))q_n^{-1}) = \kappa(Tq_n^{-1}) = -\beta(T) = -\beta(Tq_n^{-1}).$$

Furthermore, $N((T + S(I-P))q_n^{-1}) = N(T + S(I-P))/N(T)$. Thus,

$$\alpha((T + S(I-P))q_n^{-1}) + \alpha(T) = \alpha(T + S(I-P)).$$

We also have that $\kappa((T + S(I-P))q_n^{-1}) = \kappa(T + S(I-P)) - \alpha(T)$. Thus,

$$\kappa(T + S(I-P)) = \alpha(T) - \beta(T) = \kappa(T).$$

Applying Proposition 6.4.5 to SP and $T + S(I-P)$ with $N(SP) \supset M \supset W$, the result follows, i.e. $R(T + S)$ is closed and

$$\kappa(T + S) = \kappa(T + S(I-P)) = \kappa(T).$$

◇

Proposition 6.5.4
Let X be complete and let $T \in \mathcal{A}_\alpha(X,Y)$ be closed with $T(0)$ topologically complemented in Y. Suppose S is continuous and strictly singular with $D(S) \supset \overline{D(T)}$, $S(0) \subset T(0)$, $R(T+S) \cap D(A)$ is relatively closed in $D(A)$, $Y = R(T+S) + D(A)$ and $P\overline{R(T+S)} \subset \overline{R(T+S)}$, where P is a continuous projection from Y onto $N(A)$, Then,

(a) $T + S \in \mathcal{A}_\alpha$ is open and has closed range, and

(b) If $\|S'\| < \infty$ and S' is strictly singular, then $\kappa(T+S) = \bar{\kappa}(T+S) = \bar{\kappa}(T) = \kappa(T)$.

PROOF
The proof is similar to that of Proposition 6.3.5. As in Theorem 6.3.5, T is open and $R(T)$ is closed.

(a) By Proposition 6.5.1, $T + S \in \mathcal{A}_\alpha$, and thus, and $(T+S)' \in \mathcal{A}_\beta$ is open. Furthermore, since S is continuous and $D(S) \supset \overline{D(T)}$, $T + S$ is closed and $T' + S' = (T+S)'$. Thus, applying Theorem 3.3.9, $T + S$ is open and has closed range.

(b) Now T' and S' satisfy the hypotheses of Proposition 6.1.2 (b). Thus,

$$\kappa(T+S) = \bar{\kappa}(T+S) = -\kappa((T+S)') = -\kappa(T'+S') = -\kappa(T') = \bar{\kappa}(T) = \kappa(T).$$

◇

Proposition 6.5.5

Let X be complete and let $T \in \mathcal{A}_\beta(X,Y)$ be closed with a continuous regulariser B. Suppose S is continuous and strictly singular with $D(S) \supset \overline{D(T)}$, $S(0) \subset T(0)$, $R(S') \subset D(B')$ and $J_Y S$ is strictly cosingular with $dim Q_T S(0) < \infty$. If $(T+S)(0)$ and $N(T+S)$ are closed in $R(T+S)$ and $D(T+S)$, respectively, and $R(T+S) \subset D(B)$, then

(a) $T+S \in \mathcal{A}_\beta$ is open and has closed range, and

(b) If $||S'|| < \infty$ and S' is strictly singular, then $\kappa(T+S) = \bar{\kappa}(T+S) = \bar{\kappa}(T) = \kappa(T)$.

PROOF

Applying Propositions 6.5.2 and 6.1.2 (b), the result follows by arguments analogous to those used in Proposition 6.5.4.

◇

In Sections 6.3 and 6.5, we considered stability of Atkinson properties and of the index under perturbation by *continuous* and *continuous* strictly singular relations. Using the graph norm, we show that stability properties can be extended to strictly singular operators which are relatively bounded.

Definition 6.5.6 *A relation S is said to be* **T-precompact, T-compact or T-strictly singular** *if $S|_{D(T)}$ is precompact, compact or strictly singular with respect to the graph norm defined on $D(T)$:*

$$||x||_T := ||x|| + ||Tx||$$

Proposition 6.5.7

Let X and Y be complete, and let $T \in \mathcal{A}_\alpha(X,Y)$ be closed with $T(0)$ topologically complemented in Y. If S satisfies $D(S) \supset \overline{D(T)}$, $S(0) \subset T(0)$ and $P(R(T+S)) \subset R(T+S)$, where P is a continuous projection from Y onto $N(A)$, and if S is $T-$strictly singular and $T-$bounded, then, $T+S \in \mathcal{A}_\alpha(X,Y)$ and $\kappa(T+S) = \kappa(T)$.

PROOF

Since X and Y are complete, and T is closed, $X_T := (D(T), ||.||_T)$ is also a Banach space. Let $T_T, S_T \in LR(X_T, Y)$ be defined $T_T x := Tx$ and $S_T x := Sx$ for $x \in X_T$. Clearly S_T is continuous with $||S_T|| \leq 1$. Thus $T_T + S_T$ is closed.

We now show that $T_T \in \mathcal{A}_\alpha(X_T, Y)$. Let A denote the left regulariser of T given by $A := T|_M^{-1} P_R$ where M and P_R are as in Theorem 6.2.6. Let $A_T \in LR(Y, X_T)$ be defined $A_T y := Ay$. Then for $y \in D(A)$,

$$\begin{aligned} ||A_T y||_T &= ||Ay||_X + ||TAy||_Y \\ &\leq ||A||\,||y|| + ||TA||\,||y|| \\ &= (||A|| + ||TA||)\,||y||. \end{aligned}$$

since A and $TA = P_R|_{D(A)} + TT|_M^{-1} - TT|_M^{-1}$ are continuous. Thus A_T is continuous. If K denotes the continuous finite rank projection operator associated with $T \in \mathcal{A}_\alpha(X,Y)$ and the regulariser A (as in the Characterisation Theorem 6.2.6), then the operator $K_T \in LR(X_T)$ defined by $K_T x := Kx$ is also precompact ($T|_{R(K)}$ is continuous, $||K_T x||_T = ||Kx||_X + ||TK_T x||_Y \leq ||K|| \, ||x|| + ||T|_{R(K)}|| \, ||K|| \, ||x||$ and, hence, K_T is precompact since it is continuous and has finite rank). Since $N(T_T) = N(T)$ and $R(T_T) = R(T)$, and A_T and K_T satisfy the conditions of Theorem 6.2.6, it follows that $T_T \in \mathcal{A}_\alpha(D_T, Y)$.

Now T_T and S_T satisfy the hypotheses of Theorem 6.5.1. Thus $R(T+S) = R(T_T + S_T)$ is topologically complemented in Y. Since $N(T+S) = N(T_T + S_T)$ is finite dimensional, the result follows.

$\diamond$

6.6 Further Notes and Remarks

The work in this chapter arose from notes on generalised inverses of operators in operator ranges by T. Alvarez [3] which were discussed in seminars with R.W. Cross, and from fruitful conversations with R.W. Cross on some of the particular constructions applied in the theory developed here. The definitions for $\alpha-$Atkinson and $\beta-$Atkinson relations given here are due to the author, as are the theorems for multivalued Atkinson relations. Communications by L. Labuschagne on errata in earlier arguments for the characterisation theorems were invaluable in the development of the proofs presented for Theorems 6.2.6 and 6.2.7. Previous literature on Atkinson operators and generalised inverses which were referenced are Cross [35], González and Onieva [62], Labuschagne [84], and Muller-Horrig [111], and also Goldberg [60] and Taylor and Lay [136]. Communication from L. Labuschagne on errata was significant for the improvement of arguments in this chapter.

In the characterisation of $\beta-$Atkinson relations, Theorem 6.2.7, we have the condition that either $R(T') \subset D(B')$ or $D(B'T') \in \mathcal{C}(D(T'))$. In the proofs of perturbation theorems, in order to show $T + S \in \mathcal{A}_\beta$, we include in the hypotheses the condition that $R(S') \subset D(B')$ so that $R((S+T)') \subset D(B')$ (and hence, $D(B'(S+T)') \in \mathcal{C}(D((T+S)')$ and Proposition 5.4.1 is applicable for (ii) $\Rightarrow$ (i) of Theorem 6.2.7). If it can be shown that $D(B'(S+T)') \in \mathcal{C}(D((T+S)')$, as for example in Proposition 6.4.2, then the condition $R(S') \subset D(B')$ may be omitted from the perturbation theorems.

Also in the characterisation of $\beta-$Atkinson relations, Theorem 6.2.7, we may simplify the proof that $B' = (I-K)'(T_M^{-1})'$ as follows: In Proposition 2.6.7. (b) replace (1) and (2) with $R(T') = X'$ and $D(S') = Z'$ by $R(T') = N(T)^\perp$ and $D(S') = S(0)^\perp$, respectively. In the first line of the proof of 2.6.7 (b) replace $X' = R(T')$ by $R((ST)') \subset N(ST)^\perp \subset N(T)^\perp \subset R(T')$. This variation of Proposition 2.6.7 (b) implies that $B' = (I-K)'(T_M^{-1})'$.

The results of Section 6.2 can also be found in the paper by Alvarez, Cross and Wilcox [5]. Comments on further details are given within the body of the chapter.

Chapter 7

Spectral Theory and the Invariant Subspace Problem

7.1 Introduction and Definitions

The *invariant subspace problem* for linear operators refers to the following very general question:

> Does a continuous linear operator $T : X \to X$ defined on a Banach space have a closed non-trivial invariant subspace?

This question generalises the question about the existence of eigenvalues of an operator. While the spectrum of an operator may be non-empty, it may not contain any eigenvalues. For compact operators this question is simplified by the fact that the spectrum consists only of eigenvalues and the complex number zero. However, it is still the case that the point spectrum may be empty and the spectrum may only contain the number zero (see Example 7.2.10(2)). In 1973, Lomonosov [99] gave a general theorem concerning the existence of a common invariant subspace for the family of operators which commute with a compact operator. We conclude this chapter with a consideration of the Invariant Subspace Problem in the context of multivalued operators.

As in the theory for single-valued operators in infinite-dimensional space, we begin with the definition of the *resolvent* of a linear relation. Throughout this chapter, X will denote a normed linear space over the complex field $\mathbb{C}$. For $T \in LR(X)$, we abbreviate the relation $\lambda I_{D(T)} - T$ by $\lambda - T$.

Definitions 7.1.1 *Let* $T \in LR(X)$, *and let* $\lambda \in \mathbb{C}$. *The* **resolvent** $R(\lambda, T)$ *and* **complete resolvent** T_λ *of* T *are defined as follows:*

$$\begin{aligned} R(\lambda,T) &:= (\lambda - T)^{-1}, \text{ and} \\ T_\lambda &:= (\lambda - \tilde{T})^{-1} \text{ respectively.} \end{aligned}$$

The **resolvent set** *of $T \in LR(X)$ is defined by*

$$\begin{aligned} \rho(T) &:= \{ \lambda \in \mathbb{C} \mid R(\lambda, T) \ \textit{is densely-defined, continuous and single-valued} \} \\ &= \{ \lambda \in \mathbb{C} \mid T_\lambda \ \textit{is everywhere defined and single-valued} \}. \end{aligned}$$

The **spectrum** *of T is the complement of $\rho(T)$:*

$$\sigma(T) := \mathbb{C} \setminus \rho(T).$$

Remarks 7.1.2

It follows from the definition and the State diagram for Linear Relations that

$$\rho(T) = \{\lambda \in \mathbb{C} \mid R(\lambda, T) \in I_1 \cup II_1\} = \{\lambda \in \mathbb{C} \mid T_\lambda \in I_1\}.$$

Thus, for an arbitrary relation $T \in LR(X)$, we may investigate the properties of its spectrum or resolvent by passing, where necessary, to the completion of X and the completion of T.

We conclude the introduction with further definitions for the decomposition of the spectrum.

Definitions 7.1.3 *A scalar $\lambda \in \mathbb{C}$ such that $N(\lambda - T) \neq \{0\}$ is called an* **eigenvalue** *of T. If λ is an eigenvalue of $T \in LR(X)$, then the non-trivial subspace $N(\lambda - T)$ is called the* **eigenspace** *of T corresponding to λ, and the quantity $\alpha(\lambda - T)$ is called the* **geometric multiplicity** *of λ.*

The **point spectrum**, *$P_\sigma(T)$, the* **residual spectrum**, *$R_\sigma(T)$, and the* **continuous spectrum**, *$C_\sigma(T)$, of $T \in LR(X, Y)$, are defined as follows :*

$$\begin{aligned} P_\sigma(T) &:= \{\lambda \in \mathbb{C} \ : \ \lambda - T \in I_3 \cup II_3 \cup III_3 \}, \\ R_\sigma(T) &:= \{\lambda \in \mathbb{C} \ : \ \lambda - T \in III_1 \cup III_2 \}, \ \textit{and} \\ C_\sigma(T) &:= \{\lambda \in \mathbb{C} \ : \ \lambda - T \in I_2 \cup II_2 \}. \end{aligned}$$

Remarks 7.1.4

Clearly these subsets provide a disjoint decomposition of the spectrum. The point spectrum, $P_\sigma(T)$, consists of the eigenvalues of T, the residual spectrum, $R_\sigma(T)$, consists of $\lambda \in \mathbb{C}$ such that $\lambda - T$ is injective but does not have dense range, and the continuous spectrum, $C_\sigma(T)$, consists of $\lambda \in \mathbb{C}$ such that $\lambda - T$ is injective, has dense range but is not open.

7.2 Some Spectral Properties of Linear Relations

In this section we review generalisations of the spectral theory of linear operators, namely

The resolvent set $\rho(T)$ of a linear relation is open (and hence its the spectrum is closed),

the resolvent equation holds for linear relations,

the family of operators $\{ T_\lambda \mid \lambda \in \rho(T) \}$ is holomorphic,

the resolvent set may be empty, and

there are sufficient conditions for the spectrum $\sigma(T)$ to be nonempty.

Proposition 7.2.1 *Let $T \in LR(X)$. Then $\rho(T)$ is an open set.*

PROOF

We may clearly assume that $\rho(T)$ is non-empty. Let $\lambda \in \rho(T)$. Choose $\mu \in \mathbb{C}$ such that $|\mu - \lambda| < \gamma(\lambda - T)$. Then by Proposition 3.5.4, $\mu - T = (\mu - \lambda)I + (\lambda - T)$ is open, injective and has dense range. Hence $\mu \in \rho(T)$.

$\diamond$

Proposition 7.2.2 (The resolvent equation) *Let $T \in LR(X)$ and let $\lambda, \mu \in \rho(T)$. Then*

$$T_\mu - T_\lambda = (\lambda - \mu)T_\mu T_\lambda.$$

PROOF

Without loss of generality, we assume that X is complete and that T is closed. Let $x \in X$. Then

$$\begin{aligned}
(\lambda - \mu)T_\mu T_\lambda x &= T_\mu(\lambda - \mu)T_\lambda x \\
&= T_\mu((\lambda - \mu)T_\lambda x + T(0)) \quad \textit{since} \quad T_\mu T(0) = T_\mu T_\mu^{-1}(0) = T_\mu(0) = \{0\} \\
&= T_\mu(\lambda - T - \mu + T)T_\lambda x \\
&= T_\mu T_\lambda^{-1} T_\lambda x - T_\mu T_\mu^{-1} T_\lambda x \\
&= T_\mu(x + T_\lambda^{-1}(0)) - (T_\lambda x + T_\mu(0)) \\
&= T_\mu x + T_\mu T(0) - T_\lambda x \\
&= T_\mu x - T_\lambda x.
\end{aligned}$$

$\diamond$

Corollary 7.2.3 *Let $T \in LR(X)$ and let $\lambda, \mu \in \rho(T)$. Then*

$$T_\lambda T_\mu = T_\mu T_\lambda.$$

$\diamond$

We recall the following well-known result from Operator Theory :

Lemma 7.2.4 (cf. Taylor [136] IV.1.4.) *Suppose a (single-valued) operator T has the property that the series $\sum_{n=0}^{\infty} T^n$ converges in the uniform operator topology. Then $I - T$ is invertible and*

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

Theorem 7.2.5 *Let $T \in LR(X)$ and let $\lambda \in \rho(T)$. Then*

$$\lim_{\mu \to \lambda} (\mu - \lambda)^{-1}(T_\mu - T_\lambda) = -T_\lambda^2.$$

PROOF

Without loss of generality, we assume that X is complete and that T is closed. By Proposition 7.2.1 there exists a neighbourhood of λ contained in $\rho(T)$. Let $\mu \neq \lambda$ be in such a neighbourhood. Since T_μ and T_λ commute (Corollary 7.2.3), we have from the resolvent equation, Proposition 7.2.2, that

$$(\mu - \lambda)^{-1}(T_\mu - T_\lambda) + T_\lambda^2 = (T_\lambda - T_\mu)T_\lambda.$$

Thus,

$$||(\mu - \lambda)^{-1}(T_\mu - T_\lambda) + T_\lambda^2|| \leq ||T_\lambda - T_\mu|| \, ||T_\lambda||. \tag{7.1}$$

Let $|\lambda - \mu| < ||T_\lambda||^{-1}$. Then

$$\sum_{n=0}^{\infty} |\lambda - \mu|^n ||T_\lambda||^n < \infty.$$

By Lemma 7.2.4, $I - (\lambda - \mu)T_\lambda$ is invertible, and

$$(I - (\lambda - \mu)T_\lambda)^{-1} = \sum_{n=0}^{\infty} (\lambda - \mu)^n T_\lambda^n \tag{7.2}$$

For $x \in D(T)$ we have

$$\begin{aligned} T_\lambda^{-1}(I - (\lambda - \mu)T_\lambda)x &= T_\lambda^{-1}(x - (\lambda - \mu)T_\lambda x) \\ &= T_\lambda^{-1}x - (\lambda - \mu)T_\lambda^{-1}T_\lambda x \\ &= T_\lambda^{-1}x - (\lambda - \mu)(x + T_\lambda^{-1}(0)) \\ &= (\lambda - T)x - (\lambda - \mu)(x + T(0)) \\ &= (\mu - T)x. \end{aligned}$$

Thus,

$$T_\mu = (I - (\lambda - \mu)T_\lambda)^{-1}T_\lambda. \tag{7.3}$$

It follows from (7.2) and (7.3) that

$$T_\mu = \sum_{n=0}^{\infty} (\lambda - \mu)^n T_\lambda^{n+1}. \tag{7.4}$$

Thus,

$$T_\mu - T_\lambda = T_\lambda \sum_{n=1}^{\infty} (\lambda - \mu)^n T_\lambda^n$$

and

$$||T_\mu - T_\lambda|| \leq ||T_\lambda|| \sum_{n=1}^{\infty} |\lambda - \mu|^n ||T_\lambda||^n. \tag{7.5}$$

Letting $\mu \to \lambda$ in (7.1) and applying (7.5), the result follows.

$\diamond$

Corollary 7.2.6 *The family* $\{ T_\lambda \ : \ \lambda \in \rho(T) \}$ *is holomorphic.*

Corollary 7.2.7 *If* $\lambda \in \rho(T)$*, then* $|\lambda - \mu| < ||T_\lambda||^{-1}$ *implies that* $T_\mu = \sum_{n=0}^{\infty} (\lambda - \mu)^n T_\lambda^{n+1}$.

Proposition 7.2.8 $\sigma(T) = \sigma(T')$.

PROOF

This follows from the State diagram from Closed Linear relations.

$\diamond$

Proposition 7.2.9 *Let T be open, injective and have dense range. Then*

$$\sigma(T) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \geq \gamma(T)\}.$$

PROOF

Choose λ such that $0 < |\lambda| < \gamma(T)$. Then by Corollary 3.5.4 $\lambda - T$ is open, injective and has dense range. Thus $\lambda \in \rho(T)$.

◇

Example 7.2.10

(1) Let X be a nontrivial normed linear space, and let T be the linear relation with graph $G(T) = X \times X$. Then $T = T^{-1}$, $||T|| = 0$ and $T_\lambda(0) = \tilde{X}$ for any $\lambda \in \mathbb{C}$, and hence T is bounded and $\rho(T) = \rho(T^{-1}) = \emptyset$.

(2) Let $K : l_2 \longrightarrow l_2$ be defined by $Kx := (0, x_1, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots)$ for $x = (x_1, x_2, x_3, \ldots) \in l_2$. Then K is compact with $\sigma(K) = \{0\}$. Let M be non-trivial closed subspace such that $M \cap R(K) = \{0\}$, and define $K_2 \in LR(l_2)$ by $K_2 x := Kx + M$. Then K_2 is closed. Now let $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then $\lambda \in \rho(K)$, $\lambda - K$ is surjective and for $m \in M$, $m \neq 0$, there exists $x \in X$ such that $(\lambda - K)x = m$. Thus $(\lambda - K_2)x = M = (\lambda - K_2)(0)$, i.e. $\lambda \in P_\sigma(K_2)$, and $\sigma(K_2) \supset \rho(K) = \mathbb{C} \setminus \{0\}$. Since $\sigma(K_2)$ is closed, $\sigma(K_2) = \mathbb{C}$.

Lemma 7.2.11 *Let T be continuous and densely defined. Then*

$$\lim_{\substack{\lambda \in \rho(T) \\ |\lambda| \to \infty}} ||T_\lambda|| = 0.$$

PROOF

Without loss of generality we assume that X is complete and T is closed. For $\lambda, \mu \in \rho(T)$, μ fixed, it follows from the resolvent equation, Proposition 7.2.2, that

$$(T_\lambda - T_\mu)(\mu - T) = (\mu - \lambda)T_\lambda T_\mu(\mu - T)$$

if and only if

$$T_\lambda(\mu - T) - I_{D(T)} = (\mu - \lambda)T_\lambda I_{D(T)}.$$

Since $D(T)$ is dense, it follows that $||T_\lambda I_{D(T)}|| = ||T_\lambda|_{D(T)}|| = ||T_\lambda||$. And, since $(\mu - T)(0) \subset D(T_\lambda)$ it follows from Corollary 2.3.13 that

$$|(\mu - \lambda)| \, ||T_\lambda|| \leq 1 + ||T_\lambda|| \, ||\mu - T||$$

Thus, since $||\mu - T|| < \infty$,

$$(1 - \tfrac{||\mu - T||}{|(\mu - \lambda)|})||T_\lambda|| \leq \tfrac{1}{|(\mu - \lambda)|}.$$

Letting $|\lambda| \to \infty$ it follows that $\lim_{|\lambda| \to \infty} ||T_\lambda|| = 0$.

◇

Theorem 7.2.12 *Let X be a non-zero normed space, and let $T \in LR(X)$ be continuous and densely defined. Then the $\sigma(T)$ is non-empty.*

PROOF

Without loss of generality, we assume that X is complete and T is closed. Thus $D(T) = X$. Suppose $\rho(T) = \mathbb{C}$. Then by Theorem 7.2.5 we have for $x \in X$, $x' \in X'$ that

$$\lim_{\mu \to \lambda} \frac{x'T_\mu x - x'T_\lambda x}{\mu - \lambda} = -x'T_\lambda^2 x.$$

Thus, the single-valued function $f(\lambda) = x'T_\lambda x$ is an entire analytic function. Furthermore,

$$||f(\lambda)|| \leq ||x'|| \; ||T_\lambda|| \; ||x||.$$

Since $\lim_{|\lambda| \to \infty} ||T_\lambda|| = 0$ (Lemma 7.2.11), it follows from Liouville's theorem that $f(\lambda) = 0$ for all $\lambda \in \mathbb{C}$. Since $x' \in X'$ was arbitrary, it follows that $T_\lambda x = 0$ for all $x \in X$. Thus $X = N(T_\lambda) = (\lambda - T)(0) = T(0)$, and hence,

$$0 = T_\lambda(\lambda - T)(0) = T_\lambda(\lambda - T)x = x$$

for all $x \in X$ (since T_λ is injective), which contradicts our assumption that X is non-trivial.

◇

7.3 The Spectrum of a Linear Selection

Lemma 7.3.1 *Let $T \in LR(X, Y)$. Then A is a linear selection of T if and only if $\lambda - A$ is a selection of $\lambda - T$.*

PROOF

If A is a selection of T, then $\forall x \in D(T)$ we have

$$Tx = Ax + T(0).$$

Thus,

$$(\lambda - T)x = \lambda x - Ax + T(0) = (\lambda - A)x + (\lambda - T)(0).$$

Since $\lambda - A$ is single-valued, $\lambda - A$ is a selection of $\lambda - T$. The converse follows from what has just been shown, using

$$A = \lambda - (\lambda - A) \text{ and } T = \lambda - (\lambda - T).$$

◇

Proposition 7.3.2 *Let $T \in LR(X, Y)$. If A is a linear selection of T, then $P_\sigma(T) = P_\sigma(A)$.*

PROOF

This follows from the fact that $N(\lambda - T) = N(\lambda - A)$.

◇

Proposition 7.3.3 *Let $T \in LR(X,Y)$. If A is a linear selection of T and $A \neq T$ then $\rho(A) \subset P_\sigma(T)$.*

PROOF
Since $A \neq T$, $T(0) \neq \{0\}$. Since $\sigma(T) = \sigma(\tilde{T})$ and $\sigma(A) = \sigma(\tilde{A})$, we assume without loss of generality that X is complete and T and A are closed.

Suppose $\lambda \in \rho(A)$. Then $\lambda - A$ is surjective and for $y \in T(0)$, $y \neq 0$, there exists $x \in X$ such that $(\lambda - A)x = y$. Thus $(\lambda - T)x = T(0) = (\lambda - T)(0)$, i.e. $\lambda \in P_\sigma(T)$.

◇

Remarks 7.3.4

We note that the completion $\tilde{A}$ of a linear selection A, as in Proposition 7.3.3 above, may not be single-valued.

7.4 The Augmented spectrum and the Möbius transform

Proposition 7.4.1 *Let $\mu \in \rho(T)$, and let $\lambda \neq \mu$. Then*

$$N(\lambda - T) = N((\mu - \lambda)^{-1} - (\mu - T)^{-1}).$$

In particular, λ is an eigenvalue of T if and only if $(\mu - \lambda)^{-1}$ is an eigenvalue of $(\mu - T)^{-1}$, and the corresponding eigenvalues have equal geometric multiplicities.

PROOF
We first note that $x \in N((\mu - \lambda)^{-1} - (\mu - T)^{-1})$ if and only if $x = (\mu - \lambda)(\mu - T)^{-1}x$. The latter implies that $x \in R((\mu - T)^{-1}) = D(\mu - T)$ and, hence, it suffices to consider the case $x \in D(T)$, $x \neq 0$:

$$\begin{aligned}
x \in N(\lambda - T) \quad &\Leftrightarrow \quad (\lambda - T)x = (\lambda - T)(0) = T(0) \\
&\Leftrightarrow \quad (\mu - T)x = (\mu - \lambda)x + T(0) \\
&\Leftrightarrow \quad x + (\mu - T)^{-1}(0) = (\mu - \lambda)(\mu - T)^{-1}x + (\mu - T)^{-1}(0) \\
&\Leftrightarrow \quad x = (\mu - \lambda)(\mu - T)^{-1}x \\
&\Leftrightarrow \quad (I - (\mu - \lambda)(\mu - T)^{-1})x = 0 \\
&\Leftrightarrow \quad ((\mu - \lambda)^{-1} - (\mu - T)^{-1})x = 0.
\end{aligned}$$

◇

Definitions 7.4.2 *Let $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ endowed with the usual topology, and let $\mu \in \mathbb{C}$. The* **Möbius transform** *$\eta_\mu(\lambda)$ is defined by :*

$$\eta_\mu(\lambda) := \begin{cases} (\mu - \lambda)^{-1} & \text{if } \lambda \in \mathbb{C} \setminus \mu \\ \infty & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda = \infty \end{cases}$$

Let $T \in LR(X)$. The **augmented spectrum** *of T is defined to be the set :*

$$\bar{\sigma}(T) := \begin{cases} \sigma(T) \cup \{\infty\} & \text{if } 0 \in \sigma(T^{-1}) \\ \sigma(T) & \text{otherwise} \end{cases}$$

Remarks 7.4.3

The Möbius transform is a topological homeomorphism from $\mathbb{C}_\infty$ onto itself. Note that $\{\infty\} \not\subset \bar{\sigma}(T)$ if and only if $\tilde{T}$ is a bounded linear operator.

Theorem 7.4.4 *Let $T \in LR(X)$ and suppose $D(T) \neq \{0\}$ and $\mu \in \rho(T)$. Then*

$$\eta_\mu(\bar{\sigma}(T)) = \sigma(T_\mu).$$

PROOF

Without loss of generality, we assume that X is complete and T is closed. Let $\lambda \in \mathbb{C}$, $\lambda \neq \mu$, and let $S := (\mu - \lambda)((\mu - \lambda)^{-1} - T_\mu)$. Then

$$\begin{aligned} \lambda - T &= (\mu - T) - (\mu - \lambda)I \\ &= (I - (\mu - \lambda)T_\mu)(\mu - T) \quad (\textit{since} \quad D(T_\mu) = X \ \textit{and} \ T_\mu(0) = 0) \\ &= S(\mu - T). \end{aligned} \tag{7.6}$$

Suppose $\lambda \in \rho(T)$. Then S is injective since

$$\begin{aligned} Sx = 0 \quad &\Rightarrow \quad (\mu - \lambda)^{-1}x = T_\mu x \\ &\Rightarrow \quad (\mu - \lambda)^{-1}(\mu - T)x = x + (\mu - T)(0) = x + T(0) \\ &\Rightarrow \quad (\mu - T)x = (\mu - \lambda)x + T(0) \\ &\Rightarrow \quad (\lambda - T)x = T(0) = (\lambda - T)(0) \\ &\Rightarrow \quad x = 0 \quad (\textit{since } \lambda - T \textit{ is injective}). \end{aligned}$$

It follows from (7.6) that for $\lambda \in \rho(T)$, we have $X = R(\lambda - T) \subset R(S)$. Thus S is surjective, and, hence, S is open. Thus $(\mu - \lambda)^{-1} \in \rho(T_\mu)$.

Conversely, let $(\mu - \lambda)^{-1} \in \rho(T_\mu)$. For $x \in D(T)$, it follows from (7.6) that

$$||(\lambda - T)x|| \; = \; ||S(\mu - T)x|| \; \geq \; \gamma(S(\mu - T))\, d(x, T_\mu S^{-1}(0)).$$

Since $T_\mu S^{-1}(0) = T_\mu(0) = 0$, it follows that

$$||(\lambda - T)x|| \; \geq \; \gamma(S(\mu - T))||x||.$$

Now $\gamma(S(\mu - T)) \geq \gamma(S)\gamma(\mu - T)$ (Proposition 2.3.11). Thus, since $\gamma(S) > 0$, $(\mu - \lambda)^{-1} \in \rho(T_\mu)$ and $\gamma(\mu - T) > 0$ $(\mu \in \rho(T))$, it follows that $\lambda - T$ is open and injective. From (7.6) it follows that $\lambda - T$ is surjective. Thus $\lambda \in \rho(T)$.

◇

7.5 The Polynomial Spectral Mapping Theorem

Proposition 7.5.1 *Let $T \in LR(X)$. The relations $\lambda - T$ and $\mu - T$ commute.*

PROOF
Clearly

$$D((\mu - T)(\lambda - T)) = D(T^2) = D((\lambda - T)(\mu - T)).$$

For $x \in D(T^2)$ we have

$$\begin{aligned}
(\lambda - T)(\mu - T)x &= (\lambda - T)(\mu x - Tx) \\
&= (\lambda - T)\{\mu x - y \mid y \in Tx, y \in D(T)\} \\
&= (\{\lambda\mu x - \lambda y - \mu Tx + Ty \mid y \in Tx, y \in D(T)\} \\
&= \lambda\mu x - (\lambda + \mu)Tx + T^2 x \\
&= (\mu - T)(\lambda - T)x.
\end{aligned}$$

◇

Proposition 7.5.2 *Let X be complete, and let $S, T \in LR(X)$ be closed bounded below and surjective. Then ST has the same properties.*

PROOF
Clearly ST is injective. Thus, since $\gamma(ST) \geq \gamma(S)\gamma(T)$ (Proposition 2.3.11), it follows that ST is bounded below. Since $ST(X) = S(TX) = SX = X$ it follows that ST is surjective.

We now show that ST is closed. Let $\{x_n, z_n\}_{n \in \mathbb{N}}$ be a sequence in $G(ST)$ such that $(x_n, z_n) \to (x, z)$. For each n there exists y_n such that $(x_n, y_n) \in G(T)$ and $(y_n, z_n) \in G(S)$. Since S^{-1} is continuous and single-valued, $y_n = S^{-1} z_n$ converges to some $y \in X$. Since S is closed, it follows that $(y_n, z_n) \to (y, z) \in G(S)$. Since $(x_n, y_n) \to (x, y)$ and T is closed, it follows that $(x, y) \in G(T)$. Thus $(x, z) \in G(ST)$.

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Theorem 7.5.3 *Let X be a normed linear space, and let $T \in LR(X)$. Then for any complex polynomial p we have*

$$\sigma(p(\tilde{T})) = p(\sigma(\tilde{T})).$$

PROOF
Fix $\lambda \in \mathbb{C}$, and let

$$\mu - p(\lambda) := c \prod_{j=1}^{n} (\alpha_j - \lambda).$$

Then

$$\mu I - p(T) = c\prod_{j=1}^{n}(\alpha_j I - T). \tag{7.7}$$

Without loss of generality, we assume X is complete and T is closed.

Let $\mu \in \sigma(p(T))$. If $\alpha_j \in \rho(T)$ for all $j = 1, 2, \ldots, n$, then, by Proposition 7.5.2, $p(T) - \mu$ would be bounded below and surjective, contradicting the assumption that $\mu \in \sigma(p(T))$. Thus $\exists j,\ 1 \leq j \leq n$ such that $\alpha_j \in \sigma(T)$. Since $p(\alpha_j) = \mu$, it follows that $\mu \in p(\sigma(T))$.

Conversely suppose that $\mu \in p(\sigma(T))$. Then $\mu = p(\lambda)$ for some $\lambda \in \sigma(T)$. Thus $\lambda = \alpha_j$ for some j such that $1 \leq j \leq n$. Since the factors commute, (Proposition 7.5.1), we may assume that $j = 1$.

Suppose $\alpha_1 - T$ is bounded below. Then it cannot be surjective ($\alpha_1 = \lambda \in \sigma(T)$). From (7.7) it follows that $\mu - p(T)$ cannot be surjective. Thus $\mu \in \sigma(p(T))$.

On the other hand, if $\alpha_j - T$ is surjective for every $j,\ 1 \leq j \leq n$, then by the Open Mapping theorem, $\alpha_j - T$ is open for every $j,\ 1 \leq j \leq n$. Since $\alpha_1 \in \sigma(T)$, $\alpha_1 - T$ cannot be injective. Thus $\mu - p(T)$ is not injective. It follows that $\mu \in \sigma(p(T))$.

◇

7.6 The Domain of Iterates of a Linear Relation

Theorem 7.6.2 applies the Baire property for linear relations, Theorem 3.1.6, and is a generalisation of a theorem due to Lennard [92]. It gives a condition for an arbitrary set of polynomials of a relation to share a common dense domain (see also Example 3.1.5).

Notation 7.6.1 *Let $T \in LR(X)$. We define the set $\Sigma(T)$ by :*

$$\Sigma(T) \ := \ \{S \in LR(X) \mid S = \alpha T + \beta I,\ \alpha, \beta \in I\!K,\ \ and\ \ S(D(T))\ \ is\ \ dense\ \}.$$

Theorem 7.6.2 *Let X be complete and let $T \in LR(X)$ be closed with $D(T)$ dense in X. Suppose $\rho(T) \neq \emptyset$ and $\{S_n\}$ is any sequence of operators in $\Sigma(T)$. Then*

(a) $\bigcap_{n=1}^{\infty} S_1 S_2 \ldots S_n D(T^n)$ is dense in X. In particular, $\bigcap_{n=1}^{\infty} D(T^n)$ is dense in X.

(b) If $R(T)$ is dense, then $\bigcap_{n=1}^{\infty} T^n D(T^n)$ is dense in X.

PROOF

Let $\mu \in \rho(T)$, let $(X_0, ||\text{-}||_0) := (X, ||\text{-}||_X)$, and let $X_n := D(T^n), n \in I\!N$, with $||x||_n := ||x|| + ||(\mu - T)x|| + \ldots + ||(\mu - T)^n x||$ for $x \in X_n$.

Since T is closed, each $(X_n, ||\text{-}||_n)$ is a complete normed linear space. Let n be fixed, and let $x \in X_n$. T and I map $(X_n, ||\text{-}||_n)$ onto $(X_{n-1}, ||\text{-}||_{n-1})$, and clearly $||Ix||_{n-1} \leq ||x||_n$. Furthermore,

$$\begin{aligned}
||Tx||_{n-1} &\leq ||(\mu - T)x||_{n-1} \ + \ |\mu|\,||x||_{n-1} \\
&= ||(\mu - T)x|| + ||(\mu - T)^2 x|| + \ldots + ||(\mu - T)^n x|| + |\mu|\,||x||_{n-1} \\
&\leq ||x||_n + |\mu|||x||_n \\
&= (1 + |\mu|)||x||_n.
\end{aligned}$$

Thus, T and I are continuous as elements of $LR(X_n, X_{n-1})$, and hence, $S := \beta I + \alpha T$ maps $(X_n, ||\cdot||_n)$ continuously into $(X_{n-1}, ||\cdot||_{n-1})$.

We now show that $S(X_n)$ is dense in X_{n-1} when $S(D(T))$ is dense in X.
Let $y \in X_{n-1} = D(T^{n-1})$, and fix $\epsilon > 0$. Since S commutes with $\mu - T$, it follows that for all $x \in X_n$ we have

$$\begin{aligned}
||y - Sx||_{n-1} &= ||y - Sx|| + ||(\mu - T)(y - Sx)|| \\
&\quad + \ldots + ||(\mu - T)^{n-2}(y - Sx)|| + ||(\mu - T)^{n-1}(y - Sx)|| \\
&= ||T_\mu^{n-1}((\mu - T)^{n-1}y - S(\mu - T)^{n-1}x)|| + ||T_\mu^{n-2}((\mu - T)^{n-1}y - S(\mu - T)^{n-1}x)|| \\
&\quad + \ldots + ||T_\mu(\mu - T)^{n-1}y - S(\mu - T)^{n-1}x)|| + ||(\mu - T)^{n-1}y - S(\mu - T)^{n-1}x|| \\
&\leq (||T_\mu||^{n-1} + ||T_\mu||^{n-2} + \ldots + ||T_\mu|| + 1)||(\mu - T)^{n-1}y - S(\mu - T)^{n-1}x|| \\
&= B_n||(\mu - T)^{n-1}y - S(\mu - T)^{n-1}x||,
\end{aligned}$$

where $B_n := ||T_\mu||^{n-1} + ||T_\mu||^{n-2} + \ldots + ||T_\mu|| + 1$.

Consider $(\mu - T)^{n-1}y \subset X$. Since $S(D(T))$ is dense, there exists $z \in D(T)$ such that

$$d((\mu - T)^{n-1}y, Sz) < \tfrac{\epsilon}{B_n}.$$

Now T_μ maps X onto $D(T)$. Let $x_0 := T_\mu^{n-1}z$. Then

$$x_0 = ((\mu - T)^{-1})^{n-1}z = ((\mu - T)^{n-1})^{-1}z \in X_n = D(T^n),$$

and

$$\begin{aligned}
||y - Sx_0||_{n-1} &\leq B_n||(\mu - T)^{n-1}y - S(\mu - T)^{n-1}x_0|| \\
&= B_n||(\mu - T)^{n-1}y - S(z + T^{n-1}(0))|| \\
&= B_n||(\mu - T)^{n-1}y - Sz + T^n(0)|| \\
&< B_n.\frac{\epsilon}{B_n} = \epsilon
\end{aligned}$$

Thus (a) follows from the Baire Property of Linear Relations. In particular, if $\alpha = 0$ and $\beta = 1$ in $S = \beta I + \alpha T$ then $\bigcap_{n=1}^{\infty} D(T^n)$ is dense in X. Similarly, (b) follows from (a) with $\alpha = 1$ and $\beta = 0$ in $S = \beta I + \alpha T$.

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7.7 The Invariant Subspace Problem

Saveliev [126] extended Lomonosov's theorem [99] to include multivalued linear operators, and gave a proof for linear relations whose multivalued parts are finite-dimensional by applying fixed point methods for multivalued operators. He then applied this theorem for multivalued operators to prove a new variant of Lomonosov's theorem for single-valued linear operators.

In this section we give alternative proofs and extend these recent results. We also make the distinction between *left* and *right commutation* for linear relations - only right commutation is considered in [126], where two multivalued operators are said to *commute* if they *right commute* in the sense given below.

Definitions 7.7.1

*Let $T \in LR(X)$. A set M of a normed linear space X is said to be $T-$***invariant** *if $T(M) \subset M$. M is said to be* **weakly $T-$invariant** *if $T(M) \cap M \neq \emptyset$.*

Let $S \in LR(X)$. We say that S **left commutes** *with T if $G(ST) \subset G(TS)$, and that S* **right commutes** *with T if $G(TS) \subset G(ST)$. We define the* **left commutant** *and* **right commutant**, *denoted $Comm_L(T)$ and $Comm_R(T)$, respectively, as follows :*

$$\begin{aligned} Comm_L(T) &:= \{S \in LR(X) \mid S \ left \ commutes \ with \ T \ and \ D(T) = D(S)\ \}, \\ Comm_R(T) &:= \{S \in LR(X) \mid S \ right \ commutes \ with \ T \ and \ D(T) = D(S)\ \}. \end{aligned}$$

We note that if T is single-valued then there is no distinction between the *left* and *right* commutativity. The invariant subspace problem is concerned with the existence of *nontrivial* closed subspaces which are invariant under T. When T is multivalued, the subspace $T(0)$ is always weakly $T-$invariant. In this sense, space $T(0)$ is trivially weakly invariant. Thus, in the context of multivalued relations, the invariant subspace problem should be interpreted as the question about existence of an invariant subspace distinct from $\{0\}$, $T(0)$, and X.

Examples 7.7.2

(1) If $\lambda \in P_\sigma(T)$ then the eigenspace $N(\lambda - T)$ is $T-$invariant.

(2) The orbit $\{x, Tx, T^2x, \dots\}$ is a $T-$invariant. If X is non-separable and $T(0)$ is a separable subspace of X, then the closed subspace generated by $\{x, Tx, T^2x, \dots\}$ is a non-trivial invariant subspace.

(3) The sets $L_x := \{Sx \mid S \in Comm_L(T)\ \}$ and $R_x := \{Sx \mid S \in Comm_R(T)\ \}$ are $S-$invariant for every $S \in Comm_L(T)$ and $S \in Comm_R(T)$, respectively (these sets may be trivial, though).

Proposition 7.7.3 *Let T, $S \in LR(X)$, and suppose P_T and P_S are single-valued projections with kernels $T(0)$ and $S(0)$, respectively.*

(a) If $S \in Comm_L(T)$ and $x \in D(T)$, then

$$P_S S P_T T x \ \subset \ P_T T P_S S x + P_T T S(0) + T(0) + S(0).$$

(b) If $S \in Comm_R(T)$ and $x \in D(T)$, then

$$P_T T P_S S x \ \subset \ P_S S P_T T x + P_S S T(0) + S(0) + T(0).$$

PROOF

(a) Clearly

$$G(P_T ST) \subset G(P_T TS),$$

and, since $P_T T$ is a selection of T,

$$G(SP_T T) \subset G(ST) \subset G(TS). \tag{7.8}$$

Let $y \in SP_T Tx$ for some $x \in D(T)$. Then, applying (7.8), $y \in TSx$, and, since $P_T T$ is a selection of T, $y \in P_T TSx + T(0)$. Thus,

$$SP_T Tx \subset P_T TSx + T(0).$$

Since $Sx = P_S Sx + S(0)$,

$$\begin{aligned} P_S SP_T Tx &\subset P_S P_T TSx + P_S T(0) \\ &= P_S P_T TP_S Sx + P_S P_T TS(0) + P_S T(0) \\ &\subset P_T TP_S Sx + P_T TS(0) + T(0) + S(0), \end{aligned}$$

which is what we needed to show.

(b) This follows by interchanging S and T in (a).

$\diamond$

Proposition 7.7.4 *Let* $T,\ S \in LR(X)$.

(a) If $S \in Comm_L(T)$, *and* P *is a single-valued projection with kernel* $TS(0) + T(0) + S(0)$, *then*

$$PSPT = PTPS.$$

(b) If $S \in Comm_R(T)$, *and* P *is a single-valued projection with kernel* $ST(0) + T(0) + S(0)$, *then*

$$PSPT = PTPS.$$

PROOF

By substituting both P_T and P_S in Proposition 7.7.3 with the projection P, similar arguments apply, and we have for $x \in D(T)$:

$$PSPTx \subset PTPSx + PTS(0) + T(0) + S(0). \tag{7.9}$$

Since PS and PT are single-valued , the desired equality follows on applying P to both sides of (7.9).

(b) This follows by interchanging S and T in part (a).

$\diamond$

Theorem 7.7.5 (Lomonosov) *Let $S \in B(X)$. If S commutes with an operator $T \in B(X)$, $T \neq \lambda I$ and T in turn commutes with a nonzero compact operator, then there exists a nontrivial closed $S-$invariant subspace.*

Notation 7.7.6 *We will use the following notation:*

$$BR_0(X,Y) := \{ \, T \in LR(X,Y) \mid dim\, T(0) < \infty \;\; and \;\; T \;\; is \;\; continuous \, \}$$

Theorem 7.7.7 *Suppose $T \in BR_0(X)$, and λI is not a selection of T.*

(a) If there exists a nonzero compact relation $K \in Comm_L(T) \cap BR_0(X)$, then there exists a nontrivial, closed, weakly $S-$invariant subspace for any $S \in Comm_L(T) \cap BR_0(X)$.

(b) If there exists a nonzero compact relation $K \in Comm_R(T) \cap BR_0(X)$, then there exists a nontrivial, closed, weakly $S-$invariant subspace for any $S \in Comm_R(T) \cap BR_0(X)$.

PROOF

(a) Suppose $S \in Comm_L(T) \cap BR_0(X)$ and there exists a nonzero compact relation $K \in Comm_L(T) \cap BR_0(X)$. Let P be a single-valued projection with kernel $N(P) = TK(0) + TS(0) + T(0) + S(0) + K(0)$. Since T, S, $K \in BR_0(X)$, the projection P is continuous, and so are the single-valued relations PK, PT, and PS. By Proposition 7.7.4, $PK \in Comm_L(PT)$, and $PS \in Comm_L(PT)$. Thus, by Lomonosov's Theorem 7.7.5, there exists a nontrivial closed $PS-$invariant subspace, M. For such M we have

$$S(M) \subset M + TK(0) + TS(0) + T(0) + S(0) + K(0) = M + N(P).$$

Thus, M is a weakly $S-$invariant subspace.

(b) The proof is similar to (a).

◇

Corollary 7.7.8 *Suppose $S, T, K \in B(X)$ such that K is compact, $T \neq \lambda I$, and there exists a topologically complemented subspace N such that $T(N) \subset N$, $(TK - KT)x \in N$, $x \in X$, and $ST = TS$. Then there exists a nontrivial closed weakly $S-$invariant subspace.*

PROOF

Let $K_2 \in LR(X)$ denote the compact relation defined by

$$K_2 x := Kx + N, \quad x \in X.$$

Then we have

$$TK_2x = TKx + T(N) \subset TKx + N = KTx + N = K_2Tx.$$

Letting P denote a continuous projection with kernel $N(P) = N$, it follows that $PK \in Comm_R(T)$. Furthermore, by Proposition 7.7.4, $PK \in Comm_R(PT)$ and $PS \in Comm_R(PT)$, and hence, there exists a non-trivial $PS-$invariant subspace M. It follows that $S(M) \subset M + N$ and M is weakly $S-$invariant.

$\diamond$

Corollary 7.7.9 *Suppose $S, T, K \in B(X)$ such that $T \neq \lambda I$, K is compact, and there exists a topologically complemented subspace N such that $(TK - KT)\,x \in N$, $x \in X$, and $K(N) \subset N$. If $ST = TS$, then there exists a nontrivial closed weakly $S-$invariant subspace.*

PROOF

Let $T_2 \in LR(X)$ denote the relation defined by

$$T_2 x := Tx + N, \quad x \in X.$$

Then we have

$$KT_2 x = KTx + K(N) \subset KTx + N = TKx + N = T_2 Kx.$$

Letting P denote a continuous projection with kernel $N(P) = N$, it follows that $K \in Comm_L(T)$. Furthermore, as in Proposition 7.7.4, $PK \in Comm_L(PT)$ and $PS \in Comm_L(PT)$, and hence, there exists a non-trivial $PS-$invariant subspace M. It follows that $S(M) \subset M + N$ and M is weakly $S-$invariant.

$\diamond$

7.8 Further Notes and Remarks

The material in this chapter continues the work presented in Cross [35]. Section 7.3. is due to the author, as well as the proof of Theorem 7.6.2 for multivalued operators. Burlando [25] showed that the single-valued case of the latter (due to Lennard [92]) can be generalised to give an analogous theorem for paracomplete operators which have non-empty *essential resolvent* (essential spectra and resolvents are discussed in the next chapter). It is not known whether this result holds when the operator is multivalued.

The definitions given in Section 7.7 are based on those given in Saveliev [126], and the proofs of theorems given here are due to the author. The first examples of operators without invariant subspaces were found independently by P.Enflo and C.Read (see Beauzamy [18] for further references and counterexamples). A survey of work on the invariant subspace problem and references to significant contributions can be found in Abramovich, Aliprantis and Burkinshaw [1] and the paper, *Some aspects of the invariant subspace problem*, by Enflo and Lomonosov in [71].

In 1999, Read [123] provided a counterexample to the question whether every continuous strictly singular operator has an invariant subspace. Since every continous operator on the space of Gowers and Maurey is of the form form $\lambda I + S$, where S is strictly singular, a positive answer to the latter question would have revealed the first known example of an infinite dimensional Banach space such that every continuous operator on it had an invariant subspace (see Androulakis and Schlumprecht [6]). This question is still open for the Gowers and Maurey space.

Chapter 8

The Essential Spectra of Linear Relations

8.1 Introduction and Definitions

H. Weyl showed that the limit points of the spectrum (i.e. all points of the spectrum, except isolated eigenvalues of finite multiplicity) of a bounded symmetric transformation on a Hilbert space are invariant under perturbation by compact symmetric operators (cf. Riesz and Sz-Nagy [124]). In the modern theory of linear operators, the essential spectra are subsets of the spectrum which are stable under perturbation by small and relatively compact operators. In this chapter we apply results which were given in Chapter 5 to show that theory for the essential spectra known for linear operators can be extended naturally to linear relations.

Definitions 8.1.1 *The* **essential resolvents**, $\rho_{ei}(T)$ *for* $i = 1,2,3,4,5$, *of* $T \in LR(X)$ *are defined as follows :*

$$\begin{aligned}
\rho_{e1}(T) &:= \{\, \lambda \in \mathbb{C} \mid (\lambda - \tilde{T}) \in \Phi_+ \cup \Phi_- \} \\
\rho_{e2}(T) &:= \{\, \lambda \in \mathbb{C} \mid (\lambda - \tilde{T}) \in \Phi_+ \} \\
\rho_{e3}(T) &:= \{\, \lambda \in \mathbb{C} \mid (\lambda - \tilde{T}) \in \Phi \} \\
\rho_{e4}(T) &:= \{\, \lambda \in \mathbb{C} \mid (\lambda - \tilde{T}) \in \Phi \text{ and } \kappa(\lambda - \tilde{T}) = 0 \} \\
\rho_{e5}(T) &:= \bigcup \rho_{e1}^{(n)}(T) \text{ where } \rho_{e1}^{(n)}(T) \text{ is a component of } \rho_{e1}(T) \\
&\qquad\qquad \text{and } \rho_{e1}^{(n)}(T) \cap \rho(T) \neq \emptyset
\end{aligned}$$

The **essential spectra**, $\sigma_{ei}(T)$, $i = 1,2,3,4,5$, *of* $T \in LR(X)$ *are the respective complements of the essential resolvents :*

$$\sigma_{ei}(T) := \mathbb{C} \setminus \rho_{ei}(T), \quad i = 1,2,3,4,5.$$

We also define

$$\begin{aligned} \rho'_{e2}(T) &:= \{ \lambda \in \mathbb{C} \mid (\lambda - \tilde{T}) \in \Phi_- \} \\ \sigma'_{e2}(T) &:= \mathbb{C} \setminus \rho'_{e2}(T) \end{aligned}$$

Remarks 8.1.2

Applying properties of Fredholm relations, we may equivalently define the essential resolvents as follows:

$$\begin{aligned} \rho_{e1}(T) &= \{ \lambda \in \mathbb{C} \mid (\lambda - T) \in \mathcal{F}_+ \cup \mathcal{F}_- \} \\ \rho_{e2}(T) &= \{ \lambda \in \mathbb{C} \mid (\lambda - T) \in \mathcal{F}_+ \} \\ \rho_{e3}(T) &= \{ \lambda \in \mathbb{C} \mid (\lambda - T) \in \mathcal{F} \} \\ \rho_{e4}(T) &= \{ \lambda \in \mathbb{C} \mid (\lambda - T) \in \mathcal{F} \textit{ and } \bar{\kappa}(\lambda - \tilde{T}) = 0 \} \\ &= \{ \lambda \in \mathbb{C} \mid (\lambda - T) \in \mathcal{F} \textit{ and } \alpha(\lambda - \tilde{T}) \ = \beta(\lambda - \tilde{T}) \ < \ \infty \} \end{aligned}$$

Clearly we have that $\rho_{ei}(T) \supset \rho_{ej}(T)$ for $i < j < 4$, and, thus, $\sigma_{ei}(T) \subset \sigma_{ej}(T)$ for $i < j < 4$. We will see later that $\rho_{e4}(T) \supset \rho_{e5}(T)$. It also follows from the definitions that $\rho_{ei}(T) = \rho_{ei}(\tilde{T}), \ i = 1, 2, 3, 4, 5$.

Applying the equivalences

$$\begin{aligned} T \in \Phi_+ \quad &\Leftrightarrow \quad T \in \mathcal{F}_+, \ \textit{and} \\ T \in \Phi_- \quad &\Leftrightarrow \quad T \in \mathcal{F}_-, \end{aligned}$$

we may investigate the properties of the essential spectra and the essential resolvents of $T \in LR(X)$ by passing to the completion of X and that of T (Corollaries 5.2.11 and 5.8.5). Therefore,

in the sequel we assume that X and Y are Banach spaces and T is closed.

8.2 Basic Properties of the Essential Spectra

We begin this section by showing that the various essential spectra are closed, and then illustrate some characteristic properties. In the single-valued case, the set $\bigcap_{P \in \mathcal{K}_T} \sigma(T + K)$ is referred to as the *Weyl essential spectrum*. Proposition 8.2.4 shows that $\sigma_{e4}(T)$ can be characterised in terms of the Weyl essential spectrum in the Multivalued case as well (cf. Edmunds and Evans [49]). We conclude this section by giving properties of the quantities $\alpha(\lambda - T)$, $\beta(\lambda - T)$ and $\kappa(\lambda - T)$ for λ in the essential spectra, and deduce in Proposition 8.2.9 the inclusions

$$\sigma_{e1}(T) \subset \sigma_{e2}(T) \subset \sigma_{e3}(T) \subset \sigma_{e4}(T) \subset \sigma_{e5}(T) \subset \sigma(T).$$

Proposition 8.2.5 is included here for application in Proposition 8.2.9 and is based on the single-valued analogue given in Goldberg [60].

Proposition 8.2.1 *For $i = 1, 2, 3, 4, 5$, $\sigma_{ei}(T)$ is closed.*

PROOF

Suppose $\lambda \in \rho_{ei}(T)$, $i = 1,2,3,4,5$. Since $R(\lambda - T)$ is closed, it follows from the Open Mapping Theorem, 3.3.7, that $\gamma(\lambda - T) > 0$. If $\lambda - T \in \mathcal{F}_+$ and $|\mu| < \gamma(\lambda - T)$, then by Theorem 5.4.2, $\mu + \lambda - T \in \mathcal{F}_+$. Similarly, if $\lambda - T \in \mathcal{F}_-$ and $|\mu| < \gamma(\lambda - T')$, then by Theorem 5.7.1, $\mu + \lambda - T \in \mathcal{F}_-$. Thus, $\rho_{e1}(T)$, $\rho_{e2}(T)$ and $\rho_{e3}(T)$ are open. Furthermore, by Theorem 6.1.1, $\kappa(\mu + \lambda - T) = \kappa(\lambda - T)$, i.e. $\rho_{e4}(T)$ is open. Since each component of $\rho_{e1}(T)$ is open, so is $\rho_{e5}(T)$.

◇

Proposition 8.2.2 *Let $T \in LR(X)$. Then*

(a) $\sigma_{ei}(T') = \sigma_{ei}(T)$ for $i = 1, 3, 4, 5$

(b) $\sigma_{e2}(T') = \sigma'_{e2}(T)$

PROOF

(a) Suppose $\lambda \in \rho_{ei}(T)$, $i = 1,3,4$. By Proposition 2.7.6, $\alpha(\lambda - T') = \beta(\lambda - T)$ since $R(\lambda - T)$ is closed. By the Closed Range Theorem 3.3.8, $R(\lambda - T')$ if and only if $R(\lambda - T)$ is closed and, since $\lambda - T$ is open, $\beta(\lambda - T') = \alpha(\lambda - T)$. Thus, the result holds for $i = 1, 3$ and 4. Since $\rho_{e1}(T) = \rho_{e1}(T')$ and $\rho(T) = \rho(T')$, it follows that $\rho_{e1}^{(n)}(T') = \rho_{e1}^{(n)}(T)$, i.e. the result holds for $i = 5$.

(b) follows from the reasons given in (a).

◇

Proposition 8.2.3 *$\lambda \in \sigma_{e2}(T)$ if and only if $\lambda - T$ has a singular sequence.*

PROOF

Since $\lambda \in \sigma_{e2}(T)$ if and only if $\lambda - T \notin \mathcal{F}_+$, the result follows from Theorem 5.9.3.

◇

Proposition 8.2.4

$$\sigma_{e4}(T) \;=\; \bigcap_{P \in \mathcal{K}_T} \sigma(T + K),$$

where $\mathcal{K}_T := \{K \in LR(X) \mid K$ is compact and $K(0) \subset \overline{T(0)}\;\}$.

PROOF

We show first that $\sigma_{e4}(T) \subset \bigcap_{K \in \mathcal{K}_T} \sigma(T + K)$. Suppose $\lambda \notin \bigcap_{K \in \mathcal{K}_T} \sigma(T + K)$. Then there exists $K \in \mathcal{K}_T$ such that $\lambda \in \rho(T + K)$. Thus $\lambda \in \rho_{e4}(T + K)$. By Propositions 5.4.2 and 5.7.1, $\lambda - T = \lambda - T - K + K \in \Phi$, and by Theorem 6.1.2,

$$\kappa(\lambda - T) = \kappa(\lambda - T - K + K) = \kappa(\lambda - T - K).$$

Thus , $\lambda \in \rho_{e4}(T)$, i.e. $\lambda \notin \sigma_{e4}(T)$.

Conversely, suppose $\lambda \in \rho_{e4}(T)$. Then $R(\lambda - T)$ is closed, and $\alpha(\lambda - T) = \beta(\lambda - T) = n$, say. Let $\{x_1, \ldots, x_n\}$ and $\{y_1', \ldots, y_n'\}$ be bases for $N(\lambda - T)$ and $R(\lambda - T)^{\perp} = N(\lambda - T')$, respectively. Choose $x_j' \in X'$ and $y_j \in X$, $j = 1, \ldots, n$ such that

$$\begin{aligned} x_j' x_k &= \delta_{jk}, \quad \textit{and} \\ y_j' y_k &= \delta_{jk}, \end{aligned}$$

where $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jk} = 1$ if $j = k$, and define $K \in LR(X)$ as follows:

$$Kx := \sum_{k=1}^{n} (x_k' x)\, y_k, \qquad x \in X$$

Then $dim\, R(K) < \infty$ and

$$||Kx|| \le (\sum_{k=1}^{n} ||x_k'||\, ||y_k||)\, ||x||.$$

By Proposition 5.2.5, it follows that K is a compact operator. By Propositions 5.4.2 and 5.7.1, it follows that $\lambda - (T + K) \in \Phi$ and by Theorem 6.1.2, $\kappa(\lambda - (T+K)) = \kappa(\lambda - T)$.

Without loss of generality, assume $\lambda = 0$. Now if $x \in N(T)$, then $x = \sum_{k=1}^{n} a_k x_k$ and $x_j'(x) = a_j$, $1 \le j \le n$. On the other hand, if $x \in N(K)$, then $x_j'(x) = 0$. Thus $N(T) \cap N(K) = 0$.

Similarly, if $y \in R(K)$, then $y = \sum_{k=1}^{n} a_k y_k$ and $y_j'(y) = a_j$, $1 \le j \le n$, and if $y \in R(T)$, then $y_j'(y) = 0$. Thus $R(K) \cap R(T) = 0$

Next, suppose $x \in N(T + K)$. Then $Tx = -Kx + T(0)$. It follows from the argument above, that $Tx = T(0)$, i.e. $x \in N(T)$. Thus, $x = \sum_{k=1}^{n} a_k x_k$ and $x_k'(x) = a_k$, $1 \le k \le n$. Since $Kx = \sum_{k=1}^{n} (x_k' x) y_k = 0$, it follows that $x_k'(x) = 0$, $1 \le k \le n$, and hence $x = 0$. Thus, $\alpha(T + K) = 0 = \beta(T + K)$, i.e. $0 \in \rho_{e4}(T + K)$.

◇

Proposition 8.2.5 *Let X and Y be complete, and suppose $T \in \Phi_+ \cup \Phi_-$ and $S \in LR(X,Y)$ satisfies $D(S) \supset D(T)$, $S(0) = \overline{S(0)} \subset \overline{T(0)}$, and $||S|| < \gamma(T)$. Then $\exists\ \nu > 0$ such that $\alpha(T + \lambda S)$ and $\beta(T + \lambda S)$ are constant in the annulus $0 < |\lambda| < \nu$.*

PROOF

We first assume $\alpha(T) < \infty$. Let $\lambda \neq 0$ and let $x \in N(T + \lambda S)$. Then

$$Tx \supset -\lambda Sx,$$

whence

$$\begin{aligned} Sx &\subset R(T) =: R_1, \quad \textit{and} \\ x &\in S^{-1} R_1 =: D_1. \end{aligned}$$

Thus

$$\begin{aligned} -\lambda Sx &\subset Tx \subset TD_1 =: R_2, \quad \textit{and} \\ x &\in S^{-1}R_2 =: D_2. \end{aligned}$$

Proceeding in this way, we obtain

$$R_{k+1} := TD_k, \quad \textit{where} \quad D_k := S^{-1}R_k.$$

Clearly

$$R_1 \supset R_2 \supset \dots \quad \text{and} \quad D_1 \supset D_2 \supset \dots$$

It follows from the construction of these sequences of subspaces that

$$N(T+\lambda S) \subset \bigcap_{k=1}^{\infty} D_k. \tag{8.1}$$

By induction, we have that R_n are closed subspaces of Y, and D_n are relatively closed subspaces of $D(S)$: from the hypothesis, R_1 is closed, and, hence, since S is continuous, and $S(0)$ is closed, D_1 is relatively closed in $D(S)$; if R_k and D_k are closed and relatively closed, respectively, then, since $T|_{D_k} \in \Phi_+ \cup \Phi_-$, it follows that $R_{k+1} = TD_k$ is closed, and, since S is continuous, and $S(0)$ is closed, $D_{k+1} = S^{-1}R_{k+1}$ is relatively closed in $D(S)$.

Define

$$\begin{aligned} X_1 &:= \bigcap_{k=1}^{\infty} D_k, \quad \textit{and} \\ Y_1 &:= \bigcap_{k=1}^{\infty} R_k. \end{aligned}$$

Then, by the definitions of R_k and D_k, it follows that

$$TX_1 \subset Y_1 \quad \text{and} \quad SX_1 \subset Y_1.$$

Now define T_1 and S_1 by :

$$T_1 := T|_{D(T)\cap X_1}, \quad \textit{and} \quad S_1 := S|_{D(T)\cap X_1}.$$

Then $R(T_1) \subset Y_1$ and $R(S_1) \subset Y_1$, and since T is closed and X_1 is relatively closed in $D(S)$ and hence also in $D(T)$, T_1 is a closed relation. To see that T_1 is surjective, let $y \in Y_1 = \bigcap_{n=1}^{\infty} TD_n$. Then for each $n \geq 1$, there exists $x_n \in D_n$ such that $y \in Tx_n$. Since $\alpha(T) < \infty$ and $D_n \supset D_{n+1}$, there exists k_0 such that for $k \geq k_0$,

$$N(T) \cap D_{k_0} = N(T) \cap D_k,$$

and for $x_k \in D_k$, and $x_{k_0} \in D_{k_0}$,

$$x_k - x_{k_0} \in N(T) \cap D_{k_0} = N(T) \cap D_k \subset D_k.$$

From this it follows that

$$x_{k_0} \in \bigcap_{k \geq k_0} D_k = X_1, \quad \textit{and} \quad y \in T x_{k_0}.$$

i.e. T_1 is surjective. By the Open Mapping Theorem 3.3.7, T_1 is open.

By Theorem 3.5.2, Propositions 5.4.2 and 5.7.1, and by Theorem 6.1.1 , $\exists\, \nu > 0$ such that for $|\lambda| < \nu$ we have

$$\kappa(T + \lambda S) \;=\; \kappa(T). \tag{8.2}$$

Since

$$\beta(T_1 + \lambda S_1) \;\leq\; \beta(T_1) \;=\; \bar{\beta}(T_1) \;=\; 0, \tag{8.3}$$

it follows that $\beta(T_1 + \lambda S_1) = 0$, and hence

$$\alpha(T_1 + \lambda S_1) \;=\; \kappa(T_1 + \lambda S_1) \;=\; \kappa(T_1) \;=\; \alpha(T_1). \tag{8.4}$$

By (8.1), it follows that for $\lambda \neq 0$,

$$N(T + \lambda S) = N(T_1 + \lambda S_1). \tag{8.5}$$

In particular, $\alpha(T + \lambda S) = \alpha(T_1 + \lambda S_1)$. By (8.2), (8.3), (8.4) and (8.5) it follows that $\alpha(T + \lambda S)$ and $\beta(T + \lambda S)$ are constant in the annulus $0 < |\lambda| < \nu$.

If $\alpha(T) = \infty$, then $\beta(T) < \infty$, and the result is obtained by passing to the conjugates.

$\diamond$

Proposition 8.2.6 *Let X be complete and let $T \in LR(X)$ be closed. If $\rho_{ei}^{(n)}(T)$ is a component of $\rho_{ei}(T)$, $i = 1, 2, 3$, then $\alpha(\lambda - T)$ and $\beta(\lambda - T)$ have constant values, n_1 and n_2, respectively, $n_1, n_2 \in I\!N \cup \{\infty\}$, except perhaps at isolated points where*

$$\alpha(\lambda - T) > n_1 \;\textit{and}\; \beta(\lambda - T) > n_2.$$

PROOF

We first prove the result for the quantities $\alpha(\lambda - T)$. Since any component of an open set in $\mathbb{C}$ is open, we have that $\rho_{ei}^{(n)}(T)$ are open sets. We first consider the case $\rho_{e1}^{(n)}(T)$. If $\alpha(\lambda - T) = \infty$ for all $\lambda \in \rho_{e1}^{(n)}(T)$, then we are done. Now suppose $\alpha(\lambda - T) < \infty$ for some $\lambda \in \rho_{e1}^{(n)}(T)$, define $\alpha(\lambda) := \alpha(\lambda - T)$, and choose λ_0 such that $\alpha(\lambda_0) = n_1$ is the smallest non-negative integer attained by $\alpha(\lambda)$ on $\rho_{e1}^{(n)}(T)$. Suppose $\alpha(\lambda') \neq n_1$ for some λ'. Since $\rho_{e1}^{(n)}(T)$ is connected , there exists an arc Λ in $\rho_{e1}^{(n)}(T)$ with endpoints λ_0 and λ'. Since $\lambda - T \in \Phi_+ \cup \Phi_-$, it follows from Proposition 8.2.5 that for each $\mu \in \Lambda$ there exists an open ball B_μ contained in $\rho_{e1}^{(n)}(T)$ such that $\alpha(\lambda)$ is constant on $B_\mu \setminus \{\mu\}$. Since Λ is compact, there exists a finite set of points $\lambda_1, \lambda_2, \ldots, \lambda_n = \lambda'$ such that $B_{\lambda_0}, B_{\lambda_1}, \ldots, B_{\lambda_n}$ cover Λ, and, for $0 \leq i \leq n-1$,

$$B_{\lambda_i} \cap B_{\lambda_{i+1}} \neq \emptyset. \tag{8.6}$$

It follows from Theorem 3.5.2 that $\alpha(\lambda) \leq \alpha(\lambda_0)$ for λ sufficiently close to λ_0. Thus, since $\alpha(\lambda_0)$ is the minimum value attained by $\alpha(\lambda)$ on $\rho_{e1}^{(n)}(T)$, it follows that $\alpha(\lambda) = \alpha(\lambda_0)$ for λ sufficiently close to λ_0. Since $\alpha(\lambda)$ is constant for all $\lambda \neq \lambda_0$ in B_{λ_0}, this constant must be $\alpha(\lambda_0)$. Similarly $\alpha(\lambda)$ is constant on $B_{\lambda_i} \setminus \{\lambda_i\}$ for $1 \leq i \leq n$. Thus, by (8.6) that $\alpha(\lambda) = \alpha(\lambda_0)$ for all $\lambda \in B_{\lambda'} \setminus \{\lambda'\}$ and $\alpha(\lambda') > n_1$.

To see that the result holds for $\beta(\lambda - T)$, we pass to the conjugate of T and apply the above, and the equality

$$\alpha(\lambda - T') = \beta(\lambda - T).$$

The proofs for $\rho_{e2}^{(n)}(T)$ and $\rho_{e3}^{(n)}(T)$ are similar.

◇

Proposition 8.2.7 *$\lambda \in \rho_{e5}(T)$ if and only if $\lambda \in \rho_{e4}(T)$ and a deleted neighbourhood of λ lies in $\rho(T)$.*

PROOF

Suppose $\lambda \in \rho_{e5}(T)$. Then, by definition, λ lies in a component $\rho_{e1}^{(n)}(T)$ of $\rho_{e1}(T)$ which intersects $\rho(T)$. Let C be such a component. Clearly $C \cap \rho(T)$ is open.

Since $\mu \in C \cap \rho(T)$ implies $\alpha(\mu - T) = \beta(\mu - T) = \kappa(\mu - T) = 0$, it follows from Theorem 6.1.1 that $\kappa(\lambda - T) = 0$ for $\lambda \in C$ when λ is sufficiently close to μ, and, hence for all $\lambda \in C$. Applying Proposition 8.2.6, we see that $\alpha(\lambda - T) = \beta(\lambda - T) = 0$ for all except some isolated points, say λ_j where $\alpha(\lambda_j - T) > 0$ and $\beta(\lambda_j - T) > 0$. Thus if $\lambda \in \rho_{e5}(T)$, then either $\lambda \in \rho(T)$ or λ is one of these isolated points in $\rho_{e4}(T)$.

Clearly the converse is true.

◇

Corollary 8.2.8 *If $\rho_{e4}(T)$ is connected and $\rho(T) \neq \emptyset$, then $\rho_{e5}(T) = \rho_{e4}(T)$.*

PROOF

Since $\rho(T) \subset \rho_{e4}(T)$, it follows from the hypothesis and Proposition 8.2.6 that $\alpha(\lambda - T) = \beta(\lambda - T) = 0$ for all $\lambda \in \rho_{e4}(T)$ except perhaps at isolated points, i.e. a deleted neighbourhood of λ lies in $\rho(T)$. The result follows from Proposition 8.2.7.

◇

Proposition 8.2.9

$$\sigma_{e1}(T) \subset \sigma_{e2}(T) \subset \sigma_{e3}(T) \subset \sigma_{e4}(T) \subset \sigma_{e5}(T) \subset \sigma(T)$$

PROOF

Clearly

$$\rho_{e1}(T) \supset \rho_{e2}(T) \supset \rho_{e3}(T) \supset \rho_{e4}(T).$$

The remaining inclusions follow from Proposition 8.2.7.

$\diamond$

Proposition 8.2.10 *The index is constant in each connected component $\rho_{ek}^{(n)}(T)$ of $\rho_{ek}(T), k = 1, 2, 3, 4, 5$.*

PROOF

Clearly the result holds for $\rho_{e4}^{(n)}(T)$, and it follows from Proposition 8.2.7 that the result hold for $\rho_{e5}^{(n)}(T)$.

Let λ and λ' be distinct points in $\rho_{ek}^{(n)}(T), k = 1, 2, 3$. Let Λ be an arc in $\rho_{ek}^{(n)}(T)$ with endpoints λ and λ'. By Theorem 6.1.1, there exists $\epsilon > 0$ such that $\kappa(\mu - T) = \kappa(\lambda - T)$ for any μ such that $|\mu - \lambda| < \epsilon$. Clearly the open balls $B(\lambda)$, $\lambda \in \Lambda$ cover Λ. Since Λ is compact, a finite number of these balls suffices to cover Λ. Since each of these balls overlap, it follows that $\kappa(\lambda - T) = \kappa(\lambda' - T)$,

.

$\diamond$

8.3 Perturbation of the Essential Spectra

In this short section, we apply perturbation theorems for Fredholm relations to verify the stability properties of the essential spectra under small and compact perturbation. In particular we arrive at a generalisation of Weyl's theorem for linear operators.

Theorem 8.3.1 *Let X and Y be complete, and let $T \in LR(X)$ be closed. Suppose $S \in LR(X)$ is $T-$compact with $T-$bound $b < 1$, and $D(S) \supset \overline{D(T)}$ and $S(0) \subset T(0)$. Then for $i = 1, 2, 3, 4$*

$$\sigma_{ei}(T + S) = \sigma_{ei}(T).$$

If additionally ρ_{e4} is connected and neither $\rho(T)$ nor $\rho(T + S)$ are empty, then

$$\sigma_{e5}(T + S) = \sigma_{e5}(T).$$

PROOF

By Corollary 2.8.7, the norms $||\text{-}||_T$ and $||\text{-}||_{\lambda-T}$ are equivalent and hence, S is $(\lambda - T) - compact$. Let $G_{\lambda-T}$ denote the graph operator from space $X_{\lambda-T} := (X, ||x||_{\lambda-T})$ into X. Suppose $\lambda - T \in \Phi_\pm$. Clearly $R(TG_{\lambda-T}) = R(T)$, and as subsets of the set X, we have $N(TG_{\lambda-T}) = N(T)$. By Proposition 2.8.4, $(\lambda - T)G_{\lambda-T}$ is open, and hence $(\lambda - T)G_{\lambda-T} \in \Phi_\pm$. Thus, by Propositions 5.4.2 and 5.7.1, it follows that $(\lambda - T) - S = \lambda - (T + S) \in \Phi_\pm$ and by Theorem 6.1.2, $\kappa(\lambda - (T + S)) = \kappa(\lambda - T)$.

On the other hand, suppose $\lambda - (T + S) \in \Phi_\pm$. By the equivalence of the norms $||\text{-}||_T$ and $||\text{-}||_{\lambda-(T+S)}$ (Proposition 2.8.6 and Corollary 2.8.7), it follows that S is $(\lambda - (T + S)) - \textit{compact}$. Arguing as before, it follows that $\lambda - T \in \Phi_\pm$ and $\kappa(\lambda - T) = \kappa(\lambda - (T + S))$.

Thus $\rho_{ei}(T + S) = \rho_{ei}(T)$ for $i = 1, 2, 3, 4$. It follows from the additional hypotheses, Corollary 8.2.8, and what has just been proved that

$$\rho_{e5}(T) = \rho_{e4}(T) = \rho_{e4}(T + S) = \rho_{e5}(T + S).$$

$\diamond$

8.4 Functions of the Essential Spectra

Theorem 8.4.1 below is analogous to the Theorem 7.4.4 on the Möbius transform of the spectrum.

Theorem 8.4.1 *Let X be complete and let $T \in LR(X)$ be closed. Suppose $\mu \in \rho(T)$. Then for $i = 1, 2, 3, 4, 5$*

$$\lambda \in \sigma_{ei}(T) \Leftrightarrow (\mu - \lambda)^{-1} \in \sigma_{ei}(T_\mu).$$

PROOF
Let S be as in Theorem 7.4.4, i.e.

$$S := (\mu - \lambda)((\mu - \lambda)^{-1} - T_\mu) .$$

We have that

$$\lambda - T = S(\mu - T).$$

Since T is closed, so is $\lambda - T$, and since $R(\mu - T) = X$ it follows that

$$R(\lambda - T) = R(S). \tag{8.7}$$

Since T_μ is single valued,

$$\alpha(\lambda - T) = \dim T_\mu S^{-1}(0) \leq \dim S^{-1}(0) = \alpha(S).$$

Thus, $S \in \Phi_\pm$ implies that $\lambda - T \in \Phi_\pm$, i.e. $(\mu - \lambda)^{-1} \in \rho_{ei}(T_\mu)$ implies that $\lambda \in \rho_{ei}(T)$ for $i = 1, 2, 3$. Applying Proposition 2.1.4 we have

$$\begin{aligned}
(\mu - T)S &= (\mu - T)(\mu - \lambda)((\mu - \lambda)^{-1} - T_\mu) \\
&= (\mu - T) - (\mu - \lambda)(\mu - T)(\mu - T)^{-1} \\
&= (\mu - T) - (\mu - \lambda)(I + (\mu - T)(\mu - T)^{-1} - (\mu - T)(\mu - T)^{-1}) \\
&= \lambda - T + (\mu - \lambda)(TT^{-1} - TT^{-1}) \\
&= \lambda - T.
\end{aligned}$$

Thus, since $\kappa(\mu - T)$and $\kappa(S)$ are finite and $D(S) = X$, it follows from Corollary 2.7.4 that

$$\kappa(\lambda - T) = \kappa(S) + \kappa(\mu - T) - \dim(S(0) \cap N(\mu - T)). \tag{8.8}$$

In particular, if $(\mu - \lambda)^{-1} \in \rho_{e4}(T_\mu)$ then $\kappa(S) = 0$, and, since $\mu \in \rho(T)$, we have $\kappa(\mu - T) = 0 = \alpha(\mu - T)$. Thus $\kappa(\lambda - T) = 0$, i.e. $\lambda \in \rho_{e4}(T)$. Applying Proposition 8.2.7, it follows that the forward implication also holds for $i = 5$.

For the reverse implication, it follows from (8.7) that if $\lambda - T \in \Phi_-$, then $S \in \Phi_-$, i.e. $(\mu - \lambda)^{-1} - T_\mu \in \Phi_-$. Now suppose $\lambda - T \in \Phi_+$. Then there exists $M \in \mathcal{C}(D(\lambda - T))$ such that $(\lambda - T)|_M$ is injective. As in Theorem 7.4.4 it follows that $S|_M$ is injective, and hence $\alpha(S) < \infty$. Thus, $S \in \Phi_+$, and consequently $(\mu - \lambda)^{-1} - T_\mu \in \Phi_+$. We have

$$\lambda \in \rho_{ei}(T) \;\Rightarrow\; (\mu - \lambda)^{-1} \in \rho_{ei}(T_\mu) \quad \text{for } i = 1, 2, 3.$$

Now if $\lambda \in \rho_{e4}(T)$ then $\kappa(\lambda - T) = 0$, and since $\alpha(\mu - T) = \kappa(\mu - T) = 0$ it follows from (8.8) that $0 = \kappa(S) = \kappa((\mu - \lambda)^{-1} - T_\mu)$. Thus $(\mu - \lambda)^{-1} \in \rho_{e4}(T_\mu)$. Another application of Proposition 8.2.7 shows that the converse is true for $i = 5$.

◇

Theorem 8.4.2 *Let X be complete and let $T, S \in LR(X)$ be closed. Suppose $\mu \in \rho(T) \cap \rho(S)$ and $T_\mu - S_\mu$ is compact. Then for $i = 1, 2, 3, 4$*

$$\sigma_{ei}(S) = \sigma_{ei}(T).$$

If additionally $\rho_{e4}(S)$ is connected then equality holds for $i = 5$ as well.

PROOF

For $i = 1, 2, 3, 4$ it follows from Theorem 8.4.1, that

$$\lambda \in \sigma_{ei}(T) \quad \Leftrightarrow \quad (\lambda - \mu)^{-1} \in \sigma_{ei}(T_\mu),$$

and

$$\lambda \in \sigma_{ei}(S) \quad \Leftrightarrow \quad (\lambda - \mu)^{-1} \in \sigma_{ei}(S_\mu),$$

and by Theorem 8.3.1,

$$\sigma_{ei}(T_\mu - (T_\mu - S_\mu)) \;=\; \sigma_{ei}(T_\mu).$$

Applying Proposition 8.2.7 shows that the result it true for $i = 5$ under the additional hypotheses.

◇

8.5 Further Notes and Remarks

In Cross [35], the subset σ_{e1} is investigated as the *essential spectrum* of a linear relation. The definitions in this chapter are based on the classifications given in Edmunds and Evans [49] for single-valued operators. The properties investigated above extend those given in [35], and generalise some of the properties reviewed in [49].

For simplicity, we have assumed that the spaces on which the relations are defined are complete, and that the operators are closed. Fredholm properties are, however, stable under more general conditions (reviewed in Chapter 5; cf. Cross [35] for the case σ_{e1}). Thus, proofs for σ_{ei}, $i = 1,2,3$ do not necessarily require assumptions of completeness. The index may not be stable under perturbation, though, and hence, generalisations which weaken assumptions of completeness for σ_{ei}, $i = 4,5$ would have to proceed with considerations similar to those applied in Chapter 6 for Atkinson relations (cf. Labuschagne [84]).

Other subsets of the spectrum of a linear operator have also been investigated for stability under perturbation, for example the *Browder essential spectrum* defined by :

$$\sigma_b(T) := \bigcup\{ \sigma(T+K) \mid TK = KT \ \ and \ \ K \ \ is \ \ compact \}.$$

It is possible that such investigations may be extended to multivalued linear operators by the methods employed in this work.

Bibliography

[1] Y.A. Abramovich, C.D. Aliprantis and O. Burkinshaw, *The Invariant subspace Problem: Some recent advances*, Rend. Istit. Mat. Univ. Trieste Suppl. Vol. XXIX, 3-79 (1998)

[2] N.I. Akhiezer and I.M. Glazman, *Theory of Linear Operators in Hilbert Space, Volume I.* translated from Russian by M. Nestell, Frederick Ungar Publishing Co., New York, 1961

[3] T. Alvarez, *Notes on the Perturbation of Atkinson type Operators in Operator Ranges*, unpublished notes (1998)

[4] T. Alvarez and M. Gonzâlez, *Paracomplete normed spaces and Fredholm Theory,* Rend. del Cirolo. Mat. di Palermo Serie II. Tomo XLVIII 257-264 (1999)

[5] T. Alvarez, R.W. Cross and D.L. Wilcox, *Multivalued Fredholm type operators with abstract generalised inverses*, to appear, J. Math. Anal. Appl.

[6] G. Androulakis and T. Schlumprecht, *Strictly Singular, non-compact operators exist on the space of Gowers and Maurey,* preprint, February 2001, Banach space archives, http://www.math.okstate.edu/ alspach/banach

[7] R. Arens, *Operational Calculus of linear relations*, Pacific J. Math., 11, 9-23 (1961)

[8] R.P. Agarwal, M. Meehan and D. O'Regan, *Fixed point theory and applications* , Cambridge University Press, 2001

[9] W.B Arveson, *Ten lectures on Operator Algebras*, CBMS Regional Conference Series in Mathematics 55, AMS, Providence, R.I. (1984)

[10] M.Atiyah and I. Singer, *The index of elliptic operators I and V,* Ann. of Math. , 87, 484-530 (1968) and 93, 139-149 (1971), respectively

[11] F.V. Atkinson, *On relatively regular operators,* Acta Sci. Mat Szeged 15, 38-56 (1953) [*Russian*]

[12] F.V. Atkinson, *The normal solvability of linear equations in normed spaces*, Mat. Sbornik 28 (7) 3 - 14 (1951) [*Russian*]

[13] J.P. Aubin, and A. Cellina, *Differential Inclusions*, Grundlehren der mathematischen Wissenschaften 264, Springer-Verlag, Berlin, 1984

[14] J.P. Aubin and H. Frankowska, *Setvalued Analysis*, Birkhauser, Boston, 1990

[15] S. Banach, *Théorie des opérationes linéaires,* Warsaw, 1932

[16] S. Banach and S. Mazur, *Zur theorie der lineare Dimension,* Studia Math., 4, 100-112 (1933)

[17] B. Beauzamy, *Introduction to Banach spaces and their geometry*, North-Holland, Amsterdam, 1982

[18] B. Beauzamy, *Introduction to Operator Theory and Invariant Subspaces*, North-Holland, Amsterdam, 1988

[19] T. Bermudez and M. Gonzalez, *Almost regular operators are regular*, Bull. Korean Math. Soc. 38, 1, 205-210 (2001)

[20] C.Bessaga, *Every reflexive Banach space is homeomorphic to a Hilbert space*, Bull. Pol. Acad. Sci., 15 (1967)

[21] C. Bessaga and A. Pełczyński, *Spaces of Continuous functions (IV)*, Studia Math., 19, 53-62 (1960)

[22] N. Bourbaki, *Elements of Mathematics, General Topology, Chapters 1-4*, Springer-Verlag, Berlin, 1989

[23] F.E. Browder, *On the spectral theory of elliptic differential operators I.*, Math. Ann. 142, 22-130 (1961)

[24] F.E. Browder, *Functional Analysis and partial differential equations II.*, Math. Ann. 14, 81-226 (1962)

[25] L. Burlando, *On the Iterates of a Paracomplete Operator*, J. Operator Theory, 36, 357-377 (1996)

[26] S.L. Campbell (editor), *Recent applications of generalised inverses*, Research Notes in Math. 66, Pitman Advanced Publishing, London, 1982

[27] P. Casazza and T. Shura, *Tsirelson's space*, Lecture notes in Math. 1363, Springer-Verlag, Berlin, 1989

[28] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience Publication, Wiley and Sons, Toronto, 1983

[29] E.A. Coddington, *Multivalued Operators and boundary value problems*, Lecture Notes in Math. 183, Springer-Verlag, Berlin, 1971

[30] E.A. Coddington, *Self-adjoint subspace extensions of nondensely defined symmetric operators*, Adv. in Math., 14, 309-332 (1974)

[31] E.A. Coddington, *Self-adjoint problems for nondensely defined ordinary differential operators and their eigenfunction expansions*, Adv. in Math., 15 , 1-40 (1975)

[32] E.A. Coddington, *Extension theory of formally normal and symmetric subspaces*, Memoirs of the A.M.S. 134, A.M.S. Providence, Rhode Island, 1973

[33] E.A. Coddington and H.S.V. De Snoo, *Positive Self-adjoint extensions of positive symmetric subspaces*, Math. Z., 159, 203-214 (1978)

[34] E.A. Coddington and A. Dijksma, *Adjoint subspaces in Banach spaces with applications to ordinary differential subspaces*, Ann. Mat. Pura Appl. (IV), CXVII, 1-118 (1978)

[35] R.W. Cross, *Multivalued Linear Operators*, Marcel-Dekker, New York, 1998

[36] R.W. Cross, *On the Perturbation of Unbounded Linear Operators with Topologically Complemented Ranges*, J. Funct. Anal. 92 , 468-473 (1990)

[37] R.W. Cross, *Unbounded strictly singular operators*, Nederl. Akad. Wetensch. Indag. Math. Ser. A, 91, 245-248, (1988)

[38] R.W. Cross, *Unilateral quasi-regularizers of closed operators*, Math. Proc. Camb. Phil. Soc., 87, 471 - 480 (1980)

[39] R.W. Cross, *Notes on Multivalued linear projections*, unpublished notes

[40] R.W. Cross, M.I. Ostrovskii and V.V. Shevchik, *Operator Ranges in Banach Spaces I*, Math. Nachr., 173, 91-114 (1995)

[41] R.W. Cross and D.L. Wilcox, *Multivalued Linear Projections*, preprint.

[42] R.W. Cross and D.L. Wilcox, *Perturbation theorems for Atkinson relations,* in preparation

[43] M.M. Day, *Normed linear space* Springer-Verlag, Berlin, 1958

[44] J. Diestel, *Geometry of Banach spaces - selected topics,* Lecture notes in Mathematics, 485, Springer-Verlag, 1975

[45] J. Diestel, *Sequences and Series in Banach spaces*, Springer-Verlag, Berlin, 1983

[46] J. Dieudonne, *History of Functional Analysis*, North-Holland, Amsterdam, 1981

[47] S. Djordjević, *On continuity of the Browder essential approximative spectrum*, Ann. Soc. Math. Polonae, Series I: Comm. Math XXXVI, 69-73 (1996)

[48] N. Dunford and J.T. Schwartz, *Linear Operators, Parts I and II*, Interscience Publishers, New York, 1958 and 1963, respectively

[49] D.E. Edmunds and W.D. Evans, *Spectral Theory and Differential Operators*, Clarendon, Oxford, Oxford, 1987

[50] P.Enflo, *A counter-example to the approximation problem in Banach spaces,* Acta Math., 130, 309-317 (1973)

[51] J. Esterle *Mittag-Leffler methods in the theory of Banach Algebras and a new approach to Michael's probelem*, Contemporary Mathematics, vol 32, 107 - 129 (1984)

[52] A. Favini and A. Yagi, *Degenerate differential equations in Banach spaces*, Marcel Dekker Inc., New York, 1999

[53] I. Fredholm, *Sur une classe d'equations fonctionelles,* Acta math., 27, 365-390 (1903)

[54] A. Georgescu and I. Oprea, *Bifurcation Theory from the Applications point of view*, Monografii Matematice 51, Facultatea de Matemmatică, Universitatea de vest din Timişoara, 1994

[55] I.C. Gohberg and S. Goldberg, *Basic Operator Theory*, Birkhauser, Basel, 1981

[56] I.C. Gohberg, S. Goldberg and M.A. Kaashoek, *Classes of Linear Operators, Volume I*, Birkhauser Verlag, Basel, 1990

[57] I.C. Gohberg and M.C. Krein, *The basic propositions on defect numbers, root numbers and indices of linear operators*, Uspehi Mat. Nauk 12, 2 (74), 43-118 (1957) [*Russian*]

[58] I.C. Gohberg, A.S. Markus and I.A. Fel'dman, *On normally solvable operators and related ideals*, Izv. Moldav. Filial Akad. Nauk SSSR 10 (76) 51-69 (1960) [*Russian*]

[59] I.C. Gohberg and A.S. Markus, *Two theorems on the opening of subspaces of Banach space*, Usephi Mat. Nauk 14, 5 (89), 135- 140 (1959) [*Russian*]

[60] S. Goldberg, *Unbounded Linear Operators*, McGraw-Hill, New York, 1966

[61] M.A. Goldman, *On the stability of the property of normal solvabiity of linear equations* Dokl. Akad. Nauk SSSR, 100, 201-204 (1955) [*Russian*]

[62] M. González and V.M. Onieva, *Atkinson Operators in Locally Convex Spaces*, Math.Z. 190, 505-517 (1985)

[63] M. González and V. M. Onieva, *On the instability of non-semi-Fredholm operators under Compact Perturbations*, J. Math. Anal. Appl., 114, 450-457 (1986)

[64] D.B. Goodner, *Projections in normed linear spaces*, Trans. Amer. Math. Soc. 69, 89-108 (1950)

[65] L. Gorniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Kluwer, 1999

[66] W.T. Gowers and B. Maurey, *The Unconditional Basic Sequence problem*, Trans. Amer. Math. Soc. 6, 851-874 (1993)

[67] M. Gromov, *Partial differential relations*, Springer-Verlag, Berlin, 1986

[68] K. Gustafson and J. Weidmann, *On the Essential Spectrum*, J. Math. Anal. Appl., 25, 121-127 (1969)

[69] P.R. Halmos, *Measure Theory,* van Nostrand, Toronto, 1950

[70] R. Harte, *Invertibility and Singularity for Bounded Linear Operators*, Marcel Dekker, New York, 1988

[71] W.B. Johnson and J. Lindenstrauss, editors, *Handbook of the Geometry of Banach Spaces, Vol 1,* Elsevier, Amsterdam, 2001

[72] M.I. Kadec, *A proof of the topological equivalence of all separable and infinite-dimensional Banach spaces*, Funct. Anal. Appl., 1, 53-62 (1967)

[73] M.J. Kascic, *Polynomials in linear relations I, II,* Pacific J. Math. 24, 291-295 (1968); 29, 593-607 (1969)

[74] T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators,* J.Analyse Math. 6, 273-322 (1958)

[75] T. Kato, *Perturbation Theory for linear operators*, Springer-Verlag, Berlin, 1966

[76] O.D. Kellog, *Foundations of Potential Theory,* Grundlehren der mathematischen Wissenschaften 31, Springer-Verlag, Berlin, 1929

[77] M. Kline, *Mathematical Thought from Ancient to Modern Times,* Oxford University Press, New York, 1972

[78] M.A. Krasnosel'skii and P.P. Zabreiko, *Geometrical Methods of Nonlinear Analysis,* Grundlehren der mathematischen Wissenschaften 263, Springer-Verlag, Berlin, 1984, translated from original Russian edition by Nauka, Moscow, 1975

[79] M.G. Krein and M.A. Krasnosel'skii, *Stability of the index of an unbounded operator*, Mat. Sbornik 30 (72), 219-224 (1952) [*Russian*]

[80] M.G. Krein, M.A. Krasnosel'skii and D.C. Milman, *On the defect numbers of linear operators in Banach space and on some geometric problems*, Sbornik Trud. Inst. Mat. Akad. Mauk Ukr. SSSR 11, 97 - 112 (1948) [*Russian*]

[81] V. Kordula and V. Müller *On the axiomatic theory of spectrum*, Studia Math., 119 (2), 109-128 (1996)

[82] C. Kuratowski, *Topologie, Volume I,* Polska Akademia Nauk. Monografie Matematyczne Tom XX, Warsaw, 1952

[83] J.P. Labrousse, *Les Opérateurs quasi-Fredholm, une généralisation des Opérateurs semi-Fredholm,* Rend. Circ. Math. Palermo (ser.II), 29, 161-258, 1980

[84] L.E. Labuschagne, *The Perturbation of Relatively Open Operators with Reduced Index*, Math. Proc. Cambridge Phil. Soc., 112, 385-402 (1992)

[85] L.E. Labuschagne, *Functions of operators and the classes associated with them*, Ph.D. thesis, University of Cape Town, 1988.

[86] L.E. Labuschagne, *On the minimum modulus of an arbitrary linear operator*, Quaestiones Math. 14, 77-91 (1991)

[87] E. Lacey, *The isometric theory of classical Banach spaces,* Springer-Verlag, Berlin, 1974

[88] S.J. Lee, *Operator parts of differential subspaces with applciations to nonlinear problems,* in Trends in Theory and Practice of Nonlinear differential equations (V. Lakshmikantham, editor), Marcel-Dekker, New York, 1984, 315-321

[89] S.J. Lee and M.Z. Nashed, *Least-square solutions to multi-valued linear operator equations in Hilbert spaces,* J. Approx. Theory, 8, 380-391, (1983)

[90] S.J. Lee and M.Z. Nashed, *Normed Linear Relations: domain decomposability, adjoint subspaces, and selections*, Linear Algebra Appl. 153, 135-159 (1991)

[91] C. Lennard, *A Generalization of Baire's Category Theorem*, J. Math. Anal. Appl., 168, 367-371 (1992)

[92] C. Lennard, *A Baire Category Theorem for the Domains of Iterates of Linear Operator*, Rocky Mountain J. Math., Vol. 24, 2, Spring (1994)

[93] J. Lindenstrauss, *On a theorem of Murray and Mackey*, Anais de Acad. Brasileira Cien. 39, 1-5 (1967)

[94] J. Lindenstrauss, *On the subspaces of Banach spaces without quasicomplements*, Israel J. Math., 6, 36-38 (1968)

[95] J. Lindenstrauss, *Some aspects of the theory of Banach spaces,* Adv. Math., 5, 159-180 (1970)

[96] J. Lindenstrauss and A. Pełczyński, *Contributions to the theory of the classical Banach spaces,* J. Funct. Anal., 8, 2, 225 - 249 (1971)

[97] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces, vol. I and vol. II,* Springer-Verlag, Berlin, 1977 and 1979

[98] J. Lindenstrauss and L. Tzafriri, *On the complemented subspace problem,* Israel J. Math., 9, 263-269 (1971)

[99] V.I. Lomonosov, *Invariant subspaces of the family of operators that commute with a completely continuous operator,* Fuktsional. Anal. i Prilozhen 7 (3), 55-56 (1973)

[100] G. Mackey, *Note on the theorem of Murray,* Bull. Amer. Math. Soc. 52 , 322-325, 1946

[101] M. Mbekhta, *On the generalised resolvent in Banach spaces,* J. Math. Anal. Appl. 189, 362 - 377 (1995)

[102] M. Mbekhta and V. Muller, *On the axiomatic theory of spectrum. II,* Studia Math. 119 (2) , 129–147 (1996)

[103] R. Mennicken and B. Sagraloff, *Eine Verallgemainerung des Satzes von abgeschlossenen Werterbereich in lokalkonvexen Räumen,* Manuscripta Math. 18, 109 -146 (1976)

[104] R. Mennicken and B. Sagraloff, *On Banach's closed range theorem,* Archiv. Math. 33, 461-465 (1979)

[105] E. Michael, *Continuous Selections I, II, III,* Annals of Math. , 63 361-381; 64 562-580; 65 375-390.

[106] S.G. Mikhlin, *Integral Equations,* translated from Russian by A.H. Armstrong, Pergamon Press, Londdon, 1957

[107] A.A. Milutin, *Isomorphism of spaces of continuous functions on compacta of cardinality continuum,* Teoria Funktsii, Funkts. Anal. i. Priloz, 2, 150-156 (1966) [Russian]

[108] H. Minkowski, *Geomtrie der Zahlen,* Leipzig, Teubner, 1896

[109] R.H. Moore and M.Z. Nashed, *Approximations to generalised inverses of linear operators,* SIAM J. Appl. Math., 27, 1-16 (1974)

[110] F.J. Murray, *Quasicomplements and closed projections in reflexive Banach spaces,* Trans. Amer. Math.Soc. 58 77-95 (1945)

[111] V. Muller-Horrig, *Zur Theori der Semi-fredholm-Operatoren met stetig projitziertem Kern und Bild,* Math. Nachr.99, 185-197 (1980)

[112] M.Z. Nashed (editor), *Generalized inverses and applications,* Academic Press, New York, 1976

[113] M.Z. Nashed, *Operator parts and generalised inverses of linear manifolds with applications,* in Trends in Theory and Practice of Nonlinear differential equations (V. Lakshmikantham, editor), Marcel-Dekker, New York, 1984, 395-412

[114] N.K. Nikol'skij, *Functional Analysis I,* Springer-Verlag, Berlin, 1988

[115] M.I. Ostrovskii, *Topologies on the set of all subspaces of a Banach space and related questions of Banach space geometry,* Quaestiones Math. 17, No.3, 259-319 (1994)

[116] M.I. Ostrovskii, *Classes of a Banach spaces stable and unstable with respect to the opening,* Quaestiones Math. 19, 191-210 (1996)

[117] R.S. Palais, *The Index Theorem,* Princeton University Press, 1965

[118] A. Pełczyński, *Projections in certain Banach spaces,* Studia Math. 19, 209-228 (1960)

[119] A. Pełczyński, *On strictly singular and strictly cosingular operators I.,* Bull. Acad. Polonaise des Sciences, serie des sciences math., astr. et phys. - vol. XIII, 1, 31-41 (1965)

[120] A. Pietsch, *Homomorphismen in lokalkonvexen Vektorräumen*, Math. Nachr. 22, 162-174 (1960)

[121] B. Randrianantoanina, *One-complemented subspaces of real sequence spaces*, preprint, Feb 1997, Banach space archives, http://www.math.okstate.edu/ alspach/banach

[122] B. Randrianantoanina, $1-$*complemented subspaces of spaces with* $1-$*unconditional bases*, preprint May 1996, Banach space archives, http://www.math.okstate.edu/ alspach/banach

[123] C.J. Read, *Strictly Singular operators and the invariant subspace problem*, Studia Math. 132 (3), 203 - 227 (1999)

[124] F. Riesz and B. Sz.-Nagy, *Functional Analysis* translated from 2nd French edition by L.F. Boron, Frederick Ungar Publishing Co., New York, 1955

[125] R.T. Rockafellar, *Convex Analysis*, Princeton Landmarks in Mathematics and Physics, New Jersey, 1970

[126] P. Saveliev, *Lomonosov's Invariant Subspace Theorem for Multivalued Linear Operators*, preprint, August 2000, Banach space archives, http://www.math.okstate.edu/ alspach/banach

[127] P. Saveliev, *Fixed points and selections of set-valued maps on spaces with convexity*, preprint

[128] M. Schechter, *Invariance of the Essential Spectrum*, Bull. Am. Math. Soc. 71 265-267 (1965)

[129] M. Schechter, *On the essential spectrum of an arbitrary linear operator I.*, J. Math. Anal. Appl. 13, 205 - 215 (1966)

[130] M. Schechter, *Principles of Functional Analysis*, Academic Press, New York, 1971

[131] C. Schmoeger, *Relatively Regular Operators and a Spectral Mapping Theorem*, J. Math. Anal. Appl., 175, 315 - 320 (1993)

[132] A. Simonič, *A construction of Lomonosov Functions and Application to the Invariant Subspace Problem*, Pac. J. Math., 175, 1, 257-270 (1996)

[133] I. Singer, *Bases in Banach spaces I and II*, Springer-Verlag, Berlin, 1970 and 1981

[134] B. Sz.-Nagy, *Perturbations des transformation linéaires fermées,* Acta. Sci. Math., Szeged, 4, 125-137 (1951)

[135] A.E. Taylor, *Theorems on Ascent, Descent, Nullity and Defect of Linear Operators*, Math. Annalen 163, 18-49 (1966)

[136] A.E. Taylor and D.C. Lay, *Introduction to Functional Analysis, 2nd Ed.*, Wiley, New York, 1979

[137] E.C. Titchmarsh, *Some theorems on perturbation theory I, II, II, IV and V*, Proc. Roy. Soc. London Ser. A., 200, 34-46 (1949), 201, 473-479 (1950), 207, 321-328-46 (1951), 210, 30-47 (1951), and J. Analyse Math. 4. 187 - 208 (1954 / 1956), respectively

[138] N. Tomczak-Jaegermann, *From finite- to infinite-dimensional phenomena in geometric functional analysis*, Documenta Mathematics, Extra Volume ICM 1998 II, 731-742

[139] F.G. Tricomi, *Integral Equations*, Interscience Publishers, New York, 1957

[140] V.G. Troitsky, *Lomonosov's theorem cannot be extended to chains of four operators*, preprint, September 1998, Los Alamos archives, http://xxx.lanl.gov

[141] B.S. Tsirelson, *Not every Banach space contains* l_p *or* c_0, Funct. Anal. Appl., 8, 138-141 (1974)

[142] R.J. Whitley, *Strictly Singular Operators and their Conjugates*, Trans. Amer. Math. Soc. 113, 252-261 (1964)

[143] A. Wilansky, *Functional Analysis,* Blaisdell, New York, 1964

[144] S. Willard, *General Topology*, Addison-Wesley, Reading, Mass. , 1970

[145] P. Wojtaszczyk, *Banach spaces for analysts*, Cambridge studies in advanced mathematics, 1991

[146] B. Yood, *Properties of linear transformations preserved under addition of a completely continuous transformation*, Duke Math. J., 18, 599-612 (1951)

[147] K. Yosida, *Functional Analysis*, Grundlehren der mathematischen Wissenschaften 123, Springer-Verlag, Berlin, 1965

[148] P.P. Zabreyko, et al., *Integral Equations*, Noordhof Publishing, Leyden, 1975

[149] W. Zelazko, *An axiomatic approach to joint spectra I*, Studia Math., T. LXIV, 249-261 (1979)