Interval AR(1) Modelling of South African Stock Market Prices

by

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B.Sc.(Hon.)

Submitted to the Department of Statistical Sciences, University of Cape Town
in partial fulfillment of the requirements for the degree of

Masters of Science in Mathematics of Finance

at the

UNIVERSITY OF CAPE TOWN

December 2005

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10 December 2005

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Abstract

Stock Market Forecasting using classical Time Series Analysis makes use only of closing prices, which is partial information market. A plethora of other information, for instance the readily available data on low and high stock prices is usually ignored. An Interval-valued Time Series Analysis approach is introduced and contrasted against the classical Time Series Analysis in utilising this additional stock market information. This approach could lead to a rich source of information for optimal financial decision making about the likely direction of the market prices. By its very nature interval analysis offers vast possibilities for accomodating this type of low and high price data, to make better forecasts. In this dissertation an attempt is made to model and estimate parameters in a context of uncertain input data, using interval analysis techniques. Though the results are not very convincing, this approach offers hope for future research under more realistic Time Series models.

Xhosa Translation

Xa kuthekelelwa amaxabiso kwinkonzo zemali, kusetyenziswa izixhobo eziqheleileyo, ikakhulu kuphelelwa ekusetyenzisweni amaxabiso avela xa zivalwayo ezinkonzo zemali. Inkathazo yoluhlobo lokuthekelela kukuba incukacha eninzi yezinkonzo ayityenziswa, ingakumbi awona maxabiso aphantsi, nawona maxabiso aphezulu ethi ifikelele kuwo imali. Lomqulu uqwalasela indlela entsha ekunokuthi isetyenziswe ngayo lencukacha, ikwenza oku ngokuvelela isixhobo esingaqhelekanga kwinzululwazi zencubeko kwinkonzo zemali. Ithemba ke ngokwenza oku lelokuba ekuhambeni kwexesha siyakuphuhliswa esisixhobo sibeluqilimba, sithi xa sisetyenziswa nezo seziqhelekile sinike umfutho ingakumbi kwabasenza kwezinkonzo zemali.

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Declaration

1. Plagiarism is to use another's work and pretend that it is one's own. I know that plagiarism is wrong and therefore my work does not contain any plagiarism.

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ACKNOWLEDGEMENTS

My heartfelt gratitude goes to Professor Renkuan Guo, my dissertation supervisor, for his guidance, encouragement, patience and confidence in my abilities. I am also profusely grateful to Professor C. G. Troskie, of the Department of Statistical Sciences for conceiving and suggesting the idea of using Interval analysis in the financial markets, for his wisdom and sense of justice, and lastly for the provision of financial data, used in this dissertation.

My thanks also goes to the Department of Mathematics and Applied Mathematics for hosting the MSc in Mathematics of Finance programme and for accommodating me.

I would forever be grateful to my family for sacrifices, material and moral support throughout the time I spent away in Cape Town, and afterwards during my sojourns to Cape Town trying to finalise this dissertation; I would also like to thank them for always being there for me.

Chapter 1

Introduction

In today's volatile financial market trading, it is desirable to use ideas based on interval analysis proposed by Moore [1962] for providing some reliable interval ranges of stock prices. The goal of this dissertation is to empirically explore financial time series interval modelling on the Johannesburg Securities Exchange of South Africa(JSE) data.

1.1 Problem Statements for the Mini-Dissertation

Interval analysis was introduced by Ramon E. Moore at about 1959-1962 as a tool for automatic control of the errors in computed results, that arise from input errors, rounding errors during computation, and truncation errors from using numerical approximations to mathematical problems. An important application is the enclosure of ranges of functions, which enables us, for instance to solve equations with interval coefficients, to prove existence of specific solutions, and to solve global optimisation problems. Interval analysis techniques are used, in this dissertation, in a context where the time series models stock market prices/indices. It is also the time first that an attempt is made in using the techniques to estimate parameters using low and high share prices. The financial market models are statistical in nature, thus must be data-driven. It follows that the merit of a financial model should be judged on whether the observed market dynamics have been reasonably captured.

1.2 Objectives of the Mini-Dissertation

The primary goals for this mini-dissertation are to explore, both empirically and theoretically, the following:

1. Understanding of the theoretical foundation of the interval-valued time series and

2. Empirically modelling the interval-valued *AR(1)* and the classical *AR(1)* model. In order to achieve the second goals, the four objectives for this mini-dissertation are defined as follows:

Objective 1: Using the ideas of interval analysis to split the time series of Absa, Anglo Prices and JSE-Over Index (from 06 May 1987 to 15 September 2004) into nine distinct samples.

Objective 2: Estimating the low-high classical *AR(1)* parameters using the nine distinct samples of Absa, Anglo and JSE-Over index.

Objective 3: Performing interval *AR(1)* modelling under metric d_G using the nine distinct samples of Absa, Anglo and JSE-Over index.

Objective 4: Comparing the modelling results between classical and interval valued forecasts of low high share prices/index.

1.3 Structure of the Mini-Dissertation

Chapter 1 is used to state the aims of the mini-dissertation. Chapter 2 discusses the interval arithmetic. the support function and the metric in the interval space. particularly. a new generalized metric d_G is proposed. Chapter 3 is used to review the interval random variables and the concept of interval stochastic processes. Chapter 4 the classical *AR(p)* time-series analysis is reviewed. Chapter 5 an interval *AR(1)* model is proposed and particularly the related theory of *AR(1)* under d_G metric is developed. In Chapter 6, the empirical modelling efforts are carried on three JSE stock prices, Absa, Anglo and JSE-Over from two aspects: using a naive classical *AR(1)* fitting of Low-High prices and from filtered intervals and using interval *AR(1)* under d_G metric modelling on them. Then in terms of five-day "forecasting" , the naive low-high price intervals and interval AR(1) under d_G price intervals are compared. Chapter 7 is used to state a few remarks of the empirical modeling evidence.

The bibliography lists the references used for the completion of this dissertation.

The appendix includes the Mathematica programing codes used to calculate the various parameters required for the analysis.

Chapter 2

A Review of Theoretical Foundations of Interval Analysis

2.1 Uncertainties of randomness and imprecision

The interval random variable theory serves the purpose that the data is constituted of interval-valued observations due to the randomness uncertainty and imprecision uncertainty (hard uncertainty). Uncertainties are intrinsically rooted in environmental problems. An uncertainty is said to be *soft* if the available information concerning the value of the critical pollution threshold can be modelled by a unique (additive) probability measure. In contrast the hard uncertainty arises when the environmental variables concerned have both randomness and imprecision so that the information available is not adequate to be modelled by a unique probability measure. "The concept of hard uncertainty seems to be particularly relevant in the discussion of many environmental risks. Indeed, concerning most of these questions the experts consider that the current state of scientific knowledge is not sufficient to measure precisely the incurred risk." Situations of regular uncertainty can be associated with the concept of closed random intervals. Indeed closed random intervals define a system of upper and lower probability measures which are the upper and lower envelopes of a (convex) class of probability measures (Dempster [1968]).

The interval random variable theory is linked to the interval analysis. R. E. Moore [1979] pointed out that uncertainty in initial data would be a more accurate description for many applications than "error" in initial data. In practical situations, the data observations collected depend upon the measuring equipment, what is often guaranteed is the upper bound Δ_i on the measurement error

$$\Delta y_i \triangleq \widetilde{y}_i - y_i.$$

This upper bound is characterized by the equipment itself, i.e., $|\Delta y_i| \le \Delta_i$. Once the measurements are collected, what we really know is that the actual (unknown) value y_i belongs to the interval $\overline{y}_i = \left[y_i^l, y_i^u\right]$, where

$$y_i^l \triangleq \widetilde{y}_i - \Delta_i \text{ and } y_i^u \triangleq \widetilde{y}_i + \Delta_i.$$

The operations and properties with respect to interval-valued numbers, called *interval arithmetic*, was developed in the modern version in R. E. Moore's work [1966]. Therefore it is necessary to have a basic review of the contents of interval arithmetic.

2.2 Interval arithmetic

Definition 1 *The real-valued closed set*

$$\overline{x} = \left[x^l, x^u\right] = \left\{x | x^l \le x \le x^u, x^l, x^u \in \mathbb{R}, x^l \le x^u\right\}$$

is said to be a closed interval number, x^l and x^u are denoted as the lower endpoint and upper endpoint respectively.

$$\mathcal{P}_{\mathrm{IV}}(\mathbb{R}) \triangleq \left\{\overline{x} = \left[x^l, x^u\right] | x^l, x^u \in \mathbb{R}, x^l \le x^u\right\}$$

is said to be the collection of all the closed interval numbers on $\mathbb{R}$. If $x^l \equiv x^u = x$, then $\left[x^l, x^u\right] = [x, x]$ is said to be a point interval number. If $0 < x^l \le x^u$, $\left[x^l, x^u\right]$ is said to be a positive closed interval number; similarly, if $x^l \le x^u < 0$, $\left[x^l, x^u\right]$ is said to be a negative closed interval number.

Definition 2 *For all $\overline{x} = \left[x^l, x^u\right] \in CI(\mathbb{R})$, the quantities*

$$m_{\overline{x}} \triangleq \frac{x^l + x^u}{2},$$

$$W_{\overline{x}} \triangleq x^u - x^l,$$

and

$$|\overline{x}| \triangleq \left|x^l\right| \vee |x^u| = \max\{|x^l|, |x^u|\} \tag{2.1}$$

are the mid-value, the width value and the absolute value of the closed interval $\overline{x} = \left[x^l, x^u\right]$ *respectively. Note that the absolute value of an interval-valued number is defined as the maximum of the absolute values of the lower and upper endpoints of the interval-valued number.*

Definition 3 *Assume that*

$$\Box(x, y) = x \Box y$$

defines an algebraic relation on $\mathbb{R}$, *then for* $\forall \overline{x} = \left[x^l, x^u\right]$, $\overline{y} = \left[y^l, y^u\right] \in CI(\mathbb{R})$, *the expansion operation on* $CI(\mathbb{R})$ *is defined as*

$$\begin{aligned} \Box(\overline{x}, \overline{y}) &= \left[x^l, x^u\right] \Box \left[y^l, y^u\right] \\ &\triangleq \left\{\overline{z} \mid \exists (x, y) \in \overline{x} \times \overline{y}, z = x \Box y\right\}. \end{aligned}$$

Particularly, the algorithm operations on $CI(\mathbb{R})$ *are defined as*

$$\overline{x} + \overline{y} = \left[x^l, x^u\right] + \left[y^l, y^u\right] \triangleq \left[x^l + y^l, x^u + y^u\right],$$

$$\overline{x} - \overline{y} = \left[x^l, x^u\right] - \left[y^l, y^u\right] \triangleq \left[x^l - y^u, x^u - y^l\right],$$

$$\overline{x} \cdot \overline{y} = \left[x^l, x^u\right] \cdot \left[y^l, y^u\right] \triangleq \left[\wedge_{\overline{x}\cdot\overline{y}}, \vee_{\overline{x}\cdot\overline{y}}\right]$$

where

$$\wedge_{\overline{x}\cdot\overline{y}} \triangleq \min\left(x^l y^l, x^l y^u, x^u y^l, x^u y^u\right)$$

and

$$\vee_{\overline{x}\cdot\overline{y}} \triangleq \max\left(x^l y^l, x^l y^u, x^u y^l, x^u y^u\right)$$

and

$$\frac{\overline{x}}{\overline{y}} = \frac{[x^l, x^u]}{[y^l, y^u]} \triangleq \left[x^l, x^u\right] \cdot \left[\frac{1}{y^u}, \frac{1}{y^l}\right]$$

where

$$0 \notin \left[y^l, y^u\right]$$

respectively. The "$\vee$" and "$\wedge$" operation on $CI(\mathbb{R})$ are defined as

$$\overline{x} \vee \overline{y} = \left[x^l, x^u\right] \vee \left[y^l, y^u\right] \triangleq \left[x^l \vee y^l, x^u \vee y^u\right]$$

and

$$\overline{x} \wedge \overline{y} = \left[x^l, x^u\right] \wedge \left[y^l, y^u\right] \triangleq \left[x^l \wedge y^l, x^u \wedge y^u\right]$$

respectively. The "$\cup$" and "$\cap$" operation on $CI(\mathbb{R})$ are defined as

$$\overline{x} \cup \overline{y} = \left[x^l, x^u\right] \cup \left[y^l, y^u\right] \triangleq \left[x^l \wedge y^l, x^u \vee y^u\right]$$

and

$$\overline{x} \cap \overline{y} = \left[x^l, x^u\right] \cap \left[y^l, y^u\right] \triangleq \left[x^l \vee y^l, x^u \wedge y^u\right]$$

respectively, if $\overline{x} \cap \overline{y} \neq \emptyset$.

Definition 4 *For $\forall \overline{x} \in CI(\mathbb{R})$, the mid-point, absolute value, and width values are defined by*

$$m\left(\overline{x}_1\right) \triangleq \frac{x^l + x^u}{2},$$

$$|\overline{x}| \triangleq \max\left(\left|x^l\right|, |x^u|\right),$$

and

$$W\left(\overline{\mathbf{x}}\right) \triangleq x^u - x^l$$

respectively.

Definition 5 $\overline{\mathbf{X}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$, $\overline{x}_j = \left[x_j^l, x_j^u\right] \in CI(\mathbb{R})$, $j = 1, 2, \ldots, n$ *is said to be a closed interval vector of dimension n. Accordingly, the mid-point, norm, and width values are*

$$m\left(\overline{\mathbf{X}}\right) \triangleq \left(m\left(\overline{x}_1\right), m\left(\overline{x}_2\right), \ldots, m\left(\overline{x}_n\right)\right),$$

$$\|\overline{\mathbf{X}}\| \triangleq \bigvee_{k=1}^{n} |\overline{x}_k|$$

and

$$W\left(\overline{\mathbf{X}}\right) \triangleq \bigvee_{k=1}^{n} W\left(\overline{x}_k\right)$$

respectively.

Definition 6 $\overline{A} = (\overline{a}_{ij})_{n \times n}$ *is a closed interval matrix if its elements* $\overline{a}_{ij} \in CI(\mathbb{R})$*. Accordingly, the mid-point, norm, and width values are defined by*

$$m\left(\overline{A}\right) \triangleq \left(m\left(a_{ij}\right)\right)_{n \times n},$$

$$\|\overline{A}\| \triangleq \bigvee_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|\right)$$

and

$$W\left(\overline{A}\right) \triangleq \bigvee_{i,j=1}^{n} W\left(a_{ij}\right)$$

respectively.

2.2.1 Closed convex set

Definition 7 *Let* $\mathbb{U}$ *be an Euclidean space and a set* $A \subseteq \mathbb{U}$ *. A is convex on* $\mathbb{U}$ *if and only if for* $\forall u_1, u_2 \in A$ *and for* $\forall \kappa \in [0,1]$*,* $\kappa u_1 + (1-\kappa) u_2 \in A$*. Set A is closed on* $\mathbb{U}$ *if and only if for* $\forall u_n \in A$*,* $n = 1, 2, \ldots,$

$$\lim_{n \to \infty} u_n = u \in A.$$

If any set A on $\mathbb{U}$ *is both closed and convex, A is a closed convex set on* $\mathbb{U}$*.*

Theorem 8 *A bounded set A is a closed convex set on* $\mathbb{R}$ *if and only if A is a closed interval number on* $CI(\mathbb{R})$*.*

2.2.2 Derivatives of interval-valued function

Definition 9 *Assume that* $f^l(t) : \mathbb{T} \to \mathbb{R}$ *and* $f^u(t) : \mathbb{T} \to \mathbb{R}$ *and*

$$\overline{f}(t) = \left[f^l(t), f^u(t)\right]$$

is an interval function on $\mathbb{T} \subseteq \mathbb{R}$ *. If these real-valued functions* $f^l(t)$ *and* $f^u(t)$ *are differentiable at* $t_0 \in T$*, i.e.,* $\frac{d}{dt} f^l(t_0)$ *and* $\frac{d}{dt} f^u(t_0)$ *exist, interval function* $\overline{f}(t)$ *is differentiable at* $t_0 \in T$*. The derivative of the interval function* $f(t)$ *at* t_0 *is defined as*

$$\frac{d}{dt}\overline{f}(t_0) = \left[\frac{d}{dt} f^l(t_0) \wedge \frac{d}{dt} f^u(t_0), \frac{d}{dt} f^l(t_0) \vee \frac{d}{dt} f^u(t_0)\right].$$

If $\frac{d}{dt} f^l(t_0) \leq \frac{d}{dt} f^u(t_0)$ *then* $\overline{f}(t)$ *is said to be differentiable at* t_0 *same-orderly, then*

$$\frac{d}{dt}\overline{f}(t_0) = \left[\frac{d}{dt} f^l(t_0), \frac{d}{dt} f^u(t_0)\right], \ t_0 \in T;$$

If $\frac{d}{dt} f^l(t_0) \geq \frac{d}{dt} f^u(t_0)$ *then* $f(t)$ *is said to be differentiable at* t_0 *reverse-orderly, then*

$$\frac{d}{dt}\overline{f}(t_0) = \left[\frac{d}{dt} f^u(t_0), \frac{d}{dt} f^l(t_0)\right], t_0 \in \mathbb{T}$$

Definition 10 *For interval function* $\overline{f}(t) = \left[f^l(t), f^u(t)\right]$ *on* $\mathbb{T}$*, if the derivative of* $\overline{f}(t)$*,*

$$\frac{d}{dt}\overline{f}(t) = \left[\frac{d}{dt} f^l(t) \wedge \frac{d}{dt} f^u(t), \frac{d}{dt} f^l(t) \vee \frac{d}{dt} f^u(t)\right]$$

exists for $\forall t \in \mathbb{T}$*, then*$\overline{f}(t)$ *is differentiable on* $\mathbb{T}$*. Similarly,* $\overline{f}(t)$ *is differentiable same-orderly on* $\mathbb{T}$ *if*

$$\frac{d}{dt}\overline{f}(t) = \left[\frac{d}{dt} f^l(t), \frac{d}{dt} f^u(t)\right], \forall t \in \mathbb{T}$$

and $f(t)$ *is differentiable reverse-orderly on* $\mathbb{T}$ *if*

$$\frac{d}{dt}\overline{f}(t) = \left[\frac{d}{dt} f^u(t), \frac{d}{dt} f^l(t)\right], \forall t \in \mathbb{T}.$$

2.2.3 Further discussion on derivative of an interval-valued function

A further question is that given the interval function $F(\overline{x})$, where $\overline{x} \triangleq [x^l, x^u]$, $x^l < x^u$, $x^l, x^u \in \mathbb{R}$ i.e., a real-valued interval, what is the definition of the derivative of $F(\overline{x})$, $F'(\overline{x})$? For example, consider $F(\overline{x}) = [1,2]\cdot\overline{x}^2$, if the interval $[1,2]$ is "interpreted as a single point value that is known only to lie in the interval $[1,2]$, then the logical derivative would be $2 \cdot [1,2] \cdot \overline{x}$. This is because $f(x) = ax^2$ for any $a \in [1,2]$, so $f'(x) = 2ax$ for the *same* a in $[1,2]$. When I speak of 'interval derivative' or 'interval Jacobi matrix', this is what I mean", see B. Kearfott [2001].

V. Kreinovich, Hung T. Nguyen, G. P. Dimuro and B. Bedregal [2003] developed a new differential formalism for interval-valued functions.

Definition 11 *An interval function F is a finite sequence of pairs $(x_i, \overline{y}_i)$, $i = 1, 2, \cdots, n$, where for each i, x_i is a real number, i.e., $x_i \in \mathbb{R}$, $x_1 < x_2 < \cdots < x_n$, and $\overline{y}_i = \left[y_i^l, y_i^u\right]$ is a nondegenerate interval ($y_i^l < y_i^u$, $y_i^l, y_i^u \in \mathbb{R}$).*

Definition 12 *A function $f : \mathbb{R} \to \mathbb{R}$ is said to belong to an interval function $F = \{(x_1, \overline{y}_1), (x_2, \overline{y}_2), \cdots$ i.e., $f \in F$, if f is continuously differentiable and for each i, $f(x_i) \in \overline{y}_i$.*

Definition 13 *Let F be an interval function and $[a, b]$ be an interval. A derivative of F,*

$$F'([a,b]) \triangleq \bigcap_{f \in F} f'([a,b])$$

where $f'(x)$ is the the first order derivative of $f(x)$, and

$$f'([a,b]) \triangleq \{f'(x) | x \in [a,b]\},$$

where $f'([a,b])$ is the range *of the derivative $f'(x)$ over the interval $[a,b]$.*

Remark 14 *The notation $F'([a,b])$ looks like the notation of a range for a real-valued function and shares some properties of range, say, the range is inclusion-monotonic by which $[a,b] \subseteq [c,d] \Rightarrow f'([a,b]) \subseteq f'([c,d])$. Therefore, derivative $F'([a,b])$ is inclusion-monotonic, i.e., $[a,b] \subseteq [c,d] \Rightarrow F'([a,b]) \subseteq F'([c,d])$. Furthermore, if $[a,b] \cap [c,d] \neq \varnothing$, $[a,b] \cup [c,d] = [a,d]$,* *an interval, then $F'([a,b]\cup[c,d)\supseteq F'([a,b])\cup F'([c,d])$. But it is not a range, if an interval is narrow enough, $F'([a,b])=\varnothing$.*

Definition 15 *Let $F=\{(x_1,\overline{y}_1),(x_2,\overline{y}_2),\cdots,(x_n,\overline{y}_n)\}$ be an interval function, and let s be a real-valued number. Define a interval function*

$$F-sx\triangleq\{(x_1,\overline{y}_1-sx_1),(x_2,\overline{y}_2-sx_2),\cdots,(x_n,\overline{y}_n-sx_n)\}$$

where

$$\overline{y}-c\triangleq[y^l-c,y^u-c]$$

for the given interval $\overline{y}=[y^l,y^u]$, $c\in\mathbb{R}$.

Lemma 16 *(Kreinovich et al) For all F and any interval $[a,b]$, $s\in F'([a,b])$ if and only if*

$$0\in(F-sx)'([a,b]).$$

Theorem 17 *(Kreinovich et al) For all F and any interval $[a,b]$, let i_l and j_u be the first and last index of the values inside $[a,b]$. Then*

$$F'([a,b])=\left[(F')^l,(F')^u\right]$$

where

$$(F')^l\triangleq\min_{i_l\le i\le j\le j_u}\triangle_{ij}^u$$
$$(F')^u\triangleq\max_{i_l\le i\le j\le j_u}\triangle_{ij}^l$$
$$\triangle_{ij}^l\triangleq\frac{y_i^l-y_j^u}{x_j-x_i}$$
$$\triangle_{ij}^u\triangleq\frac{y_i^u-y_j^l}{x_j-x_i}$$

and

$$[p,q]\triangleq\{x|p\le x\ \&\ x\le q\}$$

such that if $p>q$, $[p,q]=\varnothing$. See Kreinovich et al for proof.

2.3 Support function

A compact convex set A in $\mathbb{R}^d$ is uniquely determined by its support function

$$s_A(u) = \sup_{\alpha \in A} \langle u, \alpha \rangle , \, u \in \mathbb{S}^{d-1}$$

where $\mathbb{S}^{d-1}$is the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$ and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^d$. It is noticed that

$$s_{\beta A + \gamma B}(u) = \beta s_A(u) + \gamma s_B(u), u \in \mathbb{S}^{d-1}$$

Denote σ_A the Steiner point of a convex set A,

$$\sigma_A = d \int_{\mathbb{S}^{d-1}} u s_A(u) \mu(du)$$

where μ is the normalized $(d-1)$-dimensional Lebesgue measure on the unit sphere $\mathbb{S}^{d-1}$ (i.e., $\mu\left(\mathbb{S}^{d-1}\right) = 1$), then $\sigma_A \in A$ and

$$\sigma_{\beta A + \gamma B} = \beta \sigma_A + \gamma \sigma_B$$

Furthermore, for any given convex set A, set $A_o = A - \sigma_A$ has its Steiner point at the origin, i.e.,

$$\sigma_{A - \sigma_A} = o.$$

Example 18 *Let* $\overline{a} = \left[a^l, a^u\right]$, $a^l \leq a^u$, $a^l, a^u \in \mathbb{R}^1$. *the support function for the two directions* $\{1, -1\} = \mathbb{S}^{d-1}$

$$s_{\overline{a}} = \begin{cases} a^u & \text{if} \quad u = 1 \\ -a^l & \text{if} \quad u = -1 \end{cases}$$

2.4 Metric in an Interval Space

Metric plays an important role in describing the distance between two interval numbers in the interval space and therefore determines the formality of the optimal criterion when seeking statistical estimation of parameters involved in the interval models.

2.4.1 Moore's distances for interval space

R.E. Moore [1966] defines the distance between two intervals $\overline{a} = [a^l, a^u]$ and $\overline{b} = [b^l, b^u]$

$$d_M\left(\overline{a}, \overline{b}\right) = \max\left(\left|a^l - b^l\right|, \left|a^u - b^u\right|\right)$$

and pointed out that

$$d\left(\overline{a}, \overline{b}\right) = d\left(\overline{b}, \overline{a}\right),$$

and

$$d\left(\overline{a}, \overline{b}\right) = 0$$

if and only if $\overline{a} = \overline{b}$.

If $\overline{a}, \overline{b}, \overline{c} \in \mathbb{I}(\mathbb{R})$, then

$$d\left(\overline{a}, \overline{b}\right) \leq d\left(\overline{a}, \overline{c}\right) + d\left(\overline{c}, \overline{b}\right).$$

Moore and later Diamond [1988] also pointed out that the distance

$$D_2\left(\overline{a}, \overline{b}\right) = \sqrt{\left(a^l - b^l\right)^2 + \left(a^u - b^u\right)^2}$$

is also a metric for an interval space.

2.4.2 Hausdorff metric

The Hausdorff metric in the compact convex interval space is defined as

$$d_H\left(\overline{a}, \overline{b}\right) = \max\left(\delta\left(\overline{a}, \overline{b}\right), \delta\left(\overline{b}, \overline{a}\right)\right)$$

where

$$\delta(A, B) = \sup_{x \in A}\left(\inf_{y \in B} |x - y|\right).$$

For all $\overline{a}, \overline{b} \in P_{\mathbb{IV}}(\mathbb{R})$, the expression of $d_H\left(\overline{a}, \overline{b}\right)$ depends upon the relative position of $\overline{a}$ and $\overline{b}$.

Case (i) if $\overline{a} < \overline{b}$, $a^l < a^u < b^l < b^u$, i.e., $\overline{a} \cap \overline{b} = \varnothing$, then

$$\begin{aligned}
\delta\left(\overline{a}, \overline{b}\right) &= \sup_{x \in \overline{a}} \left(\inf_{y \in \overline{b}} |x - y| \right) \\
&= \sup_{x \in \overline{a}} \left(\left| x - b^l \right| \right) \\
&= \left| a^l - b^l \right|
\end{aligned}$$

and

$$\begin{aligned}
\delta\left(\overline{b}, \overline{a}\right) &= \sup_{y \in \overline{b}} \left(\inf_{x \in \overline{a}} |x - y| \right) \\
&= \sup_{y \in \overline{b}} \left(|a^u - y| \right) \\
&= |a^u - b^u|
\end{aligned}$$

hence

$$d_H\left(\overline{a}, \overline{b}\right) = \max\left(\left| a^l - b^l \right|, |a^u - b^u| \right)$$

Case (ii) If $\overline{a} > \overline{b}$, $b^l < b^u < a^l < a^u$

$$\begin{aligned}
\delta\left(\overline{a}, \overline{b}\right) &= \sup_{x \in \overline{a}} \left(\inf_{y \in \overline{b}} |x - y| \right) \\
&= \sup_{x \in \overline{a}} \left(|x - b^u| \right) \\
&= |a^u - b^u|
\end{aligned}$$

and

$$\begin{aligned}
\delta\left(\overline{b}, \overline{a}\right) &= \sup_{y \in \overline{b}} \left(\inf_{x \in \overline{a}} |x - y| \right) \\
&= \sup_{y \in \overline{b}} \left(\left| a^l - y \right| \right) \\
&= \left| a^l - b^l \right|
\end{aligned}$$

hence

$$d_H\left(\overline{a}, \overline{b}\right) = \max\left(|a^u - b^u|, \left| a^l - b^l \right| \right) = \max\left(\left| a^l - b^l \right|, |a^u - b^u| \right)$$

Case (iii) If $\overline{a} \leq \overline{b}$, $a^l < b^l < a^u < b^u$, i.e., $\overline{a} \cap \overline{b} \neq \varnothing$, then

$$\begin{aligned}
\delta\left(\overline{a}, \overline{b}\right) &= \sup_{x \in \overline{a}} \left(\inf_{y \in \overline{b}} |x - y| \right) \\
&= \sup_{x \in \left[a^l, b^l\right]} \left(\left| x - b^l \right| \right) \vee \sup_{x \in \left[b^l, a^u\right]} (0) \\
&= \left| a^l - b^l \right|
\end{aligned}$$

and

$$\begin{aligned}
\delta\left(\overline{b}, \overline{a}\right) &= \sup_{y \in \overline{b}} \left(\inf_{x \in \overline{a}} |x - y| \right) \\
&= \sup_{y \in [a^u, b^u]} \left(|a^u - y| \right) \vee \sup_{y \in \left[b^l, a^u\right]} (0) \\
&= |a^u - b^u|
\end{aligned}$$

hence

$$d_H\left(\overline{a}, \overline{b}\right) = \max\left(\left| a^l - b^l \right|, |a^u - b^u| \right)$$

Case (iv) If $\overline{b} \leq \overline{a}$, $b^l < a^l < b^u < a^u$, then the expression for d_H can be developed by switching a and b in Case (iii),

$$d_H\left(\overline{a}, \overline{b}\right) = \max\left(\left| a^l - b^l \right|, |a^u - b^u| \right)$$

Case (v) If $\overline{a} \cap \overline{b} = \overline{a}$, i.e., $b^l < a^l < a^u < b^u$, then

$$\delta\left(\overline{a}, \overline{b}\right) = \sup_{x \in \overline{a}} \left(\inf_{y \in \overline{b}} |x - y| \right) = \sup_{x \in \overline{a}} (0) = 0$$

and

$$\begin{aligned}
\delta\left(\overline{b},\overline{a}\right) &= \sup_{y\in\overline{b}}\left(\inf_{x\in\overline{a}}|x-y|\right)\\
&= \sup_{y\in\left[b^l,a^l\right]}\left(\inf_{x\in\overline{a}}|x-y|\right)\vee\sup_{y\in\left[a^l,a^u\right]}\left(\inf_{x\in\overline{a}}|x-y|\right)\vee\sup_{y\in\left[a^u,b^u\right]}\left(\inf_{x\in\overline{a}}|x-y|\right)\\
&= \sup_{y\in\left[b^l,a^l\right]}\left(\left|a^l-y\right|\right)\vee 0\vee\sup_{y\in\left[a^u,b^u\right]}\left(\left|a^u-y\right|\right)\\
&= \max\left\{\left|a^l-b^l\right|,\left|a^u-b^u\right|\right\}
\end{aligned}$$

Thus

$$d_H\left(\overline{a},\overline{b}\right)=\max\left\{0,\max\left\{\left|a^l-b^l\right|,\left|a^u-b^u\right|\right\}\right\}=\max\left\{\left|a^l-b^l\right|,\left|a^u-b^u\right|\right\}$$

Case (vi) If $\overline{a}\cap\overline{b}=\overline{b}$, i.e., $a^l<b^l<b^u<a^u$, then by switching a and b in (v), we obtain

$$d_H\left(\overline{a},\overline{b}\right)=\max\left\{\left|a^l-b^l\right|,\left|a^u-b^u\right|\right\}$$

Therefore, in general, the Moore's metric for interval space is nothing but the Hausdorff metric, i.e.,

$$d_M\left(\overline{a},\overline{b}\right)=d_H\left(\overline{a},\overline{b}\right),\ \forall\ \overline{a},\overline{b}\in P_{IV}\left(\mathbb{R}\right)$$

2.4.3 L_2-metric

The representation of non-empty convex sets by the corresponding support functions suggests a L_2-metric on the space of convex sets in terms of the L_2-metric on the space of Lebesgue integrable functions on $\mathbb{S}^{d-1}$. The L_2-norm

$$\|A\|_2=\|s_A\|_2=\left\{d\int_{\mathbb{S}^{d-a}}\left|s_A(u)\right|^2\mu(du)\right\}^{\frac{1}{2}}$$

and thus the L_2-metric is defined as

$$\delta_2(A,B)=\|s_A-s_B\|_2=\left\{d\int_{\mathbb{S}^{d-a}}\left|s_A(u)-s_A(u)\right|^2\mu(du)\right\}^{\frac{1}{2}}.$$

As an extension, a L_p-metric can be defined as

$$\delta_p(A,B) = \begin{cases} \{\sup_{u\in\mathbb{S}^{d-1}} |s_A(u) - s_A(u)|^p\}^{\frac{1}{p}} & p\in[1,\infty) \\ \{\sup_{u\in\mathbb{S}^{d-1}} |s_A(u) - s_A(u)|\} & p=\infty \end{cases}$$

Example 19 *The distance between two intervals,* $\overline{a} = [a^l, a^u]$ *and* $\overline{b} = [b^l, b^u]$.

$$\begin{aligned} & \delta_2(\overline{a},\overline{b}) \\ = & \left\|s_{\overline{a}} - s_{\overline{b}}\right\|_2 \\ = & \left\{d \times \int_{\mathbb{S}^{d-a}} \left|s_{\overline{a}}(u) - s_{\overline{b}}(u)\right|^2 \mu(du)\right\}^{\frac{1}{2}} \\ = & \sqrt{1 \times \sum_{\{1,-1\}} \left|s_{\overline{a}}(u) - s_{\overline{b}}(u)\right|^2 \times \frac{1}{2}} \\ = & \sqrt{\frac{1}{2} \times \left[\left|s_{\overline{a}}(1) - s_{\overline{b}}(1)\right|^2 + \left|s_{\overline{a}}(-1) - s_{\overline{b}}(-1)\right|^2\right]} \\ = & \sqrt{\frac{1}{2} \times \left[|a^u - b^u|^2 + |(-a^l) - (-b^l)|^2\right]} \\ = & \sqrt{\frac{1}{2} \times \left[|a^u - b^u|^2 + |a^l - b^l|^2\right]} \\ = & \sqrt{\frac{1}{2}} D_2(\overline{a},\overline{b}) \end{aligned}$$

which shows that L_2*-metric is nothing but* $\frac{1}{\sqrt{2}}$ *times Diamond's* D_2 *distance.*

Remark 20 *Geometrically speaking,* L_2*-metric is just the distance in the 2-dimensional space with the horizontal axis* x *as the lower end value* a^l *and the vertical axis* y *as the upper value* a^u *of an interval* $\overline{a} = \left[a^l, a^u\right]$. *In other words, the coordinate system* $(x,y) \triangleq (a^l, a^u)$. *The set of the points representing intervals will be half plane* $\{(x,y) | x \leq y,\ x,y \in \mathbb{R}\}$.

2.4.4 Bertoluzza metric in the interval space

Bertoluzza et al [1995] developed a metric in the context of fuzzy sets space. However, the similar distance for intervals can be stated as follows.

Definition 21 *Assume that* $\forall\overline{a} \in \mathcal{P}_{\mathbb{IV}}(\mathbb{R})$ *and the support function corresponding to* $\overline{a}$ *is de-*

noted as $s_{\overline{a}}(\cdot)$ *and* $\mathbb{S}^{d-1} = \{1, -1\}$, *furthermore, define a function*

$$\eta(\lambda, x) = \lambda \vartheta_{\mathbb{S}^{d-1}}(x) + (1-\lambda)(1 - \vartheta_{\mathbb{S}^{d-1}}(x))$$

where

$$\vartheta_{\mathbb{S}^{d-1}}(x) = \begin{cases} 1 & if \quad x = 1 \\ 0 & if \quad x = -1 \end{cases}$$

then we may define a norm

$$\begin{aligned} v_B(\overline{a}) &= \|\overline{a}\|_B \\ &= \sqrt{\int_0^1 \left[\int_{\mathbb{S}^{d-1}} \eta(\lambda, x) \left| s_{\overline{a}}(x) \right| \mu(dx) \right]^2 d\varpi(\lambda)} \end{aligned}$$

Definition 22 *For all* $\overline{a}, \overline{b} \in \mathcal{P}_{\mathbb{IV}}(\mathbb{R})$, *the distance between* $\overline{a}$*and* $\overline{b}$

$$\begin{aligned} d_B\left(\overline{a}, \overline{b}\right) &= v_B\left(\overline{a} - \overline{b}\right) = \left\|\overline{a} - \overline{b}\right\|_B \\ &= \sqrt{\int_0^1 \left[\int_{\mathbb{S}^{d-1}} \eta(\lambda, x) \left| s_{\overline{a}}(x) - s_{\overline{b}}(x) \right| \mu(dx) \right]^2 d\varpi(\lambda)} \end{aligned}$$

Example 23 *Let* $d\varpi(\lambda) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \lambda^{\alpha-1} (1-\lambda)^{\beta-1} d\lambda$, $\lambda \in (0,1)$, $\alpha, \beta > 0$ *(Beta density). Then*

$$\begin{aligned} v(\overline{a}) &= \|\overline{a}\|_B = \left\{ \int_0^1 \frac{1}{4} \left[\lambda a^l + (1-\lambda) a^u \right]^2 d\varpi(\lambda) \right\}^{\frac{1}{2}} \\ &= \left\{ \int_0^1 \frac{1}{4} \left[\lambda^2 (a^l)^2 + 2\lambda a^l a^u (1-\lambda) + (1-\lambda)^2 (a^u)^2 \right] d\varpi(\lambda) \right\}^{\frac{1}{2}} \end{aligned}$$

Now,

$$
\begin{aligned}
& \int_0^1 \lambda^2 (a^l)^2 d\varpi\left(\lambda\right) \\
&= (a^l)^2 \int_0^1 \lambda^2 \frac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)} \lambda^{\alpha-1}\left(1-\lambda\right)^{\beta-1} d\lambda \\
&= (a^l)^2 \frac{\Gamma\left(\alpha+2\right)\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha+2+\beta\right)\Gamma\left(\alpha\right)} \int_0^1 \frac{\Gamma\left(\alpha+2+\beta\right)}{\Gamma\left(\alpha+2\right)\Gamma\left(\beta\right)} \lambda^{(\alpha+2)-1}\left(1-\lambda\right)^{\beta-1} d\lambda \\
&= (a^l)^2 \frac{\Gamma\left(\alpha+2\right)\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha+2+\beta\right)\Gamma\left(\alpha\right)} = \frac{(a^l)^2\left(\alpha+1\right)\alpha}{\left(\alpha+\beta+1\right)\left(\alpha+\beta\right)}
\end{aligned}
$$

$$
\begin{aligned}
& \int_0^1 2a^l a^u \lambda\left(1-\lambda\right) d\varpi\left(\lambda\right) \\
&= 2a^l a^u \int_0^1 \lambda\left(1-\lambda\right) \frac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)} \lambda^{\alpha-1}\left(1-\lambda\right)^{\beta-1} d\lambda \\
&= 2a^l a^u \frac{\Gamma\left(\alpha+\beta\right)\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)\Gamma\left(\alpha+\beta+2\right)} \int_0^1 \frac{\Gamma\left(\alpha+1+\beta+1\right)}{\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)} \lambda^{(\alpha+1)-1}\left(1-\lambda\right)^{(\beta+1)-1} d\lambda \\
&= 2a^l a^u \frac{\Gamma\left(\alpha+\beta\right)\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)\Gamma\left(\alpha+\beta+2\right)} = \frac{2a^l a^u \alpha\beta}{\left(\alpha+\beta+1\right)\left(\alpha+\beta\right)}
\end{aligned}
$$

and finally

$$
\begin{aligned}
& \int_0^1 (a^u)^2\left(1-\lambda\right)^2 d\varpi\left(\lambda\right) \\
&= \int_0^1 (a^u)^2\left(1-\lambda\right)^2 \frac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)} \lambda^{\alpha-1}\left(1-\lambda\right)^{\beta-1} d\lambda \\
&= (a^u)^2 \frac{\Gamma\left(\alpha+\beta\right)\Gamma\left(\beta+2\right)}{\Gamma\left(\beta\right)\Gamma\left(\alpha+\beta+2\right)} \int_0^1 \frac{\Gamma\left(\alpha+\beta+2\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta+2\right)} \lambda^{\alpha-1}\left(1-\lambda\right)^{(\beta+2)-1} d\lambda \\
&= (a^u)^2 \frac{\Gamma\left(\alpha+\beta\right)\Gamma\left(\beta+2\right)}{\Gamma\left(\beta\right)\Gamma\left(\alpha+\beta+2\right)} = \frac{(a^u)^2\left(\beta+1\right)\beta}{\left(\alpha+\beta+1\right)\left(\alpha+\beta\right)}
\end{aligned}
$$

Hence

$$\begin{aligned} v(\overline{a}) &= \|\overline{a}\|_B = \left\{ \int_0^1 \frac{1}{4}\left[\lambda a^l + (1-\lambda) a^u\right]^2 d\varpi(\lambda) \right\}^{\frac{1}{2}} \\ &= \left\{ \frac{(\alpha+1)\alpha\left(a^l\right)^2 + 2\alpha\beta a^l b^l + (\beta+1)\beta(a^u)^2}{4(\alpha+\beta+1)(\alpha+\beta)} \right\}^{\frac{1}{2}} \end{aligned}$$

Theorem 24 *Assume that* $([0,1], \mathcal{B}([0,1]), \varpi)$ *is a probability space and* $\overline{a}$, $\overline{b}$, $\overline{c} \in \mathcal{P}_{\mathbb{IV}}(\mathbb{R})$. *The distance between two intervals,* $\overline{a} = [a^l, a^u]$ *and* $\overline{b} = [b^l, b^u]$.

$$d_B\left(\overline{a}, \overline{b}\right) \triangleq \sqrt{\int_0^1 \left[\int_{\mathbb{S}^{d-1}} \eta(\lambda, x)\left|s_{\overline{a}}(x) - s_{\overline{b}}(x)\right| \mu(dx)\right]^2 d\varpi(\lambda)}$$

is a metric, i.e., (ii) $d_B\left(\overline{a}, \overline{b}\right) = d_B\left(\overline{b}, \overline{a}\right)$*; (i)* $d_B(\overline{a}, \overline{a}) = 0$*; and (iii)*

$$d_B\left(\overline{a}, \overline{b}\right) \le d_B(\overline{a}, \overline{c}) + d_B\left(\overline{c}, \overline{b}\right).$$

Proof.

$$\begin{aligned} d_B\left(\overline{a}, \overline{b}\right) &\triangleq \sqrt{\int_0^1 \left[\int_{\mathbb{S}^{d-1}} \eta(\lambda, x)\left|s_{\overline{a}}(x) - s_{\overline{b}}(x)\right| \mu(dx)\right]^2 d\varpi(\lambda)} \\ &= \frac{1}{2}\sqrt{\int_0^1 \left[\lambda\left|a^l - b^l\right| + (1-\lambda)\left|a^u - b^u\right|\right]^2 d\varpi(\lambda)} \end{aligned}$$

Therefore, (i) and (ii) is obvious. Let us investigate whether (iii) holds. Notice that

$$\begin{aligned} & \left[\lambda\left|a^l - b^l\right| + (1-\lambda)\left|a^u - b^u\right|\right]^2 \\ = & \left\{\lambda\left|a^l - c^l + c^l - b^l\right| + (1-\lambda)\left|a^u - c^u + c^u - b^u\right|\right\}^2 \\ \le & \left\{\lambda\left[\left|a^l - c^l\right| + \left|c^l - b^l\right|\right] + (1-\lambda)\left[\left|a^u - c^u\right| + \left|c^u - b^u\right|\right]\right\}^2 \\ \le & \left[\lambda\left|a^l - c^l\right| + (1-\lambda)\left|c^l - b^l\right|\right]^2 + \left[\lambda\left|c^l - b^l\right| + (1-\lambda)\left|c^l - b^l\right|\right]^2 \end{aligned}$$

Hence,

$$\begin{aligned}
& \left[d_B\left(\overline{a},\overline{b}\right)\right]^2 \\
= & \int_0^1 \left[\lambda\left|a^l-b^l\right|+(1-\lambda)\left|a^u-b^u\right|\right]^2 d\varpi(\lambda) \\
\leq & \int_0^1 \left[\lambda\left|a^l-c^l\right|+(1-\lambda)\left|c^l-b^l\right|\right]^2 d\varpi(\lambda) \\
& +\int_0^1 \left[\lambda\left|c^l-b^l\right|+(1-\lambda)\left|c^l-b^l\right|\right]^2 d\varpi(\lambda) \\
= & \left[d_B\left(\overline{a},\overline{c}\right)\right]^2+\left[d_B\left(\overline{c},\overline{b}\right)\right]^2 \\
\leq & \left[d_B\left(\overline{a},\overline{c}\right)+d_B\left(\overline{c},\overline{b}\right)\right]^2
\end{aligned}$$

which leads to the triangle inequality

$$d_B\left(\overline{a},\overline{b}\right)\leq d_B\left(\overline{a},\overline{c}\right)+d_B\left(\overline{c},\overline{b}\right).$$

Therefore distance d_B is a metric. ■

Lemma 25 *Assume that* $d\varpi(\lambda)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\lambda^{\alpha-1}(1-\lambda)^{\beta-1}d\lambda$, $\lambda\in(0,1)$, $\alpha,\beta>0$. *Then*

$$\begin{aligned}
& d_B\left(\overline{a},\overline{b}\right) \\
= & \left\{\frac{(\alpha+1)\alpha\left|a^l-b^l\right|^2+2\alpha\beta\left|a^l-b^l\right|\left|a^u-b^u\right|+(\beta+1)\beta\left|a^u-b^u\right|^2}{4(\alpha+\beta+1)(\alpha+\beta)}\right\}^{\frac{1}{2}}.
\end{aligned}$$

Proof. Notice that

$$\begin{aligned}
& \left[\lambda\left|a^l-b^l\right|+(1-\lambda)\left|a^u-b^u\right|\right]^2 \\
= & \lambda^2\left|a^l-b^l\right|^2+2\lambda(1-\lambda)\left|a^l-b^l\right|\left|a^u-b^u\right|+(1-\lambda)^2\left|a^u-b^u\right|^2
\end{aligned}$$

hence

$$\begin{aligned}
&\int_0^1 \lambda^2 d\varpi(\lambda) \\
&= \int_0^1 \lambda^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \lambda^{\alpha-1}(1-\lambda)^{\beta-1} d\lambda \\
&= \frac{\Gamma(\alpha+2)\Gamma(\alpha+\beta)}{\Gamma(\alpha+2+\beta)\Gamma(\alpha)} \int_0^1 \frac{\Gamma(\alpha+2+\beta)}{\Gamma(\alpha+2)\Gamma(\beta)} \lambda^{(\alpha+2)-1}(1-\lambda)^{\beta-1} d\lambda \\
&= \frac{\Gamma(\alpha+2)\Gamma(\alpha+\beta)}{\Gamma(\alpha+2+\beta)\Gamma(\alpha)} = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \lambda(1-\lambda) d\varpi(\lambda) \\
&= \int_0^1 \lambda(1-\lambda) \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \lambda^{\alpha-1}(1-\lambda)^{\beta-1} d\lambda \\
&= \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+2)} \int_0^1 \frac{\Gamma(\alpha+1+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \lambda^{(\alpha+1)-1}(1-\lambda)^{(\beta+1)-1} d\lambda \\
&= \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+2)} = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)}
\end{aligned}$$

and finally

$$\begin{aligned}
&\int_0^1 (1-\lambda)^2 d\varpi(\lambda) \\
&= \int_0^1 (1-\lambda)^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \lambda^{\alpha-1}(1-\lambda)^{\beta-1} d\lambda \\
&= \frac{\Gamma(\alpha+\beta)\Gamma(\beta+2)}{\Gamma(\beta)\Gamma(\alpha+\beta+2)} \int_0^1 \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha)\Gamma(\beta+2)} \lambda^{\alpha-1}(1-\lambda)^{(\beta+2)-1} d\lambda \\
&= \frac{\Gamma(\alpha+\beta)\Gamma(\beta+2)}{\Gamma(\beta)\Gamma(\alpha+\beta+2)} = \frac{(\beta+1)\beta}{(\alpha+\beta+1)(\alpha+\beta)}
\end{aligned}$$

so that

$$
\begin{aligned}
& \left[d_B\left(\overline{a},\overline{b}\right)\right]^2 \\
= & \int_0^1 \left[\lambda\left|a^l-b^l\right|+(1-\lambda)\left|a^u-b^u\right|\right]^2 d\varpi(\lambda) \\
= & \frac{(\alpha+1)\,\alpha\left|a^l-b^l\right|^2+2\alpha\beta\left|a^l-b^l\right|\left|a^u-b^u\right|+(\beta+1)\,\beta\left|a^u-b^u\right|^2}{4(\alpha+\beta+1)(\alpha+\beta)}
\end{aligned}
$$

which gives the conclusion by square-rooting both sides. ■

Example 26 *Let* $\overline{a}_1=\left[a_1^l,a_1^u\right]=[1,2]$, $\overline{b}_1=\left[b_1^l,b_1^u\right]=[0,3]$ *and* $\overline{a}_2=\left[a_2^l,a_2^u\right]=[0,2]$, $\overline{b}_2=\left[b_2^l,b_2^u\right]=[1,3]$, *then* $\left|a_1^l-b_1^l\right|=|1-0|=1$. $\left|a_1^u-b_1^u\right|=|2-3|=1, \left|a_2^l-b_2^l\right|=|0-1|=1$, $\left|a_2^u-b_2^u\right|=|3-1|=2$. *It is obvious that the Hausdorff distance*

$$
\begin{aligned}
d_H\left(\overline{a}_1,\overline{b}_1\right) &= \max\left(\left|a_1^l-b_1^l\right|,\left|a_1^u-b_1^u\right|\right)=\max(1,1)=1 \\
d_H\left(\overline{a}_2,\overline{b}_2\right) &= \max\left(\left|a_2^l-b_2^l\right|,\left|a_2^u-b_2^u\right|\right)=\max(1,1)=1
\end{aligned}
$$

The L_2*-distance*

$$
\begin{aligned}
\delta_2\left(\overline{a}_1,\overline{b}_1\right) &= \sqrt{\frac{1}{2}\left(\left|a_1^l-b_1^l\right|^2+\left|a_1^u-b_1^u\right|^2\right)}=1 \\
\delta_2\left(\overline{a}_2,\overline{b}_2\right) &= \sqrt{\frac{1}{2}\left(\left|a_2^l-b_2^l\right|^2+\left|a_2^u-b_2^u\right|^2\right)}=1
\end{aligned}
$$

Assume the Beta density $d\varpi(\lambda)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\lambda^{\alpha-1}(1-\lambda)^{\beta-1}\,d\lambda$, $\lambda\in(0,1)$. $\alpha,\beta>0$. *Thus*

$$
\begin{aligned}
& d_G\left(\overline{a}_1,\overline{b}_1\right) \\
= & \left\{\frac{(\alpha+1)\,\alpha\left|a_1^l-b_1^l\right|^2+2\alpha\beta\left|a_1^l-b_1^l\right|\left|a_1^u-b_1^u\right|+(\beta+1)\,\beta\left|a_1^u-b_1^u\right|^2}{4(\alpha+\beta+1)(\alpha+\beta)}\right\}^{\frac{1}{2}} \\
= & \left\{\frac{(\alpha+1)\,\alpha+2\alpha\beta+(\beta+1)\,\beta}{4(\alpha+\beta+1)(\alpha+\beta)}\right\}^{\frac{1}{2}}=\frac{1}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
& d_G\left(\overline{a}_2, \overline{b}_2\right) \\
= & \left\{\frac{(\alpha+1)\,\alpha\left|a_2^l-b_2^l\right|^2+2\alpha\beta\left|a_2^l-b_2^l\right|\left|a_2^u-b_2^u\right|+(\beta+1)\,\beta\left|a_2^u-b_2^u\right|^2}{4\,(\alpha+\beta+1)\,(\alpha+\beta)}\right\}^{\frac{1}{2}} \\
= & \left\{\frac{(\alpha+1)\,\alpha+2\alpha\beta+(\beta+1)\,\beta}{4\,(\alpha+\beta+1)\,(\alpha+\beta)}\right\}^{\frac{1}{2}}=\frac{1}{2}
\end{aligned}
$$

2.4.5 A general metric d_G in interval space

In this subsection a new and general metric is proposed, denoted by d_G, which is an extension to the metric developed by Bertoluzza *et al.*

Definition 27 *Assume that $([0,1], \mathcal{B}([0,1]), \varpi)$ is a probability space. For all $\overline{a}, \overline{b} \in \mathcal{P}_{\mathbb{IV}}(\mathbb{R})$, the distance between $\overline{a}$ and $\overline{b}$*

$$d_G\left(\overline{a}, \overline{b}\right)=\int_0^1\left[\lambda\delta_2\left(\overline{a}, \overline{b}\right)+(1-\lambda)\left|m_{\overline{a}}-m_{\overline{b}}\right|\right] d\varpi(\lambda)$$

where L_2-distance

$$\delta_2\left(\overline{a}, \overline{b}\right)=\sqrt{\frac{1}{2}\times\left[\left|a^u-b^u\right|^2+\left|a^l-b^l\right|^2\right]}$$

and the mid-value of interval $\overline{x}=\left[x^l, x^u\right]$

$$m_{\overline{x}}=\frac{1}{2}\left(x^l+x^u\right)$$

Remark 28 *The distance between mid-values of intervals*

$$
\begin{aligned}
& \left|m_{\overline{a}}-m_{\overline{b}}\right| \\
= & \left|\frac{1}{2}\left(a^l+a^u\right)-\frac{1}{2}\left(b^l+b^u\right)\right| \\
= & \left|\frac{1}{2}\left[\left(a^l-b^l\right)+\left(a^u-b^u\right)\right]\right|
\end{aligned}
$$

Theorem 29 *Assume that $([0,1], \mathcal{B}([0,1]), \varpi)$ is a probability space and $\overline{a}$, $\overline{b}$, $\overline{c} \in \mathcal{P}_{\mathbb{IV}}(\mathbb{R})$. The generalized distance between two intervals, $\overline{a}=\left[a^l, a^u\right]$ and $\overline{b}=\left[b^l, b^u\right]$, $d_G\left(\overline{a}, \overline{b}\right)$ is a metric* *in the interval space.*

Proof. That Property (i) $d_G(\overline{a},\overline{a}) = 0$ and Property (ii) $d_G(\overline{a},\overline{b}) = d_G(\overline{b},\overline{a})$ hold is easy to verify. Now let us investigate whether (iii) the triangle inequality holds

$$d_G(\overline{a},\overline{b}) \leq d_G(\overline{a},\overline{c}) + d_G(\overline{c},\overline{b})$$

where $\overline{a},\overline{b},\overline{c} \in \mathcal{P}_{\mathrm{IV}}(\mathbb{R})$. Notice that

$$d_G(\overline{a},\overline{b}) = \int_0^1 \left[\lambda\delta_2(\overline{a},\overline{b}) + (1-\lambda)\left|m_{\overline{a}} - m_{\overline{b}}\right|\right] d\varpi(\lambda)$$

since δ_2 is a L-metric hence the triangle inequality holds

$$\delta_2(\overline{a},\overline{b}) \leq \delta_2(\overline{a},\overline{c}) + \delta_2(\overline{c},\overline{b}) \text{ for } \forall\overline{a},\overline{b},\overline{c} \in \mathcal{P}_{\mathrm{IV}}(\mathbb{R})$$

also,

$$\left|m_{\overline{a}} - m_{\overline{b}}\right| \leq \left|m_{\overline{a}} - m_{\overline{c}}\right| + \left|m_{\overline{c}} - m_{\overline{b}}\right|$$

therefore

$$\begin{aligned}
& \lambda\delta_2(\overline{a},\overline{b}) + (1-\lambda)\left|m_{\overline{a}} - m_{\overline{b}}\right| \\
\leq\ & \lambda\delta_2(\overline{a},\overline{c}) + (1-\lambda)\left|m_{\overline{a}} - m_{\overline{c}}\right| + \lambda\delta_2(\overline{c},\overline{b}) + (1-\lambda)\left|m_{\overline{c}} - m_{\overline{b}}\right|
\end{aligned}$$

which leads to

$$\begin{aligned}
& \int_0^1 \left[\lambda\delta_2(\overline{a},\overline{b}) + (1-\lambda)\left|m_{\overline{a}} - m_{\overline{b}}\right|\right] d\varpi(\lambda) \\
\leq & \int_0^1 \left[\lambda\delta_2(\overline{a},\overline{c}) + (1-\lambda)\left|m_{\overline{a}} - m_{\overline{c}}\right|\right] d\varpi(\lambda) \\
& + \int_0^1 \left[\lambda\delta_2(\overline{c},\overline{b}) + (1-\lambda)\left|m_{\overline{c}} - m_{\overline{b}}\right|\right] d\varpi(\lambda)
\end{aligned}$$

i.e.,

$$\delta_2\left(\overline{a},\overline{b}\right) \leq \delta_2\left(\overline{a},\overline{c}\right) + \delta_2\left(\overline{c},\overline{b}\right) \text{ for } \forall \overline{a},\overline{b},\overline{c} \in \mathcal{P}_{\nabla}\left(\mathbb{R}\right)$$

■

Lemma 30 *Assume that* $d\varpi\left(\lambda\right) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\lambda^{\alpha-1}\left(1-\lambda\right)^{\beta-1}d\lambda$, $\lambda \in (0,1)$, $\alpha,\beta > 0$. *Then*

$$\begin{aligned}
& d_G\left(\overline{a},\overline{b}\right) \\
= & \int_0^1 \left[\lambda\delta_2\left(\overline{a},\overline{b}\right) + (1-\lambda)\left|m_{\overline{a}_1} - m_{\overline{b}_1}\right|\right] d\varpi\left(\lambda\right) \\
= & \frac{\alpha}{\alpha+\beta}\delta_2\left(\overline{a},\overline{b}\right) + \frac{\beta}{\alpha+\beta}\left|m_{\overline{a}} - m_{\overline{b}}\right|.
\end{aligned}$$

Proof.

$$\begin{aligned}
& d_G\left(\overline{a},\overline{b}\right) \\
= & \int_0^1 \left[\lambda\delta_2\left(\overline{a},\overline{b}\right) + (1-\lambda)\left|m_{\overline{a}} - m_{\overline{b}}\right|\right] d\varpi\left(\lambda\right) \\
= & \int_0^1 \left[\lambda\delta_2\left(\overline{a},\overline{b}\right) + (1-\lambda)\left|m_{\overline{a}_1} - m_{\overline{b}_1}\right|\right] \frac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\lambda^{\alpha-1}\left(1-\lambda\right)^{\beta-1}d\lambda \\
= & \delta_2\left(\overline{a},\overline{b}\right)\int_0^1 \lambda\frac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\lambda^{\alpha-1}\left(1-\lambda\right)^{\beta-1}d\lambda \\
& + \left|m_{\overline{a}} - m_{\overline{b}}\right|\int_0^1 (1-\lambda)\frac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\lambda^{\alpha-1}\left(1-\lambda\right)^{\beta-1}d\lambda \\
= & \delta_2\left(\overline{a},\overline{b}\right)\frac{\Gamma\left(\alpha+1\right)\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha+\beta+1\right)\Gamma\left(\alpha\right)}\int_0^1 \frac{\Gamma\left(\alpha+\beta+1\right)}{\Gamma\left(\alpha+1\right)\Gamma\left(\beta\right)}\lambda^{(\alpha+1)-1}\left(1-\lambda\right)^{\beta-1}d\lambda \\
& + \left|m_{\overline{a}} - m_{\overline{b}}\right|\frac{\Gamma\left(\beta+1\right)\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha+\beta+1\right)\Gamma\left(\beta\right)}\int_0^1 \frac{\Gamma\left(\alpha+\beta+1\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta+1\right)}\lambda^{\alpha-1}\left(1-\lambda\right)^{(\beta+1)-1}d\lambda \\
= & \delta_2\left(\overline{a},\overline{b}\right)\frac{\Gamma\left(\alpha+1\right)\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha+\beta+1\right)\Gamma\left(\alpha\right)} + \left|m_{\overline{a}} - m_{\overline{b}}\right|\frac{\Gamma\left(\beta+1\right)\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha+\beta+1\right)\Gamma\left(\beta\right)} \\
= & \frac{\alpha}{\alpha+\beta}\delta_2\left(\overline{a},\overline{b}\right) + \frac{\beta}{\alpha+\beta}\left|m_{\overline{a}} - m_{\overline{b}}\right|.
\end{aligned}$$

■

Example 31 *Let* $\overline{a}_1 = \left[a_1^l, a_1^u\right] = [1,2]$, $m_{\overline{a}_1} = \frac{3}{2}$, $\overline{b}_1 = \left[b_1^l, b_1^u\right] = [0,3]$, $m_{\overline{b}_1} = \frac{3}{2}$ *and* $\overline{a}_2 = \left[a_2^l, a_2^u\right] = [0,2]$, $m_{\overline{a}_2} = 1$, $\overline{b}_2 = \left[b_2^l, b_2^u\right] = [1,3]$, $m_{\overline{b}_2} = 2$. *then* $\left|a_1^l - b_1^l\right| = |1-0| = 1$, $|a_1^u - b_1^u| = |2-3| = 1$ *and* $\left|m_{\overline{a}_1} - m_{\overline{b}_1}\right| = 0.\left|a_2^l - b_2^l\right| = |0-1| = 1$. $|a_2^u - b_2^u| = |3-2| = 1$ *and* $\left|m_{\overline{a}_2} - m_{\overline{b}_2}\right| = |1-2| = 1$.

$$\begin{aligned}
\delta_2\left(\overline{a}_1, \overline{b}_1\right) &= \sqrt{\frac{1}{2} \times \left[|a_1^u - b_1^u|^2 + \left|a_1^l - b_1^l\right|^2\right]} = 1 \\
\delta_2\left(\overline{a}_2, \overline{b}_2\right) &= \sqrt{\frac{1}{2} \times \left[|a_2^u - b_2^u|^2 + \left|a_2^l - b_2^l\right|^2\right]} = 1
\end{aligned}$$

Therefore, with the assumption that $d\varpi(\lambda) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\lambda^{\alpha-1}(1-\lambda)^{\beta-1}\, d\lambda$. $\lambda \in (0,1)$. $\alpha.\beta > 0$,

$$\begin{aligned}
d_G\left(\overline{a}_1, \overline{b}_1\right) &= \frac{\alpha}{\alpha+\beta}\delta_2\left(\overline{a}_1, \overline{b}_1\right) + \frac{\beta}{\alpha+\beta}\left|m_{\overline{a}_1} - m_{\overline{b}_1}\right| \\
&= \frac{\alpha}{\alpha+\beta} \times 1 + \frac{\beta}{\alpha+\beta} \times 0 \\
&= \frac{\alpha}{\alpha+\beta}
\end{aligned}$$

while

$$\begin{aligned}
d_G\left(\overline{a}_2, \overline{b}_2\right) &= \frac{\alpha}{\alpha+\beta}\delta_2\left(\overline{a}_2, \overline{b}_2\right) + \frac{\beta}{\alpha+\beta}\left|m_{\overline{a}_2} - m_{\overline{b}_2}\right| \\
&= \frac{\alpha}{\alpha+\beta} \times 1 + \frac{\beta}{\alpha+\beta} \times 1 \\
&= 1
\end{aligned}$$

Therefore, $d_G\left(\overline{a}_2, \overline{b}_2\right) > d_G\left(\overline{a}_1, \overline{b}_1\right)$ *which coincides with common sense.*

Chapter 3

Random Interval Variable and Random Interval Processes

3.1 Random interval variables and distributions

Definition 32 *(Random interval) Let $(\Omega, \mathcal{A}, P)$ be a probability space and*

$$\mathcal{P}_{\mathbb{IV}}(\mathbb{R}) \triangleq \{[a,b] | a,b \in \mathbb{R},\ a \leq b\}$$

the set of all the closed intervals on $\mathbb{R}$. The mapping from Ω to $\mathcal{P}_{\mathbb{IV}}(\mathbb{R})$

$$\begin{aligned} &\overline{\zeta} : \Omega \to \mathcal{P}_{\mathbb{IV}}(\mathbb{R}) \\ &\omega \to \overline{\zeta}(\omega) = \left[\zeta^l(\omega), \zeta^u(\omega)\right] \\ &\zeta^l(\omega) \leq \zeta^u(\omega), \forall \omega \in \Omega \end{aligned}$$

is said to be a random interval if $\zeta^l(\omega)$ and $\zeta^u(\omega)$ both are random variables on $(\Omega, \mathcal{A}, P)$.

Remark 33 *Here, the random interval concept involves the statistical ordering issue since both $\zeta^l(\omega)$ and $\zeta^u(\omega)$ are random variables on $(\Omega, \mathcal{A}, P)$. The meaning and implications of conditions $\zeta^l(\omega) \leq \zeta^u(\omega), \forall \omega \in \Omega$ should be discussed clearly. Starting from statistical ordering discussions, the definition of expectation of the interval-valued random variable $\overline{\zeta}(\omega)$, $E[\overline{\zeta}]$, and the definition of variance of the interval-valued random variable $\overline{\zeta}(\omega)$, $\mathrm{Var}\left[\overline{\zeta}\right]$ should be* *obtainable, and even the characteristic function* $\phi_{\overline{\zeta}}(\theta) = E\left[\exp\left(i\theta\overline{\zeta}\right)\right]$ *should be defined.*

3.2 Random interval stochastic process

Definition 34 *A random interval stochastic process* $\left\{\overline{\zeta}_t, t \in \mathbb{T}\right\}$ *on probability space* $(\Omega, \mathcal{F}, P)$ *is the mapping from* Ω *to* $\mathcal{P}_{\mathbb{IV}}(\mathbb{R})$

$$
\begin{aligned}
&\overline{\zeta} : \Omega \to \mathcal{P}_{\mathbb{IV}}(\mathbb{R}) \\
&\omega \to \overline{\zeta}_t(\omega) = \left[\zeta_t^l(\omega), \zeta_t^u(\omega)\right] \\
&\zeta_t^l(\omega) \leq \zeta_t^u(\omega), \forall\omega \in \Omega
\end{aligned}
$$

for $\forall t$ *uniformly on* $\mathbb{T}$, *where*

$$\mathcal{P}_{\mathbb{IV}}(\mathbb{R}) = \{[a, b] | a, b \in \mathbb{R},\ a \leq b\}$$

the class of all the closed intervals in $\mathbb{R}$.

Definition 35 $\left\{\overline{\zeta}_t, t \in \mathbb{T}\right\}$ *is said to be a Gaussian random interval stochastic process on probability space* $(\Omega, \mathcal{F}, P)$ *if and only if*

$$\zeta_t(\omega) \triangleq \left[\zeta_t^l(\omega), \zeta_t^u(\omega)\right]$$

and both $\left\{\zeta_t^l, t \in \mathbb{T}\right\}$ *and* $\{\zeta_t^u, t \in \mathbb{T}\}$ *are Gaussian (normal) processes.*

Definition 36 $\left\{\overline{\xi}_t, t \in \mathbb{T}\right\}$ *is said to be a lognormal random interval stochastic process on probability space* $(\Omega, \mathcal{F}, P)$ *if and only if*

$$\log \xi_t(\omega) \triangleq \left[\log \xi_t^l(\omega), \log \xi_t^u(\omega)\right]$$

and both $\left\{\log \xi_t^l, t \in \mathbb{T}\right\}$ *and* $\{\log \xi_t^u, t \in \mathbb{T}\}$ *are Gaussian (normal) processes.*

3.3 Variance-covariance

The Aumann-expectation $\mathbb{E}^{(A)}$ of a random compact convex set is defined as

$$\mathbb{E}^{(A)}X \triangleq \{E\xi : \xi \text{ is a selector of } X,\ E\|\xi\| < \infty\},$$

where a random vector ξ is a selector of X if $\xi \in X$ with probability one. The Aumann-expectation $\mathbb{E}^{(A)}$ is alternatively defined by the Bochner expectation of the support function of X

$$s_{E[X]}(u) = E[s_X(u)], \forall u \in \mathbb{S}^{d-1}.$$

Example 37 *Let $\overline{X} \triangleq [X^l, X^u]$ be a random interval. then its Bochner expectation is*

$$s_{E[\overline{X}]}(u) = E[s_{\overline{X}}(u)], \forall u \in \mathbb{S}^{d-1}.$$

Notice that $\mathbb{S}^{d-1} = \{+1, -1\}$,

$$\begin{aligned} & s_{E[\overline{X}]}(u) \\ = \ & E[s_{\overline{X}}(u)] \\ = \ & \begin{cases} E[s_{\overline{X}}(+1)] = E\left[X^u\right] & \text{if} \quad u = +1 \\ E[s_{\overline{X}}(-1)] = E\left[-X^l\right] & \text{if} \quad u = -1 \end{cases} \end{aligned}$$

3.4 Semi-scalar product and cross-covariance

Definition 38 *For $\forall \overline{a}, \overline{b} \in \mathcal{P}_{\mathbb{IV}}(\mathbb{R})$, function*

$$\langle s_{\overline{a}}, s_{\overline{b}} \rangle = \langle \overline{a}, \overline{b} \rangle = d \int_{\mathbb{S}^{d-1}} s_{\overline{a}}(u)\, s_{\overline{b}}(u)\, \mu(du)$$

is called the semi-scalar product between intervals $\overline{a}$ and $\overline{b}$.

Lemma 39 *Semi-scalar product is only positive homogeneous, i.e., for $\forall \beta > 0$,*

$$\langle \beta \overline{a}, \overline{b} \rangle = \beta \langle \overline{a}, \overline{b} \rangle.$$

Proof. For all $\beta > 0$, in accordance with interval arithmetic, $\beta\overline{a} = \beta\left[a^l, a^u\right] = \left[\beta a^l, \beta a^u\right]$ and also, since $\beta > 0$,

$$s_{\beta\overline{a}} = \beta s_{\overline{a}}$$

then

$$\begin{aligned}
& \left\langle \beta s_{\overline{a}}, s_{\overline{b}} \right\rangle \\
= & \left\langle s_{\beta\overline{a}}, s_{\overline{b}} \right\rangle \\
= & \ d\int_{\mathbb{S}^{d-1}} s_{\beta\overline{a}}(u)\, s_{\overline{b}}(u)\, \mu(du) \\
= & \ d\int_{\mathbb{S}^{d-1}} \beta s_{\overline{a}}(u)\, s_{\overline{b}}(u)\, \mu(du) \\
= & \ \beta\left(d\int_{\mathbb{S}^{d-1}} s_{\overline{a}}(u)\, s_{\overline{b}}(u)\, \mu(du)\right) \\
= & \ \beta\left\langle s_{\overline{a}}, s_{\overline{b}} \right\rangle
\end{aligned}$$

■

Remark 40 *Assume that* $\forall\beta < 0$,

$$\beta\overline{a} = \left[\beta a^u, \beta a^l\right]$$

which leads

$$\left\langle \beta\overline{a}, \overline{b} \right\rangle = \beta\left(a^u b^l + a^l b^u\right)$$

while

$$\left\langle \overline{a}, \overline{b} \right\rangle = \left(a^l b^l + a^u b^u\right)$$

therefore

$$\left\langle \beta\overline{a}, \overline{b} \right\rangle \neq \beta\left\langle \overline{a}, \overline{b} \right\rangle \ \textit{for}\ \forall\beta < 0.$$

Definition 41 *Let* $\overline{X}$ *and* $\overline{Y}$ *be two interval random variables, then*

$$cov\left(\overline{X}, \overline{Y}\right) = E\left\langle s_{\overline{X}} - s_{E\left[\overline{X}\right]}, s_{\overline{Y}} - s_{E\left[\overline{Y}\right]} \right\rangle$$

is called a covariance between two interval random variables $\overline{X}$ *and* $\overline{Y}$*. Also,*

$$var\left(\overline{X}\right) = cov\left(\overline{X}, \overline{X}\right)$$

is called a variance of random interval $\overline{X}$*. Furthermore,*

$$cor\left(\overline{X}, \overline{Y}\right) = \frac{cov\left(\overline{X}, \overline{Y}\right)}{\sqrt{var\left(\overline{X}\right) var\left(\overline{Y}\right)}}$$

is called a correlation between two interval random variables $\overline{X}$ *and* $\overline{Y}$*.*

Definition 42 *Assume that* $\left\{\overline{X}_t, t \in \mathbb{T}\right\}$ *be an interval stochastic process, then* $\forall \tau_1, \tau_2 \in \mathbb{T}$*,* $cor\left(\overline{X}_{\tau_1}, \overline{X}_{\tau_2}\right)$ *is called an auto-covariance and furthermore*

$$cor\left(\overline{X}_{\tau_1}, \overline{X}_{\tau_2}\right) = \frac{cov\left(\overline{X}_{\tau_1}, \overline{X}_{\tau_2}\right)}{\sqrt{var\left(\overline{X}_{\tau_1}\right) var\left(\overline{X}_{\tau_2}\right)}}$$

is called an auto-correlation function of $\overline{X}_t$ *between* τ_1 *and* τ_2*.*

Assume that a discrete-time indexed interval stochastic process $\left\{\overline{X}_n | \overline{X}_n \triangleq \left[X_t^l, X_t^u\right], n \in \mathbb{N}\right\}$ is defined on $(\Omega, \mathcal{A}, P)$. Also, assume that $E\left[\overline{X}\right]$ is the Bochner expectation

$$\mathbb{E}^B\left[\overline{X}\right] = E\left[s_{\overline{X}}(\cdot)\right].$$

Then,

$$\begin{aligned}
& cov\left(\overline{X}_n, \overline{X}_{n+k}\right) \\
= \ & E\left\langle s_{\overline{X}_n} - s_{E\left[\overline{X}_n\right]}, s_{\overline{X}_{n+k}} - s_{E\left[\overline{X}_{n+k}\right]}\right\rangle \\
= \ & E\left\{d\int_{\mathbb{S}^{d-1}}\left(s_{\overline{X}_n} - s_{E\left[\overline{X}_n\right]}\right)\left(s_{\overline{X}_{n+k}} - s_{E\left[\overline{X}_{n+k}\right]}\right)\mu(du)\right\} \\
= \ & \frac{1}{2}E\left[\sum_{u\in\{1,-1\}}\left(s_{\overline{X}_n}(u) - s_{E\left[\overline{X}_n\right]}(u)\right)\left(s_{\overline{X}_{n+k}}(u) - s_{E\left[\overline{X}_{n+k}\right]}(u)\right)\right] \\
= \ & \frac{1}{2}E\left[\left(s_{\overline{X}_n}(+1) - s_{E\left[\overline{X}_n\right]}(+1)\right)\left(s_{\overline{X}_{n+k}}(+1) - s_{E\left[\overline{X}_{n-k}\right]}(+1)\right)\right] \\
& +\frac{1}{2}E\left[\left(s_{\overline{X}_n}(-1) - s_{E\left[\overline{X}_n\right]}(-1)\right)\left(s_{\overline{X}_{n+k}}(-1) - s_{E\left[\overline{X}_{n+k}\right]}(-1)\right)\right] \\
= \ & \frac{1}{2}E\left[\left(X_n^u - E\left[X_n^u\right]\right)\left(X_{n+k}^u - E\left[X_{n+k}^u\right]\right) + \left(X_n^l - E\left[X_n^l\right]\right)\left(X_{n+k}^l - E\left[X_{n+k}^l\right]\right)\right].
\end{aligned}$$

Assuming that

$$E\left[X_n^l\right] = \mu_n^l \text{ and } E\left[X_n^u\right] = \mu_n^u ,$$

then

$$\begin{aligned}
& cov\left(\overline{X}_n, \overline{X}_{n+k}\right) \\
= \ & \frac{1}{2}E\left[\left(X_n^u - \mu_n^u\right)\left(X_{n+k}^u - \mu_{n+k}^u\right) + \left(X_n^l - \mu_n^l\right)\left(X_{n+k}^l - \mu_{n+k}^l\right)\right] \\
= \ & \frac{1}{2}\left[cov\left(X_n^u, X_{n+k}^u\right) + cov\left(X_n^l, X_{n+k}^l\right)\right].
\end{aligned}$$

In the same spirit,

$$\begin{aligned}
var\left(\overline{X}_n\right) &= \frac{1}{2}\left[var\left[X_n^l\right] + var\left[X_n^u\right]\right] \\
&= \frac{\left(\sigma_n^l\right)^2 + \left(\sigma_n^u\right)^2}{2}
\end{aligned}$$

Finally,

$$cor\left(\overline{X}_n, \overline{X}_{n+k}\right) = \frac{cov\left(X_n^u, X_{n+k}^u\right) + cov\left(X_n^l, X_{n+k}^l\right)}{\sqrt{\left(\left(\sigma_n^l\right)^2 + \left(\sigma_n^u\right)^2\right)\left(\left(\sigma_{n+k}^l\right)^2 + \left(\sigma_{n+k}^u\right)^2\right)}}.$$

3.5 The Fréchet-principle

It is a well-known fact that for a random variable Z defined on the probability space $(\Omega, \mathcal{A}, P)$, the expectation $\mathbb{E}[Z]$ minimizes $\mathbb{E}\left[(Z-a)^2\right]$ and

$$var[Z] = \mathbb{E}\left[(Z - \mathbb{E}[Z])^2\right].$$

Fréchet proposed the expectation $\mathbb{E}^{\delta}[Z]$ for a random variable Z in a metric space (M, δ) as a solution of the problem

$$\mathbb{E}\left\{\left[\delta\left(Z, \mathbb{E}^{\delta}[Z]\right)\right]^2\right\} = \inf_{a \in M} \mathbb{E}\left\{[\delta(Z, a)]^2\right\}$$

It should be emphasised that $\mathbb{E}^{\delta}[Z]$ is not necessarily unique. Also, it is clear that $\mathbb{E}\left\{[\delta(Z, a)]^2\right\}$ is merely the expectation of random variable $[\delta(Z, a)]^2$ (which will be denoted as $\delta^2(Z, a)$ from now on). Accordingly, the generalised variance based on the Fréchet-principle is

$$var^{\delta}[Z] = \mathbb{E}\left[\delta^2\left(Z, \mathbb{E}^{\delta}[Z]\right)\right].$$

Theorem 43 *The Aumann-expectation $\mathbb{E}^{(A)}$ is a Fréchet-expectation with respect to metric δ_2. See Körner [1995] for proof.*

Chapter 4

A Review of Classical AR(p) Time-Series Analysis

4.1 Stationarity Assumption

In finance it is the asset or stock return that is usually the subject of time series analysis, and therefore in anticipation of this context the time series will now be denoted by $\{r_t\}$ instead of using the more general notation $\{X_t\}$. For the review that follows see Troskie et al [2003].

Definition 44 *Strict Stationarity. A stochastic process $\{r_t\}$ is said to be strictly stationary if the joint distribution of*

$$(r_{t_1}, \ldots, r_{t_k})$$

is identical to that of

$$(r_{t_1+t}, \ldots, r_{t_k+t})$$

for all t. Thus the joint distribution of $(r_{t_1}, \ldots, r_{t_k})$ is invariant under time shift.

Definition 45 *Weak Stationarity. A time series $\{r_t\}$ is said to be weakly stationary if both the mean of r_t,*

$$E(r_t) = \mu$$

and the covariances between r_t and r_{t-l}

$$cov(r_t, r_{t-l}) = \gamma_l$$

are time invariant and finite for all l.

In practice, weak stationarity implies that if we observe a time series $\{r_t, t = 1, \ldots, T\}$ then the time plot of the data would show that the r_t values fluctuate with constant variation around a constant level.

In the time plots of the log returns and simple returns of Absa and the Anglo daily time series there is a clear indication of mean stationarity, but can we say the same of the variances and covariances.

Explicitly in the condition of weak stationarity, we assume that the first two moments of r_t are finite. If r_t is strictly stationary and its first two moments are finite then r_t is also weakly stationary. The converse is not true in general. However, if the time series is normally distributed, then weak stationarity is equivalent to strict stationarity. In this study we will be mainly concerned with weakly stationary series.

The covariance $cov(r_t, r_{t-l}) = \gamma_l$ is called the lag-l autocovariance of r_t. It has two important properties:

(i) $\gamma_0 = var\,(r_t)$;

(ii) $\gamma_l = \gamma_{-l}$.

In the finance literature, it is common to assume that the asset or stock return series is weakly stationary. This assumption is often checked empirically. Figures 4-1, ?? and 4-3 show the graphs of log-returns of Absa, Anglo share prices and JSE-Over index

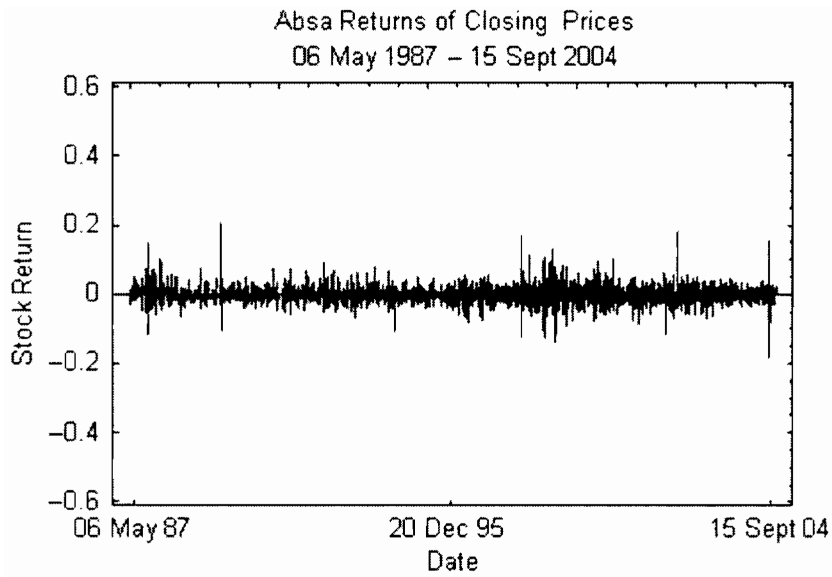

Figure 4-1: Plot of observed Absa Daily Log-Returns

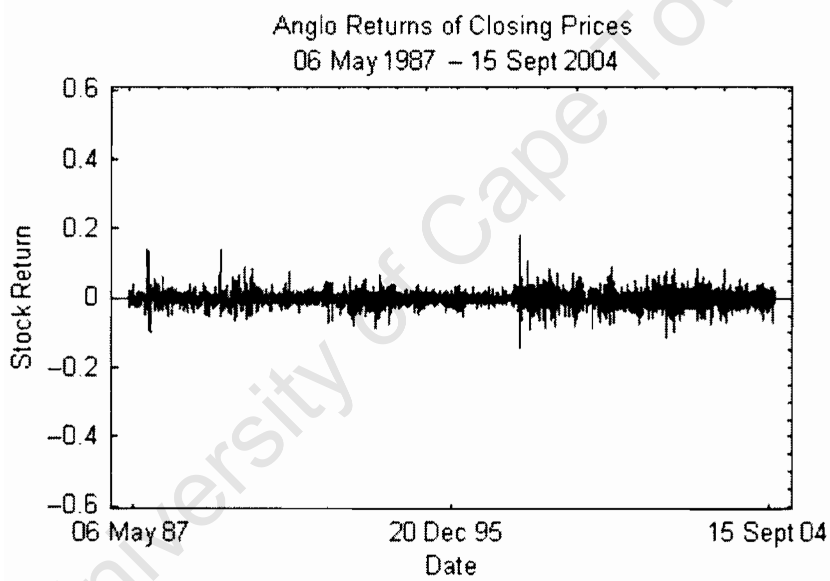

Figure 4-2: Plot of observed Anglo Daily Log-Returns

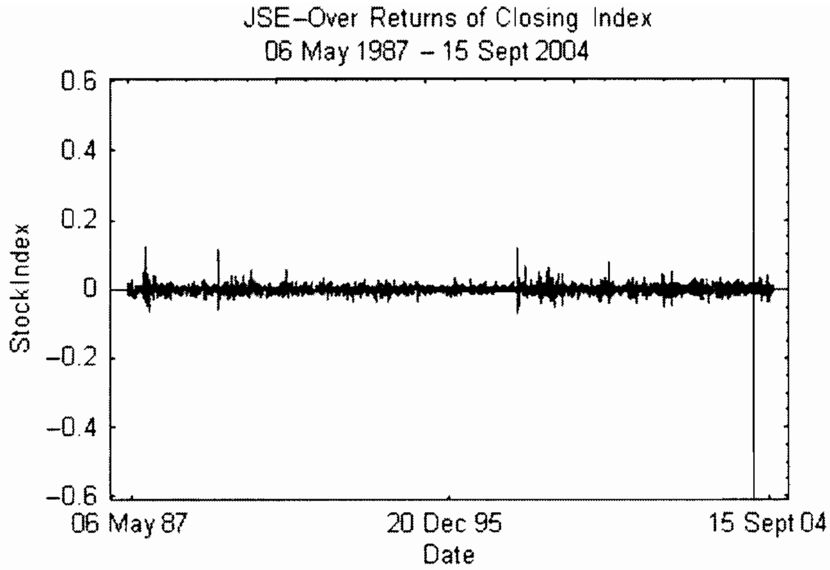

Figure 4-3: Plot of observed JSE-Over Daily Log-Returns

4.1.1 Autocorrelation

The correlation between two random variables X and Y is defined as

$$\begin{aligned}\rho_{XY} &= \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}} \\ &= \frac{E(X-E(X))(Y-E(Y))}{\sqrt{E(X-E(X))^2E(Y-E(Y))^2}}\end{aligned}$$

and if a sample $(x_i, y_i,\ i = 1,\ldots,T)$ is available then a consistent estimator of ρ_{XY} is

$$\hat{\rho}_{XY} = \frac{\sum_{i=1}^{T}(x_i-\bar{x})(y_i-\bar{y})}{\{\sum_{i=1}^{T}(x_i-\bar{x})^2\sum_{i=1}^{T}(y_i-\bar{y})^2\}^{1/2}}.$$

Consider now a weakly stationary return series r_t. For a time series $\{r_t\}_{t=l+1}^{T}$ with $l \in \mathbb{Z}^+$, the linear dependence between r_t and r_{t-l} is now of interest. The correlation coefficient between r_t and r_{t-l} is defined as

$$\rho_l = \frac{cov(r_t, r_{t-l})}{\sqrt{var(r_t)var(r_{t-l})}} = \frac{cov(r_t, r_{t-l})}{var(r_t)} = \frac{\gamma_l}{\gamma_0}$$

and is called the lag-l autocorrelation. From the definition, it is noticed that

$$\begin{aligned} \rho_0 &= 1 \\ \rho_l &= \rho_{-l} \\ -1 &\leq \rho_l \leq 1. \end{aligned}$$

A weakly stationary series is not correlated if and only if $\rho_l = 0$ for all $l > 0$.

For a given sample of returns $\{r_t\}_{t=1}^{T}$ the lag-1 sample autocorrelation of r_t is

$$\hat{\rho}_1 = \frac{\sum_{t=2}^{T}(r_t - \bar{r})(r_{t-1} - \bar{r})}{\sum_{t=1}^{T}(r_t - \bar{r})^2}$$

Under some general conditions $\hat{\rho}_1$ is a consistent estimator of ρ_1. If $\{r_t\}$ is an independent and identically distributed ($i.i.d.$) sequence and $E(r_t)^2 < \infty$ then

$$\hat{\rho}_1 \overset{d}{\sim} N(0, 1/T)$$

which can be used to test the hypotheses

$$H_0 : \rho_1 = 0 \text{ vs } H_1 : \rho_1 \neq 0.$$

The lag-l sample autocorrelation is defined as

$$\hat{\rho}_l = \frac{\sum_{t=l+1}^{T}(r_t - \bar{r})(r_{t-l} - \bar{r})}{\sum_{t=1}^{T}(r_t - \bar{r})^2}, \; 0 \leq l < T - 1.$$

Under the same conditions stated above

$$\hat{\rho}_l \overset{d}{\sim} N(0, 1/T)$$

for any fixed positive integer l.

More generally, if $\{r_t\}_{t=1}^T$ is a weakly stationary time series satisfying

$$r_t = \mu + \sum_{i=0}^{k} \psi_j e_{t-i} \text{ , where } \psi_0 = 1$$

and $\{e_t\}$ is a Gaussian white noise series (i.e. $e_t \sim N(0,1)$) then

$$\hat{\rho}_l \overset{d}{\sim} N(0, (1 + 2\sum_{i=0}^{k} \rho_i^2)/T) \text{ for } l > k.$$

The above formula is referred to as *Bartlett's formula.*

In finite samples $\hat{\rho}_l$ is a biased estimator of ρ_l. The bias is of the order $1/T$, which can be substantial if T is small. In most financial calculations T is relatively large so that the bias is not serious.

4.2 Portmanteau Test

Financial applications often require to test jointly that several autocorrelations of r_t are zero. Box and Pierce [1970] proposed the Portmanteau statistic

$$Q^*(m) = T \sum_{l=1}^{m} \hat{\rho}_l^2$$

as a test for

$$\begin{aligned} H_0 &: \rho_1 = \cdots = \rho_m = 0 \\ & \quad vs \\ H_a &: \rho_i \neq 0 \text{ for some } i \in \{1, \ldots, m\}. \end{aligned}$$

Under the assumption that $\{r_t\}$ is an *i.i.d.* sequence with specific moment conditions

$$Q^*(m) \overset{d}{\sim} \chi_m^2.$$

Degrees of freedom $(m-1)$	8	9	10	11	12
Absa: $Q(m)$	7.7190	12.6817	13.1224	14.8638	15.2802
Anglo: $Q(m)$	8.6465	8.7376	10.6528	11.3011	11.4364
JSE-Over: $Q(m)$	15.7126	15.7504	15.7618	15.7699	16.5657
$\chi^2_{0.05}(m-1)$	15.5073	16.9190	18.3070	19.6751	21.0261

Table 4.1: Modified Box-Pierce statistic for Monthly Log Returns

Ljung and Box [1978] modify the $Q^*(m)$ statistic as follows

$$Q(m) = T(T+2)\sum_{l=1}^{m} \hat{\rho}_l^2/(T-l).$$

This increases the power in finite samples. Simulation studies suggest that the choice of $m \approx \log(T)$ provides better power performance.

The function $\{\hat{\rho}_1, \hat{\rho}_2, \ldots \hat{\rho}_t, \ldots\}$ is called the *sample autocorrelation function* (ACF) of the return r_t. It plays an important role in linear time series analysis. The sample ACF captures the dynamic time and stochastic structure of the time series process. In the graphs we give the sample autocorrelation $ACF's$ for the monthly log returns of Absa, Anglo and the JSE-Over Index. The sample $ACFs$ of the three series as displayed in Figures 4-4, 4-5 and 4-6 suggest that the autocorrelations of the monthly log returns are very close to zero and not statistically significant (outside the 5% limits drawn on the figures). None of the Ljung-Box $Q(m)$ statistics are significant as given in Table 4.1.

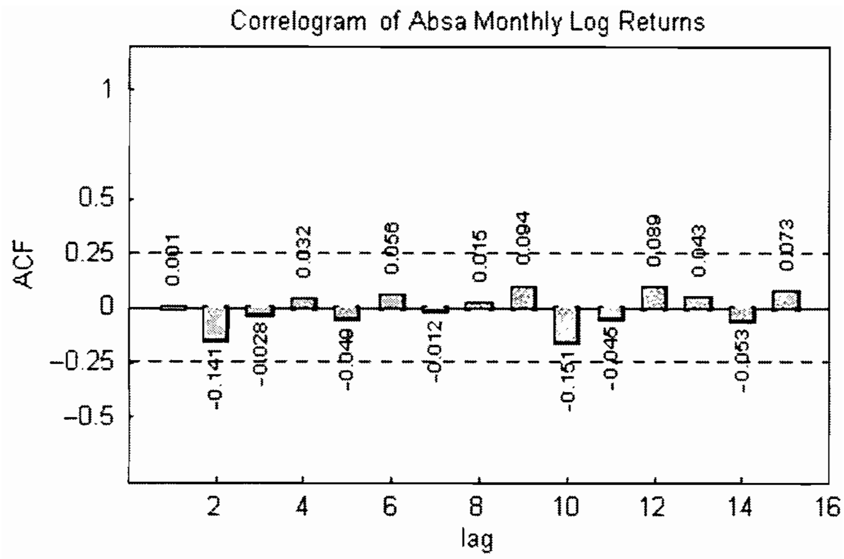

Figure 4-4: ACF of Absa Monthly Log-Returns on closing prices

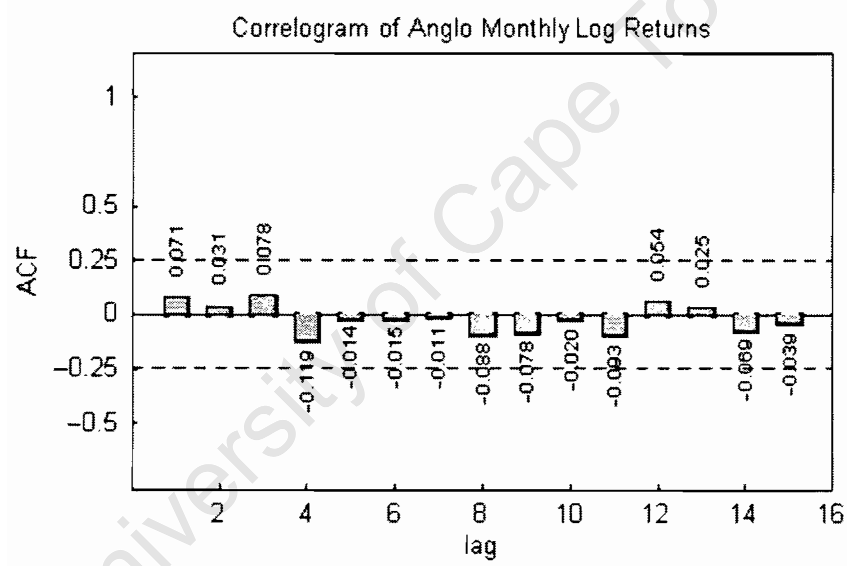

Figure 4-5: ACF of Anglo Monthly Log-Returns on closing prices

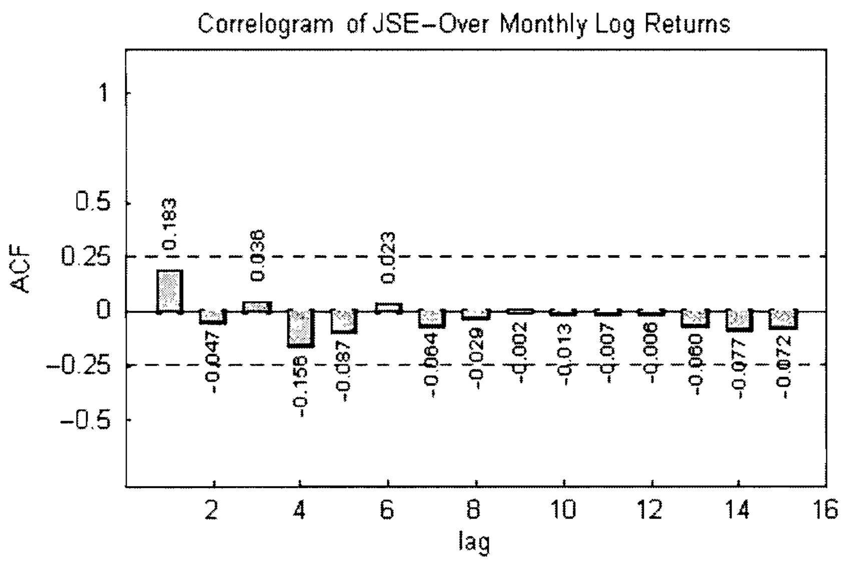

Figure 4-6: ACF of JSE-Over Monthly Log-Returns on closing index levels

4.3 AR(p) processes

AR(p), autoregressive process with order p, is a very useful stochastic process for representing a particular event occurrence sequence. An AR(p) process or time-series can be expressed by

$$\Phi_p(B)\widetilde{y}_t = \varepsilon_t$$

where $\widetilde{y}_t = y_t - \mu$ and B is the backshift operator such that $By_t = y_{t-1}$ and a (p-step) transfer function is defined as

$$\Phi_p(B) = 1 - \phi_1 B - \varphi_2 B^2 - \cdots - \varphi_p B^p.$$

The error process $\{\varepsilon_t,\ t \in \mathbb{N}\}$ is in general assumed to be a weak stationary process with

$$E[\varepsilon_t] = 0,\ E\left[\varepsilon_t^2\right] = \sigma_\varepsilon^2,$$

and the autocovariance of the error process $cov\left[\varepsilon_t, \varepsilon_{t+h}\right] = f(h)$ is independent of t. It is obvious that AR(p) model contains $p+2$ unknown parameters $\mu,\ \phi_1,\ \varphi_2,\ \cdots,\ \phi_p,\ \sigma_\varepsilon^2$ to be estimated from data.

In time series analysis, the typical concepts involved are to be reviewed. For a weakly stationary process the autocovariance at lag k is

$$\gamma_k = cov\left(y_t, y_{t+k}\right) = E\left[\left(y_t - \mu\right)\left(y_{t+k} - \mu\right)\right]$$

and the autocorrelation at lag k is

$$\rho_k = \frac{cov\left(y_t, y_{t+k}\right)}{\sqrt{var\left(y_t\right) var\left(y_{t+k}\right)}} = \frac{\gamma_k}{\sigma_y^2} = \frac{\gamma_k}{\gamma_0}$$

since for a stationary process

$$\gamma_0 = \sigma_y^2$$

which implies $\rho_0 = 1$. Correspondingly, a sample autocovariance estimate and sample autocorrelation estimate can be constructed as

$$\widehat{\gamma}_k = \frac{1}{N} \sum_{t=1}^{N-k} \left(y_t - \overline{y}\right)\left(y_{t+k} - \overline{y}\right)$$

and

$$\widehat{\rho}_t = \frac{\widehat{\gamma}_k}{\widehat{\gamma}_0}$$

respectively. Notice that

$$\overline{y} = \frac{1}{N} \sum_{t=1}^{N} y_t$$

and

$$\widehat{\gamma}_0 = \frac{1}{N} \sum_{t=1}^{N} \left(y_t - \overline{y}\right)^2$$

are calculated in the usual way.

4.3.1 Identifying $AR(p)$ Models in Practice

In applications the order p of an $AR(p)$ process is not known. It must be specified empirically. This task is referred to as the *order determination* of an $AR(p)$ process and has been extensively studied in the time series literature. Two general approaches are available. The first approach is to use the partial autocorrelation function and the second approach uses the information criterion function.

4.3.2 The $AR(p)$ Process and Regression Analysis

Assuming that our series $(r_1, \ldots, r_T)$ is indeed generated by an AR process, we need to find its order p and the values of the parameters $\phi_1, \ldots, \phi_p$ to describe the process. If we know the order p then we can use regression analysis to estimate the parameters $\phi_1, \ldots, \phi_p$ since the equation

$$r_t = \phi_1 + \phi_1 r_{t-1} + \ldots + \phi_p r_{t-p} + e_t$$

has precisely the form of a linear statistical regression model.

Note, however that the regressors (explanatory variables) $(r_{t-1}, \ldots, r_{t-p})$ are stochastic variables. If the e_t are white noise, or even Gaussian white noise, then an individual e_t represents random shock, which is added to the process at time t, and is independent of random variables at previous time points. Hence the regressors in a particular equation are independent of the error term. Thus we may estimate

$$\boldsymbol{\phi}_p = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix}$$

by the least squares (LS) method.

Replacing the random variables by the observed values we obtain

$$\begin{aligned} r_{p+1} &= \phi_0 + \phi_1 r_p + \ldots + \phi_p r_1 + e_{p+1} \\ r_{p+2} &= \phi_0 + \phi_1 r_{p+1} + \ldots + \phi_p r_2 + e_{p+2} \\ &\vdots \\ r_T &= \phi_0 + \phi r_{T-1} + \ldots + \phi_p r_{T-p} + e_T \end{aligned}$$

or in matrix notation

$$\mathbf{y}_p = \mathbf{X}_p \boldsymbol{\phi}_p + \mathbf{e} \tag{1}$$

where

$$\mathbf{r}_p = \begin{pmatrix} r_{p+1} \\ r_{p+2} \\ \vdots \\ r_T \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} e_{p+1} \\ e_{p+2} \\ \vdots \\ e_T \end{pmatrix} \tag{2}$$

and

$$\mathbf{X}_p = \begin{pmatrix} 1 & r_p & r_{p-1} & \cdots & r_1 \\ 1 & r_{p+1} & r_p & \cdots & r_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{T-1} & r_{T-2} & \cdots & r_{t-p} \end{pmatrix}.$$

If the e_t and thus the r_t are normally distributed, the LS estimator of $\boldsymbol{\phi}$, with estimate

$$\hat{\boldsymbol{\phi}}_p = (\mathbf{X}_p'\mathbf{X}_p)^{-1}\mathbf{X}_p'\mathbf{r}_p$$

is consistent and asymptotically normally distributed. That is,

$$\sqrt{T}(\hat{\boldsymbol{\phi}}_p - \boldsymbol{\phi}_p) \rightarrow^d N(\mathbf{0}, \boldsymbol{\Sigma})$$

with the variance - covariance matrix

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma^2/T & 0 \\ 0 & \mathbf{\Sigma}_{\hat{\boldsymbol{\phi}}_p} \end{pmatrix}$$

and

$$\mathbf{\Sigma}_{\hat{\boldsymbol{\phi}}_p} = \sigma^2 \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{p-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{p-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{p-1} & \gamma_{p-2} & \gamma_{p-3} & \cdots & \gamma_0 \end{bmatrix}.$$

The variance - covariance matrix Σ can be consistently estimated by

$$\hat{\mathbf{\Sigma}} = \hat{\sigma}^2(\mathbf{X}_p'\mathbf{X}_p)^{-1}$$

where

$$\hat{\sigma}^2 = \frac{(\mathbf{r}_p - \mathbf{X}_p\hat{\boldsymbol{\phi}}_p)'(\mathbf{r}_p - \mathbf{X}_p\hat{\boldsymbol{\phi}}_p)}{T-(2p+1)},$$

is the LS estimate of the variance σ^2 of the white noise process e_t. Note that we use $T-(2p+1)$ rather than $T-(p+1)$ since there are only $T-p$ complete observations with which to estimate the $(p+1)$ parameters in $\boldsymbol{\phi}$.

Note that ϕ_0 is not the mean of the process r_t. Rather

$$\mu = \frac{\phi_0}{1-\phi_1-\cdots-\phi_p}.$$

Thus two possible estimates of the process mean μ are

$$\bar{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$$

and

$$\hat{\mu} = \frac{\hat{\phi}_0}{1-\hat{\phi}_1-\cdots-\hat{\phi}_p}$$

where $\hat{\phi}_0, \hat{\phi}_1, \ldots, \hat{\phi}_p$ are the least squares estimates. Under general conditions (e.g. the r_t

are normally distributed) the corresponding estimates are consistent and have identical asymptotic normal distributions. For instance

$$\sqrt{T}(\hat{\mu} - \mu) \rightarrow^d N[0, \sigma^2(1 - \phi_1 - \cdots - \phi_p)^{-2}].$$

Furthermore the estimators $\bar{r}$ and $\hat{\mu}$ are asymptotically independent of the estimators $\hat{\phi}_1, \ldots, \hat{\phi}_p$. This property is one reason why the estimation of ϕ_p is often discussed in terms of zero mean processes. In practice the sample mean may be subtracted from the data prior to estimating the other parameters. In this case the asymptotic distribution theory is not affected.

4.3.3 Estimation

For a specified $AR(p)$ model the conditional least squares method starting with the $(p+1)th$ observation as discussed above is often used to estimate the parameters.

Let

$$r_t = \phi_0 + \phi_1 r_{t-1} + \ldots + \phi_p r_{t-p} + e_t, t = p+1, \ldots, T \qquad (3)$$

then the estimates are obtained by minimizing

$$\sum_{t=p+1}^{T} e_t^2$$

giving the least squares estimates $(\hat{\phi}_0, \hat{\phi}_1, \ldots, \hat{\phi}_p)$ (see also (1) and (2)).

The fitted model is

$$\hat{r}_t = \hat{\phi}_0 + \hat{\phi}_1 r_{t-1} + \ldots + \hat{\phi}_p r_{t-p}, \qquad (4)$$

and the estimated residual

$$\hat{e}_t = r_t - \hat{r}_t.$$

The series $\{\hat{e}_t\}_{t=p+1}^{T}$ is called the residual series from which we obtain

$$\hat{\sigma}^2 = \frac{1}{T - 2p - 1} \sum_{t=p+1}^{T} \hat{e}_t^2. \qquad (5)$$

4.3.4 Model Checking

A fitted model must be examined carefully to check for possible model inadequacy. If the model is adequate then the residual series should behave as a white noise. The ACF and the Ljung-Box statistics of the residuals can be used to check the closeness of $\{\hat{e}_t\}$ to a white noise. The Akaike Information Criterion AIC and Schwarz Criterion SC must also be examined.

For an $AR(p)$ process the Ljung-Box statistic $Q(m)$ follows asymptotically a chi-squared distribution with $m-p$ degrees of freedom. Here the number of degrees of freedom is modified to signify that, p, AR coefficients have been estimated.

4.3.5 Forecasting

Forecasting is an important application of time series analysis. Suppose we have a time index h and we are interested in forecasting r_{h+l} for $l \geq 1$. The index h is called the forecast origin and the positive integer l the forecast horizon. Let $\hat{r}_h(l)$ be the forecast of r_{h+l} using the minimum square error loss function. In other words the forecast is chosen such that

$$E[r_{h+l} - \hat{r}_h(l)]^2 \leq E(r_{h+l} - g)^2$$

where g is a function of the information available at time h (inclusive).
We refer to $\hat{r}_h(l)$ as the *l-step* ahead forecast of r_t at the forecast origin h.

1-Step Ahead Forecast

From the $AR(p)$ model, we have

$$r_{h+1} = \phi_0 + \phi_1 r_h + \ldots + \phi_p r_{h+1-p} + e_{h+1}$$

Under the minimum squared error loss function, the point forecast of r_{h+1} given the model and observations up to time h is the conditional expectation

$$\begin{aligned} r_h(1) &= E(r_{h+1}/r_h, r_{h-1}, \ldots) = \phi_0 + \phi_1 r_h + \phi_2 r_{h-1} + \ldots + \phi_p r_{h+1-p} \\ &= \phi_0 + \sum_{i=1}^{p} \phi_i r_{h+1-i}. \end{aligned} \tag{6}$$

The forecast error is

$$e_h(1) = r_{h+1} - r_h(1) = e_{h+1}.$$

Consequently the variance of the one-step-ahead forecast is

$$\begin{aligned} var(e_h(1)) &= var(e_{h+1}) \\ &= \sigma^2. \end{aligned}$$

If e_t is normally distributed then a 95% 1-step forecast interval is given by

$$r_{h+1} \in r_h(1) \pm 1.96\sigma. \tag{7}$$

Notice that both the estimates $r_h(1)$ and the confidence interval given by (7) depend on the population parameters $(\phi_0, \phi_1, \ldots, \phi_p)$ and σ^2. In practice, these need to be estimated by $(\hat{\phi}_0, \hat{\phi}_1, \ldots, \hat{\phi}_p)$ and $\hat{\sigma}^2$.

2-Step Ahead Forecast

Next consider the forecast of r_{h+2} at the forecast origin h. From the $AR(p)$ model we have

$$r_{h+2} = \phi_0 + \phi_1 r_{h+1} + \ldots + \phi_p r_{h+2-p} + e_{h+2}.$$

Taking conditional expectations

$$r_h(2) = E(r_{h+2}/r_h, r_{h-1}, \ldots) = \phi_0 + \phi_1 r_{h+1} + \phi_2 r_h + \ldots + \phi_p r_{h+2-p}.$$

But r_{h+1} is not known since it is an observation after the horizon h. Estimating r_{h+1} by its conditional expectation $r_h(1)$

$$r_h(1) = E(r_{h+1}/r_h, r_{h-1}, \ldots) = \phi_0 + \sum_{i=1}^{p} \phi_i r_{h+1-i}$$

we obtain

$$r_h(2) = E(r_{h+2}/r_h, r_{h-1}, \ldots) = \phi_0 + \phi_1 r_h(1) + \phi_2 r_h + \ldots + \phi_p r_{h+2-p}.$$

The forecast error is

$$\begin{aligned}
e_h(2) &= r_{h+2} - r_h(2) \\
&= r_{h+2} - (\phi_0 + \phi_1 r_h(1) + \phi_2 r_h + \ldots + \phi_p r_{h+2-p}) \\
&= \phi_0 + \phi_1 r_{h+1} + \phi_2 r_h + \ldots + \phi_p r_{h+2-p} + e_{h+2} - (\phi_0 + \phi_1 r_h(1) + \phi_2 r_h + \ldots + \phi_p r_{h+2-p}) \\
&= \phi_1 [r_{h+1} - r_h(1)] + e_{h+2} \\
&= \phi_1 e_{h+1} + e_{h+2}\ .
\end{aligned}$$

Thus

$$var[e_h(2)] = (1 + \phi_1^2)\sigma^2$$

so that

$$\begin{aligned}
& r_{h+2} - r_h(2) \quad \text{has mean} \\
E[r_{h+2} - r_h(2)] &= 0 \ \text{ and variance } (1 + \phi_1^2)\sigma^2.
\end{aligned}$$

If e_t is normally distributed then a 2-step ahead confidence interval is given by

$$r_{h+2} \in r_h(2) \pm 1.96\sqrt{(1 + \phi_1^2)\sigma^2}. \tag{8}$$

To compute the actual interval we need estimates for ϕ_i and σ^2. It is customary to use $\hat{\phi}_i$ and $\hat{\sigma}^2$.

Notice that

$$var(e_h(1)) = var(e_{h+1}) = \sigma^2$$

$$\begin{aligned} var[e_h(2)] &= (1+\phi_1^2)\sigma^2 \\ &\geq var(e_h(1)) = \sigma^2, \end{aligned}$$

meaning that, as the forecast horizon increases the uncertainty in forecast also increases. This increase makes sense since we are more uncertain about r_{h+2} than r_{h+1}at the time index h.

l-Step Ahead Forecast

In general we have

$$r_{h+l} = \phi_0 + \phi_1 r_{h+l-1} + \cdots + \phi_p r_{h+l-p} + e_{h+l} \tag{9}$$

and proceeding exactly as above we get the l-step ahead forecast as the conditional expectation

$$r_h(l) = E(r_{h+l}/r_h, r_{h-1}, \ldots) = \phi_0 + \phi_1 r_h(l-1) + \phi_2 r_h(l-2) + \ldots + \phi_p r_h(l-p).$$

The l-step ahead forecast error is

$$\begin{aligned} e_h(l) &= r_{h+l} - r_h(l), \text{ with variance} \\ var(e_h(l)) &= (1+\phi_1^2+\phi_2^2+\ldots+\phi_{l-1}^2)\sigma^2. \end{aligned}$$

The forecast can be computed by recursively computing forecasts

$$\begin{aligned} &r_h(1) \\ &r_h(2) \text{ and recursively} \\ &r_h(3), r_h(4), \ldots, r_h(l-1) \end{aligned} \tag{10}$$

Notice however, that all the conditional forecasts above are functions of the parameters $(\phi_0, \phi_1, \ldots, \phi_p)$ and σ^2. Starting with $r_h(1)$, if these parameters are estimated by

$$(\hat{\phi}_0, \hat{\phi}_1, \ldots, \hat{\phi}_p) \text{ and } \hat{\sigma}^2$$

then

$$\hat{r}_h(1) = \hat{\phi}_0 + \hat{\phi}_1 r_h + \ldots + \hat{\phi}_p r_{h+1-p}, \text{ see (9).} \tag{11}$$

If

$$\hat{\boldsymbol{\phi}} = \begin{pmatrix} \hat{\phi}_0 \\ \hat{\phi}_1 \\ \vdots \\ \hat{\phi}_p \end{pmatrix} \text{ and } \mathbf{r}_h = \begin{pmatrix} 1 \\ r_h \\ \vdots \\ r_{h+1-p} \end{pmatrix} \tag{12}$$

then

$$\hat{r}_h(1) = \mathbf{r}_h' \hat{\boldsymbol{\phi}} \tag{13}$$

and assuming $\mathbf{r}_h$ is fixed (given) the variance of $\hat{r}_h(1) = \mathbf{r}_h' \hat{\boldsymbol{\phi}}$ is

$$var(\hat{r}_h(1)) = \mathbf{r}_h' var(\hat{\boldsymbol{\phi}}) \mathbf{r}_h.$$

Furthermore if

$$\hat{r}_h(2) = \hat{\phi}_0 + \hat{\phi}_1 \hat{r}_h(1) + \hat{\phi}_2 r_h + \ldots + \hat{\phi}_p r_{h+2-p},$$

then

$$var(\hat{r}_h(2)) = var(\hat{\phi}_0 + \hat{\phi}_1 \hat{r}_h(1) + \hat{\phi}_2 r_h + \ldots + \hat{\phi}_p r_{h+2-p}) \tag{14}$$

which already becomes quite a complicated expression.

Thus for all practical purposes we use as forecasts recursively from (10)

$$\{\hat{r}_h(1), \hat{r}_h(2), \hat{r}_h(3), \ldots, \hat{r}_h(l-1)\}$$

where in $\{\hat{r}_h(i), i = 1, \ldots, l-1\}$ we have used the estimates $(\hat{\phi}_0, \hat{\phi}_1, \ldots, \hat{\phi}_p)$ for $(\phi_0, \phi_1, \ldots, \phi_p)$.

As forecast error we use

$$\hat{var}(e_h(l)) = (1 + \hat{\phi}_1^2 + \hat{\phi}_2^2 + \ldots + \hat{\phi}_{l-1}^2)\hat{\sigma}^2.$$

which will be sufficient for large sample sizes.

4.3.6 $AR(1)$ Model. First-Order Autoregressive Model

The $AR(1)$ process is given by

$$r_t = \phi_0 + \phi_1 r_{t-1} + e_t \tag{15}$$

with e_t assumed to be white noise with

$$E(e_t) = 0 \text{ and } var(e_t) = \sigma^2.$$

Assuming that the process is weakly stationary, we have

$$\begin{aligned} E(r_t) &= \mu,\ var(r_t) = \gamma_0 \text{ and} \\ cov(r_t r_{t-j}) &= \gamma_j \end{aligned}$$

where μ and γ_0 are constant and γ_j is a function of j and not t.

Now the mean is

$$E(r_t) = \phi_0 + \phi_1 E(r_{t-1}) \ .$$

since $E(e_t) = 0$. Under the stationarity condition $E(r_t) = E(r_{t-1}) = \mu$ and hence

$$\mu = \phi_0 + \phi_1\mu \text{ or } E(r_t) = \mu = \frac{\phi_0}{1-\phi_1}.$$

This result has two implications. First the mean of r_t exists if $\phi_1 \neq 1$. Second, the mean of r_t is zero if and only if $\phi_0 = 0$. Thus for a stationary $AR(1)$ process, the constant term ϕ_0 is related to the mean of r_t and $\phi_0 = 0$ implies $E(r_t) = 0$.

Next using $\phi_0 = (1 - \phi_1)\mu$ the $AR(1)$ model can be rewritten as

$$r_t - \mu = \phi_1(r_{t-1} - \mu) + e_t. \tag{16}$$

By repeated substitutions we get

$$\begin{aligned} r_t - \mu &= e_t + \phi_1 e_{t-1} + \phi_1^2 e_{t-2} + \cdots\cdots \\ &= \sum_{i=0}^{\infty} \phi_1^i e_{t-i}. \end{aligned} \tag{17}$$

Thus $r_t - \mu$ is a linear function of e_{t-i} for $i \geq 0$. Using this property and the independence of the series $\{e_t\}$, we obtain $E(r_t - \mu)e_{t+1} = 0$. By the stationarity assumption

$$cov(r_{t-1}, e_t) = E[(r_{t-1} - \mu)e_t] = 0.$$

This result can also be seen from the result that r_{t-1} occurred before time t and e_t does not depend on any past information. Taking the square, then the expectation of (16), we obtain

$$var(r_t) = \phi_1^2 var(r_{t-1}) + \sigma^2, \text{ since } cov(r_{t-1}, e_t) = 0.$$

Thus, under stationarity $var(r_t) = var(r_{t-1})$ so that (noting that $var(e_t) = \sigma^2$) we have

$$var(r_t) = \frac{\sigma^2}{1 - \phi_1^2}$$

provided that $\phi_1^2 < 1$. The requirement of $\phi_1^2 < 1$ results from the fact that the variance of the random variable r_t is bounded and non-negative. Consequently the weak stationarity of an $AR(1)$ process implies that $-1 < \phi_1 < 1$.

But, if $-1 < \phi_1 < 1$ then by (16) and the independence of e_t, we can show that the mean and variance of r_t are finite. In addition, by the Cauchy-Schwartz inequality, all the covariances of r_t are finite. Therefore, the $AR(1)$ model is weakly stationary.

Thus, in summary, the necessary and sufficient condition for the $AR(1)$ process to be stationary is $|\phi_1| < 1$.

Autocorrelation Function of an $AR(1)$ Model

Multiplying equation (16) by e_t, using the independence between e_t and r_{t-1} and taking expectation, we obtain

$$\begin{aligned} E[e_t(r_t - \mu)] &= E[e_t(r_{t-1} - \mu)] + E(e_t^2) \\ &= E(e_t^2) = \sigma^2. \end{aligned}$$

Thus multiplying (16) by $(r_{t-l} - \mu)$ and taking expectations, we have

$$\begin{aligned} E[(r_t - \mu)(r_{t-l} - \mu)] &= E[\phi_1(r_{t-1} - \mu)(r_{t-l} - \mu) + E[e_t(r_{t-l} - \mu)] \\ &= \phi_1\gamma_1 + \sigma^2 \text{ for } l = 0 \\ &= \phi_1\gamma_{l-1} \text{ for } l > 0, \end{aligned}$$

where we use $\gamma_l = \gamma_{-l}$. Consequently, for a weak stationary $AR(1)$ model in equation (32)

$$var(r_t) = \gamma_0, \text{ and } \gamma_l = \phi_1\gamma_{l-1} \text{ for } l > 0.$$

Thus the ACF of ρ_l satisfies

$$\rho_l = \phi_1\rho_{l-1} \text{ for } l \geq 0.$$

Because $\rho_0 = 1$ we have

$$\rho_l = \phi_1^l, \tag{18}$$

thus the ACF of a weakly stationary $AR(1)$ series decays exponentially with rate ϕ_1 after starting at $\rho_0 = 1$.

Alternative Derivation of the ACF for the $AR(1)$ Model.

Now since from (17)

$$r_t - \mu = e_t + \phi_1 e_{t-1} + \phi_1^2 e_{t-2} + \phi_1^3 e_{t-3} + \dots,$$

the variance is

$$\begin{aligned}
\gamma_0 &= E(r_t - \mu)^2 \\
&= E(e_t + \phi_1 e_{t-1} + \phi_1^2 e_{t-2} + \phi_1^3 e_{t-3} + \ldots)^2 \\
&= (1 + \phi_1^2 + \phi_1^4 + \phi_1^6 + \cdots\cdots)\sigma^2 \\
&= \sigma^2/(1 - \phi_1^2),
\end{aligned}$$

while the lth autocovariance is

$$\begin{aligned}
\gamma_l &= E(r_t - \mu)(r_{t-l} - \mu) \\
&= E(e_t + \phi_1 e_{t-1} + \phi_1^2 e_{t-2} + \phi_1^3 e_{t-3} + \ldots) \\
&\quad \times (e_{t-l} + \phi_1 e_{t-l-1} + \phi_1^2 e_{t-l-2} + \phi_1^3 e_{t-l-3} + \ldots) \\
&= [\phi_1^l + \phi_1^{l+2} + \phi_1^{l+4} + \ldots]\sigma^2 \\
&= \phi_1^l[1 + \phi_1^2 + \phi_1^4 + \ldots]\sigma^2 \\
&= [\phi_1^l/(1 - \phi_1^2)]\sigma^2.
\end{aligned}$$

The autocorrelation function is therefore

$$\begin{aligned}
\rho_j &= \frac{\gamma_j}{\gamma_0},\ \rho_0 = 1 \\
&= \phi_1^l.
\end{aligned} \tag{19}$$

By using the Lag operator $Lr_t = r_{t-1}$ and $L^q r_t = r_{t-q}$ we obtain from (16)

$$\begin{aligned}
(1 - \phi_1 L)(r_t - \mu) &= e_t\ , \quad \text{or,} \\
\Phi(L)(r_t - \mu) &= e_t, \text{ with } \Phi(L) = 1 - \phi_1 L
\end{aligned} \tag{20}$$

Let $x = 1/L$ then we obtain the characteristic equation $\Phi(\frac{1}{x}) = 0$ of the $AR(1)$ model. The characteristic equation plays an important role in identifying Linear Time Series Processes.

The characteristic equation is

$$\Phi(\frac{1}{x}) = x - \phi_1 = 0 \tag{21}$$

and the only solution of the characteristic equation

is

$$x = \phi_1. \tag{22}$$

But for the process to be stationary $|\phi_1| < 1$. Thus the solution of the characteristic equation requires $|\phi_1| < 1$ for the process to be stationary, or, the solution must be within the unit circle. Note that some writers use as characteristic equation, with $z = L$,

$$\Phi_1(z) = 1 - \phi_1 z = 0 \tag{23}$$

with solution $z = 1/\phi_1$ and $|z| = |1/\phi_1| > 1$, requires $|\phi_1| < 1$ for the process to be stationary. Thus the solutions to the characteristic equation $\Phi_1(z) = 0$, must all lie outside the unit circle.

Chapter 5

An Interval-valued AR(1) model under metric d_G

The weakly stationary assumptions for the interval-valued time series are

$$\begin{aligned} E\left[\bar{\varepsilon}_t\right] &= [0,0] \\ V\left[\bar{\varepsilon}_t\right] &= \overline{\sigma^2} = \left[\sigma^{2.l}, \sigma^{2.u}\right] \in \mathbb{R}_+^2 \\ Cov\left[\bar{\varepsilon}_t, \bar{\varepsilon}_{t+h}\right] &= f(h) \text{ independent of } t \end{aligned}$$

$$\bar{y}_t = \bar{\mu} + \bar{\phi}_1 \bar{y}_{t-1} + \bar{\varepsilon}_t$$

where

$$\bar{y}_t = \left[y_t^l, y_t^u\right], t = 0, 1, 2, \cdots, N$$

are interval-valued observations and

$$\bar{\phi}_1 = \left[\phi_1^l, \phi_1^u\right]$$

is the interval-valued parameter.

5.1 Estimation of AR(1) parameters

The estimation is effectively converted into a common optimization problem.

$$\min_{(\overline{\mu},\overline{\phi}_1)} \sum_{t=1}^{N} d_G\left(\overline{y}_t, \overline{\mu} + \overline{\phi}_1 \overline{y}_{t-1}\right)$$

In this set-up we let

$$\overline{x}_t = \overline{\mu} + \overline{\phi}_1 \overline{y}_{t-1}$$

then

$$d_G\left(\overline{y}_t, \overline{x}_t\right) = \frac{\alpha}{\alpha+\beta}\delta_2\left(\overline{y}_t, \overline{x}_t\right) + \frac{\beta}{\alpha+\beta}\left|m_{\overline{y}_t} - m_{\overline{x}_t}\right|$$

where

$$\delta_2\left(\overline{y}_t, \overline{x}_t\right) = \sqrt{\frac{1}{2}\left[\left|y_t^u - x_t^u\right|^2 + \left|y_t^l - x_t^l\right|^2\right]}$$

and

$$m_{\overline{y}_t} = \frac{1}{2}(y_t^l + y_t^u)$$

$$m_{\overline{x}_t} = \frac{1}{2}(x_t^l + x_t^u)$$

with

$$x_t^l = \mu^l + \min(\phi_1^l y_{t-1}^l, \phi_1^l y_{t-1}^u, \phi_1^u y_{t-1}^l, \phi_1^u y_{t-1}^u)$$

and

$$x_t^u = \mu^u + \max(\phi_1^l y_{t-1}^l, \phi_1^l y_{t-1}^u, \phi_1^u y_{t-1}^l, \phi_1^u y_{t-1}^u)$$

Then the optimisation algorithm implemented in the form of Mathematica code in the Appendix solves the optimisation problem

$$\min_{(\overline{\mu},\overline{\phi}_1)} \sum_{t=1}^{N} d_G\left(\overline{y}_t, \overline{\mu} + \overline{\phi}_1 \overline{y}_{t-1}\right)$$

for interval parameter estimates

$$\left(\overline{\mu}, \overline{\phi}_1\right) = \left(\left[\mu^l, \mu^u\right], \left[\phi_1^l, \phi_1^u\right]\right)$$

given a choice of a combination of parameters α and β.

5.2 Variance-covariance for estimators

The basic idea here is in terms of Taylor's expansion to obtain an asymptotic variance-covariance. Denote the estimated parameter of $\left(\overline{\mu}, \overline{\phi}_1\right)$ as $\left(\widetilde{\overline{\mu}}, \widetilde{\overline{\phi}}_1\right)$

$$\begin{aligned}
& f\left(\widetilde{\overline{\mu}}, \widetilde{\overline{\phi}}_1\right) - f\left(\overline{\mu}, \overline{\phi}_1\right) \\
= \; & \frac{df}{d\left(\overline{\mu}, \overline{\phi}_1\right)}\left[\left(\widetilde{\overline{\mu}}, \widetilde{\overline{\phi}}_1\right) - \left(\overline{\mu}, \overline{\phi}_1\right)\right] + \frac{d^2 f}{d^2\left(\overline{\mu}, \overline{\phi}_1\right)}\left[\left(\widetilde{\overline{\mu}}, \widetilde{\overline{\phi}}_1\right) - \left(\overline{\mu}, \overline{\phi}_1\right)\right]^2 + o\left(\left|\left(\widetilde{\overline{\mu}}, \widetilde{\overline{\phi}}_1\right) - \left(\overline{\mu}, \overline{\phi}_1\right)\right|\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
& E\left[\left(\left(\widetilde{\overline{\mu}}, \widetilde{\overline{\phi}}_1\right) - \left(\overline{\mu}, \overline{\phi}_1\right)\right)^2\right] \\
= \; & \left[\frac{d^2 f}{d^2\left(\overline{\mu}, \overline{\phi}_1\right)}\right]^{-1}\left\{E\left[f\left(\widetilde{\overline{\mu}}, \widetilde{\overline{\phi}}_1\right) - f\left(\overline{\mu}, \overline{\phi}_1\right)\right] - E\left[\frac{df}{d\left(\overline{\mu}, \overline{\phi}_1\right)}\left[\left(\widetilde{\overline{\mu}}, \widetilde{\overline{\phi}}_1\right) - \left(\overline{\mu}, \overline{\phi}_1\right)\right]\right]\right\}
\end{aligned}$$

Also,

$$\mathrm{VAR}\, X = E\left[X^2\right] - (EX)^2$$

We need to explore the link between $f\left(\widetilde{\overline{\mu}}, \widetilde{\overline{\phi}}_1\right) - f\left(\overline{\mu}, \overline{\phi}_1\right)$ and $\sum_{t=1}^{N} d_G\left(\widetilde{\overline{y}}_t, \overline{\mu} + \overline{\phi}_1 \overline{y}_{t-1}\right)$. As a matter of fact, $d_G\left(\widetilde{\overline{\mu}} + \widetilde{\overline{\phi}}_1 \overline{y}_{t-1}, \overline{\mu} + \overline{\phi}_1 \overline{y}_{t-1}\right)$ can be evaluated and therefore

$$\sum_{t=1}^{N} d_G\left(\widetilde{\overline{\mu}} + \widetilde{\overline{\phi}}_1 \overline{y}_{t-1}, \overline{\mu} + \overline{\phi}_1 \overline{y}_{t-1}\right)$$

can be obtained. Finally,

$$\left(\widetilde{\overline{\mu}} - \overline{\mu}, \widetilde{\overline{\phi}}_1 - \overline{\phi}_1\right) \overset{d}{\sim} N\left(\left(\overline{0}, \overline{0}\right), \mathrm{VAR}\left[\left(\widetilde{\overline{\mu}}, \widetilde{\overline{\phi}}_1\right) - \left(\overline{\mu}, \overline{\phi}_1\right)\right]\right)$$

asymptotically.

Chapter 6

Interval Modelling of three JSE Stock Prices

6.1 Johannesburg Securities Exchange (JSE) South Africa

This section gives a brief summary of the background of the JSE Securities Exchange South Africa (JSE). The main source of information for this summary is the official JSE world wide web site (http://www.jse.co.za).

In 1886, the Witwatersrand goldfields, the richest of their kind in the world at that time, were discovered in Johannesburg. Little more than a year later, the JSE Securities Exchange South Africa was founded in 1887 as the Johannesburg Stock Exchange (JSE) to enable new mines and their financiers to raise funds for the development of the booming mining industry. Today, the majority of listed companies on the JSE are non-mining organisations.

6.1.1 Features of the JSE

The JSE is the only stock exchange for equities and other securities in South Africa. However, the JSE has been classified as an "emerging market" when compared to the global stock markets. The South African economy is characterised by both established first world fundamentals as well as third world features. Since Monday 10 June 1996, all trades on the JSE are conducted through the automated trading system, *JET* (JSE Equities Trading).

The JSE is a self-regulatory organisation that is governed by a set of rules drawn up by the elected JSE Committee. The JSE is a member of the Fédération Internationale des Bourses de Valuers (FIBV) and was granted designation status by the Japanese Securities Dealers Association effective on 14 December 1994. On 1 March 1995 the JSE was included in the Morgan Stanley index for emerging markets and on 7 April 1995 was included in the IFC Emerging Markets Global and Investable Indices.

6.1.2 Roles of the JSE in South Africa

The main functions of the JSE are:

1. To provide an orderly primary and secondary markets for trades in equities and other securities in order to create new investment opportunities in South Africa.
2. To create liquidity, thus ensuring the primary market fulfils its function of raising new investment capital.
3. To re-channel cash resources into productive economic activities, thus allowing for the raising of primary capital required to develop the country's economy.
4. To make the services it provides to be accessible to the entire nation.
5. To ensure that the nation is suitably informed of the advantages and risks of share ownership.

6.1.3 Re-structuring of the JSE

In anticipation of the fundamental changes to the South African political and economic environment that was to follow the first democratic election in 1994, the JSE applied to the South African Parliament for a re-structuring. In September 1995, the Stock Exchanges Control Amendment Act was approved by Parliament, which allowed for the re-structuring of the JSE. The restructuring plan, as approved by the JSE Committee, was aimed at ensuring that:

1. The stock exchange de-regulates efficiently and successfully,

2. The stock exchange contributes towards the needs of the new political and economic regime, and
3. The JSE's attractiveness to local and foreign investors will be further enhanced.

The JSE was re-structured without bias towards any particular business or social community. The restructuring has impacted on:

1. Membership in the JSE,
2. Trading principles and systems,
3. Clearing and settlement,
4. Transfer and registration,
5. Capital requirements of member firms, and
6. Financial structure of the JSE.

The overall benefits to the economy from this re-structuring were suggested as follows:

1. A listing on the JSE enables substantial amounts of capital to be raised for the financing of new businesses, expansion of existing businesses, and the creation of new employment opportunities.
2. To allow for speculative buying and selling of shares for individuals who are knowledgeable about the performance of selected shares.

6.1.4 Trading Hours on the JSE

The trading days on the JSE are from Monday to Friday, but excluding the public holidays as declared by the South African Parliament. The continuous trading hours on a trading day are from 09H00 to 17H00 (South African time). The JET system operates every trading day from 08H25 until 18H00, with pre-opening sessions applicable from 08H25 to 09H00, and the runoff session applicable from 17H00 to 18H00.

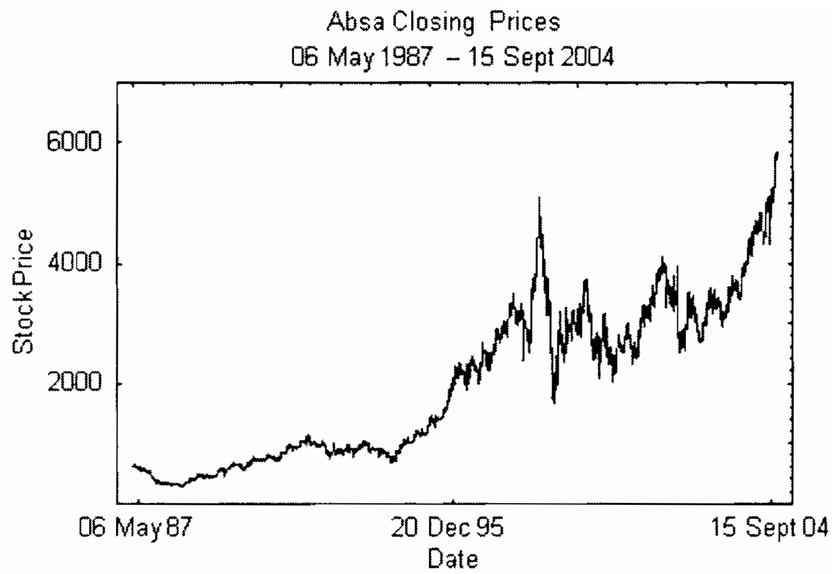

Figure 6-1: Plot of observed Absa Daily Price Time Series

6.2 Three JSE stock prices used in the mini-dissertation

6.2.1 The Absa shares

Absa is one of South Africa's largest providers of financial services to the business market. In 2003 Absa was chosen as South Africa's bank of the year and also the most admired financial services brand in South Africa. Absa's share price has shown steady growth over the years.

Absa's attractiveness to both domestic and international investors was further confirmed by the interest shown by Barclays Bank (UK) in acquiring a substantial stake in Absa. At the time of the writing of this dissertation the process of this acquisition process was in advanced stages. Figure 6-1 displays the almost exponential growth of the Absa share price between 06 May 1987 and 15 September 2004.

6.2.2 The Anglo shares

Anglo American with its subsidiaries, joint ventures and associates is a global leader in the mining and natural resources sectors. It has a significant and focused interest in platinum

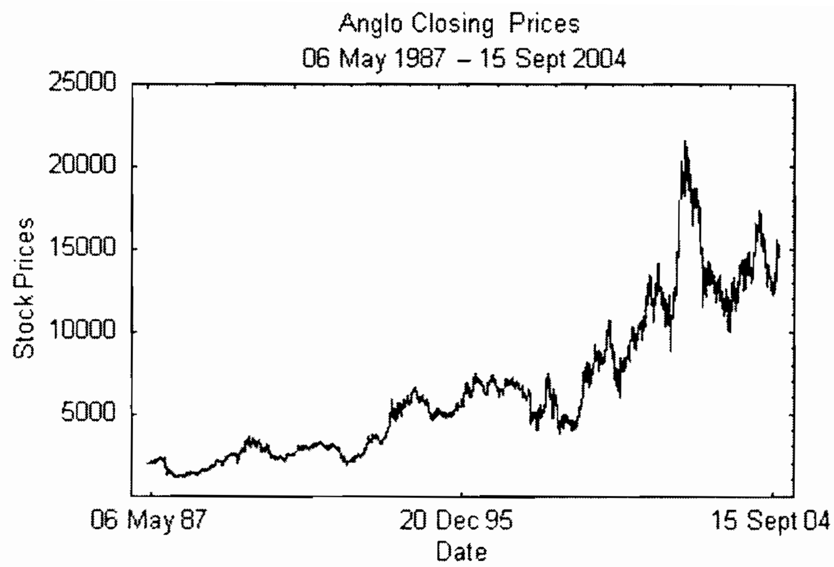

Figure 6-2: Plot of observed Anglo Daily Price Time Series

group metals, paper and packaging and ferrous metals. It has also a financial as well as technical strength. Figure 6-2 displays the time series growth of the Anglo share price between 06 May 1987 and 15 September 2004.

6.2.3 The JSE Overall index

This section contains a brief summary of the financial index measuring the overall performance of the JSE over a trading day. The main source of information for this summary is the official FTSE/JSE Africa Index Series world wide web site (http://ftse.jse.co.za).

In financial markets, financial indices are published by the regulators to summarise the price movements of a group of listed shares on the market. The primary functions of these financial indices are:

1. To describe the market at a point in time in terms of price levels, dividend yield and earnings yields.

2. To enable investors and traders to compare the performance of a particular listed share with the performance of a market index.

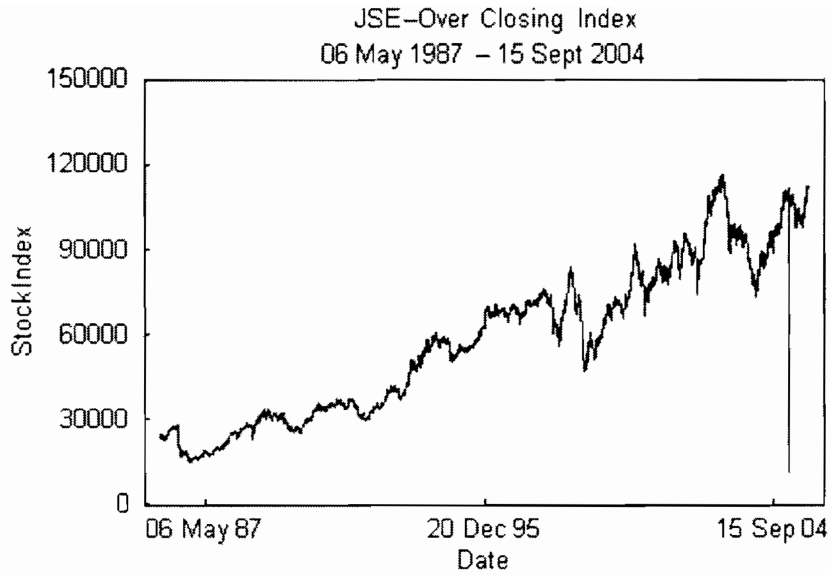

Figure 6-3: Plot of observed JSE-Over Daily Price Time Series

3. To provide an indication of the investors' confidence in the economy and the market's expectations for the different sectors within the market as well as the overall market.

On the JSE, the Johannesburg Stock Exchange Overall (**JSE-Over**) Index measures the performance of the overall equity market. According to JSE's "old" (Prior to 24 June 2002) sector classification system, the JSE is comprised of 5 sectors and 53 sub-sectors. The JSE-Over index is a *weighted arithmetic* index, where the weights are the market capitalisation (or market cap) of each constituent security. Figure 6-3 displays the daily JSE-Over index pattern between 06 May 1987 and 15 September 2004.

6.3 Classical AR(1) Modeling of Low and High Prices

The basic idea is whether or not an interval prediction can be offered for any stock prices. One of the naive approaches is to directly use the low-high values of the stock price/index to form low-high price interval and then take the estimated low-high price/index pairs to form a low-high interval series and finally use these pairs to estimate parameters for forecasting. It should be pointed out that the classical time series techniques make use of stationary time series.

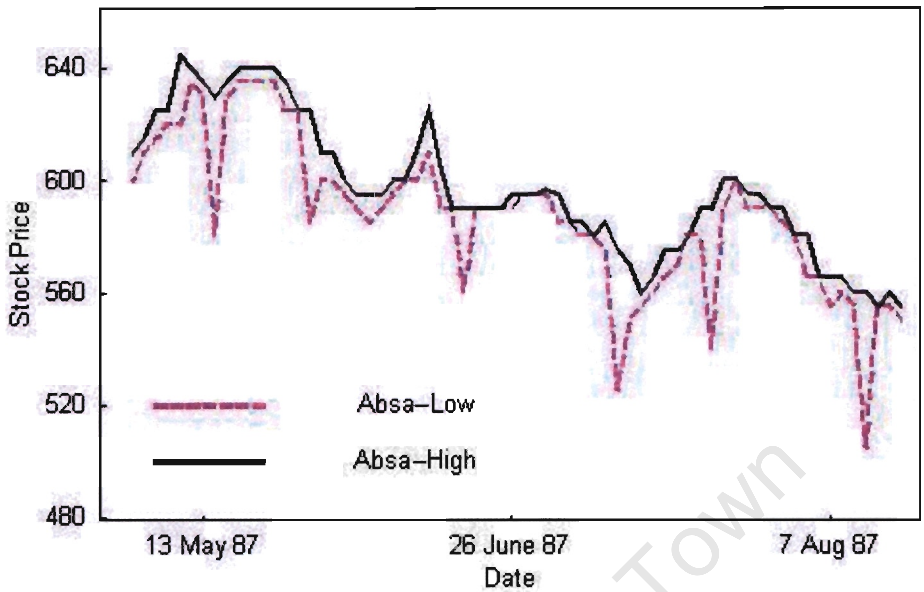

Figure 6-4: Time plot of daily low and high Absa prices

However, from price theory, the low and high prices of a stock are in general not stationary. The new approach adopted in this dissertation is to apply classical time-series analysis techniques and later use the interval AR(1) and the general metric d_G to provide a comparison base for the two models. The steps involved in parameter estimation for each observed series will involve establishing whether the data series is stationary or not, and if not, performing necessary transformations so that estimation is conducted on stationary time series. In this analysis only 3 month of daily time series for the three stocks are used.

6.3.1 Absa low-high interval

We begin with a series of Absa low-high prices observed between 06 May 1987 and 07 August 1987 as given in Figure 6-4.

The first objective is to decide on a reasonable - but tentative - ARMA(p, q) model with the

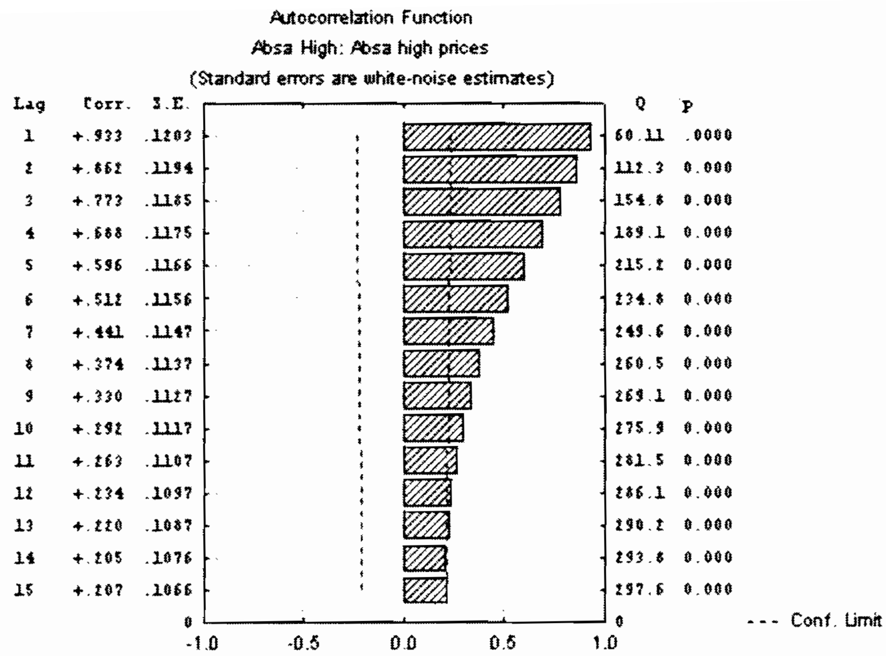

Figure 6-5: ACF of high Absa prices

aid of the autocorrelation and partial autocorrelation functions (ACF and PACF). Having done so, the next step will be to estimate the parameters for that model using the method of least squares. Next, the model will be critically checked for adequacy. If the model is inadequate in some way, the nature of inadequacy will be considered. Finally, the model will be used to compute forecasts of Absa high prices for up to 5 days using 1-step ahead forecasts. This approach will also be applied separately for low Absa prices, as well. The ACF and PACF of the high Absa series are given in Figures 6-5, and 6-6 respectively.

From Figure 6-5 the ACF values decay exponentially and the PACF has a spike only at lag 1. All other values of PACF lie within the interval of $\pm\frac{2}{\sqrt{66}} = \pm 0.2462$. This analysis suggests a tentative AR(1) model given below with parameters estimated using the least squares method,

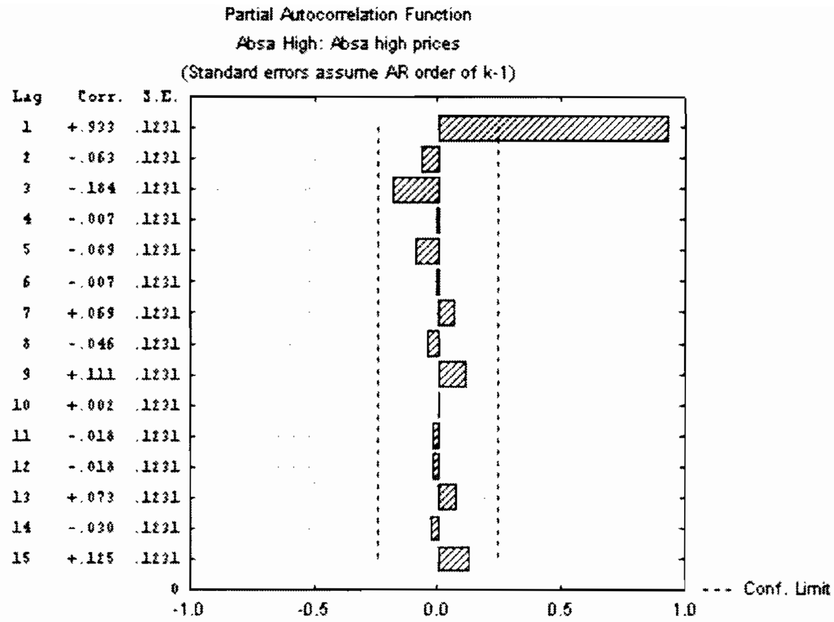

Figure 6-6: PACF of high Absa prices

$$\begin{aligned} r_t &= 12.3546 + 0.977915 r_{t-1},\ \hat{\sigma} = 7.1460 \\ &\quad (22.2353) \quad (0.03717) \\ &\quad (0.5556) \quad (26.3092) \end{aligned}$$

with the standard errors and t-statistics in brackets below the estimates. The next step is to check the model for adequacy. This is achieved by first checking the graph of residues - differences between actual and model prices and such graph is given in Figure 6-7.

A more useful tool for model diagnostics is the ACF of residuals as a check for the independence of the noise terms in the model. For true white noise, the sample autocorrelations are approximately uncorrelated and normally distributed with zero means and variance $1/T$. The ACF for the high Absa prices is given in Figure 6-8. Comparing the observed values with the two times standard errors gives no reason to question the adequacy of the fitted model.

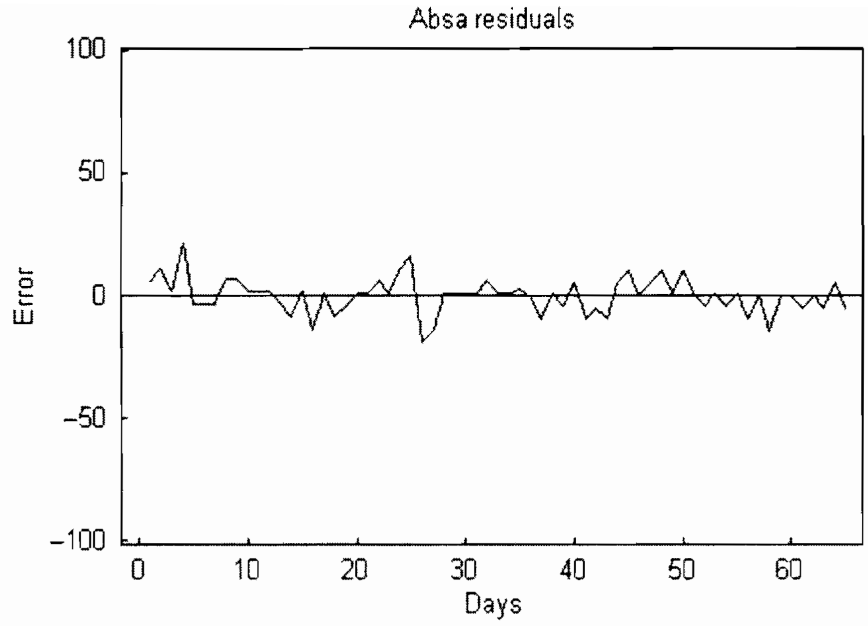

Figure 6-7: Residuals of high Absa series under AR(1) model

Degrees of freedom $(m-1)$	6	7	8	9	10	11	12
$Q^*(m)$	3.340	4.298	5.111	5.113	5.150	6.537	6.585
$\chi^2_{0.05}(m-1)$	12.592	14.067	15.507	16.919	18.307	19.675	21.026

Table 6.1: Modified Box-Pierce statistic for high Absa prices

The graphical evidence of serial independence of residual terms can further be reinforced by the modified Box-Pierce statistics as shown in Table 6.1. Note that none of the modified Box-Pierce statistics are significant at the 5% level and thus the conclusion to fit the AR(1) is a correct one.

The 1-day up to 5-day forecasts of high Absa series are depicted in Figure 6-9. The values of the forecasts are given in Table 6.2 together with forecast standard deviations (errors). The forecast origin is day 66 (7 August 1987). As the forecast horizon increases, the uncertainty in the forecasts also increases, as indicated by increasing standard errors.

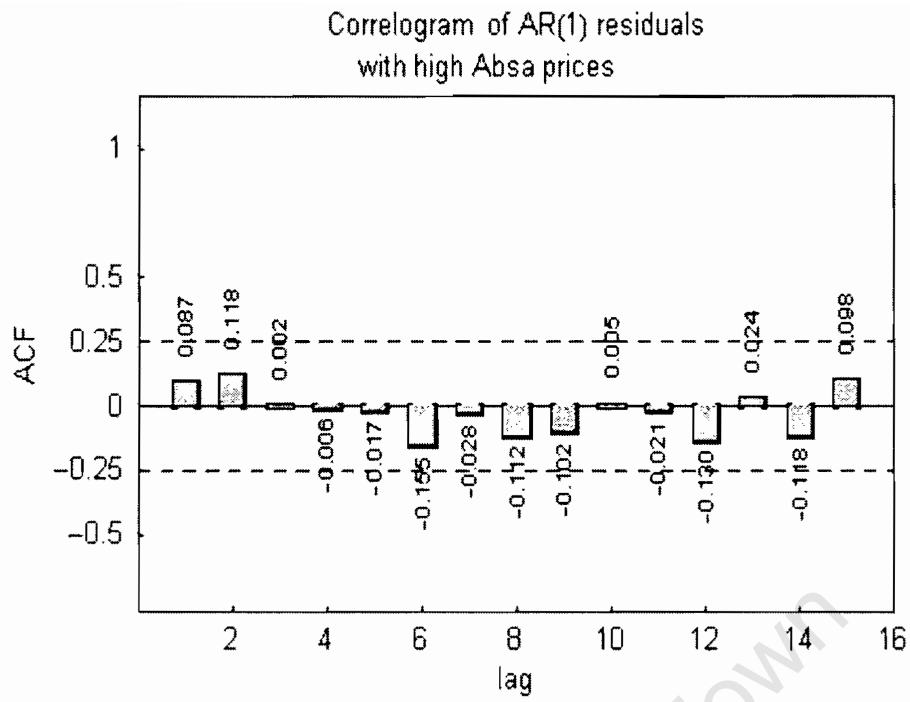

Figure 6-8: ACF of the residuals of high Absa prices

Forecast	555.097	555.193	555.286	555.377	555.466
Standard error	(7.146)	(9.995)	(12.108)	(13.830)	(15.296)
Day	67	68	69	70	71

Table 6.2: Forecasts of high Absa prices

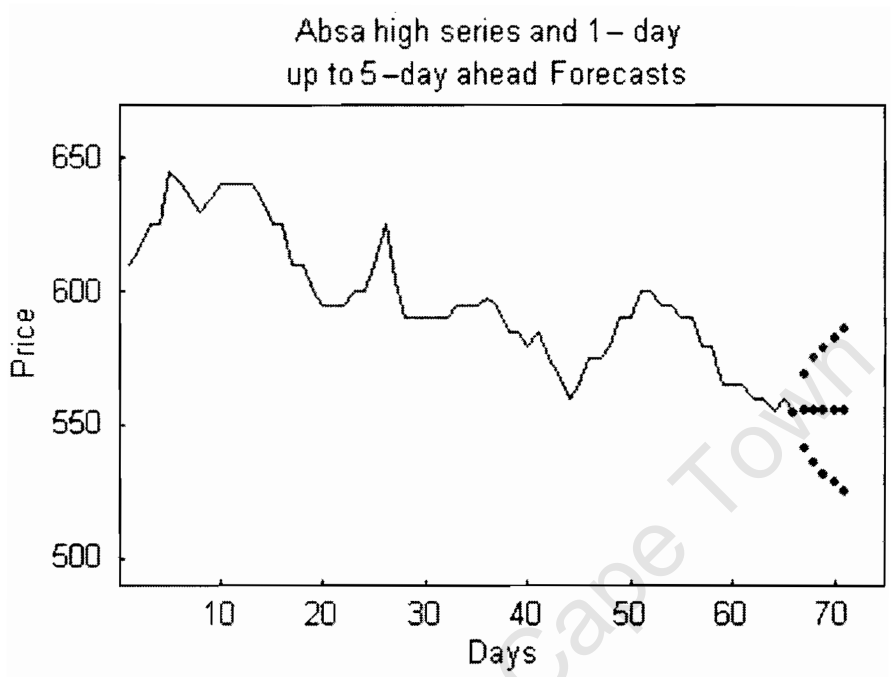

Figure 6-9: Forecast of high Absa prices with the original series

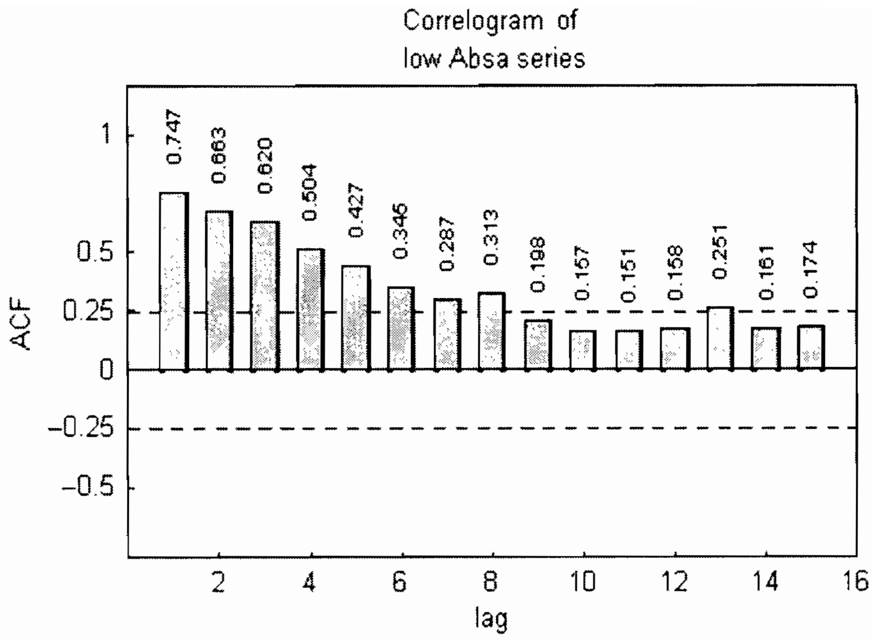

Figure 6-10: ACF of low Absa prices

Parameter Estimation Of Absa Low Prices

The focus now shifts from high Absa prices to low Absa prices. The ACF and PACF of the low Absa series is given in Figures 6-10 and 6-11 respectively.

The ACF and PACF in Figures 6-10 and 6-11 suggest an AR(1) model. The ACF decays almost exponentially except that only few values notably at lag 8 and lag 13 allow for some slight deviations from the exponential decay trend. The PACF has a significant spike at lag 1, and beyond that lag, almost all values fall within the 95% confidence band except for values at lag 13 and lag 14. The AR(1) model is given below with the least squares parameter estimates,

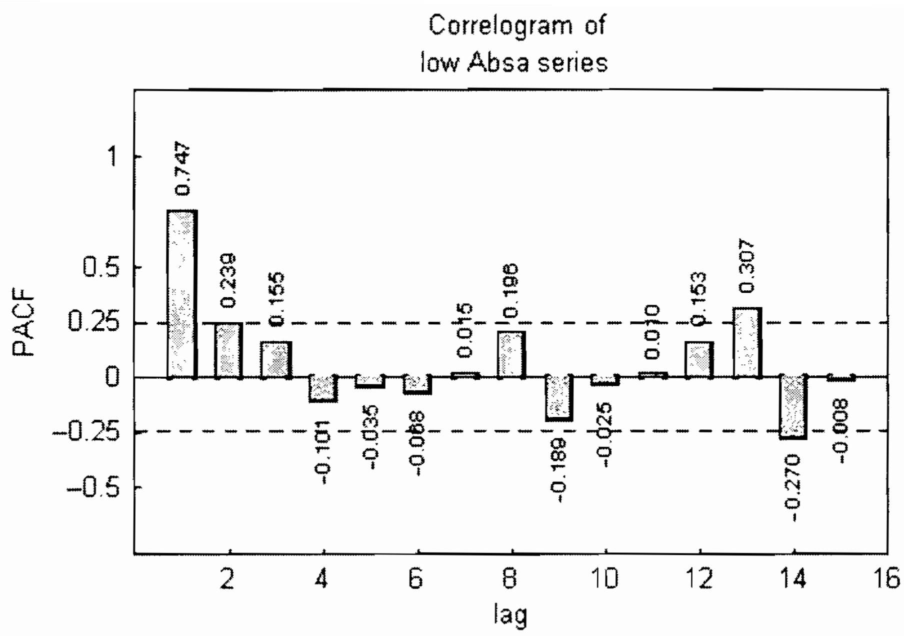

Figure 6-11: PACF of low Absa prices

$$
\begin{aligned}
r_t &= 134.8423 + 0.7695 r_{t-1}, \ \hat{\sigma} = 17.9004 \\
& \quad (48.9411) \quad (0.08311) \\
& \quad (2.7552) \quad (9.2579)
\end{aligned}
$$

where the standard errors and t-statistics are shown in brackets.

Figure 6-12 is a graph of residuals. From the graph there does not seem to be any pattern in the residuals. and the residuals appear to be independent of each other. This influence is further confirmed by Figures 6-13 and 6-14, where no strong correlations show up at any lags.

The one day ahead forecasts of the low Absa prices with the origin at day 66 (07 August 1987) and the horizon days 67 up to 71 (14 August 1987) are represented in Figure 6-15

Table 6.3 gives the tabular representation of the1-day ahead forecasts of the low prices together with the corresponding standard errors.

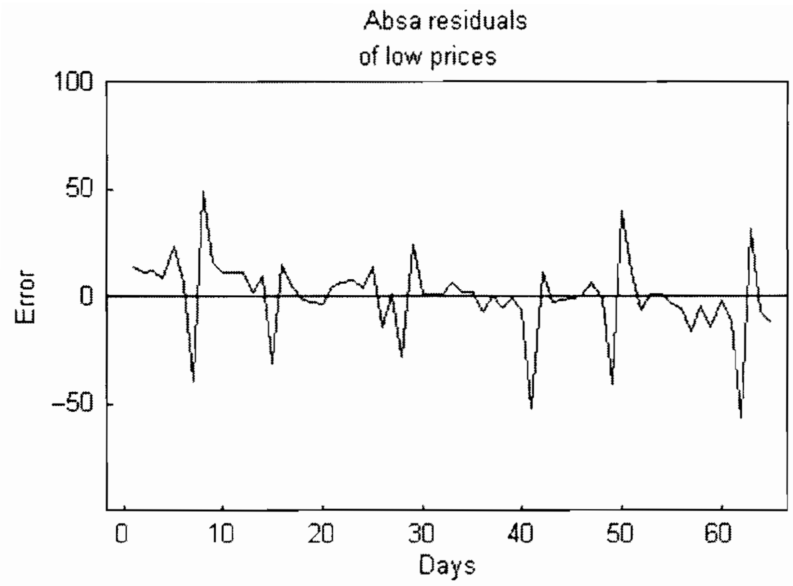

Figure 6-12: Residuals of low Absa series under AR(1) model

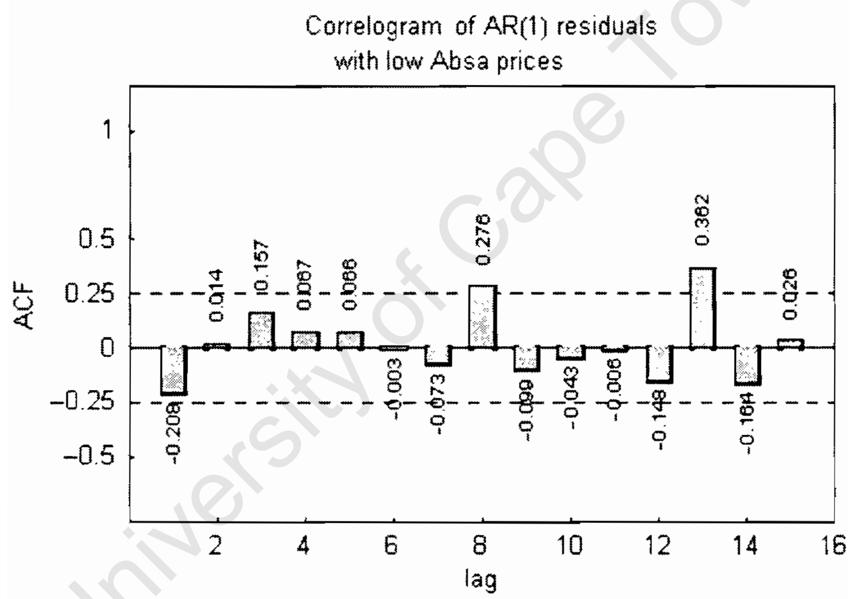

Figure 6-13: ACFof residuals of low Absa prices

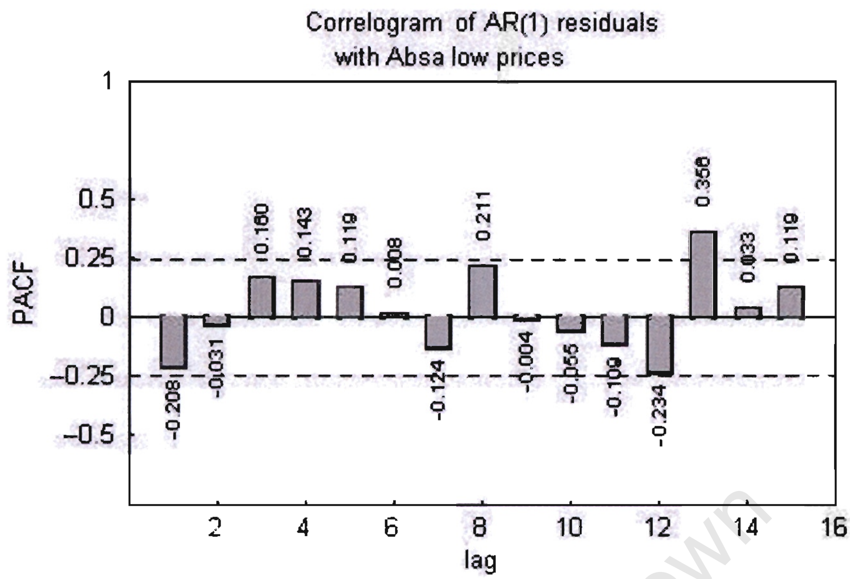

Figure 6-14: PACF of residuals of low Absa prices

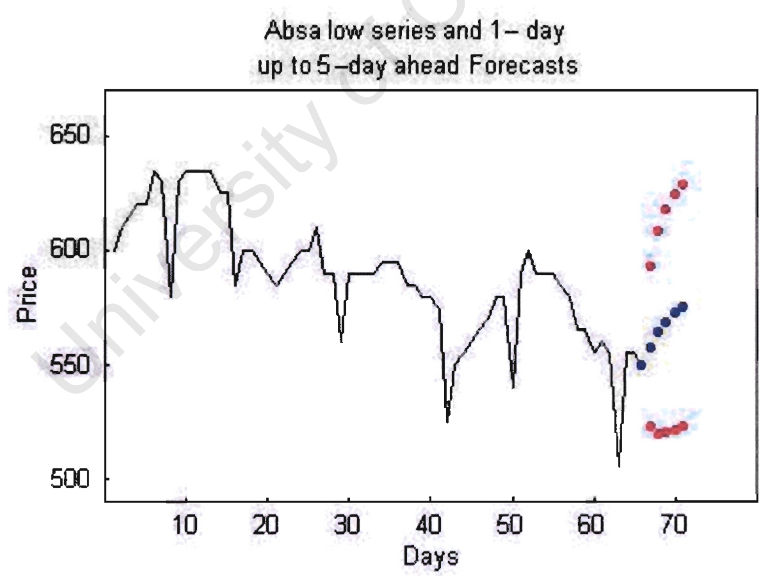

Figure 6-15: Forecast of low Absa price with the original series

Forecast	558.044	564.234	568.997	572.662	575.482
Standard error	(17.90)	(22.586)	(24.949)	(26.248)	(26.988)
Day	67	68	69	70	71

Table 6.3: Forecasts of low Absa prices

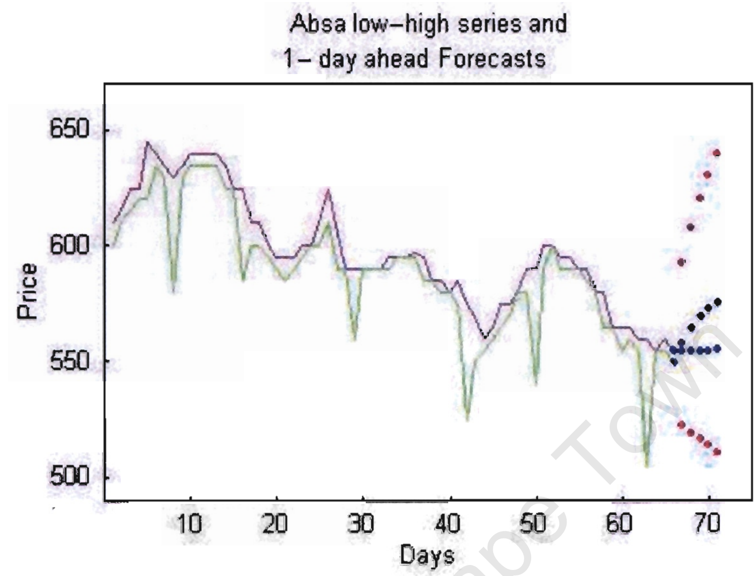

Figure 6-16: Forecast of low-high Absa prices

What follows next is a combination of the low and high forecasts in one graph Figure 6-16, and giving the upper limit of the low-high interval as the upper limit of the confidence intervals of the high estimates, and inversely, the lower limit of the low-high interval as the lower limit of the confidence intervals of the low prices.

The low-high forecasts and the low-high confidence interval are given in Table 6.4.

From Table 6.4 and Figure 6-16 it can be noticed that uncertainty increases as time the forecast horizon increases, this increasing uncertainty being demonstrated by the widening confidence interval as forecast horizon increases. All the 5 actual closing Absa prices and the actual low-high pairs in the forecast region are enclosed by the corresponding forecast confidence intervals in the forecast region.

Forecast Dates	Actual Intervals	AR(1) Intervals	Confidence Intervals of AR(1) Intervals	Closing
10/08/87	[550, 550]	[555.10, 558.05]	[522.96, 593.13]	550
11/08/87	[550, 555]	[555.19, 564.23]	[519.97, 608.50]	555
12/08/87	[555, 560]	[555.29, 569.00]	[517.15, 620.85]	555
13/08/87	[555, 560]	[555.38, 572.66]	[514.20, 631.12]	555
14/08/87	[560, 565]	[555.47, 575.48]	[511.09, 639.87]	560

Table 6.4: Forecasts of low-high intervals of Absa prices

6.3.2 Anglo Low-High interval

The data used in this subsection for interval parameter estimation under the AR(1) model is for the period 25/04/1995 to 03/08/1995 - a period of 3 months. The graph of low-high series of Anglo prices is given in Figure 6-17.

The analysis is again carried out first on the high Anglo prices. The autocorrelations of the observed series of high Anglo prices is shown in Figure 6-18.

The ACF of the time series dies out exponentially and this decay suggests a tentative AR(p) model for the observed data series. The order suggested by the PACF leads to an AR(1) model, since there is only one spike at lag 1, as shown in Figure 6-19.

The least squares method gives the specific model

$$\begin{aligned} r_t &= 591.9094 + 0.88096 r_{t-1}, \ \hat{\sigma} = 53.7758 \\ & \quad (265.0704) \ \ (0.05303) \\ & \quad (2.2330) \ \ (16.6122) \end{aligned}$$

with the corresponding standard errors and t-statistics given, respectively, within the brackets. The errors of the high Anglo prices from the AR(1) model values are given in Figure 6-20.

The residual values suggest some serial independence and can therefore be regarded as white noise values. This inference can be confirmed by the ACF and the PACF values of the residuals depicted in Figures 6-21 and 6-22 respectively.

The 1-day ahead estimates performed up to a 5 day horizon with origin at day 70 (03 August 1995) and horizons at days 71, 72, 73, 74, and 75 are depicted graphically in Figure 6-23.

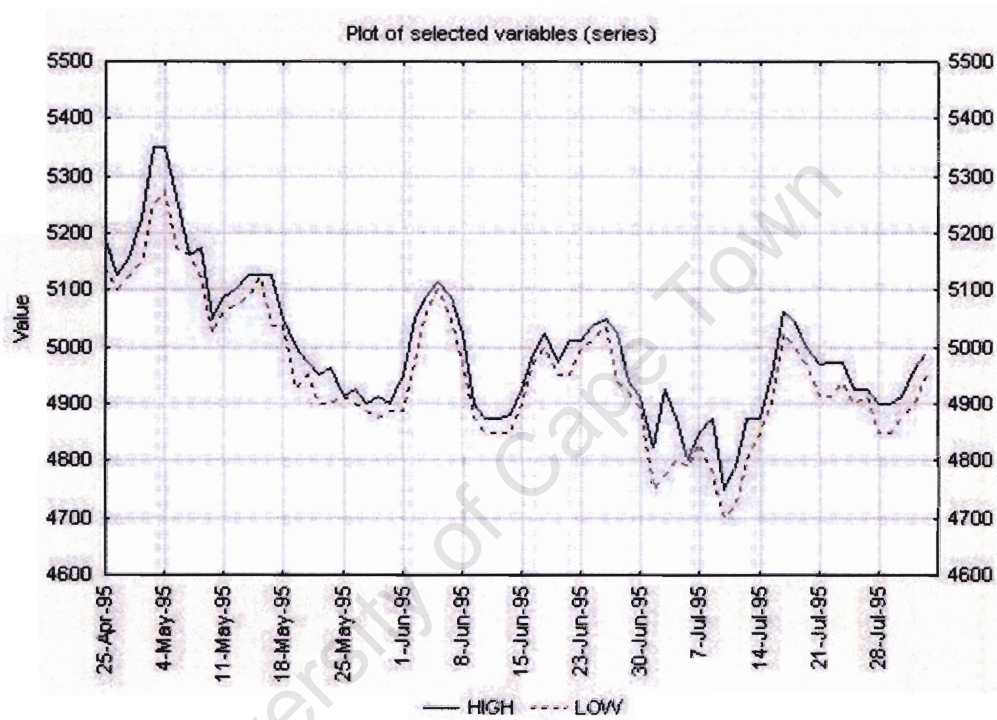

Figure 6-17: Time plot of daily low-high Anglo prices

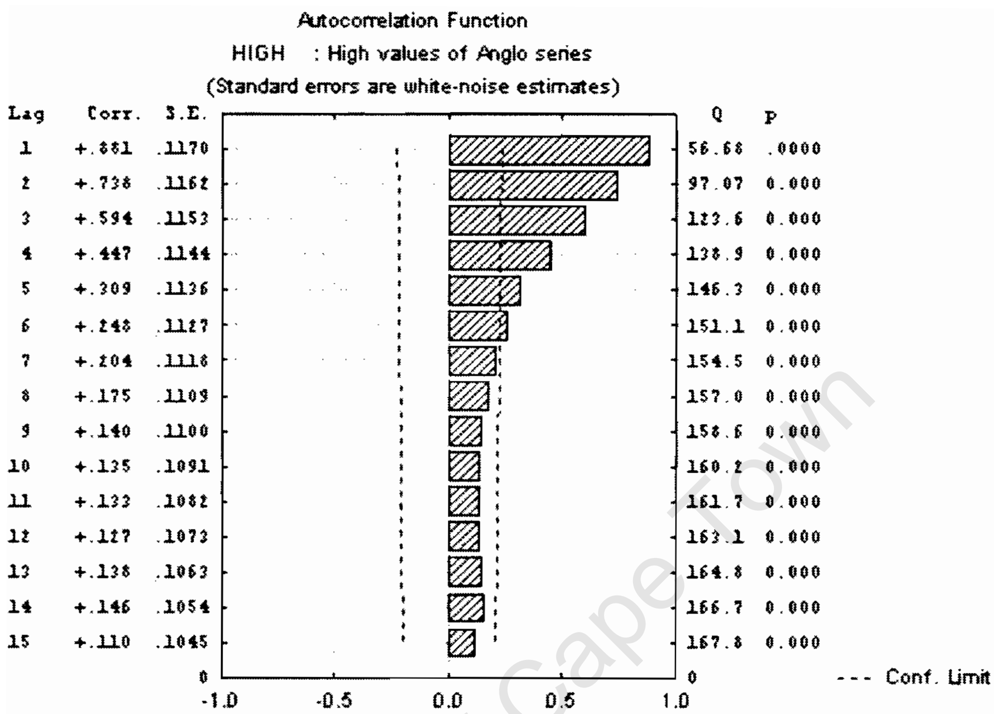

Figure 6-18: ACF of high Anglo prices

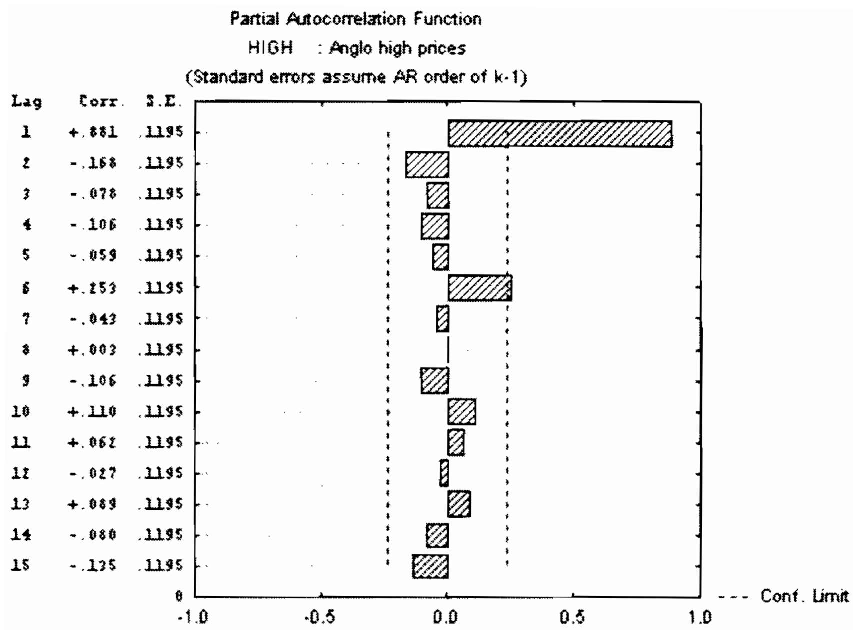

Figure 6-19: PACF of high Anglo prices

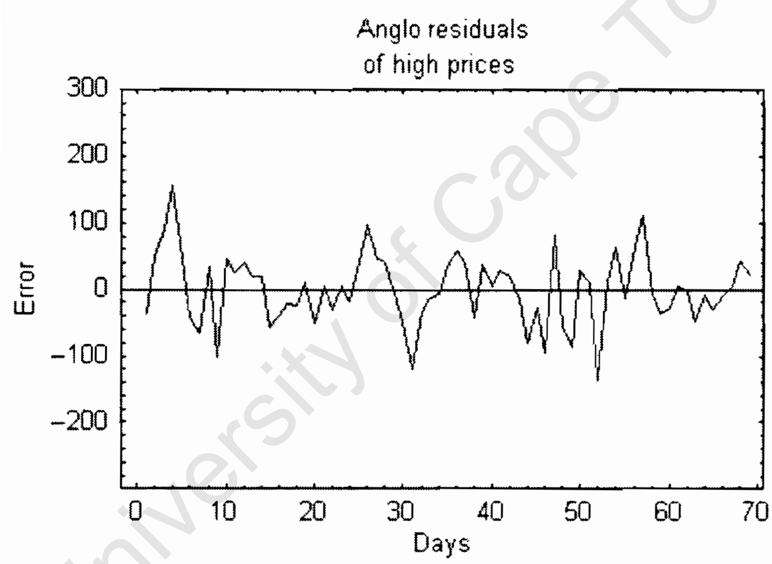

Figure 6-20: Residuals of high Anglo series under AR(1) model

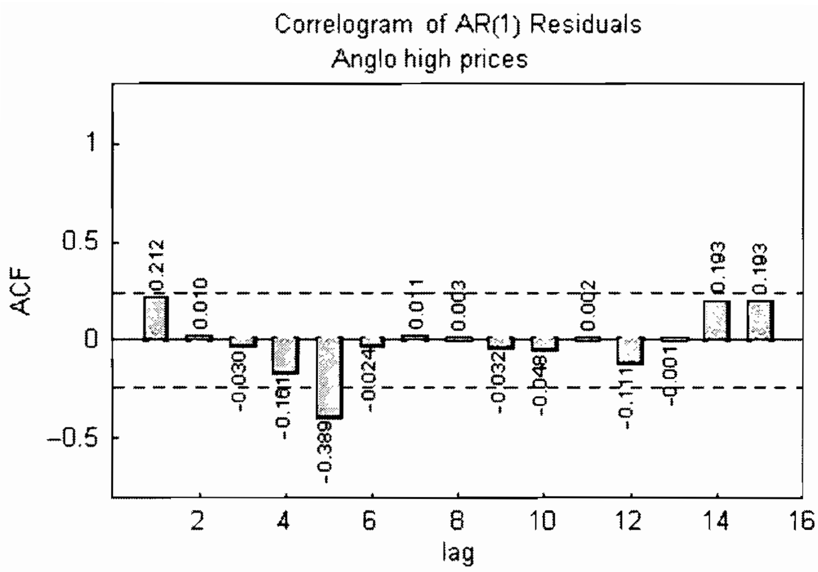

Figure 6-21: ACF of residuals of high Anglo prices

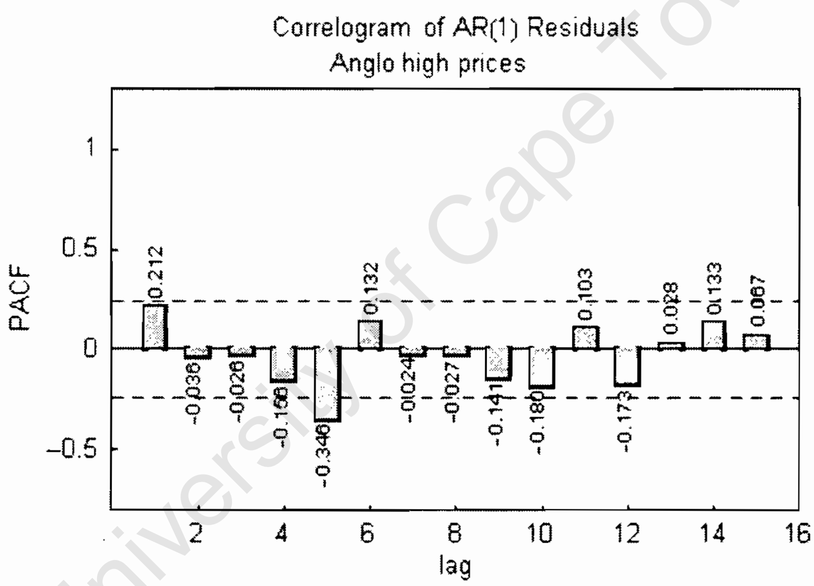

Figure 6-22: PACF of residuals of high Anglo prices

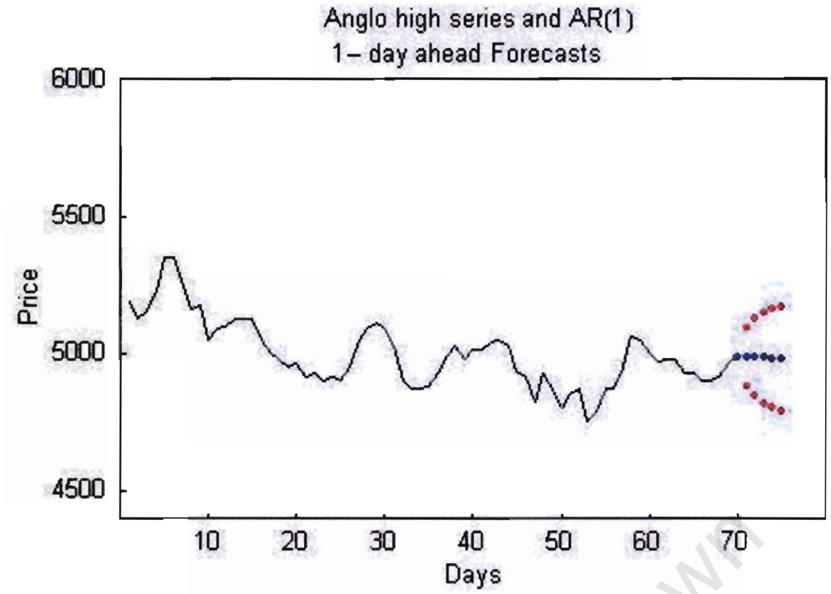

Figure 6-23: Forecast of high Anglo prices

Forecast	4986.16	4984.53	4983.10	4981.84	4980.73
Standard error	(53.776)	(71.667)	(85.910)	(98.106)	(108.946)
Day	71	72	73	74	75

Table 6.5: Forecasts of high Anglo prices

Again it emerges clearly in the graph that increasing the forecast horizon increases the uncertainty in forecast. The forecasts are shown in Table 6.5 together with the corresponding uncertainty measures as given by the standard errors in brackets.

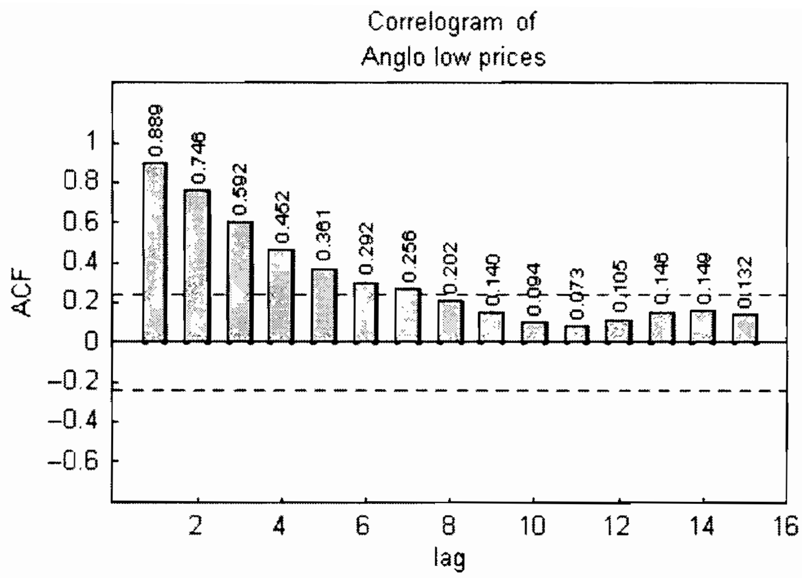

Figure 6-24: ACF of low Anglo prices

Parameter Estimation Of Anglo Low Prices

The process of model identification is now applied to low prices of Anglo. Figures 6-24 and 6-25 give the ACF and PACF of the low Anglo prices, respectively.

The combination suggests an AR(1) model with the estimated model, given with standard errors and t-statistics in brackets by

$$
\begin{aligned}
r_t &= 549.0713 + 0.8886 r_{t-1}, \ \hat{\sigma} = 52.0145 \\
& \quad (255.0839) \ \ (0.05147) \\
& \quad (2.1525) \ \ (17.2662)
\end{aligned}
$$

The graph of residuals of low Anglo price series is given in Figure 6-26.

The ACF of the low prices of Anglo is given in Figure 6-27.

The Box-Pierce Q-statistic of the residuals gives values

$$Q(8) = 15.3749 \quad Q(9) = 15.3764 \quad Q(10) = 17.3960$$

which are not quite significant at the 5% significance level. The Q-statistics together with

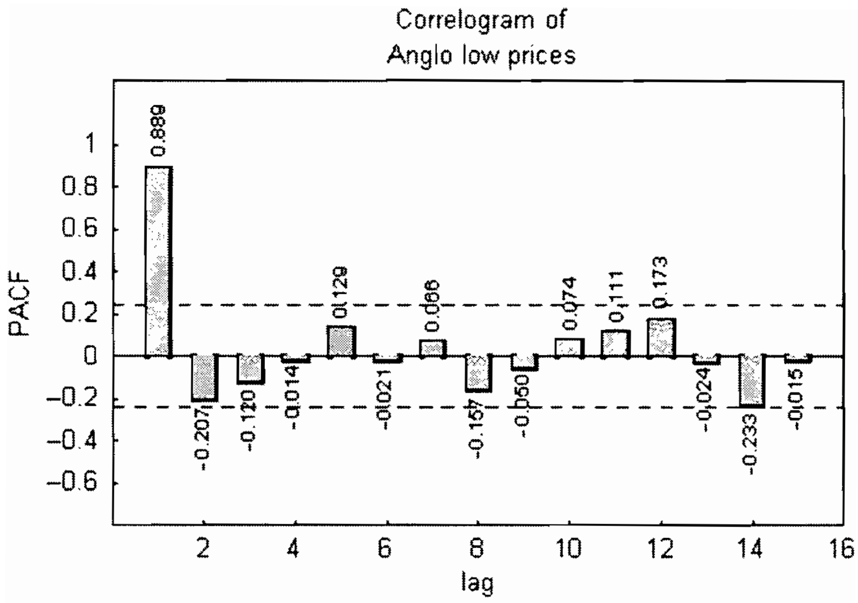

Figure 6-25: PACF of low Anglo prices

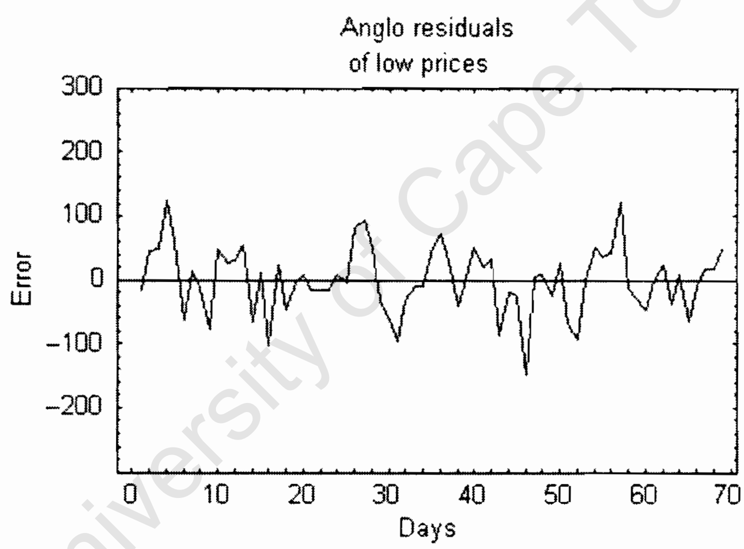

Figure 6-26: Residuals of low Anglo series under AR(1) model

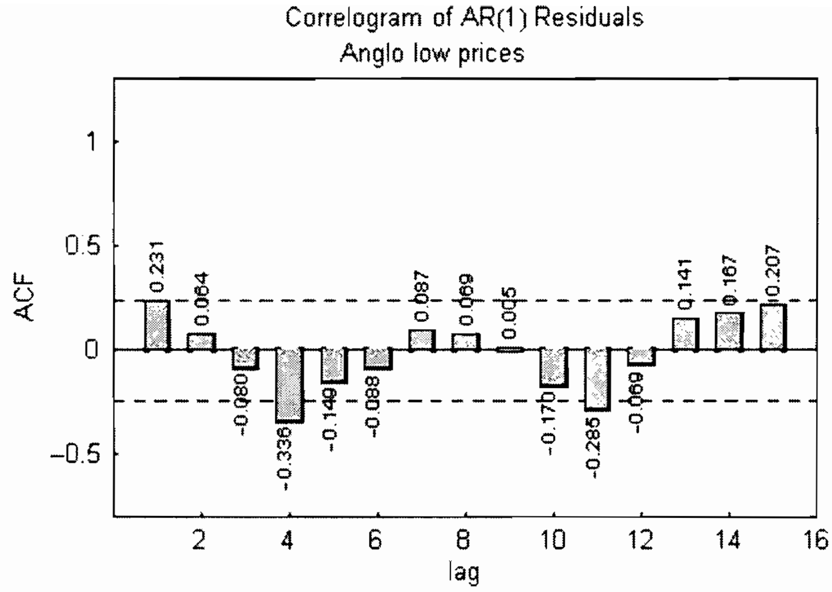

Figure 6-27: ACF of the residuals of low Anglo prices

Forecast	4947.80	4945.85	4944.12	4942.58	4941.21
Standard error	(52.015)	(69.584)	(80.803)	(88.664)	(94.411)
Day	71	72	73	74	75

Table 6.6: Forecasts of low Anglo prices

information from the graph of residuals suggest that the residuals can be regarded as white noise. The forecasts are shown in Table 6.6, together with the corresponding uncertainty measures given as the standard errors in brackets

Combining the low-high forecasts of Anglo prices and their confidence intervals yields the graphical picture given in Figure 6-28.

The low-high forecasts and the low-high confidence interval of Anglo are given in Table 6.7. Again all the 5 actual closing Anglo prices and the actual low-high pairs for the forecast period are enclosed by the forecast confidence in the forecast region.

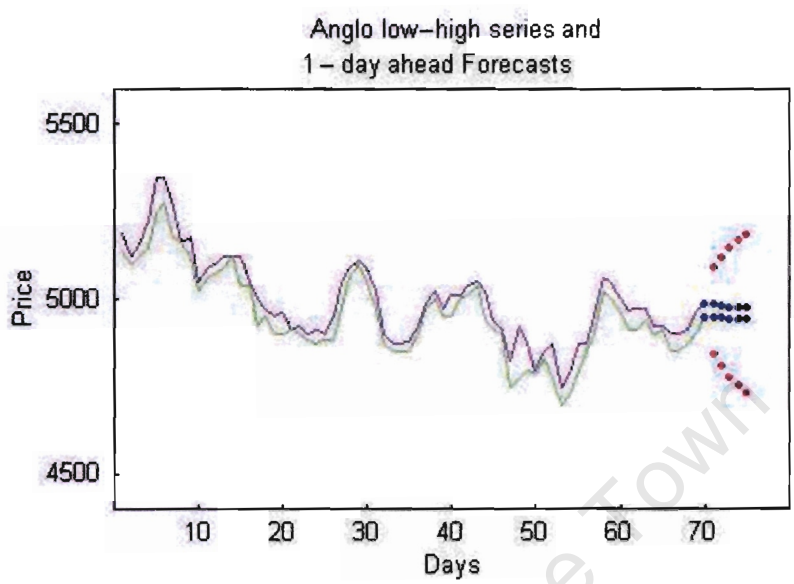

Figure 6-28: Forecast of low-high Anglo prices

Forecast Dates	Actual Intervals	AR(1) Intervals	Confidence Intervals of AR(1) Intervals	Closing
04/08/95	[4975, 5000]	[4947.81, 4986.16]	[4845.96, 5091.56]	4988
07/08/95	[4963, 5000]	[4945.86, 4984.54]	[4809.47, 5125.00]	5000
08/08/95	[4994, 5013]	[4944.12, 4983.11]	[4785.75, 5145.65]	5000
10/08/95	[4988, 5000]	[4942.59, 4981.85]	[4768.80, 5159.65]	4938
11/08/95	[4925, 4950]	[4941.22, 4980.74]	[4756.16, 5169.54]	4950

Table 6.7: Forecasts of the low-high intervals of Anglo prices

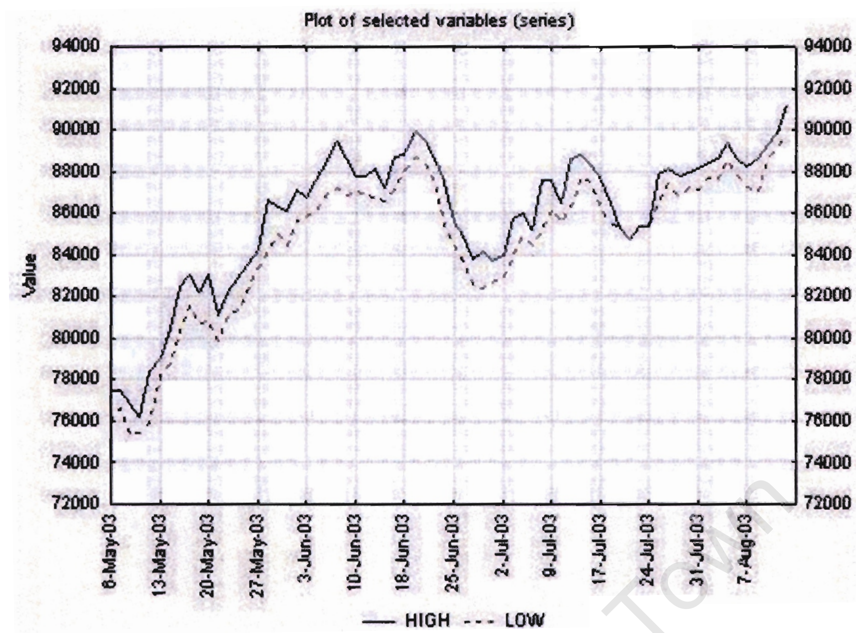

Figure 6-29: Daily low-high JSE-Over index levels

6.3.3 JSE-Over Low-High interval

The interval parameter estimation for the low-high JSE-Over index levels under the classical AR(1) model is performed on a JSE-Over series for the period between 06 May 2003 to 13 August 2003. The low-high time series is given in Figure 6-29.

The ACF of the low index values, given in Figure 6-30, dies fairly quickly and it suggests an autoregressive model for the data. Furthermore the PACF values of the observed series given in Figure 6-31. All the PACF values beyond lag 1 are within the $\frac{\pm 2}{\sqrt{70}} = 0.239$ confidence band, strongly suggesting an AR(1) model.

The least squares estimates of the model parameters are given by the model with standard errors and t-statistics in brackets below.

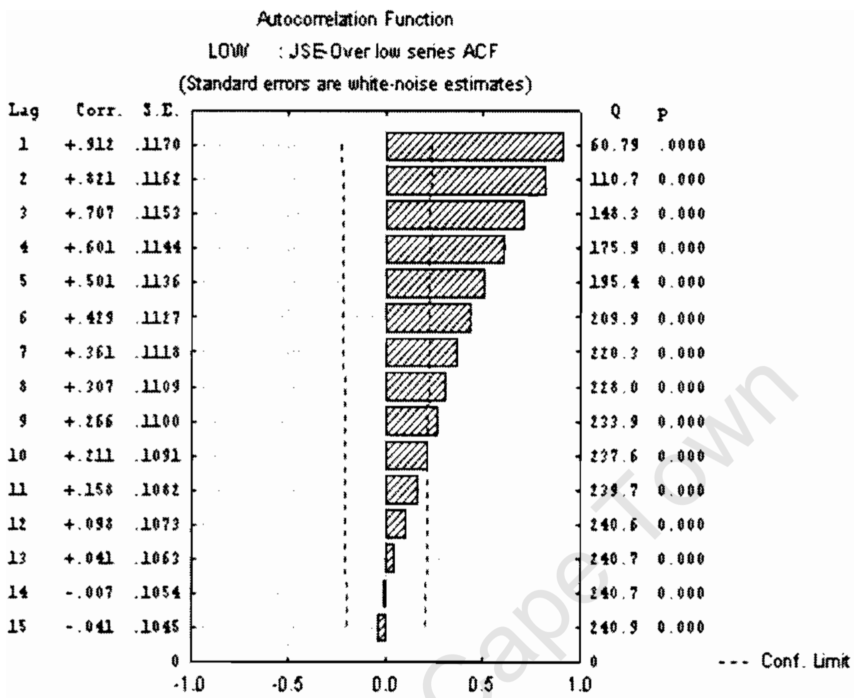

Figure 6-30: ACF of low JSE-Over index series

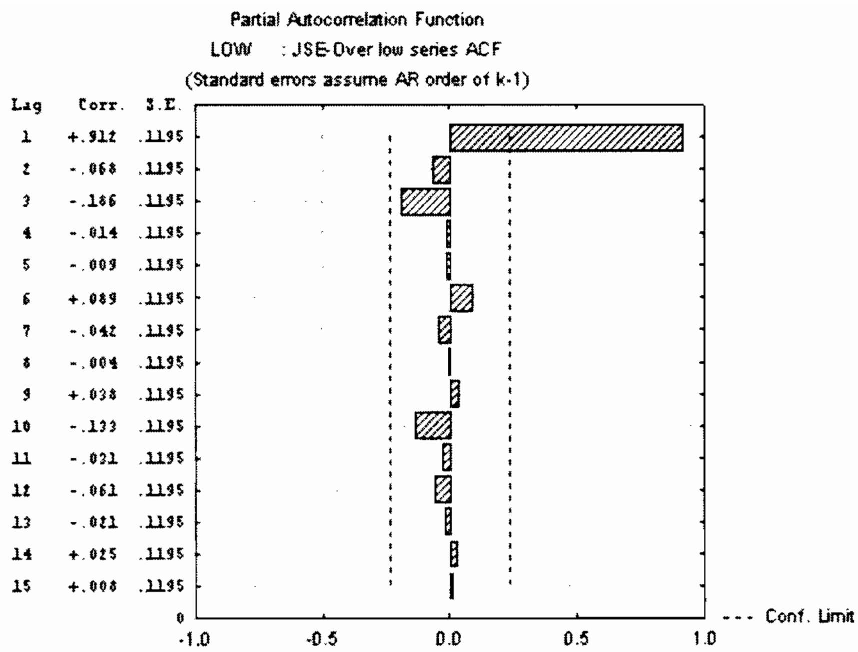

Figure 6-31: PACF of low JSE-Over index series

$$
\begin{aligned}
r_t &= 4994.76 + 0.94331 r_{t-1}, \ \hat{\sigma} = 796.264 \\
& \quad (2353.53) \ \ (0.02781) \\
& \quad (2.1222) \ \ (33.91827)
\end{aligned}
$$

The graph of residuals of the low index values of JSE-Over is given in Figure 6-32.

The residuals from Figure 6-32 appear independent. This view is confirmed by the ACF values of residuals given in Figure 6-33, where all the ACF values of the residuals are within the $\frac{\pm 2}{\sqrt{70}} = 0.239$ limits. The modified Box-Pierce statistics for the ACFs of the residuals are given in Table 6.8.

All the statistics are not significant at the 95% level, and it is not unreasonable to suggest that the residuals behave like white noise. The AR(1) forecasts of the low JSE-Over index

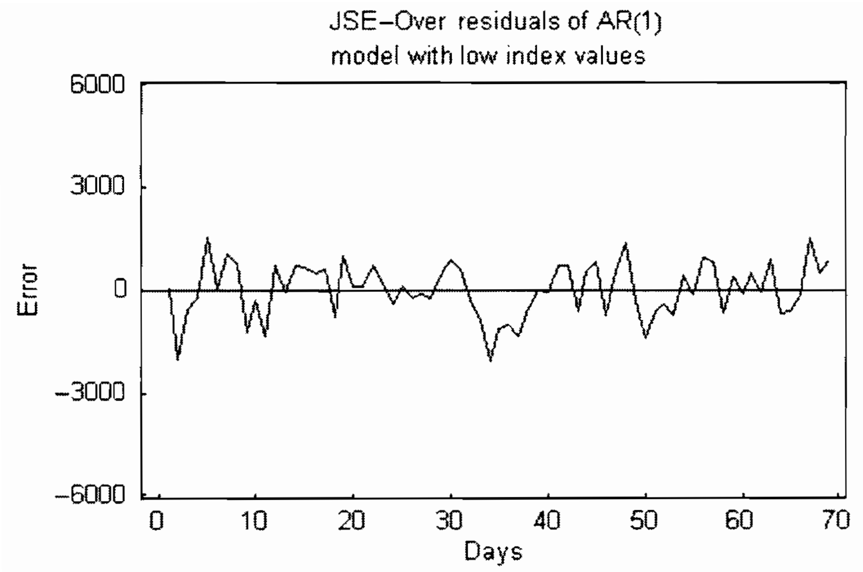

Figure 6-32: Residuals of low JSE-Over series under AR(1) model

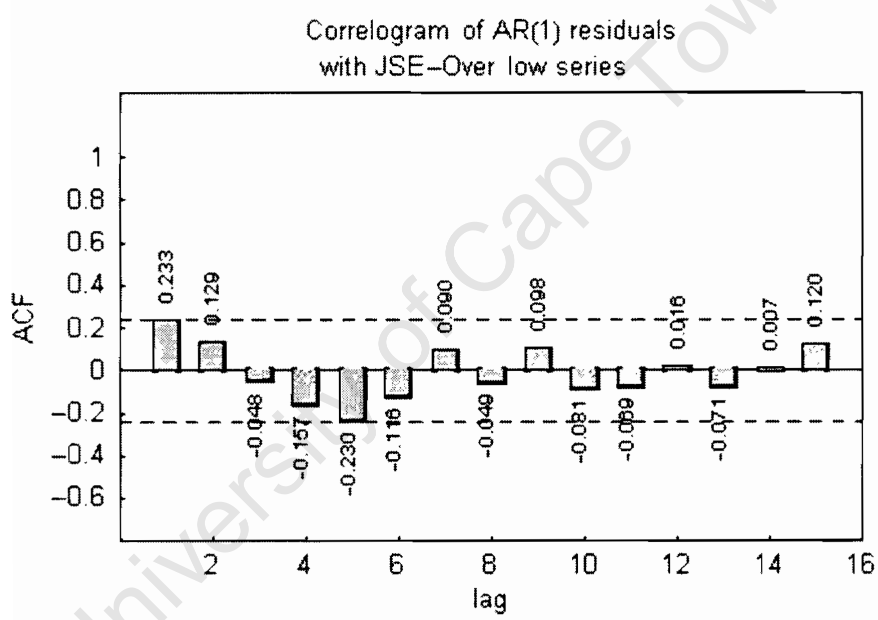

Figure 6-33: ACF of residuals of low JSE-Over index series

Degrees of freedom $(m-1)$	8	9	10	11	12
$Q^*(m)$	14.085	14.641	15.053	15.074	15.515
$\chi^2_{0.05}(m-1)$	15.507	16.919	18.307	19.675	21.026

Table 6.8: Modified Box-Pierce statistic for low JSE-Over series

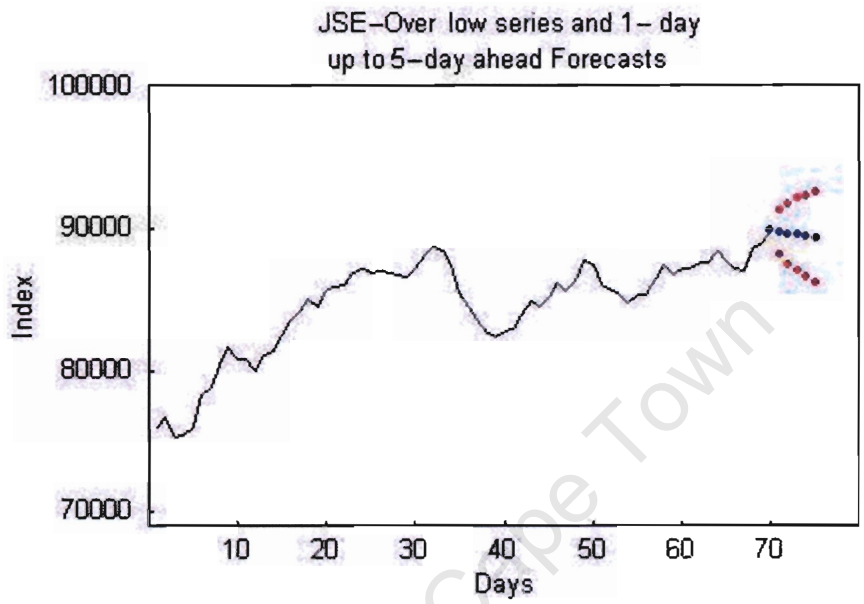

Figure 6-34: Forecasts of low JSE-Over index levels with the original series

levels with the origin at day 70 (13 August 2003) and horizons of 1 up to 5 days are given in Figure 6-34.

The forecasts are shown in Table 6.9 together with the corresponding uncertainty measures as given by the standard errors in brackets.

Forecast	89745.20	89652.10	89564.20	89481.30	89403.10
Standard error	(796.26)	(1094.63)	(1303.94)	(1465.25)	(1595.14)
Day	71	72	73	74	75

Table 6.9: Forecasts of low JSE-Over index level

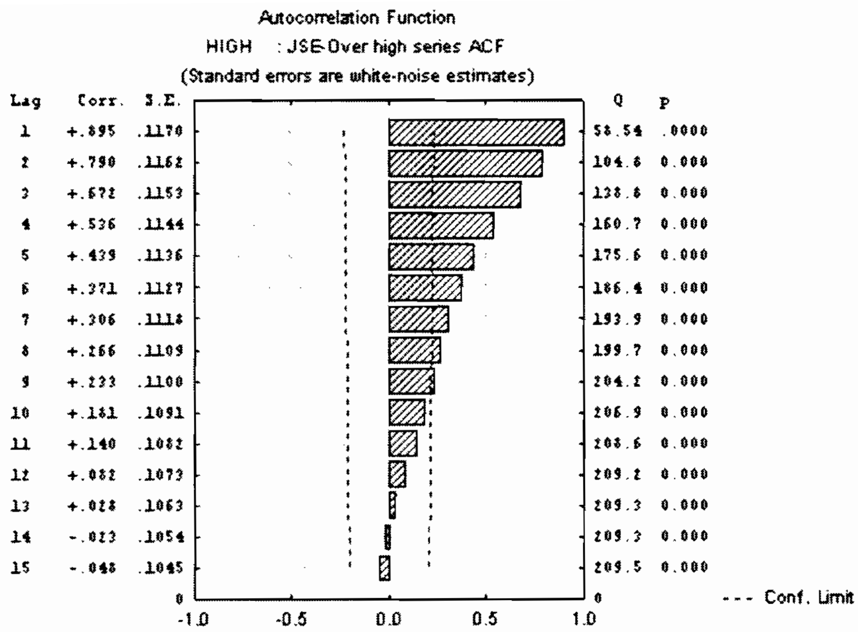

Figure 6-35: ACF of high JSE-Over index series

Parameter estimation of JSE-Over High Index values

The values of high JSE-Over index values lead to the ACF and PACF values given in Figures 6-35 and 6-36 and this information suggests the AR(1) model.

Model adequacy is checked graphically using Figure 6-37.

Two times standard errors of $\hat{r}_1 = \frac{2\times 0.928835}{\sqrt{70}} = 0.222$, so that the lag 1 autocorrelation value is not significantly different from zero. For values of k two times standard error of $\hat{r}_k$ is approximately $\frac{2}{\sqrt{70}} = 0.239$, so only the autocorrelation value at lag 7 is significant and falls beyond two standard errors of zero. This value alone should not lead to the rejection of the null hypothesis of independence of the residuals. The prescribed AR(1) model is given by

$$\begin{aligned} r_t &= 6302.09 + 0.9288 r_{t-1},\ \hat{\sigma} = 795.229 \\ & \quad (3044.74) \quad (0.03547) \\ & \quad (2.0698) \quad (26.1834) \end{aligned}$$

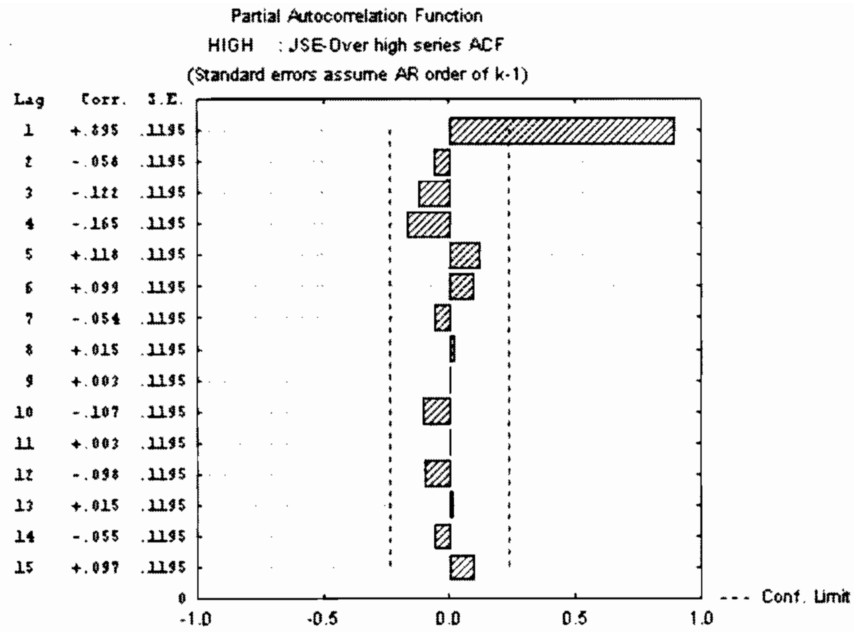

Figure 6-36: PACF of high JSE-Over index series

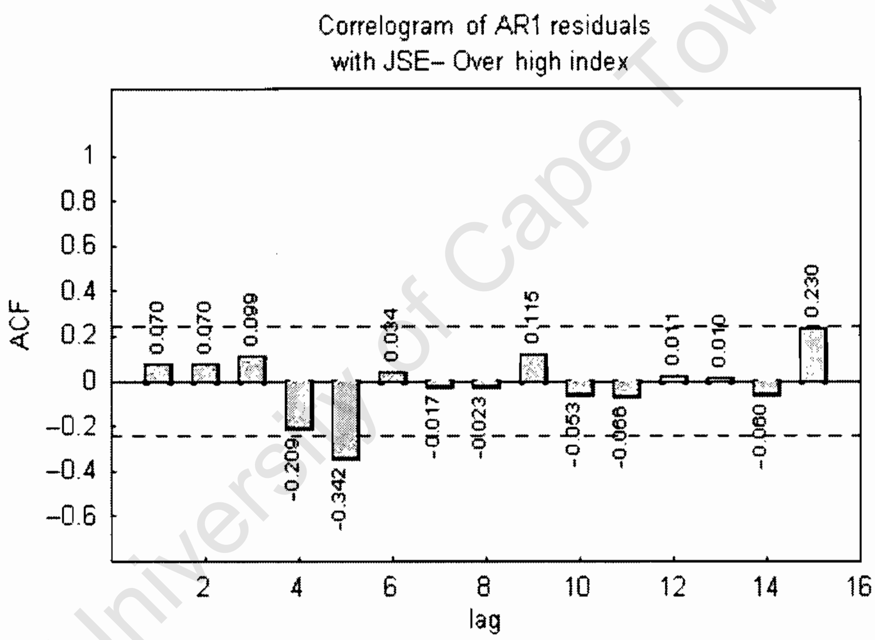

Figure 6-37: ACF of residuals of high JSE-Over index series

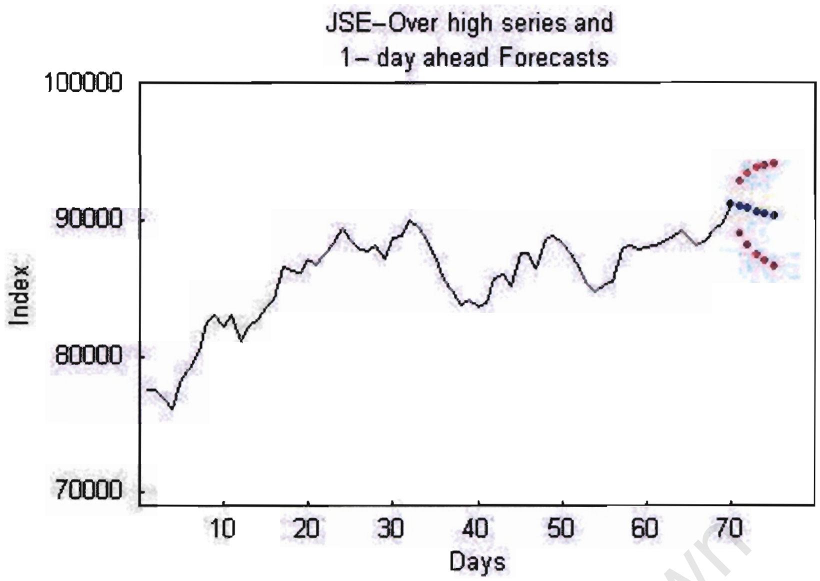

Figure 6-38: Forecasts of high JSE-Over index level

Forecast	91001.60	90827.60	90665.90	90515.80	90376.30
Standard deviation	(795.23)	(1085.34)	(1284.00)	(1438.44)	(1550.84)
Day	71	72	73	74	75

Table 6.10: Forecasts of high JSE-Over index level

The forecast of high JSE-Over index levels with the origin at day 70 and horizons at days $71, 72, 73, 74$,and 75 are given in Figure 6-38.

The forecasts are shown on Table 6.10 together with the corresponding uncertainty measures as given by the standard errors in brackets.

Putting together on one graph the values of low-high forecasts and their confidence intervals of JSE-Over index leads to Figure 6-39.

The low-high forecasts and the low-high confidence intervals of JSE-Over index levels are given in Table 6.11.

As in the previous two cases of Absa and Anglo series, all the 5 actual closing Anglo prices and the actual low-high pairs in the forecast region are enclosed by the forecast confidence in the forecast region.

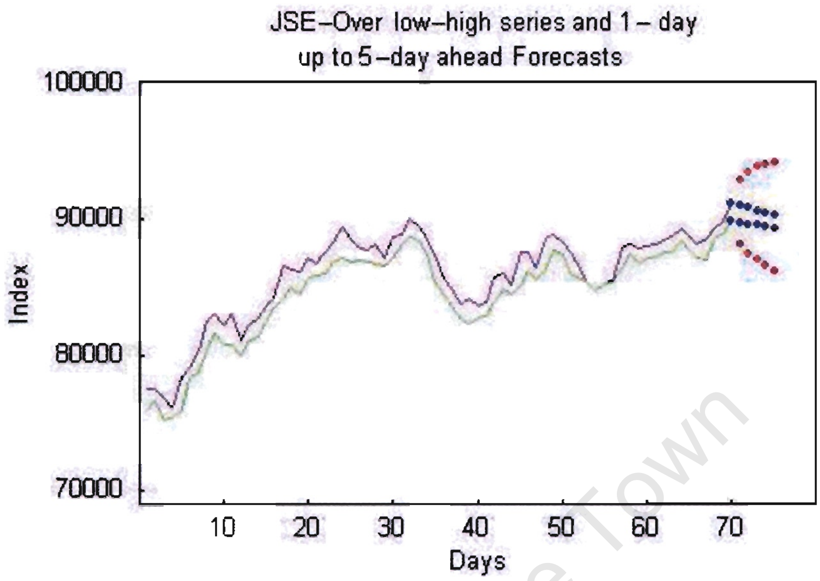

Figure 6-39: Forecasts of low-high JSE-Over index level

Forecast Dates	Actual Intervals	AR(1) Intervals	Confidence Intervals of AR(1) Intervals	Closing
14/08/2003	[89907, 91023]	[89745.20, 91001.60]	[88184.60, 92560.30]	89907
15/08/2003	[89907, 90775]	[89652.10, 90827.60]	[87506.60, 92954.90]	90187
18/08/2003	[89700, 90412]	[89564.20, 90665.90]	[87008.50, 93182.60]	90412
19/08/2003	[90412, 91380]	[89481.30, 90515.80]	[86609.40, 93325.30]	91253
20/08/2003	[91253, 92081]	[89403.10, 90376.30]	[86276.60, 93416.00]	91564

Table 6.11: Forecasts of low-high JSE-Over index level

6.4 Interval AR(1) Analysis Results under d_G metric

6.4.1 Absa interval AR(1) analysis results under d_G metric

Tables 6.12, 6.13, and 6.14 give interval parameters computed from optimising the following expression $\min_{(\overline{\mu},\overline{\phi}_1)} \sum_{t=1}^{N} d_G\left(\overline{y}_t, \overline{\mu} + \overline{\phi}_1 \overline{y}_{t-1}\right)$.

Absa Share Parameter Estimates with $(\alpha = 2.0, \beta = 1.5)$		
Period	$[\mu^l, \mu^u]$	$[\phi_1^l, \phi_1^u]$
06/05/87 - 08/06/87	[1.00039801, 1.00040975]	[0.99842457, 0.99843685]
06/05/87 - 08/07/87	[0.99981432, 0.99981437]	[0.99834741, 0.99834741]
06/05/87 - 07/08/87	[0.99980813, 0.99980815]	[0.99831964, 0.99831964]
25/04/95 - 26/05/95	[1.00017787, 1.00018481]	[0.99930542, 0.99930543]
25/04/95 - 28/06/95	[1.00000942, 1.00000979]	[0.99941870, 0.99941870]
25/04/95 - 28/07/95	[1.00017147, 1.00017155]	[0.99929252, 0.99929252]
06/05/03 - 04/06/03	[0.99815280, 0.99897767]	[1.00549481, 1.00549481]
06/05/03 - 07/07/03	[0.99869331, 0.99927681]	[1.00388696, 1.00388696]
06/05/03 - 07/08/03	[0.99935060, 0.99964059]	[1.00193174, 1.00193174]

Table 6.12: Interval parameter estimates of Absa series

Table 6.12 gives the interval parameter estimates for 9 sets of Absa daily price series. The data used for estimating the first set of parameters is taken from a period of 1 month (22 days)-period 06 May 1987 to 08 June 1987. The second and third set of parameters utilise data over a period of 2 and 3 months (44 and 65 days), respectively; from the top half of the overall Absa share prices of data. The 4^{th}, 5^{th}, and 6^{th}sets of parameters are calibrated from observations of 1 month, 2 months and 3 months, respectively. The three data sets are taken from around the middle of the overall Absa share prices data provided. The last three sets of parameters the 7^{th}, 8^{th}, and 9^{th} are also calibrated from observed data of 1 month, 2 months and 3 months, respectively. The three data sets used are taken from almost the bottom end of the overall Absa share prices data provided.

6.4.2 Anglo interval AR(1) analysis results under d_G metric

Table 6.13 gives the interval parameter estimates for 9 data sets of Anglo daily prices. The data used for estimating the first set of parameters is taken from period of 1 month (22 days)-period 06 May 1987 to 08 June 1987. The second and third sets of parameters utilise observed data over a period of 2 and 3 months (44 and 65 days), respectively; from the first half of the overall Anglo share prices data. The 4^{th}, 5^{th}, and 6^{th} sets of parameters are calibrated from sets of 1 month, 2 months and 3 months (70 days), respectively. The three sets of data are taken from around the middle of the overall Anglo share prices data provided. The remaining sets of parameters the 7^{th}, 8^{th}, and $9^{th}$are also calibrated from data of 1 month, 2 months and

Anglo Share Parameter Estimates with ($\alpha = 2.0, \beta = 1.5$)		
Period	$[\mu^l, \mu^u]$	$[\phi_1^l, \phi_1^u]$
06/05/87 - 08/06/87	[0.99965736, 0.99981037]	[1.00101924, 1.00101924]
06/05/87 - 08/07/87	[0.99972024, 0.99984517]	[1.00083218, 1.00083218]
06/05/87 - 07/08/87	[0.99967888, 0.99982228]	[1.00095522, 1.00095522]
25/04/95 - 26/05/95	[0.99995608, 0.99995846]	[0.99969197, 0.99969742]
25/04/95 - 28/06/95	[3.33649330, 7.91845180]	[0.99941685, 0.99941685]
25/04/95 - 28/07/95	[16.8153975, 21.2237408]	[0.99661687, 0.99661687]
06/05/03 - 04/06/03	[0.99604530, 0.99771010]	[1.00901413, 1.01026250]
06/05/03 - 07/07/03	[1.00001684, 1.00001729]	[0.99962757, 0.99962757]
06/05/03 - 07/08/03	[1.00001976, 1.00001977]	[0.99988855, 0.99988856]

Table 6.13: Interval parameter estimates of Anglo series

JSE-Over Parameter Estimates with ($\alpha = 2.0, \beta = 1.5$)		
Period	$[\mu^l, \mu^u]$	$[\phi_1^l, \phi_1^u]$
06/05/87 - 08/06/87	[1.00000785, 1.00001418]	[0.99995782, 0.99995783]
06/05/87 - 08/07/87	[0.99972373, 0.99984710]	[1.00082181, 1.00082181]
06/05/87 - 07/08/87	[0.99944042, 0.99969030]	[1.00166455, 1.00166455]
25/04/95 - 26/05/95	[1.00012038, 1.00016606]	[0.99935771, 0.99935771]
25/04/95 - 28/06/95	[1.00027222, 1.00027310]	[0.99893489, 0.99893489]
25/04/95 - 28/07/95	[1.00000096, 1.00000174]	[0.99999481, 0.99999481]
06/05/03 - 04/06/03	[608.828648, 872.078188]	[0.99637617, 0.99637617]
06/05/03 - 07/07/03	[0.99954713, 0.99974936]	[1.00134709, 1.00134709]
06/05/03 - 07/08/03	[0.99910776, 0.99950619]	[1.00265412, 1.00265412]

Table 6.14: Interval parameter estimates of JSE-Over Samples

3 months, respectively. The three data sets are taken from around the later end of the overall Anglo share prices provided.

6.4.3 JSE-Over interval AR(1) analysis results under d_G metric

Table 6.14 gives the interval parameter estimates for 9 sets of JSE-Over daily index. The data series used for estimating the first set of parameters is taken from period of 1 month, period 06 May 1987 to 08 June 1987 (22 days). The second and third sets of parameter estimates utilise series over a period of 2 and 3 months (44 and 65 days), respectively; from the later half of the overall JSE-Over index data. The 4^{th}, 5^{th}, and 6^{th} sets of parameters are calibrated from data of 1 month, 2 months and 3 months, respectively. The three sets data the 7^{th}, 8^{th}, and

Absa Interval Series	**Metric**	7.3871	7.6310	7.8835	8.3883	8.8932	9.3981	9.6506	9.9030
06/05/87–	$\frac{\alpha}{\alpha+\beta}$	0.0040	0.1000	0.2000	0.4000	0.6000	0.8000	0.9000	1.0000
07/08/87	$\frac{\beta}{\alpha+\beta}$	0.9960	0.9000	0.8000	0.6000	0.4000	0.2000	0.1000	0.0000

Table 6.15: Changes in the metric with respect to weights using Absa data

Anglo Interval Series	**Metric**	39.9646	40.9930	42.0999	43.1659	44.1397	44.6343	45.1363
05/04/95–	$\frac{\alpha}{\alpha+\beta}$	0.0040	0.2000	0.4000	0.6000	0.8000	0.9000	1.0000
28/07/95	$\frac{\beta}{\alpha+\beta}$	1.0000	0.8000	0.6000	0.4000	0.2000	0.1000	0.0000

Table 6.16: Changes in the metric with respect to weights using Anglo data

9^{th} are taken from around the middle of the overall JSE-Over index data provided. The last three sets of parameters are again calibrated from data of 1 month, 2 months and 3 months, respectively. The three sets are taken from around May 2003 up to around mid August 2003, within the overall JSE-Over index levels provided.

The choice of the combination of parameters $\alpha = 2.0$ and $\beta = 1.5$ is only shown for illustration purposes, from among many possible combinations. In fact other combinations tried gave results that were not materially different to the ones obtained in Tables 6.12, 6.13, and 6.14. The changes in the metric with respect to changes in α and β and therefore with respect to the changes in the weights $\frac{\alpha}{\alpha+\beta}$ and $\frac{\beta}{\alpha+\beta}$ are given in Tables 6.15, 6.16 and 6.17. The effect of choosing α and β in such a way that as $\frac{\alpha}{\alpha+\beta}$ approaches 1 and therefore $\frac{\beta}{\alpha+\beta}$ approaches 0, is that the fitted model intervals get tighter and conversely as $\frac{\alpha}{\alpha+\beta}$ approaches 0 and therefore $\frac{\beta}{\alpha+\beta}$ approaches 1, the fitted model intervals get wider.

JSE-Over Interval Series	**Metric**	744.402	763.684	768.339	787.410	807.382	817.420	827.570
06/05/2003–	$\frac{\alpha}{\alpha+\beta}$	0.004	0.200	0.400	0.600	0.800	0.900	1.000
07/08/2003	$\frac{\beta}{\alpha+\beta}$	0.996	0.800	0.600	0.400	0.200	0.100	0.000

Table 6.17: Changes in the metric with respect to weights using JSE-Over data

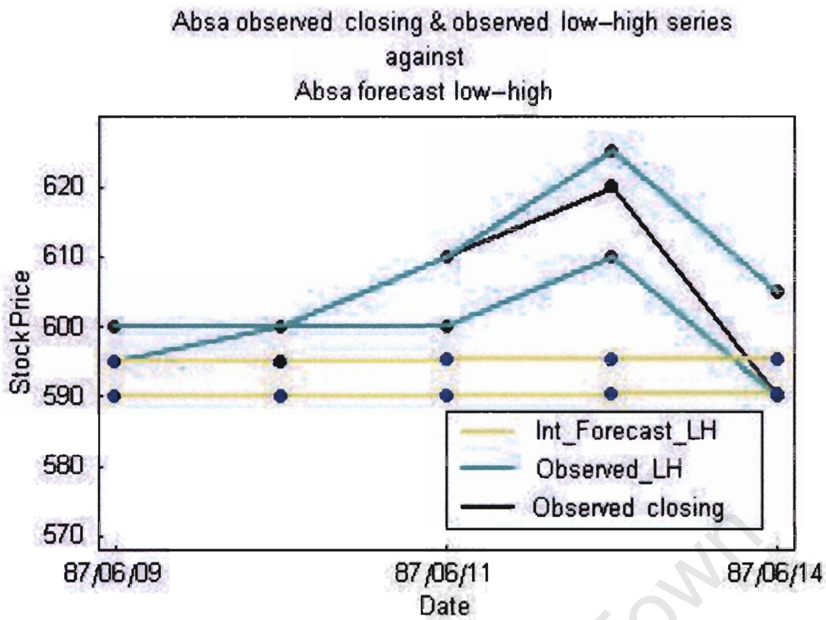

Figure 6-40: 5-day low-high forecast of Absa prices from Interval AR(1)

6.5 5-day Low-High Forecasting and Comparisons

Figure 6-40 gives 5-day low-high forecast following 1 month (22-day) series from 6 May 1987 to 8 June 1987.

From Figure 6-40 it can be seen that the forecast is bad in that the interval of low-high prices does not enclose the actual closing share price, nor does it enclose the actual low-high of the share price. The next Figure 6-41, gives the plot of the 5-day forecast following a different one month series, from 4 April 1995 to 5 May 1995.

The forecast matches the actual closing and the actual low-high closing prices only on the first day of forecast, and on the second day and beyond it overestimates both the actual low-high values. However, the interval-valued AR(1) performs better than classical low-high forecast.

Figure 6-42 depicts the plot of 5-day forecast of low-high Absa prices from both the classical and the interval-valued AR(1) models. The sample used is a 3 month sample starting on 5 May 1987 to 7 August 1987. Both models perform fairly well, however the interval-valued forecast performs slightly better than the classical model in that it encloses all the actual closing prices

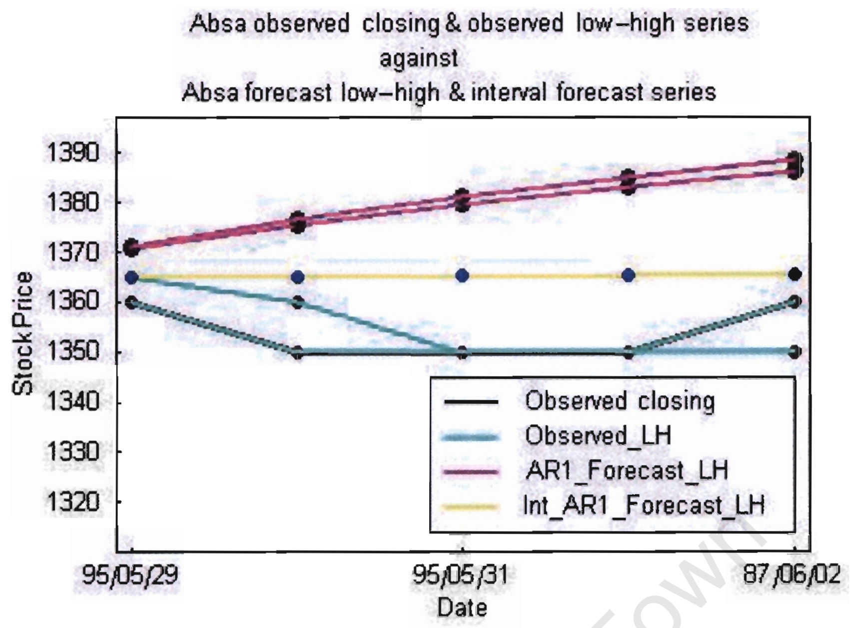

Figure 6-41: 5-day low-high forecast of Absa prices from Interval AR(1)

and most of low-high values.

Figure 6-43 gives forecasts from a 2 month sample (taken from the middle of the Absa Time Series), the forecast looks perfect but the interval-AR(1) model performs slightly better than its classical counterpart.

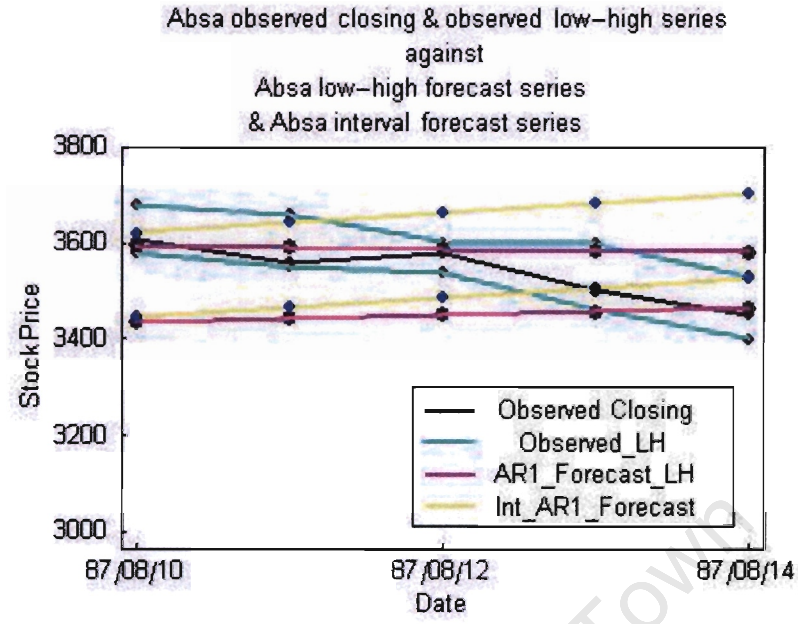

Figure 6-42: 5-day low-high forecast of Absa prices from Interval AR(1)

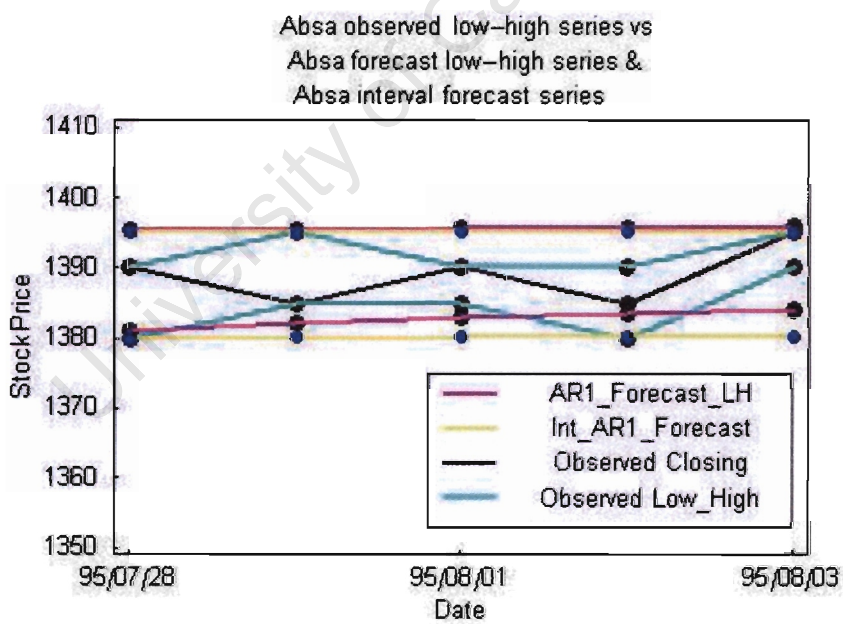

Figure 6-43: 5-day low-high forecast of Absa prices from Interval AR(1)

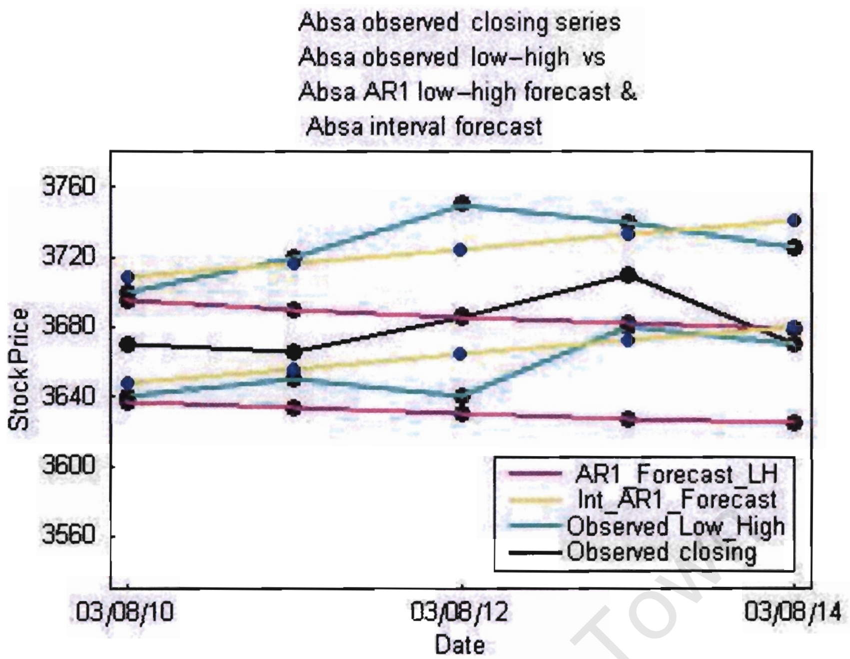

Figure 6-44: 5-day low-high forecast of Absa prices from Interval AR(1)

Figure 6-44 is the forecast from a 3 month sample taken from the middle of the Absa time series.

The interval-valued AR(1) forecast seems to track the actual low-high Absa values, better than its classical counterpart.

Figure 6-45 gives 5-day forecast of the 1 month Anglo sample from the beginning of the time series. The forecasts do not look good for both models, however in comparative terms the interval AR(1) forecast is marginal sample, winner.

Figure 6-46 gives the 5-day forecast of Anglo prices from a 2 month series, taken between 05 May 1987 to 06 July 1987. The forecast looks good for the first 3 days and performs badly from the fourth day onwards.

Figure 6-47 gives a 5-day forecast of Anglo prices from a 2 month series-that starts at the middle of the time series. The forecast looks perfect and again the interval-valued AR(1) performs better than its classical counterpart.

Figure 6-48 gives the 5-day forecast of Anglo prices from a 2 month data series that starts

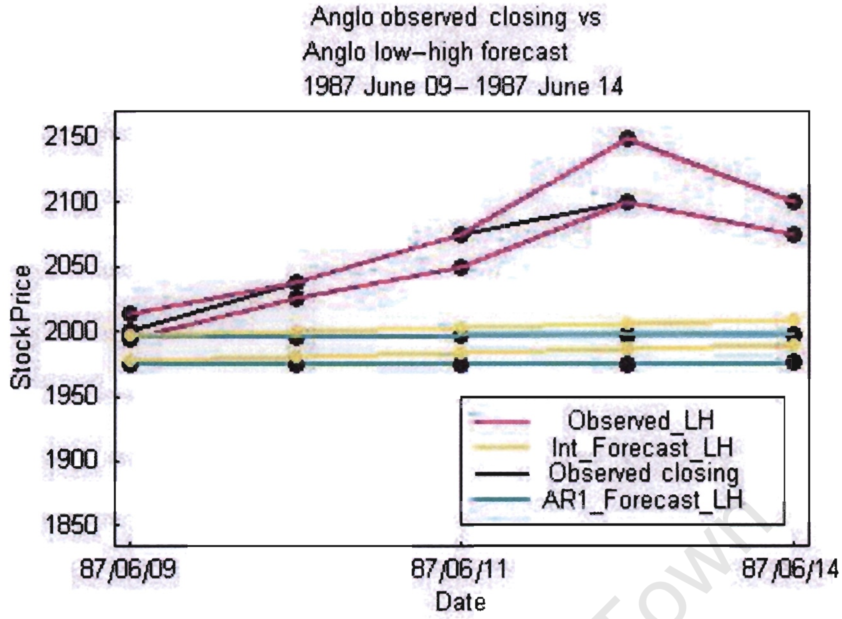

Figure 6-45: 5-day low-high forecast of Anglo prices from Interval AR(1)

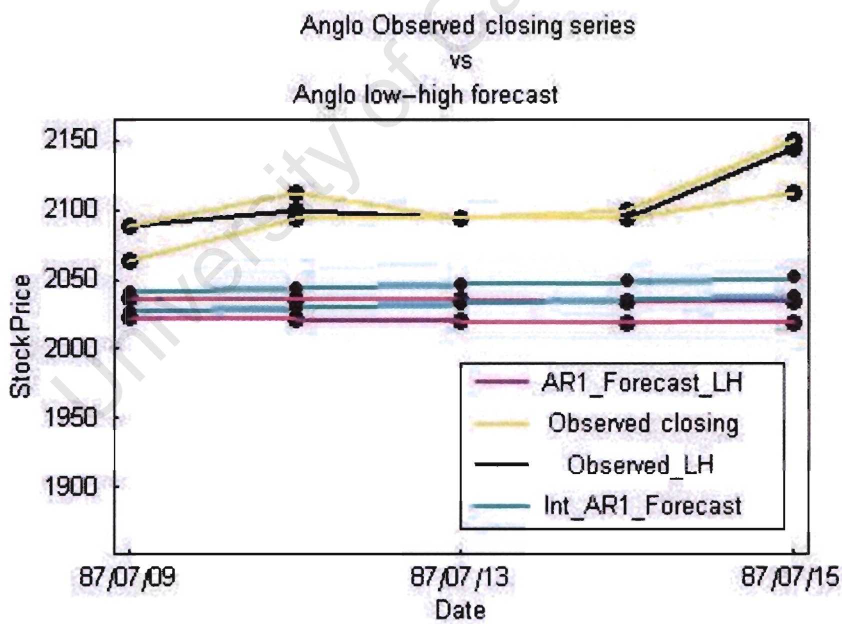

Figure 6-46: 5-day low-high forecast of Anglo prices from Interval AR(1)

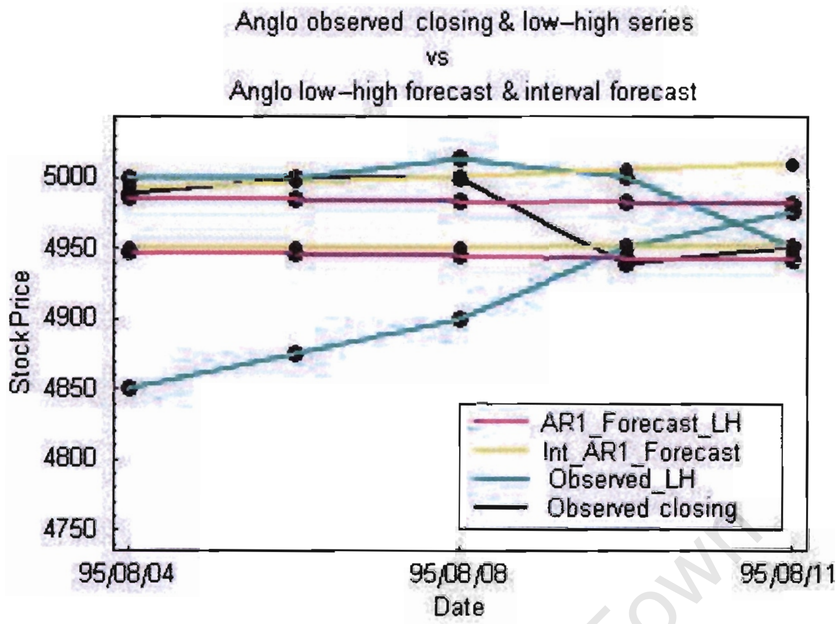

Figure 6-47: 5-day low-high forecast of Anglo prices from Interval AR(1)

at the later end of the overall Anglo time series. The forecast looks perfect, however there is no real difference in terms of performance between the interval-valued AR(1) and its classical counterpart.

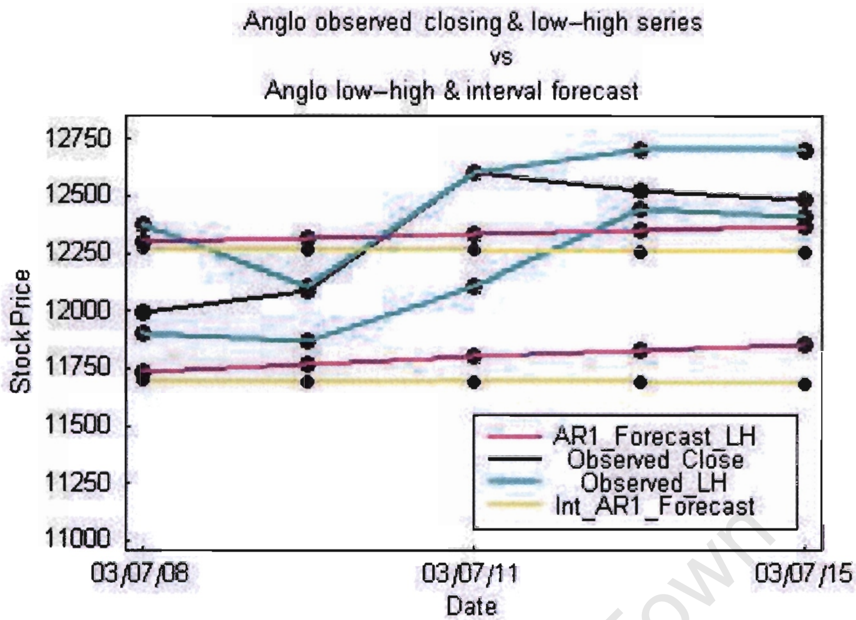

Figure 6-48: 5-day low-high forecast of Anglo prices from Interval AR(1)

Figure 6-49 gives a 5-day forecast of the JSE-Over index with parameters from a 1 month time series. The interval-valued AR(1) model gives the same results as the classical low-high AR(1) model.

Figure 6-50 gives a 5-day forecast of the JSE-Over Index using parameters from a 3 month sample. The forecast does not give good results at all, but again the interval AR(1) model performs better than its classical counterpart.

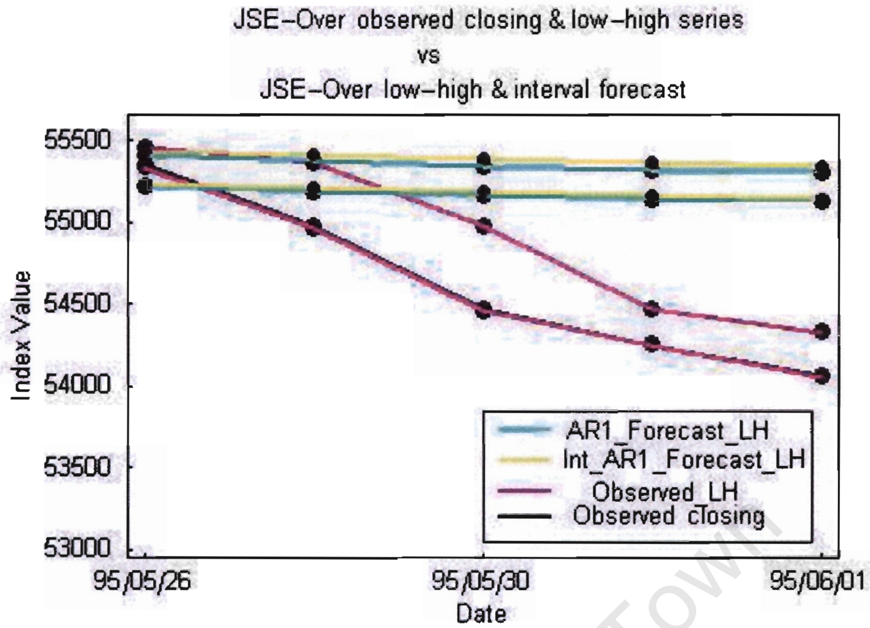

Figure 6-49: 5-day low-high forecast of JSE-Over Index values from Interval AR(1)

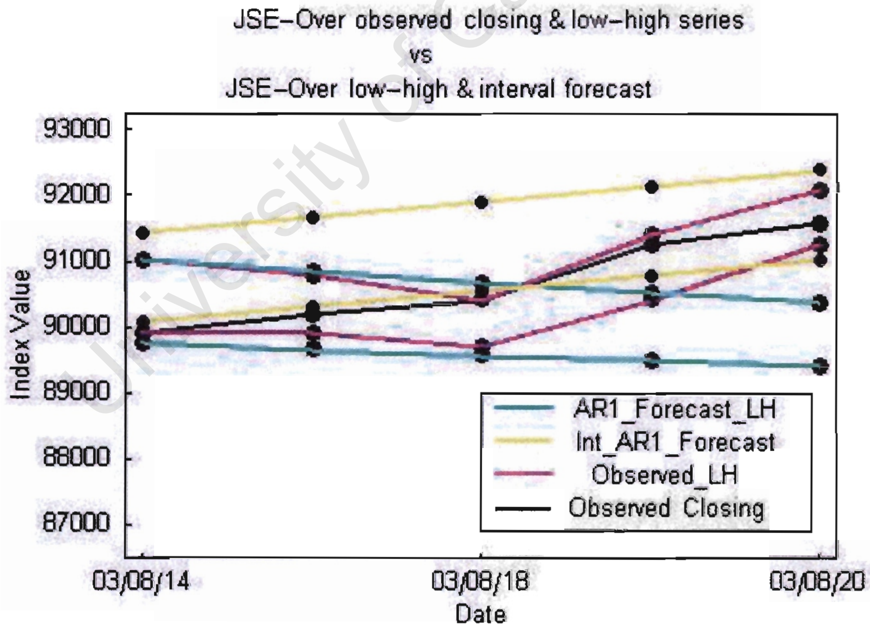

Figure 6-50: 5-day low-high forecast of JSE-Over Index values from Interval AR(1)

Chapter 7

Concluding Remarks

In this mini-dissertation we propose a new methodology for forecasting low and high market prices of traded financial instruments. This is a novel and fresh approach in analyzing financial data. We have contrasted Interval Analysis and traditional Time Series Analysis. The analysis of this research was confined to the AR(1) model, and comparisons were made in the forecasting power of the two methodologies. There is no conclusive evidence from the results as to whether the interval-valued AR(1) model is a better prediction method than the classical AR(1) model used to model lower and higher stock prices. As a matter of fact in standard time-series analysis AR(1) modelling of stock prices is far from realistic and adequate. Extension of this work could include modelling by traditional AR(p) as well as the interval AR(p) under d_G.

The AR(1) model is known, as seen from the time series analysis of the given to be a bad model for predicting of stock prices. A more general ARMA(p, q) with $p > 1, q \geq 1$ needs to be investigated. It is clear from the Time Series analysis of both low and high stock and index levels that AR(1) model is not an appropriate model for market prices but can be used to model market returns. Rather than using one method or the other it would a good idea to use any tool to complement the already existing techniques. For the traditional Time Series modelling the current state of the technology is mostly confined to single Time Series analysis tools, which makes it rather tedious to handle time series for low prices and then separately go through the same estimation and forecasting process for time series of high prices/index values.

The outstanding challenges in the interval analysis toolkit are to develop optimisation tools that can deal with interval parameter estimation for two dimensions and beyond, so that Interval

AR(p) for $p \geq 2$, can be estimated. Research in this direction would hopefully improve the power of interval forecasts.

Bibliography

[1] Alamo, T., Bravo, J.M., and Camacho, E.F. "Guaranteed state estimation by zonotopes" IEEE (2003), 5831-5836

[2] Bertoluzza, C. Salas, A. and Corral, N. "On a New Class of Distances between Fuzzy Numbers" Mathware Soft Comput. 2 (1995) 71-84

[3] Bowerman, B.L., and O'Connell, R.T. " Time Series Forecasting: Unified Concepts and Computer Implementation". Second Edition, PWS Publishers (1987).

[4] Corliss, G. F: "Tutorial on Validated Scientific Computing Using Interval Analysis". www.eng.mu.edu/corlissg/PARA04, (June 2004)

[5] Dempster, A. P. " Upper and Lower Probabilities Generated By a Random Closed Interval", the Annals of Mathematical Statistics, 1968, Vol. 30, No 3, 957-966

[6] Diamond, P. "Interval-Valued Random Functions and the Kriging of Intervals", Mathematical Geology, Vol. 20, No 3. 1988, 145-165

[7] Dimitrova, N.S., Markov, S.M., and Popova, E.D. "Extended Interval Arithmetics: New Results and Applications". Computer Arithmetic and Enclosure Methods, Elselvier Sci. Publishers:1992, 225-232

[8] Ferson, S. and Kreinovich V. "A New Cauchy-Based Black-Box Technique for Uncertainty in Risk Analysis"

[9] Gau, C. and Stadtherr, M.A. "Reliable Nonlinear Parameter Estimation using Interval Analysis: Error-in-Variable Approach". PSE (2000)

[10] Granvilliers, L., Cruz, J. and Barahona, P. "Parameter Estimation using Interval Computations"

[11] Gutowski, M.W. "Interval Straight Line Fitting". Available at URL: http://arXiv:math.org/abs/cs/0108163 v1, (2001)

[12] Gutowski, M.W. "Interval Experimental Data Fitting", Available at gutowski@ifpan.edu.pl

[13] Hickey, T. and Ju, Q. and van Emden, M.H. "Interval Arithmetic: from Principle to Implementation". Available at URL: http://www.cs.brandeis.edu/~tim/Papers/cp98.ps.gz

[14] Jaulin, L. "Interval Constraint Propagation with Application to Bounded-error Estimation". Automatica 36:10(2000), 1547-1552

[15] Jaulin, L., Kiffer, M., Didrit, O., and Walter, E. "Applied Interval Analysis". Springer-Verlag, Berlin, 2001

[16] Jaulin, L., and Walter, E. "Set Inversion via Interval Analysis for Nonlinear Bounded-error Estimation". Automatica 29:4(1993), 1053-1064

[17] Jaulin, L., and Walter, E. "Guaranteed Parameter Bounding for Nonlinear Models with Uncertain Experimental Factors". Automatica 35:5 (1999), 849-856

[18] Kearfott, B. " Interval Frequently Asked Questions". http://www.cs.utep.edu/interval-comp/FAQ (2001)

[19] Keiper, J. B. "Interval Arithmetic in Mathematica". Wolfram Research, Inc.

[20] Körner, R., 1995 "Variance of Compact Convex Random Sets". Working Paper, Institut for Stochastik, Benard-von-Cottes-Str. 2005599, Freiberg

[21] Kreinovich, V., Nguyen, H. T., Dimuro, G. P. and Bedregal, B. "A New Differential Formalism for Interval-Valued Functions and Its Potential Use in Detecting 1-D Landscape Features". Reliable Computing (to appear)

[22] Maeder, R.E. "Interval Plotting and Global Optimization Part 1", The Mathematica Journal 7:3 (1999)

[23] Maeder, R.E. "Interval Plotting and Global Optimization Part 2", The Mathematica Journal 7:4 (2000)

[24] Markov, S "Least square approximations under input data, Contributions to Computer Arithmetic and Self-Validating Numerical Methods" (Ed. C. P. Ulrich), IMACS Annals on Computing and Applied Mathematics, vol. 7, J. C. Baltzer Sci. Publ., Basel, 1990, 133-147.

[25] Moore, R. E., "Interval Analysis", Prentice-Hall, Inc. 1966

[26] Moore, R. E., "Methods and Applications of Interval Analysis", SIAM, Philadelphia, 1979.

[27] Neumaier, A., and Pownuk, A. "Linear Systems with Large Uncertainties, with Application to Truss Structures". http://www.mat.univie.ac.at/~nuem/

[28] Neumaier, A. and Schneider, T. 2001. "Estimation of Parameters and Eigenmodes of Multivariate Autoregressive Models". *ACM Trans. Math. Softw. 27*, 27-57.

[29] Shaw, S. " Forecasting". s.c.shaw@maths.bath.ac.uk, 2001.

[30] Troskie, C.G., and Clark, A.E, "Financial Time Series Models", Lecture Notes, July 2003. Available from:cgtros@stats.uct.ac.za

[31] van Emden, M.K., "Computing Functional and Relational Box Consistency by Structured Propagation in Atomic Constraint Systems",arXiv:cs:PL/0106008 v1 2001

Chapter 8

Appendix: Mathematica Programs for Interval AR(1) Parameter Estimations

This section of the appendix provides a listing of the Mathematica codes written to perform the following:

1. Optimisation procedure of the least squares minimisation for low-high classical AR(1) parameter estimation.
2. Optimisation procedure of the general metric for interval-valued AR(1) parameter estimation.
3. Global optimisation using interval techniques.

8.1 Main Optimisation Routine

```
ar1Parameter[xx_List]:=
    Module[{x,y,XX,XY,b},
    {x,y}=Transpose[Partition[xx,2,1]];
    XX=Sum[Transpose[{{1,x[[i]]}}].{{1,x[[i]]}},{i,Length[x]}]/Length[x];
```

```
XY=Sum[{1,x[[i]]}y[[i]],{i,Length[y]}]/Length[y];
b=Inverse[XX].XY;
b//N
]
ar1Forecast[x_,n_]:=NestList[(a+b#)&,Last[x],n]/.{a->ar1Parameter[x][[1]],
b->ar1Parameter[x][[2]]}
```

8.2 Codes for Minimising the Sum of the Metric $d_G\left(\overline{y}_t, \overline{\mu} + \overline{\phi}_1\overline{y}_{t-1}\right)$

```
dG[x_Interval,{ml_,mu_},{fl_,fu_},y_Interval,alpha_beta_]:=
Module[{d2,mid},
d2=Sqrt[(1/2)Abs[x[[1,2]-mu-Max[fl*y[[1,1]],fl*y[[1,2]],
fu*y[[1,1]],fu*y[[1,2]]]]^2
+[(1/2)Abs[x[[1,1]-ml-Min[fl*y[[1,1]],fl*y[[1,2]],
  fu*y[[1,1]],fu*y[[1,2]]]]^2];
mid=Abs[(1/2)(x[[1,1]]+x[[1,2]])
-(1/2)(ml+Min[fl*y[[1,1]],fl*y[[1,2]],fu*y[[1,2]]
+mu+Max[fl*y[[1,1]],fl*y[[1,2]],fu*y[[1,1]],
fu*y[[1,2]]]);
(alpha/(alpha+beta))*d2+(beta/(alpha+beta))*mid
]
dGpara[xl_List,xh_List,a_,b_]:={ml,mu,fl,fu}/.NMinimize[{Apply[Plus,
dG[#[[1]],{ml,mu},{fl,fu},#[[2]],a,b]&/@Transpose[{Drop[Interval
/@Transpose[{xl,xh}],1],Drop[Interval/@Transpose[{xl,xh}],-1]}]],
ml<=mu,fl<=fu},{ml,mu,fl,fu}][[2]]/.{x1_,x2_,y1_,y2_}->{Interval[{x1,x2}],Interval[{y1,y2}]}
S[a_,b_,tseries_]:=Apply[Plus,Map[(a+b#[[1]]-#[[2]])^2&,
Transpose[{Drop[tseries,-1],Rest[tseries]}]]]
lsPara[tseries_]:=
EnclosingRectangle[IntervalMinimum[
S[a,b,tseries],{a,0.75,1.0},{b,0.75,1.35},
```

```
Tolerance->0.001][[2]]
```

8.3 Codes for Interval Optimisation

```
BeginPackage["Intervals`"]
    IntervalPlot::usage = "IntervalPlot[expr,{x,x0,x1},opts...] plots expr as a
        function of x."
    IntervalPlot3D::usage = "IntervalPlot[expr,{x,x0,x1},{y,y0,y1},opts...] plots
        expr as a function of x and y."
    IntervalMinimum::usage = "IntervalMinimum[expr,{x,x0,x1},...,opts..] finds
        the global minimum of expr as a function of the given variables."
    IntervalMinimumConstrained::usage = "IntervalMinimum[expr,{x,x0,x1},...,constraint, opts..] finds
        the global minimum of expr as a function of the given variables.
        Search is restricted to those regions satisfying constraint."
    RectangleGraph::usage = "RectangleGraph[l] gives a Graphics or Graphics3D
        of the list of pairs or triplets of intervals."
    RectangleGraphColor::usage = "RectangleGraphColor[l] gives a Graphics or Graphics3D
        of the list of pairs or triplets of intervals. Components are colored
        randomly."
    EnclosingRectangle::usage = "EnclosingRectangle[l] gives a list of intervals
        that enclose the given list or lists of intervals."
    FeasiblePoint::usage = "FeasiblePoint->p is an option of IntervalMinimumConstrained giving an
initial feasible point."
    Options[IntervalPlot] = { Tolerance -> 0.01 }
    Options[IntervalPlot3D] = { Tolerance -> 0.05 }
    Options[IntervalMinimum] = { Tolerance -> 0.001 }
    Options[IntervalMinimumConstrained] = { Tolerance -> 0.001,
        FeasiblePoint -> Automatic }
    Begin["`Private`"]
    Needs["Utilities`FilterOptions`"]
```

```
Needs["PriorityQueue`"]
(* works for intervals or lists of intervals *)
length[i_] := Max[i]-Min[i]
(* split an interval into two *)
split[i_Interval]:=
 With[{m=(Min[i]+Max[i])/2}, {Interval[{Min[i],m}],Interval[{m,Max[i]}]}]
(* split Cartesian products *)
split[l_List] := Distribute[split/@l,List,List]
(* normalize *)
SetAttributes[interval, Listable]
interval[i_Interval] := i
interval[i_?NumericQ] := Interval[i]
interval[_] := Interval[{-Infinity,Infinity}] (* junk *)
(* midpoint *)
SetAttributes[mid,Listable]
mid[i_Interval]:= (Min[i]+Max[i])/2
(* avoid invisible rectangles/cuboids *)
rect[l:{xi_,yi_}]/;length[yi]==0 := Line[{Min/@l, Max/@l}]
rect[{xi_,yi_,zi_}]/;length[zi]==0 :=
 With[{x0=Min[xi],x1=Max[xi],y0=Min[yi],y1=Max[yi],z=Min[zi]},
 Polygon[{{x0,y0,z},{x1,y0,z},{x1,y1,z},{x0,y1,z}}]
 ]
(* infinite boundaries *)
infcolor=RGBColor[1,0,0];
huge=10.0^6;
rect[{xi__,yi_}]/;Min[yi]==-Infinity||Max[yi]==Infinity :=
 {infcolor, rect[{xi, yi /. DirectedInfinity[d:(-1|1)] :> d huge}]}
(* 1D *)
rect[l:{x_}] := Line[{{Min[x],0},{Max[x],0}}]
(* 2D *)
```

```
rect[l:{_,_}] := Rectangle[Min/@l, Max/@l]
(* 3D *)
rect[l:{_,_,_}] := Cuboid[Min/@l, Max/@l]
RectangleGraph[l:{{_}..},opts___?OptionQ] :=
 Graphics[{Thickness[0.01],rect/@l}, {opts}]
RectangleGraphColor[l:{{_}..},opts___?OptionQ] :=
 Graphics[{Thickness[0.01],{Hue[Random[]],rect[#]}&/@l}, {opts}]
RectangleGraph[l:{{_,_}..},opts___?OptionQ] :=
 Graphics[{Thickness[0],rect/@l}, {opts}]
RectangleGraphColor[l:{{_,_}..},opts___?OptionQ] :=
 Graphics[{Thickness[0],{Hue[Random[]],rect[#]}&/@l}, {opts}]
RectangleGraph[l:{{_,_,_}..,opts___?OptionQ}] :=
 Graphics3D[{EdgeForm[Thickness[0]],rect/@l}, {opts}]
EnclosingRectangle[l_List] := Apply[IntervalUnion, Transpose[l], {1}]
IntervalPlot[expr_, {x_Symbol,x0_,x1_}, opts___?OptionQ]:=
 With[{tol = (x1-x0)Tolerance /. {opts} /. Options[IntervalPlot]},
 Module[{finals},
 finals = refine[Function[x,expr],
 Interval/@N[{{x0,x1}}], tol];
 Show[RectangleGraph[finals],
 Evaluate[FilterOptions[Graphics,opts]],
 PlotRange->{{x0,x1},Automatic}, Axes->True]
 ]]
IntervalPlot3D[expr_,{x_Symbol,x0_,x1_},{y_Symbol,y0_,y1_},opts___?OptionQ]:=
 With[{tol = (x1-x0)Tolerance /. {opts} /. Options[IntervalPlot3D]},
 Module[{finals},
 finals = refine[Function[{x,y},expr],
 Interval/@N[{{x0,x1},{y0,y1}}], tol];
 Show[RectangleGraph[finals],
     Evaluate[FilterOptions[Graphics3D,opts]],
```

```
 PlotRange->{{x0,x1},{y0,y1},Automatic}, Axes->True,
 BoxRatios->{1,1,0.5}]
 ]]
refine[f_, init_, tol_] :=
 Module[{new = {init, {}}, finals = h[], i, j},
 While[new =!= {},
 {i,new} = new; (* dequeue one *)
 j = interval[f@@i];
 If[ length[j]<tol || Max[length/@i] < tol,
 finals = h[Append[i,j], finals]
 ,(*else*)
 Scan[ (new = {#, new})&, split[i] ] (* queue new ones *)
 ]
 ];
 List@@Flatten[finals, Infinity, h]
 ]
IntervalMinimum[expr_, ranges:{_Symbol,_,_}.., opts___?OptionQ] :=
 With[{vars = First/@{ranges},n = Length[{ranges}],
 tol = Tolerance /. {opts} /. Options[IntervalMinimum],
 dom = N[Interval /@ Rest /@ {ranges}]},
 With[{f = Function@@Hold[vars, expr]},
 Module[{div = MakeQueue[Min[#1[[2]]] < Min[#2[[2]]]&], fs,
 minmax = Infinity, fin = h[], new},
 EnQueue[div, {dom, f@@dom}];
 While[!EmptyQueue[div],
 new = DeQueue[div]; (* get smallest one *)
 If[ Min[new[[2]]] >= minmax, Continue[] ]; (* no longer in the game *)
 (* split it *)
 new = split[new[[1]]];
 fs = interval[Apply[f, new, {1}]];
```

```
new = Transpose[{new, fs}];
(* update minmax *)
minmax = Min[Max /@ fs, minmax];
(* stopping criteria *)
Scan[
If[ Min[#[[2]]] < minmax, (* requeue *)
If[ length[#[[2]]] < tol || length[#[[1,1]]] < tol,
fin = h[#, fin], (* no further division *)
EnQueue[div, #] (* divide further *)
]]&, new ];
];
DeleteQueue[div];
fin = List@@Flatten[fin, Infinity, h];
fin = Select[fin, Min[#[[2]]]<minmax&]; (* throw away non-minima *)
{new, fs} = Transpose[fin];
{Interval[{Min[fs], minmax}], new}
]]]
IntervalMinimumConstrained[expr_, ranges:{_Symbol,_,_}.., cond_, opts___?OptionQ] :=
With[{vars = First/@{ranges},n = Length[{ranges}],
tol = Tolerance /. {opts} /. Options[IntervalMinimumConstrained],
dom = N[Interval /@ Rest /@ {ranges}]},
With[{f = Function@@Hold[vars, expr], t=Function@@Hold[vars, cond]},
Module[{div = MakeQueue[Min[#1[[2]]] < Min[#2[[2]]]&], fs,
fx = Infinity, fin = h[], new, in, out, ok, fp},
fp = FeasiblePoint /. {opts} /. Options[IntervalMinimumConstrained];
If[ t@@fp, fx = f@@fp ]; (* feasible point *)
EnQueue[div, {dom, f@@dom}];
While[!EmptyQueue[div],
new = DeQueue[div]; (* get smallest one *)
If[ Min[new[[2]]] > fx, Continue[] ]; (* no longer in the game *)
```

```
(* split it *)
new = split[new[[1]]];
(* apply constraint *)
states = Apply[t, new, {1}];
in = Flatten[Position[states, True, {1}, Heads->False]];
out = Flatten[Position[states, False, {1}, Heads->False]];
ok = Complement[Range[Length[states]], out]; (* not discarded *)

(* update fx, the value of f at a feasible point *)
fx = Min[Apply[f, mid/@Part[new, in], {1}], fx];
new = Part[new, ok];
(* range intervals *)
fs = interval[Apply[f, new, {1}]];
new = Transpose[{new, fs}];
(* stopping criteria *)
Scan[
If[ Min[#[[2]]] <= fx, (* requeue *)
If[ length[#[[2]]] < tol || length[#[[1,1]]] < tol,
fin = h[#, fin], (* no further division *)
EnQueue[div, #] (* divide further *)
]]&, new ];
];
DeleteQueue[div];
fin = List@@Flatten[fin, Infinity, h];
fin = Select[fin, Min[#[[2]]] <= fx&]; (* throw away non-minima *)
fin = Select[fin, t@@#[[1]]=!=False&]; (* not outside feasible region *)
{new, fs} = Transpose[fin];
{Interval[{Min[fs], fx}], new}
]]]
End[]
```

```
Protect[IntervalPlot, IntervalPlot3D, IntervalMinimum,
 IntervalMinimumConstrained, RectangleGraph, EnclosingRectangle]
EndPackage[]
```

8.4 Codes for PriorityQueue

```
BeginPackage["PriorityQueue`"]
 MakeQueue::usage = "MakeQueue[pred] creates an empty priority queue with
   the given ordering predicate. The default predicate is Greater."
 CopyQueue::usage = "CopyQueue[q] makes a copy of the priority queue q."
 DeleteQueue::usage = "DeleteQueue[q] frees the storage used for q."
 EmptyQueue::usage = "EmptyQueue[q] is True if the priority queue q is empty."
 EnQueue::usage = "EnQueue[a, item] inserts item into the priority queue q."
 TopQueue::usage = "TopQueue[q] returns the largest item in the priority queue q."
 DeQueue::usage = "DeQueue[q] removes the largest item from the priority queue q.
   It returns the item removed."
 PriorityQueue::usage = "PriorityQueue[...] is the print form of priority queues."
 Begin["`Private`"]
 SetAttributes[queue, HoldAll]
 SetAttributes[array, HoldAllComplete]
 makeArray[n_] := array@@Table[Null, {n}]
 MakeQueue[pred_:Greater] :=
  Module[{ar,n=0},
  ar = makeArray[2];
  queue[ar, n, pred]
  ]
 CopyQueue[queue[a0_,n0_,pred_]] :=
  Module[{ar=a0,n=n0},
  queue[ar, n, pred]
  ]
```

```
EnQueue[q:queue[ar_,n_,pred_], val_] :=
 Module[{i,j},
 If[ n == Length[ar], (* extend (double size) *)
 ar = Join[ar, makeArray[Length[ar]]] ];
 n++;
 ar[[n]] = val; i = n;
 While[ True, (* restore heap *)
 j = Floor[i/2];
 If[ j < 1 || pred[ar[[j]], ar[[i]]], Break[] ];
 {ar[[i]], ar[[j]]} = {ar[[j]], ar[[i]]};
 i = j;
 ];
 q
 ]
EmptyQueue[queue[ar_,n_,pred_]] := n == 0
TopQueue[queue[ar_,n_,pred_]] := ar[[1]]
DeQueue[queue[ar_,n_,pred_]] :=
 Module[{i,j,res=ar[[1]]},
 ar[[1]] = ar[[n]]; ar[[n]] = Null; n--;
 j = 1;
 While[ j <= Floor[n/2], (* restore heap *)
 i = 2j;
 If[ i < n && pred[ar[[i+1]], ar[[i]]], i++ ];
 If[ pred[ar[[i]], ar[[j]]],
 {ar[[i]], ar[[j]]} = {ar[[j]], ar[[i]]} ];
 j = i
 ];
 res
o ]
DeleteQueue[queue[ar_,n_,pred_]] := (ClearAll[ar,n];)
```

```
queue/:Normal[q0_queue] :=
 Module[{l={}, q=CopyQueue[q0]},
 While[!EmptyQueue[q], AppendTo[l, TopQueue[q]]; DeQueue[q]];
 DeleteQueue[q];
 l
 ]
Format[q_queue/:EmptyQueue[q]] := PriorityQueue[]
Format[q_queue] := PriorityQueue[TopQueue[q], "\[TripleDot]"]
End[]
Protect[ MakeQueue, CopyQueue, DeleteQueue, EmptyQueue,
 EnQueue, TopQueue, DeQueue, PriorityQueue ]
EndPackage[]
```