26

Power Constructs and Propositional Systems

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Abstract

Propositional systems are deductively closed sets of sentences phrased in the language of some propositional logic. The set of systems of a given logic is turned into an algebra by endowing it with a number of operations, and into a relational structure by endowing it with a number of relations. Certain operations and relations on systems arise from some corresponding base operation or relation, either on sentences in the logic or on propositional valuations. These operations and relations on systems are called power constructs. The aim of this thesis is to investigate the use of power constructs in propositional systems. Some operations and relations on systems that arise as power constructs include the Tarskian addition and product operations, the contraction and revision operations of theory change, certain multiple-conclusion consequence relations, and certain relations of verisimilitude and simulation. The logical framework for this investigation is provided by the definition and comparison of a number of multiple-conclusion logics, including a paraconsistent three-valued logic of partial knowledge.

Contents

Preface

Alfred Tarski presented his *calculus of deductive systems* in a series of papers between 1930 and 1936. The systems in question are sets of sentences from some formal language, that are closed under certain inference rules. This means that a system contains all the elements that can be inferred from it. Tarski's purpose was to facilitate the investigation of certain metamathematical properties of systems. To this end, he defines some relations and operations on systems:

- The inclusion of a system Γ in a system Δ indicates that Δ is *logically stronger* than Γ;
- equality between systems coincides with *logical identity*;
- the *product* of two systems is their intersection;
- the *addition* of the systems Γ and Δ is the set of all sentences that may be inferred by the union of Γ and Δ.
- the *negation* of a system Γ is obtained by forming, for each element $\gamma \in \Gamma$, the smallest system containing its negation $\neg\gamma$, and then taking the intersection of these sets.

In this thesis I will be concerned specifically with propositional languages, and accordingly, with sets of propositional formulas, or *formula sets*, and the propositional systems they give rise to by closure under the inference rules of some propositional logic.

There are many more useful relations and operations on formula sets, in addition to those listed above:

- A formula set Δ is called a *multiple-conclusion consequence* of a formula set Γ if, whenever all the elements of Γ are true, then at least one element of Δ is true.
- A *verisimilar* relation orders formula sets according to their closeness to the truth.
- A formula set Γ *explains* a formula set Δ if, whenever at least one element of Γ is true, then at least one element of Δ is true.
- A formula set may be *simulated* by another formula set in a different logic.
- A formula set may be *contracted* by another formula set.
- A formula set may be *revised* by another formula set.

Leaving propositional systems aside for a moment, let us turn to the concept of a *power construct.* Given any set of elements A, one can form its *power set* $\mathcal{P}(A)$, the set of all subsets of A. Further, if A is endowed with some operation f, one can define a *power operation* f^+, which is based on f, and operates on elements of $\mathcal{P}(A)$. Similarly, given any relation r defined on A, one can define a *power relation* of the same arity on $\mathcal{P}(A)$. A power construct is defined as any such power operation or relation, obtained by powering some base operation or relation.

Power constructs abound in propositional systems. The aim of this thesis is to point out and discuss existing power constructs in propositional systems, and explore new ones. Most of the operations and relations mentioned above can be defined as non-trivial power constructs. This includes Tarski's product and addition operations, contraction and revision operations, as well as certain multiple-conclusion consequence, verisimilitude, and simulation relations.

The outline of the thesis is as follows: Consequence relations can be studied either semantically (model-theoretically), or syntactically (proof-theoretically), or abstractly. Chapter 1 describes the syntactic and abstract approaches to the study of consequence relations, and relates the two approaches to each other. The relations of logical and explanatory strength are characterized in terms of an abstract consequence operator. The chapter concludes with a semantic definition of the consequence relation of a multiple-conclusion version of classical propositional logic, abbreviated MCC.

In Chapter 2 the semantic definition of consequence of Chapter 1 is refined, and applied to a multiple-conclusion logic, MCK, based on Kleene's strong three-valued truth tables. A proof system for MCK is given, and soundness and adequacy results proved. MCK is presented as a logic of growth in information. Unlike classical logic, it is not explosive, which makes it suitable as reasoning mechanism of an agent presented with conflicting information.

A further refinement of the consequence relation of MCK yields the logic $\diamond$MCK, which is equivalent to the paraconsistent Logic of Paradox. The consequence relation of $\diamond$MCK is strictly stronger than that of MCK, and strictly weaker than that of MCC. This provides a new semantics for the Logic of Paradox.

In Chapter 3 the calculus of deductive systems, and the concept of power constructs, are introduced formally. The calculus of deductive systems of a certain class of logics is then characterized as a *power algebra of theories.* Two alternative algebraic characterizations of calculi of deductive systems are also given. Namely, the calculus of $\diamond$MCK-systems is characterized using the Lindenbaum/Tarski construction, and the calculus of MCK-systems is characterized model-theoretically as an *algebra of meanings.* The chapter concludes with a discussion on the simulation of one logic by another.

Chapter 4 deals with relations of verisimilitude. I show that power relations may be used to define a parameterized *theorylike* order on formula sets. The order is based on a likeness order on propositional valuations that incorporates a notion of relevance to the truth. These ideas are also translated to the three-valued logic $\diamond$MCK.

In Chapter 5 I show that the theory change operations of contraction, expansion and revision can be characterized as power operations in an algebra of theories enriched with a set of unary power operations. I also show that revised belief sets can be described in terms of a theorylike order on formulas, as defined in Chapter 4.

Parts of this thesis have been published, or presented at conferences. Namely, part of Chapter 4 appeared in (Britz & Brink, 1995) and (Brink & Britz, 1999), part of Chapters 3 and 5 appeared in (Britz, 1999), and part of Chapter 3 appeared in (Brink *et al.*, 1993). Conference contributions include "A new semantics for the Logic of Paradox", read at the 42nd Annual Conference of the South African Mathematical Society, and "Relating, and operating on, formula sets", read by co-author Chris Brink at the 4th International Seminar on Relational Methods in Logic, Algebra and Computer Science.

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Chapter 1

Introduction

Logical consequence relations can be described either semantically (model-theoretically), or syntactically (proof-theoretically), or abstractly, by a set of postulates. In this chapter I characterize a certain class of consequence relations syntactically and abstractly, and show how these characterizations relate to each other. This is done in Sections 1.2 and 1.3. The results presented in these sections are general, in that they do not depend on any specific logic. The chapter concludes with a simple case study, which shows how these general results apply to a specific logic, in this case, a multiple-conclusion version of propositional logic called MCC. First, however, I define some terminology that will be used in this and subsequent chapters.

1.1 Preliminaries

The formal language L of classical propositional logic, or PC, is built up from a denumerable set of sentential symbols $L_0 = \{p_1, p_2, ...\}$, the binary connectives $\wedge$ and $\vee$, the unary connective $\neg$, and the parentheses (and). The elements of L are called *well-formed formulae*, or wffs. The set L is defined inductively, as follows:

1. Every element of L_0 is a wff.

2. If α and β are wffs, then so are $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$ and $(\neg \alpha)$.

Where there is no ambiguity, the parentheses will be omitted from a wff. The symbols α and β are used here as metalinguistic variables that range over L. I will continue to use lower case Greek symbols α, β, γ, δ, ... as variables to refer to wffs from L. To refer to sets of wffs, also called *formula sets*, I will use upper case Greek symbols Γ, Δ, Σ, ... The empty set is denoted by $\emptyset$. The *power set* of L is the set of all subsets of L, and is written $\mathcal{P}(L)$.

The connectives of L can also be viewed as operations on wffs, with $\neg\alpha$ being the outcome of applying the unary operation $\neg$ to the wff α, and $\alpha \wedge \beta$ and $\alpha \vee \beta$ being the respective outcomes of applying the binary operations $\wedge$ and $\vee$ to the wffs α and β. Hence L may be viewed as an abstract algebra, generated by L_0. Wójcicki (1988) mentions that this perspective is due to Lindenbaum, and first stated in print by Tarski (1935). As is commonly done, for example in (Wójcicki, 1988), I will identify the language L with the abstract algebra $(L, \wedge, \vee, \neg)$ associated with it.

Any algebra $\mathcal{A} = (A, f_0^2, f_1^2, f_2^1)$, with base set A and operations f_0^2, f_1^2 and f_2^1 of the same arity as $\wedge$, $\vee$ and $\neg$ respectively, is said to be of the same *similarity* as L. The class of all such algebras is called the similarity class of L. Further, if h_o is an n-ary mapping from L_0 into A, then there is a homomorphism h, extending h_0, from L into A. Namely, h is defined by:

$$\begin{aligned} h(\alpha) &= h_0(\alpha) \text{ if } \alpha \in L_0; \\ h(\alpha \wedge \beta) &= f_0^2(h(\alpha), h(\beta)); \end{aligned}$$

$$\begin{aligned} h(\alpha \vee \beta) &= f_1^2(h(\alpha), h(\beta)); \\ h(\neg\alpha) &= f_2^1(h(\alpha)). \end{aligned}$$

This means L is a *free algebra* in its similarity class, with set of free generators L_0. Wójcicki (1988) gives an exposition of the basic theory of logical calculi, including this result. The following definitions are standard terminology in textbooks on lattice theory such as (Davey & Priestley, 1990), and are listed here for later reference.

Definition 1.1 *A preorder on a set A is a binary relation $\leq$ on A which is reflexive and transitive.*

Definition 1.2 *A partial order on a set A is a binary relation $\leq$ on A which is reflexive, transitive and antisymmetric.*

Lemma 1.3 Zorn's Lemma. *If every chain in a partially ordered set $(A, \leq)$ has an upper bound in A, then the set A contains a maximal element.*

Definition 1.4 *An equivalence relation on a set A is a binary relation on A which is reflexive, symmetric and transitive.*

Definition 1.5 *A congruence relation on an algebra $\mathcal{A}$ is an equivalence relation on its base set which preserves the operations of $\mathcal{A}$.*

Definition 1.6 *A meet-semilattice is a structure $(A, \wedge)$ such that $\wedge$ is a binary operation which is associative and commutative.*

Definition 1.7 *A lattice is a structure $(A, \wedge, \vee)$ such that*

1. *A is a non-empty, partially ordered set;*

2. *for every $x, y \in A$, their meet $x \wedge y$ and join $x \vee y$ exist, where the meet of two elements is their greatest lower bound, and the join of two elements is their least upper bound.*

A lattice A has a *zero* element $0 \in A$ if $x = x \vee 0$ for every $x \in A$, and has a *unit* element $1 \in A$ if $x = x \wedge 1$ for every $x \in A$.

Definition 1.8 *A distributive lattice is a lattice $(A, \wedge, \vee)$ which satisfies the distributive laws:*

$$(\forall x, y, z \in A)\,[x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)]\,;$$
$$(\forall x, y, z \in A)\,[x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)]\,.$$

Definition 1.9 *A De Morgan lattice, also called a quasi-Boolean algebra, is a structure $(A, \wedge, \vee, \neg, 0, 1)$ with unary operation $\neg$, zero element 0 and unit element 1 such that*

1. *$(A, \wedge, \vee)$ is a distributive lattice;*

2. $\neg\neg\alpha = \alpha$;

3. $\neg(\alpha \wedge \beta) = \neg\alpha \vee \neg\beta$;

4. $\neg(\alpha \vee \beta) = \neg\alpha \wedge \neg\beta$.

Definition 1.10 *A Kleene lattice is a De Morgan lattice $(A, \wedge, \vee, \neg, 0, 1)$ such that*

$$\alpha \wedge \neg\alpha \wedge (\beta \vee \neg\beta) = \alpha \wedge \neg\alpha.$$

Definition 1.11 *A Heyting algebra, also called a Brouwerian algebra, is a structure $(A, \wedge, \vee, \Rightarrow, \neg, 0, 1)$such that*

1. *$(A, \wedge, \vee, \neg, 0, 1)$ is a De Morgan lattice;*

2. $\alpha \Rightarrow \beta$ *is the greatest element in* A *such that* $\alpha \wedge (\alpha \Rightarrow \beta) \leq \beta$*.*

Definition 1.12 *A Boolean algebra is a structure* $(B, \wedge, \vee, \neg, 0, 1)$ *such that*

1. $(B, \wedge, \vee)$ *is a distributive lattice;*

2. $\alpha \vee 0 = \alpha$ *and* $\alpha \wedge 1 = \alpha$ *for all* $\alpha \in B$*;*

3. $\alpha \vee \neg\alpha = 1$ *and* $\alpha \wedge \neg\alpha = 0$ *for all* $\alpha \in B$*.*

The two-element Boolean algebra $\mathcal{B}_2 = \{\{t, f\}, \wedge, \vee, \neg, f, t\}$, with base set $\{t, f\}$, meet $\wedge$, join $\vee$, complement $\neg$, zero element f, and unit element t, is of the same similarity as L. Any mapping $v_0 : L_0 \to \{t, f\}$ can therefore be extended to a homomorphism from L to $\mathcal{B}_2$. That is, any *truth assignment* v_0 of truth values to sentential symbols can be extended to a *valuation* $v : L \to \{t, f\}$, such that:

1. If $\alpha \in L_0$, then $v(\alpha) = v_0(\alpha)$;

2. $v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$;

3. $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$;

4. $v(\neg\alpha) = \neg v(\alpha)$.

Let Val_0 denote the set of truth assignments $\{v_0 : L_0 \to \{t, f\}\}$, and let Val_2 denote the set of homomorphisms $\{v : L \to \{t, f\}\}$. A valuation v *satisfies* a wff γ iff $v(\gamma) = t$. A valuation v which satisfies every element of a set of wffs Γ is called a *model* of Γ. I will call a valuation which satisfies at least one element of Γ a *co-model* of Γ. The set of models of Γ is denoted $Mod_{\text{PC}}(\Gamma)$, and its set of co-models $CoMod_{\text{PC}}(\Gamma)$. Γ is called *satisfiable* if it has a model, and *co-satisfiable* if it has a co-model. A *tautology* is a wff which is satisfied by every valuation, and a *contradiction* is a wff which is not satisfied by any valuation. Given any formula set Γ and wff β, if β is satisfied by every valuation which satisfies Γ, we say β is a *semantic consequence* of Γ, and write $\Gamma \models_{\text{PC}} \beta$. If Γ is a singleton set $\{\alpha\}$, we write $\alpha \models_{\text{PC}} \beta$. If $\alpha \models_{\text{PC}} \beta$ and $\beta \models_{\text{PC}} \alpha$, α and β are called *semantically equivalent*, written $\alpha \approx_{\text{PC}} \beta$.

The Lindenbaum/Tarski construction provides a standard way of turning the set of equivalence classes under $\approx_{\text{PC}}$ into an algebra. Since $\approx_{\text{PC}}$ is a congruence relation on L, we may form the quotient set $L/\approx_{\text{PC}}$, consisting of equivalence classes of elements of L. Let $[\alpha]$ denote the $\approx_{\text{PC}}$-equivalence class of α, let 1 denote the equivalence class of tautologies, and 0 the equivalence class of contradictions. Also, being a congruence relation, $\approx_{\text{PC}}$ preserves the operations on L. The meet, join and complement operations on $L/\approx_{\text{PC}}$ can therefore be defined by:

$$\begin{aligned} [\alpha] \wedge [\beta] &= [\alpha \wedge \beta]; \\ [\alpha] \vee [\beta] &= [\alpha \vee \beta]; \\ \neg [\alpha] &= [\neg \alpha]. \end{aligned}$$

This yields a Boolean algebra $\mathcal{L}_{\text{PC}} = (L/\approx_{\text{PC}}, \wedge, \vee, \neg, 0, 1)$, called the *Lindenbaum algebra* (also referred to as the Lindenbaum/Tarski algebra) for classical propositional logic. The partial order $\leq$ on $L/\approx_{\text{PC}}$ is:

$$[\alpha] \leq [\beta] \text{ iff } \alpha \models_{\text{PC}} \beta.$$

The Boolean algebra $\mathcal{B}_2 = \{\{t, f\}, \wedge, \vee, \neg, f, t\}$ can also be used to define a *matrix semantics* for PC. More specifically, the tuple $\langle \mathcal{B}_2, \{t\} \rangle$ *determines* the logic PC. The set $\{t\}$ is called the set of *designated* elements of $\mathcal{B}_2$. The semantic consequence relation for PC can then be reformulated as a *matrix consequence* relation: Given any formula set Γ and wff β, we say β is a matrix consequence of Γ if, for any homomorphism $v : L \to \mathcal{B}_2$, $v(\beta) \in \{t\}$ whenever $v(\Gamma) \subseteq \{t\}$. I will not present any general results concerning matrix consequence relations, but will only refer to specific cases as they occur. The interested reader may consult (Wójcicki, 1988).

To conclude this section, I define a number of other standard concepts that will be used in subsequent chapters.

Definition 1.13 *A filter in a lattice* $(A, \wedge, \vee)$ *is a non-empty set of elements from* A *which is upward closed under the partial order of the lattice, and closed under finite meets.*

For each $x \in A$, the upclosure of x, that is, the set $\{y : x \leq y\}$, is a filter, called the *principal* filter generated by x. A filter is called *proper* if it does not coincide with A. A proper filter of A is called an *ultrafilter* if the only filter properly containing it, is A itself. A filter F is called *prime* if it is proper and, for any $x, y \in F$, $x \vee y \in F$ implies $x \in F$ or $y \in F$. In a Boolean algebra, the notions of an ultrafilter and a prime filter coincide.

1.2 Proof systems

Consequence relations are traditionally defined as relations between $\mathcal{P}(L)$ and L. This is understandable in view of the emphasis placed on proof theory, where one is interested in the single consequences of a set of premisses. But, more generally, a consequence relation can be defined as a binary relation on $\mathcal{P}(L)$. It is then referred to as a *multiple-conclusion consequence relation*. The roots of this idea can be found in (Gentzen, 1969; Carnap, 1943; Kneale, 1956; Gabbay, 1971; Scott, 1974; Shoesmith & Smiley, 1978). Avron (1991b; 1992) uses multiple-conclusion consequence relations on finite multisets as a general framework within which to investigate logics. Since then, they have been used sporadically in a number of diverging application areas. For example, Wisniewski (1991; 1994) uses a multiple-conclusion consequence relation to address the problem of the reducibility of questions in erotetic logic, and Miller (1996) uses it in the specification logic Forum. Gentzen-style sequent calculi are also formulated in terms of sets of premisses and conclusions. I will use the term 'consequence relation' to refer to multiple-conclusion consequence relations; when I refer to a single-conclusion consequence relation, I will state so explicitly.

Consequence relations are characterized in three ways: semantic, syntactic or abstract. The semantic characterization depends on the logic in question, hence I will defer any discussion of it until Section 1.4 and Chapter 2. For the present, I will focus on the syntactic and abstract characterizations of consequence relations.

The syntactic characterization of a consequence relation is as a derivability relation in a formal proof system. So we first need to clarify what is meant by a formal proof system and a derivation. The proof system I will describe is in the Hilbert-Frege style, but the notion of a derivation was adapted by Shoesmith & Smiley (1978) to allow for multiple-conclusion inferences.

Definition 1.14 *An inference rule is an ordered pair of sets of formulae, written in the form*

$$\Gamma(p_1,\ldots,p_n)/\Delta(p_1,\ldots,p_n)=\frac{\gamma_1(p_1,\ldots,p_n),\gamma_2(p_1,\ldots,p_n),\ldots,\gamma_m(p_1,\ldots,p_n)}{\delta_1(p_1,\ldots,p_n),\delta_2(p_1,\ldots,p_n),\ldots,\delta_k(p_1,\ldots,p_n)},$$

where $\Gamma=\{\gamma_i : 1\leq i\leq m\}$, $\Delta=\{\delta_j : 1\leq j\leq k\}$, *and where each* $\gamma_i(p_1,\ldots,p_n)$ *and* $\delta_j(p_1,\ldots,p_n)$ *is a wff from L built up from sentential symbols chosen from* $p_1,\ldots,p_n$.

An inference rule is actually a schema for an infinite set of rules, one for each uniform substitution of wffs for the symbols $p_1, ..., p_n$. $\gamma_1, ..., \gamma_m$ are called the *premisses*, and $\delta_1, ..., \delta_k$ the *conclusions* of the rule. Rules with an empty premiss set or conclusion set are allowed.

A set of inference rules constitutes a *formal proof system.* Usually, especially in single-conclusion logics, the definition of a formal proof system also allows for a set of axioms. But an axiom is just an inference rule with an empty premiss set, and no special provision need therefore be made for it.

An inference rule $\Phi(p_1, ..., p_n) / \Psi(p_1, ..., p_n)$ is *applicable* to a formula set Σ if there exists a uniform substitution of wffs $\alpha_1, ..., \alpha_n$ for the symbols $p_1, ..., p_n$ such that $\Phi(\alpha_1, ..., \alpha_n) \subseteq \Sigma$. That is, every element of Φ, with each p_i replaced by α_i, is in Σ. A *derivation* from Γ to Δ in a formal proof system $\mathcal{S}$ will be defined in 1.15 as a rooted tree with nodes labelled by wffs. I will use the terms child, parent, ancestor, descendant, root and leaf as is customary in standard texts on graph theory such as (West, 1996). A *branch* in a tree is a sequence of wffs, that is an ordered list, starting at the root and ending at a leaf. If the order of wffs and number of occurrences of each wff in a branch is not important, a branch can also be regarded as a set of wffs. The length of a branch is the number of nodes appearing in it. An *initial subsequence* of a branch Σ with length n is a sequence of wffs Σ' which differs from Σ only in that the last m, where $m < n$, elements of Σ have been removed. To expand a branch Σ with α means to form a new tree by adding α as a child to the leaf of Σ. This tree has a branch $(\Sigma; \alpha)$ having the initial subsequence Σ followed by the leaf α. The *branching factor* of a tree is the maximum number of children of any node in the tree. Both the length of each branch and the branching factor are assumed to be finite.

Definition 1.15 *A proof tree from* Γ *in a formal proof system* $\mathcal{S}$ *is created as follows:*

1. Create an unlabelled root node. Then continue to expand the tree by any finite sequence of repetitions of steps 2 and 3.

2. Choose any branch Σ in the tree. (Initially, there is only one, consisting of the root node.) Pick any element α from Γ. Expand Σ with α to obtain a new tree which differs from the old tree only in that the branch Σ has been replaced by $(\Sigma; \alpha)$.

3. Choose any branch Σ in the tree (initially, there is only one), and pick any inference rule $\Phi(p_1, ..., p_n) / \Psi(p_1, ..., p_n)$ from $\mathcal{S}$ applicable to Σ, to obtain $\Psi(\alpha_1, ..., \alpha_n)$. Suppose $\Psi(\alpha_1, ..., \alpha_n) = \{\psi_1, ..., \psi_k\}$. If $k > 0$, add k new children $\psi_1, ..., \psi_k$ to the leaf of the branch Σ to obtain a new tree which differs from the old tree only in that the branch Σ has now been replaced by k branches $(\Sigma; \psi_1), ..., (\Sigma; \psi_k)$. If $k = 0$, then mark the branch *closed*.

Let $l(T)$ denote the set of leaves of the tree T, excluding the leaves on closed branches. An *initial subtree* of a tree T is a tree T' such that it is possible to obtain T from T' by repeated application of steps 2 and 3 of Definition 1.15.

Definition 1.16 *A derivation from Γ to Δ in a formal proof system $\mathcal{S}$ is a proof tree T starting from Γ, and with $l(T) \subseteq \Delta$. We write $\Gamma \vdash_{\mathcal{S}} \Delta$, and say Δ is derivable from Γ in $\mathcal{S}$, and $\vdash_{\mathcal{S}}$ is the derivability relation of $\mathcal{S}$.*

Note that, if each inference rule of a proof system has a single conclusion, any derivation will consist of a single branch. The notion of a derivation then coincides with the well-known concept of a proof in a single-conclusion logic, where a derivation from Γ to α is a sequence of wffs starting from Γ and terminating in α. Such a proof system will be called a single-conclusion proof system, and its derivability relation a single-conclusion derivability relation.

1.3 Abstraction

The abstract characterization of a consequence relation is by means of a set of properties, and is related to the syntactic characterization by a theorem of Łoś & Suszko (1958). The theorem was initially formulated for single-conclusion consequence relations, but I state and prove it below for the more general case.

Definition 1.17 *An abstract consequence relation* $\vdash$ *is any binary relation over* $\mathcal{P}(L)$ *satisfying the conditions:*

1. If $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \vdash \Delta$. (sharing)
2. If $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$, then $\Gamma' \vdash \Delta'$. (weakening)
3. If $\Gamma, \phi \vdash \Delta$ and $\Gamma \vdash \phi, \Delta$ then $\Gamma \vdash \Delta$. (cut)
4. $\Gamma \vdash \Delta$ iff there exist finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \vdash \Delta'$. (finiteness)
5. For any substitution $e : L_0 \to L$, if $\Gamma \vdash \Delta$ then $e(\Gamma) \vdash e(\Delta)$. (uniform substitution)

For singleton sets, I will drop the set notation, and thus write $\gamma \vdash \Delta$ instead of $\{\gamma\} \vdash \Delta$, and $\Gamma \vdash \delta$ instead of $\Gamma \vdash \{\delta\}$.

Theorem 1.18 *A binary relation* $\vdash$ *is an abstract consequence relation iff there exists a formal proof system* $\mathcal{S}$ *such that* $\vdash$ *is the derivability relation for* $\mathcal{S}$.

Proof. Given a proof system $\mathcal{S}$ with derivability relation $\vdash_{\mathcal{S}}$, we first check that $\vdash_{\mathcal{S}}$ satisfies the conditions of an abstract consequence relation.

1. Sharing: Suppose $\alpha \in \Gamma \cap \Delta$. Construct a proof tree consisting of a single branch of elements from Γ terminating in α. This constitutes a derivation from Γ to Δ.

2. Weakening: Any tree establishing $\Gamma \vdash_{\mathcal{S}} \Delta$ also establishes $\Gamma' \vdash_{\mathcal{S}} \Delta'$.

3. Cut: Suppose $\Gamma, \phi \vdash_{\mathcal{S}} \Delta$ and $\Gamma \vdash_{\mathcal{S}} \phi, \Delta$. Append a derivation from Γ, ϕ to Δ to every branch of a derivation from Γ to ϕ, Δ which terminates in ϕ. The resulting tree is a derivation from Γ to Δ.

4. Finiteness: A derivation is finite by definition.

5. Uniform substitution: This property follows from the definition of an inference rule and its applicability, which allows for uniform substitution of variables.

Conversely, let $\vdash$ be a relation which satisfies the conditions of Definition 1.17. We have to find a proof system $\mathcal{S}$ with derivability relation $\vdash_{\mathcal{S}}$ such that $\vdash_{\mathcal{S}}=\vdash$. Let $\mathcal{S}$ be the set of all proof rules Γ/Δ such that Γ and Δ are finite, and $\Gamma \vdash \Delta$. Note that this includes the case where Γ or Δ is empty. By the finiteness property of $\vdash$, it is sufficient to show that $\Gamma \vdash \Delta$ iff $\Gamma \vdash_{\mathcal{S}} \Delta$ for finite Γ and Δ.

Left to right: Suppose $\Gamma \vdash \Delta$. Then it is an inference rule, and hence there is a derivation from Γ to Δ.

Right to left: Suppose $\Gamma \vdash_{\mathcal{S}} \Delta$. The proof proceeds by induction on the number of steps followed in a derivation T from Γ to Δ. I will show that for each initial subtree T_j of T created by steps 2 and 3 of Definition 1.15, $\Gamma \vdash l(T_j)$.

Base case: If step 2 is executed first to obtain a tree with single leaf $\alpha \in \Gamma$, then $\Gamma \vdash \alpha$ by the sharing property. If step 3 is executed first, it means some inference rule $\emptyset/\{\psi_1, ..., \psi_k\}$ is applied to obtain a tree with leaves $\psi_1, ..., \psi_k$. By the definition of $\mathcal{S}$, $\emptyset \vdash \{\psi_1, ..., \psi_k\}$, and by weakening, $\Gamma \vdash \{\psi_1, ..., \psi_k\}$.

Inductive step: Let T_j be an initial subtree of T. Assume that for each proper initial subtree T_i of T_j, $\Gamma \vdash l(T_i)$. T_j is formed by an application of either step 2 or step 3 of Definition 1.15 to an initial subtree T_{j-1}. Suppose it is step 2. That is, some branch Σ in T_{j-1} is replaced by a branch $(\Sigma; \alpha)$, where $\alpha \in \Gamma$. By the sharing property, $\Gamma \vdash l(T_j)$.

Else, step 3 is applied to T_{j-1}. That is, choose some branch Σ of T_{j-1} with leaf σ, and some inference rule

$$\Phi/\Psi = \frac{\phi_1(p_1, ..., p_n), ..., \phi_m(p_1, ..., p_n)}{\psi_1(p_1, ..., p_n), ..., \psi_k(p_1, ..., p_n)}$$

applicable to Σ. This means there are wffs $\alpha_1, ..., \alpha_n$ such that $\Phi(\alpha_1, ..., \alpha_n) \subseteq \Sigma$. Let $\phi_1 \in \Phi$, and let T_{ϕ_1} be the tree obtained from T_{j-1} by removing all descendents of ϕ_1. By the inductive hypothesis, $\Gamma \vdash l(T_{\phi_1})$. This can also be written $\Gamma \vdash (l(T_{\phi_1}) - \phi_1), \phi_1$, where $l(T_{\phi_1}) - \phi_1$ is the set obtained by removing ϕ_1 from $l(T_{\phi_1})$. By weakening, $\Gamma, \Phi - \phi_1 \vdash (l(T_{\phi_1}) - \phi_1), \phi_1, \Psi$. By weakening the inference rule $\Phi \vdash \Psi$, we have $\Gamma, \Phi \vdash (l(T_{\phi_1}) - \phi_1), \Psi$. By the cut property, $\Gamma, \Phi - \phi_1 \vdash (l(T_{\phi_1}) - \phi_1), \Psi$. Now let $\phi_2 \in \Phi - \phi_1$. As above, $\Gamma \vdash (l(T_{\phi_2}) - \phi_2), \phi_2$. By weakening, $\Gamma, \Phi - \phi_1 - \phi_2 \vdash (l(T_{\phi_1}) - \phi_1), (l(T_{\phi_2}) - \phi_2), \phi_2, \Psi$. By another application of cut, $\Gamma, \Phi - \phi_1 - \phi_2 \vdash (l(T_{\phi_1}) - \phi_1), (l(T_{\phi_2}) - \phi_2), \Psi$. Repeating this process m times, once for each $\phi \in \Phi$, we obtain $\Gamma \vdash \bigcup_{\phi \in \Phi} \{l(T_\phi) - \phi\}, \Psi$. Since each T_ϕ was obtained from T_{j-1} by removing the descendents of ϕ, every other leaf of T_ϕ is a leaf of T_{j-1}. Therefore $\bigcup_{\phi \in \Phi} \{l(T_\phi) - \phi\} \subseteq l(T_{j-1}) - \sigma$. Thus $\Gamma \vdash l(T_j)$ by weakening. This proves the inductive step. Since T is a derivation from Γ to Δ, $l(T) \subseteq \Delta$. Hence $\Gamma \vdash \Delta$.

□

The result of Theorem 1.18 can also be stated in terms of *consequence operators*. Like so many other other fundamental concepts in formal logic, the notion of an abstract consequence operator is due to Tarski. The formulation I adopt is from (Rybakov, 1997).

Definition 1.19 *(Tarski, 1935) An abstract consequence operator is a mapping $\mathcal{C} : \mathcal{P}(L) \to \mathcal{P}(L)$ satisfying the conditions:*

1. $\Gamma \subseteq \mathcal{C}(\Gamma)$ (inclusion)

2. $\mathcal{C}(\mathcal{C}(\Gamma)) \subseteq \mathcal{C}(\Gamma)$. (closure)

3. If $\Gamma \subseteq \Delta$ then $\mathcal{C}(\Gamma) \subseteq \mathcal{C}(\Delta)$. (monotony)

4. $\mathcal{C}(\Gamma) = \bigcup \{\mathcal{C}(\Delta) : \Delta \subseteq \Gamma$ and Δ is finite$\}$. (finiteness)

5. $(\forall e : L_0 \to L)\ e(\mathcal{C}(\Gamma)) \subseteq \mathcal{C}(e(\Gamma))$. (uniform substitution)

Definition 1.20 *Given any formal proof system $\mathcal{S}$ with single-conclusion derivability relation $\vdash$, the syntactic consequence operator $Cn_{\vdash} : \mathcal{P}(L) \to \mathcal{P}(L)$ for $\mathcal{S}$ is defined by*

$$Cn_{\vdash}(\Gamma) = \{\alpha : \Gamma \vdash \alpha\}.$$

Theorem 1.21 *(Rybakov, 1997) A mapping $\mathcal{C} : P(L) \to P(L)$ is an abstract consequence operator iff $\mathcal{C}$ is the syntactic consequence operator of some single-conclusion formal proof system.* □

Theorem 1.21 implies that, given any abstract consequence operator $\mathcal{C} : P(L) \to P(L)$, one can define a single-conclusion derivability relation $\vdash_{\mathcal{C}}$ by:

$$\Gamma \vdash_{\mathcal{C}} \alpha \text{ iff } \alpha \in \mathcal{C}(\Gamma).$$

Then $\alpha \in \mathcal{C}(\Gamma)$ iff $\Gamma \vdash_{\mathcal{C}} \alpha$ iff $\alpha \in Cn_{\vdash_{\mathcal{C}}}(\Gamma)$. Thus, given any abstract consequence operator $\mathcal{C}$, we have $\mathcal{C}_{\vdash_{\mathcal{C}}} = \mathcal{C}$.

Definition 1.20 assumes that all the proof rules of $\mathcal{S}$ have single conclusions. In the remainder of this section I will show how this restriction can be relaxed.

The consequence operator of Definition 1.20 operates on formula sets that are thought of conjunctively, as a set of *premisses.* If α is a consequence of Γ, then there is a proof tree in which the elements of Γ appear in a single branch with leaf α. In a multiple-conclusion logic, the consequence operator has a natural counterpart, which I will call an *antecedence* operator. (A similar suggestion was recently made by Zwart (1998).) An antecedence operator operates on formula sets that are thought of disjunctively, as a set of alternatives or *options.* If α is an antecedent of Δ, then there is a proof tree from α such that each branch terminates in an element of Δ.

The next definition generalizes Definition 1.20.

Definition 1.22 *Given any formal proof system $\mathcal{S}$ with derivability relation $\vdash$, its syntactic consequence operator $Cn_{\vdash} : \mathcal{P}(L) \to \mathcal{P}(L)$ is defined by*

$$Cn_{\vdash}(\Gamma) = \{\alpha : \Gamma \vdash \alpha\},$$

and its syntactic antecedence operator $An_{\vdash} : \mathcal{P}(L) \to \mathcal{P}(L)$ is defined by

$$An_{\vdash}(\Delta) = \{\alpha : \alpha \vdash \Delta\}.$$

$Cn_{\vdash}(\Gamma)$ is called the *propositional system* (or just *system*, for short) which Γ gives rise to, and $An_{\vdash}(\Delta)$ the *propositional co-system* which Δ gives rise to. Since $Cn_{\vdash}(\Gamma) = \Gamma$ if and only if $\Gamma = \{\phi : \Gamma \vdash \phi\}$, a system is also called a *deductively closed* formula set or *deductive system.* $\{Cn_{\vdash}(\Gamma) : \Gamma \in \mathcal{P}(L)\}$ is the base set of Tarski's calculus of propositional systems (Tarski, 1930a;

Tarski, 1930b; Tarski, 1935), which is partially ordered by the relation of *logical strength*:

Definition 1.23 *Given any formula sets* Γ *and* Δ, Γ *is logically stronger than* Δ *(*Δ *is logically weaker than* Γ*), written* $\Delta \leq_{Cn} \Gamma$, *iff* $Cn_{\vdash}(\Delta) \subseteq Cn_{\vdash}(\Gamma)$.

Theorem 1.24 *(Wójcicki, 1988) The set of abstract consequence operators of L forms a complete lattice under the relation of logical strength.* □

Definition 1.25 *Let* Γ *and* Σ *be formula sets in a logic with syntactic consequence operator* Cn. Γ *and* Σ *are called* $\equiv_{Cn}$*-equivalent iff they give rise to the same system, that is, if they have the same deductive closure:*

$$\Gamma \equiv_{Cn} \Sigma \text{ iff } Cn(\Gamma) = Cn(\Sigma).$$

The set $An_{\vdash}(\Delta)$ is useful in applications of multiple-conclusion logic. For example, in erotetic logic (Belnap & Steel, 1976; Harrah, 1984; Wisniewski, 1994), one of the prerequisites for a question q to be reducible to a question s, is that every direct answer to q has to imply some direct answer to s. This defines an *explanatory* relation on direct answer sets. Let Γ be the set of direct answers to q, and Δ be the set of direct answers to s. Then,

$$\begin{aligned} \Gamma \text{ explains } \Delta \quad & \text{iff} \quad (\forall \gamma \in \Gamma)\,[\gamma \vdash \Delta] \\ & \text{iff} \quad \Gamma \subseteq An_{\vdash}(\Delta) \\ & \text{iff} \quad An_{\vdash}(\Gamma) \subseteq An_{\vdash}(\Delta). \end{aligned}$$

The set of propositional co-systems is therefore partially ordered by the relation of explanatory strength:

Definition 1.26 *Given any formula sets* Γ *and* Δ, Γ *explains* Δ, *written* $\Delta \leq_{An} \Gamma$, *iff* $An_{\vdash}(\Gamma) \subseteq An_{\vdash}(\Delta)$.

Definition 1.27 *Declare two formula sets $\equiv_{An}$-equivalent iff they give rise to the same co-system:*

$$\Gamma \equiv_{An} \Delta \textit{ iff } An(\Gamma) = An(\Delta).$$

Thus $\Gamma \equiv_{An} \Delta$ if they are equivalent as answer or explanation sets. We will encounter the antecedence operator again in subsequent chapters. The next theorem shows that syntactic antecedence and consequence operators are characterized by the same set of properties, namely those of Definition 1.19.

Theorem 1.28

(i) A mapping $\mathcal{A} : P(L) \to P(L)$ is an abstract consequence operator if and only if $\mathcal{A}$ is the syntactic antecedence operator of some formal proof system.

(ii) A mapping $\mathcal{A} : P(L) \to P(L)$ is an abstract consequence operator if and only if $\mathcal{A}$ is the syntactic consequence operator of some formal proof system.

Proof. (i) Given a proof system $\mathcal{S}$ with derivability relation $\vdash$ and syntactic antecedence operator An, we have to check that An satisfies the properties of an abstract consequence operator. By Theorem 1.18, $\vdash$ is an abstract consequence relation, and the properties of Definition 1.17 therefore apply to it.

1. Inclusion: If $\alpha \in \Gamma$, then $\alpha \vdash \Gamma$ by sharing. Therefore $\alpha \in An(\Gamma)$, and hence $\Gamma \subseteq An(\Gamma)$.

2. Closure: If $\alpha \in An(An(\Gamma))$ then $\alpha \vdash An(\Gamma)$. By the finiteness of $\vdash$, there is a finite subset $\{\gamma_1, ..., \gamma_n\}$ of $An(\Gamma)$ such that $\alpha \vdash \{\gamma_1, ..., \gamma_n\}$.

By weakening, $\alpha \vdash \{\gamma_1, ..., \gamma_n\}, \Gamma$. For each γ_i, since $\gamma_i \in An(\Gamma)$, $\gamma_i \vdash \Gamma$. By weakening, $\gamma_1, \alpha \vdash \{\gamma_2, ..., \gamma_n\}, \Gamma$. By cut, $\alpha \vdash \{\gamma_2, ..., \gamma_n\}, \Gamma$. Similarly, by another $n-1$ applications of cut, $\alpha \vdash \Gamma$. Hence $\alpha \in An(\Gamma)$.

3. Monotony: If $\Gamma \subseteq \Delta$ and $\alpha \in An(\Gamma)$, then $\alpha \vdash \Gamma$, and so, by monotony, $\alpha \vdash \Delta$. Hence $\alpha \in An(\Delta)$.

4. Finiteness: It follows from the monotony of An that

$$\bigcup \{An(\Delta) : \Delta \subseteq \Gamma \text{ and } \Delta \text{ is finite}\} \subseteq An(\Gamma).$$

   Conversely, it follows from the finiteness of $\vdash$ that, for any $\alpha \in An(\Gamma)$, there exists a finite $\Delta \subseteq \Gamma$ such that $\alpha \in An(\Delta)$. Hence $\alpha \in \bigcup \{An(\Delta) : \Delta \subseteq \Gamma \text{ and } \Delta \text{ is finite}\}$.

5. Uniform substitution: Let $\alpha \in An(\Gamma)$ and $e : L_0 \to L$. Then $e(\alpha) \in e(An(\Gamma))$ and $\alpha \vdash \Gamma$. By the uniform substitution property of $\vdash$, $e(\alpha) \vdash e(\Gamma)$, hence $e(\alpha) \in An(e(\Gamma))$.

Conversely, let $\mathcal{A}$ be an abstract consequence operator. Let S be the proof system with inference rules all α/Ψ such that Ψ is finite and $\alpha \in \mathcal{A}(\Psi)$. Denote its derivability relation by $\vdash$, and its corresponding antecedence relation by An. I will show that, for any formula set Δ, $\mathcal{A}(\Delta) \subseteq An(\Delta)$, and $An(\Delta) \subseteq \mathcal{A}(\Delta)$.

Suppose $\alpha \in \mathcal{A}(\Delta)$. Then, by the finiteness of $\mathcal{A}$, there is a finite $\Delta_0 \subseteq \Delta$ such that $\alpha \in \mathcal{A}(\Delta_0)$. By the inference rule α/Δ_0, $\alpha \vdash \Delta_0$. By monotony, $\alpha \vdash \Delta$. Therefore $\alpha \in An(\Delta)$.

Conversely, suppose $\alpha \in An(\Delta)$, that is, $\alpha \vdash \Delta$. The proof proceeds by induction on the number of steps followed in a derivation T from α to Δ. I will show that each initial subtree T_j of T created by steps 2 and 3 of Definition 1.15 preserves the property $\alpha \vdash l(T_j)$.

Base case: If step 2 is executed first to obtain a tree with single leaf α, then $\alpha \vdash \alpha$ by sharing. If step 3 is executed first, it means some inference rule $\emptyset/\{\psi_1,...,\psi_k\}$ is applied to obtain a tree with leaves $\psi_1,...,\psi_k$. By the definition of $\mathcal{S}$, $\emptyset \vdash \{\psi_1,...,\psi_k\}$, and by weakening, $\alpha \vdash \{\psi_1,...,\psi_k\}$.

Inductive step: Let T_j be an initial subtree of T. Assume that for each proper initial subtree T_i of T_j, $\alpha \vdash l(T_i)$. T_j is formed by an application of either step 2 or step 3 of Definition 1.15 to T_{j-1}. Suppose it is step 2. That is, some branch Σ in T_{j-1} is replaced by a branch $(\Sigma;\alpha)$. By the sharing property, $\alpha \vdash l(T_j)$. Else, step 3 is applied to T_{j-1}. That is, choose some branch Σ of T_{j-1} with leaf σ, and some inference rule

$$\Phi/\Psi = \frac{\phi_1(p_1,...,p_n)}{\psi_1(p_1,...,p_n),...,\psi_k(p_1,...,p_n)}$$

applicable to Σ. Let T_{ϕ_1} be the tree obtained from T_{j-1} by removing all descendents of ϕ_1. By the inductive hypothesis, $\alpha \vdash l(T_{\phi_1})$. This can also be written $\alpha \vdash l(T_{\phi_1}) - \phi_1, \phi_1$, where $l(T_{\phi_1}) - \phi_1$ is the set obtained by removing ϕ_1 from $l(T_{\phi_1})$. By weakening, $\alpha \vdash l(T_{\phi_1}) - \phi_1, \phi_1, \Psi$. Also, by weakening the inference rule $\phi_1 \vdash \Psi$, $\alpha, \phi_1 \vdash l(T_{\phi_1}) - \phi_1, \Psi$. By the cut property, $\alpha \vdash l(T_{\phi_1}) - \phi_1, \Psi$. Since T_{ϕ_1} was obtained from T_{j-1} by removing the descendents of ϕ_1, $l(T_{\phi_1}) - \phi_1 \subseteq l(T_{j-1}) - \sigma$. Therefore $\alpha \vdash l(T_j)$ by weakening. This proves the inductive step. Since T is a derivation from α to Δ, $l(T) \subseteq \Delta$. Hence $\alpha \vdash \Delta$.

(ii) The proof is similar to that of (i), and of Theorem 1.21. □

Theorem 1.28 states that any given abstract consequence operator $\mathcal{C}$ is also the syntactic consequence operator of some formal proof system. As on page 14, let $\vdash_{\mathcal{C}}$ denote the derivability relation of such a proof system. Then

$$\Gamma \vdash_{\mathcal{C}} \alpha \text{ iff } \alpha \in Cn_{\vdash_{\mathcal{C}}}(\Gamma) \text{ iff } \alpha \in \mathcal{C}(\Gamma).$$

Theorem 1.28 states that a given abstract consequence operator $\mathcal{A}$ is also the syntactic antecedence operator of some formal proof system. Let $\vdash_{\mathcal{A}}$ denote the derivability relation of such a proof system. Then

$$\alpha \vdash_{\mathcal{A}} \Delta \text{ iff } \alpha \in An_{\vdash_{\mathcal{A}}}(\Delta) \text{ iff } \alpha \in \mathcal{A}(\Delta).$$

Given two arbitrary abstract consequence operators $\mathcal{C}$ and $\mathcal{A}$, it is not in general possible to find a single formal proof system with derivability relation $\vdash$ such that $\mathcal{C} = Cn_{\vdash}$ and $\mathcal{A} = An_{\vdash}$. However, if we only consider abstract consequence operators that are *compatible* in a sense made precise below, this is possible. This is proved in Theorem 1.31 below.

Definition 1.29 *An ordered pair of abstract consequence operators $(\mathcal{C}, \mathcal{A})$ is called compatible if, for any formula sets Γ and Δ,*

$$(\Gamma \vdash_{\mathcal{C}} \delta \textit{ implies } \Gamma \vdash_{\mathcal{A}} \delta) \textit{ and } (\gamma \vdash_{\mathcal{A}} \Delta \textit{ implies } \gamma \vdash_{\mathcal{C}} \Delta).$$

Definition 1.30 *Let $(\mathcal{C}, \mathcal{A})$ be a pair of compatible consequence operators. Define the relation $\vdash_{\mathcal{CA}}$ by:*

$$\Gamma \vdash_{\mathcal{CA}} \Delta \textit{ iff } \Gamma \vdash_{\mathcal{C}} \Delta \textit{ and } \Gamma \vdash_{\mathcal{A}} \Delta.$$

The compatibility condition ensures that the intersection $\vdash_{\mathcal{CA}} = \vdash_{\mathcal{C}} \cap \vdash_{\mathcal{A}}$ contains at least all those pairs (Γ, δ) such that $\Gamma \vdash_{\mathcal{C}} \delta$, and all pairs (γ, Δ) such that $\gamma \vdash_{\mathcal{A}} \Delta$.

Definitions 1.22 and 1.30 enable us to define a consequence relation on the one hand and two consequence operators on the other, in terms of each other.

Theorem 1.31 *Let $(\mathcal{C}, \mathcal{A})$ be a pair of compatible abstract consequence operators. Then $\vdash_{\mathcal{CA}}$ is a consequence relation. Further, $Cn_{\vdash_{\mathcal{CA}}} = \mathcal{C}$ and $An_{\vdash_{\mathcal{CA}}} = \mathcal{A}$.*

Proof. The intersection of any two abstract consequence relations is again an abstract consequence relation. This is proved by checking that $\vdash_{\mathcal{CA}}$ satisfies the conditions of Definition 1.17.

1. Sharing: If $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \vdash_{\mathcal{C}} \Delta$ and $\Gamma \vdash_{\mathcal{A}} \Delta$. Hence $\Gamma \vdash_{\mathcal{CA}} \Delta$.

2. Weakening: Suppose $\Gamma \vdash_{\mathcal{CA}} \Delta$ and $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. By the definition of $\vdash_{\mathcal{CA}}$, $\Gamma \vdash_{\mathcal{C}} \Delta$ and $\Gamma \vdash_{\mathcal{A}} \Delta$. By the monotony of $\mathcal{C}$ and $\mathcal{A}$, $\Gamma' \vdash_{\mathcal{C}} \Delta'$ and $\Gamma' \vdash_{\mathcal{A}} \Delta'$. Hence $\Gamma' \vdash_{\mathcal{CA}} \Delta'$.

3. Cut: Suppose $\Gamma, \phi \vdash_{\mathcal{CA}} \Delta$ and $\Gamma \vdash_{\mathcal{CA}} \phi, \Delta$. By the definition of $\vdash_{\mathcal{CA}}$, $\Gamma, \phi \vdash_{\mathcal{C}} \Delta$ and $\Gamma, \phi \vdash_{\mathcal{A}} \Delta$ and $\Gamma \vdash_{\mathcal{C}} \phi, \Delta$, and $\Gamma \vdash_{\mathcal{A}} \phi, \Delta$. By the cut rule for $\vdash_{\mathcal{C}}$ and $\vdash_{\mathcal{A}}$, $\Gamma \vdash_{\mathcal{C}} \Delta$ and $\Gamma \vdash_{\mathcal{A}} \Delta$. Hence $\Gamma \vdash_{\mathcal{CA}} \Delta$.

4. Finiteness: Let $\Gamma \vdash_{\mathcal{CA}} \Delta$. Then $\Gamma \vdash_{\mathcal{C}} \Delta$ and $\Gamma \vdash_{\mathcal{A}} \Delta$. By finiteness of $\mathcal{C}$ and $\mathcal{A}$, there exist finite subsets Γ_0 and Γ_1 of Γ, and finite subsets Δ_0 and Δ_1 of Δ such that $\Gamma_0 \vdash_{\mathcal{C}} \Delta_0$ and $\Gamma_1 \vdash_{\mathcal{A}} \Delta_1$. Hence, by weakening, $\Gamma_0, \Gamma_1 \vdash_{\mathcal{CA}} \Delta_0, \Delta_1$.

5. Uniform substitution: Let $\Gamma \vdash_{\mathcal{CA}} \Delta$ and $e : L_0 \to L$. Then $\Gamma \vdash_{\mathcal{C}} \Delta$ and $\Gamma \vdash_{\mathcal{A}} \Delta$. By uniform substitution of $\mathcal{C}$ and $\mathcal{A}$, $e(\Gamma) \vdash_{\mathcal{C}} e(\Delta)$ and $e(\Gamma) \vdash_{\mathcal{A}} e(\Delta)$. Hence $e(\Gamma) \vdash_{\mathcal{CA}} e(\Delta)$.

This completes the proof that $\vdash_{\mathcal{CA}}$ is a consequence relation. Next,

$$
\begin{aligned}
\gamma \in Cn_{\vdash_{\mathcal{CA}}}(\Gamma) \quad & \text{iff} \quad \Gamma \vdash_{\mathcal{CA}} \gamma \\
& \text{iff} \quad \Gamma \vdash_{\mathcal{C}} \gamma \text{ and } \Gamma \vdash_{\mathcal{A}} \gamma \text{ (by Definition 1.30)} \\
& \text{iff} \quad \gamma \in \mathcal{C}(\Gamma) \text{ (by Definition 1.29).}
\end{aligned}
$$

Similarly,

$$\begin{aligned} \delta \in An_{\vdash_{\mathcal{CA}}}(\Delta) \quad & \text{iff} \quad \delta \vdash_{\mathcal{CA}} \Delta \\ & \text{iff} \quad \delta \vdash_{\mathcal{C}} \Delta \text{ and } \delta \vdash_{\mathcal{A}} \Delta \text{ (by Definition 1.30)} \\ & \text{iff} \quad \delta \in \mathcal{A}(\Delta) \text{ (by Definition 1.29).} \end{aligned}$$ □

1.4 A case study

In the previous section, no mention was made of any particular logic, apart from fixing the propositional language L and the properties of a consequence relation. The definition of a propositional logic with language L, also requires either a proof system, or a model-theoretic description of semantic consequence. In this section I will follow the latter approach in the definition of a multiple-conclusion extension of classical propositional logic, abbreviated MCC. The logic MCC is a tuple $(L, \models_{\text{MCC}})$, where $\models_{\text{MCC}} \subseteq \mathcal{P}(L) \times \mathcal{P}(L)$ is a multiple-conclusion semantic consequence relation which coincides with classical semantic consequence on $\mathcal{P}(L) \times L$. The main result of this section is Theorem 1.37, which shows that $\models_{\text{MCC}}$ is in fact an abstract consequence relation in the sense of Definition 1.17.

Valuations, models and co-models in MCC are defined as set out in Section 1.1. Given any formula set Γ, its set of models is denoted by $Mod_{\text{MCC}}(\Gamma)$, and its set of co-models by $CoMod_{\text{MCC}}(\Gamma)$.

Definition 1.32 *For any formula sets Γ and Δ in* MCC, *Δ is a semantic consequence of Γ, written $\Gamma \models_{\text{MCC}} \Delta$, iff $Mod_{\text{MCC}}(\Gamma) \subseteq CoMod_{\text{MCC}}(\Delta)$.*

In Sections 1.2 and 1.3, consequence relations were defined syntactically and abstractly. These two descriptions were related in Theorem 1.18. Theorem 1.37 relates that result to Definition 1.32. We do not actually have to exhibit a set of proof rules for MCC, since Theorem 1.18 guarantees its existence. It is sufficient to show that the semantic consequence relation has all the properties of an abstract consequence relation.

(*Remark*: An alternative route would be to reformulate Definition 1.32 as a matrix consequence relation. One would then apply a general result relating multiple-conclusion matrix consequence relations to abstract consequence relations. More specifically, the result needed is the following: Every consequence relation determined by a finite set of finite matrices (in this case a single matrix), is a finitary, abstract consequence operator as defined in 1.19. The reader can consult (Wójcicki, 1988) for more details, in particular Theorems 3.1.3 and 4.1.7.)

The semantic property which corresponds to the finiteness property of an abstract consequence relation, is that of *compactness*, proved in theorem 1.35.

Definition 1.33 *A logic is called compact if, whenever every finite subset of a formula set* Γ *is satisfiable, then so is* Γ.

Corollary 1.36 derives the finiteness property from compactness. The following terminology, which will be discussed in more detail in Section 3.1, will come in handy: The power operation $\neg^+$ of the negation operation $\neg$ is defined by

$$\neg^+(\Delta) = \{\neg\delta : \delta \in \Delta\}.$$

Lemma 1.34 *For any formula sets* Γ *and* Δ, $\Gamma \models_{\text{MCC}} \Delta$ *iff* $\Gamma \cup \neg^+\Delta$ *is unsatisfiable.*

Proof.

$$\Gamma \models_{\text{MCC}} \Delta \quad \text{iff} \quad (\forall v \in Val_2)\,[(\exists\gamma \in \Gamma)\,[v(\gamma) = f] \ \text{ or } \ (\exists\delta \in \Delta)\,[v(\delta) = t]]$$

$$\text{iff} \quad (\forall v \in Val_2)\left(\exists \alpha \in \Gamma \cup \neg^{+}\Delta\right)[v(\alpha) = f]$$

$$\text{iff} \quad \Gamma \cup \neg^{+}\Delta \text{ is unsatisfiable.}$$ □

Theorem 1.35 MCC *is compact.*

Proof. The proof is taken from Enderton (1972). Suppose every finite subset of Σ is satisfiable. Let $\alpha_1, \alpha_2, \ldots$ be an enumeration of elements of L. Let $\Delta_0 = \Sigma$, and for each $n \geq 0$,

$$\Delta_{n+1} = \begin{bmatrix} \Delta_n \cup \alpha_{n+1} \text{ if every finite subset of this set is satisfiable;} \\ \Delta_n \text{ otherwise.} \end{bmatrix}$$

Then every finite subset of each Δ_n is satisfiable.

Let $\Delta = \bigcup_n \Delta_n$. Then every finite subset of Δ is a finite subset of some Δ_n, and is therefore satisfiable. Define a truth assignment v by:

$$v(p_i) = t \text{ iff } p_i \in \Delta.$$

Extend v to a valuation on all wffs. Then, for any ϕ, $v(\phi) = t$ iff $\phi \in \Delta$. Therefore Δ is satisfiable. Since $\Sigma \subseteq \Delta$, so is Σ. □

Corollary 1.36 *If* $\Gamma \models_{\text{MCC}} \Delta$ *then there exist finite subsets* $\Gamma_0 \subseteq \Gamma$ *and* $\Delta_0 \subseteq \Delta$ *such that* $\Gamma_0 \models_{\text{MCC}} \Delta_0$.

Proof. The proof uses Lemma 1.34. Suppose $\Gamma_0 \not\models \Delta$ for every finite $\Gamma_0 \subseteq \Gamma$. That is, $\Gamma_0 \cup \neg^{+}\Delta$ is satisfiable for every finite $\Gamma_0 \subseteq \Gamma$. Let Σ be any finite subset of $\Gamma \cup \neg^{+}\Delta$. Then $\Sigma = \Gamma_0 \cup \neg^{+}\Delta_0$ for some finite $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$. Therefore Σ is satisfiable. By compactness, $\Gamma \cup \neg^{+}\Delta$ is satisfiable. Hence $\Gamma \not\models \Delta$.

Next, suppose $\Gamma \not\models \Delta_0$ for every finite $\Delta_0 \subseteq \Delta$. That is, $\Gamma \cup \neg^{+}\Delta_0$ is satisfiable for every finite $\Delta_0 \subseteq \Delta$. Let Σ be any finite subset of $\neg^{+}\Gamma \cup \Delta$.

Then $\Sigma = \neg^+\Gamma_0 \cup \Delta_0$ for some finite $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$. Therefore $\neg^+\Sigma$ is satisfiable. By compactness, $\Gamma \cup \neg^+\Delta$ is satisfiable. Hence $\Gamma \not\models \Delta$. □

Corollary 1.36 uses compactness to prove finiteness. The converse also holds: Given a formal proof system and a completeness result, compactness can be proved from finiteness. Since I have not given any formal proof system for the logic under consideration, I will not prove that this is the case. This route will be followed for the logic MCK, which is the topic of Chapter 2.

Theorem 1.37 *There exists a derivability relation $\vdash_{\text{MCC}}$ such that, for any formula sets Γ and Δ, $\Gamma \models_{\text{MCC}} \Delta$ iff $\Gamma \vdash_{\text{MCC}} \Delta$.*

Proof. We have to check that the conditions of Definition 1.17 hold for $\models_{\text{MCC}}$.

1. Sharing: Suppose $\Gamma \cap \Delta \neq \emptyset$. Let $\alpha \in \Gamma \cap \Delta$. Any model of Γ is a model of α, and hence a co-model of Δ.

2. Weakening: Suppose $\Gamma \models_{\text{MCC}} \Delta$ and $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. Any model of Γ' is a model of Γ and hence a co-model of Δ. Since every co-model of Δ is a co-model of Δ', the result follows.

3. Cut: Suppose $\Gamma, \phi \models_{\text{MCC}} \Delta$ and $\Gamma \models_{\text{MCC}} \phi, \Delta$. Let v be a model of Γ. Then v is a co-model of $\{\phi\} \cup \Delta$. So either v is a co-model of ϕ or v is a co-model of Δ. If v is a co-model of ϕ, it is a model of ϕ and hence a model of $\Gamma \cup \{\phi\}$. So v is a co-model of Δ. Therefore $\Gamma \models_{\text{MCC}} \Delta$.

4. Finiteness: This follows from Corollary 1.36.

5. Uniform substitution: Let $e : L_0 \to L$, $v \in Val_2$ and $\gamma \in L$. Define the truth assignment $v_0 \circ e : L_0 \to \{t, f\}$ by:

$$v_0 \circ e\,(p_i) = v\,(e\,(p_i))\,.$$

Extend $v_0 \circ e$ to a homomorphism $v \circ e : L \to \{t, f\}$. Then v is a model of $e(\gamma)$ iff $v \circ e$ is a model of γ. Suppose $\gamma \models_{\text{MCC}} \delta$. Then every model of γ is a model of δ. Let u be a model of $e(\gamma)$. Then $u \circ e$ is a model of γ. Hence $u \circ e$ is a model of $e(\delta)$. Thus u is a model of δ, and $e(\gamma) \models_{\text{MCC}} e(\delta)$. Now suppose $\Gamma \models_{\text{MCC}} \Delta$. By finiteness, there exists finite subsets $\{\gamma_1, ..., \gamma_n\} \subseteq \Gamma$ and $\{\delta_1, ..., \delta_m\} \subseteq \Delta$ such that $\{\gamma_1, ..., \gamma_n\} \models_{\text{MCC}} \{\delta_1, ..., \delta_m\}$. By the definition of a valuation, u is a model of $\{\gamma_1, ..., \gamma_n\}$ iff u is a model of $\gamma_1 \wedge ... \wedge \gamma_n$, and u is a co-model of $\{\delta_1, ..., \delta_m\}$ iff u is a model of $\delta_1 \vee ... \vee \delta_m$. Let $\gamma = \gamma_1 \wedge ... \wedge \gamma_n$ and let $\delta = \delta_1 \vee ... \vee \delta_m$. Then $\gamma \models_{\text{MCC}} \delta$ as above, and therefore $e(\gamma) \models_{\text{MCC}} e(\delta)$. The result follows by weakening. $\square$

The relations of logical and explanatory strength were defined in terms of syntactic consequence and antecedence operators, in Definitions 1.23 and 1.26 respectively. Theorem 1.28 characterizes these relations in terms of abstract consequence operators. Let Cn_{MCC} denote the syntactic consequence operator of MCC. The next theorem gives the semantic counterpart of the syntactic characterization of logical strength.

Theorem 1.38 *For any formula sets Γ and Σ, $Cn_{\text{MCC}}(\Gamma) \subseteq Cn_{\text{MCC}}(\Sigma)$ iff $Mod_{\text{MCC}}(\Gamma) \supseteq Mod_{\text{MCC}}(\Sigma)$.*

Proof. Suppose $Cn_{\text{MCC}}(\Gamma) \subseteq Cn_{\text{MCC}}(\Sigma)$. Let $\gamma \in Cn_{\text{MCC}}(\Gamma)$. Then $\Gamma \vdash_{\text{MCC}} \gamma$ and hence, by Theorem 1.37, $\Gamma \models_{\text{MCC}} \gamma$. Thus every model of Γ is a model of γ. Every model of Γ is therefore also a model of $Cn_{\text{MCC}}(\Gamma)$. Since $\Gamma \subseteq Cn_{\text{MCC}}(\Gamma)$, the converse also holds. Therefore $Mod_{\text{MCC}}(\Gamma) = Mod_{\text{MCC}}(Cn_{\text{MCC}}(\Gamma))$. Similarly, $Mod_{\text{MCC}}(\Sigma) = Mod_{\text{MCC}}(Cn_{\text{MCC}}(\Sigma))$. Now let $v \in Mod_{\text{MCC}}(\Sigma)$. Then $v \in Mod_{\text{MCC}}(Cn_{\text{MCC}}(\Sigma)) \subseteq Mod_{\text{MCC}}(Cn_{\text{MCC}}(\Gamma)) = Mod_{\text{MCC}}(\Gamma)$.

Conversely, suppose $Cn_{\text{MCC}}(\Gamma) \not\subseteq Cn_{\text{MCC}}(\Sigma)$. Then there exists a wff α such that $\alpha \in Cn_{\text{MCC}}(\Gamma)$ and $\alpha \notin Cn_{\text{MCC}}(\Sigma)$. Therefore $\Gamma \models_{\text{MCC}} \alpha$ and $\Sigma \not\models_{\text{MCC}} \alpha$. So there is some valuation v such that $v \in Mod_{\text{MCC}}(\Sigma)$ but $v \notin Mod_{\text{MCC}}(\alpha)$. Hence $v \notin Mod_{\text{MCC}}(\Gamma)$. Therefore $Mod_{\text{MCC}}(\Gamma) \not\supseteq Mod_{\text{MCC}}(\Sigma)$.□

Corollary 1.39 *For any formula sets Γ and Σ, $Cn_{\text{MCC}}(\Gamma) = Cn_{\text{MCC}}(\Sigma)$ iff $Mod_{\text{MCC}}(\Gamma) = Mod_{\text{MCC}}(\Sigma)$.* □

Thus the logically strongest system is the set of all wffs, which has no models, and the logically weakest system is the set of tautologies, with set of models Val_2.

Let An_{MCC} denote the syntactic antecedence operator of MCC. The equivalence relations $\equiv_{Cn}$ and $\equiv_{An}$ are not the same, for example, the sets $\{p, q\}$ and $\{p, p \wedge q\}$ are $\equiv_{Cn}$-equivalent, but they are not $\equiv_{An}$-equivalent.

Theorem 1.40 *For any formula sets Γ and Δ, $An_{\text{MCC}}(\Gamma) \subseteq An_{\text{MCC}}(\Delta)$ iff $CoMod_{\text{MCC}}(\Gamma) \subseteq CoMod_{\text{MCC}}(\Delta)$.*

Proof. Suppose $An_{\text{MCC}}(\Gamma) \subseteq An_{\text{MCC}}(\Delta)$. Let $\gamma \in An_{\text{MCC}}(\Delta)$. Then $\gamma \vdash_{\text{MCC}} \Delta$ and hence, by Theorem 1.37, $\gamma \models_{\text{MCC}} \Delta$. Thus every co-model of γ is a co-model of Δ. Every co-model of $An_{\text{MCC}}(\Delta)$ is therefore also a co-model of Δ. Since $\Delta \subseteq An_{\text{MCC}}(\Delta)$, the converse also holds. Therefore $CoMod_{\text{MCC}}(\Delta) = CoMod_{\text{MCC}}(An_{\text{MCC}}(\Delta))$. Similarly, $CoMod_{\text{MCC}}(\Gamma) = CoMod_{\text{MCC}}(An_{\text{MCC}}(\Gamma))$. Now let $v \in CoMod_{\text{MCC}}(\Gamma)$. Then $v \in CoMod_{\text{MCC}}(An_{\text{MCC}}(\Gamma)) \subseteq CoMod_{\text{MCC}}(An_{\text{MCC}}(\Delta)) = Mod_{\text{MCC}}(\Delta)$.

Conversely, suppose $An_{\text{MCC}}(\Gamma) \not\subseteq An_{\text{MCC}}(\Delta)$. Then there exists a wff α such that $\alpha \in An_{\text{MCC}}(\Gamma)$ and $\alpha \notin An_{\text{MCC}}(\Delta)$. Therefore $\alpha \models_{\text{MCC}} \Gamma$ and $\alpha \not\models_{\text{MCC}} \Delta$. So there is some valuation v such that $v \in Mod_{\text{MCC}}(\alpha)$ but $v \notin CoMod_{\text{MCC}}(\Delta)$. Hence $v \notin CoMod_{\text{MCC}}(\Gamma)$. Therefore $CoMod_{\text{MCC}}(\Gamma) \not\subseteq CoMod_{\text{MCC}}(\Delta)$. □

Corollary 1.41 *For any formula sets* Γ *and* Δ, $An_{\text{MCC}}(\Gamma) = An_{\text{MCC}}(\Delta)$ *iff* $CoMod_{\text{MCC}}(\Gamma) = CoMod_{\text{MCC}}(\Delta)$.

Definition 1.26 orders co-systems according to their explanatory strength: The weakest co-system is the set of all wffs, with set of co-models Val_2, and the strongest co-system is the set of contradictions, which has no co-models. In terms of the example of erotetic logic mentioned on page 16, this means that a smaller direct answer set is stronger, allowing fewer direct answers, while a weaker answer set allows more possible answers to a question.

To summarize, the three semantic relations on formula sets described in this section are:

(i) Consequence: $\Gamma \models_{\text{MCC}} \Delta$ iff $Mod_{\text{MCC}}(\Gamma) \subseteq CoMod_{\text{MCC}}(\Delta)$.

(ii) Logical strength: $\Delta \leq_{Cn} \Gamma$ iff $Mod_{\text{MCC}}(\Gamma) \subseteq Mod_{\text{MCC}}(\Delta)$.

(iii) Explanatory strength: $\Delta \leq_{An} \Gamma$ iff $CoMod_{\text{MCC}}(\Gamma) \subseteq CoMod_{\text{MCC}}(\Delta)$.

These relations may also be defined syntactically, as relations between systems and/or co-systems. They then read:

(i) Consequence: $\Gamma \vdash_{\text{MCC}} \Delta$ iff $Cn_{\text{MCC}}(\Gamma) \cap An_{\text{MCC}}(\Delta) \neq \emptyset$.

(ii) Logical strength: $\Delta \leq_{Cn} \Gamma$ iff $Cn_{\text{MCC}}(\Delta) \subseteq Cn_{\text{MCC}}(\Gamma)$.

(iii) Explanatory strength: $\Delta \leq_{An} \Gamma$ iff $An_{\text{MCC}}(\Gamma) \subseteq An_{\text{MCC}}(\Delta)$.

These relations can all be refined in useful ways. In Chapter 2 I will consider a refinement of the semantic consequence relation of Definition 1.32. Chapter 4 deals with refinements to the relations of logical strength and, to a lesser extent, explanatory strength.

Chapter 2

Three-valued logics of partial knowledge

In Chapter 1 consequence relations were characterized abstractly, as binary relations on $\mathcal{P}(L)$, and syntactically, as derivability relations in formal proof systems. In Definition 1.32, the semantic consequence relation of the logic MCC was defined an example of a consequence relation. This relation, or rather, its single-conclusion restriction, is the standard definition of semantic consequence, both for two-valued and for many-valued logics. There are, however, some natural refinements of this definition, that trivialise in two-valued logics, but are useful in many-valued logics. In this chapter, I will define two logics, called MCK and $\diamond$MCK respectively, that each employ a refined notion of consequence. MCK and $\diamond$MCK are both multiple-conclusion three-valued propositional logics based on Kleene's strong three-valued truth tables; as such, they are logics of partial information, where partial information about a proposition may be due to either lack of knowledge or undefinedness.

The consequence relation of $\diamond$MCK is an example of a power relation between formula sets, based on a structure which I will call a semi-bilattice of truth values. This point will be elaborated in Chapter 3, where I will discuss power constructs in more detail.

2.1 Background

Lack of information about the truth of certain statements can only be dealt with indirectly in classical logic. Tarski's semantic account of truth (Tarski, 1931) formalized what it means to say "p is true" or "p is false", but provides no way of asserting explicitly that no information about p is available. Thus statements about lack of information are made through omissions from that which we do know. A knowledge base indicates which statements are true; their negations are false, and everything else is unknown.

Logics of partial information change the semantics (and, correspondingly, the proof theory) of classical logic in order to deal with the expression of lack of information in a more direct fashion, while remaining as close as possible to classical logic. Two approaches suggested in the literature to achieve this goal are supervaluations and many-valued semantics.

Van Fraassen (1966; 1969) proposed the use of supervaluations as a semantic way of dealing with non-denoting terms. On this view, propositions may lack a truth value. A compound proposition, of which some constituents may lack a truth value, is assigned a truth value only if any assignment of truth values to the variables lacking truth values, would assign that value to it. For example, if p lacks a truth value and q is false, then the proposition $q \vee p \vee \neg p$ is true, since any classical assignment of a truth value to p would make $p \vee \neg p$ true. Supervaluations yield the same contradictions and tautologies as classical propositional logic, but differ in another important aspect from classical valuations - they are non-truthfunctional. For example, $p \vee \neg p$ is true, even if p lacks a truth value, but if both p and q lack a truth value, then so does $p \vee q$.

An alternative way to deal with truth value gaps, is to introduce a third truth value which is on a par with the classical truth values. Many different three-valued logics have been proposed, assigning different meanings to the third truth value. To name but a few, there are the three-valued logics of Łukasiewicz (1930), translated in (Łukasiewicz, 1967) (to deal with future contingent sentences , or with undetermined events, as in (Slupecki, 1964)), Bochvar (1938), translated in (Bochvar, 1981) (to deal with meaningless sentences), Reichenbach (1944) (to deal with causal anomalies in quantum mechanics), and Kleene (1952) (to deal with undetermined propositions, and applied to recursive function theory).

The advent of the computer and information sciences gave rise to a number of new applications of three-valued logics of partial knowledge. For example, the logic of partial functions LPF of Blamey (1986) is used in the axiomatization of the formal specification system VDM (Barringer *et al.*, 1984; Jones, 1986; Gibbins, 1988). LPF is used to reason about program correctness involving undefinedness. Blikle (1991), together with Konikowska & Tarlecki (1991), use a three-valued logic of McCarthy (1967) for the same purpose. A three-valued Kripke-Kleene semantics is also used to reason about logic programs (Fitting, 1985; Fitting, 1991b; Liu & Moore, 1998), and Bergstra & Ponse (1998) use Kleene logic to provide an operational semantics to ACP process algebra. Avron (1991a; 1991b) characterizes a number of the three-valued logics I have mentioned here by studying their consequence relations. For a more general account of many-valued logics, the reader can consult (Urquhart, 1986; Bolc & Borowik, 1992; Malinowski, 1993).

Logics with four or more truth values have also been proposed for computerized reasoning. Belnap's (1977a; 1977b) four-valued logic has become a standard reference in this regard. It is defined semantically in terms of a four-element quasi-Boolean algebra, with elements true (t), both (b), none (n) and false (f). The intuition of the four values is to indicate which subset of the set of classical truth values $\{t, f\}$ is assigned to each atom in a knowledge base, with $v(p_i) = n$ indicating that the truth assignment v assigns no classical truth value to p_i, and $v(p_i) = b$ indicating that v assigns both the classical truth values t and f to p_i. The partial order associated with the lattice is called the *truth order* $\leq_t$. f is the minimum element with respect to the truth order, t is the maximum element, and b and n are incomparable. As follows:

$$\begin{tikzcd} & t & \\ b \arrow[ur] & & n \arrow[ul] \\ & f \arrow[ul] \arrow[ur] & \end{tikzcd}$$

This yields the abstract algebra $\mathcal{B}_4 = \{\{t, b, n, f\}, \wedge, \vee, \neg\}$, with operations defined by the following tables:

$\wedge$	t	b	n	f
t	t	b	n	f
b	b	b	f	f
n	n	f	n	f
f	f	f	f	f

$\vee$	t	b	n	f
t	t	t	t	t
b	t	b	t	b
n	t	t	n	n
f	t	b	n	f

$\neg$	
t	f
b	b
n	n
f	t

Belnap uses these truth tables to define a logic for reasoning amidst inconsistencies. It turns out that the resulting logic is exactly the relevance logic of first degree entailments FDE of Anderson & Belnap (1975). The entailment relation $\rightarrow_{\text{FDE}} \subseteq L \times L$ of FDE is characterized as follows by the four-element quasi-Boolean algebra $\mathcal{B}_4$:

$$\phi \rightarrow_{\text{FDE}} \psi \text{ iff for every homomorphism } v : L \rightarrow \mathcal{B}_4,\ v(\phi) \leq_t v(\psi).$$

The idea of providing FDE with a four-valued semantics in a quasi-Boolean algebra (also called a De Morgan lattice, see Definition 1.9), originated with Dunn (1966; 1976). A number of other relevance logics have since been given a four-valued semantics (Restall, 1995). A number of different axiomatizations also exist for FDE, both in the Hilbert style and as a Gentzen style sequent calculus (Pynko, 1995a; Font, 1997).

I have thus far not mentioned another central theme in Belnap's exposition, namely that of knowledge approximation. On this view, the four truth values t, b, n and f are partially ordered by a *knowledge order* $\leq_k$. According to the knowledge order, n is the minimum element, b is the maximum element, and t and f are incomparable. This lattice is isomorphic to the power set lattice of $\{t, f\}$, ordered by set inclusion. Thus the order is one of growth in information. Endowing $\mathcal{B}_4$ with the knowledge order, yields the following four-valued structure, called a *bilattice*:

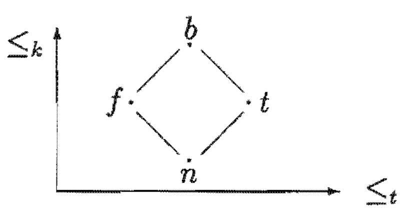


Figure 2.1: The Belnap bilattice

Following Scott (1970; 1972), Belnap argues that the knowledge order is crucial to the choice of connectives in a logic. Namely, the connectives should be *monotone* with respect to the knowledge order:

Definition 2.1 *An n-ary connective f in a logic characterized by an abstract algebra $\mathcal{A}$ with corresponding n-ary operation f is monotone with respect to a given partial order $\leq_k$ on elements of $\mathcal{A}$ if it preserves the order $\leq_k$. That is, whenever $a_1 \leq_k b_1$, ..., $a_n \leq_k b_n$, then also $f(a_1, ..., a_n) \leq_k f(b_1, ..., b_n)$.*

Put more simply, the definition says that no increase in the truth value of sentential symbols with respect to the knowledge order, can result in a decrease (again with respect to the knowledge order) in the truth value of any wff. The intuition behind the monotony condition is that an increase in information about the truth of an atom can not result in a decrease in information about the truth of a compound wff. The monotony condition determined Belnap's choice of connectives: $\wedge$, $\vee$ and $\neg$ are all monotone with respect to the knowledge order on the four-valued bilattice.

Ginsberg (1988) generalized the four-valued bilattice to bilattices with more than four values. Arieli & Avron (1996) recently showed that many useful logics with bilattice semantics can be charaterized in terms of the four values. There are many different applications of logics based on bilattice theory, especially in an artificial intelligence context. Belnap's original motivation was to create a tool to reason about inconsistent knowledge bases. A number of authors have since built on his approach, with the same aim in mind (Kaluzhny & Muravitsky, 1993; Lakemeyer, 1996; Arieli & Avron, 1997). Another fruitful application area of bilattice theory is in the semantics of logic programming (Fitting, 1985; Fitting, 1991a; Fitting, 1994; Stark, 1996; Ruet & Fages, 1997).

Returning to three-valued logics, the strong three-valued truth tables of Kleene (1952) was motivated by the undeterminedness of certain propositions, but, as Kleene himself noted, the third (undetermined) truth value is also susceptible of another meaning - that of the unknown. It is this reading that has led to the definition of LPF (Barringer *et al.*, 1984; Blamey, 1986; Gibbins, 1988). The logics MCK and $\diamond$MCK, which I will define in Section 2.2, are similar in aim to LPF, but they only have the operations of the strong Kleene system. The logic KL, which I define below, is ususally referred to as Kleene logic, and is based on Kleene's strong truth tables.

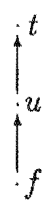
The language of KL is the propositional language L of Section 1.1. Its set of truth values is $\{t, u, f\}$. The abstract algebra for KL is the Kleene lattice $\mathcal{K}_3 = \{\{t, u, f\}, \wedge, \vee, \neg\}$, with base set $\{t, u, f\}$ and operations $\wedge$, $\vee$ and $\neg$ defined by the following tables:

$\wedge$	t	u	f
t	t	u	f
u	u	u	f
f	f	f	f

$\vee$	t	u	f
t	t	t	t
u	t	u	u
f	t	u	f

$\neg$	
t	f
u	u
f	t

The partial order associated with $\mathcal{K}_3$ is the *truth order* depicted below, and defined by: $\alpha \leq_t \beta$ iff $\alpha = \alpha \wedge \beta$.



The algebra $\mathcal{K}_3$ is of the same similarity as L (see Section 1.1). Any assignment of elements of $\mathcal{K}_3$ to sentential symbols can therefore be extended to a homomorphism $v : L \to \mathcal{K}_3$. Let Val_3 denote the set of all such valuations. The semantic consequence relation of KL is defined as follows:

Definition 2.2 *Given any formula set* Γ *and wff* α, $\Gamma \models_{\text{KL}} \alpha$ *iff for every valuation* $v \in Val_3$, $v(\alpha) = t$ *whenever* $v(\Gamma) \subseteq \{t\}$.

As in the case of the four-valued bilattice, the elements of $\mathcal{K}_3$ can also be endowed with a second order. Namely, the structure $(\{t, u, f\}, \cdot)$, with the meet operation defined by the table given below, is a meet-semilattice, as defined in 1.6.

·	t	u	f
t	t	u	u
u	u	u	u
f	u	u	f

The partial order associated with the meet-semilattice is the *knowledge order* $\leq_k$, defined by:

$$\alpha \leq_k \beta \text{ iff } \alpha = \alpha \cdot \beta.$$

Endowing the base set of $\mathcal{K}_3$ with both the truth order and the knowledge order, one obtains the following structure, which I will call a *semi-bilattice*:

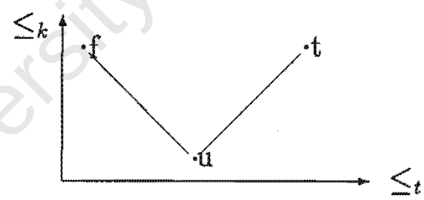


Figure 2.2: The Kleene semi-bilattice

In the next two sections I will discuss the model theory and proof theory of the logics MCK and ◇MCK, both of which are based on the three-valued semi-bilattice. In Section 2.4 I will comment on some properties of these logics, notably that of paraconsistency.

2.2 Model theory

The strong truth tables of Kleene do not fix a unique logic. In order to do that, we also need a syntactic notion of derivability, or a semantic consequence relation. A standard way to define a semantic consequence relation employs the notion of a logical matrix. In its simplest form, a logic is determined by a single matrix which consists of an abstract algebra of the same similarity as the language of the logic, and a set of designated elements. For example, the determining matrix of the logic PC, presented in Section 1.1, is the tuple $\langle \mathcal{B}_2, \{t\} \rangle$. More generally, a logic is determined by a class of matrices $\mathcal{M}$. The matrix consequence relation determined by $\mathcal{M}$ is defined as the intersection of the matrix consequence relations determined by each matrix. A detailed account of matrix consequence relations can be found in (Wójcicki, 1988).

Definition 2.3 *Let $M = \langle \mathcal{A}, D \rangle$ be a logical matrix. The matrix consequence relation $\models_M$ determined by M is defined by:*

$$\Gamma \models_M \Delta \quad \textit{iff} \quad \textit{for any homomorphism } v \textit{ from } L \textit{ to } \mathcal{A},$$
$$\textit{if } v(\Gamma) \subseteq D \textit{ then } v(\Delta) \cap D \neq \emptyset.$$

Definition 2.4 *Let $\mathcal{M} = \{\langle \mathcal{A}_i, D_i \rangle : i \in I\}$ be a class of matrices. Then*

$$\Gamma \models_{\mathcal{M}} \Delta \textit{ iff for every } M \in \mathcal{M},\ \Gamma \models_M \Delta.$$

Semantic equivalence of wffs in a logic with a matrix consequence relation $\models_{\mathcal{M}}$ is then defined as in classical logic:

$$\phi \approx_{\mathcal{M}} \psi \text{ iff } \phi \models_{\mathcal{M}} \psi \text{ and } \psi \models_{\mathcal{M}} \phi.$$

To take the definition of a matrix consequence relation into account, the terminology of Section 1.1 generalizes as follows. In a logic determined by a single matrix, a valuation v which assigns a designated value to each element of a formula set Γ, is called a *model* of Γ. If Γ has a model, it is called *satisfiable*. A wff ϕ is called a *tautology* if it always takes on a designated value, that is, if every valuation is a model of ϕ. In a logic determined by a class of matrices $\mathcal{M} = \{\langle \mathcal{A}, D_i \rangle : i \in I\}$, v is a model of Γ if it assigns a designated value to each element of Γ in each determining matrix $\langle \mathcal{A}, D_i \rangle$. Note that the concept of a model has no meaning if the abstract algebras of the matrices in $\mathcal{M}$ differ. A tautology is similarly defined as a wff which takes on a designated value in each matrix $\langle \mathcal{A}, D_i \rangle$.

The abstract algebra $\mathcal{K}_3$ defined in the previous section forms the basis of any definition of matrix consequence based on Kleene's strong truth tables. Designating only t yields the logic KL, with determining matrix $\langle \mathcal{K}_3, \{t\} \rangle$ and single-conclusion semantic consequence relation of Definition 2.2. Designating both t and u yields the logic of paradox LP (Priest, 1979), with determining matrix $\langle \mathcal{K}_3, \{t, u\} \rangle$ and single-conclusion semantic consequence relation defined as follows:

Definition 2.5 $\Gamma \models_{\text{LP}} \alpha$ *iff for every valuation* v, $v(\alpha) \in \{t, u\}$ *whenever* $v(\Gamma) \subseteq \{t, u\}$.

I will defer a discussion of the intuition behind LP until Section 2.4.

Definition 2.6 *Let* $\mathcal{A}$ *be any algebra of the same similarity as* L, *and let* $\leq$ *be a partial order on its base set* A. *Let* Val_A *be the set of valuations, i.e. homomorphisms, from* L *to* $\mathcal{A}$. *The pointwise partial order* $\leq$ *on* Val_A *is defined as follows: For any* $v, w \in Val_A$,

$$v \leq w \text{ iff } (\forall p_i \in L_0)\,[v(p_i) \leq w(p_i)]\,.$$

The knowledge order $\leq_k$ on the Belnap bilattice of figure 2.1, and that on the Kleene semi-bilattice of figure 2.2, are examples of a partial order on an algebra of truth values. The knowledge order on the Boolean algebra $\mathcal{B}_2$, which determines classical propositional logic, is the discrete order, which leaves f and t unrelated.

Definitions 2.2 and 2.5 both ignore to some extent the truth order on the elements of $\mathcal{K}_3$. The former regards u and f as equally untrue, while the latter regards u and t as equally true. The truth order gives an indication of how close a wff is to being true or false. This information is lost if a simple partition between designated and non-designated elements of the algebra is used. The truth order of a logic can be used in one of two ways: either define a corresponding connective in the logic, or use it at the meta-level in the definition of semantic consequence.

Lukasiewicz (1967) followed the former route. To obtain his logic, add a binary connective $\rightarrow_{\text{L}}$ to L, and define a corresponding binary operation in $\mathcal{K}_3$ as follows:

$\rightarrow_{\text{L}}$	t	u	f
t	t	u	f
u	t	t	u
f	t	t	t

The truth order is used in Lukasiewicz logic in that $v(\phi \rightarrow_{\text{L}} \psi) = t$ iff $v(\phi) \leq_t v(\psi)$. Unlike the connectives of KL, $\rightarrow_{\text{L}}$ is not monotone with respect to the knowledge order $\leq_k$, in the sense of Definition 2.1. My motivation for requiring the connectives to be monotone with respect to the knowledge order is similar as for Belnap's four-valued logic. Suppose, for example, that the truth values of both p_i and p_j are unknown. So, according to the truth table for $\rightarrow_{\text{L}}$, $p_i \rightarrow_{\text{L}} p_j$ is true. Now suppose the truth value of p_i becomes known, say p_i is true. In accordance with the truth table for $\rightarrow_{\text{L}}$, this increase in information forces a retraction of the fact that $p_i \rightarrow_{\text{L}} p_j$ is true. $p_i \rightarrow_{\text{L}} p_j$ now becomes undefined. This means that an increase in information about atoms causes a decrease in information about certain composite wffs. This is not the case with material implication, which is monotone. The property of monotony in three-valued logics is studied by Blamey (1986).

The second option mentioned above, is to use the truth order at the meta-level in the definition of a semantic consequence relation. This option is followed in Definition 2.7 below. This defines a three-valued multiple-conclusion logic based on Kleene's strong truth tables, abbreviated MCK. This is, in effect, what Belnap did to arrive at the logic FDE as a useful four-valued logic for reasoning about knowledge bases. A similar relation was used by Gibbins (1988) to give semantics to the sequent calculus LPF3. It also receives a brief reference in (Avron, 1991b). The minimum and maximum truth values in Definition 2.7 are taken with respect to the truth order of the logic, with $\min_{\leq_t} \emptyset = t$ and $\max_{\leq_t} \emptyset = f$.

Definition 2.7 *For any formula sets* Γ *and* Δ, Δ *is a semantic consequence of* Γ *in* MCK, *written* $\Gamma \models_{\text{MCK}} \Delta$, *iff*

$$(\forall v \in Val_3)\left[\min_{\leq_t} \{v(\gamma) : \gamma \in \Gamma\} \leq_t \max_{\leq_t} \{v(\delta) : \delta \in \Delta\}\right].$$

This yields a logically weaker relation, in the sense of Definition 1.23, than that of Definition 1.32. If only classical, two-valued valuations are considered, the refinement trivialises, so that Definition 2.7 coincides with Definition 1.32.

Consider again the semantic consequence relations of KL and LP, defined in 2.2 and 2.5 respectively. Generalized to multiple-conclusion consequence relations, they read as follows:

Definition 2.8 *Given any formula sets* Γ *and* Δ, $\Gamma \models_{\text{KL}} \Delta$ *iff* $\forall v \in Val_3$, $v(\Delta) \cap \{t\} \neq \emptyset$ *whenever* $v(\Gamma) \subseteq \{t\}$.

Definition 2.9 *Given any formula sets* Γ *and* Δ, $\Gamma \models_{\text{LP}} \Delta$ *iff* $\forall v \in Val_3$, $v(\Delta) \cap \{t, u\} \neq \emptyset$ *whenever* $v(\Gamma) \subseteq \{t, u\}$.

Since the single-conclusion restrictions of these relations coincide with Definitions 2.2 and 2.5 respectively, I will use the names KL and LP to refer to the respective multiple-conclusion relations as well as their single-conclusion restrictions. The respective determining matrices of KL and LP are $\langle \mathcal{K}_3, \{t\} \rangle$ and $\langle \mathcal{K}_3, \{t, u\} \rangle$. Unlike these logics, MCK is not determined by a single matrix, but by both these matrices. Further, the sets $\{t\}$ and $\{t, u\}$ are precisely the proper filters (defined on page 7) of $\mathcal{K}_3$. MCK could therefore also have been defined as the matrix consequence relation obtained from the following set of matrices:

$$\mathcal{M}_{\text{MCK}} = \{\langle \mathcal{K}_3, f \rangle : f \text{ is a proper filter of } \mathcal{K}_3\}.$$

Finally, the relation $\models_{\text{MCK}}$ can also be formulated as follows: Δ is a semantic consequence of Γ iff, whenever every element of Γ is true, then at least one element of Δ is true, and whenever every element of Δ is false, then at least one element of Γ is false. This is the formulation used by Gibbins (1988) to define the logic LPF3.

The following three theorems compare the logical strength of the consequence relations of MCK, KL, LP and MCC.

Theorem 2.10 $\Gamma \models_{\text{MCK}} \Delta$ *iff* $\Gamma \models_{\text{KL}} \Delta$ *and* $\Gamma \models_{\text{LP}} \Delta$.

Proof. This follows from the fact that the consequence relation of MCK is determined by the matrices for KL and LP. □

Theorem 2.11 *If* $\Gamma \models_{\text{KL}} \Delta$ *then* $\Gamma \models_{\text{MCC}} \Delta$.

Proof. Suppose $\Gamma \models_{\text{KL}} \Delta$. Then $v(\Delta) \cap \{t\} \neq \emptyset$ whenever $v(\Gamma) \subseteq \{t\}$. Let $v \in Val_2$ and suppose $v \in Mod_{\text{MCC}}(\Gamma)$. Then $v(\Gamma) \subseteq \{t\}$, and hence, since $v \in Val_3$, $v(\Delta) \cap \{t\} \neq \emptyset$. So $v \in CoMod(\Delta)$. □

Theorem 2.12 *If* $\Gamma \models_{\text{LP}} \Delta$ *then* $\Gamma \models_{\text{MCC}} \Delta$.

Proof. Suppose $\Gamma \models_{\text{LP}} \Delta$. Then $v(\Delta) \cap \{t, u\} \neq \emptyset$ whenever $v(\Gamma) \subseteq \{t, u\}$. Let $v \in Val_2$ and suppose $v \in Mod_{\text{MCC}}(\Gamma)$. Then, since $v \in Val_3$, $v(\Delta) \cap \{t, u\} \neq \emptyset$. But $v(\Delta) \cap \{u\} = \emptyset$ since $v \in Val_2$. So $v(\Delta) \cap \{t\} \neq \emptyset$. That is, $v \in CoMod(\Delta)$. □

Corollary 2.13 *If* $\Gamma \vDash_{\text{MCK}} \Delta$ *then* $\Gamma \vDash_{\text{MCC}} \Delta$. □

We still have to check that $\vDash_{\text{MCK}}$ is an abstract consequence relation (but also read the remark on page 23). This can be done directly, as was done for the relation $\vDash_{\text{MCC}}$ in Section 1.4, or, alternatively, we can define a syntactic consequence relation for MCK, and then prove that Definition 2.7 is its semantic counterpart. It then follows from Theorem 1.18 that the semantic consequence relation is an abstract consequence relation. I will follow the latter approach in Section 2.3.

Definition 2.7 takes the truth order seriously, but ignores the knowledge order. A consequence relation can be weakened by incorporating aspects of *preferential reasoning* that are based on the knowledge order. This approach is followed, for example, by Arieli & Avron (1996), who use the knowledge order as a preference relation to reduce the set of models considered in defining a consequence relation for a four-valued logic with bilattice semantics. Namely, only the models that are $\leq_k$-minimal are considered. Thus Γ entails Δ if every $\leq_k$-minimal model of Γ is a model of Δ. This yields a non-monotonic consequence relation, which fails the weakening condition of Definition 1.17. The idea behind this approach is that as little knowledge as possible about Γ should be assumed.

Alvarado & Núñez (1997) also proposed that the knowledge order be used to weaken the consequence relation, this time of a three-valued logic based on Kleene's strong truth tables. Their proposal is that Γ entails Δ if every model of Γ is below some model of Δ in the knowledge order. The idea behind this approach is that Γ should entail Δ if, for every world u in which every element of Γ is true, there exists some world which is approximated by u, in which some element of Δ is true. This then yields a modal logic of knowledge and belief, with accessibility relation determined by $\leq_k$.

The knowledge order is already implicitly present in the semantic consequence relation of KL, as the next theorem shows.

Theorem 2.14 *Given any formula sets* Γ *and* Δ,

$$\Gamma \models_{\text{KL}} \Delta \textit{ iff } (\forall v \in Val_3)\left[\begin{array}{l}\min\limits_{\leq_t}\{v(\gamma):\gamma\in\Gamma\} \leq_t \max\limits_{\leq_t}\{v(\delta):\delta\in\Delta\} \;\; \textit{or} \\ \min\limits_{\leq_t}\{v(\gamma):\gamma\in\Gamma\} \leq_k \max\limits_{\leq_t}\{v(\delta):\delta\in\Delta\}.\end{array}\right]$$

Proof. Suppose $\Gamma \models_{\text{KL}} \Delta$, and let $v \in Val_3$. Then, by Definition 2.8, $v(\Delta) \cap \{t\} \neq \emptyset$ whenever $v(\Gamma) \subseteq \{t\}$. Suppose $\min_{\leq_t}\{v(\gamma):\gamma\in\Gamma\} \not\leq_k \max_{\leq_t}\{v(\delta):\delta\in\Delta\}$. Then $\min_{\leq_t}\{v(\gamma):\gamma\in\Gamma\} = f$ and $\max_{\leq_t}\{v(\delta):\delta\in\Delta\} = t$. Hence $\min_{\leq_t}\{v(\gamma):\gamma\in\Gamma\} \leq_t \max_{\leq_t}\{v(\delta):\delta\in\Delta\}$.

Conversely, suppose either $\min_{\leq_t}\{v(\gamma):\gamma\in\Gamma\} \leq_t \max_{\leq_t}\{v(\delta):\delta\in\Delta\}$ or $\min_{\leq_t}\{v(\gamma):\gamma\in\Gamma\} \leq_k \max_{\leq_t}\{v(\delta):\delta\in\Delta\}$. Suppose also $v(\Gamma) \subseteq \{t\}$. Then, for both inequalities, $\max_{\leq_t}\{v(\delta):\delta\in\Delta\} = t$. Hence $v(\Delta) \cap \{t\} \neq \emptyset$. □

Thus Δ is a semantic consequence of Γ in KL iff $\max_{\leq_t}\{v(\delta):\delta\in\Delta\}$ does not represent a decrease in either truth or knowledge from $\min_{\leq_t}\{v(\gamma):\gamma\in\Gamma\}$.

Definition 2.7 strengthens this relation by disallowing an increase in knowledge as sufficient reason for Δ to be a consequence of Γ. In particular, if $v(\gamma) = u$ and $v(\delta) = f$, then $\gamma \nvDash_{\text{MCK}} \delta$, whereas this does not show that $\gamma \nvDash_{\text{KL}} \delta$.

Definition 2.7 can be strengthened by taking into account the knowledge order, without strengthening it so much that it coincides with the consequence relation of KL. This is achieved by combining Definition 2.7 with the relation of Alvarado & Núñez mentioned above. We thus obtain the following relation, describing a logic which I will call $\diamond$MCK:

Definition 2.15 *For any formula sets Γ and Δ, Δ is a semantic consequence of Γ in $\diamond$MCK, written $\Gamma \vDash_{\diamond\text{MCK}} \Delta$, iff*

$$(\forall v \in Val_3)\,(\exists w \in Val_3 \geq_k v)\left[\min_{\leq_t}\{v(\gamma) : \gamma \in \Gamma\} \leq_t \max_{\leq_t}\{w(\delta) : \delta \in \Delta\}\right].$$

This relation is strictly stronger than $\vDash_{\text{MCK}}$ in the sense of Definition 1.23. For example, the law of the excluded middle $\vDash_{\diamond\text{MCK}} q \vee \neg q$ holds in $\diamond$MCK, but not in MCK. On the other hand, it is strictly weaker than $\vDash_{\text{MCC}}$. For example, $p \wedge \neg p \vDash_{\text{MCC}} q$, but $p \wedge \neg p \nvDash_{\diamond\text{MCK}} q$. I will defer further comparisons of these relations until Section 2.4, after I have presented the proof theory of MCK and $\diamond$MCK.

A remark about the name $\diamond$MCK: $\diamond$ is traditionally read as a modal possibility operation. This is also the purpose of incorporating the knowledge order into Definition 2.15. It strengthens Definition 2.7 by weakening the criterion that $\max_{\leq_t}\{v(\delta) : \delta \in \Delta\}$ be at least as true as $\min_{\leq_t}\{v(\gamma) : \gamma \in \Gamma\}$ is, to demand instead only that there must be some valuation w, which is compatible with v but possibly more informative, such that $\max_{\leq_t}\{w(\delta) : \delta \in \Delta\}$ is at least as true as $\min_{\leq_t}\{v(\gamma) : \gamma \in \Gamma\}$ is. However, as I will show in the next section, $\vDash_{\diamond\text{MCK}}$ does not define a truly modal logic. In fact, the next theorem states that the consequence relation of $\diamond$MCK coincides precisely with the consequence relation of LP. But, since the semantic intuition of the two logics differ, I prefer not to call $\diamond$MCK by the name LP.

Theorem 2.16 *For any formula sets* Γ *and* Δ, $\Gamma \models_{\diamond\text{MCK}} \Delta$ *iff* $\Gamma \models_{\text{LP}} \Delta$.

Proof. Suppose $\Gamma \not\models_{\diamond\text{MCK}} \Delta$. Then there exists some $v \in Val_3$ such that either $v(\Gamma) \subseteq \{t\}$ and $(\forall w \geq_k v)[w(\Delta) \cap \{t\} = \emptyset]$, or $v(\Gamma) \subseteq \{t, u\}$ and $(\forall w \geq_k v)[w(\Delta) \subseteq \{f\}]$. If the latter holds, it follows immediately that $\Gamma \not\models_{\text{LP}} \Delta$. Else $v(\Gamma) \subseteq t$. If $v(\Delta) \subseteq \{f\}$, it also follows immediately that $\Gamma \not\models_{\text{LP}} \Delta$. Else $v(\Delta) \cap \{u\} \neq \emptyset$. Pick any $w \in Val_2$ such that $v \leq_k w$. Then $w(\Delta) \subseteq \{f\}$, and, by the monotony of the connectives with respect to $\leq_k$, $w(\Gamma) \subseteq t$. Hence $\Gamma \not\models_{\text{LP}} \Delta$.

Conversely, suppose $\Gamma \not\models_{\text{LP}} \Delta$. Then there exists some $v \in Val_3$ such that $v(\Gamma) \subseteq \{t, u\}$ and $v(\Delta) \subseteq \{f\}$. Hence, by the monotony of the connectives with respect to $\geq_k$, $(\forall w \geq_k v)[w(\Delta) \subseteq \{f\}]$. Thus $\Gamma \not\models_{\diamond\text{MCK}} \Delta$. □

The following lattice summarizes the relative strength of the semantic consequence relations discussed and introduced in this section, and related in Theorems 2.10 to 2.13 and 2.16.

$$\begin{tikzcd}
 & \models_{\text{MCC}} & \\
\models_{\text{KL}} \arrow[ur] & & \models_{\diamond\text{MCK}} = \models_{\text{LP}} \arrow[ul] \\
 & \models_{\text{MCK}} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

2.3 Proof theory

In this section I will define proof systems for MCK and $\diamond$MCK, and prove that they are sound and adequate with respect to the semantic consequence relations of Definitions 2.7 and 2.15 respectively. This is not strictly speaking necessary for $\diamond$MCK, as existing proof systems for LP could easily be adapted to allow for multiple conclusions, but it will be instructive to compare the two similarly defined proof systems.

Recall from Theorem 1.18 that a relation $\vdash$ is an abstract consequence relation (as defined in 1.17) if and only if there exists a formal proof system with derivability relation $\vdash$. The syntactic derivability relation $\vdash_{\text{MCK}}$ defined by the formal proof system for MCK in 2.17 below, is therefore an abstract consequence relation. The Completeness Theorem 2.25 then proves that $\vdash_{\text{MCK}}$ coincides with the semantic consequence relation $\models_{\text{MCK}}$ of Definition 2.7.

The proof system for $\diamond$MCK is similarly defined in 2.27. Its completeness is proved in Theorem 2.31.

Definition 2.17 *The formal proof system of* MCK *consists of the following inference rules:*

1. $p, \neg p \ / \ q, \neg q$
2. $p \ / \ \neg\neg p$
3. $\neg\neg p \ / \ p$
4. $p \wedge q \ / \ p$
5. $p \wedge q \ / \ q$
6. $\neg p \ / \ \neg(p \wedge q)$
7. $\neg q \ / \ \neg(p \wedge q)$
8. $p, q \ / \ p \wedge q$
9. $\neg(p \wedge q) \ / \ \neg p, \neg q$

The proof system given above is in terms of the conjunction and negation connectives. Disjunction and material implication are then introduced as follows:

$$\begin{aligned} \phi \vee \psi &= \neg(\neg\phi \wedge \neg\psi); \\ \phi \supset \psi &= \neg\phi \vee \psi. \end{aligned}$$

The next three lemmas illustrate the use of tree proofs, as defined in 1.15, to prove theorems in or about the proof system of Definition 2.17. Let

$$\begin{array}{c} \gamma_1 \\ \vdots \\ \gamma_n \end{array}$$

depict a finite branch in a tree proof with elements $\gamma_1, ..., \gamma_n$, and let

$$\begin{array}{c} \phi \\ \vdots \\ \psi \end{array}$$

depict a finite branch with first element ϕ and final element ψ.

Lemma 2.18 *$\Gamma \vdash_{\mathrm{MCK}} \phi \wedge \psi$ iff $\Gamma \vdash_{\mathrm{MCK}} \phi$ and $\Gamma \vdash_{\mathrm{MCK}} \psi$.*

Proof. Left to right: Suppose $\Gamma \vdash_{\mathrm{MCK}} \phi \wedge \psi$. Since proof trees are finite, there exist $\gamma_1, ..., \gamma_n \in \Gamma$ such that $\gamma_1, ..., \gamma_n \vdash_{\mathrm{MCK}} \phi \wedge \psi$. The following trees are proofs of $\Gamma \vdash_{\mathrm{MCK}} \phi$ and $\Gamma \vdash_{\mathrm{MCK}} \psi$.

$$\begin{array}{ccc} \begin{array}{l} \gamma_1 \\ \vdots \\ \gamma_n \\ | \\ \phi \wedge \psi \\ | \text{— by rule 4} \\ \phi \end{array} & \qquad & \begin{array}{l} \gamma_1 \\ \vdots \\ \gamma_n \\ | \\ \phi \wedge \psi \\ | \text{— by rule 5} \\ \psi \end{array} \end{array}$$

Right to left: Suppose $\Gamma \vdash_{\text{MCK}} \phi$ and $\Gamma \vdash_{\text{MCK}} \psi$. Since proof trees are finite, there exist $\gamma_1, ..., \gamma_m \in \Gamma$, and $\gamma_{m+1}, ..., \gamma_n \in \Gamma$ such that $\gamma_1, ..., \gamma_m \vdash_{\text{MCK}} \phi$ and $\gamma_{m+1}, ..., \gamma_n \vdash_{\text{MCK}} \psi$. The sets $\{\gamma_1, ..., \gamma_m\}$ and $\{\gamma_{m+1}, ..., \gamma_n\}$ need not be disjunct. The following tree is a proof of $\Gamma \vdash_{\text{MCK}} \phi \wedge \psi$.

$$\begin{array}{c}
\gamma_1 \\
\vdots \\
\gamma_n \\
| \\
\phi \\
| \\
\psi \\
| \text{— by rule 8} \\
\phi \wedge \psi
\end{array}$$

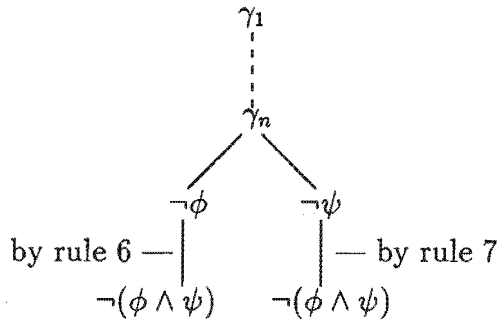
□

Lemma 2.19 $\Gamma \vdash_{\text{MCK}} \neg(\phi \wedge \psi)$ *iff* $\Gamma \vdash_{\text{MCK}} \neg\phi, \neg\psi$.

Proof. Left to right: Suppose $\Gamma \vdash_{\text{MCK}} \neg(\phi \wedge \psi)$. Then there exist $\gamma_1, ..., \gamma_n \in \Gamma$ such that $\gamma_1, ..., \gamma_n \vdash_{\text{MCK}} \neg(\phi \wedge \psi)$. The following is a proof of $\Gamma \vdash_{\text{MCK}} \neg\phi, \neg\psi$.

$$\begin{array}{ccc}
 & \gamma_1 & \\
 & \vdots & \\
 & \gamma_n & \\
 & | & \\
 & \neg(\phi \wedge \psi) & \\
 / & & \backslash \text{— by rule 9} \\
\neg\phi & & \neg\psi
\end{array}$$

Right to left: Suppose $\Gamma \vdash_{\text{MCK}} \neg\phi, \neg\psi$. Then there exist $\gamma_1, ..., \gamma_n \in \Gamma$ such that $\gamma_1, ..., \gamma_n \vdash_{\text{MCK}} \neg\phi, \neg\psi$. The following is a proof of $\Gamma \vdash_{\text{MCK}} \neg(\phi \wedge \psi)$.

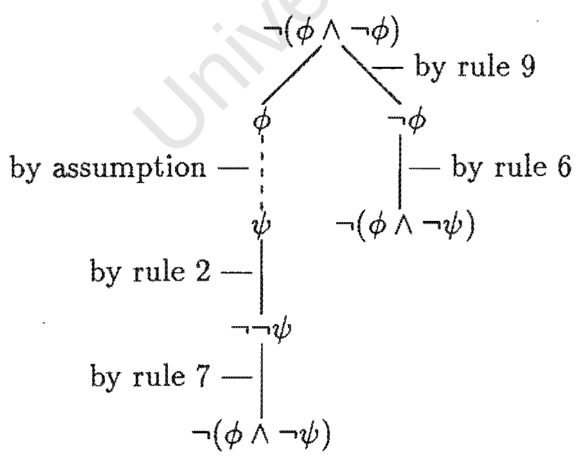


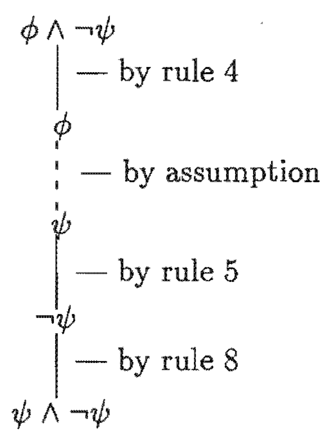
□

Lemma 2.20 *For any wffs ϕ and ψ, $\phi \vdash_{\mathrm{MCK}} \psi$ iff $\phi \vee \neg\phi \vdash_{\mathrm{MCK}} \phi \supset \psi$ and $\neg(\phi \supset \psi) \vdash_{\mathrm{MCK}} \psi \wedge \neg\psi$.*

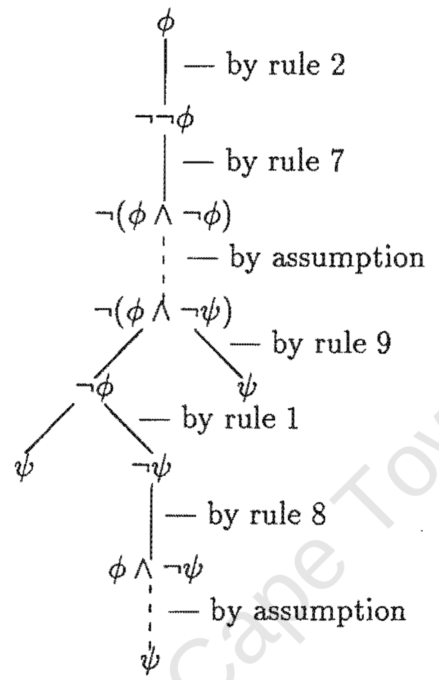
Proof. By eliminating the disjunction and material implication connectives, the right-hand side can be written as follows: $\neg(\phi \wedge \neg\phi) \vdash_{\mathrm{MCK}} \neg(\phi \wedge \neg\psi)$ and $\phi \wedge \neg\psi \vdash_{\mathrm{MCK}} \psi \wedge \neg\psi$.

Left to right: Suppose $\phi \vdash_{\mathrm{MCK}} \psi$. The following are proof trees for $\neg(\phi \wedge \neg\phi) \vdash_{\mathrm{MCK}} \neg(\phi \wedge \neg\psi)$ and $\phi \wedge \neg\psi \vdash_{\mathrm{MCK}} \psi \wedge \neg\psi$ respectively.





Right to left: Suppose $\neg(\phi \wedge \neg\phi) \vdash_{\mathrm{MCK}} \neg(\phi \wedge \neg\psi)$ and $\phi \wedge \neg\psi \vdash_{\mathrm{MCK}} \psi \wedge \neg\psi$. The following is a proof of $\phi \vdash_{\mathrm{MCK}} \psi$.



□

Lemma 2.24 will be used in Theorem 2.25. The proofs of Lemma 2.24 and Theorem 2.25 are loosely based on similar proofs in (Urquhart, 1986). But first, I introduce some new terminology.

Definition 2.21 *Let $\vdash$ be any abstract consequence relation. A formula set Γ is called $\vdash$-inconsistent if there exists a wff ϕ such that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$.*

Definition 2.22 *Let $\vdash$ be any abstract consequence relation. A formula set Γ is called $\vdash$-prime if, for any formula set Δ such that $\Gamma \vdash \Delta$, there is some element $\phi \in \Delta$ such that $\Gamma \vdash \phi$.*

This definition can also be formulated as follows:

Definition 2.23 *A consequence relation $\vdash^*$ is called Γ-prime if, for any Δ such that $\Gamma \vdash^* \Delta$, there is some element $\phi \in \Delta$ such that $\Gamma \vdash^* \phi$.*

Which formulation of primeness is used in a proof, depends on how one wants to apply it. In some proofs, an extension of a formula set Γ to a prime formula set Γ' which contains Γ, is called for. Definition 2.22 is then appropriate. In other proofs, such as Lemma 2.24 below, an extension of the consequence relation is called for, making Definition 2.23 more appropriate.

Lemma 2.24 *For any formula sets Γ and Δ, if $\Gamma \nvdash_{\text{MCK}} \Delta$, then there is a Γ-prime consequence relation $\vdash^*$ extending $\vdash_{\text{MCK}}$ such that $\Gamma \nvdash^* \Delta$.*

Proof. Consider the family $\mathcal{F}$ of abstract consequence relations that are logically stronger than $\vdash_{\text{MCK}}$, such that for every $\vdash \in \mathcal{F}$, $\Gamma \nvdash \Delta$. By Theorem 1.24, the supremum of any chain of elements of $\mathcal{F}$ exists. This supremum is also in $\mathcal{F}$, so by Zorn's lemma, $\mathcal{F}$ contains a maximal element. Let $\vdash^*$ denote such a maximal element. I will show that $\vdash^*$ is Γ-prime. Suppose it is not. Then, by finiteness, there exists a finite set of wffs Σ such that $\Gamma \vdash^* \Sigma$, but $\Gamma \nvdash^* \alpha$ for any $\alpha \in \Sigma$. Let $\psi \in \Sigma$ and let $\vdash_1$ be the derivability relation of the proof system obtained by adding the proof rule Γ/ψ to the proof rules of $\vdash^*$. By maximality of $\vdash^*$, $\Gamma \vdash_1 \Delta$. I will show that every proof tree from Γ using inference rules of $\vdash_1$, is also a proof tree from $\Gamma \cup \{\psi\}$ using inference rules of $\vdash^*$. It then follows that $\Gamma, \psi \vdash^* \Delta$.

The proof is by induction on the size of the proof tree. For the base case, consider any proof tree $\Gamma \vdash_1 \alpha$ which consists of a single leaf node α. Then α is either a premiss (that is, $\alpha \in \Gamma$), or α is the conclusion of a proof rule with an empty premiss set, or the premiss set Γ is empty and $\alpha = \psi$. In all three these cases, the tree is also a proof of $\Gamma, \psi \vdash^* \alpha$.

Next, suppose that every proof tree from Γ using inference rules of $\vdash_1$ with fewer than n nodes, is also a proof from $\Gamma \cup \{\psi\}$ using inference rules of $\vdash^*$. Consider any such proof tree for $\Gamma \vdash_1 \Upsilon$ with fewer than n nodes and set of leaves Υ. By the induction hypothesis, this is also a proof of $\Gamma, \psi \vdash^* \Upsilon$. The proof tree for $\Gamma \vdash_1 \Upsilon$ can grow in three ways: (i) by appending a premiss to a leaf of the tree, (ii) by applying a proof rule of $\vdash^*$ to a branch, and (iii) by applying the proof rule Γ/ψ to a branch. I will show that each of these steps is also a valid step in the expansion of the tree for $\Gamma, \psi \vdash^* \Upsilon$.

In what follows, λ denotes a leaf in a proof tree. $\Upsilon - \lambda$ denotes the formula set $\Upsilon - \{\lambda\}$ if there is only one leaf labelled λ, and Υ if there is more than one leaf labelled λ. (i) Appending a premiss $\gamma \in \Gamma$ to a leaf λ, will result in a proof tree for $\Gamma \vdash_1 \Upsilon - \lambda, \gamma$ The proof tree for $\Gamma, \psi \vdash^* \Upsilon$ can be expanded in the same way. (ii) Applying any proof rule Ω/Λ other than Γ/ψ to a branch with leaf λ of the tree, will result in a proof tree for $\Gamma \vdash_1 \Upsilon - \lambda, \Lambda$. The same proof rule can be applied to the proof tree for $\Gamma, \psi \vdash^* \Upsilon$, resulting in a proof tree for $\Gamma, \psi \vdash^* \Upsilon - \lambda, \Lambda$. (iii) Applying the proof rule Γ/ψ to a branch with leaf λ, appends ψ to the branch. The resulting tree is a proof of $\Gamma \vdash_1 \Upsilon - \lambda, \psi$. Since ψ is in the premiss set $\Gamma \cup \{\psi\}$, the proof tree for $\Gamma, \psi \vdash^* \Upsilon$ can be expanded in the same way, resulting in a proof of $\Gamma, \psi \vdash^* \Upsilon - \lambda, \psi$.

This shows that each step whereby a proof tree from Γ with fewer than n nodes can be expanded, using inference rules of $\vdash_1$, is also a valid step for the expansion of the proof tree from $\Gamma \cup \{\psi\}$ using inference rules of $\vdash^*$. Hence a proof tree for $\Gamma, \psi \vdash^* \Delta$ exists.

By weakening $\Gamma \vdash^* \Sigma$, we obtain $\Gamma \vdash^* \Delta, \Sigma$, and hence $\Gamma \vdash^* \psi, \Delta, \Sigma - \{\psi\}$, and by weakening $\Gamma, \psi \vdash^* \Delta$, we obtain $\Gamma, \psi \vdash^* \Delta, \Sigma - \{\psi\}$. By the cut rule, $\Gamma \vdash^* \Delta, \Sigma - \{\psi\}$. Repeating this argument for all $\alpha \in \Sigma$, we get $\Gamma \vdash^* \Delta$, a contradiction. Therefore $\vdash^*$ is Γ-prime. $\square$

Theorem 2.25 *For any formula sets Γ and Δ, $\Gamma \vdash_{\text{MCK}} \Delta$ iff $\Gamma \models_{\text{MCK}} \Delta$.*

Proof. All instances of the proof rules are elements of $\models_{\text{MCK}}$, as can easily be checked by truth tables. For example, suppose $v(\neg(\phi \wedge \psi)) = t$, then

$v(\phi \wedge \psi) = f$, so $v(\phi) = f$ or $v(\psi) = f$. Hence $v(\neg\phi) = t$ or $v(\neg\psi) = t$. Next, suppose $v(\neg\phi) = f$ and $v(\neg\psi) = f$. Then $v(\phi) = t$ and $v(\psi) = t$. So $v(\phi \wedge \psi) = t$. Hence $v(\neg(\phi \wedge \psi)) = f$. Therefore $\vdash_{\mathsf{MCK}}$ is contained in $\models_{\mathsf{MCK}}$.

Conversely, suppose $\Gamma \nvdash_{\mathsf{MCK}} \Delta$. I will show that $\Gamma \not\models_{\mathsf{MCK}} \Delta$. By Lemma 2.24, there exists a Γ-prime consequence relation $\vdash^*$ extending $\vdash_{\mathsf{MCK}}$ in which $\Gamma \nvdash^* \Delta$. Define a function $v_0 : L_0 \to \{t, u, f\}$ as follows:

$$v_0(p) = \begin{bmatrix} t \text{ iff } (\Gamma \vdash^* p \text{ and } \Gamma \nvdash^* \neg p), \\ f \text{ iff } (\Gamma \vdash^* \neg p \text{ and } \Gamma \nvdash^* p), \\ u \text{ otherwise.} \end{bmatrix}$$

Extend v_0 to a valuation $v \in Val_3$. I will prove by induction on the length of wffs that, for any wff ϕ,

$$v(\phi) = \begin{bmatrix} t \text{ iff } (\Gamma \vdash^* \phi \text{ and } \Gamma \nvdash^* \neg\phi), \\ f \text{ iff } (\Gamma \vdash^* \neg\phi \text{ and } \Gamma \nvdash^* \phi), \\ u \text{ otherwise.} \end{bmatrix} \tag{2.1}$$

Suppose the claim 2.1 holds for all wffs with fewer than n connectives. Let ϕ be a wff with at most n connectives. Suppose $\phi = \neg\psi$. By the induction hypothesis, the claim holds for ψ. Therefore,

$$\begin{aligned} v(\neg\psi) &= \begin{bmatrix} t \text{ iff } v(\psi) = f, \\ f \text{ iff } v(\psi) = t, \\ u \text{ otherwise} \end{bmatrix} \\ &= \begin{bmatrix} t \text{ iff } (\Gamma \vdash^* \neg\psi \text{ and } \Gamma \nvdash^* \psi), \\ f \text{ iff } (\Gamma \vdash^* \psi \text{ and } \Gamma \nvdash^* \neg\psi), \\ u \text{ otherwise} \end{bmatrix} \\ &= \begin{bmatrix} t \text{ iff } (\Gamma \vdash^* \phi \text{ and } \Gamma \nvdash^* \neg\phi), \\ f \text{ iff } (\Gamma \vdash^* \neg\phi \text{ and } \Gamma \nvdash^* \phi), \\ u \text{ otherwise.} \end{bmatrix} \end{aligned}$$

The claim therefore holds for ϕ. Else $\phi = \psi \wedge \xi$, where the number of connectives in ψ and ξ are smaller than n. By the induction hypothesis, 2.1 holds for ψ and for ξ. Therefore,

$$\begin{aligned}
v(\psi \wedge \xi) = t \quad &\text{iff} \quad v(\psi) = t \text{ and } v(\xi) = t \\
&\text{iff} \quad (\Gamma \vdash^* \psi \text{ and } \Gamma \nvdash^* \neg\psi) \text{ and } (\Gamma \vdash^* \xi \text{ and } \Gamma \nvdash^* \neg\xi) \\
&\text{iff} \quad \Gamma \vdash^* \psi \wedge \xi \text{ and } \Gamma \nvdash^* \neg\psi, \neg\xi \\
&\text{iff} \quad \Gamma \vdash^* \psi \wedge \xi \text{ and } \Gamma \nvdash^* \neg(\psi \wedge \xi)\,.
\end{aligned}$$

The second last line above follows by Lemma 2.18, the prime property and weakening. The last line follows by Lemma 2.19.

$$\begin{aligned}
v(\psi \wedge \xi) = f \quad &\text{iff} \quad v(\psi) = f \text{ or } v(\xi) = f \\
&\text{iff} \quad (\Gamma \vdash^* \neg\psi \text{ and } \Gamma \nvdash^* \psi) \text{ or } (\Gamma \vdash^* \neg\xi \text{ and } \Gamma \nvdash^* \xi) \\
&\text{iff} \quad \Gamma \vdash^* \neg\psi, \neg\xi \text{ and } \Gamma \nvdash^* \psi \wedge \xi \text{ and} \\
&\qquad\ (\Gamma \vdash^* \neg\psi \text{ or } \Gamma \nvdash^* \xi) \text{ and } (\Gamma \nvdash^* \psi \text{ or } \Gamma \vdash^* \neg\xi) \\
&\text{iff} \quad \Gamma \vdash^* \neg(\psi \wedge \xi) \text{ and } \Gamma \nvdash^* \psi \wedge \xi.
\end{aligned}$$

The third line above follows by the prime property and Lemma 2.18. The last line requires some further explanation. Suppose first that Γ is $\vdash^*$-consistent. $\Gamma \vdash^* \neg\psi, \neg\xi$ implies $\Gamma \vdash^* \neg\psi$ or $\Gamma \vdash^* \neg\xi$, and hence $\Gamma \vdash^* \neg\psi$ or $\Gamma \nvdash^* \xi$. Similarly, $\Gamma \vdash^* \neg\psi, \neg\xi$ implies $\Gamma \nvdash^* \psi$ or $\Gamma \vdash^* \neg\xi$. Next, suppose that Γ is inconsistent. $\Gamma \nvdash^* \psi \wedge \xi$ implies $\Gamma \nvdash^* \psi$ or $\Gamma \nvdash^* \xi$, and hence $\Gamma \vdash^* \neg\psi$ or $\Gamma \nvdash^* \xi$. Similarly, $\Gamma \nvdash^* \psi \wedge \xi$ implies $\Gamma \nvdash^* \psi$ or $\Gamma \vdash^* \neg\xi$. Therefore $\Gamma \vdash^* \neg\psi, \neg\xi$ and $\Gamma \nvdash^* \psi \wedge \xi$ together imply $(\Gamma \vdash^* \neg\psi$ or $\Gamma \nvdash^* \xi)$ and $(\Gamma \nvdash^* \psi$ or $\Gamma \vdash^* \neg\xi)$. Equation 2.1 therefore holds for all wffs.

I will now show that $\Gamma \nvdash_{\text{MCK}} \Delta$. There are two cases. Case 1: Γ is $\vdash^*$-consistent. Let $\phi \in \Gamma$. By the sharing property, $\Gamma \vdash_{\text{MCK}} \phi$, and hence $\Gamma \vdash^* \phi$. Γ is $\vdash^*$-consistent, so $\Gamma \nvdash^* \neg\phi$. By 2.1, $v(\phi) = t$. Let $\psi \in \Delta$. By the definition of $\vdash^*$, $\Gamma \nvdash^* \Delta$, therefore $\Gamma \nvdash^* \psi$. By 2.1, $v(\psi) \neq t$. Thus, $\forall \phi \in \Gamma$, $v(\phi) = t$ and $\forall \psi \in \Delta$, $v(\psi) \neq t$.

Case 2: Γ is $\vdash^*$-inconsistent. Let $\phi \in \Gamma$. By sharing, $\Gamma \vdash_{\text{MCK}} \phi$, and hence $\Gamma \vdash^* \phi$. By 2.1, $v(\phi) \neq f$. Let $\psi \in \Delta$. By the definition of $\vdash^*$, $\Gamma \nvdash^* \Delta$, therefore $\Gamma \nvdash^* \psi$. Since Γ is $\vdash^*$-inconsistent, $\Gamma \vdash^* \psi, \neg\psi$, and hence $\Gamma \vdash^* \neg\psi$. Therefore $v(\psi) = f$. Thus $\forall \phi \in \Gamma$, $v(\phi) \neq f$ and $\forall \psi \in \Delta$, $v(\psi) = f$. Therefore $\Gamma \nvDash_{\text{MCK}} \Delta$. □

Recall from Definition 1.17 that the finiteness property of an abstract consequence relation $\vdash$ states that $\Gamma \vdash \Delta$ iff there exist finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \vdash \Delta'$. The consequence relation of MCK satisfies this property:

Corollary 2.26 MCK *satisfies the finiteness property.*

Proof. Tree proofs are finite by definition. The result follows from the Completeness Theorem for MCK and Theorem 1.18. □

I will now discuss the proof system of the logic $\diamond$MCK, which was defined in 2.15. The proof system and completeness result for $\diamond$MCK resemble that of MCK closely. The only difference between the proof systems is in the first inference rule.

Definition 2.27 *The formal proof system of* $\diamond$MCK *consists of the following inference rules:*

1. $\emptyset \;/\; q, \neg q$

2. $p \;/\; \neg\neg p$

3. $\neg\neg p \;/\; p$

4. $p \wedge q \;/\; p$

5. $p \wedge q \;/\; q$

6. $\neg p \;/\; \neg (p \wedge q)$

7. $\neg q \;/\; \neg (p \wedge q)$

8. $p, q \;/\; p \wedge q$

9. $\neg (p \wedge q) \;/\; \neg p, \neg q$

Disjunction and material implication are introduced as for MCK. Lemmas 2.18, 2.19 and 2.24 all hold if $\vdash_{\text{MCK}}$ is replaced by $\vdash_{\diamond\text{MCK}}$. They are proved in the same way, with $\vdash_{\text{MCK}}$ replaced by $\vdash_{\diamond\text{MCK}}$.

Lemma 2.28 $\Gamma \vdash_{\diamond\text{MCK}} \phi \wedge \psi$ *iff* $\Gamma \vdash_{\diamond\text{MCK}} \phi$ *and* $\Gamma \vdash_{\diamond\text{MCK}} \psi$. □

Lemma 2.29 $\Gamma \vdash_{\diamond\text{MCK}} \neg (\phi \wedge \psi)$ *iff* $\Gamma \vdash_{\diamond\text{MCK}} \neg\phi, \neg\psi$. □

Lemma 2.30 *For any formula sets* Γ *and* Δ*, if* $\Gamma \nvdash_{\diamond\text{MCK}} \Delta$*, then there is a* Γ*-prime consequence relation* $\vdash^*$ *extending* $\vdash_{\diamond\text{MCK}}$ *such that* $\Gamma \nvdash^* \Delta$. □

Theorem 2.31 *For any formula sets* Γ *and* Δ*,* $\Gamma \vdash_{\diamond\text{MCK}} \Delta$ *iff* $\Gamma \vDash_{\diamond\text{MCK}} \Delta$.

Proof. All instances of the proof rules are elements of $\vDash_{\diamond\text{MCK}}$, as can easily be checked by truth tables. Consider, for example, proof rule 1. Let $v \in Val_3$. Then $\min_{\leq_t} \{v(\gamma) : \gamma \in \emptyset\} = t$ by definition. If $v(q) = t$ or $v(q) = f$, let $w = v$. Then $\max_{\leq_t} \{w(q), w(\neg q)\} = t$. Else, if $v(q) = u$, let $w = v$ except that $w(q) = t$. Then $v \leq_k w$ and $\max_{\leq_t} \{w(q), w(\neg q)\} = t$. Therefore $\emptyset \vDash_{\diamond\text{MCK}} q, \neg q$.

Conversely, suppose $\Gamma \nvdash_{\diamond\text{MCK}} \Delta$. I will show that $\Gamma \nvDash_{\diamond\text{MCK}} \Delta$. By Lemma 2.30, there exists a Γ-prime consequence relation $\vdash^*$ extending $\vdash_{\diamond\text{MCK}}$ such that $\Gamma \nvdash^* \Delta$. Define a function $v_0 : L_0 \to \{t, u, f\}$ exactly as in Theorem 2.25, and extend v_0 to the valuation of equation 2.1.

Let $\psi \in \Delta$. By rule 1, $\emptyset \vdash_{\diamond\mathrm{MCK}} \psi, \neg\psi$. Therefore $\emptyset \vdash^* \psi, \neg\psi$. By weakening, $\Gamma \vdash^* \psi, \neg\psi$. By definition of $\vdash^*$, $\Gamma \nvdash^* \Delta$, and so $\Gamma \nvdash^* \psi$. Therefore, by the primeness of $\vdash^*$, $\Gamma \vdash^* \neg\psi$. By equation 2.1, $v(\psi) = f$. So $\max_{\leq_t} \{v(\psi) : \psi \in \Delta\} = f$. Let $w \geq_k v$. By the monotony of the connectives of $\diamond$MCK, $w(\psi) = f$. Now let $\phi \in \Gamma$. Then $v(\phi) = t$ or $v(\phi) = u$. There is therefore no $w \geq_k v$ such that $\min_{\leq_t} \{v(\phi) : \phi \in \Gamma\} \leq_t \max_{\leq_t} \{v(\psi) : \psi \in \Delta\}$. Hence $\Gamma \nvdash_{\diamond\mathrm{MCK}} \Delta$. $\square$

In this section, I presented proof systems for the logics MCK and $\diamond$MCK, in Definitions 2.17 and 2.27. The proof systems resemble each other closely, the only difference being in the first axiom. There are two other closely related proof systems: Replacing the first axiom in Definition 2.17 by

1. $p, \neg p / q$

yields a proof system for KL, while dropping the first axiom in Definition 2.17 altogether, yields a proof system for FDE. (See (Pynko, 1995a) for a different Hilbert style axiomatization of LP and FDE, which proves this point.) So we have the following lattice of syntactic consequence relations:

$$\begin{tikzcd}
 & \vdash_{\mathrm{MCC}} & \\
\vdash_{\mathrm{KL}} \arrow[ur] & & \vdash_{\diamond\mathrm{MCK}} = \vdash_{\mathrm{LP}} \arrow[ul] \\
 & \vdash_{\mathrm{MCK}} \arrow[ul] \arrow[ur] & \\
 & \vdash_{\mathrm{FDE}} \arrow[u] &
\end{tikzcd}$$

2.4 Paraconsistency

In this section, I discuss some interesting and useful properties of MCK and ◇MCK, most notably that of *paraconsistency*, defined in 2.32 below. The roots of contemporary paraconsistent logic dates back to the work of Lukasiewicz (1910), translated in (Łukasiewicz, 1971), Vasil'év (1910) and Jaśkowski (1948), translated in (Jaśkowski, 1969). Since the 1950's, many formal systems of paraconsistent logic have been defined. The collection *Paraconsistent Logic: Essays on the Inconsistent* (Priest *et al.*, 1989) surveys the field of paraconsistent logic until the early eighties.

Recall from Definition 2.21 that a formula set Γ is inconsistent if there exists a wff ϕ such that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$. In classical logic, there are no non-trivial inconsistent deductive systems; the only inconsistent deductive system is L. That is, if Γ is inconsistent, then $Cn(\Gamma) = L$. A logic which has this property, is called *explosive*. In an explosive logic, all inconsistent formula sets therefore belong to the same $\equiv_{Cn}$-equivalence class, as defined in 1.25.

Definition 2.32 *A logic is called paraconsistent if there exist inconsistent formula sets that are not $\equiv_{Cn}$-equivalent to L.*

The proof-theoretic aim of paraconsistent logics is to provide a framework for reasoning about systems that may be inconsistent. Well-known examples of inconsistencies are the set-theoretic paradoxes such as Russell's paradox: Let $X = \{x : x \notin x\}$. Then both $X \in X$ and $X \notin X$. Inconsistencies also have to be dealt with in data bases and in legal systems. Formalisms such as theory change (which I will discuss in Chapter 5) deal with inconsistencies in knowledge bases by avoiding them, and by removing them once they are located. Paraconsistent logics, on the other hand, reason non-explosively in the presence of inconsistencies.

I will call a formula set Δ *co-inconsistent* if there exists a wff ϕ such that $\phi \vdash \Delta$ and $\neg\phi \vdash \Delta$. The following definition then provides a natural dual to the notion of paraconsistency.

Definition 2.33 *A logic is called pathological if there exist co-inconsistent formula sets that are not $\equiv_{An}$-equivalent to L.*

Paraconsistency and pathologicality are characterized by the absence of certain inference rules. In a (multiple-conclusion) paraconsistent logic,

$$p, \neg p / \emptyset$$

is not a valid rule of inference, while in a pathological logic,

$$\emptyset / q, \neg q$$

is not a valid rule of inference. This puts KL in the class of explosive, pathological logics, LP and ◇MCK in the class of paraconsistent, non-pathological logics, and MCK and FDE in the class of paraconsistent, pathological logics.

There are several paraconsistent logics based on Kleene's strong truth tables. Examples are Sobociński logic (Sobociński, 1952), the logic of paradox LP (Priest, 1979; Priest, 1987) , the semi-relevant logic RM_3 (Anderson & Belnap, 1975; Dunn, 1986), the equivalent system $\mathrm{RM}_3^{\supset}$ (Avron, 1986), and the logics ◇MCK and MCK defined in 2.7 and 2.15. All these logics, except MCK, are determined by a three-valued matrix in which the third truth value is designated. The consequence relation of MCK is not determined by a single matrix $\langle \mathcal{K}_3, D \rangle$ - it is the intersection of the consequence relations determined by $\langle \mathcal{K}_3, \{t\} \rangle$ and $\langle \mathcal{K}_3, \{t, u\} \rangle$. This point was also made on page 41.

Quoting da Costa (1974), Arieli and Avron (1996) state that no standard three-valued logic in which the non-classical value is not designated, is paraconsistent. It is not clear what is meant by 'standard', but I assume these are logics based on Kleene's strong truth tables. This would make the paraconsistent, three-valued extension $\mathrm{C}_{0.2}$ of the calculus C_1 (da Costa, 1974; Mortensen, 1989), in which the third truth value is not designated, non-standard. However, MCK still provides a counter-example to Arieli and Avron's statement.

LP was introduced to accommodate the logical paradoxes (Priest, 1979). These include both the set-theoretical paradoxes, such as Russell's paradox, and the semantic paradoxes, such as the Liar's paradox. A wff in LP can be either true (and not false), or false (and not true), or paradoxical (both true and false). This yields a three-valued logic based on Kleene's truth tables, but in which the third truth value indicates that a wff is paradoxical, as opposed to being undefined or undetermined. Since a paradoxical wff is true (as well as false), the third truth value in the matrix for LP is also designated. The semantic consequence relation of LP was defined in 2.5, and its multiple-conclusion counterpart in 2.9. The semantic consequence relation of $\diamond$MCK was defined in 2.15, and in Theorem 2.16 I showed that LP and $\diamond$MCK are in fact the same. Therefore, although the intuition behind these two logics differ, technically all results obtained for LP, also apply to $\diamond$MCK.

MCK is pathological, in the sense of Definition 2.33. But, given an inconsistent formula set, all the classical tautologies follow. In this respect, MCK resembles the logic RM (Anderson & Belnap, 1975).

Theorem 2.34 *Given any inconsistent formula set* Γ *and classical tautology* ϕ, $\Gamma \models_{\mathrm{MCK}} \phi$.

Proof. By monotony with respect to the knowledge order $\leq_k$, no classical tautology can be false in MCK. By the inconsistency of Γ, $\min_{\leq_t} \{v(\gamma) : \gamma \in \Gamma\}$ can't be true. Hence $\Gamma \models_{\text{MCK}} \phi$ by Definition 2.7. $\square$

As mentioned on page 38, in a logic determined by a class of matrices, a tautology is a wff which is assigned a designated value by each valuation in each determining matrix. So in $\diamond$MCK, a tautology is a wff which is never false, and in both KL and MCK, a tautology is a wff which is always true. Neither KL nor MCK have any tautologies, since the valuation which assigns the value u to all sentential symbols, will also assign the u to any wff.

In $\diamond$MCK, the situation is less bleak. The completeness result 2.31 assures us that a wff in $\diamond$MCK is always designated if and only if it is a theorem. Theorem 2.35, stated in (Priest, 1979) for LP, further assures us that $\diamond$MCK has the same theorems as classical logic, and Theorem 2.36 shows that material implication in $\diamond$MCK can be read as a semantic entailment operation.

Because MCK is pathological, it is not possible to define a connective in MCK which functions as a semantic entailment operation. For if $\rightarrow$ is intended as such an entailment operation, then the Deduction Theorem should hold. That is, if $\phi \vdash_{\text{MCK}} \psi$ then $\vdash_{\text{MCK}} \phi \rightarrow \psi$. And this is not possible, since then $\phi \rightarrow \psi$ would have to be a tautology.

Theorem 2.35 *$\diamond$MCK and MCC have the same tautologies.*

Proof. Every tautology in LP is a tautology in MCC, since the latter has a stronger consequence relation.

Conversely, suppose $\models_{\text{MCC}} \phi$. Then $(\forall w \in Val_2)\,[w(\phi) = t]$. Let $v \in Val_3$. Then $v \leq_k w$ for some $w \in Val_2$, and $w(\phi) = t$. Therefore $\models_{\diamond\text{MCK}} \phi$. $\square$

Theorem 2.36 *Given any formula set Γ and wffs ϕ and ψ in $\diamond$MCK, if $\Gamma, \phi \vdash_{\diamond\text{MCK}} \psi$ then $\Gamma \vdash_{\diamond\text{MCK}} \phi \supset \psi$.*

Proof. Suppose $\Gamma, \phi \vdash_{\diamond\text{MCK}} \psi$. By proof rule 1, $\vdash_{\diamond\text{MCK}} \phi, \neg\phi$. By weakening, $\Gamma, \phi \vdash_{\diamond\text{MCK}} \neg\phi, \psi$ and $\Gamma \vdash_{\diamond\text{MCK}} \phi, \neg\phi, \psi$. By the cut rule, $\Gamma \vdash_{\diamond\text{MCK}} \neg\phi, \psi$. By Lemma 2.29, $\Gamma \vdash_{\diamond\text{MCK}} \neg(\phi \wedge \neg\psi)$. By the definition of $\supset$, $\Gamma \vdash_{\diamond\text{MCK}} \phi \supset \psi$. □

In PC, a formula set is consistent if and only if it has a model. The trademark of paraconsistent logics is that inconsistent formula sets can have models. Theorems 2.37 and 2.38 relate consistency in ◇MCK and MCK respectively to the existence of a valuation which makes every element in Γ true.

Theorem 2.37 *A formula set Γ in ◇MCK is consistent if and only if there exists some $v \in Val_3$ such that $v(\Gamma) \subseteq \{t\}$.*

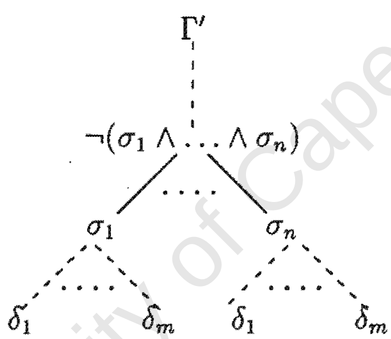
Proof. Suppose Γ is inconsistent, that is, suppose there exists a wff ϕ such that $\Gamma \vdash_{\diamond\text{MCK}} \phi$ and $\Gamma \vdash_{\diamond\text{MCK}} \neg\phi$. Let v be any valuation. If $v(\Gamma) \subseteq \{t\}$, then $v(\phi) = t$ and $v(\neg\phi) = t$, which is impossible, since $v(\phi) = t$ iff $v(\neg\phi) = f$. Therefore $\Gamma \nsubseteq \{t\}$.

Conversely, suppose Γ is consistent. I will construct a valuation v such that $v(\Gamma) \subseteq \{t\}$. Let $\Delta = \{\neg\phi : \phi \in \Gamma\}$. Then $\Gamma \nvdash_{\diamond\text{MCK}} \Delta$. For suppose this is not so. Then there exist $\phi_1, ..., \phi_m \in \Delta$ such that $\Gamma \vdash_{\diamond\text{MCK}} \phi_1, ..., \phi_m$. By Lemma 2.29, $\Gamma \vdash_{\diamond\text{MCK}} \neg(\neg\phi_1 \wedge ... \wedge \neg\phi_m)$. But since for each ϕ_i, $\Gamma \vdash_{\diamond\text{MCK}} \neg\phi_i$, by Lemma 2.28 we have $\Gamma \vdash_{\diamond\text{MCK}} \neg\phi_1 \wedge ... \wedge \neg\phi_m$. This contradicts the consistency of Γ. Therefore $\Gamma \nvdash_{\diamond\text{MCK}} \Delta$.

Let $\alpha_1, \alpha_2, ...$ be an enumeration of all wffs in L. Form a chain of formula sets $\Gamma_0, \Gamma_1, ...,$ where

$$\begin{aligned} \Gamma_0 &= \Gamma, \\ \Gamma_{i+1} &= \left[\begin{array}{l} \Gamma_i \cup \{\alpha_{i+1}\} \text{ if } \Gamma_i, \alpha_{i+1} \nvdash_{\diamond\text{MCK}} \Delta, \\ \Gamma_i \text{ otherwise.} \end{array} \right] \end{aligned}$$

Then for all i, $\Gamma_i \nvdash_{\diamond\text{MCK}} \Delta$. Let Γ' be the union of all the elements in this chain. By finiteness, $\Gamma' \nvdash_{\diamond\text{MCK}} \Delta$. I will show that Γ' is prime, as defined in 2.23. Suppose Γ' is not prime. Then there exists some Σ such that $\Gamma' \vdash_{\diamond\text{MCK}} \Sigma$ but there is no element $\sigma \in \Sigma$ such that $\Gamma' \vdash_{\diamond\text{MCK}} \sigma$. By finiteness, there exists finite $\Sigma_0 \subseteq \Sigma$ such that $\Gamma' \vdash_{\diamond\text{MCK}} \Sigma_0$. Say $\Sigma_0 = \{\sigma_1, ..., \sigma_n\}$. Since Γ' is maximal with respect to the property that Δ cannot be deduced, $\Gamma', \sigma_j \vdash_{\diamond\text{MCK}} \Delta$ for each $\sigma_j \in \Sigma_0$. By finiteness, there exists finite $\Delta_0 \subseteq \Delta$ such that $\Gamma', \sigma_j \vdash_{\diamond\text{MCK}} \Delta_0$. The following is a proof tree for $\Gamma' \vdash_{\diamond\text{MCK}} \Delta_0$, where $\Delta_0 = \{\delta_1, ..., \delta_m\}$. This contradicts the assumption that Γ' is not prime.



Define a function $v_0 : L_0 \to \{t, u, f\}$ as follows:

$$v_0(p) = \left[\begin{array}{l} t \text{ iff } (\Gamma' \vdash_{\diamond\text{MCK}} p \text{ and } \Gamma' \nvdash_{\diamond\text{MCK}} \neg p), \\ f \text{ iff } (\Gamma' \vdash_{\diamond\text{MCK}} \neg p \text{ and } \Gamma' \nvdash_{\diamond\text{MCK}} p), \\ u \text{ otherwise.} \end{array}\right]$$

Extend v_0 to a valuation $v \in Val_3$. As in the proof of Theorem 2.31, it can be proved by induction on the number of connectives in a wff that, for any wff ψ,

$$v(\psi) = \left[\begin{array}{l} t \text{ iff } (\Gamma' \vdash_{\diamond\text{MCK}} \psi \text{ and } \Gamma' \nvdash_{\diamond\text{MCK}} \neg\psi), \\ f \text{ iff } (\Gamma' \vdash_{\diamond\text{MCK}} \neg\psi \text{ and } \Gamma' \nvdash_{\diamond\text{MCK}} \psi), \\ u \text{ otherwise.} \end{array}\right]$$

Let $\gamma \in \Gamma$. Since $\Gamma' \nvdash_{\diamond\text{MCK}} \Delta$ and $\neg\gamma \in \Delta$, $v(\gamma) = t$. Hence $v(\Gamma) \subseteq \{t\}$, as required. □

One of the 'nice-to-have' properties of MCK and ◇MCK that I have used, and will use in Chapter 3, is that all their connectives are monotone with respect to the knowledge order. This is also a shortcoming since, consequently, both MCK and ◇MCK are *expressively incomplete*. That is, there are some truth tables that are not realized by any wff in the logic. For example, there is no wff ϕ in MCK or ◇MCK which has the following truth table:

p	ϕ
f	t
u	t
t	t

It can easily be made expressively complete for monotone truth tables by the addition of the constants t and u to the language, but to make it expressively complete in general, the language L has to be enriched with a non-monotonic connective such as the definedness operation of Barringer, Cheng & Jones (1984), or Łukasiewicz implication.

Chapter 3

An algebraic perspective

The content of this chapter is best described by the title of the thesis, for it deals with *power constructs* and *propositional systems.* More specifically, it deals with power constructs in Tarski's calculus of deductive systems. Power constructs are introduced in Section 3.1 as operations and relations on some power set $\mathcal{P}(A)$, that derive from some corresponding base operations and relations defined on the set A. The calculus of deductive systems was already mentioned in the preface, and again alluded to on page 15. It is the topic of Section 3.2.

The main result of Section 3.2 is Theorem 3.20, which characterizes the calculus of systems of a certain class of logics as a *power algebra.* In Section 3.3 I apply this result to the calculus of LP-systems. I also discuss the algebraic characterization of LP in terms of Kleene lattices. In Section 3.4 I apply Theorem 3.20 to the calculus of MCK-systems. I show that MCK is not algebraizable, but that the calculus of MCK-systems is isomorphic to a semantically defined structure called a *lattice of meanings.* I also show that the consequence relation of $\diamond$MCK is a power relation in this lattice of meanings.

In Section 3.5, I define a *simulation* between formula sets in two different logics as a power relation of the pointwise knowledge order on valuations.

3.1 Power constructs

Let A be any set. Its power set $\mathcal{P}(A)$ forms a Boolean algebra of sets $(\mathcal{P}(A), \cap, \cup, ')$, with operations intersection, union and complementation. Conversely, Stone (1936) proved that any Boolean algebra is isomorphic to a field of sets; a study of Boolean algebras is thus essentially a study of the calculus of sets. Apart from the set-theoretic operations defined on $\mathcal{P}(A)$, each operation f defined on A gives rise to the following *power operation* f^+ on $\mathcal{P}(A)$.

Definition 3.1 *Let $f : A^n \to A$ and $X_1, ..., X_n \subseteq A$. Then $f^+ : \mathcal{P}(A)^n \to \mathcal{P}(A)$ is defined by*

$$f^+(X_1, ..., X_n) = \{y \in A : (\exists x_1 \in X_1) ... (\exists x_n \in X_n) [f(x_1, ..., x_n) = y]\}.$$

Similarly, given any n+1-ary relation r on A, one can define an n-ary power operation $r^\dagger$ on $\mathcal{P}(A)$.

Definition 3.2 *Let $r \subseteq A^{n+1}$ and $X_1, ..., X_n \subseteq A$. Then $r^\dagger : \mathcal{P}(A)^n \to \mathcal{P}(A)$ is defined by*

$$r^\dagger(X_1, ..., X_n) = \{y \in A : (\exists x_1 \in X_1) ... (\exists x_n \in X_n) [(x_1, ..., x_n, y) \in r]\}.$$

Given any n-ary relation r on a set A, one can also define an n-ary *power relation* r^+ on $\mathcal{P}(A)$. In particular, the binary power relation of a binary relation is defined as follows:

Definition 3.3 *Let $r \subseteq A^2$. The power relation of r is the relation $r^+ \subseteq \mathcal{P}(A)^2$, defined by: $r^+ = r_0^+ \cap r_1^+$, where*

$$(X_1, X_2) \in r_0^+ \quad \textit{iff} \quad (\forall x_1 \in X_1)(\exists x_2 \in X_2)[(x_1, x_2) \in r], \textit{ and}$$
$$(X_1, X_2) \in r_1^+ \quad \textit{iff} \quad (\forall x_2 \in X_2)(\exists x_1 \in X_1)[(x_1, x_2) \in r].$$

The relations r_0^+ and r_1^+ are called weak power relations.

For any algebra $\mathcal{A} = (A, \{f_i\}_{i \in I})$ with set of operations $\{f_i\}_{i \in I}$, the *power algebra* of $\mathcal{A}$ is a Boolean algebra over $\mathcal{P}(A)$, endowed with the power operations $\{f_i^+\}_{i \in I}$, in addition to the usual set-theoretic operations. The power algebra of $\mathcal{A}$ is written $\mathcal{P}(\mathcal{A}) = \left(\mathcal{P}(A), \cap, \cup, ', \{f_i^+\}_{i \in I}\right)$.

A *relational structure* $(A, \{r_i\}_{i \in J})$ consists of the set A with set of relations $\{r_i\}_{i \in J}$ defined on it. The power algebra of this structure is the Boolean algebra $\left(\mathcal{P}(A), \cap, \cup, ', \left\{r_i^\uparrow\right\}_{i \in J}\right)$, endowed with the power operations $\left\{r_i^\uparrow\right\}_{i \in J}$, in addition to the usual set-theoretic operations. Jónsson & Tarski (1951; 1952) investigate the properties of Boolean algebras endowed with additional (additive) operators, abbreviated BAO's, the power algebra of a relational structure being an example of a BAO. Jónsson (1993) gives a more recent survey of the field. Power algebras are also referred to as complex algebras, for example in (Grätzer, 1979; Goldblatt, 1989), due to their application in group theory.

In analogy to the formation of a power algebra, one can construct the *power structure* of a relational structure $(A, \{r_i\}_{i \in J})$. This is the relational structure $\left(\mathcal{P}(A), \{r_i^+\}_{i \in J}\right)$, consisting of the power set $\mathcal{P}(A)$, with power relations $\{r_i^+\}_{i \in J}$ defined on it.

The use of power relations is relatively new, the first references being (Smithson, 1971), where they are used in fixed point theory, (Plotkin, 1976; Smyth, 1978), where they are used in domain theory, and (Whitney, 1977), where they feature in a universal algebraic context. The first general investigation into the properties of power structures, is that of Grätzer & Whitney (1984). Brink (1993) gives an overview of power structures and their applications, and establish some universal-algebraic properties. The following results from this paper will be used later in this chapter.

Theorem 3.4 *(Brink, 1993) For any algebra $\mathcal{A}$ and congruence relation $\cong$ over $\mathcal{A}$, $\cong^+$ is a congruence relation over $\mathcal{P}(\mathcal{A})$.* □

Theorem 3.5 *(Brink, 1993) For any algebra $\mathcal{A}$ and congruence relation $\cong$ over $\mathcal{A}$, the power algebra $\mathcal{P}(\mathcal{A}/\cong)$ of the quotient algebra $\mathcal{A}/\cong$ is isomorphic to the quotient algebra $\mathcal{P}(\mathcal{A})/\cong^+$ of the power algebra with respect to the power congruence.* □

The first applications of power constructs to logic are due to Brink (1984; 1986). One such application is in Hyperboolean Modal Logic, abbreviated HBML, axiomatized by Goranko & Vakarelov (1999). In the possible worlds semantics of HBML, the set of worlds forms a power algebra, called a hyperboolean algebra. The modalities of HBML correspond to the operations of this algebra. Goranko & Vakarelov show that hyperboolean algebras provide an algebraic semantics for HBML.

Other applications of power constructs to logic are in the semantics of the relevance logic $R^{\neg}$ (Brink, 1989a), in verisimilitude (Brink & Heidema, 1987), which is the topic of Chapter 4, and in theory change, which is the topic of Chapter 5.

In (Brink *et al.*, 1993) power constructs are generalized to *fuzzy power constructs*. In fuzzy set theory, introduced by Zadeh (1965), the characteristic function of a set X is extended to a membership function μ_X from the universal set A to the interval $[0,1]$. For any $y \in A$, $\mu_X(y)$ indicates the extent or degree to which z is an element of X, with $\mu_X(y) = 1$ indicating that y is definitely a member of X, and $\mu_X(y) = 0$ indicating that y is definitely not a member of X. A *crisp* set X is characterized by the property that $\mu_X(A) \subseteq \{0,1\}$. Let $\mathcal{P}_f(A)$ denote the fuzzy power set of A. The *fuzzy set extension principle* (Zadeh, 1975; Dubois & Prade, 1980) extends Definition 3.1 to a crisp power operation on $\mathcal{P}_f(A)$:

Definition 3.6 *Let $f : A^n \to A$ and $X_1, ..., X_n \subseteq A$. The crisp n-ary power operation $f^+ : \mathcal{P}_f(A)^n \to \mathcal{P}_f(A)$ is defined by*

$$\mu_{f^+(X_1,...,X_n)}(y) = \sup\{\min(\mu_{X_1}(x_1), ...\mu_{X_n}(x_n)) : f(x_1, ..., x_n) = y\}.$$

The fuzzy set extension principle can be generalized to include both crisp and fuzzy power operations and relations on $\mathcal{P}_f(A)$. The generalization employs a standard concept in fuzzy set theory, namely *α-cuts*: Given any fuzzy binary relation ρ on A and $\alpha \in [0,1]$, the α-cut R_α is a crisp binary relation on A, defined by:

$$R_\alpha = \{(x,y) : \mu_\rho(x,y) \geq \alpha\}.$$

Definition 3.7 *Let ρ be a fuzzy binary relation on A. The fuzzy power relation ρ^+ on $\mathcal{P}_f(A)$ is defined by:*

$$\mu_{\rho^+}(X_1, X_2) = \sup\{\alpha : X_1 R_\alpha^+ X_2\}.$$

If ρ is a crisp relation, then Definition 3.7 coincides with Definition 3.3 on crisp sets. Further, if ρ is a crisp operation, then it coincides with Definition 3.6, and with Definition 3.1 on crisp sets.

Let Q_α denote the set-theoretic complement of R_α on A:

$$Q_\alpha = \{(x,y) : \mu_\rho(x,y) < \alpha\}.$$

The following relation does not coincide with that of Definition 3.7, but it also generalizes both Definitions 3.3 and 3.6:

$$\mu_{\rho^+}(X_1, X_2) = \inf\left\{\alpha : X_1 Q_\alpha^+ X_2\right\}.$$

Fuzzy sets and systems form a vast research field, with a journal dedicated to it. The fuzzy power constructs presented here generalize their crisp counterparts in a natural way. Their usefulness merits further investigation, but I will not diverge in this direction here.

The following results are stated here for later use, and can be found in standard textbooks on Universal Algebra, such as (Grätzer, 1979; Burris & Sankappanavar, 1981):

Definition 3.8 *Let* $f : \mathcal{A} \to \mathcal{B}$ *be a homomorphism. The kernel of* f*, written* $\ker(f)$*, is defined by*

$$\ker(f) = \{\langle a, b\rangle \in A \times A : f(a) = f(b)\}.$$

Theorem 3.9 Homomorphism Theorem. *Let* $f : \mathcal{A} \to \mathcal{B}$ *be a homomorphism onto* $\mathcal{B}$*. Then there is an isomorphism* $i : \mathcal{A}/\ker(f) \to \mathcal{B}$ *defined by* $f = i \circ v$*, where* $v : \mathcal{A} \to \mathcal{A}/\ker(f)$ *is the natural map defined by* $v(a) = a/\ker(f)$. □

Theorem 3.10 Second Isomorphism Theorem. *If* r *and* s *are congruence relations on* $\mathcal{A}$*, and* $r \subseteq s$*, then the map* $h : (A/r)/(s/r) \to A/s$*, defined by*

$$h((a/r)/(s/r)) = a/s$$

is an isomorphism from $(\mathcal{A}/r)/(s/r)$ *to* $(\mathcal{A}/s)$. □

3.2 The calculus of systems

Tarski (Tarski, 1930a; Tarski, 1930b; Tarski, 1935) defined his calculus of deductive systems as part of an investigation into the properties of certain metamathematical concepts, such as the axiomatizability, irreducibility, completeness, and atomicity of systems. A Tarskian system is defined in terms of an abstract consequence operator, as defined in 1.19. The set of systems is turned into an algebra by the addition of certain operations on systems. Tarski does not fix the language of the logic, but I will assume that the language is L, as defined in Section 1.1.

Definition 3.11 *(Tarski, 1935) The Tarski calculus of deductive systems is an algebra* $(\{Cn(\Gamma) : \Gamma \subseteq L\}, \cdot, \dot{+}, ^{-})$*, such that, for any systems* Φ *and* Ψ*,*

1. $\Phi \cdot \Psi = \Phi \cap \Psi$;

2. $\Phi \dot{+} \Psi = Cn(\Phi \cup \Psi)$;

3. $\bar{\Phi} = \bigcap_{\alpha \in \Phi} Cn(\{\neg\alpha\})$.

The calculus of (deductive) systems is, as Tarski noted, closely related to an algebraic version of intuitionistic logic. It is in fact a De Morgan lattice (see Definition 1.9). The addition of an operation $\Rightarrow$ defined by

$$\Phi \Rightarrow \Psi = Cn\left(\bigcup_{\Gamma \subseteq L} \{\Gamma : Cn(\Phi) \cdot Cn(\Gamma) \subseteq Cn(\Psi)\}\right)$$

turns it into a Heyting algebra (see Definition 1.11), which is the algebra of intuitionistic logic.

The equivalence relation $\equiv_{Cn}$ of Definition 1.25 divides $\mathcal{P}(L)$ into equivalence classes, such that Γ is $\equiv_{Cn}$-equivalent to Σ iff $Cn(\Gamma) = Cn(\Sigma)$. There is therefore a one-one correspondence between elements of the calculus of systems, and elements of $P(L)/\equiv_{Cn}$.

The positive fragment of the calculus of systems is obtained by dropping the negation operation. This yields the algebra $(\{Cn(\Gamma) : \Gamma \subseteq L\}, \cdot, \dot{+})$, called the *positive calculus of systems.* Brink & Heidema (1987) define a certain power algebra which is isomorphic to the positive calculus of systems in classical propositional logic. This result is published in (Brink & Rewitzky, 1999), and generalized in Theorem 3.20 below.

As I mentioned in Section 1.1, the language L can be viewed as an abstract algebra, with operations $\wedge$, $\vee$ and $\neg$. As such, it has a power algebra $(\mathcal{P}(L), \wedge^+, \vee^+, \neg^+)$, with power operations $\wedge^+$, $\vee^+$ and $\neg^+$. Following Definition 3.1, these power operations are defined as follows:

Definition 3.12 *For any formula sets Γ and Σ, the power connectives $\wedge^+$, $\vee^+$ and $\neg^+$ are defined by:*

$$\begin{aligned} \Gamma \wedge^+ \Sigma &= \{\phi \wedge \psi : \phi \in \Gamma \textit{ and } \psi \in \Sigma\}; \\ \Gamma \vee^+ \Sigma &= \{\phi \vee \psi : \phi \in \Gamma \textit{ and } \psi \in \Sigma\}; \\ \neg^+\Gamma &= \{\neg\phi : \phi \in \Gamma\}. \end{aligned}$$

Definition 3.13 *(Pynko, 1995a) Given any consequence operator Cn, the property of conjunction, PC, is defined as:*

$$\textit{For all } \phi, \psi \in L,\ Cn(\phi \wedge \psi) = Cn(\phi, \psi).$$

The property of disjunction, PDI, is defined as:

$$\textit{For all } \Gamma \cup \{\phi, \psi\} \subseteq L,\ Cn(\Gamma, \phi \vee \psi) = Cn(\Gamma, \phi) \cap Cn(\Gamma, \psi).$$

The properties of De Morgan, PDM, are defined as: For all $\phi, \psi \in L$,

1. $Cn(\neg\neg\phi) = Cn(\phi)$;

2. $Cn(\neg(\phi \vee \psi)) = Cn(\neg\phi \wedge \neg\psi)$;

3. $Cn(\neg(\phi \wedge \psi)) = Cn(\neg\phi \vee \neg\psi)$.

Lemma 3.14 *(Pynko, 1995a) The Logic of First Degree Entailments,* FDE, *is the weakest logic satisfying PC, PDI and PDM.* □

Lemma 3.15 *(Pynko, 1995b) The Logic of Paradox,* LP, *is the strongest paraconsistent logic satisfying PC, PDI and PDM.* □

Lemma 3.16 MCK *satisfies PC, PDI and PDM.*

Proof. Suppose $\phi \wedge \psi \vdash_{\text{MCK}} \delta$. Then $\phi, \psi \vdash_{\text{MCK}} \delta$ by rule 8 of Definition 2.17. Conversely, suppose $\phi, \psi \vdash_{\text{MCK}} \delta$. Then $\phi \wedge \psi \vdash_{\text{MCK}} \delta$ by rules 4 and 5. Therefore MCK satisfies PC.

Next, suppose $\Gamma, \phi \vee \psi \vdash_{\text{MCK}} \delta$. By rule 2, $\Gamma, \phi \vdash_{\text{MCK}} \neg\neg\phi$. By rule 6, $\Gamma, \phi \vdash_{\text{MCK}} \neg(\neg\phi \wedge \neg\psi)$. By the definition of disjunction, $\Gamma, \phi \vdash_{\text{MCK}} \phi \vee \psi$. Hence $\Gamma, \phi \vdash_{\text{MCK}} \delta$. By a similar argument, $\Gamma, \psi \vdash_{\text{MCK}} \delta$. Conversely, suppose $\Gamma, \phi \vdash_{\text{MCK}} \delta$ and $\Gamma, \psi \vdash_{\text{MCK}} \delta$. By rule 9 and the definition of disjunction, $\Gamma, \phi \vee \psi \vdash_{\text{MCK}} \neg\neg\phi, \neg\neg\psi$. By rule 3, $\Gamma, \phi \vee \psi \vdash_{\text{MCK}} \phi, \psi$. By the assumption, $\Gamma, \phi \vee \psi \vdash_{\text{MCK}} \delta$. Therefore MCK satisfied PDI.

PDM (1) follows from rules 2 and 3. PDM (2) and (3) follow from the definition of disjunction as well as rules 2 and 3. □

Recall from the discussion on page 57 that proof rules 2 to 9 of Definition 2.17 are shared by KL, LP, MCK and FDE. Since these are the only rules used in the proof of Lemma 3.16, it also constitutes a proof that each of these logics satisfy PC, PDI and PDM.

Lemma 3.17 *Given any consequence operator Cn which satisfies PC and PDI, and formula sets Γ and Σ,*

(i) $Cn(\Gamma_1 \wedge^+ \Sigma_1) = Cn(\Gamma_1 \cup \Sigma_1)$;

(ii) $Cn(\Gamma_1 \vee^+ \Sigma_1) = Cn(\Gamma_1) \cap Cn(\Gamma_2)$.

Proof. The results follow from the finiteness property, PC and PDI. $\square$

Lemma 3.18 *For any consequence operator Cn satisfying PC and PDI, $\equiv_{Cn}$ is a congruence relation on the algebra $(\mathcal{P}(L), \wedge^+, \vee^+)$.*

Proof. We have to check that $\equiv_{Cn}$ preserves power conjunctions and power disjunctions. Suppose $\Gamma_1 \equiv_{Cn} \Gamma_2$ and $\Sigma_1 \equiv_{Cn} \Sigma_2$. Let $\phi \in Cn(\Gamma_1 \wedge^+ \Sigma_1)$. Then $\phi \in Cn(\Gamma_1 \cup \Sigma_1)$ by Lemma 3.17. So there exist some finite $\Gamma_0 \subset \Gamma_1$ and $\Sigma_0 \subseteq \Sigma_1$ such that $\phi \in Cn(\Gamma_0 \cup \Sigma_0)$. $\Gamma_0 \subseteq Cn(\Gamma_2)$ and $\Sigma_0 \subseteq Cn(\Sigma_2)$, hence $\phi \in Cn(\Gamma_2 \cup \Sigma_2)$. So $\phi \in Cn(\Gamma_2 \wedge^+ \Sigma_2)$. Therefore $Cn(\Gamma_1 \wedge^+ \Sigma_1) \subseteq Cn(\Gamma_2 \wedge^+ \Sigma_2)$. By a similar argument $Cn(\Gamma_2 \wedge^+ \Sigma_2) \subseteq Cn(\Gamma_1 \wedge^+ \Sigma_1)$. Therefore $\Gamma_1 \wedge^+ \Sigma_1 \equiv_{Cn} \Gamma_2 \wedge^+ \Sigma_2$.

Next, suppose $\psi \in Cn(\Gamma_1 \vee^+ \Sigma_1)$. Then $\psi \in Cn(\Gamma_1) \cap Cn(\Sigma_1)$ by Lemma 3.17. Therefore $\psi \in Cn(\Gamma_2) \cap Cn(\Sigma_2)$. Hence $\psi \in Cn(\Gamma_2 \vee^+ \Sigma_2)$. Therefore $Cn(\Gamma_1 \vee^+ \Sigma_1) \subseteq Cn(\Gamma_2 \vee^+ \Sigma_2)$. By a similar argument, $Cn(\Gamma_2 \vee^+ \Sigma_2) \subseteq Cn(\Gamma_1 \vee^+ \Sigma_1)$. Therefore $\Gamma_1 \vee^+ \Sigma_1 \equiv_{Cn} \Gamma_2 \vee^+ \Sigma_2$. $\square$

The relation $\equiv_{Cn}$ does not, incidentally, preserve power negations in all logics. For example, in PC $\{p, p \wedge q\} \equiv_{\text{PC}} \{p \wedge q\}$, but $\{\neg p, \neg(p \wedge q)\}$ is not $\equiv_{\text{PC}}$-equivalent to $\{\neg(p \wedge q)\}$. This means that $\equiv_{Cn}$ is not in general a congruence relation over the algebra $(\mathcal{P}(L), \wedge^+, \vee^+, \neg^+)$.

Since $\equiv_{Cn}$ is a congruence on $(\mathcal{P}(L), \wedge^+, \vee^+)$, we may form its quotient algebra, called the *algebra of theories.* Thus *theories* are $\equiv_{Cn}$-equivalence classes in $\mathcal{P}(L)$. I will sometimes refer to a theory Γ as shorthand for the $\equiv_{Cn}$-equivalence class of Γ.

Definition 3.19 *The algebra of theories in a logic with consequence operator Cn is the structure $(P(L)/\equiv_{Cn}, \wedge^+, \vee^+)$, with meet operation $\wedge^+$ and join operation $\vee^+$ defined by:*

$$
\begin{aligned}
[\Gamma] \wedge^+ [\Sigma] &= [\Gamma \wedge^+ \Sigma]; \\
[\Gamma] \vee^+ [\Sigma] &= [\Gamma \vee^+ \Sigma].
\end{aligned}
$$

The next theorem generalizes a similar result for PC by Brink & Heidema (1989).

Theorem 3.20 *Let Cn be any consequence operator satisfying PC and PDI. There is an order-reversing isomorphism between the positive calculus of systems $(\{Cn(\Gamma) : \Gamma \subseteq L\}, \cdot, \dot{+})$ and the algebra of theories $(\mathcal{P}(L)/\equiv_{Cn}, \wedge^+, \vee^+)$.*

Proof. Cn is an order-reversing homomorphism from $(\mathcal{P}(L), \wedge^+, \vee^+)$ to the positive calculus of systems, with meets and joins in $(\mathcal{P}(L), \wedge^+, \vee^+)$ corresponding to joins and meets in $(\{Cn(\Gamma) : \Gamma \subseteq L\}, \cdot, \dot{+})$. The algebra of theories $(\mathcal{P}(L)/\equiv_{Cn}, \wedge^+, \vee^+)$ of Definition 3.19 is the quotient algebra of the power algebra $(\mathcal{P}(L), \wedge^+, \vee^+)$, obtained by factoring out under the kernel of Cn. The Homomorphism Theorem (see 3.9) therefore applies. □

Brink & Heidema (1989) show that the positive calculus of systems in PC is isomorphic to an algebra obtained via the Lindenbaum/Tarski construction. Recall from Section 1.1 that the Lindenbaum algebra for PC is a Boolean algebra of equivalence classes of wffs $\mathcal{L}_{PC} = (L/\approx_{PC}, \wedge, \vee, \neg, 0, 1)$. The semantic equivalence relation $\approx_{PC}$ was defined in terms of the semantic consequence relation $\models_{PC}$. However, since the completeness result for PC states that $\phi \models_{PC} \psi$ iff $\phi \vdash_{PC} \psi$, we may also regard $\approx_{PC}$ as a syntactic equivalence relation, defined in terms of the syntactic consequence relation $\vdash_{PC}$. The partial order on elements of $\mathcal{L}_{PC}$ then becomes that of syntactic consequence:

$$[\alpha] \leq [\beta] \text{ iff } \alpha \vdash_{PC} \beta.$$

Let F be a filter (as defined in 1.13) in $\mathcal{L}_{PC}$. Then F is an element of $\mathcal{P}(L/\approx_{PC})$. Since the elements of $L/\approx_{PC}$ are equivalence classes of wffs, we may form the union over F of its elements. This yields a formula set $\cup(F)$, defined by:

$$\cup(F) = \bigcup_{x \in F} x.$$

Lemmas 3.21 and 3.22 show that we may think of filters in $\mathcal{L}_{PC}$ as representing deductive systems.

Lemma 3.21 *For any filter F in $\mathcal{L}_{PC}$, the formula set $\cup(F)$ is deductively closed. Further, given any deductively closed formula set Γ, the set $\Gamma/\approx_{PC} = \{[\gamma] : \gamma \in \Gamma\}$ is a filter in $\mathcal{L}_{PC}$.*

Proof. Suppose $\cup(F) \vdash_{PC} \psi$. By finiteness, there exist $\phi_1, ..., \phi_n$ such that $\phi_1, ..., \phi_n \vdash_{PC} \psi$. Therefore $\phi_1 \wedge ... \wedge \phi_n \vdash_{PC} \psi$. Hence $[\psi] \in F$. So $\psi \in \cup(F)$.

To prove the second part of the lemma, suppose $\Gamma = \{\gamma : \Gamma \vdash_{PC} \gamma\}$. Let $F = \{[\gamma] : \gamma \in \Gamma\}$. We have to show that F is a filter. Suppose $x \in F$ and $x \leq z$. Then $x = [\gamma]$ for some $\gamma \in \Gamma$ and $z = [\delta]$ for some wff δ. So $\gamma \vdash_{PC} \delta$. Hence $[\delta] \in F$. Next, suppose $[\alpha_1], ..., [\alpha_m] \in F$. Let $y = [\alpha_1] \wedge ... \wedge [\alpha_m] = [\alpha_1 \wedge ... \wedge \alpha_m]$. Then $\alpha_1, ..., \alpha_m \in \Gamma$, and hence $\alpha_1 \wedge ... \wedge \alpha_m \in \Gamma$. Therefore $y \in F$. □

Lemma 3.22 *There is a one-one correspondence between filters in $\mathcal{L}_{PC}$ and deductively closed formula sets.*

Proof. Let Γ be any deductively closed formula set in L. Then $\cup(\Gamma/\approx_{\mathrm{PC}}) = \bigcup_{x\in\Gamma/\approx_{\mathrm{PC}}} x = \Gamma$. Next, let F be any filter in $\mathcal{L}_{\mathrm{PC}}$. Then $\cup(F)/\approx_{\mathrm{PC}} = \{[\alpha] : \alpha \in \cup(F)\} = \{[\alpha] : \alpha \in \bigcup_{x\in F} x\} = F$. □

Theorem 3.23 *(Gericke, 1963) The set of filters of a distributive lattice is a distributive lattice under set inclusion.* □

Since $\mathcal{L}_{\mathrm{PC}}$ is a Boolean algebra, $(L/\approx_{\mathrm{PC}}, \wedge, \vee)$ is a distributive lattice. By Theorem 3.23, the set of filters in $\mathcal{L}_{\mathrm{PC}}$ is also a distributive lattice. Let $\mathcal{F}(L/\approx_{\mathrm{PC}})$ denote the set of filters of $\mathcal{L}_{\mathrm{PC}}$. The filter lattice of $\mathcal{L}_{\mathrm{PC}}$ is the structure $\mathcal{F}(\mathcal{L}_{\mathrm{PC}}) = (\mathcal{F}(L/\approx_{\mathrm{PC}}), \cap, +)$. Its meet operation $\cap$ takes the intersection of two filters, and its join operation $+$ generates a filter from the union of two filters. Its bottom element is the filter $\{[1]\}$, consisting only of the equivalence class of tautologies, and its top element is the filter $L/\approx_{\mathrm{PC}}$, representing the inconsistent deductive system.

Theorem 3.24 *(Brink & Rewitzky, 1999) Let $(L, \wedge, \vee)$ be a distributive lattice. Then meets and joins in the filter lattice $(\mathcal{F}(L), \cap, +)$ are the power operations of joins and meets in $(L, \wedge, \vee)$. That is,*

$$\begin{aligned} F \cap G &= F \vee^{+} G, \\ F + G &= F \wedge^{+} G. \end{aligned}$$

Theorem 3.25 *(Brink & Rewitzky, 1999) The positive calculus of systems $(\{Cn_{\mathrm{PC}}(\Gamma) : \Gamma \subseteq L\}, \cdot, \dot{+})$ is isomorphic to the distributive filter lattice $(\mathcal{F}(L/\approx_{\mathrm{PC}}), \vee^{+}, \wedge^{+})$.* □

A formula set Γ_0 is called a *base* for Γ if $Cn_{\mathrm{PC}}(\Gamma_0) = \Gamma$. Every formula set which is $\equiv_{\mathrm{PC}}$-equivalent to Γ_0, is therefore also a base for Γ. If there exists a base for Γ which is finite, then Γ is called finitely axiomatizable. The finitely axiomatizable systems correspond to *principal* filters in the filter lattice, that is, filters generated by a single element. The algebra of finitely axiomatizable systems is therefore isomorphic to the principal filter lattice.

A third characterization of the positive calculus of systems in PC is via the semantics of the logic. Associate with each formula set Γ its set of models $Mod_{\text{PC}}(\Gamma)$, and call this the *meaning* of Γ. Two formula sets have the same meaning if and only if they have the same deductive closure, and so, after factoring out under this equivalence, there a one-one correspondence between the elements of $\mathcal{P}(L)/\equiv_{\text{PC}}$ and the set of meanings of formula sets $M_{\text{PC}} = \{Mod_{\text{PC}}(\Gamma) : \Gamma \subseteq L\}$. Theorem 3.26 below assures us that meanings of formula sets in PC are closed under both intersection and union, and that the map Mod_{PC} which takes any formula set to its meaning is a homomorphism. Theorem 3.27 then applies the Homomorphism Theorem (see 3.9) to prove that the power algebra of theories of Definition 3.19 is isomorphic to the *distributive lattice of meanings.*

Theorem 3.26 *For any formula sets* $\Gamma_1, \Gamma_2 \in \mathcal{P}(A)$,

$$\begin{aligned} Mod_{\text{PC}}(\Gamma \wedge^{+} \Sigma) &= Mod_{\text{PC}}(\Gamma) \cap Mod_{\text{PC}}(\Sigma); \\ Mod_{\text{PC}}(\Gamma \vee^{+} \Sigma) &= Mod_{\text{PC}}(\Gamma) \cup Mod_{\text{PC}}(\Sigma). \end{aligned}$$

Theorem 3.27 *The algebra of theories* $(\mathcal{P}(L)/\equiv_{\text{PC}}, \wedge^{+}, \vee^{+})$ *is isomorphic to the distributive lattice of meanings* $\mathcal{M}_{\text{PC}} = (M_{\text{PC}}, \cap, \cup)$. □

3.3 The algebra of LP

The calculus of systems in LP is the structure $(\{Cn_{\text{LP}}(\Gamma) : \Gamma \subseteq L\}, \cdot, \dot{+})$ defined in 3.11. This calculus can also be cast as a power algebra of theories.

Theorem 3.28 *There is an order-reversing isomorphism between the positive calculus of systems* $(\{Cn_{\text{LP}}\Gamma : \Gamma \subseteq L\}, \cdot, \dot{+})$ *and the algebra of theories* $(\mathcal{P}(L) / \equiv_{\text{LP}}, \wedge^+, \vee^+)$.

Proof. This follows from Lemma 3.15 and Theorem 3.20. □

The algebraic characterization of several relevance logics, including FDE, which was discussed on page 32, are as algebras based on a De Morgan lattice. This makes De Morgan lattices central to the study of relevance logics. De Morgan lattices are also central to the study of many-valued logics. The algebraic characterization of both Łukasiewicz logics and Post logics, namely as Łukasiewicz algebras and Post algebras respectively, are based on Heyting algebras, defined in 1.11. In turn, Heyting algebras are based on De Morgan lattices.

In view of the above, it is no surprise that De Morgan lattices also play an important role in the algebraic characterization of LP. Recall from Sections 1.1 and 2.2 that valuations are homomorphisms from the free word algebra L to some specified similar algebra. In the case of LP, this algebra is the Kleene lattice $\mathcal{K}_3$, defined on page 35, which is also a De Morgan lattice. $\mathcal{K}_3$ has the same role in the variety of Kleene lattices as $\mathcal{B}_2$ has in the variety of Boolean algebras:

Theorem 3.29 *(Kalman, 1958) The variety of Kleene lattices is generated by the three-element algebra* $\mathcal{K}_3$. □

Pynko (1995b) characterizes LP algebraically in terms of Kleene lattices. The Lindenbaum/Tarski construction outlined in Section 1.1 does not suffice for the algebraic characterization of LP, since the relation of semantic equivalence of wffs in LP is not a congruence on the free word algebra L. For example, $p \wedge (q \vee \neg q) \approx_{\text{LP}} p$, but it is not the case that $\neg(p \wedge (q \vee \neg q)) \approx_{\text{LP}} \neg p$.

Blok & Pigozzi (1989) give a very general framework for the algebraization of a logic, which encompasses the Lindenbaum/Tarski construction. Informally, a logic with derivability relation $\vdash$ is algebraizable if there exists a class of algebras K such that $\vdash$ can be interpreted in the equational consequence relation of K, and *vice versa*. For a detailed exposition of algebraizability, the reader can consult (Blok & Pigozzi, 1989; Font & Jansana, 1996).

One characterization of algebraizable logics is in terms of the *Leibniz operator*.

Definition 3.30 *Let $\mathcal{A}$ be an algebra and $F \subseteq A$. The Leibniz operator $\Omega_{\mathcal{A}}$ maps subsets of the base set A to congruence relations on $\mathcal{A}$. Namely, $\Omega_{\mathcal{A}}(F)$ is the largest congruence relation on $\mathcal{A}$ compatible with F in the following sense:*

$$(\forall \alpha, \beta \in A)\,[\textit{if } \alpha \in F \textit{ and } \langle \alpha, \beta \rangle \in \Omega_{\mathcal{A}}(F) \textit{ then } \beta \in F].$$

Definition 3.30 is obtained as a theorem in (Blok & Pigozzi, 1989), and can therefore be taken as the definition of the Leibniz operator, as I have done here. Note also that the use of the Greek symbol Ω for the Leibniz operator marks an exception from the notational convention set out in Section 1.1.

Definition 3.31 *Let $\langle \mathcal{A}, F \rangle$ be a matrix, with algebra $\mathcal{A}$ of the same similarity as L, and let $\vdash$ be the derivability relation of a formal proof system S. F is called an S-filter on $\mathcal{A}$ if the matrix consequence relation determined by $\langle \mathcal{A}, F \rangle$, as defined in 2.3, is logically weaker than the derivability relation of S.*

Thus F is an S-filter on $\mathcal{A}$ if and only if F is closed under all the inference rules of S. The S-filters on L are precisely the deductive systems of the calculus of systems, and the set of tautologies T of S is an S-filter in the word algebra L.

Theorem 3.32 *(Blok & Pigozzi, 1989) Let S be a formal proof system and K a variety of algebras. The following two conditions are equivalent:*

(i) *S is algebraizable with equivalent semantics K.*

(ii) *For every algebra $\mathcal{A}$, the Leibniz operator $\Omega_{\mathcal{A}}$ is an isomorphism between the lattice of S-filters and the lattice of K-congruences of $\mathcal{A}$.* □

Theorem 3.33 *(Blok & Pigozzi, 1989) A formal proof system S is algebraizable if and only if the Leibniz operator satisfies the following two conditions.*

(i) *Ω_L is injective and order-preserving on the positive calculus of systems.*

(ii) *Ω_L preserves unions of directed subsets in the positive calculus of systems.* □

The *Tarski congruence* of a logic with consequence operator Cn is the largest congruence relation Θ over the free word algebra L which is compatible with every deductive system of the logic, or equivalently, if $\langle\alpha,\beta\rangle \in \Theta$, then $Cn(\alpha) = Cn(\beta)$.

Lemma 3.34 *(Font & Jansana, 1996) For any logic with consequence operator Cn, its Tarski congruence relation $\Theta = \bigcap\{\Omega_L(\Gamma) : \Gamma = Cn(\Gamma)\}$.* □

Lemma 3.35 *(Pynko, 1995b) The Tarski congruence of* LP *is the following congruence relation on the free word algebra L:*
$\Theta = \{\ \langle\alpha,\beta\rangle : Cn_{\text{LP}}(\alpha) = Cn_{\text{LP}}(\beta)$ *and* $Cn_{\text{LP}}(\neg\alpha) = Cn_{\text{LP}}(\neg\beta)\}$. □

We can now form the quotient algebra of L under the congruence relation Θ, to obtain the 'Lindenbaum' algebra $\mathcal{L}_{\text{LP}} = (L/\Theta, \wedge, \vee, \neg)$.

Lemma 3.36 *(Pynko, 1995b) $\mathcal{L}_{\text{LP}}$ is a De Morgan lattice.* □

Theorem 3.37 *$\mathcal{L}_{\text{LP}}$ is a Kleene lattice.*

Proof. We know that $\mathcal{L}_{\text{LP}}$ is a De Morgan lattice. The result follows from the observation that the identity $\alpha \wedge \neg\alpha \wedge (\beta \wedge \neg\beta) = \alpha \wedge \neg\alpha$ also belongs to Θ. □

Theorems 3.23 and 3.37 tell us that the filter lattice of $\mathcal{L}_{\text{LP}}$, the structure $\mathcal{F}(\mathcal{L}_{\text{LP}}) = (\mathcal{F}(L/\Theta), \cap, +)$, is a distributive lattice. The power algebra of $\mathcal{L}_{\text{LP}}$ is the structure $\mathcal{P}(\mathcal{L}_{\text{LP}}) = (\mathcal{P}(L/\Theta), \wedge^+, \vee^+, \neg^+)$. By Theorem 3.5, this algebra is isomorphic to the quotient algebra $\mathcal{P}(L)/\Theta^+$. Since Θ^+ is a proper subrelation of $\equiv_{\text{LP}}$, the Second Isomorphism Theorem (see 3.10) applies. Thus the quotient algebra $\mathcal{P}(L/\Theta)/(\equiv_{\text{LP}}/\Theta^+)$ is isomorphic to the algebra of theories $\mathcal{P}(L)/\equiv_{\text{LP}}$.

Theorem 3.38 *(Pynko, 1995b) Let $\mathcal{K}$ be any Kleene lattice and F a proper inconsistent filter in $\mathcal{K}$. Then the matrix $(\mathcal{K}, F)$ determines a matrix consequence relation for* LP. □

Theorems 3.37 and 3.38 characterize LP in terms of Kleene lattices. However, this is not sufficient for the variety of Kleene lattices to be an equivalent algebraic semantics for LP. Font & Jansana (1996) point out that LP is not algabraizable. They also discuss a number of related concepts and algebraic characterizations of sentential logics, but I will not pursue them here.

3.4 The lattice of meanings in MCK

Without further ado, we can state that:

Theorem 3.39 MCK *is not algebraizable.*

Proof. The matrix consequence relation of MCK is determined by the matrices $\langle \mathcal{K}_3, \{t\}\rangle$ and $\langle \mathcal{K}_3, \{t, u\}\rangle$. This point was made on page 41. Both $\{t\}$ and $\{t, u\}$ are therefore MCK-filters. $\mathcal{K}_3$ is a simple algebra, that is, it has no congruence relations other than the identity relation $I_{\mathcal{K}_3}$ and the universal relation $U_{\mathcal{K}_3}$. Thus $\Omega_{\mathcal{K}_3}(\{t, u, f\}) = U_{\mathcal{K}_3}$ and $\Omega_{\mathcal{K}_3}(\{t\}) = \Omega_{\mathcal{K}_3}(\{t, u\}) = I_{\mathcal{K}_3}$. This means that $\Omega_{\mathcal{K}_3}$ is not injective on the MCK-filters of $\mathcal{K}_3$. The result follows from Theorem 3.32. □

Despite this negative result, the positive calculus of MCK-systems can be characterized in alternative ways.

Theorem 3.40 *There is an order-reversing isomorphism between the positive calculus of systems* $(\{Cn_{\text{MCK}}(\Gamma) : \Gamma \subseteq L\}, \cdot, \dot{+})$ *and the algebra of theories* $(\mathcal{P}(L) / \equiv_{Cn_{\text{MCK}}}, \wedge^{+}, \vee^{+})$.

Proof. The result follows from Lemma 3.16 and Theorem 3.20. □

The second characterization is as a *distributive lattice of meanings.* Recall that the meaning of a formula set in PC was defined on page 80 as its set of models. There is a one-one correspondence between meanings of formula sets and deductive systems in PC. The same definition of the meaning of a formula set does not suffice for MCK, as MCK is determined by the class of matrices $\{\langle \mathcal{K}_3, F\rangle : F \text{ is a proper filter in } \mathcal{K}_3\}$, as opposed to a single matrix. There is therefore no one-one correspondence between elements of some subset of $\mathcal{P}(Val_3)$ and deductive systems in MCK. Consider, for example, the wffs q and $(p \vee \neg p) \supset q$. The same valuations assign the value t to both these wffs, namely those v such that $v(q) = t$. But, since $q \notin Cn_{\text{MCK}}((p \vee \neg p) \supset q)$, $Cn_{\text{MCK}}(q) \neq Cn_{\text{MCK}}((p \vee \neg p) \supset q)$. Both determining matrices of MCK has to be taken into account in the definition of the meaning of a formula set in

MCK. This is the case in Definition 3.41 below. Lemma 3.42 then states that two formula sets have the same meaning if and only if they axiomatize the same deductive system.

Definition 3.41 *The meaning of any formula set* Γ *in* MCK *is a tuple* $m_{\mathsf{MCK}}(\Gamma) = \langle m_0(\Gamma), m_1(\Gamma)\rangle$*, where*

$$\begin{aligned} m_0(\Gamma) &= \{v : v(\Gamma) \subseteq \{t\}\}; \\ m_1(\Gamma) &= \{v : v(\Gamma) \subseteq \{t, u\}\}. \end{aligned}$$

Lemma 3.42 *For any formula sets* Γ *and* Σ*,* $Cn_{\mathsf{MCK}}(\Gamma) = Cn_{\mathsf{MCK}}(\Sigma)$ *iff* $m_{\mathsf{MCK}}(\Gamma) = m_{\mathsf{MCK}}(\Sigma)$*.*

Proof. $Cn_{\mathsf{MCK}}(\Gamma) = Cn_{\mathsf{MCK}}(\Sigma)$ iff $\Gamma \vdash_{\mathsf{MCK}} \phi$ for each $\phi \in Cn_{\mathsf{MCK}}(\Sigma)$, and $\Sigma \vdash_{\mathsf{MCK}} \psi$ for each $\psi \in Cn_{\mathsf{MCK}}(\Gamma)$. That is,

$$\begin{aligned} &(\forall \phi \in Cn_{\mathsf{MCK}}(\Sigma))\,(\forall v \in Val_3) \min\{v(\gamma) : \gamma \in \Gamma\} \leq v(\phi), \text{ and} \\ &(\forall \psi \in Cn_{\mathsf{MCK}}(\Gamma))\,(\forall v \in Val_3) \min\{v(\delta) : \delta \in \Sigma\} \leq v(\psi). \end{aligned}$$

Or, equivalently, $(\forall v \in Val_3)\,[\min\{v(\gamma) : \gamma \in \Gamma\} = \min\{v(\delta) : \delta \in \Sigma\}]$. For any valuation v, this minimum is t, u or f. In the first case, $v \in m_0(\Gamma) \cap m_1(\Gamma)$ and $v \in m_0(\Sigma) \cap m_1(\Sigma)$. In the second case, $v \in m_1(\Gamma) - m_0(\Gamma)$ and $v \in m_1(\Sigma) - m_0(\Sigma)$. In the third case, $v \notin (m_0(\Gamma) \cup m_1(\Gamma))$ and $v \notin (v_0(\Sigma) \cup m_1(\Sigma))$. □

Definition 3.43 *Let* M_{MCK} *denote the set of meanings of formula sets in* MCK*. Define the operations* $\Cap$ *and* $\Cup$ *on elements of* M_{MCK} *by:*

$$\begin{aligned} m_{\mathsf{MCK}}(\Gamma) \Cap m_{\mathsf{MCK}}(\Sigma) &= \langle m_0(\Gamma) \cap m_0(\Sigma), m_1(\Gamma) \cap m_1(\Sigma)\rangle; \\ m_{\mathsf{MCK}}(\Gamma) \Cup m_{\mathsf{MCK}}(\Sigma) &= \langle m_0(\Gamma) \cup m_0(\Sigma), m_1(\Gamma) \cup m_1(\Sigma)\rangle. \end{aligned}$$

Lemma 3.44 shows that these operations are well-defined in that there exist formula sets with meanings $m_{\text{MCK}}(\Gamma)\Cap m_{\text{MCK}}(\Sigma)$ and $m_{\text{MCK}}(\Gamma)\Cup m_{\text{MCK}}(\Sigma)$ respectively.

Lemma 3.44 *For any formula sets Γ and Σ in* MCK*,*

(i) $m_0(\Gamma) \cap m_0(\Sigma) = m_0(\Gamma \wedge^+ \Sigma)$;

(ii) $m_1(\Gamma) \cap m_1(\Sigma) = m_1(\Gamma \wedge^+ \Sigma)$;

(iii) $m_0(\Gamma) \cup m_0(\Sigma) = m_0(\Gamma \vee^+ \Sigma)$;

(iv) $m_1(\Gamma) \cup m_1(\Sigma) = m_1(\Gamma \vee^+ \Sigma)$.

Proof. (i)

$v \in m_0(\Gamma) \cap m_0(\Sigma)$
iff $(\forall\phi \in \Gamma)[v(\phi) = t]$ and $(\forall\phi \in \Sigma)[v(\phi) = t]$
iff $(\forall\phi \in \Gamma \wedge^+ \Sigma)[v(\phi) = t]$
iff $v \in m_0(\Gamma \wedge^+ \Sigma)$.

(ii)

$v \in m_1(\Gamma) \cap m_1(\Sigma)$
iff $(\forall\phi \in \Gamma)[v(\phi) \in \{t,u\}]$ and $(\forall\phi \in \Sigma)[v(\phi) \in \{t,u\}]$
iff $(\forall\phi \in \Gamma \wedge^+ \Sigma)[v(\phi) \in \{t,u\}]$
iff $v \in m_1((\Gamma \wedge^+ \Sigma))$.

(iii)

$v \in m_0(\Gamma) \cup m_0(\Sigma)$
iff $(\forall\phi \in \Gamma)[v(\phi) = t]$ or $(\forall\phi \in \Sigma)[v(\phi) = t]$
iff $(\forall\phi \in \Gamma \vee^+ \Sigma)[v(\phi) = t]$
iff $v \in m_0((\Gamma \vee^+ \Sigma))$.

(iv) $v \in m_1(\Gamma) \cup m_1(\Sigma)$

iff $(\forall \phi \in \Gamma)[v(\phi) \in \{t, u\}]$ or $(\forall \phi \in \Sigma)[v(\phi) \in \{t, u\}]$

iff $(\forall \phi \in \Gamma \vee^+ \Sigma)[v(\phi) \in \{t, u\}]$

iff $v \in m_1((\Gamma \vee^+ \Sigma))$ □

Theorem 3.45 *There is an order-reversing isomorphism between the positive calculus of systems* $(\{Cn_{\mathsf{MCK}}(\Gamma) : \Gamma \subseteq L\}, \cdot, \dotplus)$ *and the distributive lattice of meanings* $\mathcal{M}_{\mathsf{MCK}} = (M_{\mathsf{MCK}}, \Cap, \Cup)$.

Proof. Lemma 3.44 shows that m_{MCK} is a homomorphism from the power algebra $(\mathcal{P}(L), \wedge^+, \vee^+)$ to the lattice of meanings of formula sets $(M_{\mathsf{MCK}}, \Cap, \Cup)$. Lemma 3.42 shows that $\equiv_{Cn_{\mathsf{MCK}}}$ is the kernel of m_{MCK}. The Homomorphism Theorem (see 3.9) therefore applies. The result then follows from Theorem 3.40. □

As I explained on page 16, a formula set Σ may also be construed as an explanation set Σ describing the co-system $An_{\mathsf{MCK}}(\Sigma)$ of Definition 1.22. Recall that the equivalence relation $\equiv_{An_{\mathsf{MCK}}}$ on $\mathcal{P}(L)$, defined in 1.27, is determined by the antecedence operator An_{MCK}. Let X' denote the set-theoretical complement of X in Val_3.

Definition 3.46 *The composite map* $m_{\mathsf{MCK}} \circ \neg^+ : \mathcal{P}(L) \to M_{\mathsf{MCK}}$ *is defined by:*

$$\begin{aligned}(m_{\mathsf{MCK}} \circ \neg^+)(\Gamma) &= m_{\mathsf{MCK}}(\neg^+\Gamma) \\ &= \langle m_0(\neg^+\Gamma), m_1(\neg^+\Gamma)\rangle.\end{aligned}$$

Definition 3.47 *The dual meaning of any formula set* Γ *in* MCK *is a tuple* $m^d_{\mathsf{MCK}}(\Gamma) = \langle m^d_0(\Gamma), m^d_1(\Gamma)\rangle$, *where*

$$m^d_0(\Gamma) = m_1(\neg^+\Gamma)'$$

$$\begin{aligned} &= \{v : v(\Gamma) \cap \{t\} \neq \emptyset\}; \\ m_1^d(\Gamma) &= m_0(\neg^+\Gamma)' \\ &= \{v : v(\Gamma) \cap \{t, u\} \neq \emptyset\}. \end{aligned}$$

Lemma 3.48 *For any formula sets* Γ *and* Σ*, the following are equivalent:*

(i) $An_{\mathsf{MCK}}(\Gamma) = An_{\mathsf{MCK}}(\Sigma)$.

(ii) $m_{\mathsf{MCK}} \circ \neg^+(\Gamma) = m_{\mathsf{MCK}} \circ \neg^+(\Sigma)$.

(iii) $m^d_{\mathsf{MCK}}(\Gamma) = m^d_{\mathsf{MCK}}(\Sigma)$.

Proof. $An_{\mathsf{MCK}}(\Gamma) = An_{\mathsf{MCK}}(\Sigma)$ iff $\phi \vdash_{\mathsf{MCK}} \Gamma$ for each $\phi \in An_{\mathsf{MCK}}(\Sigma)$, and $\psi \vdash_{\mathsf{MCK}} \Sigma$ for each $\psi \in An_{\mathsf{MCK}}(\Gamma)$. That is,

$$(\forall \phi \in An_{\mathsf{MCK}}(\Sigma))(\forall v \in Val_3)\, v(\phi) \leq \max\{v(\gamma) : \gamma \in \Gamma\} \text{ and}$$
$$(\forall \psi \in An_{\mathsf{MCK}}(\Gamma))(\forall v \in Val_3)\, v(\psi) \leq \max\{v(\delta) : \delta \in \Sigma\}.$$

Or, equivalently, $(\forall v \in Val_3) \max\{v(\gamma) : \gamma \in \Gamma\} = \max\{v(\delta) : \delta \in \Sigma\}$. For any such v, this maximum is f, u or t. In the first case, $v \in m_0 \circ \neg^+(\Gamma) \cap m_1 \circ \neg^+(\Gamma)$, and $v \in m_0 \circ \neg^+(\Sigma) \cap m_1 \circ \neg^+(\Sigma)$. In the second case, $v \in m_1 \circ \neg^+(\Gamma) - m_0 \circ \neg^+(\Gamma)$ and $v \in m_1 \circ \neg^+(\Sigma) - m_0 \circ \neg^+(\Sigma)$. In the third case, $v \notin m_0 \circ \neg^+(\Gamma) \cup m_1 \circ \neg^+(\Gamma)$ and $v \notin m_0 \circ \neg^+(\Sigma) \cup m_1 \circ \neg^+(\Sigma)$. Therefore (i) is equivalent to (ii).

Further, $m_{\mathsf{MCK}} \circ \neg^+(\Gamma) = m_{\mathsf{MCK}} \circ \neg^+(\Sigma)$ iff ($m_0 \circ \neg^+(\Gamma) = m_0 \circ \neg^+(\Sigma)$ and $m_1 \circ \neg^+(\Gamma) = m_1 \circ \neg^+(\Sigma)$) iff ($m_0 \circ \neg^+(\Gamma)' = m_0 \circ \neg^+(\Sigma)'$ and $m_1 \circ \neg^+(\Gamma)' = m_1 \circ \neg^+(\Sigma)'$) iff $m^d_{\mathsf{MCK}}(\Gamma) = m^d_{\mathsf{MCK}}(\Sigma)$. □

Lemma 3.49 *For any formula sets* Γ *and* Σ *in* MCK*, the map* $m_{\mathsf{MCK}} \circ \neg^+ : \mathcal{P}(L) \to M_{\mathsf{MCK}}$ *is a homomorphism, with*

$$\begin{aligned} m_{\mathsf{MCK}} \circ \neg^+(\Gamma \wedge^+ \Sigma) &= m_{\mathsf{MCK}} \circ \neg^+(\Gamma) \Cup m_{\mathsf{MCK}} \circ \neg^+(\Sigma); \\ m_{\mathsf{MCK}} \circ \neg^+(\Gamma \vee^+ \Sigma) &= m_{\mathsf{MCK}} \circ \neg^+(\Gamma) \Cap m_{\mathsf{MCK}} \circ \neg^+(\Sigma). \end{aligned}$$

Proof.

$$\begin{aligned}
m_{\mathsf{MCK}} \circ \neg^{+}\left(\Gamma \wedge^{+} \Sigma\right) &= m_{\mathsf{MCK}}\left(\neg^{+}\left(\Gamma \wedge^{+} \Sigma\right)\right) \\
&= \left\langle m_0\left(\neg^{+}\left(\Gamma \wedge^{+} \Sigma\right)\right), m_1\left(\neg^{+}\left(\Gamma \wedge^{+} \Sigma\right)\right)\right\rangle \\
&= \left\langle m_0\left(\neg^{+}\Gamma \vee^{+} \neg^{+}\Sigma\right), m_1\left(\neg^{+}\Gamma \vee^{+} \neg^{+}\Sigma\right)\right\rangle \\
&= m_{\mathsf{MCK}}\left(\neg^{+}\Gamma\right) \Cup m_{\mathsf{MCK}}\left(\neg^{+}\Sigma\right) \\
&= m_{\mathsf{MCK}} \circ \neg^{+}(\Gamma) \Cup m_{\mathsf{MCK}} \circ \neg^{+}(\Sigma).
\end{aligned}$$

$$\begin{aligned}
m_{\mathsf{MCK}} \circ \neg^{+}\left(\Gamma \vee^{+} \Sigma\right) &= m_{\mathsf{MCK}}\left(\neg^{+}\left(\Gamma \vee^{+} \Sigma\right)\right) \\
&= \left\langle m_0\left(\neg^{+}\Gamma \wedge^{+} \neg^{+}\Sigma\right), m_1\left(\neg^{+}\Gamma \wedge^{+} \neg^{+}\Sigma\right)\right\rangle \\
&= m_{\mathsf{MCK}}\left(\neg^{+}\Gamma\right) \Cap m_{\mathsf{MCK}}\left(\neg^{+}\Sigma\right) \\
&= m_{\mathsf{MCK}} \circ \neg^{+}(\Gamma) \Cap m_{\mathsf{MCK}} \circ \neg^{+}(\Sigma).
\end{aligned}$$

□

It follows from Lemmas 3.48 and 3.49 that $\equiv_{An_{\mathsf{MCK}}}$ is a congruence relation on the power algebra $(\mathcal{P}(L), \wedge^{+}, \vee^{+})$. We may therefore form the quotient algebra $(\mathcal{P}(L)/\equiv_{An_{\mathsf{MCK}}}, \wedge^{+}, \vee^{+})$, called the *algebra of co-theories* in MCK.

Theorem 3.50 *There is an order-reversing isomorphism between the algebra of co-theories* $(\mathcal{P}(L)/\equiv_{An_{\mathsf{MCK}}}, \wedge^{+}, \vee^{+})$ *and the distributive lattice of meanings* $(M_{\mathsf{MCK}}, \Cap, \Cup)$.

Proof. Lemma 3.49 shows that the map $m_{\mathsf{MCK}} \circ \neg^{+}$ is a homomorphism, and Lemma 3.48 shows that $\equiv_{An_{\mathsf{MCK}}}$ is the kernel of $m_{\mathsf{MCK}} \circ \neg^{+}$. The result then follows from the Homomorphism Theorem. □

Corollary 3.51 *The algebra of theories* $(\mathcal{P}(L)/\equiv_{Cn_{\mathsf{MCK}}}, \wedge^{+}, \vee^{+})$ *is isomorphic to the algebra of co-theories* $(\mathcal{P}(L)/\equiv_{An_{\mathsf{MCK}}}, \wedge^{+}, \vee^{+})$, *the map* $\neg^{+}$ *being an order-reversing isomorphism.* □

I conclude this section with an algebraic characterization of some of the consequence relations of Chapter 2 in the algebra of meanings $(M_{\mathsf{MCK}}, ⋒, ⋓)$. The order reversing isomorphism $\neg^+$ from the algebra of co-theories to the algebra of theories makes it possible to describe these relations purely in terms of the latter, instead of in terms of both theories and co-theories.

Theorem 3.52 *The following three conditions are equivalent:*

(i) $\Gamma \vdash_{\mathsf{KL}} \Delta$.

(ii) $\Gamma \wedge^+ \neg^+ \Delta$ is inconsistent.

(iii) $m_0 (\Gamma \wedge^+ \neg^+ \Delta) = \emptyset$.

Proof. $m_0 (\Gamma \wedge^+ \neg^+ \Delta) = \emptyset$ iff for all $v \in Val_3$, $v (\Gamma \wedge^+ \neg^+ \Delta) \not\subseteq \{t\}$. The equivalence of (ii) and (iii) then follows from Lemma 2.38. Also, for any $v \in Val_3$,

$$\begin{aligned} v \left(\Gamma \wedge^+ \neg^+ \Delta\right) \not\subseteq \{t\} \quad & \text{iff} \quad v (\Gamma) \not\subseteq \{t\} \text{ or } v \left(\neg^+ \Delta\right) \not\subseteq \{t\} \\ & \text{iff} \quad v (\Gamma) \not\subseteq \{t\} \text{ or } v (\Delta) \cap \{t\} \neq \emptyset \\ & \text{iff} \quad v (\Gamma) \subseteq \{t\} \text{ implies } v (\Delta) \cap \{t\} \neq \emptyset. \end{aligned}$$

Therefore (iii) is equivalent to (i). □

The semantic consequence relation of $\diamond$MCK, defined in 2.15, is another example of a power relation:

$$\begin{aligned} \Gamma \models_{\diamond\mathsf{MCK}} \Delta \quad \text{iff} \quad & (\forall v \in m_0 (\Gamma)) \left(\exists w \in m_0^d (\Delta)\right) [v \leq_k w] \text{ and} \\ & (\forall v \in m_1 (\Gamma)) \left(\exists w \in m_1^d (\Delta)\right) [v \leq_k w] \\ \text{iff} \quad & m_0 (\Gamma) (\leq_k)_0^+ \, m_0^d (\Delta) \text{ and } m_1 (\Gamma) (\leq_k)_1^+ \, m_1^d (\Delta). \end{aligned}$$

This is an example of a power relation between sets of a dual nature. The meaning of Γ is the tuple $\langle m_0 (\Gamma), m_1 (\Gamma)\rangle$, and the dual meaning of Δ is the tuple $\langle m_0^d(\Delta), m_1^d(\Delta)\rangle$. The consequence relation, when phrased as a power relation of the knowledge order $\leq_k$ defined by the Kleene semi-bilattice, relates the meaning of Γ to the dual meaning of Δ. This construction is quite general, and can be applied to any logic with a bilattice semantics, provided the meaning and dual meaning of formula sets are suitably defined.

Let $M_{\text{KL}} = \{Mod_{\text{KL}}(\Gamma) : \Gamma \subseteq L\}$ and $M_{\text{LP}} = \{Mod_{\text{LP}}(\Gamma) : \Gamma \subseteq L\}$ denote the set of meanings of formula sets in KL and LP respectively.

Lemma 3.53 *For any formula sets Γ and Σ, $Cn_{\text{KL}}(\Gamma) = Cn_{\text{KL}}(\Sigma)$ iff $m_0(\Gamma) = m_0(\Sigma)$.*

Proof. $Cn_{\text{KL}}(\Gamma) = Cn_{\text{KL}}(\Sigma)$ iff $Mod_{\text{KL}}(\Gamma) = Mod_{\text{KL}}(\Sigma)$ iff $\{v : v(\Gamma) \subseteq \{t\}\} = \{v : v(\Sigma) \subseteq \{t\}\}$ iff $m_0(\Gamma) = m_0(\Sigma)$. □

Theorem 3.54 *The algebra of theories $(\mathcal{P}(L)/\equiv_{\text{KL}}, \wedge^+, \vee^+)$ is isomorphic to the distributive lattice of meanings $\mathcal{M}_{\text{KL}} = (M_{\text{KL}}, \cap, \cup)$.*

Proof. The proof is similar to that of Theorem 3.45. □

Lemma 3.55 *For any formula sets Γ and Σ, $Cn_{\text{LP}}(\Gamma) = Cn_{\text{LP}}(\Sigma)$ iff $m_1(\Gamma) = m_1(\Sigma)$.*

Proof. $Cn_{\text{LP}}(\Gamma) = Cn_{\text{LP}}(\Sigma)$ iff $Mod_{\text{LP}}(\Gamma) = Mod_{\text{LP}}(\Sigma)$ iff $\{v : v(\Gamma) \subseteq \{t, u\}\} = \{v : v(\Sigma) \subseteq \{t, u\}\}$ iff $m_1(\Gamma) = m_1(\Sigma)$. □

Theorem 3.56 *The algebra of theories $(\mathcal{P}(L)/\equiv_{\text{LP}}, \wedge^+, \vee^+)$ is isomorphic to the distributive lattice of meanings $\mathcal{M}_{\text{LP}} = (M_{\text{LP}}, \cap, \cup)$.*

Proof. The proof is similar to that of Theorem 3.45. □

3.5 Simulation

In the previous section, the consequence relation of $\diamond$MCK was characterized as a power relation on the meanings of formula sets. This illustrated that the power construction of Definition 3.3, based on a pointwise knowledge order on valuations, can be viewed as a consequence relation. This section explores the application of the power construction, again based on a pointwise knowledge order on valuations, to the meanings of formula sets in two different logics. I will show that this relation can be viewed as a *simulation* relation. A simulation is defined as a mapping between the algebras of theories of the respective logics. I will show that the algebra of theories of certain three-valued logics, such as KL, is simulated by the algebra of theories of MCC.

Simulation relations have their origins in the algebraic theory of automata (Ginzburg, 1968) and program semantics (Milner, 1971). Bisimulation, a relation of simulation equivalence, was introduced by Park (1981), and used by Bergstra & Klop (1988) and Hennessy (1988) to give the semantics of processes in terms of labelled graphs or transition systems. Bisimulations have also been used as a translation mechanism between process algebras and modal logic, and between modal logic and first order logic (Van Benthem *et al.*, 1994).

A *labelled transition system* is a triple $(S, I, \longrightarrow)$, where S is a set of states, I is an index or label set, and $\longrightarrow \subseteq S \times I \times S$ is a ternary relation. The relation $\longrightarrow$ can also be written as a set of binary relations over S, indexed by elements of I. This leads to the following definition:

Definition 3.57 *A transition structure is a pair $\mathcal{S} = (S, \{R_i\}_{i \in I})$, where I is an index set, and each $R_i \subseteq S \times S$ is a binary transition relation.*

Definition 3.58 *(Bergstra & Klop, 1988) A simulation of a transition struc-*

ture $\mathcal{S} = (S, \{R_i\}_{i\in I})$ by a transition structure $\mathcal{S}' = (S', \{R_i'\}_{i\in I})$ is a binary relation $\sim \subseteq S \times S'$ such that, for any $\alpha \in S$, $\beta \in S'$ and $i \in I$, if $\alpha \sim \beta$ and $\alpha R_i \alpha'$ then $\exists \beta' \in S'$ such that $\beta R_i' \beta'$ and $\alpha' \sim \beta'$.

The following picture shows a simulation between the transition structures $\mathcal{S} = (S, R)$ and $\mathcal{S}' = (S', R')$.

$$\begin{tikzcd}
S & S' \\
\alpha \arrow[r, "\sim"] \arrow[d, "R"'] & \beta \arrow[d, "R'"] \\
\alpha' \arrow[r, "\sim"] & \beta'
\end{tikzcd}$$

To illustrate the idea of one logic simulating another, I will use the logics MCC and KL, defined in 1.32 and 2.8 respectively. Recall that the algebra of theories in a logic with consequence operator Cn, $(\mathcal{P}(L)/\equiv_{Cn}, \wedge^+, \vee^+)$, was defined in 3.19.

Definition 3.59 *Let Cn be the consequence operator of a logic satisfying PC and PDI. The transition structure induced by Cn is the structure*

$$(\mathcal{P}(L)/\equiv_{Cn}, \Rightarrow_{Cn}).$$

Its base set is that of the algebra of theories $(\mathcal{P}(L)/\equiv_{Cn}, \wedge^+, \vee^+)$, and it has a single transition relation, defined by the natural order on this algebra:

$$x \Rightarrow_{Cn} y \text{ iff } x \vee^+ y = y.$$

The transition structure of KL is the structure $(\mathcal{P}(L)/\equiv_{\text{KL}}, \Rightarrow_{\text{KL}})$. By Theorem 3.20, there is an order-reversing isomorphism between $(\mathcal{P}(L)/\equiv_{\text{KL}}, \Rightarrow_{\text{KL}})$ and the structure $(\{Cn_{\text{KL}}(\Gamma) : \Gamma \subseteq L\}, \subseteq)$, induced by the positive calculus of KL-systems of Definition 3.11. Similarly, the transition structure of MCC is the structure $(\mathcal{P}(L)/\equiv_{\text{MCC}}, \Rightarrow_{\text{MCC}})$, which is isomorphic to the structure $(\{Cn_{\text{MCC}}(\Gamma) : \Gamma \subseteq L\}, \subseteq)$.

The simulation relation between the transition structures of two logics is induced by a knowledge order $\leq_k$ on the truth values that formulas in the logic can assume. This order is defined for any logic with a bilattice semantics (Ginsberg, 1988), and the same idea therefore applies to any such logic.

The relation $\leq_k^+$ between sets of valuations is the power order of the pointwise knowledge order on valuations defined in 2.6.

Definition 3.60 *For any formula sets* Γ *in* KL *and* Δ *in* MCC, *the relation* $\leq_k^+$ *is defined by:*

$$\begin{aligned}
\Gamma \leq_k^+ \Delta \quad &\textit{iff} \quad Mod_{\text{KL}}\Gamma \leq_k^+ Mod_{\text{MCC}}\Delta \\
&\textit{iff} \quad (\forall v \in Mod_{\text{KL}}\Gamma)(\exists w \in Mod_{\text{MCC}}\Delta)[v \leq_k w] \;\; \textit{and} \\
&\qquad\; (\forall w \in Mod_{\text{MCC}}\Delta)(\exists v \in Mod_{\text{KL}}\Gamma)[v \leq_k w].
\end{aligned}$$

In Section 3.4 I showed that the consequence relation of $\diamond$MCK can be defined as a power relation in the lattice of meanings of MCK. Definition 3.60 is another example of a power relation on sets of valuations, but this time between different logics.

As explained in Chapter 3, the knowledge order $\leq_k$ signifies an increase in information. The power relation $\leq_k^+$ lifts this order to sets of valuations, or theories. If $\Gamma \leq_k^+ \Delta$, then every model of Γ approximates some model of Δ, and every model of Δ is approximated by some model of Γ. Γ can be viewed as an approximation of Δ, since an increase in information in Γ can result in Δ.

Recall that the connectives of KL are monotone with respect to the knowledge order, in the sense defined in 2.1. This means that an increase in knowledge can never cause a retraction of the truth or falsity of any wff. The following lemma is a consequence of this property.

Lemma 3.61 *For any formula sets* Γ *in* KL *and* Δ *in* MCC, $\Gamma \leq_k^+ \Delta$ *iff* $Mod_{\mathrm{KL}}\Gamma \cap Val_2 = Mod_{\mathrm{MCC}}\Delta$.

Proof. Suppose $\Gamma \leq_k^+ \Delta$. Let $v \in Mod_{\mathrm{KL}}\Gamma \cap Val_2$. Then $v \in Mod_{\mathrm{MCC}}\Delta$ by $\leq_k^+$. Next, let $w \in Mod_{\mathrm{MCC}}\Delta$. Then $(\exists v \in Mod_{\mathrm{KL}}\Gamma)\,[v \leq_k w]$. Hence, by the monotony of the connectives, $w \in Mod_{\mathrm{KL}}\Gamma$.

Conversely, suppose $Mod_{\mathrm{KL}}\Gamma \cap Val_2 = Mod_{\mathrm{MCC}}\Delta$. Let $v \in Mod_{\mathrm{KL}}\Gamma$. Let $w \in Val_2$ such that $v \leq_k w$. By the monotony of the connectives, $w \in Mod_{\mathrm{KL}}\Gamma$, and hence $w \in Mod_{\mathrm{KL}}\Delta$. Finally, let $w \in Mod_{\mathrm{MCC}}\Delta$. Then $w \in Mod_{\mathrm{KL}}\Gamma$, and $w \leq_k w$. Hence $\Gamma \leq_k^+ \Delta$. □

Corollary 3.62 *If* $\Gamma \leq_k^+ \Sigma$ *and* $\Gamma \leq_k^+ \Delta$ *then* $\Sigma \equiv_{\mathrm{MCC}} \Delta$. □

Theorem 3.63 $\leq_k^+$ *is a simulation of* $(\mathcal{P}(L)\,/\equiv_{\mathrm{KL}}, \Rightarrow_{\mathrm{KL}})$ *by* $(\mathcal{P}(L)\,/\equiv_{\mathrm{MCC}}, \Rightarrow_{\mathrm{MCC}})$.

Proof. Let Γ and Γ' be formula sets in KL, let Δ be a formula set in MCC, and suppose $\Gamma \leq_k^+ \Delta$ and $\Gamma \Rightarrow_{\mathrm{KL}} \Gamma'$. Then $Mod_{\mathrm{KL}}\Gamma \cap Val_2 = Mod_{\mathrm{MCC}}\Delta$ by Lemma 3.61. Let $\Delta' = \Gamma'$. Then $Mod_{\mathrm{KL}}\Gamma' \cap Val_2 = Mod_{\mathrm{MCC}}\Delta'$, that is, $\Gamma' \leq_k^+ \Delta'$. Also, $Mod_{\mathrm{KL}}\Gamma \subseteq Mod_{\mathrm{KL}}\Gamma'$, so $Mod_{\mathrm{KL}}\Gamma \cap Val_2 \subseteq Mod_{\mathrm{KL}}\Gamma' \cap Val_2$. Therefore, by Lemma 3.61, $Mod_{\mathrm{MCC}}\Delta \subseteq Mod_{\mathrm{MCC}}\Delta'$, that is, $\Delta \Rightarrow_{\mathrm{MCC}} \Delta'$. □

Theorem 3.63 is illustrated below.

$$\begin{tikzcd}
\mathrm{KL} & \mathrm{MCC} \\
\Gamma \arrow[r, "\leq_k^+"] \arrow[d, "\Rightarrow\mathrm{KL}"'] & \Delta \arrow[d, "\Rightarrow\mathrm{MCC}"] \\
\Gamma' \arrow[r, "\leq_k^+"] & \Delta'
\end{tikzcd}$$

Informally, the theorem states that the relation of logical strength is preserved under an increase in information. The simulation relation $\leq_k^+$ signifies an increase in information, while the relation $\Rightarrow$ is that of logical strength. The models of a theory Γ in KL are three-valued, and therefore represent partial information, whereas the models of a theory Δ in MCC are two-valued, and represent complete information. The simulation relation indicates that there is a special relation between the consequence operators of KL and MCC. Namely, if Γ approximates Δ, then, for any Γ' such that $\Gamma \Rightarrow_{\mathrm{KL}} \Gamma'$, there is some corresponding Δ' approximated by Δ, such that $\Delta \Rightarrow_{\mathrm{MCC}} \Delta'$.

Lemma 3.61 uses the fact that the connectives of KL are monotone with respect to the knowledge order. Other three-valued logics based on Kleene's strong truth tables often have an additional, non-monotonic connective. Examples are the implication connective $\rightarrow_{\mathrm{L}}$ of the three-valued logic Ł$_3$ of Łukasiewicz, and the unary definedness connective of Barringer, Cheng & Jones (1984). I will briefly consider the simulation of such logics by MCC.

Let K be any three-valued logic based on Kleene's strong truth tables, and with semantic consequence relation that of Definition 1.32, where $Mod_{\mathrm{K}}\Gamma = \{v : v(\Gamma) \subseteq \{t\}\}$ for any formula set Γ in K. Note that this excludes logics such as MCK and LP, but includes logics such as Ł$_3$.

Lemma 3.61 does not in general hold for K. Consider, for example, the wffs $p \rightarrow q$, $\neg p \vee q$ and $p \vee \neg p$ in Ł$_3$. It is easy to check that $p \rightarrow q \leq_k^+ \neg p \vee q$ and $p \rightarrow q \leq_k^+ p \vee \neg p$, but $\neg p \vee q$ is not logically equivalent to $p \vee \neg p$.

Let $\mathcal{P}_0(L)$ denote the set of finite subsets of L. The elements $\mathcal{P}_0(L)/\equiv_{\mathrm{K}}$ are finitely axiomatizable theories over L.

Theorem 3.64 $\leq_k^+$ *is a simulation of* $(\mathcal{P}_0(L)/\equiv_{\mathrm{K}}, \Rightarrow_{\mathrm{K}})$ *by* $(\mathcal{P}_0(L)/\equiv_{\mathrm{MCC}}, \Rightarrow_{\mathrm{MCC}})$.

Proof. Suppose $\Gamma \leq_k^+ \Delta$ and $\Gamma \Rightarrow_{\mathrm{K}} \Gamma'$. Since Γ, Δ and Γ' are finite, there exist wffs α, β and α' such that $\alpha \equiv_{\mathrm{K}} \Gamma$, $\beta \equiv_{\mathrm{K}} \Delta$, and $\alpha' \equiv_{\mathrm{K}} \Sigma$, and $\alpha \leq_k^+ \beta$ and $\alpha \Rightarrow_{\mathrm{K}} \alpha'$. Define an equivalence relation on valuations as follows: Let $v \approx w$ iff they agree on all propositional variables appearing in α, β or α'. Let $\bar{v} = \{w : w \approx v\}$. If $v \in Mod_{\mathrm{K}}\alpha'$, then $\bar{v} \subseteq Mod_{\mathrm{K}}\alpha'$. For each $\bar{v} \subseteq Mod_{\mathrm{K}}\alpha'$, pick any classical valuation $z \geq_k v$, and let $\phi_{\bar{v}}$ be a wff with $Mod_{\mathrm{K}}\phi_{\bar{v}} = \bar{z}$. Let $\beta' = \beta \vee (\bigvee_{\bar{v}} \phi_{\bar{v}})$. Then $\beta \Rightarrow_{\mathrm{MCC}} \beta'$, and $\alpha' \leq_k^+ \beta'$. □

The result obtained in Theorem 3.64 raises the question: Under what circumstances is one logic simulated by another? The simulation relation is defined in terms of a knowledge order on valuations, so the same knowledge order would need to be used by both logics (although both logics need not use every truth value). The characterization of the class of logics simulated by MCC (or, more generally, any logic) is an open question, which may merit further investigation.

Chapter 4

Theorylikeness

Verisimilitude, or truthlikeness, concerns the ordering of theories according to their closeness to the truth. The word 'theory' is not used here in the exact sense of Definition 3.19, but in a broad sense, accommodating the many perspectives in the literature on verisimilitude. Originally, the notion is due to Popper (1963), for whom it was a necessary ingredient in his philosophy that science makes progress by discarding one scientific theory in favour of another which is closer to the truth. On Popper's definition, one theory is closer to the truth than another if and only if it has more true consequences and fewer false consequences. However, Miller (1974) and Tichý (1974) later showed that Popper's ordering on theories was flawed in that all theories that have some false consequences are incomparable. The similarity approach to truthlikeness started at this time, with the proposal that truthlikeness be described by means of a similarity relation between theories. The outcome of this line of investigation is summarized in the books by Oddie (1986) and Niiniluoto (1987a), the survey paper by Brink (1989b), and the collection edited by Kuipers (1987b). Developments in verisimilitude since then are surveyed by Niiniluoto (1998) and Zwart (1998).

The problem of verisimilitude, as conceived by Popper, deals with ordering theories relative to a theory which pronounces upon the truth or falsity of all facts. This *complete truth assumption* has been shared by many, but not all, subsequent authors on verisimilitude. Dropping the complete truth assumption yields a parameterized, *theorylike* order, where two theories are ordered relative to a given third theory. Some examples of such parameterized orders in the literature are Kuipers's (1987b; 1992) 'naive' and 'refined' structuralism, Niiniluoto's (1987a) truthlikeness for singular sentences, and van Benthem's (1987) investigation into the link between verisimilar orders and conditional assertions. In this chapter I will show that such a theorylike order can be defined as a power relation.

In an artificial intelligence context, the problem of choosing between possibly conflicting information lead to the development of research areas such as non-monotonic reasoning and theory change. In this context, theories may also be ordered relative to given background information. In Chapter 5, I will deal with the topic of theory change and its relation to theorylikeness.

I will refer mainly to the following two existing approaches to truthlikeness: the structuralist approach of Kuipers (1987b; 1992), and the power relations approach of Brink & Heidema (1987), and Burger & Heidema (1994). Section 4.1 describes Kuipers's structuralist approach, as it applies to propositional logic. In Section 4.2, I generalize the power relations approach to truthlikeness by dropping the complete truth assumption. This yields a parameterized order of theorylikeness. This order is then compared to the 'refined' theorylike order of Kuipers (1992). I will show that the power relations approach escapes certain shortcomings of the structuralist approach. Section 4.3 describes the theorylike power order obtained in Section 4.2 syntactically, first in terms of certain relevance criteria on the consequence relation, and then in terms of closure operators. In Section 4.4 I discuss many-valuedness in verisimilitude, with reference to the logic $\diamond$MCK of Chapter 2.

4.1 Structuralism

Kuipers's 'naive' structuralist definition of truthlikeness is formulated in terms of set-theoretic structures, and applies readily to a model-theoretic treatment of truthlikeness in classical propositional logic. The description given below is restricted to this application, with structures from the outset being identified with classical propositional truth assignments. A *constituent* of a formula set Γ is defined as a maximally consistent set of literals Σ such that $Mod_{PC}\Sigma \subseteq Mod_{PC}\Gamma$. Thus each model v of Γ corresponds to the constituent of Γ having v as model.

The structuralist approach does not make the complete truth assumption; it defines a preorder on $\mathcal{P}(L)$ relative to an arbitrary, fixed formula set Γ, which may be viewed as a description of an incomplete truth. The models of Γ represent *empirical possibilities*, and elements of $Val_2 - Mod_{PC}\Gamma$, empirical *im*possibilities. The latter is not necessarily describable by any formula set, as the following argument by Brink & Heidema (1989) shows: Let $\aleph_0$ be the cardinality of the natural numbers, and let $c = 2^{\aleph_0}$. The Lindenbaum algebra $\mathcal{L}_{PC}$ has cardinality $\aleph_0$, hence $\mathcal{P}(\mathcal{L}_{PC})$ has cardinality c. On the other hand, $\mathcal{P}(Val_2)$ has cardinality 2^c. Thus not every subset of Val_2 is the set of models of some formula set.

A *theory approximation* Φ is a formula set, together with the claim $Mod_{PC}\Phi = Mod_{PC}\Gamma$. (Kuipers calls Φ a theory, but this term is already used in this thesis to refer to an equivalence class of wffs, in the sense of Definition 1.25.) Thus the theory approximation Φ is true iff it describes exactly the same empirical possibilities as Γ. In approximating Γ, Φ can make two kinds of mistakes: it can exclude some empirical possibilities, and it can include some empirical impossibilities. These two sets are given by $Mod_{\mathrm{PC}}\Gamma - Mod_{\mathrm{PC}}\Phi$ and $Mod_{\mathrm{PC}}\Phi - Mod_{\mathrm{PC}}\Gamma$ respectively. Ψ is said to be closer to Γ than Φ is if it makes fewer mistakes of each kind.

Definition 4.1 *Given any formula sets Γ, Φ and Ψ, Ψ is at least as close to Γ as Φ is, written $\Phi \sqsubseteq^{ns}_{\Gamma} \Psi$, iff*

(Ni) $Mod_{\mathrm{PC}}\Gamma - Mod_{\mathrm{PC}}\Psi \subseteq Mod_{\mathrm{PC}}\Gamma - Mod_{\mathrm{PC}}\Phi$, and

(Nii) $Mod_{\mathrm{PC}}\Psi - Mod_{\mathrm{PC}}\Gamma \subseteq Mod_{\mathrm{PC}}\Phi - Mod_{\mathrm{PC}}\Gamma$.

The superscript ns in $\sqsubseteq^{ns}_{\Gamma}$ indicates that the definition is obtained by a 'naive structuralist' approach. The following equivalent formulations of Definition 4.1 are derived in (Kuipers, 1987a; Kuipers, 1992) and (Zwart, 1998). The first is in terms of the *symmetric difference* operation:

Definition 4.2 *For any sets X and Y, their symmetric difference $X -_s Y$ is defined as:*

$$X -_s Y = Y -_s X = (X - Y) \cup (Y - X).$$

The union of the sets $Mod_{\mathrm{PC}}\Gamma - Mod_{\mathrm{PC}}\Psi$ and $Mod_{\mathrm{PC}}\Psi - Mod_{\mathrm{PC}}\Gamma$ describes the total mistakes of Φ in approximating Γ.

Lemma 4.3 $\Phi \sqsubseteq^{ns}_{\Gamma} \Psi$ *iff* $Mod_{\mathrm{PC}}\Gamma -_s Mod_{\mathrm{PC}}\Psi \subseteq Mod_{\mathrm{PC}}\Gamma -_s Mod_{\mathrm{PC}}\Phi$. □

The second equivalent formulation is in terms of the *matches* made by Φ and Ψ respectively. An *instantial* match made by Φ is an empirical possibility which is correctly included by Φ, and an *explanatory* match made by Φ is an empirical impossibility which is correctly excluded by Φ. The next lemma then states that every instantial match made by Φ is also made by Ψ, and every explanatory match made by Φ is also made by Ψ.

Lemma 4.4 $\Phi \sqsubseteq_{\Gamma}^{ns} \Psi$ *iff*

(Ni′) $Mod_{\text{PC}}\Phi \cap Mod_{\text{PC}}\Gamma \subseteq Mod_{\text{PC}}\Psi \cap Mod_{\text{PC}}\Gamma$.

(Nii′) $(Mod_{\text{PC}}\Phi \cup Mod_{\text{PC}}\Gamma)' \subseteq (Mod_{\text{PC}}\Psi \cup Mod_{\text{PC}}\Gamma)'$.

The third equivalent formulation is formulated syntactically:

Lemma 4.5 $\Phi \sqsubseteq_{\Gamma}^{ns} \Psi$ *iff* $Cn(\Gamma) \cap Cn(\Phi) \subseteq Cn(\Psi)$ *and* $An(\Phi) \cap An(\Gamma) \subseteq An(\Psi)$. □

The 'refined' structuralist approach to truthlikeness generalizes the naive approach, but now the order is based on an underlying ternary relation of *structurelikeness*, which indicates which of two structures is more similar to a third. In the context of propositional logic, the relation determines which of two valuations is more similar to a third valuation. Kuipers (1992) proposes that the relation of structurelikeness also be defined in terms of the symmetric difference operation.

For any valuation $v \in Val_2$, let v_0 denote its restriction to L_0. Then $v_0^{-1}(t) = \{p_i \in L_0 : v(p_i) = t\}$.

Definition 4.6 *For any* $v, w \in Val_2$, *let*

$$\begin{aligned} v -_s w &= v_0^{-1}(t) -_s w_0^{-1}(t) \\ &= \left(v_0^{-1}(t) - w_0^{-1}(t)\right) \cup \left(w_0^{-1}(t) - v_0^{-1}(t)\right) \\ &= \{p_i \in L_0 : v(p_i) \neq w(p_i)\}. \end{aligned}$$

Definition 4.7 *For any* $u, v, w \in Val_2$, w *is at least as similar to* v *as* u *is, iff* $v -_s w \subseteq v -_s u$.

Thus the symmetric difference $v -_s w$ is the set of sentential symbols on which v and w differ, and w is at least as similar to v as u is iff w agrees with v on all those sentential symbols where u agrees with v.

A second component in the refined structuralist approach is that of relatedness of structures. This condition concerns the domain of the structure-likeness relation. Since the domain of the symmetric difference operation of Definition 4.6 is Val_2, this component trivialises in the context of propositional logic.

Definition 4.8 *Given any formula sets Γ, Φ and Ψ, Ψ is at least as close to Γ as Φ is, written $\Phi \sqsubseteq^{rs}_{\Gamma} \Psi$, iff*

(Ri) $(\forall u \in Mod_{PC}\Phi)\,(\forall z \in Mod_{PC}\Gamma)\,(\exists w \in Mod_{PC}\Psi)\,[w -_s z \subseteq u -_s z]$, and

(Rii) $(\forall w \in Mod_{PC}\Psi - (Mod_{PC}\Phi \cup Mod_{PC}\Gamma))\,(\exists u \in Mod_{PC}\Phi - Mod_{PC}\Gamma)$
$(\exists z \in Mod_{PC}\Gamma)\,[w -_s z \subseteq u -_s z]$.

The refined structuralist truthlike order of Definition 4.8 states that Ψ is at least as close to Γ as Φ is, iff every approximation of an instantial match made by Φ can be improved upon by an approximate instantial match made by Ψ, and every explanatory mistake made by Ψ is an improvement of some explanatory mistake made by Φ. (Rii) may be reformulated as:

(Rii′) $(\forall w \in Mod_{PC}\Psi - Mod_{PC}\Gamma)\,(\exists u \in Mod_{PC}\Phi - Mod_{PC}\Gamma)$
$(\exists z \in Mod_{PC}\Gamma)\,[w -_s z \subseteq u -_s z]$.

Zwart (1998) divides the similarity approach to truthlikeness into two strategies: *content* proposals and *likeness* proposals. Content proposals place the emphasis on Popper's intuition of increasing the number of true consequences of a theory, and accordingly, on the logical strength of the theory.

Likeness proposals place the emphasis on the truth or falsity of the atoms of the language, and accordingly, on every model of the theory. This distinction is already alluded to by Kuipers (1987b) when he says:

> Hence, one problem is to define judgements of the form "structure x is closer to the actual possibility than y", the other problem is to define "the set of structures A, or theory A, is closer to the set of empirical possibilities than B".

In this categorization, Kuipers's 'naive' definition of truthlikeness is a content proposal, while his 'refined' definition is a likeness proposal. The next section, which deals with the power relations approach to verisimilitude, also belongs to the category of likeness proposals.

4.2 Power relations

In this section, I extend the power relations approach introduced by Brink & Heidema (1987), by exhibiting a parameterized theorylike partial order on propositional formula sets. That is, formula sets are ordered relative to an arbitrary third formula set, which may be viewed as a description of the truth. In Section 4.3, I show that this order is equivalent to an order proposed by Ryan (1992) based on the notion of natural consequence. I also give an algorithmic, syntactic description of the order.

The truthlikeness proposal of Brink & Heidema (1987; 1989) orders formula sets in classical propositional logic relative to the truth, which is assumed to be complete. In (Brink & Heidema, 1987), the language is finite, while in (Brink & Heidema, 1989) the language has denumerably many sentential symbols. Recall from Section 1.1 that L_0 denotes the set of sentential symbols of the language L. Brink & Heidema assume that the truth is describable by the constituent L_0.

By Definition 2.6, the pointwise partial truth order on Val_2 is given by:

$$v \leq_t w \text{ iff } (\forall p_i \in L_0)\left[v\left(p_i\right) \leq_t w\left(p_i\right)\right].$$

In the power approach to truthlikeness, this relation is used to order valuations according to their closeness to the truth. A truthlike order on formula sets is obtained by lifting this relation to a power order on sets of valuations, as defined in Section 3.1:

Definition 4.9 *(Brink & Heidema, 1987) For any formula sets* Γ *and* Δ,

$$\begin{aligned} \Gamma \leq_t^+ \Delta \quad & \textit{iff} \quad Mod_{\mathrm{PC}}\Gamma \leq_t^+ Mod_{\mathrm{PC}}\Delta \\ & \textit{iff} \quad (\forall v \in Mod_{\mathrm{PC}}\Gamma)(\exists w \in Mod_{\mathrm{PC}}\Delta)[v \leq_t w] \;\; \textit{and} \\ & \qquad (\forall w \in Mod_{\mathrm{PC}}\Delta)(\exists v \in Mod_{\mathrm{PC}}\Gamma)[v \leq_t w]. \end{aligned}$$

In view of Theorems 3.20 and 3.27, this order may be viewed either as an order on deductive systems, or as an order on theories (that is, $\equiv_{\mathrm{PC}}$-equivalence classes of formula sets), or as an order on meanings of theories.

The first step in extending the definition to a theorylike order, is to parameterize the pointwise truth order on valuations, so that valuations are ordered relative to an arbitrary third valuation. The order then states that w is closer to v than u is iff w agrees with v everywhere u agrees with v.

Definition 4.10 *For any valuations* v, w *and* z,

$$v \leq_z w \textit{ iff } (\forall p_i \in L_0)\left[\textit{if } v\left(p_i\right) = z\left(p_i\right) \textit{ then } w\left(p_i\right) = z\left(p_i\right)\right].$$

This order was proposed by Britz & Brink (1995), but it is not difficult to see that it coincides with Kuipers's structurelikeness relation based on the relation of symmetric difference of Definition 4.6:

$$v \leq_z w \text{ iff } w -_s z \subseteq v -_s z.$$

The resulting power order then becomes:

Definition 4.11 *For any formula sets* Γ *and* Δ,

$$\Gamma \leq_z^+ \Delta \quad \textit{iff} \quad (\forall v \in Mod_{\text{PC}}\Gamma)\,(\exists w \in Mod_{\text{PC}}\Delta)\,[v \leq_z w] \ \textit{and}$$
$$(\forall w \in Mod_{\text{PC}}\Delta)\,(\exists v \in Mod_{\text{PC}}\Gamma)\,[v \leq_z w].$$

Definition 4.10 orders valuations relative to a single valuation. In Definition 4.22 below, I will generalize this definition to an order on valuations relative to a formula set. But first we need to establish some preliminary results and terminology.

An important concept in the ordering of theories is that of relevance. Schurz & Weingartner (1987) noted that the case against Popper's original proposal depends upon introducing certain irrelevancies as consequences in the construction of its argument. By placing certain relevance criteria on the classical deductive consequence relation, these irrelevant consequents can be disallowed. The notion of relevance which I will use, coincides with that of Ryan (1991), where it is called natural consequence.

Definition 4.12 *Two valuations* u *and* w *are called i-equivalent, written* $u \equiv_i w$, *iff* $(\forall p_j : j \neq i)\, u(p_j) = w(p_j)$.

Definition 4.13 *A formula set* Φ *is isotone in a sentential symbol* p_i *iff* $\forall u \in Mod_{\text{PC}}\Phi$ *and* $\forall w \in Val_2$ *such that* $u \equiv_i w$, *if* $u(p_i) \leq_t w(p_i)$ *then* $w \in Mod_{\text{PC}}\Phi$.

Definition 4.14 *A formula set* Φ *is antitone in a sentential symbol* p_i *iff* $\forall u \in Mod_{\text{PC}}\Phi$ *and* $\forall w \in Val_2$ *such that* $u \equiv_i w$, *if* $w(p_i) \leq_t u(p_i)$ *then* $w \in Mod_{\text{PC}}\Phi$.

Informally, Φ is isotone in p_i iff increasing the truth value of p_i preserves satisfaction of Φ, and it is antitone in p_i iff decreasing the truth value of p_i preserves satisfaction of Φ. Note that isotonicity and antitonicity are defined model-theoretically, and as such are not syntax-dependent concepts. However, if we restrict ourselves to single wffs, the variables in which a wff is isotone (antitone) can be characterized syntactically. This characterization is stated, but not proved, in (Ryan, 1991):

Theorem 4.15 (i) *A wff ϕ is isotone in a sentential symbol p_i iff there exists some wff ψ, written in disjunctive normal form and logically equivalent to ϕ, such that p_i does not occur negatively in ψ.*

(ii) *A wff ϕ is antitone in a sentential symbol p_i iff there exists some wff ψ, written in disjunctive normal form and logically equivalent to ϕ, such that p_i does not occur positively in ψ.*

Proof. I will prove (i). The proof of (ii) is similar. Suppose ϕ is isotone in p_i, and that ϕ is written in disjunctive normal form, where no disjunct is a contradiction. Say $\phi = \delta_1 \vee ... \vee \delta_n$. Suppose p_i occurs negatively in δ_j, for some $j \leq n$. Then $\delta_j = \gamma \wedge \neg p_i$ for some conjunction of literals γ, where p_i does not occur positively in γ. Consider the wff ψ, syntactically identical to $\delta_1 \vee ... \vee \delta_n$, except that δ_j has been replaced by γ, that is, $\neg p_i$ has been deleted from δ_j.

I will show that ψ is semantically equivalent to ϕ. Let $v(\phi) = t$. Then $v(\delta_k) = t$ for some $k \leq n$. If $k = j$, then $v(\gamma) = t$, else $v(\delta_k) = t$ trivially. Hence $v(\psi) = t$. Conversely, let $v(\psi) = t$. If $v(\delta_k) = t$ for some $k \leq n$, then $v(\phi) = t$. Else $v(\gamma) = t$ and $v(\neg p_i) = f$. Let $w \equiv_i v$ with $w(p_i) = f$. Then $w(\gamma) = t$ and $w(\neg p_i) = t$. Therefore $w(\phi) = t$. Since ϕ is isotone in p_i and $w \equiv_i v$, $v(\phi) = t$. This argument shows that the deletion of any negative occurrence of p_i from ϕ yields a wff which is semantically equivalent to ϕ. Since there are only finitely many negative occurrences of p_i in ϕ, they may all be deleted from ϕ to obtain a wff which is semantically equivalent to ϕ.

Conversely, suppose ϕ is written in a disjunctive normal form containing no negative occurrences of p_i, say $\phi = \delta_1 \vee ... \vee \delta_n$. I will first show that each disjunct δ_j of ϕ is isotone in p_i. Let δ_j be such a disjunct, and suppose that $v(\delta_j) = t$, $w \equiv_i v$ and $v(p_i) \leq_t w(p_i)$. There are three cases:

1. Suppose p_i does not occur in δ_j. Then v and w assign the same truth value to all sentential symbols occurring in δ_j. Hence $w(\delta_j) = t$.

2. Suppose $\delta_j = p_i$. Then $w(p_i) = v(p_i) = t$.

3. Suppose $\delta_j = \gamma \wedge p_i$, where p_i does not occur in γ. Then $w(\delta_j) = t$ by (i) and (ii) above.

The proof now proceeds by structural induction on the number of disjuncts δ_j in ϕ. If $\phi = \delta_1$, then ϕ is a primitive conjunction, and hence is isotone in p_i by the argument given above. Now suppose that the claim holds for every wff in disjunctive normal form with fewer than n disjuncts. Let $\phi = \delta_1 \vee ... \vee \delta_n$. Suppose $v(\phi) = t$, $w \equiv_i v$ and $v(p_i) \leq_t w(p_i)$. Then either $v(\delta_1 \vee ... \vee \delta_{n-1}) = t$ or $v(\delta_n) = t$. By the induction hypothesis, the claim holds for both $\delta_1 \vee ... \vee \delta_{n-1}$ and δ_n. Therefore either $w(\delta_1 \vee ... \vee \delta_{n-1}) = t$ or $w(\delta_n) = t$. Hence $w(\phi) = t$, which proves that ϕ is isotone in p_i. □

For example, $p \wedge q$ is isotone in both p and q, but antitone in neither; $p \vee \neg q$ is isotone in p and antitone in q; p is isotone in itself, and both isotone and antitone in q, and $p \vee \neg p$ and $p \wedge \neg p$ are both isotone and antitone in both p and q.

Two sets that I will frequently refer to, are:

$$\begin{aligned} \Phi_+ &= \{p_i : \Phi \text{ is not antitone in } p_i\}; \\ \Phi_- &= \{p_i : \Phi \text{ is not isotone in } p_i\}. \end{aligned}$$

Informally, $p_i \in \Phi_+$ iff it is not the case that decreasing the truth value of p_i always preserves satisfaction of Φ. Similarly, $p_i \in \Phi_-$ iff it is not the case that increasing the truth value of p_i always preserves satisfaction of Φ. If Φ is the singleton set $\{\phi\}$, then Φ_+ is the set of all sentential symbols that occur positively at least once in every disjunctive normal form of ϕ. Φ_- are those sentential symbols that occur negatively at least once in every disjunctive normal form of ϕ. These sets are used in Definition 4.16 below.

Definition 4.16 *For any formula set Γ and valuations u and w,*

$$\begin{aligned} u \preceq_\Gamma w \quad \textit{iff} \quad & (\forall p_i \in \Gamma_+)\,[\textit{if } u(p_i) = t \textit{ then } w(p_i) = t] \textit{ and} \\ & (\forall p_i \in \Gamma_-)\,[\textit{if } u(p_i) = f \textit{ then } w(p_i) = f]. \end{aligned}$$

Lemma 4.17 *If $Mod_{\mathrm{PC}}\Gamma = \{v\}$, then $u \preceq_\Gamma w$ iff $u \leq_v w$.*

Proof. Γ is logically equivalent to the set of all literals satisfied by v. Thus $v(p_i) = t$ iff $p_i \in \Gamma_+$, and $v(p_i) = f$ iff $p_i \in \Gamma_-$. Therefore, $u \preceq_\Gamma w$ iff $(\forall p_i)$ [if $v(p_i) = t$ and $u(p_i) = t$ then $w(p_i) = t$], and $(\forall p_i)$ [if $v(p_i) = f$ and $u(p_i) = f$ then $w(p_i) = f$]. That is, $u \leq_v w$. $\square$

Lemma 4.18 *$\preceq_\Gamma$ is a preorder.*

Proof. Reflexivity follows directly from Definition 4.16. To prove transitivity, let $u \preceq_\Gamma v$ and $v \preceq_\Gamma w$. That is, $(\forall p_i \in \Gamma_+)$ [if $u(p_i) = t$ then $v(p_i) = t$], $(\forall p_i \in \Gamma_-)$ [if $u(p_i) = f$ then $v(p_i) = f$], $(\forall p_i \in \Gamma_+)$ [if $v(p_i) = t$ then $w(p_i) = t$], and $(\forall p_i \in \Gamma_-)$ [if $v(p_i) = f$ then $w(p_i) = f$]. Therefore,

$(\forall p_i \in \Gamma_+)$ [if $u(p_i) = t$ then $w(p_i) = t$] and $(\forall p_i \in \Gamma_-)$ [if $u(p_i) = f$ then $w(p_i) = f$]. That is, $u \preceq_\Gamma w$. □

Definition 4.19 below maps any given formula set Γ and valuation u, to a formula set Γ_u. Lemma 4.20 states that Γ_u describes the set of valuations that are closer to Γ than u is, according to the order $\preceq_\Gamma$. This upclosure is then used in Definition 4.22 to obtain the final order on valuations which will be used to define the parameterized power order on formula sets.

Definition 4.19 *For any formula set Γ and valuation u, the formula set Γ_u is defined by:*

$$\Gamma_u = \{p_i : p_i \in \Gamma_+ \text{ and } u(p_i) = t\} \cup \{\neg p_i : p_i \in \Gamma_- \text{ and } u(p_i) = f\}.$$

Lemma 4.20 *$Mod_{PC}(\Gamma_u)$ is the $\preceq_\Gamma$-upclosure of u.*

Proof. A valuation v satisfies Γ_u iff for any $p_i \in \Gamma_+$ such that $u(p_i) = t$, $v(p_i) = t$, and for any $p_i \in \Gamma_-$ such that $u(p_i) = f$, $v(p_i) = f$. That is, $Mod_{PC}(\Gamma_u) = \{v : (\forall p_i \in \Gamma_+)$ [if $u(p_i) = t$ then $v(p_i) = t$] and $(\forall p_i \in \Gamma_-)$ [if $u(p_i) = f$ then $v(p_i) = f$]$\} = \{v : u \preceq_\Gamma v\}$. □

Lemma 4.21 *For finite Γ, $Mod_{PC}(\Gamma)$ is $\preceq_\Gamma$-upclosed.*

Proof. Suppose $u \preceq_\Gamma w$ and $u \in Mod_{PC}(\Gamma)$. By Definition 4.16, $(\forall p_i \in \Gamma_+)$ [if $u(p_i) = t$ then $w(p_i) = t$], and $(\forall p_i \in \Gamma_-)$ [if $u(p_i) = f$ then $w(p_i) = f$]. Therefore, for each p_i such that $u(p_i) \neq w(p_i)$, one of two cases hold: either $u(p_i) = t$ and Γ is antitone in p_i, or $u(p_i) = f$ and Γ is isotone in p_i. Let v be $\equiv_i$-equivalent to u, and $v(p_i) \neq u(p_i)$. Then $v \in Mod_{PC}(\Gamma)$. Repeating this argument for every p_i such that $u(p_i) \neq w(p_i)$, we get $w \in Mod_{PC}(\Gamma)$. □

Lemma 4.21 holds for any finite formula set Γ. A related property of formula sets is that of *convexity.* Γ is called convex if it is $\leq_z$-upclosed for all $z \in Mod_{\text{PC}}(\Gamma)$. That is, if $v \in Mod_{\text{PC}}(\Gamma)$ and $v \leq_z w$, then $w \in Mod_{\text{PC}}(\Gamma)$. Convexity is a desirable property in Kuipers's refined theorylike order, and I will return to it later in this section.

The next definition contracts the order $\preceq_\Gamma$, so that the models of Γ are all maximal and equivalent. For non-models of Γ, the order coincides with that of Definition 4.16.

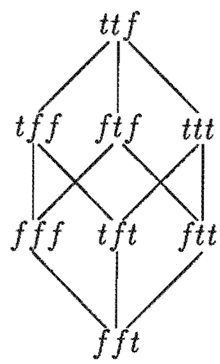
Definition 4.22 *For any formula set* Γ *and valuations* u *and* w*,*

$$u \leq_\Gamma w \text{ iff } w \in Mod_{\text{PC}}(\Gamma) \cup Mod_{\text{PC}}(\Gamma_u).$$

$Mod_{\text{PC}}(\Gamma) \cup Mod_{\text{PC}}(\Gamma_u)$ is the set of models of $\Gamma \vee^+ \Gamma_u$. If Γ has a single model v then Γ is equivalent to a formula set Γ', in which each sentential symbol appears exactly once, either negated or unnegated. The sets Γ'_+ and Γ'_- therefore forms a partition of the literals in Γ. Namely, they are the subsets of positive and negative literals in Γ'. Γ'_u is the set of those literals satisfied by u. Hence w is closer to v than u is iff w agrees with v on at least those sentential symbols where u agrees with v. This yields the same order on valuations as defined in 4.10, which is therefore a special case of Definition 4.22.

Example 4.23 Consider the formula set $\Gamma = \{p \vee q, \neg r\}$ in the propositional language generated by the variables p, q and r. The models of Γ is the set $Mod_{\text{PC}}(\Gamma) = \{ttf, tff, ftf\}$. Further, $\Gamma_+ = \{p, q\}$ and $\Gamma_- = \{r\}$. The pre-order $\preceq_\Gamma$ on valuations relative to Γ is obtained as follows: Let, for example, $u = fff$ and $v = tff$. To see if $u \preceq_\Gamma v$, we apply Definition 4.16. Namely, we check whether, if $u(p) = t$ then $v(p) = t$, if $u(q) = t$ then

$v(q) = t$, and if $u(r) = f$ then $v(r) = f$. Since this is the case, $u \preceq_\Gamma v$. Any two valuations can be ordered in this fashion, giving rise to the pre-order below. Note that the set of models of Γ is $\preceq_\Gamma$-upclosed, as proved in Lemma 4.21. □



Now that we have an order on valuations, it is easy to obtain a theorylike power order on formula sets as an instance of Definition 3.3:

Definition 4.24 *Given any formula sets Φ, Ψ and Γ, Ψ is at least as close to Γ as Φ is, written $\Phi \leq_\Gamma^+ \Psi$, iff*

(Ti) $(\forall v \in Mod_{PC}\Phi)(\exists w \in Mod_{PC}\Psi)[v \leq_\Gamma w]$, and

(Tii) $(\forall w \in Mod_{PC}\Psi)(\exists v \in Mod_{PC}\Phi)[v \leq_\Gamma w]$.

(Tii) is equivalent to:

$$(\forall w \in Mod_{PC}\Psi - Mod_{PC}\Gamma)(\exists v \in Mod_{PC}\Phi - Mod_{PC}\Gamma)[v \leq_\Gamma w].$$

Thus (Tii) may justifiably be read: Every explanatory mistake made by Ψ is an improvement of some explanatory mistake made by Φ.

I conclude this section by relating the theorylike order of Definition 4.24 to Kuipers's refined theorylike order of Definition 4.8. As I noted on page 106, the pointwise order on valuations of Definition 4.10 coincides with that of Definition 4.8. Technically, the difference in approach in the formulation of the two orders is that Definition 4.24 orders valuations relative to an arbitrary formula set, and then lifts this order to a power relation, whereas Definition 4.8 incorporates the parameterization in the power relation itself. The next lemma and example clarify the relationship between the two orders further.

Lemma 4.25 *For any valuations u and v, and formula set Γ, the following relationships hold:*

(i) If $u \leq_\Gamma v$ then $(\exists z \in Mod_{\mathrm{PC}}\Gamma)\,[u \leq_z v]$.

(ii) If $(\forall z \in Mod_{\mathrm{PC}}\Gamma)\,[u \leq_z v]$ then $u \leq_\Gamma v$.

Proof. (i) Suppose $u \leq_\Gamma v$. If $v \in Mod_{\mathrm{PC}}\Gamma$, then let $z = v$. Else, $u \preceq_\Gamma v$. Let $z \in Mod_{\mathrm{PC}}\Gamma$ such that $v \preceq_\Gamma z$. Define z' by:

$$z'(p_i) = \begin{bmatrix} z(p_i) & \text{if } p_i \in \Gamma_+ \cup \Gamma_- \\ v(p_i) & \text{otherwise.} \end{bmatrix}$$

Then $z' \in Mod_{\mathrm{PC}}\Gamma$, and $u \leq_z v$. To see this, let $p_i \in L_0$. If $p_i \notin \Gamma_+ \cup \Gamma_-$, then $z'(p_i) = v(p_i)$ by definition. Else, if $p_i \in \Gamma_+$ and $u(p_i) = z'(p_i) = t$, then, since $u \preceq_\Gamma v$ and $v \preceq_\Gamma z$, $v(p_i) = z(p_i) = z'(p_i) = t$. Else, if $p_i \in \Gamma_+$ and $u(p_i) = z'(p_i) = f$, then, since $v \preceq_\Gamma z$, $v(p_i) = z(p_i) = z'(p_i) = f$. Else, if $p_i \in \Gamma_-$ and $u(p_i) = z'(p_i) = f$, then, since $u \preceq_\Gamma v$, $v(p_i) = z(p_i) = z'(p_i) = f$. Finally, if $p_i \in \Gamma_-$ and $u(p_i) = z'(p_i) = t$, then, since $v \preceq_\Gamma z$, $z'(p_i) = z(p_i) = v(p_i) = t$.

(ii) Suppose that $(\forall z \in Mod_{\mathrm{PC}}\Gamma)\,[u \leq_z v]$. Then $(\forall z \in Mod_{\mathrm{PC}}\Gamma)\,(\forall p_i)$ [if $u(p_i) = z(p_i)$ then $v(p_i) = z(p_i)$]. If $p_i \in \Gamma_+$ then $(\exists z \in Mod_{\mathrm{PC}}\Gamma)\,[z(p_i) = t]$, and if $p_i \in \Gamma_-$ then $(\exists z \in Mod_{\mathrm{PC}}\Gamma)\,[z(p_i) = f]$. Therefore $(\forall p_i \in \Gamma_+)$ [if $u(p_i) = 1$ then $v(p_i) = 1$], and $(\forall p_i \in \Gamma_-)$ [if $u(p_i) = 0$ then $v(p_i) = 0$]. Thus $u \leq_\Gamma v$. □

The next example shows that the converse of neither of the implications in Lemma 4.25 hold:

Example 4.26 Let $\Gamma = \{p \vee q, \neg r\}$ be the formula set of Example 4.23. Let $u = tft$, $v = fff$ and $z = ftf$. Then $z \in Mod_{\mathrm{PC}}\Gamma$ and $u \leq_z v$, but $u \not\leq_\Gamma v$. Therefore it is not the case that $(\exists z \in Mod_{\mathrm{PC}}\Gamma)\,[u \leq_z v]$ implies $u \leq_\Gamma v$. Further, let $w = fft$. Then $w \leq_\Gamma u$, but $w \not\leq_z u$. Therefore it is not the case that $w \leq_\Gamma u$ implies $(\forall z \in Mod_{\mathrm{PC}}\Gamma)\,[w \leq_z u]$. □

Lemma 4.25 and Example 4.26 provide some insight into the difference between the orders of Definitions 4.8 and 4.24. Recall from page 112 that Γ is convex if it is $\leq_z$-upclosed for all $z \in Mod_{\mathrm{PC}}(\Gamma)$. It follows from (i) that, for any convex formula set Γ, (Tii) implies (Rii). For suppose $(\forall w \in Mod_{\mathrm{PC}}\Psi)$ $(\exists v \in Mod_{\mathrm{PC}}\Phi)\,[v \leq_\Gamma w]$. By Lemma 4.25, $(\forall w \in Mod_{\mathrm{PC}}\Psi)\,(\exists v \in Mod_{\mathrm{PC}}\Phi)$ $(\exists z \in Mod_{\mathrm{PC}}\Gamma)\,[v \leq_z w]$. Since Γ is convex, (Rii) follows. On the other hand, in Example 4.26, (Rii) holds for $\Phi = \{p \wedge \neg q \wedge r\}$ and $\Psi = \{\neg p \wedge \neg q \wedge \neg r\}$, whereas (Tii) does not. Thus (Tii) distinguishes between more formula sets than (Rii) does. Intuitively, (Rii) says that every explanatory mistake v made by Ψ is not as bad as some explanatory mistake u made by Φ. The criterion used to determine this, is the existence of a model z of Γ which is closer to v than to u. Kuipers (1997) later proposed that this criterion be strengthened to require that z be a model of $\Gamma - \Phi$. This strengthens condition (Rii), but not sufficiently to coincide with (Tii).

Schurz & Weingartner (1987) argue that irrelevant consequences of the theories to be ordered should be disallowed in determining a verisimilar order. Their proposal is content-based, and the truth is assumed to be complete. I will argue that, not only should irrelevant consequences of the theories to be ordered be disallowed, but so should irrelevant consequences of the theory relative to which the order is defined.

The instantial clause (Ri) seeks to relate every approximate match of Φ relative to each constituent of Γ, irrespective of the relevance of each of its elements. The next two examples illustrate that this sometimes lead to counter-intuitive results. (The relatedness criterion does not apply in this respect, even for non-propositional structures, since it only captures local relevance between structures, and not relevance to Γ as a whole.)

As a first example, consider the formula set $\Gamma = \{p \wedge q\}$, phrased in a propositional language generated by the variables p, q and r. The value assigned to r by any valuation v is irrelevant to whether v is a model of Γ, and should therefore also be irrelevant to how close v is to being a model of Γ. This is, however, not the case. For suppose we want to order the wffs $\phi = p \wedge \neg q \wedge r$ and $\psi = \neg p \wedge \neg q \wedge \neg r$ relative to Γ. Intuitively, ϕ should be closer to Γ than ψ is, since ϕ agrees with Γ on p, whereas ψ disagrees on every atomic claim made by Γ. But, since $tft -_s ttf \not\subseteq fff -_s ttf$, (Ri) pronounces that $\phi \not\sqsubseteq_\Gamma^{rs} \psi$. This is because (Ri) looks at each constituent of Γ separately when seeking to improve the approximate instantial match fff. And while every sentential symbol in the language is relevant when considering a single constituent of Γ, some parts of the constituent (in this case, the truth value of r) may actually be irrelevant. This irrelevance of r is only expressed by Γ as a whole.

As a second example, consider the formula set $\Gamma = \{p \vee q, \neg r\}$ of Example 4.23 again. Let $\phi = p \wedge \neg q \wedge r$ and $\psi = p \wedge q \wedge r$. Let $u = tft$, $w = ttt$, and $z = tff$. Then $u \leq_\Gamma w$, but $u \not\leq_z w$. Thus $\phi \leq_\Gamma^+ \psi$ but $\phi \not\sqsubseteq_\Gamma^{rs} \psi$. The fact that Γ is isotone in q makes the latter counter-intuitive. Since q does not appear negated in Γ, an increase in the truth value of q should make ttt at least as close to being a model of Γ than tft is.

The theorylike order of Definition 4.24 escapes these kinds of counter-intuitive results, since the relevance of each literal to Γ is taken into account in the order on valuations relative to Γ.

4.3 Syntactic perspective

Ryan (1991; 1992) defines an order on valuations based on a *natural consequence* relation, and suggests its power order as an order of verisimilitude. I will show that the natural consequence relation coincides with the order on valuations obtained in Definition 4.22. Superficially, the verisimilar order based on the natural consequence relation has the appearance of a content-based order, but Theorem 4.33 shows that it is equivalent to the likeness order of Definition 4.24. In Corollary 4.40, I give an alternative syntactic description of Definition 4.24, which is computationally more efficient than the syntactic description in terms of natural consequences.

In (Ryan, 1991), the natural consequence relation is defined as a binary relation on the language of some logic. I will assume that the logic is PC, and define the relation between $\mathcal{P}(L)$ and L.

Definition 4.27 *ψ is a natural consequence of Φ, written $\Phi \models^{\natural} \psi$, iff $Mod_{\text{PC}}(\Phi) \subseteq Mod_{\text{PC}}(\psi)$, $\psi_+ \subseteq \Phi_+$ and $\psi_- \subseteq \Phi_-$.*

The purpose of this definition is to rule out irrelevancies in the consequence relation. Note, for example, that if α is a natural consequence of Φ and p_i does not occur positively (negatively) in Φ, then it also does not occur positively (negatively) in α. This prohibits the introduction of irrelevant disjuncts in consequences. The order on valuations is now defined as follows:

Definition 4.28 $u \sqsubseteq_\Gamma w$ *iff every natural consequence of Γ satisfied by u is also satisfied by w.*

Definition 4.29 *(Ryan, 1992) For any formula sets* Γ, Φ *and* Ψ,

$$\Phi \sqsubseteq^{+}_{\Gamma} \Psi \quad \textit{iff} \quad (\forall v \in Mod_{\mathrm{PC}}\Phi)\,(\exists w \in Mod_{\mathrm{PC}}\Psi)\,[v \sqsubseteq_{\Gamma} w] \;\; \textit{and}$$
$$(\forall w \in Mod_{\mathrm{PC}}\Psi)\,(\exists v \in Mod_{\mathrm{PC}}\Phi)\,[v \sqsubseteq_{\Gamma} w]\,.$$

Theorem 4.33 below states that the order on valuations of Definitions 4.22 and 4.28 coincide. The theorylike order of Definitions 4.24 and 4.29 therefore also coincide.

Lemma 4.30 *Given any formula set* Γ *and wff* ϕ *such that* $\Gamma \models^{\natural} \phi$, $Mod_{\mathrm{PC}}\phi$ *is a* $\preceq_{\Gamma}$*-upclosed superset of* $Mod_{\mathrm{PC}}\Gamma$.

Proof. Let $\Gamma \models^{\natural} \phi$. Then $Mod_{\mathrm{PC}}\Gamma \subseteq Mod_{\mathrm{PC}}\phi$. Let $u \in Mod_{\mathrm{PC}}\phi$ and $u \preceq_{\Gamma} w$. We have to show that $w \in Mod_{\mathrm{PC}}\phi$. By Definition 4.16, $(\forall p_i \in \Gamma_{+})$ [if $u(p_i) = t$ then $w(p_i) = t$] and $(\forall p_i \in \Gamma_{-})$ [if $u(p_i) = f$ then $w(p_i) = f$]. By Definition 4.28, $\phi_{+} \subseteq \Gamma_{+}$ and $\phi_{-} \subseteq \Gamma_{-}$. Therefore, $(\forall p_i \in \phi_{+})$ [if $u(p_i) = t$ then $w(p_i) = t$] and $(\forall p_i \in \phi_{-})$ [if $u(p_i) = f$ then $w(p_i) = f$]. That is, $u \preceq_{\phi} w$. It follows from Lemma 4.21 that that $w \in Mod_{\mathrm{PC}}\phi$. □

Lemma 4.31 *Every consequence of* Γ *whose meaning is* $\preceq_{\Gamma}$*-upclosed, is a natural consequence of* Γ.

Proof. Let ϕ be any consequence of Γ such that $Mod_{\mathrm{PC}}\phi$ is a $\preceq_{\Gamma}$-upclosed set. Suppose ϕ is not a natural consequence of Γ. Then either $\phi_{+} \not\subseteq \Gamma_{+}$ or $\phi_{-} \not\subseteq \Gamma_{-}$. Say $\phi_{+} \not\subseteq \Gamma_{+}$, then there exists some p_i such that Γ is isotone in p_i but ϕ is not. Since ϕ is not isotone in p_i, there exist valuations u and w such that $u \equiv_i w$ and $u(p_i) < w(p_i)$ and $u \in Mod_{\mathrm{PC}}\phi$ and $w \notin Mod_{\mathrm{PC}}\phi$. Since $Mod_{\mathrm{PC}}\phi$ is $\preceq_{\Gamma}$-upclosed, $u \not\preceq_{\Gamma} w$. It follows from the definition of $\preceq_{\Gamma}$ that $(\exists p_j \in \Gamma_{+})\,[u(p_j) = t$ and $w(p_j) = f]$ or $(\exists p_j \in \Gamma_{-})\,[u(p_j) = f$ and $w(p_j) = t]$. Since $u \equiv_i w$, $i = j$, and since $u(p_i) < w(p_i)$, $p_i \in \Gamma_{-}$, contradicting the fact that Γ is isotone in p_i. Therefore ϕ must be a natural consequence of Γ. The proof if $\phi_- \not\subseteq \Gamma_-$ is similar. $\square$

Lemma 4.32 *Every natural consequence of* Γ *satisfied by* u *is a consequence of* $\Gamma \vee^+ \Gamma_u$.

Proof. Let $\Gamma \sqsubseteq^\natural \phi$. By Lemma 4.30, $Mod_{\text{PC}}\phi$ is a $\preceq_\Gamma$-upclosed set. Since $\Gamma \models \phi$, $Mod_{\text{PC}}\Gamma \subseteq Mod_{\text{PC}}\phi$. Further, $u \in Mod_{\text{PC}}\Gamma \subseteq Mod_{\text{PC}}\phi$, and Γ_u is the $\preceq_\Gamma$-upclosure of u, therefore $Mod_{\text{PC}}\Gamma_u \subseteq Mod_{\text{PC}}\phi$. Thus $Mod_{\text{PC}}\Gamma \cup Mod_{\text{PC}}\Gamma_u = Mod_{\text{PC}}(\Gamma \vee^+ \Gamma_u) \subseteq Mod_{\text{PC}}\phi$. Hence $\Gamma \vee^+ \Gamma_u \models \phi$. $\square$

Theorem 4.33 *For any formula set* Γ *and valuations* u *and* w, $u \leq_\Gamma w$ *iff* $u \sqsubseteq_\Gamma w$.

Proof. Suppose $u \leq_\Gamma w$, that is, $w \in Mod_{\text{PC}}\Gamma \cup Mod_{\text{PC}}\Gamma_u$. Let ϕ be a natural consequence of Γ satisfied by u. Then $Mod_{\text{PC}}\Gamma \cup Mod_{\text{PC}}\Gamma_u \subseteq Mod_{\text{PC}}\phi$ by Lemma 4.32. Therefore $w \in Mod_{\text{PC}}\phi$. So $u \sqsubseteq_\Gamma w$.

Conversely, suppose $u \sqsubseteq_\Gamma w$. Let $\psi \in \Gamma \vee^+ \Gamma_u$. Then $\Gamma \sqsubseteq^\natural \psi$ and $u \in Mod_{\text{PC}}(\Gamma_u) \subseteq Mod_{\text{PC}}(\Gamma \vee^+ \Gamma_u) \subseteq Mod_{\text{PC}}\psi$. Therefore $w \in Mod_{\text{PC}}\psi$. Hence $w \in Mod_{\text{PC}}(\Gamma \vee^+ \Gamma_u)$. $\square$

Consequently, instead of considering all the natural consequences of a formula set as required in Definition 4.28, we need only consider the logically strongest natural consequence of Γ, namely $\Gamma \vee^+ \Gamma_u$, when determining which of two valuations are in closer agreement with Γ. This eliminates the need to calculate all the natural consequences of Γ, as well as check whether every natural consequence of Γ satisfied by u is also satisfied by w. Instead, we only have to check whether $u \in Mod_{\text{PC}}(\Gamma \vee^+ \Gamma_u)$ implies $w \in Mod_{\text{PC}}(\Gamma \vee^+ \Gamma_u)$.

Burger & Heidema (1994) describe the truthlike order on formula sets of Definition 4.11 in terms of two closure operators $\uparrow$ and $\downarrow$ on $\mathcal{P}(Val_2)$.

The upclosure of Φ, written $\uparrow \Phi$, is the $\leq_t$-upclosure of $Mod_{\mathrm{PC}}\Phi$, and the downclosure of Φ, written $\downarrow \Phi$, is the $\leq_t$-downclosure of $Mod_{\mathrm{PC}}\Phi$. That is,

$$\begin{aligned} \uparrow \Phi &= \{v : (\exists u \in Mod_{\mathrm{PC}}\Phi)\,[u \leq_t v]\}\,; \\ \downarrow \Phi &= \{u : (\exists v \in Mod_{\mathrm{PC}}\Phi)\,[u \leq_t v]\}\,. \end{aligned}$$

The next theorem shows that, if Φ is finite, then the sets $\uparrow \Phi$ and $\downarrow \Phi$ are the meanings of some formula sets, denoted by $\triangle\Phi$ and $\triangledown\Phi$ respectively. That is, $Mod_{\mathrm{PC}} \triangle \Phi = \uparrow \Phi$ and $Mod_{\mathrm{PC}} \triangledown \Phi = \downarrow \Phi$. A finite formula set is logically equivalent to the conjunction of its elements - any discussion of finite formula sets may therefore be formulated in terms of single wffs.

Theorem 4.34 *(Burger & Heidema, 1994) The upclosure and downclosure of any given wff ϕ in disjunctive normal form are the respective meanings of wffs $\triangle\phi$ and $\triangledown\phi$, that are obtained as follows:*

1. *If ϕ is a contradiction, then so are $\triangle\phi$ and $\triangledown\phi$. Else:*
2. *Replace all negative literals in ϕ with the tautology to obtain $\triangle\phi$.*
3. *Replace all positive literals in ϕ with the tautology to obtain $\triangledown\phi$.* □

Call a wff *positive* if it can be written in disjunctive normal form using only positive literals, and *negative* if it can be written in disjunctive normal form using only negative literals. Burger & Heidema (1994) show that $\triangle\phi$ is the logically strongest positive wff that is logically weaker than ϕ. That is, $\phi \models_{\mathrm{PC}} \triangle\phi$ and for any positive ψ such that $\phi \models_{\mathrm{PC}} \psi$, $\triangle\phi \models_{\mathrm{PC}} \psi$. $\triangledown\phi$ is the logically strongest negative wff that is logically weaker than ϕ. That is, $\phi \models_{\mathrm{PC}} \triangledown\phi$ and for any negative ψ such that $\phi \models_{\mathrm{PC}} \psi$, $\triangledown\phi \models_{\mathrm{PC}} \psi$.

Theorem 4.35 *(Burger & Heidema, 1994) For any finite formula sets Φ and Ψ,*

$$\begin{aligned}\Phi \leq_t^+ \Psi \quad & \textit{iff} \quad Mod_{\mathrm{PC}}\Phi \subseteq Mod_{\mathrm{PC}} \bigtriangledown \Psi \textit{ and } Mod_{\mathrm{PC}}\Psi \subseteq Mod_{\mathrm{PC}} \bigtriangleup \Phi \\ & \textit{iff} \quad Mod_{\mathrm{PC}} \bigtriangledown \Phi \subseteq Mod_{\mathrm{PC}} \bigtriangledown \Psi \textit{ and } Mod_{\mathrm{PC}} \bigtriangleup \Psi \subseteq Mod_{\mathrm{PC}} \bigtriangleup \Phi. \quad \square\end{aligned}$$

The order of Definition 4.24 can similarly be described syntactically, in terms of closure operators. The closure operators become parameterized, since the order of Definition 4.16 is with respect to some formula set.

Definition 4.36 *For any formula sets Γ and Φ, let*

$$\begin{aligned}\uparrow_\Gamma \Phi &= \{v : (\exists u \in Mod_{\mathrm{PC}}\Phi)\,[u \leq_\Gamma v]\}\,; \\ \downarrow_\Gamma \Phi &= \{u : (\exists v \in Mod_{\mathrm{PC}}\Phi)\,[u \leq_\Gamma v]\}\,.\end{aligned}$$

Lemma 4.37 *$\uparrow_\Gamma$ and $\downarrow_\Gamma$ are closure operators.*

Proof. The proof follows from the reflexivity and transitivity of $\leq_\Gamma$. $\square$

The next theorem guarantees the existence of formula sets $\bigtriangleup_\gamma\phi$ and $\bigtriangledown_\gamma\phi$ such that $Mod_{\mathrm{PC}} \bigtriangleup_\gamma \phi = \uparrow_\gamma \phi$ and $Mod_{\mathrm{PC}} \bigtriangledown_\gamma \phi = \downarrow_\gamma \phi$.

Theorem 4.38 *Given any wffs ϕ and γ in disjunctive normal form, $\uparrow_\gamma \phi$ and $\downarrow_\gamma \phi$ are the meanings of some formula sets $\bigtriangleup_\gamma\phi$ and $\bigtriangledown_\gamma\phi$, that are obtained as follows:*

1. *If ϕ is a contradiction, then so are $\bigtriangleup_\gamma\phi$ and $\bigtriangledown_\gamma\phi$. Else:*

2. *Replace every positive literal in ϕ in which γ is antitone but ϕ not, with the tautology. Replace further every negative literal in which γ is isotone but ϕ not, with the tautology, to obtain $\bigtriangleup_\gamma\phi$.*

3. Replace every positive literal in ϕ in which γ is isotone but ϕ not, with the tautology. Replace further every negative literal in which γ is antitone but ϕ not, with the tautology, to obtain $\bigtriangledown_\gamma\phi$.

Proof. If ϕ is a contradiction, the result is immediate. For the remainder of the proof we assume that ϕ is consistent. Let ξ be the wff obtained in (2) above. I will first show that $\uparrow_\gamma \phi \subseteq Mod_{\mathrm{PC}}\xi$, and then that $Mod_{\mathrm{PC}}\xi \subseteq \uparrow_\gamma \phi$. Let $w \in \uparrow_\gamma \phi$. Then $\exists u \in Mod_{\mathrm{PC}}\phi$ with $u \preceq_\gamma w$. Hence,

$$(\forall p_i \in \gamma_+)\,[\text{if } u\,(p_i) = t \text{ then } w\,(p_i) = t] \text{ and}$$
$$(\forall p_j \in \gamma_-)\,[\text{if } u\,(p_j) = f \text{ then } w\,(p_j) = f]\,.$$

In the formation of ξ, all the positive occurrences of sentential symbols in which ϕ is not antitone and γ is antitone, were removed. Thus ξ is antitone in these symbols. ξ is in disjunctive normal form because ϕ is. Therefore $\xi_+ \subseteq \gamma_+$. Hence,

$$(\forall p_i \in \xi_+)\,[\text{if } u\,(p_i) = t \text{ then } w\,(p_i) = t] \text{ and}$$
$$(\forall p_j \in \xi_-)\,[\text{if } u\,(p_j) = f \text{ then } w\,(p_j) = f]\,.$$

In the formation of ξ, literals are replaced by the tautology. ξ is therefore logically weaker than ϕ. Therefore $u \in Mod_{\mathrm{PC}}\phi \subseteq Mod_{\mathrm{PC}}\xi$. ξ is a natural consequence of itself, so it follows from Lemma 4.30 that $Mod_{\mathrm{PC}}\xi$ is $\preceq_\xi$-upclosed. Hence $w \in Mod_{\mathrm{PC}}\xi$, and $\uparrow_\gamma \phi \subseteq Mod_{\mathrm{PC}}\xi$.

Next, suppose $\phi = \delta_1 \vee \ldots \vee \delta_n$, where each disjunct δ_i is some primitive conjunction, say $\delta_i = l_1 \wedge \ldots \wedge l_m$. Consider any such conjunction δ_i. Replace any positive literal $+p_j$ in δ_i in which γ is antitone and in which ϕ is not antitone, with the tautology, to form ϕ'. Let $w \in Mod_{\mathrm{PC}}\phi'$. If $w \in Mod_{\mathrm{PC}}\phi$ then $w \in \uparrow_\gamma \phi$. Otherwise, if $w \notin Mod_{\mathrm{PC}}\phi$, then w is j-equivalent to some $u \in Mod_{\mathrm{PC}}\delta_i \subseteq Mod_{\mathrm{PC}}\phi$. Since $p_j \notin \gamma_+$ and $u\,(p_j) = t$, $u \preceq_\gamma w$. Therefore

$w \in \uparrow_\gamma \phi$. The argument for the deletion of a negative literal is the same. ξ is formed by a finite number of such deletions, and each deletion preserves the property $Mod_{\text{PC}}\phi' \subseteq \uparrow_\gamma \phi$, where ϕ' is a wff obtained ϕ by a single deletion from a formula obtained by (zero or more) previous deletions from ϕ. Hence $Mod_{\text{PC}}\xi \subseteq \uparrow_\gamma \phi$.

The proof of (3) above is similar to the proof of (2). □

Theorem 4.39 below shows that the theorylike order on formula sets of Definition 4.24 can be described in terms of the up- and downclosure operators of Definition 4.36. The theorem does not hold if Φ is contradictory, and is therefore formulated and proved only for finite, consistent formula sets.

Theorem 4.39 *For any finite, consistent formula sets Φ, Ψ, and Γ,*

$$\Phi \leq_\Gamma^+ \Psi \quad \textit{iff} \quad (Mod_{\text{PC}}\Psi \cap Mod_{\text{PC}}\Gamma \neq \emptyset \textit{ or } Mod_{\text{PC}} \bigtriangledown_\Gamma \Phi \subseteq Mod_{\text{PC}} \bigtriangledown_\Gamma \Psi) \textit{ and}$$
$$(Mod_{\text{PC}} \bigtriangleup_\Gamma \Psi \subseteq Mod_{\text{PC}} \bigtriangleup_\Gamma \Phi \cup Mod_{\text{PC}}\Gamma)\,.$$

Proof. Left to right: Supppose $\Phi \leq_\Gamma^+ \Psi$ and $Mod_{\text{PC}}\Psi \cap Mod_{\text{PC}}\Gamma = \emptyset$. Let $u \in Mod_{\text{PC}} \bigtriangledown_\Gamma \Phi$. Then $\exists v \in Mod_{\text{PC}}\Phi$ with $u \preceq_\Gamma v$. Since $\Phi \leq_\Gamma^+ \Psi$ by assumption, $\exists w \in Mod_{\text{PC}}\Psi$ with $v \leq_\Gamma w$. That is, $w \in Mod_{\text{PC}}\Gamma \cup Mod_{\text{PC}}\Gamma_v$. Since $Mod_{\text{PC}}\Psi \cap Mod_{\text{PC}}\Gamma = \emptyset$ by assumption, $w \notin Mod_{\text{PC}}\Gamma$. Thus $w \in Mod_{\text{PC}}\Gamma_v$, that is, $v \preceq_\Gamma w$. By Lemma 4.18, $u \preceq_\Gamma w$. Hence $u \in Mod_{\text{PC}} \bigtriangledown_\Gamma \Psi$.

Next, suppose $\Phi \leq_\Gamma^+ \Psi$, and let $w \in Mod_{\text{PC}} \bigtriangleup_\Gamma \Psi$. Then $\exists v \in Mod_{\text{PC}}\Psi$ with $v \preceq_\Gamma w$. So $\exists u \in Mod_{\text{PC}}\Phi$ with $u \leq_\Gamma v$. Thus $u \preceq_\Gamma v$ or $v \in Mod_{\text{PC}}\Gamma$. If $u \preceq_\Gamma v$ then $u \preceq_\Gamma w$ by Lemma 4.18, and therefore $w \in Mod_{\text{PC}} \bigtriangleup_\Gamma \Phi$. Else $v \in Mod_{\text{PC}}\Gamma$, and hence, by Lemma 4.21, so is w. Therefore $w \in Mod_{\text{PC}} \bigtriangleup_\Gamma \Phi \cup Mod_{\text{PC}}\Gamma$.

Right to left: Conversely, suppose that either $Mod_{\text{PC}}\Psi \cap Mod_{\text{PC}}\Gamma \neq \emptyset$ or $Mod_{\text{PC}} \bigtriangledown_\Gamma \Phi \subseteq Mod_{\text{PC}} \bigtriangledown_\Gamma \Psi$, and that $Mod_{\text{PC}} \bigtriangleup_\Gamma \Psi \subseteq Mod_{\text{PC}} \bigtriangleup_\Gamma \Phi \cup Mod_{\text{PC}}\Gamma$.

Let $u \in Mod_{PC}\Phi$. If $Mod_{PC}\Psi \cap Mod_{PC}\Gamma \neq \emptyset$ then $\exists v \in Mod_{PC}\Psi \cap Mod_{PC}\Gamma$. Since $v \in Mod_{PC}\Gamma$, $u \leq_\Gamma v$. Else $Mod_{PC}\Psi \cap Mod_{PC}\Gamma = \emptyset$ and $Mod_{PC} \bigtriangledown_\Gamma \Phi \subseteq Mod_{PC} \bigtriangledown_\Gamma \Psi$. So $u \in Mod_{PC}\Phi \subseteq Mod_{PC} \bigtriangledown_\Gamma \Phi \subseteq Mod_{PC} \bigtriangledown_\Gamma \Psi$. Therefore $\exists v \in Mod_{PC}\Psi$ with $u \preceq_\Gamma v$. Hence $u \leq_\Gamma v$.

Now let $w \in Mod_{PC}\Psi$. Since $Mod_{PC} \bigtriangleup_\Gamma \Psi \subseteq Mod_{PC} \bigtriangleup_\Gamma \Phi \cup Mod_{PC}\Gamma$ by assumption, $w \in Mod_{PC} \bigtriangleup_\Gamma \Phi$ or $w \in Mod_{PC}\Gamma$. If $w \in Mod_{PC} \bigtriangleup_\Gamma \Phi$ then $\exists u \in Mod_{PC}\Phi$ with $u \preceq_\Gamma w$, and so $u \leq_\Gamma w$. Else $w \in Mod_{PC}\Gamma$, and $u \leq_\Gamma w$ for any $u \in Mod_{PC}\Phi$. □

Corollary 4.40 *For any finite, consistent formula sets Γ, Φ and Ψ,*

$$\Phi \leq_\Gamma^+ \Psi \quad \textit{iff} \quad Cn\left((\bigtriangleup_\Gamma\Phi) \vee^+ \Gamma\right) \subseteq Cn\left(\bigtriangleup_\Gamma\Psi\right) \ \textit{and}$$
$$\left((\bigtriangledown_\Gamma\Psi) \wedge^+ \Gamma \textit{ is consistent or } Cn\left(\bigtriangledown_\Gamma\Psi\right) \subseteq Cn\left(\bigtriangledown_\Gamma\Phi\right)\right).$$

Proof. The proof follows from Theorems 3.27 and 4.39. □

Theorem 4.35 is a special case of Theorem 4.39. For let Γ be the set of all the sentential symbols of a finitely generated propositional language. Then $Mod_{PC}\Gamma \subseteq Mod_{PC} \bigtriangleup_\Gamma \Phi$, and hence $Mod_{PC} \bigtriangleup_\Gamma \Psi \subseteq Mod_{PC} \bigtriangleup_\Gamma \Phi \cup Mod_{PC}\Gamma$ iff $Mod_{PC} \bigtriangleup_\Gamma \Psi \subseteq Mod_{PC} \bigtriangleup_\Gamma \Phi$. Further, if $Mod_{PC}\Psi \cap Mod_{PC}\Gamma \neq \emptyset$ then $\bigtriangledown_\Gamma\Psi$ is tautologous, and hence $Cn\left(\bigtriangledown_\Gamma\Psi\right) \subseteq Cn\left(\bigtriangledown_\Gamma\Phi\right)$. Therefore $((\bigtriangledown_\Gamma\Psi) \wedge^+ \Gamma$ is consistent or $Cn\left(\bigtriangledown_\Gamma\Psi\right) \vdash Cn\left(\bigtriangledown_\Gamma\Phi\right))$ iff $Cn\left(\bigtriangledown_\Gamma\Psi\right) \subseteq Cn\left(\bigtriangledown_\Gamma\Phi\right)$. The additional conditions in the formulation of Theorem 4.39 as opposed to Theorem 4.35 are needed because the order is a power relation of $\leq_\Gamma$, whereas the order used in the up- and downclosure operators, is $\preceq_\Gamma$. Special provision therefore has to be made for the fact that Γ may have models that are not $\preceq_\Gamma$-equivalent, but are $\leq_\Gamma$-equivalent.

Theorem 4.38 and Corollary 4.40 provide the following algorithm to calculate which of any two wffs ϕ and ψ is closer to a third wff γ:

1. Write ϕ, ψ and γ in disjunctive normal form.

2. Calculate γ_+, γ_-, ϕ_+, ϕ_-, ψ_+ and ψ_-.

3. Derive the wffs $\nabla_\gamma\phi$, $\nabla_\gamma\psi$, $\Delta_\gamma\phi$ and $\Delta_\gamma\psi$ as described in Theorem 4.38.

4. Check whether $Cn\,((\Delta_\Gamma\Phi) \vee^+ \Gamma) \subseteq Cn\,(\Delta_\Gamma\Psi)$ and either $(\nabla_\gamma\psi) \wedge \gamma$ is consistent or $Cn\,(\nabla_\Gamma\Psi) \subseteq Cn\,(\nabla_\Gamma\Phi)$.

In calculating the sets γ_+, γ_-, ϕ_+, ϕ_-, ψ_+ and ψ_- above, we cannot assume that literals do not appear redundantly in any wff. That is, we cannot depend upon the syntactical form of a wff when calculating its monotonicities. The existence of an efficient algorithm to determine the monotonicities of a wff is an open question.

4.4 Three-valued likeness

In both the structuralist and the power approach to verisimilitude, the absence of the complete truth assumption lead to a parameterized, theorylike order, which is defined relative to a description of an incomplete truth. Niiniluoto (1987b) suggests that situations where the incomplete truth assumption applies, be described formally in a *semantically indeterminate* language. In a propositional language, this means that wffs are allowed to be neither true nor false. Since the third truth value u of $\diamond$MCK may be interpreted as 'of indeterminate value', its semantics allows for semantic indeterminism. In this section I extend the power approach to truthlikeness to the logic $\diamond$MCK. Although $\diamond$MCK has the same axiomatization as LP, they assign a different interpretation to the third truth value. It is thus with good reason that I use the name $\diamond$MCK here, instead of LP.

This section hinges on the observation that there is a one-one correspondence between consistent sets of literals in L and three-valued valuations. An advantage of a three-valued approach to verisimilitude, is that a three-valued valuation can distinguish between variables that are *asserted*, variables that are *denied*, and variables that are *irrelevant*. Given any consistent set of literals Γ, let z_Γ be the valuation defined by:

$$z_\Gamma(p_i) = \left[\begin{array}{l} t \text{ if } p_i \in \Gamma; \\ f \text{ if } \neg p_i \in \Gamma; \\ u \text{ otherwise.} \end{array}\right]$$

Conversely, any three-valued valuation z gives rise to a consistent set of literals:

$$\Gamma_z = \{p_i : z(p_i) = t\} \cup \{\neg p_i : z(p_i) = f\}.$$

The meaning of Γ_z can be described in terms of the pointwise knowledge order on Val_3:

$$\begin{aligned} Mod_{\diamond\mathsf{MCK}}\Gamma_z &= \{v \in Val_3 : v(\Gamma_z) \subseteq \{t,u\}\} \text{ by def. 2.9 and Theorem 2.16} \\ &= \{v \in Val_3 : (\forall p_i \in \Gamma_z)[v(p_i) \in \{t,u\}] \text{ and} \\ &\quad (\forall \neg p_i \in \Gamma_z)[v(p_i) \in \{f,u\}]\} \\ &= \{v \in Val_3 : (\forall p_i)[z(p_i) \leq_k v(p_i) \text{ or } v(p_i) \leq_k z(p_i)]\}. \end{aligned}$$

Call two valuations v and w *compatible at* p_i if $v(p_i) \leq_k w(p_i)$ or $w(p_i) \leq_k v(p_i)$, and call v and w compatible if they are compatible on all sentential symbols. Then $Mod_{\diamond\mathsf{MCK}}\Gamma_z$ is the set of all valuations that are compatible with z. The symmetric difference operation of Definition 4.6 generalizes to Val_3 as follows: For any valuation $v \in Val_3$, let v_0 denote its restriction to L_0.

Definition 4.41 *For any $v, w \in Val_3$,*

$$v -_s w = \left(v_0^{-1}(t) \cap w_0^{-1}(f)\right) \cup \left(v_0^{-1}(f) \cap w_0^{-1}(t)\right).$$

Thus $v -_s w$ is the set of all p_i such that v and w are *in*compatible at p_i.

Definition 4.42 below orders three-valued valuations relative to a consistent set of literals Γ. Lemma 4.43 shows that this definition implies that the closer a valuation is to Γ, the fewer incompatibilities it has with z_Γ. The notion of relevance to Γ is captured by the fact that only the elements of Γ determine the order.

Definition 4.42 *For any $v, w, z \in Val_3$,*

$$v \leq_z w \text{ iff } (\forall \phi \in \Gamma_z)\left[v(\phi) \leq_t w(\phi)\right].$$

If $v, w, z \in Val_2$, then Definition 4.42 coincides with Definition 4.10, hence the use of the same notation in the two definitions.

Lemma 4.43 *Given any v, w and z in Val_3, if $v \leq_z w$ then $w -_s z \subseteq v -_s z$.*

Proof. Suppose $v \leq_z w$. We have to show that v is incompatible with z at p_i whenever w is. If w is incompatible with z at p_i, then $w(p_i) = t$ or $w(p_i) = f$. Suppose $w(p_i) = t$. Then $z(p_i) = f$. So $\neg p_i \in \Gamma_z$, and $w(\neg p_i) = f$. Therefore $v(\neg p_i) = f$, and hence v is incompatible with z at p_i. Now suppose $w(p_i) = f$. Then $z(p_i) = t$. So $p_i \in \Gamma_z$, and $w(p_i) = f$. Therefore $v(p_i) = f$, and hence v is incompatible with z at p_i. □

The converse of Lemma 4.43 does not hold. For example, let $\Gamma = \{p\}$, $w(p) = u$ and $v(p) = t$. Then $w -_s z_\Gamma = v -_s z_\Gamma = \emptyset$, but $v(p) \not\leq_t w(p)$. Definition 4.42 forces more informative valuations to be preferred to less informative ones. In the example just given, w is less informative than v

is, and should therefore not be preferred to it. Definition 4.42 is therefore based on two principles: it prefers valuations that are more informative, and it prefers valuations with fewer incompatibilities with z.

As in the two-valued case of Definition 4.11, the order of Definition 4.42 can be lifted to a power order on formula sets. The power order prefers Ψ to Φ iff every approximate instantial match of Φ is improved upon by some approximate instantial match of Ψ (in the sense of increasing the compatibility and informativeness), and every explanatory mistake made by Ψ is an improvement of some explanatory mistake made by Φ.

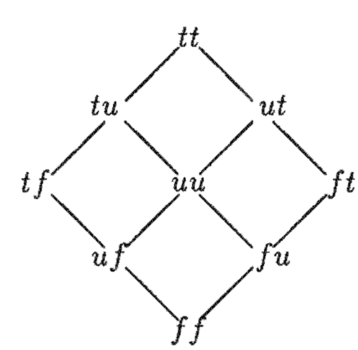
Definition 4.44 *Given any formula sets Φ and Ψ, and consistent set of literals Γ, Ψ is at least as close to Γ as Φ is, written $\Phi \leq^3_\Gamma \Psi$, iff*

(Ti′) $(\forall v \in Mod_{\diamond\mathrm{MCK}}\Phi)\,(\exists w \in Mod_{\diamond\mathrm{MCK}}\Psi)\,[v \leq_{z_\Gamma} w]$, *and*

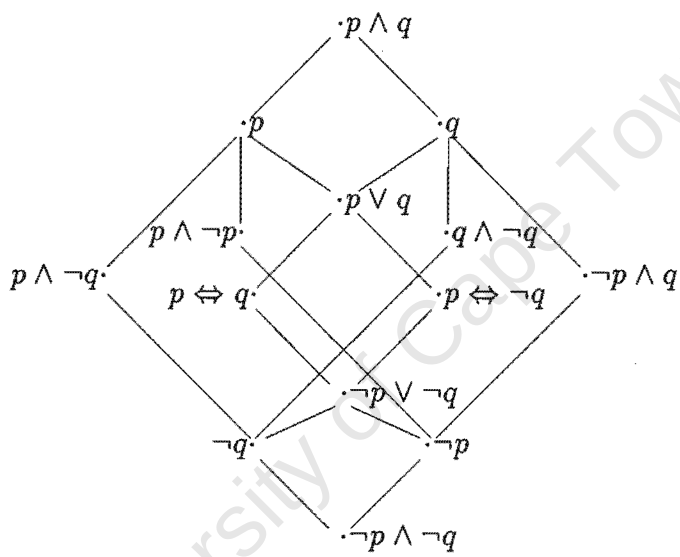
(Tii′) $(\forall w \in Mod_{\diamond\mathrm{MCK}}\Psi)\,(\exists v \in Mod_{\diamond\mathrm{MCK}}\Phi)\,[v \leq_{z_\Gamma} w]$.

This theorylike order on formula sets in ◇MCK does not coincide with the theorylike order on formula sets in PC of Definition 4.24, as the following examples show.

Example 4.45 Let $\Gamma = \{p, q\}$ in the propositional language generated by p and q. The order $\leq_{z_\Gamma}$ on valuations is given by the following Hasse diagram:

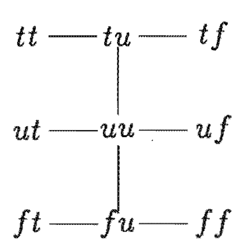


The theorylike order $\leq^3_\Gamma$ on wffs is shown below. In order to simplify the diagram, not all $\equiv_{\diamond\text{MCK}}$-equivalence classes are represented. For example, the tautologies, such as $p \vee \neg p$, are $\leq^3_\Gamma$-equivalent to $p \Leftrightarrow q$, as are $\neg p \vee q$ and $p \vee \neg q$. $p \wedge (\neg p \vee q)$ is $\leq^3_\Gamma$-equivalent to p (it is not $\equiv_{\diamond\text{MCK}}$-equivalent or $\leq^3_\Gamma$-equivalent to $p \wedge q$). Similarly, $q \wedge (\neg q \vee p)$ is $\leq^3_\Gamma$-equivalent to q, and so on.



Unlike in PC, inconsistent formula sets in $\diamond$MCK are not all equivalent to L, and unrelated by $\leq^3_\Gamma$ to every consistent formula set. For example, $p \wedge \neg p$ and $q \wedge \neg q$ are unrelated by $\leq^3_\Gamma$, whereas $p \wedge \neg p \leq^+_\Gamma q \wedge \neg q$ and $q \wedge \neg q \leq^+_\Gamma p \wedge \neg p$ vacuously. Also, $\neg p \leq^3_\Gamma p \wedge \neg p$, but, since $p \wedge \neg p$ is inconsistent, $\neg p$ and $p \wedge \neg p$ are unrelated by $\leq^+_\Gamma$. $\square$

Example 4.46 Let $\Sigma = \{p\}$ in the propositional language generated by p and q. The order on three-valued valuations is the following:



This order on valuations reflects the irrelevance of q to Σ. The resulting theorylike order $\leq^3_\Sigma$ correspondingly does not distinguish between, for example, p and $p \wedge q$, or between q and $q \wedge \neg q$. □

Definition 4.44 can be adopted for three-valued logics other than ◇MCK as well. If, for example, ◇MCK is replaced by KL, the following theorem holds.

Theorem 4.47 *Given any formula sets Φ and Ψ, and set of literals Γ, $\Phi \leq^3_\Gamma \Psi$ in* KL *iff $\Phi \leq^+_\Gamma \Psi$ in* PC.

Proof. Suppose $\Phi \leq^3_\Gamma \Psi$. Let $v \in Mod_{\text{PC}}\Phi$. Then $v \in Mod_{\text{KL}}\Phi$. So $\exists w \in Mod_{\text{KL}}\Psi$ such that $v \leq_{z_\Gamma} w$. That is, $(\forall \phi \in \Gamma)\left[v\left(p_i\right) \leq_t w\left(p_i\right)\right]$. Define w' by:

$$w'\left(p_i\right) = \begin{bmatrix} w\left(p_i\right) \text{ if } w\left(p_i\right) = t \text{ or } w\left(p_i\right) = f; \\ v\left(p_i\right) \text{ if } w\left(p_i\right) = u. \end{bmatrix}$$

Then $w \leq_k w'$ and $w' \in Val_2$, and hence $w' \in Mod_{\text{PC}}\Psi$. Also, $(\forall p_i \in \Gamma_+)$ [if $v\left(p_i\right) = t$ then $w'\left(p_i\right) = t$] and $(\forall p_i \in \Gamma_-)$ [if $v\left(p_i\right) = f$ then $w'\left(p_i\right) = f$]. Hence $v \leq_\Gamma w'$. This proves (Ti).

Next, let $w \in Mod_{\text{PC}}\Psi$. Then $w \in Mod_{\text{KL}}\Psi$. So $\exists v \in Mod_{\text{KL}}\Phi$ such that $v \leq_{z_\Gamma} w$. That is, $(\forall \phi \in \Gamma)\left[v\left(p_i\right) \leq_t w\left(p_i\right)\right]$. Define v' by:

$$v'\left(p_i\right) = \begin{bmatrix} v\left(p_i\right) \text{ if } v\left(p_i\right) = t \text{ or } v\left(p_i\right) = f; \\ w\left(p_i\right) \text{ if } v\left(p_i\right) = u. \end{bmatrix}$$

Then $v \leq_k v'$ and $v' \in Val_2$, and hence $v' \in Mod_{\text{PC}}\Phi$. Also, $(\forall p_i \in \Gamma_+)$ [if $v'\left(p_i\right) = t$ then $w\left(p_i\right) = t$] and $(\forall p_i \in \Gamma_-)$ [if $v'\left(p_i\right) = f$ then $w\left(p_i\right) = f$]. Hence $v' \leq_\Gamma w$. Therefore $\Phi \leq^+_\Gamma \Psi$. This proves (Tii).

Conversely, suppose $\Phi \leq^+_\Gamma \Psi$. Let $v \in Mod_{\text{KL}}\Phi$. Define v' by:

$$v'\left(p_i\right) = \begin{bmatrix} v\left(p_i\right) \text{ if } v\left(p_i\right) = t \text{ or } v\left(p_i\right) = f; \\ z_\Gamma\left(p_i\right) \text{ if } v\left(p_i\right) = u \text{ and } z_\Gamma\left(p_i\right) \neq u; \\ t \text{ if } v\left(p_i\right) = u \text{ and } z_\Gamma\left(p_i\right) = u. \end{bmatrix}$$

Then $v \leq_k v'$ and $v' \in Mod_{\text{PC}}\Phi$ and $v \leq_{z_\Gamma} v'$. Since $\Phi \leq_\Gamma^+ \Psi$, $\exists w \in Mod_{\text{PC}}\Psi$ such that $v' \leq_\Gamma w$. Therefore $v' \leq_{z_\Gamma} w$, and hence $v \leq_{z_\Gamma} w$. Since $w \in Mod_{\text{KL}}\Psi$, this proves (Ti′).

Finally, let $w \in Mod_{\text{KL}}\Psi$. Define w' by:

$$w'(p_i) = \begin{bmatrix} w(p_i) \text{ if } w(p_i) = t \text{ or } w(p_i) = f; \\ z_\Gamma(\neg p_i) \text{ if } w(p_i) = u \text{ and } z_\Gamma(p_i) \neq u; \\ t \text{ if } w(p_i) = u \text{ and } z_\Gamma(p_i) = u. \end{bmatrix}$$

Then $w \leq_k w'$ and $w' \in Mod_{\text{PC}}\Psi$ and $w' \leq_{z_\Gamma} w$. So $\exists v \in Mod_{\text{PC}}\Phi$ such that $v \leq_\Gamma w'$. Therefore $v \leq_{z_\Gamma} w'$, and hence $v \leq_{z_\Gamma} w$. Since $v \in Mod_{\text{KL}}\Phi$, this proves (Tii′). □

I conclude this section by suggesting an application of the order of Definition 4.44. As explained above, a three-valued valuation z represents a consistent set of literals, which may in turn represent the observed outcome of some experiment. An outcome does not provide complete information, thus each p_i may be true, false or unknown. The experiment may be repeated, producing different results, forming a set of observations Γ. This set is construed disjunctively, as a co-system. This is where it differs from the formula sets considered thus far - empirically possible worlds are construed syntactically, not model-theoretically. The question then is which theory approximation is most likely to have the range of elements of Γ as possible outcome. Different theories may be formalized and tested against the observed outcome. Although none may produce Γ exactly, they can be ordered according to their informativeness and their incompatibilities with elements of Γ. Since $\diamond$MCK is paraconsistent, inconsistencies in theory approximations are accommodated naturally, unlike in classical logic.

Chapter 5

Theory change

Logics of *theory change* or *belief revision*, as put forward initially by Alchourrón, Gärdenfors & Makinson (1985), and hence called the AGM approach, deal with changing information. A change operation operates on a deductive system and a wff, and yields another deductive system. The three kinds of changes considered are *expansion* (adding new information), *revision* (adding new information while retaining consistency) and *contraction* (removing information, while at the same time retaining as much information as possible). The behaviour of these operations are characterized by sets of postulates.

Various representation results exist for logics of belief revision, including representations in terms of remainder sets (Alchourrón & Makinson, 1981; Alchourrón *et al.*, 1985), epistemic entrenchment (Gärdenfors, 1984; Gärdenfors & Makinson, 1988), systems of spheres (Grove, 1988), and preferential models (Katsuno & Mendelzon, 1991). In this chapter I present another representation of logics of belief revision, as a *calculus of belief sets.* A calculus of belief sets is obtained by enriching a calculus of deductive systems, as defined in 3.11, with certain a set of unary operations, indexed by elements of L.

The calculus of belief sets is isomorphic to an *algebra of theory change*, obtained by enriching the algebra of theories of Definition 3.19 with a corresponding indexed set of unary operations.The theory change operations arise as power operations of the conjunction and disjunction connectives of the underlying logic. Expansion is characterized as a power disjunction and contraction as a power conjunction operation.

The next section gives the necessary background to logics of belief revision. A number of calculi of belief sets are presented in Sections 5.2 and 5.3.

5.1 The AGM approach

The AGM approach to theory change deals with ways of incorporating belief changes into a *belief set*. Belief sets are taken to be deductively closed sets of formulas, that is, they are the Tarskian deductive systems of Definition 3.11. Let $\mathcal{B}$ denote the set of belief sets:

$$\mathcal{B} = \{Cn(\Gamma) : \Gamma \subseteq L\}.$$

In logics of belief revision, the idea is to accommodate changes to a belief set in such a way that consistency is retained. The consequence operator Cn of the logic is assumed to satisfy the four conditions of Definition 1.19, as well as:

5. If $\beta \in Cn(\Gamma, \alpha)$ then $\alpha \supset \beta \in Cn(\Gamma)$. (deduction)

A number of additional assumptions are also made about the connectives. It is assumed that $\wedge$, $\vee$, $\neg$, $\supset$, and $\leftrightarrow$ are available in the logic, and that they behave as Boolean connectives in the following sense:

1. $\wedge$ and $\vee$ distribute over each other.

2. The De Morgan properties hold.

3. $\supset$ is material implication.

4. $\leftrightarrow$ is the biconditional defined from material implication.

5. For any wff ϕ, $\phi \vee \neg\phi \in Cn(\emptyset)$ and $Cn(\phi \wedge \neg\phi) = L$.

This means that the fragment of the logic containing only the above-mentioned connectives coincides with PC, and that the corresponding Lindenbaum algebra of equivalence classes of wffs defined in Section 1.1, is the Boolean algebra $\mathcal{L}_{\text{PC}}$. It also means that the logic is explosive, and that the only inconsistent belief set is the universal set L. Unlike in paraconsistent logics, inconsistencies cannot be accommodated in non-trivial belief sets. Logics of belief revision address this problem by avoiding inconsistencies altogether.

The three kinds of belief changes considered are expansion, contraction and revision. $\Gamma + \phi$ reads "Γ expanded with ϕ", $\Gamma * \phi$ reads "Γ revised with ϕ", and $\Gamma - \phi$ reads "Γ contracted with ϕ". The expansion of a belief set with a wff is defined uniquely: For any belief set Γ and wff ϕ,

$$\Gamma + \phi = Cn(\Gamma \cup \{\phi\}).$$

Expansion does not generally retain consistency in a belief set. If $\phi \in \Gamma$, then $\Gamma + \neg\phi$ is inconsistent. This raises the question of how a belief set Γ should be expanded by a wff ϕ, while at the same time retaining consistency. An operation intended as an answer to this question, is called a revision operation. A set of postulates captures the properties usually required of a revision operation. The AGM postulates for revision are:

K$*$1 Closure: For every ϕ, $\Gamma * \phi$ is a belief set.

K*2 Success: $\phi \in \Gamma * \phi$.

K*3 Expansion1: $\Gamma * \phi \subseteq \Gamma + \phi$.

K*4 Expansion2: If $\neg\phi \notin \Gamma$ then $\Gamma + \phi \subseteq \Gamma * \phi$.

K*5 Consistency preservation: $\Gamma * \phi$ is inconsistent iff $\vdash \neg\phi$.

K*6 Extensionality: If $\vdash \phi \leftrightarrow \psi$ then $\Gamma * \phi = \Gamma * \psi$.

K*7 Conjunction 1: $\Gamma * (\phi \wedge \psi) \subseteq (\Gamma * \phi) + \psi$.

K*8 Conjunction 2: If $\neg\psi \notin \Gamma * \phi$ then $(\Gamma * \phi) + \psi \subseteq \Gamma * (\phi \wedge \psi)$.

These postulates are discussed in (Alchourrón *et al.*, 1985), as well as in survey papers on belief revision, such as (Gärdenfors & Rott, 1995).

Contracting a belief set Γ with a wff ϕ amounts to a minimal weakening of Γ such that ϕ can no longer be derived. The AGM postulates K-1 to K-8 provide a yardstick with which to compare and categorize contraction operations. They are:

K-1 Closure: For every ϕ, $\Gamma - \phi$ is a belief set.

K-2 Inclusion: $\Gamma - \phi \subseteq \Gamma$.

K-3 Vacuity: If $\phi \notin \Gamma$ then $\Gamma - \phi = \Gamma$.

K-4 Success: If $\nvdash \phi$ then $\phi \notin \Gamma - \phi$.

K-5 Recovery: $\Gamma \subseteq (\Gamma - \phi) + \phi$.

K-6 Extensionality: If $\vdash \phi \leftrightarrow \psi$ then $\Gamma - \phi = \Gamma - \psi$.

K-7 Conjunction 1: $(\Gamma - \phi \cap \Gamma - \psi) \subseteq \Gamma - (\phi \wedge \psi)$.

K-8 Conjunction 2: If $\phi \notin \Gamma - (\phi \wedge \psi)$ then $\Gamma - (\phi \wedge \psi) \subseteq \Gamma - \phi$.

A standard way to define a revision operation in terms of a given contraction operation, is by means of the *Levi identity*. The revision of a belief set Γ with a wff ϕ is obtained by retracting the beliefs in Γ that are inconsistent with ϕ, and expanding the resulting belief set with ϕ:

$$\Gamma * \phi = (\Gamma - \neg\phi) + \phi.$$

Theorem 5.1 *(Alchourrón* et al., *1985) If the Levi identity is used to define a revision operation from a contraction operation which satisfies K-1 to K-4 and K-6, the revision operation thus obtained satisfies K*1 to K*6. If, in addition, the contraction operation also satisfies K-7, the revision operation satisfies K*7, and if the contraction operation also satisfies K-8, the revision operation satisfies K*8.* □

5.2 Algebra of theory change

Theorem 3.20 states that the positive calculus of systems $(\mathcal{B}, \cdot, \dot{+})$ of Definition 3.11 is isomorphic to the power algebra of theories $(P(L) / \equiv_{Cn}, \wedge^+, \vee^+)$ of Definition 3.19. It is more convenient notationally to work with elements of $\mathcal{P}(L) / \equiv_{Cn}$ than with elements of $\mathcal{B}$, but, noting that one can move freely between the two notations, I will usually talk about 'a belief set', instead of 'a $\equiv_{Cn}$-equivalence class of wffs'. I will show that, if the calculus of systems is enriched by a certain set of unary operations, the power conjunction and disjunction operations can be viewed as theory change operations: expansion is characterized by a power disjunction and contraction by a power conjunction. This yields a calculus of belief sets, isomorphic to an algebra of theory change.

Alchourrón, Gärdenfors & Makinson (1985) describe the contraction of a belief set Γ with a wff ϕ syntactically in terms of the maximal subsets of

Γ that fail to imply ϕ. I will show that the power disjunction of belief sets yields an algebraic description of contraction. To contract Γ with ϕ, a belief set containing $\neg\phi$ is picked out by a *rejection function*, and disjoined to Γ.

Definition 5.2 *A rejection function is a function $R : \mathcal{B} \times L \to \mathcal{B}$ such that, for any belief set Γ and wff ϕ:*

1. *$R(\Gamma, \phi)$ is a belief set containing $\neg\phi$.*
2. *$R(\Gamma, \phi)$ is inconsistent iff $\phi \in Cn(\emptyset)$.*
3. *If $\phi \notin \Gamma$ then $\Gamma \subseteq R(\Gamma, \phi)$.*
4. *If $\vdash \phi \leftrightarrow \psi$ then $R(\Gamma, \phi) = R(\Gamma, \psi)$.*

The rejection function may also be viewed as a set of unary rejection operations on $\mathcal{B}$, indexed by elements of L, that is $\{R_\phi : \phi \in L\}$. If $\phi \in \Gamma$ then $R(\Gamma, \phi)$ is a belief set which has no models in common with Γ. Any subset S of Γ containing ϕ, and any subset S' of $R(\Gamma, \phi)$ containing $\neg\phi$, also have no models in common, since $Mod\Gamma \subseteq ModS \subseteq Mod\phi$ and $Mod(R(\Gamma, \phi)) \subseteq ModS' \subseteq Mod\neg\phi$. Disjoining Γ and $R(\Gamma, \phi)$ yields a contraction of Γ with ϕ. This claim is supported by Theorems 5.9, 5.12 and 5.14.

Definition 5.3 *Let Γ be a belief set. Define the operation $-_R$, called the contraction defined by the rejection function R, as follows: For any wff ϕ, $\Gamma -_R \phi = \Gamma \vee^+ R(\Gamma, \phi)$.*

Lemma 5.4 *For any rejection function R, $-_R$ satisfies postulates K-1 to K-6 for contraction.*

Proof. Let Γ be a belief set and ϕ a wff.

K-1. The power disjunction of any two belief sets is a belief set, so $\Gamma \vee^+ R(\Gamma, \phi)$ is a belief set.

K-2. $\Gamma \vee^+ R(\Gamma, \phi) \geq \Gamma$ in the algebra of theories, therefore, by Theorem 3.20, $\Gamma -_R \phi \subseteq \Gamma$.

K-3. If $\phi \notin \Gamma$ then $\Gamma -_R \phi = \Gamma \vee^+ R(\Gamma, \phi) = \Gamma$.

K-4. Suppose $\phi \in \Gamma \vee^+ R(\Gamma, \phi)$. Then $\phi \in R(\Gamma, \phi)$. But $\neg\phi \in R(\Gamma, \phi)$, so $R(\Gamma, \phi)$ is inconsistent. Therefore $\phi \in Cn(\emptyset)$.

K-5. Let $\mu \in \Gamma$. Then $\mu \wedge \phi = (\mu \vee \neg\phi) \wedge \phi \in (\Gamma \vee^+ R(\Gamma, \phi)) \wedge^+ Cn(\phi)$. Therefore $\mu \in (\Gamma \vee^+ R(\Gamma, \phi)) \wedge^+ Cn(\phi) = (\Gamma \vee^+ R(\Gamma, \phi)) + \phi$.

K-6. If $\vdash \phi \leftrightarrow \psi$, then $R(\Gamma, \phi) = R(\Gamma, \psi)$ from the definition of R. □

This shows that $\Gamma \vee^+ R(\Gamma, \phi)$ can with justification be called a contraction of Γ with ϕ.

We now turn to a particular rejection function, namely the function R_p defined by:

$$R_p(\Gamma, \phi) = \begin{bmatrix} Cn(\neg\phi) \text{ if } \phi \in \Gamma; \\ \Gamma \text{ if } \phi \notin \Gamma. \end{bmatrix}$$

Lemma 5.5 *For any belief set Γ and wff ϕ, if $\phi \in \Gamma + \neg\phi$ then $\phi \in \Gamma$.*

Proof. Let $\Gamma + \neg\phi \vdash \phi$. Then $\Gamma \vdash \neg\phi \supset \phi$, so $\Gamma \vdash \phi$. □

Lemma 5.6 *For any belief set Γ and wff ϕ, if $\phi \in \Gamma$ then $\phi \in \Gamma - \neg\phi$.*

Proof. Let $\phi \in \Gamma$. If Γ is consistent, $\neg\phi \notin \Gamma$, and therefore $\Gamma - \neg\phi = \Gamma$ by K-3. If Γ is inconsistent, $\neg\phi \in \Gamma$, and therefore $\Gamma = \Gamma - \neg\phi + \neg\phi$ by K-5. The result follows from Lemma 5.5. □

Lemma 5.7 *For any belief set* Γ*, wff* ϕ *and contraction function* $-$ *satisfying K-1 to K-6,* $\Gamma - \phi \leq \Gamma \vee^{+} R_p(\Gamma, \phi) \leq \Gamma \vee^{+} Cn(\neg\phi)$.

Proof. If $\phi \notin \Gamma$, then $\Gamma - \phi = \Gamma = \Gamma \vee^{+} R_p(\Gamma, \phi) \leq \Gamma \vee^{+} Cn(\neg\phi)$. Else $\phi \in \Gamma$. Let $\mu \in \Gamma \vee^{+} Cn(\neg\phi)$. Then $\mu \in \Gamma$ and $\neg\phi \vdash \mu$. By K-5, $\Gamma - \phi + \phi = \Gamma$. Therefore $\mu \in \Gamma - \phi + \phi \subseteq \Gamma - \phi + \neg\mu$. Hence $\mu \in \Gamma - \phi$ by Lemma 5.5. □

Lemma 5.7 states that the contraction defined by R_p is the maximum (that is, it yields the smallest belief set) of all contraction functions satisfying K-1 to K-6. The contraction defined by R_p has another useful property. If its arguments are restricted to those pairs (Γ, ϕ) such that $\phi \in \Gamma$, then $-_{R_p}$ is *monotone* in both arguments, with respect to the natural order $\subseteq$ on the calculus of belief sets. This is proved in Lemma 5.8. The AGM syntax-based representation of the contraction defined by R_p is as the intersection of all the maximally consistent subsets of Γ that fail to imply ϕ, and is called the *full meet* contraction function.

Lemma 5.8 *For any belief sets* Γ *and* Ψ*:*

(i) *If* $\phi \in \Gamma$ *and* $\Gamma \subseteq \Psi$*, then* $\Gamma -_{R_p} \phi \subseteq \Psi -_{R_p} \phi$.

(ii) *If* $\phi \in \Gamma$ *and* $Cn(\phi) \subseteq Cn(\psi)$*, then* $\Gamma -_{R_p} \phi \subseteq \Gamma -_{R_p} \psi$.

Proof.

(i) If $\phi \in \Gamma$ and $\Gamma \subseteq \Psi$, then $R_p(\Gamma, \phi) = R_p(\Psi, \phi) = Cn(\neg\phi)$. So $\Gamma \vee^{+} R_p(\Gamma, \phi) \geq \Psi \vee^{+} R_p(\Psi, \phi)$.

(ii) If $Cn(\phi) \subseteq Cn(\psi)$ then $Cn(\neg\psi) \subseteq Cn(\neg\phi)$, so $\Gamma \vee^{+} R_p(\Gamma, \phi) \geq \Gamma \vee^{+} R_p(\Gamma, \psi)$. □

Theorem 5.9 *A contraction function satisfies K-1 to K-6 iff it is defined by a rejection function.*

Proof. A proof from right to left is given by Lemma 5.4. For a proof from left to right, suppose $-$ is a contraction function satisfying K-1 to K-6. Define the function $R_{-} : \mathcal{B} \times L \to \mathcal{B}$ by

$$R_{-}(\Gamma, \phi) = Cn(\neg\phi) \wedge^{+} (\Gamma - \phi).$$

We have to check that $R_{-}(\Gamma, \phi)$ is well-defined. $\neg\phi \in R_{-}(\Gamma, \phi)$ because $Cn(\neg\phi) \subseteq R_{-}(\Gamma, \phi)$. Also, if $\phi \in Cn(\emptyset)$ then $\neg\phi$ is a contradiction and so $R_{-}(\Gamma, \phi)$ is inconsistent. Else $R_{-}(\Gamma, \phi)$ is consistent since $\phi \notin \Gamma - \phi$ by K-4, and so $\phi \notin (\Gamma - \phi) + \neg\phi$ by Lemma 5.5. Finally, suppose $\phi \notin \Gamma$. Then $\Gamma = \Gamma - \phi \subseteq \Gamma - \phi \wedge^{+} Cn(\neg\phi)$. □

Lemma 5.10 *For any wff ϕ, the operations of contraction with ϕ and power disjunction with a rejection of ϕ are interdefinable.*

Proof. Let Γ be a belief set and ϕ a wff. If $-$ is a contraction operation, let R_{-} denote the rejection function defined by $-$, and if R is a rejection function, let $-_{R}$ denote the contraction defined by R.

$$\begin{aligned}
\Gamma -_{R_{-}} \phi &= \Gamma \vee^{+} R_{-}(\Gamma, \phi) \\
&= \Gamma \vee^{+} \left(Cn(\neg\phi) \wedge^{+} (\Gamma - \phi)\right) \\
&= \left(\Gamma \vee^{+} Cn(\neg\phi)\right) \wedge^{+} \left(\Gamma \vee^{+} (\Gamma - \phi)\right) \\
&= \Gamma - \phi \wedge^{+} \left(\Gamma \vee^{+} Cn(\neg\phi)\right) \\
&= \Gamma - \phi \text{ by Lemma 5.7.}
\end{aligned}$$

$$\Gamma \vee^{+} R_{-_{R}}(\phi) = \Gamma \vee^{+} \left(Cn(\neg\phi) \wedge^{+} (\Gamma -_{R} \phi)\right)$$

$$
\begin{aligned}
&= \left(\Gamma \vee^{+} Cn(\neg\phi)\right) \wedge^{+} \left(\Gamma \vee^{+} (\Gamma -_R \phi)\right) \\
&= \left(\Gamma \vee^{+} Cn(\neg\phi)\right) \wedge^{+} \left(\Gamma \vee^{+} \Gamma \vee^{+} R(\Gamma, \phi)\right) \\
&= \Gamma \vee^{+} R(\Gamma, \phi). \qquad \square
\end{aligned}
$$

The properties required of a rejection function by Definition 5.2 are sufficient to prove K-1 to K-6, which shows just how weak these postulates are. K-7 and K-8 are the only postulates that address the interaction between the connectives in L and the contraction function. In order to obtain a representation result for syntax-based contraction functions that satisfy K-1 to K-8, a *preference relation* can be used, the most familiar being the orders on the maximal subsets of Γ that fail to imply ϕ (Alchourrón *et al.*, 1985), and the epistemic entrenchment relations on elements of L (Gärdenfors, 1988). I will follow a similar route here, and place additional criteria on the rejection function R.

Definition 5.11 *A rejection function R is called a K-7 rejection function if for any belief set Γ and wffs ϕ and ψ, $R(\Gamma, \phi \wedge \psi) \leq R(\Gamma, \phi) \vee^{+} R(\Gamma, \psi)$.*

The order on elements of $\mathcal{B}$ is set containment, therefore the expression $R(\Gamma, \phi \wedge \psi) \leq R(\Gamma, \phi) \vee^{+} R(\Gamma, \psi)$ can be written $R(\Gamma, \phi) \cap R(\Gamma, \psi) \subseteq R(\Gamma, \phi \wedge \psi)$. Definition 5.11 goes part of the way in turning conjunctions in L into set intersections in $\mathcal{B}$. Any wff which is rejected both when rejecting ϕ and when rejecting ψ, must also be rejected when rejecting $\phi \wedge \psi$.

Theorem 5.12 *A contraction function satisfies K-1 to K-7 iff it is defined by a K-7 rejection function.*

Proof. Right to left: Let R be a K-7 rejection function. By Theorem 5.9, the contraction function defined by R satisfies K-1 to K-6. To prove K-7, let $\mu \in \Gamma -_R \phi \cap \Gamma -_R \psi$, that is, $\mu \in (\Gamma \vee^+ R(\Gamma, \phi)) \cap (\Gamma \vee^+ R(\Gamma, \psi))$. Then $\mu = \gamma \vee \alpha_1 \vee \alpha_2$ for some $\gamma \in \Gamma$, $\alpha_1 \in R(\Gamma, \phi)$ and $\alpha_2 \in R(\Gamma, \psi)$. So $\alpha_1 \vee \alpha_2 \in R(\Gamma, \phi) \vee^+ R(\Gamma, \psi) \geq R(\Gamma, \phi \wedge \psi)$. Hence $\mu \in \Gamma \vee^+ R(\Gamma, \phi \wedge \psi) = \Gamma -_R (\phi \wedge \psi)$.

Conversely, let $-$ be a contraction function satisfying K-1 to K-7. Define a canonical K-7 rejection function R_- as in Theorem 5.9: $R_-(\Gamma, \phi) = Cn(\neg\phi) \wedge^+ (\Gamma - \phi)$.

$$
\begin{aligned}
& R_-(\Gamma, \phi) \vee^+ R_-(\Gamma, \psi) \\
= \; & \big(Cn(\neg\phi) \wedge^+ (\Gamma - \phi)\big) \vee^+ \big(Cn(\neg\psi) \wedge^+ (\Gamma - \psi)\big) \\
= \; & \left(\begin{array}{l} Cn(\neg\phi \vee \neg\psi) \wedge^+ ((\Gamma - \phi) \vee^+ (\Gamma - \psi)) \wedge^+ \\ (Cn(\neg\phi) \vee^+ (\Gamma - \psi)) \wedge^+ (Cn(\neg\psi) \vee^+ (\Gamma - \phi)) \end{array} \right) \\
= \; & Cn(\neg(\phi \wedge \psi)) \wedge^+ \big((\Gamma - \phi) \vee^+ (\Gamma - \psi)\big) \text{ by Lemma 5.7} \\
\subseteq \; & Cn(\neg(\phi \wedge \psi)) \wedge^+ (\Gamma - (\phi \wedge \psi)) \text{ by K-7} \\
= \; & R_-(\Gamma, \phi \wedge \psi).
\end{aligned}
$$

Therefore R_- is a well-defined K-7 rejection function, which proves the theorem. □

Definition 5.13 *A rejection function R is called a K-8 rejection function if, for any belief set Γ and wffs ϕ and ψ, $\phi \notin R(\Gamma, \phi \wedge \psi)$ implies $R(\Gamma, \phi) \leq R(\Gamma, \phi \wedge \psi)$.*

The definition says that, if ϕ is rejected when rejecting $\phi \wedge \psi$, then any other wff rejected when rejecting $\phi \wedge \psi$, must also be rejected when rejecting ϕ. Definitions 5.11 and 5.13 work together to turn conjunctions in L into set intersection in $\mathcal{B}$. For suppose R is both a K-7 and a K-8 rejection function, and both $\phi \notin R(\Gamma, \phi \wedge \psi)$ and $\psi \notin R(\Gamma, \phi \wedge \psi)$. Then $R(\Gamma, \phi) \cap R(\Gamma, \psi) = R(\Gamma, \phi \wedge \psi)$, that is, $R(\Gamma, \phi \wedge \psi) = R(\Gamma, \phi) \vee^+ R(\Gamma, \psi)$. It will of course not always be the case that both $\phi \notin R(\Gamma, \phi \wedge \psi)$ and $\psi \notin R(\Gamma, \phi \wedge \psi)$, but at the least the combination of Definitions 5.11 and 5.13 serve to sandwich the rejection of a conjunction between the intersection of the conjuncts on the one hand, and one of the conjuncts on the other hand.

Theorem 5.14 *A contraction function satisfies K-1 to K-6 and K-8 iff it is defined by a K-8 rejection function.*

Proof. Right to left: Let R be a K-8 rejection function. To prove K-8, suppose $\phi \notin \Gamma -_R (\phi \wedge \psi)$, i.e. $\phi \notin \Gamma \vee^+ R(\Gamma, \phi \wedge \psi)$, i.e. either $\phi \notin \Gamma$ or $\phi \notin R(\Gamma, \phi \wedge \psi)$. If $\phi \notin \Gamma$ then $\Gamma -_R (\phi \wedge \psi) = \Gamma = \Gamma -_R \phi$. Else $\phi \notin R(\Gamma, \phi \wedge \psi)$. Let $\mu \in \Gamma -_R (\phi \wedge \psi)$, i.e. $\mu \in \Gamma \vee^+ R(\Gamma, \phi \wedge \psi)$. Then $\mu = \gamma \vee \alpha$ for some $\gamma \in \Gamma$ and $\alpha \in R(\Gamma, \phi \wedge \psi)$. So $\alpha \in R(\Gamma, \phi)$ by Definition 5.13. Therefore $\mu \in \Gamma \vee^+ R(\Gamma, \phi) = \Gamma -_R \phi$.

Conversely, let $-$ be a contraction function satisfying K-1 to K-6 and K-8. Define a canonical K-8 rejection function R_- as in Theorem 5.12: $R_-(\Gamma, \phi) = Cn(\neg\phi) \wedge^+ (\Gamma - \phi)$. Suppose $\phi \notin R_-(\Gamma, \phi \wedge \psi)$. Then $\phi \notin \Gamma - (\phi \wedge \psi)$. Therefore

$$\begin{aligned} R_-(\Gamma, \phi \wedge \psi) &= Cn(\neg\phi \vee \neg\psi) \wedge^+ (\Gamma - (\phi \wedge \psi)) \\ &\subseteq Cn(\neg\phi \vee \neg\psi) \wedge^+ (\Gamma - \phi) \\ &\subseteq Cn(\neg\phi) \wedge^+ (\Gamma - \phi) \\ &= R_-(\Gamma, \phi). \end{aligned}$$

This proves that R_- is a well-defined K-8 rejection function, which proves the theorem. □

Theorems 5.9, 5.12 and 5.14 mutually characterize contraction and rejection functions. Existing representation results relating contraction functions to, for example, selection functions, epistemic entrenchment relations and systems of spheres, therefore also apply to rejection functions.

Theorem 5.16 below makes the relation between rejection functions and selection functions explicit, but first, some comments on the algebraic characterization of contractions in PC: Theorem 3.25 states that the distributive filter lattice $\mathcal{F}(\mathcal{L}_{PC})$ is isomorphic to the positive calculus of systems in PC. Thus belief sets correspond to filters in $\mathcal{L}_{PC}$. The principal filters in $\mathcal{L}_{PC}$ give rise to a maximum contraction function, resulting in the smallest possible belief set. This contraction is defined by the rejection function R_p defined on page 139, and corresponds to the AGM full meet contraction function.

At the other end of the spectrum, the ultrafilters in $\mathcal{L}_{PC}$ give rise to minimal contractions, resulting in the largest possible belief sets. Let Γ be a belief set and $\phi \in \Gamma$. The AGM syntax-based representation of a minimal contraction is as a maximally consistent subset of Γ which fails to imply ϕ, called a *maxi-choice* contraction function. As pointed out by Grove (1988), there is a one-one correspondence between complete and consistent belief sets containing $\neg\phi$ and maximal subsets of Γ that fail to imply ϕ. With each maxi-choice contraction $\Gamma - \phi$, Grove associates a complete and consistent extension of $\neg\phi$. In the filter lattice, this translates to the observation that each ultrafilter F containing $\neg\phi$ defines a minimal contraction $\Gamma \vee^+ F$, and each minimal contraction $-$ defines an ultrafilter $(\Gamma - \phi) \wedge^+ \{\neg\phi\}$. In terms of systems of spheres, this corresponds to the Stalnaker assumption that there is a unique maximally consistent extension of ϕ which is closest to Γ. This is the ultrafilter disjoined to Γ. The smaller the filter $R(\Gamma, \phi)$ is, the coarser is the corresponding Grove system of spheres, with the coarsest system corresponding to $R(\Gamma, \phi)$ being a principal filter.

In the semantic characterization of contraction functions, each model u

of $\neg\phi$ has a maxi-choice contraction $\Gamma - \phi$ associated with it, obtained by adding u to the models of Γ. All maxi-choice contractions arise in this way. This translates to the observation that each model of $\neg\phi$ defines a minimal contraction $\Gamma \vee^{+} F$, where F is the unique ultrafilter having u as model. Every minimal contraction arises in this way, but different models of $\neg\phi$ may give rise to the same minimal contraction.

Let $\Gamma\perp\phi$ denote the set of all maximally consistent subsets of Γ that fail to imply ϕ, also called *remainder sets*, and let $f(\neg\phi)$ denote the set of all complete and consistent extensions of $\neg\phi$. As mentioned above, there is a one-one correspondence between $\Gamma\perp\phi$ and $f(\neg\phi)$. This correspondence is made explicit in the following lemma:

Lemma 5.15 *Let Γ be a belief set and $\phi \in \Gamma$, with $\phi \notin Cn(\emptyset)$. Each element of $\Gamma\perp\phi$ has exactly one complete and consistent extension containing $\neg\phi$. Conversely, each $\Delta \in f(\neg\phi)$ is the extension of exactly one element of $\Gamma\perp\phi$, namely $\Gamma \cap \Delta$.*

Proof. Let $\Sigma \in \Gamma\perp\phi$. Σ must have at least one complete and consistent extension containing $\neg\phi$, for else we would have $\phi \in \Sigma$. Next, suppose Σ_1 and Σ_2 are two distinct complete and consistent extensions of Σ such that $\neg\phi \in \Sigma_1$ and $\neg\phi \in \Sigma_2$. Then there must be some wff ψ on which they differ. Say $\psi \in \Sigma_1$ and $\neg\psi \in \Sigma_2$. Therefore $\neg\phi \supset \psi \notin \Sigma_2$ and $\neg\phi \supset \neg\psi \notin \Sigma_1$. But then $\Sigma \nvdash \neg\phi \supset \psi$ and $\Sigma \nvdash \neg\phi \supset \neg\psi$. Since $\neg\phi \supset \neg\psi \equiv_{Cn} (\neg\phi \supset \psi) \supset \phi$, $\Sigma \nvdash (\neg\phi \supset \psi) \supset \phi$. By the Deduction Theorem, $\Sigma, \neg\phi \supset \psi \nvdash \phi$, contradicting the maximality of Σ.

Conversely, let $\Delta \in f(\neg\phi)$. Then Δ is an extension of $\Gamma\cap\Delta$ and $\Gamma\cap\Delta \subseteq \Gamma$ and $\Gamma \cap \Delta \nvdash \phi$. To see that $\Gamma \cap \Delta \in \Gamma\perp\phi$, suppose $\Gamma \cap \Delta \subsetneq \Psi \subseteq \Gamma$. Then $\exists\psi \in \Psi$ such that $\psi \notin \Gamma \cap \Delta$. So $\neg\psi \in \Delta$, and $\psi \supset \phi \in \Gamma \cap \Delta \subseteq \Psi$.

Therefore $\phi \in \Psi$. Hence $\Gamma \cap \Delta \in \Gamma \perp \phi$. Finally, suppose $\Sigma_1, \Sigma_2 \in \Gamma \perp \phi$, and $\Sigma_1, \Sigma_2 \subseteq \Delta$. Then $\Sigma_1 \cup \Sigma_2 \subseteq \Gamma$ and $\Sigma_1 \cup \Sigma_2 \nvdash \phi$, so $\Sigma_1 = \Sigma_2$. $\square$

Theorem 5.9 gives a syntactic representation of contractions in terms of rejection functions. On the other hand, Alchourrón, Gärdenfors & Makinson (1985) give a syntactic representation of contractions in terms of remainder sets. A contraction of Γ with ϕ is expressed as the nonempty intersection of elements from $\Gamma \perp \phi$. Conversely, if s is a selection function which picks out a nonempty subset of $\Gamma \perp \phi$, then $\bigcap s(\Gamma \perp \phi)$ describes a contraction of Γ with ϕ. Lemma 5.15 provides a method to translate directly between rejection functions and remainder sets:

Theorem 5.16 *Let Γ be a belief set and $\phi \in \Gamma$, with $\phi \notin Cn(\emptyset)$.*

(i) *For any rejection function R, $\Gamma -_R \phi = \bigcap s(\Gamma \perp \phi)$, where $s(\Gamma \perp \phi) = \{\Gamma \cap \Delta : \Delta \in f(\neg\phi) \text{ and } R(\Gamma, \phi) \subseteq \Delta\}$.*

(ii) *For any selection function s, $\bigcap s(\Gamma \perp \phi) = \Gamma \vee^+ R(\Gamma, \phi)$, where $R(\Gamma, \phi) = \bigcap\{\Psi \in f(\neg\phi) : \Gamma \cap \Psi \in s(\Gamma \perp \phi)\}$.*

Proof. (i) Let R be a rejection function.

$$\begin{aligned}
\Gamma -_R \phi &= \Gamma \vee^+ R(\Gamma, \phi) \\
&= \Gamma \cap R(\Gamma, \phi) \text{ by Lemma 3.17} \\
&= \Gamma \cap \left(\bigcap\{\Delta \in f(\neg\phi) : R(\Gamma, \phi) \subseteq \Delta\}\right) \\
&= \bigcap\{\Gamma \cap \Delta : \Delta \in f(\neg\phi) \text{ and } R(\Gamma, \phi) \subseteq \Delta\}.
\end{aligned}$$

(ii) Let s be a selection function.

$$\begin{aligned}
\bigcap s(\Gamma \perp \phi) &= \bigcap\{\Gamma \cap \Psi : \Psi \in f(\neg\phi) \text{ and } \Gamma \cap \Psi \in s(\Gamma \perp \phi)\} \\
&= \Gamma \cap \left(\bigcap\{\Psi \in f(\neg\phi) : \Gamma \cap \Psi \in s(\Gamma \perp \phi)\}\right) \\
&= \Gamma \vee^+ \bigcap\{\Psi \in f(\neg\phi) : \Gamma \cap \Psi \in s(\Gamma \perp \phi)\}.
\end{aligned}$$

The result follows from Lemma 3.17. □

I have shown in this section that the power disjunction of a belief set and a rejection of a wff may be regarded as the contraction of the wff from a belief set. The power conjunction of two belief sets, one of which is of the form $Cn(\phi)$, may similarly be regarded as the *expansion* of a belief set with ϕ. The addition of a belief ϕ to a belief set Γ is written $\Gamma + \phi$, which is defined in the calculus of belief sets as $\Gamma \dotplus Cn(\phi)$. Theorem 3.20 shows that this operation can also be written as the power conjunction of two belief sets: $\Gamma + \phi = \Gamma \dotplus Cn(\phi) = Cn(\Gamma \cup Cn(\phi)) = \Gamma \wedge^{+} Cn(\phi)$.

The Levi identity can be used to define a revision operation in terms of contraction and expansion:

$$\begin{aligned}
\Gamma * \phi &= (\Gamma - \neg\phi) + \phi \\
&= \left(\Gamma \vee^{+} R_{-}(\Gamma, \neg\phi)\right) \wedge^{+} Cn(\phi) \\
&= \left(\Gamma \wedge^{+} Cn(\phi)\right) \vee^{+} \left(R_{-}(\Gamma, \neg\phi) \wedge^{+} Cn(\phi)\right) \\
&= \left(\Gamma \wedge^{+} Cn(\phi)\right) \vee^{+} R_{-}(\Gamma, \neg\phi) \\
&= (\Gamma + \phi) \vee^{+} R_{-}(\Gamma, \neg\phi).
\end{aligned}$$

Theorem 5.1 applies to the revision operation $*$: If $-$ satisfies K-1 to K-4 and K-6, then $*$ satisfies K$*$1 to K$*$6. If, in addition, $-$ also satisfies K-7, $*$ satisfies K$*$7, and if $-$ also satisfies K-8, $*$ satisfies K$*$8. In the special cases where either $\phi \in \Gamma$ or $\neg\phi \notin \Gamma$, the expression $(\Gamma + \phi) \vee^{+} R_{-}(\Gamma, \neg\phi)$ obtained above simplifies to $\Gamma - \neg\phi$ and $\Gamma + \phi$ respectively.

To summarize, in this section I have shown that the theory change operations of contraction, expansion and revision can be characterized algebraically as operations in a power algebra of theories, endowed with a set of unary rejection operations $\{R_\phi\}_{\phi \in L}$. This yields the algebra of theory change $\left(P(L)/\equiv, \wedge^{+}, \vee^{+}, \{R_\phi\}_{\phi \in L}\right)$.

Equivalently, the theory change operations are characterized in the calculus of belief sets $\left(\mathcal{B}, \cdot, \dot{+}, \{R_\phi\}_{\phi\in L}\right)$, which is the positive calculus of deductive systems endowed with a set of unary rejection operations. In particular, the contraction of a belief set Γ with a wff ϕ can be written as the power disjunction of Γ and a belief set containing $\neg\phi$, and the expansion of a belief set Γ with a wff ϕ can be written as the power conjunction of Γ and the belief set generated by ϕ.

The two fundamental operations that can be performed on any belief set are conjunction and disjunction. Model-theoretically, conjunction represents the elimination of models, and disjunction represents the introduction of models. Theory change operations arise as special instances of conjoining and disjoining belief sets.

- The expansion of a belief set with a wff is accomplished by conjoining an acceptance of the sentence. In the algebra of theory change, the only acceptance function maps wffs to their deductive closure.

- The contraction of a belief set with a wff is accomplished by disjoining a rejection of the wff. To this end, a rejection function is introduced in the algebra of theories. The rejection function maps any wff ϕ to a belief set containing $\neg\phi$.

5.3 Extensions

The algebraic characterization of contraction presented in the previous section can be extended in various ways, making allowance for multiple contractions (Fuhrmann & Hansson, 1994; Fuhrmann, 1997), erasure operations (Katsuno & Mendelzon, 1992), and base contractions (Hansson, 1989; Fuhrmann, 1991). Some results in these directions are given below.

Multiple contractions

The original AGM approach to theory change only allowed theory change operations that operated on a belief set and single wff to deliver a changed belief set. One generalization of this approach is to allow *multiple contractions.* The idea is to generalize the definition of a contraction function to operate on a belief set and a formula set, instead of a belief set and a single wff. Two variations of multiple contractions considered are *package* contractions and *choice* contractions. In the former, a belief set is contracted by all the elements of a formula set, while in the latter, a belief set is contracted by at least one element from the formula set. An algebra for theory change can be used to model both package and choice contractions. If the condition that belief sets are deductively closed is dropped, package contractions emerge as a generalized form of base contraction. I will briefly discuss base contractions later on in this section, while I will consider choice contractions here.

Fuhrmann & Hansson (1994) motivate a generalization of postulates K-1 to K-6 to accommodate choice contractions with arbitrary formula sets. The following postulates are adapted from (Fuhrmann & Hansson, 1994), with the additional condition added that the sets to be removed are deductively closed. This does not have any real impact on the representation result obtained below, and is done to clarify the presentation.

Let Γ, Φ and Ψ be belief sets. A choice contraction is a function $- : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ such that:

C-1 Closure: $\Gamma - \Phi$ is a belief set.

C-2 Inclusion: $\Gamma - \Phi \subseteq \Gamma$.

C-3 Vacuity: If $\Phi \not\subseteq \Gamma$ then $\Gamma - \Phi = \Gamma$.

C-4 Success: If $\Phi \neq Cn(\emptyset)$, then $\Phi \nsubseteq \Gamma - \Phi$.

C-5 Recovery: If $\Phi \subseteq \Gamma$ then $(\Gamma - \Phi) + \Phi = \Gamma$.

C-6 Extensionality: If $\Phi = \Psi$ then $\Gamma - \Phi = \Gamma - \Psi$.

A corresponding generalization is required in the definition of a rejection function which defines a choice contraction function. Choice rejection functions are defined in 5.17 below. Theorem 5.18 then states that postulates C-1 to C-6 are satisfied precisely by those contraction functions that can be defined by a choice rejection function. However, unlike in Lemma 5.10, the choice contraction $-_{R_-}$, defined by the choice rejection function R_- defined from the choice contraction $-$, does not give us back the contraction $-$. This is because Lemma 5.7 does not generalize to choice contractions: there is no unique maximum choice contraction.

Definition 5.17 *A choice rejection function is a function $R : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ such that, for any belief sets Γ and Φ:*

1. *$R(\Gamma, \Phi)$ is a belief set which intersects $\neg^+\Phi$.*

2. *$R(\Gamma, \Phi)$ is inconsistent iff $\Phi = Cn(\emptyset)$.*

3. *If $\Phi \nsubseteq \Gamma$ then $\Gamma \subseteq R(\Gamma, \Phi)$.*

4. *If $\Phi = \Psi$ then $R(\Gamma, \Phi) = R(\Gamma, \Psi)$.*

Theorem 5.18 *A contraction function satisfies C-1 to C-6 iff it is defined by a choice rejection function.*

Proof. Left to right: Suppose $-$ is a contraction function satisfying C-1 to C-6. Define the function $R_- : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ by:

$$R_-(\Gamma, \Phi) = \begin{bmatrix} Cn\left(\neg^+(\Phi - \Gamma) \wedge^+ (\Gamma - \Phi)\right) \text{ if } \Phi \neq Cn(\emptyset); \\ L \text{ if } \Phi = Cn(\emptyset). \end{bmatrix}$$

We have to check that $R_-(\Gamma,\Phi)$ is well-defined: $\Phi \cap (\Gamma - \Phi)' = \emptyset$ iff $\Phi \subseteq \Gamma - \Phi$ iff $\Phi = Cn(\emptyset)$ iff $R_-(\Gamma,\Phi) = L$. Else there exists some $\phi \in \Phi$ such that $\neg\phi \in R(\Gamma,\Phi)$. Therefore $\neg^+\Phi \cap R_-(\Gamma,\Phi) \neq \emptyset$. Conditions (1) and (2) therefore hold. Next, suppose $\Phi \not\subseteq \Gamma$. Then $\Gamma = \Gamma - \Phi \geq \neg^+(\Phi - \Gamma) \wedge^+ (\Gamma - \Phi) = R_-(\Gamma,\Phi)$. Condition (3) therefore holds. Finally, condition (4) holds because $-$ is syntax-independent.

Right to left: The proof of C-1 to C-6 mimics that of Lemma 5.4. □

Erasures

Katsuno & Mendelsohn (1992) argue that certain modifications to a knowledge base do not adhere to the AGM postulates for revision. They propose an alternative set of postulates for a class of modifications called *updates*. A corresponding set of postulates governs the *erasure* of information from a knowledge base. The postulates for update are in the same relation to the postulates for erasure, as the postulates for revision are to the postulates for contraction. In the notation used thus far, the postulates for erasure are:

E-1 Closure: $\Gamma -_e \phi$ is a belief set.

E-2 Inclusion: $\Gamma -_e \phi \subseteq \Gamma$.

E-3 Vacuity: If $\neg\phi \in \Gamma$ then $\Gamma -_e \phi = \Gamma$.

E-4 Success: If $\phi \notin Cn(\emptyset)$, then $\phi \notin \Gamma -_e \phi$.

E-5 Recovery: If $\phi \in \Gamma$ then $(\Gamma -_e \phi) + \phi = \Gamma$.

E-6 Extensionality: If $\vdash \phi \leftrightarrow \psi$ then $\Gamma -_e \phi = \Gamma -_e \psi$.

E-8 Disjunction: $(\Gamma_1 \vee^+ \Gamma_2) -_e \phi = (\Gamma_1 -_e \phi) \vee^+ (\Gamma_2 -_e \phi)$.

The first six postulates for contraction are the same as the first six postulates for erasure, except in that the vacuity postulate is weaker. The disjunction postulate for erasure applies to a disjunction of belief sets, whereas the disjunction postulate for contraction applies to a disjunction of wffs.

As in the case of contractions, the erasure of ϕ from Γ can be modelled in an algebra of theory change as the power disjunction of Γ with a rejection of ϕ. The rejection function defined by an erasure operation has to change to reflect the changes in the postulates. Condition 3 of Definition 5.2 differs from condition 3 below, and a fifth condition is added:

Definition 5.19 *An erasure rejection function is a function $R_e : \mathcal{B} \times L \to \mathcal{B}$ such that, for any belief set Γ and wff ϕ:*

1. *$R_e(\Gamma, \phi)$ is a belief set containing $\neg\phi$.*
2. *$R_e(\Gamma, \phi)$ is inconsistent iff $\phi \in Cn(\emptyset)$.*
3. *If $\neg\phi \in \Gamma$ then $\Gamma \subseteq R_e(\Gamma, \phi)$.*
4. *If $\vdash \phi \leftrightarrow \psi$ then $R_e(\Gamma, \phi) = R_e(\Gamma, \psi)$.*
5. *$R_e(\Gamma_1 \vee^+ \Gamma_2, \phi) = R_e(\Gamma_1, \phi) \vee^+ R_e(\Gamma_2, \phi)$.*

Theorem 5.20 *An erasure function satisfies E-1 to E-6 and E-8 iff it can be defined by an erasure rejection function.*

Proof. Let R be an erasure rejection function. Define the operation $-_R$: $\mathcal{B} \times L \to \mathcal{B}$ by:

$$\Gamma -_R \phi = \Gamma \vee^+ R(\Gamma, \phi).$$

E-1, E-2, E-4, E-5 and E-6 are proved as in Lemma 5.4. To prove E-3, let $\neg\phi \in \Gamma$. Then $\Gamma -_R \phi = \Gamma \vee^+ R(\Gamma, \phi) = \Gamma$ by property 3 of an erasure rejection function. This leaves E-8:

$$\begin{aligned}
(\Gamma_1 \vee^+ \Gamma_2) -_R \phi &= (\Gamma_1 \vee^+ \Gamma_2) \vee^+ R\,(\Gamma_1 \vee^+ \Gamma_2, \phi) \\
&= \Gamma_1 \vee^+ \Gamma_2 \vee^+ R_e\,(\Gamma_1, \phi) \vee^+ R_e\,(\Gamma_2, \phi) \\
&= \Gamma_1 \vee^+ R_e\,(\Gamma_1, \phi) \vee^+ \Gamma_2 \vee^+ R_e\,(\Gamma_2, \phi) \\
&= (\Gamma_1 -_R \phi) \vee^+ (\Gamma_2 -_R \phi).
\end{aligned}$$

Conversely, let $-$ be an erasure operation. Define the function $R_- : \mathcal{B} \times L \to \mathcal{B}$ by:

$$R_-\,(\Gamma, \phi) = Cn\,(\neg\phi) \wedge^+ (\Gamma - \phi).$$

R_- has properties 1 to 4 of Definition 5.19. This is checked as in the proof of Theorem 5.9. Further:

$$\begin{aligned}
R_-\,(\Gamma_1 \vee^+ \Gamma_2, \phi) &= Cn\,(\neg\phi) \wedge^+ ((\Gamma_1 \vee^+ \Gamma_2) - \phi) \\
&= Cn\,(\neg\phi) \wedge^+ ((\Gamma_1 - \phi) \vee^+ (\Gamma_2 - \phi)) \\
&= (Cn\,(\neg\phi) \wedge^+ (\Gamma_1 - \phi)) \vee^+ (Cn\,(\neg\phi) \wedge^+ (\Gamma_2 - \phi)) \\
&= R_-\,(\Gamma_1, \phi) \vee^+ R_-\,(\Gamma_2, \phi).
\end{aligned}$$

This shows that R_- is a well-defined erasure rejection function, which proves the theorem. □

Base contractions

In Section 5.2, belief sets were defined as deductively closed elements of $\mathcal{P}(L)$. One can do away with this requirement and define belief sets simply as elements of $\mathcal{P}(L)$. A *belief base* for a belief set Γ is any set of wffs Φ such that $Cn\,(\Phi) = \Gamma$, that is, $\Phi \equiv_{Cn} \Gamma$.

A rejection function as defined in 5.2 operates on belief sets. Unlike the postulates for contraction, rejection functions can easily be adapted to operate on belief bases.

Definition 5.21 *A base rejection function is a function $R_b : \mathcal{P}(L) \times L \to \mathcal{P}(L)$ such that, for any belief base Γ and wff ϕ:*

1. $\neg\phi \in R_b(\Gamma, \phi)$.

2. $R_b(\Gamma, \phi)$ *is inconsistent iff* $\phi \in Cn(\emptyset)$.

3. *If* $\phi \notin Cn(\Gamma)$ *then* $\Gamma \subseteq R_b(\Gamma, \phi)$.

4. *If* $\vdash \phi \leftrightarrow \psi$ *then* $R_b(\Gamma, \phi) = R_b(\Gamma, \phi)$.

Of these, condition 3 may call for debate. Its intention is clear: If ϕ cannot be derived from the knowledge base, nothing in it should change when rejecting ϕ. This can be phrased in various ways, but does not influence the essence of a rejection function, which is to reject a wff in a consistent way. The base contraction $-_b$ obtained from Definition 5.21 is defined by:

$$\Gamma -_b \phi = \Gamma \vee^+ R_b(\Gamma, \phi).$$

This function does not satisfy the inclusion postulate for contractions, but it does satisfy the weaker condition of *deductive inclusion* (Fuhrmann, 1997):

C-2′ Deductive inclusion: $\Gamma \vdash \Gamma -_b \phi$.

Instead of finding a suitable definition for base rejection functions, based on a set of postulates governing base contractions, Definition 5.21 can be regarded as the fundamental concept, and used to find a suitable set of postulates to govern base contractions.

5.4 Preferential reasoning

Non-monotonic reasoning is concerned with *defeasible* inference relations, that do not satisfy the monotony or weakening property of Definition 1.17.

An increase in information may thus lead to a retraction of previously accepted premisses. Non-monotonic logics are characterized abstractly by *cumulative consequence relations*, and semantically by *preferential model structures.* These two approaches are related by Makinson (1989), and by Kraus, Lehmann & Magidor (1990).

The idea to study non-monotonic logics by their consequence relations, originated with Gabbay (1985), while their semantic characterization was advocated by Shoham (1988). Different classes of non-monotonic logics may thus be characterized both abstractly, by the properties of their consequence relations, and semantically, by the properties of their preferential model structures.

A preferential model structure can be defined more generally, but for the purpose of the ensuing discussion, it suffices to regard it simply as a preorder on the set of propositional valuations Val_2. That is, a *preference relation* $\leq$ is a reflexive and transitive relation over Val_2. (For technical reasons, some authors have assumed irreflexivity, but I will not do so here.) In addition, $\leq$ is assumed to satisfy the following *smoothness* condition: For any belief set Φ and any $w \in Mod(\Phi)$, there is some $v \in Mod(\Phi)$ such that $v \leq w$ and v is $\leq$-minimal in $Mod(\Phi)$. So $Mod(\Phi)$ has no infinitely descending chains. The intuition behind a preference relation is that it orders valuations according to how close they are to being a model of some belief set Γ, with the models of Γ being $\leq_\Gamma$-minimal in the order.

Let $<$ denote the strict counterpart of $\leq$, that is, $v < w$ iff $v \leq w$ and $w \not\leq v$. A preference relation $\leq_\Gamma$ is called Γ-*faithful* if it satisfies the following properties:

1. If $v, w \in Mod(\Gamma)$ then $v \not<_\Gamma w$.

2. If $v \in Mod(\Gamma)$ and $w \notin Mod(\Gamma)$ then $v <_\Gamma w$.

3. If $\Gamma \equiv_{Cn} \Delta$ then $\leq_\Gamma = \leq_\Delta$.

Preference relations have been used to link non-monotonic reasoning to belief revision (Makinson & Gärdenfors, 1991; Makinson, 1993), and to verisimilitude (Ryan & Schobbens, 1995). Boutilier & Becher (1995) use preference relations to link belief revision to *abductive* reasoning, the process of finding a plausible explanation for a given set of observations. The antecedence operator of Definition 1.22 induces a relation of explanatory strength on formula sets, defined in 1.26. These concepts may be useful in a multiple-conclusion formulation of abduction, since they accommodate the disjunctive treatment of observations. This would be appropriate in cases where observations represent possible alternatives, for example in experimental observations.

Katsuno & Mendelzon (1991) use preference relations to characterize the AGM postulates for belief revision semantically. Their results are obtained for finite belief sets, that can be represented by single wffs. Katsuno & Mendelson also discuss a number of revision operations proposed in the literature (Borgida, 1985; Dalal, 1988; Satoh, 1988; Winslett, 1988) in terms of preference relations.

Theorem 5.22 *(Katsuno & Mendelzon, 1991) A revision operation $\circ$ satisfies postulates K*1 to K*8 for finite belief sets iff there exists, for each finite belief set Γ, a Γ-faithful total preorder $\leq_\Gamma$ such that $Mod(\Gamma \circ \phi) = Min(Mod(\phi), \leq_\Gamma)$.* □

Here, $Min(Mod(\phi), \leq_\Gamma)$ denotes the set of $\leq_\Gamma$-minimal models of ϕ. To obtain a representation result in terms of preorders that are not total, postulate K*8 has to be replaced by the weaker K*9 and K*10. These postulates are rephrased here in terms of belief sets instead of single wffs:

K*9 If $\phi \in \Gamma * \psi$ and $\psi \in \Gamma * \phi$ then $\Gamma * \phi = \Gamma * \psi$.

K*10 $\Gamma * (\phi \vee \psi) \subseteq (\Gamma * \phi) \wedge^{+} (\Gamma * \psi)$.

Theorem 5.23 *(Katsuno & Mendelzon, 1991) A revision operation $\circ$ satisfies postulates K*1 to K*7, K*9 and K*10 for finite belief sets iff there exists, for each finite belief set Γ, a Γ-faithful preorder $\leq_\Gamma$ such that $Mod(\Gamma \circ \phi) = Min(Mod(\phi), \leq_\Gamma)$.* □

Consider the preorder on valuations of Definition 4.22, and used to define a theorylike power order on formula sets. Its converse $\leq_\Gamma^{\smile}$ is a Γ-faithful preorder, as Lemma 5.24 shows. Therefore, by Theorem 5.23, it defines a revision operation $\circ$ which satisfies postulates K*1 to K*7, K*9 and K*10.

Lemma 5.24 *$\leq_\Gamma^{\smile}$ is a Γ-faithful preorder.*

Proof. Let Γ be any belief set. Let $\leq_\Gamma^{\smile}$ be the converse of the preorder of Definition 4.22. We first check that $\leq_\Gamma^{\smile}$ is Γ-faithful:

1. If $v, w \in Mod(\Gamma)$ then $v \leq_\Gamma^{\smile} w$ and $w \leq_\Gamma^{\smile} v$ by Definition 4.22, so $v \not<_\Gamma^{\smile} w$.

2. If $v \in Mod(\Gamma)$ and and $w \notin Mod(\Gamma)$ then $v \leq_\Gamma^{\smile} w$ by Definition 4.22, and $v \not\geq_\Gamma w$ by Lemma 4.21. Hence $w \not\leq_\Gamma^{\smile} v$. So $v <_\Gamma^{\smile} w$.

3. If $\Gamma \equiv_{Cn} \Delta$ then $\leq_\Gamma^{\smile} = \leq_\Delta^{\smile}$, since the order is syntax-independent. □

For every belief set Γ, the revision operation $\circ$ obtained from the preference relation $\leq_\Gamma^{\smile}$ has the property that $Mod(\Gamma \circ \phi) = Min\left(Mod(\phi), \leq_\Gamma^{\smile}\right)$. Syntactically, $\Gamma \circ \phi$ can be described in terms of the weak power relation

$(\leq_\Gamma^{\smile})_1^+$ of the preference relation $\leq_\Gamma^{\smile}$:

$$\begin{aligned}
\Gamma \circ \phi &= \left\{\psi : Min\left(Mod(\phi), \leq_\Gamma^{\smile}\right) \subseteq Mod(\psi)\right\} \\
&= \left\{\psi : \left(\forall v \ \leq_\Gamma^{\smile}\text{-minimal in } Mod(\phi)\right) (\exists w \in Mod(\psi))\, [w = v]\right\} \\
&= \left\{\psi : \left(\forall v \ \leq_\Gamma^{\smile}\text{-minimal in } Mod(\phi)\right) (\exists w \in Mod(\phi \wedge \psi))\, [w = v]\right\} \\
&= \left\{\psi : (\forall v \in Mod(\phi))\, (\exists w \in Mod(\phi \wedge \psi)) \left[w \leq_\Gamma^{\smile} v\right]\right\} \\
&= \left\{\psi : \phi \wedge \psi (\leq_\Gamma^{\smile})_1^+ \phi\right\}.
\end{aligned}$$

This provides the final example of the use of power constructs in reasoning about propositional systems. Namely, the revision of a system Γ is defined in terms of the power relation of a parameterized preference relation on valuations.

5.5 Concluding remarks

The algebra of theory change $(\mathcal{P}(L)/\equiv, \wedge^+, \vee^+, \{R_\phi\}_{\phi \in L})$ is a natural extension to Tarski's calculus of deductive systems. It characterizes the operations of theory intersection, addition and contraction algebraically. It also provides an algebraic framework for the description of a parameterized verisimilar order on theories, as well as a simulation relation between different logics. It is general enough to accommodate diverse aspects of consequence relations such as paraconsistency, partial knowledge, many-valuedness and multiple conclusions.

The operations for intersection, addition and contraction, and the relations for theorylikeness and simulation all arise by lifting an operation or relation on elements of a set to a power operation or relation on sets. Certain consequence relations and belief revision operations also arise in this way. In some of the case studies and examples considered in the thesis, the power construction is not the most direct means of achieving a result, but in most cases, it is. This shows that the power construction is an integral feature of deductive systems.

Some of the results obtained are inconclusive, for example, the results on simulation, and on the use of the power construction in the definition of multiple-conclusion consequence relations. It would be instructive to see if new logics, and new relationships between logics, can be established by these means. However, most of the results obtained, in particular the establishment of a parameterized theorylike order and the algebraic characterization of theory change operations, confirm that the power construction provides a natural and elegant mechanism to address a diverse range of problems. The use of an algebra of theory change as a calculus to reason about revisions and contractions, merits further investigation. The generalization of power constructs to first order logics was not considered here, and also merits further investigation.

References

Alchourrón, C.E., & Makinson, D. 1981. New Studies in Deontic Logic. *Pages 125–148 of:* Hilpinen, R. (ed), *New Studies in Deontic Logic.* Reidel, Dordrecht.

Alchourrón, C.E., Gärdenfors, P., & Makinson, D. 1985. On the logic of theory change: Partial meet functions for contraction and revision. *Journal of Symbolic Logic*, **50**, 510–530.

Alvarado, M., & Núñez, G. 1997. Formalization of knowledge and belief based on Kleene's strong logic. *Logic Journal of the IGPL*, **5**(6), 869–870.

Anderson, A.R., & Belnap, N.D. 1975. *Entailment: The Logic of Relevance and Necessity.* Princeton University Press.

Arieli, O., & Avron, A. 1996. Reasoning with logical bilattices. *Journal of Logic, Language and Information*, **5**(1), 25–63.

Arieli, O., & Avron, A. 1997. Four-valued diagnoses for stratified knowledge bases. *Pages 1–17 of: Computer science logic.* Lecture Notes in Computer Science, vol. 1258. Springer-Verlag, Berlin.

Avron, A. 1986. On an implication connective of RM. *Notre Dame Journal of Formal Logic*, **27**, 201–209.

Avron, A. 1991a. Natural 3-valued logics - characterization and proof theory. *Journal of Symbolic Logic*, **56**(1), 276–294.

Avron, A. 1991b. Simple consequence relations. *Information and computation*, **92**, 105–139.

Avron, A. 1992. Axiomatic systems, deduction and implication. *Journal of Logic and Computation*, **2**(1), 51–98.

Barringer, H., Cheng, J.H., & Jones, C.B. 1984. A logic covering undefinedness in program proofs. *Acta Informatica*, **21**, 251–269.

Belnap, N.D. 1977a. How a computer should think. *Pages 30–56 of:* Ryle, G. (ed), *Contemporary aspects of philosophy.* Oriel Press.

Belnap, N.D. 1977b. A useful four-valued logic. *Pages 7–73 of:* Epstein, G., & Dunn, J.M. (eds), *Modern Uses of Multiple-Valued Logic.* D. Reidel, Dordrecht.

Belnap, N.D., & Steel, T.B. 1976. *The logic of questions and answers.* Yale University Press, London.

Bergstra, J.A., & Klop, J.W. 1988. Process theory based on bisimulation semantics. *Pages 50–122 of: Linear time, branching time and partial order in logics and models of concurrency.* Lecture Notes in Computer Science, vol. 354. Springer-Verlag, Berlin.

Bergstra, J.A., & Ponse, A. 1998. Kleene's three-valued logic and process algebra. *Information Processing Letters*, **67**, 95–103.

Blamey, S. 1986. Partial logic. *Pages 1–70 of:* Gabbay, D., & Guenthner, F. (eds), *Handbook of Philosophical Logic*, vol. III. D. Reidel, Dordrecht.

Blikle, A. 1991. Three-valued predicates for software specification and validation. *Fundamenta Informaticae*, **14**, 387–410.

Blok, W.J., & Pigozzi, D. 1989. Algebraizable logics. *Memoirs of the American Mathematical Society*, **77**(396), 1–78.

Bochvar, D.A. 1981. On a three-valued calculus and its application to analysis of the paradoxes of the classical extended functional calculus. *History of Philosophical Logic*, **2**, 87–112. (Originally published in Russian in: *Matématičéskij Sbornik*, 4(46), no.2, 1938, 287–308).

Bolc, L., & Borowik, P. 1992. *Many-Valued Logics*. Springer-Verlag.

Borgida, A. 1985. Language features for flexible handling of exceptions in information systems. *ACM Transactions on Database Systems*, **10**, 563–603.

Boutilier, C., & Becher, V. 1995. Abduction as belief revision. *Artificial Intelligence*, **77**, 43–94.

Brink, C. 1984. Second-order Boolean algebras. *Questiones Mathematicae*, **7**, 93–100.

Brink, C. 1986. Power structures and logic. *Questiones Mathematicae*, **9**, 69–94.

Brink, C. 1989a. $\mathrm{R}^{\neg}$-algebras and $\mathrm{R}^{\neg}$-model structures as power constructs. *Studia Logica*, **48**, 85–109.

Brink, C. 1989b. Verisimilitude: views and reviews. *History and Philosophy of Logic*, **10**, 181–201.

Brink, C. 1993. Power structures. *Algebra Universalis*, **30**, 177–216.

Brink, C., & Britz, K. 1999. A note on truthlikeness and irrelevance. *In:* Gerbrandy, J., Marx, M., de Rijke, M., & Venema, Y. (eds), *JFAK: Essays dedicated to Johan van Benthem on the occasion of his 50th birthday.* Vossiuspers AUP/Amsterdam University Press.

Brink, C., & Heidema, J. 1987. A versimilar ordering of theories phrased in a propositional language. *British Journal for the Philosophy of Science*, **38**, 533–549.

Brink, C., & Heidema, J. 1989. *A verisimilar ordering of propositional theories: The infinite case.* Technical Report TR-ARP-1/89. Research School of Social Sciences, Australian National University.

Brink, C., & Rewitzky, I. 1999. *Power structures and program semantics.* Studies in Logic, Language and Information. European Foundation for Logic, Language and Information. To appear.

Brink, C., Britz, K., & Melton, A. 1993. A note on fuzzy power relations. *Fuzzy Sets and Systems*, **54**, 115–117.

Britz, K. 1999. A power algebra for theory chance. *Journal of Logic, Language and Information*, **8**(4), 429–443.

Britz, K., & Brink, C. 1995. Computing verisimilitude. *Notre Dame Journal of Formal Logic*, **36**(2), 30–43.

Burger, I.C., & Heidema, J. 1994. Comparing theories by their positive and negative contents. *British Journal for the Philosophy of Science*, **45**, 605–630.

Burris, S., & Sankappanavar, H.P. 1981. *A Course in Universal Algebra.* Springer-Verlag, New York.

Carnap, R. 1943. *Formalization of logic.* Harvard University Press, Cambridge, Massachusetts.

da Costa, N.C.A. 1974. Theory of inconsistent formal systems. *Notre Dame Journal of Formal Logic*, **15**, 497–510.

Dalal, M. 1988. Investigations into a theory of knowledge base revision: preliminary report. *Pages 475–479 of: Proceedings AAAI-88.*

Davey, B.A., & Priestley, H.A. 1990. *Introduction to Lattices and Order.* Cambridge Mathematical Textbooks. Cambridge University Press.

Dubois, D., & Prade, H. 1980. *Fuzzy Sets and Systems: Theory and Applications.* Academic Press, London.

Dunn, J.M. 1966. *The algebra of intensional logics.* Ph.D. thesis, University of Pittsburgh.

Dunn, J.M. 1976. Intuitive semantics for first degree entailments and coupled trees. *Philosophical Studies*, **29**, 149–168.

Dunn, J.M. 1986. Relevance logic and entailment. *Pages 117–224 of:* Gabbay, D., & Guenthner, F. (eds), *Handbook of philosophical logic*, vol. III. D. Reidel, Dordrecht.

Enderton, H.B. 1972. *A mathematical introduction to logic.* Academic Press, London.

Fitting, M. 1985. A Kripke-Kleene semantics for logic programs. *Journal of Logic Programming*, **4**, 295–312.

Fitting, M. 1991a. Bilattices and the semantics of logic programming. *Journal of Logic Programming*, **11**, 91–116.

Fitting, M. 1991b. Kleene's logic, generalized. *Journal of Logic and Computation*, **1**(6), 797–810.

Fitting, M. 1994. Kleene's three valued logics and their children. *Fundamenta Informaticae*, **20**, 113–131.

Font, J.M. 1997. Belnap's four-valued logic and De Morgan lattices. *Logic Journal of the IGPL*, **5**(3), 413–440.

Font, J.M., & Jansana, R. 1996. *A general algebraic semantics for sentential logics.* Lecture Notes in Logic, no. 7. Springer-Verlag, Berlin.

Fuhrmann, A. 1991. Theory contraction through base constaction. *Journal of Philosophical Logic*, **29**, 175–203.

Fuhrmann, A. 1997. *An Essay on Contractions.* CSLI, Stanford.

Fuhrmann, A., & Hansson, S.O. 1994. A survey of multiple contractions. *Journal of Logic, Language and Information*, **3**, 39–76.

Gabbay, D.M. 1971. Semantic proof of the Craig interpolation theorem for intuitionistic logic and extensions, Parts I and II. *Pages 391–410 of: Logic Colloquium '69.* North-Holland, Amsterdam.

Gabbay, D.M. 1985. Theoretical foundations for non-monotonic reasoning in expert systems. *Pages 439–457 of:* Apt, K.R. (ed), *Proceedings NATO Advanced study institute on logics and models of concurrent systems.* Springer, Berlin.

Gärdenfors, P. 1984. Epistemic importance and minimal changes of belief. *Australasian Journal of Philosophy*, **62**, 136–157.

Gärdenfors, P. 1988. *Knowledge in flux: Modelling the dynamics of epistemic states.* MIT Press, Cambridge, Massachusetts.

Gärdenfors, P., & Makinson, D. 1988. Revisions of knowledge systems using epistemic entrenchment. *Pages 83–95 of:* Vardi, M. (ed), *Proceedings of the Second Conference on Theoretical Aspects of Reasoning about Knowledge.* Morgan Kaufmann, Los Altos, California.

Gärdenfors, P., & Rott, H. 1995. Belief revision. *Pages 35–132 of:* D.M. Gabbay, C.J. Hogger, & Robinson, J.A. (eds), *Handbook of logic in artificial intelligence and logic programming*, vol. 4. Clarendon Press, Oxford.

Gentzen, G. 1969. The collected papers of Gerhard Gentzen. North-Holland, Amsterdam.

Gericke, H. 1963. *Lattice Theory.* George G. Harrap & Co. Ltd.

Gibbins, P.F. 1988. VDM: Axiomatising its propositional logic. *The Computer Journal*, **31**(6), 510–516.

Ginsberg, M.L. 1988. Multivalued logics: a uniform apporoach to reasoning in artificial intelligence. *Computing Intelligence*, **4**, 265–316.

Ginzburg, A. 1968. *Algebraic theory of automata.* Academic Press, London.

Goldblatt, R. 1989. Varieties of Complex Algebras. *Annals of Pure and Applied Logic*, **44**(3), 173–242.

Goranko, V., & Vakarelov, D. 1999. Hyperboolean algebras and hyperboolean modal logic. *Journal of Applied Non-classical Logics.* To appear.

Grätzer, G. 1979. *Universal Algebra.* Springer-Verlag, New York.

Grätzer, G., & Whitney, S. 1984. Infinitary varieties of structures closed under the formation of complex structures. *Colloquium Mathematicum*, **48**, 1–5.

Grove, A. 1988. Two modellings for theory change. *Journal of Philosophical Logic*, **17**, 157–170.

Hansson, S. O. 1989. New operators for theory change. *Theoria*, **55**, 114–132.

Harrah, D. 1984. The logic of questions. *Pages 715–764 of:* Gabbay, D., & Guenthner, F. (eds), *Handbook of Philosophical Logic*, vol. II. D. Reidel, Dordrecht.

Hennessey, M. 1988. *Algebraic theory of processes.* Foundations of Computing Series. MIT Press, Cambridge, Massachusetts.

Jaśkowski, S. 1969. Propositional calculus for contradictory deductive systems. *Studia Logica*, **24**, 143–157. (Originally published in Polish in: *Studia Scientarium Torunensis*, Sec. A II, 1948, 55–77).

Jones, C.B. 1986. *Systematic software development using VDM.* Prentice-Hall, Englewood Cliffs, New Jersey.

Jónsson, B. 1993. A survey of Boolean algebras with operators. *Pages 239–286 of:* Rosenberg, I.G., & Sabidussi, G. (eds), *Algebras and Orders.* Kluwer Academic Publishers, Dordrecht.

Jónsson, B., & Tarski, A. 1951. Boolean Algebras with Operators, Part I. *American Journal of Mathematics*, **73**, 891–939.

Jónsson, B., & Tarski, A. 1952. Boolean Algebras with Operators, Part II. *American Journal of Mathematics*, **74**, 127–162.

Kalman, J.A. 1958. Lattices with involution. *Transactions of the American Mathematical Society*, **87**, 485–491.

Kaluzhny, Y., & Muravitsky, A.Y. 1993. A knowledge representation based on the Belnap four-valued logic. *Journal of applied non-classical logics*, **3**(2), 189–203.

Katsuno, H., & Mendelzon, A.O. 1991. Knowledge base revision and minimal change. *Artificial Intelligence*, **52**, 263–294.

Katsuno, H., & Mendelzon, A.O. 1992. On the difference between updating a knowledge base and revising it. *Pages 183–203 of:* Gärdenfors, P. (ed), *Belief revision*. Cambridge University Press.

Kleene, S. C. 1952. *Introduction to Metamathematics*. North-Holland, Amsterdam.

Kneale, W.C. 1956. The province of logic. *Pages 237–261 of:* Lewis, H.D. (ed), *Contemporary British Philosophy, third series*. George Allan & Unwin Ltd, London.

Konikowska, B., Tarlecki, A., & Blikle, A. 1991. A three-valued logic for software specification and validation. *Fundamenta Informaticae*, **14**, 411–453.

Kraus, S., Lehmann, D., & Magidor, M. 1990. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, **44**, 167–207.

Kuipers, T.A. 1987a. A structuralist approach to truthlikeness. *Pages 79–99 of:* Kuipers, T.A. (ed), *What is closer-to-the-truth?* Rodopi, Amsterdam.

Kuipers, T.A. (ed). 1987b. *What is closer-to-the-truth?* Rodopi, Amsterdam.

Kuipers, T.A. 1992. Naive and refined truth approximation. *Synthese*, **93**, 299–341.

Kuipers, T.A. 1997. The dual foundation of qualitative truth approximation. *Erkenntnis*, **47**, 145–179.

Lakemeyer, G. 1996. Limited reasoning in first-order knowledge bases with full introspection. *Artificial Intelligence*, **84**(1-2), 209–255.

Liu, F.F., & Moore, D.H. 1998. An implementation of Kripke-Kleene semantics. *Journal of Information Sciences*, **108**, 31–50.

Łos, J., & Suszko, R. 1958. Remarks on sentential logics. *Indagationes Mathematicae*, **20**, 177–183.

Łukasiewicz, J. 1967. Philosophical remarks on many-valued systems of propositional logic. *Pages 40–65 of:* McCall, S. (ed), *Polish Logic: 1920 - 1939.* Clarendon Press, Oxford. (Originally published in German in: *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie*, Cl. iii, 23, 1930, 51–77).

Łukasiewicz, J. 1971. On the principle of contradiction in Aristotle. *Review of Metaphysics*, **24**, 485–509. (Originally published in German in: *Bull. Inter. de l'Académie des Sciences de Cracovie*, 1910, 15–38).

Makinson, D. 1989. General theory of cumulative inference. *Pages 1–18 of:* Reinfrank, M., de Kleer, J., & Ginsberg, M.L. (eds), *Nonmonotonic reasoning.* Lecture Notes in Artificial Intelligence, vol. 346. Springer-Verlag, Berlin.

Makinson, D. 1993. Five faces of minimality. *Studia Logica*, **52**, 339–379.

Makinson, D., & Gärdenfors, P. 1991. Relations between the logic of theory change and non-monotonic logic. *Pages 185–205 of:* Fuhrmann, A., & Morreau, M. (eds), *The Logic of Theory Change.* Lecture notes in Artificial Intelligence, vol. 465. Springer-Verlag, Berlin.

Malinowski, G. 1993. *Many-Valued Logics.* Oxford University Press.

McCarthy, J. 1967. A basis for a mathematical theory of computation. *Pages 33–70 of:* Brafford, P., & Hirschberg, D. (eds), *Computer programming and formal systems.* North-Holland, Amsterdam.

Miller, D. 1974. Popper's qualitative theory of verisimilitude. *British Journal for the Philosophy of Science*, **25**, 166–177.

Miller, D. 1996. Forum: A multiple-conclusion specification logic. *Theoretical Computer Science*, **165**(1), 201–232.

Milner, R. 1971. An algebraic definition of simulation between programs. *In: Proceedings of the Second International Joint conference on Artificial Intelligence. British Computing Society.*

Mortensen, C. 1989. Paraconsistency and C_1. *Pages 289–305 of:* Priest, G., Routley, R., & Norman, J. (eds), *Paraconsistent Logic: Essays on the Inconsistent.* Philosophia Verlag GmbH, München.

Niiniluoto, I. 1987a. *Truthlikeness.* D. Reidel, Dordrecht.

Niiniluoto, I. 1987b. Verisimilitude with indefinite truth. *Pages 187–195 of:* Kuipers, T.A. (ed), *What is closer-to-the-truth?* Rodopi, Amsterdam.

Niiniluoto, I. 1998. Verisimilitude: The third period. *British Journal for the Philosophy of Science*, **49**, 1–29.

Oddie, G. 1986. *Likeness to Truth.* D. Reidel, Dordrecht.

Park, D. 1981. Concurrency and automata on infinite sequences. *Pages 167–183 of: Theoretical computer science: 5th GI-conference, Karlsruhe.* Lecture Notes in Computer Science, vol. 104. Springer-Verlag, Berlin.

Plotkin, G.D. 1976. A Powerdomain Construction. *SIAM Journal of Computing*, **5**, 452–487.

Popper, K.R. 1963. *Conjectures and Refutations.* Routledge and Kegan, London.

Priest, G. 1979. The logic of paradox. *Journal of Philosophical Logic*, **8**, 219–241.

Priest, G. 1987. *In Contradiction.* Nijhoff international philosophy series, vol. 39. Martinus Nijhoff Publishers, Dordrecht.

Priest, G., Routley, R., & Norman, J. (eds). 1989. *Paraconsistent Logic: Essays on the Inconsistent.* Philosophia Verlag GmbH, München.

Pynko, A.P. 1995a. Characterizing Belnap's logic via De Morgan's laws. *Mathematical Logic Quarterly*, **41**(4), 442–454.

Pynko, A.P. 1995b. On Priest's logic of paradox. *Journal of Applied Non-Classical Logics*, **5**(2), 219–225.

Reichenbach, H. 1944. *Philosophic foundations of quantum mechanics.* California University Press.

Restall, G. 1995. Four-valued semantics for relevant logics (and some of their rivals). *Journal of Philosophical Logic*, **24**(2), 139–160.

Ruet, P., & Fages, F. 1997. Combining explicit negation and negation by failure via Belnap's logic. *Theoretical Computer Science*, **171**(1-2), 61–75.

Ryan, M.D. 1991. Defaults and revision in structured theories. *Pages 362–373 of: Proceedings of the Sixth IEEE Symposium on Logic in Computer Science (LICS)*. IEEE, Amsterdam.

Ryan, M.D. 1992. *Ordered Theory Presentations*. Ph.D. thesis, Department of Computing, Imperial College, London SW72BZ.

Ryan, M.D., & Schobbens, P. 1995. Belief revision and verisimilitude. *Notre Dame Journal of Formal Logic*, **36**(1), 15–29.

Rybakov, V.V. 1997. *Admissibility of logical inference rules*. North-Holland, Amsterdam.

Satoh, K. 1988. Nonmonotonic reasoning by minimal belief revision. *Pages 455–462 of: Proceedings International Conference on Fifth Generation Computer Systems*. Ohm, Tokyo.

Schurz, G., & Weingartner, P. 1987. Verisimilitude defined by relevant consequence-elements. *Pages 47–77 of:* Kuipers, A.F. (ed), *What is closer-to-the-truth?* Rodopi, Amsterdam.

Scott, D. 1970. Outline of a mathematical theory of computation. *Pages 169–176 of: Proceedings of the Fourth Annual Princeton Conference on Information Sciences and Systems*.

Scott, D. 1972. Continuous lattices. *Pages 97–136 of: Toposes, algebraic geometry and logic.* Lecture Notes in Mathematics, vol. 274. Springer-Verlag, New York.

Scott, D. 1974. Rules and derived rules. *Pages 147–161 of:* Stenlund, S. (ed), *Logical theory and semantical analysis.* D. Reidel, Dordrecht.

Shoesmith, D.J., & Smiley, T.J. 1978. *Multiple-conclusion logic.* Cambridge University Press.

Shoham, Y. 1988. *Reasoning about change.* MIT Press, Cambridge, Massachusetts.

Slupecki, J. 1964. An attempt of intuitionistic interpretation of three-valued Lukasiewicz logic. *In: Rozprawy Logiczne.* PWN, Warszawa.

Smithson, R.E. 1971. Fixed points of order preserving multifunctions. *Proceedings of the American Mathematical Society*, **28**, 304–310.

Smyth, M.B. 1978. Power Domains. *Journal of Computer and System Sciences*, **16**, 23–36.

Sobociński, B. 1952. Axiomatization of a partial system of three-value calculus of propositions. *Journal of Computing Systems*, **1**, 23–55.

Stark, R.F. 1996. From logic programs to inductive definitions. *Pages 453–481 of: Logic: from foundations to applications.* Oxford University Press, Yew York.

Stone, M.H. 1936. The theory of representation for Boolean algebras. *Transactions of the American Mathematical Society*, **40**, 37–111.

Tarski, A. 1930a. Fundamental concepts of the methodology of the deductive sciences. *Pages 60–109 of: Logic, Semantics, Metamathematics: Papers from 1923 to 1938.* Oxford University Press, London, 1956.

Tarski, A. 1930b. On some fundamental concepts of metamathematics. *Pages 30–37 of: Logic, Semantics, Metamathematics: Papers from 1923 to 1938.* Oxford University Press, London, 1956.

Tarski, A. 1931. The concept of truth in formalized languages. *Pages 152–295 of: Logic, Semantics, Metamathematics: Papers from 1923 to 1938.* Oxford University Press, London, 1956.

Tarski, A. 1935. On the foundations of the calculus of systems. *Pages 342–383 of: Logic, Semantics, Metamathematics: Papers from 1923 to 1938.* Oxford University Press, London, 1956.

Tichý, P. 1974. On Popper's definitions of verisimilitude. *British Journal for the Philosophy of Science*, **25**, 155–160.

Urquhart, A. 1986. Many-valued logic. *Pages 71–116 of:* Gabbay, D., & Guenthner, F. (eds), *Handbook of Philosophical Logic*, vol. III. D. Reidel, Dordrecht.

Van Benthem, J. 1987. Verisimilitude and conditionals. *Pages 103–128 of:* Kuipers, T.A.F. (ed), *What is closer-to-the-truth?* Rodopi, Amsterdam.

Van Benthem, J., van Eijck, J., & Stebletsova, V. 1994. Modal logic, transition systems and processes. *Journal of Logic and Computation*, **4**(5), 811–855.

Van Fraassen, B.C. 1966. Singular terms, truth-value gaps, and free logic. *Journal of Philosophy*, **63**, 481–495.

Van Fraassen, B.C. 1969. Presuppositions, supervaluations and free logic. *In:* Lambert, K. (ed), *The Logical Way of Doing Things.* Yale, U.P.

West, D.B. 1996. *Introduction to graph theory.* Prentice Hall, New York.

Whitney, S. 1977. *Théories linéaires.* Ph.D. thesis, Université Laval, Québec.

Winslett, M. 1988. Reasoning about action using a possible models approach. *Pages 89–93 of: Proceedings AAAI-88.*

Wisniewski, A. 1991. Erotetic arguments: a preliminary analysis. *Studia Logica*, **50**(2), 261–274.

Wisniewski, A. 1994. On the reducibility of questions. *Erkenntnis*, **40**(2), 265–284.

Wójcicki, R. 1988. *Theory of logical calculi: basic theory of consequence operations.* Kluwer Adademic Publishers, Dordrecht.

Zadeh, L.A. 1965. Fuzzy sets. *Information and Control*, **8**, 338–353.

Zadeh, L.A. 1975. The concept of a linguistic variable and its application to approximate reasoning, Part 1. *Information Sciences*, **8**, 199–249.

Zwart, S.D. 1998. *Approach to the truth: verisimilitude and truthlikeness.* Ph.D. thesis, Rijksuniversiteit Groningen, The Netherlands.