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Long-Term Portfolio Construction

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DECLARATION

Declaration

I, **Oskar Musilika**, , hereby declare that the work on which this thesis is based is my original work (except where acknowledgements indicate otherwise) and that neither the whole work nor any part of it has been, is being, or is to be submitted for another degree in this or any other university. I authorize the University to reproduce for the purpose of research either the whole or any portion of the contents in any manner whatsoever.

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My appreciation to all the helpers, I randomly encountered throughout my whole life up to now. From the primary school cleaner who advised me to stop playing around too much, go to class, and take my life seriously, to the high school teachers and students, to University Professors and close friends, to the man who leads one of the notable international corporations Berkshire Hathaway. Moreover, great appreciation for my grandmothers' constant support and appreciation for hard work, self-knowledge and discipline and perseverance.

ABSTRACT

Financial analyst commonly advice individual investors with a long investment horizon to invest in portfolios comprised more of equities. This advice is usually coupled with the practice of shifting the investor's portfolio from risky asset holdings towards bonds and cash as the investor's target date gets closer. This view rests on the notion that equities tend to be less risky over the long horizon and that stock returns exhibit mean reversion overtime. The purpose of this dissertation is to find the optimal asset allocation over various investment horizons; and investigate how the optimal asset allocation changes over the long investment horizon. The study uses data from South Africa's financial market covering the period December 2001 to December 2014. The mean-variance framework generated the optimal asset allocation over 12 investment horizons. The study finds that, over 90 percent of the portfolio should be vested into fixed-income South African bonds, with little over 5 percent equities allocation, over longer investment periods. In addition, the study found evidence of time diversification on the JSE all shares index and the presence of mean reversion properties for the all shares index. With these conclusions, implications and recommendations are suggested.

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CHAPTER 1: INTRODUCTION

This chapter highlights key findings of the single-period mean-variance framework, and then briefly discusses the shortcomings concerning that model. The chapter then introduces the debate on long-term (or multi-period) portfolio choice, followed by the discussion leading up to the multi-period asset allocation for retirement planning. The rest of the chapter consists of the following sections: section 1.1, is the background to modern portfolio theory i.e. the single-period and multi-period portfolio choice. Section 1.2 states the research problem, and the purpose of the study. Section 1.3 states the study's limitations and scope, while section 1.4 concludes the chapter.

1.1 INTRODUCTION/BACKGROUND

The objective of this dissertation is to find the optimal portfolio asset allocation over various investment horizons, for a typical life-cycle investor planning for retirement. The investor decides to put up a lump-sum amount today aiming to maximize expected utility of wealth at retirement. To find the best portfolio allocation for this investor the literature has suggested various methods of analysis depending on assumptions underlying the investor problem and asset return distribution. In most academic studies, the subject of asset allocation or portfolio selection is linked to the study of modern portfolio theory.

Modern portfolio theory started with the seminal work of Markowitz (1952), in his '*Portfolio Selection*' paper. Portfolio selection is the practice of allocating wealth[1] optimally among investment assets or financial securities. Markowitz emphasizes the notion that investment involves a trade-off between risk and reward; where risk is defined as the variance (or standard deviation) of historical returns and reward, the expected (mean) asset returns.

The risk-reward framework suggests that all mean-variance (M-V) investors would invest in the same risky efficient portfolio of securities located on the efficient frontier. The efficient frontier is by definition the locus of all non-dominant portfolios in the M-V space. Any rational investor would choose to own portfolios located on the efficient frontier, and any other portfolios not located on the efficient frontier are sub-optimal.

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[1]Wealth is the market value of the investors' assets measured in Rands. Where the author talks about end of period wealth, this refers to the real wealth of the investor at the retirement date.

Markowitz, further points out the importance of portfolio diversification to reduce portfolio risk. Portfolio diversification works when investors increase their portfolio holdings of assets that have a negative or near zero correlation between them in the portfolio. This stems from the view that the risk of the portfolio may not be equal to the weighted-sum of the risk of individual assets constituting the portfolio. Hence, to reduce the risk of the portfolio the investor must invest his/her wealth across several asset classes or individual securities.

Following the remarkable work of Markowitz, Tobin (1958) introduced the "*Separation or Mutual Fund Theorem*". "Separation", because investment assets are classified into different asset classes depending on the perceived risk of the assets. Tobin's Theorem rests on the assumptions that, investors are risk-averse[2], that they hold the same expectations with respect to expected returns, variance-covariance of the asset returns, and that there is at least one riskless asset. Thus, if these assumptions hold then all risk-averse investors will choose a linear combination of the two asset classes i.e. the riskless asset and the optimal risky portfolio located on the efficient frontier[3]. Depending on the degree of risk-aversion, investors may choose to leverage their holdings of the risky efficient portfolio (assuming no borrowing constraints).

The work of Markowitz-Tobin laid the foundation for most practical investment decision-making problems in finance. Both academics and financial practitioners acknowledge the significance of the model they presented. However, the application of that model is limited to a one-period investment horizon and therefore does not consider looking beyond a single investment period. This is limiting because most practical investment decision-making problems are concerned about asset allocation over multiple-periods or long investment horizons instead of a single period.

For example, individuals planning for retirement would be concerned about their wealth levels at the retirement date[4]. In addition, institutional investors such as pension funds, university endowment funds and charitable foundations often deal with long-term investment decision-making problems. Hence, the investment decision-making problem faced by individuals investing

[2] The concept of risk-aversion is discussed in theoretical overview section of Chapter 2.
[3] The portfolio with the highest Sharpe ratio (the portfolio's excess return over the risk-free rate divided by the standard deviation of the market portfolio), see Sharpe (1964).
[4] In this study, wealth means financial wealth; this exclude human wealth, thus wealth and financial wealth means the same in this context.

in pension or retirement funds is a long-term investment problem, that requires the investor to look beyond a single period Fischer (1983)**.**

In practice, investors are concerned about wealth beyond a single-period. Some reason that, investors want to keep a smooth consumption pattern that is; they do not want their standard of living or value of their assets to decrease, see Sorensen and Whitta-Jacobsen (2010). According to Campbell, Chan, and Viceira (2003), long-term investors are concerned about "the productivity of wealth, as well as shocks to wealth itself".

The traditional M-V framework is a simplified one-period investment model. Hence, the model assumes a constant equity risk premium i.e. defined by constant interest rate, inflation rate (constant opportunity set). In the real world investor's opportunity set changes with future movements in interest rates, and beliefs about expected long-term asset returns.

Many studies recognized this interpretation, for example Merton (1969), and Brennan, Schwartz, and Lagnado (1997), Campbell and Viceira (2002)**.** Merton (1969) observed that changing investment opportunities over the long horizon may affect the optimal porfolio choice for long-term investors. So the difference between single-period and multi-period optimal portfolio choice is the assumption about opportunity set. Brennan, et al. (1997)**,** concluded that an investor with "non-logarithmic" utility function will be concerned about hedging against shifts in the future investment opportunity set, because for a long-term investor, "a drop in interest rates may be as important for his future welfare as a substantial reduction in his current wealth".

The Markowitz-Tobin model limitations has been known since Smith (1967), Mossin (1968), Samuelson (1969), and Merton (1969, 1971, 1973). Merton (1969) is the most cited paper on multi-period portfolio problem analysis. Merton investigated the multi-period portolio choice problem for an investor planning for lifetime consumption and portfolio strategy in a continuous time framework, where he assumed security prices follows a diffusion process[5]. But the Merton model is difficult to solve in closed-form and therefore not easily tractable. According to Campbell, et al. (2003), "Solutions to the Merton model were only available in trivial cases where it reduces to the

[5] A random walk (Brownian motion) sometimes called a stochastic process, see Wooldridge (2013).

static model". This made that model less favorable in solving practical long-term investment problems.

Multi-period portfolio selection problems have been rigorously studied, following the limitations of the Merton model. Most papers extend the traditional single-period framework to the multi-period case, either attempting to solve the problem analytically or numerically. These studies differ across methodologies and assumptions the underlying asset return distribution and investor preferences or utility functions. These studies include, Mossin (1968), Kim and Omberg (1996), Campbell and Viceira (1999), Watcher (2002), Campbell, et al. (2003), Barberis (2000), Balduzzi and Lynch (1999), Brennan, et al. (1997) etc.

A common feature in these studies is the assumption of asset return predictability and non-constant investment opportunities over multiple investment horizons. Asset return predictability is based on the empirical evidence provided by Fama and Schwert (1977), Campbell (1987), Fama and French (1988), Campbell and Shiller (1988), Glosten, et al. (1993) etc.

These authors argue that asset returns are predictable, because asset returns have predictable transitory components i.e. the time-varying risk premiums—due to changes in economic variables fundamental to the asset return dynamics. Hence, the primary argument in these studies is that systematic revisions in asset allocation helps to improve portfolio returns. According to Lee (1990), "portfolios that are revised in response to changes in conditional forecasts of expected equity returns should *ex post* generate higher overall average returns than passive (buy-hold) portfolios, which naively assume constant expected equity returns".

The assumption of asset return predictability is significant for tactical portfolio allocation[6], which requires constant rebalancing of the portfolio overtime. However, the practice of portfolio rebalancing may be a costly practice—high transactions cost—to retail of individual investors. Individual investors would incur greater transactions cost if they choose to rebalance their portfolios overtime. Nevertheless, for the case of an individual planning for retirement; would be

[6] Tactical asset allocation, the portfolio manager continuously rebalances the portfolio in line with changes economic variables to enhance portfolio returns. This is a costly procedure from the individual investors' perspective; hence, institutional investors usually practice it.

important to assume that asset returns are predictable? Perhaps in a dynamic portfolio rebalancing optimization context.

This dissertation investigates the long-term optimal portfolio asset allocation for a typical life-cycle individual investor planning for retirement. The study, assumes asset returns are not fully predictable because of estimation errors inherent in asset returns dynamics. This fact makes it difficult to adopt the assumption of asset return predictability, to solve life-cycle investment problems. After all, it is well established in the literature that *ex post* returns are the best predictor of *ex ante* return performance.

The primary reason for studying optimal portfolio asset allocation is that, portfolio selection accounts for over 90 percent of asset portfolio performance, see for example Brinson, Hood, and Beebower (1986), Ibbotson and Kaplan (2000) and others. This result is perhaps the propounding reason why we are overly concerned about solving the asset allocation problem.

There is a number of empirical work on the topic concerning long-term optimal portfolio allocation. Most of these studies use the M-V model to investigate the optimal portfolio construction over several investment horizons. Fama (1970), showed that the traditional M-V framework can be used to assess multi-period portfolio problems. For example, Chopra & Ziemba (1993), Gunthorpe and Levy (1994), Tang and Lee (1997), etc. used the M-V framework to assess optimal portfolio allocation over multiple investment horizons. Others like Alles and Athanassokos (2006), and Hickman, Hugh, Byrd, Beck, and Terpening (2001) use the re-sampling (bootstrapping) techniques.

The prime focus of these studies is the relationship between optimal portfolio allocation and the investment horizon. Some primarily focused on testing the risk-investment horizon relationship. This is the concept of time diversification often alluded to in the literature; see for example Kritzman (1994). Kritzman defines time diversification as the notion that above average returns tend to offset below average returns over the long investment horizon, and therefore implying the existence of mean reversion in asset returns.

The notion of time diversification has shaped some of the publicly known ideas on investing. For example, the commonly held advice adopted by financial practitioners of recommending young investors to invest more in the risky asset class or equities for the long-term is based on the notion of time diversification. Some studies for example, Siegel (1998) observed that equities tend to be less risky over the long-term, and suggests young investors to hold portfolios invested more in equities (risky assets), and shift their portfolio holdings into safer assets like bonds and cash when near retirement date. Other studies conclude that longer investment horizon requires a lower asset allocation to risky assets Jacquier, Kane and Marcus (2005), and Alles and Athanassokos (2006).

The literature on the relationship between portfolio allocation and time has received considerable attention, and the conclusions drawn are often mixed. Nevertheless, the commonly held view is that there is a relationship between optimal portfolio allocation and the investment horizon i.e. stocks tend to be less risky over the long horizon. This relationship depends on the assumption underlying asset return distribution. Lee (1990), argues that, the "risk reduction benefits from diversification overtime are not as a consequence of risk pooling but rather non-stationarity in asset return distribution".

This study makes three contributions to the literature on long-term optimal portfolio allocation. Firstly, the study uses monthly asset return data from South Africa. Most empirical work focus on the long-term portfolio allocation using data from the United States (US) and other developed economies, because the US and other countries have long enough historical datasets extending over many years. Studies outside the US are rare because of the limited span of historical data records. In addition, the paper investigate the risk involved in investigating wealth on the JSE over time (for 40 years). To see what happens to the standard deviation of the JSE overtime.

Secondly, the risky asset class consists of two measures i.e. the Style growth and value equity indexes. Hence, the study contributes to the literature by investigating how the optimal asset allocation differs between the two risky assets overtime. For example, if over the long horizon the life-cycle investor's portfolio is more invested in growth or value portfolios, this would provide evidence regarding the common "value investors" belief that over the long-term value share are a better investment than the so called "hot stocks" or growth stocks.

Finally, unlike most of the studies on the topic under consideration, the study uses a money market index as a proxy for short-term rates (cash). Most studies use Treasury bills return data as the measure for the cash component of asset allocation. Most studies in South Africa use this measure for short-term or cash investments[7].

In summary, the prime focus of this paper is long-term portfolio construction, for an investor planning for lifetime consumption at the retirement date. The primary objective is to find the optimal asset allocation over various investment horizons and investigate the empirical evidence concerning the concept of time diversification for the case of South Africa.

1.2 PROBLEM STATEMENT AND PURPOSE OF THE STUDY

Consider an investor who has some amount (or level) of wealth, the investor wants to allocate this this wealth into some investment vehicles, broadly classified into asset classes; equities, bonds, and cash. The investor wants to enjoy maximum expected utility of wealth at some point in time in the future i.e. the retirement date. Therefore, he must decide on the best asset allocation strategy that will yield the highest expected return to meet his/her consumption needs at the retirement date.

The investor decides how much to invest, how long (the investment horizon), what asset classes to consider, and what return is earned in order to achieve maximum tax benefits over his or her life cycle i.e. how much of that return will be submitted as investment costs (transaction costs and taxes). The investors would usually have a great amount of control over these factors except, the expected returns, which are highly unpredictable.

In this case, the investor's objective is to find the best mix of asset classes over different holding periods that would achieve maximum expected utility of wealth given expected return and risk upon the retirement or terminal date.

Conventional wisdom suggests that young investors should hold more stocks earlier in their portfolios and gradually reduce their stock holdings to safer assets as they approach the retirement date. This is the traditional optimal portfolio allocation advice quoted by most practitioners.

Upon investigation of long-term optimal portfolio construction, the study aims to investigate how the optimal portfolio composition changes over the long investment horizon. That is, to observe

[7] Auret and Vivian (2014)

the differences (if any) in portfolio compositions over different investment horizons. With this objective in mind, the study attempts to provide answers to the following research questions:

a) For an investor planning for retirement, what is the optimal portfolio composition over the selected investment period(s)? That is to find out the best combination of equities, bonds and money market funds (or cash) over the selected investment horizon.
b) Given that we know the optimal portfolio composition over the selected investment horizon, how does that portfolio composition change over time? What happens to the portfolio over the long horizon i.e. is the portfolio more invested equities at the shorter (longer) horizon or more invested in bonds or cash at shorter (longer) horizon?
c) What investment implications does the findings derived from “a)” cum “b)”, suggest about the relationship (if any) between optimal portfolio composition and the investment horizon? If for example we find that at longer investment horizon the optimal portfolio is more invested in less risky assets (bonds and cash), what inferences can be drawn about optimal asset allocation and the investment horizon? Is there conclusive evidence to support the notion of time diversification cited in the literature?

The first question stems from the traditional industry practice of finding the optimal asset allocation for an investor with a long-term outlook on his/her wealth level. This question is important because it would show the optimal portfolio composition at different investment horizon(s), compared to the traditional practice of just finding the portfolio composition at a single one-year investment horizon common in the Markowitz-Tobin paradigm. For example finding the optimal asset allocation for one year, five years, ten years etc. Hansson and Persson (2000), also attempted to provide answers to this question using the traditional mean-variance framework and monthly equities and Treasury bills return data from the US.

The second question is concerned about the relationship between portfolio allocation and the investment horizon. This is the core question; it would attempt to provide evidence concerning the industry practice of recommending younger people to invest in risky assets earlier (and then tilt their portfolios towards safer assets—away from risky assets—as the people approach retirement). Therefore, with this question, the study attempts to provide empirical evidence concerning the concept of time diversification, which implies a relationship between portfolio allocation and the investment horizon. Chapter 3 contains the method used to provide empirical evidence concerning these questions.

1.3 STUDY LIMITATIONS AND RESEARCH SCOPE

This study is limited to the South African context, in the sense that is uses Johannesburg securities exchange asset return data. More specifically the study is limited to using four local indices: FTSE/JSE Style growth index, FTSE/JSE Style value index, BEASSA All bond index, and the money market index (the Alexander Forbes money market index). The study uses monthly data covering the period from December 2001 to December 2014.

For the purpose of this study, the concept "investment horizon" refers to a period of time an investor commits him/herself to holding certain assets over "that" period regardless of the market vicissitudes. That is to say, the investor is committed to hold a portfolio invested in a specified asset class, for investment purposes. The investors' objective and commitment to investing certain level of wealth should be fixed overtime, and not change unpredictably when market phenomenon go against the investors desires. This limitation stems from the concept of investment itself.

The study is not concerned about the implications of return predictability and intertemporal hedging demand for optimal portfolio choice. This has received considerable attention in the literature concerning dynamic portfolio strategies. Instead, the study investigates multi-period optimal asset allocation using the M-V analysis. Intermediate consumption between the current period the end or terminal period is important in life-cycle portfolio analysis. However, the nature of the problem this study aim to address does not require intermediate consumption.

The concept of human capital as an asset, whose returns are derived from wages and other form of compensation plays an important role in asset allocation overtime. Human capital (labor income), may act as a hedge against portfolio losses (portfolio reparation) over the investment period. Labor income however, complicates the multi-period model, in the sense that it may not be easily tractable. Hence, the author advices future studies to consider the implications of human capital on portfolio construction.

Transaction costs and tax implications on the investors' terminal wealth are ignored. Transaction costs are relevant when dealing with dynamic optimization; portfolio optimization that deals with frequent portfolio rebalancing. Taxes are relevant to this study, but this paper rather suggests future studies to consider the implications of taxes on long-term portfolio choice. For example how the future tax rates may reduce the consumer's terminal wealth level.

The norm in the investment community in asset allocation involves an investor who makes annuity (or regular monetary contributions) payments to a fund, and the entrusted portfolio manager then decides how the total value of the fund would be allocated overtime. The focus of this study is a life-cycle investor who decides to put up a lump sum or cash amount today. That is, an investor who decides to invest a certain amount of money today, of say R100. Hence, portfolio construction in this context not about regular savings contributed to a fund or Unit Trust. Future studies are encouraged to consider the life practice of asset allocation and annuity contributions—in a dynamic portfolio framework, which include portfolio rebalancing overtime. Finally, the study is not concerned about how the investor chooses to construct his/her portfolio after the retirement date.

1.4 CONCLUSION

In conclusion, despite the limitations of the Markowitz-Tobin mean-variance model, that framework is still the basis for most financial problems concerned with asset allocation decision-making. Many studies have attempted to extend the single-period model to multi-period model, for example Merton (1969). Since then a rich dynamic literature on multi-period optimal portfolio choice has emerged, mostly considering the issue of asset return predictability.

The objective of the study is to find the optimal asset allocation for several investment horizons. Further, the paper aims to investigate how the optimal portfolio allocation changes over the investment horizon. The answers to these questions have implications on the life cycle investment for retirement considerations overtime. In the sense that proper guidance avail to the investor putting up some wealth level, in order to maximize expected utility of wealth at the retirement date.

The study uses South African asset class returns data covering the period December 2001 to December 2014; and JSE all shares index data from January 1960 to December 2008. These data together with the mean-variance optimization will help provide answers to the research questions concerning asset allocation over the various investment horizons, and provide findings on the concept of time diversification for the South African case.

The rest of the dissertation is organized in the following fashion: Chapter 2 contains the theoretical framework and the literature review. The theoretical framework section describes some key finance concepts and definitions of asset returns. Furthermore, the section covers the details of portfolio construction in a single-period framework. This framework is described in vector notations, and the model is expanded to cover the *Mutual fund theorem* proposed by Tobin (1958). Following the theoretical framework section is the literature review on some of the closest studies, starting with the development of modern portfolio theory of Markowitz (1952), and further review studies that attempted to expand the Markowitz-Tobin model to multi-periods. This section also covers a short literature review on asset allocation for South Africa, and then discusses the issue concerning estimation error. This is then followed by a review of the literature covering the relationship between asset allocation and the investment horizon i.e. time diversification.

Chapter 3 contains the method of analysis used to provide answers to the research questions stated in section 1.2 of this chapter. In this section, the data description is provided, and graphical representations of the data is covered. Further, that section covers the descriptive or summary statistics of the data and highlights return computations used to compute the data that is used for analysis in chapter 4. In addition, the chapter highlights the issue of estimation error noted by Chopra and Ziemba (1993).

Chapter 4 contains the mean-variance optimization results and interpretations of these results. The first section of the chapter focuses on time diversification evidence on the JSE all shares index, by plotting the standard deviation of the JSE over different investment horizons. This section is then followed by the mean-variance analysis, to find the optimal portfolio asset allocation over several investment horizons using rolling period returns. In addition, the chapter contains the scenario analysis section, which aims compare different portfolio allocation overtime. Finally, chapter 5 concludes the dissertation with implications of the results and with suggestions for further study recommendations.

CHAPTER 2: THEORETICAL FRAMEWORK AND LITERATURE REVIEW

2.1 INTRODUCTION

The objectives of this chapter is to highlight the theoretical framework underlying most modern portfolio theory studies, and review the literature of studies closest to the topic under study. Section 2.2 is the theoretical framework section; first providing definitions to key concepts in the finance literature, and secondly, stating the single-period model based on Markowitz (1952), Markowitz (1959), and Tobin (1958) analyses. Section 2.3 is the literature review, which summarizes selected papers on optimal portfolio allocation over multi-periods. This part of the summary consists of early work attempting to extend the single-period model to the multi-period model. The last section of the literature review focuses on studies concerning the relationship between investment horizon and optimal portfolio allocation. Finally, section 2.4 concludes the chapter.

2.2 THEORETICAL FRAMEWORK

The section gives an overview on the Markowitz (1952), Markowitz (1959) portfolio selection model and then the extended version of Tobin (1958)'s model. First, the section explains the popular finance theory jargons. These concepts are important in understanding the rest of the paper.

2.2.1 Concepts definitions

2.2.1.1 Expected utility and Risk aversion

The following explanation is adopted directly from Campbell and Viceira (2002), Varian (1992), and Pratt (1964), and Arrow (1965).

Let us consider the case where the lottery space consists solely of gambles with money prizes. We know that if the consumer's choice behavior satisfies the various required axioms, we can find a representation of utility that has the expected utility property. The expected utility property says that the utility of the lottery is the expectation of the utility from its prizes. This means that we can describe the consumer's behavior over all money gambles if we only know this particular representation of his utility function for money.

For example, assume the consumer has expected utility of a gamble $p \cdot x \oplus (1-p) \cdot y$ we just look at $pu(x) + (1-p)u(y)$, where p represent the objective probability, u is the utility and x and y are random payoff.

This construction is illustrated in figure 2.1 for an assumed probability value of ½ (or $p = 1/2$). Notice that in this example the consumer prefers to get the expected value of the gamble. That is, the utility of the lottery $u(p \cdot x \oplus (1-p) \cdot y)$ is less than the utility of the expected value of the lottery, $px + (1-p)y$. Such behavior is called risk aversion. A consumer may also be risk loving; in such a case, the consumer prefers a lottery to its expected value.

If a consumer is risk-averse over some region, the chord drawn between any two points of the graph of his utility function in this region must lie below the function. This is equivalent to the mathematical definition of a concave function. Hence, concavity of the expected utility function is equivalent to risk aversion.

It is often convenient to have a measure of risk aversion. Intuitively, the more concave the expected utility function, the more risk-averse the investor. Thus, we might think we could measure risk aversion by the second derivative of the expected utility function. However, this definition is variant to changes in the expected utility function: if we multiply the expected utility function by 2, the consumer's behavior does not change, but our proposed measure of risk aversion does change. However, if we normalize the second derivative by dividing by the first, we get a reasonable measure, known as the Arrow-Pratt measure of (absolute) risk aversion Arrow (1965) and Pratt (1964):

$$A(W) = -\frac{U''(W)}{U'(W)} \qquad 2.1$$

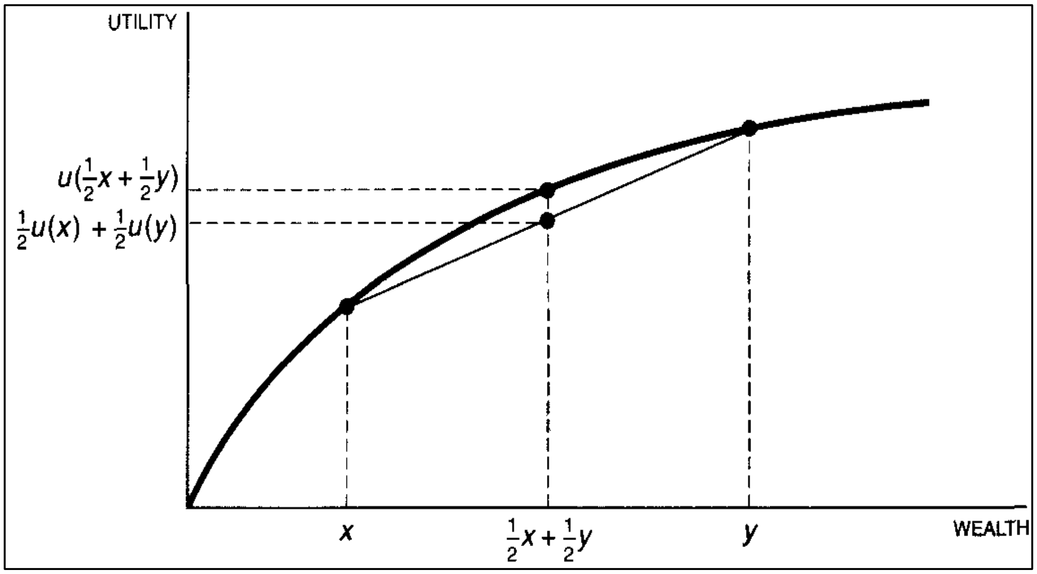
If the individual is risk-averse the, $U''(W) < 0$, the coefficient of absolute risk-aversion is positive. $A(W)$ depends on the individual's wealth level and may increase or decrease as the wealth level changes. $A(W)$, provides information about the decision-maker's attitude towards risk, not about particular numerical representations of them.

In addition, we define the coefficient of relative risk aversion as:

$$R(W) = WA(W) = -W\frac{U''(W)}{U'(W)} \qquad 2.2$$

Relative risk aversion, $R(W)$, conveys information about the investors attitude towards investing a fraction of his wealth in risky assets. The greater the value of $R(W)$ the more the investor is less willing to risk greater portion of his wealth.

Figure 2.1: Expected utility of a gamble (lottery)



Source: Varian (1992)

The expected utility of the gamble is $\frac{1}{2}u(x) + \frac{1}{2}u(y)$. The utility of the expected value of the gamble is $u(1/2x + 1/2y)$. In the case depicted above, the utility of the expected value is higher than the expected utility of the gamble, so the consumer is risk averse. The pain of losing is greater than the joy of wining.

The curvature of the utility function represents the investors risk attitude or willingness to take risk i.e. risk aversion. The slope of the utility function represents the investor's marginal utility i.e. the addition to utility for small changes in wealth. Marginal utility declines with increases in wealth levels.

The concavity of the utility function implies that standard finance investors always prefer a sure amount to a gamble with the same expected value. This view is contrary to that of the seminal findings of Kanneman and Tversky (1979) that investors are not standard finance investors, but behavioral investors: their utility is not described well as a function of wealth, and they are not always risk averse.

2.2.1.2 Utility functions

The following explanation is adopted from Campbell and Viceira (2002). Tractable models of portfolio choice require assumptions about the form of the utility function, and possibly distributional assumptions about asset returns. Three alternatives sets of assumptions produce simple results that are consistent with those of mean-variance analysis:

a) Investors have quadratic utility defined over wealth. In this case $U(W_{t+1}) = aW_t + 1 - bW^{2t} + 1$. Under this assumption, maximizing expected utility is equivalent to maximizing a linear combination of mean and variance. No distributional assumptions are needed on asset returns. Quadratic utility implies that absolute risk aversion and relative risk aversion are increasing in wealth.
b) Investors have negative exponential utility, $U(W_{t+1}) = -exp(-\theta W_{t+1})$, and asset returns are log-normally distributed. Exponential utility implies that absolute risk aversion is a constant θ, while relative risk aversion increases in wealth.
c) Investors have power utility, $U(Wt+1) = W_{t+1}^{1-\gamma}/(1-\gamma)$, and asset returns are log-normally distributed. Power utility implies that absolute risk aversion is declining in wealth, while relative risk aversion is a constant γ. The limit as γ approaches one is log utility: $U(W_{t+1}) = \log(W_{t+1})$.

We have already argued that absolute risk aversion should decline, or at the very least should not increase with wealth. This rules out the assumption on quadratic utility, and favors power utility over exponential utility. The power utility property of constant relative risk aversion is inherently attractive, and is required to explain the stability of financial variables in the face of secular economic growth.

The choice between exponential and power utility also implies distributional assumptions on returns. Exponential utility produces simple results if asset returns are normally distributed, while power utility produces simple results if asset returns are log-normally distributed (i.e. if their logs are normal).

The assumption of normal returns is appealing for some purposes, but it is inappropriate for the study of long-term portfolio choice because it cannot hold at more than one time horizon. If returns are normally distributed at a monthly frequency, then two-month returns are not normal because they are the product of two successive normal returns and sums of normal, not products of normal, are themselves normal. The assumption of lognormal returns, on the other hand, can hold at every time horizon since products of lognormal random variables are themselves lognormal. In addition, lognormal random variables can never be negative so the assumption of log-normality is consistent with the limited liability feature of most financial assets.

The assumptions of lognormal returns run into another difficulty. It does not carry over straightforwardly from individual assets to portfolios. A portfolio is a linear combination of individual assets; if each asset return is lognormal, the portfolio return is a weighted average of lognormal returns, which is not lognormal. This difficulty can be avoided by considering short time intervals. As the time interval shrinks, the non-log-normality of the portfolio return diminishes, and it disappears altogether in the limit of continues time.

2.2.1.3 Asset prices and returns

The following explanations follows from Campbell, et al. (1997). Let P_t denote the current price of an asset at date t. Then we define the simple net returns between $t-1$ and t (no dividends) as:

$$R_t = \frac{P_t}{P_{t-1}} - 1 \qquad 2.3$$

Gross (or arithmetic) returns, same asset and same period is simply $R_t + 1$, or $\frac{P_t}{P_{t-1}}$[8]. Gross returns over k periods from $t-k$ to k be:

$$\left(1 + R_t(k)\right) = (1 + R_t) \cdot (1 + R_{t-1}) \dots . (1 + R_{t-k+1}) \qquad 2.4$$

$$= \frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}} \cdot \frac{P_{t-2}}{P_{t-3}} \dots . \frac{P_{t-k+1}}{P_{t-k}} = \frac{P_t}{P_{t-k}}$$

Net (compound or multi-period) return over the same k periods is $R_t(k)$. For comparability, usually report annualized returns (geometric average):

$$\text{Annualized } R_t(k) = \left[\prod_{j=0}^{k-1}\left(1 + R_{t-j}\right)\right]^{\frac{1}{k}} - 1 \qquad 2.5$$

[8] The price here refers to the market price, not necessarily the security's value. Generally a security's "intrinsic" value or "fundamental" value is given by the discounted value of its expected future cash flows. For this value we could write $P_t = \sum_{k=1}^{T} E_t(C_{t+k})\delta^k$, where δ is the appropriate discount factor, and T represents the holding horizon.

For small single-period returns, quick and coarse approximation (arithmetic average) often used ($Taylor\ expansion$):

Annualized $R_t(k) \approx \frac{1}{k}[\sum_{j=0}^{k-1} R_{t-j}]$ 2.6

The continuously compounded return, or log return, $\boldsymbol{r_t}$ of an asset is

$r_t = \log(1 + \mathrm{R_t})$ 2.7

$= \log\left(\frac{\mathrm{P_t}}{\mathrm{P_{t-1}}}\right) = p_t - p_{t-1}$, where $p_t = \log\mathrm{P_t}$

Advantage: additive multi-period returns using 2.4 above

$$r_t(k) = \log(1 + R_t(k))$$

$$= \log\left(\prod_{j=0}^{k-1}\left(1 + R_{t-j}\right)\right)$$

$$= \sum_{j=0}^{k-1} \log(1 + R_{t-j})$$

$$= r_t + r_{t-1} + \ldots + r_{t-k+1} \qquad 2.8$$

In addition to computational convenience, it will be seen that it is easier to derive time-series properties additive processes than of multiplicative processes.

Let D_t denote dividend/cash payments just before t. Net simple return between $t-1$ and t becomes

$R_t = \frac{P_t + D_t}{P_{t-1}} - 1$ 2.9

Let R_{0t} denote return on some reference asset. Simple excess return (risk premium) on asset *i* is

$Z_{it} = R_{it} - R_{0t}$ 2.10

Alternatively, log excess return is:

$z_{it} = r_{it} - r_{0t}$, 2.11

, where lowercase letters denote logs of upper case letters.

The above exposition follows from Campbell, et al. (1997) and Hassan (2014).

2.2.1.4 Portfolios

Consider a setting with N securities or assets, indexed by $i = 1, 2, \ldots., N$. Simply, a portfolio say w is an $-vector$ $(w_1, w_2, \ldots\ldots.. w_N)$, such that $\sum_{i-1}^{N} w_i = 1$, where the w_i represents the proportion of available funds invested in security i. If short sales are not allowed, then the asset weights are restricted to positive values (i.e. $0 \leq w_i \leq 1$). If a given portfolio has $w_K < 0$ for some security k, then the portfolio has a "short" position in that security i.e. the investor has borrowed security k. The investor will short sell a security if he believes the price of that security or asset will be drop or decrease, and if it decreases, then the investor would be able to buy the security back at a much lower price and give it back to the owner of the asset (including any dividends incurred during the period). The difference between the price at which the investor sold the security and the price, which he bought it back, is the investor's profit.

The one-period return on a portfolio depends on the returns of the securities it takes positions on. Specifically, it is the weighted average (we ignore the time subscript when we are in a one-period setting),

$$R_w = w_1 R_1 + \ldots + w_N R_N \qquad 2.12$$

2.2.1.5 Uncertainty

Uncertainty is represented simply by a sample space (set of possible "states" of the world"), $\Omega = \{\omega_1, \ldots\ldots, \omega_M\}$, with an associated set of probabilities $Q = \{q_1, \ldots\ldots q_M\}$, such that $\sum_{j=1}^{M} q_j = 1$, and $0 < q_j < 1$ for all $j = 1, \ldots\ldots, M$. By the expected return on each security, we mean:

$$ER_i = q_1 R_1(\omega_1) + \ldots + q_M R_i(\omega_M) \qquad 2.13$$

2.2.1.6 Expected portfolio return

This is the weighted average of individual securities expected return, with the weights determined by the portfolio's allocation to each security:

$$ER_i = \omega_1 ER_1 + \ldots + \omega_N ER_N \qquad 2.14$$

Let μ denote the vector of expected returns $(\mu_1, \ldots\ldots., \mu_N)$, where $\mu_i = ER_i$. We can then write a portfolio's expected return more compactly as $\boldsymbol{w'\mu}$.

2.2.1.7 Portfolio variance

Let **V** denote the variance-covariance matrix

$$V = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \cdots & \sigma_{NN} \end{bmatrix} \quad 2.15$$

Where

$$\sigma_{ij} = Cov(R_i, R_j) \text{ for } i \neq j \quad 2.16$$

$$\sigma_{ii} = \sigma^2 = Var(R_i)$$

Then for a portfolio w, we have

$$Var(R_w) = E(R_w - ER_w)^2 \quad 2.17$$

$$= \mathbf{w}'\mathbf{V}\mathbf{w}$$

2.2.1.8 Portfolio Diversification

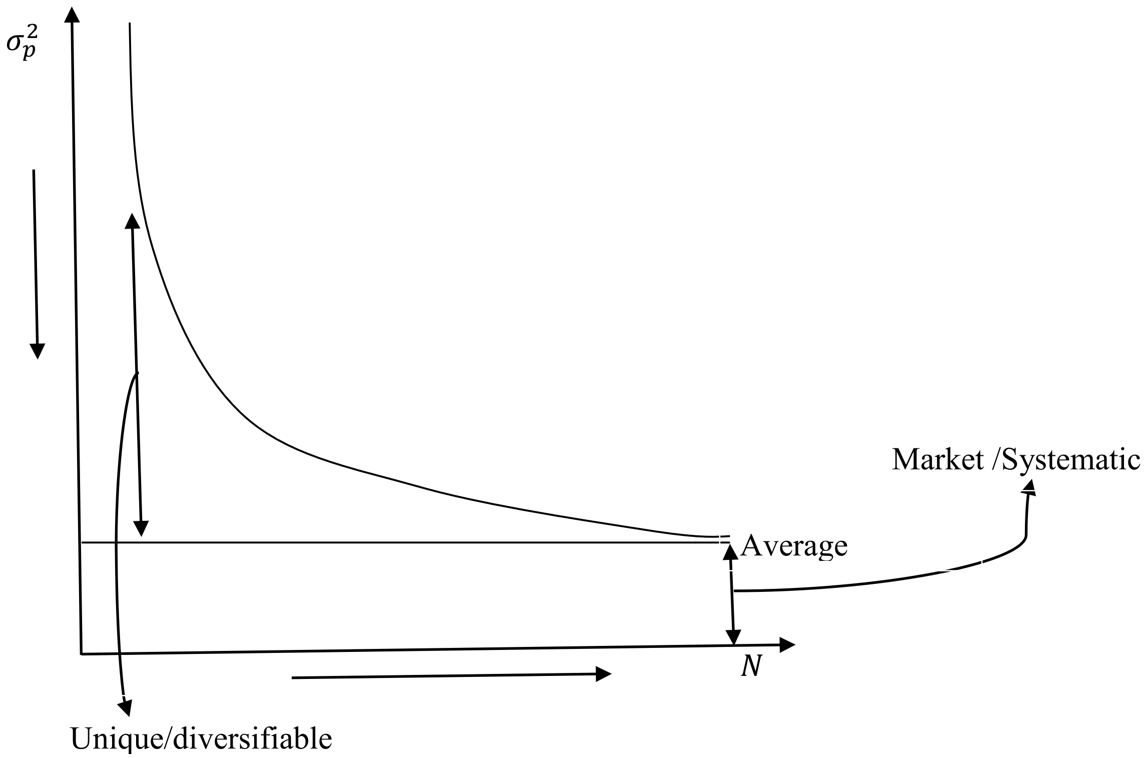
Portfolio diversification means reducing the portfolio risk by increasing the number of securities or assets making up the portfolio. This follows, because the risk of the portfolio is generally different from the weighted-average sum of the risk of the individual securities constituting the portfolio. Hence, by increasing the number of securities in the portfolio, the investor reduces portfolio risk; this is the benefit of diversification cited in the literature.

We can write

$$\mathbf{w}'\mathbf{V}\mathbf{w} = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} \quad 2.18$$

$$= \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j \neq i}^{N} w_i w_j \sigma_{ij}$$

Figure 2.2: Diversification



Letting $w_i = \frac{1}{N}\ \forall_i = 1, \ldots, N$. The graph above shows that, as we arbitrarily increase N (the number of securities in the portfolio), the variance of the portfolio does not approach zero, it approaches the average covariance. Hence, the contribution to the portfolio variance from the variance of individual securities' returns becomes negligible (only the covariance matter).

Figure 2.2 further, shows that asset specific risk can be reduced through diversification (increasing N); the only risk that the portfolio would be exposed to is the market risk, which cannot be diversified away. Asset specific risk can be diversified away because the investor has control over the assets, which would be invested in, that the investor could reduce this risk by investing in industries, or companies, which have near zero or negative correlation. Market risk is not diversifiable because it is subject to exogenous factors (factors beyond the control of the securities firm) risk exposures beyond the control of assets of companies, for example interest rates, inflation expectations and many other idiosyncratic factors.

2.2.2 Portfolio theory (single-period model)

The single-period mean-variance model is based on the seminal work of Markowitz (1952), Markowitz (1959) and Tobin (1958)[9]. In this model, investors are trying to find the optimal portfolio choice in the mean-variance space.

2.2.2.1 The opportunity set and the efficient frontier

The efficient frontier is a locus of all non-dominated portfolios in the mean-standard deviation space; and by definition, any "rational" mean-variance investor would choose to hold portfolios located on that efficient frontier. Any portfolios not located on the efficient frontier are sub-optimal.

The fundamental idea in this section is that: the expected return to a portfolio is the weighted average of the expected returns of the individual assets making up the portfolio. However, the same conclusion does not hold for the variance of the portfolio. The variance of the portfolio is generally smaller than the weighted average of the variances of the individual assets making up the portfolio. Therein lies the benefits of diversification.

We illustrate this assertion, starting with the simplified portfolio of two assets only. The investor's objective is to maximize a function, $U(\mu_p, \sigma_p)$, U is the expected utility function with the conditions that: $U' > 0$ (positive marginal utility) and $U'' < 0$ (diminishing marginal utility). The investor likes expected return (μ_p) and dislikes standard deviation (σ_p). For a given asset (or portfolio) A is said to mean-variance dominate an asset (or portfolio) B, if $\mu_A \geq \mu_B$ and simultaneously $\sigma_A < \sigma_B$, or if $\mu_A > \mu_B$, while $\sigma_A \leq \sigma_B$.

Suppose we only have to two risky securities. Using the definition of the correlation coefficient,[10] we have the variance of the portfolio as:

$$Var(R_w) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2, \qquad \text{where } \rho_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2} \qquad 2.19$$

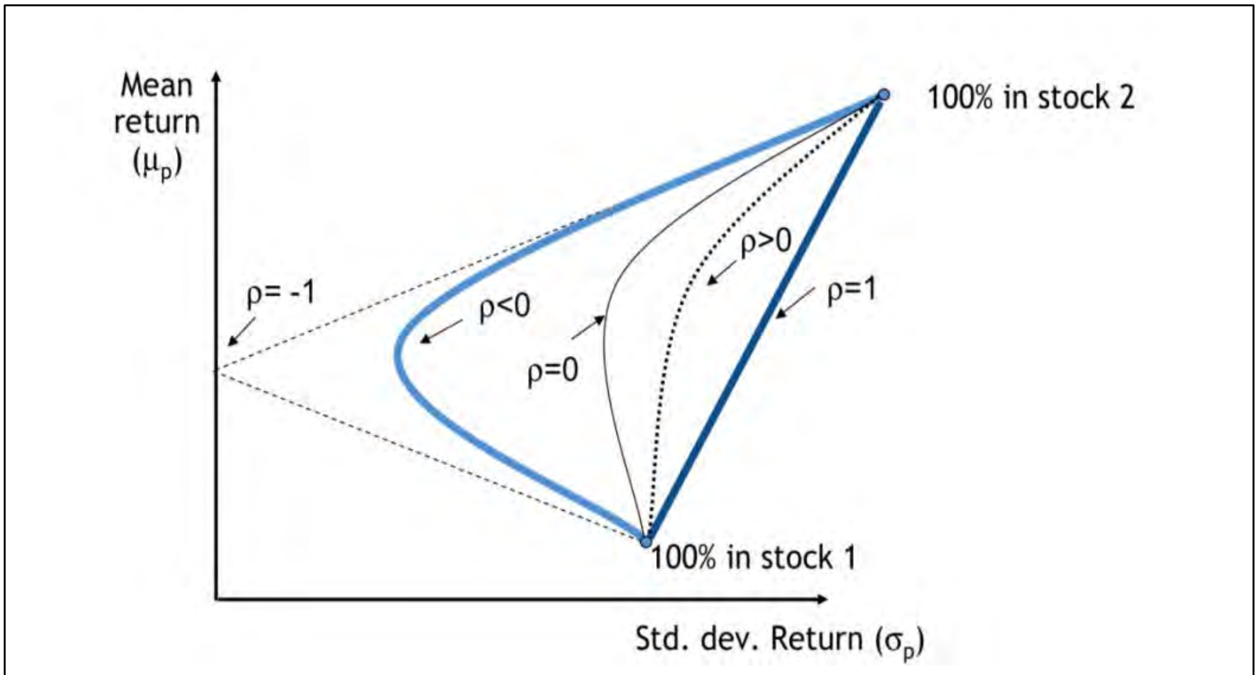
The opportunity set representing the possible mean-return and volatility pairs from simply varying the weights the portfolio attaches to each of the two securities forms a triangle. The extreme cases correspond to perfect correlation (positive or negative). Points inside the triangle correspond to

[9] We use variance, standard deviation and volatility interchangeably.

[10] The correlation coefficient measures the short-term co-movement between two assets.

different portfolio weights and imperfectly correlated returns. It is important to note how, as we move from the extreme cases of perfect correlation, the risk-return trade-off improves for a mean-variance investor. Figure 2.3 below shows this description and it is adopted from Danthine and Donaldson (2005).

Figure 2.3: The efficient frontier: two imperfectly correlated risky assets



Source: Danthine and Donaldson (2005)

When the portfolio comprises of two imperfectly correlated risky assets, the standard deviation of the portfolio is necessarily smaller than it would be if the two constituent assets were perfectly correlated (i.e. $\sigma_p \leq w_1\sigma_1 + (1 - w_1)\sigma_2$).

Figure 2.3 shows the different efficient frontier for different levels of correlations. The further away from $+1$, the more to the left is the efficient frontier. Other portfolios in fact dominate the diagram makes clear that in this case; some portfolios made up of assets 1 and 2, and hence, not all portfolios are efficient.

We distinguish the minimum variance frontier from the efficient frontier. In figure 2.3 above, all portfolios between A and B belong to the minimum variance frontier, that is they correspond to the combination of assets with minimum variance for all arbitrary levels of expected returns. However, certain levels of expected returns are not efficient targets since higher levels of returns can be obtained for identical levels of risk. Thus, portfolio C is minimum variance, but is not efficient, being dominated by portfolio D—assuming positive amounts of both assets, A and B are held.

The optimal portfolio of risky securities will differ for investors with different degrees of risk-aversion, even if we have only one efficient set. There will be one efficient frontier, applicable to all investors, if investors have the same expectations regarding expected returns and the variance-covariance matrix. The efficient set is the frontier of the opportunity set and it gives the highest mean-return for a given level of volatility, or lowest level of volatility for a given level of expected return.

2.2.2.2 Mean-variance analysis

The following discussion is on the mathematical method to identify the vector of portfolio weights associated with a given point (any point) on the mean-variance efficient frontier, corresponding to a pre-specified level of expected return, see Danthine and Donaldson (2005) and Hassan (2014)**.**

Consider an investor confronted with a set of risky securities to invest in, and all with different expected returns and standard deviation (and assuming no inside information relevant for the future value of any security). The investor's problem is to choose the relative weights for the individual assets optimally in a portfolio if the investor wants to maximize expected return for any level of standard deviation. Alternatively, minimize the standard deviation of the portfolio returns given a level of required return. We will use the variance or standard deviation as the measure of "volatility".

We now consider the case with N risky securities. We assume investors are risk-averse and mean-variance optimizers i.e. they want to maximize mean-return for a given level of volatility or minimize variance for a given level of expected return. This means that their indifference curves in mean-return—standard deviation space are convex, because investors prefer more wealth to less—they like expected return and dislike volatility. The steepness of such indifference curves represents the investor's degree of risk aversion, the more risk averse the steeper the curve, because more risk-averse investors would want to take fewer risks.

With N risky assets (no riskless asset), the mean-variance optimizing investor's problem is:

$$\min_{\mathbf{w}} \mathbf{w}'\mathbf{V}\mathbf{w} \qquad \text{Subject to}$$

$$\mathbf{w}'\boldsymbol{\mu} = \mu_a \qquad 2.20$$

$$\mathbf{1}'\mathbf{w} = 1 \qquad 2.21$$

To solve the above constrained optimization problem, we form a Langrangian and use the first order conditions for minimum. We can use the Langrangian multiplier because we do not have an inequality constraint, otherwise we would apply the Kuhn-Tucker conditions. The Langrangian function is:

$$\mathrm{L} = \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w} + \lambda_1(\mu_a - \mathbf{w}'\boldsymbol{\mu}) + \lambda_2(1 - \mathbf{1}'\mathbf{w}) \qquad 2.22$$

The first order conditions (FOCs) are:

$$\frac{\partial L}{\partial \mathbf{w}} = 0 \leftrightarrow \mathbf{V}\mathbf{w}_{\mathbf{a}} - \lambda_1\boldsymbol{\mu} - \boldsymbol{\lambda_2}\mathbf{1} = 0 \qquad 2.23$$

$$\frac{\partial L}{\partial \lambda_1} = 0 \leftrightarrow \boldsymbol{\mu}'\mathbf{w}_{\boldsymbol{a}} = \mu_{\boldsymbol{a}} \qquad 2.24$$

$$\frac{\partial L}{\partial \lambda_2} = 0 \leftrightarrow \mathbf{1}'\mathbf{w}_a = 1 \qquad 2.25$$

Rearranging 23

$$\mathbf{V}\mathbf{w} = \lambda_1\boldsymbol{\mu} + \lambda_2\mathbf{1} \qquad 2.26$$

$$\leftrightarrow \mathbf{w} = \boldsymbol{\lambda_1}\mathbf{V}^{-\mathbf{1}}\boldsymbol{\mu} + \boldsymbol{\lambda_2}\mathbf{V}^{-1} \qquad 2.27$$

Two simple steps. First, multiply each side of 26 by $\boldsymbol{\mu}$ (using FOC 2 above):

$$\mathbf{w}'\boldsymbol{\mu} = \boldsymbol{\mu}'\mathbf{w} = \lambda_1(\boldsymbol{\mu}'\boldsymbol{V}^{-1}\boldsymbol{\mu}) + \lambda_2(\boldsymbol{\mu}'\boldsymbol{V}^{-1}\mathbf{1}) \qquad 2.28$$

From the first constraint, $\boldsymbol{\mu}'\mathbf{w} = \mu_a$, to obtain

$$\mu_a = \lambda_1(\boldsymbol{\mu}'\mathbf{V}^{-\mathbf{1}}\boldsymbol{\mu}) + \lambda_2(\boldsymbol{\mu}'\mathbf{V}^{-\mathbf{1}}\mathbf{1}) \qquad 2.29$$

Second, multiply each side of 26 by $\mathbf{1}$ (using FOC 3 above)

$$\mathbf{1}'\boldsymbol{w} = \lambda_1(\mathbf{1}'\mathbf{V}^{-\mathbf{1}}\boldsymbol{\mu}) + \lambda_2(\mathbf{1}'\mathbf{V}^{-\mathbf{1}}\mathbf{1}) \underbrace{=}_{\mathbf{1}'\mathbf{w}_{\mathbf{a}=1}} 1 \qquad 2.30$$

Now observe 29 and 30 are clear scalar equations. Some matrix products occur repeatedly. Let

$$\boldsymbol{\mu}'\mathbf{V}^{-\mathbf{1}}\mathbf{1} = \mathbf{1}'\mathbf{V}^{-\mathbf{1}}\boldsymbol{\mu} = A$$

$$\boldsymbol{\mu}'\mathbf{V}^{-\mathbf{1}}\boldsymbol{\mu} = B$$

$$\mathbf{1}'\mathbf{V}^{-\mathbf{1}}\mathbf{1} = C$$

Re-write and re-arrange equations 29 and 30, so that:

$$\mu_a = \lambda_1 B + \lambda_2 A \leftrightarrow \lambda_1 = \frac{\mu_a}{B} - \lambda_2 \frac{A}{B}$$

$$1 = \lambda_1 A + \lambda_2 C \leftrightarrow \lambda_2 = \frac{1}{C} - \lambda_1 \frac{A}{C}$$

And therefore giving,

$$\lambda_1 = \frac{\mu_a C - A}{D}$$

$$\lambda_2 = \frac{B - \mu_a A}{D}$$

$$D = BC - A^2$$

Hence, from 26

$$\boldsymbol{w}_a = \frac{C\mu_a - A}{D}\mathbf{V}^{-1}\boldsymbol{\mu} + \frac{B - \mu_a}{D}\mathbf{V}^{-1}\mathbf{1}$$

$$= \ldots..$$

$$= \frac{1}{D}[B(\mathbf{V}^{-1}\mathbf{1}) - A(\mathbf{V}^{-1}\boldsymbol{\mu})] + \frac{1}{D}[C(\mathbf{V}^{-1}\boldsymbol{\mu}) - A(\mathbf{V}^{-1}\mathbf{1})]\mu_a$$

$$= g + \boldsymbol{h}\mu_a$$

Where,

$$\underbrace{\boldsymbol{g} = \frac{1}{D}[B(\mathbf{V}^{-1}\mathbf{1}) - A(\mathbf{V}^{-1}\boldsymbol{\mu})]}_{N\,X\,1}$$

$$\underbrace{\boldsymbol{h} = \frac{1}{D}[C(\mathbf{V}^{-1}\boldsymbol{\mu}) - A(\mathbf{V}^{-1}\mathbf{1})]}_{1\,X\,N}$$

To compute $\boldsymbol{g}$ and $\boldsymbol{h}$ we need data on expected asset returns and variance-covariance matrix (assuming a long only constraint).

However this is only one point on the efficient frontier (the risky portfolio), associated with the desired expected returns (log returns) μ_a, i.e. we identified the portfolio weights to pin the point (σ_a, μ_a). Moreover, it does not end there….

The above derivation of the efficient frontier only considers one risky asset (no riskless asset). To expand the model we introduce the riskless asset class to the model. Now consider a setting with N risky assets asset stated above with a riskless asset, with a deterministic return R_f. The mean-variance optimizing investor's problem is,

$$\min_{\mathrm{w}} \mathbf{w}'\mathbf{V}\mathbf{w} \qquad \text{subject to} \qquad 2.31$$

$$\mathbf{w}'\boldsymbol{\mu} + (1 - \mathbf{1}'\mathbf{w})R_f = \mu_p \qquad 2.32$$

The portfolio weights on risky assets need not add to one, since available funds are spread over the combination of N risky securities, and the riskless asset with allocation $1 - \mathbf{1'w}$. The basic procedure is the same as with the single "$N - risky\ asset\ class$". Form the Langrangian function set the derivative with respect to $\mathbf{w}$ to zero and solve.

$$L = \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w} + \lambda(\mu_p - \mathbf{w}'\boldsymbol{\mu} - (1 - \mathbf{1}'\mathbf{w})R_f) \qquad 2.33$$

The first order conditions (FOCs):

$$\frac{\partial L}{\partial \mathbf{w}} = \mathbf{V}\mathbf{w} - \lambda\boldsymbol{\mu} + R_r\mathbf{1} \qquad 2.34$$

$$= 0 \leftrightarrow Vw = \lambda\boldsymbol{\mu} - R_f\mathbf{1} = \lambda\left(\boldsymbol{\mu} - R_f\mathbf{1} = \lambda(\boldsymbol{\mu} - R_{\boldsymbol{f}}\mathbf{1})\right)$$

$$\leftrightarrow \mathbf{w} = \lambda\mathbf{V}^{-1}(\boldsymbol{\mu} - R_f\mathbf{1}) \qquad 2.35$$

Notice that $(\boldsymbol{\mu} - R_f\mathbf{1})$ is the vector of excess returns.

From the constraint (or differentiating L with respect to λ and setting to zero),

$$\frac{\partial L}{\partial \lambda} = 0 \leftrightarrow \mathbf{w}'\boldsymbol{\mu} + R_f - (\mathbf{1}'\mathbf{w})R_f - \mu_p = 0 \qquad 2.36$$

$$\leftrightarrow \mathbf{w}'(\boldsymbol{\mu} - \mathbf{1}R_f) = \mu_p - R_f \qquad 2.37$$

FOC in equation 2.34 multiplied by $(\boldsymbol{\mu} - \mathbf{1}R_f)$, implies

$$\mathbf{w}'(\boldsymbol{\mu} - \mathbf{1}R_f) = \lambda(\boldsymbol{\mu} - \mathbf{1}R_f)'\mathbf{V}^{-1}(\boldsymbol{\mu} - R_f\mathbf{1}) \qquad 2.38$$

Combining equation 2.38 and 2.36

$$\lambda(\boldsymbol{\mu} - \mathbf{1}R_f)'\mathbf{V}^{-1}(\boldsymbol{\mu} - R_f\mathbf{1}) = (\mu_p - R_f) \qquad 2.39$$

$$\leftrightarrow \lambda = \frac{(\mu_p - R_f)}{(\boldsymbol{\mu} - R_f\mathbf{1})'\mathbf{V}^{-1}(\boldsymbol{\mu} - R_f\mathbf{1})}$$

Substitute for λ in 2.34 to get:

$$\mathbf{w}_p = \underbrace{\frac{(\mu_p - R_f)}{(\boldsymbol{\mu} - R_f\mathbf{1})'\mathbf{V}^{-1}(\boldsymbol{\mu} - R_f\mathbf{1})}}_{:c_p, (1 \times 1)} \underbrace{\mathbf{V}^{-1}(\boldsymbol{\mu} - R_f\mathbf{1})}_{:=\overline{\mathbf{w}}, N \times 1}$$

$$= c_p\bar{\mathbf{w}} \qquad 2.40$$

Introducing the riskless asset, all minimum variance efficient portfolios are a combination of a given risky asset portfolio, with weights proportional to $\bar{\mathbf{w}}$ and the riskless asset. This is the "*Separation Theorem*", suggested by Tobin (1958). "Separation" because it implies that optimal portfolio of risky assets can be identified separately from an investor's knowledge of risk preferences, Danthine and Donaldson (2005).

All investors who hold the same expectations about the covariance matrix and regardless of their risk appetites would choose the same relative holdings of risky securities i.e. the same risky asset portfolio. Changes in risk appetite or target rate of expected return change only the value of c_p. This affects $\mathbf{1}'\mathbf{w}_p$, and therefore $1 - \mathbf{1}'\mathbf{w}_p$, but has no effect on $\bar{\mathbf{w}}$.

More risk-averse positions will have larger and positive $(1 - \mathbf{1}'\mathbf{w})$, and be located in expected return deviation space and along the capital market line (CML) between the intercept (which is determined by the risk-free rate) and the tangency portfolio. Comparatively riskier positions will have $(1 - \mathbf{1}'\mathbf{w}) < 0$, and be located to the northwest of the tangency portfolio along the CML. The latter represents a short position in the riskless asset, which is equivalent to borrowing at the risk-free rate, and adding the funds to the amount available for positions in risky assets.

We can obtain the portfolio vector (with weights summing to one) from $\bar{\mathbf{w}}$ by dividing each of its elements by their sum, $\mathbf{1}'\bar{\mathbf{w}}$,

$$\mathbf{w}_{\mathrm{q}} = \frac{1}{\mathbf{1}'\mathbf{V}^{-1}(\boldsymbol{\mu} - R_f\mathbf{1})}\mathbf{V}^{-1}(\boldsymbol{\mu} - R_f\mathbf{1}) \qquad 2.41$$

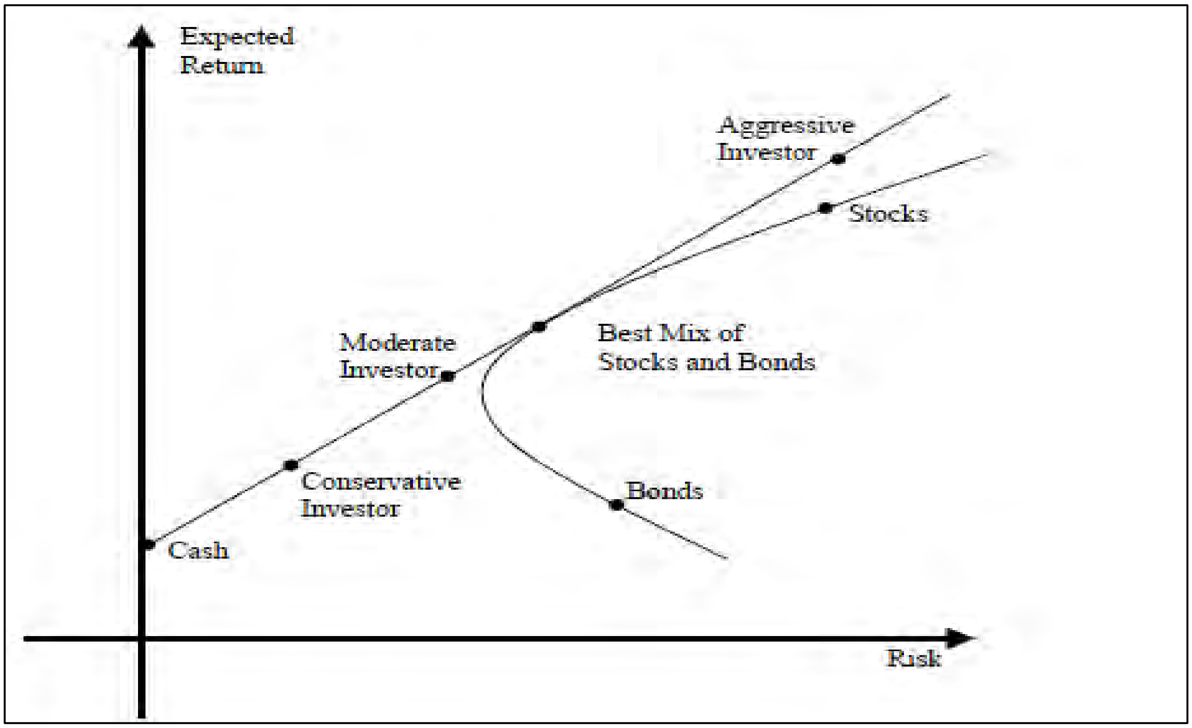
Portfolio q is the tangency portfolio for a given covariance matrix and risk-free asset. Further, we define the Sharpe ratio for any two assets or portfolio p as,

$$S_{rp} = \frac{\mu_p - R_f}{\sigma_p}, \qquad 2.42$$

, an informal measure of expected excess return per unit risk. Note that the tangency portfolio is the portfolio with maximum Sharpe ratio of all portfolios of risky assets only.

The expected return and risk diagram below is adopted from Campbell and Viceira (2002). The diagram illustrates the important results from the Markowitz-Tobin model. The horizontal axis shows the risk (standard deviation) of asset class returns and on the vertical axis is the expected return. Three asset classes (i.e. stocks, bonds and cash) are considered; the point labeled the best "mix of stocks and bonds" represent the risky optimal portfolio. Cash, on the bottom left part of the diagram offers a lower mean-return and has zero risk (near zero risk—riskless[11]). Bonds offer a lower mean-return and has lower risk; Stocks understood to offer a higher mean-return accompanied by a higher standard deviation.

Figure 2.4: The mean-standard deviation diagram



Source: Campbell and Viceira (2002)

Figure 2.4 shows the set of expected returns and risk that can be achieved by combining stocks and bonds in a risky portfolio. When cash is added to the portfolio of risky assets, the set of expected return and risk that can be achieved is a straight line on the diagram connecting cash to the risky portfolio i.e. the "best mix of stocks and bonds". An investor who cares only about the expected return and risk or standard deviation of his portfolio will choose the point on the capital market line (or efficient frontier) that is tangent to the curved line. Point on this line offers the highest expected return for any given risk. The efficient frontier and the curved line form a point of tangency; this point is the tangency portfolio ("best mix of stocks and bonds").

[11]In the presence of inflation risk, nominal money market investments are not necessarily riskless in real terms, but this short-term inflation risk is small enough that it is conventional to ignore it.

In the absence of borrowing constraints, aggressive investors marked on the diagram would leverage the "best mix of stocks and bonds". Ultimately, all investors concerned about expected return and standard deviation will hold the same tangency portfolio of risky assets ("best mix of stocks and bonds"). Conservative investors would choose to combine this risky portfolio with cash to achieve a point on the mean-variance efficient frontier that is low down and to the left; moderate investors will reduce their cash holdings, moving up to the right. Nevertheless, none of these investors should change their relative proportions of risky assets in the tangency portfolio. This conclusion follows from Tobin (1958).

2.3 LITERATURE REVIEW

The concept of asset allocation, defined as "the set of weights of broad asset classes within a portfolio", Stewart, Piros and Heisler (2011), is rooted in Markowitz (1952)'s seminal *portfolio section* paper and Tobin (1958)'s *Mutual fund theorem* concept.

Markowitz (1952)**,** defined the portfolio selection problem in terms of risk (standard deviation or variance) and expected return trade-off for a single-period investment problem. Tobin (1958)**,** introduced the mutual fund theorem, as the practice where risk-averse investors with the same asset returns expectations, variances (and covariance's), would invest in the same risky asset class regardless of their risk appetite.

Following the remarkable contribution by Markowitz and Tobin was the seminal work of Sharpe (1964), Lintner (1965), Fama (1968), etc. These studies gave birth to the capital asset pricing model (CAPM) of expected asset returns. The CAPM predicts that an asset or portfolio return is explained by its residual return and the market excess return; defined as the difference between the return on the market portfolio and the riskless asset; which is proportional to the risk of the asset relative to the market (i.e. beta). Most of these contributions are based on the single-period mean-variance analysis principles.

However, in practice portfolio investment problems look beyond a single-period (or multi-period) investment horizons. This gap in the literature has since then fueled interest among practitioners and academics, with studies focused mainly on the maximization of expected utility of terminal wealth or the expressions of general models of consumption and investment decisions in continuous time. Studies by Smith (1967), Mossin (1968), Merton (1969), Samuelson (1969), Fama (1970), and Hakansson (1971), sum up the earlier contributions concerning multi-period consumption and investment decision-making problems.

Most of these studies show that the conditions upholding the single-period model might not hold in the multi-period context. For example, Mossin (1968)**,** showed that for logarithmic and power utility investors, the maximization of the expected utility of the single-period returns leads to the same portfolio decisions as maximization of the expected utility of terminal wealth. Since then, the appropriateness of the one-period portfolio model analysis has been under serious investigation.

One of the major setbacks of the traditional model is that, investment opportunity set are held constant overtime. In a multi-period context, the investment opportunity set varies with a change in time or investment period. These changes are well documented in the literature and are due to changes in interest rates, inflation expectations, and other important economic variables.

Changing investment opportunity set has important implications for optimal portfolio choice for long-term investors. Merton (1969), observed that this may be what distinguishes the multi-period model from the single-period case. Merton (1969), considered the problem of an investor planning his lifetime consumption and portfolio strategy in a continuous time framework where asset prices are described by a diffusion process. However, the model he suggested has notable difficulties. Campbell and Viceira (2002), pointed out one such problem, that the Merton model is difficult to solve in a closed-form and therefore not easily tractable to apply it to investment problem analysis.

The seminal work of Merton (1969), and Samuelson (1969), described the conditions under which multi-period investors would choose the same portfolios as single-period investors—the so-called Merton-Samuelson conditions. According to the Merton-Samuelson conditions, the myopic portfolio choice is optimal if investors have no labor income and investment opportunity set is constant overtime. This result changes, if investors assume a relative risk aversion parameter equal to one. Hence, the buy-and-hold or myopic portfolio strategy would be optimal under changing opportunity set over the long-term.

Violations of the Merton-Samuelson conditions may result in investment horizon-effects. Investment horizon effects arises when investors would opt to hedge their portfolios against losses over the investment horizon, by tilting (and therefore demanding) the asset allocation to specific assets classes in accordance with the direction of the state variables in question. These state-variables include assets dividend yields, short-term interest rates, long-term real rates, expected inflation etc.

Most studies on multi-period investments focus on the relationship between inter-temporal consumption and optimal portfolio choice. For example, an investor has wealth in period one, allocates a proportion of that wealth to consumption during period one, and invest the difference (wealth less consumption) in some (portfolio) investments. During period one, the portfolio investment would yield the investor an uncertain return on invested wealth at the beginning of period two (or end of period one). During period two, the investor behaves in a similar manner—consumes a proportion of period two wealth and invests the difference. This is a simplified version of a typical investor, who would continue to behave in this predictable fashion until he/she dies, were his wealth would be shared among his heirs. This inter-temporal investor behavior case is well illustrated by Merton (1969), Fama (1970), Hakansson (1971), Elton and Gruber (1974) etc.

Since the work of Merton (1969), and Merton (1973)**,** most studies on inter-temporal portfolio choice, have been leaning towards the issue of return predictability; following the empirical evidence provided by Fama and Schwert (1977), Campbell and Shiller (1988), Glosten, et al. (1993)**,** etc. that suggest that asset returns are predictable.

The literature on the implications of asset return predictability on multi-period optimal portfolio choice is rich. With varying studies in model developments (numerical and analytical), studies for example, Kim and Omberg (1996), Campbell and Viceira (1999), Barberis (2000)**,** Watcher (2002), Campbell and Viceira (2002), Campbell, et al. (2003), etc. use analytical models of dynamic choice. While, Brennan, et al. (1997)**,** Balduzzi and Lynch (1999), etc. use numerical methods.

One of the acknowledged studies attempting to extend the standard portfolio selection model to the multi-period case is the work of Li and Ng (2000). They derived an analytical expression of the efficient frontier for the multi-period portfolio selection, suggesting that it will enhance investor's understanding of the trade-off between the expected terminal wealth and the risk.

Many of these studies solve the optimal portfolio problem analytically—and not using empirical data to draw conclusions; and investigate the impact of investment horizon on optimal portfolio allocation. These studies assume asset returns are predictable, and the investor is concerned about wealth at any point of his lifetime. This point of view is relevant in portfolio construction.

However, this paper is concerned about an investor planning for wealth consumption at the (specified) retirement date. The objective of this investor is to maximize utility of wealth at the retirement date (and is not concerned about intermediate period consumption). The study therefore aims to investigate the effect (if any) of the investment horizon on optimal asset allocation. A similar study was done by Gunthorpe and Levy (1994), Hansson and Persson (2000), Alles and Athanassokos (2006)**,** etc.

The author, as in many other studies on this topic assumes *ex post* returns records are the best predictor of *ex ante* returns. This assumption is necessary in light of the traditional models of expected (mean) returns and asset return variability framework[12]. Fama (1970), showed that the traditional mean-variance framework can be used to assess multi-period portfolio problems.

There are a number of studies investigating the relationship between optimal portfolio construction and the investment horizon. This relationship is referred to as time diversification; see Kritzman (1994). The notion of time diversification is based on the idea that risky or equities tend to perform better than other assets classes like bonds and cash over longer investment periods.
Time diversification has two sides; the one aspect of time diversification is concerned with the view that equity risk declines with an increase in investment horizon. The other is the practice of recommending to young people to allocate high proportions of their portfolios to stocks and reduce these proportions as they age.

The first aspects of time diversification implies that equities tends to be less risky over the long-term. This is the case if we define risk in the traditional academician sense—variance or standard deviation—then risk will decrease with the increase in investment horizon. Stewart, et al. (2011), the expected asset returns is constant per period, but the variance (or standard deviation) decreases with the increase in investment horizon.

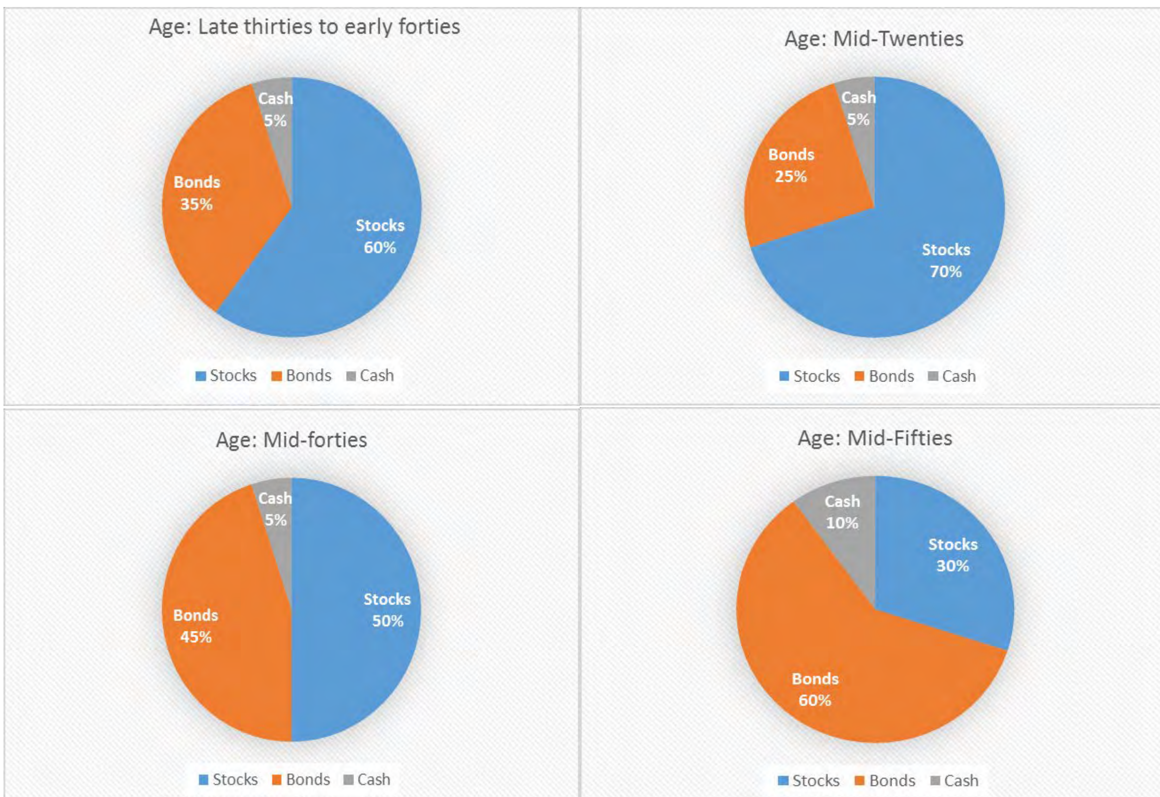
Siegel (1998), who suggests young investors to hold portfolios more invested in risky assets, and shift that portfolio holding to safer assets, as they get closer to the retirement date, raises the second

[12] The life-cycle investor makes his portfolio allocation decisions based on expected mean returns given risk (variability in asset returns).

aspect of time diversification. Inherent in Siegel (1998)'s observation is the view that equities tend to mean-revert overtime. Mean-reversion is the tendency for equities (or asset) prices to be move towards their trend or long-term average levels.

The following figure is adopted from Malkiel (1996) and Hickman, et al. (2001). The figure shows that young people's investment portfolio comprises more of stocks than bonds and cash at an early age. Also depicted is the view that older people ought to hold more bonds (long-dated fixed income assets). This is the common advise old school financial planners commonly practice.

Figure 2.5: Reference Portfolios



Source: Malkiel (1996), and Hickman, et al. (2001)

The aim of this study is to use data from South Africa's financial market to see if the time diversification aspect holds and the implication for the life-cycle investor faced with South African asset class returns. Studies in this line of work are rare for South Africa.

Rudman (2009), investigated the optimal asset allocation as a means of minimizing the investment risk, drawdown risk and longevity risk associated with an investment linked living annuity. His findings concludes that retirees have to consider, among other factors, the required standard of living, the need to withdraw one third of the retirement capital and life expectancy before investing in an investment linked living annuity.

Bradfield and Munro (2015), considered two methodological approaches to establish a strategic foreign allocation weight. Their first approach considers the strategic role of foreign investment in South African global balanced portfolios by using a mean-variance efficient frontier framework over a long-term period. Their second approach uses a non-parametric procedure. Their results conclude that both the mean-variance and the non-parametric methodology yield compelling evidence for the foreign allocation to be set at the maximum allowable bound of 25 percent.

The most popular asset class returns data used in studies from South Africa are; the Alexander Forbes money market index (for short-term interest or cash returns), BEASSA All bond index (long-term fixed-income assets), and JSE data for equities. See for example, Shapiro (2012), and Auret and Vivian (2014).

The analyses of Gunthorpe and Levy (1994) used the mean-variance framework (quadratic program) to find the vector of optimal portfolio composition weights. They found that portfolio composition changes with the investment horizon, regardless of whether returns are nonstationary or stationary. Moreover, the longer the investment horizon, the greater the proportion of safer assets held in the portfolio.

Tang and Lee (1997), examined the impact of investment horizon on portfolio composition, in three Asian Pacific stock markets: Japan, Hong Kong and Korea. They found that in all three markets; the longer the investment horizon, the larger is the proportion of defensive stocks that should be included in the optimal portfolio.

Albrecht (1998), rejects the view that low-risk assets grow more attractive at longer investment horizons. His argument is based on the observation that the long-term standard deviation of returns can be misleading as an indicator of risk—asserting, “Investments with identical standard deviations can have entirely different long-term risk characteristics even if all instantaneous investment returns are normally distributed”.

Employing expected utility of wealth and mean-variance utility examples, Van Eaton and Conover (1998), showed that the effect of a longer horizon on risky-asset allocation is consistent with increasing or decreasing optimal equity allocations, even under the assumption of constant relative risk aversion.

The study of Hansson and Persson (2000), employed a nonparametric bootstrap approach to investigate whether in a mean-variance efficient portfolio, the weights for US stocks and Treasury-bills change in a systematic manner with the investment horizon. They found significant gains from time diversification; the weights in an efficient portfolio were significantly larger for long investment horizons than a one-year horizon.

Sanfilipo (2003), investigated the relative performance of stocks and bonds over the investment horizon using data from the French market. A matched block bootstrap approach to account for estimation risk was employed in the standard return-risk analysis—with risk measured as the semi-variance. His findings suggests that bonds holdings should be avoided in the long-term due to the effects of time diversification.

Alles and Athanassokos (2006), challenged the common wisdom illustrated in figure 2.1. Their study used datasets from the Canadian financial market and employed the resampling-bootstrapping technique. Their results show that investment outcomes at short investment horizons can be very different from long horizon outcomes. They found evidence supporting the notion of time diversification.

The study by In, et al. (2011), used a new approach based on the wavelet analysis, which decomposes the returns of a particular investment strategy across multiple investment horizons. They used this approach to examine portfolio allocation between value and growth stocks over investment horizons. Their key findings point out that, gains from pursuing the value strategy is impacted by the approach used to classify value and growth stock returns. As the time scale increases, the optimal mean allocation of investors' tilts heavily away from growth stocks, particularly for lower and moderate levels of risk aversion.

Some of the studies use the mean-variance framework and the bootstrapping (and resampling) procedure to analyze the effects of investment horizon and optimal portfolio construction. However, the mean-variance framework inputs are subject to estimation errors. Errors in expected mean returns, variance-covariances. These errors are mostly the results of the underlying assumption regarding asset return distribution and the appropriate measure of risk.

Chopra and Ziemba (1993), observed that under standard assumptions, the inputs in the mean-variance framework are subject to errors in measurement—means, and variance-covariances. This is the "dark side" of portfolio optimization. Therefore, misspecification of the parameters of the return distributions (or estimation risk) in equities, bonds and cash returns may yield erroneous results. Further, the relative impact of errors in means is more important than errors in variance and covariances. Grauer and Shen (2000), showed that correcting for estimation error, particularly in mean-returns can improve investment performance substantially.

Blume (1974), observed that expected returns are subject to estimation error. Blume documented the biases inherent in using the sample arithmetic and geometric mean of one-period returns to assess long-term expected rates of return. He showed that if one-period returns are independently, identically distributed (i.i.d), both the geometric and arithmetic means if used to estimate long-term rates of return might yield biased results.

Elton and Gruber (1974), evaluated portfolio performance based on geometric mean of future multi-period returns, and on selecting portfolios based on expected utility of multi-period returns. They concluded that, portfolio decisions based on multi-period returns are identical to portfolio decisions based on one-period returns.

In another seminal paper, Elton and Gruber (1974), showed that if returns are log-normally distributed, the portfolio with the highest geometric mean must lie on the efficient frontier in (arithmetic) mean-variance space. This view contrasts Hakansson (1971)'s, that maximizing geometric mean will lead to the selection of portfolios, which are not on the efficient frontier in (arithmetic) mean-variance space.

Stutzer (2004), observed that when asset returns are log-normally distributed, the portfolio which maximizes the geometric mean lies on the efficient frontier in (arithmetic) mean-variance space; asserting that the standard mean-variance analysis is consistent with certain types of multi-period problems and with problems involving the maximization of long-term growth.

To account for the errors in measurement in expected returns, these studies employ the bootstrapping or resampling methodology, for example Hansson and Persson (2000). The problem with resampling is that it removes the serial correlation inherent in returns distributions overtime.

This study contributes to literature in the sense that it uses financial markets data from South Africa. The dissertation uses the mean-variance analysis with rolling period returns i.e. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 years of investment. In most cases, studies use longer investment horizons to analyze asset allocation for retirement planning portfolio construction. Because of data limitations, this study is limited to the above investment horizons.

2.4 CONCLUSION

In summary, many studies have been conducted on the investors' consumption and optimal investment portfolio over multi-periods. Most paperwork assume asset returns are predictable and hence, non-constant investment opportunities over multi-periods. One of the most important contribution on this topic is the work of Merton (1969). Although, the model provided is insightful, it is not easily tractable. However, many studies still advocate the traditional Markowitz-Tobin model to solve practical investment problems over several investment horizons.

There is considerable work done on the relationship between investment horizon and optimal portfolio allocation. Studies often investigate the notion of time diversification alluded by Kritzman (1994). Studies of this nature for the case of South Africa are rare (if nonexistent). Despite this limitation, work has been done using data from developed markets and other emerging markets. These studies use the mean-variance framework complimented by the bootstrapping or resampling methodology. The common conclusion in most studies support the notion of time diversification. That is, investors with long horizons should invest more in risky assets, because on average equities tend to outperform bonds in long-term. The implications of the presence of time diversification are important for a life-cycle investors' asset allocation portfolio policy. One of the implications is that younger investors with many years to go before retirement should consider holding a portfolio more invested in equities or risky assets, because they have a longer life expectancy and the ability to make up for poor investment results.

This study contributes to the present literature on time diversification, using financial market data from the South African environment. The investigation uses rolling period returns for the assets considered, and portfolio analysis is conducted for 12 investment horizons.

CHAPTER 3: RESEARCH METHODOLOGY

3.1 INTRODUCTION

This chapter consists of four important sections, including this section. The following section (section 3.2) describes the sample data, used in the study, descriptive summary statistics, and return computations. Following that section, is section 3.3, which discusses the method used to analyze the asset return data, to provide answers to the research questions stated above. Finally, section 3.4 concludes the chapter.

3.2 SAMPLE DATA

Total return index data (capital plus dividends), covering the period from December 2001 to December 2014 were obtained from I-Net Bridge (through the University of Cape Town I-Net data terminals)—157 observations. This data source includes the FTSE/JSE Style growth and the FTSE/JSE Style value total return indexes, the Bond Exchange Actuarial Society of South Africa (henceforth, BEASSA) All Bond total return index (ALBI), as well as the Alexander Forbes money market total return index data[13].

The dataset also includes the JSE All Share index (henceforth, JSE ALSI) return data for the period January 1960 to December 2008. This set of data is from the data calculations of Firer and Staunton (2002). This dataset cover a longer period than the above asset return data. For the purpose of this study, that dataset is used to investigate the risk of investing in the JSE ALSI overtime. The aim is to see what happens to the risk (as defined by the standard deviation) of the JSE ALSI as the investment horizon rises.

To examine the risk of investing on the JSE, allows us to see the implications of putting all your money (or wealth for retirement purposes) on the JSE as the investment horizon rises. This investigation is necessary to see if there is evidence of time diversification on the JSE ALSI. If the risk of investing on the JSE were declining, as found in most studies in the literature, this would imply evidence of time diversification.

[13] An index return is a change in value of a portfolio over some holding period. The return on an index is calculated as the weighted average of the returns of the individual assets constituting the index. The Johannesburg Stock Exchange provides better details on how these indexes are calculated, (www.jse.ca.za). Style means "An investment strategy that groups companies by apparent different rates of return."

The following description of the asset return indices is adopted from the Association for Savings & Investment South Africa (2010). Since the study addresses portfolio construction from the South African investor's perspective asset returns are measured in Rand currency.

3.2.1 FTSE/JSE Style growth total return index

This is a weighted-average of companies listed on South Africa's Johannesburg Stock Exchange, whose market value of shares is anticipated to rise. This index consists of a portfolio of companies varying across industries and its objective is to seek maximum capital appreciation through investment in growth companies. Growth companies specializes in the trading on high price-to-earnings ratios (PE ratios) listed shares. PE ratios measures how much investors are willing to pay for a stock relative to the profitability of the underlying business. The higher the company's share price relative to earnings the higher the PE ratio, hence companies with high PE ratios are classified as growth companies in capital markets.

Determination of a company as "growth share" takes into consideration the company's sustainable earnings growth based on a combination of: a) the 2 year historical earnings growth and, b) 1-year consensus I-Net forecasts.

3.2.2 FTSE/JSE Style value total return index

The Style value total return index is a weighted-average of companies whose underlying book (or intrinsic) value is judged to be above the value determined by the market—undervalued company shares. The index contains portfolio of stocks of JSE listed companies across industries that offer the opportunity for expanding revenue or earnings growth, especially companies trading below book value. The primary investment objective of the index is to seek out medium to long-term capital appreciation. The portfolio seek out "value" situations by typically investing in shares with low relative PE ratios as well as shares that are trading at a discount to their net asset value. The index frequently offer a higher than FTSE/JSE All Share Index average level of income.

Determination of shares as "value shares" takes into consideration: a) the current PE trading at a discount to the average PE of the market, b) the dividend yield of the company significantly exceeding the dividend yield of the market.

These risky assets indexes are selected because; value stocks tend to perform better over the long-term relative to growth stocks. Therefore looking at the asset allocation between growth and value indexes would simultaneously provide empirical evidence concerning the performance of these distinguished categories of equities over the long-term. Implicit in this argument is the belief that, value equities are a long-term investment and growth equities short-term investment vehicles. For example, for an investor with a 3 to 5 year investment outlook, it is often suggested that growth equities are the best approach. Alternatively, investors with 10 years or more investment outlook, value equities are the appropriate investment vehicle.

3.2.3 BEASSA All Bond (ALBI) total return index

The all bond index (or also known as the fixed-interest portfolio), is a weighted-average of the top 20 listed bonds ranked by market capitalization and liquidity. Market capitalization measures the average size of the individual asset relative to the average size of the whole asset or securities market. Liquidity refers to the ease with which the financial asset can be traded for solid cash on the market.

The ALBI index comprises of bonds, fixed deposits and other long-term government interest-bearing securities. The portfolios may invest in short, intermediate and long-dated bonds. The composition of the underlying investment is actively managed and will change over time to reflect the manager's assessment of interest rate trends. This portfolio offer the potential for capital growth, together with a regular and high level of income.

3.2.4 Alexander Forbes money market total return index

This is an index or a portfolio of short-term rate securities calculated on an interest accrual basis and reflects a monthly interest component in cash returns. To arrive at that figure, the daily interest rates are averaged to obtain monthly rates, which is used in the calculation of the effective term yield for the n-months. The monthly performance is then calculated from that effective term yield. The performance for that particular month, which is based on the average of the past n-months performances, which is used to calculate an index value. The index seek to maximize short-term interest income, preserve the portfolio's capital and provide immediate liquidity. This is achieved by investing in money market instruments with a maturity of less than one year while the average maturity of the underlying assets may not exceed 90 days.

Other studies, especially in the US use the 91 days Treasury bill as the measure for short-term interest or cash returns. Studies such as Shapiro (2012), used the Alexander Forbes money market index as a proxy for short-term interest rates (or cash), for the case of South Africa.

Fixed-income securities i.e. bonds and cash are an important consideration in long-term portfolio construction, because they provide a stable rate of return on investment overtime compared to their equities counterparts. Auret and Vivian (2014), conducted a study using some of these variables, in their comparative study of returns of various financial asset classes in South Africa.

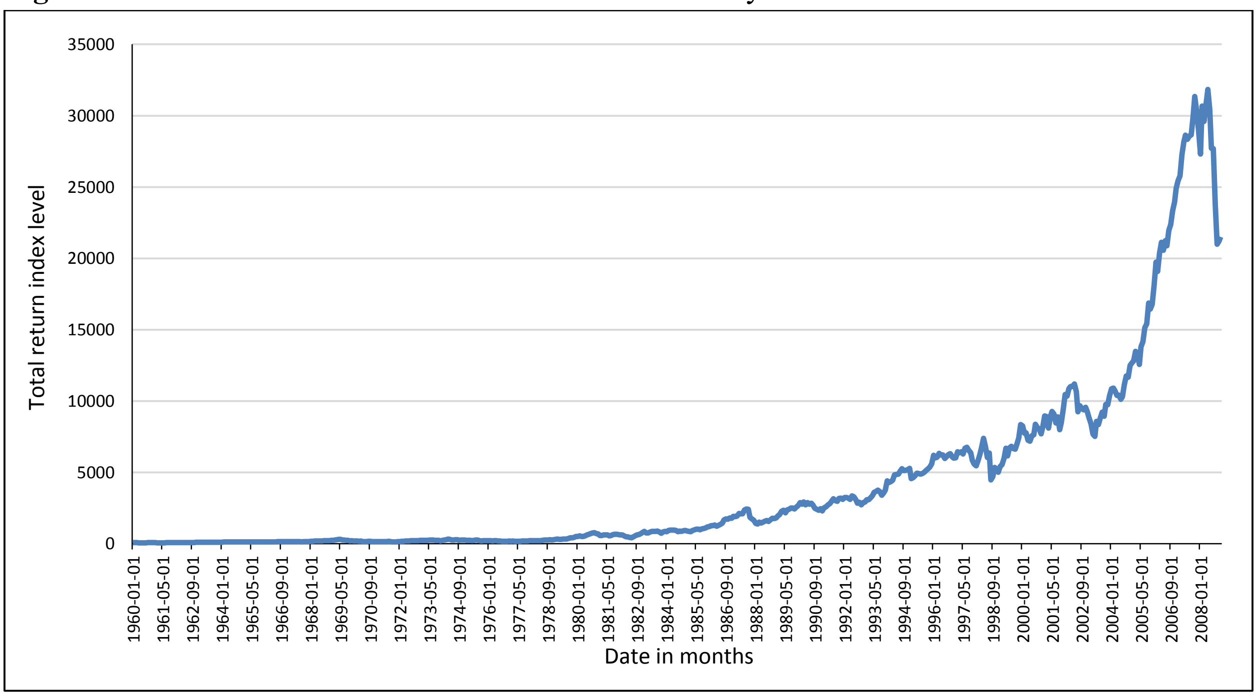
3.2.5 FTSE/JSE All Share total return index

The FTSE/JSE ALSI is the benchmark stock index for South African active portfolios and contains 165 listed companies, based on market capitalization as of December 2014 Bloomberg (2016). The listed securities varies across the range of large, mid and small capitalization shares; and these shares are known to offer medium to long-term capital growth. The index is, a market capitalization weighted, index and represents 99% of the full market value of all the listed companies on the JSE. Nearly one-fourth of the index is comprised of firms in the basic resources sector.

Historical data or *ex post returns* are the best estimates of future *ex ante* return distributions. The study uses the historical total return index series for each of the above assets to compute rolling period for different investment horizons. The difficulty in obtaining substantial data means that optimal portfolio allocation would be limited to few holding periods. However, if longer datasets could be obtained, the holding period could be increased and further analysis could be done to give reliable results[14].

Figure 3.1 shows the JSE ALSI total returns index covering the period January 1960 to December 2008, a period of 48 years. The same data was used by Firer and Mcleod (1999), and Firer and Staunton (2002).

Figure 3.1: JSE All Shares total return index 1960 January to 2008 December



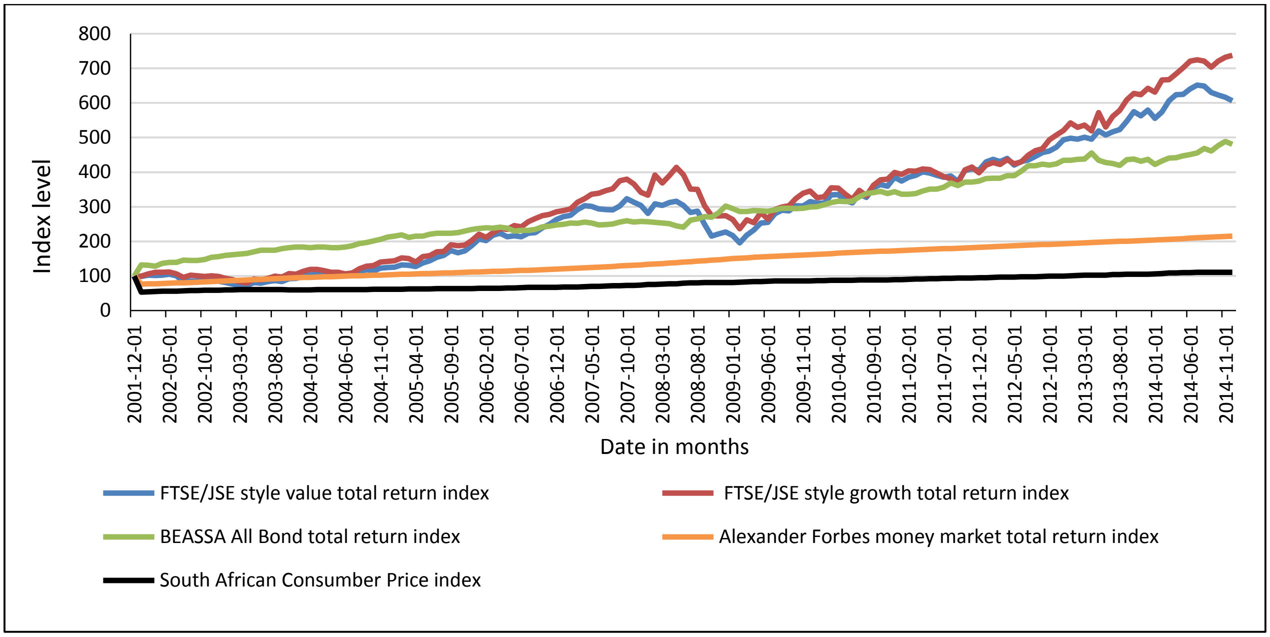
Source: Firer and Staunton (2002)**,** Firer and Mcleod (1999)

The JSE ALSI maintained a relatively flat level since the 1960's until the beginning of 1980, where it started to rise significantly. Overall, the index is trending upward with market declines, which may be observed during major market events.

[14] In order to compare equities and fixed-income bonds and cash assets, the data interval should be as long as possible in order to smoothen out the incidence of statistical noise.

Figure 3.2 shows the total return indexes for the FTSE/JSE Style growth (Style growth) and FTSE/JSE Style value (Style value), the BEASSA All bond index (bonds), the Alexander Forbes money market index (cash), as well as the South African consumer price index (CPI).

Figure 3.2: FTSE/JSE Style Growth and FTSE/JSE Style Value indexes, BEASSA all bond index, Alexander Forbes money market index[15] and the CPI index levels



Source: I-Net Bridge

The Style growth is well above the CPI level. This implies that an investment in this index portfolio over the period reviewed would have outperformed the inflation level. Style growth shares tend to outperform Style value equities overtime. Moreover, the Style value index (and the bonds, and cash indexes), also relatively outperformed the CPI level. Overtime and investment in cash return index would produce a well above inflation return. However, this return would be reduced when factoring in transaction cost (and other significant charges). Ultimately, an investment in equities would be a wonderful alternative.

From 2001 until the end of 2005, South African bonds outperformed South African Style equities (value and growth). Since then, Style shares have been outperforming bonds and cash asset classes.

[15] The values of the Alexander Forbes money market index have been scaled down, because the index level is greater than all the other assets index level. To scale it down, the original index values is divided by 100.

During the financial crisis period of 2008, an investment in equities would still have outperformed cash. The view that a portfolio should be tilted to riskless asset to avoid losses during possible crisis period is not supported by the data reflected on the graph, because a drop in equities would still give the investor good returns relative to cash. Cash investment would be ideal for investors wanting to keep a constant income level just above inflation.

3.2.5 Return computations

The literature differentiates between simple net returns (or gross returns) and continuously compounded returns; see Chapter 2, section 2.3. For the purpose of this study, it is convenient to work with continuously compounded returns. In *Appendix A*, it is shown that there is little difference between simple returns and continuously compounded (or log) returns. Most studies use continuously compounded returns, Hansson and Persson (2000).

To calculate asset returns, let R_{it} be the monthly total return on asset i at time t. Then the continuously compounded monthly return, r_{it} is defined as:

$$r_{it} = ln(1 + R_{it}) = ln\left(\frac{P_{it}}{P_{it-1}}\right) = \ln(P_{it}) - \ln(P_{it-1}) \qquad 3.1$$

Where, $Ln(\,)$ is the natural logarithmic function. r_{it} is the continuously compounded growth rate in prices, P_{it-1} and P_{it} of asset i between months $t-1$ and t, respectively. This is different from R_{it}, which is the simple growth rate in prices of asset i between months $t-1$ and t, without any compounding i.e. $\frac{P_{it}}{P_{it-1}}$.

Stewart, et al. (2011), discussed the relationship between continuously compounded returns (in equation 3.1) and simple gross returns. In that textbook they showed that, the continuously compounded return equals the logarithm of the gross returns, and the gross return is the exponential of the continuously compounded return. “If the gross return has a lognormal distribution, then the continuously compounded return is normal and vice-versa”. Following Stewart, et al. (2011), the expected values of these two measures of return are related by:

$$\mu_i \equiv \alpha_i + {}^{1}/_{2}\,\sigma_i^2 \qquad 3.2$$

Where, $\mu_i$is the expected return on the ith asset; α_i is the average return on the ith asset, calculated as $\frac{1}{T}\sum r_{it}$; and is the variance on the asset(s), σ_i^2[16]. "μ_i", is defined as the logarithm of the expected continuously compounded return plus one-half the variance of the continuously compounded return. The key difference between these measures of expected returns is that "ln-natural logarithm" is a non-linear function.

Since, we are working with monthly data, the multi-period continuously compounded returns on the given assets is the sum of the returns in each period. However, this practice would be inaccurate if we worked with simple gross or net returns. Gross returns work best for single-period model and the case under study requires logarithmic returns.

According to Stewart, et al. (2011), it is best to work with continuously compounded returns if we want to relate the mean-variance model to more general multi-period problems. For short investment horizons, the logarithmic and gross returns are approximately the same.

The logarithm of a sum is not equal to the sum of logarithms. That is, the continuously compounded portfolio return is not exactly a weighted average of the continuously compounded asset returns. It is approximately equal to that sum and volatility adjustments Campbell and Viceira (2002), and Stewart, et al. (2011) demonstrate this assertion. The continuously compounded portfolio return is approximated by:

$$\mu_P(T) = \sum_i w_i r_i + {}^1\!/_2 \sum_i w_i \sigma_i^2 T - {}^1\!/_2\, \sigma_P^2 T \quad 3.3$$

Where, μ_P and σ_P^2 are the portfolio expected returns and variance, respectively. $\sum$ is the summation sign—operator; w_i is the weight assigned to the ith asset; r_{it} is the continuously compounded asset return, and T is the investment horizon[17]. Thus, the portfolio return is given by:

$$\sum_i w_i \sigma_i T + {}^1\!/_2 \sum_i w_i \sigma_i^2 T - {}^1\!/_2\, \sigma_P^2 T \equiv (\mu_P - {}^1\!/_2\, \sigma_P^2) T \quad 3.4$$

Solving for μ_P, gives us the variance adjusted portfolio return:

$$\mu_P = \sum_i w_i (\alpha_i + {}^1\!/_2\, \sigma_i^2) = \sum_i w_i \mu_i \quad 3.5$$

[16] The variance of the asset returns is simply the sample variance calculated over the T-th investment horizon. That is the variance of the asset obtained from the table of summary statistics multiplied by the "T" horizon, $\sigma_i^2 T$.
[17] From probability theory, the expected value of a weighted average is the same as the weighted average of the expected values.

The variance of portfolio returns takes into account each asset's own variance and also the covariance between each pair of assets. Applying the standard result for the variance of a weighted average gives:

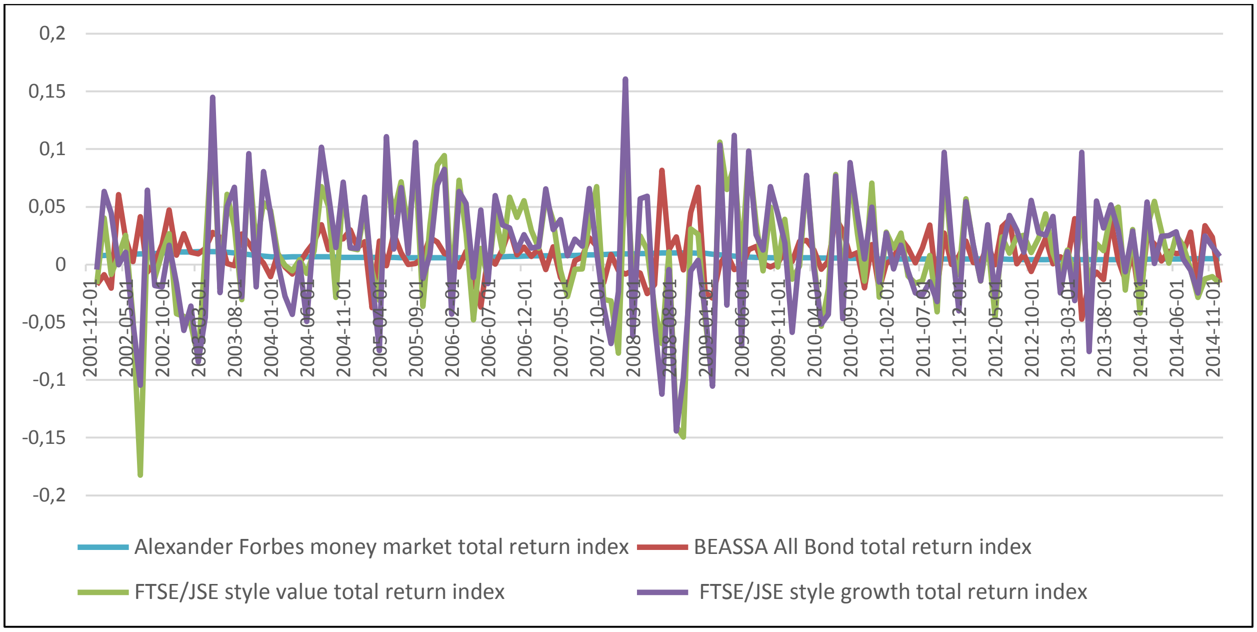
$$\sigma_P^2 T \equiv [\sum_i \sum_j w_i w_j \sigma_{ij}]T \qquad 3.6$$

Where, σ_{ij} denotes the covariance between the returns on assets i and j. If $i = j$ the covariance is equal to the variance of asset i. Note that the continuously compounded portfolio return is linear in the portfolio weights whereas the variance is quadratic because it involves squares and cross products of the weights.

The average annualized return is the sample average of the geometric average holding period returns. The average holding period return is the sample average of the compounded total returns Lee (1990). According to Lee (1990), at any date, the holding period return is the compounded return over *N*-periods of the single-period returns. For example, the holding period returns for 1 year is the monthly-compounded returns of all single-period returns over 12 months. Moreover, the holding period returns for 5 years is the monthly-compounded returns of all single-period returns over 60 months, and so on. Lee (1990), used logarithms of holding period real returns over rolling investment horizons of fixed length to compute the optimal mean-variance efficient portfolios.

Figure 3.3, shows the monthly continuously compounded real asset total returns for the JSE style growth and value indexes, the South African all bond index and the Alexander Forbes money market index (henceforth, cash return index).

Figure 3.3: Monthly continuously compounded total asset returns



FTSE/JSE All share Style Value index---style value
FTSE/JSE All share Style Value index---style growth
BEASSA Albi total return index---all bonds
Alexander Forbes money market index---cash

The continuously compounded return on cash is near zero. This implies that investment in cash over the covered period would produce little returns after considering inflation and other factors that may reduce the returns on cash (e.g. transactions cost, and taxes).

The average total returns for the style value from December 2002 to December 2014 is about 0.01155 (or 1.16 percent). For the same period, the average returns for the style growth is 0.01280 (or 1.28 percent). For the less risky assets, the average returns for ALBI is 0.008175 (or 0.8175 percent) and for the cash return index is 0.006684 (0.6684 percent).

It is evident from the figure that the total asset returns on assets for the style growth and style value indexes is uncertain (and fluctuates rapidly over the covered period). Observing the returns on these assets over key event-dates i.e. financial crises periods, both indexes yield negative returns. For example during the 2000-2003, “dot-com” bubble and the 2008-09 financial crisis and 2010-11 Euro debt crisis, a significant drop in asset returns can be observed. Except for the bonds and cash components, which shows relatively stable (and positive) returns under crisis conditions.

Over crisis periods, the style value index seems to be declining more than the style growth total return index. Under “normal” economic conditions, both total returns behave about the same, yielding little over 5 percent over the period considered.

However, the returns on South African all bonds—the green line—shows more instability compared to the cash returns. Perhaps, that is why the cash components of assets is considered riskless and the bonds relatively risky. Inflation average about 6 percent over this period, STATS SA (2015).

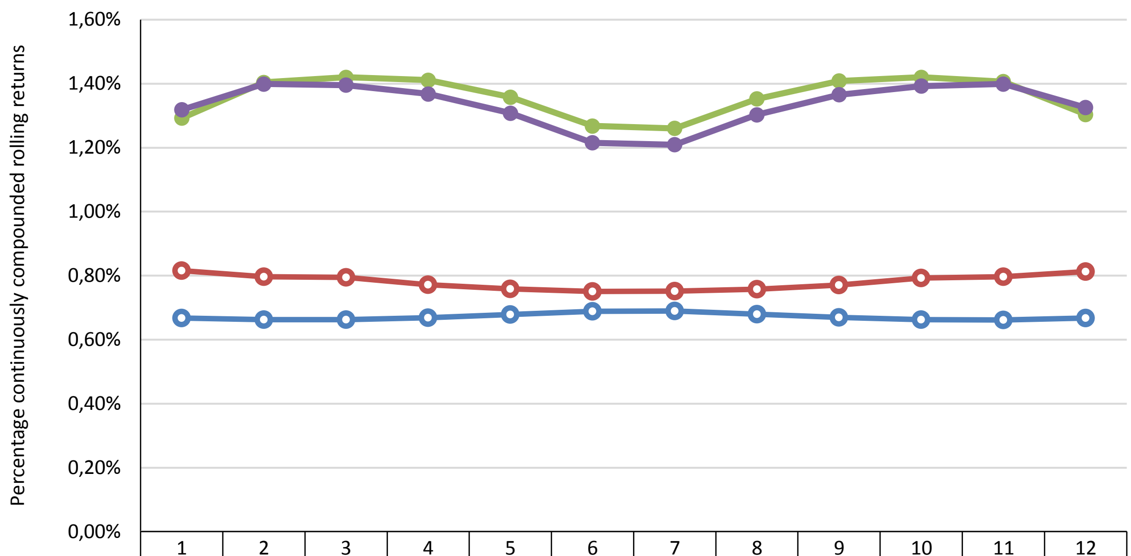
The returns depicted by Figure 3.3, may be considered point-to-point asset returns over the December 2002 until December 2014 period. The disadvantage of this way of interpreting returns is that, the results will entirely depend on the end-date, and excludes returns behavior between the periods. We cannot use point-to-point returns to evaluate returns of these fund indexes (style value, style growth, bonds and cash index) to construct portfolios.

Instead, we calculate rolling period returns for each of the four assets, for 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 years, using the monthly data describe above. Rolling returns break a performance period into many smaller overlapping periods.

Rolling returns offer a more effective measure because they provide a more accurate and in-depth picture of a fund index performance, rather, than “point-in-time” results. Furthermore, rolling returns account for the fact that investors typically do not invest at the beginning of the current five or 10-year periods, but instead are investing over many periods.

So, instead of assuming that an investment was made on January 1, rolling returns calculate all of the periods starting not only in January, but also in February, March, and April etc. For example, breaking a five-year period into a series of rolling periods, January of year 1 through December of year 1, February of year 1 through January of year 2, and so on and so forth.

Figure 3.4: Average rolling 1 to 12 year returns for SA asset classes from 2002 to 2014

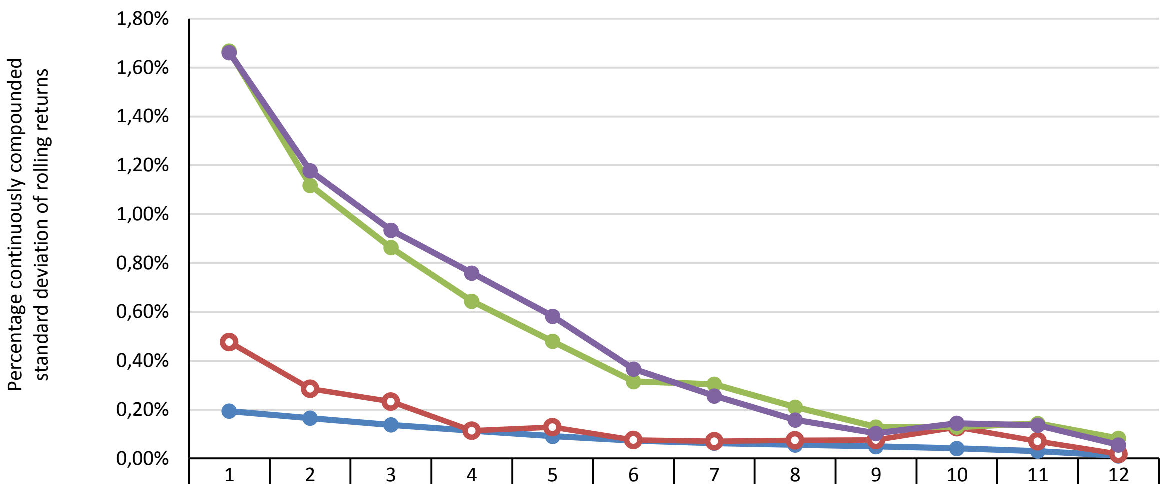


	1	2	3	4	5	6	7	8	9	10	11	12
Cash	0,67%	0,66%	0,66%	0,67%	0,68%	0,69%	0,69%	0,68%	0,67%	0,66%	0,66%	0,67%
Bonds	0,82%	0,80%	0,79%	0,77%	0,76%	0,75%	0,75%	0,76%	0,77%	0,79%	0,80%	0,81%
Style value	1,29%	1,40%	1,42%	1,41%	1,36%	1,27%	1,26%	1,35%	1,41%	1,42%	1,41%	1,30%
Style growth	1,32%	1,40%	1,40%	1,37%	1,31%	1,22%	1,21%	1,30%	1,37%	1,39%	1,40%	1,33%

Notes: number from 1 to 12 indicates the years for the rolling period. These calculations were done in Microsoft excel.
FTSE/JSE All share Style Value index---style value
FTSE/JSE All share Style Value index---style growth
BEASSA Albi total return index---all bonds
Alexander Forbes money market index---cash

Equities rolling period returns display a "butterfly" like picture, while bonds and cash show a relatively constant rolling return over the 12 years rolling periods.

Figure 3.5: Standard deviation rolling 1 to 12 year returns for SA asset classes from 2002 to 2014



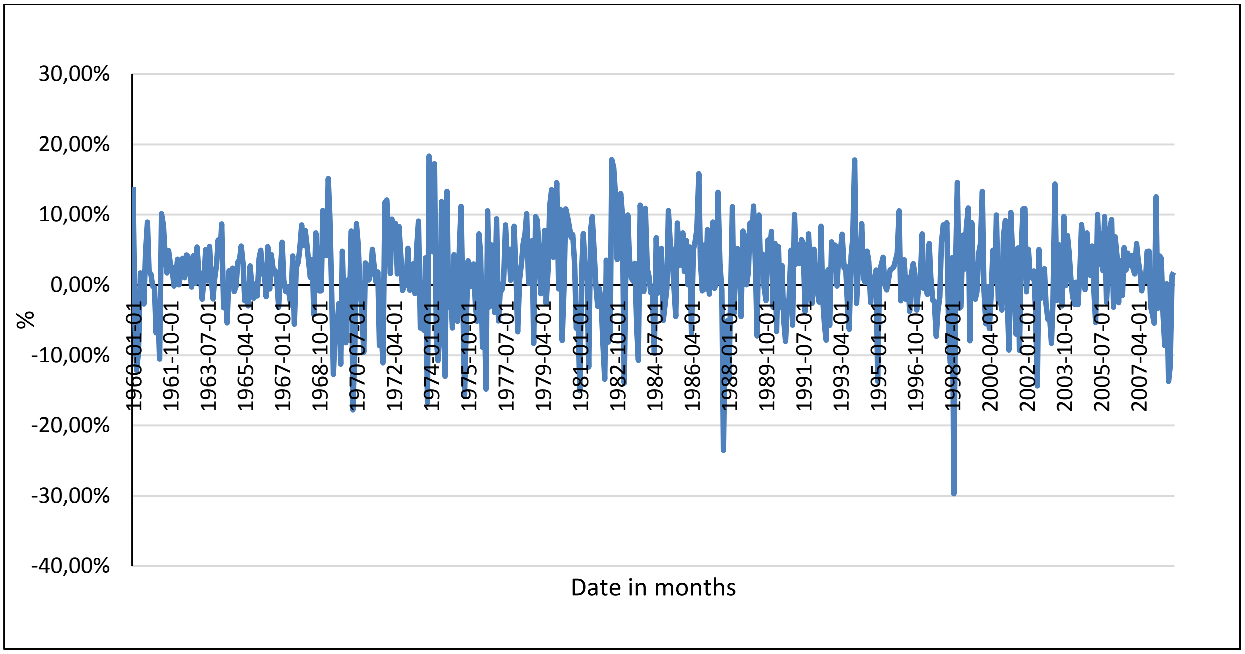
	1	2	3	4	5	6	7	8	9	10	11	12
Cash	0,19%	0,17%	0,14%	0,11%	0,09%	0,07%	0,06%	0,06%	0,05%	0,04%	0,03%	0,01%
Bonds	0,48%	0,28%	0,23%	0,11%	0,13%	0,08%	0,07%	0,07%	0,08%	0,13%	0,07%	0,02%
Style value	1,67%	1,12%	0,86%	0,64%	0,48%	0,32%	0,30%	0,21%	0,13%	0,13%	0,14%	0,08%
Style growth	1,66%	1,18%	0,94%	0,76%	0,58%	0,37%	0,26%	0,16%	0,10%	0,14%	0,14%	0,06%

Notes: These are average monthly rolling continuously compounded returns. For example, an average monthly five-year rolling return accounts for all of the five-year returns beginning at a given inception date and advancing one month sequentially. This method allows an investor to evaluate the consistency of a fund's performance over time—including the vicissitudes of market cycles.

FTSE/JSE All share Style Value index---style value
FTSE/JSE All share Style Value index---style growth
BEASSA Albi total return index---all bonds
Alexander Forbes money market index---cash

The standard deviation of all assets (based on rolling period returns) declines overtime. At shorter investment horizons, the standard deviation of bonds and cash is lower compared to style value and growth equities.

Figure 3.6: JSE ALSI log returns—1960 January until 2008 December



Notes: JSE ALSI log returns, are calculated as the natural logarithm of gross returns.

3.2.6 Descriptive statistics and correlation matrix

Table 3.1 below shows the descriptive summary statistics for the average monthly rolling period asset returns (log-returns) for 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 years.

Measures of central tendency such as the mean, variance (or standard deviation[18]) tells us about the variation of the dataset. Skewness characteristics of the data illustrate the direction of variation of the dataset. Skewness looks at the symmetry (or lack of it) of some data distribution i.e. if the distribution looks the same on the left and right from the centre point.

Kurtosis describes the shape of a random variable's probability distribution. Higher (positive) Kurtosis means small tails exhibited by the probability density function (PDF); whereas lower (negative) Kurtosis means fatter tails. Positive Kurtosis relates to a relatively peaked distribution. Negative Kurtosis relates to a relatively flat distribution.

The average returns for cash over the 12 period is about 0.0067 (0.67 percent). For bonds it is about 0.0078 (or 0.78 percent) over 12 periods. The Style value has an average return of about

[18] $\sigma_{annual} = \sqrt{12} \cdot \sigma_{monthly}$

0.01358 (1.36 percent), and the Style growth 0.01333 (1.33 percent). The standard deviation of all asset categories decreases with the investment horizon, as illustrated by Figure 3.5 above.

The Style value has a Skewness of about negative 1.01 and a Kurtosis of negative 1.018 in the first year; on average the Kurtosis is about -0.71 and an average Skewness of about -0.11. The Style growth asset has an average Skewness of about -0.13 and a Kurtosis of about -0.71, over the 12 periods. The cash return index has an average negative Skewness of about 0.0511 and a Kurtosis of about negative 0.99.

A normal distribution usually produce a Kurtosis statistic of about zero. As the Kurtosis statistic departs further from zero, a positive value indicates the possibility of a leptokurtic distributions (i.e. too tall). Any positive Kurtosis statistic observed in Table 3.1, would imply that the distribution is leptokurtic (too tall). Otherwise, the Kurtosis statistic would have known that the distribution was platykurtic (too flat).

Table 3.1: Descriptive summary statistics for the average monthly asset—rolling log returns

Period (in years)	Cash									
	Mean	Standard Deviation	Variance	Kurtosis	Skewness	Minimum	Maximum	Sum	Count	Confidence Level(95,0%)
1	0,006680	0,001939	3,7603E-06	-0,9801	0,5081	0,004291	0,01066	0,9686	145	0,0003183
2	0,006621	0,001652	2,7275E-06	-1,2914	0,1555	0,004396	0,00956	0,8805	133	0,0002833
3	0,006622	0,001376	1,8932E-06	-1,3058	-0,2818	0,004486	0,00857	0,8012	121	0,0002477
4	0,006684	0,001134	1,2868E-06	-0,8751	-0,7854	0,004581	0,00794	0,7286	109	0,0002154
5	0,006790	0,000916	8,3909E-07	-0,1609	-1,1544	0,004781	0,00762	0,6586	97	0,0001846
6	0,006890	0,000728	5,2934E-07	-0,3655	-0,6904	0,005168	0,00782	0,5856	85	0,0001569
7	0,006895	0,000633	4,0125E-07	-0,8465	0,3613	0,005821	0,00803	0,5034	73	0,0001478
8	0,006799	0,000560	3,1361E-07	-0,9181	0,6631	0,006097	0,00787	0,4147	61	0,0001434
9	0,006692	0,000494	2,4446E-07	-1,1367	0,5755	0,006126	0,00762	0,3279	49	0,0001420
10	0,006625	0,000417	1,7382E-07	-1,4474	0,3465	0,006119	0,00732	0,2451	37	0,0001390
11	0,006617	0,000305	9,3206E-08	-1,4361	-0,0645	0,006161	0,00707	0,1654	25	0,0001260
12	0,006677	0,000115	1,3126E-08	-1,1199	-0,2464	0,006483	0,00684	0,0868	13	0,0000692
	Bonds									
	Mean	Standard Deviation	Variance	Kurtosis	Skewness	Minimum	Maximum	Sum	Count	Confidence Level(95,0%)
1	0,008153	0,004765	2,2704E-05	-2,5159E-01	-6,9755E-02	-2,5779E-03	2,0111E-02	1,1822	145	0,0007821
2	0,007970	0,002848	8,1139E-06	-5,4858E-01	2,0422E-02	1,1468E-03	1,4905E-02	1,0600	133	0,0004886
3	0,007945	0,002332	5,4380E-06	4,5487E-02	5,0124E-01	2,3751E-03	1,4343E-02	0,9613	121	0,0004197
4	0,007717	0,001697	2,8792E-06	6,8698E-01	8,9079E-01	4,8851E-03	1,2923E-02	0,8411	109	0,0003222
5	0,007587	0,001288	1,6601E-06	2,2175E-01	7,3071E-01	5,4020E-03	1,1245E-02	0,7359	97	0,0002597
6	0,007501	0,000759	5,7646E-07	-7,9414E-02	4,5785E-01	5,9515E-03	9,4564E-03	0,6376	85	0,0001638
7	0,007512	0,000703	4,9383E-07	2,1201E+00	1,3508E+00	6,2725E-03	9,6422E-03	0,5484	73	0,0001640
8	0,007572	0,000748	5,5997E-07	-4,2381E-01	1,3534E-01	6,0156E-03	9,2724E-03	0,4619	61	0,0001917
9	0,007701	0,000756	5,7221E-07	-1,1040E+00	-3,1766E-01	6,2517E-03	8,9838E-03	0,3774	49	0,0002173
10	0,007922	0,000723	5,2340E-07	-1,6416E+00	2,0531E-01	6,8269E-03	9,1164E-03	0,2931	37	0,0002412
11	0,007969	0,000712	5,0670E-07	-1,0748E+00	6,2614E-01	7,1771E-03	9,3078E-03	0,1992	25	0,0002938
12	0,008128	0,000181	3,2895E-08	2,3793E+00	8,9119E-01	7,8281E-03	8,5709E-03	0,1057	13	0,0001096
	Style value									
	Mean	Standard Deviation	Variance	Kurtosis	Skewness	Minimum	Maximum	Sum	Count	Confidence Level(95,0%)
1	0,012925	0,01668	0,0002781	1,0181618	-1,0145723	-0,0377516	0,0470284	1,8741	145	0,002737
2	0,014038	0,01118	0,0001251	-0,4596895	-0,2279430	-0,0141337	0,0361438	1,8670	133	0,001918
3	0,014200	0,00864	0,0000746	-1,0010082	0,3794161	-0,0009103	0,0316221	1,7183	121	0,001554
4	0,014109	0,00644	0,0000415	-0,4488633	0,6657541	0,0044069	0,0300314	1,5379	109	0,001223
5	0,013579	0,00480	0,0000230	-0,6734507	0,3073725	0,0055360	0,0245016	1,3172	97	0,000967
6	0,012677	0,00315	0,0000099	-1,4741964	-0,2774640	0,0068119	0,0171047	1,0776	85	0,000680
7	0,012606	0,00304	0,0000093	-1,4066549	0,0348480	0,0077324	0,0184620	0,9202	73	0,000710
8	0,013526	0,00211	0,0000044	-0,6158508	-0,1572373	0,0086850	0,0179447	0,8251	61	0,000540
9	0,014084	0,00129	0,0000017	-0,0428791	-0,3152048	0,0108161	0,0167991	0,6901	49	0,000371
10	0,014198	0,00129	0,0000017	-0,5237869	-0,6392033	0,0115669	0,0162995	0,5253	37	0,000430
11	0,014066	0,00144	0,0000021	-1,1462959	-0,2003314	0,0119370	0,0163892	0,3516	25	0,000594
12	0,013041	0,00083	0,0000007	-1,7642117	0,1042092	0,0119694	0,0142681	0,1695	13	0,000502
	Style growth									
	Mean	Standard Deviation	Variance	Kurtosis	Skewness	Minimum	Maximum	Sum	Count	Confidence Level(95,0%)
1	0,01318	0,01662	0,0002761	1,4065	-1,2564	-0,04200	0,04397	1,9117	145	0,0027274
2	0,01399	0,01178	0,0001387	-0,7089	-0,3305	-0,01029	0,03519	1,8606	133	0,0020197
3	0,01396	0,00935	0,0000874	-0,9686	0,2358	-0,00164	0,03233	1,6886	121	0,0016830
4	0,01368	0,00759	0,0000576	-0,6665	0,2684	-0,00037	0,02938	1,4910	109	0,0014415
5	0,01308	0,00582	0,0000339	-0,6455	0,2190	0,00364	0,02664	1,2686	97	0,0011739
6	0,01215	0,00366	0,0000134	-1,4701	-0,0791	0,00624	0,01816	1,0331	85	0,0007895
7	0,01209	0,00256	0,0000066	-1,0206	0,1865	0,00747	0,01786	0,8829	73	0,0005975
8	0,01303	0,00158	0,0000025	-0,0416	0,1265	0,00990	0,01715	0,7949	61	0,0004048
9	0,01365	0,00103	0,0000011	-0,9032	-0,2165	0,01194	0,01584	0,6689	49	0,0002954
10	0,01392	0,00144	0,0000021	-0,6964	-0,6761	0,01110	0,01600	0,5152	37	0,0004808
11	0,01399	0,00137	0,0000019	-1,1033	-0,1143	0,01170	0,01636	0,3498	25	0,0005647
12	0,01326	0,00056	0,0000003	-1,6919	0,0703	0,01246	0,01404	0,1723	13	0,0003372

Notes: All returns used are continuously compounded monthly asset returns. "Count" is the number of months corresponding to each year period.
FTSE/JSE All share Style Value index---style value
FTSE/JSE All share Style Value index---style growth

BEASSA Albi total return index---all bonds
Alexander Forbes money market index---cash

Table 3.2 shows the correlation matrix between four asset returns. The correlations represent the average of correlations between cash, bonds, and equities returns over rolling periods. The correlation values are calculated for 12 investment periods, using monthly rolling period data.

Table 3.2: Real total returns of assets Correlation matrix

Periods (in years)		Cash	Bonds	Style value	Style growth
1	Cash	1			
	Bonds	0,301497676	1		
	Style value	-0,492018973	-0,289888461	1	
	Style growth	-0,517696469	-0,340686327	0,936211788	1
		Cash	Bonds	Style value	Style growth
2	Cash	1			
	Bonds	0,145518011	1		
	Style value	-0,420248787	0,005041046	1	
	Style growth	-0,446809757	-0,131907546	0,937518884	1
		Cash	Bonds	Style value	Style growth
3	Cash	1			
	Bonds	0,022992264	1		
	Style value	-0,258703027	0,093018216	1	
	Style growth	-0,27915064	0,001390105	0,944938978	1
		Cash	Bonds	Style value	Style growth
4	Cash	1			
	Bonds	-0,062662003	1		
	Style value	-0,03530814	0,21887817	1	
	Style growth	-0,059664168	0,099861002	0,964616645	1
		Cash	Bonds	Style value	Style growth
5	Cash	1			
	Bonds	-0,04512785	1		
	Style value	0,218964676	0,029026793	1	
	Style growth	0,21505876	0,017420154	0,982750056	1
		Cash	Bonds	Style value	Style growth
6	Cash	1			
	Bonds	0,17390911	1		
	Style value	0,615877608	0,005790705	1	
	Style growth	0,693786476	0,169140324	0,956140252	1
		Cash	Bonds	Style value	Style growth
7	Cash	1			
	Bonds	0,766582745	1		
	Style value	0,468378379	0,006800376	1	
	Style growth	0,57091744	0,141752695	0,963323033	1

		Cash	Bonds	Style value	Style growth
8	Cash	1			
	Bonds	0,829337225	1		
	Style value	0,368443919	0,354518633	1	
	Style growth	0,347145955	0,296031982	0,960357964	1
		Cash	Bonds	Style value	Style growth
9	Cash	1			
	Bonds	0,85733689	1		
	Style value	-0,073674686	0,086938074	1	
	Style growth	-0,426599584	-0,211459094	0,845232506	1
		Cash	Bonds	Style value	Style growth
10	Cash	1			
	Bonds	0,949729046	1		
	Style value	-0,607504567	-0,47691413	1	
	Style growth	-0,787797999	-0,671384534	0,94412224	1
		Cash	Bonds	Style value	Style growth
11	Cash	1			
	Bonds	0,920517731	1		
	Style value	-0,778540045	-0,877533162	1	
	Style growth	-0,819942624	-0,894969686	0,977462346	1
		Cash	Bonds	Style value	Style growth
12	Cash	1			
	Bonds	0,600525279	1		
	Style value	-0,820823512	-0,542140846	1	
	Style growth	-0,8447376	-0,731692779	0,892745175	1

Notes: These correlations are calculated from the rolling period log returns.
FTSE/JSE All share Style Value index---style value
FTSE/JSE All share Style Value index---style growth
BEASSA Albi total return index---all bonds
Alexander Forbes money market index---cash

In all investment periods, there is a strong correlation between Style value shares and Style growth shares, of about 0.94, on average. This means that Style growth and value equities are highly correlated. The strong correlation between the value and growth total return index could be found in the underlying stocks comprising these indices. A portfolio consisting of both type of indexes is less likely to be well diversified. Since, a strong correlation implies a high co-movement between the assets over the short-term. However, it is evident that even in the long-term the two asset indexes tend to be highly correlated[19].

[19] To test the long-term co-movement of asset, the reader may employ canonical econometric technique of cointegration.

The correlation between bonds and equities is mostly negative over the investment periods, on average the correlation between bonds and Style growth (value) shares is about -0.18 (-0.11). Cash and bonds have a positive correlation of about 0.45, on average. However, for these two relatively riskless assets they have a negative correlation in the 4^{th} and 5^{th} years. The correlation between cash and equities is mostly negative.

3.3 METHOD OF ANALYSIS

This section describes the mean-variance optimization technique used to find the optimal weights for 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 years of investment. Section 3.3.1 contains the description of the method of analysis used to find the optimal portfolio allocation over multiple periods.

3.3.1 Multi-period mean-variance asset allocation

The mean-variance analysis as noted above has its roots in seminal work of Markowitz (1952) and Tobin (1958). The literature proposes the mean-variance framework as the appropriate method to solve long-term optimal portfolio asset allocation. Many studies for example, Gunthorpe and Levy (1994), Tang and Lee (1997), Hansson and Persson (2000), In, et al. (2011), etc. used this method coupled with re-sampling techniques to investigate optimal portfolio allocation over multi-period investment horizons.

Suppose we have a risk averse South African investor planning for retirement. The investor has some wealth level RW_0 (or ZAR100) available to invest today. The objective is to find the optimal combination of assets to invest this wealth among three asset classes and one risk-free asset.

We model the investor's behavior with some utility function. That is, the life-cycle investor has expected utility of end-of-period wealth described by a particular utility function. Standard economic models of investor preference propose the power-utility function, because it provides solutions independent of the investor's wealth level. This class of utility function is widely applied in many academic finance works, because it exhibits constant relative risk aversion (CRRA):

$$U(W) = \frac{W^{\lambda}}{\lambda} \qquad \gamma < 1 \qquad 3.7$$

, where "λ" (=1-γ), represents the investor's level of risk aversion.

Further, we assume that the life-cycle investors' asset allocation is determined by the mean-return and the variance-covariance during the investment period. Alternatively, the investor's end-of-period wealth is determined by *ex post* historical returns. Therefore, the investor wants to find the portfolio weight(s) for assets that minimizes the trade-off between expected value of returns and variance of returns. The investor chooses a variance-return combination that maximizes the investor's end-of-period welfare[20]:

$$\underset{\boldsymbol{w}}{\text{Min}}\ \boldsymbol{w}'\mathbf{V}\boldsymbol{w} - \boldsymbol{\lambda}\boldsymbol{\mu}'\boldsymbol{w} \qquad 3.8$$

Subject to, $\boldsymbol{I}'\boldsymbol{w} = 1$

$$\boldsymbol{w} \geq 0$$

Where,

$\boldsymbol{w}$ = is the column vector of asset weights or shares of assets in the portfolio

$\mathbf{V}$ = is the variance-covariance matrix of returns

$\boldsymbol{\mu}$ = is the column vector of expected asset returns for the investment horizon under consideration

$\boldsymbol{\lambda}$ = (Lambda—$\boldsymbol{\lambda} > 0$) is the coefficient of risk aversion

$\boldsymbol{I}$ = is the vector of ones

We use vector notations to indicate that the case under study involves more than one asset.

The first restriction or the investor's budget constraint implies that the life-cycle investor's wealth is fully invested in the portfolio of assets. The non-negativity or long-only restriction on the second constraint implies, the investor takes a long position in the assets and hence, no short selling of securities at any given investment horizon.

The coefficient of risk aversion (λ) measures how the investor is willing to commit a proportion of his portfolio towards the risky asset class. For example, a high-risk averse investor (with low Lambda) would choose to hold the minimum-variance portfolio. Alternatively, low levels of risk aversion would lead the investor to choose the portfolio with a high Lambda.

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[20] The solution to the expected return maximization is a dual to the solution of variance minimization. Hence, for simplicity the author solves the minimization problem. The measure of risk used is the asset's standard deviation measured from historical returns data.

According to equation 3.8, the investor will split his ZAR100 portfolio investment between risky (equities) and relatively riskless assets (bonds or cash). Observing the asset weight vector at different investment horizons, would provide answers to the research questions (a, b and c) stated in section 1.2 of chapter 1.

The investor has no pre-determined target rate of return for which to aim for; thus, there is no investor shortfall constraint. The author does not assume a target rate of return at retirement, because of uncertainty that the "target rate" or return may not be achieved. Targeting would be relevant if the investor's goal is to achieve some mid-life obligations i.e. college tuition fees, large physical asset purchases like a beach house or a Lamborghini car. In that case, the investor derives utility from expected portfolio return exceeding the target return level. Hence, the investor view wealth as a target, and any portfolio performance falling below that target would dissatisfy the investor (i.e. disutility).

To examine the optimal portfolio allocation at various horizons, the risk aversion parameter (which determines the value of Lambda), is held constant at every investment period[21]. This is to assume that the investors risk attitude does not change overtime. In reality, this may not be the case. Hence, this assumption could be relaxed, and allow the parameter to assume different values to observe how the optimal portfolio changes over the long-term. Changing, the risk aversion parameter at different investment horizons allows the observer to see how the asset weights changes over the investment horizon and the important implications thereof.

The literature highlights conditions under which the above mean-variance framework is consistent with expected utility framework, see for example Sharpe (1991), Markowitz (1992). Thus, the above minimization problem is consistent with the optimization problem for an investor with a negative exponential utility function. Hansson and Persson (2000), observed that the above optimization method is an accurate approximation even if the investor's preferences are described by other utility functions and asset returns are not jointly normally distributed. Campbell (2011), observed that, "Utility assumptions are not needed to justify mean-variance analysis".

[21] This implies that the portfolio will change with respect to returns and variance at the selected investment periods, but the portfolio will have the same risk aversion or same risk price.

3.3.2 Estimation error

The mean-variance optimization problem is a function of the implied parameters, which is the expected mean-returns, and variance-covariances[22]. Chopra and Ziemba (1993), observed that under standard assumptions these inputs are subject to errors in measurement—expected returns, and variance-covariances. This is the "dark side" of portfolio optimization. Therefore, misspecification of the parameters of the return distributions (or estimation risk) in equities, bonds and cash returns may yield inaccurate results.

The analyses of Chopra and Ziemba (1993), shows that the relative impact of errors in means, variances and covariances depends on the investors' risk tolerance. Given a high level of risk tolerance, errors in means may have a significant impact on the optimal weights relative to the errors in variances and covariances. "Even though errors in means are more important than those in variances and covariances, the difference in importance diminishes with a decline in risk tolerance"[23].

The study of Hansson and Persson (2000), Sanfilipo (2003), and In, et al. (2011), suggested using the bootstrapping or re-sampling techniques to account for estimation errors in estimated return parameters. Bootstrapping involves resampling the historical asset return data (with or without replacement) over several different periods.

The following example (or imaginary experiment) describes the principles behind bootstrapping. Picture in your mind a sheet of paper with a table of historical asset returns data. Where the columns represent the different assets (or asset classes) i.e. equities, bonds, cash; and the rows are the historical monthly date or time periods. Now cut that sheet of paper into strips (horizontally) so that each month corresponding to the asset return is a strip. Place those strips in a round pot or bowl and mix them up nicely. Draw a strip from the pot (randomly), and write down the asset returns for each of the assets, put that strip back in the pot and repeat the process (many times) by

[22] If monthly parameters are employed we get a portfolio composition corresponding to monthly data. If annual parameters are employed we get an optimal composition corresponding to annual data. By comparing the optimal compositions corresponding to various horizons, one can draw conclusions about the implications of investment horizons.

[23] The literature differentiates between risk aversion and risk tolerance. Risk aversion is the willingness to take risk, measured as the curvature of the utility function. Risk tolerance is the willingness to take risk measured as the inverse of risk aversion Stewart, et al. (2011).

drawing more strips. After many times of drawing, you would have a new table of simulated asset returns streams. This is the fundamental idea behind the commonly used bootstrapping or resampling techniques.

There are pros and cons pertaining to this practice. Firstly, there are behaviours in some assets returns that cannot be properly modelled by any distribution function. That is to say, there may be a reason why a return that is plausible under a parametric distribution in a certain range is in reality unachievable. Or multiple peaks may exist that cannot be captured by a distribution function model. Secondly, any relationship existing between the asset returns will be preserved, in contrast with parametric distribution function models presented in the rest of this document where covariation among various transformations of assets returns are assumed to be linear. The ability to model nonlinear covariation among asset returns is particularly interesting if one believes that these relationships changes under certain market conditions i.e. the tendency for equity markets to move together during a financial crisis. It is also helpful when there is not a clear linear pattern in covariation of returns. These advantages are highlighted in Singh (1981)

However, one of the shortcomings of using bootstrapping techniques is that the technique does not account for the serial correlation between each block generated when bootstrapping, Sanfilipo (2003). In, et al. (2011), found little difference between portfolio allocation with original (not resampled) data series and bootstrapped mean returns. Therefore, to answer the above research questions we use the mean-variance framework, despite its well-established limitations.

3.4 CONCLUSION

In conclusion, the study uses monthly asset (rolling) returns data, for the period December 2001 to December 2014. Data for the FTSE/JSE Style growth and FTSE/JSE Style value indexes, South African All bond index and money market index data (Alexander Forbes) was obtained from I-Net Bridge. The money market index is assume to be risk-free. The study also uses the Firer and Staunton (2002), and Firer and Mcleod (1999) JSE all shares index data from January 1960 to December 2008.

Despite the well-established limitations of the mean-variance framework, academics and practitioners advocate its use in solving optimal portfolio construction problems, under various assumptions about asset return distribution over investment horizons. The basic mean-variance framework does not support the view that asset allocation depends on the investment horizon. In this case, we use the mean-variance optimization technique to find the trade-off between expected mean-return and the variance of returns over the various investment horizons.

Volatility adjustments, adjusts the level of the portfolio return for the difference in expected value between gross returns and continuously compounded returns. Given that the basic portfolio optimization model does not change overtime, the study adopts the optimization objective function—which is a function of time and risk aversion level—for asset allocation. The emphasis inherent in this model is that risk does not decline over the investment horizon, and it is the level of risk aversion that declines with an increase in the investment horizon.

CHAPTER 4: EMPIRICAL RESULTS

4.1 INTRODUCTION

The previous chapter discussed the mean-variance optimization problem model, and highlighted the issue of estimation error inherent in expected asset returns and variance-covariance inputs. The primary objective of this chapter is to carry out the data analysis to find the optimal portfolio asset allocation for 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 investment horizons. Furthermore, the paper attempts to find the risk involved in investing on the JSE all shares index for 1, 5, 10, 15, 20, 25, 30, 35, and 40 year periods.

The reader is cautioned to interpret the following empirical results with some degree of skepticism. For the following reasons a) because the dataset is not long enough to make strong conclusions on the results; studies of this nature are customarily carried out with dataset covering longer periods i.e. 30 or 40 years of monthly data; b) the case considers an investor with constant risk aversion overtime; the investor's risk attitude can change overtime. This assumption that the investor should maintain the same risk attitude over the different investment horizon seem intuitive and relevant. In addition c), the limitations inherent in applying the mean-variance model over several investment horizons.

The rest of the chapter has three subsections. Section 4.2 is an investigation concerning the risk involved in investing on the JSE ALSI, section 4.3 presents the optimization results solved using the solver function in Microsoft excel 2013. The last section (section 4.4) concludes the chapter.

4.2 TIME DIVERSIFICATION ON THE JSE ALL SHARES INDEX

Evidence suggesting time diversification generally implies that, a larger portion of an investor's wealth should be invested in risky assets over longer investment horizons.

In order to test for time diversification effects and examine its implications on asset allocation for retirement planning over longer investment horizons, it is necessary to understand and model the return generating process for the underlying asset return series. We investigate time diversification evidence on the JSE ALSI because we want to see if it is ideal to invest an individual's wealth on the JSE ALSI over longer investment horizons, and if there is a relationship between investment horizon and risk.

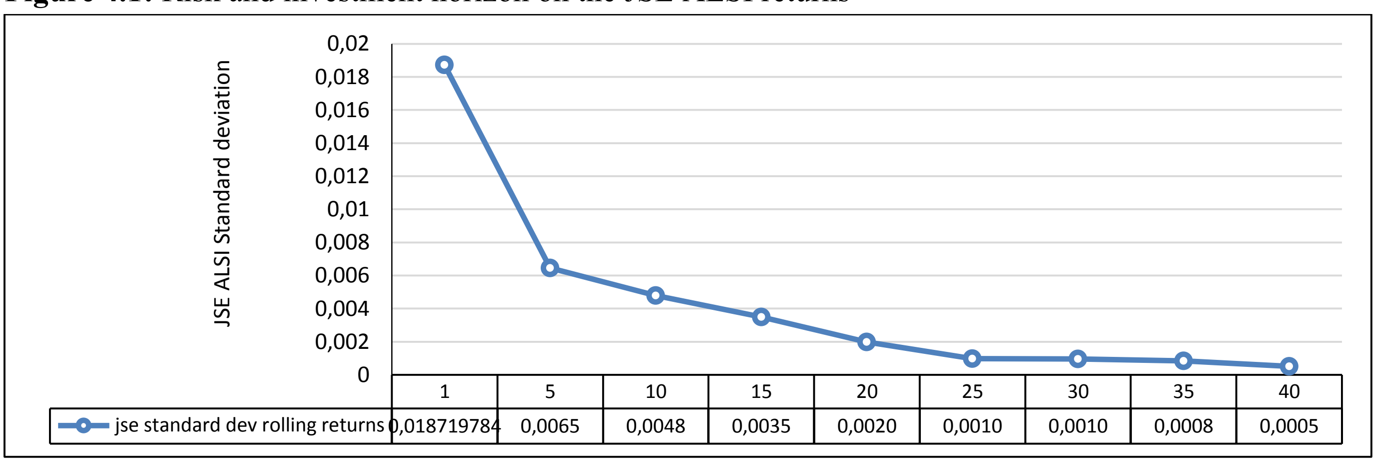
This study measures risk in terms of the standard deviation of asset returns. The standard deviation is the most widely used measure of risk; however, defining risk in terms of standard deviation would not reflect the true risk of an asset (or portfolio), and hence, conclusions drawn from the study should be interpreted with caution.

The standard deviation helps in measuring the historical volatility of asset (or portfolio) returns, and how portfolio returns have fluctuated relative to the mean-returns of assets. Thus, the standard deviation may be highly correlated with market volatility. This makes it difficult to isolate individual risk strategy from general market risk. Therefore, investors should not rely strictly on its use as a measure of risk.

Nevertheless, to examine the time diversification notion on the JSE, we plot the asset's standard deviation over different investment periods and observe what happens to the asset's standard deviation overtime[24]. The standard deviation of assets changes with the square root of time. Hence, we divide the asset's standard deviation with the square root of time. This result is presented in figure 4.1 below.

To calculate the predicted standard deviations at each horizon, by dividing the corresponding rolling-period standard deviation by the square root of the corresponding number of years in the investment horizon.

Figure 4.1: Risk and investment horizon on the JSE ALSI returns



[24] With time diversification, we are attempting to find out if there is a relationship between investment horizon and risk. That is, to check if risk decreases or increases over the investment horizon. The plot is basically based in the ratio of the standard deviation to the square root of the time horizon.

Note: The asset returns are rolling period returns (i.e. 1, 5, 10, 15, 20, 25, 30, 35, and 40 rolling periods). This conclusion on time diversification holds even in the case of point-to-point asset returns.

The JSE ALSI risk decreases with time; this implies that the JSE ALSI is less risky at longer horizons than at shorter investment periods. Lengthening the investment horizon reduces risk; stocks benefit the most from an increase in investment horizon. This result could be attributed to the mean reversion property in asset returns. Mean reversion implies that, the variance of stocks do not grow linearly with time[25]. Next, we test for the mean reversion property in asset returns using the model proposed by Lo and Mackinlay (1988).

4.2.1 Variance ratio test for mean reversion

The variance ratio test popularized by Lo and Mackinlay (1988)[26] is based on the variance of log returns for different investment horizons. The essential idea behind the variance ratio test is that, if a series is a "random walk", then successive elements in the series are uncorrelated and thus the variance of the sum of elements in the series should grow in proportion to the number of terms. If the asset returns series follow a "random walk" then there is no mean reversion (or aversion) is the underlying series.

The variance ratio is calculated as follows:

$$VR(q) = \frac{Var(r_t(q))}{qVar(r_1)} \approx 1 + 2\sum_{j=1}^{q}\left(1 - \frac{j}{q}\right)\hat{\rho}(j) \qquad 4.1$$

Where $VR(q)$ is the variance ratio; $r_t(q) = \sum_{j=1}^{q} r_{t-j+1}$, and r_t is the log returns (natural logarithm of price ratios) over some period. $Var(r_t)$ is the variance of returns and $\hat{\rho}(j)$ is the sample autocorrelation of the one-period return at lag j. Assuming returns are serially uncorrelated, $\hat{\rho}(j)$ should be close to zero at all lags.

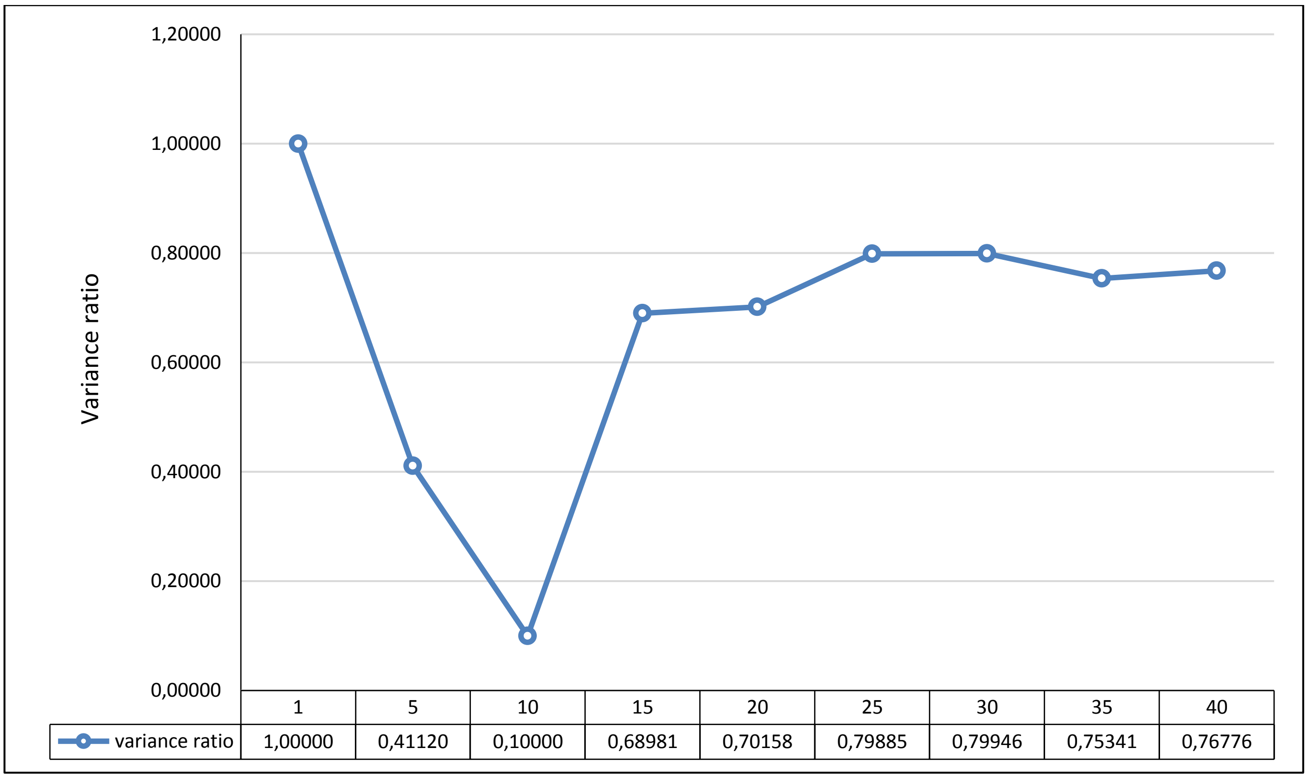
If the JSE ALSI return series follows a random walk i.e. returns are i.i.d, then $Var(rt(q) = qVar(rt)$ and therefore, $VR(q) = 1$. If the JSE ALSI returns are mean reverting (averting) then

[25] The presence of mean reversion in asset returns is characterised by the absence of serial correlations in asset returns.
[26] Details on the computations of variance ratio test can be found in Lo and Mackinlay (1988).

$VR(q)$ is less than (more than) one. The ratios are calculated for 1, 5, 10, 15, 20, 25, 30, 35, and 40 year periods.

Figure 4.2 shows a V-shaped pattern in mean reversion for the JSE ALSI returns. The variance ratio starts at 0.4112 for year 5 it then reaches a minimum of 0.1000 after 10 years, and then rises to 0.6898 for the 15-year returns. This may be considered as evidence that the JSE ALSI returns exhibit mean reversion over longer periods i.e. weak equity returns tend to be offset by strong equity performance overtime.

Figure 4.2: Variance ratio for the JSE ALSI returns-January 1960 to December 2008



Note: Lo and Mackinlay (1988), variance ratio test. Values smaller (greater) than one suggest mean reversion (aversion). The variance for each period are annualized by multiplying the monthly variance by the number of period observations—i.e. variance of one year monthly returns is multiplied by 12, variance of 5 year monthly returns is multiplied by 60 and so on. The data used is collected from the work of Firer and Staunton (2002), and Firer and Mcleod (1999).

A similar test was done using data from a group of countries including South Africa (using real equities) by Wang (2012). Wang (2012), found the following figures for South Africa:

		Number of aggregated years		
	2	4	8	16
South Africa	1,06	0,92	0,69	0,56

These results are quoted from the Doctoral thesis dissertation of Wang (2012). The data used to calculate these figure were obtained from the Ibbotson & Associate data network for the annual period 1900 to 2007.

From those results, the variance ratio is less than one; implying mean reversion is asset returns. These results aid the findings that the JSE ALSI tend to be less risky over longer investment horizons.

4.3 MULTI-PERIOD MEAN-VARIANCE OPTIMIZATION RESULTS

The objective of the mean-variance analysis is to maximize the expected portfolio return, subject to the investor's budget constraint and the long-only constraint in risky asset positions. This objective function is implemented for 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 investment horizons. The same application would be suitable for data covering longer periods. Note however, that asset returns are rolling period returns[27]. To find the optimal portfolio asset allocation for several investment periods, the study uses the objective function outlined in (equation 3.8).

The theory suggest that higher (lower) levels of risk aversion lead to larger (lower) proportions of investment in the risk free assets. The investor planning for retirement is mostly concerned about how much money (wealth) he/she will end up with, to sustaining h/her life when in retirement.

Considering the case of a pensioner planning for retirement. Naturally, such an individual is risk averse, by assumption. Given that the pensioner has three assets available for investment, plus a risk-free asset (cash), the pensioner would choose the minimum risk portfolio given expected returns.

The following table shows the optimal portfolio combination for the investor planning for retirement. Three assets are considered to construct a three-asset portfolio over twelve investment horizons.

[27] In reality, the rolling returns would match the investors' investment horizon. For example, ten-year investment horizon, ten-year rolling period returns. The data at hand is not long enough for that kind of practice.

Table 4.1: Optimal portfolio asset allocation over different investment horizons

Investment horizon	Optimal Portfolio allocation			Total weights (∑=1)	Portfolio returns (μ_p)	Portfolio Standard deviation (σ_p)	Sharpe ratio*
Years	Bonds	Style value	Style growth				
1	80%	0%	20%	100%	11,0%	0,48%	6,21
2	94%	0%	6%	100%	10,0%	0,32%	31,55
3	99%	0%	1%	100%	9,6%	0,27%	35,77
4	97%	0%	3%	100%	9,5%	2,17%	4,37
5	97%	3%	0%	100%	9,3%	1,56%	5,97
6	92%	8%	0%	100%	9,5%	1,17%	8,14
7	97%	3%	0%	100%	9,2%	0,88%	10,45
8	97%	0%	3%	100%	9,3%	0,94%	9,91
9	99%	0%	1%	100%	9,3%	0,90%	10,28
10	97%	0%	3%	100%	9,7%	0,98%	9,94
11	98%	2%	0%	100%	9,7%	0,80%	12,06
12	93%	2%	5%	100%	11,0%	0,65%	17,04

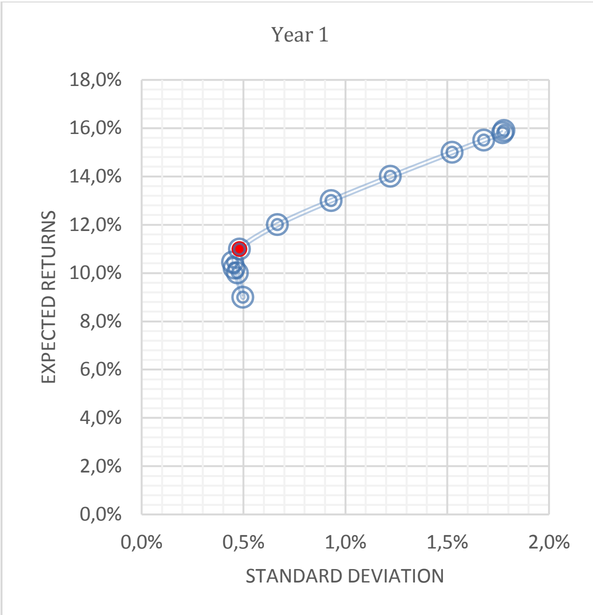
Notes: These results are computed using rolling period (nominal) returns using Microsoft excel solver function with different optimization inputs for each investment period. *The Sharpe ratio is the portfolio average returns less the risk-free rate (average cash return) divided by the standard deviation of portfolio returns. The portfolio with the highest Sharpe ratio is the portfolio with the lowest risk given expected returns.

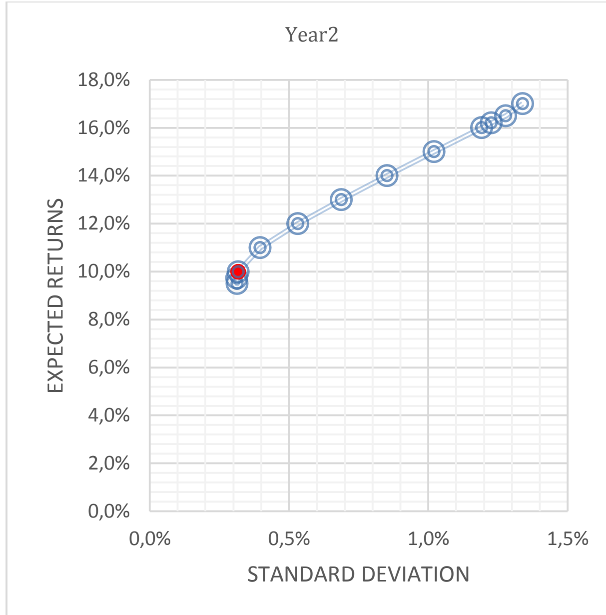
From the table it is obvious the greatest portfolio allocation is in bonds or fixed-income assets over the investment horizons. This is because our investor (or pension planner) is risk averse, and is mostly interested in investing in a portfolio that would yield the lowest risk given expected returns.

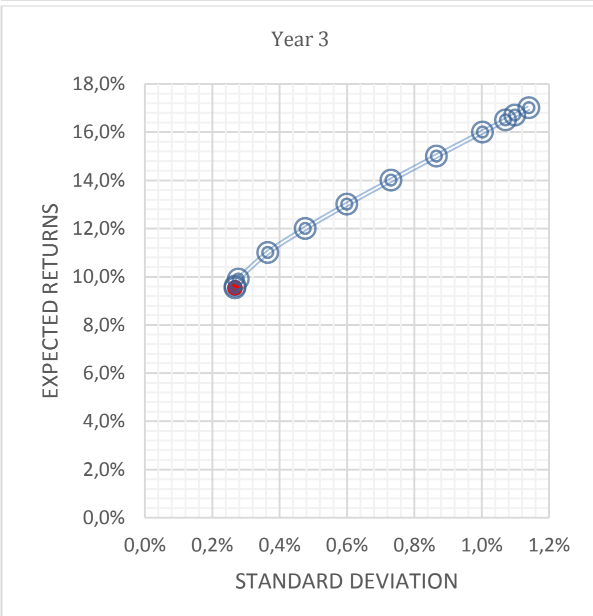
During year one, the portfolio consist of 80 percent fixed-income assets and 20 percent growth shares. This is the portfolio with the highest expected return compared to other asset combinations (i.e. 11 percent). From year two, the portfolio shifted heavily towards bonds (with over 90 percent allocation to bonds) and lower asset allocation to equities i.e. 6 percent. However, this portfolio has the second highest Sharpe ratio among other efficient portfolios over the long investment horizon. The portfolio with the highest Sharpe ratio has a greater fixed-income bond index allocation of 99 percent and a 1 percent allocation to growth equity index.

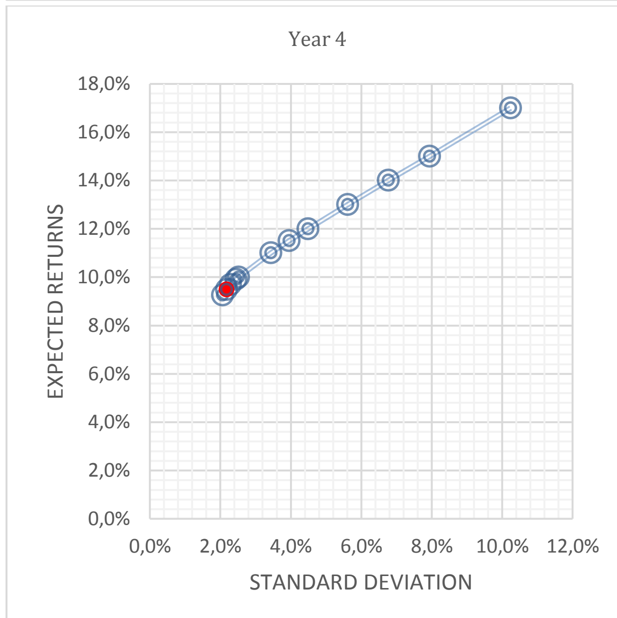
Between year five, and seven, the portfolio allocation was between bonds and value stocks. After the 12th year, the portfolio was invested in both bonds (93 percent), Style value (2 percent), and Style growth (5 percent).

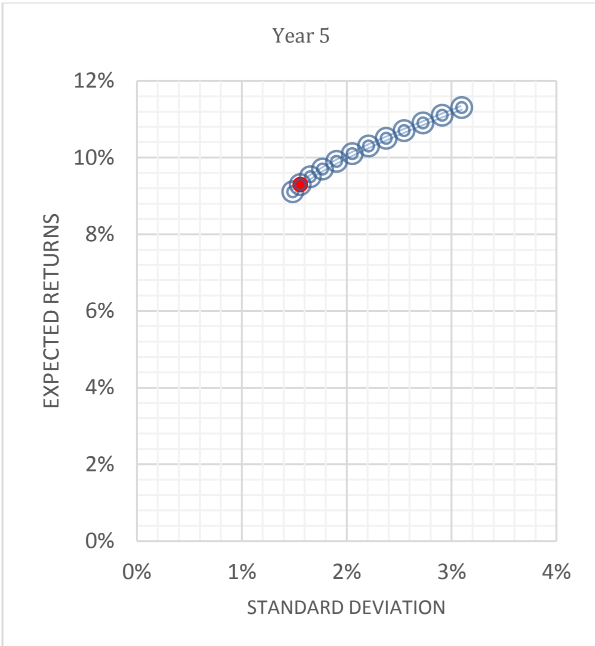
Figure 4.3: Efficient frontiers for twelve investment horizons (year 1 to year 12)

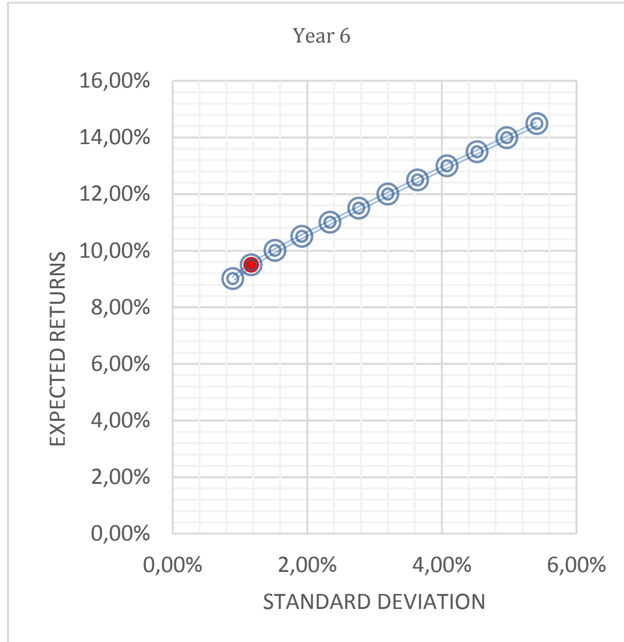

Year 1
EXPECTED RETURNS
18,0%
16,0%
14,0%
12,0%
10,0%
8,0%
6,0%
4,0%
2,0%
0,0%
0,0%
0,5%
1,0%
1,5%
2,0%
STANDARD DEVIATION

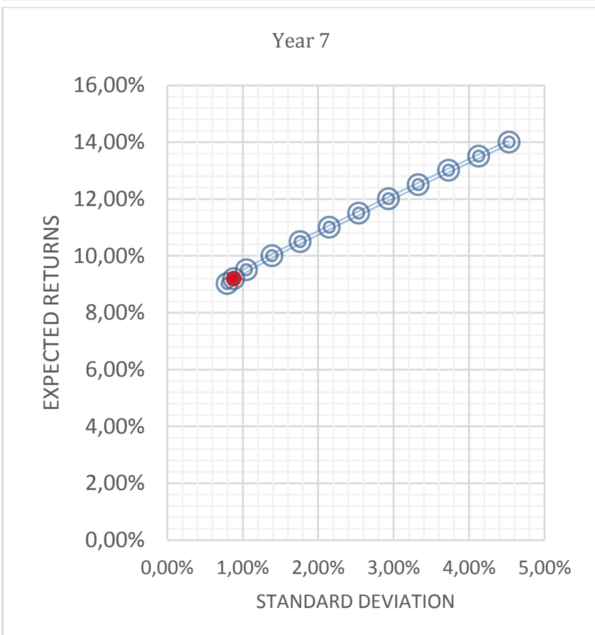

Year2
EXPECTED RETURNS
18,0%
16,0%
14,0%
12,0%
10,0%
8,0%
6,0%
4,0%
2,0%
0,0%
0,0%
0,5%
1,0%
1,5%
STANDARD DEVIATION

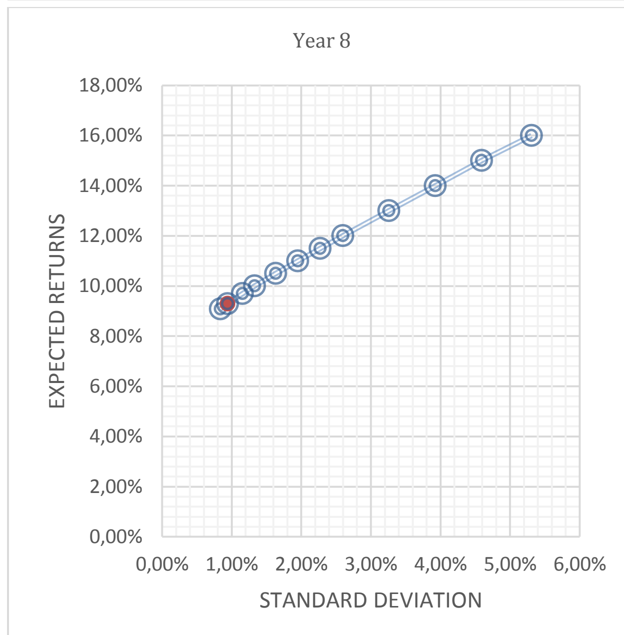

Year 3
EXPECTED RETURNS
18,0%
16,0%
14,0%
12,0%
10,0%
8,0%
6,0%
4,0%
2,0%
0,0%
0,0%
0,2%
0,4%
0,6%
0,8%
1,0%
1,2%
STANDARD DEVIATION

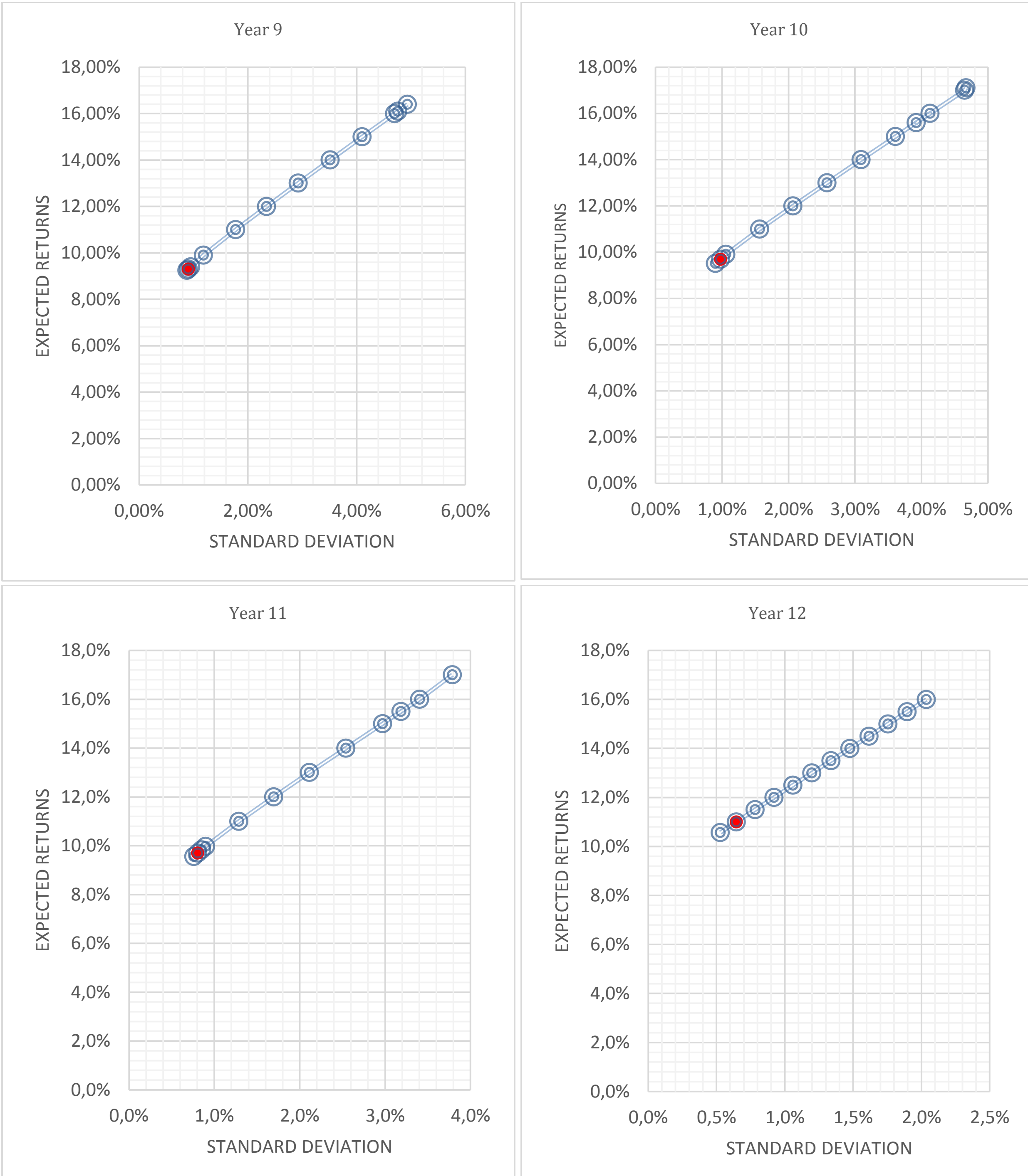

Year 4
EXPECTED RETURNS
18,0%
16,0%
14,0%
12,0%
10,0%
8,0%
6,0%
4,0%
2,0%
0,0%
0,0%
2,0%
4,0%
6,0%
8,0%
10,0%
12,0%
STANDARD DEVIATION


Year 5
EXPECTED RETURNS
12%
10%
8%
6%
4%
2%
0%
0%
1%
2%
3%
4%
STANDARD DEVIATION


Year 6
EXPECTED RETURNS
16,00%
14,00%
12,00%
10,00%
8,00%
6,00%
4,00%
2,00%
0,00%
0,00%
2,00%
4,00%
6,00%
STANDARD DEVIATION


Year 7
EXPECTED RETURNS
16,00%
14,00%
12,00%
10,00%
8,00%
6,00%
4,00%
2,00%
0,00%
0,00%
1,00%
2,00%
3,00%
4,00%
5,00%
STANDARD DEVIATION

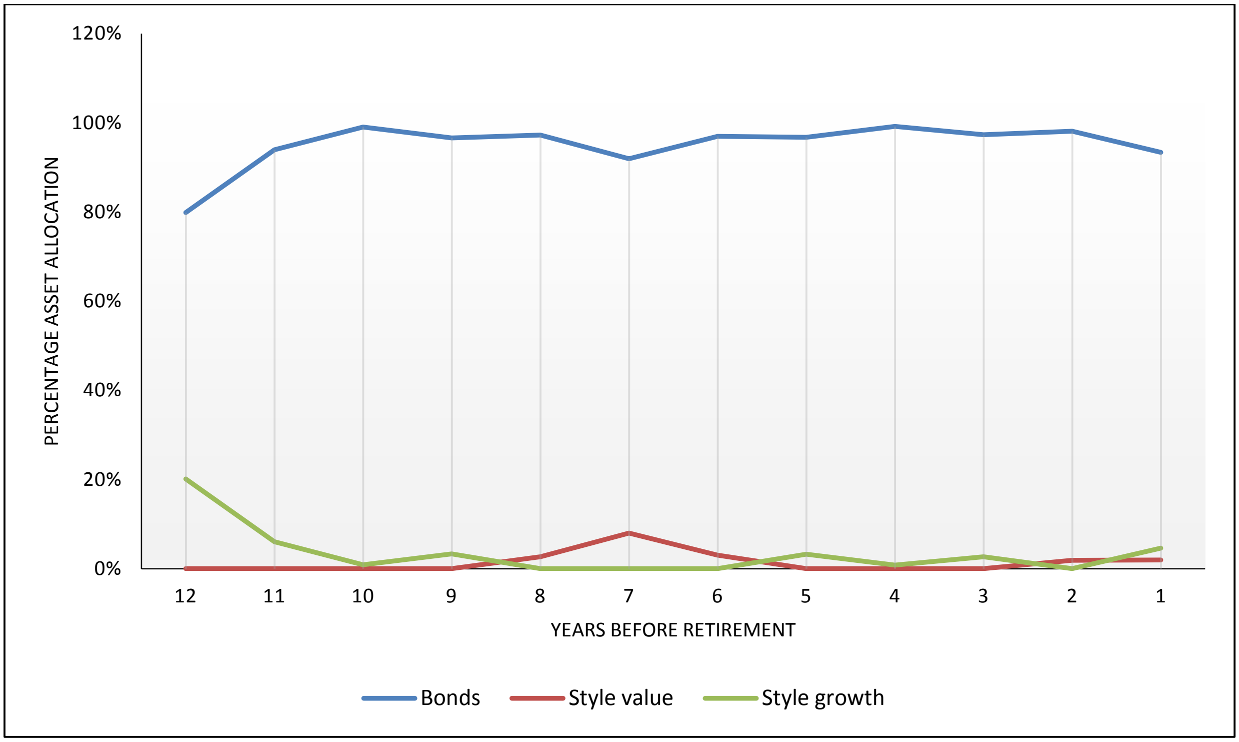

Year 8
EXPECTED RETURNS
18,00%
16,00%
14,00%
12,00%
10,00%
8,00%
6,00%
4,00%
2,00%
0,00%
0,00%
1,00%
2,00%
3,00%
4,00%
5,00%
6,00%
STANDARD DEVIATION



Notes: the “red” button indicate on each “efficient frontier” shows the minimum-variance portfolio. All other portfolios are inferior to this point.

Figure 4.3 shows the efficient frontier portfolio for each investment horizon. The efficient frontier portfolio is marked in “red”. This is the portfolio with the highest Sharpe ratio.

Figure 4.4: Optimal asset allocation and the investment horizon



Notes: In practice optimal portfolios often, reflect corner solutions with 0 percent weights for most assets or in the extreme 100 percent allocated to a single asset, for instance zero percent weights in bonds and cash at all investment horizon.

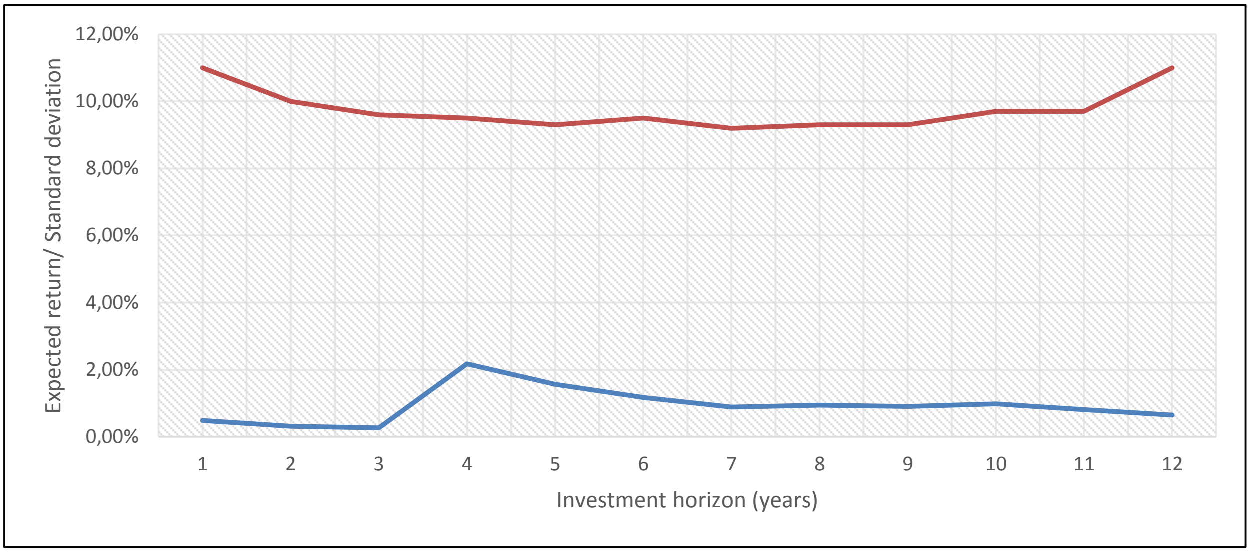
Figure 4.4 shows how the optimal portfolio composition changes overtime. When the investor has 12 years to go before retirement, the optimal portfolio composition is 80 percent investment in bonds and 20 percent in equities. Overall, the portfolio is mostly invested in the bonds index, with less than 10 percent investment in equities after year one.

These results demonstrate that a risk averse investor planning for retirement would invest most of his/her assets in fixed-income bonds and combining this with small equity allocation. The investor maintains his greater asset positions in bonds over the long term. However, the investor does not shy away from risky equities over the horizon.

For an investor putting up a lump-sum amount for retirement, the best portfolio allocation that would produce greater returns would be a portfolio comprising, mostly of bonds and with little equity allocation.

Figure 4.5 shows that the portfolio standard deviation (or risk) decreases overtime. However, the portfolio returns are relatively constant averaging about 9 percent.

Figure 4.5: Portfolio return and standard deviation (risk) over 12-investment horizon



Notes: These are rolling (nominal) portfolio returns. Portfolio standard deviation is calculated as the square root of the variance.

Assuming the investor has 12 years to go before retirement, his/her entire portfolio is best invested in bonds for greater (lower) expected portfolio returns (portfolio risk) at retirement date. However, the investor can aim to maximize returns for some time (say until 10 years before retirement); and change this objective to minimizing portfolio risk when he gets closer to the retirement date. This practice would be ideal for an investor concerned about poor portfolio performance near the target date. Otherwise, the investor can be aggressive and not be concerned about getting wiped-out at any given time over the long investment horizon.

4.4 CONCLUSION

The chapter aimed to provide answers to the research questions stated above. Firstly, to find the optimal portfolio allocation over different investment horizons; secondly, to see how the optimal portfolio composition changes with the investment horizon. Lastly, we interpret the implications of the results.

In summary, the act of investing on the JSE ALSI may be considered less risky overtime; provided the investment horizon is long enough. The standard deviation of the all shares index decreases with the square root of time. This result is supported by the concept of mean reversion property in the all shares index overtime. Mean reversion implies that lower equity performance tend to be offset by greater equity performance over longer periods of investment. Evidence of the presence of mean reversion is estimated using the variance ratio test. The variance ratio test looks at the variance of continuously compounded asset returns over different periods. The wisdom that long-term investors should invest greater proportions of their wealth in equities is therefore supported in this study.

The optimal portfolio allocation implies greater investment in bonds index (South African All Bonds' fixed-income index) coupled with small portions invested in equities over 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 investment horizons. The highest equity allocation of 20 percent can be observed when the investor has long investment outlook, in our case 12 years to go. The allocation to equities changes overtime, as we try to find the minimum-variance portfolio. The investor (retirement planner) portfolio allocation presented below hold regardless of the level of risk aversion. That is to say, if we alter the investor's level of risk aversion to reflect changes is his/her risk attitude, we would still end up with the same minimum variance portfolios, signaling higher allocation to fixed-income assets with lower allocation to equities over the period studied.

These findings contrast with extreme case scenario, were one would observe 100 percent equity allocation to equities with lower allocations to bonds. The results above should be treated with caution because a) the number of assets are too few and exclude a good proxy for a foreign asset; b) time period considered is too short to rely on the conclusions to make good investment decisions, and c) the risk-free asset considered changed slightly across investment horizons. We use the risk-free proxy to calculate excess returns and excess portfolio returns.

CHAPTER 5: SUMMARY AND CONCLUSION

5.1 INTRODUCTION

This chapter aims to summarize the findings of the study and conclude the dissertation paperwork. The basis of the conclusion is an in-depth summary of the dissertation work mainly surrounding the methodology, and empirical findings and implications thereof. The chapter contains three sections. Section 5.2 summarizes the findings and the implications of the results. Section 5.3 presents recommendations for further studies, and finally section 5.4, which concludes the chapter.

5.2 FINDINGS AND IMPLICATIONS

Asset management practitioners commonly advice retail investors with long-term investment horizon to invest their wealth in portfolios comprised more of risky assets or equities over long investment periods. This advice is usually coupled with the practice of shifting ones portfolio from risky asset holdings towards bonds and cash as the investor's target date gets closer, to avoid huge unforeseeable losses. This view is based on the notion that equities tend to be less risky over the long-term and that stock returns exhibit mean reversion properties or time diversification overtime.

The purpose of this study is to find the optimal asset allocation over various investment horizons for an investor aiming to maximize the expected utility of wealth at the retirement date. In addition, the study investigates how the optimal asset allocation changes over the long investment horizon. The study uses data from South Africa's financial market covering the period December 2001 to December 2014. The mean-variance analysis finds the optimal asset allocation over 12 investment horizons.

The study found that the optimal portfolio allocation that would yield the highest expected portfolio return is the portfolio that is more than ninety percent invested in bonds with over five percent equity allocation over the long-term. This is also the portfolio with the highest Sharpe ratio among alternative asset allocation portfolios, suggested in the community. The optimal portfolio choice does not change much overtime i.e. the portfolio shifts between growth stocks at a one-year period to value stocks overtime, before shifting into value shares. The portfolio allocation with equities is coupled with greater investment in bonds. Ultimately, the portfolio is more invested in bonds over the long-term. The results contrast with popular view that, the portfolio should be shifted into safer assets when the investor get closer to the target or retirement date. The results indicate holding relatively safer assets i.e. bonds overtime.

These findings challenges the results of Alles and Athanassokos (2006), that stocks are the optimal investment option for a life cycle investor with a long investment horizon. A peculiar result is that for all levels of risk aversion the “bonds” weights was still larger for the longer investment horizon, these findings contrast the findings highlighted by Hansson and Persson (2000).

Time diversification implies higher equity allocation overtime. The results does not support the view that a portfolio composition should be more invested in equities over a longer investment outlook. Therefore, the results does not support the evidence of time diversification, because of the higher allocation to fixed income assets over the long investment horizon. In addition, these results do not contradict the presence of time diversification observed on the JSE i.e. the presence of mean reversion on the JSE all shares index.

The implications of the study are twofold. Firstly, the investor aiming to minimize portfolio risk given maximum expected portfolio returns over the long-term should consider great wealth allocation to bonds with little over five percent in equities overtime. This advice is applicable if the investor is investing a lump sum amount, and the measure of risk is the standard deviation of asset returns. These results may not hold under different measures of risk and or if the investor makes regular annuity (saving) contributions over the long investment horizon.

Secondly, longer investment horizon imply lower risk aversion coefficient as the investment horizon increases. This implies that the investor becomes more willing to take risk at longer periods of investment than at short investment horizons. Therefore, the view that investors should tilt their portfolio holdings to cash and bonds (safer assets) when near the target date is unnecessary. The investor can therefore hold less risky assets (fixed-income assets) combined with few equity holdings even before the target date or even in retirement.

5.3 RECOMMENDATIONS FOR FUTURE STUDIES

Going forward, future studies should focus on the more practical case of an investor saving for retirement—and therefore investing hi/her savings at regular intervals in a dynamic portfolio framework. This is important because investors live in the present moment and are concerned about what happens to their portfolios at regular intervals over the long-term, and not simply at some target date, which they may not reach, because they can die before that date comes.

The findings of this study may not hold if we assume different measures of risk—a measure different from the standard deviation of asset returns—overtime. Future studies are advised to use different measures of risk to find the optimal portfolio allocation over longer investment horizons. It is highly recommended that a long span dataset be used for this endeavor.

Throughout this investigation, the key assumption that binds together these findings is that, the investors risk attitude does not change overtime. In reality, this may not be the case. Hence, this assumption could be relaxed, and allow the parameter to assume different values to observe how the optimal portfolio changes over the long-term. Changing, the risk aversion parameter at different investment horizons allows the observer to see how the asset weights changes over the investment horizon and the important implications thereof.

Lastly, it is recommended that futures studies explore other asset allocation methodologies highlighted in the literature. For example, the application of the model for strategic asset allocation proposed by Campbell and Viceira (2002), to the South African financial markets data. This would provide a different perspective on asset allocation, considering the asset return predictability implications, and the effects of changing investment opportunity set over the long investment horizons.

5.4 CONCLUSION

The objective of this dissertation was to construct the optimal portfolio strategy for an investor planning for retirement. The investor decides to invest some level of wealth (independent of other earnings factors—"lump sum") over longer periods. To this end the paper provided answers to the research questions: a) what is the optimal portfolio allocation for an investor planning for retirement?, b) how does the optimal portfolio composition change over the long investment horizon, and c) what are the portfolio policy implications for the results/findings in a) and b)?

In an attempt to provide answers to these questions, we computed 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 rolling period (nominal) returns, with South Africa's financial market data covering the period December 2001 to December 2014. Prior to our main investigation, the study investigated the presence of time diversification evidence on the JSE all shares index for 1, 5, 10, 15, 20, 25, 30, 35, and 40 years. The study found evidence of mean reversion asset return property and results supporting the time diversification concept on the JSE all shares index.

The paper employed the mean-variance analysis (considering different scenarios) to find the optimal portfolio allocations over several investment horizons. Measuring risk as the standard deviation of asset returns. The results suggests greater wealth allocation into bonds and with little over five percent allocation in equities over the given investment period. From these findings, the dissertation conclude that, the optimal portfolio construction for an investor making retirement plans, is a portfolio invested in fixed-income bonds (over 90 percent) and equities (5 percent or more). This strategy protects the investor from getting completely wiped out, should a likely crisis event take place over the span of investment period.

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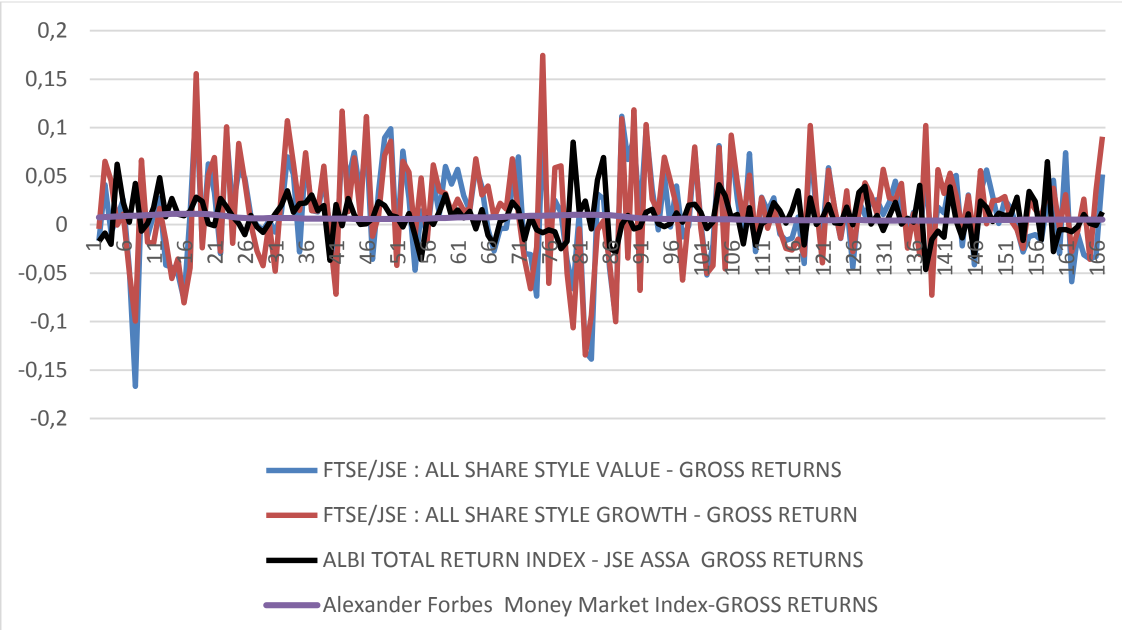
APPENDICES

APPENDIX A: ASSET RETURNS

GRAPH A1: SIMPLE GROSS RETURN



GRAPH A2: LOGARITHMIC RETURNS

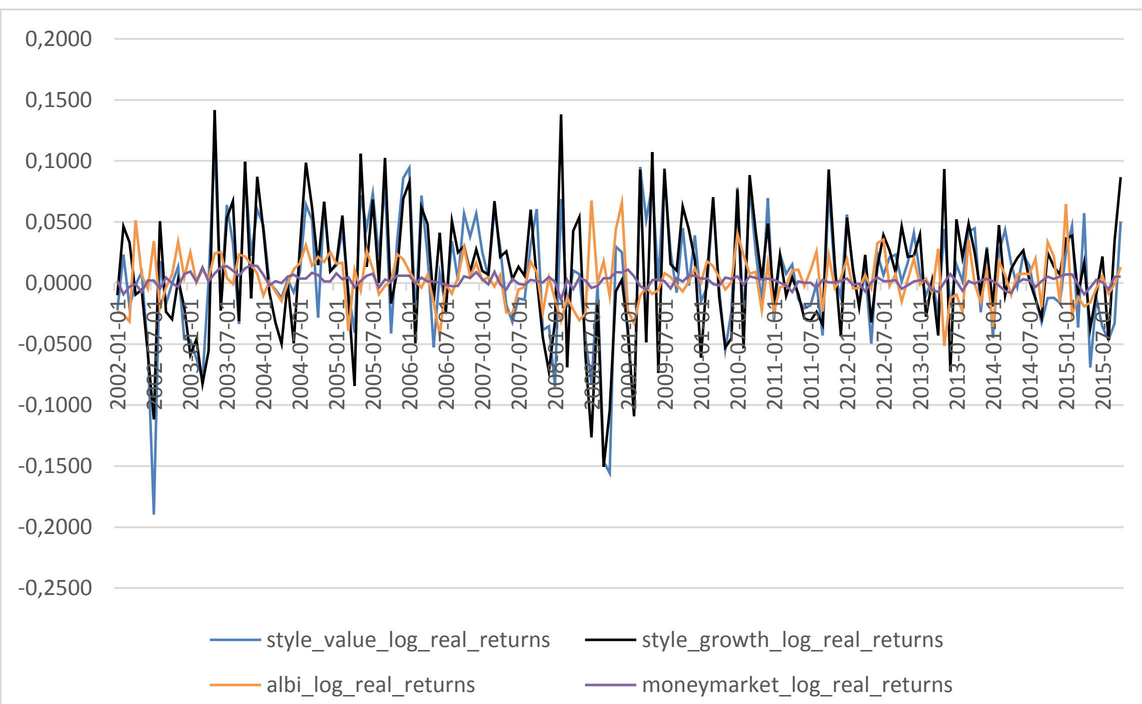


Table A1: Summary statistics

1-year							
Alexander Forbes money market total return index		BEASSA All Bond total return index		FTSE/JSE style value total return index		FTSE/JSE style growth total return index	
Mean	0,006563	Mean	0,008110	Mean	0,011515	Mean	0,012889
Standard Error	0,000153	Standard Error	0,000403	Standard Error	0,001363	Standard Error	0,001288
Standard Deviation	0,001917	Standard Deviation	0,005037	Standard Deviation	0,017020	Standard Deviation	0,016084
Sample Variance	0,000004	Sample Variance	0,000025	Sample Variance	0,000290	Sample Variance	0,000259
Kurtosis	-0,863077	Kurtosis	2,356771	Kurtosis	0,384283	Kurtosis	1,565031
Skewness	0,626846	Skewness	-0,705457	Skewness	-0,807053	Skewness	-1,234237
Range	0,006366	Range	0,035750	Range	0,084780	Range	0,085974
Minimum	0,004291	Minimum	-0,015639	Minimum	-0,037752	Minimum	-0,042004
Maximum	0,010657	Maximum	0,020111	Maximum	0,047028	Maximum	0,043969
Sum	1,023860	Sum	1,265088	Sum	1,796406	Sum	2,010719
Count	156	Count	156	Count	156	Count	156
Confidence Level(95,0%)	0,000303	Confidence Level(95,0%)	0,000797	Confidence Level(95,0%)	0,002692	Confidence Level(95,0%)	0,002544

2-years							
Alexander Forbes money market total return index		BEASSA All Bond total return index		FTSE/JSE style value total return index		FTSE/JSE style growth total return index	
Mean	0,006360	Mean	0,007795	Mean	0,012071	Mean	0,013670
Standard Error	0,000132	Standard Error	0,000273	Standard Error	0,000956	Standard Error	0,000881
Standard Deviation	0,001650	Standard Deviation	0,003415	Standard Deviation	0,011939	Standard Deviation	0,011006
Sample Variance	0,000003	Sample Variance	0,000012	Sample Variance	0,000143	Sample Variance	0,000121
Kurtosis	-1,214443	Kurtosis	13,014345	Kurtosis	-0,335688	Kurtosis	-0,464856
Skewness	0,425672	Skewness	-1,996929	Skewness	-0,242084	Skewness	-0,273059
Range	0,005164	Range	0,030544	Range	0,053084	Range	0,045481
Minimum	0,004396	Minimum	-0,015639	Minimum	-0,016940	Minimum	-0,010290
Maximum	0,009559	Maximum	0,014905	Maximum	0,036144	Maximum	0,035191
Sum	0,992122	Sum	1,215943	Sum	1,883090	Sum	2,132466
Count	156	Count	156	Count	156	Count	156

Confidence Level(95,0%)	0,000261	Confidence Level(95,0%)	0,0005	Confidence Level(95,0%)	0,001888	Confidence Level(95,0%)	0,001741

5-years							
Alexander Forbes money market total return index		BEASSA All Bond total return index		FTSE/JSE style value total return index		FTSE/JSE style growth total return index	
Mean	0,005997	Mean	0,007280	Mean	0,01100	Mean	0,013359
Standard Error	0,000100	Standard Error	0,000196	Standard Error	0,00055	Standard Error	0,000401
Standard Deviation	0,001253	Standard Deviation	0,002446	Standard Deviation	0,00692	Standard Deviation	0,005013
Sample Variance	0,000002	Sample Variance	0,000006	Sample Variance	0,00005	Sample Variance	0,000025
Kurtosis	-1,834676	Kurtosis	49,598229	Kurtosis	5,14353	Kurtosis	-0,086347
Skewness	0,063800	Skewness	-5,127505	Skewness	-1,63091	Skewness	-0,035087
Range	0,003091	Range	0,029505	Range	0,04144	Range	0,023130
Minimum	0,004533	Minimum	-0,015639	Minimum	-0,01694	Minimum	0,003508
Maximum	0,007624	Maximum	0,013866	Maximum	0,02450	Maximum	0,026637
Sum	0,935589	Sum	1,135751	Sum	1,71673	Sum	2,084015
Count	156	Count	156	Count	156	Count	156
Confidence Level(95,0%)	0,000198	Confidence Level(95,0%)	0,00039	Confidence Level(95,0%)	0,001095	Confidence Level(95,0%)	0,000793

10-years							
Alexander Forbes money market total return index		BEASSA All Bond total return index		FTSE/JSE style value total return index		FTSE/JSE style growth total return index	
Mean	0,005549	Mean	0,007241	Mean	0,010029	Mean	0,012916
Standard Error	0,000067	Standard Error	0,000183	Standard Error	0,000462	Standard Error	0,000228
Standard Deviation	0,000841	Standard Deviation	0,002283	Standard Deviation	0,005767	Standard Deviation	0,002848
Sample Variance	0,000001	Sample Variance	0,000005	Sample Variance	0,000033	Sample Variance	0,000008
Kurtosis	-1,153441	Kurtosis	65,613889	Kurtosis	10,300783	Kurtosis	0,482595
Skewness	0,311506	Skewness	-6,386160	Skewness	-2,990030	Skewness	-0,756216
Range	0,002788	Range	0,029505	Range	0,033240	Range	0,014787
Minimum	0,004533	Minimum	-0,015639	Minimum	-0,016940	Minimum	0,003508
Maximum	0,007321	Maximum	0,013866	Maximum	0,016299	Maximum	0,018295
Sum	0,865567	Sum	1,129598	Sum	1,564568	Sum	2,014941

Count	156	Count	156	Count	156	Count	156
Confidence Level(95,0%)	0,00013	Confidence Level(95,0%)	0,000361	Confidence Level(95,0%)	0,000912	Confidence Level(95,0%)	0,000450

12-years							
Alexander Forbes money market total return index		BEASSA All Bond total return index		FTSE/JSE style value total return index		FTSE/JSE style growth total return index	
Mean	0,00549	Mean	0,007178	Mean	0,009909	Mean	0,012977
Standard Error	0,00006	Standard Error	0,000180	Standard Error	0,000453	Standard Error	0,000226
Standard Deviation	0,00075	Standard Deviation	0,002248	Standard Deviation	0,005663	Standard Deviation	0,002819
Sample Variance	0,00000	Sample Variance	0,000005	Sample Variance	0,000032	Sample Variance	0,000008
Kurtosis	-1,58053	Kurtosis	69,152257	Kurtosis	10,983481	Kurtosis	0,684652
Skewness	0,05279	Skewness	-6,624661	Skewness	-3,141195	Skewness	-0,849588
Range	0,00230	Range	0,029505	Range	0,033071	Range	0,014787
Minimum	0,00453	Minimum	-0,015639	Minimum	-0,016940	Minimum	0,003508
Maximum	0,00684	Maximum	0,013866	Maximum	0,016131	Maximum	0,018295
Sum	0,85629	Sum	1,119817	Sum	1,545737	Sum	2,024371
Count	156	Count	156	Count	156	Count	156
Confidence Level(95,0%)	0,000119	Confidence Level(95,0%)	0,000356	Confidence Level(95,0%)	0,000896	Confidence Level(95,0%)	0,000446

APPENDIX B: MEAN-VARIANCE INPUTS

For each investment period, we have a variance-covariance matrix.

Table 4.1: Variance-covariance matrix for different investment horizons

	1			
	Cash	Bonds	Style value	Style growth
Cash	3,76027E-06	2,78576E-06	-1,59107E-05	-1,66806E-05
Bonds	2,78576E-06	2,27039E-05	-2,30345E-05	-2,69731E-05
Style value	-1,59107E-05	-2,30345E-05	0,000278097	0,000259417
Style growth	-1,66806E-05	-2,69731E-05	0,000259417	0,00027609
	2			
	Cash	Bonds	Style value	Style growth
Cash	2,72749E-06	6,84564E-07	-7,76147E-06	-8,6891E-06
Bonds	6,84564E-07	8,11392E-06	1,6058E-07	-4,42441E-06
Style value	-7,76147E-06	1,6058E-07	0,000125058	0,000123454
Style growth	-8,6891E-06	-4,42441E-06	0,000123454	0,000138657
	3			
	Cash	Bonds	Style value	Style growth
Cash	1,8932E-06	7,37735E-08	-3,07386E-06	-3,59141E-06
Bonds	7,37735E-08	5,438E-06	1,87314E-06	3,03106E-08
Style value	-3,07386E-06	1,87314E-06	7,45704E-05	7,62983E-05
Style growth	-3,59141E-06	3,03106E-08	7,62983E-05	8,7429E-05
	4			
	Cash	Bonds	Style value	Style growth
Cash	1,28677E-06	-1,20611E-07	-2,58073E-07	-5,13858E-07
Bonds	-1,20611E-07	2,87917E-06	2,39305E-06	1,28649E-06
Style value	-2,58073E-07	2,39305E-06	4,15177E-05	4,71899E-05
Style growth	-5,13858E-07	1,28649E-06	4,71899E-05	5,76443E-05
	5			
	Cash	Bonds	Style value	Style growth
Cash	8,39087E-07	-5,3262E-08	9,62389E-07	1,14738E-06
Bonds	-5,3262E-08	1,66012E-06	1,79449E-07	1,30728E-07
Style value	9,62389E-07	1,79449E-07	2,30222E-05	2,7464E-05
Style growth	1,14738E-06	1,30728E-07	2,7464E-05	3,3923E-05

	6			
	Cash	Bonds	Style value	Style growth
Cash	5,29336E-07	9,6067E-08	1,41253E-06	1,84761E-06
Bonds	9,6067E-08	5,76464E-07	1,38597E-08	4,70059E-07
Style value	1,41253E-06	1,38597E-08	9,93741E-06	1,10326E-05
Style growth	1,84761E-06	4,70059E-07	1,10326E-05	1,33979E-05

	7			
	Cash	Bonds	Style value	Style growth
Cash	4,0125E-07	3,41238E-07	9,03284E-07	9,26085E-07
Bonds	3,41238E-07	4,93834E-07	1,45493E-08	2,55089E-07
Style value	9,03284E-07	1,45493E-08	9,26915E-06	7,51038E-06
Style growth	9,26085E-07	2,55089E-07	7,51038E-06	6,55752E-06

	8			
	Cash	Bonds	Style value	Style growth
Cash	3,1361E-07	3,47542E-07	4,34931E-07	3,07254E-07
Bonds	3,47542E-07	5,59968E-07	5,59209E-07	3,50115E-07
Style value	4,34931E-07	5,59209E-07	4,44332E-06	3,19947E-06
Style growth	3,07254E-07	3,50115E-07	3,19947E-06	2,49794E-06

	9			
	Cash	Bonds	Style value	Style growth
Cash	2,44459E-07	3,20652E-07	-4,70527E-08	-2,16904E-07
Bonds	3,20652E-07	5,72215E-07	8,49479E-08	-1,64494E-07
Style value	-4,70527E-08	8,49479E-08	1,6685E-06	1,12275E-06
Style growth	-2,16904E-07	-1,64494E-07	1,12275E-06	1,05752E-06

	10			
	Cash	Bonds	Style value	Style growth
Cash	1,73821E-07	2,86462E-07	-3,26522E-07	-4,73661E-07
Bonds	2,86462E-07	5,234E-07	-4,44805E-07	-7,00471E-07
Style value	-3,26522E-07	-4,44805E-07	1,66197E-06	1,75526E-06
Style growth	-4,73661E-07	-7,00471E-07	1,75526E-06	2,07971E-06

	11			
	Cash	Bonds	Style value	Style growth
Cash	9,32061E-08	2,00046E-07	-3,42217E-07	-3,42466E-07
Bonds	2,00046E-07	5,06699E-07	-8,99368E-07	-8,71556E-07
Style value	-3,42217E-07	-8,99368E-07	2,07299E-06	1,92536E-06
Style growth	-3,42466E-07	-8,71556E-07	1,92536E-06	1,87165E-06

	12			
	Cash	Bonds	Style value	Style growth
Cash	1,31263E-08	1,24786E-08	-7,80689E-08	-5,39977E-08
Bonds	1,24786E-08	3,28949E-08	-8,16269E-08	-7,40415E-08
Style value	-7,80689E-08	-8,16269E-08	6,8915E-07	4,13492E-07
Style growth	-5,39977E-08	-7,40415E-08	4,13492E-07	3,1129E-07

Notes: These results are computed from rolling period returns. The excel sheet attached has these computations.