An Introduction to Interest Rate Jumps at Deterministic Times

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy in the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.

October 18, 2022

Abstract

The observation of jumps in empirical interest-rate data has prompted the inclusion of these jumps in recent term-structure models. This dissertation focusses on explaining the effects of jumps that occur at known times on the pricing of bonds. Filipovic (2009) affirms that the transition from the physical measure to the risk-neutral measure is key to the pricing of bonds and other financial instruments. Jumps in the interest rate at known times add a layer of complexity to this measure-change process. A simplified version of the term-structure model proposed by Kim and Wright (2014) is employed to analyse the effect of the jumps on the one-year point on the yield curve. Jumps at deterministic times are found to have a material effect on the one-year yield with an increasing effect as time approaches a deterministic jump date.

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Chapter 1

Introduction

The pricing of bonds and other interest-rate instruments in the face of interest-rate uncertainty is crucial to the existence of functional financial markets. Moreover, interest rates have implications on the value of assets which are not explicitly dependent on interest rates (equities for example) because of the relation of the interest rate to the assets' discount factors. As such, the development of term-structure models has been a key area of study in the realm of quantitative finance since its inception.

Early term-structure models, although useful, only reflected limited real-world market characteristics. While stochastic volatility of interest rates has been thoroughly studied in response to strong market evidence thereof, interest-rate jumps have received relatively little application in term-structure models despite the widespread acknowledgement of their empirical presence. Moreover, most term-structure models in the literature that allow for jumps assume that they occur at random times with Poisson intensities characterised by jump-diffusion processes. While some jumps do occur at random times, Andersen *et al.* (2007) argue jumps at specific known times, such as times corresponding to key public announcements, are more material to explaining market movements. This supports the notion of less-researched models that model jumps occurring at deterministic times but of random magnitudes such as the papers by Piazzesi (2010) and Kim and Wright (2014), which provide the key source material for this dissertation.

The primary goal of this dissertation is to provide a detailed description of how deterministic-time jumps are incorporated into a term-structure model. Moreover we look to investigate the resulting effects of deterministic jumps on bond yields. As such, this dissertation will first summarise key aspects and results from traditional non-jump term-structure models and then extend these to allow for jumps.

When pricing bonds with a term-structure model, a change from the physical measure to the risk-neutral measure is required. This process, however, provides an additional level of complexity with the market price of risk when jumps are introduced. This will be expounded on in this dissertation. Another key component of this dissertation will involve the use of a simplified version of the affine term-structure model presented by Kim and Wright (2014) to investigate the nature of jump-inclusive bond yields. In particular, the one-year point on the yield curve will be simulated over a two-year time horizon under both the non-jump and jump models to show both how bond yields change when jumps are introduced and, moreover, how the behaviour of the yield changes as a known jump date is approached. The dissertation will proceed as follows: Chapter 2 will provide an overview of key aspects and results from traditional non-jump term-structure models which will be referenced later (when the technical implications of these jumps are explained). Chapter 3 provides an overview of interest-rate jumps and thereafter delves into the technical aspects of a model with jumps. Chapter 4 contains the methodology and results of the yield simulations. Chapter 5 concludes.

Chapter 2

Term-Structure Models

As is common in the literature, we deem term-structure models as those which model the relationship between bond prices (and thus interest rates) and their maturities. Interest-rate movements are a substantial source of risk for all financial institutions because of the importance of interest rates in the pricing of almost all financial instruments. As such, the literature on term-structure models is extensive. In this chapter, we focus on introducing short rate models without jumps to provide the building blocks for the jump-inclusive short rate models used further into the dissertation. It is also worth noting that specifying the short rate dynamics is not the only way of obtaining a term-structure model. Another popular approach is to model forward rates, such as in the HJM framework of Heath *et al.* (1992) or the market model framework by Brace *et al.* (1997). For an extension of forward rate models to allow for jumps, see the works of Fontana *et al.* (2020) or Glasserman and Kou (2003).

2.1 Short Rate Model Framework

Short rate models are some of the earliest term-structure models developed and yet, they are still highly relevant today. Key to most term-structure models is the requirement that bond prices be arbitrage free. An intuitive motivation for this is that if an arbitrage opportunity existed, it would be immediately exploited by financial institutions and thus be driven away through market forces. For a more rigorous motivation of this logic, see Filipovic (2009, Ch.4). Short rate models are created by specifying (usually stochastic) dynamics of the short rate, denoted $r(t)$. If the dynamics of the short rate are deterministic, the time-T value of a continuous investment of one unit made at time t, $C(t,T)$, in an account that earns the short rate, is known in advance as:

$$C(t,T) = \exp\left(\int_t^T r(s)\, ds\right).$$

Thus, if there is no arbitrage, the time-t price of a bond which pays one unit at time T, is the inverse of the terminal value of the continuous investment. That is:

$$P(t,T) = \exp\left(-\int_t^T r(s)\,ds\right).$$

When stochasticity in the short rate is introduced, arbitrage-free prices are harder to determine. In general, assumptions must be made to navigate the added complexity. The first assumption when moving into the stochastic realm with short rate models (still without jumps) is that the short rate according to a continuous diffusion process with dynamics:

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW_t,$$

where μ and σ are functions of time and the short rate, and W_t is a Brownian Motion under the relevant measure. In general, μ and σ can be stochastic functions, in the sense of depending on other state processes; we, however, will focus on one-factor models, where μ and σ are deterministic functions of t and $r(t)$, ensuring that $r(t)$ is the only stochastic variable of the model. The dynamics can be assumed under any measure, but for the purpose of explaining the transition to the risk-neutral measure, we will assume dynamics under the physical measure, denoted $\mathbb{P}$, for this dissertation.

The first result utilised is the Fundamental Theorem of Asset Pricing (FTAP), taken from Filipovic (2009), which states that if an Equivalent Martingale Measure (EMM), say $\mathbb{Q}$, exists, then the model is arbitrage free. The EMM must be equivalent to the physical measure, $\mathbb{P}$, in the sense of agreeing on zero-probability events. In order to characterise the EMM $\mathbb{Q}$ we directly apply the following specification of Girsanov's Theorem:

Theorem 2.1. *Let $\mathbb{P}$ be a probability measure on $\mathcal{F}_T$, W_t be a $\mathbb{P}$-Brownian Motion and assume that $\lambda(t)$ is a predictable process. If we define a measure $\mathbb{Q}$ on $\mathcal{F}_T$ by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\int_0^T \lambda(t)dW_t - \frac{1}{2}\int_0^T ||\lambda(t)||^2 dt\right),$$

then:

- *$\mathbb{Q}$ is a probability measure on $\mathcal{F}_T$.*
- *$\tilde{W}_t = W_t - \int_0^t \lambda_s ds$ is a $\mathbb{Q}$-Brownian Motion.*

The theorem relies on Novikov's condition, a well-known integrability check on the process.

Using the above theorem, one of our key assumptions is the existence of an EMM $\mathbb{Q}$ of the form described in Girsanov's Theorem under which discounted asset prices are martingales. The market price of risk process, $\lambda(t)$, will be discussed further later. We will call $\mathbb{Q}$ the risk-neutral measure as it is the measure under which the riskless bank account is the numeraire. This allows us to obtain the time-t price of a bond with maturity T by taking the conditional expectation of the discounted payoff under the EMM, $\mathbb{Q}$. Since the payoff of a bond is $P(T,T) = 1$, by the FTAP:

$$P(t,T) = E^{\mathbb{Q}}\left(\exp\left(-\int_t^T r(s)ds\right)|\mathcal{F}_t\right), \tag{2.1}$$

where $\mathcal{F}_t$ has the usual definition of the natural filtration of information up until time t.

One of the earliest short rate models was developed by Vasicek (1977) which assumes the short rate evolves according to an Ornstein-Uhlenbeck process specifically with $\mathbb{P}$-dynamics:

$$dr(t) = \alpha(\theta_{\mathbb{P}} - r(t))dt + \sigma dW_t, \tag{2.2}$$

where $\theta_{\mathbb{P}}$ is the long-term mean to which $r(t)$ reverts, α reflects the rate of mean reversion and σ is the volatility of the process. It captures the mean-reverting property of short rates and is tractable because the resulting short-rate distribution is Gaussian. Demonstrating the measure change process without jumps, we apply Girsanov's Theorem to the dynamics in Equation (2.2) to obtain the $\mathbb{Q}$-dynamics of the short rate:

$$\begin{aligned} dr(t) &= \alpha(\theta_{\mathbb{P}} - r(t))dt + \sigma dW_t \\ &= \alpha(\theta_{\mathbb{P}} - r(t))dt + \sigma(d\tilde{W}_t + \lambda_t dt) \\ &= \alpha\left(\theta_{\mathbb{P}} + \frac{\sigma\lambda(t)}{\alpha} - r(t)\right)dt + \sigma d\tilde{W}_t. \end{aligned}$$

The above is true in general. In the case where $\lambda(t)$ is constant, we can simplify the above to:

$$dr(t) = \alpha\left(\theta_{\mathbb{Q}} - r(t)\right)dt + \sigma d\tilde{W}_t,$$

where

$$\theta_{\mathbb{Q}} = \theta_{\mathbb{P}} + \frac{\sigma\lambda}{\alpha}. \tag{2.3}$$

Regardless of how $\lambda(t)$ is specified, only the drift of the short rate process changes with a change of measure while the short rate volatility remains constant. In the case above, only the mean reversion level changes. When we change measure with the inclusion of jumps later in this dissertation, the short rate volatility does indeed remain constant, but adjustments are made to the jump size distribution (in addition to the drift) to account for the change.

Additionally, Brigo and Mercurio (2006) show, under the standard Vasicek model, that the conditional distribution at the current time t of the short rate at some future time T is Gaussian under the physical measure with:

$$\begin{aligned} E^{\mathbb{P}}\left(r(T)|\mathcal{F}_t\right) &= r(t)\exp(-\alpha(T-t)) + \theta_{\mathbb{P}}\left(1-\exp(-\alpha(T-t))\right), \\ Var^{\mathbb{P}}\left(r(T)|\mathcal{F}_t\right) &= \frac{\sigma^2}{2\alpha}\left(1-\exp(-2\alpha(T-t))\right). \end{aligned} \tag{2.4}$$

This conditional distribution under $\mathbb{P}$ will be used for our simulations under the physical measure in Chapter 4.

It is also important to define Affine Term-Structure (ATS) models (of which the Vasicek model is a special case) as this will be used when computing bond prices in Chapter 4. A model possesses an ATS if bond prices are of the functional form:

$$P(t,T) = \exp(A(t,T) + B(t,T)r(t)),$$

where $A(t,T)$ and $B(t,T)$ are deterministic functions. Crucially, bond prices under ATS models are exponential affine functions of the short rate. Additionally, Duffie and Kan (1996) prove that a model with risk-neutral short rate dynamics of the following form:

$$dr(t) = \tilde{\mu}(t, r(t))dt + \tilde{\sigma}(t, r(t))d\tilde{W}_t,$$

and that both $\tilde{\mu}(t, r(t))$ and $\tilde{\sigma}^2(t, r)$ are affine functions of the short rate as well. i.e.,

$$\tilde{\mu}(t, r(t)) = a(t)r(t) + b(t),$$

$$\tilde{\sigma}^2(t, r(t)) = c(t)r(t) + d(t),$$

then the model possesses an ATS. ATS models are generally favourable, because they provide a simple, closed-form solution to bond prices and because of this, have been well-researched. The closed form bond price functions for both the standard Vasicek (1977) model and the simplified Kim and Wright (2014) model will be presented in Chapter 3.

Chapter 3

Interest Rate Jumps

3.1 A Motivation for Deterministic-Time Interest Rate Jumps

Popular interest-rate models tend to assume that either the short or the forward rate dynamics follow a continuous diffusion process. Kim and Wright (2014) state that this carries an implicit assumption that public announcements and news are released continuously, which is problematic as large bunches of material information are released on predetermined dates at public announcements. Monetary policy announcements are a great example of this. Monetary Policy Committees (MPCs) meet at predetermined dates to evaluate the state of the economy and decide on the level of the relevant state's base interest rate amongst other macroeconomic considerations. Because the base interest rate plays a large role in determining the interbank rate and other prevailing interest rates in an economy, the release of the results of a MPC meeting generally draws much attention from the market. Indeed, Piazzesi (2001) states that monetary policy announcements often cause a large reaction in the bond market. Should the base interest rate be changed, historical data tells us that other interest rates and linked market variables follow quickly, resulting in what empirically looks like a jump. Supporting this claim, Das (2002) postulates that the kurtosis of short-term rates is inconsistent with a standard continuous diffusion model. Supporting the notion that these jumps need to be explicitly modeled, jumps corresponding to monetary policy announcement dates are widely cited as level-shifts which can materially affect the short rate path, and thus bond prices.

Andersen *et al.* (2007) view the reporting of macroeconomic statistics like unemployment or inflation as extremely important for interest-rate movements. Specifically, Kim and Wright (2014) argue that such data releases strongly influence the market's expectations about future monetary policy, the future levels of macroeconomic demand for financial assets and even future levels of inflation. Even though our proposed model only models jumps in the short rate, the short rate is theoretically an implicit function of these other variables and is thus influenced by these reports. Other variables are explicitly allowed to jump in the full model developed by Kim and Wright (2014) which allows for jumps in a state vector of which the short rate is an affine function. This method maintains the tractability which affine term structure models provide while incorporating additional key information.

While announcements are clearly key drivers in the two cases of jumps outlined above, they are not the only cause of jumps at predetermined times. Interbank interest-rate jumps occur regularly at month end (before the end of the maintenance period for deposits). Many banks fail to keep sufficient reserves as specified by regulation during the month and thus require borrowings to meet month-end requirements. Fontana *et al.* (2020) suggest this leads to upward pressure on interest rates at month-end as there is increased liquidity pressure in the interbank borrowings market. These month-end jumps are perhaps better labelled as spikes because they are temporary increases which revert to pre-jump levels when the maintenance period ends. These spikes should not have large implications on bond prices with medium to long times-to-maturity, because the integral of the rate over longer time periods should not be significantly affected by one-day spikes. Thus, this dissertation will focus rather on the jumps resulting in level shifts rather than the aforementioned spikes. For more background on these month-end interest-rate spikes, refer to the works of Fontana *et al.* (2020), who refer to these spikes as 'type II' jumps rather than the typical 'type I' level-shift jumps.

Clear empirical evidence of jumps in the interest-rate markets at predetermined dates is enough to motivate the development of jump-inclusive models. The motivation is strengthened by the existence of other empirical properties in the market that standard continuous term structure models cannot capture. Notably, Kim and Wright (2014) find that bond yield volatilities differ significantly on announcement days. Not only is the volatility higher on these days, but the volatility term structure has a different shape. Additionally, Fleming and Remolona (1999) conclude that volatility is hump-shaped in maturity on macroeconomic announcement days with peaks at intermediate maturities. This represents the impact of the announcement on future monetary policy and, as such, the risk premia of bonds with different maturities.

Considering the empirical importance of interest-rate jumps, some models have been developed which allow for jumps but they generally assume the jumps occur at random times with Poisson arrival processes. Although some jumps do occur at random times in the market, many of the interest-rate jumps occur on days of public announcements. Both Fleming and Remolona (1999) and Kim and Wright

(2014) argue that the important empirical features can be sufficiently captured by models which allow for jumps at deterministic times but of random sizes. Consequently, this dissertation draws on the Kim and Wright (2014) model that allows for jumps at deterministic times in a tractable manner. For studies of interest-rate models with random-time jumps, see, e.g., Glasserman and Kou (2003), Backwell and Hayes (2021) and Jiao *et al.* (2017).

3.2 The Deterministic-Time Jump Framework

Kim and Wright (2014) provide a generalised version of a short rate model by allowing a state vector (of which the short rate is an affine function) to jump at deterministic times rather than the short rate explicitly. We will be utilising a simplified, one-dimensional version Kim and Wright (2014) model such that it is effectively a jump-inclusive Vasicek model. The $\mathbb{P}$-dynamics are defined as:

$$dr(t) = \alpha(\theta_{\mathbb{P}} - r(t))dt + \sigma dW_t + JdN_t, \tag{3.1}$$

where α, $\theta_{\mathbb{P}}$ and σ maintain their meaning from the standard Vasicek model specified earlier. N_t is a deterministic counting process which starts at 0 and counts in increments of one at each known jump time, T_i, all of which are known at inception. J is the random variable corresponding to the jump size which we specify, under $\mathbb{P}$, such that $J \sim N(\mu_{\mathbb{P}}, \sigma^2_{\mathbb{P}})$.

Without jumps, the change of measure process can be performed with the well-known Girsanov's Theorem as in Chapter 2. When jumps are introduced, the process becomes more involved. We thus take a more manual approach to establish the risk-adjustment with jumps from first principles. It is important to note that the counting process is the same under $\mathbb{P}$ and $\mathbb{Q}$ because the jump times are the same under both measures. This follows from the fact that the probability of a jump under $\mathbb{P}$ at a deterministic time is one, and by equivalence it is too under $\mathbb{Q}$. While the jump times are the same, the jump size random variable, J, will change to act as a risk-adjustment mechanism for the jump term. To illustrate this, we start by effecting a change of measure with the Radon-Nikodym density, f, where:

$$f_t = E^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t\right).$$

Piazzesi (2010) shows that this density must also be allowed to jump at deterministic times in order for the measure change to include a jump risk adjustment. We implement this adaptation by adjusting the dynamics of the density in a similar way we did the short rate, but now include a multiplicative jumping term, γ, rather than an arithmetic one. N_t has the same specification as in the short rate dynamics. Using the jump counting process, N_t, consider the following $\mathbb{P}$-dynamics:

$$\frac{df_t}{f_{t-}} = \lambda(t)dW_t + \gamma dN_t. \tag{3.2}$$

Note that if the jump part of the above equation is removed, we recover the same f_t, labelled as $\frac{d\mathbb{Q}}{d\mathbb{P}}$ in Theorem 2.1. The jump term, γ, can have an arbitrary representation but needs to fulfill the following conditions to ensure that f_t is a strictly positive martingale:

1. $\gamma > -1$ so that the density remains positive after a jump.
2. $E_{\mathbb{P}}(\gamma) = 0$, because f is required to be a $\mathbb{P}$-martingale.

To see that γ is required to be greater than -1, consider the jump size of the density, $f_{t^-}\gamma$, and the resulting density value after the jump, $f_{t^-}(1+\gamma)$, which is required to be greater than zero. In order to ensure the martingale property holds through the jump time, we follow the logic from Piazzesi (2010) to see the necessity of requirement two:

$$\begin{aligned} E_{t^-}(f_t) &= E_{t^-}(f_{t^-}(1+\gamma_t)) \\ &= f_{t^-}\,(1 + E_{t^-}(\gamma_t)) \\ &= f_{t^-} \quad \text{if requirement 2 holds.} \end{aligned}$$

Now, let

$$J = \mu_{\mathbb{P}} + \sigma_{\mathbb{P}} Z$$

so that Z is a standard normal variable under the physical measure, i.e, $Z \sim N(0,1)$ under $\mathbb{P}$. A convenient specification of γ used by both Piazzesi (2010) and Kim and Wright (2014) which fits these requirements is:

$$\gamma = \exp\left(-\beta Z - \frac{1}{2}\beta^2\right) - 1, \tag{3.3}$$

for some constant β. This form assists in controlling the short rate's jump distribution after a measure change. We show this by finding the Moment Generating Function (MGF) of J under $\mathbb{Q}$. Applying the Abstract Bayes' Theorem, for some constant k,

$$\begin{aligned} E^{\mathbb{Q}}\left(\exp(kJ)|\mathcal{F}_{t^-}\right) &= \frac{E^{\mathbb{P}}\left(f_t \exp(kJ)|\mathcal{F}_{t^-}\right)}{f_{t^-}} \\ &= E^{\mathbb{P}}\left(\frac{f_t}{f_{t^-}}\exp(kJ)|\mathcal{F}_{t^-}\right) \\ &= E^{\mathbb{P}}\left((\gamma+1)\exp(kJ)|\mathcal{F}_{t^-}\right). \end{aligned}$$

The above follows from $\frac{f_t}{f_{t^-}} = (\gamma + 1)$. Continuing,

$$\begin{aligned} E^{\mathbb{Q}}\left(\exp(kJ)|\mathcal{F}_{t^-}\right) &= E^{\mathbb{P}}\left(\exp\left(-\beta Z - \frac{1}{2}\beta^2 + k(\mu_{\mathbb{P}} + \sigma_{\mathbb{P}} Z)\right)|\mathcal{F}_{t^-}\right) \\ &= \exp\left(k\mu_{\mathbb{P}} - \frac{1}{2}\beta^2\right) E^{\mathbb{P}}\left(\exp((k\sigma_{\mathbb{P}} - \beta)Z)|\mathcal{F}_{t^-}\right) \\ &= \exp\left(k\mu_{\mathbb{P}} - \frac{1}{2}\beta^2 + \frac{1}{2}(k\sigma_{\mathbb{P}} - \beta)^2\right) \\ &= \exp\left(k(\mu_{\mathbb{P}} - \sigma_{\mathbb{P}}\beta) + \frac{1}{2}k^2\sigma_{\mathbb{P}}^2\right). \end{aligned}$$

The final line shows that the jump random variable maintains normality (as it has the MGF of a normal random variable, which is used to write the second last line) but that its mean is shifted under a change of measure with respect to the constant, β, utilised in Equation (3.3). This constant, β, crucially acts as a market price of jump risk parameter. The variance of the jump random variable, however, stays constant. This is comforting as it acts as a jump analog for our standard change of numeraire invariance law. In addition, when changing from $\mathbb{P}$-dynamics to $\mathbb{Q}$-dynamics under the usual Vasicek model, there is a shift in the long-term mean reversion parameter. This reflects the market's aversion to movements of a particular direction in the interest rate. The shift of the jump distribution's mean acts as a similar risk adjustment process whereby the market implicitly specifies an unfavourable jump direction and prices that in accordingly. So, under $\mathbb{Q}$,

$$J \sim N(\mu_{\mathbb{Q}}, \sigma_{\mathbb{P}}^2),$$

where

$$\mu_{\mathbb{Q}} = \mu_{\mathbb{P}} - \beta\sigma_{\mathbb{P}}. \tag{3.4}$$

Apart from this jump distribution change, the standard Girsanov Theorem adjustment with the normal market price of risk process, $\lambda(t)$, applies to the drift term under the change of measure.

3.2.1 Bond Pricing

One added complexity in bond pricing noted by Piazzesi (2010) is that the inclusion of jumps at deterministic times makes a model time inhomogenous. In particular, the time to maturity does not solely define the bond price. Intuitively, it should also be affected by the number of jumps before maturity and the timing of these jumps. This is because these factors affect the riskiness of the bond and thus should have a risk-adjusted effect on the price. Even with this added complexity as well as the discontinuities implied by the jumps, Kim and Wright (2014) have found that the model conveniently maintains an affine structure which is crucial for development of a closed-form solution to bond prices with the added complexities.

Although our model is simple, it still manages to capture the time inhomogeneity and the crucial risk-adjustment in the jump distribution when changing measure. Given the short-rate dynamics in Equation (3.1) and the measure-change dynamics in Equation (3.2), Kim and Wright (2014) show that the resulting bond price (i.e., Equation (2.1)) can be expressed as:

$$P(t,T,\mathcal{T}) = \exp\left(\bar{A}(t,T,\mathcal{T}) + B(t,T)r(t)\right), \tag{3.5}$$

where $\mathcal{T}$ represents the vector of future jump dates, T_i, and where

$$\begin{aligned} \bar{A}(t,T,\mathcal{T}) &= A(t,T) + \\ &\sum_{T_i \in (t,T)} \left[\frac{-1}{\alpha}(1 - e^{-\alpha(T-T_i)})\mu_{\mathbb{Q}} + \frac{1}{2\alpha^2}\left(1 - e^{-\alpha(T-T_i)}\right)^2 \sigma_P^2 \right], \\ A(t,T) &= \frac{\sigma^2\left(4\exp(-\alpha(T-t)) - \exp\left(-2\alpha(T-t)\right) + 2\alpha(T-t) - 3\right)}{4\alpha^3} \\ &+ \theta_{\mathbb{Q}}\frac{\exp(-\alpha(T-t)) - 1 + \alpha(T-t)}{\alpha}, \\ B(t,T) &= \frac{1}{\alpha}\left(\exp(-\alpha(T-t)) - 1\right). \end{aligned} \tag{3.6}$$

Note that the sum must include all jump times between the time of pricing, t, and the bond maturity, T. Note also that this sum fully accounts for the jump aspect of the model, that is, for the standard Vasicek model:

$$P(t,T) = \exp\left(A(t,T) + B(t,T)r(t)\right). \tag{3.7}$$

Chapter 4

Simulation under the Physical Measure

Although one prices bonds under the risk-neutral measure, in order to simulate bond prices over time, we require a path of the short rate generated under the physical measure $\mathbb{P}$ to input into our bond pricing formula. Effectively, at each particular point in time, we use the risk-neutral measure, but the physical simulation point $r(t)$ as an input into the bond pricing formula. Another, perhaps more intuitive, way of explaining the relationship between the two measures in the simulation process is that we move through time with $\mathbb{P}$-dynamics, but at each point in time, we take a cross section using $\mathbb{Q}$-dynamics to price the bond. Using the model defined in Chapter 3, we simulate short rate paths under the physical measure using both the non-jump (standard Vasicek (1977)) and jump (simplified Kim and Wright (2014)) models with the parameters specified in Table 4.1 below.

Parameter	**Value**
r_0	0.05
α	0.1
$\theta_{\mathbb{P}}$	0.05
σ	0.01
$\sigma_{\mathbb{P}}$	0.01
$\mu_{\mathbb{P}}$	0
β	$-\frac{1}{4}$
λ	-0.3
dt	$\frac{1}{360}$
T_i	$\frac{i}{4}; i \in \{1, 2, ..., 12\}$

Tab. 4.1: Parameter values for short rate simulations.

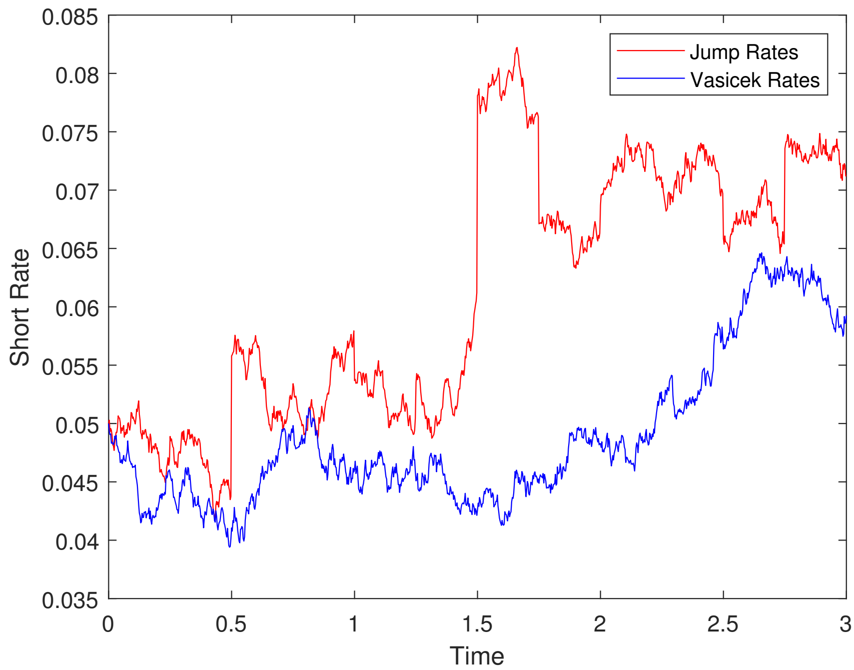


Fig. 4.1: A simulation of short rate paths with and without jumps.

We also assumed 360-day years for numerical convenience and that jumps occurred every 90 days over a three-year period. For the standard Vasicek simulations, we use the exact moments from the conditional distribution of the future short rate, stated in Equation (2.4), for the recursive formula:

$$r(t+dt) = \exp(-\alpha dt)r(t) + \theta(1-\exp(-\alpha dt)) + Z\sqrt{\frac{\sigma^2}{2\alpha}(1-\exp(-2\alpha dt))},$$

where Z is generated from the standard normal distribution. For the jump-inclusive simulation, we use the recursive formulae below:

$$r(t+dt) = \exp(-\alpha dt)r(t) + \theta(1-\exp(-\alpha dt)) + X\sqrt{\frac{\sigma^2}{2\alpha}(1-\exp(-2\alpha dt))} + \sigma_{\mathbb{P}}Y, \tag{4.1}$$

$$r(t+dt) = \exp(-\alpha dt)r(t) + \theta(1-\exp(-\alpha dt)) + Z\sqrt{\frac{\sigma^2}{2\alpha}(1-\exp(-2\alpha dt))}, \tag{4.2}$$

where X, Y and Z are independently generated random numbers from the standard normal distribution. At a jump time, Equation (4.1) is used. Otherwise, Equation (4.2) is used. These formulae generate the paths in Figure 4.1 above.

The jump model is clearly capturing the jumping nature of interest rates at deterministic time points. For example, the short rate jumps upwards by 2.2% about

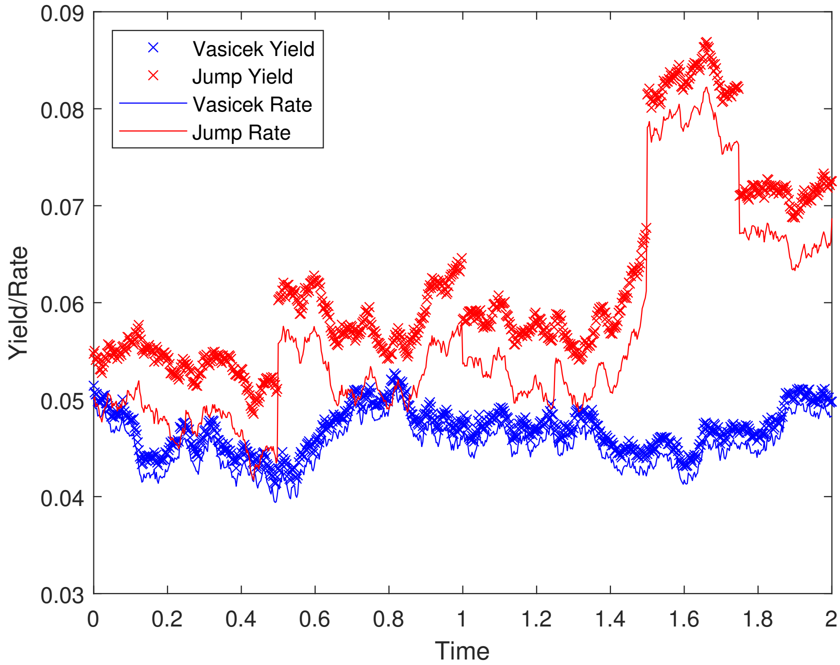


Fig. 4.2: Simulated yields vs simulated rates.

halfway through the three-year period. Short rate path simulations are done over one extra year relative to the bond yields, because jump times are only simulated over a three-year period and we need jump times between time t and time $t+1$ for jumps to have an effect on the price of a bond with a one-year time-to-maturity at time t. In other words, with jump times from time 0 to time 3, the latest time we can price a one-year bond, with the effect of jumps, is at time 2. Using our short rate simulations, we calculate the one-year point on the yield curve over a two-year period for both models. This is done, as mentioned above, by calculating the bond price, via Equation (3.5) and Equation (3.7), at each successive time point with the current short rate level as an input into the bond pricing formula which yields the plot in Figure 4.2 above.

The yields are higher than the short rate at each time point. Since the current short rate level is the short end of the yield curve, a one-year yield above the short rate implies the yield curve is upward sloping until the one-year point. We can also see that, for these chosen paths, the difference between jump model yields and rates is consistently larger than the difference between Vasicek yields and rates. This implies that the jump-inclusive yield curve is steeper than the Vasicek yield curve. It also shows that the risk adjustment with jumps is contributing to an increased yield. To better investigate this, we simulate 10,000 short rate paths and

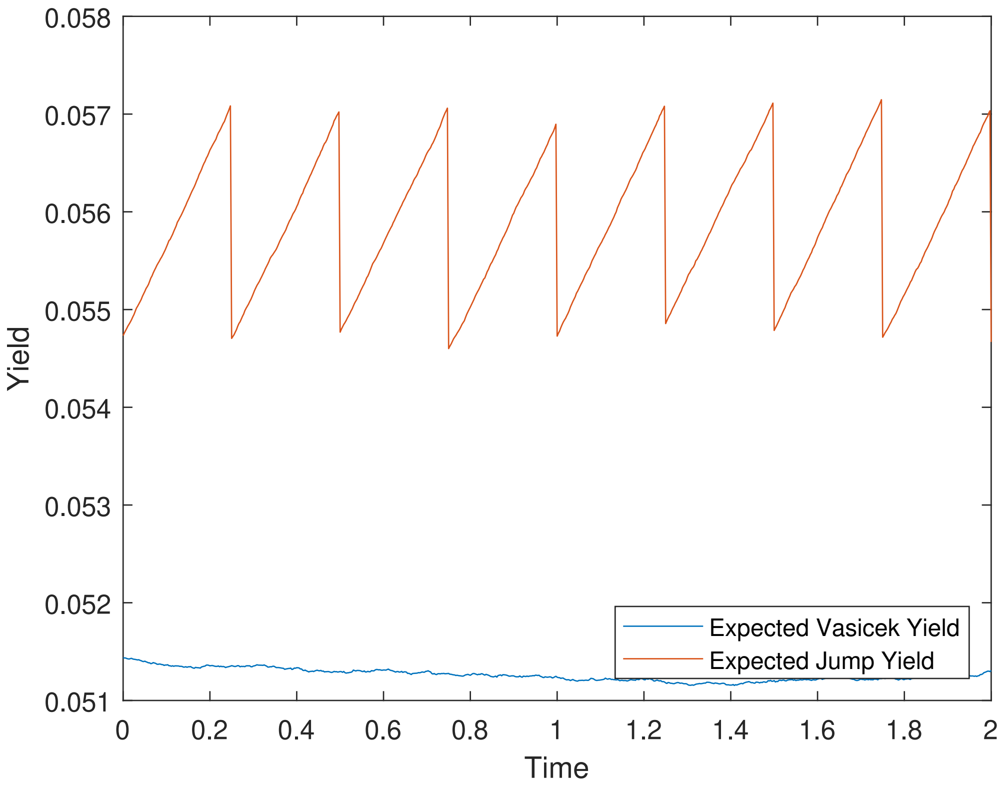


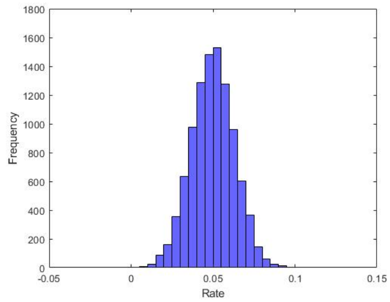
Fig. 4.3: Differences between the yields.

the corresponding one-year yields under both models (the information presented in Figure 4.2 above). This allows us to obtain estimates for expected one-year yields at each point in time over the two-year period by taking the mean of the yields at each cross-section. Specifically, to estimate the expected one-year yield at time t, we calculate the mean of all 10,000 time-t yields. This gives us insight into how the jump-inclusive yields behave relative to non-jump yields, particularly when jump times are approached. The numerical estimates for the expected yields under both models are plotted in Figure 4.3 above.

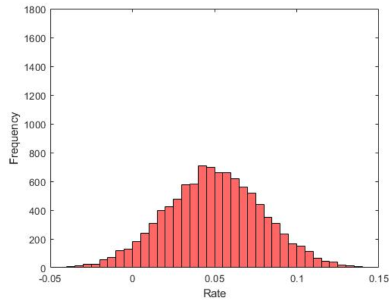
Clearly, the expected yield implied by the jump-inclusive model is higher than that of the Vasicek model at each time point. This is because of the parameters we assigned in Table 4.1 - specifically, our values for $\mu_{\mathbb{P}}$ and β. We assumed under the physical measure that the mean of the jumps would be zero - that in reality we could not tell which direction the short rate was more likley to jump. Under the risk-neutral measure, the mean gets adjusted as per Equation (3.4) by β which we are assuming to be negative. This means we are assuming the market finds upward jumps in the short rate unfavourable and thus require to be compensated for accepting this risk. If we look into the bond pricing formula in Equation (3.6), this logic is mirrored as we see that the positive $\mu_{\mathbb{Q}}$ decreases bond prices and thus increases yields. In other words, the model can reflect the opposite kind of jump-risk aversion.

The potentially more interesting part of the plot is the sawtooth behaviour of the jump yield through time. It is clear that as we approach a jump date, the yield increases until the jump occurs at which point, the yield drops back down. Intuitively, the closer one gets to a jump date, the more the integral of the short rate is affected by a jump. To better think about this behaviour, consider a three-month bond with three months until a jump date. Since there will be no jumps in the short rate through the life of the bond (and thus no jump risk), its yield will be the same as that under the Vasicek model. On the contrary, if we hold a one-year bond, which is going to jump tomorrow, the jump will significantly influence the average level of the short rate throughout the life of the bond. In this way, we can truly see the importance of the time-inhomogeneity as a result of these jumps at known times. Clearly the time until the next jump has a strong influence on bond yields under this model.

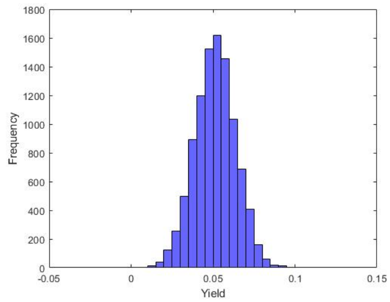
Lastly, from the 10,000 simulations of short rate paths and one-year yield points over a two-year period, we inspect the resulting empirical distributions of the short rates and one-year yields at time 2 under both no-jump and jump-inclusive models which are plotted in Figure 4.4 below. Looking at the similarity of the respective short rate and one-year yield histograms (under both models), we see that the observation made from Figure 4.2 regarding the one-year yield point being greater than the short rate is borne out by a large sample. The one-year yield point was greater than the short rate, because of the risk premium which can be attributed to the regular market price of risk parameter, λ, and the parameter controlling the market price of jump risk, β. Additionally, we observe that the distributions of the short rate and yield under the jump-inclusive model have near-identical empirical means, but more variation than those under the no-jump model. Given that the inclusion of jumps to a model intuitively adds another source of noise, the increased variation under the jump-inclusive model is not surprising. The similarity of the means between the two models can be attributed to the mean parameter of the jump-size distribution, $\mu_{\mathbb{P}}$, was set to zero.



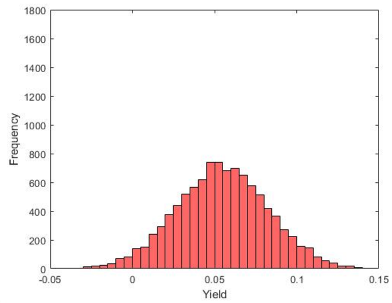
(a) No-jump short rate distribution.



(b) Jump-inclusive short rate distribution.



(c) No-jump one-year yield distribution.



(d) Jump-inclusive one-year yield distribution.

Fig. 4.4: Comparing time-2 rate and yield distributions.

One key property that we ignored from the Kim and Wright (2014) model is the state dependence of jumps which can be an intuitively appealing aspect of the model. For example, if interest rates are very high, one might expect the Monetary Policy Committee to decrease rates at some future date at a higher probability than increasing them. Including this functionality would require the mean of the jump size distribution to be given by some function of the current short rate level. This would be a natural extension of this introductory piece on deterministic-time jumps.

Chapter 5

Conclusion

The motivation for including interest rate jumps in term-structure models is clear. Well-researched scholars have presented strong empirical evidence supporting the notion that jumps in interest rates do occur. While interest rate jumps can be modelled at random times with Poisson arrival processes, this dissertation focusses on incorporating jumps at deterministic times coinciding with public announcements and data releases. In particular, the process of adding jumps at known times to an existing affine term-structure model is illustrated. This is achieved by introducing a traditional short rate model setup and thereafter, drawing on the work by Kim and Wright (2014) and Piazzesi (2010) to show the addition of the jump term in the short rate dynamics, and of a jump-risk term to the Radon-Nikodym process that controls the change from the physical to risk-neutral measure. Technically important, is that under the influence of jumps, we cannot merely apply Girsanov's Theorem when changing measure in the risk-adjustment process. This dissertation takes a manual approach of the measure change process to show that the mean of the jump random variable in our dynamics is shifted in the risk-adjustment process according to a parameter that controls the market price of jump risk.

Once the risk-adjustment process is correctly executed, a closed form solution for bond prices from Kim and Wright (2014) is obtained as the model maintains its affine structure. It does not however maintain its time homogeneity. To investigate the nature of the inhomogeneity, short rate paths were simulated under the physical measure using both the initial Vasicek model and the jump-inclusive model. Using the closed form solution of Kim and Wright (2014), one-year bond yield paths were obtained and compared. Time inhomogeneity results from the fact that the time until the next relevant deterministic jumps changes through time. In particular, as we approached a known jump date, bond yields increased because the jump would have a greater impact on the average level of the short rate over the life of the bond.

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