TOPICS IN THE ALGEBRAIC THEORY OF HIGHER DEGREE FORMS

by

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ABSTRACT

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Let $d \geq 2$ be an integer and let F be a field. A form of degree d over F is a polynomial of homogeneous degree d with coefficients in F. In degree d=2 there is an extensive theory of quadratic forms. We consider forms of degree d>2.

The following are among the new results we have proved:

1. A **nonsingular** form over a field of characteristic zero has **nonzero Hessian**. This was proved by Harrison in degree d=3. We use some basic algebraic geometry and rational differential forms to give a proof valid in all degrees $d \geq 2$.

2. The formal differences of **split** forms constitute an ideal in the **Grothendieck ring** of higher degree forms. This generalises a well–known result for quadratic forms to higher degree.

3. In the monoid of equivalence classes of nondegenerate forms of degree $d \geq 3$, with the tensor product operation, the submonoid generated by the equivalence classes of the **hyperbolic** forms is free. This is a small step towards answering a question posed by Harrison.

4. Let the base field have characteristic zero. In every odd degree $d \geq 3$ there are no nontrivial families of **additive invariants** of the forms of degree d. In every even degree d there is a nontrivial family of additive invariants of the forms of degree d. The most familiar example is the family of discriminants of quadratic forms. Our proof involves the **symbolic method** for representing invariants of the forms of degree d.

5. We give a new proof, in characteristic zero, that a **nonsingular** form of degree $d \geq 3$ has a zero **Lie algebra**. Our proof involves a certain **Schur functor** and invokes the basis theorem of Akin, Buchsbaum and Weyman.

6. Let F be a field of characteristic zero and let $K|F$ be a field extension. Let f be a form of degree $d \geq 3$ in more than three indeterminates with coefficients in F. Then if f is equivalent over K to a **hyperbolic** form, f must already be equivalent to a **hyperbolic** form over F. Compare this with the degree 2 case where, for example, the forms $\Sigma(x_k^2+y_k^2)$ are hyperbolic over $\mathbb{C}$ but not over $\mathbb{R}$.

INTRODUCTION

The theory of **quadratic** forms has a long history. For centuries quadratic forms have been of interest in number theory, and they are important in geometry. The work of Witt, Pfister and Knebusch has lead to a rich algebraic theory of quadratic forms..

By comparison the algebraic theory of **higher degree** forms is scanty. Harrison **[HAR1]** showed that there is a Grothendieck ring of higher degree forms. As an abelian group it is freely generated by the isomorphism classes of nondegenerate, indecomposable forms. At this level the algebra is simple. But there are indecomposable forms in every dimension and the problem of classifying them up to isomorphism is daunting.

Let d be an integer >2. Let F be a field. We assume throughout that $d! \neq 0$ in F. Let V be a vector space of finite dimension over F. Let $\theta: V^d \to F$ be d–multilinear and symmetric. We discuss several topics in the algebraic theory of such higher degree forms.

Chapter One is introductory. No new results are proved.

We make use of the symmetric algebra functor S(_) from the category of free modules M of finite rank over a fixed commutative ring R, to the category of graded Hopf–algebras over R. S(M) is the quotient of the tensor algebra of M over R, by the 2–sided ideal generated by the set $\{m \otimes n - n \otimes m : m,n \in M\}$.

In particular, the graded Hopf–algebra structure on $S(V^*)$ enables us to establish isomorphisms between the following F–vector spaces:

$\bigodot^d(V)$ the space of symmetric d–multilinear forms $\theta: V^d \to F$;

$S_d(V^*)$ the grade d component of $S(V^*)$;

$F_d[x_1,\ldots,x_n]$ the F–vector space of forms of degree d, having chosen a basis $x_1,\ldots,x_n$ of V^*.

We introduce the notion of symmetric space of degree d due to Harrison **[HAR1]**. This is a pair (V,θ) where $\theta \epsilon \bigodot^d(V)$, generalising Witt's quadratic spaces. Following Harrison we explain how isomorphism classes of symmetric spaces correspond to $GL_n(V)$–equivalence classes in $S_d(V^*)$ and to $GL_n(F)$–equivalence classes in $F_d[x_1,\ldots,x_n]$. Further we give Harrison's definitions of the direct sum of symmetric spaces and the corresponding orthogonal sum of forms of degree d, and his proof that these operations are compatible with the $GL(V)$, resp. $GL_n(F)$–action.

Finally we include a proof of the proposition that if R is an integral domain with more than d elements; and $p \epsilon R[t_1,\ldots,t_n]$ has degree $\leq d$ in each t_i; and $p(r_1,\ldots,r_n)=0$ for all $r_1,\ldots,r_n \epsilon R$; then $p=0$. This theorem is used repeatedly in the sequel.

Chapter Two is divided into three parts.

(1) In the first part we discuss the properties: nondegeneracy; nonzero Hessian; and nonsingularity of a symmetric space.

These definitions and most of the results are due to Harrison.

Using basic algebraic geometry we give proofs that as is well–known, changing the base field to the algebraic closure k of F, the subsets of: nondegenerate forms; forms with nonzero Hessian; and nonsingular forms are all Zariski open in $\bigodot^d(V \otimes_F k)$.

We give a proof based on the formal Chain Rule for polynomial maps, of the well–known result that, in 19–th century terminology, the Hessian is a **covariant** of weight 2.

We introduce the hyperbolic forms, suggested by my supervisor Dr K.R.Hughes, which are higher degree analogues of the hyperbolic symmetric bilinear forms. In dimension ≥ 3 and degree ≥ 3 the hyperbolic forms are nondegenerate, but have zero Hessian. Harrison **[HAR2 p521]** states that nondegenerate forms with zero Hessian are hard to find! Harrison **[HAR2, Prop1.1]** proved that, when the ground field has characteristic zero, any nonsingular symmetric space of degree 3 has nonzero Hessian.

We extend this result to all degrees $d \geq 3$, using basic algebraic geometry and rational differential forms.

(2) In the second section we introduce Harrison's definitions of indecomposability, and the centre of a symmetric space of degree ≥ 3. We give some examples of symmetric spaces, and we compute their centres. Then it is clear that in a given degree $d \geq 3$ there are nondegenerate indecomposable symmetric spaces in each dimension. We also compute the centres of the hyperbolic forms.

(3) We define the notion of split symmetric space directly extending the field case of the definition in degree 2, due to Knebusch **[MH p12]**, to higher degree. We show that in dimension ≥ 3 all the hyperbolic forms are split, and we give an example showing that in each degree ≥ 3 there are split forms which are not hyperbolic. We state Harrison's definition of the tensor product of symmetric spaces, and we include a proof that the tensor product of nondegenerate symmetric spaces is nondegenerate, a special case of Harrison's more general result **[HAR1 p128]**. We prove that the direct sum of nondegenerate split symmetric spaces is split; and the tensor product of a nondegenerate split symmetric space and a nondegenerate symmetric space is split. Then it is clear that the set of formal differences of split symmetric spaces is an ideal in the Grothendieck ring. Harrison **[HAR1 p136]** raised the question of the structure of the monoid of isomorphism classes of absolutely indecomposable nondegenerate symmetric spaces under tensor product. We use our calculation of the centres of the hyperbolic forms to show that the submonoid generated by their isomorphism classes is free.

Chapter Three is concerned with **families of additive invariants.**

In the theory of quadratic forms the invariants

d_n = discriminant of quadratic forms in n indeterminants play a major role.

(d_n) is a family of additive invariants:

Let f(resp. g) be a quadratic form in p(resp. q) indeterminants $x_1,\dots,x_p$ (resp. $y_1,\dots,y_q$) Let $f\perp g(x_1,\dots,x_p,y_1,\dots,y_q) = f(x_1,\dots,x_p)+g(y_1,\dots,y_q)$.
Then $d_{p+q}(f\perp g) = d_p(f).d_q(g)$.

We give an example of a family of nonconstant additive invariants in each even degree.

We prove that there are no nontrivial families of additive invariants in odd degree, relying on the lucid treatment of the symbolic method by Grosshans et al **[GRS]**, of which we require only the case of a single symmetric tensor of step d (or a form of degree d).

Chapter Four is concerned with the Lie theory of a d–multilinear form $\theta: V^d \to F$, and is divided into three parts.

(1) In the first part we define the algebraic isometry group of a **symmetric** d–multilinear form θ, and its Lie algebra. We prove that equivalent forms have conjugate algebraic isometry groups and Lie algebras. We show that the Lie algebra of a direct sum of forms is the direct sum of the Lie algebras of the forms. We reduce the problem of calculating the isometry group of a form to the indecomposable case. We show that the connected component of the identity in the isometry group of a direct sum of forms, is the direct product of the connected components of the identity in their various isometry groups.

(2) In the second part we calculate the algebraic isometry groups of the hyperbolic forms, and their Lie algebras. Assuming characteristic zero, we compute their radicals and show that they are 2–step solvable.

In the course of our calculations we encounter the space of (d+1)–multilinear forms $\varphi: V^{d+1} \to F$ which satisfy:

(a) $\varphi(v_1,\dots,v_d,v_{d+1})$ is symmetric in $v_1,\dots,v_d$

(b) $\sum_{i=1}^{d+1} \varphi(v_1,\dots,\hat{v}_i,\dots,v_{d+1},v_i) = 0$, the $\hat{}$ denoting omission.

Given a symmetric (d+1)–multilinear form θ: $V^{d+1} \to F$ and M in the Lie algebra of θ, the form ψ:$V^{d+1} \to F$ defined by

$\psi(v_1,\ldots,v_{d+1}) = \theta(v_1,\ldots,v_d,Mv_{d+1})$ satisfies (a),(b) above.

(3) In the third part we show that the space of forms satisfying 2(a),(b) above is the space $L_\lambda(V^*)$ where λ is the Young diagram

with d rows and L_λ is the Schur functor of Akin et al **[ABW II.1.3]**.

In the proof we consider actions of the group of permutations of 1,...,d on the letter–place algebra with letters $a_1,\ldots,a_{d+1}$ and places 1,...,n; and we appeal to the Basis Theorem for $L_\lambda(V^*)$, due to Akin et al **[ABW II.2.16]**.

In the rest of the chapter we assume characteristic zero.

We deduce from a proposition of Schneider **[SCHN Prop 2]** the following:

Let M be a nilpotent endomorphism in the Lie algebra of θ such that $M^r \neq 0$, $M^{r+1}=0$ with $r \geq 1$ so that we have a flag $V=M^0V \supseteq MV \supseteq \ldots \supseteq M^rV \supseteq \{0\}$.

(*) Then if $v_i \in M^{r_i}V$ for i=1,...,d and $\Sigma r_i \geq dr$, we have $\theta(v_1,\ldots,v_{d+1}) = 0$.

We prove this independently of Schneider's proposition. The starting point of the proof is the observation that the form ψ of (2) above is in the space $L_\lambda(V^*)$ above. Then we use the dual basis of a basis of V w.r.t. which the matrix of M is in Jordan canonical form, and obtain from it a standard basis of $L_\lambda(V^*)$. We express ψ as a linear combination of these basis forms. From the identity

$$\psi(v_1,\ldots,v_{d-1},Mv_d,v_{d+1}) = \psi(v_1,\ldots,v_{d-1},Mv_{d+1},v_d)$$

we obtain relations between the coefficients and deduce that certain of them are zero.

We also give a Lie algebraic proof of Schneider's result **[SCHN Theorem]** that when θ above is nonsingular, its algebraic isometry group is finite.

We also include a short proof of the proposition (*) above in the case of r **odd**.

Chapter Five is a miscellany.

(1) Our initial formulation of the main result of the first part was a higher degree analogue of the Weak Hasse–Minkowski Theorem:

Let (V,θ) be a symmetric space over an algebraic number field F. Let p be a prime of F, finite or infinite. Let F_p be the completion of F at the prime p. Let (V_p,θ_p) be the symmetric space obtained from (V,θ) by the base change $F \to F_p$. Suppose that for all primes p of F, (V_p,θ_p) is isomorphic to a hyperbolic symmetric space. Then (V,θ) itself is isomorphic to a hyperbolic symmetric space.

But we later realised that when the base field F has characteristic zero and $K|F$ is any field extension: given a (d+1)–multilinear form $\phi\colon W^{d+1} \to F$ with $d \geq 2$, if the extended form $\phi_K\colon W \otimes_F K \to K$ is equivalent to a hyperbolic form, then ϕ must itself be equivalent to a hyperbolic form. Compare with the quadratic case: $\Sigma(x_k^2+y_k^2)$ is hyperbolic over $\mathbb{C}$ as

$x^2+y^2 = (x+iy)(x-iy)$, but is not hyperbolic over $\mathbb{R}$.

The proof makes use of our calculation of the Lie algebras of the hyperbolic forms and their radicals.

(2) The second part concerns the lift of a form on the residue field of a complete discrete valuation ring (DVR) to a form on the ring. Let R be a complete DVR with maximal ideal M and residue field k. In degree d=2 this lift is unique up to equivalence. We show that in each degree $d \geq 3$ there exist two nonsingular symmetric d–multilinear forms $(R^3)^d \to R$ which reduce mod M to the same form $k^3 \to k$, but have nonisomorphic centres and hence cannot be equivalent, even after a base change $R \to L$ where L is any field containing R.

(3) In the last part we prove a result we obtained while trying to emulate Kneser's [KNE] beautiful treatment of composition of binary quadratic forms in the ternary cubic case. We had hoped that we could improve on Waterhouse's approach [WAT2].

We proved that: If F is a field of characteristic zero, with algebraic closure K, then any nonsingular ternary cubic form in u_1,u_2,u_3 is of the form $f = \det(\Sigma u_i \alpha_i)$ with $\alpha_1,\alpha_2,\alpha_3 \epsilon M_3(K)$. The starting point of the proof is Van den Bergh's result **[VDB Thm 3.2]** that the Clifford algebra C_f of a nonsingular ternary cubic form f has a rank 3 linearization: i.e. there is an F–algebra homomorphism φ: $C_f \to M_3(K)$.

CHAPTER ONE

Throughout this chapter:

$d > 1$ is a fixed positive integer.

F is the ground field and $d! \neq 0$ in F.

V is a vector space over F of finite dimension n.

A is a commutative F–algebra.

$V_A = V \otimes_F A$

$V_A^* = \mathrm{Hom}_A(V_A, A)$

$S(V_A^*)$ is the symmetric algebra of the free A–module V_A^* of rank n, and is a graded Hopf–algebra. See **[ABW I.3]**.

Given a positive integer k

$\odot^k(V)$ is the F–vector space of symmetric k–multilinear forms $\theta\colon V^k \to F$; and

G_k is the group of permutations of $1, \dots, k$.

We will see that there are natural group actions:

$GL(V)$ acts on the left on $S(V_A^*)$ as graded Hopf–algebra automorphisms; and for each positive integer k, G_k acts on the left on $S_k(V_A^*)$ commuting with the $GL(V)$ action.

Apart from introducing the required terminology:

(a) We fill in some details, following **[ABW I.3]**, not given in **[HAR1 §1]**: we explain the polarisation isomorphism between

the F–vector space $F_d[x_1, \dots, x_n]$ of **forms of degree d** over F;

and the F–vector space of symmetric d–multilinear forms $\theta\colon V^d \to F$.

(b) We include a proof of the well–known theorem that:

If R is an integral domain with more than d elements ; $p(t_1,\dots,t_n)$ is a polynomial of degree $\leq d$ in each t_i ; with coeficients in R ; and

$p(r_1,\dots,r_n)=0$ for all $r_1,\dots,r_n$ in R ;

then $p=0$.

We will then have achieved the main results of this chapter i.e. **given that $d!\neq 0$ in F** we can:

(1) identify $p\in S_d(V^*)$ with its polarisation $\theta_p \in \odot^d(V)$.

(2) identify $p\in \bigoplus_{k=0}^{d} S_k(V^*)$ with the induced polynomial map $f(p)\colon V \to F$.

We summarize the contents of this chapter.

In §1 we define a G_d–action on the space of d–multilinear maps $f\colon V^d \to F$, and we let G_d act on $(V^*)^{\otimes d}$ so that the canonical isomorphism between these spaces is a G_d–isomorphism.

We can then define the isomorphic spaces of symmetric d–multilinear maps $\theta\colon V^d \to F$, and symmetric tensors in $(V^*)^{\otimes d}$.

In §2 we introduce the symmetric algebra functor $S(\_)$ from finite dimensional vector spaces over F to graded Hopf–algebras over F.

In §§3–6 we use the comultiplication in the Hopf–algebra $S(V^*)$ to describe the polarisation isomorphism of $S_d(V^*)$ with the space of symmetric tensors in $(V^*)^{\otimes d}$, or what comes to the same thing, the space $\odot^d(V)$ of symmetric d–multilinear maps $\theta\colon V^d\colon \to F$.

In §§9 and 10 we describe a method of polarising which will be useful in some of our later computations.

In §§11–13 we describe some $GL(V)$–actions and we see that polarisation is a $GL(V)$–map.

In §14 we define the notion of symmetric space of degree d. The isomorphism classes of symmetric spaces of degree d were used by Harrison [HAR1] to construct the Grothendieck ring of degree d forms. In §§ 17–19 we define the addition of two such isomorphism classes. A symmetric space of degree d on the vector space V is, when $d!\neq 0$ in F, the same thing as an element of $S_d(V^*)$. Isomorphic symmetric spaces of degree d over the same vector space V, will have the corresponding elements of $S_d(V^*)$ GL(V)–equivalent.

In §20 we prove the theorem stated in (b) above. An immediate consequence of this result is that if the ground field F is infinite and V has dimension n, then $S(V^*)$ may be identified with the affine coordinate ring of the affine algebraic group A^n_F. This motivates the Hopf–algebra structure on $S(V^*)$ in the general case.

1. Let V be an n–dimensional vector space over the base field F. We will define what is meant by: a symmetric tensor in $(V^*)^{\otimes d}$; and a symmetric d–multilinear map $f:V^d \to F$. We will see that the F–vector space of these symmetric tensors is canonically isomorphic to the F–vector space of these symmetric d–multilinear maps.

We will identify the following F–vector spaces by the canonical isomorphisms:

the space of all d–multilinear maps $f:V^d \to F$

$\cong (V^{\otimes d})^* \cong (V^*)^{\otimes d}$

To define symmetry we introduce some left G_d–actions.

First we explain the left action of G_d on the space of d–multilinear maps $f: V^d \to F$ given in (*) below.

$GL(V^d)$ acts on the left on the space of d–multilinear maps $f:V^d \to F$ by

$\sigma.f = f\circ\sigma^{-1}$ for σ in $GL(V^d)$.

We have a representation $r: G_d \to GL(V^d)$ with

$$r(\pi)(v_1,\dots,v_d) = (v_{\pi^{-1}(1)},\dots,v_{\pi^{-1}(d)}).$$

Check:

$$\begin{aligned} r(\rho\pi)(\dots,v_i,\dots) &= (\dots,v_{\pi^{-1}(\rho^{-1}(i))},\dots) \\ &= r(\rho)(\dots,v_{\pi^{-1}(i)},\dots) \\ &= r(\rho)\circ r(\pi)(\dots,v_i,\dots). \end{aligned}$$

Therefore G_d acts linearly on the left on the F–vector space of all d–multilinear maps $f{:}V^d \to F$, by means of r and the above left action of $GL(V^d)$:

(*) $\pi.f = f\circ(\,r(\pi)\,)^{-1}$ i.e $\pi.f(\dots,v_i,\dots) = f(\dots,v_{\pi(i)},\dots)$.

We define a left action of G_d on $(V^*)^{\otimes d}$ by requiring the canonical isomorphism to be a G_d–isomorphism.

We now describe this action in terms of coordinates. Let $v_1,\dots,v_n$ be a basis of V over F, and let $x_1,\dots,x_n$ be the dual basis of V^*.

For each map α: $\{1,\dots,d\} \to \{1,\dots,n\}$ we have $x_{\alpha1}\otimes\dots\otimes x_{\alpha d} \in (V^*)^{\otimes d}$ and these form a basis of $(V^*)^{\otimes d}$ over F.

Let $t = \Sigma(t\,|\,\alpha1\dots\alpha d)x_{\alpha1}\otimes\dots\otimes x_{\alpha d} \in (V^*)^{\otimes d}$.

Under the canonical isomorphism t corresponds to the d–multilinear map

$f{:}\ V^d \to F$ with $f(\,v_{\alpha1},\dots,v_{\alpha d}\,) = (t\,|\,\alpha1\dots\alpha d)$.

Let π be in G_d. $\pi.f$ corresponds to $\pi.t$, and

$\pi.f(\,v_{\alpha1},\dots,v_{\alpha d}\,) = f(\,v_{\alpha(\pi1)},\dots,v_{\alpha(\pi d)}\,)$

so $\pi.t = \Sigma(\,t\,|\,\alpha(\pi1)\dots\alpha(\pi d)\,)x_{\alpha1}\otimes\dots\otimes x_{\alpha d}$.

A d–multilinear map $f{:}\ V^d \to F$ is called **symmetric** if $\pi.f = f$ for all π in G_d.

A tensor t in $(V^*)^{\otimes d}$ is called **symmetric** if $\pi.t = t$ for all π in G_d. Since the canonical isomorphism is a G_d–isomorphism, the space $\bigodot^d(V)$ of symmetric d–multilinear forms

$\theta{:}\ V^d \to F$ is isomorphic to the space of symmetric tensors t in $(V^*)^{\otimes d}$.

In §§2–8 we show that, when $d!\neq 0$ in F, there is a canonical isomorphism between the F–vector space $S_d(V^*)$ –which after a choice of basis, say $x_1,\dots,x_n$ of V^*, is isomorphic to the space $F_d[x_1,\dots,x_n]$ of **forms of degree d** in $x_1,\dots,x_n$ over F –and the space of symmetric tensors in $(V^*)^{\otimes d}$. By §1 this gives an isomorphism $S_d(V^*) \cong \bigodot^d(V)$ which is called the **polarisation** process. Any $p \in S_d(V^*)$ defines a polynomial map $f(p)\colon V \to F$. When θ in $\bigodot^d(V)$ is the polarisation of p we will see that $f(p)(v) = \theta(v,\dots,v)$.

2. Let R be a ring. We denote by $S(\_)$ the symmetric algebra functor from the category of finitely generated free R–modules to the category of commutative graded R–Hopf algebras. Akin et al [**ABW §I.1,§I.3**] describe this latter category and the functor S.

Each $p \in S(V^*)$ defines a polynomial map $f(p)\colon V \to F$, as follows.
Let $v \in V$. We define the map $e_v\colon S(V^*) \to F$ by requiring

(i) $e_v(p) = p(v)$ when $p \in V^*$

(ii) e_v is an F–algebra homomorphism.

Given $p \in S(V^*)$, we define the map $f(p)\colon V \to F$ by $f(p)(v)=e_v(p)$.

3. Following Akin et al [**ABW §I.3**] we show that there is an isomorphism $p \mapsto \theta_p$ from the F–vector space $S_d(V^*)$ to the F–vector space of symmetric tensors in $(V^*)^{\otimes d}$, and that $\theta_p(v,\dots,v)=f(p)(v)$.

We first give a coordinate–free description of this isomorphism $p \mapsto \theta_p$ and its inverse

$\theta \mapsto p_\theta$.

Given a symmetric tensor $\theta \in (V^*)^{\otimes d} = (S_1(V^*))^{\otimes d}$, to get p_θ we apply the **component**

$(V^*)^{\otimes d} \to S_d(V^*)$ of **d–fold multiplication:**

$$\begin{array}{l} S(V^*)^{\otimes d} \\ \downarrow m \otimes 1^{\otimes(d-2)} \\ S(V^*)^{\otimes(d-1)} \\ \downarrow m \otimes 1^{\otimes(d-3)} \\ \vdots \\ \downarrow m \\ S(V^*) \end{array}$$

Given $p \epsilon S_d(V^*)$, to obtain the symmetric tensor θ_p we multiply p by $1/d!$ and apply the **component** $S_d(V^*) \to (V^*)^{\otimes d}$ of **d–fold comultiplication:**

$$\begin{array}{l} S(V^*) \\ \downarrow \Delta \\ S(V^*)^{\otimes 2} \\ \downarrow \Delta \otimes 1 \\ \vdots \\ \downarrow \Delta \otimes 1^{\otimes(d-2)} \\ S(V^*)^{\otimes d} \end{array}$$

This latter map is called the **polarisation** process.

We now describe the above maps in terms of coordinates.

4. Let $v_1,\dots,v_n$ be a basis of V, and let $x_1,\dots,x_n$ be the dual basis of V^*.

In this chapter α will denote a map $\alpha\colon \{1,\dots,d\} \to \{1,\dots,n\}$, and i will denote an n–tuple $(i_1,\dots,i_n)$ of nonnegative integers with sum d. We define the **content** $c(\alpha)$ of α to be $(i_1,\dots,i_n)$, where for $k=1,\dots,n$ i_k is the number of instances of k in the list $\alpha 1,\dots,\alpha d$. Clearly if $c(\alpha)=(i_1,\dots,i_n)$ then $i_1+\dots+i_n=d$.

5. First suppose that $\theta = \sum_\alpha \theta(v_{\alpha 1},\dots,v_{\alpha d})x_{\alpha 1}\otimes\dots\otimes x_{\alpha d}$ is a symmetric tensor.

Then $\theta(v_{\alpha 1},\dots,v_{\alpha d}) \in F$ and $x_{\alpha 1}\dots x_{\alpha d} \in S_d(V^*)$ depend only on the content of α. The number of α with a given content $(i_1,\dots,i_n)$ is

$$\begin{bmatrix} d \\ i_1 \end{bmatrix}\begin{bmatrix} d-i_1 \\ i_2 \end{bmatrix}\cdots\begin{bmatrix} d-i_1-\dots-i_{n-1} \\ i_n \end{bmatrix} = \frac{d!}{i_1!\dots i_n!}.$$

So the image of θ under d–fold multiplication is

$$p_\theta = \sum_i \frac{d!}{i_1!\dots i_n!}(p_\theta|i_1\dots i_n)x_1^{i_1}\dots x_n^{i_n}$$

where $(p_\theta|i_1\dots i_n)=\theta(v_{\alpha 1},\dots,v_{\alpha d})$ for any α of content $(i_1,\dots,i_n)$.

6. Secondly, we describe the polarisation process.

Since $d!\neq 0$ in F, any p in $S_d(V^*)$ is of the form

$$p = \sum_i \frac{d!}{i_1!\dots i_n!}(p|i_1\dots i_n)x_1^{i_1}\dots x_n^{i_n}.$$

Now

$$\begin{aligned}\Delta(x_1^{i_1}\dots x_n^{i_n}) &= (x_1\otimes 1 + 1\otimes x_1)^{i_1}\dots(x_n\otimes 1 + 1\otimes x_n)^{i_n} \\ &= \sum_{j_1=1}^{i_1}\dots\sum_{j_n=1}^{i_n} \begin{bmatrix} i_1 \\ j_1 \end{bmatrix}\cdots\begin{bmatrix} i_n \\ j_n \end{bmatrix} x_1^{i_1-j_1}\dots x_n^{i_n-j_n}\otimes x_1^{j_1}\dots x_n^{j_n}\end{aligned}$$

so the component in $S_{d-1}(V^*)\otimes V^*$ of $\Delta(x_1^{i_1}\dots x_n^{i_n})$ is

$$\sum_{k=1}^{n} i_k . x_1^{i_1}\dots x_k^{i_k-1}\dots x_n^{i_n}\otimes x_k = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}(x_1^{i_1}\dots x_n^{i_n})\otimes x_k.$$

The polarisation process therefore involves

$$
\begin{array}{ll}
S_d(V^*) & p \mapsto (1/d!)p \\
\Delta\downarrow & \mapsto (1/d!) \sum_{\alpha d=1}^{n} \frac{\partial p}{\partial x_{\alpha d}} \\
S_{d-1}(V^*) \otimes V^* & \\
\Delta\otimes 1\downarrow & \mapsto (1/d!) \sum_{\alpha d=1}^{n} \sum_{\alpha(d-1)=1}^{n} \frac{\partial}{\partial x_{\alpha(d-1)}} \frac{\partial}{\partial x_{\alpha d}} p \otimes x_{\alpha(d-1)} \otimes x_{\alpha d} \\
(S_{d-2}(V^*) \otimes V^*) \otimes V^* & \\
\Delta\otimes 1\downarrow & \\
\vdots & \\
\Delta\otimes 1\downarrow & \mapsto (1/d!) \sum_{\alpha} \frac{\partial}{\partial x_{\alpha 1}} \cdots \frac{\partial}{\partial x_{\alpha d}} p \otimes x_{\alpha 1} \otimes \dots \otimes x_{\alpha d} \\
(V^*)^{\otimes d} &
\end{array}
$$

Since $\frac{\partial}{\partial x_{\alpha 1}} \cdots \frac{\partial}{\partial x_{\alpha d}}$ depends only on the content of α we have a symmetric tensor.

Let $p = \sum_i \frac{d!}{i_1! \dots i_n!} (p \mid i_1 \dots i_n) x_1^{i_1} \dots x_n^{i_n}$.

Then, as $\frac{\partial}{\partial x_{\alpha 1}} \cdots \frac{\partial}{\partial x_{\alpha d}} x_1^{i_1} \dots x_n^{i_n}$

$= 0$ if α does not have content i, and

$= i_1! \dots i_n!$ otherwise;

the polarisation of p is

$$
\begin{aligned}
\theta_p &= (1/d!) \sum_i (p \mid i_1 \dots i_n) \frac{d!}{i_1! \dots i_n!} \sum_{c(\alpha)=i} i_1! \dots i_n! . x_{\alpha_1} \otimes \dots \otimes x_{\alpha_d} \\
&= \sum_i (p \mid i_1 \dots i_n) \sum_{c(\alpha)=i} x_{\alpha(1)} \otimes \dots \otimes x_{\alpha(d)} .
\end{aligned}
$$

This is a symmetric tensor which on d–fold multiplication gives back p.

7. Notice that $\theta_p(v_{\alpha_1},\ldots,v_{\alpha_d}) = (p|i_1\ldots i_n)$ where $(i_1,\ldots,i_n)$ is the content of α, as in §5 with θ and p_θ.

8. It follows from §6 that, regarding θ_p as a d–multilinear map $\theta_p\colon V^d \to F$, $\theta_p(v,\ldots,v) = f(p)(v)$ the evaluation of p at v.

9. Akin et al [**ABW §I.1,§I.3**] show that all of §§**1–8** remains true even if F is just a commutative ring, V is a free F module of rank n, and $V^* = \mathrm{Hom}_F(V,F)$ which is also a free F–module of rank n.

In particular let $F \to A$ be a base change, where A is a commutative F–algebra. If V is a vector space of dimension n over F then $V_A = V \otimes_F A$ is a free A–module of rank n. We have a natural homomorphism $S(V^*) \to S(V_A^*)$ of graded Hopf–algebras and a commutative square

9.1
$$\begin{array}{ccc} S_d(V^*) & \longrightarrow & S_d(V_A^*) \\ \downarrow & & \downarrow \\ (V^*)^{\otimes d} & \longrightarrow & (V_A^*)^{\otimes d} \end{array}$$

in which the vertical arrows are the polarisations. The square commutes because polarisation is defined in terms of d–fold comultiplication and $S(V^*) \to S(V_A^*)$ is a homomorphism of graded Hopf–algebras. We expand slightly on this.

Let A be a commutative F–algebra.

Then V_A is a free A–module of rank n and so is $V_A^* = \mathrm{Hom}_A(V \otimes_F A, A)$.

Given $x \in V^*$ we have $x\otimes 1\colon V \otimes_F A \to F \otimes_F A \cong A$ so $x\otimes 1 \in S_1(V_A^*)$. Hence we have an F–linear map $V^* \to S(V_A^*)$ which extends to a graded Hopf–algebra homomorphism $S(V^*) \to S((V \otimes_F A)^*)$.

We will write p_A, θ_A for the images of $p \in S_d(V^*)$, $\theta \in (V^*)^{\otimes d}$ under the above homomorphisms.

In particular let $A = F[x_1,\dots,x_n]$, where $x_1,\dots,x_n$ is the dual basis of the basis $v_1,\dots,v_n$ of V. Then by §§5,7

9.2
$$\begin{aligned} p(x_1,\dots,x_n) &= \sum_{\alpha} \theta_p(v_{\alpha 1},\dots,v_{\alpha d}).x_{\alpha 1}\dots x_{\alpha d} \\ &= (\theta_p)_A(v,\dots,v) \text{ where } v = x_1v_1+\dots+x_nv_n. \end{aligned}$$

10. The definition of the polarisation process in Hopf–algebra language is useful in theory, as we saw in §9. To compute it we will often use the following method. Let $p \in S_d(V^*)$. Let A denote the ring of polynomials in $\lambda_1,\dots,\lambda_d$ with coefficients in F. Then $\theta_p(v_1,\dots,v_d) =$ the coefficient of $\lambda_1\dots\lambda_d$ in $(d!)^{-1}.f(p_A)(\lambda_1v_1+\dots+\lambda_dv_d)$

Proof: Since §8 is still valid for p_A and $\theta_{p_A} = (\theta_p)_A$, the above coefficient of $\lambda_1\dots\lambda_d =$ the coefficient of $\lambda_1\dots\lambda_d$ in
$(d!)^{-1}.(\theta_p)_A(\lambda_1v_1+\dots+\lambda_dv_d,\dots,\lambda_1v_1+\dots+\lambda_dv_d)$
$= (d!)^{-1}\sum_{\pi\in G_d} \theta_p(v_{\pi 1},\dots,v_{\pi d}) = \theta_p(v_1,\dots,v_d)$ by symmetry of θ_p. □

A choice of basis for V gives a preferred basis for $S(V^*)$ consisting of the monomials in the dual basis of V^*. A change of basis of V then results in a change of basis for $S(V^*)$. This leads us to define some natural GL(V)–actions, and we see that the diagram 9.1 above is a diagram of GL(V)–maps. §§11–13 contain the details.

11. The choice of basis $v_1,\dots,v_n$ of V with dual basis $x_1,\dots x_n$ of V^* defines an isomorphism $\varphi: V \longrightarrow F^n$ such that: if $e_1,\dots,e_n$ is the usual basis of F^n; and $r_1,\dots,r_n$ is the dual basis of $(F^n)^*$, then $\varphi(v_i) = e_i$ and $r_i\circ\varphi = x_i$. Let $p \in S(V^*)$ with
$p = \Sigma(p_\varphi|i_1\dots i_n)x_1^{i_1}\dots x_n^{i_n}$.

Then
$$\begin{aligned} p &= \Sigma(p_\varphi|i_1\dots i_n)(r_1\circ\varphi)^{i_1}\dots(r_n\circ\varphi)^{i_n} \\ &= S(\varphi^*)(\Sigma(p_\varphi|i_1\dots i_n)r_1^{i_1}\dots r_n^{i_n}). \end{aligned}$$

Let $w_1,\dots,w_n$ be another choice of basis of V with dual basis $y_1,\dots,y_n$ of V^*. This choice defines another isomorphism $\psi\colon V \to F^n$ and $p = S(\psi^*)(\,\Sigma(p_\psi|i_1\dots i_n)r_1^{i_1}\dots r_n^{i_n}\,)$.
Therefore $\Sigma(p_\psi|i_1\dots i_n)r_1^{i_1}\dots r_n^{i_n} = S(\,(\psi^*)^{-1}\circ\varphi^*\,)\,(\,\Sigma(p_\varphi|i_1\dots i_n)r_1^{i_1}\dots r_n^{i_n}\,)$.
The diagram

$$\begin{tikzcd} F^n \arrow[rr, "\sigma"] & & F^n \\ & V \arrow[ul, "\varphi"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes, and $(\psi^*)^{-1}\circ\varphi^* = (\varphi\circ\psi^{-1})^* = (\sigma^{-1})^*$.

12. We see that the coefficients $(p_\varphi|i_1\dots i_n)$ transform by the natural left action of $\sigma\epsilon GL_n(F)$ on $S((F^n)^*)$ by $S(\,(\sigma^*)^{-1}\,)$. More generally $\sigma\epsilon GL(V)$ acts on the left on $S(V^*)$ by the graded Hopf–algebra automorphism $S(\,(\sigma^*)^{-1}\,)$. When $F \longrightarrow A$ is a base change by a commutative F–algebra A, then $\sigma\epsilon GL(V)$ acts on the left on $S(V_A^*)$ by the graded Hopf–algebra automorphism $S(\,((\sigma\otimes 1)^*)^{-1}\,)$.

13. $\sigma\epsilon GL(V)$ acts on the left on $\{\, f\colon V^d \to F,\ f \text{ d–multilinear} \,\}$ by $(\sigma.f)(v_1,\dots,v_d) = f(\sigma^{-1}v_1,\dots,\sigma^{-1}v_d)$. We transfer this to a GL(V)–action on $(V^*)^{\otimes d}$ by requiring that the canonical isomorphism be a GL(V)–isomorphism. The G_d–actions defined in §1 commute with these GL(V)–actions, so the spaces of symmetric d–multilinear maps, and symmetric tensors in $(V^*)^{\otimes d}$ are invariant under the GL(V)–action.

Let $F \longrightarrow A$ be a change of base by a commutative F–algebra. $\sigma\epsilon GL(V)$ acts on the left on $\{\, f\colon (V_A^*)^d \to A : f \text{ d–multilinear w.r.t. } A \,\}$ by
$(\sigma.f)(m_1,\dots,m_d) = f(\,(\sigma\otimes 1)^{-1}m_1,\dots,(\sigma\otimes 1)^{-1}m_d\,)$.
We transfer this to a GL(V)–action on $(V_A^*)^{\otimes d}$ by requiring that the canonical isomorphism be a GL(V)–map. There are actions of the group G_d, of permutations of $1,\dots,d$, on these spaces generalising those of §1. These G_d–actions commute with the

GL(V)–actions, so the A–submodules of: symmetric tensors in $(V_A^*)^{\otimes d}$; and symmetric maps f: $(V_A)^d \longrightarrow A$ are invariant under these GL(V)–actions. With these GL(V) actions, the diagram 9.1 above is a diagram of GL(V)–maps.

We summarise **§§14–19.**

Let V be a vector space of finite dimension n over the base field F. Suppose that $d! \neq 0$ in F. We have seen that the symmetric d–multilinear forms θ: $V^d \longrightarrow F$ correspond 1–1 by an F–linear isomorphism to the $p \in S_d(V^*)$. When we choose a basis of V we can specify $p \in S_d(V^*)$ as a homogeneous degree d polynomial in $F_d[\, r_1,\dots,r_n \,]$. We call elements of $F_d[\, r_1,\dots,r_n \,]$ **forms of degree d.** In the other direction, a degree d form in $r_1,\dots r_n$ only determines $p \in S_d(V^*)$ up to GL(V)–equivalence. We define the notion of **symmetric space of degree d over F.** We show that an isomorphism class of symmetric spaces of degree d is the same as a GL(V)–equivalence class in $S_d(V^*)$ or a $GL_n(F)$–equivalence class of forms of degree d in $F_d[\, r_1,\dots,r_n \,]$. We then define the **direct sum of symmetric spaces of degree d** and the corresponding **orthogonal sum of forms of degree d.**

14. Let V be an n–dimensional vector space over F. Let θ: $V^d \longrightarrow F$ be a symmetric d–multilinear form. Then we say that (V,θ) is a **symmetric space of degree d.** We say that two symmetric spaces (V,θ), (V',θ') of degree d over F are **isomorphic** if there is a vector space isomorphism φ: $V \longrightarrow V'$ such that $\theta'(\, \varphi(v_1),\dots,\varphi(v_d)\,) = \theta(v_1,\dots,v_d)$.
We continue to assume that $d! \neq 0$ in F.

15. Given any $p \in S_d((F^n)^*)$, (F^n,θ_p) is a symmetric space of degree d.
For $\sigma \in GL_n(F)$ $\theta_{\sigma.p} = \sigma.\theta_p$ by §13, so the symmetric spaces (F^n,θ_p) , $(F^n,\theta_{\sigma.p})$ corresponding to equivalent polynomials are isomorphic.

16. Given a symmetric space (V,θ) of degree d, there is a unique $p\epsilon S_d(V^*)$ with $\theta_p = \theta$. After a choice of coordinates $\varphi: V \to F^n$ we have $S((\varphi^{-1})^*)p \in S_d((F^n)^*)$. A different choice of coordinates will by §11 result in a form of degree d in the same $GL_n(F)$–equivalence class.

17. Given two symmetric spaces (V_1,θ_1), (V_2,θ_2) of degree d over F, the map $\theta_1 \oplus \theta_2\colon (V_1 \oplus V_2)^d \to F$ with $\theta_1 \oplus \theta_2[(v_{11},v_{21}),\ldots,(v_{1d},v_{2d})] = \theta_1(v_{11},\ldots,v_{1d}) + \theta_2(v_{21},\ldots,v_{2d})$ is d–multilinear and symmetric. We **define** $(V_1,\theta_1)\oplus(V_2,\theta_2)$ to be $(V_1\oplus V_2,\theta_1\oplus\theta_2)$. Now $\theta_1\oplus\theta_2((v_1,v_2),\ldots,(v_1,v_2)) = \theta_1(v_1,\ldots,v_1) + \theta_2(v_2,\ldots,v_2)$. Therefore if we choose bases $x_1,\ldots,x_n$ of V_1^* and $y_1,\ldots,y_m$ of V_2^* then $x_1,\ldots,x_n,y_1,\ldots,y_m$ is a basis of $(V_1\oplus V_2)^*$ and $p_{\theta_1\oplus\theta_2}(x_1,\ldots,x_n,y_1,\ldots,y_m) = p_{\theta_1}(x_1,\ldots,x_n) + p_{\theta_2}(y_1,\ldots,y_m)$, by §9.2.

18. When $p_1(x_1,\ldots,x_n)$, $p_2(y_1,\ldots,y_m)$ are two forms of degree d with coefficients in F, the **notation:** $p_1 \perp p_2$ is used for the form $p_1(x_1,\ldots,x_n) + p_2(x_{n+1},\ldots,x_{n+m})$. Here $p_1 \epsilon S_d((F^n)^*)$, $p_2 \epsilon S_d((F^m)^*)$ and $p_1\perp p_2 \epsilon S_d((F^n \oplus F^m)^*)$. Let (F^n,θ_1), (F^m,θ_2) be the symmetric spaces corresponding to p_1, p_2 respectively. Then $(F^n,\theta_1)\oplus(F^m,\theta_2)$ is the symmetric space corresponding to $p_1 \perp p_2$.

Next we show that we can add two isomorphism classes of symmetric spaces of degree d (resp. equivalence classes of forms of degree d) to get another isomorphism class of symmetric spaces of degree d (resp. equivalence class of forms of degree d).

19. If for i=1,2 $(V_i,\theta_i) \cong (V'_i,\theta_i')$ then $(V_1,\theta_1)\oplus(V_2,\theta_2) \cong (V_1',\theta_1')\oplus(V_2',\theta_2')$ by taking the direct sum of the two isomorphisms. If $\sigma_i.p_i = p_i'$ for i=1,2 where $\sigma_1 \epsilon GL_n(F)$, $\sigma_2 \epsilon GL_m(F)$ then $\sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \epsilon\, GL_{n+m}(F)$ has $\sigma.(p_1 \perp p_2) = p_1' \perp p_2'$. Later in **2§3.2** we define the product of two isomorphism classes of symmetric spaces.

We conclude with an important result which we will often use in the sequel.

20. Proposition: Let R be an integral domain with more than d elements. Let $p \in R[t_1,\ldots,t_n]$. If p has degree $\leq d$ in each t_i and $p(r_1,\ldots,r_n)=0$ for all $r_1,\ldots,r_n \in R$, then $p=0$.

Proof: We use the Vandermonde argument explained by Rowen in [**ROW** p129]. Let $c_0,c_1,\ldots,c_d$ be $d+1$ distinct elements of R. The proof will be by induction on n. Suppose $n=1$. Then $p = a_0+a_1t_1+\ldots+a_dt_1^d$. Setting $t_1 = c_i$ $i=0,\ldots,d$ we get

$$\begin{bmatrix} 1 & c_0 & \cdots & c_0{}^d \\ 1 & c_1 & \cdots & c_1{}^d \\ \vdots & & \ddots & \vdots \\ 1 & c_d & \cdots & c_d{}^d \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{bmatrix} = 0 \quad .$$

Since $\det(c_i^j) = \prod_{0\leq k<l\leq d} (c_l-c_k) \neq 0$ we conclude that $a_i=0$ for $i=0,\ldots,d$. Suppose that the proposition is true for polynomials in $t_1,\ldots,t_k$. Let $p = p_0 + p_1t_{k+1} +\ldots+ p_dt_{k+1}^d$ where for $i=0,\ldots,d$ p_i is a polynomial in $t_i,\ldots,t_k$; and each p_i has degree $\leq d$ in t_j for each $j=1,\ldots,k$. For each c_i $i=0,\ldots,d$; and all $r_1,\ldots,r_k \in R$ $p(r_1,\ldots,r_k,c_i) = \sum_{j=0}^{d} p_j(r_1,\ldots,r_k)c_i^j = 0$. We conclude as in the case $n=1$ above that $p_j(r_1,\ldots,r_k)=0$ for all $j=0,\ldots,d$ and all $r_1,\ldots,r_k \in R$. Therefore $p_j=0$ for $j=0,\ldots,d$ and $p=0$. □.

21. It follows from §20 and §3 that when F is a field and $d! \neq 0$ in F:

(i) if (V,θ) is a symmetric space of degree d and $\theta(v,\ldots,v)=0$ for all v, then $\theta=0$.

(ii) $p \in S_d(V^*)$ may be identified with the polynomial map $f(p)\colon V \longrightarrow F$ of §2.

CHAPTER TWO

This chapter is divided into three parts. We outline the contents of each part.

(1) In the first part we discuss the conditions: **nondegeneracy**, **nonzero Hessian** and **nonsingularity** of a symmetric space. Most of the results are due to Harrison [**HAR1**], [**HAR2**].

The above three conditions are invariant under the GL(V)–action on the set of symmetric spaces (V,θ) of degree d.

Over the algebraic closure of the ground field, the subsets of the affine algebraic variety of all symmetric d–multilinear forms on a given vector space, consisting of:

(a) the nondegenerate forms

(b) the forms with nonzero Hessian

(c) the nonsingular forms

are all Zariski open.

We will see that, in 19th century terminology [**TUR p206**], the Hessian of a form is a covariant of weight 2.

We introduce the **hyperbolic** symmetric (d+1)–multilinear forms defined for each $d \geq 1$ and for each vector space V, which for d=1 are the classical hyperbolic bilinear forms. We show that the hyperbolic forms are nondegenerate, but that when $d \geq 2$ and $\dim(V) \geq 2$ they have zero Hessian.

Harrison [**HAR2, Prop1.1**] proved that, when the ground field has characteristic zero, any nonsingular symmetric space of degree 3 has nonzero Hessian. The degree 2 case is trivial. We extend this result to all degrees $d \geq 2$.

(2) In the second part we introduce the condition of indecomposability, and the centre of a symmetric space of degree $d \geq 3$, both defined by Harrison [**HAR1 p128,p133**].

We give some examples of symmetric spaces and we compute their centres. It is then clear that there are nondegenerate indecomposable symmetric spaces in each dimension.

We compute the centres of the hyperbolic symmetric spaces of degree $d \geq 3$.

(3) In the third part we introduce the notion of split symmetric space— a direct higher degree generalisation, in the field case, of Knebusch's definition [**MH p12**]. We show easily that the hyperbolic symmetric spaces are split, and give an example in each degree $d \geq 3$ of a split space which is not hyperbolic. We prove that in a fixed degree $d \geq 3$, the direct sum of nondegenerate split symmetric spaces is split, and the tensor product of a nondegenerate split symmetric space and any nondegenerate symmetric space is a nondegenerate split symmetric space. Then it is clear that in the Grothendieck ring of degree d forms, the set of formal differences of isomorphism classes of split spaces is an ideal.

Harrison raised the question of the structure of the monoid of absolutely indecomposable nondegenerate symmetric spaces under tensor product [**HAR1 p136**]. In our final result we use our calculation of the centres of the hyperbolic symmetric spaces to show that the submonoid generated by the hyperbolics is free.

1 Let V be a vector space of dimension n over the ground field F. Suppose that $d! \neq 0$ in F. Let $\theta\colon V^d \to F$ be a symmetric d–multilinear form.

We will define the condition of nondegeneracy of θ and show that it is $GL(V)$–invariant.

1.1 Definition: We say that θ is **nondegenerate** if for any nonzero $v \epsilon V$ the symmetric $(d-1)$–multilinear form $\theta_v: V^{d-1} \to F$ defined by $\theta_v(v_2,\ldots,v_d) = \theta(v,v_2,\ldots,v_d)$ is nonzero. Otherwise we say that θ is **degenerate**.

1.2 Proposition: Let $\sigma \epsilon GL(V)$. Then θ is nondegenerate iff $\sigma.\theta$ is nondegenerate.

Proof: $\theta_v(v_2,\ldots,v_d) = \theta(v,v_2,\ldots,v_d) = \sigma.\theta(\sigma v,\sigma v_2,\ldots,\sigma v_d) = (\sigma.\theta)_{\sigma v}(\sigma v_2,\ldots,\sigma v_n)$ and as v_i runs through V so does σv_i. □

Because of the isomorphisms

$$\bigodot^d(V) \cong S_d(V^*) \cong F_d[x_1,\ldots,x_n]$$

the second isomorphism depending on a choice of basis $x_1,\ldots,x_n$ of V^*, there are corresponding conditions of nondegeneracy of $p \epsilon S_d(V^*)$, and of forms of degree d in $x_1,\ldots,x_n$.

We loosely state the next proposition before its formal proof: Let $p \epsilon S_d(V^*)$ have polarisation θ. Let $v \epsilon V$. Let $p_v \epsilon S_{d-1}(V^*)$ have polarisation θ_v. Then $d.p_v$ is the directional derivative of p in the direction v.

1.3 Proposition: Let $v_1,\ldots,v_n$ be a basis of V with dual basis $x_1,\ldots,x_n$ of V^*. Let $v=\Sigma a_i v_i$. Under the isomorphism of **1§3** let $p \leftrightarrow \theta$, $p_v \leftrightarrow \theta_v$. Then

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} p = d.p_v.$$

In particular, if $\theta_{v_n}=0$, then p is a polynomial in $x_1,\ldots,x_{n-1}$.

Proof: Let $B = F[x_1,\dots,x_n,\lambda]$. Then

$\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} p$ = the coefficient of λ in $p(x_1+a_1\lambda,\dots,x_n+a_n\lambda)-p(x_1,\dots,x_n)$

= (by 1§9.2) the coefficient of λ in $\theta_B(\dots,\Sigma v_i \otimes (x_i+a_i\lambda),\dots) - \theta_B(\dots,\Sigma v_i \otimes x_i,\dots)$

Since $\Sigma v_i \otimes (x_i+a_i\lambda) = (\Sigma v_i \otimes x_i)+v\otimes\lambda$ and θ is symmetric this coefficient is

$d.\theta_A(v\otimes 1,\Sigma v_i\otimes x_i,\dots,\Sigma v_i\otimes x_i)$, where $A = F[x_1,\dots,x_n]$

$= d.(\theta_v)_A(\,\Sigma v_i\otimes x_i,\dots,\Sigma v_i\otimes x_i\,)$

$= d.p_v$ by 1§9.2 □

Now we show that the generic symmetric d–multilinear form on V is nondegenerate.

1.4 Proposition: In the affine algebraic variety of all symmetric d–multilinear forms θ, the subset of all nondegenerate forms is Zariski open.

Proof: Fix bases of V and of the space of (d–1)–multilinear forms on V. θ is nondegenerate iff the map $v \mapsto \theta_v$ is a monomorphism.

With our choice of bases this linear map is represented by a matrix whose entries depend linearly on θ. θ is nondegenerate iff one of the nxn minor determinants of this matrix is nonzero. □

1.5 Now we discuss the Hessian of the generic form of degree d in $r_1,\dots,r_n$. It is an example of what in 19–th century terminology is called a **covariant** of weight **2**.

A covariant of the form of degree d in $r_1,\dots,r_n$ is a polynomial in the indeterminate coefficients of the form and $r_1,\dots r_n$; which has a certain kind of $GL_n(F)$–invariance.

To simplify the discussion let us assume that the base field is infinite so that by **1§21(ii)** we can identify a polynomial with its induced polynomial map.

Let $G: V_{n,d} \times F^n \to F$, where $V_{n,d}$ is the space of all polynomials of degree d in $r_1,\dots,r_n$ and G is a polynomial map.

$GL_n(F)$ acts on the left on the F–algebra of all such maps G by

$$(\sigma.G)(p,v) = G(\sigma^{-1}.p,\sigma^{-1}v) = G(p\circ\sigma,\sigma^{-1}v).$$

Here we identify polynomials p with the induced polynomial maps $p\colon F^n \to F$, and we have the picture

$$F^n \xrightarrow{\sigma} F^n \xrightarrow{p} F$$
$$\sigma^{-1}v \mapsto v$$

G is a **covariant** of **weight** $g\epsilon\mathbb{Z}$ if $\sigma.G = (\det\sigma)^g G$ for all $\sigma\epsilon GL_n(F)$.

1.6 Definition: Let $p\epsilon F[r_1,\ldots,r_n]$. the **Hessian** $H(p;r_1,\ldots,r_n)$ of p is the polynomial

$$\det\left(\frac{\partial^2 p}{\partial r_j \partial r_i}\right).$$

When $d>2$ the Hessian is a polynomial in $r_1,\ldots,r_n$ and the coefficients of p.

Given $p\epsilon S_d(V^*)$ we can only define the Hessian as above after a choice of coordinates. The next proposition states that the Hessian is a covariant of weight 2. Once this is known it is clear that the condition of nonzero Hessian does not depend on the choice of basis.

1.7 Proposition: Let $\sigma\epsilon GL_n(F)$. Then

$$H(\sigma^{-1}.p;r_1,\ldots,r_n) = (\det\sigma)^2 H(p;\sigma^* r_1,\ldots,\sigma^* r_n).$$

Proof: The proof is based on the formal Chain Rule for polynomial maps. We set up the notations to state this Chain Rule.

Suppose that $p_i\epsilon F[r_1,\ldots,r_n]$ for $i=1,\ldots,m$.

Let $P=\begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix}$. We define the derivative matrix P' of P by $P' = \left(\frac{\partial p_i}{\partial r_j}\right)$.

Suppose that $q_i\epsilon F[s_1,\ldots,s_k]$, $i=1,\ldots,n$. Let $Q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$. Then for $p\epsilon F[r_1,\ldots,r_n]$, $p\circ Q$ stands for $p(q_1,\ldots,q_n)\epsilon F[s_1,\ldots,s_k]$.

With P as above we let $P\circ Q = \begin{pmatrix} p_1\circ Q \\ \vdots \\ p_m\circ Q \end{pmatrix}$ and we let $P'\circ Q = \left(\frac{\partial p_i}{\partial r_j}\circ Q\right)$.

Now we can state the formal Chain Rule: $(P\circ Q)' = (P'\circ Q).Q'$.

For $p\epsilon F[r_1,\ldots,r_n]$, $\sigma\epsilon GL_n(F)$ we have the following equalities of $n\times n$ matrices:

(1) $$\left(\frac{\partial^2(\sigma^{-1}.p)}{\partial r_j\partial r_i}\right) = \begin{pmatrix} \frac{\partial}{\partial r_1}(\sigma^{-1}.p) \\ \vdots \\ \frac{\partial}{\partial r_n}(\sigma^{-1}.p) \end{pmatrix}' = ((\sigma^{-1}.p)'^{t})'.$$

Now $(\sigma^{-1}.p)(r_1,\ldots,r_n)=p(\overset{*}{\sigma} r_1,\ldots,\overset{*}{\sigma} r_n)$ and $\overset{*}{\sigma} r_i = \sum_{j=1}^{n} \sigma_{ij}r_j$.

(2) Let $A = \begin{pmatrix} \sum_{j=1}^{n} \sigma_{1j}r_j \\ \vdots \\ \sum_{j=1}^{n} \sigma_{nj}r_j \end{pmatrix}$ then

(3) $A' = \sigma$ and

(4) $\sigma^{-1}.p = p\circ A$.

(5) $(\sigma^{-1}.p)' = (p\circ A)'$ (by (4))$= (p'\circ A).A'$ (by the Chain Rule) $= (p'\circ A).\sigma$ (by (3))

and $(\sigma^{-1}.p)'^{t}$ is a column with i–th entry $\sum_{j=1}^{n} \left(\frac{\partial p}{\partial r_j}\circ A\right)\sigma_{ji}$.

Let

(6) $$B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} \sigma_{j1}r_j \\ \vdots \\ \sum_{j=1}^{n} \sigma_{jn}r_j \end{pmatrix}.$$ Then

(7) $B' = \sigma^t$ and

(8) $$(\sigma^{-1}.p)'^{t} = B\circ \begin{pmatrix} \frac{\partial p}{\partial r_1} \\ \vdots \\ \frac{\partial p}{\partial r_n} \end{pmatrix} \circ A.$$

Now $\left(\frac{\partial^2(\sigma^{-1}.p)}{\partial r_j\partial r_j}\right) = (\sigma^{-1}.p)'^{t'}$ (by (1)) $= \sigma^t.\left[\left(\frac{\partial^2 p}{\partial r_i\partial r_j}\right)\circ A\right].\sigma$ (by (8), the Chain Rule,(3)and (7)). Taking determinants and recalling the definition of A in (2) we get the result. □

Now we can define the condition of nonzero Hessian for a symmetric d–multilinear form.

1.8 Definition: Let $v_1,\dots,v_n$ be a basis of V with dual basis $x_1,\dots,x_n$ of V^*. We say that the symmetric d–multilinear form θ: V→ F has **nonzero Hessian** when

$$\det\left(\frac{\partial^2}{\partial x_j \partial x_i} p_\theta \right) \neq 0.$$

By §1.7 this condition does not depend on the choice of basis, and the condition of nonzero Hessian is invariant under the action of GL(V).

It is easy to see that a degenerate form has zero Hessian.

1.9 Proposition: If θ has nonzero Hessian then θ is nondegenerate.

Proof: Suppose that θ is degenerate. Then there is a basis $u_1,\dots,u_n$ of V, with dual basis $x_1,\dots,x_n$ of V^*, such that $\theta_{u_n}=0$. By §3 $\frac{\partial}{\partial x_n} p_\theta = 0$. Therefore the matrix $\left(\frac{\partial^2}{\partial x_j \partial x_i} p_\theta \right)$ has zero n–th row and so has zero determinant. □

Harrison [HAR2 p521] states that nondegenerate forms with zero Hessian are hard to find. We will see that the higher degree analogues of the symmetric bilinear hyperbolic forms have zero Hessian when in n≥3 variables, but are nondegenerate.

We now define the hyperbolic forms.

1.10 Let d be an integer ≥1. Let $\bigodot^d(V)$ denote the vector space of all symmetric d–multilinear forms θ: V^d→ F on the F–vector space V.

Put $W = V \oplus (\bigodot^d(V))$ and define ψ: W^{d+1}→ F by

$$\psi[(v_1,\theta_1),\dots,(v_{d+1},\theta_{d+1})] = \sum_{i=1}^{d+1} \theta_i(v_1,\dots,\hat{v}_i,\dots,v_{d+1}),$$ where $\hat{v}_i$ indicates that v_i is omitted.

We call these forms **hyperbolic**. When d=1, ψ is the well–known hyperbolic symmetric bilinear form.

A simple check shows that ψ is symmetric, mutilinear and nondegenerate:

(i) ψ **is symmetric:** Let π be a permutation of $1,\dots,d+1$.

$$\psi[(v_{\pi 1},\theta_{\pi 1}),\dots,(v_{\pi(d+1)},\theta_{\pi(d+1)})]$$
$$=\sum_{i=1}^{d+1}\theta_{\pi i}[v_{\pi 1},\dots,\hat{v}_{\pi i},\dots,v_{\pi(d+1)}]$$
$$=\sum_{i=1}^{d+1}\theta_{\pi i}[v_1,\dots,v_{\pi(i)-1},v_{\pi(i)+1},\dots,v_{d+1}] \text{ (each } \theta_{\pi(i)} \text{ is symmetric)}$$
$$=\psi[(v_1,\theta_1),\dots,(v_{d+1},\theta_{d+1})] \text{ (addition in F is commutative)}$$

(ii) ψ **is multilinear:** One can easily prove this using only multilinearity of each $\theta\epsilon \bigodot^d(V)$, but for ease of notation we appeal to symmetry of ψ and check:

$$\psi[(a.v'+b.v'',\ a.\theta'+b.\theta''),(v_1,\theta_1),\dots,(v_d,\theta_d)]$$
$$= a\theta'(v_1,\dots,v_d) + \sum_{i=1}^{d} a\theta_i(v',v_1,\dots,\hat{v}_i,\dots,v_d) + b\theta''(v_1,\dots,v_d) +$$
$$+\sum_{i=1}^{d} b\theta_i(v'',v_1,\dots,\hat{v}_i,\dots,v_d) \quad \text{(each } \theta_i \text{ is multilinear)}$$
$$= a\psi[(a.v'+b.v'',a.\theta'+b.\theta''),(v_1,\theta_1),\dots,(v_d,\theta_d)] +$$
$$+\ b\psi[(v'',\theta''),(v_1,\theta_1),\dots,(v_d,\theta_d)]$$

(iii) ψ **is nondegenerate:** Suppose that for some $(v,\theta)\epsilon W$ we have

$$\psi[(v,\theta),(v_1,\theta_1),\dots,(v_d,\theta_d)]=0 \text{ for all } (v_1,\theta_1),\dots,(v_d,\theta_d)\ \epsilon\ W.$$

Let $v_1=0$ and let θ_1 be nondegenerate. It follows that $v=0$.

Now let $\theta_1=\dots=\theta_d=0$ to see that $\theta=0$. □

We now choose convenient bases of V and $\bigodot^d(V)$ to get a form of degree $d+1$ corresponding to ψ, so that we can compute that ψ has zero Hessian.

1.11 Let $u_1,\dots,u_n$ be a basis of V. Let $(i_1,\dots,i_n)$ be an n–tuple of nonnegative integers with sum d.

Define: $\theta_{i_1\dots i_n}\ \epsilon \bigodot^d(V)$ by requiring

$$\theta_{i_1\dots i_n}(u_{\alpha 1},\dots,u_{\alpha d}) = \begin{cases} 1 & \text{if } \alpha \text{ has content } (i_1,\dots,i_n).\\ 0 & \text{otherwise}\end{cases}$$

Then the $\theta_{i_1\ldots i_n}$ form a basis of $\bigodot^d(V)$, and $(u_1,0),\ldots,(u_n,0);\ldots,(0,\theta_{i_1\ldots i_n}),\ldots$ form a basis of $V\oplus\bigodot^d(V)$.

Let $r_1,\ldots,r_n;\ldots,s_{i_1\ldots i_n},\ldots$ denote the dual basis.

We calculate p_ψ as at the end of 1§9.2

Let $A = F[r_1,\ldots,r_n;\ldots,s_{i_1\ldots i_n},\ldots]$.

$$p_\psi(r_1,\ldots,r_n;\ldots,s_{i_1\ldots i_n},\ldots)$$
$$= \psi_A[\ldots,(\sum_{i=1}^{n} e_i\otimes r_i\,,\ \sum_{i_1+\ldots i_n=d} \theta_{i_1\ldots i_n}\otimes s_{i_1\ldots i_n}),\ldots]$$
$$= (d+1)\sum_i \sum_\alpha \theta_{i_1\ldots i_n}(e_{\alpha 1},\ldots,e_{\alpha d})\, s_{i_1\ldots i_n} r_{\alpha 1}\ldots r_{\alpha d}$$

(the first sum is over all n–tuples i of nonnegative integers $(i_1,\ldots,i_n)$ with sum d; and the second sum is over all maps α: $\{1,\ldots,d\}\to\{1,\ldots,n\}$)

$$= (d+1)\sum_i \frac{d!}{i_1!\ldots i_n!}\, s_{i_1\ldots i_n} r_1^{i1}\ldots r_n^{in}$$

because the $\theta_{i_1\ldots i_n}$ are symmetric, and we calculate the number of α with a given content as in 1§5.

Note that p_ψ is homogeneous of degree d+1 with degree 1 in the $s_{i_1\ldots i_n}$ and degree d in $r_1,\ldots,r_n$. Also $p_\psi=0$ unless $(d+1)!\neq 0$ in F.

1.12 Claim: When $d\geq 2$ and $\dim V\geq 2$, ψ has zero Hessian.

Proof: We will use the dual basis of §1.11 to show that p_ψ has zero Hessian.

p_ψ is linear in the $s_{i_1\ldots i_n}$, so $\dfrac{\partial^2}{\partial s_{j_1\ldots j_n}\partial s_{i_1\ldots i_n}} p_\psi = 0$ and the matrix of second order partial derivatives of p_ψ has $\dim_F(\bigodot^d(V)) = \begin{bmatrix} n+d-1 \\ n-1 \end{bmatrix}$ of its columns all contained in a subspace, over the field $F(r_1,\ldots,r_n;\ldots,s_{i_1\ldots i_n},\ldots)$, of dimension $\leq n$. Therefore the wedge product of these columns will be zero whenever $\begin{bmatrix} n+d-1 \\ n-1 \end{bmatrix} > n$, which is the case iff $d\geq 2$ and $n\geq 2$. □

Now we discuss nonsingularity of a symmetric d–multilinear θ: $V^d :\to F$; or the corresponding $p \in S_d(V^*)$; or the form $f \in F_d[x_1,\ldots,x_n]$ defined by p and a choice of basis $x_1,\ldots,x_n$ of V^*.

First we define the singular locus of θ over an extension of the base field, and see how this locus changes when we replace θ by an equivalent form. The singular locus of θ over an extension field is defined so as to be compatible with the usual definition of the singular locus of the form f as the set of common zeros of its first order partial derivatives.

Let (V,θ) be a symmetric space of degree d over F. Let K be a field containing the ground field F.

The homomorphism of graded Hopf–algebras $S(V^*) \to S((V \otimes_F K)^*)$ defined in 1§9 is a monomorphism.

1.13 Definition: **The singular locus of θ over K**

$S_K(\theta) = \{ w \in V \otimes_F K : \theta_K(u,w,\ldots,w) = 0 \text{ for all } u \in V \otimes_F K \}$

with θ_K as defined in 1§§9.

For $\sigma \in GL(V)$ let $\sigma_K \in GL(V \otimes_F K)$ be the unique K–linear automorphism of $V \otimes_F K$ with $\sigma_K(v \otimes a) = \sigma(v) \otimes a$ for all $v \in V, a \in K$.

1.14 Proposition: $S_K(\sigma.\theta) = \sigma_K S_K(\theta)$.

Proof: $w \in S_K(\sigma.\theta)$ means that $(\sigma.\theta)_K(u,w,\ldots,w)=0$ for all $u \in V \otimes_F K$. By **1§§9,13** $(\sigma.\theta)_K = \sigma_K.\theta_K$ so $w \in S_K(\sigma.\theta)$ means that for all $u \in V \otimes_F K$

$$0 = \sigma_K.\theta_K(u,w,\ldots,w)$$
$$= \theta_K(\sigma_K^{-1}u, \sigma_K^{-1}w, \ldots, \sigma_K^{-1}w).$$

As u runs through $V \otimes K$ so does $\sigma_K^{-1}u$, therefore $w \in S_K(\sigma.\theta)$ means $\sigma_K^{-1}w \in S_K(\theta)$.□

1.15 Let $v_1,\ldots,v_n$ be a basis of V with dual basis $x_1,\ldots,x_n$ of V^*. Then $v_1 \otimes 1,\ldots,v_n \otimes 1$ is a basis of $V \otimes K$ over K with dual basis $x_1 \otimes 1,\ldots,x_n \otimes 1$ of $(V \otimes_F K)^* \cong V^* \otimes_F K$.

Proposition: Suppose that $d!\neq 0$ in F. Then $w \epsilon S_K(\theta)$ iff for all $i=1,\ldots,n$ w is a zero of $\frac{\partial}{\partial(\overline{x_i \otimes 1})^p}\theta_K$.

Proof: $w \epsilon S_K(\theta) \Leftrightarrow \theta_K(v_i \otimes 1, w,\ldots,w) = 0$ for all $i=1,\ldots,n$

since θ_K is K–multilinear and $v_1 \otimes 1,\ldots,v_n \otimes 1$ span $V \otimes_F K$ over K.

The result follows from **§1.3**, and the assumption that $d!\neq 0$ in F.□

Let k be the algebraic closure of the ground field F.

1.16 Definition: θ is **nonsingular** if $S_k(\theta) = \{0\}$ and **singular** otherwise.

Next we show that there is a homogeneous polynomial G in the coefficients of the form f of degree d in $x_1,\ldots,x_n$, such that $G(f)=0$ iff f is singular.

The proof avoids classical elimination theory and appeals instead to elementary algebraic geometry.

1.17 Proposition: Let V be a vector space of dimension **n** over k, and suppose that $d!\neq 0$ in k. Then in the projective algebraic variety on the vector space of all symmetric d–multilinear forms $\theta: V^d \to k$, the set $\{ [\theta] : \theta \text{ is singular} \}$ is Zariski closed of codimension 1.

Proof: Let $x_1,\ldots,x_n$ be a basis of V^*.

Let W denote the vector space of all polynomials of homogeneous degree d–1 in $x_1,\ldots,x_n$ with coefficients in k.

Let $\mathbb{P}(W)$ denote the projective space of all lines in W.

Let X denote the product of n copies of $\mathbb{P}(W)$.

Let $Y = \{ ([f_1],\ldots,[f_n]) \epsilon X \ : \text{ for some } a \epsilon \mathbb{P}^{n-1} \ f_i(a)=0 \text{ for all } i=1,\ldots,n \ \}$

Claim: Y is irreducible of codimension 1 in X.

Before we prove the claim we will deduce **Prop 1.17** from it.

Since $d! \neq 0$, **Prop 1.15** applies:

θ is singular iff $\frac{\partial}{\partial x_i} p_\theta$ $i=1,\dots,n$ have a common nontrivial zero in V.

Having proved the claim, it follows by **[SHA Thm 3′ p56]** that Y is the set of zeros of an irreducible polynomial, homogeneous in each of $[f_1],\dots,[f_n]$. Substituting $\frac{\partial}{\partial x_i} p_\theta$ for f_i, we get a polynomial in the coefficients of p_θ, and θ is singular iff this polynomial vanishes.

Proof of the claim: First we show that Y is closed. Consider the projection map

$X \times \mathbb{P}^{n-1} \to X$. Let Z be the closed subset of $X \times \mathbb{P}^{n-1}$ consisting of all $([f_1],\dots,[f_n], a)$ with $f_i(a)=0$ for all $i=1,\dots,n$. Then Y is the image of Z under the projection map. By **[SHA p45 Thm3]** Y is closed.

Next we apply **[SHA p61 Thm8]** to show that Z, and hence Y, is irreducible: We consider the projection map

$Z \to \mathbb{P}^{n-1}$ and observe that

(a) it maps Z onto $\mathbb{P}^{n-1}$

(b) given $a \in \mathbb{P}^{n-1}$ the fibre over a is the product of n hyperplanes in $\mathbb{P}(W)$, so that all fibres have the same dimension and are irreducible.

Finally we apply **[SHA p60 Thm7]** to show that Y has codimension 1 in X.

Let X′ be the product of n–1 copies of $\mathbb{P}(W)$.

Consider the projection $X \to X'$: $([f_1],[f_2],\dots,[f_n]) \mapsto ([f_2],\dots,[f_n])$ and let

π: $Y \to X'$ be its restriction to Y.

Let $x' = ([f_2],\dots,[f_n]) \in X'$ and let $Z(f_2,\dots,f_n) = \{ a \in \mathbb{P}^{n-1} : f_i(a)=0 ,i=2,\dots,n \}$. By **[SHA p57 Cor5]** $Z(f_2,\dots,f_n) \neq \emptyset$ so $([f_2],[f_2],\dots,[f_n]) \in \pi^{-1}(x')$ and $\pi^{-1}(x') \neq \emptyset$. Consider the dimension of $\pi^{-1}(x')$. There are two cases:

Case 1: $Z(f_2,\dots,f_n)$ has dimension ≥ 1.

Then by **[SHA p57 Cor5]** $\pi^{-1}(x')$ is isomorphic to $\mathbb{P}(W)$.

Case 2: $Z(f_2,\dots,f_n)$ has dimension 0.

Then $\pi^{-1}(x')$ is isomorphic to a union of finitely many hyperplanes in $\mathbb{P}(W)$. By **[SHA p60 Thm7] Case 2** is generic and $\dim Y - (n-1)\dim\mathbb{P}(W) = \dim\mathbb{P}(W) - 1$.□

We will refer again to [SHA] for results on rational differential forms in the proof of the next proposition.

1.18 Proposition: Suppose that the base field F has characteristic zero. If the symmetric d–multilinear form $\theta: V^d \to F$ is nonsingular, then θ has nonzero Hessian.

Remark: Harrison [HAR2 Prop1.1] proved this proposition in degree d=3.

Proof: Let k be the algebraic closure of the ground field F. With basis and coordinate map $\varphi: V \otimes_F k \to k^n$ as in §1.15 we have a form $p \in k[r_1,\ldots,r_n]$ associated with θ_k as in 1§11.

By §1.15 we must show that if the $\frac{\partial}{\partial r_i} p$ $i=1,\ldots,n$ have no nonzero common zero in k^n, then $\det\left(\frac{\partial^2 p}{\partial r_j \partial r_i} \right) \neq 0$.

Suppose that the $\frac{\partial p}{\partial r_i}$ $i=1,\ldots,n$ have no common zero in $\mathbb{P}^n$. Then we may define a regular map

$$\chi: \mathbb{P}^{n-1} \to \mathbb{P}^{n-1} \text{ by } \chi(x) = \left(\frac{\partial p}{\partial r_1}(x) : \ldots : \frac{\partial p}{\partial r_n}(x) \right).$$

By **[SHA p51 Thm8, remark after Def2 on p49, p60 Thm7]** χ is a finite mapping with image $\mathbb{P}^{n-1}$. The induced homomorphism

$\chi^*: k(\mathbb{P}^{n-1}) \to k(\mathbb{P}^{n-1})$ on the function fields is an embedding. Let $y_1,\ldots,y_n$ be homogeneous coordinates on $\mathbb{P}^{n-1}$. Then

$$\chi^*\left(\frac{y_i}{y_1}\right) = \frac{\partial p}{\partial r_i} \Big/ \frac{\partial p}{\partial r_1}.$$

Let $\Omega(\mathbb{P}^{n-1})$ denote the graded $k(\mathbb{P}^{n-1})$–vector space of rational differential forms on $\mathbb{P}^{n-1}$. By **[SHA Thm1 p167]** $\chi^*: \Omega^{n-1}(\mathbb{P}^{n-1}) \to \Omega^{n-1}(\mathbb{P}^{n-1})$ is an embedding. Therefore

$$0 \neq \chi^*\left(d\left(\frac{y_2}{y_1}\right) \wedge \ldots \wedge d\left(\frac{y_n}{y_1}\right) \right) =$$

$$= d\left(\frac{\partial p}{\partial r_2} \Big/ \frac{\partial p}{\partial r_1} \right) \wedge \ldots \wedge d\left(\frac{\partial p}{\partial r_n} \Big/ \frac{\partial p}{\partial r_1} \right)$$

$$= \det\left(\frac{\partial}{\partial (r_j/r_1)} \left[\frac{\partial p}{\partial r_i} \Big/ \frac{\partial p}{\partial r_1} \right] \right)_{2 \leq i,j \leq n} d\left(\frac{r_2}{r_1}\right) \wedge \ldots \wedge d\left(\frac{r_n}{r_1}\right)$$

and so the above determinant is nonzero.

For $2 \leq j \leq n$ $\frac{\partial}{\partial(r_j/r_1)} = r_1\frac{\partial}{\partial r_j}$ and

$$\frac{\partial}{\partial r_j}\left[\frac{\partial p}{\partial r_i} / \frac{\partial p}{\partial r_1}\right]$$
$$= \left(\frac{\partial^2 p}{\partial r_j \partial r_i}\cdot\frac{\partial p}{\partial r_1} - \frac{\partial p}{\partial r_i}\cdot\frac{\partial^2 p}{\partial r_j \partial r_1}\right) / \left(\frac{\partial p}{\partial r_1}\right)^2.$$

Therefore

$$(*) \qquad \det\left[\frac{\partial^2 p}{\partial r_i \partial r_i}\cdot\frac{\partial p}{\partial r_1} - \frac{\partial^2 p}{\partial r_j r_1}\cdot\frac{\partial p}{\partial r_i}\right] \neq 0$$
$$2 \leq i,j \leq n.$$

We now perform some elementary row and column operations on the Hessian matrix $\left(\frac{\partial^2 p}{\partial r_j \partial r_i}\right)$, $1 \leq i,j \leq n$, after which we will see by (*) that its determinant is nonzero.

Replace the first column C_1 by $1/(d-1).[\, r_1C_1+\ldots+r_nC_n\,]$. Then by Euler's theorem the new first column is $(\ldots\frac{\partial p}{\partial r_i}\ldots)^t$ $i=1,\ldots n$.

Next for $2 \leq i \leq n$ replace the i–th row R_i by $\frac{\partial p}{\partial r_1}.R_i - \frac{\partial p}{\partial r_i}.R_1$ this yields the matrix with $C_1^t = \left(\frac{\partial p}{\partial r_1},0,\ldots,0\right)$ and with (1,1)–minor equal to the matrix of (*) above. The determinant of our new matrix is therefore equal to (*), hence the Hessian of p is nonzero.□

2 In the second part we discuss **indecomposability** and the **centre** of a form. We also consider some examples of symmetric spaces.

Fix the base field F and the integer $d \geq 3$. Let (V,θ) be a symmetric space of degree d over F. We recall Harrison's definitions **[HAR1 p128,p133]**.

2.1 Definitions: The symmetric space (V,θ) is **decomposable** if

$(V,\theta) \cong (V_1,\theta_1) \oplus (V_2,\theta_2)$ with θ_1 and θ_2 nonzero.

Otherwise we say that θ is **indecomposable.**

The **centre** of (V,θ) consists of all endomorphisms f of V for which

$$\theta(\, fv_1,v_2,v_3,\ldots,v_d\,) = \theta(\, v_1,fv_2,v_3,\ldots,v_d)$$

for all $v_1,\ldots,v_d$ in V. We will denote the centre of θ by $\mathbf{Z(\theta)}$.

This centre is an F–subalgebra of End(V) and when θ is nondegenerate the structure of the centre decides the decomposability of θ.

2.2 Harrison **[HAR1 Prop4.1]** proved the following:

Let (V,θ) be nondegenerate. Then

(i) $Z(\theta)$ is a commutative F–algebra.

(ii) $Z(\theta)$ has no idempotents except 0,1 iff (V,θ) is indecomposable.

Since $Z(\theta)$ is a commutative Artin ring by **[AM Thm8.7]**, $Z(\theta)$ is a finite product of commutative Artin local rings. Further, a commutative Artin local ring has a unique prime ideal which is the set of all nilpotent elements in the ring.

It is easy to see that equivalent forms have conjugate centres.

2.3 Proposition: Let $\sigma\epsilon GL(V)$. Then $Z(\sigma.\theta) = \sigma.Z(\theta).\sigma^{-1}$.

Proof: $f\epsilon Z(\sigma.\theta)$ means that for all $v_1,\ldots,v_d \epsilon V$

$$(\sigma.\theta)(fv_1,v_2,\ldots,v_d)=(\sigma.\theta)(v_1,fv_2,\ldots,v_d)$$

$$\Leftrightarrow \theta(\,(\sigma^{-1}f\sigma)\sigma^{-1}v_1,\sigma^{-1}v_2,\ldots,\sigma^{-1}v_d) = \theta(\sigma^{-1}v_1,(\sigma^{-1}f\sigma)\sigma^{-1}v_2,\ldots,\sigma^{-1}v_d)$$

for all $v_1,\ldots,v_d \epsilon V$.

Since $\sigma^{-1}v_1,\ldots,\sigma^{-1}v_d$ run through V as $v1,\ldots,v_d$ do we have:

$f\epsilon Z(\sigma.\theta) \Leftrightarrow \sigma^{-1}f\sigma\epsilon Z(\theta)$.□

Next we show that the centre of a direct sum of nondegenerate forms is the direct product of their centres. Harrison's result that the centre of a nondegenerate decomposable form has idempotents other than 0,1 is an immediate consequence.

2.4 **Proposition:** Suppose that (V,θ) is nondegenerate and

$$(V,\theta) \cong (V_1,\theta_1)\oplus \ldots \oplus(V_k,\theta_k).$$

Then $$Z(\theta) \cong Z(\theta_1)\times \ldots \times Z(\theta_k).$$

Proof: By **Prop2.3** we may assume that $(V,\theta) = (V_1,\theta_1)\oplus \ldots \oplus(V_k,\theta_k)$. Since (V,θ) is nondegenerate so are all the (V_i,θ_i). If the result holds for $k=2$, then an easy induction argument establishes it for all $k\geq 2$. So we assume that $k=2$.

Let $f \epsilon Z(\theta_1 \oplus \theta_2)$. Write f as a matrix w.r.t $V = V_1 \oplus V_2$:

$$f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \text{ where } f_{ij}: V_j \to V_i.$$

We will show that when $i \neq j$ $f_{ij}=0$, and that $f_{ii} \epsilon Z(\theta_i)$ for $i=1,2$.

For all $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \ldots, \begin{bmatrix} x_d \\ y_d \end{bmatrix}$ in $V_1 \oplus V_2$

$$\theta_1(f_{11}x_1 + f_{12}y_1, x_2, \ldots, x_d) + \theta_2(f_{21}x_1 + f_{22}y_1, y_2, \ldots, y_d)$$

$$= \theta_1(x_1, f_{11}x_2 + f_{12}y_2, x_3, \ldots, x_d) + \theta_2(y_1, f_{21}x_2 + f_{22}y_2, y_3, \ldots, y_d).$$

Set $y_1, \ldots, y_d$ all zero to see that $f_{11} \epsilon Z(\theta_1)$. Similarly $f_{22} \epsilon Z(\theta_2)$. To see that $f_{12}=0$, set $x_1=0, y_3=0$ and invoke nondegeneracy of θ_1. Similarly, $f_{21}=0$.□

2.5 Harrison [HAR1 4.3] proved the following: Let $F \to R$ be a base extension by a flat commutative F–algebra R. Let (V,θ) be a nondegenerate symmetric space of degree $d \geq 3$ over F. Let (V_R, θ_R) denote the symmetric space obtained from (V,θ) by extending θ to

$$\theta_R = \theta \otimes 1 : V_R = V \otimes R \to F \otimes R \cong R.$$

Then $Z(\theta_R) \cong Z(\theta) \otimes_F R$.

Our next result is that, roughly speaking, if $\theta: V^d \to F$ is symmetric d–multilinear and nondegenerate, and $f \epsilon Z(\theta)$, then the singular locus of θ over any extension field is invariant under f.

2.6 Proposition: Let (V,θ) be a nondegenerate symmetric space of degree $d \geq 3$. Let $f \epsilon Z(\theta)$, and let K be a field containing F. Then $f \otimes 1(S_K(\theta)) \subseteq S_K(\theta)$.

Proof: Let $a \epsilon S_K(\theta)$. Then for all $x \epsilon V \otimes_F K$ $\theta_K(a, \ldots, a, x) = 0$. By **§2.5** $f \otimes 1 \epsilon Z(\theta_K)$ so for all $x \epsilon V \otimes_F K$

$$\theta_K((f \otimes 1)a, \ldots, (f \otimes 1)a, x)$$

$$= \theta_K(a, \ldots, a, (f \otimes 1)^{d-1} x) = 0.$$

□

2.7 Harrison [**HAR1 p137**] proved that the centre of a nonsingular symmetric space of degree $d \geq 3$ contains no nonzero nilpotents. Suppose that (V,θ) is nonsingular.

Let $(V,\theta) \cong (V_1,\theta_1) \oplus \ldots \oplus (V_k,\theta_k)$

be the decomposition of (V,θ) as a sum of indecomposables. Then each (V_i,θ_i) is nonsingular and by **2.4**

$$Z(\theta) \cong Z(\theta_1) \times \ldots \times Z(\theta_k).$$

Each $Z(\theta_i)$ is a commutative Artin local ring with no nilpotents. The unique prime ideal of $Z(\theta_i)$ is $\{0\}$, so $Z(\theta_i)$ is a field.

We close this part with some examples of symmetric spaces, of which we will compute their centres to decide their indecomposability. We will conclude that there are indecomposable symmetric spaces in each degree $d \geq 3$ and each dimension. This is in marked contrast to the degree 2 case, where we have diagonalizability.

Examples:

(i) Let A be an associative finite dimensional F–algebra with unity. Let $T: A \to F$ be F–linear and such that

$T(ab)$ is a nondegenerate symmetric bilinear form.

For example:

(a) $A=K$ with $K|F$ a finite seperable field extension and $T=\mathrm{Trace}_{K|F}$. Harrison's example [**HAR1 Prop4.5**] is more general.

(b) $A = M_n(F)$ and $T=\mathrm{trace}$.

Define (A,θ) by $\theta(a_1,\ldots,a_d) = T(\Sigma a_{\pi 1} \cdots a_{\pi d})$

the sum over all permutations π of $1,\ldots,d$.

An easy check shows that θ is symmetric and d–multilinear.

***θ* is also nondegenerate:** For suppose that $\theta(a_1,\ldots,a_d)=0$ for all $a_2,\ldots,a_d$.

Set $a_3=\ldots a_d=1$. By symmetry and nondegeneracy of $T(ab)$, and since $d! \neq 0$, it follows that $a_1=0$.

Consider the centre of θ.

Claim: Let $f\epsilon Z(\theta)$. Then $fa = ac+ca$ where $c\epsilon A$ commutes with all elements of $[A,A] = \{xy-yx: x,y\epsilon A\}$.

Proof: For all $a_1,\ldots,a_d$ in A

$$\theta(fa_1,a_2,a_3,\ldots,a_d) = \theta(a_1,fa_2,a_3,\ldots,a_d)$$

Set $a_4=\ldots=a_d=1$. Since $T(ab)$ is symmetric we have

$$\begin{aligned}
&\theta(fa_1,a_2,a_3,1,\ldots,1) \\
&= (d-3)!\{ 3T[(fa_1)a_2a_3] + 3T[a_2(fa_1)a_3] \} \\
&= (d-3)!\{ 3T[a_1(fa_2)a_3] + 3T[(fa_2)a_1a_3] \} = \theta(a_1,fa_2,a_3,1,\ldots,1).
\end{aligned}$$

As $d!\neq 0$ we have for all a_1,a_2,a_3 in A

(*) $T[(fa_1)a_2a_3 + a_2(fa_1)a_3] = T[a_1(fa_2)a_3 + (fa_2)a_1a_3]$.

By nondegeneracy of $T(ab)$ we have: for all a_1,a_2 in A

$$(fa_1)a_2 + a_2(fa_1) = a_1(fa_2) + (fa_2)a_1.$$

Set $a_2=1$ and put $(1/2)f(1) = c$. Then $fa = ac+ca$. Substituting in (*) we get after some cancelation,

$c[a_1,a_2]=[a_1,a_2]c$ and we have proved the claim.□

In case (i)(a) $A = K$ is commutative and it is clear from the **Claim** that $Z(\theta)=K$.

In case (i)(b) $A = M_n(F) = \mathfrak{gl}n(F)$, as a Lie algebra and we have $[A,A] = \mathfrak{sl}n(F)$, the matrices with zero trace. The subring of $M_n(F)$ generated by the matrices with zero trace is the full ring $M_n(F)$. Since the centre of $M_n(F)$ is F, we have $Z(\theta)=F$.

Since these θ are all nondegenerate we conclude that each θ is indecomposable.

2.8 Remark: There are degree $d\geq 3$ indecomposable symmetric spaces in each dimension.

2.9 Example(ii): Let $W_1=V$, $W_2=\bigodot^{d-1}(V)$ where $d\geq 3$. We define the hyperbolic degree d symmetric space $(W_1\oplus W_2, \psi)$ by

$$\psi[(x_1,\theta_1),\ldots,(x_d,\theta_d)] = \sum_{i=1}^{d} \theta_i(x_1,\ldots,\hat{x}_i,\ldots,x_d)$$

as in §1.10, where we checked that ψ is symmetric d–multilinear and nondegenerate.

2.10 Next we compute the centre of ψ. Write

$f\in Z(\psi)$ as a matrix $\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ of linear maps $f_{ij}\colon W_j \to W_i$.

By §2.6 $f(S_F(\psi)) \subseteq S_F(\psi)$. Since $W_2 = S_F(\psi)$, we must have $f_{12}=0$. As

$$\psi[f\begin{bmatrix}0\\ \theta\end{bmatrix},\begin{bmatrix}x_2\\ 0\end{bmatrix},\begin{bmatrix}x_3\\ 0\end{bmatrix},\ldots,\begin{bmatrix}x_d\\ 0\end{bmatrix}] = \psi[\begin{bmatrix}0\\ \theta\end{bmatrix}, f\begin{bmatrix}x_2\\ 0\end{bmatrix},\begin{bmatrix}x_3\\ 0\end{bmatrix},\ldots,\begin{bmatrix}x_d\\ 0\end{bmatrix}]$$

we have

(*) $\quad (f_{22}\theta)(x_2,\ldots,x_d) = \theta(f_{11}x_2,x_3,\ldots,x_d)$.

As $\psi[f\begin{bmatrix}x_1\\ 0\end{bmatrix},\begin{bmatrix}x_2\\ 0\end{bmatrix},\begin{bmatrix}x_3\\ 0\end{bmatrix},\ldots,\begin{bmatrix}x_d\\ 0\end{bmatrix}] = \psi[\begin{bmatrix}x_1\\ 0\end{bmatrix},f\begin{bmatrix}x_2\\ 0\end{bmatrix},\begin{bmatrix}x_3\\ 0\end{bmatrix},\ldots,\begin{bmatrix}x_d\\ 0\end{bmatrix}]$

we have $(f_{21}x_1)(x_2,\ldots,x_d) = (f_{21}x_2)(x_1,x_3,\ldots,x_d)$.

Therefore the d–multilinear form $\alpha\colon V^d \to F$ defined by:

$$\alpha(x_1,\ldots,x_d) = (f_{21}x_1)(x_2,\ldots,x_d)$$

is symmetric and determines f_{21}.

Since $\psi[f\begin{bmatrix}x_1\\ 0\end{bmatrix},\begin{bmatrix}x_2\\ 0\end{bmatrix},\ldots,\begin{bmatrix}x_{d-1}\\ 0\end{bmatrix},\begin{bmatrix}0\\ \theta\end{bmatrix}]$ is symmetric in x_1, x_2 we have

$$\theta(f_{11}x_1,x_2,\ldots,x_{d-1}) = \theta(x_1,f_{11}x_2,x_3,\ldots,x_{d-1})$$

for all $x_1,\ldots,x_{d-1}$ in V, and all $\theta\in\bigodot^{d-1}(V)$.

If $d-1\geq 3$ we can take θ as in Example (i) and conclude that f_{11} is multiplication by a scalar. We are left with the case $d-1=2$. In this case we have θ corresponding to some symmetric matrix B, and for all x_1, x_2 in V

$$(f_{11}x_1)^tBx_2 = x_1^tB(f_{11}x_2) \text{ hence } f_{11}^tB = Bf_{11}.$$

Taking $B=I$ we get $f_{11}^t=f_{11}$. So f_{11} is a symmetric matrix which commutes with all symmetric matrices, thus f_{11} is a scalar in this case also.

From (*) above we see that $f_{22} = f_{11}$ a scalar.

Given $\lambda \epsilon F$ and $\alpha \epsilon \bigodot^d(V)$, define

$$f\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda\theta + \alpha(x,-,\dots,-) \end{bmatrix}.$$

We have seen that every element of $Z(\psi)$ must be of this form, and now we check that $f \epsilon Z(\psi)$:

$$\psi[\, f\begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix}, \dots, \begin{bmatrix} x_d \\ \theta_d \end{bmatrix}\,]$$

$$= \lambda\theta_1(x_2,\dots,x_d) + \alpha(x_1,x_2,\dots,x_d) + \sum_{i=2}^{d} \lambda\theta_i(x_1,\dots,\hat{x}_i,\dots,x_d)$$

$$= \lambda \sum_{i=1}^{d} \theta_i(x_1,\dots,\hat{x}_i,\dots,x_d) + \alpha(x_1,\dots,x_d)$$

which is symmetric in $x_1,\dots,x_d$. Therefore $f \epsilon Z(\psi)$.

As an F–vector space the centre of ψ is $F \oplus \bigodot^d(V)$ and the multiplication is

$$\begin{bmatrix} \lambda & 0 \\ \theta & \lambda \end{bmatrix}\begin{bmatrix} \mu & 0 \\ \varphi & \mu \end{bmatrix} = \begin{bmatrix} \lambda\mu & 0 \\ \theta\mu + \lambda\varphi & \lambda\mu \end{bmatrix}.$$

So the centre of ψ is $F * \bigodot^d(V)$, the trivial extension of F by $\bigodot^d(V)$, which is the unique prime ideal of the centre.

3 In this part we introduce the concept of **orthogonality** in a symmetric space. **Split** symmetric spaces are those with a self–orthogonal subspace. We define the tensor product of two symmetric spaces of degree d.. Harrison explains in **[HAR1 §§2,3]** how the Grothendieck ring of forms of degree d over F is constructed from the set of isomorphism classes of nondegenerate symmetric spaces of degree d, by means of the direct sum **(1§17)** and the tensor product **(§3.2)**. The elements of this Grothendieck ring are formal differences of isomorphism classes of nondegenerate symmetric spaces of degree d. The significance of **Prop3.3** is that the set of formal differences of isomorphism classes of nondegenerate **split** symmetric spaces is an ideal in the Grothendieck ring.

Fix the base field F and the integer $d \geq 3$, and suppose that $d! \neq 0$ in F. Let (V,θ) be a symmetric space of degree d over F.

3.1 Let W be a subspace of V. Harrison **[HAR1 p128]** defined the **orthogonal** subspace $W^{\perp}$ to W by

$$W^{\perp} = \{ x \in V: \ \theta(w,x,x_3,\ldots,x_d)=0 \quad \text{for all } \ w \in W \ \text{ and all } \ x_3,\ldots,x_d \in V \ \}$$

A symmetric space (V,θ) is **split** if there is a subspace W of V with $W^{\perp} = W$.

Remark: This definition of a split space is an extension of the field case of the definition of split symmetric bilinear forms, due to Knebusch, **[MH p12]** to degrees ≥ 3.

None of the spaces in the example (i)(a) above is split as

$a_1 \perp a_2$ means that $T(a_1a_2a_3\ldots a_d) = 0$ so $a_1a_2=0$ on putting $a_4=\ldots=a_d=1$ and by nondegeneracy of $T(ab)$.

In degree 2 over a field of characteristic $\neq 2$

(i) the split spaces coincide with the hyperbolic spaces

(ii) the direct sum of nondegenerate split spaces is split, and the tensor product of a nondegenerate space with a nondegenerate split space is split.

The hyperbolic space $(V \oplus \odot^d(V) ,\psi)$ of §1.10 has $\odot^d(V)$ self–orthogonal :

$\psi[(0,\theta),(0,\theta_2),(x_3,\theta_3),\ldots,(x_d,\theta_d)] =0$; and

$\psi[(x,\theta),(0,\theta_2),(x_3,\theta_3),\ldots,(x_d,\theta_d)] = 0$ for all $\theta_2,\ldots,\theta_d$, and all $x_3,\ldots x_d$ just means that

$\theta_2(x,x_3,\ldots,x_d)=0$ for all θ_2 and all $x_3,\ldots,x_d$, so we can take θ_2 nondegenerate to get $x = 0$.

In degree $d \geq 3$, the split forms are not all hyperbolic. We give an example to show this.

Example: Let $e_1,\ldots,e_n$ be the usual basis of F^n. Define (F^n,θ) by

$\theta(e_{\alpha 1},\ldots,e_{\alpha d}) = 0$

unless some $\alpha(i)=1$ and, for some $2\leq k\leq n$, all the other $\alpha(i)=k$ in which case

$\theta(e_1,e_k,\ldots,e_k) = 1/d$.

The corresponding form of degree d is

$p = x_1x_2^{d-1} + \ldots + x_1x_n^{d-1}$. In the Hessian matrix $\left(\frac{\partial^2 p}{\partial x_j \partial x_i}\right)$ the first row is

$(d-1).(0,x_2^{d-2},\ldots,x_n^{d-2})$;

the first column is the transpose of this;

the diagonal is $(d-1)(d-2)x_1.(0,x_2^{d-3},\ldots,x_n^{d-3})$;

and all other entries are 0. Let us calculate the Hessian determinant by expanding along the first row:

$$H(p;x) = x_1^{n-1}.\sum_{i=2}^{n} \pm(x_i^{d-2})^2(x_2\ldots\hat{x}_i\ldots x_n)^{d-3}.$$

Since $2(d-2)>d-3$ the monomials in this sum are all distinct and $H(p;x)\neq 0$. We conclude by §1.9 and §1.12 that θ is nondegenerate, and not hyperbolic.

$\theta(e_1,e_1,x_3,\ldots,x_d)=0$ for all $x_3,\ldots,x_d$ in V, so $Fe_1 \subseteq (Fe_1)^{\perp}$.

Suppose that $x = a_1e_1+\ldots+a_ne_n \in (Fe_1)^{\perp}$.

This means that $\theta(e_1,x,x_3,\ldots,x_d) = 0$ for all $x_3,\ldots,x_d$ in F^n.

For $i\geq 2$ set $x_3=\ldots=x_d=e_i$ to get $a_i=0$.

Therefore $(Fe_1)^{\perp} = Fe_1$.

3.2 Let (V,θ) , (W,φ) be symmetric spaces of degree d over F. We will define the symmetric space $(V\otimes W,\theta\otimes\varphi)$ also of degree d over F. Let $v_1,\ldots,v_n$ be a basis of V and let $w_1,\ldots,w_m$ be a basis of W. Then the set of $v_i\otimes w_j$ for $1\leq i\leq n$, $1\leq j\leq m$ is a basis of $V\otimes W$. For each map $\alpha:\{1,\ldots,d\}\to\{1,\ldots,n\}$ and each map $\beta:\{1,\ldots,d\}\to\{1,\ldots,m\}$ let

$$\theta\otimes\varphi(v_{\alpha 1}\otimes w_{\beta 1},\ldots,v_{\alpha d}\otimes w_{\beta d}) = \theta(v_{\alpha 1},\ldots,v_{\alpha d}).\varphi(w_{\beta 1},\ldots,w_{\beta d}).$$

This extends uniquely to a d–multilinear map

$\theta\otimes\varphi$: $(V\otimes W)^d \to F$ such that for $x_1,\ldots,x_d$ in V, and $y_1,\ldots,y_d$ in W

$\theta\otimes\varphi(\, x_1\otimes y_1,\ldots,x_d\otimes y_d \,) = \theta(x_1,\ldots,x_d).\varphi(y_1,\ldots,y_d).$

It is easy to check that this tensor product operation is compatible with the relation of isomorphism.

Next we check that when θ and φ are both nondegenerate so also is $\theta\otimes\varphi$. This is the field case of Harrison's more general result **[HAR1 p128 second paragraph]**. For suppose

$z = \Sigma\, a_{ij}v_i\otimes w_j$ has $(\theta\otimes\varphi)_z = 0.$

Then for all $x_2,\ldots,x_d \in V$, and all $y_2,\ldots,y_d \in W$ we have

$$\begin{aligned} 0 &= \theta\otimes\varphi(\, \Sigma a_{ij}v_i\otimes w_j, x_2\otimes y_2,\ldots,x_d\otimes y_d \,) \\ &= \Sigma\, a_{ij}\,\theta(v_i,x_2,\ldots,x_d)\varphi(w_j,y_2,\ldots,y_d) \\ &= \varphi(\, \Sigma_j[\, \Sigma_i a_{ij}\,\theta(v_i,x_2,\ldots,x_d)\,]w_j,y_2,\ldots,y_d \,). \end{aligned}$$

Since φ is nondegenerate and $w_1,\ldots,w_m$ are linearly independent

$$\Sigma_i a_{ij}\,\theta(v_i,x_2,\ldots,x_d) = 0 \quad \text{for } j=1,\ldots,m;$$

$$\text{i.e}\quad \theta(\Sigma_i a_{ij}v_i,x_2,\ldots,x_d) = 0 \quad \text{for } j=1,\ldots,m.$$

Since θ is nondegenerate and $v_1,\ldots,v_n$ are linearly independent each $a_{ij}=0$.

Harrison **[HAR1 Prop4.2]** proved that when (V,θ) and (W,φ) are nondegenerate then $Z(\theta\otimes\varphi) \cong Z(\theta)\otimes Z(\varphi)$.

Harrison **[HAR1 Prop2.1]** showed that with $\oplus$ as in **1§17** and $\otimes$ just defined, the Grothendieck construction may be applied to obtain a ring whose elements are formal differences of isomorphism classes of nondegenerate symmetric spaces of degree d. A consequence of the next proposition is that the set of formal differences of isomorphism classes of nondegenerate split symmetric spaces is an ideal in this ring.

3.3 Proposition: Let d be an integer ≥ 3. Then

(i) the direct sum of two nondegenerate split symmetric spaces of degree d is split;

(ii) the tensor product of a nondegenerate split symmetric space of degree d and a nondegenerate symmetric space of degree d is split.

Proof: (i) Suppose that (V_i,θ_i) i=1,2 are split. For i=1,2 there are subspaces M_i of V_i with $M_i^{\perp} = M_i$. We will show that $(M_1 \oplus M_2)^{\perp} = (M_1 \oplus M_2)$. For i=1,2 let $m_i,n_i \in M_i$; $v_{i3},\ldots,v_{id} \in V_i$. Then

$$(\theta_1 \oplus \theta_2)[\,(m_1,m_2),(n_1,n_2),(v_{13},v_{23}),\ldots,(v_{1d},v_{2d})\,]$$

$$= \theta_1(m_1,n_1,v_{13},\ldots,v_{1d}) + \theta_2(m_2,n_2,v_{23},\ldots,v_{2d}) = 0, \text{ since } M_i \subseteq M_i^{\perp} \text{ for } i=1,2.$$

Therefore $M_1 \oplus M_2 \subseteq (M_1 \oplus M_2)^{\perp}$.

Suppose $(v_1,v_2) \in (M_1 \oplus M_2)^{\perp}$. Then for $m_1 \in M_1$; $v_3,\ldots,v_d \in V_1$ we have

$$0 = (\theta_1 \oplus \theta_2)[\,(v_1,v_2),(m_1,0),(v_3,0),\ldots,(v_d,0)\,]$$

$$= \theta_1(v_1,m_1,v_3,\ldots,v_d) \text{ so } v_1 \in M_1^{\perp} = M_1. \text{ Similarly } v_2 \in M_2. \square$$

(ii) Suppose that (V,θ) is a nondegenerate split symmetric space of degree d, having a self–orthogonal subspace M. Let (W,φ) be a nondegenerate symmetric space of degree d. We will show that the tensor product of these symmetric spaces has $M \otimes W$ self–orthogonal.

First we will show that $M \otimes W \subseteq (M \otimes W)^{\perp}$. By multilinearity of $\theta \otimes \varphi$ and since the $m \otimes w$ with $m \in M, w \in W$ span $M \otimes W$; we need only calculate that for

$$m_1,m_2 \in M;\ v_3,\ldots,v_d \in V;\ w_1,w_2,\ldots,w_d \in W$$

$$\theta \otimes \varphi(m_1 \otimes w_1, m_2 \otimes w_2, v_3 \otimes w_3,\ldots,v_d \otimes w_d) = \theta(m_1,m_2,v_3,\ldots,v_d)\varphi(w_1,\ldots,w_d) = 0$$

since $M \subseteq M^{\perp}$.

Secondly, we show that $(M\otimes W)^{\perp} \subseteq M\otimes W$. Let $w_1,\dots,w_n$ be a basis of W. Suppose that $\sum_{i=1}^{n} v_i\otimes w_i \in (M\otimes W)^{\perp}$. Then for $m\in M$; $v'_3,\dots,v'_d\in V$; and any map $\alpha:\{2,\dots,d\}\to\{1,\dots,n\}$

$$\begin{aligned}
0 &= \theta\otimes\varphi\left(\sum_{i=1}^{n} v_i\otimes w_i\ , m\otimes w_{\alpha 2}, v'_3\otimes w_{\alpha 3},\dots,v'_d\otimes w_{\alpha d}\right)\\
&= \sum_{i=1}^{n} \theta(v_i,m,v'_3,\dots,v'_d).\varphi(w_i,w_{\alpha 2},\dots,w_{\alpha d})\\
&= \varphi\left(\sum_{i=1}^{n} \theta(v_i,m,v'_3,\dots,v'_d).w_i\ , w_{\alpha 2},\dots,w_{\alpha d}\right).
\end{aligned}$$

Since φ is nondegenerate we conclude that

$$\sum_{i=1}^{n} \theta(v_i,m,v'_3,\dots,v'_d).w_i = 0.$$

Now $w_1,\dots,w_n$ is a basis of W; $m\in M$ and $v'_3,\dots,v'_d \in V$ are arbitrary.

Therefore all the v_i are in $M^{\perp} = M$ so $\Sigma v_i\otimes w_i \in M\otimes W$.□

The final result of this chapter is an application of our calculation of the centres of the hyperbolic forms. We show that in the monoid of isomorphism classes of nondegenerate symmetric spaces of a given degree d, with the tensor product operation; the submonoid generated by the isomorphism classes of the hyperbolic spaces of degree d is free.

3.4 Proposition: Let $(F^n\oplus[\ \odot^{d-1}(F^n)\ ],\psi_n)$ be the hyperbolic symmetric space defined in §1.10. If $\psi_{m_1}\otimes\dots\otimes\psi_{m_q} \cong \psi_{n_1}\otimes\dots\otimes\psi_{n_p}$ with $m_1\leq\dots\leq m_q$ and $n_1\leq\dots\leq n_p$, then $p=q$ and $m_i=n_i$, for all $1\leq i\leq p$.

Proof: By Harrison's result [HAR1 Prop4.2] and §2.10

$$Z(\ \psi_{n_1}\otimes\dots\otimes\psi_{n_p}\ ) \cong \bigotimes_{i=1}^{p} Z(\psi_{n_i}) \cong \bigotimes_{i=1}^{p} [F^*\odot^{n_i}(F^{n_i})].$$

We will prove more generally that if

$$\bigotimes_{i=1}^{p} (F^*V_i) \cong \bigotimes_{j=1}^{q} (F^*W_j)$$

with $\dim_F V_1\leq\dots\leq\dim_F V_p$ and $\dim_F W_1\leq\dots\leq\dim_F W_q$, then $p=q$ and $\dim_F V_i=\dim_F W_i$ for all i.

Let $A_i = F^*V_i$. Then A_i is a $\mathbb{Z}$–graded F–algebra with

$A_{i_0} = F$, $A_{i_1} = V_i$; and $A_{i_j} = 0$ in all other grades j.

Let $B = \bigotimes_{i=1}^{p} A_i$.

Being a tensor product of $\mathbb{Z}$–graded F–algebras, B is a $\mathbb{Z}$–graded F–algebra. As an F–vector space

$$B = \oplus (A_{1_{\alpha 1}} \otimes \ldots \otimes A_{p_{\alpha p}})$$

where the direct sum is over all maps α: $\{1,\ldots,p\} \to \{0,1\}$.

The $\mathbb{Z}$–grading is $B_j = \oplus (A_{1_{\beta 1}} \otimes \ldots \otimes A_{p_{\beta p}})$

where the direct sum is taken over all maps which take the value 1 exactly j times.

Note that:

(i) In B $x_{1_{\alpha 1}} \otimes \ldots \otimes x_{p_{\alpha p}} \cdot y_{1_{\beta 1}} \otimes \ldots \otimes y_{p_{\beta p}} = 0$

iff for some i: $\alpha i = \beta i = 1$ or $x_{i_{\alpha i}} = 0$ or $y_{i_{\beta i}} = 0$.

(ii) B is zero in grade greater than p.

(iii) B is generated as an F–algebra by its grade 1 elements.

(iv) B is a local ring with maximal ideal M, say.

(v) For $k \geq 1$ M^k is spanned by the homogeneous elements of B of grade $\geq k$.

(vi) For $k \geq 1$ $\dim_F M^k/M^{k+1}$

$$= \sum_{1 \leq i_1 < \ldots < i_k \leq p} n_{i_1} \ldots n_{i_k} \text{, where } n_j = \dim_F V_j$$

= the coefficient of t^{p-k} in $(t+n_1)\ldots(t+n_p)$.

Suppose that C is the tensor product of the F^*W_i for $1 \leq i \leq q$, where W_i is an F–vector space of dimension m_i. Then C is a local ring with maximal ideal N, say.

If $B \cong C$ then for each nonnegative integer k $M^k/M^{k+1} \cong N^k/N^{k+1}$ so that

$(t+n_1)\ldots(t+n_p) = (t+m_1)\ldots(t+m_q)$. □

CHAPTER THREE

In this chapter we study **families of additive invariants** of the forms of a given degree d over a field of characteristic zero. Our main result is **2.7** that there are no nontrivial families of additive invariants when the degree is odd.

In **2.6** we give an example of a nontrivial family of additive invariants in each even degree. We have not attempted to find all families of additive invariants in even degree.

First we summarise the required background material on invariants of a form.

1 We give a brief account of the symbolic method for describing invariants of a single form of homogeneous degree d in $x_1,\dots,x_n$. This is the theory of Grosshans et al **[GRS]** in the special case of $L=L^-$ with all the letters of L "belonging to" $S_d(V^*)$. In this case $\mathrm{Super}[L|P] = \mathrm{Sym}[L|P]$. The underlying algebra of this Hopf–algebra is the algebra of polynomials in the $(a|i)$ where $a\epsilon L$, $i\epsilon P$. This is the letter–place algebra of Desarmenien et al **[DKR p66]**.

1.1 Let the ground field F have characteristic zero, Grosshans et al **[GRS bottom p50]** comment on the necessity of this condition. By **1§21(ii)**, if U is a finite dimensional F–vector space, we can identify the F–algebra $S(U^*)$ with the F–algebra of polynomial maps

φ: U→ F.

Let V be a vector space over F with basis $v_1,\dots,v_n$ and let $x_1,\dots,x_n$ be the dual basis of V^*. Let $W = S_d(V^*)$. Let $F[W]$ denote the F–algebra of polynomial maps

φ: W→ F.

We define what is meant by an **invariant** of the form of degree d.

1.2 GL(V) acts on the left on F[W] as follows: for $\varphi\epsilon F[W]$ and $\sigma\epsilon GL(V)$

$(\sigma.\varphi)(p) = \varphi(\sigma^{-1}.p) = \varphi(\, S\,(\sigma^*)p\,)$.

We are interested in the **GL(V)–invariants**: $\varphi \epsilon F[W]$ such that for some integer g

$\sigma.\varphi = (\det\sigma)^g\varphi$ for all $\sigma\epsilon GL(V)$. We call g the **weight** of φ.

We describe the GL(V)–action on F[W].

1.3 The monomials $x_1^{i_1}\dots x_n^{i_n}$ of degree d form a basis of $S_d(V^*)$. Given $f\epsilon S_d(V^*)$ we have

$$f = \sum_i \frac{d!}{i_1!\dots i_n!} t_{i_1\dots i_n}(f)x_1^{i_1}\dots x_n^{i_n},\ \text{and the } t_{i_1\dots i_n} \text{ form a basis of } W^*.$$

Since F is infinite we can, by **1§20**, identify F[W] with the F–algebra of polynomials in these $t_{i_1\dots i_n}$.

Given a map α: $\{1,\dots,d\}\to\{1,\dots,n\}$ we define $t_\alpha = t_{i_1\dots i_n}$ where $(i_1,\dots,i_n)$ is the **content** of α (1§4). Setting $x_\alpha = x_{\alpha 1}\dots x_{\alpha d}$ we can express

$f = \sum_\alpha t_\alpha(f)x_\alpha$ which is the image under d–fold multiplication of the symmetric tensor

$$\sum_\alpha t_\alpha(f)x_{\alpha 1}\otimes\dots\otimes x_{\alpha d}.$$

1.4 To calculate $\sigma.t_\beta$ we must first compute the action of σ^{-1} on $f\epsilon S_d(V^*)$. We have

$$\sigma^{-1}.f = S_d(\sigma^*)(\sum_\alpha t_\alpha(f)x_\alpha)$$
$$= \sum_\alpha t_\alpha(f)\sum_\beta \sigma_{\alpha\beta}x_\beta \quad (\text{ where}$$
$$\sigma_{\alpha\beta} = \sigma_{\alpha 1\beta 1}\cdots\sigma_{\alpha d\beta d})$$
$$= \sum_\beta(\sum_\alpha \sigma_{\alpha\beta}t_\alpha)(f)x_\beta.$$

Now we check that the above coefficients of the x_β are symmetric. For any permutation π of 1,...,d

$$\sum_\alpha \sigma_{\alpha,\beta\circ\pi}t_\alpha = \sum_\alpha{}' \sigma_{\alpha\circ\pi,\beta\circ\pi}t_{\alpha\circ\pi}$$
$$= \sum_\alpha \sigma_{\alpha\beta}t_\alpha \quad \text{recalling the definition of } \sigma_{\alpha\beta}.$$

Therefore $\sigma^{-1}.f = \sum_i \frac{d!}{i_1!\dots i_n!}[\sum_\alpha \sigma_{\alpha\beta}t_\alpha]x_1^{i_1}\dots x_n^{i_n}$ where β is arbitrary with content i.

Hence $\sigma.t_\beta = \sum_\alpha \sigma_{\alpha\beta}t_\alpha$.

1.5 We define the **letter–place algebra**. Let L be a countably infinite set whose elements will be called **letters**. Let $P = \{1,\dots,n\}$ where $n=\dim_F V$. The elements of P will be called **places**. Let $[L|P] = \{ (a|i) : a\epsilon L, i\epsilon P \}$. Elements of $[L|P]$ will be called **letter–places.**

Let **Sym[L|P]** denote the graded Hopf–algebra of polynomials in the letter–places with coefficients in F.

1.6 We describe an action of $GL_n(F)$ on the letter–place algebra. Let $GL_n(F)$ act on Sym[L|P] as graded Hopf–algebra automorphisms such that

$$\text{for } \sigma\epsilon GL_n(F) \text{ and } (a|j)\epsilon[L|P]\text{: } \sigma.(a|j) = \sum_{i=1}^{n} \sigma_{ij}(a|i).$$

1.7 We define the **umbral linear functional** U: $\mathrm{Sym}[L|P]\to F[W]$. The image of φ under **U will be denoted $<U,\varphi>$ and is calculated as follows:**

(i) **U is F–linear.**

(ii) **On a monomial of the form $\varphi = (a|1)^{i_1}\dots(a|n)^{i_n}$**

$<U,\varphi> = 0$ unless the degree $i_1+\dots+i_n=d$ in which case

$<U,\varphi> = t_{i_1\dots i_n}$.

(iii) On the general monomial φ, we write $\varphi = \varphi_1\dots\varphi_r$ such that for distinct letters $a_1,\dots,a_r$ each φ_i is a monomial in the $(a_i|1),\dots,(a_i|n)$ and then

$<U,\varphi> = <U,\varphi_1>\dots<U,\varphi_r>$ where each factor $<U,\varphi_i>$ is calculated as in (ii) above.

The letters of L are called in, 19–th century terminology, **equivalent symbols** since (ii) holds for all letters a.

1.8 It follows easily from §1.4 and §1.6 that the umbral linear functional is a $GL_n(F)$–map.

Since the umbral linear functional is clearly surjective, and $GL_n(F)$ is completely reducible; (quoting from Grosshans et al **[GRS p49]**) it follows that an invariant in F[W] is the image under the umbral linear functional of an invariant in the letter–place algebra Sym[L|P].

1.9 The Standard Basis Theorem yields an explicit basis for Sym[L|P] as a vector space over F, given linear orderings of the sets L and P. The $GL_n(F)$–invariants of Sym[L|P] can then be specified in terms of this basis. We assume from now on that L is linearly ordered, and that P is linearly ordered by $1<\ldots<n$.

To state the Standard Basis Theorem we need to:

(i) Define the **Laplace pairing**

$\mathrm{Ext}(L)\times\mathrm{Ext}(P)\to \mathrm{Sym}[L|P]$ denoted $(w,u)\mapsto (w|u)$

where Ext(L), Ext(P) denote the exterior algebras generated by L, P respectively.

(ii) Explain what is meant by a **standard Young diagram** with entries from a linearly ordered set.

1.10 The **Laplace pairing** is defined by requiring

(a) bilinearity

(b) if $w\in\mathrm{Ext}(L)$, $u\in\mathrm{Ext}(P)$ are monomials of different lengths then $(w|u) = 0$

(c) if $w=a_1\ldots a_k$, with $a_1,\ldots,a_k \in L$ and $u=j_1\ldots j_k$, with $j_1,\ldots,j_k \in P$ then

$(w|u) = \det(\, (a_p|j_q)\,)$.

1.11 The Laplace expansions (see Doubilet et al **[DRS p194]**) for calculating a determinant can be concisely stated in terms of comultiplication in the Hopf–algebras Ext(L) and Ext(P). See Akin et al **[ABW I.2]** for details of the comultiplication on an exterior algebra.

We state the relevant results of Grosshans et al **[GRS 2§2]** in our special case.

Let w,u be monomials of the same length in Ext(L),Ext(P) respectively.

(i) Suppose that $u=u'u''$, and $\Delta w = \Sigma\, w_{(1)} \otimes w_{(2)}$. Then

$(w|u'u'') = \Sigma\, \mathrm{sgn}(|w_{(2)}|.|u'|)(w_{(1)}|u')(w_{(2)}|u'')$ where $|\_|$ denotes the length of a monomial, and for $n \in \mathbb{Z}$ $\mathrm{sgn}(n)=(-1)^n$.

(ii) Similarly, if $w = w'w''$, and $\Delta u = \Sigma u_{(1)} \otimes u_{(2)}$ then

$(w'w''|u) = \Sigma\, \mathrm{sgn}(|w''|.|u_{(1)}|)(w'|u_{(1)})(w''|u_{(2)})$.

1.12 Let A be a linearly ordered set. A **Young diagram** D on A is a sequence of words $(w_1,\dots,w_g)$ in the alphabet A with $\lambda_1=\mathrm{length}(w_1)\geq\dots\geq\lambda_g=\mathrm{length}(w_g)$. We then say that D has **shape** $(\lambda_1,\dots,\lambda_g)$. If $w_i = x_{i1}\dots x_{i\lambda_i}$ we picture D as:

$$\begin{array}{l} x_{11}x_{12}\cdots\cdots x_{1\lambda_1} \\ x_{21}x_{22}\cdots x_{2\lambda_2} \\ \quad\vdots \\ x_{g1}\cdots x_{g\lambda_g} \end{array}$$

We call $x_{i1}x_{i2}\dots x_{i\lambda_i}$ the i–th row of D and we call

$$\begin{array}{l} x_{1j} \\ x_{2j} \\ \vdots \end{array} \quad \text{the } j\text{–th column of D.}$$

We say that D is **standard** if:

(i) when a precedes b in the same row of D then $a<b$;

(ii) when a precedes b in the same column of D then $a\leq b$.

1.13 Definition: Let $D = (w_1,\dots,w_g)$ and $E = (u_1,\dots u_g)$ be Young diagrams of the same shape in the alphabets L, P respectively. We define **Tab(D|E)** ϵ Sym[L|P] by $\mathrm{Tab}(D|E) = (w_1|u_1)\dots(w_g|u_g)$.

1.14 The **Standard Basis Theorem** asserts that the Tab(D|E) where D, E are standard Young diagrams of the same shape in the alphabets L, P respectively, constitute a basis for Sym[L|P] as an F–vector space.

1.15 We can now describe the $GL_n(F)$–invariants of Sym[L|P]: a basis for the F–vector space of these invariants is given by the Tab(D|E) where E has , say, g rows all equal to 12.....n and D is standard of the same shape as E. This Tab(D|E) has weight g.

1.15.1 **Remark:** Unless each letter in D occurs exactly d times, by the definition of the umbral linear functional in §1.7, we have $<U,Tab(D,E)> = 0$.

1.16 By §§1.8,1.14,1.15 a $GL_n(F)$–invariant $\varphi \epsilon F[W]$ of weight g is a linear combination of the $<U, Tab(D|E)>$ with D|E as in §1.15.

2 We now discuss families of additive invariants of forms of degree d. Let the ground field have characteristic zero.

2.1 **Definition:** Suppose that for each positive integer n I_n is an invariant of the form of degree d in $x_1,\dots,x_n$. The family of invariants $I_1,I_2,I_3,\dots$ is said to be **additive** if when f (resp g) is a form of degree d in $x_1,\dots,x_p$ (resp $x_1,\dots,x_q$) then

$$I_{p+q}(f\perp g) = I_p(f).I_q(g).$$

The family of discriminants of the quadratic forms is the most familiar example.

2.2 We will consider invariants I of the form of degree d in $x_1,\dots,x_p,x_{p+1},\dots,x_{p+q}$ and calculate

$I(f\perp g)$ where f is a form of degree d in $x_1,\dots,x_p$, and g is a form of degree d in $x_1,\dots,x_q$.

Recall $(f\perp g)(x_1,\dots,x_{p+q}) = f(x_1,\dots,x_p)+g(x_{p+1},\dots,x_{p+q})$.

Note that

$$\begin{aligned}
&(t_{i_1\dots i_p i_{p+1}\dots i_{p+q}})(f\perp g)\\
&= t_{i_1\dots i_p}(f) \quad \text{if} \quad i_{p+1}=\dots=i_{p+q}=0\\
&= t_{i_1\dots i_q}(g) \quad \text{if} \quad i_1=\dots=i_p=0\\
&= 0 \quad \text{otherwise.}
\end{aligned}$$

2.3 Let D, E be Young diagrams of the same shape in L, P respectively; where E is the diagram

$$\begin{matrix} 1 \dots p\ (p+1) \dots (p+q) \\ \vdots \\ 1 \dots p\ (p+1) \dots (p+q) \end{matrix}$$

with g equal rows; and D is the diagram $(w_1,\dots,w_g)$.

We calculate $<U, \mathrm{Tab}(D|E)> (f_{1g})$.

We will see that if this is nonzero then for $i=1,\dots,g$ we can write $w_i = w_i' w_i''$ such that: w_i' has length p, w_i'' has length q and the diagrams

$D' = (w_1',\dots,w_g')$ and $D'' = (w_1'',\dots,w_g'')$ have no letters in common.

Let $r=\binom{p+q}{p}$, and for $i = 1,\dots,g$ let the component in $\mathrm{Ext}^p(L)\otimes\mathrm{Ext}^q(P)$ of

Δw_i be $\sum_{\alpha i=1}^{r} w_{i,\alpha i}' \otimes w_{i,\alpha i}''$.

By §1.11(ii)

$$\mathrm{Tab}\left(\begin{array}{c|c} w_1 & 1 \dots p\ (p+1) \dots (p+q) \\ \vdots & \vdots \\ w_g & 1 \dots p\ (p+1) \dots (p+q) \end{array}\right)$$

$$= \prod_{i=1}^{g} (\ w_i|\ 1 \dots p\ (p+1) \dots (p+q)\ )$$

$$= \prod_{i=1}^{g} \sum_{\alpha i=1}^{r} (-1)^{pq}(w_{i,\alpha i}'|1\dots p)(\ w_{i,\alpha i}''|(p+1)\dots(p+q)\ )$$

$$= (-1)^{pqg} \sum_{\alpha} \mathrm{Tab}\left(\begin{array}{c|c} w_{1,\alpha 1}' & 1\dots p \\ \vdots & \\ w_{g,\alpha g}' & 1\dots p \end{array}\right) \mathrm{Tab}\left(\begin{array}{c|c} w_{1,\alpha 1}'' & (p+1)\dots(p+q) \\ \vdots & \\ w_{g,\alpha g}'' & (p+1)\dots(p+q) \end{array}\right)$$ the sum taken

over all maps α: $\{1,\dots,g\}\to\{1,\dots,r\}$. We will denote

$$\begin{array}{c|c} w_{1\alpha 1}' & 1\dots p \\ \vdots & \vdots \\ w_{g\alpha g}' & 1\dots p \end{array} \quad \text{and} \quad \begin{array}{c|c} w_{1\alpha 1}'' & (p+1)\dots(p+q) \\ \vdots & \vdots \\ w_{g\alpha g}'' & (p+1)\dots(p+q) \end{array} \quad \text{by}$$

$D_\alpha'|E_1$ and $D_\alpha''|E_2$ respectively. It follows by §2.2 that when D_α', D_α'' have a letter in common then $< U, \mathrm{Tab}(D_\alpha'|E_1)\mathrm{Tab}(D_\alpha''|E_2) > (f_{1g}) = 0$.

Suppose that D'_α and D''_α have no letters in common. Then L is the disjoint union of subsets L' and L'' such that the entries of D'_α, D''_α come from L', L'' respectively. We can write

$$\mathrm{Tab}(\,D'_\alpha|E_1\,) = \sum_i T'_i \;,\quad \mathrm{Tab}(\,D''_\alpha|E_2\,) = \sum_j T''_j \quad \text{where}$$

(a) each T'_i is a monomial in the $(a|1)^{i_1}\ldots(a|p)^{i_n}$ with a in L' and $i_1+\ldots+i_n = d$

(b) each T''_j is a monomial in the $(a|p+1)^{j_1}\ldots(a|p+q)^{j_q}$ with a in L'' and $j_1+\ldots+j_q=d$.

Then

$$\begin{aligned}
&<U,\ \mathrm{Tab}(D'_\alpha|E_1)\mathrm{Tab}(D''_\alpha|E_2)> (f \perp g)\\
&= <U,\ \sum_i\sum_j T'_iT''_j> (f \perp g)\\
&= \sum_i\sum_j <U,\ T'_iT''_j> (f \perp g)\\
&= \sum\sum <U,T'_i>(f)<U,S''_j>(g) \quad \text{see notes (i),(ii) below}\\
&= <U,\mathrm{Tab}(D'_\alpha|E')>(f).<U,\mathrm{Tab}(D''_\alpha|E'')>(g) \quad \text{see note (iii) below}
\end{aligned}$$

Notes:

(i) We obtain S''_j from T''_j by substituting $(a|k)$ for $(a|p+k)$ throughout.

(ii) In $<U,T'_i>$, $<U,\mathrm{Tab}(D'_\alpha|E')$ the letters of L' belong to the form of degree d in $x_1,\ldots,x_p$; and in $<U,S''_j>$, $<U,\mathrm{Tab}(D''_\alpha|E'')>$ the letters of L'' belong to the form of degree d in $x_1,\ldots,x_q$.

(iii) $E' = E_1$ is the diagram with g rows all equal to $1\ldots p$, and E'' is the diagram with g equal rows all equal to $1\ldots q$.

2.3.1 We conclude that

$$\begin{aligned}
&<U,\ \mathrm{Tab}(D|E)> (f \perp g)\\
&= (-1)^{pqg}\sum_\alpha{}' <U,\ \mathrm{Tab}(D'_\alpha|E')>(f).<U,\ \mathrm{Tab}(D''_\alpha|E'')>(g)
\end{aligned}$$

where Σ' indicates the sum over all α such that D'_α, D''_α have no letters in common.

We stress that the letters of D, D'_α, D''_α belong to the forms of degree d in $p+q$, p, and q variables respectively.

2.4 Let E be as defined at the start of §2.3. Let D have the same shape as E, and let the entries of D be equivalent letters all belonging to the form of degree d in $x_1,\dots,x_{p+q}$. Recall **remark 1.15.1** that $\langle U,\mathrm{Tab}(D|E)\rangle=0$, unless each letter in D occurs exactly d times. Suppose that the letters of D form the set $\{a_1,\dots,a_r\}$ of distinct letters of L.

For each permutation π of $1,\dots,r$ let $\pi.D$ denote the diagram obtained from D by replacing each a_i by $a_{\pi^{-1}(i)}$. Since $a_1,\dots,a_r$ are equivalent letters for each such π $\langle U,\mathrm{Tab}(\pi.D|E) = \langle U,\mathrm{Tab}(D|E)\rangle$.

2.5 The absence of nontrivial families of additive invariants in odd degree will be deduced from the following elementary lemma.

We keep the notations of §2.4.

Lemma: Suppose that d is odd. Suppose that a,b are distinct letters of D such that a appears in a row of D iff b appears in that row. Then $\langle U,\mathrm{Tab}(D|E)\rangle = 0$.

Proof: Since a,b are equivalent letters, if we let D' denote the diagram obtained from D by interchanging a and b throughout then $\langle U,\mathrm{Tab}(D|E)\rangle = \langle U,\mathrm{Tab}(D'|E)\rangle$. But

$(\dots b\dots a\dots|1\dots n) = -(\dots a\dots b\dots|1\dots n)$ therefore

$\langle U,\mathrm{Tab}(D'|E)\rangle = (-1)^d\langle U,\mathrm{Tab}(D|E)\rangle$ □

2.6 We are now ready to prove that there are no nontrivial families of additive invariants in odd degree. First we compute an example showing that there are nonconstant additive invariants in each even degree.

Example: Let $D_n|E_n$ be a pair of Young diagrams in L, P respectively, where each diagram has d equal rows with:

the rows of $D_n = a_1\dots a_n$

the rows of $E_n = 1\dots n$.

Let $I_n = (1/n!).\langle U,\mathrm{Tab}(D_n|E_n)\rangle$. When d is odd and $n \geq 2$ we know by **lemma2.5** that $I_n = 0$.

Suppose that d is even.

If f is a form of degree d in one variable x: $f = t_{1\ldots 1}(f)x^d$ then $I_1(f) = \langle U,(a|1)^d\rangle(f) = t_{1\ldots 1}(f)$ so $I_1 \neq 0$.

We will use the method of **2.3** to show that $I_1,I_2,I_3,\ldots$ is a family of additive invariants i.e. that

$I_{p+q}(f\perp g) = I_p(f).I_q(g)$ when f, g are forms of degree d in $x_1,\ldots,x_p$; $x_1,\ldots,x_q$ respectively.

By Akin et al **[ABW I.2]** the component of $\Delta(a_1\ldots a_{p+q})$ in $\mathrm{Ext}^p(L)\otimes\mathrm{Ext}^q(P)$ is $\sum_\sigma \mathrm{sgn}(\sigma)a_{\sigma 1}\ldots a_{\sigma p}\otimes a_{\sigma(p+1)}\ldots a_{\sigma(p+q)}$ where the sum is over all permutations σ of $1,\ldots,p+q$ such that $\sigma 1<\ldots<\sigma p$ and $\sigma(p+1)<\ldots<\sigma(p+q)$. Such a permutation is called a **(p,q)–shuffle** of $1,\ldots,p+q$, and there are $\binom{p+q}{p}$ of them. The splittings of D into D'_α , D''_α with no common letters are determined by these (p,q)–shuffles σ. By the final equation of §**2.3**

$$I_{p+q}(f\perp g)$$

$$= 1/(p+q)!\,.\,\langle U,\mathrm{Tab}(D_{p+q}|E_{p+q})\rangle(f\perp g)$$

$$=(-1)^{pqd}\sum(\mathrm{sgn}\,\sigma)^d\langle U,\mathrm{Tab}\left(\begin{matrix}a_{\sigma 1}\ldots a_{\sigma p}\\ \vdots\cdots\vdots\\ a_{\sigma 1}\ldots a_{\sigma p}\end{matrix}\middle|\begin{matrix}1\ldots p\\ \vdots\cdots\vdots\\ 1\ldots p\end{matrix}\right)\rangle(f).$$

$$.\langle U,\mathrm{Tab}\left(\begin{matrix}a_{\sigma(p+1)}\ldots a_{\sigma(p+q)}\\ \vdots\cdots\vdots\\ a_{\sigma(p+1)}\ldots a_{\sigma(p+q)}\end{matrix}\middle|\begin{matrix}1\ldots q\\ \vdots\cdots\vdots\\ 1\ldots q\end{matrix}\right)\rangle(g)$$ the sum over all (p,q)–shuffles σ. Since d is even and for each σ

$$<U,\mathrm{Tab}\begin{pmatrix} a_{\sigma 1}\cdots a_{\sigma p} & 1\ldots p \\ \vdots\cdots\cdots\vdots & \vdots\cdots\vdots \\ a_{\sigma 1}\cdots a_{\sigma p} & 1\ldots p \end{pmatrix}> = p!I_p\,, \text{ and}$$

$$<U,\mathrm{Tab}\begin{pmatrix} a_{\sigma(p+1)}\cdots a_{\sigma(p+q)} & 1\ldots q \\ \vdots\cdots\cdots\vdots & \vdots\cdots\vdots \\ a_{\sigma(p+1)}\cdots a_{\sigma(p+q)} & 1\ldots q \end{pmatrix}> = q!I_q \text{ it is clear that}$$

$$I_{p+q}(f\perp g) = I_p(f).I_q(g).$$

Now we prove the main result of this chapter.

2.7 Proposition: Let F be a field of characteristic zero. When d is odd and $I_1, I_2, I_3, \ldots$ is a family of additive invariants of the forms of degree d with coefficients in F then all the I_n are constant.

We first prove the following lemma.

2.8 Lemma: If $I_1, I_2, I_3, \ldots$ is a family of additive invariants of the forms of degree d then all the nonzero I_n have the same weight.

Proof: Suppose $I_p \neq 0$ and $I_q \neq 0$. Then there are forms f; g of degree d in $x_1, \ldots, x_p$ and $x_1, \ldots, x_q$ respectively with $I_p(f) \neq 0$ and $I_q \neq 0$. For all $A \in GL_p(F)$, $B \in GL_q(F)$ we have

$$\begin{aligned}
&(\det A)^{g_p}.(\det B)^{g_q} I_p(f).I_q(g) \\
&= I_p(A.f).I_q(B.g) \\
&= I_{p+q}(A.f \perp B.g) \\
&= I_{p+q}\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.(f\perp g) \right) \\
&= \left(\det\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right)^{g_{p+q}} I_{p+q}(f\perp g) \\
&= (\det A)^{g_{p+q}}(\det B)^{g_{p+q}} I_p(f).I_q(g), \text{ so } g_p = g_{p+q} = g_q
\end{aligned}$$

□

Proof of Prop2.7: Suppose that I_k has weight $g>0$ and $I_k(f) \neq 0$. Then for every positive integer N $I_{Nk}(f\perp f\perp \dots \perp f) \neq 0$. By **lemma2.8** and **§1.16**

$I_{Nk} = \sum_i \langle U, Tab(D_i | E)\rangle$ where E has g rows all equal to 1 2 ... Nk. Suppose that D is one of the D_i, with $\langle U, Tab(D|E)\rangle(f\perp \dots \perp f) \neq 0$. Suppose $D = (w_1, \dots, w_g)$. Then by an induction argument based on **§2.3.1**, it must be possible for each $i=1,\dots,g$ to write

$w_i = \pm w_{i1} \dots w_{iN}$ with all w_{ij} of length k, and such that the diagrams

$D_1 = (w_{11}, \dots, w_{g1}), \dots, D_N = (w_{1N}, \dots, w_{gN})$ have no letters in common. Each of the above diagrams D_j has g rows and k columns, and each letter in it is repeated exactly d times. Up to renaming the letters, there are only finitely many possibilities for such diagrams. Therefore for large N there must be distinct i, j such that D_i, D_j are the same except for the naming of the letters. But then by **Lemma 2.5**

$\langle U, Tab(D|E)\rangle = 0$, a contradiction. □

2.9 Remark: We have not attempted to find all the families of additive invariants in even degree.

CHAPTER FOUR

In this chapter we study the Lie theory associated with a d–multilinear form θ: $V^d \to F$, where F is an infinite field and V is a finite dimensional vector space over F. We briefly review our reference works.

[SER] gives a rapid account of the Lie algebra of an algebraic matrix group.

[BOR] develops the Lie theory of an algebraic group in analogy with the analytic Lie theory.

[CHE1] is an earlier treatment which deals explicitly with algebraic subgroups of some GL(V), where V is a finite dimensional vector space over the base field.

[WAT1] gives a more general theory in functorial language, but does not contain all the results we need.

[CHE2], [BOU] contain proofs of a proposition, which we will apply in Chapter 5, concerning the behaviour of the radical of a Lie algebra under change of base field.

This chapter consists of three parts. We describe the contents of each part.

Let V be a fixed vector space of finite dimension over the base field F.

(1) In **§1.1** we define algebraic subgroups of GL(V), and their Lie algebras. In **§1.3** we define for a d–multilinear form θ: $V^d \to F$, the group $G(\theta)=\{\sigma \epsilon GL(V) : \sigma.\theta=\theta\}$ of **isometries** of θ and we show in **§1.4** that it is an algebraic subgroup of GL(V). In **§1.5** we characterise the Lie algebra of θ. In **§1.7** we characterise $G(\theta)$ and its Lie algebra for symmetric θ. In **§1.6** we show that for $\tau \epsilon GL(V)$, $G(\tau.\theta)$ (resp. its Lie algebra) is the conjugate by τ of $G(\theta)$ (resp. its Lie algebra). In **§1.8** we show that the Lie algebra of a direct sum of forms is the direct sum of the Lie algebras of these forms. In the remainder of this part we reduce the problem of calculating the isometry group of a form to the case of an indecomposable form. We can then deduce that if

$$(V,\theta) = \bigoplus_{i=1}^{k} (V_i,\theta_i) \text{ with the } (V_i,\theta_i) \text{ indecomposable, then}$$

$$G(\theta)_0 = \prod_{i=1}^{k} G(\theta_i)_0$$ where for an algebraic group G, G_0 is the connected component containing the identity.

(2) In the second part we calculate the isometry groups and Lie algebras of the hyperbolic forms. In §§2.1–2.4 we calculate the algebraic isometry groups of the hyperbolic forms. In §2.6 we calculate the radicals of these algebraic groups, and show that they are 2–step solvable. Here we need to assume characterictic zero. In §§2.7–2.12 we calculate the Lie algebras of the hyperbolic forms. In §2.13, assuming characteristic zero, we calculate the radicals of these Lie algebras rederiving 2–step solvability. In §2.2 and §2.11 we encounter, in the course of our calculations, the space of (d+1)–multilinear forms

$\varphi\colon V^{d+1} \to F$ which satisfy

(a) $\varphi(v_1,\dots,v_d,v_{d+1})$ is symmetric in $v_1,\dots,v_d$

(b) $\sum_{i=1}^{d+1} \varphi(v_1,\dots,\hat{v}_i,\dots,v_{d+1},v_i) = 0.$

This turns out to be the space $L_\lambda(V^*)$ where λ is the diagram

$\begin{array}{ll}\square & \square \\ \square & \\ \vdots & \\ \square & \end{array}$ with d rows, and L_λ is the Schur functor defined by Akin et al **[ABW II§1.3]**.

We defer the proof of this to the third part.

(3) In the third part we consider Schneider's result **[SCHN Prop2]** that:

If $\theta\colon V^{d+1} \to F$ is a symmetric (d+1)–multilinear form where $d \geq 2$ and $(d+1)! \neq 0$ in F; $\sigma \in G(\theta)$ is unipotent with $(\sigma-1)^r \neq 0$, $(\sigma-1)^{r+1} = 0$, $r \geq 1$; then the image of $(\sigma-1)^r$ is a subspace of singular points of θ. Our main result **Prop3.4** follows easily from Schneider's proposition. We give an independent proof, valid only in characteristic zero, which is elementary but rather long. Our proof is of interest as it involves the space of forms $L_\lambda(V^*)$ defined at the end of (2) above. We use the Lie algebra of θ, rather than $G(\theta)$ itself.

We observe that if M is in the Lie algebra of θ , then the form

$\psi(v_1,\dots,v_d,v_{d+1}) = \theta(v_1,\dots,v_d,Mv_{d+1})$ satisfies conditions **(2)(a),(b)** above.

In **§§3.9–3.17** we prove that the space of these forms is $L_\lambda(V^*)$ as claimed. By **[BOR II(7.3) (1)]** or **[CHE1 II§13 Prop1, II§14 Prop5]** when in characteristic zero:

unipotents $u \epsilon G(\theta)$ correspond 1–1 by $M = \log(1+(u-1))$, $u = \exp(M)$ to nilpotents M in the Lie algebra of $G(\theta)$. Furthermore

$(u-1)^r \neq 0$, $(u-1)^{r+1} = 0$ iff $M^r \neq 0$, $M^{r+1} = 0$; and

$(u-1)^r$ has the same image as M^r.

We use the **Basis Theorem** for $L_\lambda(V^*)$ due to Akin et al **[ABW II§2.16]**) to set up our proof. Not only do we need $(d+1)! \neq 0$, we also appeal at crucial moments in the proof to the fact that a negative rational number cannot equal a positive rational number. We also give a characteristic zero proof of Schneider's result that when θ is nonsingular, $G(\theta)$ is finite. We use the result **[CHE1 II§14 Prop3]** which implies that when M is in the Lie algebra of θ, so are the separable and nilpotent components of the additive Jordan decomposition of M. By **Prop3.4** we know that the nilpotent component is zero. We show in **Prop3.6** that the separable component of M is also zero. This means that $G(\theta)$ has zero Lie algebra. Therefore by **[BOR I (3.3)Cor]** or **[CHE1 II§8Thm5]** $G(\theta)$ is finite. As a matter of interest, in **§3.7** we give a short proof that if M is a nilpotent in the Lie algebra of θ and $M^r \neq 0$, $M^{r+1}=0$ with r **odd**, then the image of M^r is a subspace of singular points of θ.

1 We describe algebraic subgroups of GL(V) and their Lie algebras.

Fix the base field F. Let V be a vector space of dimension n over F. First note that GL(V) has the structure of an affine algebraic variety over F. If we specify a basis of V then $M \epsilon GL(V)$ is represented by a matrix $(m_{ij}) \epsilon A_F^{n^2}$ with $\det(m_{ij}) \neq 0$.

1.1 Definitions: An **algebraic group** is an algebraic variety which is also a group, such that the group operations are morphisms of algebraic varieties. An **algebraic matrix group** is an algebraic group which is a closed subgroup of some $GL_n(F)$.

An **algebraic subgroup of GL(V)** is a subgroup which after a choice of basis for V may be identified with an algebraic matrix group.

An algebraic subgroup of GL(V) has an associated Lie algebra. The Lie algebra of GL(V) is denoted $\mathfrak{gl}(V)$; as an F–vector space it is End(V), and the Lie bracket is $[M,N]=MN-NM$.

Let G be an algebraic subgroup of GL(V). We describe the Lie algebra of L(G) of G. Let $F[\epsilon]$ denote the F–algebra generated by ϵ with $\epsilon^2=0$. Suppose that G is defined by the equations $p_\lambda, \lambda\epsilon\Lambda$. Then L(G) consists of all $M\epsilon \mathrm{End}(V)$ such that $I+\epsilon M$ in $\mathrm{End}(V\otimes_F F[\epsilon])$ satisfies all the equations p_λ; and L(G) is a Lie subalgebra of $\mathfrak{gl}(V)$: see **[SER pLA1.3(v)]**.

We will prove that the direct product of two algebraic groups is an algebraic group, with Lie algebra isomorphic to the direct sum of their Lie algebras. In the more general theory expounded by Waterhouse **[WAT1]**, if G_1 and G_2 are affine group schemes represented by the Hopf–algebras A_1 and A_2 respectively, then $G_1\times G_2$ is an affine group scheme represented by the Hopf–algebra $A_1\otimes A_2$.

1.2 Proposition: Let G_1, G_2 be algebraic subgroups of $GL(V_1)$, $GL(V_2)$. Then $G_1\times G_2$ is isomorphic to an algebraic subgroup of $GL(V_1\oplus V_2)$ with $L(G_1\times G_2) \cong L(G_1)\oplus L(G_2)$.

Proof: Under the correspondence $(g_1,g_2)\ \epsilon\ G_1\times G_2 \leftrightarrow \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}\ \epsilon\ GL(V_1\oplus V_2)$ with $g_i\epsilon G_i$ $i=1,2$; we see that $G_1\times G_2$ is an algebraic matrix group. Given M in $\mathrm{End}(V_1\oplus V_2)$ we have $I+\epsilon M$ in $(G_1\times G_2)(k[\epsilon])$ iff

$$I+\epsilon M = \begin{pmatrix} I+\epsilon M_1 & 0 \\ 0 & I+\epsilon M_2 \end{pmatrix}$$ with $I+\epsilon M_i$ in $G_i(F[\epsilon])$ for $i=1,2$.□

Next we define, for a d–multilinear form θ, the group $G(\theta)$. We show that it is an algebraic subgroup of GL(V) and characterise its Lie algebra $L(\theta)$.

1.3 Let V be a vector space of finite dimension over the ground field F. Let $\theta: V^d \to F$ be a d–multilinear form. $\sigma\epsilon GL(V)$ acts as usual on the left on the space of such forms θ by

$$\sigma.\theta(v_1,\dots,v_d) = \theta(\sigma^{-1}v_1,\dots,\sigma^{-1}v_d).$$

We define $G(\theta) = \{\sigma\epsilon GL(V) : \sigma.\theta = \theta\}$, the stabilizer of θ, and call it the group of **isometries of θ.**

1.4 Proposition: $G(\theta)$ is an algebraic subgroup of GL(V).

Proof: Let $w_1,\dots,w_n$ be a basis of V. $\sigma\epsilon GL(V)$ is represented w.r.t this basis by a matrix (σ_{ij}) and $\sigma.\theta=\theta$ iff for all $v_1,\dots,v_d$ in V:

$\theta(\sigma^{-1}v_1,\dots,\sigma^{-1}v_d) = \theta(v_1,\dots,v_d)$. For each $v_1,\dots,v_n$ this condition is that a polynomial in the σ_{ij} and $(\det\sigma)^{-1}$ must be zero. □

1.5 Proposition: The Lie subalgebra $L(\theta)$ of $\mathfrak{gl}(V)$ consists of all $M\epsilon End(V)$ such that

$$\sum_{i=1}^{d} \theta(v_1,\dots,v_{i-1},Mv_i,v_{i+1},\dots,v_d) = 0 \text{ for all } v_1,\dots,v_d \text{ in V.}$$

Proof: First note: with $\epsilon^2=0$ and $M\epsilon End(V)$ the inverse of $I+\epsilon M$ in $GL(V\otimes F[\epsilon])$ is $I-\epsilon M$. Let $\theta_\epsilon: (V\otimes_F F[\epsilon])^d \to F[\epsilon]$, be the extension of θ. Since $V\otimes_F F[\epsilon]$ is spanned by V over $F[\epsilon]$, $M\epsilon L(\theta)$ means that : for all $v_1,\dots,v_d$ in V

$\theta_\epsilon((I-\epsilon M)v_1,\dots,(I-\epsilon M)v_d) = \theta(v_1,\dots,v_d)$. The result follows easily from $\epsilon^2=0$ and the multilinearity of θ. □

We show that the algebraic isometry groups and Lie algebras of equivalent forms are conjugate.

1.6 Proposition: Let $\tau\epsilon GL(V)$. Then

(i) $G(\tau.\theta) = \tau.G(\theta).\tau^{-1}$

(ii) $L(\tau.\theta) = \tau.L(\theta).\tau^{-1}$

Proof: (i) $\sigma \in G(\tau.\theta)$ iff $\tau.\theta(\sigma^{-1}v_1,\dots,\sigma^{-1}v_d) = \tau.\theta(v_1,\dots,v_d)$ for all $v_1,\dots,v_d$ in V. Now

$$\tau.\theta(\sigma^{-1}v_1,\dots,\sigma^{-1}v_d) = \theta[\,(\tau^{-1}\sigma^{-1}\tau)\tau^{-1}v_1,\dots,(\tau^{-1}\sigma^{-1}\tau)\tau^{-1}v_d]$$ and the $\tau^{-1}v_i$ run through V as the v_i do. Therefore

$$\sigma \in G(\tau.\theta) \text{ iff } (\tau^{-1}\sigma^{-1}\tau)^{-1} = \tau^{-1}\sigma\tau \in G(\theta) \text{ iff } \sigma \in \tau.G(\theta).\tau^{-1}.$$

The proof of (ii) is similar. □

We characterise the algebraic isometry group and Lie algebra of a symmetric d–multilinear form.

1.7 Proposition: Let $\theta\colon V^d \to F$ be a symmetric d–multilinear form. Suppose that $d! \neq 0$ in F.

(i) $\sigma \in G(\theta)$ iff $\theta(\sigma v,\dots,\sigma v) = \theta(v,\dots,v)$ for all $v \in V$.

(ii) $M \in L(\theta)$ iff $\theta(Mv,v,\dots,v) = 0$ for all $v \in V$.

Proof: (i) The form $\sigma^{-1}.\theta - \theta$ is symmetric and d–multilinear with

$$(\sigma^{-1}.\theta - \theta)(v,\dots,v) = \theta(\sigma v,\dots,\sigma v) - \theta(v,\dots,v) = 0$$ for all $v \in V$. Therefore by **1§21(i)** we have σ^{-1} in V □

(ii) By (i) $M \in L(\theta)$ iff $\theta[\,(I-\epsilon M)v,\dots,(I-\epsilon M)v\,] = \theta(v,\dots,v)$ for all $v \in V$. By the multilinearity of θ and since $\epsilon^2=0$, this condition is equivalent to

$$\sum_{i=1}^{d} \theta(v,\dots,Mv,\dots,v) = 0$$ for all $v \in V$, where Mv is in the i–th place.

The result follows by symmetry of θ. □

Our next result is that the Lie algebra of a direct sum of nondegenerate symmetric spaces of degree $d \geq 3$ is the direct sum of their Lie algebras.

1.8 Proposition: Let (V,θ) be a nondegenerate symmetric space of degree $d \geq 3$ over a field F with $d! \neq 0$ in F. Suppose that $(V,\theta) = \bigoplus_{i=1}^{r}(V_i,\theta_i)$. Then $L(\theta) = \bigoplus_{i=1}^{r} L(\theta_i)$.

Proof: Let r=2. We write endomorphisms of $V_1 \oplus V_2$ as matrices $\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ where each M_{ij}: $V_j \to V_i$ is a linear map. By §1.7(ii) such an endomorphism is in $L(\theta)$ iff

$$\theta_1 \oplus \theta_2\left(\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \ldots, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) = 0$$ for all v_1 in V_1, and all v_2 in V_2. By definition of the direct sum of θ_1 and θ_2 (**1§17**) and their multilinearity, this means that

$$\theta_1(M_{11}v_1,v_1,\ldots,v_1) + \theta_2(M_{22}v_2,v_2,\ldots,v_2) + \theta_1(M_{12}v_2,v_1,\ldots,v_1) + \theta_2(M_{21}v_1,v_2,\ldots,v_2) = 0$$

for all v_1 in V_1, and all v_2 in V_2. Since $d \geq 3$ each of these four terms has a different content in v_1,v_2, so by 1§20 each term is identically zero. The vanishing of the first 2 terms means that M_{11} and M_{22} are in the Lie algebras of θ_1, θ_2 respectively. Since θ is nondegenerate so are θ_1 and θ_2. The vanishing of the 3rd term, an application of **1§21(i)** and the nondegeneracy of θ_1 yields $M_{12}=0$. Similarly $M_{21}=0$. The rest of the proof is a simple induction argument. □

1.9 Suppose that (V,θ) is a nondegenerate symmetric space of degree $d \geq 3$ over a field F in which $d! \neq 0$. We will reduce the calculation of $G(\theta)$ to the indecomposable case. Harrison **[HAR1 p129]** made the

Definition: A subspace of V is called a **summand** of (V,θ) if $V = W \oplus W^{\perp}$. Harrison proved **[HAR1 penultimate sentance in the proof of Prop 2.3]** that when (V,θ) is nondegenerate:

$V = W_1 \perp \ldots \perp W_s$ where the W_i are nonzero indecomposable summands of (V,θ); and that any nonzero indecomposable summand of (V,θ) is one of these W_i.

We note the simple

Lemma 1: Let $\sigma \in G(\theta)$ and let W be a nonzero indecomposable summand of (V,θ). Then σW is a nonzero indecomposable summand of (V,θ). □

In the above orthogonal sum

$V = W_1 \perp \ldots \perp W_s$, we group together the summands (W_i,θ) which are in the same isomorphism class:

$V = U_1 \perp \ldots \perp U_t$, where t is the number of distinct isomorphism classes of the (W_i,θ), and U_j is the orthogonal sum of all the (W_i,θ) in the j–th isomorphism class. If $\sigma \epsilon G(\theta)$ then by **Lemma 1** σW_i is one of the W_j and of course $(\sigma W_i,\theta) \cong (W_i,\theta)$. We have proved

Lemma 2: $G(V,\theta)$ is the direct product of the above $G(U_i,\theta)$ for $1 \leq i \leq t$.□

We have reduced to the case $V = W_1 \perp \ldots \perp W_r$ where the W_i are the nonzero indecomposable summands of V, and all the (W_i,θ) are isomorphic, say to (W,ϕ). In this case we have

Lemma 3: $G(V,\theta) \cong G_r \bowtie [G(W,\phi)]^r$ a semi–direct product of the group G_r of permutations of $1,\ldots r$; with the direct product of r copies of $G(W,\phi)$. The group operation is:

$$(\pi;\sigma_1,\ldots,\sigma_r).(\rho;\tau_1,\ldots,\tau_r) = (\pi \circ \rho \; ; \; \sigma_{\rho 1} \circ \tau_1 \,,\ldots,\; \sigma_{\rho r} \circ \tau_r).$$

Proof: For each $i=1,\ldots,r$ fix an isomorphism $\tau_i : (W,\phi) \to (W_i,\theta)$. Let $\sigma \epsilon G(V,\theta)$. By **Lemma 1 and [HAR1 penultimate sentence of Prop2.3]** there is a permutation π of $1,\ldots,r$ with $\sigma W_i = W_{\pi i}$. For each i let $\sigma_i = (\tau_{\pi i})^{-1}.\sigma.\tau_i$, so that the following diagrams of isomorphisms of symmetric spaces commute:

$$\begin{array}{ccc} (W_i,\theta) & \xrightarrow{\sigma} & (W_{\pi i},\theta) \\ \tau_i \uparrow & & \uparrow \tau_{\pi i} \\ (W,\phi) & \xrightarrow{\sigma_i} & (W,\phi) \end{array}$$

Consider the composite $\sigma.\sigma'$ of σ, σ' in $G(V,\theta)$, where

σ is defined by $(\pi;\, \sigma_1,\ldots,\sigma_r)$, and

σ' is defined by $(\pi';\, \sigma_1',\ldots,\sigma_r')$.

Then the following diagrams of isomorphisms of symmetric spaces commute

$$\begin{tikzcd} W_i \arrow[r, "\sigma'"] & W_{\pi' i} \arrow[r, "\sigma"] & W_{\pi(\pi' i)} \\ W \arrow[u, "\tau_i"] \arrow[r, "\sigma_i"'] & W \arrow[u, "\tau_{\pi' i}"] \arrow[r, "\sigma_{\pi i}"'] & W \arrow[u, "\tau_{\pi(\pi' i)}"] \end{tikzcd}$$

hence the result. □

Suppose that the nondegenerate symmetric space (V,θ) has:

$V = W_1 \perp \ldots \perp W_s$ where the W_i are the nonzero, indecomposable summands of (V,θ). By the above lemmas; since a finite product of irreducible affine varieties is an irreducible affine variety [SHA p24 Thm3]; and by the characterisation of the connected component of an algebraic group containing the identity as the unique irreducible algebraic subgroup of finite index, [WAT1 5.2] or [BOR (1.2)], we have:

Proposition: The connected component of $G(V,\theta)$ containing the identity is equal to the direct product of the connected components of the $G(W_i,\theta)$ containing the identity.□

Since the Lie algebra of an algebraic group is equal to the Lie algebra of its connected component containing the identity, [BOR (3.3)Cor], we have another proof of **Prop1.8.**

2 In the second part we calculate the algebraic isometry groups and Lie algebras of the hyperbolic forms. We will apply our knowledge of these Lie algebras in Chapter 5 when we study hyperbolicity in a change of base field. An interesting feature of these calculations is the appearance of the space of multilinear forms $\varphi: V^{d+1} \to F$ with $\lambda = \begin{array}{ll} \square & \square \\ \square & \\ \vdots & \\ \square & \end{array}$ –shaped symmetry. This is the space $L_\lambda(V^*)$ of Akin et al [ABW II§1.3].

First we calculate $G(\psi)$ for the hyperbolic forms ψ.

2.1 Let $d\geq 2$ be an integer. We suppose that $(d+1)!\neq 0$ in F. Let V be a finite dimensional vector space over F. In **2§1.10** we defined the symmetric (d+1)–multilinear hyperbolic form

$$\psi: [\, V\oplus(\odot^d(V))\,]^{d+1} \to F.$$

We will write endomorphisms of $V \oplus(\odot^d(V))$ as matrices

$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of linear maps where A, D are endomorphisms of V, $\odot^d(V)$ respectively; and $B: \odot^d(V)\to V$, $C: V\to \odot^d(V)$ are linear maps. Let

$\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be in $G(\psi)$. We will show that:

(1) $B = 0$ which implies that $A\in GL(V)$.

(2) $D\theta = A.\theta$.

(3) Identifying a linear map $C: V \to \odot^d(V)$ with a (d+1)–multilinear form $\Gamma: V^{d+1} \to F$ by $C(v_{d+1})(v_1,\dots,v_d) = \Gamma(v_1,\dots,v_d,v_{d+1})$ the constraint on C is that the form

$\varphi: V^{d+1} \to F$ defined by

$\varphi(v_1,\dots,v_{d+1}) = \Gamma(v_1,\dots,v_d,A^{-1}v_{d+1}) = C(A^{-1}v_{d+1})(v_1,\dots,v_d)$ must satisfy

(a) $\varphi(v_1,\dots,v_d,v_{d+1})$ is symmetric in $v_1,\dots,v_d$

(b) $\sum_{i=1}^{d+1} \varphi(v_1,\dots,\hat{v}_i,\dots,v_{d+1},v_i) = 0.$

Let $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be in $G(\psi)$. Then σ leaves the singular locus of ψ over F invariant. Now $\odot^d(V)$ is the singular locus of ψ over F. Therefore $B=0$. Since $B=0$, A is invertible. This proves (1).

We have for all $v \in V$ and all $\theta \in \odot^d(V)$

$$\psi\left(\begin{pmatrix} v \\ \theta \end{pmatrix}, \dots, \begin{pmatrix} v \\ \theta \end{pmatrix}\right) = \psi\left(\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}\begin{pmatrix} v \\ \theta \end{pmatrix}, \dots, \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}\begin{pmatrix} v \\ \theta \end{pmatrix}\right)$$

i.e $(d+1)\theta(v,\dots,v) = (d+1)(Cv)(Av,\dots,Av) + (d+1)(D\theta)(Av,\dots,Av)$. Since $(d+1)! \neq 0$ we have, by **1§20** that for all v and all θ:

(i) Collecting terms of degree 1 in θ, degree d in v

$$(D\theta)(Av,\dots,Av) = \theta(v,\dots,v)$$

$\Leftrightarrow$ $(D\theta)(v,\dots,v) = \theta(A^{-1}v,\dots,A^{-1}v)$, which proves (2).

(ii) Collecting terms of degree d+1 in v, degree 0 in θ,

$$(Cv)(Av,\dots,Av) = 0.$$

2.2 We multilinearise the identity (ii) by the method of **1§10**. Let $\lambda_1,\dots,\lambda_{d+1}$ be indeterminants and consider

$$C\Big(\sum_{i=1}^{d+1} \lambda v\Big)\Big(\dots, A\Big(\sum_{i=1}^{d+1} \lambda_i v_i\Big), \dots\Big)$$

$= \sum_{\alpha} \lambda_{\alpha 1} \cdots \lambda_{\alpha(d+1)} C(v_{\alpha 1})(Av_{\alpha 2}, \dots, Av_{\alpha(d+1)})$. We compute the coefficient of $(1/d!)\lambda_1 \dots \lambda_{d+1}$, assisted by the symmetry of the $C(v_i)$ to get

$$\sum_{i=1}^{d+1} C(v_i)(Av_1,\dots,\hat{Av_i},\dots,Av_{d+1}) = 0 \quad \text{for all } v_1,\dots,v_{d+1} \text{ or}$$

$$\sum_{i=1}^{d+1} C(A^{-1}v_i)(v_1,\dots,\hat{v}_i,\dots,v_{d+1}) = 0 \quad \text{for all } v_1,\dots,v_{d+1}.$$ We identify a linear map

$C: V \to \odot^d(V)$ with a $(d+1)$–multilinear form

$\Gamma: V^{d+1} \to F$ by $\Gamma(v_1,\dots,v_{d+1}) = C(v_{d+1})(v_1,\dots,v_d)$. Then we have:

$\Gamma(v_1,\dots,v_d,v_{d+1})$ is symmetric in $v_1,\dots,v_d$, and

$$\sum_{i=1}^{d+1} \Gamma(v_1,\dots,v_i,\dots,v_{d+1},A^{-1}v_i) = 0.$$

Therefore an element $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ of $G(\psi)$ can be identified with a pair (A,φ) where $A \in GL(V)$ and

$\varphi(v_1,\ldots,v_{d+1}) = \Gamma(v_1,\ldots,v_d,A^{-1}v_{d+1}) = C(A^{-1}v_{d+1})(v_1,\ldots,v_d)$ is such that:

(a) $\varphi(v_1,\ldots,v_d,v_{d+1})$ is symmetric in $v_1,\ldots,v_{d+1}$

(b) $\sum_{i=1}^{d+1} \varphi(v_1,\ldots,\hat{v}_i,\ldots,v_{d+1},v_i) = 0$. We have now shown (3).

We will prove later in **Prop3.12** of part **3** that this is the space $L_\lambda(V^*)$ where λ is the Young diagram $\begin{array}{ll}\square & \square\\ \square & \\ \vdots & \\ \square & \end{array}$ and L_λ is the corresponding Schur functor defined by Akin et al **[ABW II§1.3]**.

From (A,φ) we recover $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ by

$(D\theta)(v_1,\ldots,v_d) = \theta(A^{-1}v_1,\ldots,A^{-1}v_d)$

$(Cv_{d+1})(v_1,\ldots,v_d) = \varphi(v_1,\ldots,v_d,Av_{d+1})$.

2.3 Now that we have identified $G(\psi)$, as a set, with $GL(V) \times L_\lambda(V^*)$ we can interpret matrix multiplication as a multiplication in this set.

To multiply $(A_1,\varphi_1).(A_2,\varphi_2)$ we calculate the product of the corresponding matrices $\begin{pmatrix} A_1 & 0 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} A_2 & 0 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1A_2 & 0 \\ C_1A_2+D_1C_2 & D_1D_2 \end{pmatrix}$ which corresponds, say, to (A_1A_2,φ) where

$$\begin{aligned}
&\varphi(v_1,\ldots,v_d,\ A_1A_2v_{d+1})\\
&= (C_1A_2 + D_1C_2)(v_{d+1})\ (v_1,\ldots,v_d)\\
&= C_1(A_2v_{d+1})(v_1,\ldots,v_d) + D_1(C_2v_{d+1})(v_1,\ldots,v_d)\\
&= \varphi_1(v_1,\ldots,v_d,A_1A_2v_{d+1}) + C_2(v_{d+1})(A_1^{-1}v_1,\ldots,A_1^{-1}v_d)\\
&= \varphi_1(v_1,\ldots,v_d,A_1A_2v_{d+1}) + \varphi_2(A_1^{-1}v_1,\ldots,A_1^{-1}v_d,A_1^{-1}A_1A_2v_d) \text{ and therefore}
\end{aligned}$$

$\varphi = \varphi_1 + A_1.\varphi_2$. We have proved:

2.4 Proposition: Let ψ: $[\, V \oplus(\odot^d(V))]^{d+1} \to F$ be the hyperbolic symmetric (d+1)–multilinear form defined in **2§1.10**. With $\lambda = \begin{array}{l}\square\square\\ \vdots\\ \square\end{array}$ we will see(**Prop3.2** below) that $L_\lambda(V^*)$ is the space of (d+1)–multilinear forms φ: $V^{d+1} \to F$ such that:

(a) $\varphi(v_1,\ldots,v_d,v_{d+1})$ is symmetric in $v_1,\ldots,v_d$;

(b) $\sum_{i=1}^{d+1} \varphi(v_1,\ldots,\hat{v}_i,\ldots,v_{d+1},v_i) = 0$. Then the algebraic group $G(\psi)$ is isomorphic to a semidirect product $GL(V) \ltimes L_\lambda(V^*)$. (A,φ) corresponds to $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ where

$D\theta = A.\theta$,

$C(v_{d+1})(v_1,\ldots,v_d) = \varphi(v_1,\ldots,v_d,Av_{d+1})$. The group law is:

$(A_1,\varphi_1).(A_2,\varphi_2) = (A_1A_2,\varphi_1+A_1.\varphi_2)$

where as usual $(A_1.\varphi_2)(v_1,\ldots,v_{d+1}) = \varphi_2(A_1^{-1}v_1,\ldots,A_1^{-1}v_{d+1})$. $\square$

Next we will show that these algebraic groups $G(\psi)$ are connected. Then we will compute their radicals and show that they are 2–step solvable.

2.5 Since a product of irreducible varieties is irreducible, it follows that $G(\psi)$ is connected. We can also deduce this as follows:

Let G be an algebraic group. Suppose that $G = G_1 \ltimes G_2$ is a semidirect product of connected algebraic groups G_1, G_2. Then G is connected.

Proof: Let $(g_1,g_2)\epsilon G$ and let e_1, e_2 be the identity elements of G_1, G_2 respectively. Since G_1 is connected, (g_1,g_2) is in the same connected component of $G_1 \ltimes G_2$ as (e_1,g_2), and since G_2 is connected (e_1,g_2) is in the same connected component of $G_1 \ltimes G_2$ as (e_1,e_2). The connected components of an algebraic group are disjoint **[BOR (1.2)]**.$\square$

GL(V) is connected since it is open in A^{n^2}, and $L_\lambda(V^*)$ is connected since it is an A^N. Therefore from the above proposition we conclude that $G(\psi)$ is connected.

2.6 We keep the notations of **Prop2.4.**

Proposition: Suppose that the ground field F has characteristic zero. Then the radical of $G(\psi)$ is $F^* \ltimes L_\lambda(V^*)$, which is 2–step solvable.

Proof: First we show that the radical R of GL(V) is F^*. Clearly R contains F^*.

$$\begin{tikzcd}[column sep=small]
 & GL(V) \arrow[dl, dash] \arrow[dr, dash] & & & & \mathfrak{gl}(V) \arrow[dl, dash] \arrow[dr, dash] & \\
SL(V) \arrow[ddr, dash] & & R \arrow[d, dash, "*"] & & \mathfrak{sl}(V) \arrow[ddr, dash] & & L(R) \arrow[d, dash] \\
 & & F^* \arrow[dl, dash] & \xrightarrow{L} & & & F \arrow[dl, dash] \\
 & \{1\} & & & & \{0\} &
\end{tikzcd}$$

$\mathfrak{sl}(V)$ is simple [JAC p137 Thm6], so $\mathfrak{sl}(V) \cap L(R)=\{0\}$. Since F has characteristic zero, $\mathfrak{gl}(V) = F \oplus \mathfrak{sl}(V)$. Therefore for dimension reasons $L(R) = F = L(F^*)$. By **[CHE1 II§12Cor1]** $R=F^*$. Now let R denote the radical of $GL(V) \ltimes L_\lambda(V^*)$. Let

π: $GL(V) \ltimes L_\lambda(V^*) \to GL(V)$ be the projection. π is an epimorphism of algebraic groups, therefore it maps a connected normal solvable subgroup onto a connected normal solvable subgroup. The closure of a normal solvable subgroup is a closed normal solvable subgroup **[WAT1 4.3]**. Since the closure of an irreducible subspace is irreducible, it follows that the closure of $\pi(R)$ is contained in F^*, the radical of GL(V). Therefore R is contained in $F^* \ltimes L_\lambda(V^*)$. By §2.5 $F^* \ltimes L_\lambda(V^*)$ is connected ; and it is clearly a closed subgroup of $GL(V) \ltimes L_\lambda(V^*)$. Since F^* is normal in GL(V) it is clear that $F^* \ltimes L_\lambda(V^*)$ is normal in $GL(V) \ltimes L_\lambda(V^*)$. The first derived subgroup of $F^* \ltimes L_\lambda(V^*)$ is contained in $\{1\} \times L_\lambda(V^*)$. Since

$(1,\varphi_1).(1,\varphi_2)=(1,\varphi_1+\varphi_2)$, $\{1\} \times L_\lambda(V^*)$ is abelian. □

We will now calculate the Lie algebras of the hyperbolic forms. Then we will calculate the radicals of these Lie algebras. We will apply these results later when we study hyperbolicity under a change of base field.

2.7 Definition/Notation:

We keep the notations of **2.1** and make the following definition. Let $r \geq 1$ be an integer, let

$f: V^r \to F$ be a polynomial map, let $A \epsilon \mathrm{End}(V)$, and let $(v_1,\ldots,v_r) \epsilon V^r$. Then we define **the directional derivative of f at $(v_1,\ldots,v_r)$ in the direction $(Av_1,\ldots,Av_r)$** to be the coefficient of ϵ, where $\epsilon^2=0$, in

$f(v_1+\epsilon Av_1,\ldots,v_r+\epsilon Av_r)-f(v_1,\ldots,v_r)$. We denote this polynomial map by

$d_A f : V^r \to F$.

2.7.1 **Remark:** Suppose that $f: V^r \to F$ is r–multilinear. Then

$$d_A f(v_1,\ldots,v_r) = \sum_{i=1}^{r} f(v_1,\ldots,v_{i-1},Av_i,v_{i+1},\ldots,v_r).$$

2.8 **Proposition:** The conditions for $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \epsilon \mathrm{End}(\, V \oplus (\odot^d(V))\,)$ to be in the Lie algebra of ψ are:

(1) $B = 0$.

(2) $D\theta = -d_A \theta$

(3) The (d+1)–multilinear form $\varphi: V^{d+1} \to F$ defined by

$\varphi(v_1,\ldots,v_d,v_{d+1}) = C(v_{d+1})(v_1,\ldots,v_d)$ must satisfy:

(a) φ is symmetric in $v_1,\ldots,v_d$;

(b) $\sum_{i=1}^{d+1} \varphi(v_1,\ldots,\hat{v}_i,\ldots,v_{d+1},v_i) = 0$

i.e. $\varphi \epsilon L_\lambda(V^*)$ (see Prop3.12 below).

So as an F–vector space we can identify the Lie algebra of ψ with $\mathrm{End}(V) \oplus L_\lambda(V^*)$.

The Lie bracket is

$$[\,(A_1,\varphi_1),(A_2,\varphi_2)\,] = (\,[A_1,A_2]\,,\, d_{A_2}\varphi_1 - d_{A_1}\varphi_2\,)$$

Proof: Suppose that $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is in the Lie algebra of $G(\psi)$. This means that $I+\epsilon M$ is in the algebraic matrix group of the extension of ψ to $V \otimes_F F[\epsilon]$, so that by **2.1** $B=0$. We have proved (1).

We will again identify linear maps

$C: V \to \odot^d(V)$ with $(d+1)$–multilinear forms

$\varphi: V^{d+1} \to F$ by $\varphi(v_1,\ldots,v_d,v_{d+1}) = C(v_{d+1})(v_1,\ldots,v_d)$. By **Prop1.7** for $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ to be in the Lie algebra of ψ means that for all v,θ:

$$0 = \psi\left[\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}\begin{pmatrix} v \\ \theta \end{pmatrix}, \begin{pmatrix} v \\ \theta \end{pmatrix}, \ldots, \begin{pmatrix} v \\ \theta \end{pmatrix} \right]$$

$= d\theta(Av,v,\ldots,v) + D\theta(v,\ldots,v) + C(v)(v,\ldots,v)$. By 1§20 we have

(i) $D\theta(v,\ldots,v) = -d\theta(Av,v,\ldots,v)$ (terms of degree d in v, 1 in θ)

(ii) $C(v)(v,\ldots,v) = 0$ (terms of degree d+1 in v, 0 in θ)

2.9 We polarise (i) by the method of 1§10: substitute $\sum_{i=1}^{d} \lambda_i v_i$ for v, where the λ_i are indeterminate, and find the coefficient of $(1/d!)\lambda_1\ldots\lambda_d$. Since $D\theta$ is symmetric we get

$D\theta(v_1,\ldots,v_d)$

$= (1/d!)(-d \sum_\sigma \theta(Av_{\sigma 1},v_{\sigma 2},\ldots,v_{\sigma d})$ (summing over all permutations of 1,...,d)

$= (d/d!)(d-1)! \sum_{i=1}^{d} \theta(Av_i,v_1,\ldots,\hat{v}_i,\ldots,v_d)$ because θ is symmetric

$= -d_A\theta(v_1,\ldots,v_d)$ by **2.8.1** and symmetry of θ. This proves (2).

2.10 We polarise (ii) in the same way: We let $\lambda_1,\ldots,\lambda_{d+1}$ be indeterminate and compute the coefficient of $\lambda_1\ldots\lambda_{d+1}$ in $C(\Sigma\lambda_i v_i)(\ldots,\Sigma\lambda_i v_i,\ldots)$ which is

$\sum_\sigma C(v_{\sigma 1})(v_{\sigma 2},\ldots,v_{\sigma(d+1)})$ summing over all permutations σ of 1,...,d+1

$= d! \sum_{i=1}^{d+1} C(v_i)(v_1,\ldots,\hat{v}_i,\ldots,v_{d+1})$ since each $C(v_i)$ is symmetric. It follows that the (d+1)–multilinear form

$\varphi: V^{d+1} \to F$ corresponding to C as in (3) must satisfy:

(a) $\varphi(v_1,\ldots,v_d,v_{d+1})$ is symmetric in $v_1,\ldots,v_d$

(b) $\sum_{i=1}^{d+1} \varphi(v_1,\ldots,\hat{v}_i,\ldots,v_{d+1},v_i) = 0$

i.e. $\varphi \in L_\lambda(V^*)$ (see **Prop3.12**).

So as an F–vector space we can identify the Lie algebra of ψ with $\mathrm{End}(V)\oplus L_\lambda(V^*)$.

2.11 We must still compute the Lie bracket. Suppose that $\begin{pmatrix} A_i & 0 \\ C_i & D_i \end{pmatrix}$ $i=1,2$ are in the Lie algebra of ψ. Now

$\begin{pmatrix} A_1 & 0 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} A_2 & 0 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1A_2 & 0 \\ C_1A_2+D_1C_2 & D_1D_2 \end{pmatrix}$ and if φ is the (d+1)–multilinear form corresponding to $C_1A_2+D_1C_2$

$\varphi(v_1,\ldots,v_d,v_{d+1}) = (C_1A_2+D_1C_2)(v_{d+1})\,(v_1,\ldots,v_d)$. Let φ_1, φ_2 similarly correspond to C_1, C_2. Then

$$\begin{aligned}
&(C_1A_2+D_1C_2)(v_{d+1})\,(v_1,\ldots,v_d)\\
&= C_1(A_2v_{d+1})\,(v_1,\ldots,v_d) + D_1(\,C_2(v_{d+1})\,)(v_1,\ldots,v_d)\\
&= \varphi_1(v_1,\ldots,v_d,A_2v_{d+1}) - \sum_{i=1}^{d} C_2(v_{d+1})(A_1v_i,v_1,\ldots,\hat{v}_i,\ldots,v_d)\\
&= \varphi_1(v_1,\ldots,v_d,A_2v_{d+1}) - \sum_{i=1}^{d} \varphi_2(A_1v_i,v_1,\ldots,\hat{v}_i,\ldots,v_d,v_{d+1}).
\end{aligned}$$

We can now calculate the Lie bracket:

$$\begin{aligned}
&[\,(A_1,\varphi_1),(A_2,\varphi_2)\,]\\
&= (\,[A_1,A_2],\varphi\,) \text{ where}\\
&\varphi(v_1,\ldots,v_{d+1})\\
&= \varphi_1(v_1,\ldots,v_d,A_2v_{d+1}) - \sum_{i=1}^{d} \varphi_2(A_1v_i,v_1,\ldots,\hat{v}_i\ldots,v_d,v_{d+1})\\
&- \varphi_2(v_1,\ldots,v_d,A_1v_{d+1}) + \sum_{i=1}^{d} \varphi_1(A_2v_i,v_1,\ldots,\hat{v}_i,\ldots,v_d,v_{d+1})
\end{aligned}$$

$= d_{A_2}(\varphi_1) - d_{A_1}(\varphi_2)$. This completes the proof of **Prop2.8.** □

2.12 Let L denote the Lie algebra of $G(\psi)$. Assuming that the ground field has characteristic zero, we compute the radical of L. The projection $L \to \mathfrak{gl}(V)$ is an epimorphism of Lie algebras, and a solvable ideal is mapped onto a solvable ideal by a Lie algebra epimorphism. The argument at the beginning of the proof of **Prop2.6** shows that the radical of $\mathfrak{gl}(V)$ is F. Therefore the radical of L is contained in the ideal

$M = F \oplus L_\lambda(V^*)$. Now $[M,M] \subseteq \{0\}\oplus L_\lambda(V^*)$ which is abelian. We have proved:

Proposition: Let the ground field F have characteristic zero. Then the radical of the Lie algebra of ψ is $F\oplus L_\lambda(V^*)$ and is 2–step solvable.

3 In the third and last part of this chapter, we assume characteristic zero and prove a proposition which is implied by a result of Schneider **[SCHN Prop2]**. Working in characteristic zero enables us to use Lie algebra methods and, we believe, simplify Schneider's proof.

Schneider's **Prop2** is equivalent to the following: Let

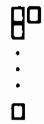
(a) $\theta\colon V^{d+1} \to F$ be a symmetric (d+1)–multilinear form,

(b) $M \in End(V)$ be a nilpotent element of the Lie algebra of θ such that for some integer $r \geq 1$, $M^r \neq 0$, $M^{r+1} = 0$.

Then the image of M^r consists entirely of singular points of θ.

The main steps in our proof are

(1) The form $\psi(v_1,\dots,v_d,v_{d+1}) = \theta(v_1,\dots,v_d,Mv_{d+1})$ satisfies **2.2(a),(b)** above.

(2) We prove as promised in §2.2 that the space of (d+1)–multilinear forms satisfying **2.2(a),(b)** is the space $L_\lambda(V^*)$, where $\lambda =$  with d rows.

(3) We note the Basis Theorem for $L_\lambda(V^*)$ due to Akin et al **[ABW II.2.16]**. We choose an ordered basis for V w.r.t. which the matrix representing M is in Jordan canonical form. We use the dual of this basis to define a standard basis of $L_\lambda(V^*)$.

(4) By symmetry of θ: $\psi(v_1,\dots,v_{d-1},Mv_d,v_{d+1}) = \psi(v_1,\dots,Mv_{d-1},v_d)$ and we have relations between the coefficients in the expression of ψ as a linear combination of the standard basis forms.

(5) We deduce from these relations that certain coefficients are zero, whence the result.

We state Schneider's Prop 2.

3.1 Let (V,θ) be a symmetric space of degree $(d+1)\geq 3$ over a field F in which $(d+1)! \neq 0$ then:

[SCHN Prop2] Let $\sigma \neq 1$ be a unipotent element of $G(\theta)$, such that $(\sigma-1)^r \neq 0$, $(\sigma-1)^{r+1} = 0$. Then $(\sigma-1)^r V$ is contained in the singular locus of θ.

In §§**3.2,3.3** we deduce our main result **Prop3.4** from Schneider's **Prop2.**

3.2 Suppose that the ground field F has characteristic zero. Let G be an algebraic group defined over F with Lie algebra L(G). Then by **[CHE1 §13Thm10,§14Prop5]**:

(i) If $\sigma\epsilon G$ is unipotent with $(\sigma-1)^r \neq 0$, $(\sigma-1)^{r+1} = 0$; then

$$M = \log\sigma = \log[1+(\sigma-1)] = \sum_{k=1}^{r} (-1)^{k-1}\frac{(\sigma-1)^k}{k} \text{ is in } L(G) \text{ and}$$

$M^r \neq 0$, $M^{r+1} = 0$.

(ii) If $M\epsilon L(G)$ is nilpotent with $M^r \neq 0$, $M^{r+1}=0$; then

$$\sigma = \exp M = \sum_{k=0}^{r} \frac{1}{k!} M^k \text{ is in G and } (\sigma-1)^r \neq 0 , (\sigma-1)^{r+1} = 0.$$

3.3 In 3.2 above, suppose that $G = G(\theta)$, where (V,θ) is a symmetric space of degree $(d+1)$ over a field F of characteristic zero. Let $M\epsilon L(\theta)$ be nilpotent with

$M^r \neq 0$, $M^{r+1} = 0$, $r\geq 1$. By **3.2(ii), 3.1** and **1§21** we have

for all $v_1,\dots,v_{d+1}$: $\theta(v_1,M^r v_2,\dots,M^r v_{d+1}) = 0$. Consider $\theta(M^{r_1}v_1,\dots,M^{r_{d+1}}v_{d+1})$ where the r_i are nonnegative integers with sum $\geq rd$, and we set $M^0=I$. By symmetry we may assume that r_1 is the least of the r_i's. If $r_1\geq 1$ we can write

$$\theta(\dots,M^{r_i}v_i,\dots)$$

$$= -\sum_{i=2}^{d+1} \theta(M^{r_1-1}v_1,M^{r_2}v_2,\dots,M^{r_i+1}v_i,\dots,M^{r_{d+1}}v_{d+1}).$$ Repeating this process, and since $M^{r+1} = 0$, we end up with an integer multiple of $\theta(v_1,M^r v_2,\dots,M^r v_{d+1})$ which is zero.

So Schneider's Prop2 implies

3.4 Proposition: Let d be an integer ≥ 2. Let V be a vector space of finite dimension n over a field F of characteristic zero. Let (V,θ) be a symmetric space of degree d+1 over F. Suppose that M is a nonzero nilpotent element of the Lie algebra of θ with

$$\{0\} = M^{r+1}V \subset M^rV \subset \ldots \subset MV \subset M^0V = V.$$

Let $r_1,\ldots,r_{d+1}$ be nonnegative integers with sum $\geq$ rd. Then if $v_i \in M^{r_i}V$ for i=1,...,d+1 we have $\theta(v_1,\ldots,v_{d+1}) = 0$.

We will give a proof of this proposition, independent of Schneider's **Prop2.**

First we continue our discussion.

We state the result on additive Jordan decomposition in a Lie algebra. Then we prove that if θ is nonsingular, the Lie algebra of θ contains no separable elements.

3.5 Let L be the Lie algebra of an algebraic subgroup of GL(V). Let $M \epsilon L$ and let $M = M_s + M_n$ be the additive Jordan decomposition of M with M_s separable and M_n nilpotent. Then by **[BOR (4.4)]** or **[CHE1 II§14Prop3]** M_s and M_n are in L.

Therefore to prove that L = {0}, we can proceed in 2 steps

(i) show that there are no nonzero separable elements in L

(ii) show that there are no nonzero nilpotents in L.

3.6 Proposition: Let F be a field of characteristic zero. Suppose that $\theta: V^{d+1} \rightarrow F$ is a nonsingular symmetric (d+1)–multilinear form with $d \geq 2$. Let M be a separable element of the Lie algebra of θ. Then M=0.

Proof: Let B be a basis of V consisting of eigenvectors of M. For $b \epsilon B$ let $\lambda(b)$ denote the eigenvalue of b. Suppose that for some $b_1 \epsilon B$ $\lambda(b_1) \neq 0$. Since θ is nonsingular there is some $b_2 \epsilon B$ with

$\theta(b_1,\ldots,b_1,b_2) \neq 0$. Since M is in the Lie algebra of θ

$d\theta(Mb_1,b_1,\ldots,b_1,b_2) + \theta(b_1,\ldots,b_1,Mb_2) = 0$ i.e.

$\{ d\lambda(b_1) + \lambda(b_2) \}\theta(b_1,\ldots,b_1,b_2) = 0$ and it follows that

$\lambda(b_2) = (-d)\lambda(b_1)$. Similarly we find $b_3 \epsilon B$ with

$\lambda(b_3) = (-d)\lambda(b_2)=(-d)^2\lambda(b_1)$, and for each $k=1,2,3,\ldots$ there is a b_k in B with $\lambda(b_k) = (-d)^{k-1}\lambda(b_1)$; which is absurd as we are in characteristic zero with $d\geq 2$.

3.7 As a matter of interest we note the following: Suppose that d is an integer ≥ 3 and θ: $V^d \to F$ is symmetric and d–multilinear. Let M be a nilpotent in the Lie algebra of θ, with $M^r \neq 0$, $M^{r+1}=0$ where r is odd. Then there is a **short proof** that all the points of M^rV are singular:

Since M is in the Lie algebra of θ

$$\theta(My_1,y_2,\ldots,y_d)+\theta(y_1,My_2,y_3,\ldots,y_d)+ \ldots+\theta(y_1,\ldots,y_{d-1},My_d) = 0$$

for all $y_1,\ldots,y_d$ in V. Taking $y_i = M^r z_i$ for $i=2,\ldots,d-1$ we have

$$\theta(My_1,M^rz_2,\ldots,M^rz_{d-1},y_d) = -\theta(y_1,M^rz_2,\ldots,M^rz_{d-1},My_d)$$

for all $y_1,z_2,\ldots,z_{d-1},y_d$ in V. Therefore

$$\begin{aligned}
&\theta(M^rx_1,M^rx_2,\ldots,M^rx_{d-1},x_d)\\
&= -\theta(M^{r-1}x_1,M^rx_2,\ldots,M^rx_{d-1},Mx_d)\\
&\quad\vdots\\
&= (-1)^r\theta(x_1,M^rx_2,\ldots,M^rx_d)\\
&= -\theta(x_1,M^rx_2,\ldots,M^rx_d)
\end{aligned}$$

since r is odd. Put $x_1=x_d$. Then by symmetry of θ

$$\begin{aligned}
0 &= 2\theta(M^rx_d,M^rx_2,\ldots,M^rx_{d-1},x_d)\\
&= 2\theta(M^rx_2,\ldots,M^rx_d,x_d)\\
&\quad\vdots\\
&= 2(-1)^r\theta(x_2,M^rx_3,\ldots,M^rx_d,M^rx_d)
\end{aligned}$$ □

3.8 Let (V,θ) be a symmetric space of degree d+1, with $d\geq 2$. Let M be any element of the Lie algebra of θ. Then the form $\psi : V^{d+1} \to F$ defined by

$$\psi(v_1,\ldots,v_d,v_{d+1}) = \theta(v_1,\ldots,v_d,Mv_{d+1})$$ satisfies:

(a) $\psi(v_1,\ldots,v_d,v_{d+1})$ is symmetric in $v_1,\ldots,v_d$

(b) $\sum_{i=1}^{d+1} \psi(v_1,\ldots,\hat{v}_i,\ldots,v_{d+1},v_i) = 0.$

In §§3.9 through 3.17 we will analyse the space of (d+1)–multilinear forms φ: $V^{d+1} \to F$. satisfying (a),(b). Then we will be equipped to prove **Prop3.4**. Our intention is to identify the above space of forms with the space $L_\lambda(V^*)$ defined by Akin et al **[ABW II§1.3]**, where λ is the diagram  with d rows.

In the next section we set up some of the symbolic method language in order to describe the space of all (d+1)–multilinear forms $\phi : V^{d+1} \to F$.

3.9 Let d be an integer ≥ 2. Let the base field F have characteristic zero, and let V be a vector space of dimension n over F. Fix a basis $\alpha_1,\ldots,\alpha_n$ of V^*.

The F–vector space of (d+1)–multilinear forms ϕ: $V^{d+1} \to F$ $\cong (V^{\otimes(d+1)})^* \cong (V^*)^{\otimes(d+1)} = (S_1(V^*))^{\otimes(d+1)}$, and $S(V^*)^{\otimes(d+1)} \cong S((V^*)^{d+1}) \cong S((V^{d+1})^*)$. Since F has characteristic zero we can identify $S((V^{d+1})^*)$ with the F–algebra $F[V^{d+1}]$ of all polynomial maps f: $V^{d+1} \to F$.

We will use the symbolic method of Grosshans et al **[GRS]**, which in this case amounts to the letter–place algebra (see Doubilet et al **[DKR]** or Doubilet et al **[DRS]**), to describe such f. We identify V with $S_1(V)$, and let $L = L^-$ be a set of negatively signed letters each belonging to one of the d+1 factors $V=S_1(V)$ of V^{d+1}. Let $P=P^- = \{1,\ldots,n\}$ be the set of places all negatively signed. Then $\mathrm{Super}[L|P] = \mathrm{Sym}[L|P]$, since $L=L^-$ and $P=P^-$. The umbral linear functional U has

$<U,(a|j)>(v_1,\ldots,v_d) = \alpha_j(v_i)$ when a belongs to the i–th factor of V^{d+1}.

Let $a_1,\ldots,a_{d+1}$ be letters of L such that for i=1,...,n a_i belongs to the i–th factor of V^{d+1}.

Let $L_0 = \{a_1,\ldots,a_{d+1}\}$. Then restricting the umbral linear functional, we have an isomorphism

U: $\mathrm{Sym}[L_0|P] \to F[V^{d+1}]$.

Let G_{d+1} be the group of permutations of $1,\ldots,d+1$. In the next section, **§3.10**, we define a G_{d+1}–action on $\mathrm{Sym}[L_0|P]$ such that with the usual G_{d+1}–action on $F[V^{d+1}]$, the above isomorphism is a G_d–isomorphism. In **§3.11** we use the Standard Basis Theorem for $\mathrm{Sym}[L_0|P]$ to express a multilinear $\phi \in F[V^{d+1}]$ in terms of a standard basis. In **§3.12** we formulate the conditions **§3.8(a),(b)** in terms of the G_{d+1}–action.

3.10 Recall the representation $\mathrm{r} : G_{d+1} \to GL(V^{d+1})$ of 1§1 and let G_{d+1} act on the left on the F–algebra $F[V^{d+1}]$ of polynomial maps $f : V^{d+1} \to F$ by

$\pi.f = f o r(\pi)^{-1} = f o r(\pi^{-1})$ for $\pi \epsilon G_{d+1}$ and $f \epsilon F[V^{d+1}]$. Then

$(\pi.f)(\ldots,v_i,\ldots) = f(\ldots, v_{\pi i},\ldots)$.

G_{d+1} acts as F–algebra automorphisms on the left on $\mathrm{Sym}[L_0|P]$ by

$\pi.(a_i|j) = (a_{\pi i}|j)$. We have

$$\begin{aligned}
&<U, \{ \pi.[(a_{i_1}|j_1)^{r_1}\ldots(a_{i_k}|j_k)^{r_k}] \}> (v_1,\ldots,v_{d+1})\\
&= <U, \{ (a_{\pi i_1}|j_1)^{r_1}\ldots(a_{\pi i_k}|j_k)^{r_k} \}> (v_1,\ldots,v_{d+1})\\
&= [\alpha_{j_1}(v_{\pi i_1})]^{r_1}\ldots[\alpha_{j_k}(v_{\pi i_k})]^{r_k}\\
&= <U, [(a_{i_1}|j_1)^{r_1}\ldots(a_{i_k}|j_k)^{r_k}]> o r(\pi)^{-1}(v_1,\ldots,v_{d+1})\\
&= \{ \pi.<U,(a_{i_1}|j_1)^{r_1}\ldots(a_{i_k}|j_k)^{r_k}> \} (v_1,\ldots,v_{d+1}).
\end{aligned}$$

Therefore with the above G_{d+1}–actions, the umbral linear functional provides a G_{d+1}–isomorphism $\mathrm{Sym}[L_0|P] \cong F[V^{d+1}]$.

3.11 Order L_0 by $a_1<\ldots<a_{d+1}$ and P by $1<\ldots<n$. By the Standard Basis Theorem, under our identification $\mathrm{Sym}[L_0|P] = F[V^{d+1}]$, a **(d+1)–multilinear** form $\phi: V^{d+1} \to F$ is identified with an element of $\mathrm{Sym}[L_0|P]$ of the form

$\Sigma\, c_{ij}\mathrm{Tab}(D_i|E_j)$ where

(i) $D_i|E_j$ is a pair of standard Young tableaux of the same shape with entries in L_0, P respectively;

(ii) each D_i has $d+1$ entries and each letter of L_0 appears exactly once in D_i.

3.12 We formulate **3.8(a),(b)** in terms of the G_{d+1}–action. Let G_d denote the subgroup of G_{d+1} consisting of all π with $\pi(d+1)=d+1$. For $1 \le i \le d$ let

$s_{i,d+1} \in G_{d+1}$ be the **swop** of i and d+1, and let

$s_{d+1,d+1}$ be the identity.

Proposition: The F–vector space of all (d+1)–multilinear forms $\varphi : V^{d+1} \to F$ such that

(a) $\pi.\varphi = \varphi$ for all $\pi \in G_d$ (i.e $\varphi(v_1,\ldots,v_d,v_{d+1})$ is symmetric in $v_1,\ldots,v_d$)

(b) $\sum_{i=1}^{d+1} s_{i,d+1}.\varphi = 0$ (which given (a) means that $\sum_{i=1}^{d+1} \varphi(v_1,\ldots,\hat{v}_i,\ldots,v_{d+1},v_i) = 0$.)

is isomorphic to the image of the linear map $d_\lambda : \Lambda^\lambda(V^*) \to S_{\tilde\lambda}(V^*)$ defined by Akin et al **[ABW II§1.3]**, where $\lambda = \begin{array}{l}\square\square\\ \square\\ \vdots\\ \square\end{array}$ with d rows. We say that such forms have $\begin{array}{l}\square\square\\ \square\\ \vdots\\ \square\end{array}$ –shaped symmetry.

3.13 Before proving the proposition we describe d_λ in this special case.

$\tilde\lambda = \begin{array}{l}\square\square\ldots\square\\ \square\end{array}$ with d columns

$\Lambda^\lambda(V^*) = \Lambda^2(V^*)\otimes\Lambda^1(V^*)\otimes\ldots\otimes\Lambda^1(V^*)$ (with d $\Lambda^1(V^*)$ factors) $= \Lambda^2(V^*)\otimes(V^*)^{\otimes d}$

$S_{\tilde\lambda}(V^*) = S_d(V^*)\otimes V^*$.

We will calculate d_λ by the direct method of Akin et al **[ABW top of p222]**. For $x_1,\ldots,x_{d+1}$ in V^* we calculate $d_\lambda[\,(x_1\Lambda x_{d+1})\otimes x_2\otimes\ldots\otimes x_d\,]$ as follows:

Picture $(x_1\Lambda x_{d+1})\otimes x_2\otimes\ldots\otimes x_d$ as

$$\begin{array}{ll} x_1 & x_{d+1}\\ x_2 & \\ \vdots & \\ x_d & \end{array}$$

We multilinearise $x_1\Lambda x_{d+1}$ using the comultiplication Δ of the Hopf–algebra $\Lambda(V^*)$.

$\Delta(\, x_1\Lambda x_{d+1}\,)$

$= (x_1\otimes 1 + 1\otimes x_1).(x_{d+1}\otimes 1 + 1\otimes x_{d+1})$

$= (x_1 \Lambda x_{d+1}) \otimes 1 + x_1 \otimes x_{d+1} - x_{d+1} \otimes x_1 + 1 \otimes (x_1 \Lambda x_{d+1})$. The multilinear terms are $x_1 \otimes x_{d+1} - x_{d+1} \otimes x_1$. We take $(x_1 \otimes x_{d+1}) \otimes x_2 \otimes \ldots \otimes x_d - (x_{d+1} \otimes x_1) \otimes x_2 \otimes \ldots \otimes x_d$ and picture it as

$$\begin{array}{l} x_1\ x_{d+1} \\ x_2 \\ \vdots \\ x_d \end{array} \quad - \quad \begin{array}{l} x_{d+1}\ x_1 \\ x_2 \\ \vdots \\ x_d \end{array}$$

then we multiply in $S(V^*)$ down each column and tensor the results together ending up in $S_d(V^*) \otimes V^*$:

$$d_\lambda[\, (x_1 \Lambda x_{d+1}) \otimes x_2 \otimes \ldots \otimes x_d \,] = (x_1 x_2 \ldots x_d) \otimes x_{d+1} - (x_{d+1} x_2 \ldots x_d) \otimes x_1 \ .$$

3.14 **Proof of Prop3.12 :** The proof extends over **§§3.14–3.16**. Suppose φ satisfies **3.12(a),(b)**. We have

$\psi = \Sigma\, c_{ij} \mathrm{Tab}(D_i | E_j)$ as above in **§3.11**. By (a) and the way G_{d+1} acts on $\mathrm{Sym}[L_0 | P]$

$\varphi = \Sigma\, c_{ij} \frac{1}{d!} \sum_{\pi \epsilon G_d} \mathrm{Tab}(\pi.D_i | E_j)$. Recall that $\pi \epsilon G_d \subset G_{d+1}$ fixes d+1 , and the tableau $\pi.D_i$ is obtained from D_i by replacing each entry a_k of D_i by $a_{\pi k}$.

3.15 Suppose the pair $D|E$ satisfies **3.11(i),(ii)**. Suppose that for $i<j\leq d$, D has the letters a_i, a_j in the same row, the r–th row say. Let $s \epsilon G_d$ denote the **swop of i and j**. The group G_d splits up into left cosets $G_d/\{1,s\}$ modulo the subgroup $\{1,s\}$. For each coset $\{\pi, \pi s\}$, $\pi.D$ and $\pi s.D$ differ only in the r–th row, in which the only difference is that $a_{\pi i}$ and $a_{\pi j}$ are swopped. Therefore

$\mathrm{Tab}(\, \pi.D | E\,) + \mathrm{Tab}(\, \pi s.D | E) = 0$ and so

$1/d! \sum_{\pi \epsilon G_d} \mathrm{Tab}(\pi.D | E) = 0.$

3.16 We now have

(*) $\varphi = \Sigma\, c_{ij}\ 1/d! \sum_{\pi \epsilon G_d} \mathrm{Tab}(\, \pi.D_i | E_j\,)$ where D_i is standard and for all $1 \leq p,q \leq d$ D_i does not have a_p, a_q in the same row. There are only two such standard D_i with the required content:

(1) $\begin{array}{ll} a_1 & a_{d+1} \\ a_2 & \\ \vdots & \\ a_d & \end{array}$

(2) $\begin{array}{l} a_1 \\ a_2 \\ \vdots \\ a_{d+1} \end{array}$

Suppose that D is one of the tableaux (1), (2). Note that if $s_{i,d+1}\pi = s_{j,d+1}\rho$ with $\pi, \rho \in G_d$ then evaluating at $d+1$ we get i=j, and hence $\pi = \rho$. Therefore

$$\sum_{i=1}^{d+1} 1/d! \sum_{\pi \in G_d} \mathrm{Tab}(\ s_{i,d+1}.(\pi.D) \mid E\ )$$

(**) $$= 1/d! \sum_{\pi \in G_{d+1}} \mathrm{Tab}(\ \pi.D \mid E\ ).$$

If D is the tableau (2) above and E is the following standard tableau with entries from P:

$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_{d+1} \end{array}$ then the sum (**) is $(d+1)\alpha_{p_1} \odot \ldots \odot \alpha_{p_{d+1}} \neq 0$.

If D is the tableau (1) then, for any standard tableau E of the same shape with entries from P, we see that the sum (**) is zero by an argument similar to **3.15**, this time splitting G_{d+1} up into left cosets modulo the subgroup $\{1, s_{1,d+1}\}$. So D must be the tableau

$\begin{array}{ll} a_1 & a_{d+1} \\ a_2 & \\ \vdots & \\ a_d & \end{array}$

and we can express φ as a linear combination

$\varphi = \Sigma\, c(E).\varphi(E)$ where the sum is over all standard E of the same shape as D with entries from P , and

$$\varphi(E) = 1/d! \sum_{\pi \in G_d} \mathrm{Tab}(\ \pi.D \mid E\ ).$$

3.17 When E is the standard tableau

$$\begin{array}{ll} p_1 & p_{d+1} \\ p_2 & \\ \vdots & \\ p_d & \end{array}$$

with entries from P, then

$$\begin{aligned}
&\psi(E) \\
&= 1/d!\sum_{\pi\epsilon G_d} \mathrm{Tab}(\ \pi.D|E\ ) \\
&= 1/d!\sum_{\pi\epsilon G_d} \{\ (a_{\pi 1}|p_1)(a_{d+1}|p_{d+1})(a_{\pi 2}|p_2)\ldots(a_{\pi d}|p_d)\ - \\
&(a_{\pi 1}|p_{d+1})(a_{d+1}|p_1)(a_{\pi 2}|p_2)\ldots(a_{\pi d}|p_d)\ \} \\
&= 1/d!\sum_{\pi\epsilon G_d} (a_{\pi 1}|p_1)(a_{\pi 2}|p_2)\ldots(a_{\pi d}|p_d)(a_{d+1}|p_{d+1}) \\
&-1/d!\sum_{\pi\epsilon G_d} (a_{\pi 1}|p_{d+1})(a_{\pi 2}|p_2)\ldots(a_{\pi d}|p_d)(a_{d+1}|p_1)
\end{aligned}$$

which we identify with the (d+1)–multilinear map

$$\varphi(E) = (\alpha_{p_1}\odot\ldots\odot\alpha_{p_d})\otimes\alpha_{p_{d+1}} - (\alpha_{p_{d+1}}\odot\alpha_{p_2}\odot\ldots\odot\alpha_{p_d})\otimes\alpha_{p_1}\ .$$

A simple check shows that each $\varphi(E)$ above satisfies **3.12(a),(b)**. By the Standard Basis Theorem of Akin et al [**ABW II.2.16**], the $\psi(E)$ with E standard form a basis of $L_\lambda(V^*)$. This concludes the proof of **Prop3.12.** □

3.18 Proof of proposition 3.4:

We follow the five steps outlined at the start of this part.

Step 1: We define ψ as in **3.8** and note that it satisfies the conditions **(a),(b)** of **Prop3.12.**

Step 2: Was accomplished by proving **Prop3.12.**

Step 3: We apply the Standard Basis Theorem for $L_\lambda(V^*)$ using a convenient basis of V^*. Since M is nilpotent with $M^r \neq 0$, $M^{r+1}=0$ there is a basis of V of the form

$$B = \{\ v_1, Mv_1, \ldots, M^{r_1}v_1; v_2, Mv_2, \ldots, M^{r_2}v_2; \ldots; v_p, Mv_p, \ldots, M^{r_p}v_p\ \}\ \text{ with } r_1=r\geq r_2\geq\ldots\geq r_p.$$

We order the basis B by

$M^{j_1}v_{i_1} < M^{j_2}v_{i_2}$ means $i_1<i_2$, or else $i_1=i_2$ and $j_1<j_2$. Notice that if $i_1<i_2$ and $M^j v_{i_2} \neq 0$, then $M^j v_{i_1} \neq 0$. Let B^* be the dual ordered basis of V^*. To simplify the notation we will often denote elements of the basis B by $b_1,b_2,\ldots$etc , and then $b^*_1,b^*_2,\ldots$etc will denote the corresponding elements of the dual basis B^*.

We will use the linearly ordered set B as our set of places.

3.19 Some **definitions** and **notations:** If E is the standard tableau

$$\begin{array}{ll} b_1 & b_{d+1} \\ b_2 & \\ \vdots & \\ b_d & \end{array}$$

with entries from B we define

$$\varphi(E) = (b^*_1 \odot \ldots \odot b^*_d) \otimes b^*_{d+1} - (b^*_{d+1} \odot b^*_2 \odot \ldots \odot b^*_d) \otimes b^*_1$$

We have $\psi = \Sigma c(E).\varphi(E)$ where the sum is over all standard

$\begin{array}{ll} \square & \square \\ \square & \\ \vdots & \\ \square & \end{array}$ –shaped tableaux E with entries in B, and the coefficients

c(E) are in the ground field F. If $b \epsilon B$ and $b=M^j v_i$ we say that b has **power** j and **vector** v_i. Given **a** tableau E with entries from B, the **power of the tableau** E is the sum of the powers of the entries of E; and the **vector tableau** of E is the tableau obtained from E by replacing each entry of E by its vector. Given elements $b_1,\ldots,b_q$ of B we let

$$|b_1,\ldots,b_q| \;=\; b^*_1 \odot \ldots \odot b^*_q(b_1,\ldots b_q) .$$

We will prove **Prop3.4** by showing that $c(E)=0$ for all standard tableaux E of power $rd-1$

Step 4: We have the identity

$\psi(v_1,\ldots,Mv_d,v_{d+1}) = \psi(v_1,\ldots,Mv_{d+1},v_d)$ immediate from the definition of ψ and the symmetry of θ.

Step 5: From the identity of **Step 4** we deduce various relations between the coefficients c(E).

The proof is broken up into a number of lemmas.

Suggestion to the reader: At the first reading assume that d=2, so that $\lambda = \begin{smallmatrix}\square\square\\\square\end{smallmatrix}$.

In higher degrees the extra difficulty is only notational.

Lemma 1: Let E be the following standard tableau with entries from B:

$$\begin{matrix} b_1 & b_{d+1} \\ b_2 & \\ \vdots & \\ b_d & \end{matrix}$$

Then for $a_1,\ldots,a_{d+1}$ in B

$\psi(E)(a_1,\ldots,a_{d+1}) = 0$ unless the content of $(a_1,\ldots,a_{d+1})$ equals the content of E and either

(a) $b_{d+1} = a_{d+1}$ when $\psi(E) = |b_1,\ldots,b_d|$ or

(b) $b_1 = a_{d+1}$ when $\psi(E) = -|b_{d+1},b_2,\ldots,b_d|$.

Since E is standard at most one of (a) or (b) holds.

Proof: Immediate from the definitions. □

Lemma 2: Let E be the following standard tableau with entries from B

$$\begin{matrix} a_1 & a_{d+1} \\ a_2 & \\ \vdots & \\ a_d & \end{matrix}$$

Then

$$\psi(a_1,\ldots,a_{d+1}) = |a_1,\ldots,a_d|\,c(E).$$

Proof: Suppose that F is the following standard tableau with the same content as E:

$$\begin{matrix} b_1 & b_{d+1} \\ b_2 & \\ \vdots & \\ b_d & \end{matrix}$$

Then since the least entry in F is b_1, and the least entry in E is a_1; $a_1=b_1$. As $a_1<a_{d+1}$, $b_1<a_{d+1}$; so if $\varphi(F)(a_1,\ldots,a_{d+1}) \neq 0$ we must, by **Lemma 1**, have $b_{d+1} = a_{d+1}$. From this and the fact that E and F are standard with the same content it follows that E=F.□

Lemma 3: Let E be the following standard tableau with entries from B:

$$\begin{array}{ll} b_1 & b_{d+1} \\ b_2 & \\ \vdots & \\ b_d & \end{array}$$

Suppose that $Mb_{d+1}=0$. Then $c(E)=0$.

Proof: By **Lemma 2** $|b_1,\ldots,b_d|c(E) = \psi(b_1,\ldots,b_d,b_{d+1}) = \theta(b_1,\ldots,b_d,Mb_{d+1}) = 0$.□

Lemma 4: Let E be the following standard tableau with entries from B:

$$\begin{array}{ll} b_1 & b_{d+1} \\ b_2 & \\ \vdots & \\ b_{i-1} & \\ Mb_i & \\ b_{i+1} & \\ \vdots & \\ b_d & \end{array}$$

Suppose that $b_1<b_i$. Then there is a unique standard tableau F of the same shape as E, with first row $b_1\ b_i$, consisting of

$$\begin{array}{ll} b_1 & b_i \\ \vdots & \\ b_{i-1} & \\ b_{i+1} & \\ \vdots & \\ b_d & \end{array}$$

and with the entry Mb_{d+1} inserted somewhere below b_1. Furthermore,

$$|b_1,\ldots,\hat{b}_i,\ldots,b_d,Mb_i|c(E) = |b_1,\ldots,\hat{b}_i,\ldots,b_d,Mb_{d+1}|c(F).$$

Proof: By **Lemma 2**

$$|b_1,\ldots,\hat{b}_i,\ldots,b_d,Mb_i|\,c(E)$$
$$= \psi(b_1,\ldots,\hat{b}_i,\ldots b_d,Mb_i,b_{d+1})$$
$$= \theta(b_1,\ldots,\hat{b}_i,\ldots,b_d,Mb_i,Mb_{d+1})$$
$$= \theta(b_1,\ldots,\hat{b}_i,\ldots,b_d,Mb_{d+1},Mb_i)$$
$$= \psi(b_1,\ldots,\hat{b}_i,\ldots,b_d,Mb_{d+1},b_i)$$
$$= |b_1,\ldots,\hat{b}_i,\ldots,b_d,Mb_{d+1}|\,c(F)$$ by **Lemma 2** again. □

Lemma 5: Let E be the following standard tableau with entries from B:

$$\begin{array}{ll} Mb_1 & b_{d+1} \\ b_2 & \\ \vdots & \\ b_d & \end{array}$$

Suppose that $Mb_{d+1} \neq 0$.

Then the following tableau, F_1 is standard

$$\begin{array}{ll} b_1 & Mb_{d+1} \\ b_2 & \\ \vdots & \\ b_d & \end{array}$$

and for each $i=2,\ldots,d$ there is a unique standard tableau F_i

$$\begin{array}{ll} b_1 & b_i \\ \vdots & \\ \hat{b}_i & \\ \vdots & \\ b_d & \end{array}$$

with the entry Mb_{d+1} inserted somewhere below b_1. Furthermore

$$|Mb_1,b_2,\ldots,b_d| = -|b_2,\ldots,b_d,Mb_{d+1}| \sum_{i=1}^{d} c(F_i).$$

Proof: By **Lemma 2**

$$\psi(Mb_1,b_2,\ldots,b_d,b_{d+1}) = |Mb_1,b_2,\ldots,b_d|\,c(E)$$
$$= \theta(Mb_1,b_2,\ldots,b_d,Mb_{d+1})$$
$$= \theta(b_2,\ldots,b_d,Mb_{d+1},Mb_1)$$
$$= \psi(b_2,\ldots,b_d,Mb_{d+1},b_1)$$
$$= \Sigma\, c(F)\varphi(F)(b_2,\ldots,b_d,Mb_{d+1},b_1).$$

Note that since $b_1<Mb_1$ and $b_{d+1}<Mb_{d+1}$, and E is standard, b_1 is less than all of $b_2,\ldots,b_d,Mb_{d+1}$. Therefore any standard F with the same shape and content as E must be one of $F_1,\ldots,F_d$. The result now follows from **Lemma 1**, noting that condition (b) pertains.

Lemma 6: Let E be the following standard tableau with entries from B, and power $\geq$ rd–1:

$$\begin{array}{ll} b_1 & b_{d+1} \\ b_2 & \\ \vdots & \\ b_d & \end{array}$$

Then c(E)=0 , unless E has power rd–1 and the vector of each entry is not killed by M^r.

Proof: By **Lemma 3** and **Lemma 4**, if $c(E) \neq 0$ then c(E) is a rational linear combination of nonzero c(F)'s where the tableaux F have the same vector content and the same power as E, and have the form

$$\begin{array}{ll} a_1 & a_{d+1} \\ a_2 & \\ \vdots & \\ a_d & \end{array}$$

where a_1 has zero power and the power of a_{d+1} is $\leq$r–1. The maximal power of such an F is rd–1. So if c(E)$\neq$0 and has power $\geq$rd–1, then E and all the F's must have power rd–1. Therefore the vector of a_{d+1} has power r–1 and the vectors of $a_2,\ldots,a_d$ have power r. By **Lemma 3** the vector of a_{d+1} is not killed by M^r, and since $a_2,\ldots,a_d$ have power r their vectors are not killed by M^r. By definition of the ordered basis B, and since $a_1 \leq a_2$, the vector of a_1 is also not killed by M^r. Because the vector contents of the tableaux E and the

F's are equal, the vectors of E are not killed by M^r. □

From now on:

(a) we **fix the vector content** of our tableaux

(b) we assume that all these vectors are **not killed by M^r**

(c) we assume that all tableaux have **power rd–1**.

Lemma 7: Consider all the standard tableaux E with entries from B:

$$\begin{array}{ll} b_1 & b_{d+i} \\ b_2 & \\ \vdots & \\ b_d & \end{array}$$

which have a certain fixed vector content, and power rd–1. There is a rational constant C_0 such that:

if $Mb_{d+i} \neq 0$, and b_1 has power i, then c(E) is a **positive rational** multiple of $(-1)^i C_0$.

Proof: Let F be such a tableau:

$$\begin{array}{ll} a_1 & a_{d+1} \\ a_2 & \\ \vdots & \\ a_d & \end{array}$$

with vector tableau

$$\begin{array}{ll} w_1 & w_{d+1} \\ w_2 & \\ \vdots & \\ w_d & \end{array}$$

We will argue by induction on the power of a_1. Suppose that a_1 has power 0 so that $a_1 = w_1$. Then for i=2,...,d $a_i = M^r w_i$, and $a_{d+1} = M^{r-1} w_{d+1}$. Any other standard tableau F_i:

$$\begin{array}{ll} b_1 & b_{d+1} \\ b_2 & \\ \vdots & \\ b_d & \end{array}$$

with the same vector content; power $rd-1$; and for which b_1 has power 0; must have: $b_1=a_1=w_1$; first row $w_1 \ M^{r-1}w_i$ for some $2\leq i\leq d$ for which $w_1<M^{r-1}w_i$; and first column $w_1,M^rw_2,\ldots,M^rw_{i-1},M^rw_{i+1},\ldots,M^rw_d$ together with the entry M^rw_{d+1} inserted somewhere below w_1. By **Lemma 4** it follows that there is a constant C_0 such that for each of the above standard tableaux F, c(F) is a positive rational multiple of C_0. **Lemma 5** enables us to prove the result in the case b_1 of power $i+1$, having established it for b_i of power i. □

We now complete the proof of the proposition by showing that **the constant C_0 of Lemma 7 must be zero.** We need to distinguish a number of cases.

Suppose that the fixed vector content of our tableaux is $w_1\leq w_2\leq\ldots\leq w_d\leq w_{d+1}$.

Case I $\quad w_1<w_2$.

The tableaux

E_1:

$$\begin{array}{ll} M^rw_1 & M^{r-1}w_3 \\ w_2 & \\ M^rw_4 & \\ \vdots & \\ M^rw_{d+1} & \end{array}$$

E_2:

$$\begin{array}{ll} M^rw_1 & w_2 \\ M^{r-1}w_3 & \\ M^rw_4 & \\ \vdots & \\ M^rw_{d+1} & \end{array}$$

are standard of power $rd-1$. Now

$\psi(w_2,M^{r-1}w_3,M^rw_4,\ldots,M^rw_{d+1},M^rw_1) = 0$ by **Lemma 3** and

$= -|w_2,M^{r-1}w_3,M^rw_4,\ldots,M^rw_{d+1}|(\ c(E_1)+c(E_2)\ )$ by **Lemmas 1,3**. Since $c(E_1)+c(E_2)=0$, and by **Lemma 7** both $c(E_1)$ and $c(E_2)$ are positive rational multiples of $(-1)^rC_0$,we have $C_0=0$.

Case II: $w=w_1=w_2<w_3$.

The tableaux

E_1:
$$\begin{matrix} M^{r-1}w & w_3 \\ M^r w & \\ M^r w_4 & \\ \vdots & \\ M^r w_{d+1} & \end{matrix}$$

E_2:
$$\begin{matrix} M^{r-1}w & Mw_3 \\ M^{r-1}w & \\ M^r w_4 & \\ \vdots & \\ M^r w_{d+1} & \end{matrix}$$

are both standard of power rd–1.

$$|M^{r-1}w,M^rw,M^rw_4,\ldots,M^rw_{d+1}|\,c(E_1)$$
$$= \psi(M^{r-1}w,M^rw,M^rw_4,\ldots,M^rw_{d+1},w_3) \quad \text{by } \textbf{Lemma 2}$$
$$= \theta(M^{r-1}w,M^rw,M^rw_4,\ldots,M^rw_{d+1},Mw_3)$$
$$= \psi(M^{r-1}w,Mw_3,M^rw_4,\ldots,M^rw_{d+1},M^{r-1}w)$$
$$=-|M^{r-1}w,Mw_3,M^rw_4,\ldots,M^rw_{d+1}|\,c(E_2) \quad \text{by } \textbf{Lemmas 1,3}.$$

If $r=1$ then by **Lemma 3** $c(E_2)=0$, so $c(E_1)=0$ and by **Lemma 7** $C_0=0$. If $r>1$ then since by **Lemma 7** both $c(E_1)$ and $c(E_2)$ are positive rational multiples of $(-1)^rC_0$, we have $C_0=0$.

Case III: $w=w_1=w_2=w_3$.

We distinguish 3 subcases depending on $2r-1 \pmod 3$.

(a) $2r-1 = 3i$.

Then by **Lemmas 1,3** $\psi(M^iw,M^iw,M^rw_4,\ldots,M^rw_{d+1},M^iw) = 0$, because the only standard tableaux with the required content have first row: $M^iw \ \ M^rw_j$ for some $4\leq j\leq d+1$.

(b) $2r-1 = 3i+1$.

Firstly suppose that $i+1>r-1$. Since $2r-1=3i+1$, we have $i+1>r-1$ iff $4>r$. The only integer r with $1\leq r<4$ and $2r-1 = 1 \pmod 3$ is $r=1$. But then the tableau

$$\begin{matrix} w & M^{r-1}w \\ M^r w & \\ M^r w_4 & \\ \vdots & \\ M^r w_{d+1} & \end{matrix}$$

is not standard.

Next we suppose that $i+1\leq r-1$. By **Lemma 3**, the tableau E:

$$\begin{matrix} M^i w & M^{i+1}w \\ M^i w & \\ M^r w_4 & \\ \vdots & \\ M^r w_{d+1} & \end{matrix}$$

is the only possible standard tableau with that content and $c(E)\neq 0$. Therefore

$$-|M^i w, M^{i+1}w, M^r w_4, \ldots, M^r w_{d+1}|\, c(E)$$

$$= \psi(M^i w, M^{i+1}w, M^r w_4, \ldots, M^r w_{d+1}, M^i w) \quad \text{by } \textbf{Lemmas 1,3}$$

$$= \psi(M^{i+1}w, M^{i+1}w, M^r w_4, \ldots, M^r w_{d+1}, M^{i-1}w)$$

$$= -|M^{i+1}w, M^{i+1}w, M^r w_4, \ldots, M^r w_{d+1}|\, c(F) \quad \text{by } \textbf{Lemma 1} \text{ where the tableau F:}$$

$$\begin{matrix} M^{i-1}w & M^{i+1}w \\ M^{i+1}w & \\ M^r w_4 & \\ \vdots & \\ M^r w_{d+1} & \end{matrix}$$

is the only possible tableau of that shape and content with $c(F)\neq 0$. It follows from **Lemma 7** that $C_0=0$.

(c) $2r-1 = 3i+2$.

Let E be the tableau:

$$\begin{array}{ll} M^i w & M^{i+1}w \\ M^{i+1}w & \\ M^r w_4 & \\ \vdots & \\ M^r w_{d+1} & \end{array}$$

We must have $i+1<r$, since if $i+1\geq r$ then $0\geq r$.

$$|M^i w, M^{i+1}w, M^r w_4, \ldots, M^r w_{d+1}|\, c(E)$$

$$= \psi(M^i w, M^{i+1}w, M^r w_4, \ldots, M^r w_{d+1}, M^i w) \quad \text{by } \textbf{Lemma 1}$$

$$= \psi(M^i w, M^{i+2}w, M^r w_4, \ldots, M^r w_{d+1}, M^i w)$$

$$= -|M^i w, M^{i+2}w, M^r w_4, \ldots, M^r w_{d+1}|\, c(F) \quad \text{by } \textbf{Lemmas 1,3}$$

where F is the tableau:

$$\begin{array}{ll} M^i w & M^{i+2}w \\ M^i w & \\ M^r w_4 & \\ \vdots & \\ M^r w_{d+1} & \end{array}$$

which is the only possible standard tableau with that content and $c(F)\neq 0$.

If $i+2 = r$ then by **Lemmas 3,7** $C_0=0$.

If $i+2 < r$ then by **Lemma7** $C_0=0$.

This completes the proof of Proposition 3.4. □

3.20 By **§§3.4;3.5;3.6**; and **[BOR I(3.3)Cor]** or **[CHE1 II§8 Thm5]** it follows that when θ is nonsingular then $G(\theta)$ is finite.

CHAPTER FIVE

A Miscellany

This chapter consists of three rather disparate parts, motivated by a desire to generalise fundamental results on quadratic forms. The first two concern the behaviour of forms under a change of the base. The third part arose from an attempt to understand composition of higher degree forms. Our initial formulation of the main result of the first part was an extension to higher degree of the weak Hasse–Minkowski Theorem **[SCHA p193 lemma7.4(i)]**, which states that if f is a quadratic form defined over an algebraic number field F and if for all primes p of F, including the infinite ones, f is equivalent to a hyperbolic form over the completion of F at the prime p, then f is equivalent to a hyperbolic form over F.

(1) In the first part we prove the following: Let the base field F have characteristic zero, and let K|F be a field extension. Let $\phi : W^{d+1} \to F$ be a symmetric (d+1)–multilinear form with $d \geq 2$, and let $\phi_K : W_K{}^{d+1} \to K$ be the extended form. Then if ϕ_K is equivalent over K to a hyperbolic form, the form ϕ must itself be equivalent over F to a hyperbolic form.

The higher degree analogue of the weak Hasse–Minkowski theorem is an immediate consequence.

Compare with the quadratic case, where e.g. $\Sigma(x_i^2+y_i^2)$ is not hyperbolic over $\mathbb{R}$ but becomes hyperbolic over $\mathbb{C}$.

The proof uses the following two general results about the extension K|F :

(i) **[WAT1 p94]**, or **[CHE1 p192]** : Let V be a finite dimensional vector space over F. Let G be an algebraic subgroup of GL(V). Let G_K be the algebraic subgroup of $GL(V \otimes_F K)$ obtained from G by extension of scalars. Let L(G), $L(G_K)$ be the Lie algebras of G, G_K respectively. Then $L(G_K) = L(G) \otimes_F K$.

This result does not require F to have characteristic zero.

(ii) [BOU 5.6] or [CHE2 p107] : Let L be a Lie algebra over F with radical L_r. Then the radical of the Lie algebra $L \otimes_F K$ over K, is $L_r \otimes_F K$.

This result requires F to have characteristic zero.

We also apply our calculation, 4§§2.7–2.12, of the Lie algebras of the hyperbolic forms and their radicals. Recall that to compute these radicals we needed to be in characteristic zero.

(2) In the second part R is a complete discrete valuation ring with maximal ideal M and residue field k. We consider the base change $R \to R/M=k$.

Notation: If N is an R–module and $n \in N$, we set

$n(\text{mod } MN) = \bar{n} \in N/MN.$

Let $\theta : (R^n)^d \to R$ be a symmetric d–multilinear form. Then we define the symmetric d–multilinear form

$\bar{\theta} : (k^n)^d \to k$ by

$\bar{\theta}(\bar{v}_1,\dots,\bar{v}_d) = \theta(v_1,\dots,v_d) \ (\text{mod } M) \in k.$

It is easy to see that this does not depend on the choice of $v_1,\dots,v_d$. When d=2 it is known, [SCHA p26], that if φ, ϕ are two such nonsingular symmetric bilinear forms with $\bar{\varphi} = \bar{\theta}$ then φ and ϕ are equivalent over R.

We give an example for each $d \geq 3$ of two nonsingular symmetric d–multilinear forms $\varphi, \phi : (R^3)^d \to R$, with $\bar{\varphi} = \bar{\theta}$, but the centre of φ not isomorphic to the centre of θ.

(3) The third part derives from an attempt to follow M.Kneser's **[KNE]** treatment of composition of binary quadratic forms in the ternary cubic case. As W.C.Waterhouse points out in his paper **[WAT2]** on composition of forms, Kneser's approach via Clifford algebras does not automatically generalize to higher degree. Clifford algebras of higher degree forms are not well behaved. Apart from the Clifford algebras of quadratic forms, only the structure of the Clifford algebras of binary cubics is known **[HAI]**. We hoped to use Van den Bergh's **[VDB]** result on the existence of a rank 3 linearization of the Clifford algebra of a nonsingular ternary cubic form to study composition of such forms.

What we could prove is that if F is a field of characteristic zero with algebraic closure K, then any nonsingular ternary cubic form f over F is of the form

$f(u_1,u_2,u_3) = \det(u_1\alpha_1 + u_2\alpha_2 + u_3\alpha_3)$ with $\alpha_1,\alpha_2,\alpha_3 \in M_3(K)$.

1 We state the main proposition of this section.

1.1 **Proposition:** Let F be a field of characteristic zero. Let (W,θ) be a symmetric space over F of degree ≥ 3, and let W have dimension >2 over F. Let $K|F$ be a field extension. Suppose that the symmetric space (W_K, θ_K) is isomorphic to a hyperbolic symmetric space over K. Then (W,θ) is isomorphic to a hyperbolic symmetric space over F.

See the Appendix at the end of this chapter for a proof of the case dim W = 2.

Proof: There is no loss of generality in assuming that for some F–vector space V,

$W = V \oplus (\odot^d(V))$, where $\dim_F(V) > 1$, ψ is the hyperbolic symmetric (d+1)–multilinear form defined in 2§1.10, and

there is some $\sigma \in GL(W_K)$ with $\theta_K = \sigma.\psi_K$.

We must show that there exists a $\rho \in GL(W)$ with $\rho.\theta = \psi$.

We will break up the proof into a series of lemmas.

First we characterize those symmetric (d+1)–multilinear forms on W which are equivalent to the hyperbolic form ψ.

1.2 **Lemma:** Suppose that $\varphi : W^{d+1} \to F$ is a nondegenerate, symmetric, (d+1)–multilinear form such that

(a) $W = U \oplus S$ with $\dim_F S = \dim_F \odot^d(U)$

(b) $\varphi(s,s,w_3,\ldots,w_{d+1}) = 0$ for all $s \in S$, all $w_3,\ldots,w_{d+1} \in W$

(c) $\varphi(u,\ldots,u) = 0$ for all $u \in U$.

Then φ is equivalent to the hyperbolic form ψ.

Proof: Since F has characteristic zero we can, by 1§13, prove the lemma by showing that the polynomial maps $W \to F$ corresponding to φ, ψ as in 1§5, are equivalent.

Let $v_1,\ldots,v_n$ be a basis of V and let $x_1,\ldots,x_n$ be the dual basis of V^*. Then the $(x_1^{\odot i_1})\odot\ldots\odot(x_n^{\odot i_n})$ where $(i_1,\ldots,i_n)$ is an n–tuple of nonnegative integers with sum d, form a basis of $\odot^d(V)$. Let

$(t|i_1...i_n)$ be the corresponding element of the dual basis of $(\odot^d(V))^*$. Now

$\psi(..., (v,\theta),...) = (d+1)\theta(v,...,v)$. With the basis

$x_1,...,x_n;...,(t|i_1...i_n),...$ of $(V \oplus \odot^d(V))^*$ the polynomial map corresponding to ψ is

$(d+1)\Sigma (t|i_1...i_n) x_1^{i_1}...x_n^{i_n}$ where the sum is over all n–tuples $(i_1,...,i_n)$ with sum d.

Let

$\{s_{i_1...i_n} \mid (i_1,...,i_n)$ is an n–tuple of nonnegative integers with sum d $\}$ be a basis of S^*, and let $u_1,...,u_n$ be a basis of U^*. Together these form a basis of $(U\oplus S)^*$.

By (a), (b) and **1§21**, with this choice of basis the polynomial map $W \to F$ corresponding to φ is of the form

$\Sigma\, l_{i_1...i_n}(s)u_1^{i_1}...u_n^{i_n}$ where the sum is over all n–tuples of nonnegative integers $(i_1,...,i_n)$ with sum d, and each $l_{i_1...i_n}(s)$ is a linear form in the $s_{j_1...j_n}$.

If the forms were not linearly independent then φ would be degenerate. So these forms are linearly independent, and we can make an invertible linear change of variables

$l_{i_1...i_n}(s) \mapsto (d+1)(t|i_1...i_n)$

$u_j \mapsto x_j$

completing the proof of the lemma.□

We isolate condition (b) of **Lemma 1.2**.

1.3 **Definition:** Let $\phi: W^{d+1} \to F$ be a (d+1)–multilinear form. Then we define

$T(\phi) = \{ t \in W : \phi(t,t,w_3,...,w_{d+1}) = 0$ for all $w_3,...,w_{d+1} \in W \}$.

1.4 It is easy to see, by an argument like that of 2§1.14, that

for all $\tau \in GL(W)$, $T(\tau\phi) = \tau T(\phi)$

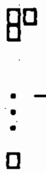
We investigate $T(\psi)$ for the hyperbolic form ψ, and characterize it in terms of the Lie algebra of ψ.

1.5 Lemma: (i) $T(\psi) = \{ (0,\theta) : \theta \in \odot^d(V) \}$ and in terms of its Lie algebra

(ii) $T(\psi) = \{ w \in W : [L(\psi)_r , L(\psi)_r].w = 0 \}$

Proof: $(v,\theta) \in T(\psi) \Rightarrow \psi[(v,\theta),(v,\theta),(0,\theta_3),(v,0),\ldots(v,0)]=0$ for all θ_3, and so $v=0$. Clearly $(0,\theta)\in T(\psi)$ so we have proved (i). In **4§2.12** we calculated the first derived algebra of the radical of $L(\psi)$:

$[L(\psi)_r,L(\psi)_r]$ consists of all endomorphisms of W of the form $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ such that the map $W^{d+1} \to F$ defined by $C(w_{d+1})(w_1,\ldots,w_d)$ has



–shaped symmetry. Clearly $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \theta \end{pmatrix} = 0$. Suppose that $v\neq0$. Then since $\dim(V)>1$, there is a basis $v_1,\ldots,v_n=v$ of V with $n>1$. Let $x_1,\ldots,x_n$ be the dual basis of V^*. Consider the element of the first derived algebra of the radical of $L(\psi)$ corresponding to the form

$(x_1\odot\ldots\odot x_1)\otimes x_n - (x_n\odot x_1\odot\ldots\odot x_1)\otimes x_1$. Then $(Cv)(v_1,\ldots,v_1)=1\neq0$. So when $v\neq0$ (v,θ) is not killed by $[L(\psi)_r , L(\psi)_r]$ and we have proved (ii).□

Now we turn our attention to the form θ: $W^{d+1} \to F$ for which we have some $\sigma\in GL(W_K)$ such that $\theta_K = \sigma.\psi_K$.

1.6 Lemma: $T(\theta_K) = \{ w \epsilon W_K : [L(\theta_K)_r , L(\theta_K)_r].w = 0 \}$

Proof: By §1.4

$T(\theta_K) = \sigma.T(\psi_K)$. By **4§1.6(ii)** $L(\theta_K) = \sigma.L(\psi_K).\sigma^{-1}$. Also conjugation by σ maps the lattice of solvable ideals of $L(\theta_K)$ isomorphically onto the lattice of solvable ideals of $L(\psi_K)$. Therefore

$$L(\psi_K)_r = \sigma^{-1}.L(\theta_K)_r.\sigma \text{ and}$$

$$[L(\theta_K)_r , L(\theta_K)_r].w = 0$$

$$\text{iff } \sigma^{-1}.[L(\theta_K)_r , L(\theta_K)_r].\sigma.\sigma^{-1}.w = 0$$

$$\text{iff } [\sigma^{-1}.L(\theta_K)_r.\sigma , \sigma^{-1}.L(\theta_K)_r.\sigma].\sigma^{-1}w = 0$$

$$\text{iff } [L(\psi_K)_r, L(\psi_K)_r].\sigma^{-1}.w = 0$$

which by **Lemma 1.5** means that $\sigma^{-1}w \epsilon T(\psi_K)$ i.e. $w \epsilon \sigma.T(\psi_K) = T(\theta_K)$ by **1.4**.□

In the next Lemma we show that for any $\rho\epsilon GL(W)$, $T(\rho.\theta)$ can be characterised similarly in terms of the radical of the Lie algebra of $\rho.\theta$.

1.7 Lemma For any $\rho\epsilon GL(W)$ $T(\rho.\theta) = \{ w\epsilon W : [L(\rho.\theta)_r , L(\rho.\theta)_r].w = 0 \}$ a subspace of W with dimension equal to $\dim_F(\odot^d(V))$.

Proof: We have for all $\rho\epsilon GL(W)$

(A) $L(\rho.\theta_K)_r = L(\rho.\theta)_r \otimes_F K$ **[CHE2 p107 Prop3]** or **[BOU §5.6]**.

(B) $L(\rho.\theta_K) = L(\rho.\theta) \otimes_F K$ **[CHE1 p129 Prop2]**.

Define the linear map

$$\alpha : W \to \mathrm{Hom}_F([L(\rho.\theta)_r , L(\rho.\theta)_r] , W)$$

by $\alpha(w)(f) = f(w)$. We will prove the lemma by showing that $\ker\alpha = T(\rho.\theta)$.

Applying $-\otimes_F K$ we obtain

$\alpha\otimes 1 : W_K \to \mathrm{Hom}_F(\, [\, L(\rho.\theta)_r\, ,\, L(\rho.\theta)_r\,]\, ,\, W\,) \otimes_F K$. Now

$\mathrm{Hom}_F(\, [L(\rho.\theta)_r\, ,\, L(\rho.\theta)_r]\, ,W\,) \otimes_F K$

$\cong \mathrm{Hom}_K(\, [L(\rho.\theta)_r\, ,\, L(\rho.\theta)_r]\otimes_F K\, ,\, W_K\,)$

$\cong \mathrm{Hom}_K(\, [L(\, (\rho.\theta)_K\,)_r\, ,\, L(\, (\rho.\theta)_K\,)_r]\, ,\, W_K\,)$ by (A) and (B).

By **Lemma 1.6** the kernel of $\alpha\otimes 1$ is $T(\, (\rho.\theta)_K\,)$. Since $K|F$ is flat, $\ker(\alpha\otimes 1)=\ker\alpha\otimes_F K$. It follows that $\ker\alpha \subseteq T(\rho.\theta)$. The canonical embedding

$W \to W_K$ maps $T(\rho.\theta)$ into $T(\, (\rho.\theta)_K\,)$. If $\ker\alpha \neq T(\rho.\theta)$ then we contradict

$\dim_F\ker(\alpha) = \dim_K\ker(\alpha\otimes 1) = \dim_K T(\, (\rho.\theta)_K\,)$.□

We have established that

for every $\rho\in GL(W)$ the form $\rho.\theta$ satisfies (a), (b) of **Lemma 1.2.**

Now we try to find $\rho\in GL(W)$ such that $\rho.\theta$ also satisfies condition (c) of **Lemma 1.2.**

In **4§§2.7–2.12** we calculated $L(\psi_K)$. We found in **4§2.10** that if

$X\in L(\psi_K)$ then, by **Lemma 2**, $X.T(\psi_K) \subseteq T(\psi_K)$. By **4§1.6(ii)** and **[CHE1 p129, Prop2]** we have

$L(\theta_K) = L(\sigma.\psi_K) = \sigma.L(\psi_K).\sigma^{-1}$ and $L(\theta_K) = L(\theta)\otimes_F K$. Also by **1.1**

$T(\theta_K) = T(\sigma.\psi_K) = \sigma.T(\psi_K)$ so

$X\in L(\theta_K)$

$\Leftrightarrow X\in\sigma.L(\psi_K).\sigma^{-1}$

$\Leftrightarrow \sigma^{-1}X\sigma \in L(\psi_K)$

$\Rightarrow \sigma^{-1}X\sigma.T(\psi_K) \subseteq T(\psi_K)$

$\Rightarrow X.\sigma T(\psi_K) \subseteq \sigma T(\psi_K)$

$\Rightarrow X.T(\theta_K) \subseteq T(\theta_K)$. Now $L(\theta_K) = L(\theta)\otimes_F K$ and by **Lemma 1.7**

$T(\theta_K) = T(\theta)\otimes_F K$, and $T(\theta)$ is a subspace of W. Therefore

$X.T(\theta) \subseteq T(\theta)$ for all $X\in L(\theta)$, and we have a homomorphism of Lie algebras

$L(\theta) \to \mathcal{gl}(\,W/T(\theta)\,).$

1.8 Lemma The above canonical homomorphism of Lie algebras is an epimorphism.

Proof: By 4§2.11 we know that the canonical homomorphism of Lie algebras $L(\psi_K) \to \mathcal{gl}(\,W_K/T(\psi_K)\,)$ is an epimorphism. The following square commutes:

$$\begin{array}{ccc} \sigma.L(\theta_K).\sigma^{-1}=L(\psi_K) & \longrightarrow & \mathcal{gl}(\,W_K/\sigma T(\theta_K)\,) = \mathcal{gl}(\,W_K/T(\psi_K)\,) \\ \uparrow & & \downarrow \\ L(\theta_K) & \longrightarrow & \mathcal{gl}(\,W_K/T(\theta_K)\,) \end{array}$$

The horizontal maps are the canonical ones; the upward map is $f \mapsto \sigma.f.\sigma^{-1}$; and the downward map is $\bar{f} \mapsto \bar{\sigma}^{-1}.\bar{f}.\bar{\sigma}$. All the homomorphisms in the square, except perhaps the bottom one, are known to be epimorphisms, so the bottom one is too. But

$L(\theta_K) \cong L(\theta)\otimes_F K$ and $\mathcal{gl}(\,W_K/T(\theta_K)\,) \cong \mathcal{gl}(\,W/T(\theta)\,)\otimes_F K$

so the bottom epimorphism comes from applying $\underline{\quad}\otimes_F K$ to the homomorphism

$L(\theta) \to \mathcal{gl}(\,W/T(\theta)\,)$. Since $K|F$ is faithfully flat the proof is finished. □

1.9 Let U be a subspace of W complementary to $T(\theta)$:

$W = U \oplus T(\theta)$. We will write endomorphisms of W as matrices

$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ where A, D are endomorphisms of U, $T(\theta)$ respectively, and $C\colon U \to T(\theta)$ is a linear map. By **Lemma 1.8** there is some $\begin{pmatrix} 1 & 0 \\ C & D \end{pmatrix} \in L(\theta)$. So

for all $u \in U$, $t \in T(\theta)$

$$\theta[\begin{pmatrix} 1 & 0 \\ C & D \end{pmatrix}\begin{pmatrix} u \\ t \end{pmatrix}, \begin{pmatrix} u \\ t \end{pmatrix}, \ldots, \begin{pmatrix} u \\ t \end{pmatrix}] = 0$$

$$\Leftrightarrow \theta[\begin{pmatrix} u \\ Cu+Dt \end{pmatrix}, \begin{pmatrix} u \\ t \end{pmatrix}, \ldots, \begin{pmatrix} u \\ t \end{pmatrix}] = 0$$

$$\Leftrightarrow \theta[\begin{pmatrix} u \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} u \\ 0 \end{pmatrix}] + \theta[\begin{pmatrix} 0 \\ Cu+Dt \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} u \\ 0 \end{pmatrix}] + d\theta[\begin{pmatrix} u \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} u \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t \end{pmatrix}] = 0$$ because

F has characteristic zero and $T(\theta)$ is a subspace of W, so that **1§21(i)** applies. Since F has characteristic zero, we deduce from **1§20** that the terms of degree 1 in t, and d in u sum to zero:

$\theta[\begin{pmatrix}0\\ Dt+dt\end{pmatrix},\begin{pmatrix}u\\ 0\end{pmatrix},\ldots,\begin{pmatrix}u\\ 0\end{pmatrix}] = 0$. Since ψ_K is nondegenerate θ_K is too, hence so is θ. By definition of $T(\theta)$ and after an application of **1§21(i)** we conclude that

$Dt=-dt$. Therefore we have

$\begin{pmatrix}1 & 0\\ C & -d\end{pmatrix} \in L(\theta)$. Let

$\rho = \begin{pmatrix}1 & 0\\ -(d+1)^{-1}C & 1\end{pmatrix} \in GL(W)$. Note that

$T(\rho.\theta) = \rho T(\theta) = T(\theta)$. Also

$L(\rho.\theta) = \rho.L(\theta).\rho^{-1}$, so we have

$\rho\begin{pmatrix}1 & 0\\ C & -d\end{pmatrix}\rho^{-1} = \begin{pmatrix}1 & 0\\ 0 & -d\end{pmatrix} \in L(\rho.\theta)$. It follows that for all $u \in U$

$(\rho.\theta)[\begin{pmatrix}1 & 0\\ 0 & -d\end{pmatrix}\begin{pmatrix}u\\ 0\end{pmatrix},\begin{pmatrix}u\\ 0\end{pmatrix},\ldots,\begin{pmatrix}u\\ 0\end{pmatrix}] = 0$ i.e.

$(\rho.\theta)[\begin{pmatrix}u\\ 0\end{pmatrix},\ldots,\begin{pmatrix}u\\ 0\end{pmatrix}] = 0$ for all $u \in U$.

This completes the proof of the proposition. □

2 Let R be a complete discrete valuation ring with parameter π. Scharlau **[SCHA p26,Thm6.13]** shows that a symmetric bilinear form over the residue field $R/\pi R$ lifts uniquely up to equivalence, to a symmetric bilinear form over R. This is not true in degree $d \geq 3$ as the following example shows.

2.1 Example: Consider the following symmetric d–multilinear forms

$\varphi,\theta : (R^3)^d \to R$ defined below.

Let e_1,e_2,e_3 be the usual basis of R^3.

Define

$$\begin{aligned}\varphi(e_{\alpha 1},\ldots,e_{\alpha d}) &= 1 \text{ if } \alpha 1=\ldots=\alpha d\\ &= 0 \text{ otherwise.}\end{aligned}$$

Define for $\alpha 1 \leq \ldots \leq \alpha d$

$$\begin{aligned}\theta(e_{\alpha 1},\ldots,e_{\alpha d}) &= 1 \text{ if } \alpha 1=\ldots=\alpha d\\ &= \pi \text{ if } \alpha 1=1,\ \alpha 2=\ldots=\alpha d=2\\ &= \pi \text{ if } \alpha 1=2,\ \alpha 2=\ldots=\alpha d=3\\ &= 0 \text{ otherwise.}\end{aligned}$$

Clearly φ and θ are equal mod πR. It is easy to see that φ is nonsingular, and that the projections onto the submodules Re_i are in the centre of φ. We will show that θ is nonsingular and has trivial centre. Then it follows that φ and θ cannot be equivalent.

2.2 We show that θ is nonsingular. The quotient field K of R is complete w.r.t. the unique extension of the given valuation to K. Let Ω be the algebraic closure of K. Then **[LANG p291,Prop4]** the valuation extends uniquely to Ω. If θ is singular then there exists

$x=(x_1,x_2,x_3) \in \Omega^3$ with $x \neq 0$ and

(1) $0 = \theta(x,\ldots,x,e_1) = x_1^{d-1} + \pi x_2^{d-1}$

(2) $0 = \theta(x,\ldots,x,e_2) = x_2^{d-1} + (d-1)\pi x_1 x_2^{d-2} + \pi x_3^{d-1}$

(3) $0 = \theta(x,\ldots,x,e_3) = x_3^{d-1} + (d-1)\pi x_2 x_3^{d-2}$.

If $x_2=0$ then by (1), $x_1=0$ and so by (2) $x_3=0$. Suppose that $x_2 \neq 0$. Then

(4) $(x_1/x_2)^{d-1} + \pi = 0$

(5) $1 + (d-1)\pi(x_1/x_2) + \pi(x_3/x_2)^{d-1} = 0$

(6) $(x_3/x_2)^{d-1} + (d-1)\pi(x_3/x_2)^{d-2} = 0$

If $(x_3/x_2) = 0$ then by (5) and as $d-1 \neq 0$, we get $\mathrm{ord}_\pi(x_1/x_2)<0$. But we also have by (4) that $\mathrm{ord}_\pi(x_1/x_2)>0$. Therefore $x_3/x_2 \neq 0$ and hence from (6)

(7) $x_3/x_2 + (d-1)\pi = 0$, with $d-1 \neq 0$. By (4) $\mathrm{ord}_\pi(x_1/x_2) > 0$ and by (7) $\mathrm{ord}_\pi(x_3/x_2)>0$: together these contradict (5). It follows that $x_2=0$, so as observed above $x_1=x_2=x_3=0$.

2.3 Let L be any field extension of K. Let $\theta_L\colon L^3 \to L$ denote the extension of θ. We will show that the centre of θ_L is L. This is elementary linear algebra. Taking coordinates w.r.t. the usual ordered basis e_1,e_2,e_3 of L^3 we will represent endomorphisms of L^3 by matrices. Suppose that $A = (a_{ij}) \epsilon M_3(L)$ is in the centre of θ_L.

$\theta_L(Ae_1,e_2,e_3,\ldots,e_3) = \theta_L(e_1,Ae_2,e_3,\ldots,e_3) = 0$, so

(1) $a_{31}=0$.

$0 = \theta_L(Ae_1,e_3,e_1,\ldots,e_1) = \theta_L(e_1,Ae_3,e_1,\ldots,e_1)$, so

(2) $a_{13}=0$.

$0 = \theta_L(e_1,Ae_2,e_3,e_2,\ldots,e_2) = \theta_L(e_1,e_2,Ae_3,e_2,\ldots,e_2)$, so

(3) $a_{23}=0$.

$\theta_L(Ae_1,e_3,e_3,\ldots,e_3) = \theta_L(e_1,Ae_3,e_3,\ldots,e_3) = 0$, so

(4) $a_{21}=0$.

$\theta_L(Ae_1,e_2,e_1,\ldots,e_1) = \theta_L(e_1,Ae_2,e_1,\ldots,e_1)$ so if $d=3$ $\pi a_{21}=a_{12}$, and if $d>3$ $0=a_{12}$; but since by (4) $a_{21}=0$

(5) $a_{12}=0$.

$\theta_L(Ae_2,e_3,e_2,e_3,\ldots,e_3) = \theta_L(e_2,Ae_3,e_2,e_3,\ldots,e_3)$ so if $d=3$ $\pi a_{32}=\pi a_{13}+a_{23}$, and if $d>3$ $\pi a_{32}=0$; but since by (2) $a_{13}=0$, and by (3) $a_{23}=0$, in any case

(6) $a_{32}=0$.

$\theta_L(Ae_1,e_2,e_2,\ldots,e_2) = \theta_L(e_1,Ae_2,e_2,\ldots,e_2)$ so $\pi a_{11}+a_{21}=\pi a_{22}$ and so by (4)

(7) $a_{11}=a_{22}$.

$\theta_L(Ae_2,e_3,e_3,\ldots,e_3) = \theta_L(e_2,Ae_3,e_3,\ldots,e_3)$ so $\pi a_{22}+a_{32}=\pi a_{33}$ and so by (6)

(8) $a_{22}=a_{33}$.

We conclude that the centre of θ_L is L. □

3 The main result of this part is that any nonsingular ternary cubic form f over a field of characteristic zero is of the form

$f(u_1,u_2,u_3) = \det(u_1\alpha_1+u_2\alpha_2+u_3\alpha_3)$ where $\alpha_1,\alpha_2,\alpha_3$ are 3×3 matrices with entries in the algebraic closure of the ground field.

Let F be the ground field, and let K be the algebraic closure of F. Let F be of characteristic zero.

3.1 Van den Bergh **[VDB §3 Thm2]** showed that given any nonsingular ternary cubic form f, over a field of characteristic zero, its Clifford algebra has a rank 3 linearization. This ammounts to finding three 3×3 matrices $\alpha_1,\alpha_2,\alpha_3$ with entries in the algebraic closure of the ground field, such that

$(u_1\alpha_1 + u_2\alpha_2 + u_3\alpha_3)^3 = f(u_1,u_2,u_3)I$ for all u_1,u_2,u_3 in the ground field.

3.1.1 **Remark:** Notice that if, for a nonempty Zariski open subset of points $(u_1,u_2,u_3)\epsilon A^3$, the minimal polynomial of $u_1\alpha_1+u_2\alpha+u_3\alpha_3$ were equal to its characteristic polynomial then we would have

$f(u_1,u_2,u_3) = \det(u_1\alpha_1 + u_2\alpha_2 + u_3\alpha_3)$ for all u_1,u_2,u_3 in the ground field.

3.2 **Definitions:** Let $f(t_1,...,t_n)$ be a form of degree d with coefficients in F. Let $F\{x_1,...,x_n\}$ denote the free noncommutative F–algebra generated by the indeterminates $x_1,...,x_n$. Let I_f denote the 2–sided ideal of $F\{x_1,...,x_n\}$ generated by

$\{ (\Sigma u_i x_i)^3 - f(u_1,...,u_n) : (u_1,...,u_n)\epsilon F^n \}$. The **Clifford algebra** C_f of the form f is $F\{x_1,...,x_n\}/I_f$.

A rank m linearization of C_f is an F–algebra homomorphism

$\varphi : C_f \to M_m(K)$.

As stated in **3.1**, Van den Bergh showed that for every nonsingular ternary cubic form f over F, C_f has a rank 3 linearization

$\varphi : C_f \to M_3(K)$. By **Definition 3.2** this means that there exist $\alpha_1, \alpha_2, \alpha_3 \in M_3(K)$ such that

$(u_1\alpha_1+u_2\alpha_2+u_3\alpha_3)^3 - f(u_1,u_2,u_3) = 0$ for all $u_1,u_2,u_3 \in F$, so that we can put $\varphi(x_i)=\alpha_i$.

3.3 Proposition: Let $f \in F[t_1,t_2,t_3]$ be a ternary cubic form. Suppose there are $\alpha_1, \alpha_2, \alpha_3 \in M_3(K)$ with

$(\Sigma\alpha_i u_i)^3 - f(u_1,u_2,u_3)I = 0$ for all $(u_1,u_2,u_3) \in F^3$. Then if f is not a cube in $K[t_1,t_2,t_3]$

$\det(\Sigma u_i\alpha_i) = f(u_1,u_2,u_3)$ for all $(u_1,u_2,u_3) \in F^3$.

Proof: Since $(\Sigma u_i\alpha_i)^3 - f(u_1,u_2,u_3)I = 0$ for all $(u_1,u_2,u_3) \in F^3$, and F has characteristic zero, we know, by 1§20, that this equation is in fact true for all $(u_1,u_2,u_3) \in K^3$. We will show that the minimal polynomial of $\Sigma u_i\alpha_i$ has degree 3.

Define the map $\alpha : A^3_K \to M_3(K)$ by

$\alpha(u_1,u_2,u_3) = \Sigma u_i\alpha_i$. α is a morphism of affine algebraic varieties over the algebraically closed field K. We will prove that:

(1) The subset of $M_3(K)$ consisting of matrices with minimal polynomial of degree 3 is Zariski open.

(2) The inverse image under α of this open set is not empty.

Then it follows by **Remark 3.1.1** that $\det(\Sigma u_i\alpha_i) = f(u_1,u_2,u_3)$ on a nonempty Zariski open subset of A^3_K, and hence on its Zariski closure A^3_K.

Proof of (1): Consider the pairs $X \epsilon M_n(K)$, $(x_0:\ldots:x_{n-1}) \epsilon \mathbb{P}^{n-1}$ with $x_0 + x_1X + \ldots + x_{n-1}X^{n-1} = 0$. The above matrix equation amounts to the vanishing of n^2 polynomials in the entries of X and the x_i, and these polynomials are all linear in the x_i. Therefore by **[SHA p44 Thm1]** these pairs form a Zariski closed set in $M_n(K) \times \mathbb{P}^{n-1}$. The image of this closed set under the projection onto $M_n(K)$ is exactly the set of matrices whose minimal polynomial has degree less than n. By **[SHA p45 Thm3]** this image is a Zariski closed subset of $M_n(K)$. This proves (1). □

Proof of (2): Suppose that none of the matrices $\Sigma u_i\alpha_i$ with $(u_1,u_2,u_3)\epsilon K^3$ has minimal polynomial of degree 3. We argue towards a contradiction.

Define $Y = \{ (u_1,u_2,u_3,x)\epsilon \mathbb{A}^4_K : x^3 - f(u_1,u_2,u_3) = 0 \}$.

First we show that Y is irreducible.

Y reducible $\Leftrightarrow$ $x^3 - f(u_1,u_2,u_3)$ splits into factors in $K[u_1,u_2,u_3,x]$.

One of these factors is linear in x so that $f(u_1,u_2,u_3)$ is a cube in $K[u_1,u_2,u_3]$, a contradiction. Therefore Y is irreducible and so is its Zariski open subset $U = \{ (u_1,u_2,u_3,x)\epsilon Y : x \neq 0 \}$.

Define $\beta : U \to M_3(K)$ by $(u_1,u_2,u_3,x) \mapsto (1/x)(u_1\alpha_1+u_2\alpha_2+u_3\alpha_3)$. Then every matrix in the image of β satisfies $t^3-1=0$. Since we have assumed that every matrix $u_1\alpha_1+u_2\alpha_2+u_3\alpha_3$ has minimal polynomial of degree ≤ 2, the same must be true of every matrix in the image of β. Let ω be a primitive cube root of 1. Set

$p_1(t)=(t-1)(t-\omega)$; $p_2(t)=(t-1)(t-\omega^2)$; $p_3(t)=(t-\omega)(t-\omega^2)$.

Since U is irreducible, for some $i=1,2$ or 3 we have:

every matrix in the image of β satisfies p_i.

If $A = (1/x)(u_1\alpha_1+u_2\alpha_2+u_3\alpha_3) \epsilon$ image(β), then also $(1/\omega)A \epsilon$ image(β) and both A and $(1/\omega)A$ satisfy $p_i(t)$. Therefore A satisfies two distinct monic degree 2 polynomials $p_i(t)$ and $\omega^2 p_i(t/\omega)$, so A is scalar.

The condition: $u_1\alpha_1+u_2\alpha_2+u_3\alpha_3$ is scalar defines a Zariski closed subset of A^3_K , and $f(u_1,u_2,u_3)\neq 0$ defines a Zariski open subset of A^3_K ; so $u_1\alpha_1+u_2\alpha_2+u_3\alpha_3$ must be scalar for all u_1,u_2,u_3 in K. But then $(\Sigma u_i\alpha_i)^3 = f(u_1,u_2,u_3)$ for all u_1,u_2,u_3 in K means that, applying 1§20, $f(t_1,t_2,t_3)$ is a cube in $K[t_1,t_2,t_3]$, contrary to assumption.□

APPENDIX TO PROPOSITION 1.1

Descent of hyperbolicity also holds for binary forms.

Proposition: Let $d \geq 2$ be an integer. Let F be a field of characteristic zero. Let $f(X,Y) \in F[X,Y]$ be a form of degree d+1. Let K|F be a field extension. Suppose that $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in GL_2(K)$ with $(\alpha X + \beta Y)(\gamma X + \delta Y)^d = f(X,Y)$. Then there exists $\begin{bmatrix} a & b \\ c & e \end{bmatrix} \in GL_2(F)$ with $(aX + bY)(cX + eY)^d = f(X,Y)$.

Proof: If $\delta = 0$ then $f(X,Y) = (\alpha\gamma^d X + \beta\gamma^d Y)X^d$, and $\begin{bmatrix} \alpha\gamma^d & \beta\gamma^d \\ 1 & 0 \end{bmatrix} \in GL_2(F)$ as $f \in F[X,Y]$. So the proposition is true in this case, and similarly if $\gamma = 0$. Suppose that $\gamma \neq 0$ and $\delta \neq 0$. Not both α and β are zero. By symmetry in X and Y we may suppose $\alpha \neq 0$. We have $0 \neq \alpha\gamma^d \in F$. Let $Z = X/Y$, and put $g(Z) = (Z + \beta/\alpha)(Z + \delta/\gamma)^d$. Then $g \in F[Z]$. Since F has characteristic zero the minimal polynomial of β/α over F contains no repeated roots, so it is either (a) $Z + \beta/\alpha$ or (b) $(Z + \beta/\alpha)(Z + \delta/\gamma)$.

In case (a) $\beta/\alpha \in F$ and dividing g by $Z + \beta/\alpha$ we have $(Z + \delta/\gamma)^d \in F[Z]$. Because F has characteristic zero $\gamma/\delta \in F$. **In case (b)** dividing g by $(Z + \beta/\alpha)(Z + \delta/\gamma)$ we have $(Z + \delta/\alpha)^{d-1} \in F[Z]$, with $d-1 \geq 1$. So $\delta/\gamma \in F$ because F has characteristic zero. On division of g by $(Z + \gamma/\delta)^d$ we conclude that $\beta/\alpha \in F$. In each case we have $\beta = x\alpha$, $\delta = y\gamma$ with x, y in F; and $f(X,Y) = (\alpha\gamma^d X + x\alpha\delta^d Y)(X + yY)^d$. $f \in F[X,Y]$ so $\begin{bmatrix} \alpha\gamma^d & x\alpha\delta^d \\ 1 & y \end{bmatrix} \in GL_2(F)$.□

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