The Mathematics of Insider Trading

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To My Parents

Abstract

Over the past decade the research into the topic of incorporating non-market information has accelerated. This dissertation aims to serve as a monograph of the contemporary body of research to the insider problem, under a Brownian setting in a complete market.

Firstly, the techniques of the initial enlargement of filtration are reviewed. The Brownian motion process in the original (honest investors') filtration can be re-written as a Brownian motion in the insider's filtration with a drift. A Martingale Preserving Measure can be specified and the dynamics of the asset price process as perceived by the insider explicitly formulated. The Martingale Representation Theorem for the insider's enlarged filtration and the solution for the optimal portfolio problem from the point of view of the insider will be developed. The advantages for having the insider information can be explicitly formulated in terms of additional expected utility, monetary value and relative entropy.

Secondly, the relevance of Malliavin calculus will be discussed. The techniques of Malliavin calculus allow the onerous assumptions involved in initially enlarging the market filtration to be relaxed, thus allowing a wider class of insider information to be analyzed. Progressive enlargement of filtration techniques can be applied to incorporate dynamical information, such as information regarding the timing of specific events and information that gets updated as time passes.

Thirdly, the insider problem will be considered using the forward integral approach. The effect of a large insider's influence on the honest investor's portfolio will also be considered.

Finally, the relationship between the preservation of semimartingale property, the finiteness of expected utility and the absence of arbitrage will be reviewed. The arbitrage strategies available to the insider with different types of information will also be considered.

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Chapter 1

Introduction

In the beginning...

To trace back to the origin of the field of mathematical finance, one might need to go back more than a century ago to 1900 when Louis Bachelier developed in his thesis "The Theory of Speculation" a model of option pricing. It was basically a humble model of the stock price process as a Brownian motion with drift. Option prices were calculated as expected values under the real-world measure. His work was largely ignored for half a century, until Samuelson modelled the process of stock return with a Brownian motion with drift, i.e. the stock price itself would follow a geometric Brownian motion process. This eliminated the awkward possibility of negative stock prices.

The major breakthrough, however, happened in the 1970s with the work of Black & Scholes and Merton. The celebrated Black-Scholes formula, see Black & Scholes (1973), prices European call/put options as the discounted present value of the expected payoff of the option under the risk-neutral measure, assuming the stock price process follows a geometric Brownian motion. Then Harrison & Kreps (1979) and Harrison & Pliska (1981) introduced the martingale approach to asset pricing in a frictionless market with continuous trading. The general stochastic integral is used to represent capital gains. The market is shown to be complete if and only if the price process has a martingale representation property.

Within this framework, pricing can be done under the risk-neutral measure, and hence is preference-free, since a hedging (replicating) portfolio can be constructed so that the hedged position is locally risk-free. Furthermore, the replicating portfolio is self-financing and hence in the absence of arbitrage the initial value of the option must be equal to the initial cost of the replicating portfolio. This replicating portfolio can be explicitly represented.

Is this realistic?

IT MAY BE CONVENIENT TO KEEP QUOTING OPTION PRICES IN TERMS OF BLACK-SCHOLES EQUIVALENT VOLATILITIES, BUT IT IS PROBABLY INCORRECT TO CALCULATE OPTIONS PRICES USING THE BS FORMULA. - E. DERMAN & I. KANI

As much as the formula was successful in terms of its popularity. The framework was not flawless. A large amount of explicit and implicit assumptions involved were later challenged by economists and mathematicians. An obvious area of debate (relevant to this dissertation) is the implicit assumption that there is the one and only level of information, the information known by the market. For risk-neutral pricing and hedging of derivatives, this assumption is irrelevant. But for Merton's portfolio optimization problem, see Merton (1969), this assumption had been implicitly adopted in all the subsequent work on the problem. The questions that yearned to be asked are

1. Is the assumption realistic? and
2. Does it matter?

The presence of information asymmetry is real. The so-called retail investors certainly do not have access to information to the same degree as institutional investors do in terms of the speed of receiving it as well as the amount of information to be received. There is also a significant difference in the efficiency in interpreting available information and in formulating the appropriate reaction. Every so often one can see in the media stories about alleged insider trading activities by brokers or directors of companies, etc.

Insiders? What insiders?

It must be clarified as this stage that the term "insider" does not necessarily refer to an insider in the legal sense. For example, a brokerage firm which has a research department can put together a piece of research about a particular listed company. Armed with such extra knowledge, the brokerage might have a different opinion about what the company's share price should be. The brokerage might even piece together more pertinent and specific information about specific events involving the company such as an imminent merger or acquisition announcement. Such information may not be considered as *insider information* in the legal sense as the *mosaic theory* will be applied. (The mosaic theory is a method of analysis used by security analysts to gather information about a corporation, it involves collecting public, non-public and non-material information about a company in order to determine the underlying value of the company's securities and to enable the analyst to make recommendations to clients based on that information.)[1]

But since this information is not in the possession of the market and hence is not reflected on the market prices. The brokerage is able to exploit this knowledge and expect a revaluation to the stock once the information becomes public knowledge and the share price is adjusted to reflect the piece of information. One can therefore interpret the term "insider" as a short hand for *an investor who possesses extra information not known by the market* in this dissertation.

How then, one must ask, does the framework of modelling stock or asset prices change from the view of an investor who has access to extra information, and does it matter? The intuitive answer must be that it does matter. Depending on the kind of information that is in the possession of the insider, the insider must see the stock price process to take a different path as compared to the view of those who do not have the extra information, i.e. the *honest investors*. Hence in the eyes of the insider the asset price must follow a different dynamic, i.e. a different stochastic differential equation. The optimal trading strategy must also then be different for the insider. The insider may also be able to obtain a greater amount of utility by achieving a greater trading profit or generate the same amount of trading profit but with lower amount of risk. To take this one step further, the insider might be pricing a derivative differently. The insider might even have a wider set of *attainable* portfolios and hence may be able to achieve a better hedge than the honest investor for a derivative in an incomplete market setting.

[1]See http://www.investopedia.com/terms/m/mosaictheory.asp
The CFA Institute (formerly known as AIMR) has recognized mosaic theory as a valid method of analysis. In the South African context, the mosaic theory has been entrenched in the Insider Trading Act 135 of 1998.

It is then obvious that the Black-Scholes framework is inadequate to facilitate answers to the questions listed above. An extension is needed to model markets where there exist agents with different levels of information.

Equilibrium Models

There are two main approaches for such an extension. Researchers had put forward attempts in the form of discrete- time equilibrium models from an economics perspective. These often involve a discrete one-period model which is then generalised to a multi-period setting and then a continuous time setting by setting the time period infinitely small in the limit.

It is unclear as to who initiated this approach to the problem, but academic literature commonly refer to the studies of Kyle (1985). In the paper, a sequential auction model in which there are three types of economic agents was considered: a risk neutral insider who is better informed in the sense that he has the knowledge of the future liquidation value of an asset; noise traders who trades in a random (independent) manner and a risk-neutral market maker. The market maker will set a price according to the combined order volume placed in the market by the insiders and the noise traders so that market equilibrium is restored and his expected profit is thus zero. Markets are assumed to be efficient in the semi-strong sense.

The underlying asset value is assumed to have a normal distribution (as perceived by the market maker). A single period model was first considered and it was shown that a unique equilibrium exist at which the insider's expected profit is maximised and the asset price is set at the expected final liquidation value (as far as the market maker is concerned) given the order flow information, in other words profit maximisation is achieved for the insider while market efficiency is ensured.

The equilibrium achieved in the single period setting was then generalised to a multi-period setting with a sequential auction model. In such a setting the trading prices follow a martingale process whose volatility process would reflect the rate at which information is incorporated into the trading price over time. This is then further generalised (though rather heuristically) to a continuous trading setting.

Importantly in Kyle's model, the insider's expected profit is related (indeed proportional) to the market depth or how much *camouflage* the noise traders can provide with their trading volume. Also, information is gradually (and smoothly) incorporated into the price process (in fact at a constant rate). The market as in the model possesses liquidity characteristics that are essentially consistent with that of a liquid market.

It is perhaps debatable as to whether these features are desirable, the encouraging fact is that a model is developed that can be consistent with these features. However, Kyle's model does not have a formal basis in the continuous-time setting. The assumption of asset prices having a normal distribution is also dubious - leading to the awkward possibility of negative values.

Then Back (1992) formalised the continuous-time version of the Kyle model. An extension to the equilibrium pricing rule for the market makers is solved in closed form for a general distribution of asset value by solving a Bellman equation. This is desirable as a log-normal distribution can then be applied as in the Black-Scholes framework.

The main results of the paper are

1. There exists an equilibrium in which the pricing rule of market makers is a smooth and strictly monotone function of the cumulative order of the insiders and noise traders. The form of the

optimal trading strategy is also given, which does not correlate locally with orders from noise trades and does not involve discrete orders, i.e. discrete orders of a large size is sub-optimal;

2. The equilibrium is such that the Bellman equation characterizes the insider's optima (his optimal trading strategy).

3. In equilibrium, price changes are locally proportional to order sizes. This characterizes the distribution of the cumulative order flow process.

Importantly, in such an equilibrium the insider's expectation of the price change would be zero if he were to refrain from trading. The equilibrium pricing rule (as a function of the cumulative order flow and time) has a unbiasedness property in that for a general distribution of asset prices, it is a martingale. This is a generalization of Kyle (1985) that the slope of the residual supply curve (i.e. the pricing rule) cannot vary in a deterministic version and in the normal distribution model it is a constant. For a lognormal distribution of asset prices, the equilibrium pricing rule is a geometric Brownian motion.

The martingale approach: Karatzas & Pikovsky started something special

The martingale approach to the problem of modelling investors with different level of information was, according to Amendinger (1999), initiated by Duffie & Huang (1986).

Then in Karatzas & Pikovsky (1996), the techniques of *grossissement de filtrations* (or enlargement of filtration) was used to incorporate insider information in the form of knowledge regarding the terminal prices of assets, etc. These techniques were first presented in a series of papers (see Yor (1985*a*,*b*,*c*,*d*), Jeulin (1980), Jacod (1985) and Chaleyat-Maurel & Jeulin (1985)) by the French school in the early 1980s.

Karatzas and Pikovsky studied the stochastic control problem of maximising expected utility from terminal wealth where the portfolio is allowed to be anticipative. Hence the investor is assumed to have some knowledge, either exact or with some uncertainty, about the terminal price of the stocks over a given investment horizon. Their work is pivotal in the study of the insider trading problem and was cited by almost all the subsequent literature on the problem.

Since the information is assumed to be available to the insider from the beginning of the investment period, the *initial enlargement of filtration techniques* are employed. In essence, this involves enlarging the filtration that is available to the honest trader, by incorporating the information in the possession of the insider so that in the information available to the insider is represented by the enlarged filtration. Furthermore, under certain conditions, in the Brownian framework, the Brownian motion in the honest traders' filtration can be written as a Brownian motion in the insider's filtration plus a drift term. This drift term is called the *information drift* since it represent the additional drift as observed by the insider in the asset price process as a direct result of the extra information in his/her possession.

The discovery of this drift term is significant in that it determines the evolution of the asset price process through the eyes of the insider. I.e. if this drift is defined, then one would know the dynamics of the asset price process from the view of the insider. One can then study the optimal portfolio for the insider. The pursuit of the functional form of this information drift for different types of insider information is one of the main themes of this dissertation.

Karatzas and Pikovsky studied the problem of finding the optimal admissible portfolio π^* where π is allowed to be measurable with respect to a filtration that is larger than the natural filtration, specifically one that incorporates the terminal price (if it is assumed to be known exactly) or one that incorporates the terminal price together with some noise (if the terminal price is assumed to be known with some uncertainty).

The work of Amendinger et al.

Then in Amendinger (1999), Amendinger et al. (1998) and Amendinger et al. (2003) the problem was studied in a general martingale framework, i.e. the stochastic variable that drives the dynamics of the asset price process is assumed to be a (semi)martingale.

Amendinger et al. (1998) address the question of the additional utility that can be gained by the insider by adopting a different trading strategy according to the extra information.

They showed that under the *equivalence assumption* (which will be discussed later) and with a logarithmic utility function, an explicit expression for the insiders utility gain can then be derived in terms of the relative entropy of the "real world distribution" and the conditional distribution of the random variable representing the extra information. This result improves that given by Karatzas & Pikovsky, where closed form solutions or upper bounds are obtained only in some special cases. In particular, in the case of a classical complete market model with some interval-typed information about the outcome of the price, Karatzas & Pikovsky conjectured that the additional expected utility is finite; in Amendinger et al. (1998) the quantity were explicitly given in closed-form solution under some specific conditions.

They also showed that if the extra information is a discrete random variable, the amount of utility gained by the insider by time t is equal the amount of certainty that the regular trader has gained about the random variable by time t. Moreover, the utility gain is zero if the piece of information and the prices are independent, which makes intuitive sense.

Then Amendinger et al. (2003) introduces the concept of a *martingale preserving probability measure* (MPPM) under which the honest traders' filtration and the σ-field generated by the insider information are independent. Furthermore, under this measure the semimartingale properties are preserved when passing to the enlarged filtration. The integrability of predictable process in the honest traders' filtration with respect to the semimartingales is also preserved in enlarged filtration. The $\mathbb{P}$-density of the MPPM can be constructed with the density of the conditional distribution of the insider information.

They then address the question of the "fair" value of the insider information in monetary terms, instead of in terms of utility. I.e. when the investor is presented with the opportunity to buy some extra information at a certain cost, how much should he be willing to pay for it? By buying the piece of information, the investor is able to base his investment decision on an enlarged filtration. They have given the explicit closed-form formula for this quantity for a general utility function and have given examples of the value under a number of different utility functions.

In his thesis Amendinger (1999) also showed how these would be applied in an incomplete market.

Detecting insiders

In 1998, Grorud & Pontier (1998) addresses the different assumptions involved in the initial enlargement of filtration and how they are inter-related. They also showed how the insider's optimal portfolio can be solved using the Lagrange multiplier method in a Brownian setting, parallel to the work of Amendinger et al. They also gave a test for detecting an insider based on the Neyman-Pearson test on a set of random variables representing the discrete increments of the (continuous) discounted consumption process. See also Grorud & Pontier (1999) and Denis et al. (2000) which discuss a test in a Brownian-Poisson setting.

The relevance of Malliavin Calculus

The initial enlargement of filtration technique as adopted by Karatzas & Pikovsky (1996), Amendinger (1999), Amendinger et al. (2003) and Amendinger et al. (1998), while it works, is very onerous in terms of the assumptions required. As a consequence, it is only capable of taking into account a relatively limited class of information.

Then Imkeller et al. (2001) employed Malliavin calculus techniques in order to incorporate a wider class of information such as knowledge regarding the maximum prices in the investment period. Jacod's hypothesis in Jacod (1985) (which requires the absolute continuity of the conditional law of the random variable representing the extra information) that was necessary for the existence and derivation of the information drift was replaced by absolute continuity condition involving the Malliavin derivative of the conditional law of the insider's information. See also Imkeller (2003).

They also explored conditions under which arbitrage opportunities exist. It was shown that for the specific types of information, arbitrage opportunities exist under fairly general conditions, that the quotient of the drift and volatility process governing the stock price need only be continuous.

Progressive Enlargement of Filtration and time information

Imkeller (2002) went on to show how information regarding a certain random time can be incorporated into the framework using the *progressive enlargement of filtration techniques.* This type of filtration enlargement has been dealt with thoroughly in the papers on the subject of *grossissement de filtration*: Jeulin (1980) and Yor (1985*a*,*b*,*c*,*d*).

Arbitrage opportunities exist due to the appearance of a 3-dimensional Bessel process (which has a drift that cannot be eliminated via a change of equivalent measure) in the dynamics of the asset price process from the point of view of the insider while the honest trader will continue to observe a martingale process.

Forward Integral Approach

Then Léon et al. (2003) considered the insider's optimal portfolio problem with what they called a *forward integral approach* (or anticipating integral approach). The forward integral is an extension of the Itô integral with the integrand being *anticipative*, i.e. the integrand is adapted to a filtration that is larger than the filtration with respect to which the integrator is measurable. The term "anticipative" is used because the larger filtration is often formed by "looking ahead in time" with the original filtration.

Under certain conditions, the forward integral is related to the Skorohod integral. Therefore the Malliavin calculus techniques can be employed to evaluate the forward integral. By exploiting this fact, León et al modelled the asset prices dynamics in terms of forward integrals and showed how the insider problem can then be solved. Their results reconciled with the *enlargement of filtration approach.*

Biagini & Øksendal (2005) gave a formalised framework for the approach. They also showed the converse result to the insider's optimal problem: that if the optimal portfolio exists for the insider, then the Brownian motion in the honest trader's filtration must be a semimartingale in the insider's filtration. The results of Karatzas & Pikovsky (1996) were also generalised.

Hu & Øksendal (2003) had in 2004 extended the insider's optimal problem to incorporate a further constraint. Notwithstanding the ultimate aim of profit maximization, they argued that the insider will have the additional objective of being stealth and undetected. Hence the insider would also aim to trade in a "smooth" manner, i.e. no large block trades to raise any attention. They have incorporated this "secondary objective" by introducing a penalty function that penalizes trading strategies that are not "smooth".

Then in 2006, Kohatsu-Higa & Sulem (2006) considered the large insider problem with a forward integral approach in a general partial information framework. The large insider is no longer assumed to be a price taker as in all the above-mentioned publications. Instead, the insider's trading strategy may influence the prices of the assets traded. They considered the optimal portfolio for the insider in such a situation. The generlised framework is used so that the optimal portfolio of a small insider and the honest investor can also be formulated and compared against each other.

Organisation of this dissertation

This dissertation aims to review the different approaches to the insider problem, under a Brownian setting in a complete market and is organized as follows:

In Chapter 2, the techniques of the initial enlargement of filtration are reviewed. Under the condition of Jacod's hypothesis, i.e. that the conditional law of the random variable representing the insider's information is absolutely continuous with respect to the (unconditional) law of the random variable concerned, the Brownian motion process in the original (honest investors') filtration can be re-written as a Brownian motion in the insider's filtration with a drift. Via a change of measure (to the so-called *Martingale Preserving Measure*, the drift can be eliminated. The Martingale Preserving Measure and the dynamics of the asset price process as perceived by the insider shall be defined. Examples will be given to demonstrate how to obtain the measure and the new dynamics with particular insider information.

In Chapter 3, by assuming a classical market with the stock price dynamics given by a geometric Brownian motion, the Martingale Representation Theorem for the insider's enlarged filtration is introduced. The optimal portfolio problem from the point of view of the insider is then properly defined and the solution will be developed. Then the advantages for having the insider information will be discussed in terms of additional utility, in monetary terms and in terms of the relative entropy.

In Chapter 4, the insider modelling problem is approached using the techniques of Malliavin calculus. The basic concepts of Malliavin calculus will be reviewed, including the celebrated Clark-Ocone formula. In order to apply the Malliavin calculus techniques to the insider problem, a version of the Malliavin calculus for the space of measure-valued random variables is needed. This will be developed. Examples will be given to demonstrate how these will be applied to specific insider information.

In Chapter 5, the case where the insider possesses information that would be updated (i.e. dynamic information) is considered. Specifically, two types of dynamical information will be discussed:

1. initial information on terminal values that are distorted by vanishing noise; and
2. time information.

Different techniques to those presented in the first three Chapters will be required.

In Chapter 6, the forward integral approach to the insider problem will be introduced in a partial information framework. The market where there exists a "large" insider who has the ability to influence prices will be considered. The optimization problem will be re-visited with the forward integral approach. Finally, the optimization problem will be modified to incorporate the desire for a "smooth" portfolio.

Finally in Chapter 7, the relationship between the preservation of semimartingale property, the finiteness of expected utility and the absence of arbitrage will be reviewed. The arbitrage strategies available to the insider with different types of information will also be considered.

Other areas of research

In the last decade the research into the insider problem has intensified and the body of knowledge is much wider than what will be covered in this dissertation. Other areas of research includes how to incorporate weak information, analysis of the problem in a Lévy setting and the insider problem in an incomplete market setting. These topics deserve, at the very least, a brief introduction which will be given below.

Weak Information

In most of the research mentioned above, the investor is assumed to possess knowledge regarding the outcome of a certain quantity, in most of the examples it would be the price or the value of the Brownian motion underlying the price process. Therefore the insider is assumed to have pathwise knowledge about certain quantities, i.e. the investor has knowledge about which path ω will be realized, which path will not. This type of information is called *strong information*.

There is another type of information, called *weak information*: investors possess this type of information only have knowledge about the law of certain quantities. For example, the distribution of the terminal price, this distribution may be different from that implied by the "market knowledge". It may be "superior" in the sense that it is more accurate, perhaps with a smaller variance. Where an investor has the knowledge about the terminal price but with this information distorted by noise, he is effectively possessing weak information. This approach has been studied by Baudoin (2002), Baudoin & Nguyen-Ngoc (2004).

Incomplete market

Let $\mathbb{F} = (\mathscr{F}_t)_t$ be the filtration generated by the Brownian motion underlying the asset price dynamic as seen by the "market". It was shown in Amendinger (1999) that if $\mathbb{F} \subset \mathbb{G}$ then the value (price) of a $\mathbb{F}$-attainable derivative must be the same for the $\mathbb{G}$-investor, i.e. the insider, and the $\mathbb{F}$-investor, i.e. the honest trader. Logical intuition would suggest that in a generally incomplete market setting, the insider may attach a different value to a payoff that is $\mathbb{F}$-unattainable.

Biagini & Øksendal (2006) formalised this by studying the problem of minimum variance hedging from the point of view of an insider, following the approach for the insider problem in Biagini &

Øksendal (2005) and the minimum variance hedging techniques reviewed in Schweizer (2001). It was shown that for certain forms of the enlarged filtration (i.e. for certain insider information), a claim that is not attainable for the honest investor (and hence the honest investor can only do a minimum variance hedge) can be perfectly replicated by the insider.

They also considered the mean-variance problem for the insider, i.e. given a desired expected value of a claim, what is the minimum variance? It was shown that the insider can always obtain a smaller variance than the honest trader. But the optimal portfolio may not exist for the insider.

Lévy Processes

There are also a number of publications on the insider problem in a Lévy setting, i.e. the underlying stochastic processes that govern asset prices are assumed to be Lévy processes. For example, using a forward integral approach, Di Nunno et al. (2003) studied the optimal portfolio for an insider with a logarithmic utility function. Results obtained were similar to those in the Brownian setting presented in this dissertation.

Chapter 2

Initial Enlargement of Filtration

The initial enlargement of filtrations method (or *grossissement de filtrations*) was developed by the French school in a series of intense works, e.g. Yor (1985*a*,*b*,*c*,*d*), Jeulin (1980), Jacod (1985) and Chaleyat-Maurel & Jeulin (1985) as cited by Grorud & Pontier (1998) and Imkeller et al. (2001). Its application to the modelling of insider trading was initiated by Karatzas & Pikovsky (1996).

In this chapter, a few basic results of initial enlargement of filtrations are reviewed. Using these results, the problems of insider's optimal portfolio, his utility gain and the issue of arbitrage will be tackled in the next chapter for certain types of information in a Brownian setting.

A note on notation

In the sequel, T will be used to denote the end of the investment horizon. Hence the time variable t will be an element of $[0, T]$. However, $\mathscr{T}$ will be used to indicate the corresponing time of certain *terminal information*, such as the future price of an asset.

In general, $T \neq \mathscr{T}$ so that one may be able to assume that the insider knows the value of the stock price S at time $\mathscr{T}$ in advance, where $T < \mathscr{T}$, i.e. the insider is only allowed to trade up to time T even if the information in his possession is related to a later time. But in a lot of the examples given throughout the dissertation, it will be assumed that $T = \mathscr{T}$ so that the two are used interchangeably. The distinction between the two, where it matters, will be specifically mentioned.

The information possessed by the "market" will be identified by the σ-algebra $\mathscr{F}$ or the filtration $\mathbb{F}$, where $\mathscr{F} = (\mathscr{F}_t)_t$. The alternative symbol $\mathbb{F}$ is used to denote the time-dependence of the filtration. For example, the $\mathbb{F}_T = (\mathscr{F}_t)_{t\in[0,T]}$ which is in general not same filtration as $\mathbb{F}_{\mathscr{T}} = (\mathscr{F}_t)_{t\in[0,\mathscr{T}]}$.

Similarly, the σ-algebra and the filtration identified with the information possessed by the insider which includes the extra information, are denoted by $\mathscr{G}$ or $\mathbb{G}$ respectively.

2.1 Introduction: Enlargement of Filtration

Suppose there are two types of investors with different levels of information in a continuous-time security market. The uncertainty of the security market is described by a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with the filtration $\mathbb{F}$ satisfying the usual conditions.

While the *honest investor's* information flow is modelled by the filtration $\mathbb{F}$, which also represents information known by the "market". The *insider* has in addition to the information contained in $\mathbb{F}$ some extra information which is represented by a random variable L in the sequel. For example, L can be the price of a stock at time $\mathscr{T}$ if this information is exact; the price range of a stock at time $\mathscr{T}$ if the insider know with certainty that the price will be in a certain range; or the price of a stock at time $\mathscr{T}$ distorted by some noise; or the distribution of the price at time $\mathscr{T}$. The information may be available from the beginning (at time 0), or the information may be available or become more certain only at a later stage.

For now, assume that the insider possesses information from the beginning some future price information, hence L is a $\mathbb{F}$-measurable random variable with values in the Polish space $(X, \mathscr{X})$. Then the insider's information is modelled by the initially enlarged filtration $\mathbb{G}_{\mathscr{T}} = (\mathscr{G}_t)_{t \in [0,\mathscr{T}]}$ with $\mathscr{G}_t = \mathscr{F}_t \vee \sigma(L), t \in [0, \mathscr{T}]$. Such a filtration is essentially richer than the market filtration $\mathbb{F}_{\mathscr{T}}$.

Remark 2.1.1. *More precisely, the associated right continuous filtration will be utilised, i.e.*

$$\mathscr{G}_{t+} = \cap_{s>t}(\mathscr{F}_s \vee \sigma(L), t \in [0, \mathscr{T}]),$$

The reason for such a formulation of the enlarged filtration is to apply the standard results in stochastic analysis which depend on the assumption of right-continuity of the underlying filtration. Amendinger (1999) points out that the two major reasons for this dependence are that the right-continuity of filtration allows for choosing modifications of martingales that are càdlàg; and that if the filtration is right-continuous, the debut times of progressive sets of right-continuous adapted processes are stopping times.

It was shown in Amendinger (1999) that if Assumption 2.1.5 below is satisfied for $T \in [0, \mathscr{T}]$, then for all $t \in [0, T)$, $\mathscr{G}_t = \mathscr{G}_{t+}$ (See Proposition 1.10 of Amendinger (1999)).

Under a Brownian framework, $\mathscr{F}_t = \sigma((W_s)_{s<t})$, i.e. W is a $(\mathbb{F}, \mathbb{P})$-Brownian motion. As mentioned in the introduction, to model gains an interpretation of the stochastic integral of the type $\int \pi \, dS$ is required, where S is a $\mathbb{F}$-martingale with respect to some measure $\mathbb{Q}^{\mathbf{F}_T}$ equivalent to $\mathbb{P}$. Since π is $\mathbb{G}$-predictable for the insider, i.e. the integrand π is predictable to a larger filtration than that to which the integrator dS is measurable, the integral does not have a meaning in the Itô sense. One way to give meaning to the stochastic integral is to establish the dynamics or behaviour of S or indeed W in the enlarged filtration.

As an example, let $T = \mathscr{T}$ and $\mathbb{G}$ be the filtration $\mathbb{F}$ enlarged by $L = W_{\mathscr{T}} = W_T$. Consider the stochastic integral $\int \psi \, dW$ for some $\mathbb{G}$-predictable ψ. It will be shown later that

$$\tilde{W} = W - \int \frac{W_T - W_s}{T - s} \, ds$$

is a $(\mathbb{G}, \mathbb{P})$-Brownian motion. Hence W is a $(\mathbb{G}, \mathbb{P})$-semimartingale on $[0, T)$. Indeed

$$\int \psi \, dW_s = \int \psi \, d\tilde{W}_s + \int \psi \frac{W_T - W_s}{T - s} \, ds$$

where the stochastic integral $\int \psi \, d\tilde{W}_s$ is well defined in the classical way.

If, however, W is not a $\mathbb{G}$-semimartingale, then $\int \pi \, dS$ may have no meaning. One can then see such *preservation of the semimartingales*, i.e. where martingales and hence semimartingale in the "market" filtration remain semimartingales in the enlarged filtration, is a desirable property that will make the analysis of the insider's portfolio simple. This property is known as Jacod's H' hypothesis. To state formally,

Assumption 2.1.2. *[Jacod's H' hypothesis, Jacod (1985)] Every $\mathbb{F}$-semimartingale is a $\mathbb{G}$-semimartingale, or equivalently: every $\mathbb{F}$-local martingale is a $\mathbb{G}$-semimartingale.*

However, $(\mathbb{F}, \mathbb{P})$-martingales (or semimartingales) are in general not $(\mathbb{G}, \mathbb{P})$ semimartingales. As cited by Amendinger (1999), Jeulin & Yor (1979) showed that if L is the endpoint of a $(\mathbb{F}, \mathbb{P})$-Brownian motion, i.e. $L = W_T$ and $T = \mathscr{T}$, then W is a $(\mathbb{G}, \mathbb{P})$ semimartingale as mentioned above but they have given a deterministic ψ such that $\int \psi \, dW$ is not a $(\mathbb{G}, \mathbb{P})$-semimartingale. Hence they have shown that there exist $(\mathbb{F}, \mathbb{P})$-martingales that are not $(\mathbb{G}, \mathbb{P})$-semimartingales; and there exist stochastic integrals that are well-defined with respect to $(\mathbb{F}, \mathbb{P})$ but not to $(\mathbb{G}, \mathbb{P})$.

In light of such an "inconvenience" research effort was focused on the conditions under which Assumption 2.1.2 is true. Jacod (1985) gave a sufficient condition under which this is true.

Assumption 2.1.3. *[Jacod's criterion, Jacod (1985)] There exists a σ-finite measure ν on $(X, \mathscr{X})$ such that the regular conditional distributions of L given $\mathscr{F}_t, t \in [0, T]$ are absolutely continuous with respect to ν for $\mathbb{P}$-a.a. $\omega \in \Omega$, i.e.*

$$\mathbb{P}[L \in \cdot | \mathscr{F}_t](\omega) \ll \nu(\cdot) \quad \forall t \in [0, T], \mathbb{P} - a.a. \quad \omega \in \Omega \tag{2.1.1}$$

Remark 2.1.4. *A weaker assumption is equivalent: For each t there is a ν_t so that*

$$\mathbb{P}[L \in \cdot | \mathscr{F}_t](\omega) \ll \nu_t(\cdot) \tag{2.1.2}$$

It can be shown that there is a single ν that satisfies (2.1.2) for all t, (cf. Protter (2005)), and that ν can be taken as the law of L.

The possibility that the semimartingale property is lost under enlargement will be addressed in Chapter 6, as well as a new kind of integral that is needed for the analysis in that case.

Föllmer & Imkeller (1993) introduced the following stronger assumption to facilitate the analyses.

Assumption 2.1.5. *[Föllmer & Imkeller (1993)] The regular conditional distributions of L given $\mathscr{F}_t, t \in [0, T]$ are* equivalent *with respect to the law of L for $\mathbb{P}$-a.a. $\omega \in \Omega$, i.e.*

$$\mathbb{P}[L \in \cdot | \mathscr{F}_t](\omega) \sim \mathbb{P}[L \in \cdot] \quad \forall t \in [0, T], \mathbb{P} - a.a. \quad \omega \in \Omega \tag{2.1.3}$$

Remark 2.1.6. *Intuitively, the absolute continuity assumption implies that what is an impossible outcome for an honest investor would be impossible for the insider at all times $t \in [0, T]$, since any event that has a measure of 0 under $\mathbb{P}$ conditioned on the honest investor's information as represented by $\mathscr{F}_t$ must have a measure of 0 under ν representing the insider's knowledge.*

But what is an impossible outcome for the insider is not necessarily impossible for the honest investor, since an event that has a measure of 0 under ν does not necessarily have a measure of 0 under $\mathbb{P}$ conditioned on $\mathscr{F}_t$ under the absolute continuity assumption.

Hence the insider, with the extra information in his/her possession, will be able to ignore outcomes that are impossible and concentrate on outcomes that are possible to him/her. The honest investor, on the other hand, will see more possible outcomes, i.e. the outcome is more uncertain. In the extreme, the insider's information may be exact, i.e. the insider knows the outcome with certainty, but the honest investor will still see a distribution of the outcome.

Remark 2.1.7. *The equivalence assumption is a stronger assumption. It implies that at any time $t \in [0,T]$ the insider sees the same set of possibilities as the honest investor. Obviously the insider might see certian outcome of the random quantity of concern as more probable than the honest investor would.*

To summarise, Assumption 2.1.5 implies Assumption 2.1.3, which in turn implies the H' hypothesis.

2.2 Preliminary Results

Let $\mathscr{O}(\mathbb{F})$ denote the optional σ-algebra on $\Omega \times \mathbb{R}$ and $\mathscr{P}(\mathbb{F})$ denote the predictable σ-algebra on $\Omega \times \mathbb{R}$.

The following lemma provided by Amendinger was a modification of Jacod's results in Jacod (1985) which assures a "nice" version of the conditional density process of L given $\mathscr{F}_t$ under Assumption 2.1.3.

Lemma 2.2.1 (Lemma 2.1 of Amendinger et al. (1998) and Lemme 1.8 of Jacod (1985)). *Suppose Assumption 2.1.3 is satisfied.*

1. *There exists a non-negative $\mathscr{O}(\mathbb{F})\otimes\mathscr{X}$-measurable function $(\omega, t, x) \longrightarrow q_t^x(\omega)$, which is càdlàg in t and such that*
   (a) *$\forall x \in X$, q^x is a $(\mathbb{F},\mathbb{P})$-martingale, the processes q^x, q_-^x are strictly positive on $[0,\tau^x)$[1], and $q^x = 0$ on $[\tau^x, T]$, where*
   $$\tau^x := inf\{t \geq 0 : q_{t-}^x = 0\} \wedge T;$$
   (b) *$\forall t \in [0,T]$, the measure $q_t^x\nu(dx)$ on $(X, \mathscr{X})$ is a version of the conditional distribution $\mathbb{P}[L \in dx|\mathscr{F}_t]$.*
2. *$\tau^L = T \quad \mathbb{P}-a.s.$*

In the sequel, a normalised version p^x of q^x will be used: since for all $A \in \mathscr{X}$,

$$\int_A \mathbb{P}[L \in dx] = \mathbb{P}[L \in A] = \mathbb{P}[L \in A|\mathscr{F}_0] = \int_A \mathbb{P}[L \in dx|\mathscr{F}_0] = \int_A q_0^x \nu(dx),$$

therefore one can assume that $q_0^x > 0 \quad \forall x \in X$ (where X can be chosen appropriately). Hence for all $t \in [0,T]$ and $\mathbb{P}$-a.a. ω

$$\mathbb{P}[L \in A|\mathscr{F}_t](\omega) = \int_A q_t^x(\omega)\nu(dx) = \int_A \frac{q_t^x}{q_0^x}\mathbb{P}[L \in dx]$$

So one can define the *conditional density process*

$$p_t^x(\omega) := \frac{q_t^x}{q_0^x},$$

which is equivalent to choosing the law of L as ν.

The conditional density process p_t^x plays a significant role in defining the behaviour of martingales and Brownian motions in the enlarged filtration, as the following proposition and its corollary demonstrate.

[1] $[S,T]$ is a stochastic time interval where $[S,T] = \{(\omega,t) : S(\omega) \leq t \leq T(\omega)\} \subseteq \Omega \times \mathbb{R}$.

Proposition 2.2.2. *[Théorème 2.1 of Jacod (1985) & Proposition 3.5 of Grorud & Pontier (1998)] Suppose Assumption 2.1.3 is satisfied.*

1. *For $t < T$, there exists a $\mathscr{P}(\mathbb{F})\otimes\mathscr{X}$-measurable version of the conditional density p_t^x such that for all $x \in \mathbb{R}^d$, $p_t^x(\omega)$ is a $(\mathbb{F},\mathbb{P})$-martingale and hence can be written as*

$$p_t^x = p_0^x + \int_0^t \alpha_s^x, dW_s;$$

moreover, $p_t^L > 0$ $\mathbb{P}$-a.s., for all $t \leq T$,

2. *if M_t is a $(\mathbb{F}_T,\mathbb{P})$-continuous local martingale (hence can be written as $M_0 + \int_0^t \beta_s, dW_s$ for $t \leq T$), then*

$$d\langle M, p\rangle_t = \langle \alpha, \beta\rangle_t dt$$

and the process

$$\tilde{M}_t = M_t - \int_0^t \frac{\langle \alpha, \beta\rangle_t|_{x=L}}{p_u^L} du, \quad 0 \leq t \leq T \tag{2.2.1}$$

is a $(\mathbb{G}_T,\mathbb{P})$-continuous local martingale.

Remark 2.2.3. *The above proposition can be generalised to a multi-dimensional Brownian processes in a straight forward manner, see Jacod (1985) and Grorud & Pontier (1998).*

By applying the previous proposition to the $(\mathbb{F}_T,\mathbb{P})$-Brownian motion, for which $\beta = 1$, one obtains

Corollary 2.2.4. *The process*

$$\tilde{W}_t = W_t - \int_0^t \gamma_u \, du, \quad t \in [0, T],$$

where

$$\gamma_u^i = \frac{\alpha_u^{L,i}}{p_u^L}, i = 1, \ldots, d, \tag{2.2.2}$$

is a $(\mathbb{G}_{\mathscr{T}},\mathbb{P})$-Brownian motion.

Remark 2.2.5. *The drift term γ_u in the above Corollary is called the* information drift, *as mentioned in the introduction, since it represents all the extra information prossessed by the insider.*

Jacod (1985) showed that a continuous local $\mathbb{F}$-martingale M (and hence also any continuous $\mathbb{F}$-semimartingales) remains a semimartingale for the filtration $\mathbb{G}$ under Assumption 2.1.3. Moreover, he also provided a canonical decomposition of M in $\mathbb{G}$ in terms of the conditional density process p_t^x.

However, Assumption 2.1.3 is not typically satisfied for $T \in [0, \mathscr{T}]$, but only for $T \in [0, \mathscr{T})$. Hence in this case p_t^x is only defined for $t \in [0, \mathscr{T})$. As illustrated by the following example (from Amendinger (1999)):

Example: Terminal value of Brownian motion

Let $L = W_{\mathcal{T}}$ where W is a 1-dimensional $(\mathbb{F}, \mathbb{P})$-Brownian motion. Then for all $t < \mathcal{T}$

$$\begin{aligned}\mathbb{P}[W_{\mathcal{T}} \in dx|\mathcal{F}_t] &= \mathbb{P}[W_{\mathcal{T}} - W_t + W_t \in dx|\mathcal{F}_t] \\ &= \mathbb{P}[W_{\mathcal{T}} - W_t \in dx - y]|_{y=W_t} \\ &= \frac{1}{\sqrt{2\pi(\mathcal{T} - t)}} \exp\left\{-\frac{(x - W_t)^2}{2(\mathcal{T} - t)}\right\} dx \\ &= p_t^x \mathbb{P}[W_{\mathcal{T}} \in dx],\end{aligned}$$

where $p_t^x := \sqrt{\frac{\mathcal{T}}{\mathcal{T}-t}} \exp\left\{-\frac{(x-W_t)^2}{2(\mathcal{T}-t)} + \frac{x^2}{2\mathcal{T}}\right\}, x \in \mathbb{R}$, is strictly positive for all $t < \mathcal{T}$. Hence by applying Itô's formula to $\frac{(x-W_t)^2}{(\mathcal{T}-t)}$,

$$p_t^x = \mathcal{E}\left(\int \frac{x - W_s}{\mathcal{T} - s} dW_s\right)_t. \tag{2.2.3}$$

First observe that the conditional law of L given $\mathcal{F}_t$ is only absolutely continuous to the law of L on $t \in [0, \mathcal{T})$, since at $t = \mathcal{T}$ the conditional law of L given $\mathcal{F}_{\mathcal{T}}$ is the point mass, i.e. given $\mathcal{F}_{\mathcal{T}}$ the value of L is known by assumption. For $T < \mathcal{T}$, one can apply Corollary 2.2.4 and obtain that

$$\tilde{W}_t := W_t - \int_0^t \frac{W_{\mathcal{T}} - W_s}{\mathcal{T} - s} ds, \quad t \in [0, T],$$

is a $(\mathbb{G}_T, \mathbb{P})$-Brownian motion.

Jeulin & Yor (1979) had extended $\tilde{W}$ on the interval $[0, \mathcal{T}]$ and showed that $\tilde{W}$ is also a $(\mathbb{G}_{\mathcal{T}}, \mathbb{P})$-Brownian motion and hence a $(\mathbb{F}_{\mathcal{T}}, \mathbb{P})$ semimartingale. However, as cited by Amendinger (1999), Jeulin & Yor also provided a deterministic function ϕ such that $\int_0^{\mathcal{T}} \phi^2(s)\, ds < \infty$ and $\int \phi(s)\, dW_s$ is not a $(\mathbb{G}_{\mathcal{T}}, \mathbb{P})$-semimartingale. As remarked by Amendinger, this shows the following:

1. There exist $(\mathbb{F}_{\mathcal{T}}, \mathbb{P})$-martingales that are not $(\mathbb{G}_{\mathcal{T}}, \mathbb{P})$-semimartingales;
2. There exist stochastic integrals that are well-defined with respect to $(\mathbb{F}_{\mathcal{T}}, \mathbb{P})$ but not $(\mathbb{G}_{\mathcal{T}}, \mathbb{P})$, even when the integrator is a $(\mathbb{G}_{\mathcal{T}}, \mathbb{P})$-semimartingale.

Föllmer & Imkeller (1993) investigated such an "anomally", though from a different angle. The conditional density process p_t^x or more precisely,

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{G}_t} = \frac{1}{p_t^L}, \quad 0 \leq t < \mathcal{T} \tag{2.2.4}$$

defines a new probability measure $\mathbb{Q}$ on $t \in [0, \mathcal{T})$, under which W is again a Brownian motion on $t \in [0, \mathcal{T})$ with respect to the enlarged filtration $(\mathcal{G}_t)_{0 \leq t < \mathcal{T}}$ and this would extend to the terminal time $t = \mathcal{T}$. But (for such L) $\mathcal{G}_{\mathcal{T}} = \mathcal{F}_{\mathcal{T}}$ and so $\mathbb{Q}$ may be seen as the distribution of $W_{\mathcal{T}}$. This would imply that $\mathbb{Q}$ is identical to the Wiener measure $\mathbb{P}$, thus contradicting (2.2.4), since p_t^x is not equivalent to 1.

To explain such a "paradox", as Föllmer & Imkeller put it, first notice that by (2.2.3) $p_t = p_0\mathcal{E}(Y)$, where $Y = \int \frac{(x-W_s)}{\mathcal{T}-s} dW_s$. However,

$$\mathbb{E}\left[\exp\left(\frac{1}{2}[Y]_t\right)\right] \to \infty \text{ as } t \to \mathcal{T},$$

hence the Novikov condition is not satisfied and p_t is not necessarily a martingale on $[0, \mathscr{T}]$. Indeed, (see Lemma 1 of Föllmer & Imkeller (1993)) $(p_t)_{0\leq t\leq \mathscr{T}}$ is a process such that

$$\mathbb{E}(p_t) = 1, \quad 0 \leq t < \mathscr{T},$$

and

$$\lim_{t\uparrow \mathscr{T}} p_t = 0 \quad \mathbb{P} - a.s.,$$

and therefore $(p_t)_{0\leq t\leq \mathscr{T}}$ is a martingale which is not uniformly integrable. Moreover, $(p_t)_{0\leq t\leq \mathscr{T}}$ induce on the product space

$$\bar{\Omega} = C([0, \mathscr{T})) \times \mathbb{R},$$

which is endowed with the product σ-field $\bar{\mathscr{G}}_t = \mathscr{F}_t \otimes \mathscr{B}$, a measure $\mathbb{Q}$ that is singular to $\mathbb{P}$ on $\bar{\mathbb{G}}$ but coincides with $\mathbb{P}$ on $\mathscr{F}_t$. This is the inspiration for the decoupling measure that is to be introduced in the sequel. Föllmer & Imkeller (1993) showed that Assumption 2.1.5 guarantees the existence of such a measure.

2.3 The Martingale Preserving Measure

In this section, the decoupling measure mentioned in the end of the previous section will be defined. This measure is also called the *Martingale Preserving Measure*, so named for the fact that martingales in the original filtration under $\mathbb{P}$ will remain martingales under this measure in the enlarged filtration, as will be shown.

In the previous section, it was shown that for $L = W_{\mathscr{T}}$, p_t^L failed to satisfy the Novikov condition on $[0, \mathscr{T}]$ and p_t^L failed to induce a measure on $[0, \mathscr{T}]$ such that the Brownian motion is preserved as a semimartingale in the enlarged filtration. Indeed, a Novikov-type condition on p_t^L (precisely the density of p_t^L) is sufficient for the existence of an equivalent decoupling measure, as the following proposition shows. But prior to that, an assumption on γ_t needs to be introduced, recall from Corollary 2.2.4 that γ_t is the information drift that transforms a $(\mathbb{F}, \mathbb{P})$-Brownian motion W into a $(\mathbb{G}, \mathbb{P})$-Brownian motion.

Assumption 2.3.1. *[Novikov-type condition for a generic measure $\mathbb{P}$ on the interval $[0, T]$] There exists $k > 0$ and K such that*

$$\mathbb{E}_{\mathbb{P}}\left[\exp(k\|\gamma_t\|^2)\right] < K \quad , t \in [0, T].$$

The following proposition provides the sufficient conditions for the existence of the decoupling measure.

Proposition 2.3.2 (Proposition 3.6 of Grorud & Pontier (1998)). *Suppose there exists $T \in [0, \mathscr{T})$ such that Assumption 2.1.3 and Assumption 2.3.1 are satisfied. Then there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathscr{F}_T \vee \sigma(L)$ such that under $\mathbb{Q}$, the σ-fields $\mathscr{F}_t$ and $\sigma(L)$ are independent for all $t \in [0, T]$.*

Proof. By Assumption 2.1.3, γ and $\tilde{W}$ do exist. Assumption 2.3.1 implies the existence of a $(\mathbb{G}_T, \mathbb{P})$-uniformly integrable martingale ρ_t, $t \leq T$, such that

$$d\rho_t = -\rho_t(\gamma_t, d\tilde{W}_t), \quad \rho_0 = 1.$$

Therefore ρ defines an equivalent probability measure $\mathbb{Q} = \rho_T \mathbb{P}$ on $\mathbb{G}_T$. Then

$$W_t = \tilde{W}_t + \int_0^t \gamma_u \, du, \quad t \in [0,T]$$

is a $(\mathbb{G}_T, \mathbb{Q})$-Brownian motion. Hence under $\mathbb{Q}$, W is independent of $\mathscr{G}_0$. And since $\mathbb{F}$ is generated by W, $\mathscr{G}_0$ and $\mathscr{F}_t$ are independent for $t \in [0,T]$. This in turn implies that $\mathscr{F}_t$ and $\sigma(L)$ are independent for all $t \in [0,T]$. □

The next natural step would be to define the decoupling measure. The following lemma will be needed.

Lemma 2.3.3. *[Lemma 1.2 of Amendinger (1999)] Suppose Assumption 2.1.3 is satisfied. Let* $(\omega, t, x) \longrightarrow H_t^x(\omega)$ *be a non-negative function that is measurable with respect to* $\mathscr{O}(\mathbb{F}) \otimes \mathscr{X}$. *Then the* $\mathbb{F}$*-optional projection of* $(H_t^G)_{t\in[0,T]}$ *is then given by*

$$\left(\int H_s^x p_s^x \mathbb{P}[L \in dx] \right)_{t\in[0,T]}.$$

The following proposition is an adaptation of Amendinger's (Proposition 1.6 of Amendinger (1999)) general martingale framework to the Wiener setting that defines the Martingale Preserving Measure:

Proposition 2.3.4. *[Martingale Preserving Measure] If Assumption 2.1.5 is satified on* $[0,T]$, *then*

1. $\frac{1}{p^L}$ *is a* $(\mathbb{G}_T, \mathbb{P})$*-martingale, and*
2. *the probability measure* $\mathbb{Q}_T$ *defined by*

$$\mathbb{Q}_T(A) = \int_A \frac{1}{p_T^L} \, d\mathbb{P} \quad \textit{for } A \in \mathscr{G}_T \tag{2.3.1}$$

   satisfies the following:

   (a) the σ*-algebras* $\mathscr{F}_T$ *and* $\sigma(L)$ *are independent under* $\mathbb{Q}_T$,

   (b) $\mathbb{Q}_T = \mathbb{P}$ *on* $(\Omega, \mathscr{F}_T)$, *and* $\mathbb{Q}_T = \mathbb{P}^L$ *on* $(\Omega, \sigma(L))$, *hence for* $A_T \in \mathscr{F}_T$ *and* $B \in \mathscr{X}$,

$$\mathbb{Q}_T\left[A_T \cap (L \in B)\right] = \mathbb{P}[A_T]\mathbb{P}[L \in B] = \mathbb{Q}_T[A_T]\mathbb{Q}_T[L \in B]$$

Proof. For part 2,

$$\begin{aligned}
\mathbb{Q}_T\left[A_T \cap (L \in B)\right] &= \mathbb{E}\left[\frac{1}{p_T^L} \mathbf{1}_{A_T \cap (L \in B)}\right] \\
&= \mathbb{E}\left[\mathbf{1}_{A_T} \mathbb{E}\left[\mathbf{1}_{(L\in B)} \frac{1}{p_T^L} \middle| \mathscr{F}_T\right]\right] \\
&= \int_{A_T} \mathbb{E}\left[\mathbf{1}_{(L\in B)} \frac{1}{p_T^L} \middle| \mathscr{F}_T\right](\omega) \, \mathbb{P}(d\omega).
\end{aligned}$$

But since p is $\mathscr{O}(\mathbb{F}_T) \otimes \mathscr{X}$-measurable, by Lemma 2.3.3 one obtains

$$\mathbb{E}\left[\mathbf{1}_{(L\in B)} \frac{1}{p_T^L} \middle| \mathscr{F}_T\right](\omega) = \int_B \frac{1}{p_T^x(\omega)} p_T^x(\omega) \mathbb{P}[L \in dx] = \mathbb{P}[L \in B],$$

hence

$$\begin{aligned}\mathbb{Q}_T\left[A_T \cap (L \in B)\right] &= \mathbb{P}[L \in B] \int_{A_T} \mathbb{P}(d\omega) \\ &= \mathbb{P}[A_T]\mathbb{P}[L \in B]\end{aligned}$$

Now

$$\mathbb{P}[A_T]\mathbb{P}[L \in B] = \mathbb{Q}_T[A_T]\mathbb{Q}_T[L \in B]$$

is equivalent to

$$\mathbb{E}[\mathbf{1}_{A_T}]\mathbb{E}[\mathbf{1}_{(L \in B)}] = \mathbb{E}\left[\frac{1}{p_T^L}\mathbf{1}_{A_T}\right]\mathbb{E}\left[\frac{1}{p_T^L}\mathbf{1}_{(L \in B)}\right]$$

which follows by setting $A_T = \Omega$ or $B = X$.

For part 1, fix $0 \le s \le t \le T$ and choose $A \in \mathscr{G}_s$ of the form $A = A_s \cap (L \in B)$ with $A_s \in \mathscr{F}_s$ and $B \in \mathscr{X}$. And

$$\begin{aligned}\mathbb{E}[\frac{1}{p_T^L}\mathbf{1}_A] &= \mathbb{P}[A_s]\mathbb{P}[L \in B] \\ &= \mathbb{E}\left[\mathbf{1}_{A_s}\mathbb{P}[L \in B]\right] \\ &= \int_{A_s}\int_B \frac{1}{p_s^x(\omega)} p_s^x(\omega)\mathbb{P}[L \in dx]\mathbb{P}(d\omega) \\ &= \mathbb{E}\left[\mathbf{1}_{A_s}\mathbb{E}\left[\mathbf{1}_{(L \in B)}\frac{1}{p_s^L}\middle| \mathscr{F}_s\right]\right] \\ &= \mathbb{E}\left[\frac{1}{p_s^L}\mathbf{1}_A\right].\end{aligned}$$

This can be extended to arbitrary $A \in \mathscr{G}_s$ by a monotone class argument. Hence $1/p^L$ is a $(\mathbb{G}_T, \mathbb{P})$-martingale which starts at 1 since $p_0^L = 1$. Hence $\mathbb{Q}_T$ defined in (2.3.1) is a probability measure on $(\Omega, \mathbb{G}_T)$. □

The theorem below shows that $(\mathbb{F}_T, \mathbb{P})$-martingales remain $\mathbb{F}_T$-martingales under $\mathbb{Q}_T$ defined in (2.3.1). It is rather fitting in the most obvious way that the measure $\mathbb{Q}_T$ is termed *martingale preserving probability measure under initial enlargement of filtration* in Amendinger (1999) under a general martingale framework.

Theorem 2.3.5. *[Theorem 1.7 of Amendinger (1999)] Let $\mathscr{M}(\mathbb{F}_T, \mathbb{P})$ denote the space of martingales with respect to the filtration $\mathbb{F}_T$ under the measure $\mathbb{P}$. If Assumption 2.1.5 is satisfied, then*

$$\mathscr{M}(\mathbb{F}_T, \mathbb{P}) = \mathscr{M}(\mathbb{F}_T, \mathbb{Q}_T) \subseteq \mathscr{M}(\mathbb{G}_T, \mathbb{Q}_T).$$

Moreover any $(\mathbb{F}_T, \mathbb{P})$-Brownian motion W is a $(\mathbb{G}_T, \mathbb{Q})$-Brownian motion.

Proof. Let M be a $(\mathbb{F}_T, \mathbb{P})$-martingale. For fixed $0 \le s \le t \le T, A_s \in \mathscr{F}_s$ and $B \in \mathscr{X}$,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_T}[\mathbf{1}_{A_s \cap (L \in B)} M_t] &= \mathbb{E}_{\mathbb{Q}_T}[\mathbf{1}_{A_s} M_t]\mathbb{Q}_T[L \in B] \\ &= \mathbb{E}_{\mathbb{P}}[\mathbf{1}_{A_s} M_t]\mathbb{Q}_T[L \in B] \\ &= \mathbb{E}_{\mathbb{P}}[\mathbf{1}_{A_s} M_s]\mathbb{Q}_T[L \in B]\end{aligned}$$

by the independence of $\mathscr{F}_t$ and $\sigma(L)$ under $\mathbb{Q}_T$, the fact that $\mathbb{Q}_T$ and $\mathbb{P}$ coincide on $(\Omega, \mathbb{F}_T)$ and that M is a $(\mathbb{F}_T, \mathbb{P})$-martingale. Reversing the above,

$$\mathbb{E}_{\mathbb{P}}[\mathbf{1}_{A_s} M_s]\mathbb{Q}_T[L \in B] = \mathbb{E}_{\mathbb{Q}_T}[\mathbf{1}_{A_s \cap (L \in B)} M_s]$$

and hence

$$\mathbb{E}_{\mathbb{P}}[\mathbf{1}_{A_s} M_t]\mathbb{Q}_T[L \in B] = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_{A_s} M_s]\mathbb{Q}_T[L \in B]$$

and therefore M is a $(\mathbb{Q}_T, \mathbb{G}_T)$-martingale via a monotone class argument.

Since the quadratic variation of continuous martingales can be computed pathwise, the filtration is irrelevant. Hence for all $t \in [0, T]$,

$$[W]_t^{(\mathbb{Q}_T, \mathbb{G}_T)} = [W]_t^{(\mathbb{Q}_T, \mathbb{F}_T)} = [W]_t^{(\mathbb{P}, \mathbb{F}_T)} = t$$

since $\mathbb{Q}_T = \mathbb{P}$ on $(\Omega, \mathbb{F}_T)$. Therefore W is also a $(\mathbb{G}_T, \mathbb{Q}_T)$-Brownian motion by Lévy's characterisation. □

Remark 2.3.6. *Grorud & Pontier (1998) provides a different proof for the fact that any $(\mathbb{F}_T, \mathbb{P})$-Brownian motion W remains a $(\mathbb{G}_T, \mathbb{Q})$-Brownian motion. See proposition 3.2 and lemma 3.3 of Grorud & Pontier (1998).*

Since Assumption 2.1.5 is a stronger condition than Assumption 2.1.3, semimartingales are preserved in the initially enlarged filtration under Assumption 2.1.5 as well. Hence under Assumption 2.1.5, a $(\mathbb{F}_T, \mathbb{P})$-martingale must also be a $(\mathbb{G}_T, \mathbb{P})$-semimartingale. Then via a Girsanov transformation with the measure $\mathbb{Q}_T$ as defined in (2.3.1), the $(\mathbb{G}_T, \mathbb{P})$-semimartingale is again a martingale in $(\mathbb{G}_T, \mathbb{Q}_T)$. In other words, as Amendinger put it, the martingale property is preserved under an initial filtration enlargement with a simultaneous change to the equivalent probability measure.

2.4 More Examples

In this section, examples of different types of *initial* information will be considered. It will be shown that how each of the information L satisfies or fails to satisfy Assumption 2.1.5. The Radon-Nikodym derivative of the martingale preserving measure will be derived in each case where the assumption of its existence is satisfied.

2.4.1 Terminal value of Brownian motion

In the case where the insider's information is the terminal value of the Brownian motion, then $L = W_{\mathscr{T}}$. Assume that $T = \mathscr{T}$. Assumption 2.1.5 will not be satisfied for $[0, T]$, since the conditional distribution of L given $\mathscr{F}_T$, i.e. $\mathbb{P}[L \in dx|\mathscr{F}_T]$ is the point mass at $L = W_T$. Hence the conditional distribution cannot be equivalent to the law of L at the point $t = T$. However, for $T \in [0, \mathscr{T})$, Assumption 2.1.5 is satisfied.

By Proposition 2.3.4 and (2.2.3), one can define $\mathbb{Q}_T$ by setting the Radon-Nikodym derivative as

$$\begin{aligned}\left.\frac{d\mathbb{Q}_T}{d\mathbb{P}}\right|\mathscr{G}_t &= \frac{1}{p_T^t} \\ &= \mathscr{E}\left(-\int_0^t \frac{W_T - W_s}{T - s}\, dW_s\right)\end{aligned}$$

2.4.2 Terminal value of Brownian motion distorted by noise

Suppose that at time $t = 0$, the insider possesses information about the terminal values of the Brownian motion process W that underlies the asset prices, but this information is distorted by some "noise", which introduces a degree of uncertainty to the information. Hence the random variable that represents this information is of the form:

$$L = \lambda W_{\mathscr{T}} + (1 - \lambda)\varepsilon,$$

where ε is a standard normal random variable, independent of W and $\lambda \in [0, 1]$. The "enlargement of filtration" for such information L should be performed as follows.

Denote by $\mathbb{P}_t^L$ the conditional distribution of L at time t, given $\mathscr{F}_t$, i.e.

$$\mathbb{P}_t^L(\omega, B) = \mathbb{P}[L \in B|\mathscr{F}_t](\omega).$$

This conditional distribution is a normal distribution with mean λW_t and variance

$$\lambda^2(1 - t) + (1 - \lambda)^2.$$

Thus the Radon-Nikodym derivative of $\mathbb{P}_t^L$ with respect to the Lebesgue measure is

$$q_t^x(\omega) = \frac{d\mathbb{P}_t^L(\omega, .)}{dx} = \phi_t(x, W_t(\omega))$$

with

$$\phi_t(x, y) = \frac{1}{\sqrt{2\pi[\lambda^2(T - t) + (1 - \lambda)^2]}} \exp\left(-\frac{(x - \lambda y)^2}{2[\lambda^2(T - t) + (1 - \lambda)^2]}\right),$$

hence

$$p_t^x(\omega) = \frac{\phi_t(x, y)}{\phi_0(x)} = \phi_t^*(x, y) \tag{2.4.1}$$

$$= \sqrt{\frac{\lambda^2 T + (1 - \lambda)^2}{\lambda^2(T - t) + (1 - \lambda)^2}} \exp\left(-\frac{1}{2}\left\{\frac{(x - \lambda y)^2}{\lambda^2(T - t) + (1 - \lambda)^2} - \frac{x^2}{\lambda^2 T + (1 - \lambda)^2}\right\}\right). \tag{2.4.2}$$

Now with Itô's formula

$$\begin{aligned} p_t^x =& p_0^x + \int_0^t \frac{\partial}{\partial t}\phi_s^*(x, w(s))\, ds + \int_0^t \frac{\partial}{\partial y}\phi_s^*(x, w(s))\, dW(s) \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2}{\partial y^2}\phi_s^*(x, w(s))\, ds \end{aligned}$$

therefore

$$\begin{aligned} \alpha_s^x &= \frac{\partial}{\partial y}\phi_s^*(x, w(s)) \\ &= p_t^x \frac{\lambda(x - \lambda W(s))}{\lambda^2(T - t) + (1 - \lambda)^2} \end{aligned}$$

Hence by Corollary 2.2.4,

$$\tilde{W}_t = W_t - \int_0^t \frac{\lambda(L - \lambda w(s))}{\lambda^2(T - t) + (1 - \lambda)^2}\, ds, \quad t \in [0, T],$$

is a $(\mathbb{G}_T, \mathbb{P})$-Brownian motion.

With (2.4.1), Assumption 2.1.5 is satisfied and Proposition 2.3.4 can be applied to obtain the Martingale Preserving Measure $\mathbb{Q}_T$.

In the next chapter, it will be demonstrated how these tools can be utilised in making sense of the dynamics of the insider's wealth process in terms of the insider's portfolio in the enlarged filtration, thus making the optimization of the insider's trading strategy possible.

Chapter 3

Insider Trading with Initial Enlargement of Filtration

In this chapter, the optimal portfolios are considered for two types of investors who have different levels of information at their disposal for their respective investment-consumption decision making in a general continuous-time security market. The enlargement of filtration techniques developed in the previous chapter will be applied to solve the portfolio optimization problem in the insider context.

The chapter is organised as follows. In section 1 the general framework of the model will be defined as a classical market with stock prices dynamics given by geometric Brownian motions. In section 2 a martingale representation theorem will be introduced for the enlarged filtration. The portfolio optimization problem will be defined in section 3 and a solution for the optimal wealth and consumption processes will be given. In section 4 the optimal portfolio will be discussed.

Remark 3.0.1 (A note on notation). *The portfolio optimization problem will be solved in a generalised manner for both the insider and the honest investor. Hence in the sequel, the generic filtration that represents either the information in the insider's possession and that possessed by the honest trader will be denoted by* $\mathbb{H}_T \in \{\mathbb{F}_T, \mathbb{G}_T\}$.

3.1 The Market Model

The discounted price process $S = (S^1, \ldots, S^d)'$ of d stocks is assumed to be given by the following equation:

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + (\sigma_t^i, dW_t), \quad 0 \le t \le T, S_0 \in [0, \infty)^d, i = 1, \cdots, d, \tag{3.1.1}$$

where W is d-dimensional Brownian motion and $(.,.)$ denotes the scalar product in $\mathbb{R}^d$, and a risk-free bond modelled by

$$S_t^0 = 1 + \int_0^t S_s^0 r_s \, ds. \tag{3.1.2}$$

In (3.1.1) above, $\sigma_t = (\sigma_t^{i,k})_{1 \le i,k \le d}$ is a matrix-valued process which is adapted (to $\mathbb{F}$) and invertible

$dt \otimes d\mathbb{P}$ a.s.; $\mu_t = (\mu_t^i)_{1\leq i \leq d}$ is an adapted (to $\mathbb{F}$) vector process. It is further assumed that

$$\mathbb{P}\left[\int_0^T \left(|r(t)| + \sum_{i=1}^{d} |\mu_t^i| + \sum_{i,k=1}^{d} (\sigma_t^{i,k})\right)^2 dt < \infty\right] = 1, a.s.,$$

so that (3.1.1) and (3.1.2) together represent a well-defined process.

Suppose a financial agent has a positive amount V_0 at the outset $t = 0$ to consume or invest and he would like to optimize his consumption investment strategy in the sense that he would want to set his strategy so as to maximize the utility he derives. His knowledge is modelled by a filtration $\mathbb{H}_T = (\mathscr{H}_t)_{t\in[0,T]}$. The consumption rate is denoted by c_t, which is a $\mathbb{H}$-adapted non-negative process such that $\int_0^T c_s\, ds < \infty$, $\mathbb{P} - a.s.$.

Denote by θ_t^i the number of units of the i-th asset held at time t. The investor's wealth at time t is given by

$$V_t = \sum_{i=0}^{d} \theta_t^i S_t^i = \sum_{i=0}^{d} \pi_t^i V_t$$

with $\pi_t^i = \frac{\theta_t^i S_t^i}{V_t}$ being the proportion of wealth invested in stock i, hence and the portfolio can be represented by $\pi = \{(\pi_t^i), i = 0, 1, \ldots, d\}$, with the constraint that

$$\sum_{i=0}^{d} \pi_t^i = 1.$$

The set of admissible strategies is defined as follows:

Definition 3.1.1. *An $\mathbb{H}$-**investment-consumption strategy** (π, c) is a pair consists of*

1. *an investment process π that is $\mathbb{H}$-predictable; and*
2. *a consumption process c that is $\mathbb{H}$-adapted, with $c \geq 0$, $\int_0^T c_s\, ds < \infty$;*

satisfying $\sigma'\pi \in L^2[0,T]$, $\mathbb{P}$-a.s.; and that

$$\mathbb{P}\left[\int_0^T |\pi'(t)\,(\mu(t) - r(t)\mathbf{1}_d)|\, dt < \infty\right] = 1, a.s.$$

The *"self-financing"* condition can be formulated as

$$dV_t = \sum_{i=0}^{d} \theta_t^i dS_t^i - c_t dt = \sum_{i=0}^{d} \frac{\pi_t^i V_t}{S_t^i} dS_t^i - c_t dt \tag{3.1.3}$$

hence there is no injection of funds after $t = 0$ and the change in wealth is a result of change in assets values and consumption only. Under this condition, the insider's wealth has the following dynamics (by substituting (3.1.1) and (3.1.2) into (3.1.3)):

$$dV_t = V_t\left(\pi_t^0 r_t dt + \sum_{i=1}^{d} \pi_t^i \mu_t^i dt + \sum_{i=1}^{d} \pi_t^i (\sigma_t^i, dW_t)\right) - c_t dt, \quad V_0 > 0,$$

and V_0 is a $\mathscr{H}_0$ random variable. Since $\pi_t^0 = V_t - \sum_{i=1}^{d} \pi_t^i V_t = V_t\left(1 - \sum_{i=1}^{d} \pi_t^i\right)$, the dynamics of the wealth process can be re-written as

$$dV_t = V_t\left(r_t dt + (\pi_t, \mu_t - r_t\mathbf{1})dt + (\pi_t, \sigma_t dW_t)\right) - c_t dt, \quad V_0 > 0, \quad V_0 \in \mathscr{H}_0.$$

Let $R_t = (S_t^0)^{-1}$ be the discounting process. Hence $dR_t = -R_t r_t dt$ with $R_0 = 1$, and the discounted wealth process $V_t R_t$ has the dynamics

$$\begin{aligned} d(V_t R_t) &= V_t dR_t + R_t dV_t + d\langle V_t, R_t\rangle \\ &= -R_t c_t dt + V_t R_t\left((\pi_t, \mu_t - r_t\mathbf{1})dt + (\pi_t, \sigma_t dW_t)\right) \end{aligned}$$

hence

$$V_t R_t = V_0 + \int_0^t V_s R_s(\pi_s, \mu_s - r_s\mathbf{1})ds + \int_0^t V_s R_s(\pi_s, \sigma_s dW_s) - \int_0^t R_s c_s\, ds. \tag{3.1.4}$$

From the point of view of an insider, the stochastic integral in (3.1.4) is anticipating, i.e. π_s is not predictable with respect to the filtration generated by W, rather it is predictable with respect to a larger filtration $\mathscr{G}$. Thus the stochastic integral has no meaning in the classical sense. In order to give meaning to the stochastic integral and hence (3.1.4), the initial enlargement of filtration techniques developed in the previous chapter will need to be employed.

Via a change of measure, the drift on the asset values can be eliminated:

Proposition 3.1.2. *Suppose Assumption 2.3.1 is satisfied for $\mathbb{Q}$. Define $\mathbb{Q}^* = M_T\mathbb{Q}$ with*

$$M_t = \mathscr{E}\left(-\int_0^t (\eta_s, dW_s)\right), \tag{3.1.5}$$

where

$$\eta_s = \sigma_t^{-1}(\mu_s - r_s\mathbf{1}).$$

Then the process

$$\hat{W}_t = W_t + \int_0^t \eta_s\, ds, \quad t \in [0, T]$$

is a $(\mathbb{H}, \mathbb{Q}^*)$-Brownian motion.

Proof. Since the Novikov's condition is satisfied for $\mathbb{Q}$ by Assumption 2.3.1 and W is a $(\mathbb{H}, \mathbb{Q})$-Brownian motion, the result is obtained by a direct application of the Girsanov theorem. □

Hence the discounted wealth process is under $\mathbb{Q}^*$:

$$V_t R_t = V_0 + \int_0^t R_s(\pi_s, \sigma_s d\hat{W}_s) - \int_0^t R_s c_s\, ds, \quad t \in [0, T], \tag{3.1.6}$$

and the stochastic integral in 3.1.6 can be interpreted in the classical sense, since π is $\mathbb{H}$-predictable and $\hat{W}$ is a $(\mathbb{H}, \mathbb{Q}^*)$-Brownian motion.

Remark 3.1.3. *Following the theory developed in the previous chapter, the above proposition implicitly assumed that the martingale preserving measure $\mathbb{Q}$ is defined. Recall that for the insider $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathscr{E}\left(-\int_0^T \gamma_s\, dW_s\right)$ and for the honest investor the same theory can be applied with $\gamma = 0$. Then W the $(\mathbb{F}, \mathbb{P})$-Brownian motion is also a $(\mathbb{H}, \mathbb{Q})$-Brownian motion. With the above proposition, the drift on the asset values is eliminated by defining a new Brownian motion $\hat{W}$ under a new measure $\mathbb{Q}^*$.*

Equivalently, one could have done this in one step as follows: Define a probability measure $\tilde{\mathbb{Q}}$ on $(\Omega, \mathbb{H}, \mathbb{P})$ by

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} = \tilde{M}_T = \exp\left[-\int_0^T (\gamma_s + \eta_s, dW_s) - \frac{1}{2}\int_0^T \|\gamma_s + \eta_s\|^2 \, ds\right]$$
$$= \mathscr{E}\left(-\int_0^T \gamma_s + \eta_s \, dW_s\right).$$

It can be shown that $\mathbb{Q}^*$ and $\tilde{\mathbb{Q}}$ coincides on $\mathbb{H}$, see Proposition 4.2 of Grorud & Pontier (1998) for a detailed proof.

Remark 3.1.4. *The Radon-Nikodym derivative of the measure $\mathbb{Q}^*$ with respect to $\mathbb{P}$ will be denoted as Z_T. Hence for the insider, $Z_T^{\mathbb{G}} = \frac{M_T}{p_T^L} = \tilde{M}_T$. For the honest investor $Z_T^{\mathbb{F}} = M_T$ (since $\gamma_t = 0$ and $p_T^L = 1$ implies $\mathbb{Q}_T^{\mathbb{F}} = \mathbb{P}$.*

3.2 Martingale Representation Theorem

In order to characterise the optimal portfolio under the insider's filtration, a martingale representation theorem is needed for the enlarged filtration. This is not necessarily a trivial matter even if one can assume such a theorem exist for the "normal filtration", since the enlarged filtration $\mathbb{G}_T$ is not necessarily generated by the $\mathscr{G}$-Brownian motion. The common approach is to start with a filtration that is equipped with a martingale representation theorem and extend the martingale representation to an enlarged filtration that satisfies certain assumptions. Amendinger (1999) cites that Pikovsky (1997) showed the case where the enlarged filtration is a Brownian filtration initially enlarged by a Gaussian variable plus an independent noise term. In Amendinger (2000), a martingale representation theorem is proved assuming Assumption 2.1.5 under an initial enlargement of filtration and a simultaneous change to the corresponding martingale preserving measure in a general martingale framework. Amendinger (2000) has also proven a martingale representation theorem with respect to the initially enlarged filtration and the original probability measure $\mathbb{P}$, under the condition that the local martingale underlying the original filtration is continuous.

In this section a martingale representation theorem for the enlarged filtration in the Brownian framework is given, as a variant of the result given by in Grorud & Pontier (1998).

Assumption 3.2.1. *For any $F \in L^\infty(\mathscr{F}_T)$, there exist $\varphi \in L^2(\mathbb{Q}^{\mathbb{F}_T}, \mathbb{F}_T)$ such that*

$$F = \mathbb{E}_{\mathbb{Q}^{\mathbb{F}_T}}[F] + \int_0^T \varphi_s \, dW_s.$$

Theorem 3.2.2. *[Martingale Representation Theorem] Suppose that a martingale preserving measure $\mathbb{Q}$ exist and $\mathbb{Q}^*$ is as defined in Proposition 3.1.2. Let $Z \in L^1(\Omega, \mathscr{G}_{\mathscr{T}}, \mathbb{Q}^*)$; then there exists a unique $\mathbb{G}_T$-predictable process φ such that*

$$Z = \mathbb{E}_{\mathbb{Q}^*}[Z|\mathscr{G}_0] + \int_0^T (\varphi_s, d\hat{W}_s). \tag{3.2.1}$$

Proof. By Theorem 2.3.5, W is a $(\mathbb{G}_T, \mathbb{Q})$-Brownian motion. Since W is also $\mathbb{F}$-adapted, it is also a $(\mathbb{F}_T, \mathbb{Q})$-Brownian motion. Any $(\mathbb{F}_T, \mathbb{Q})$-martingale has a representation with respect to W and by Theorem 4.33 pg 189 of Jacod & Shiryaev (2003), any local $(\mathbb{G}_T, \mathbb{Q})$-martingale has the representation property with respect to W.

By the equivalence of $\mathbb{Q}$ and $\mathbb{Q}^*$, any local $(\mathbb{G}_T, \mathbb{Q}^*)$-martingale thus has the representation property with respect to $\hat{W}$. And by the definition of M, see (3.1.5), the Radon-Nikodym derivative process of $\mathbb{Q}^*$ with respect to $\mathbb{Q}$ satisfy $M_t = M_0 - \int_0^t M_s(\eta_s, dW_s)$. Hence

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^*}[Z|\mathscr{G}_t] &= \frac{\mathbb{E}_{\mathbb{Q}}[M_T Z|\mathscr{G}_t]}{\mathbb{E}_{\mathbb{Q}}[M_T|\mathscr{G}_t]} \\ &= \frac{\mathbb{E}_{\mathbb{Q}}[M_T Z|\mathscr{G}_t]}{M_t} \\ &= \frac{\mathbb{E}_{\mathbb{Q}}[M_T Z|\mathscr{G}_t]}{1 - \int_0^t M_s(\eta_s, dW_s)}\end{aligned}$$

Moreover, the $(\mathbb{G}_T, \mathbb{Q})$-martingale representation property with respect to W gives for some $\mathscr{G}$-predictable ϕ:

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[M_T Z|\mathscr{G}_t] &= \mathbb{E}_{\mathbb{Q}}[M_T Z|\mathscr{G}_0] + \int_0^t (\phi_s, dW_s) \\ &= \mathbb{E}_{\mathbb{Q}^*}[Z|\mathscr{G}_0] + \int_0^t (\phi_s, dW_s).\end{aligned}$$

Then by Itô's formula

$$\mathbb{E}_{\mathbb{Q}^*}[Z|\mathscr{G}_t] = \mathbb{E}_{\mathbb{Q}^*}[Z|\mathscr{G}_0] + \int_0^t (\frac{\phi_s}{M_s} + \mathbb{E}_{\mathbb{Q}^*}[Z|\mathscr{G}_s]\eta_s, d\hat{W}_s).$$

then (3.2.1) follows by setting $\varphi_s = \frac{\phi_s}{M_s} + \mathbb{E}_{\mathbb{Q}^*}[Z|\mathscr{G}_s]\eta_s$ for all $s < T$. □

3.3 The Portfolio Optimization Problem

The portfolio optimization problem, also known as the *Merton's problem*, was formulated and solved by Robert Merton in Merton (1969). The problem involves an investor who has to choose between investing in a risky asset and a risk-free bond and consuming so as to maximize the utility derived from consumption and terminal wealth. The original formulation of the problem assumed constant parameters for the dynamics of the risky asset. This can be relaxed, allowing the parameters to be time-dependent.

A solution to the portfolio optimization problem will be developed in this section. Firstly, the concept of utility functions will be formally introduced. Then the optimal portfolio problem will be defined. A solution to the optimal wealth and consumption process will be developed using the Lagrange multiplier method. The existence of such a solution will be discussed. Examples will be given as to demonstrate how all the theory will be applied given specific information.

Before the portfolio optimization problem can be formulated, the concept of utility functions needs to be formally defined.

3.3.1 Utility Function

Utility functions form part of the formulation of the portfolio optimization problem. They reflect the relative trade-off in terms of utility for the investor between different objectives, such as return (in terms of wealth), consumption and risk (however defined). The following assumption is made regarding the general utility functions employed in the portfolio optimization problem in the sequel:

Assumption 3.3.1. *[Utility functions] The assumptions regarding utility functions* $U : (0, \infty) \to \mathbb{R}$ *are that they are*

1. *strictly increasing, i.e.* $U'(x) > 0 \quad \forall x > 0$,
2. *concave, i.e.* $U''(x) < 0 \quad \forall x > 0$,
3. *satisfying* $U \in C^2$*; and*
4. *the Inada conditions, i.e.*
   (a) $\lim_{x \to 0+} U'(x) = \infty$ *and*
   (b) $\lim_{x \to \infty} U'(x) = 0$.

Remark 3.3.2. *Utility functions that satisfy the above criteria includes the log-utility function:* $x \longmapsto \log x$ *and the power-utility function:* $x \longmapsto x^\alpha, \alpha \in (0, 1)$.

Remark 3.3.3. *Some of the criteria listed above have specific economic justifications. For example, utility functions are required to be strictly increasing because for any rational investor more (wealth) is always preferred to less. They must be concave according to the law of diminishing marginal utility.*

Following convention from literature, denote the inverse of U' by I; note that $I : (0, \infty) \to (0, \infty)$ and $I(0+) = \infty$ and $I(\infty) = 0$. More importantly U and I satisfy the following inequality:

$$U(I(y)) \geq U(x) + y(I(y) - x), \quad \forall \quad x \geq 0, y > 0. \tag{3.3.1}$$

In the remainder of the section, the investment strategy is optimized for general utility functions U_1 and U_2, representing the utility the investor derive from consumption and terminal wealth respectively.

3.3.2 Formulation of the problem

With the concept of utility functions in the previous subsection, the set of admissible strategies can now be defined.

Definition 3.3.4. *[Admissible Strategies] the class of* admissible investment-consumption strategies *for the filtration* $\mathbb{H}_T$*, denoted* $\mathscr{A}_{\mathbb{H}_T}(V_0)$*, is defined as the set of* $\mathscr{H}$*-investment-consumption strategies* (π, c) *(See Definition 3.1.1) such that given a non-negative initial wealth* $v_0 \in L^1(\mathscr{H}_0, \mathbb{P})$,

1. $$V_t^{v_0,\pi,c} \geq 0, \quad \mathbb{P} - a.s. \quad \forall \quad t \in [0, T], \tag{3.3.2}$$

   and

2. $$\mathbb{E}\left[\int_0^T U_1^-(R_t c_t)\, dt + U_2^-(V_T^{v_0,\pi,c} R_T) | \mathscr{H}_0\right] < \infty, \quad \mathbb{P} - a.s., \tag{3.3.3}$$

Remark 3.3.5. *A portfolio* π *that satisfy (3.3.2) is called a* tame portfolio*. Tame portfolios are portfolios for which there exists a constant* $K > -\infty$ *such that the (discounted) wealth process is greater than* K *with probability* 1 *during the trading period* $[0, T]$*. Such a "lower bound" restriction* *effectively means that "doubling schemes" are prevented. This can also be interpreted as a limit on borrowing as the investor cannot borrow without limit to finance the purchase of losing stocks in a tame portfolio, see Karatzas & Shreve (2001) for example for more details.*

Another significance of tame portfolios is their relation to the condition of absence of arbitrage. See Levental & Skorohod (1995).

The following characterization of the admissible strategies is a direct corollary of the Theorem 3.2.2.

Proposition 3.3.6. *[Characterization of Admissible Strategies][Proposition 4.4 of Grorud & Pontier (1998)] Suppose that a martingale preserving measure $\mathbb{Q}^*$ exist. And the self-financing condition is also satisfied. Let V_0 be a positive $\mathscr{H}_0$-measurable variable. Then*

1. *For an admissible strategy (π, c) and the associated final wealth $V_T^{\pi,c}$, one must have*

$$\mathbb{E}_{\mathbb{Q}^*}\left[V_T^{\pi,c} R_T + \int_0^T R_t c_t \, dt \middle| \mathscr{H}_0 \right] \leq V_0. \tag{3.3.4}$$

2. *Conversely, given*
   (a) *an initial wealth $V_0 \in L^1(\mathscr{H}_0)$,*
   (b) *$\mathbb{H}$-adapted positive consumption process c such that $\int_0^T c_s \, ds < \infty$, $\mathbb{Q}^* - a.s.$;*
   (c) *a random variable $H \in L^1(\mathbb{H}_T, \mathbb{Q}^*)$ such that*

$$\mathbb{E}_{\mathbb{Q}^*}\left[H R_T + \int_0^T R_t c_t \, dt \middle| \mathscr{H}_0 \right] = V_0,$$

there exists $\mathbb{H}$-predictable portfolio $\pi = (\pi_t)_{t \in [0,T]}$ such that (π, c) is admissible and $V_T^{\pi,c} = H$.

Proof. 1. Under $\mathbb{Q}^*$, (3.1.6) can be written as

$$dV_t R_t + R_t c_t dt = V_t R_t (\pi_t, \sigma_t d\hat{W}_t), \quad V_0 \in L^0(\mathscr{H}_0), \quad t \in [0, T].$$

Hence the right hand side is a positive $(\mathbb{H}_T, \mathbb{Q}^*)$-local martingale, hence by Fatou's lemma it is a supermartingale with initial value V_0.

2. Let

$$Y_t = \mathbb{E}_{\mathbb{Q}^*}\left[H R_T + \int_0^T R_t c_t \, dt \middle| \mathscr{H}_t \right], \quad t \in [0, T].$$

Hence $Y_0 = V_0$ by assumption and Y is a $\mathbb{H}_T$-martingale and by Theorem 3.2.2

$$Y_t = \mathbb{E}_{\mathbb{Q}^*}[Y_T | \mathscr{H}_t] + \int_0^t (\varphi_s, d\hat{W}_s), \tag{3.3.5}$$

where φ_t is a $\mathbb{H}_T$-predictable process and $Y_0 = \mathbb{E}_{\mathbb{Q}^*}[Y_T | \mathscr{H}_0] = V_0$. Set $\pi_t = R_t^{-1}(\sigma_t')^{-1}\varphi_t$ for $t \in [0, T]$ with (σ_t') denotes the transpose of (σ_t). Then π_t is $\mathbb{H}_T$-predictable. And with (π, c) as the strategy the discounted wealth equation under $\mathbb{Q}^*$ is

$$dV_t^{\pi,c} R_t + R_t c_t dt = V_t^{\pi,c} R_t (\pi_t, \sigma_t d\hat{W}_t), \quad V_0 \in L^0(\mathscr{H}_0), t \in [0, T]. \tag{3.3.6}$$

Combining (3.3.5) and (3.3.6)

$$V_t^{\pi,c}R_t + \int_0^t R_s c_s\, ds = Y_t = \mathbb{E}_{\mathbb{Q}^*}\left[HR_T + \int_0^T R_t c_t\, dt \middle| \mathscr{H}_t \right] \tag{3.3.7}$$

is a uniformly integrable $(\mathbb{H}_T, \mathbb{Q}^*)$-martingale which is equal to the conditional expectation of its terminal value. Hence

$$V_t^{\pi,c}R_t = \mathbb{E}_{\mathbb{Q}^*}\left[ZR_T + \int_t^T R_s c_s\, dt \middle| \mathscr{H}_t \right] > 0,$$

thus the strategy is admissible. Moreover, $V_T^{\pi,c} = H$, as required.

□

The optimization problem can now be formulated. Define the *total expected utility function* as

$$\mathbb{U}_t^{v_0,\pi,c} := \mathbb{E}\left[\int_0^t U_1(R_t c_t)\, dt + U_2(V_t^{v_0,\pi,c}R_t) \middle| \mathscr{H}_0 \right], t \in [0,T],$$

and let the initial wealth $v_0 \in L^1(\mathscr{H}_0, \mathbb{P})$ and hence known to the investor. The investor will then attempt to maximise $\mathbb{U}$ by choosing an investment-consumption strategy from the set of admissible strategies according to the amount of information he possesses. This is defined formally as:

Definition 3.3.7. *[Optimization Problems]*

1. *Given* $V_0 \in L^1(\mathbb{P}, \mathscr{F}_0)$*, the* ordinary investor's optimization problem *is to find:*

$$\sup_{(\pi,c)\in\mathscr{A}_{\mathbb{F}_T}(V_0)} \mathbb{U}_T^{V_0,\pi,c}.$$

2. *Given* $V_0 \in L^1(\mathbb{P}, \mathscr{G}_0)$*, the* insider's optimization problem *is to find:*

$$\sup_{(\pi,c)\in\mathscr{A}_{\mathbb{G}_T}(V_0)} \mathbb{U}_T^{V_0,\pi,c}.$$

Note that the ordinary investor's optimization problem can be considered as a special case of the insider's problem (by the choice of L).

Because of the fact that $\hat{W}$ is a $(\mathscr{H}, \mathbb{Q}^*)$-Brownian motion. For all $(\pi, c) \in \mathscr{A}_{\mathbb{H}_T}$,

$$V_t R_t + \int_0^t R_s c_s\, ds = V_0 + \int_0^t V_s R_s(\pi_s, \sigma_s d\hat{W}_s), \quad t \in [0,T]. \tag{3.3.8}$$

is a non-negative local $(\mathscr{H}, \mathbb{Q}^*)$-martingale, and hence a $(\mathscr{H}, \mathbb{Q}^*)$-supermartingale by Fatou's lemma. And because $Z_0^{\mathbb{H}_T} = 1$, therefore

$$\mathbb{E}_{\mathbb{Q}^*}\left[\int_0^T R_t c_t\, dt + R_T V_T^{v_0,\pi,c} \middle| \mathscr{H}_0 \right] = \mathbb{E}_{\mathbb{P}}\left[\int_0^T Z_t^{\mathbb{H}} R_t c_t\, dt + Z_T^{\mathbb{H}} R_T V_T^{v_0,\pi,c} \middle| \mathscr{H}_0 \right] \tag{3.3.9}$$

$$\leq V_0 \tag{3.3.10}$$

for all $(\pi, c) \in \mathscr{A}_{\mathbb{H}_T}$ by Proposition 3.3.6. This forms another constraint in the optimization problem.

As Amendinger (1999) has put it, there are really two parts to the optimization problem for an investor:

1. The first part (termed "*the Lagrangian Optimization Problem*" by Amendinger) is to maximize the total expected utility by the Lagrange multiplier method, i.e. maximize $\mathbb{U}_T^{v_0,\pi,c}$ over all $(c, H) \in \mathscr{A}_{\mathbb{H}_T}^{Lagr}$, where

$$\mathscr{A}_{\mathbb{H}_T}^{Lagr} = \left\{ (c, H) \,\middle|\, \begin{array}{l} c \text{ is a consumption rate process,} \\ H \in \mathbb{G}_T, \text{ such that} \\ \text{(3.3.3) and (3.3.4) are satisfied with } H = V_T^{\pi,c} \end{array} \right\}.$$

2. Secondly, the problem of finding an investment strategy that finances the corresponding consumption strategy such that the utility derived from the terminal wealth and consumption is maximized.

In the remainder of this section, the Lagrangian optimization problem will be dealt with, whereas the problem of finding the optimal strategy is dealt with in the next section.

The following proposition gives a characterization of the solution if the Lagrange multiplier λ exist.

Proposition 3.3.8. *[Proposition 4.3 of Amendinger (1999)] If there exists a $\mathscr{H}_0$-measurable random variable $\lambda : \Omega \to (0, \infty)$,$\mathbb{P}$-a.s. such that*

$$\mathbb{E}\left[\int_0^T Z_t^{\mathbb{H}_T} I_1\left(\lambda Z_t^{\mathbb{H}_T}\right) dt + Z_T^{\mathbb{H}_T} I_2\left(\lambda Z_T^{\mathbb{H}_T}\right) \middle| \mathscr{H}_0 \right] = V_0, \tag{3.3.11}$$

then

$$(R_t c_t^*, R_T H^*) := \left(I_1\left(\lambda Z_t^{\mathbb{H}_T}\right), I_2\left(\lambda Z_T^{\mathbb{H}_T}\right) \right)_{t \in [0,T]}$$

solves the Lagrangian optimization problem.

Proof. Firstly one must show that $(c^*, H^*) \in \mathscr{A}_{\mathbb{H}_T}^{Lagr}$. c^* is obviously a consumption process and $H^* \geq 0$ is $\mathscr{H}_T$-measurable. Let $x = 1$ and $y = \lambda Z_t^{\mathbb{H}_T}$ in (3.3.1), then

$$\begin{aligned} U_1(I_1(\lambda Z_t^{\mathbb{H}_T})) &\geq U_1(1) + \lambda Z_t^{\mathbb{H}_T}\left(I_1(\lambda Z_t^{\mathbb{H}_T}) - 1\right) \\ &\geq -|U_1(1)| - \lambda Z_t^{\mathbb{H}_T}, \end{aligned}$$

since $U'(x) > 0 \forall x > 0$ and hence $I(y) = I(\lambda Z_t^{\mathbb{H}_T}) > 0$ and similarly

$$U_2(H^*) \geq -|U_2(1)| - \lambda Z_T^{\mathbb{H}_T}.$$

And since λ is $\mathscr{H}_0$-measurable and finite $\mathbb{P}$-a.s. and that $Z^{\mathbb{H}_T}$ is a $(\mathbb{H}_T, \mathbb{P})$-supermartingale,

$$\mathbb{E}\left[\int_0^T U_1^-(R_t c_t^*)\, dt + U_2^-(R_T H^*) \middle| \mathscr{H}_0 \right] \leq T(|U_1(1)| + \lambda) + (|U_2(1)| + \lambda) < \infty,$$

$\mathbb{P}$-a.s.. Hence $(c^*, H^*) \in \mathscr{A}_{\mathbb{H}_T}^{Lagr}$.

For an arbitrary pair $(c, H) \in \mathscr{A}_{\mathbb{H}_T}^{Lagr}(V_0)$, the inequalities above imply

$$\mathbb{E}\left[\int_0^T U_1(R_t c_t^*)\,dt + U_2(R_T H^*)|\mathscr{H}_0\right] \geq \tag{3.3.12}$$

$$\geq \mathbb{E}\left[\int_0^T U_1(R_t c_t)\,dt + U_2(R_T H)|\mathscr{H}_0\right] + \lambda\mathbb{E}\left[\int_0^T Z_t^{\mathbb{H}_T} R_t(c_t^* - c_t)\,dt + Z_T^{\mathbb{H}_T} R_T(H^* - H)|\mathscr{H}_0\right], \tag{3.3.13}$$

since λ is $\mathscr{H}_0$-measurable. And therefore by (3.3.11)

$$\mathbb{E}\left[\int_0^T Z_t^{\mathbb{H}_T}(R_t c_t)\,dt + Z_T^{\mathbb{H}_T}(R_T H)\middle|\mathscr{H}_0\right] \leq V_0$$
$$= \mathbb{E}\left[\int_0^T Z_t^{\mathbb{H}_T} R_t c_t^*\,dt + Z_T^{\mathbb{H}_T} R_T H^*\middle|\mathscr{H}_0\right].$$

Hence

$$\mathbb{E}\left[\int_0^T Z_t^{\mathbb{H}_T} R_t(c_t^* - c_t)\,dt + Z_T^{\mathbb{H}_T} R_T(H^* - H)\middle|\mathscr{H}_0\right] \geq 0,$$

and (3.3.12) implies the optimality of (c^*, H^*). □

One may ask then what guarantees the existence the $\mathscr{H}_0$ random variable that satisfies (3.3.11). The following lemma from Amendinger (1999) gives a sufficient condition.

Lemma 3.3.9. *[Lemme 4.4 of Amendinger (1999)] If for $\mathbb{P}$-a.a. $\omega \in \Omega$ the functions*

$$\Psi_\omega(\lambda) := \mathbb{E}\left[\int_0^T Z_t^{\mathbb{H}_T} I_1(\lambda Z_t^{\mathbb{H}_T})\,dt + Z_T^{\mathbb{H}_T} I_2(\lambda Z_T^{\mathbb{H}_T})\middle|\mathscr{H}_0\right](\omega)$$
$$= \int_{\mathbb{D}[0,T]}\left(\int_0^T z(t) I_1\left(\lambda z(t)\right)\,dt + z(T) I_2\left(\lambda z(T)\right)\right)\mathbb{P}\left[Z^{\mathbb{H}_T} \in dz|\mathscr{H}_0\right](\omega),$$

are finite, then there exists a $\mathscr{H}_0$-measurable $\lambda \in (0,\infty)$ that satisfies (3.3.11).

The next theorem summarizes the above results.

Theorem 3.3.10. *[Theorem 4.5 of Amendinger (1999)] Suppose Assumption 2.1.5 is satisfied for $[0,T]$ and Assumption 3.2.1 is also satisfied. Further assume that there exists a $\mathscr{H}_0$-measurable random variable $\hat{\lambda} \in (0,\infty)$ such that (3.3.11) is also satisfied. Then there exist a solution to the optimization problem for the insider. The optimal discounted consumption rate and the optimal discounted terminal wealth are then given by*

$$R_t c_t^* = I_1\left(\hat{\lambda} Z_t^{\mathbb{H}_T}\right), \quad t \in [0,T],$$

and

$$R_T H^* = I_2\left(\hat{\lambda} Z_T^{\mathbb{H}_T}\right).$$

Moreover, there exist a $\mathbb{H}_T$-trading strategy π^* such that $(\pi^*, c^*) \in \mathscr{A}_{\mathbb{H}_T}(v_0)$ and

$$V_T^{v_0,\pi^*,c^*} = H^*.$$

Proof. Any pair $(\pi, c) \in \mathscr{A}_{H_T}(V_0)$ will satisfy (3.3.9). By Proposition 3.3.8,

$$\mathbb{E}\left[\int_0^T U_1(R_t c_t)\,dt + U_2\left(R_T V_T^{v_0,\pi,c}\right)\middle|\,\mathscr{H}_0\right] \le \mathbb{E}\left[\int_0^T U_1(R_t c_t^*)\,dt + U_2\left(R_T V_T^{v_0,\pi,c}\right)\middle|\,\mathscr{H}_0\right].$$

Therefore a $\mathbb{H}_T$-trading strategy π^* that finances H^* and c^* needs to be established. $\mathbb{E}_{\mathbb{Q}^{\mathbb{H}_T}}[v_0] = \mathbb{E}[v] < \infty$. Define the process v as

$$v_t := \mathbb{E}_{\mathbb{Q}^{\mathbb{H}_T}}\left[\int_0^T R_s c_s^*\,ds + R_T H^*\middle|\,\mathscr{H}_t\right], \quad t \in [0, T],$$

which is well-defined. Furthermore, v is a $(\mathbb{H}, \mathbb{Q}^{\mathbb{H}_T})$-martingale. By the martingale representation Theorem 3.2.2, there exist a process $\pi^* \in L^1_{loc}(\mathscr{H}, \mathbb{Q}^{\mathbb{H}_{\mathscr{T}}})$ such that for all $t \in [0, T]$,

$$\begin{aligned} v_t &= \mathbb{E}_{\mathbb{Q}^{\mathbb{H}_T}}\left[\int_0^T R_s c_s^*\,ds + R_T H^*\middle|\,\mathscr{H}_0\right] + \int_0^t \pi_s^*\,d\hat{W}_s \\ &= \mathbb{E}\left[\int_0^T Z_t^{\mathbb{H}_T} R_t c_t^*\,dt + Z_T^{\mathbb{H}_T} R_T H^*\middle|\,\mathscr{H}_0\right] + \int_0^t \pi_s^*\,d\hat{W}_s \\ &= v_0 + \int_0^t \pi_s^*\,dW_s, \end{aligned}$$

since $\hat{\lambda}$ satisfies (3.3.11). Set the wealth process $V(v_0, \pi^*, c^*) := v - \int R_s c_s^*\,ds$, then

$$\begin{aligned} V_T(v_0, \pi^*, c^*) &= v_T - \int_0^T R_s c_s^*\,ds \\ &= \mathbb{E}_{\mathbb{Q}^{\mathbb{H}_T}}\left[\int_0^T R_s c_s^*\,ds + R_T H^*\middle|\,\mathscr{H}_T\right] - \int_0^T R_s c_s^*\,ds \\ &= \mathbb{E}_{\mathbb{Q}^{\mathbb{H}_T}}[H^* | \mathscr{H}_T] = H^*. \end{aligned}$$

And by Proposition 3.3.11, $(\pi^*, c^*) \in \mathscr{A}_{\mathbb{H}^T}(v_0)$. This completes the proof. □

The above theorem pre-supposes the existence of λ that satisfies (3.3.11) and gives the optimal consumption function and the optimal terminal wealth as functions of λ. The obvious question to ask is: what is the value of λ?

It can be shown that if a unique $\lambda(\omega)$ exist then

$$\lambda(\phi) = \sup\left\{y \in \mathbb{R}^+ : X(y)(\phi) \ge V_0(\phi)\right\},$$

where

$$X(y)(\phi) = \mathbb{E}_P\left[\int_0^T R_t \tilde{M}_t I_1(y R_t \tilde{M}_t)\,dt + R_T \tilde{M}_T I_2(y R_T \tilde{M}_T) | \mathscr{H}_0\right](\phi). \tag{3.3.14}$$

and $\lambda(\phi)$ is $\mathscr{H}_0$-measurable. See Grorud & Pontier (1998) for more details.

3.3.3 Examples

Log utility from discounted terminal wealth, no consumption

In this example, assume $c_t = 0$ (or equivalently $U_1 = 0$), $U_2(x) = \log(x)$. If a λ exist. Hence $I_2(y) = 1/y$. Then, from (3.3.14),

$$X(y)(\phi) = \mathbb{E}_{\mathbb{P}}\left[\frac{R_T\tilde{M}_T}{yR_T\tilde{M}_T}\middle|\mathscr{H}_0\right](\phi) = V_0$$
$$\mathbb{E}_{\mathbb{P}}\left[\frac{1}{y}\middle|\mathscr{H}_0\right](\phi) = V_0$$
$$\hat{\lambda} = y = \frac{1}{V_0}$$

Therefore the optimal terminal wealth is

$$V_T^{\pi^*,c^*} = I_2(\hat{\lambda}\tilde{M}_T R_T) = \frac{V_0}{\tilde{M}_T R_T},$$

hence

$$U_2(V_T^{\pi^*,c^*}) = \log(R_T^{-1}V_0) + \int_0^T (\zeta_t + \eta_t, dW_t) + \frac{1}{2}\int_0^T \|\zeta_t + \eta_t\|^2\, dt \tag{3.3.15}$$

Therefore the optimal value of the optimization problem is

$$\mathbb{U}(V_0, \pi^*, c^*) = \log(R_T^{-1}V_0) + \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\int_0^T \|\zeta_t + \eta_t\|^2\, dt\middle|\mathscr{H}_0\right], \forall T < \mathscr{T}. \tag{3.3.16}$$

Log utility from discounted terminal wealth and consumption

In this example, $U_1(x) = U_2(x) = \log(x)$. If a λ exist, then $I_1(y) = I_2(y) = 1/y$. Then, from (3.3.14),

$$X(y)(\phi) = \mathbb{E}_{\mathbb{P}}\left[\int_0^T \frac{R_t\tilde{(M)}_t}{yR_t\tilde{(M)}_t}\, dt + \frac{R_T\tilde{M}_T}{yR_T\tilde{M}_T}\middle|\mathscr{H}_0\right](\phi) = V_0$$
$$\mathbb{E}_{\mathbb{P}}\left[\int_0^T \frac{1}{y}\, dt + \frac{1}{y}\middle|\mathscr{H}_0\right](\phi) = V_0$$
$$\frac{T+1}{y} = V_0$$
$$\hat{\lambda} = y = \frac{T+1}{V_0}$$

Therefore and the optimal consumption function is

$$c_t^* = I_1(\hat{\lambda}\tilde{M}_t R_t) = \frac{V_0}{(T+1)\tilde{M}_t R_t}, \tag{3.3.17}$$

the optimal terminal wealth is

$$V_T^{\pi^*,c^*} = I_2(\hat{\lambda}\tilde{M}_T R_T) = \frac{V_0}{(T+1)\tilde{M}_T R_T}. \tag{3.3.18}$$

In the remaining sections, the benefits for having extra information will be quantified. The theory will continue to be developed for a d-dimensional Brownian motion framework (and hence with *d risky stocks to ensure a complete market). But the examples will be given assuming $d = 1$ to simplify the notations. Although they can be easily generalised to $d > 1$. Moreover, the theory will be concentrated in the case of a logarithmic utility function, since this is the most common class of utility functions that satisfy Assumption 3.3.1. It is possible to use other classes of utility functions to develop the concepts below and draw similar conclusions, as long as they satisfy Assumption 3.3.1, although the algebra and notations tends to be more tedious.*

3.4 Optimal Portfolio

In the previous section, the solution to the optimal consumption process and the optimal wealth process were given. The next natural question to ask would be which portfolio would finance the optimal consumption process and the optimal wealth process? I.e., what is the optimal portfolio? In this section, the solution will be developed for the optimal portfolio and its explicit representation.

Karatzas & Pikovsky (1996) made a conjecture for the optimal portfolio for the insider for optimization with respect to a logarithmic utility function on terminal wealth, but no formal proof was given. Grorud & Pontier (1998) gave a proof for the case logarithmic utility is applied to both (discounted) consumption and (discounted) terminal wealth, which will be presented below. But first the result in the following lemma is needed:

Lemma 3.4.1. *Let $\xi_t = -(\zeta_t + \eta_t)$ and $N_t = \tilde{M}_t^{-1}$, then N_t satisfies the following equation:*

$$dN_t = -N_t(\xi_t, d\hat{W}_t), N_0 = 1. \tag{3.4.1}$$

and hence

$$N_t = 1 - \int_0^t N_s(\xi_s, d\hat{W}_s). \tag{3.4.2}$$

Proposition 3.4.2. *Assume that $\mathbb{Q}^*$ as defined above exist (as hence the existence of ζ_t and η_t. The optimal portfolio function is given by:*

$$\pi_t^* = (\sigma_t')^{-1}(\zeta_t + \eta_t), \tag{3.4.3}$$

where σ' is the transpose of σ.

Proof. [*Adapted from arguments given in Grorud & Pontier (1998).*] From the proof of Proposition 3.3.6, specifically (3.3.7),

$$\begin{aligned} V_t^* R_t + \int_0^t R_s c_s^* \, ds &= \mathbb{E}_{\mathbb{Q}^*}\left[V_T^* R_T + \int_0^T R_s c_s^* \, ds | \mathscr{H}_t \right] \\ &= \frac{V_0}{T+1} \mathbb{E}_{\mathbb{Q}^*}\left[1 - \int_0^T N_s(\xi_s, d\hat{W}_s) + T - \int_0^T \left[\int_0^s N_u(\xi_u, d\hat{W}_u) \right] ds | \mathscr{H}_t \right] \\ &= \frac{V_0}{T+1} \mathbb{E}_{\mathbb{Q}^*}\left[T + 1 - \int_0^T N_s(\xi_s, d\hat{W}_s) - \int_0^T \int_u^T ds N_u(\xi_u, d\hat{W}_u) | \mathscr{H}_t \right] \\ &= V_0 - \frac{V_0}{T+1}\left[\int_0^t (1 + T - s) N_s(\xi_s, d\hat{W}_s) \right], \end{aligned}$$

Compare this expression to (3.3.8),

$$R_t V_t \sigma_t' \pi_t^* = \frac{V_0(T+1-s)}{T+1} N_t(-\xi_t). \tag{3.4.4}$$

Now subtracting the consumption from 0 to t in (3.3.7):

$$\begin{aligned} V_t^* R_t &= \mathbb{E}_{\mathbb{Q}^*}\left[V_T^* R_T + \int_t^T R_s c_s^* \, ds | \mathscr{H}_t\right] \\ &= \frac{V_0}{T+1}\mathbb{E}_{\mathbb{Q}^*}\left[1 - \int_0^T N_s(\xi_s, d\hat{W}_s) + (T-t)\left[1 - \int_0^t N_s(\xi_s, d\hat{W}_s)\right] | \mathscr{H}_t\right] \\ &= \frac{V_0}{T+1}\mathbb{E}_{\mathbb{Q}^*}\left[(1+T-t)\left[1 - \int_0^t N_s(\xi_s, d\hat{W}_s)\right] | \mathscr{H}_t\right] \\ &= V_0 \frac{T+1-t}{T+1} N_t, t \in [0,T] \end{aligned}$$

Substituting that into (3.4.4), the explicit expression for the optimal portfolio process (3.4.3) follows. □

Hence a general solution to the portfolio optimization problem is obtained for the honest investor as well as the insider for additional information represented by L. In the next three section, specific examples of L will be considered, as well as the answer to the question: how much better off is the insider, really?

3.5 Additional value achieved by the insider

Note that any $\mathbb{F}_T$-investment-consumption strategy with consumption is also a $\mathbb{G}_T$-trading strategy with consumption. Amendinger shown this explicitly in Amendinger (1999). But intuitively, the *honest* investor will select his optimal investment-consumption strategy according to the filtration $\mathbb{F}_T$. The *insider* will select his optimal investment-consumption strategy according to the filtration $\mathbb{G}_T$ which is essentially larger, reflecting the larger amount of information possessed by the insider. The strategies available by considering the information encapsulated in $\mathbb{F}_T$ would obviously be available to the insider, since $\mathbb{F}_T \subseteq \mathbb{G}_T$. However, the $\mathbb{F}_T$-optimal investment-consumption strategy may not be optimal for the insider, i.e. it may not be $\mathbb{G}_T$-optimal, as has been shown in the previous section. This will be investigated in the sequel.

Since the insider has a larger amount of strategies at his disposal, hence one would intuitively expect the insider to be able achieve at the very least the same value to the optimization problem to that achieved by the honest investor, if not more. In this section, it will be shown that this is indeed the case. But first, the concept of additional value needs to be defined:

Definition 3.5.1. *The* additional value achieved by the insider *is defined as*

$$\sup_{(\pi,c)\in\mathscr{A}_{\mathbb{G}_T}(V_0)} \mathbb{U}_T^{v_0,\pi,c} - \sup_{(\pi,c)\in\mathscr{A}_{\mathbb{F}_T}(V_0)} \mathbb{U}_T^{v_0,\pi,c}.$$

Remark 3.5.2. *This is merely a convenient way to quantify "value ". May be a more precise way to put the wording would be "the additional optimal utility achieved by the insider".*

From (3.3.16), the honest investor's optimal value is

$$\begin{aligned}\mathbb{U}(V_0,\pi^*,c^*) &= \log(R_T^{-1}V_0)+\frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\int_0^T \|\eta_t\|^2\,dt|\mathscr{F}_0\right]\\ &= \log(R_T^{-1}V_0)+\frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\int_0^T \|\eta_t\|^2\,dt\right]\end{aligned}$$

since $\gamma_t = 0$ forall t for the honest investor. Hence the additional value achieved by the insider is

$$\frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\int_0^T \|\zeta_t\|^2\,dt|\mathscr{G}_0\right].$$

3.5.1 Example: terminal value of Brownian motion

Now if $L = W_{\mathscr{T}}$, $T < \mathscr{T}$, then $\zeta_t = \frac{\alpha_t}{p_t^L} = \frac{W_{\mathscr{T}}-W_t}{\mathscr{T}-t}$ and the optimal value of the optimization problem for the insider is

$$\begin{aligned}\mathbb{U}(V_0,\pi^*,c^*) &= \log(R_T^{-1}V_0)+\frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\int_0^T \left|\frac{W_{\mathscr{T}}-W_t}{\mathscr{T}-t}+\eta_t\right|^2 dt\middle|\mathscr{G}_0\right]\\ &= \log(R_T^{-1}V_0)+\frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\int_0^T \left|\frac{W_{\mathscr{T}}-W_t}{\mathscr{T}-t}\right|^2 dt\middle|\mathscr{G}_0\right]+\frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\int_0^T \frac{(\mu_t-r_t)^2}{\sigma_t^2}\,dt\right].\end{aligned}$$

And the optimal value of the optimization problem for the honest investor is

$$\mathbb{U}(V_0,\pi^*,c^*) = \log(R_T^{-1}V_0)+\frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\int_0^T \frac{(\mu_t-r_t)^2}{\sigma_t^2}\,dt\right].$$

The additional value achieved by the insider is

$$\begin{aligned}\frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\int_0^T \left|\frac{W_{\mathscr{T}}-W_t}{\mathscr{T}-t}\right|^2 dt\middle|\mathscr{G}_0\right] &= \frac{1}{2}\int_0^T \frac{1}{\mathscr{T}-\mathscr{T}}\,dt\\ &= \ln\sqrt{\frac{\mathscr{T}}{\mathscr{T}-T}}.\end{aligned}$$

Two observations can be made:

1. As $T \to \mathscr{T}$, the insider's value to the optimization problem tends to ∞, see also remark below.

2. The honest investor's value to the optimization problem is finite (by assumption imposed on μ, r and σ). And hence the additional value achieved by the insider also tends to ∞ as $T \to \mathscr{T}$.

Remark 3.5.3. *If the insider has the knowledge of W_T, he can exploit the direction at which the Brownian path will take in the future. Consider the case where μ, r and σ are fixed contants (or just deterministic), having the knowledge of W_T is equivalent to having the knowledge of the terminal asset prices.*

Then the insider's "sure fire" strategy is simple, buy when the asset prices are below the terminal price, short when the asset prices rises above the terminal price. The only other consideration is the holding period cost at r per unit of time. Since the market is assumed to be frictionless, *i.e. the insider can buy or sell at no cost for infinte volume, including the cash asset, i.e. he can borrow or lend at r for an infinite amount. An insider employing such a strategy for this kind of knowledge is expected to make infinite amount of profit in this idealized market. Hence the value to the optimization problem for the insider is also infinite.*

If, however, the knowledge the insider possesses is the share price at anytime $\mathscr{T}$ after T, the end of the trading horizon (for whatever reason he cannot trade after T), then there is an element of risk as there is uncertainty as to the prices at which his position will be liquidated at time T. As the results above has shown, this uncertainty rendered the insider with only a finite amount of utility value.

3.5.2 Example: terminal value of Brownian motion distorted by noise

Now if $L = \lambda W_{\mathscr{T}} + (1-\lambda)\varepsilon$, $T = \mathscr{T}, \lambda \in (0,1)$, then $\zeta_t^x = \frac{\alpha_t^x}{p_t^x} = \frac{\lambda(L-\lambda W_t)}{\lambda^2(T-t)+(1-\lambda)^2}$ and the optimal value of the optimization problem for the insider is

$$\begin{aligned} &\mathbb{U}^{\mathbb{G}}(v_0, \pi^*, c^*) \\ &= \log(R_T^{-1}V_0) + \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\int_0^T \left|\frac{\lambda(L-\lambda W_t)}{\lambda^2(T-t)+(1-\lambda)^2}\right|^2 dt \middle| \mathscr{G}_0\right] + \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\int_0^T \frac{(\mu_t - r_t)^2}{\sigma_t^2}\, dt\right]. \end{aligned}$$

The additional value achieved by the insider is

$$\begin{aligned} \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\int_0^T \left|\frac{\lambda(L-\lambda W_t)}{\lambda^2(T-t)+(1-\lambda)^2}\right|^2 dt \middle| \mathscr{G}_0\right] &= \frac{1}{2}\int_0^T \mathbb{E}_{\mathbb{P}}\left[\left|\frac{\lambda(\lambda(W_T - W_t)+(1-\lambda)\varepsilon)}{\lambda^2(T-t)+(1-\lambda)^2}\right|^2 \middle| \mathscr{G}_0\right] dt \\ &= \frac{1}{2}\int_0^T \frac{\lambda^2(\lambda(T-t)+(1-\lambda)^2)}{(\lambda^2(T-t)+(1-\lambda)^2)^2}\, dt \\ &= \frac{1}{2}\int_0^T \frac{\lambda^2}{(\lambda^2(T-t)+(1-\lambda)^2)}\, dt \\ &= \lambda^2 \ln\sqrt{\frac{\lambda^2 T + (1-\lambda)^2}{(1-\lambda)^2}} = \lambda^2 \ln\sqrt{1 + \frac{\lambda^2 T}{(1-\lambda)^2}} \end{aligned}$$

Again, one can make the following observations:

1. For $\lambda \in (0,1)$, the insider's value to the optimization problem is finite. As $\lambda \to 1$, i.e. the information regarding the terminal value becomes more exact, the value approaches ∞. This is the results one would expect after the analysis in the previous subsection.

2. Again the honest investor's value to the optimization problem is finite (by assumption imposed on μ, r and σ) and the additional value achieved by the insider also tends to ∞ as $\lambda \to 1$.

3.6 Monetary Value of the Insider Information

The concept of additional utility is a theoretical economic concept that may appear to be very abstract to some. Another way to express the value of the extra information to the insider is to put it in monetary terms. Amendinger et al. (2003) suggested the concept of *utility indifference* *value*. The precise definition of the concept and the way it is calculated will be presented in this section following the approach of Amendinger et al. (2003).

Consider an investor with information flow $\mathbb{F}$, who is faced with the opportunity to buy some extra information regarding the asset he is trading, we can denote this information by a random variable L, following the notation in the previous discussions. This may sound dubious (and it may well be), but it can be interpreted in the context of an investor who has to consider the spending on research conducted in-house regarding a company whose share is to be traded. Trading on such "extra" information is totally legitimate - the concept of Mosaic Theory. The investor may also be considering to purchase research from a broker or research organisation. Again, such "extra" information is considered public knowledge, although it may not be reflected on the market prices (and hence not included in $\mathbb{F}$, but nevertheless legitimate.

By procuring this extra bit of information, the investor is able to base his investment decision on an enlarged filtration $\mathbb{G}$. The question one may ask is: what is the "fair" cost of this extra information to the investor? The answer is the utility indifference value - at which the initial reduction in wealth (spent to acquire the information) will be offset by the extra utility derived from the increase in terminal wealth and consumption resulting from being able to choose an investment strategy based on the enlarged filtration. Such a value thus quantifies the informational advantage of the "insider" in monetary terms, instead of additional utility.

The following assumption is made in this section:

Assumption 3.6.1. *Suppose*

1. *there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $\mathscr{F}_T$ and $\sigma(L)$ are $\mathbb{Q}$-independent;*
2. *there is a unique probability measure $\mathbb{Q}^{\mathbb{F}} \sim \mathbb{P}$ with $\frac{d\mathbb{Q}^{\mathbb{F}}}{d\mathbb{P}}$ $\mathscr{F}_T$-measurable, $\mathbb{Q}^{\mathbb{F}} = \mathbb{P}$ on $\mathscr{F}_0$ and W is a $(\mathbb{F}, \mathbb{Q}^{\mathbb{F}})$-Brownian motion.*

Definition 3.6.2. *The* utility indifference value *of the additional information L is defined as a solution $\kappa = \kappa(v_0)$ of the equation*

$$\mathbb{U}^{\mathbb{F}}(v_0) = \mathbb{U}^{\mathbb{G}}(v_0 - \kappa). \tag{3.6.1}$$

Remark 3.6.3. *(3.6.1) can be interpreted as follows: with the cost of the extra information L at κ, the investor who aim to maximize his expected utility is indifferent between two alternatives:*

1. *to invest the initial capital v_0 optimally using the information available from $\mathbb{F}$; or*
2. *spend κ to acquire the information L and then invest the remaining capital $v_0 - \kappa$ optimally according to the information available from $\mathbb{G} = (\mathscr{F}_t \vee \sigma(L))_{t\in[0,T]}$*

A unique indifference value κ must exist if Assumption 3.3.1 is satified for the utility function(s) in the definition of $\mathbb{U}$. Under Assumption 3.3.1, κ must also be non-negative. To see this, because $\mathscr{F} \subseteq \mathscr{G}$, $\mathscr{A}_{\mathbb{F}} \subseteq \mathscr{A}_{\mathbb{G}}$ and $\mathbb{U}^{\mathbb{F}}(x) \leq \mathbb{U}^{\mathbb{G}}(x)$ forall x. Hence κ must be non-negative for the equality of the two sides of (3.6.1).

The main result of this section:

Theorem 3.6.4. *[Theorem 5.3(1) of Amendinger et al. (2003)] Suppose Assumption 3.6.1 is satisfied. For a logarithmic utility function $U(x) = \log(x)$, if $\mathbb{E}_{\mathbb{P}}\left[\log \frac{1}{Z_T^{\mathbb{G}}}\right] < \infty$ then the utility* *indifference value κ satisfies*

$$\kappa = v_0 \left(1 - \exp\left(-\mathbb{E}_{\mathbb{P}}\left[\log \frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\right]\right)\right). \tag{3.6.2}$$

Proof. Assumption 3.6.1 implies that the second part of Assumption 3.6.1 is also satisfied for the enlarged filtration $\mathscr{G}$. Since $\mathbb{U}$ is strictly increasing, κ is unique. Hence all that is required is to verify that (3.6.2) satisfies (3.6.1).

Recall that for a generic filtration $\mathbb{H}$,

$$\mathbb{U}^{\mathbb{H}}(v_0) = \log v_0 + \mathbb{E}_{\mathbb{P}}\left[\log \frac{1}{Z_T^{\mathbb{H}}}\right]. \tag{3.6.3}$$

Now

$$\mathbb{E}_{\mathbb{P}}\left[\log \frac{1}{Z_T^{\mathbb{F}}}\right] = \mathbb{E}_{\mathbb{P}}\left[\log \frac{1}{\mathbb{E}_{\mathbb{P}}\left[Z_T^{\mathbb{G}}|\mathscr{F}_T\right]}\right] \leq \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[\log \frac{1}{Z_T^{\mathbb{G}}}\middle|\mathscr{F}_T\right]\right] = \mathbb{E}_{\mathbb{P}}\left[\log \frac{1}{Z_T^{\mathbb{G}}}\right] < \infty$$

by Jensen's inequality. Hence

$$\begin{aligned} 0 &= \mathbb{U}^{\mathbb{G}}(v_0 - \kappa) - \mathbb{U}^{\mathbb{F}}(v_0) \\ &= \log(v_0 - \kappa) + \mathbb{E}_{\mathbb{P}}\left[\log \frac{1}{Z_T^{\mathbb{G}}}\right] - \log v_0 - \mathbb{E}_{\mathbb{P}}\left[\log \frac{1}{Z_T^{\mathbb{F}}}\right] \\ -\mathbb{E}_{\mathbb{P}}\left[\log \frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\right] &= \log\left(\frac{v_0 - \kappa}{v}\right) \\ \kappa &= v_0\left(1 - \exp\left(-\mathbb{E}_{\mathbb{P}}\left[\log \frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\right]\right)\right) \end{aligned}$$

as required. □

Remark 3.6.5. *In utility indifference value relating to the power utility function $U(x) = \frac{x^\gamma}{\gamma}, \gamma \in (0,1)$ and the exponential utility function $U(x) = -e^{-\gamma x}, \gamma > 0$ are given in Amendinger et al. (2003).*

3.6.1 Examples

Terminal value of Brownian motion

In this case $L = W_{\mathscr{T}}$, where $T < \mathscr{T}$ for part 1 of Assumption 3.6.1 to be satisfied. Assume that $\mathbb{E}_{\mathbb{P}}\left[\int_0^T |\eta_t|^2\, dt\right]$ is finite so that

$$\mathbb{E}_{\mathbb{P}}\left[\log\left(\frac{1}{Z_T^{\mathbb{F}}}\right)\right] = \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\int_0^T |\eta_t|^2\, dt\right] < \infty.$$

Now recall that

$$p_T^L = \exp\left[\int_0^T \frac{L - W_s}{\mathscr{T} - s}\, dW_s - \frac{1}{2}\int_0^T \left(\frac{L - W_s}{\mathscr{T} - s}\right)\, ds\right],$$

hence

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[\log p_T^L] &= -\frac{1}{2}\int_0^T \frac{1}{\mathscr{T} - s}\, ds \\ &= -\frac{1}{2}\log(\mathscr{T} - s)|_0^T \\ &= \log\sqrt{\frac{\mathscr{T}}{\mathscr{T} - T}}\end{aligned}$$

and therefore $\mathbb{E}_{\mathbb{P}}\left[\log\left(\frac{1}{Z_T^{\mathbb{G}}}\right)\right] = \mathbb{E}_{\mathbb{P}}\left[\log\left(\frac{1}{Z_T^{\mathbb{F}}}\right)\right] + \mathbb{E}_{\mathbb{P}}[\log p_T^L] < \infty$. By (3.6.2) and the fact that $Z_T^{\mathbb{G}} = \frac{Z_T^{\mathbb{F}}}{p_T^L}$, the utility indifference value for logarithmic utility case is

$$\begin{aligned}\kappa &= v_0\left(1 - \exp\left(-\log\sqrt{\frac{\mathscr{T}}{\mathscr{T} - T}}\right)\right) \\ &= v_0\left(1 - \sqrt{\frac{\mathscr{T} - T}{\mathscr{T}}}\right).\end{aligned}$$

A few observations can be made:

1. Since $T < \mathscr{T}$, $\kappa < v_0$. Hence the investor will only be willing to pay no more than his initial wealth in exchange for the information, which makes intuitive sense.

2. For fixed T, as $\mathscr{T} \to \infty$, κ tends to 0. The information regarding the value of the future value of the Brownian process becomes useless as this is pushed further out into the future beyond the investment period. The same can be said about the situation where $T \to 0$ for fixed $\mathscr{T}$. As the trading period is becomes more limited and there is a greater gap between T and $\mathscr{T}$.

3. As $T \to \mathscr{T}$, $\kappa \to v_0$. The value of the information to the investor becomes close to his entire initial wealth, this almost implies an arbitrage opportunity - that is, in the limit the investor is almost willing to pay the entire initial wealth away to obtain the exact information regarding the terminal value of the process. Hence arbitrage must exist (in the limit) if the investor is to be acting rationally. Note that the case where $T = \mathscr{T}$ is not included in this model as this will violate Assumption 3.6.1.

Terminal value of Brownian motion distorted by noise

In this case $L = \lambda W_{\mathscr{T}} + (1 - \lambda)\varepsilon$, $T = \mathscr{T}, \lambda \in (0, 1)$. Taking the same approach from the previous case, assuming that $\mathbb{E}_{\mathbb{P}}\left[\int_0^T |\eta_t|^2\, dt\right]$ is finite and recall from (2.4.1)

$$p_T^L(y) = \sqrt{\frac{\lambda^2 T + (1-\lambda)^2}{(1-\lambda)^2}}\exp\left(-\frac{1}{2}\left\{\frac{(L - \lambda y)^2}{(1-\lambda)^2} - \frac{L^2}{\lambda^2 T + (1-\lambda)^2}\right\}\right),$$

hence

$$\mathbb{E}_{\mathbb{P}}[\log p_T^L] = \frac{1}{2}\log\frac{\lambda^2 T + (1-\lambda)^2}{(1-\lambda)^2} < \infty.$$

By (3.6.2), the utility indifference value for logarithmic utility case is

$$\begin{aligned}\kappa &= v_0\left(1-\exp\left(-\frac{1}{2}\log\frac{\lambda^2 T+(1-\lambda)^2}{(1-\lambda)^2}\right)\right)\\ &= v_0\left(1-\left(\frac{(1-\lambda)^2}{\lambda^2 T+(1-\lambda)^2}\right)^{\frac{1}{2}}\right).\end{aligned}$$

Again, a few observations can be made:

1. Since $\lambda < 1$, $\kappa < v_0$. Again, the investor will only be willing to pay no more than his initial wealth in exchange for the information.
2. For fixed T, $\kappa \to 0$ as $\lambda \to 0$. In fact, if $\lambda = 0$, then $\kappa = 0$. I.e. the investor will not be willing to pay a dime for the information as it is complete noise.
3. For fixed $\lambda \in (0,1)$, κ is an increasing function of T.
4. As $\lambda \to 1$, $\kappa \to v_0$. Again, the value of the information to the investor becomes close to his entire initial wealth as the level of noise reduces and the information becomes exact. Once again, this implies an arbitrage opportunity as per the previous example. The case where $\lambda = 1$ is not included in this model as this will violate Assumption 3.6.1.

3.7 Relationship to the Relative Entropy

With a logarithmic utility function on terminal wealth, the additional utility achieved by the insider is related to the concept of relative entropy (or Kullback - Leibler divergence), of the measure $\mathbb{P}$ with respect to $\mathbb{Q}_T$, where $\mathbb{Q}_T$ is a probability measure on $\mathscr{G}_T$. More precisely, it is the *expected logarithmic utility gain* (which will be defined below) that is equal to the relative entropy. This relationship was first pointed out by Karatzas & Pikovsky (1996) under a Brownian motion setting. Amendinger et al. (1998) provided some concrete proofs for the relationship under a general martingale setting.

The Kullback - Leibler divergence (also known as information divergence, information gain) is a information theory and probability theory concept. It is a measure of the difference between two probability measures: $\mathbb{P}$ usually denote the "real world" measure and $\mathbb{Q}$ an arbitrary probability measure, and is defined as follows:

Definition 3.7.1. *For two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on $(\Omega, \mathscr{X})$, the Kullback – Leibler divergence (or relative entropy) of $\mathbb{P}$ with respect to $\mathbb{Q}$ on the set $X \in \mathscr{X}$ is defined as*

$$H_X(\mathbb{P}|\mathbb{Q}) := \begin{cases} \mathbb{E}_{\mathbb{P}}\left[\log \frac{d\mathbb{P}}{d\mathbb{Q}}|x\right] & \text{if } \mathbb{P} \ll \mathbb{Q} \text{ on } X, \\ +\infty & \text{otherwise .}\end{cases} \tag{3.7.1}$$

Note that the measure is not a metric since it is not symmetric, i.e. $H_X(\mathbb{P}|\mathbb{Q}) \neq H_X(\mathbb{Q}|\mathbb{P})$ in general, hence the term *divergence.* Nevertheless it satisfies a number of mathematical properties that characterizes a metric: it is always non-negative; and $H_X(\mathbb{P}|\mathbb{Q}) = 0$ if and only if $\mathbb{P} = \mathbb{Q}$ on X; and it is increasing in X in the sense that if $X_1 \subseteq X_2$ then $H_{X_1}(\mathbb{P}|\mathbb{Q}) \leq H_{X_2}(\mathbb{P}|\mathbb{Q})$.

Definition 3.7.2. *The insider's expected logarithmic utility gain up to time t, $a_t, t \in [0,T]$ is defined as*

$$\begin{aligned} a_t &= \sup_{\pi \in \mathscr{A}_{\mathbb{G}}(v_0)} \mathbb{U}_t^{v_0,\pi} - \sup_{\pi \in \mathscr{A}_{\mathbb{F}}(v_0)} \mathbb{U}_t^{v_0,\pi} \\ &= \sup_{\pi \in \mathscr{A}_{\mathbb{G}}(v_0)} \mathbb{E}_{\mathbb{P}}[\log V_t^{v_0,\pi}] - \sup_{\pi \in \mathscr{A}_{\mathbb{F}}(v_0)} \mathbb{E}_{\mathbb{P}}[\log V_t^{v_0,\pi}]. \end{aligned}$$

Remark 3.7.3. *Note that the above definition differ slightly from that given in Amendinger et al. (1998), where the insider's utility gain is defined in the general martingale setting as $\mathbb{E}[\frac{1}{2}\int_0^t \zeta_s^* d\langle M\rangle_s \zeta_s]$ and can be translated to the Brownian motion setting as $\mathbb{E}[\frac{1}{2}\int_0^t \|\zeta_s\|^2\, ds]$. It was pointed out in Amendinger et al. (1998), the two definitions coincides (even in the cases where the quantity is infinite) except in the special case where $\mathbb{E}[\frac{1}{2}\int_0^t \|\zeta_s\|^2\, ds] < \infty$ and $\tilde{M}$ (analogous to $\tilde{W}$) is a (local) $\mathbb{G}$-martingale on $[0,T)$ but not on $[0,T]$. However, no example was provided in Amendinger et al. (1998) for such a situation.*

It is then apparent from Remark 3.1.4 and (3.3.15) that

$$\begin{aligned} a_t = \mathbb{E}_{\mathbb{P}}\left[\frac{1}{2}\int_0^t \|\zeta_s\|^2\, ds\right] = \mathbb{E}_{\mathbb{P}}\left[\log \frac{M^T}{\tilde{M}_T}\middle| \mathscr{G}_t\right] = \mathbb{E}_{\mathbb{P}}\left[\log \frac{d\mathbb{P}}{d\mathbb{Q}}\middle| \mathscr{G}_t\right] &= H_{\mathscr{G}_t}(\mathbb{P}|\mathbb{Q}) \\ = \mathbb{E}_{\mathbb{P}}\left[\log p_t^L\right] &= H_{\mathscr{G}_t}(\mathbb{P}|\mathbb{P}_t^L) \end{aligned}$$

The larger the utility gain, the larger the relative entropy which implies a larger amount of information carried by $\mathbb{Q}$ relative to $\mathbb{P}$. This is consistent with what intuition would suggest: more (quality) information possessed by the insider implies more utility gain. Note that for the honest investor $\mathbb{Q} = \mathbb{P}$ and $H_{\mathscr{G}_t}(\mathbb{P}|\mathbb{Q}) = 0$, i.e. no utility gain, which is obviously true.

Chapter 4

Malliavin Calculus

Recall from Section 3.1 that the market is assumed to be modelled by a (d-dimensional) stock process

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + (\sigma_t^i, dW_t), \quad 0 \leq t \leq T, S_0 \in [0,\infty)^d, i = 1, \cdots, d, \tag{4.0.1}$$

where W is d-dimensional Brownian motion, and a bond process

$$S_t^0 = 1 + \int_0^t S_s^0 r_s \, ds. \tag{4.0.2}$$

In Chapter 3, it was shown how information possessed by the insider can be incorporated into the model and how optimal portfolios can be derived, and the expected utility change as a result. However, not all types of information can be handled with the techniques of initial enlargement of filtration. Jacod's hypothesis (and Assumption 2.1.3) is fundamental in the initial filtration enlargement approach in that it ensures the preservation of the semimartingale in the enlarged filtration. But in many situations, the required absolute continuity may be too onerous (i.e. not satisfied), as the following example from Imkeller (2003) illustrates.

Example 4.0.4. *If we assume that $d = 1$, μ and σ are constants, specifically, let $\mu = \frac{1}{2}$ and $\sigma = 1$, so that $S_t = \exp(W_t)$. Assume $T = \mathscr{T} =$ (This would not change the analysis as it can be easily generalised to a general T by scaling the time parameter). Let the extra information possessed by the insider be the maximum price of the stock over the investment period of $[0,1]$, i.e. let $L = \sup_{t\in[0,1]} S_t$. This is equivalent to $L = \sup_{t\in[0,1]} W_t$, since $\sup S_t = \exp(\sup W_t)$. Denote for $t \in [0,1]$*

$$L_t = \sup_{s\in[0,t]} W_s, \tilde{L}_{1-t} = \sup_{s\in[t,1]} (W_s - W_t).$$

Also denote the density function of $\tilde{L}_{1-t}$ by p_{1-t}. Then L can be re-written as

$$L = L_t \vee (W_t + \tilde{L}_{1-t}).$$

For $A \in \mathscr{B}(\mathbb{R})$,

$$\mathbb{P}(L \in A|\mathscr{F}_t)(\omega) = \mathbb{P}_t^L(\omega, \mathscr{A}),$$

as in Assumption 2.1.3.

The conditional distribution of L given $\mathscr{F}_t$ can be derived with the following reasoning: given $\mathscr{F}_t$, W_t is known, hence it is the probability of $\tilde{L}_{1-t}$ being less than or equal to $L_t - W_t$ (hence $L = L_t$)

plus the probability that $L = W_t + \tilde{L}_{1-t}$, i.e. $\tilde{L}_{1-t}$ must be greater than $L_t - W_t$. Therefore,

$$\begin{aligned}\mathbb{P}(L \in A|\mathscr{F}_t) &= \mathbb{P}(L_t \in A, \tilde{L}_{1-t} \leq L_t - W_t|\mathscr{F}_t) + \mathbb{P}(\tilde{L}_{1-t} \in A, \tilde{L}_{1-t} \geq L_t - W_t|\mathscr{F}_t) \\ &= \mathbf{1}_{\{L_t \in A\}} \cdot \mathbb{P}(\tilde{L}_{1-t} \leq L_t - W_t|\mathscr{F}_t) + \mathbb{P}(\tilde{L}_{1-t} \in A \cap [L_t - W_t, \infty)|\mathscr{F}_t) \\ &= \int_{-\infty}^{L_t - W_t} p_{1-t}(y)\, dy \cdot \delta_{L_t}(A) + \int_{A \cap [L_t - W_t, \infty)} p_{1-t}(y)\, dy\end{aligned}$$

since L_t is $\mathscr{F}_t$-measurable, and $\mathscr{F}_t$ is independent of $\tilde{L}_{1-t}$.

The important observation that can be made from the above equation is that the family of Dirac measures in the first term of the last line on the right hand side is supported on the points $L_t(\omega)$ and the law of it is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^+$. Therefore there does not exist any common reference measure ν such that $\mathbb{P}_t^L \ll \nu$ and hence Assumption 2.1.3 is not satisfied. (Because one may take ν to be the law of L, for example.)

Hence the information in Example 4.0.4 cannot be handled using the techniques developed in Chapter 1. The question is then how can the information drift γ be identified in terms of the conditional densities of the additional information? The answer is provided by the celebrated Clark-Ocone formula from the theory of Malliavin's calculus. The concepts required in order to make sense of the formula and its application to the problem of identifying the information drift are discussed in Section 4.1 and 4.2.

In order to simplify the notations and to keep the underlying workings of the theory more transparent, the dimension of the model d will be set as 1 in this section.

4.1 Malliavin Calculus: A brief overview

In this section the basic concepts of Malliavin Calculus are introduced in a concise way. For a more detailed exposition of the subject of Malliavin calculus, one can refer to Malliavin & Thalmaier (2005) and Nualart (2006). One can also refer to Øksendal (1997) for a brief introduction. The aim of this chapter is to develop the basic tools of Malliavin calculus so that they can be meaningfully applied to the problem of the mathematics of insider trading. Following the approach of Nualart (2006), the concept of Wiener chaos is introduced. The Itô-Wiener chaos decomposition ensures that a random variable (satisfying certain conditions) can be expressed in terms of an orthogonal basis in the form of Wiener chaos. The notion of Malliavin derivative DF for some random process F and its meaning in the $L^2(T)$ space are discussed. The Clark-Ocone formula is introduced and this will turn out to be a useful result in the context of the insider problem. Then the Malliavin Divergence and the Skorohod integral are discussed. The notion of the forward integral is introduced as well as its link to the Skorohod integral.

4.1.1 Wiener Chaos

The notion of Hermite polynomials and Wiener chaos are discussed below with the aim to develop a basis of the isonormal Gaussian space which is the context on which the tools of Malliavin calculus will operate.

Itô-Wiener Chaos Decomposition

Suppose that H is a separable Hilbert space with scalar product $\langle .,. \rangle_H$. Hence H has a denumerable orthonormal basis.

Definition 4.1.1 (Definition 1.1.1 of Nualart (2006)). *An* isonormal Gaussian process *is a stochastic process* $W = \{W(h), h \in H\}$ *defined in a complete probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *where* W *is a centered Gaussian family of random variables such that*

$$E\left(W(h)W(g)\right) = \langle h, g \rangle_H \text{ for all } h, g \in H.$$

We denote the nth *Hermite polynomial* by $H_n(x)$. We can define $H_n(x)$ as in Nualart (2006) by

$$H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}), n \geq 1,$$

and $H_0(x) = 1$. [1]

The relationship between Gaussian random variables and Hermite polynomials is as follows:

Proposition 4.1.2. *(Lemma 1.1.1 of Nualart (2006)) Let* X, Y *be two random variables with joint Gaussian distribution such that the marginal distributions are standard Gaussian distribution. Then for* $n, m \geq 0$ *we have*

$$E(H_n(X)H_m(Y)) = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{n!}(E(XY))^n & \text{if } n = m \end{cases}$$

Let $\mathscr{F}$ denote the σ-field generated by the set of random variables $\{W(h), h \in H\}$. We will denote by $\mathscr{H}_n : n \geq 1$ the closed linear subspace of $L^2(\Omega, \mathscr{F}, \mathbb{P})$ generated by the random variables $\{H_n(W(h)) : h \in H, \|h\|_H = 1\}$. $\mathscr{H}_0$ is the set of constants. By 4.1.2 above the subspaces $\mathscr{H}_n$ and $\mathscr{H}_m$ are orthogonal whenever $n \neq m$. The space $\mathscr{H}_n$ is called the *Wiener Chaos of order n*. The following theorem provides a decomposition of the $L^2(\Omega, \mathscr{F}, \mathbb{P})$ space.

Theorem 4.1.3. *(Theorem 1.1.1 of Nualart (2006))* $L^2(\Omega, \mathscr{F}, \mathbb{P})$ *can be decomposed into the infinite orthogonal sum of the subspaces* $\mathscr{H}_n$*:*

$$L^2(\Omega, \mathscr{F}, \mathbb{P}) = \oplus_{n=0}^{\infty} \mathscr{H}_n.$$

Multiple Wiener-Itô Integrals

Suppose the Hilbert space H is a space of the form $L^2(T, \mathscr{B}, \mu)$, where $(T, \mathscr{B})$ is a measurable space and ν is a σ-finite measure without atoms. W can then be regarded as a Gaussian measure

[1] Alternatively, we can define $H_n(x)$ iteratively as in Teichmann (2002) via

$$\begin{aligned} H_0(x) &= 1 \\ H_n(x) &= \delta H_{n-1}(x) = (\delta)^n 1 \end{aligned}$$

where δ is the integral operator(or the creation operator as in Malliavin & Thalmaier (2005)) defined as

$$\delta\phi = -\phi' + x\phi$$

and ϕ is a polynomial function. The two definitions of Hermite polynomials $\{H_n\}$ differ by a factor of n. However the results deduced are consistent.

on $(T,\mathscr{B})$, we call it the *white noise* based on ν. $W(h)$ can be interpreted as a stochastic integral[2]. Elements of $\mathscr{H}_n$ can then be expressed as multiple stochastic integrals with respect to W.

To construct the multiple stochastic integrals, we first define the (n-fold) *iterated Itô integral* $J_n(f)$ as follows

$$J_n(f) := \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} f(t_1,\ldots,t_n)\, dW(t_1)\, dW(t_2) \cdots dW(t_{n-1})\, dW(t_n) \qquad (4.1.1)$$

If f is a symmetric square integrable functions on $[0,T]^n$, denote $f \in \hat{L}^2\left([0,T]^n\right)$, we define

$$I_n(f) := \int_{[0,T]^n} f(t_1,\ldots,t_n)\, dW^{\otimes n}(t) := n! J_n(f)$$

Furthermore, the symmetrization $\tilde{f}$ of the function f is defined as

$$\tilde{f}(t_1,\ldots,t_n) = \frac{1}{n!}\sum_{\sigma} f\left(t_{\sigma(1)},\ldots,t_{\sigma(n)}\right),$$

σ running over all permutations of $\{1,\ldots,n\}$.

$I_n(f)$ has the following properties:

1. I_n is linear,
2. $I_n(f) = I_n\left(\tilde{f}\right)$,
3. $$E\left(I_n(f)\, I_q(g)\right) = \begin{cases} 0 & \text{if } n \neq q, \\ n!\left\langle \tilde{f}, \tilde{g}\right\rangle_{L^2(T^n)} & \text{if } n = q. \end{cases}$$

The following proposition provides the relationship between the Hermite polynomials and the multiple Itô integrals.

Proposition 4.1.4. *Let $H_n(x)$ be the nth Hermite polynomial, and let $h \in H = L^2(T), \|h\|_H = 1$. Then*

$$n! H_n(W(h)) = \int_{T^n} h(t_1) \cdots h(t_n)\, W(dt_1) \cdots W(dt_n).$$

Since the linear subspaces $\mathscr{H}_n$ "make up" the space of $L^2(\Omega,\mathscr{F},\mathbb{P})$, the set of Hermite polynomials $\{H_n(x) | n = 0,1,2,\ldots\}$ (and hence the multiple Itô integrals by the above proposition) forms a basis for $L^2(\Omega,\mathscr{F},\mathbb{P})$. The following theorem summarizes the ideas above and guarantees an expression for a random variable as a function of Wiener processes.

Theorem 4.1.5. *[Wiener Chaos Expansion] Any random variable $F \in L^2(\Omega,\mathscr{F},\mathbb{P})$, where $\mathscr{F}$ denotes the σ-field generated by W, can be written as a unique series of multiple stochastic integrals:*

$$F = \sum_{n=0}^{\infty} I_n(f_n).$$

$f_0 = E(F)$, and I_0 is the identity mapping and the functions $f_n \in L^2(T^n)$ are symmetric.

[2]When $H = L^2(T)$, $W(h)$ can be written as $\int_0^T h(t)\, d\omega(t)$, see Page 8 of Nualart (2006).

4.1.2 The Malliavin Derivative

In this section the Malliavin derivative operator will be defined and some of its properties are stated. We assume that W is defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and that $\mathscr{F}$ is generated by W.

Differentiation in the Malliavin sense is, roughly speaking, to differentiate a square integrable random variable $F : \Omega \to \mathbb{R}$, with respect to $\omega \in \Omega$. Denote by $C_p^\infty(\mathbb{R}^n)$ the set of infinitely differentiable functions on $\mathbb{R}^n$ with all of its partial derivatives of polynomial growth.

Definition 4.1.6. *A smooth random variable F is defined as a random variable that has the form*

$$F = f(W(h_1), \ldots, W(h_n)), \tag{4.1.2}$$

where $f \in C_p^\infty(\mathbb{R}^n)$,$h_i \in H$, $1 \leq i \leq n$.

And denote by $\mathscr{S}$ the class of smooth random variables.

The following derivatives notations are applied in the sequel: $\partial_i f = \frac{\partial f}{\partial x_i}$ and $\nabla f = (\partial_1 f, \ldots, \partial_n f)$, for $f \in C^1(\mathbb{R}^n)$.

Definition 4.1.7. *The derivative of a smooth random variable F of the form in (4.1.2) is a random variable in H given by*

$$DF = \sum_{i=1}^{n} \partial_i f(W(h_1), \ldots, W(h_n))h_i. \tag{4.1.3}$$

The operator $D : L^2(\Omega) \to L^2(\Omega : H)$ is closable[3] (cf. Proposition 1.2.1 of Nualart (2006)).

We will denote the domain of D in $L^2(\Omega)$ by $\mathbb{D}^{1,2}$, i.e. $\mathbb{D}^{1,2}$ is the closure of $\mathscr{S}$ with respect to the norm

$$\|F\|_{1,2} = \left[E(|F|^2) + E(\|DF\|_H^2)\right]^{\frac{1}{2}}.$$

The iterative derivative $D^k F$ is defined such that for $F \in \mathscr{S}$, $D^k F$ is a random variable with values in $H^{\otimes k}$. Then one can denote $\mathbb{D}^{k,2}$ the completion of the family of $\mathscr{S}$ with respect to the norm

$$\|F\|_{k,2} = \left[E(|F|^2) + \sum_{j=1}^{k} E(\|D^j F\|_{H^{\otimes j}}^2)\right]^{\frac{1}{2}}.$$

Malliavin Derivative for White Noise

Suppose $H = L^2(T, \mathscr{B}, \mu)$. The derivative of a random variable $F \in \mathbb{D}^{1,2}$ is a stochastic process $D_t F, t \in T$. In general, for $k \geq 1$ and $F \in \mathbb{D}^{1,2}$,

$$D^k F = \left\{D^k_{(t_1,\ldots,t_k)} F, t_i \in T\right\},$$

is measurable on $T^k \times \Omega$ and is defined a.e. with respect to the measure $\nu \times \mathbb{P}$.

Suppose that F is a square integrable random variable of the form

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{4.1.4}$$

[3]If an operator $T : A \to B$ (with $Dom(T)$ a dense subset of A) is closed, then $Dom(T)$ is closed in the norm $|||a||| = \|a\|_A + \|Ta\|_B$.

where the kernels $f_n \in \hat{L}^2(T^n)$. We know that all square integrable random variables can be written in this form from Theorem 4.1.5. The derivative of F can then be computed with the following proposition.

Proposition 4.1.8. *[Proposition 1.2.7 of Nualart (2006)] Let $F \in \mathbb{D}^{1,2}$ be a square integrable random variable of the form (4.1.4). Then*

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(.,t)).$$

where $f_n(.,t)$ is the function f_n with the last parameter fixed with a value of t.

Let $A \in \mathscr{B}$. Denote by $\mathscr{F}_A$ the (completed) σ-field generated by the random varibales $\{W(B) : B \subset A, B \in \mathscr{B}\}$. Then we can compute the conditional expectation with respect to $\mathscr{F}_A$ with the following result.

Lemma 4.1.9. *[Lemma 1.2.5 of Nualart (2006)] Suppose that F is a square integrable random variable of the form (4.1.4). Let $A \in \mathscr{B}$. Then*

$$E(F|\mathscr{F}_A) = \sum_{n=0}^{\infty} I_n\left(f_n \mathbf{1}_A^{\otimes n}\right).$$

By combining Proposition 4.1.8 and Lemma 4.1.9, the following result is obtained.

Proposition 4.1.10 (Proposition 1.2.8 of Nualart (2006)). *Suppose that $F \in \mathbb{D}^{1,2}$ and $A \in \mathscr{B}$. Then $E(F|\mathscr{F}_A) \in \mathbb{D}^{1,2}$ also, and*

$$D_t(E(F|\mathscr{F}_A)) = E(D_t F|\mathscr{F}_A) \mathbf{1}_A(t)$$

a.e. in $T \times \Omega$.

In the next subsection, the divergence operator will be defined. The following derivative operator will prove to be useful in the analysis:

Definition 4.1.11. *For a fix element $h \in H$, the operator D^h is defined on $\mathscr{S}$ by*

$$D^h F = \langle DF, h \rangle_H, F \in \mathscr{S}.$$

Its domain is denoted by $\mathbb{D}^{h,2}$.

4.1.3 The Divergence Operator and the Skorohod Integral

The divergence operator can be defined as the adjoint of the derivative operator defined in the previous section. In the particular case where the underlying Hilbert space H is the space of $L^2(T, \mathscr{B}, \mu)$, one can interpret the divergence operator as a stochastic integral and this is commonly referred to as the Skorohod integral. It is so named because in the case of Brownian motion it coincides with the extension of the Itô integral to anticipating integrands pioneered by Skorohod (1975). In this section, the notion of the divergence operator is first developed with respect to a Gaussian isonormal process $W = \{W(h), h \in H\}$.

Definition 4.1.12. *(Definition 1.3.1 of Nualart (2006)) The divergence operator, denoted by δ, is the adjoint of the derivative operator D. Hence $\delta : L^2(\Omega, H) \rightarrow L^2(\Omega)$ is an unbounded operator such that*

1. *The domain of* δ, *denoted by Dom*δ, *is the set of random variables* $h \in L^2(\Omega, H)$ *with values in* H *such that*

$$|E(\langle DF, h\rangle_H)| \leq c\|F\|_2,$$

for all $F \in \mathbb{D}^{1,2}$, *where* c *is some constant.*

2. *If* $v \in Dom\delta$, *then* $\delta(h) \in L^2(\Omega)$ *is characterized by*

$$E(F\delta(h)) = E(\langle DF, h\rangle_H) \tag{4.1.5}$$

for any $F \in \mathbb{D}^{1,2}$, *i.e.*

$$\langle F, \delta(h)\rangle_{L^2(\Omega)} = \langle DF, h\rangle_{L^2(\Omega,H)}.$$

The divergence operator is a linear operator.

The following proposition is a useful operating tool for a direct evaluation of the Malliavin divergence.

Proposition 4.1.13. *[Proposition 1.3.4 of Nualart (2006)] Let* $h \in H$ *and* $F \in \mathbb{D}^{h,2}$. *Then* $Fh \in Dom\delta$ *and the following holds*

$$\begin{aligned} \delta(Fh) &= FW(h) - D^hF \\ &= FW(h) - \langle DF, h\rangle_H. \end{aligned}$$

The Skorohod Integral

Suppose that $v \in H = L^2(T, \mathscr{B}, \nu)$. If $Dom\delta \subset L^2(T \times \Omega)$, then the divergence $\delta(v)$ is termed the *Skorohod integral* of the process v. The following notation is commonly used in the literature to indicate that it is an integral:

$$\delta(v) = \int_T v_t \, \delta W_t.$$

Again, any element $v \in L^2(T \times \Omega)$ has a Wiener chaos expansion of the form

$$h(t) = \sum_{n=0}^{\infty} I_n(f_n(.,t)). \tag{4.1.6}$$

by Theorem 4.1.5. The Skorohod integral of v is then defined as follows.

Proposition 4.1.14. *Let* $v \in L^2(T \times \Omega)$ *with the form in (4.1.6). Then* $v \in Dom\delta$ *if and only if*

$$\delta(v) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n) < \infty$$

in $L^2(\Omega)$.

The link between Itô stochastic integral and Skorohod integral

The Skorohod integral can be interpreted as an extension of the Itô integral that allows the integrand to be stochastic processes that are not necessarily adapted to the Brownian motion (in the integrator). The adaptability assumption is (in this context) "replaced" by regularity conditions.

However, the Skorohod integral of adapted processes coincides with the Itô integral. To state this formally:

Proposition 4.1.15. *Let L_a^2 denote the subspace of adapted processes in $L^2([0,T] \times \Omega; \mathbb{R}^d) \cong (T \times \Omega)$. Hence $L_a^2 \subset Dom\delta$, the operator δ on L_a^2 coincides with the Itô integral, i.e.,*

$$\delta(v) = \int_T v_s \, \delta W_s = \int_T v_s \, dW_s.$$

Proof. See Section 1.3.3 of Nualart (2006). The basic idea is that for the class of adpated processes $v \in L_a^2$, the term $D^v F$ in Proposition 4.1.13 would become 0. □

The Clark-Ocone formula

The following result shows that a square integrable random variable can be written as two orthogonal components: its expected value and the Itô integral of some adapted process v.

Theorem 4.1.16 (Integral Representation, Theorem 1.1.3 of Nualart (2006)). *Let $F \in L^2(\Omega)$. Then there exists a unique process $v \in L_a^2(T \times \Omega)$ such that*

$$F = E(F) + \int_T v_t \, dW_t. \tag{4.1.7}$$

For $F \in \mathbb{D}^{1,2}$, the Clark-Ocone formula states that the process v is the conditional expectation of the derivative of F given $\mathscr{F}_t$.

Proposition 4.1.17 (Clark-Ocone formula). *Let $F \in \mathbb{D}^{1,2}$, and suppose that W is a one-dimensional Brownian motion. Then*

$$F = E(F) + \int_0^T E(D_t F|\mathscr{F}_t) \, dW_t.$$

Proof of the Clark-Ocone formula. Suppose that $F = \sum_{n=0}^{\infty} I_n(f_n)$. By Proposition 4.1.8 and Proposition 4.1.9 we have

$$\begin{aligned} E(D_t F|\mathscr{F}_t) &= \sum_{n=1}^{\infty} n E(I_{n-1}(f_n(.,t))|\mathscr{F}_t) &(4.1.8)\\ &= \sum_{n=1}^{\infty} n I_{n-1} f_n(t_1, \dots, t_{n-1}, t) \mathbf{1}_{\{t_1 \vee \dots \vee t_{n-1} \leq t\}}. &(4.1.9)\end{aligned}$$

Hence by Proposition 4.1.14

$$\delta\left(E(D_t F|\mathscr{F}_t)\right) = \sum_{n=1}^{\infty} I_n(f_n) = F - E(F), \tag{4.1.10}$$

but

$$\delta(E(D_t F|\mathscr{F}_t)) = \int_T E(D_F|\mathscr{F}_t) \, dW_t, \tag{4.1.11}$$

by Proposition (4.1.15), the result follows. □

4.2 Solving for the information drift

In the previous section, the Malliavin calculus was introduced for the set of smooth random variables. In order to obtain the information drift, the calculus will be applied to $\mathbb{P}_t^L$, the conditional law of L given $\mathscr{F}_t$ which is a measure valued martingale. But first, the theoretical motivation for the use of Malliavin calculus will be given following Imkeller (2003).

4.2.1 Motivation

In determining the information drift, the objects of interest are the conditional densities p_t^L, where

$$\mathbb{P}(L \in dx|\mathscr{F}_t)(\omega) = p_t^x(\omega)\mathbb{P}(L \in dx).$$

See Chapter 2. If the Clark-Ocone formula can be applied to p_t^L, then under suitable regularity condition on L, for $t \in [0, T]$:

$$\begin{aligned} p_t^x(\cdot) &= p_0^x(\cdot) + \int_0^t \mathbb{E}(D_s p_t^x(\cdot)|\mathscr{F}_s)\, dW_s \\ &= p_0^x(\cdot) + \int_0^t D_s\mathbb{E}(p_t^x(\cdot)|\mathscr{F}_s)\, dW_s \\ &= p_0^x(\cdot) + \int_0^t D_s p_s^x(\cdot)\, dW_s, \end{aligned}$$

using the fact that p_t^x are martingales in the time parameter with respect to $\mathbb{F} = (\mathscr{F}_t)_t$.

Remark 4.2.1. *The stochastic integrand in the above equation $D_s p_s^x(\cdot)$ is referred to as a* Malliavin trace (typed object) *and is interpreted as*

$$D_s p_s^x(\cdot) = \lim_{t \downarrow s} D_s p_t^x(\cdot).$$

Recall from Proposition 2.2.2 that $p_t^x = p_0^x + \int_0^t \alpha_s^x \, dW_s$. The information drift can be re-written as

$$\gamma_t^L = \frac{\alpha_t^L}{p_t^L(\cdot)} = \frac{D_t p_t^x(\cdot)}{p_t^x(\cdot)}|_{x=L} = D_t \ln p_t^x(\cdot)|_{x=L}, t \in [0, T].$$

Hence the information drift is identified with a logarithmic Malliavin trace of the conditional density.

According to Imkeller (2003), it is possible (with some regularity on p_t^L) to interchange D_t and the Radon-Nikodym derivation $\frac{d}{d\mathbb{P}^L}$ to obtain

$$\gamma_t^L = \frac{D_t p_t^x(\cdot)}{p_t^x(\cdot)}|_{x=L} = \frac{D_t \frac{d\mathbb{P}_t^L}{d\mathbb{P}^L}(\cdot, x)}{\frac{d\mathbb{P}_t^L}{d\mathbb{P}^L}(\cdot, x)} = \frac{dD_t\mathbb{P}_t^L(\cdot)}{d\mathbb{P}_t^L(\cdot)}(x).$$

Hence in order to obtain the information drift under this setting, instead of requiring the absolute continuity of the conditional law of L with respect to its law $\mathbb{P}_L$, one needs to make sense of $D_t\mathbb{P}_t^L(\cdot)$ as well as to establish the absolute continuity (or equivalence) of $D_t\mathbb{P}_t^L(\cdot)$ with respect to the measure $\mathbb{P}_t^L$.

Since $D_t\mathbb{P}_t^L(\cdot)$ and $\mathbb{P}_t^L$ are measure-valued random variables. A measure-valued version of the Clark-Ocone formula is needed.

4.2.2 Malliavin calculus for signed measures

In this subsection, the Malliavin calculus will be applied to the space of signed measures with the aim of developing a measure-valued version of the Clark-Ocone formula. The basic measure-valued Malliavin calculus was established in Imkeller et al. (2001).

Following the notations in Imkeller (2002, 2003), let $\mathbf{M}$ be the space of signed measures on $\mathbb{R}$ equipped with its Borel sets. For $\nu \in \mathbf{M}$, $f \in C_b(\mathbb{R})$, denote $\langle \nu, f\rangle = \int f\, d\nu$. Let Φ be the mapping of the standard embedding of $\mathbf{M}$ into an infinite dimensional metrizable space, i.e.

$$\begin{aligned} \Phi : M &\to \mathbb{R}^{\mathbb{N}} \\ \mu &\to (\langle \mu, f_i\rangle)_{i\in\mathbb{N}} \end{aligned}$$

where $(f_i)_{i\in\mathbb{N}}$ is a dense subset of $C_b(\mathbb{R})$.

Let $W(h) = \int_0^1 h(s)\, dW_s$. Define the set of *smooth cylinder functions* $\mathscr{S}(\mathbf{M})$ by

$$\mathscr{S}(\mathbf{M}) = \left\{F : F = f(W(h_1), \cdots, W(h_k), x)dx, f \in C_c^{\infty}(\mathbb{R}^{k+1}), h_i \in L^2([0,T]), k \in \mathbb{N}\right\}$$

So the Malliavin derivative for smooth cylinder functions is defined by

$$D_s F = \sum_{i=1}^{k} \left(\int \partial_i f(W(h_1), \cdots, W(h_k), x)dx\right) h_i(s), s \in [0,T].$$

DF is considered an element of $L^2(\Omega \times [0,T], \mathbf{M})$ with respect to the Banach space topology. Hence $\langle DF, f\rangle = D\langle F, f\rangle, f \in C_b(\mathbb{R})$ and $DF = \Phi^{-1}((D\langle F, f_i\rangle)_{i\in\mathbb{N}}$. For $F \in \mathscr{S}(\mathbf{M})$ let

$$\|F\|_{1,2} = \mathbb{E}(|F|^2)^{\frac{1}{2}} + \mathbb{E}(\||DF|\|_2^2)^{\frac{1}{2}}$$

be a norm on $\mathscr{S}(\mathbf{M})$. Let $\mathbf{D}_{1,2}(\mathbf{M})$ denote the closure of $\mathscr{S}(\mathbf{M})$ with respect to $\|\cdot\|_{1,2}$. In a similar manner, $\mathbf{D}_{k,p}(\mathbf{M})$ can be defined for higher derivatives of order k, and replace the 2-norm by the p-norm, $p \geq 1$. Under this setting it can be shown that a measure valued version of the Clark-Ocone formula holds:

Theorem 4.2.2. *[Theorem 1.1 of Imkeller et al. (2001)] Let $F \in \mathbf{D}_{1,2}(\mathbf{M})$ Then*

$$F = \mathbb{E}(F) + \int_0^T \mathbb{E}(D_s F|\mathscr{F}_s)\, dW_s. \tag{4.2.1}$$

In order to apply (4.2.1) to measure-valued martingales, specifically $\mathbb{P}_t^L$, while minimizing regularity conditions for L, one can start by working with smooth approximations of $\mathbb{P}(.,dx)$ as follows: let

$$L_\epsilon = L + \sqrt{\epsilon}N$$

where $\epsilon > 0$ and N is a standard Gaussian variable independent of $\mathbb{F}$. Let $\mathbb{P}_t^\epsilon, t \in [0,T]$ be the family of conditional laws of L_ϵ given $\mathscr{F}_t$. Further suppose that $L \in \mathbf{D}_{1,2}$. Then it can be shown (see Imkeller et al. (2001)) that $D_t\mathbb{P}_t^\epsilon(.,dx) \in L^2([0,T];\mathbf{M})$ and

$$\mathbb{P}_t^\epsilon(.,dx) = \mathbb{P}_0^\epsilon(.,dx) + \int_0^t D_s\mathbb{P}_s^\epsilon(.,dx)\, dW_s, t \in [0,T]. \tag{4.2.2}$$

Remark 4.2.3. *In a very crude sense, adding some "noise" $\sqrt{\epsilon}N$ to the information L ensures* *that the conditional law is "smooth" in the sense that $\mathbb{P}_t^\epsilon \in \mathscr{S}(\mathbf{M})$. Then Theorem 4.2.2 can be applied to obtain (4.2.2).*

By taking $\epsilon \to 0$, the following version of the Clark-Ocone formula is obtained:

Theorem 4.2.4. *[Theorem 3.2 of Imkeller (2003)] Suppose that there exists an* **M***-valued process $k_t(.,dx), t \in [0,T]$, denoted by*

$$D_t\mathbb{P}_t^L(.,dx) = k_t(.,dx), \quad t \in [0,T],$$

such that for any $t \in [0,T], f \in C_b(\mathbb{R})$ we have

$$\mathbb{E}\left[\int_0^t \langle [D_s\mathbb{P}_s^\epsilon(.,dx) - D_s\mathbb{P}_s^L(.,dx)], f\rangle^2\, ds\right] \to 0$$

as $\epsilon \to 0$, and

$$\sup_{f\in C_b(\mathbb{R}),\|f\|\leq 1} \mathbb{E}\left[\int_0^t \langle D_s\mathbb{P}_s^L(.,dx), f\rangle^2\, ds\right] < \infty, \tag{4.2.3}$$

then for any $t \in [0,T]$

$$\mathbb{P}_t^L(.,dx) = \mathbb{P}_0^L(.,dx) + \int_0^t D_s\mathbb{P}_s^L(.,dx)\, dW_s.$$

Then Jacod's condition (2.1.3) can be replaced by the following assumption:

Assumption 4.2.5. *$D_t\mathbb{P}_t^L(.,dx)$ is absolutely continuous with respect to $\mathbb{P}_t^L(.,dx)$, $\mathbb{P}$-a.s. for $t \in [0,T]$.*

Remark 4.2.6. *If Assumption 4.2.5 is satisfied, then the quantity*

$$g_t(.,x) := \frac{dD_t\mathbb{P}_t^L(.,dx)}{d\mathbb{P}_t^L(.,dx)}(x), \quad t \in [0,T], x \in \mathbb{R}$$

is well-defined.

And hence the following modified version of Corollary 2.2.4:

Theorem 4.2.7. *[Theorem 3.3 of Imkeller (2003)] Suppose that Assumption 4.2.5 and (4.2.3) are satisfied and that*

$$\gamma_t^L = g_t(.,L) \in L^\infty([0,T]), \mathbb{P} - a.s.,$$

then

$$\tilde{W} = W - \int_0^\cdot \gamma_s^L\, ds$$

is a $\mathbb{G}$-Brownian motion, where $\mathbb{G} = (\mathscr{G}_t)_{t\in[0,T]}$ and $\mathscr{G}_t = \mathscr{F}_t \vee \sigma(L)$.

The fact that in Theorem 4.2.7 $\frac{dD_t\mathbb{P}_t^L(.,dx)}{d\mathbb{P}_t^L(.,dx)}$ is required to be an element of $L^\infty([0,T])$, i.e. that

$$\int_0^T |g_t^L|\, dt < \infty \tag{4.2.4}$$

implies the conservation of the semimartingale property is linked to the integrability of g_t^L on $[0,T]$.

Remark 4.2.8. *In Imkeller et al. (2001), the following results were proven for $T = 1$:*

1.
$$\mathbb{E}\left(\int_0^T g_t^2(.,L)\,dt\right) < \infty \Longrightarrow H(\mathbb{P}_t(.,dx)|\mathbb{P}^L) < \infty, \quad \mathbb{P}-a.s., t \in [0,T], \tag{4.2.5}$$

where $H(\mathbb{P}_t(.,dx)|\mathbb{P}^L)$ is the relative entropy of $\mathbb{P}(L \in dx|\mathscr{F}_t)$ with respect to $\mathbb{P}^L$, see Definition 3.7.1. This implies $\mathbb{P}_t^L(.,dx) \ll \mathbb{P}_L$.

2.
$$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^T g_t^2(.,L)\,dt\right)\right) < \infty \Longrightarrow \mathbb{P}_t^L(.,dx) \sim \mathbb{P}_L, \quad \mathbb{P}-a.s., t \in [0,T]. \tag{4.2.6}$$

Hence the conditions imposed on $g_t(.,L)$ in Theorem 4.2.7 are less onerous (i.e. more general) then Assumptions 2.1.3 and 2.1.5.

4.2.3 Example: Maximum value for the Brownian motion

An example of how the information drift can be obtained using the theory developed above shall be given. Suppose the insider has information regarding the maximum value to be reached by the Brownian motion process underlying the stock prices process in the interval $[0,T]$. I.e. suppose $L = \sup_{t\in[0,T]} W_t$. Admittedly this scenario is not very realistic but it nevertheless demonstrates the mechanics involved in obtaining the information drift. See also Remark 4.2.9 below.

Following the notation in Example 4.0.4, let $W_t^* := \sup_{s\in[0,t]} W_s$ and $\beta_{T-t} = \sup_{s\in[t,T]}(W_s - W_t)$. Then β_{T-t} is independent of W_t and W_t^* and

$$L = W_t^* \vee (W_t + \beta_{T-t}), t \in [0,T],$$

then the density of the law of β_{T-t} is

$$\mathbb{P}(\beta_{T-t} \in dz) = \frac{2}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{z^2}{2(T-t)}\right) \mathbf{1}_{[0,\infty)}(z), \quad z \in \mathbb{R}.$$

(see (8.3) of section 2.8 of Karatzas & Shreve (1991)) Denote this density by f_{T-t}. Now, for $f \in C_b(\mathbb{R})$

$$\begin{aligned}
\langle \mathbb{P}_t^\epsilon(.,dx), f\rangle &= \mathbb{E}(f(L_\epsilon)|\mathscr{F}_t) \\
&= \mathbb{E}\left(\int_{\mathbb{R}} p_\epsilon(y-L) f(y) dy|\mathscr{F}_t\right) \\
&= \int_{\mathbb{R}} \mathbb{E}(p_\epsilon(y-L)|\mathscr{F}_t f(y)\, dy \\
&= \langle \mathbb{E}(p_\epsilon(y-L)|\mathscr{F}_t) dy, f\rangle
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{P}_t^{\epsilon}(.,dx) &= \mathbb{E}(p_\epsilon(x-L)|\mathscr{F}_t)\,dx \\
&= \left(\mathbb{E}(p_\epsilon(x-W_t^*)\mathbf{1}_{\{L=W_t^*\}}|\mathscr{F}_t) + \mathbb{E}(p_\epsilon(x-W_t-\beta_{T-t})\mathbf{1}_{\{L=W_t+\beta_{T-t}\}}|\mathscr{F}_t)\right)dx \\
&= \left(\mathbb{E}(p_\epsilon(x-W_t^*)\mathbf{1}_{\{\beta_{T-t}\le W_t^*-W_t\}}|\mathscr{F}_t) + \mathbb{E}(p_\epsilon(x-W_t-\beta_{T-t})\mathbf{1}_{\{\beta_{T-t}\ge W_t^*-W_t\}}|\mathscr{F}_t)\right)dx \\
&= \left(p_\epsilon(x-W_t^*)\int_0^{W_t^*-W_t} f_{T-t}(y)\,dy + \int_{W_t^*-W_t}^{\infty} p_\epsilon(x-W_t-\beta_{T-t})f_{T-t}(y)\,dy\right)dx \\
&= \left(p_\epsilon(x-W_t^*)\int_0^{W_t^*-W_t} f_{T-t}(y)\,dy + \int_{-\infty}^{x-W_t^*} p_\epsilon(v)f_{T-t}(x-W_t-v)\,dy\right)dx
\end{aligned}$$

since W_t^* and W_t are $\mathscr{F}_t$-measurable and by letting $v = x - W_t - \beta_{T-t}$.

By letting $\epsilon \to 0$, $p_\epsilon(v) \to \delta_0(v)$ and therefore

$$\mathbb{P}_t(.,dx) = \delta_{W_t^*}(dx)\int_0^{W_t^*-W_t} f_{T-t}(y)\,dy + \mathbf{1}_{[W_t^*,\infty)}(x)f_{T-t}(x-W_t)dx, \tag{4.2.7}$$

Hence using the fact that $D_tW_t^* = 0$ and $df_{T-t}(x) = -\frac{x}{T-t}f_{T-t}(x)dx$

$$D_t\mathbb{P}_t(.,dx) = -\delta_{W_t^*}(dx)f_{T-t}(W_t^*-W_t) + \mathbf{1}_{[W_t^*,\infty)}(x)\frac{x-W_t}{T-t}f_{T-t}(x-W_t)dx. \tag{4.2.8}$$

Combining (4.2.7) and (4.2.8), one obtains the information drift as

$$\frac{dD_t\mathbb{P}_t(.,dx)}{d\mathbb{P}_t(.,dx)} = -\frac{f_{T-t}(W_t^*-W_t)}{\int_0^{W_t^*-W_t} f_{T-t}(y)\,dy}\mathbf{1}_{W_t^*}(x) - \mathbf{1}_{[W_t^*,\infty)}(x)\frac{x-W_t}{T-t}, \quad t\in[0,T], \tag{4.2.9}$$

Remark 4.2.9. *The case where the insider has the information regarding the maximum stock price over the trading interval, i.e.* $L = \sup_{t\in[0,T]} S_t$, *is demonstrated in Imkeller (2003).*

Chapter 5

Dynamical Information

Up to this point, the information possesses by the insider is assumed to be known by the insider at the beginning of the investment period, i.e. at $t = 0$. Specifically, it has been assumed that L is a $\mathscr{F}_{\mathscr{T}}$-measurable random variable, which is set to be $\mathscr{G}_0$-measurable by setting $\mathscr{G}_t = \mathscr{F}_t \vee \sigma(L), \forall t \in [0, T]$. But certain types of information may not be incorporated this way under the framework described. For example, information that comes from a continuous flow of knowledge, i.e. information that are being "updated" as time passes. In this chapter, two such types of information will be considered:

1. Terminal information (e.g. regarding terminal value of the Brownian motion) which is distorted by noise that is vanishing as one approaches the "revelation date", following the approach of Corcuera et al. (2004);
2. Time information, following the approach of Imkeller (2002).

5.1 Terminal information distorted by diminishing noise

In Chapter 2, we considered the case where information regarding the terminal value (i.e. at time $T = \mathscr{T}$) of the Brownian motion process underlying the price process distorted by some Gaussian random variable representating "noise" in the information, i.e. a random element in the information that serves to distort the "true" information. The level of noise was constant throughout the investment period in that case, so that the insider will be more certain about the information in his possession as the "revelation date" gets near.

Corcuera et al. (2004) presented what they termed *dynamical enlargement of filtration* to obtain a semimartingale decomposition of the $\mathbb{F}$-Brownian motion, given a filtration that is enlarged by this type of information. The conclusion that they reached is that if the rate at which the blurring noise disappears is sufficiently slow then there will be a finite additional logarithmic utility and no arbitrage.

From the discussion in the previous chapters, it is clear that the general approach for insider modelling involves finding the compensator, i.e. the information drift γ_t^* so that if W_t is a $\mathbb{F}$-Brownian motion, then $\tilde{W}_t = W_t - \int_0^t \gamma_s \, ds$ is a $\mathbb{G}$-Brownian motion. It will be shown below that for the type of information which is being discussed in this section, such a compensator does indeed exist. A discussion around the topic of arbitrage will be presented in Chapter 7.

Let $\{L_t, t \in [0,T]\}$ denote the additional information possessed by the insider at time $t \in [0,T]$. Note that L is now parameterized with a time variable t representing information that evolves through time. Assume the random variables L_t have the following general formulation:

$$L_t = f(X, Y_t),$$

where X is $\mathscr{F}_T$-measurable, $Y = (Y_t)_{t\in[0,T]}$ is independent of $\mathscr{F}_T$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a given measurable function. Then let $\mathbb{G} = (\mathscr{G}_t)_{t\in[0,T]}$ denote the "usual" augmentation of the filtration

$$(\mathscr{F}_t \vee \sigma(L_s, s \leq t))_{t\in[0,T]}.$$

Y represents the (independent) noise that distort X from the insider, hence it would also be assumed that $Y_T = 0$ and the variance of Y should decrease to zero as $t \to T$.

5.1.1 Enlarging the filtration dynamically

The following proposition is very useful in that it says, if one knows the drift for the case $L_t = X$, i.e. $\mathscr{H}_t = \mathscr{F}_t \vee \sigma(X)$, then one can obtain the drift for the case that $L_t = f(X, Y_t)$.

Proposition 5.1.1. *[Proposition 1 of Corcuera et al. (2004)] Let X be an $\mathscr{F}_T$-measurable random variable and suppose that there exists an $\mathscr{F} \vee \sigma(X)$-progressively measurable process $\gamma^* = (\gamma^*_t)_{t\in[0,T)} \in L^1([0,T])$, such that $W - \int_0^{\cdot} \gamma^*_t \, dt$ is a $(\mathscr{F} \vee \sigma(X))$-Brownian motion. Then $W - \int_0^{\cdot} \mathbb{E}(\gamma^*_t|\mathscr{G}_t)\, dt$ is a $\mathbb{G}$-Brownian motion for an appropriate version of $\mathbb{E}(\gamma^*_t|\mathscr{G}_t)$.*

Proof. Since Y is independent of $\mathscr{F}_T$, then

$$\bar{W}_t = W_t - \int_0^t \gamma^*_s \, ds \tag{5.1.1}$$

is an $\mathbb{I}$-Brownian motion, where

$$\mathbb{I} = (\mathscr{F}_t \vee \sigma(X) \vee \sigma(Y_s, s \leq t))_{t\in[0,T)}.$$

(5.1.1) implies that

$$\mathbb{E}(\bar{W}_t|\mathscr{G}_t) = W_t - \int_0^t \mathbb{E}(\gamma^*_s|\mathscr{G}_s)\, ds.$$

Obviously a $\mathbb{G}$-progressively measurable version of $\mathbb{E}(\gamma^*_s|\mathscr{G}_s)$, $s \in [0,T)$ exists. And since L_t is just a measurable function of X and Y_t, $\mathscr{G}_t \subset \mathscr{I}_t$. Hence $\mathbb{E}(\bar{W}_t|\mathscr{G}_t)$, $t \in [0,T)$ is a $\mathscr{G}$-martingale. In fact, for $0 \leq s < t < T$

$$\mathbb{E}(\mathbb{E}(\bar{W}_t|\mathscr{G}_t)|\mathscr{G}_s) = \mathbb{E}(\bar{W}_t|\mathscr{G}_s) = \mathbb{E}(\mathbb{E}(\bar{W}_t|\mathscr{I}_s)|\mathscr{G}_s) = \mathbb{E}(\bar{W}_s|\mathscr{G}_s).$$

Lévy's characterization of Brownian motion then implies the result. □

In the remainder of the section, $\gamma_t = \mathbb{E}(\gamma^*_t|\mathscr{G}_t)$. The following proposition provides an explicit formula for calculating this quantity.

Proposition 5.1.2. *[Proposition 4 of Corcuera et al. (2004)] Let $L_t = X + Y_t$, where Y_t is a continuous process with independent increments whose marginal has density q_{T-t}. Then for*

$t \in [0, T]$,

$$\gamma_t = \frac{\int_{\mathbb{R}} \gamma_t^*(x) q_{T-t}(L_t - x)\, \mathbb{P}_t^X(dx)}{\int_{\mathbb{R}} q_{T-t}(L_t - x)\, \mathbb{P}_t^X(dx)}.$$

where $\mathbb{P}_t^X(dx)$ denotes a regular version of the conditional law of X given $\mathscr{F}_t$.

Proof. For $t \in [0, T]$

$$\begin{aligned} \gamma_t &= \mathbb{E}\left(\gamma_t^*(X) | \mathscr{F}_t \vee \sigma(L_s : s \le t)\right) \\ &= \mathbb{E}\left(\gamma_t^*(X) | \mathscr{F}_t \vee \sigma(L_t) \vee \sigma(Y_t - Y_s : s \le t)\right) \\ &= \mathbb{E}\left(\gamma_t^*(X) | \mathscr{F}_t \vee \sigma(L_t)\right). \end{aligned}$$

Let $\mathbb{P}_t^{(X,L)}$ be a regular version of the conditional distribution of $(X, X + Y_t)$ given $\mathscr{F}_t$. Then for $C \in \mathscr{B}(\mathbb{R}^2)$

$$\begin{aligned} \mathbb{P}_t^{(X,L)}(C) &= \int_{\mathbb{R}^2} \mathbf{1}_C(x, x+y) q_{T-t}(y) \mathbb{P}_t^X(dx)\, dy \\ &= \int_{\mathbb{R}^2} \mathbf{1}_C(x, l) q_{T-t}(l - x) \mathbb{P}_t^X(dx)\, dl. \end{aligned}$$

Hence for $A \in \mathscr{B}(\mathbb{R})$

$$\mathbb{P}(X \in A | \mathscr{F}_t \vee \sigma(L_t)) = \frac{\int_A q_{T-t}(L_t - x)\, \mathbb{P}_t^X(dx)}{\int_{\mathbb{R}} q_{T-t}(L_t - x)\, \mathbb{P}_t^X(dx)},$$

and therefore

$$\mathbb{E}(\gamma_t^*(X) | \mathscr{F}_t \vee \sigma(L_t)) = \frac{\int_{\mathbb{R}} \gamma_t^*(x) q_{T-t}(L_t - x)\, \mathbb{P}_t^X(dx)}{\int_{\mathbb{R}} q_{T-t}(L_t - x)\, \mathbb{P}_t^X(dx)}.$$

□

5.1.2 Examples

Using the propositions established in the previous subsection, two examples from Corcuera et al. (2004) will be presented below to demonstrate the exact workings involve in obtaining the compensator for this type of dynamic information. See Corcuera et al. (2004) for a more detailed discussion.

Terminal value of Brownian motion with vanishing noise

Suppose the insider has the knowledge of the terminal value of the Brownian motion process that drives the asset prices, but this information is distorted by noise that vanishes with time.

Remark 5.1.3. *It might be more realistic (and practical) to assume that the insider has knowledge regarding the terminal prices and this information is distorted by noise. Just like the case where the insider has exact information regarding the terminal asset prices, it is possible to calculate the compensator but the algebra would be more complicated. An example of this type is presented in Corcuera et al. (2004).*

Let $L_t = X + Y_t = f(W_T) + \hat{W}_{g(T-t)}$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with $\mathbb{E}(f'(W_T)^2) < \infty$ for a reason that will become apparent later and $\hat{W}$ is a Brownian motion independent of W. Also let $g : [0,T] \to [0,+\infty]$ be a strictly increasing bounded function with $g(0) = 0$. Hence $Y_T = \hat{W}_0 = 0$ so that the noise term has decreasing variance and vanishes at $t = T$. If $L = W_T$, then $\gamma_t^* = \frac{W_T - W_t}{T-t}$ (see Chapter 1). Denote the density of the law of the Gaussian random variable $\hat{W}_{g(T-t)}$ by $\phi_{g(T-t)}$. The conditional density of W_T given $\mathscr{F}_t$ is therefore $\phi_{T-t}(W_T - x)$ where $W_t = x$. By Proposition 5.1.2 and with a change of variable to $z = W_T - W_t$,

$$\gamma_t = \frac{\int_{\mathbb{R}} z\phi_{g(T-t)}(L_t - f(W_t + z))\phi_{T-t}(z)\,dz}{(T-t)\int_{\mathbb{R}} \phi_{g(T-t)}(L_t - f(W_t + z))\phi_{T-t}(z)\,dz}.$$

Using the fact that $\frac{z}{T-t}\phi_{T-t}(z) = -\phi'_{T-t}(z)$ and integration by parts:

$$\begin{aligned}
\gamma_t &= \frac{-\int_{\mathbb{R}} \phi_{g(T-t)}(L_t - f(W_t + z))\phi'_{T-t}(z)\,dz}{\int_{\mathbb{R}} \phi_{g(T-t)}(L_t - f(W_t + z))\phi_{T-t}(z)\,dz} \\
&= \frac{-\int_{\mathbb{R}} f'(W_t + z)\phi'_{g(T-t)}(L_t - f(W_t + z))\phi_{T-t}(z)\,dz}{\int_{\mathbb{R}} \phi_{g(T-t)}(L_t - f(W_t + z))\phi_{T-t}(z)\,dz} \\
&= \frac{\int_{\mathbb{R}} \frac{(L_t - f(W_t + z)}{g(T-t)} f'(W_t + z)\phi_{g(T-t)}(L_t - f(W_t + z))\phi_{T-t}(z)\,dz}{\int_{\mathbb{R}} \phi_{g(T-t)}(L_t - f(W_t + z))\phi_{T-t}(z)\,dz} \\
&= \frac{1}{g(T-t)}\mathbb{E}\left(\hat{W}_{g(T-t)} f'(W_T)\middle| W_t, L_t\right).
\end{aligned}$$

Maximum value of Brownian motion with vanishing noise

In this example it is assumed that the insider has information regarding the maximum value of the Brownian motion process that drives the asset prices and this information is distorted by noise that vanishes with time.

Remark 5.1.4. *Again, it might be more realistic to assume that the insider has the knowledge of the maximum asset price for the investment interval concerned and this is distorted by noise. But given the results obtained here, it should be possible to obtain a compensator for the more realistic case. Once again the algebra can be expected to be considerably more complicated, but the important message is that it is possible.*

Let $X = W^* = \sup_{t\in[0,T]} W_t$, $Y_t = \hat{W}_{g(T-t)}$ independent of W and $L_t = X + Y = W^* + \hat{W}_{g(T-t)}, t \in [0,T]$. Denote

$$W_t^* = \sup_{0\le s\le t} W_s, \text{ and } \beta_{T-t} = \sup_{s\in[0,T-t]} (W_{t+s} - W_t), t \in [0,T].$$

Hence $W^* = W_t^* \vee (\beta_{T-t} + W_t)$ and for any bounded and measurable function F on $\mathbb{R}$,

$$F(W^*) = F(W_t^*)\mathbf{1}_{\{W^* = W_t^*\}} + F(\beta_{T-t} + W_t)\mathbf{1}_{\{W^* = \beta_{T-t} + W_t\}}. \tag{5.1.2}$$

Let f_{T-t} denotes the density of the running maximum of the Wiener process in the interval $[0, T-t]$, as in the previous chapter, then

$$\begin{aligned}
\mathbb{E}(f(W^*)|\mathscr{F}_t) &= \mathbb{E}(f(W_t^*)\mathbf{1}_{\{\beta_{T-t} \le W_t^* - W_t\}} + f(\beta_{T-t} + W_t)\mathbf{1}_{\{\beta_{T-t} + W_t > W_t^*\}}|\mathscr{F}_t) \\
&= f(W_t^*)\int_0^{W_t^* - W_t} f_{T-t}(x)\,dx + \int_{W_t^* - W_t}^{\infty} f(x + W_t) f_{T-t}(x)\,dx \\
&= f(W_t^*)\int_0^{W_t^* - W_t} f_{T-t}(x)\,dx + \int_{W_t^*}^{\infty} f(y) f_{T-t}(y - W_t)\,dy,
\end{aligned}$$

Hence the conditional density of X given $\mathscr{F}_t$ is

$$\mathbb{P}_t^X(dx) = \delta_{W_t^*}(dx)\int_0^{W_t^*-W_t} f_{T-t}(x)\,dx + f_{T-t}(x-W_t)\mathbf{1}_{(W_t^*,\infty)}(x)dx \tag{5.1.3}$$

And from the previous chapter γ_t^* for $\mathscr{H}_t = \mathscr{F}_t \vee \sigma(X)$ is

$$\gamma_t^*(x) = -\frac{f_{T-t}(W_t^*-W_t)}{\int_0^{W_t^*-W_t} f_{T-t}(y)\,dy}\mathbf{1}_{\{x=W_t^*\}} + \frac{x-W_t}{T-t}\mathbf{1}_{(W_t^*,\infty)}(x).$$

Hence by Proposition 5.1.2 the information drift for the insider is

$$\gamma_t = \frac{-f_{T-t}(W_t^*-W_t)\phi_{T-t}(L_t-W_t^*) + \int_{W_t^*}^{\infty} f_{T-t}(x-W_t)\frac{x-W_t}{T-t}\phi_{T-t}(L_t-x)\,dx}{\int_0^{W_t^*-W_t} f_{T-t}(x)\,dx\phi_{T-t}(L_t-W_t^*) + \int_{W_t^*}^{\infty} f_{T-t}(x-W_t)\phi_{T-t}(L_t-x)\,dx} \tag{5.1.4}$$

To establish the integrability of γ_t, first recall that

$$f_{T-t}(x-W_t)\frac{x-W_t}{T-t} = -\frac{\partial}{\partial x}f_{T-t}(x-W_t) \text{ for } x > W_t.$$

Then using integration by parts

$$\begin{aligned}
&\int_{W_t^*}^{\infty} f_{T-t}(x-W_t)\frac{x-W_t}{T-t}\phi_{g(T-t)}(L_t-x)\,dx \\
&= -\int_{W_t^*}^{\infty}\frac{\partial}{\partial x}f_{T-t}(x-W_t)\phi_{g(T-t)}(L_t-x)\,dx \\
&= f_{T-t}(W_t^*-W_t)\phi_{g(T-t)}(L_t-W_t^*) + \int_{W_t^*}^{\infty} f_{T-t}(x-W_t)\frac{\partial}{\partial x}\phi_{g(T-t)}(L_t-x)\,dx \\
&= f_{T-t}(W_t^*-W_t)\phi_{g(T-t)}(L_t-W_t^*) + \frac{1}{g(T-t)}\int_{W_t^*}^{\infty} f_{T-t}(x-W_t)(L_t-x)\phi_{g(T-t)}(L_t-x)\,dx
\end{aligned}$$

since

$$\frac{\partial}{\partial x}\phi_{g(T-t)}(L_t-x) = \frac{(L_t-x)}{T-t}\phi_{g(T-t)}(L_t-x).$$

Combining this with (5.1.4) yields

$$\begin{aligned}
\gamma_t &= \frac{1}{g(T-t)}\frac{\int_{W_t^*}^{\infty} f_{T-t}(x-W_t)(L_t-x)\phi_{g(T-t)}(L_t-x)\,dx}{\int_0^{W_t^*-W_t} f_{T-t}(x)\,dx\phi_{g(T-t)}(L_t-W_t^*) + \int_{W_t^*}^{\infty} f_{T-t}(x-W_t)\phi_{g(T-t)}(L_t-x)\,dx} \\
&= \frac{1}{g(T-t)}\mathbb{E}(Y_t\mathbf{1}_{\{W^*>W_t^*\}}|\mathscr{F}_t \vee \sigma(L_t))
\end{aligned}$$

In Chapter 7, the condition for the absence of arbitrage in these cases will be reviewed.

5.2 Time information

Another type of additional information is related to knowledge about a random time at which a specific event takes place. Examples include knowledge about the time at which the asset price will reach its maximum (this is different from knowing the maximum prices itself), or the time at which the asset price will reach a certain level for the last time within an investment horizon. To incorporate this type of information into the established framework, one has to interpret the information as coming in the form of a continuous flow, i.e. the body of knowledge is actually updated as time goes by. Hence the insider's filtration is a *progressive enlargement* of that of the regular investor's. This type of filtration enlargement has been dealt with thoroughly in the papers on the subject of *grossissement de filtration*, one can refer to Jeulin (1980) and Yor (1985*a*,*b*,*c*,*d*) for a deeper discussion of the mathematical basis for this technique.

Imkeller (2002) has presented the way these progressive enlargement of filtration techniques can be applied and examples for incorporating this type of information. These will be discussed below.

5.2.1 Enlargement of filtration with time information

In this instance, L is a random time, hence a random variable with values in the investment period $[0, T]$. Let the $\mathbb{F} = (\mathscr{F}_t)_t$ be the filtration generated by W. The filtration $\mathbb{F}$ needs to be enlarged to incorporate the information from knowing L.

Imkeller proposed that the enlarged filtration $\mathbb{G} = (\mathscr{G}_t)_{t\in[0,T]}$ is given by the progressive enlargement defined as

$$\begin{aligned}\mathscr{G}_t &= (\mathscr{F}_t \vee \sigma(L \wedge t))_+ . \\ &= \cap_{s>t} (\mathscr{F}_s \vee \sigma(L \wedge s)) .\end{aligned}$$

(See also Remark 2.1.1.) The filtration $\mathbb{G}$ is the smallest filtration satisfying the usual conditions for which L is a stopping time.

Under such an enlarged filtration, the insider would for example know that the stock price has reached its maximum for the investment period when it has done so. Note the subtlety here, this is not the same as knowing when the stock price will reach its maximum *at inception*, i.e. L is not $\mathscr{G}_0$-measurable.

5.2.2 Progressive enlargement of filtration techniques

In this subsection the progressive enlargement of filtration techniques that allows one to determine the information drift will be introduced. But first the concept of *honest time* needs to be dicussed.

Definition 5.2.1. *A random time L is an* honest time *if L is the end point (in terms of the time variable) of a predictable set. Therefore there exists a predictable set $\Gamma \subset [0, 1] \times \Omega$ such that*

$$L(\omega) = \sup \{t : (t, \omega) \in \Gamma\} ,$$

with $\sup \emptyset$ *is defined to be* 0.

Denote the right continuous version of the supermartingale $\mathbb{P}(L > t|\mathscr{F}_t), t \in [0, T]$ by Y^L. Let M^L be the martingale part of Y^L in its Doob-Meyer decomposition. Then the information drift can be obtained with the following proposition:

Proposition 5.2.2. *[See Imkeller (2002)] If L is an honest time,*

$$\tilde{W}_t = W_{L\wedge t} - \int_0^t \mathbf{1}_{[0,L]}(s) \frac{\frac{d}{dt}\langle M^L, W\rangle_s}{Y^L_{s-}}\, ds + \int_0^t \mathbf{1}_{(L,1]}(s) \frac{\frac{d}{dt}\langle M^L, W\rangle_s}{1 - Y^L_{s-}}\, ds, \quad t \in [0, T],$$

is a $\mathbb{G}$-Brownian motion process. Hence

$$\gamma_t = \mathbf{1}_{[0,L]}(s)\frac{\frac{d}{dt}\langle M^L, W\rangle_s}{Y^L_{s-}} - \mathbf{1}_{(L,1]}(s)\frac{\frac{d}{dt}\langle M^L, W\rangle_s}{1-Y^L_{s-}}, \quad t \in [0,1]. \tag{5.2.1}$$

Remark 5.2.3. *[See Imkeller (2002) & p.80 of Jeulin (1980)] Regardless or whether L is an honest time, the process*

$$\tilde{W}_t = W_{L\wedge t} - \int_0^t \mathbf{1}_{[0,L]}(s)\frac{\frac{d}{dt}\langle M^L, W\rangle_s}{Y^L_{s-}}\,ds, \quad t \in [0,T],$$

is a $\mathbb{G}$-Brownian motion process at $t < L$.

5.2.3 Progressive enlargement techniques applied to honest times

With Proposition 5.2.2 in the previous subsection, one can obtain the information drift and then establish whether this drift would lead to arbitrage opportunity for the insider or it would allow for an equivalent change in measure, so that the insider would still view the price process as a martingale under the new measure. Imkeller (2002) applied excursion theory for Brownian motion to show that the latter is indeed impossible in the cases below.

Case: Last crossing of a particular level by a Brownian motion

In this example, suppose the insider has the knowledge of when the Brownian motion process underlying the price process crosses a particular level $K \in \mathbb{R}$, in the sense that the insider will be able to recognise such an event as and when it happens. I.e., let

$$L = \sup\{0 \le t, W_t = K\}.$$

Now let $\Gamma = \{(t,\omega) : W_t(\omega) = K\}$ which is $\mathbb{F}$-previsible. Thus L is an honest time.

Intuitively, while the honest investor who observe the process at $t = L$ must view the processs $(W_{L+s} - W_L)_{s\in[0,T-L]}$ with the knowledge in $\mathscr{F}_L$ as a Brownian motion and hence has a zero drift, the insider who is able to recognise L must see the process differently.

The following result is required to obtain γ.

Proposition 5.2.4 (See Proposition 2.1 of Imkeller (2002)). *Let F_t be the distribution function of the law of $|W_t|, t \in [0,T]$. Then for $t \in [0,T]$*

$$Y^L_t = 1 - F_{T-t}(|W_t - K|).$$

Remark 5.2.5. *In Imkeller (2002), the proposition was actually proven for $t \in [0,1]$. But this can be easily generalised to $t \in [0,T]$.*

Proposition 5.2.6. *For $t \in [0,T]$, let p_t be the density of the law of $|W_t|$, and F_t be its distribution function. Then for $t \in [0,T]$,*

$$\gamma_t = -\mathbf{1}_{[0,L]}(t)\frac{p_{T-t}(|W_t - K|)}{1 - F_{T-t}(|W_t - K|)}sgn(W_t - K) - \mathbf{1}_{]L,T]}(t)\frac{p_{T-t}(|W_t - K|)}{F_{T-t}(|W_t - K|)}sgn(W_t - K). \tag{5.2.2}$$

Proof. By the Tanaka's formula (see Appendix),

$$|W_t - K| - |K| = \int_0^t sgn(W_s - K)\, dW_s + L_t^K, \quad t \in [0,T],$$

with the local time L_t^K at level K.

Hence by the Itô's formula

$$\begin{aligned} &F_{T-t}(|W_t - K|) \\ =&F_{T-t}(|W_0 - K|) + \int_0^t p_{T-t}(|W_s - K|)\, d(|W_s - K|) + \frac{1}{2}\int_0^t p'_{T-t}(|W_s - K|)\, d(|W_s - K|)^2, \end{aligned}$$

where p_{T-t} denotes the density function of the distribution function F_{T-t}.

Therefore by Proposition 5.2.4, Y_t^L has a Doob-Meyer decomposition

$$Y_t^L = -\int_0^t p_{T-t}(|W_s - K|)sgn(W_s - K)\, dW_s + A_t,$$

where A_t is some increasing predictable process null at zero. Hence

$$\frac{d}{dt}\langle M^L, W\rangle_t = -p_{T-t}(|W_t - K|)sgn(W_t - K).$$

Applying Proposition 5.2.2 yields (5.2.2). □

To show that the process allows the possibilities of arbitrage and free lunches, Imkeller (2002) showed that γ is not square integrable on a set of positive measure. This is discussed in Chapter 7.

Case: Time when the Brownian motion process attains the maximum

For $t \in [0,T]$, let the running maximum process be defined as

$$W_t^* = \sup_{0\le s\le t} W_s,$$

and

$$\varsigma = \sup\{0 \le t \le T : W_t = W_t^*\}.$$

Let $L = \tau$ be the time when W reaches its maximum in $[0,T]$, then $\varsigma = \tau$, $\mathbb{P}$-a.s. And since ς is an honest time, so is τ. Let

$$\begin{aligned} \varrho &= \sup\{0 \le t \le T : |W_t| = 0\} \\ &= \sup\{0 \le t \le T : W_t = 0\} \end{aligned}$$

Now since $(W_t^* - W_t)_{t\ge0}$ and $(|W_t|)_{t\ge0}$ have the same law (see Theorem 2.3 of Chapter VI of Revuz & Yor (1991)), hence τ and ϱ must have the same law also. Therefore the results from the previous case can be used. Let

$$L_s = \sup_{0\le h\le s}(W_{t+h} - W_t),$$

for $s \geq 0$ which given W_t must have the same law as W_s^*. Also let G_t be the distribution function of W_t^*.

Using the fact that the random variables W_t^* and $|W_t|$ are equal in law (see Proposition 3.7 of Chapter III of Revuz & Yor (1991)) as well as the equality of the laws of $(W_t^* - W_t)_{t\geq 0}$ and $(|W_t|)_{t\geq 0}$ one can then rephrase Proposition 5.2.6 as:

Proposition 5.2.7 (Proposition 3.1 of Imkeller (2002)). *For $t \in [0,T]$, let G_t and q_t denote the law and the density function of W_t^* respectively. Then for $t \in [0,T]$*

$$\gamma_t = -\mathbf{1}_{[0,\tau]}(t)\frac{q_{T-t}(W_t^* - W_t)}{1 - G_{T-t}(W_t^* - W_t)} - \mathbf{1}_{(\tau,1]}(t)\frac{q_{T-t}(W_t^* - W_t)}{G_{T-t}(W_t^* - W_t)}. \qquad (5.2.3)$$

In Chapter 7, an arbitrage strategy for the insider will be derived.

Chapter 6

The Forward Integral Approach and the Partial Information Framework

In this chapter a forward integral approach is used to model the financial market where an insider is presented. Such an approach to the insider problem has been studied in Biagini & Øksendal (2005), Hu & Øksendal (2003), Léon et al. (2003) and Kohatsu-Higa & Sulem (2006).

To understand the relevance of such an approach, consider again a classical financial market of one non-risky asset (a bond) and one risky asset (a stock) which is driven by a Brownian motion W (in a geometric Brownian motion with drift process). Denote the filtration generated by W by $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$. The interest rate r, the stock appreciation rate μ and the volatility σ are assumed to be adapted to a filtration denoted by $\mathbb{G} = (\mathscr{G}_t)_{t\in[0,T]}$. In general, no particular assumption is made regarding the relationship between $\mathbb{F}$ and $\mathbb{G}$. Hence $\mathscr{G}_t$ can be larger or smaller than $\mathscr{F}_t$ or may even be deterministic, i.e. $\mathscr{G}_t = \mathscr{F}_0$ forall t.[1]

The proportion of the wealth of an investor invested in the stock is denoted by a measurable and $\mathbb{H}$-adapted process, which represents his investment strategy. I.e. the filtration $\mathbb{H} = (\mathscr{H}_t)$ represents the information in the investor's possession.

In Chapter 3, we assumed the market consists of a risk-free asset modelled by

$$dS_0(t) = r(t,\omega)S_0(t)dt; \quad S_0(0) = 1 \tag{6.0.1}$$

and one risky stock described by

$$\frac{dS_1(t)}{S_1(t)} = \mu(t)dt + \sigma(t)dW(t), \quad 0 \le t \le T, \quad S_1(0) \in [0,\infty), \tag{6.0.2}$$

where W is a Brownian motion.

The wealth process $V^{(\pi)}(t)$ at time $t \in [0,T]$ can be written as

$$V^{(\pi)}(t) = \sum_{i=0}^{1} \theta^i(t)S_i(t) = \sum_{i=0}^{1} \pi_i(t)V(t), \tag{6.0.3}$$

[1] Up to this point, it has been assumed that $\mathscr{G}_t = \mathscr{F}_t$.

where $\theta_i(t)$ denotes the number of units of the i-th asset held, and $\pi_i(t)$ denotes the proportion of wealth V_t invested in the i-th asset at time t. Imposing the constraint that $\pi_0(t) + \pi_1(t) = 1$ for all t means that one can write $\pi_1(t)$ as $\pi(t)$ and $\pi_0(t)$ as $1 - \pi(t)$. Hence $V^{(\pi)}(t)$ is a function of (π) which represents the investment strategy.

By combining (6.0.3) with (6.0.2) and (6.0.1), the dynamics of $V^{(\pi)}(t)$ is described by the following Itô-type stochastic differential equation:

$$\frac{dV_t^{(\pi)}}{V_t^{(\pi)}} = (r(t) + (\mu(t) - r(t))\pi(t))\, dt + \sigma(t)\pi(t)dW_t, \quad t \in [0,T], \quad V(0) = v_0 > 0 \qquad (6.0.4)$$

For simplicity, assume for now $\mathbb{G} = \mathbb{F}$. For an honest investor, i.e. $\mathbb{H} = \mathbb{F}$, (6.0.4) can be solved using classical techniques. The solution can be used to solve for an optimal portfolio, see Cvitanic & Karatzas (1995) and Karatzas (1989).

From the insider's point of view, when $\mathscr{H}_t \supset \mathscr{F}_t$, the stochastic integral

$$\int_0^t \sigma(s)\pi(s)V^{(\pi)}(s)\, dW_s \qquad (6.0.5)$$

has no meaning in the (classical) Itô sense. The forward integral (see Definition 6.1.1 below) is an extension of the Itô integral that gives meaning to (6.0.5). It will be shown in the sequel that it is natural to interpret the stochastic integral in (6.0.5) as a forward integral in the insider trading framework. The forward integral is also related to the Skorohod integral under suitable conditions, hence the techniques of Malliavin calculus can be applied. See also Coviello & Russo (2006) for a discussion of forward integral techniques as they are applied to financial assets modelling that does not suppose a priori that the dynamics of the asset price is a semimartingale.

One of the main consideration in modelling insider trading is the characterisation of the optimal portfolio. To state this formally, for a given utility function $U = U(x)$, the following optimal portfolio problem is considered:

Problem 6.0.8. *Find $\pi^* \in \mathscr{A}_{\mathbb{H}}$ such that*

$$u_T^{\mathscr{G},\mathscr{H}} := \sup_{\pi \in \mathscr{A}_{\mathbb{H}}} E\left[U(V^{(\pi)}(T))\right] = E\left[U(V^{(\pi^*)}(T))\right]. \qquad (6.0.6)$$

This chapter is organized as follows: In Section 1 the forward integral is formally defined and some of the relevant properties will be briefly reviewed. In Section 2 the portfolio optimization problem will be put into context in the partial informatiom framework in more details. This is a generalized framework for a market in which different agents possess different level of information, and is used by Kohatsu-Higa & Sulem (2006) to analyse the insider problem. The relevance of the "semimartingale preservation" assumption will be reviewed following the approach of Biagini & Øksendal (2005). Section 3 is devoted to the solution of the insider's optimal portfolio and the additional utility achieved by the insider. In Section 4 a market with a large insider is studied to analyse the effect of the large insider's influence of the price process on both the honest investor's and the insider's optimal portfolio and the utility achieved with the optimal portfolio. Finally, in Section 5, the optimization problem is modified so that the resulting optimal portfolio is a "smooth" portfolio. Following Hu & Øksendal, this is achieved by the introduction of a suitable penalty function that penalises "unsmooth" portfolios.

6.1 Preliminaries on forward integrals

In this section some basic definitions and results of the forward integral are briefly reviewed. For a more detailed account of the properties of the forward intgeral, one can refer to Nualart & Pardoux (1988) and Russo & Vallois (1993, 2000).

Definition 6.1.1. *[Definition 2.1 of Biagini & Øksendal (2005)] Let $\phi : [0,T] \times \Omega \longrightarrow \mathbb{R}$ be a measurable process. The forward integral of ϕ with respect to $W(.)$ is defined by*

$$\int_0^T \phi(t,\omega)\, d^- W(t) = \lim_{\epsilon \to 0} \int_0^T \phi(t,\omega) \frac{W(t+\epsilon) - W(t)}{\epsilon}\, dt \tag{6.1.1}$$

if the limit exists in $L^1(\Omega)$, in which case ϕ is called forward integrable. *If the limit also exists in $L^2(\mathbb{P})$, then this is denoted by $\phi \in Dom_2\delta^-$.*

Remark 6.1.2 (Léon et al. (2003)). *The forward integral is also an anticipating integral, in the sense that $L^2_a([0,T] \times \Omega) \subset Dom_2\delta^-$, where $L^2_a([0,T] \times \Omega)$ is the subspace of adapted processes in $L^2([0,T] \times \Omega)$.*

Remark 6.1.3. *The forward integral of a square integrable and $\mathscr{F}_t$-adapted process coincides with its Itô integral with respect to W.*

Lemma 6.1.4 (Biagini & Øksendal (2005)). *If ϕ is càdlàg and forward integrable, then*

$$\int_0^T \phi(t,\omega)\, d^- W(t) = \lim_{\Delta t_j \to 0} \sum_j \phi(t_j).\Delta W(t_j) \tag{6.1.2}$$

Proof. Assume that $\phi(t,\omega) = \sum_{j=1}^n \phi(t_j,\omega) \mathbf{1}_{(t_j,t_{j+1}]}(t)$. Then

$$\int_0^\infty \phi(t,\omega)\, d^- W(t) = \sum_{j=1}^n \phi(t_j) \lim_{\epsilon \to 0} \int_{t_j}^{t_{j+1}} \frac{W(t+\epsilon) - W(t)}{\epsilon}\, dt$$

Now to evaluate the right hand side one needs to evaluate a function of the following form:

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_a^b \frac{W(t+\epsilon) - W(t)}{\epsilon}\, dt &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_a^b W(t+\epsilon) - W(t)\, dt \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \int_{a+\epsilon}^{b+\epsilon} W(t)\, dt - \int_a^b W(t)\, dt \right\} \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \int_b^{b+\epsilon} W(t)\, dt - \int_a^{a+\epsilon} W(t)\, dt \right\} \\
&= W(b) - W(a)
\end{aligned}$$

where a change of variable $t \to t + \epsilon$ has been used to obtain $\int_a^b W(t+\epsilon)\, dt = \int_{a+\epsilon}^{b+\epsilon} W(t)\, dt$ and the fundamental theorem of calculus was used to obtain $\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_a^{a+\epsilon} W(t)\, dt = W(a)$. Hence

$$\int_0^\infty \phi(t,\omega)\, d^- W(t) = \sum_{j=1}^n \phi(t_j) \lim_{\epsilon\to 0} \int_{t_j}^{t_{j+1}} \frac{W(t+\epsilon)-W(t)}{\epsilon}\, dt$$
$$= \sum_{j=1}^n \phi(t_j)\left(W(t_{j+1}) - W(t_j)\right).$$

□

Lemma 6.1.5. *Let $\mathscr{H}_t \supset \mathscr{F}_t$ for all t and assume that $W(t)$ is a semimartingale with respect to $\mathbb{H}$, i.e.*

$$W(t) = \hat{W}(t) + A(t); 0 \le t \le T \tag{6.1.3}$$

where $\hat{W}(t)$ is a $\mathbb{H}$-adapted Brownian motion, $A(t)$ is a $\mathbb{H}$-adapted finite variation continuous process. Let $\phi(t,\omega)$ adapted to $\mathscr{H}_t$ be forward integrable and càdlàg. Then $\int_0^T \phi\, dW(t)$ exists as a semimartingale integral and

$$\int_0^T \phi(t)\, dW(t) = \int_0^T \phi(t)\, d^- W(t). \tag{6.1.4}$$

Proof. Using (6.1.2), one obtains

$$\int_0^T \phi(t)\, d\hat{W}(t) + \int_0^T \phi\, dA(t) = \lim_{\Delta t_j \to 0} \sum_j \phi(t_j).\left(\Delta\hat{W}(t_j) + \Delta A(t_j)\right)$$
$$= \lim_{\Delta t_j \to 0} \sum_j \phi(t_j).\Delta W(t_j)$$
$$= \int_0^T \phi(t)\, d^- W(t)$$

□

This is a well-known result. It states that if W is a semimartingale in the enlarged filtration, then any measurable, bounded process ϕ that is adapted to the enlarged filtration is forward integrable and its forward integral with respect to W coincides with the Itô integral of it with respect to W.

It is important to note that (6.1.4) holds only if (6.1.3) is true. Nevertheless Biagini and Øksendal adopted the forward integral in their insider trading model in Biagini & Øksendal (2005) without explicitly assuming (6.1.3). It will be shown later in the chapter (see Remark 6.2.7) that this can be justified and it is natural to interpret the stochastic integral as a forward integral.

Definition 6.1.6. *Suppose $X(t) = X(t,\omega)$ is of the form*

$$X(t) = X(0) + \int_0^t a(s,\omega)\, ds + \int_0^t b(s,\omega)\, d^- W(s),$$

where $a(s,\omega)$ and $b(s,\omega)$ are measurable (not necessarily $\mathbb{F}$-adapted) processes such that $a(t) \in L^\infty(\mathbb{R}^+ \times \Omega)$ a.s. for all $t > 0$ and

$$\int_0^t b(s,\omega)\, d^- W(s) \text{ exists } \forall t > 0.$$

Then X is called a forward process.

The following results are essential in obtaining a solution for the portfolio optimization problem.

Theorem 6.1.7. *[Itô formula for forward processes, see Hu & Øksendal (2003) and Russo & Vallois (2000)] Let*

$$d^- X(t) = \mu(t)dt + \sigma(t)d^- W(t) \tag{6.1.5}$$

be a forward process. Let $f \in C^2(\mathbb{R})$ and define

$$Y(t) = f(X(t))$$

Then $Y(t)$ is also a forward process and

$$d^- Y(t) = f'(X(t))d^- X(t) + \frac{1}{2} f''(X(t))\sigma^2(t)dt$$

Applying the above theorem, one obtains

Corollary 6.1.8. *Let $\mu(t)$, $\sigma(t)$ be measurable processes such that*

$$\int_0^t \left(|\mu(s)| + |\sigma(s)|^2\right) ds < \infty, \text{ and}$$

$$\int_0^t \sigma(s)\, d^- W(s) < \infty \quad \forall t > 0$$

then (6.1.5) has the unique solution

$$X(t) = x \exp\left[\int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right) ds + \int_0^t \sigma(s)\, d^- W(s)\right],$$

for $X(0) = x$.

The next result establishes the relation between the forward integral and the Skorohod integral. One can refer to Russo & Vallois (1993) for related results.

Lemma 6.1.9. *[Lemma 2.2 of Kohatsu-Higa & Sulem (2006)] Suppose that $\phi : [0,T] \times \Omega \to \mathbb{R}$ belongs to $\mathbb{L}^{1,2}[0,T]$, i.e. $\phi(t) \in \mathbb{D}^{1,2}$ for all $t \in [0,T]$ and*

$$\mathbb{E}\left(\int_0^T |\phi(t)|^2\, dt + \int_0^T \int_0^T |D_u \phi(t)|^2\, du\, dt\right) < +\infty.$$

Moreover, assume that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{u-\epsilon}^{u} \phi(t)\, dt = \phi(u) \text{ for a.a. } u \in [0,T] \text{ in } \mathbb{L}^{1,2}[0,T]$$

and that $D_{t^+}\phi(t) := \lim_{s \to t^+} D_s \phi(t)$ exists uniformly in $t \in [0,T]$ in $\mathbb{L}^1((0,T) \times \Omega)$. Then the forward integral of ϕ exists and

$$\int_0^T \phi(t)\, d^- W(t) = \int_0^T \phi(t)\, \delta W(t) + \int_0^T D_{t^+}\phi(t)\, dt.$$

Moreover,

$$\mathbb{E}\left[\int_0^T \phi(t)\, d^- W(t)\right] = \mathbb{E}\left[\int_0^T D_{t^+}\phi(t)\, dt\right].$$

Proof. See page 155 of Kohatsu-Higa & Sulem (2006). □

6.2 The optimal portfolio problem for an investor with general utility

In this section, the general framework for the portfolio optimization problem is formulated following the partial information modelling presented in Kohatsu-Higa & Sulem (2006). Under this generalized framework no assumption is made regarding the relationship between $\mathscr{F}_t$, $\mathscr{G}_t$ and $\mathscr{H}_t$ in general. For ease of interpretation, one can assume $\mathscr{F}_t \subseteq \mathscr{G}_t$ so that we have an anticipative market. But from the point of view of an "honest" trader, $\mathscr{H}_t = \mathscr{F}_t$ and from the point of view of an "insider", $\mathscr{F}_t \subseteq \mathscr{G}_t \subseteq \mathscr{H}_t$.

Assume a financial market with two investment assets:

1. a risk-free asset with price dynamics

$$dS_0(t) = r(t,\omega)S_0(t)dt; \quad S_0(0) = 1 \tag{6.2.1}$$

2. a risky asset with price dynamics

$$dS(t) = S(t)[\mu(t,\omega)dt + \sigma(t,\omega)d^-W(t)]; \quad S(0) = s > 0. \tag{6.2.2}$$

where the coefficients $r(t) = r(t,\omega)$, $\mu(t) = \mu(t,\omega)$ and $\sigma(t) = \sigma(t,\omega)$ are $\mathbb{G}$-adapted, $\sigma(t)$ is càdlàg and forward integrable and

$$E\left[\int_0^T \left\{|r(t)| + |\mu(t)| + \sigma(t)^2\right\} dt\right] < \infty.$$

The optimal portfolio can be formulated in terms of the fraction of wealth invested in the risky asset at time t, $\pi(t) = \pi(t,\omega)$. This portfolio is chosen by the investor with the knowledge he or she possesses and hence $\pi(t)$ is $\mathbb{H}$-adapted.

We define the set of admissible strategies as follows:

Definition 6.2.1. *[Admissible Strategies] The set $\mathscr{A}_{\mathbb{H}}$ consists of all $\mathbb{H}$-adapted stochastic processes π such that*

1. *$\pi(t)$ is càdlàg.*
2. *$\sigma(t)\pi(t)$ is forward integrable.*
3. *$\mathbb{E}\left[\int_0^T \left(|\mu(t) - r(t)||\pi(t)| + \sigma^2(t)\pi^2(t)\right) dt\right] < \infty$.*
4. *$U'(V^{(\pi)}(T)) > 0$ a.s. and $E\left[U'(V^{(\pi)}(T))V^{(\pi)}\right] < \infty$*

Remark 6.2.2. *If conditions 1,2 and 3 of Definition 6.2.1 holds, then the dynamics of the wealth process $V(t) = V^{(\pi)}(t)$ of the insider is given by*

$$dV(t) = V(t)\left[\{r(t) + (\mu(t) - r(t))\pi(t)\} dt + \sigma(t)\pi(t)d^-W(t)\right]; \quad V(0) = v_0. \tag{6.2.3}$$

From (6.2.3), the dynamics of the discounted wealth process is then:

$$dV(t) = V(t)\left[(\mu(t) - r(t))\pi(t)dt + \pi(t)\sigma(t)d^{-}W(t)\right] \quad V(0) = v_0 > 0. \tag{6.2.4}$$

This equation can be deduced by using the Itô's formula for forward integrals (see Theorem 6.1.7). The solution is as follows:

$$V^{(\pi)}(T) = v_0 \exp\left\{\int_0^T \left((\mu(t) - r(t))\,\pi(t) - \frac{1}{2}\pi^2(t)\sigma^2(t)\right) dt + \int_0^T \pi(t)\sigma(t)\, d^{-}W(t)\right\}. \tag{6.2.5}$$

Without loss of generality, assume $v_0 = 1$. Using a logarithmic utility function on terminal (discounted) wealth, i.e. let $U(x) = \ln x$, then the optimal portfolio problem 6.0.8 becomes

Problem 6.2.3. *Find $\pi^* \in \mathscr{A}_{\mathbb{H}}$ such that*

$$\sup_{\pi \in \mathscr{A}_{\mathbb{H}}} u(\pi) = u(\pi^*), \tag{6.2.6}$$

where

$$\begin{aligned} u(\pi) &:= \mathbb{E}\left[\ln V^{(\pi)}(T)\right] \\ &:= \mathbb{E}\left[\int_0^T \left((\mu(t) - r(t))\,\pi(t) - \frac{1}{2}\pi^2(t)\sigma^2(t)\right) dt + \int_0^T \pi(t)\sigma(t)\, d^{-}W(t)\right]. \end{aligned} \tag{6.2.7}$$

6.2.1 Finding the optimal portfolio and optimal utility

In this subsection, the solution to Problem 6.2.3 will be stated and the optimized utility will also be given. It will be shown that the solutions to the Merton problem can be recovered from the results obtained under this general framework.

The following assumption is made:

Assumption 6.2.4. *The expected optimal utility is finite.*

There will be instances where the optimal utility is not finite, depending on $\mathbb{H}$, i.e. the information the investor possesses, as discussed in the previous chapters.

The following theorem provide a characterization of optimal portfolios.

Theorem 6.2.5. *[Theorem 4.1 of Kohatsu-Higa & Sulem (2006)] The following statements are equivalent:*

1. *There exists an optimal portfolio $\pi^* \in \mathscr{A}_{\mathbb{H}}$ for Problem 6.2.3.*

2. *There exists $\pi^* \in \mathscr{A}_{\mathbb{H}}$ such that the process*

$$M_{\pi^*}(t) := \mathbb{E}\left[\int_0^t \left(\mu(s) - r(s) - \sigma^2(s)\pi^*(s)\right) ds + \int_0^t \sigma(s)\, d^{-}W(s)\,\middle|\, \mathscr{H}_t\right]$$

is an $\mathbb{H}$-martingale.

3. *The function*

$$s \mapsto \mathbb{E}\left[\int_0^s \sigma(u)\, d^- W(u) \middle| \mathscr{H}_t\right]; \quad s > t$$

is absolutely continuous and there exists $\pi^* \in \mathscr{A}_{\mathbb{H}}$ such that for a.a. t, ω,

$$\frac{d}{ds}\mathbb{E}\left[\int_0^s \sigma(u)\, d^- W(u) \middle| \mathscr{H}_t\right] = -\mathbb{E}\left[\mu(s) - r(s) - \sigma^2(s)\pi^*(s)|\mathscr{H}_t\right]; \qquad a.a.\ s > t. \qquad (6.2.8)$$

Proof. See page 160 of Kohatsu-Higa & Sulem (2006). □

Theorem 6.2.6. *For an insider who possesses information represented by $\mathscr{H}_t \supseteq \mathscr{G}_t \supseteq \mathscr{F}_t$,*

1. *if $\pi \in \mathscr{A}_{\mathbb{H}}$ is optimal for the Problem 6.2.3 then the process*

$$N(t) = \int_0^t \sigma(s)\, d^- W(s)$$

is a $\mathbb{H}$-semimartingale.

2. *Furthermore, if an optimal $\pi \in \mathscr{A}_{\mathbb{H}}$ exists and*

$$\sigma(s) \neq 0 \text{ for a.a. } (s,\omega) \in [0,T] \times \Omega \qquad (6.2.9)$$

then $W(t)$ is a $\mathbb{H}$-semimartingale.

Proof. 1. From part 2 of Theorem 6.2.5, $M_{\pi^*}(t)$ is an $\mathbb{H}$-martingale where

$$M_{\pi^*}(t) := \int_0^t \left(\mu(s) - r(s) - \sigma^2(s)\pi^*(s)\right) ds + \int_0^t \sigma(s)\, d^- W(s)$$

since $\mathscr{H}_t \supset \mathscr{G}_t \supseteq \mathscr{F}_t$. The result is then a direct consequence.

2. By part 1,

$$N(t) = \int_0^t \sigma(s)\, d^- W(s)$$

is a $\mathbb{H}$-semimartingale. Then if (6.2.9) holds,

$$\int_0^t \sigma^{-1}(s) dN(s) = \int_0^t \sigma^{-1}(s)\sigma(s)\, d^- W(s) = W(t)$$

is also a $\mathbb{H}$-semimartingale.

□

Remark 6.2.7. *Hence for the optimal portfolio, the conditions underlying Lemma 6.1.5 are satisfied. And the classical martingale method to solve the optimal portfolio problem can be applied under $\mathscr{H}$. Therefore the problem is solved without assuming (6.1.3), but with only the assumption that an optimal portfolio exists.*

The significance of semimartingales in the insider context has been discussed in Chapter 2.

(6.2.8) implies that

$$\mathbb{E}\left[\sigma^2(s)\pi^*(s)|\mathscr{H}_t\right] = \mathbb{E}\left[\mu(s) - r(s)|\mathscr{H}_t\right] + \frac{d}{ds}\mathbb{E}\left[\int_0^s \sigma(u)\, d^- W(u) \middle| \mathscr{H}_t\right] \qquad \text{a.a. } s > t.$$

And since $\pi^* \in \mathscr{A}_{\mathbb{H}}$ is $\mathbb{H}$-adapted, hence

Corollary 6.2.8. *Suppose there exists $\pi^* \in \mathscr{A}_{\mathbb{H}}$ for Problem 6.2.3. Then π^* must satisfy*

$$\pi^*(t)\mathbb{E}\left[\sigma^2(t)|\mathscr{H}_t\right] = \mathbb{E}\left[\mu(t) - r(t)|\mathscr{H}_t\right] + a(t)$$

i.e.

$$\pi^*(t) = \frac{\mathbb{E}\left[\mu(t) - r(t)|\mathscr{H}_t\right] + a(t)}{\mathbb{E}\left[\sigma^2(t)|\mathscr{H}_t\right]} \tag{6.2.10}$$

where

$$a(t) := \lim_{h \to 0^+} \frac{1}{h}\mathbb{E}\left[\int_t^{t+h} \sigma(u)\, d^- W(u) \middle| \mathscr{H}_t\right] \tag{6.2.11}$$

Remark 6.2.9. *In the Merton problem (Merton (1969)), $\mathscr{F}_t = \mathscr{G}_t = \mathscr{H}_t$, hence $a(t) = 0$. And since $\mu(t)$, $r(t)$ and $\sigma(t)$ are $\mathbb{G}$-adapted, hence they are also $\mathbb{H}$-adapted. The optimal portfolio π^* in (6.2.10) reduces to*

$$\pi^*(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)},$$

exactly as expected. In (6.2.10) the expected return rate and volatility are replaced by their best estimators, i.e. their conditional expectations with respect to $\mathscr{H}_{\approx}$. The extra term $a(t)$ is linked to the anticipative nature of the general equation. Examples for the calculation of this term will be provided later in the chapter.

With the optimal portfolio obtained using (6.2.10), the value function for Problem 6.2.3 can be computed:

Theorem 6.2.10. *[Theorem 4.4 of Kohatsu-Higa & Sulem (2006)] Suppose $\sigma(t) \neq 0$ for a.a. (t,ω) and there exists $\pi^* \in \mathscr{A}_{\mathbb{H}}$ for Problem 6.2.3. The optimal expected utility is then given by*

$$u(\pi^*) = \mathbb{E}\left[\int_0^T \left\{\frac{1}{2}\frac{\mathbb{E}\left[\mu(t) - r(t)|\mathscr{H}_t\right]^2}{\mathbb{E}\left[\sigma^2(t)|\mathscr{H}_t\right]} - \frac{1}{2}\frac{a^2(t)}{\mathbb{E}\left[\sigma^2(t)|\mathscr{H}_t\right]} + D_{t+}\left(\sigma(t)\frac{\mathbb{E}\left[\mu(t) - r(t)\mathscr{H}_t\right] + a(t)}{\mathbb{E}\left[\sigma^2(t)|\mathscr{H}_t\right]}\right)\right\} dt\right]. \tag{6.2.12}$$

Proof. Substituting (6.2.10) into (6.2.7), one obtains

$$\begin{aligned}
u(\pi^*) &= \mathbb{E}\left[\int_0^T (\mu(t) - r(t))\left(\frac{\mathbb{E}\left[\mu(t) - r(t)|\mathscr{H}_t\right] + a(t)}{\mathbb{E}\left[\sigma^2(t)|\mathscr{H}_t\right]}\right)\right] \\
&\quad - \frac{1}{2}\mathbb{E}\left[\left(\frac{\mathbb{E}\left[\mu(t) - r(t)|\mathscr{H}_t\right] + a(t)}{\mathbb{E}\left[\sigma^2(t)|\mathscr{H}_t\right]}\right)^2 \sigma^2(t)\, dt\right] \\
&\quad + \mathbb{E}\left[\int_0^T \left(\frac{\mathbb{E}\left[\mu(t) - r(t)|\mathscr{H}_t\right] + a(t)}{\mathbb{E}\left[\sigma^2(t)|\mathscr{H}_t\right]}\right)\sigma(t)\, d^- W(t)\right] \\
&= \mathbb{E}\left[\int_0^T \left(\frac{(\mu(t) - r(t))\,\mathbb{E}\left[\mu(t) - r(t)|\mathscr{H}_t\right] + (\mu(t) - r(t))\,a(t)}{\mathbb{E}\left[\sigma^2(t)|\mathscr{H}_t\right]}\right) dt\right] \\
&\quad - \frac{1}{2}\mathbb{E}\left[\int_0^T \left(\frac{\mathbb{E}\left[\mu(t) - r(t)|\mathscr{H}_t\right]^2 + 2a(t)\mathbb{E}\left[\mu(t) - r(t)|\mathscr{H}_t\right] + a^2(t)}{\mathbb{E}\left[\sigma^2(t)|\mathscr{H}_t\right]}\right)\frac{\sigma^2(t)}{\mathbb{E}\left[\sigma^2(t)|\mathscr{H}_t\right]}\, dt\right] \\
&\quad + \mathbb{E}\left[\int_0^T \left(\frac{\mathbb{E}\left[\mu(t) - r(t)|\mathscr{H}_t\right] + a(t)}{\mathbb{E}\left[\sigma^2(t)|\mathscr{H}_t\right]}\right)\sigma(t)\, d^- W(t)\right].
\end{aligned}$$

Then one can make use of the following relationships:

1.

$$\mathbb{E}\left[(\mu(t) - r(t))\,\mathbb{E}\left[\mu(t) - r(t)|\mathscr{H}_t\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\mu(t) - r(t)|\mathscr{H}_t\right]^2\right];$$

2. Since $a(t)$ is $\mathbb{H}$-measurable, hence

$$\mathbb{E}\left[(\mu(t)-r(t))\,a(t)\right] = \mathbb{E}\left[\mathbb{E}\left[(\mu(t)-r(t))\,a(t)|\mathscr{H}_t\right]\right] = \mathbb{E}\left[\mathbb{E}\left[(\mu(t)-r(t))\,|\mathscr{H}_t\right]a(t)\right];$$

3.

$$\mathbb{E}\left[\frac{\sigma^2(t)}{\mathbb{E}\left[\sigma^2(t)|\mathscr{H}_t\right]}\right] = 1;$$

4. and by Lemma 6.1.9

$$\mathbb{E}\left[\int_0^T \left(\frac{\mathbb{E}\left[\mu(t)-r(t)|\mathscr{H}_t\right]+a(t)}{\mathbb{E}\left[\sigma^2(t)|\mathscr{H}_t\right]}\right)\sigma(t)\,d^-W(t)\right]$$
$$= \mathbb{E}\left[\int_0^T D_{t^+}\left(\frac{\mathbb{E}\left[\mu(t)-r(t)|\mathscr{H}_t\right]+a(t)}{\mathbb{E}\left[\sigma^2(t)|\mathscr{H}_t\right]}\sigma(t)\right)dt\right].$$

and (6.2.12) follows. □

Remark 6.2.11. *Again, if we have $\mathscr{F}_t = \mathscr{G}_t = \mathscr{H}_t$ as in the Merton problem, $a(t) = 0$. And since*

$$\pi^*(t) = \frac{\mu(t)-r(t)}{\sigma^2(t)},$$

hence

$$D_{t^+}\left(\sigma(t)\pi^*(t)\right) = 0,$$

therefore

$$u(\pi^*) = \frac{1}{2}\mathbb{E}\left[\int_0^T \frac{(\mu(t)-r(t))^2}{\sigma^2(t)}\,dt\right], \tag{6.2.13}$$

as expected.

Remark 6.2.12. *For certain choices of $\mathbb{H}$, $a(t)$ does not exist. For example, if $\mathscr{G}_{t+\delta} \subseteq \mathscr{H}_t, \delta > 0$, investors (insiders) who possess information represented by such a filtration can obtain an infinite amount of wealth and the market admits arbitrage for such investors. If $u(\pi^*)$ is infinite, then problem 6.2.3 has no solution.*

6.2.2 The insider strategy: an alternative view

In this section we look at an alternative way of solving the optimal portfolio problem from the point of view of an insider who has access to information represented by $\mathscr{H}_t \supseteq \mathscr{G}_t \supseteq \mathscr{F}_t$.

From Theorem 6.2.6, if an optimal portfolio exists, then $W(t)$ is a $\mathbb{H}$-semimartingle. Hence there exists a $\mathscr{H}_t$-adapted process $\gamma(t)$ such that

$$\hat{W}(t) = W(t) - \int_0^t \gamma(s)\,ds \tag{6.2.14}$$

is a $\mathbb{H}$-Brownian motion. Hence the dynamics of the risky asset can be re-written as

$$dS(t) = S(t)\left[(\mu(t)+\sigma(t)\gamma(t))\,dt + \sigma(t)d^-\hat{W}(t)\right]. \tag{6.2.15}$$

Recall that $\mathbb{F}$ is the filtration generated by the Brownian motion W, denote the filtration generated by $W(t)$ with $\mathbb{F}^W$ and the filtration generated by $\hat{W}(t)$ with $\mathbb{F}^{\hat{W}}$. Hence for the model in the form of (6.2.15), $\mathscr{F}_t^{\hat{W}} = \mathscr{H}_t$ since $\hat{W}(t)$ is a $\mathbb{H}$-Brownian motion. Thus the problem is reduced to the Merton problem again if we apply the theory developed thus far with $\mathscr{F}_t^{\hat{W}} = \mathscr{H}_t$. From Remarks 6.2.9 and 6.2.11, the optimal portfolio for the insider is then

$$\pi^*(t) = \frac{\mu(t) - r(t) + \sigma(t)\gamma(t)}{\sigma^2(t)} \tag{6.2.16}$$

and the optimal utility is

$$u(\pi^*) = \frac{1}{2}\mathbb{E}\left[\int_0^T \frac{(\mu(t) - r(t) + \sigma(t)\gamma(t))^2}{\sigma^2(t)}\, dt\right]. \tag{6.2.17}$$

This is equivalent to the result obtained in Chapter 3, see Section 3.4. The solution to the optimal portfolio problem from the insider's point of view for the power utility case was given in Biagini & Øksendal (2005).

Comparing (6.2.13) with (6.2.17), the difference

$$\mathbb{E}\left[\int_0^T \frac{(\mu(t) - r(t))\,\gamma(t)}{\sigma(t)} + \frac{1}{2}\gamma^2(t)\, dt\right] \tag{6.2.18}$$

is the additonal utility obtained by the insider with his ability to choose π^* from $\mathscr{A}_{\mathbb{H}}$ instead of $\mathscr{A}_{\mathbb{F}}$. Assuming a non-anticipating market, i.e. $\mu(t)$, $r(t)$ and $\sigma(t)$ are $\mathbb{F}$-adapted, i.e. $\mathscr{G}_t = \mathscr{F}_t$ and therefore

$$\mathbb{E}\left[\int_0^T \frac{(\mu(t) - r(t))\,\gamma(t)}{\sigma(t)}\, dt\right] = 0. \tag{6.2.19}$$

To see this, by (6.2.14)

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \frac{(\mu(t) - r(t))\,\gamma(t)}{\sigma(t)}\, dt\right] \\
&= \mathbb{E}\left[\int_0^T \frac{(\mu(t) - r(t))}{\sigma(t)} \left(dW(t) - d\hat{W}(t)\right)\right] \\
&= \mathbb{E}\left[\int_0^T \frac{(\mu(t) - r(t))}{\sigma(t)}\, dW(t)\right] - \mathbb{E}\left[\int_0^T \frac{(\mu(t) - r(t))}{\sigma(t)}\, d\hat{W}(t)\right] = 0.
\end{aligned}$$

Then (6.2.18) becomes

$$\frac{1}{2}\mathbb{E}\left[\int_0^T \gamma^2(t)\, dt\right].$$

This result is consistent to results obtained in Chapter 3.

6.3 The honest investor's strategy in an insider influenced market

In this section the optimal portfolio problem from the view of a small investor in a market influenced by an insider is considered following the approach of Kohatsu-Higa & Sulem (2006). Suppose the investor can only observe the stock price process S as in (6.2.2). Hence

$$\mathscr{H}_t = \sigma\left(S(u), 0 \leq u \leq t\right), \tag{6.3.1}$$

Therefore the quadratic variation process of S is given by

$$\langle S, S\rangle_t = \int_0^t \sigma^2(u) S^2(u)\, du, \quad 0 \le t \le T.$$

Hence the process $\sigma(t)$ is $\mathbb{H}$-adapted.

Assume that $\mu(t) = \mu$, $r(t) = r$ and $\sigma(t) = \sigma$. The aim is to develop the solution to the optimal portfolio problem where the stock price process S is influenced by a large insider in the sense that the drift is changed from $\mu(t) = \mu$ to $\mu(t) = \mu + \beta X$, where X is a general (smooth) $\mathscr{F}_T$-measurable random variable representing the insider's privileged information. Admittedly this is a rather simplistic model of how the insider would utilise his privileged information, but it would serve the purpose of demonstrating the effect of such influences to the optimal portfolio problem.

Remark 6.3.1. *In this section, the discussion follows the point of view of an "honest" trader whose knowledge is represented by $\mathbb{H}$. The insider's action is assumed to influence the dynamics of the asset prices and hence $\mathscr{G}_t \supset \mathscr{H}_t$.*

6.3.1 Example: Influence via trading strategy

As an example, consider the case where the insider behaviour influences the price process through the effect of his or her trading strategy. I.e. let

$$dS(t) = (\mu + \beta\pi(t))\, S(t) + \sigma S(t) d^- W(t)$$

where $\pi(t)$ represents the trading strategy of the large insider, hence its $\mathscr{G}_t$-adapted and also let $0 < \beta < \sigma^2/2$. Assume further that W is a $\mathbb{G}$-semimartingale of the form (6.2.14). Hence

$$dS(t) = S(t)\left[(\mu + \beta\pi(t) + \sigma\gamma(t))\, dt + \sigma d\hat{W}(t)\right]$$

and from the above calculations one can deduce that the optimal portfolio for the insider must be

$$\hat{\pi} = \frac{\mu - r + \sigma\gamma(t)}{\sigma^2 - 2\beta}.$$

The small honest investor can only estimate the drift based on $\mathbb{H}$ and will model the price process as

$$dS(t) = \mathbb{E}\left(\mu + \beta\hat{\pi} | \mathscr{H}_t\right) S(t)dt + \sigma S(t) d\tilde{W}(t),$$

where $\tilde{W}$ is a $\mathbb{H}$-Brownian motion. The optimal portfolio is of course

$$\pi^*(t) = \frac{\mu - r + \beta\mathbb{E}\left[\hat{\pi}(t) | \mathscr{H}_t\right]}{\sigma^2}.$$

This simple example provide some interesting insight on how the insider's influenced is introduced. For $\sigma^2/2 > \beta > 0$, the insider's influence on the drift is positive and the asset price will be driven upwards. As a result, for the insider the optimal portfolio (the proportion of wealth invested in the risky asset) $\hat{\pi}$ is greater than π^* obtained in (6.2.16), as expected. The honest investor's optimal portfolio, also comprise of a greater proportion of the risky asset than that in a market with no such influence.

6.3.2 The general case

In this section a general model for an insider influenced market is developed. Specifically, assume that $\mu(t) = \mu + \beta X,\ X \in \mathscr{F}_T$, i.e. the dynamics of the risky asset's price is given by

$$dS(t) = S(t)(\mu + \beta X)dt + \sigma S(t)d^{-}W(t), \tag{6.3.2}$$

where $\mu, \beta \in \mathbb{R}$, $\sigma > 0$. The expressions for the optimal portfolio and optimal utility are given below:

Lemma 6.3.2. *[Lemma 6.9 of Kohatsu-Higa & Sulem (2006)] The quantity $a(t)$ defined in (6.2.11) for the model (6.3.2) is given by*

$$a(t) \equiv \lim_{h\to 0^+} \frac{1}{h}\mathbb{E}\left[\sigma\left(W(t+h) - W(t)\right)|\mathscr{H}_t\right] = \sigma\mathbb{E}\left[\int_t^T \frac{D_\nu X D_t X}{\int_t^T (D_r X)^2\, dr}\,\delta W(\nu)|\mathscr{H}_t\right].$$

if the right hand side above is well defined and right-continuous in t.

Proof. From (6.3.2), one can obtain

$$S(t) = S_0 \exp\left(\mu t + btX - \frac{1}{2}\sigma^2 t + \sigma W(t)\right)$$

Hence

$$\begin{aligned}\mathscr{H}_t &= \sigma\left(\mu s + bsX - \frac{1}{2}\sigma^2 s + \sigma W(s), 0 \le s \le t\right)\\ &= \sigma\left(bsX + \sigma W(s), 0 \le s \le t\right)\end{aligned}$$

then

$$\sigma\mathbb{E}\left[W(t+h) - W(t)|\mathscr{H}_t\right] = \sigma\mathbb{E}\left[W(t+h) - W(t)|bsX + \sigma W(s), 0 \le s \le t\right].$$

Now consider the following partition of [0,t]:

$$0 = s_0 < s_1 < ... < s_n = t \text{ with the time interval } \Delta = s_{i+1} - s_i$$

and let $\mathscr{H}_t^n$ denote the σ-algebra generated by

$$\{bs_i X + \sigma W(s_i), i = 0, \dots, n\}.$$

For a smooth bounded function f

$$\begin{aligned}&\mathbb{E}\left[W(t+h) - W(t)|bs_i X + \sigma W(s_i), i = 0, \dots, n\right]\\ &= \mathbb{E}\left[(W(t+h) - W(t))\, f(bX(s_n - s_{n-1}) + \sigma(W(s_n) - W(s_{n-1})), \dots, bXs_1 + \sigma W(s_1))\right]\\ &= \mathbb{E}\left[(W(t+h) - W(t))\, f(Z)\right]\end{aligned}$$

if we let

$$Z = (bX(s_n - s_{n-1}) + \sigma(W(s_n) - W(S_{n-1})), \dots, bXs_1 + \sigma W(s_1)).$$

Now, by the duality formula and the Fubini theorem,

$$\mathbb{E}\left[(W(t+h)-W(t))\,f(Z)\right] = \mathbb{E}[\delta(\mathbf{1}_{[t,t+h]})f(Z)] \tag{6.3.3}$$

$$= \mathbb{E}[\langle Df(Z), \mathbf{1}_{[t,t+h]}\rangle] \tag{6.3.4}$$

$$= \int_t^{t+h} \mathbb{E}\left[\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(Z) b(s_i - s_{i-1}) D_u X\right] du. \tag{6.3.5}$$

On the other hand, for $t \le \tau_1 < \tau_2$

$$\int_{\tau_1}^{\tau_2} D_\nu X D_\nu f(Z)\, d\nu = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(Z) b(s_i - s_{i-1}) \int_{\tau_1}^{\tau_2} (D_\nu X)^2\, d\nu.$$

Multiplying both sides by $\frac{D_u X}{\int_{\tau_1}^{\tau_2}(D_u X)^2\, du}$ and taking expectation, this becomes

$$\mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{D_\nu X D_u X}{\int_{\tau_1}^{\tau_2}(D_u X)^2\, du} D_\nu f(Z)\, d\nu\right] = \mathbb{E}\left[\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(Z) b(s_i - s_{i-1}) D_u X\right] \tag{6.3.6}$$

$$\mathbb{E}\left[f(Z)\int_{\tau_1}^{\tau_2} \frac{D_\nu X D_u X}{\int_{\tau_1}^{\tau_2}(D_u X)^2\, du}\, \delta W(\nu)\right] = \mathbb{E}\left[\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(Z) b(s_i - s_{i-1}) D_u X\right]. \tag{6.3.7}$$

by the duality formula again. Substituting (6.3.7) into (6.3.5), one obtains

$$\mathbb{E}\left[(W(t+h)-W(t))\,f(Z)\right] = \int_t^{t+h} \mathbb{E}\left[\mathbb{E}\left[f(Z)\int_{\tau_1}^{\tau_2} \frac{D_\nu X D_u X}{\int_{\tau_1}^{\tau_2}(D_u X)^2\, du}\, \delta W(\nu)\middle|\, \mathscr{H}_u\right]\right] du.$$

since $f(Z)$ is $\mathbb{H}$-measurable. Hence the process

$$\tilde{W}_t \equiv \mathbb{E}\left[W(t)|\mathscr{H}_t\right] - \int_0^t \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{D_\nu X D_u X}{\int_{\tau_1}^{\tau_2}(D_u X)^2\, du}\, \delta W(\nu)\middle|\, \mathscr{H}_u\right] du$$

is a $\mathbb{H}$-martingale. Hence for any $t \le \tau_1 < \tau_2 \le T$

$$\lim_{h\to 0^+} \frac{1}{h}\mathbb{E}\left[\sigma\left(W(t+h)-W(t)\right)|\mathscr{H}_t\right] = \sigma\mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{D_\nu X D_t X}{\int_{\tau_1}^{\tau_2}(D_r X)^2\, dr}\, \delta W(\nu)\middle|\, \mathscr{H}_t\right]$$

Taking $\tau_1 = t$ and $\tau_2 = T$ yields the desired result. □

The following results is a straight forward application of Lemma 6.3.2 and Corollary 6.2.8:

Theorem 6.3.3. *Suppose that $S(t)$ is given by (6.3.2) and $\mathscr{H}_t$ is given by (6.3.1). Then The optimal portfolio for Problem (6.2.3) exists and is given by*

$$\pi^*(t) = \frac{\mu - r + \beta\mathbb{E}\left[X|\mathscr{H}_t\right]}{\sigma^2} + \frac{\sigma\mathbb{E}\left[\int_t^T \frac{D_\nu X D_t X}{\int_t^T (D_r X)^2\, dr}\, \delta W(\nu)\middle|\, \mathscr{H}_t\right]}{\sigma^2}$$

The optimal utility can be obtained by applying Theorem 6.2.10 to the results obatined from Lemma 6.3.2 and Theorem 6.3.3. The optimal portfolio and optimal utility for the particular case where $X = W(T)$ are worked out in detail in Kohatsu-Higa & Sulem (2006).

6.3.3 Influence through terminal value of Brownian motion

In this subsection, the particular case where $X = W(T)$ (i.e. $\mu(t) = \mu + \beta W(T)$) is considered in order to demonstrate the meaning and significance of some of the important results, following the approach of Kohatsu-Higa & Sulem (2006).

The dynamics of the prices of the risky asset are

$$dS(t) = S(t)\left(\mu + \beta W(T)\right)dt + \sigma S(t)d^- W(t). \tag{6.3.8}$$

Hence if $W(T) > 0$ in this model, then $\beta \geq 0$ implies that the insider introduces a higher appreciation rate in the stock price, this means the higher the value of the final stock price the bigger the value of the drift driving S. $\beta < 0$ is also possible. In such case the insider drives the price lower the higher the final stock price. Studies involving such a model is certainly of dubious nature. However, traders do in practice employ such "buy-to-sell" or "sell-to-buy" strategies. In what follows no assumption is being made regarding the value of β other than that $\beta \in \mathbb{R}$.

Before the expression and interpretation of $a(t)$ is given, a technical lemma on the conditional expectation of $W(T)$ is required:

Lemma 6.3.4. *[Lemma 6.5 of Kohatsu-Higa & Sulem (2006)]*

$$\mathbb{E}\left[W(T)|\mathscr{H}_t\right] = \frac{\beta T + \sigma}{\left(\beta^2 T + 2\beta\sigma\right)t + \sigma^2}\left(\beta W(T)t + \sigma W(t)\right). \tag{6.3.9}$$

Proof. Let $0 = s_0 < s_1 < \cdots < s_n = t$ and $\Delta = s_{i+1} - s_i$.

$$\begin{aligned}\mathbb{E}\left[W(T)|\mathscr{H}_t^n\right] &= \mathbb{E}\left[W(t)|\beta s_i W(T) + \sigma W(s_i), 0 \leq i \leq n\right] \\ &= \sum_{i=0}^{n-1} \kappa_i\left(\beta W(T)\Delta + \sigma\left(W(s_{i+1}) - W(s_i)\right)\right),\end{aligned}$$

for some $(\kappa_i)_{(i=0,\ldots,n-1)}$.

By computing the correlation of the above with $\beta W(T)\Delta + \sigma\left(W(s_{j+1}) - W(s_j)\right)$, one obtains

$$\beta T\Delta + \sigma\Delta = \sum_{i=0, i\neq j}^{n-1} \kappa_i\left(\beta^2\Delta^2 T + 2\sigma\beta\Delta^2\right) + \kappa_j\left(\beta^2\Delta^2 T + 2\sigma\beta\Delta^2 + \sigma^2\Delta\right).$$

This is equivalent to

$$(\beta T + \sigma)\mathbf{1}_{n\times 1} = \left((\beta T + 2\sigma)\beta\Delta\mathbf{1}_{n\times n} + \sigma^2 I_{n\times n}\right)\gamma.$$

This implies

$$\kappa_0 = \kappa_1 = \cdots = \kappa_{n-1} \equiv \kappa$$

$$\gamma = \frac{\beta T + \sigma}{(\beta^2 T + 2\beta\sigma)t + \sigma^2}.$$

and (6.3.9) follows. □

Remark 6.3.5. *Similarly one can deduce that*

$$\mathbb{E}\left[W(t)|\mathscr{H}_t\right] = \frac{\beta t + \sigma}{\left(\beta^2 T + 2\beta\sigma\right)t + \sigma^2}\left(\beta W(T)t + \sigma W(t)\right). \tag{6.3.10}$$

An expression of $a(t)$ can now be derived:

Lemma 6.3.6. *[Lemma 6.4 of Kohatsu-Higa & Sulem (2006)] Suppose that $S(t)$ satisfies (6.3.8), then $a(t)$ is given by*

$$a(t) = \frac{\sigma\beta\left(\beta W(T)t + \sigma W(t)\right)}{\left(\beta^2 T + 2\beta\sigma\right)t + \sigma^2}.$$

Proof. Since

$$D_s X = D_s W(T) = 1 \quad \forall s \in [0,T],$$

by Lemma 6.3.2,

$$\begin{aligned} a(t) &= \sigma\mathbb{E}\left[\int_t^T \frac{D_\nu X D_t X}{\int_t^T (D_r X)^2\, dr}\,\delta W(\nu)|\mathscr{H}_t\right] \\ &= \sigma\mathbb{E}\left[\frac{1}{T-t}\int_t^T \delta W(\nu)|\mathscr{H}_t\right] \\ &= \frac{\sigma}{T-t}\mathbb{E}\left[W(T) - W(t)|\mathscr{H}_t\right] \\ &= \frac{\sigma}{T-t}\left[\frac{\beta(T-t)}{\left(\beta^2 T + 2\beta\sigma\right)t + \sigma^2}\left(\beta W(T)t + \sigma W(t)\right)\right] \\ &= \frac{\sigma\beta\left(\beta W(T)t + \sigma W(t)\right)}{\left(\beta^2 T + 2\beta\sigma\right)t + \sigma^2}. \end{aligned}$$

□

Note that in this case $a(t) \neq 0$.

Remark 6.3.7. *The proof given above is more succinct than that given in Kohatsu-Higa & Sulem (2006).*

With the expression of $a(t)$, the optimal portfolio and the optimal expected utility can be evaluated.

Theorem 6.3.8 (Theorem 6.6 of Kohatsu-Higa & Sulem (2006)). *Suppose that $S(t)$ satisfies (6.3.8), then*

1. *The optimal portfolio for Problem 6.2.3 exists and is given by*

$$\pi^*(t) = \frac{\mu - r}{\sigma^2} + \frac{\beta\left(\beta W(T)t + \sigma W(t)\right)\left(\beta T + 2\sigma\right)}{\sigma^2\left(\left(\beta^2 T + 2\beta\sigma\right)t + \sigma^2\right)}, \tag{6.3.11}$$

2. *The optimal utility is finite and is given by*

$$u(\pi^*) = \frac{(\mu - r)^2 T}{2 * \sigma^2} + \frac{1}{2\gamma}\left(1 - \gamma\ln\left(1 + \frac{1}{\gamma}\right)\right), \tag{6.3.12}$$

where

$$\gamma \equiv \frac{\sigma^2}{\beta T(\beta T + 2\sigma)}.$$

Proof. (6.3.11) is obtained by combining Corollary 6.2.8 and Lemma 6.3.6.

(6.3.12) is a direct consequence of Theorem 6.2.10, together with Lemma 6.3.4 and Lemma 6.3.6:

$$-\frac{1}{2\sigma^2}\mathbb{E}\int_0^T a(s)^2\, ds = -\frac{\beta^2}{2}\int_0^T \frac{s}{\left(\beta^2 T + 2\beta\sigma\right)s + \sigma^2}\, ds.$$

by applying Lemma 6.3.6 and straight forward calculations. Moreover, by Lemma 6.3.4

$$\begin{aligned} D_{s^+}\mathbb{E}\left[W(T)|\mathscr{H}_s\right] &= \frac{(\beta T+\sigma)\beta s}{(\beta^2 T+2\beta\sigma)\,s+\sigma^2} \\ D_{s^+}a(s) &= \frac{\sigma\beta^2 s}{(\beta^2 T+2\beta\sigma)\,s+\sigma^2} \\ \Rightarrow D_{s^+}\left[\frac{\mathbb{E}\left[\mu(s)-r|\mathscr{H}_t\right]+a(s)}{\sigma}\right] &= \frac{\beta^2(\beta T+2\sigma)s}{\sigma\left((\beta^2 T+2\beta\sigma)\,s+\sigma^2\right)}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \frac{1}{2\sigma^2}\mathbb{E}\int_0^T \mathbb{E}\left[\mu(s)-r|\mathscr{H}_s\right]^2 ds &= \frac{1}{2\sigma^2}\mathbb{E}\int_0^T \left\{\mu-r+\frac{\beta^2(\beta T+\sigma)^2\left(\beta W(T)s+\sigma W(s)\right)^2}{\left((\beta^2 T+2\beta\sigma)\,s+\sigma^2\right)^2}\right\} ds \\ &= \frac{1}{2\sigma^2}\int_0^T \frac{\beta^2(\beta T+\sigma)^2\left(\beta^2 s^2 T+2\beta\sigma s^2+\sigma^2 s\right)}{\left((\beta^2 T+2\beta\sigma)\,s+\sigma^2\right)^2} ds+\frac{(\mu-r)T}{2\sigma^2} \\ &= \frac{\beta^2(\beta T+\sigma)^2}{2\sigma^2}\int_0^T \frac{s}{\left((\beta^2 T+2\beta\sigma)\,s+\sigma^2\right)} ds+\frac{(\mu-r)T}{2\sigma^2} \end{aligned}$$

and for $b>0$ one can obtain by integration

$$\begin{aligned} \int_0^T \frac{s}{\left((\beta^2 T+2\beta\sigma)\,s+\sigma^2\right)} ds &= \frac{T}{\beta^2 T+2\beta\sigma}\left(1-\frac{\sigma^2}{(\beta^2 T+2\beta\sigma)T}\ln\left(1+\frac{(\beta^2 T+2\beta\sigma)T}{\sigma^2}\right)\right) \\ &= \frac{T^2\gamma}{\sigma^2}\left(1-\gamma\ln\left(1+\frac{1}{\gamma}\right)\right). \end{aligned}$$

Combining the above with Theorem 6.2.10, one obtains

$$\begin{aligned} u(\pi^*) &= \frac{(\mu-r)T}{2\sigma^2}+\left(\frac{\beta^2(\beta T+\sigma)^2}{2\sigma^2}-\frac{\beta^2}{2}+\frac{\beta^2}{\sigma}(\beta T+2\sigma)\right)\left(\int_0^T \frac{s}{\left((\beta^2 T+2\beta\sigma)\,s+\sigma^2\right)} ds\right) \\ &= \frac{(\mu-r)T}{2\sigma^2}+\frac{\beta^2\left(\beta^2 T^2+4\beta T\sigma+2\sigma^2\right)}{2\sigma^2}\frac{T^2\gamma}{\sigma^2}\left(1-\gamma\ln\left(1+\frac{1}{\gamma}\right)\right) \\ &= \frac{(\mu-r)T}{2\sigma^2}+\frac{1}{2\gamma}\left(1-\gamma\ln\left(1+\frac{1}{\gamma}\right)\right) \end{aligned}$$

□

Remark 6.3.9. *In Kohatsu-Higa & Sulem (2006) a slightly different result to (6.3.11) was given.*

Remark 6.3.10. 1. *According to Kohatsu-Higa & Sulem (2006) the coefficient $\frac{1}{\gamma}$ can be interpreted as the insider effect om the utility of the $\mathbb{H}$-investor. This is because when $\beta \to 0$, $\gamma \to \infty$, i.e. the insider effect vanishes and the investor's utility approaches to that in the classical Merton problem.*

2. *The insider's effect can actually be quantified in terms of utility.*

6.4 Penalty Function

In this section a slight adjustment is introduced to the optimization problem in the form of an introduction of a penalty function related to the portfolio chosen, following the approach of Hu & Øksendal (2003). The effect on the insider's optimal portfolio and the optimal utility are investigated. Intuitively, the introduction of a penalty function steer the insider away from the optimal portfolio from a pure maximization of log-utility from terminal wealth objective by incorporating the effect of some competing objectives. The optimal portfolio thus obtained would therefore achieve a lower log-utility from terminal wealth than what would be the case had there been no penalty function introduced. By the monotonicity of our utility function, i.e. the log function, this implies that the terminal wealth achieved would be lower. It will be shown below that this is indeed the case in the model.

Why would one want to introduce a penalty function? Consider the particular case where the insider knows, in addition to the knowledge possessed by the market and the honest trader, the future value of W at time T_0, where $T_0 > T$. Thus his information is represented by the filtration $\mathscr{H}_t = \mathscr{F}_t \vee \sigma(W(T_0))$. The insider will utilise this filtration to optimize his portfolio. From previous discussions, the optimal portfolio for Problem 6.2.3 is

$$\pi^*(t) = \frac{\mu(t) - r(t)}{\sigma(t)} + \frac{W(T_0) - W(t)}{\sigma(t)(T_0 - t)}. \tag{6.4.1}$$

and the corresponding optimal utility $u(\pi^*)$ is given by

$$u(\pi^*) = \mathbb{E}\left[\int_0^T \left\{\frac{1}{2}\frac{(\mu(s) - r(s))^2}{\sigma^2(s)} + \frac{1}{2(T_0 - s)}\right\} ds\right]; \quad T_0 \geq T. \tag{6.4.2}$$

If $T_0 = T$, i.e. if the terminal value of the Brownian motion is known to the insider from inception, then

$$u(\pi^*) = \infty$$

according to (6.4.2). Questions were raised by Hu & Øksendal (2003) on whether this result is realistic. The insider's optimal portfolio, according to (6.4.1), should be

$$\pi^*(t) = \frac{\mu(t) - r(t)}{\sigma(t)} + \frac{W(T) - W(t)}{\sigma(t)(T - t)}. \tag{6.4.3}$$

As $t \to T$ the second term on the right-hand side of (6.4.3) converges towards the derivative of $W(t)$ at $t = T^-$. This implies more wild fluctuations in $\pi^*(t)$ as $t \to T$, since a Brownian motion is nowhere differentiable with respect to t with probability one. This means in order to achieve the theoretical optimal of infinite utility, the insider must make more wild adjustments to the portfolio as $t \to T$. This is clearly impractical: as there will be trading costs involve as well as liquidity issues in practice that are not considered in the current form of the model. This may also be undesirable from the point of view of the insider in the sense that he would not want to behave differently from the honest traders in such an obvious fashion, whose the honest trader's portfolio is just

$$\pi^*(t) = \frac{\mu(t) - r(t)}{\sigma(t)},$$

thus increasing his chance of being detected by others. Therefore the deduction of infinte utility being achievable is clearly unrealistic both from a practical and a "strategic" point of view of the insider.

Clearly there is a value (or desire) of employing a "smooth" portfolio strategy, i.e. one that does not require large and frequent adjustments to the portfolio. This desire for a "smooth" portfolio strategy can be incorporated into the portfolio optimization problem via the introduction of a penalty function that penalizes "unsmooth" portfolios. In Hu & Øksendal (2003), it was proposed that the set of admissible strategies and the portfolio optimisation problem 6.0.8 to be modified to the following for a general $\mathbb{H}$, where $\mathscr{H}_t \supset \mathscr{F}_t$:

Definition 6.4.1. *In the remainder of the section, let $\mathscr{A}_{\mathbb{H}}$ denote the space of all stochastic processes $\pi(t)$ such that the following is satisfied in addition to all the conditions in Definition 6.2.1:*

$$\mathbb{E}\left[\int_0^T |\mathbb{Q}\pi(t)|^2\,dt\right] < \infty.$$

where $\mathbb{Q} = \mathbb{Q} : \mathscr{A}_{\mathbb{H}} \to \mathbb{R}$ is some operator on the portfolio function $\pi(t)$.

Problem 6.4.2. *Find $\pi^* \in \mathscr{A}_{\mathbb{H}}$ such that*

$$u(\pi^*) = \sup_{\pi \in \mathscr{A}_{\mathbb{H}}} u(\pi), \tag{6.4.4}$$

where

$$\begin{aligned} u(\pi) &:= \mathbb{E}\left[\ln X^{(\pi)}(T) - \int_0^T |\mathbb{Q}\pi(s)|^2\,ds\right] \\ &= \mathbb{E}\left[\int_0^T \left((\mu(t) - r(t))\,\pi(t) - \frac{1}{2}\pi^2(t)\sigma^2(t)\right)dt + \int_0^T \pi(t)\sigma(t)\,d^-W(t) - \int_0^T |\mathbb{Q}\pi(t)|^2\,dt\right]. \end{aligned}$$

Specific examples of $\mathbb{Q}$ will be given later. But before that, a solution to the above problem is needed.

We have the following equivalent to Theorem 6.2.5 for Problem 6.4.2:

Theorem 6.4.3. *The following statements are equivalent:*

1. *There exists an optimal portfolio $\pi^* \in \mathscr{A}_{\mathbb{H}}$ for Problem 6.4.2.*
2. *Let $\mathbb{Q}^*$ denote the adjoint of $\mathbb{Q}$ in $L^2([0,T]\times\Omega)$. There exists $\pi^* \in \mathscr{A}_{\mathbb{H}}$ such that the process*

$$M_{\pi^*}(t) := \mathbb{E}\left[\int_0^t \left(\mu(s) - r(s) - \sigma^2(s)\pi^*(s) - \mathbb{Q}^*\mathbb{Q}\pi(s)\right)ds + \int_0^t \sigma(s)\,dW(s)|\mathscr{H}_t\right]$$

   is an $\mathbb{H}$-martingale.
3. *Suppose there exists a $\mathscr{H}_t$-measurable process $\zeta_t(s)$ such that*

$$W(t) - \int_0^t \zeta_t(s)\,ds \text{ is a } \mathbb{H}\text{-martingale}$$

   where the function

$$t \mapsto \int_0^t \zeta_t(s)\,ds$$

   is of finite variation and there exists $\pi^* \in \mathscr{A}_{\mathbb{H}}$ such that for a.a. t,ω,

$$\sigma(t)\frac{d}{dt}\mathbb{E}\left[\int_0^t \zeta_t(s)\,ds\right] = -\mathbb{E}\left[\mu(s) - r(s) - \sigma^2(s)\pi^*(s) - \mathbb{Q}^*\mathbb{Q}\pi^*(s)|\mathscr{H}_t\right]; \qquad a.a.\ s > t \tag{6.4.5}$$

Proof. See Hu & Øksendal (2003) for the proof of (2) $\iff$ (3). The proof for (1) $\Rightarrow$ (2) is given below: Suppose an optimal portfolio $\pi = \pi^* \in \mathscr{A}_{\mathbb{H}}$ exists for the insider. Let θ be another portfolio in $\mathscr{A}_{\mathbb{H}}$. Then the function

$$f(y) := u(\pi + y\theta)\ ;\ y \in \mathbb{R}$$

must be a maximum for $y = 0$ and so

$$0 = \frac{d}{dy}\left[u(\pi + y\theta)\right]_{y=0} = \theta.u'(\pi) \tag{6.4.6}$$

$$= \mathbb{E}\left[\int_0^T \left\{(\mu(t) - r(t))\theta(t) - \sigma^2(t)\pi(t)\theta(t)\right\} dt + \int_0^T \sigma(t)\theta(t)\, d^- W(t) - \int_0^T \mathbb{Q}\pi(t)\mathbb{Q}\theta(t)\, dt\right] \tag{6.4.7}$$

$$= \mathbb{E}\left[\int_0^T \left\{\mu(t) - r(t) - \sigma^2(t)\pi(t) - \mathbb{Q}^*\mathbb{Q}\pi(t)\right\}\theta(t)\, dt + \int_0^T \sigma(t)\theta(t)\, d^- W(t)\right] \tag{6.4.8}$$

since

$$\mathbb{E}\left[\int_0^T \mathbb{Q}\pi(t)\mathbb{Q}\theta(t)\, dt\right] = \mathbb{E}\left[\int_0^T \mathbb{Q}^*\mathbb{Q}\pi(t)\theta(t)\, dt\right]$$

by the definition of an adjoint. For a fixed t, fix θ to be

$$\theta(s) = \theta_0(t)\mathbf{1}_{[t,t+h]}(s);\, s \in [0,T], 0 \le t < t+h < T$$

where $\theta_0(t)$ is $\mathscr{H}_t$-measurable. Then by Lemma 6.1.9 and the duality formula,

$$\begin{aligned}\mathbb{E}\left[\int_0^T \sigma(t)\theta(t)\, d^- W(t)\right] &= \mathbb{E}\left[\int_t^{t+h} \sigma(s)\theta_0(t)\, d^- W(s)\right]\\ &= \mathbb{E}\left[\int_t^{t+h} D_{s^+}(\sigma(s)\theta_0(t))\, ds\right]\\ &= \mathbb{E}\left[\int_t^{t+h} \theta_0(t) D_{s^+}\sigma(s)\, ds\right]\\ &= \mathbb{E}\left[\theta_0(t)\int_t^{t+h} \sigma(s)\, dW(s)\right].\end{aligned}$$

Substituting this into (6.4.6), one obtains

$$\mathbb{E}\left[\left(\int_t^{t+h} \left\{\mu(s) - r(s) - \sigma^2(s)\pi(s) - \mathbb{Q}^*\mathbb{Q}\pi(s)\right\} ds + \int_t^{t+h} \sigma(s)\, dW(s)\right)\theta_0(t)\right] = 0$$

Since this holds for all such $\mathscr{H}_t$-measurable $\theta_0(t)$, hence

$$\mathbb{E}\left[M_\pi(t+h) - M_\pi(t) | \mathscr{H}_t\right] = 0.$$

□

We have the following corollary which is analogous to Corollary (6.2.8):

Corollary 6.4.4. *Suppose there exists an optimal portfolio $\pi^* \in \mathscr{A}_{\mathbb{H}}$ for Problem 6.4.2. Then π^* must satisfy*

$$\mathbb{E}\left[\pi^*(t)\sigma^2(t) + \mathbb{Q}^*\mathbb{Q}\pi^*(t)|\mathscr{H}_t\right] = \mathbb{E}\left[\mu(t) - r(t)|\mathscr{H}_t\right] + \sigma(t)\frac{d}{dt}\left(\int_0^t \zeta_t(s)\, ds\right)$$

Consider the following example of $\mathbb{Q}$ given in Hu & Øksendal (2003):

Example 6.4.5 (Example 5.6 of Hu & Øksendal (2003)). *Set*

$$\mathbb{Q}\pi(t) = \lambda(t)\sigma(t)\pi(t)$$

where $\lambda(t) \geq 0$ *is deterministic.*

This penalty function penalizes the insider for taking large positions, especially when the asset price is highly volatile. Again, consider the insider's filtration mentioned at the beginning of this section:

$$\mathscr{H}_t = \mathscr{F}_t \vee \sigma(W(T_0)) \text{ for some } T_0 > T,$$

then

$$\gamma_t(s) = \gamma(s) = \frac{W(T_0) - W(s)}{T_0 - s}$$

and by Corollary 6.4.4 the optimal portfolio is

$$\pi^*(t) = \frac{\mu(t) - r(t) + \sigma(t)\frac{W(T_0)-W(t)}{T_0-t}}{\sigma^2(t)[1+\lambda^2(t)]}, \tag{6.4.9}$$

and the corresponding optimal utility is

$$u(\pi^*) = \mathbb{E}\left[\int_0^T \left\{\frac{1}{2}\frac{\left[\frac{\mu(t)-r(t)}{\sigma(t)} + \frac{W(T_0)-W(t)}{T_0-t}\right]^2}{1+\lambda^2(t)}\right\} dt\right] \tag{6.4.10}$$

Compare to the optimal portfolio for Problem 6.2.3 where no penalty function is imposed, (6.4.9) is smaller by a factor of $[1 + \lambda^2(t)]^{-1}$. Hence a smaller position is taken reflecting the constraint imposed by the penalty function. The integrand in the optimal utility function in (6.4.10) is also smaller by a factor of $[1 + \lambda^2(t)]^{-1}$, hence a lower utility is achieved compare to the case where no penalty is imposed.

One can set $\lambda(t)$ in such a way that $u(\pi^*) < \infty$. If,

$$\lambda(t) = (T_0 - t)^{-\beta}, \quad \beta > 0,$$

then for some contants K_1 and K_2,

$$u(\pi) \leq K_1 + K_2 \int_0^T (T_0 - t)^{-1+2\beta}\, dt < \infty,$$

even if $T_0 = T$. Hence the optimal utility is finite as a result of the introduction of the penalty function.

An alternative way to ensure a "smooth" portfolio is to penalize the insider for large (and frequent) portfolio adjustments. This can be achieve by introducing a penalty function in the form of

$$\mathbb{Q}\pi(t) = \pi'(t) = \frac{d}{dt}\pi(t).$$

It can be shown that the optimal utility for the insider is finite in this case for

$$\mathscr{H}_t = \mathscr{F}_t \vee \sigma(W(T_0)) \text{ for some } T_0 > T,$$

one can refer to Hu & Øksendal (2003) for more details.

These examples of penalty functions given are geared towards the aim of ensuring a smooth portfolio. The same concepts and techniques can be applied to incorporate other objectives in the portfolio optimization process by introducing suitable penalty functions.

Chapter 7

Arbitrage Opportunities

In Chapter 3, it was shown that the insider will be able to achieve additional utility compare to the honest trader, and this additional utility can in some cases be infinite. One might ask if this immediately implies the existence of arbitrage opportunities for the insider, in a market where arbitrage is not possible for the honest trader? The answer is, unsurprisingly, that arbitrage is possible, but that depends on the type of information possessed by the insider.

In the last example of Chapter 4, we obtained the information drift γ_t^L in (4.2.9) for an insider who possesses information about the maximum value reached by the Brownian motion process underlying the price process, i.e. when $L = \sup_{t\in[0,T]} W_t$. It can be shown that

$$\int_0^T |\gamma_s^L|\, ds < \infty \quad \mathbb{P} - \text{a.s.},$$

and hence W is preserved as a semimartingale in $\mathbb{G}$. However, even with this knowledge it remains uncertain whether this is enough to prevent or establish any arbitrage opportunities.

Indeed Imkeller et al. (2001) has shown the existence of arbitrage opportunities for the insider with such information by utilising the results of Delbaen & Schachermayer (1994): the existence of arbitrage is linked to the existence of an equivalent martingale measure for the price process.

Imkeller (2002) shows the existence of arbitrage when the insider has information regarding the time when certain event takes place. Furthermore, in Imkeller (2003) it is shown that arbitrage opportunities exist when $L = \sup_{t\in[0,T]} S_t$. On that other hand, Corcuera et al. (2004) has shown that if the privileged information is blurred by noise and if the rate at which this noise vanishes is slow enough then there can still be no arbitrage for the insider and his (additional) utility remains finite.

In this chapter, the relationship between the semimartingale property, the absence of arbitrage and finiteness of utility will be briefly reviewed. The existence of arbitrage opportunities as well as the strategies for the insider to exploit the opportunities will be explored.

7.1 Finite utility, no-arbitrage and semimartingales

As demonstrated in Chapter 2, the initial enlargement of filtration approach to modelling insider behaviour initiated by Karatzas & Pikovsky (1996) essentially involves finding a measure under which the Brownian motion in the honest trader's filtration $\mathbb{F}$ will remain a semimartingale in the enlarged insider filtration $\mathbb{G}$ (the so-called *semimartingale property* or *preservation of semimartingales property*). If such a measure exists, W remains a semimartingale in $\mathbb{G}$ and the insider's additional utility is finite. Assuming a complete market, the optimal portfolio for the insider can be found. However, such a measure does not always exist, if the absolute continuity assumption (Assumption 2.1.3) or the equivalence assumption (Assumption 2.1.5) are not met.

In Chapter 4, we have seen that the above-mentioned assumptions can be relaxed in evaluating the information drift. This provides a tool for the analysis of insider information that would lead to the violation of Assumptions 2.1.3 and 2.1.5. Since the assumption is relaxed, the semimartingale property and the finiteness of the insider's additional utility can no longer be guaranteed. The natural question is then what is the sufficient condition that would provide such a guarantee?

Biagini & Øksendal (2005) studied the problem from the opposite direction. It was shown that if the optimal portfolio exists, i.e. the maximum utility is attained, then the Brownian motion W must be a semimartingale with respect to $\mathbb{G}$. See also Hu & Øksendal (2003).

In Imkeller et al. (2001) and Imkeller (2002, 2003), Imkeller and his colleagues approached the question of the existence arbitrage opportunities for the insider by utilizing results of Delbaen & Schachermayer (1994). Focusing of the existence of an equivalent (local) martingale measure for W in $\mathbb{G}$.

Building on this, Ankirchner & Imkeller (2005) provides a rigorous discussion around the matter and shows that whenever an agent's expected utility is finite, the price process S (and hence the underlying Brownian motion W) must be a semimartingale.

In this section, the relationship between the semimartingale property, the absence of arbitrage and finiteness of utility will be reviewed in a concise manner. One can refer to Ankirchner & Imkeller (2005) for further details.

7.1.1 Definitions and Preliminaries

In order to facilitate the discussion that follows, the meaning of arbitrage needs to be defined properly.

The absence of arbitrage opportunities is an important concept that underlies the contemporary theory of asset pricing. Harrison & Pliska (1981) defined an *arbitrage opportunity* as some admissible investment strategy π such that

1. the initial (time $t = 0$) value of the portfolio is 0, i.e. $V_0^{(\pi)} = 0$, and yet
2. the expected value of the terminal (time $t = T$) value of the portfolio is greater than 0, i.e. $\mathbb{E}(V_T^{(\pi)}) > 0$.

Such a strategy, if it does exist, represents a riskless plan for making profit without any investment. Since $V_t^{(\pi)} \geq 0$, there must be a non-zero probability of ending up with a positive portfolio value.

Hence the *No-Arbitrage* (NA) condition implies, in layman terms, one cannot make something out of nothing.

In the fundamental paper by Delbaen & Schachermayer (1994), it is shown that the *no free lunch with vanishing risk* (NFLVR) condition is a stronger condition than the *no-arbitrage* (NA) condition. In order to define (NFLVR) properly. A few notations are needed:

Consider once again a market consists of a risk-free asset modelled by

$$dS_0(t) = r(t)S_0(t)dt; \quad S_0(0) = 1 \tag{7.1.1}$$

and one risky stock described by

$$\frac{dS_1(t)}{S_1(t)} = \mu(t)dt + \sigma(t)dW(t), \quad 0 \le t \le T, \quad S_1(0) \in [0, \infty), \tag{7.1.2}$$

where W is a Brownian motion. The wealth process can be written as

$$V_t^\theta = \theta_0(t)S_0(t) + \theta_1(t)S_1(t).$$

Hence the investment strategy is represented by $(\theta) = (\theta_0, \theta_1)$. One can then interpret the stochastic integral process $(\theta \cdot S)$ as the gains process. Given the initial wealth $V_0 = x \in \mathbb{R}$, $V_t^\theta = x + (\theta \cdot S)_t$.

Recall the following terminology introduced in Delbaen & Schachermayer (1994):

Definition 7.1.1. *Suppose $S = (S_0, S_1)$ is a $\mathbb{R}^2$-valued semimartingale defined on the probability space $(\Omega, \mathbb{F}, \mathbb{P})$.*

1. *Let $a \ge 0$, the $\mathbb{R}^2$ valued predictable process θ is called* a-admissible *if it is S-integrable, $(\theta \cdot S) \ge -a$ almost surely and $\lim_{t\to\infty}(\theta \cdot S)$ exists almost surely.*
2. *θ is called* admissible *if it is a-admissible for some $a \ge 0$.*

Define the following sets:

$$\begin{aligned}
\mathscr{K} &= \{(\theta \cdot S)_\infty | \theta \text{ is admissible}\} \\
\mathscr{K}_a &= \{(\theta \cdot S)_\infty | \theta \text{ is a-admissible}\} \\
\mathscr{C}_0 &= \mathscr{K} - L_+^0, \\
\mathscr{C} &= \mathscr{C}_0 \cap L^\infty.
\end{aligned}$$

Then the formally (NA) and (NFLVR) are defined as follows:

Definition 7.1.2. 1. *The process S is said to satisfy the (NA) condition if*

$$\mathscr{K} \cap L_+^0 = \{0\}, \tag{7.1.3}$$

2. *the process S is said to satisfy the (NFLVR) condition if*

$$\bar{\mathscr{C}} \cap L_+^\infty = \{0\}.$$

As was mentioned above, the (NFLVR) condition is a stronger condition than the (NA) condition, as the following remark would help clarify:

Remark 7.1.3. *(7.1.3) is equivalent to*

$$\mathscr{C} \cap L_+^\infty = \{0\}.$$

The following theorem gives the relationship between the two conditions:

Theorem 7.1.4 (Delbaen & Schachermayer (1994)). *The process S satisfies the (NFLVR) condition if and only if it satisfies*

1. *(NA), and*
2. *$\mathscr{K}_1$ is bounded in L^0.*

Remark 7.1.5 (Economic interpretation of item 2., Delbaen & Schachermayer (1994)). *The boundedness of $\mathscr{K}_1$ can be interpreted as: for outcomes that have a maximal loss of 1, the profit is bounded in probability. This implies the probability of making a big profit can be estimated from above, uniformly over such outcomes.*

7.1.2 The link between the no-arbitrage condition and the semimartingale property

Having (NA) and (NFLVR) properly defined, the link between absence of arbitrage and the semimartingale property can be established. In Delbaen & Schachermayer (1994), the fundamental theorem is given as:

Theorem 7.1.6. *Let S be a semimartingale defined on $(\Omega, \mathbb{F}, \mathbb{P})$. Then S satisfies the (NFLVR) condition if and only if there exists a probability measure $\mathbb{Q} \equiv \mathbb{P}$ such that S is a (local)-martingale with respect to $\mathbb{Q}$.*

Remark 7.1.7. *Actually Delbaen & Schachermayer (1994) proved that under the stated condition S satisfies (NFLVR) if and only if there exists $\mathbb{Q}$ such that S is a sigma-martingale with respect to $\mathbb{Q}$. Sigma-martingales are related to martingales in the same way as sigma-finite measures are related to finite measures, as Delbaen and Schachermayer put it. They also remarked that for mathematical finance applications, sigma-martingales can be taken as local martingales. See Delbaen & Schachermayer (1994) for a more detailed discussion.*

This has two immediate implications:

Corollary 7.1.8. *If an asset price process S satisfies the (NFLVR) condition, then the agent must view S as a semimartingale, i.e. S remains a semimartingale in the agent's filtration.*

Corollary 7.1.9. *Since (NFLVR) is a stronger condition than (NA), in order to establish the presence or absence of arbitrage opportunities, one only needs to show whether an equivalent (local) martingale measure $\mathbb{Q} \equiv \mathbb{P}$ exists.*

7.1.3 Bounded expected utility and no-arbitrage

Ankirchner & Imkeller (2005) investigated the relationship between the (NFLVR) condition, the semimartingale property of S in the financial agent's filtration (for an insider, this is $\mathbb{G} \supset \mathbb{F}$) and bounded expected utility.

They started by proving that if an agent has bounded expected utility with respect to his investment horizon $[0, T]$, then (NFLVR) will be satisfied for simple integrands θ. By extending this to general integrands and combining this with Corollary 7.1.8, they showed that the finiteness of expected utility implies that S (and hence W) must be a semimartingale in the agent's filtration, precisely:

Theorem 7.1.10 (Corollary 1.9 of Ankirchner & Imkeller (2005)). *Denote by $\mathscr{S}$ the collection of simple integrands. Let S be an arbitrary adapted continuous process indexed by $t \in [0,T]$, $U : \mathbb{R} \to [-\infty, \infty)$ a utility function with $\lim_{x\to\infty} U(x) = \infty$ and $x > \sup\{y \in \mathbb{R} : U(y) = -\infty\}$. If $\sup_{\mathscr{S}\ni\theta adm.} \mathbb{E}\left[U(x + (\theta \cdot S)_T)\right] < \infty$, then S is a semimartingale and the expected utility maximized over general admissible integrands is either infinite or given by $\sup_{\mathscr{S}\ni\theta adm.} \mathbb{E}\left[U(x + (\theta \cdot S)_T)\right]$.*

7.2 Arbitrage opportunities for the insider

Following the notations in Imkeller et al. (2001), let $r = 0$ for all t in (7.1.1), hence we are using the risk-free stock as our numeraire. Then $S_1(t)$ can be re-written as $S(t)$.

The risky stock process becomes

$$\begin{aligned} \frac{dS(t)}{S(t)} &= \int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dW_s \\ &= \int_0^t \mu_s + \sigma_s \gamma_s\, ds + \int_0^t \sigma_s\, d\tilde{W}_s \\ &= \int_0^t \tilde{\mu}_s\, ds + \int_0^t \tilde{\sigma}_s\, d\tilde{W}_s, \end{aligned}$$

with $\tilde{\mu}_s = \mu_s + \sigma_s \gamma_s$ and $\tilde{\sigma}_s = \sigma_s$, $t \in [0,T]$ and $\tilde{W}$ is a $\mathbb{G}$-Brownian motion.

By Corollary 7.1.9, if S satisfies (NFLVR), then there must exists an equivalent (local) martingale measure $\mathbb{Q}$ under which S is a local martingale. Since we are in the Brownian world, if $\mathbb{Q}$ exists it must be unique and the Radon-Nikodým derivative has the form

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \alpha_t\, d\tilde{W}_t - \frac{1}{2}\int_0^T \alpha_t^2\, dt \right),$$

where $\alpha_t = \frac{\tilde{\mu}_t}{\tilde{\sigma}_t} = \frac{\mu_t}{\sigma_t} + \gamma_t$ and $\mathbb{Q}$ must be (mutually) absolutely continuous with respect to $\mathbb{P}$. Hence one must have

$$\int_0^T \alpha_t^2\, dt < \infty, \tag{7.2.1}$$

otherwise $\frac{d\mathbb{Q}}{d\mathbb{P}} = 0$ and $\mathbb{Q}$ cannot be equivalent to $\mathbb{P}$.

Summarizing the above, the following test for the (NFLVR) condition is obtained:

Theorem 7.2.1. *S satisfy the (NFLVR) condition if and only if (7.2.1) holds.*

7.2.1 Example: Maximum value of Brownian motion

In this case $L = \sup_{t\in[0,T]} W_t$. The information drift $\gamma(t)$ is obtained in (4.2.9) as

$$\gamma_t = -\frac{f_{T-t}(W_t^* - W_t)}{\int_0^{W_t^* - W_t} f_{T-t}(y)\, dy} \mathbf{1}_{W_t^*}(x) - \mathbf{1}_{[W_t^*,\infty)}(x)\frac{x - W_t}{T-t}, \quad t \in [0,T],$$

where f_{T-t} denotes the density of the law of $\beta_{T-t} := \sup_{s\in[t,T]}(W_s - W_t)$ and $W_t^* := \sup_{s\in[0,t]} W_s$.

Imkeller et al. (2001) showed that on a set of positive probability

$$\int_0^T \alpha_t^2\, dt = \infty.$$

Hence the (NFLVR) is not satisfied. But where is the arbitrage opportunity?

Fix $0 < T \leq 1$, let τ be the random time at which the maximum $L = \sup_{t\in[0,1]} W_t$ is taken, i.e.

$$\tau = \{t \in [0,1] : W_t = L\}$$

τ has an absolutely continuous law on $[0, 1)$ and an obvious atom at $\{1\}$, since $\mathbb{P}[\tau = 1] = \mathbb{P}\left[\sup_{t\in[0,1)} W_t \leq \sup_{t\in[1,\infty)} W_t\right] > 0$. By (4.2.9),

$$\int_0^T \gamma_t^2(.,L)\,dt \geq \int_0^T \left(\frac{f_{1-t}(L-W_t)}{\int_0^{L-W_t} f_{1-t}(y)\,dy}\right)^2 dt \geq \int_\tau^T \left(\frac{f_{1-t}(L-W_t)}{\int_0^{L-W_t} f_{1-t}(y)\,dy}\right)^2 dt.$$

Let

$$\tau_0 = \inf\{t \geq \tau : W_t = 0\}.$$

i.e. τ_0 is the first hitting time of 0 after the process has hit its maximum. Then

$$\int_0^T \gamma_t^2(.,L)\,dt \geq \int_\tau^{\tau_0\wedge T} \left(\frac{f_{1-t}(L-W_t)}{\int_0^{L-W_t} f_{1-t}(y)\,dy}\right)^2 dt \qquad (7.2.2)$$

Then according to William's theorem (see Appendix), $(W_t)_{t\in[\tau,\tau_0]}$ is a BES^3 process. See the Appendix for a brief background on Bessel processes.

The significance of Bessel processes in the financial context is highlighted in Delbaen & Schachermayer (1995) which shows that if one allows general admissible integrands (as oppose to simple admissible integrands) as trading strategies, then BES^3 admits arbitrage possibilities.

Arbitrage strategies

So it has been confirmed that arbitrage opportunities exist if $L = \sup_{t\in[0,T]} W_t$. Define the return process as gains process

$$dR = \frac{dS}{S} = \mu_t dt + \sigma_t dW_t,$$

and the gains process as $G = \int \pi\, dR$, where π_t is the proportion of wealth invested in S at time t. Suppose that $T = 1$. One can construct arbitrage strategies following Imkeller et al. (2001) as demonstrated below:

Case where $\frac{\mu_t}{\sigma_t}$ bounded from below

For the case where the ratio $\frac{\mu_t}{\sigma_t}$ is bounded from below, i.e. there exists a constant $k > 0$ such that $\frac{\mu_t}{\sigma_t} \geq -k$, for all $t \in [0,1]$, $\mathbb{P}$-a.s. This can be interpreted as there being a limit on the amount that an honest investor would short if he is to follow an optimal strategy. (Recall that the optimal strategy for the honest investor is $\frac{\mu_t}{\sigma_t}$.)

Let

$$T_t = \exp\left(\int_0^t \frac{\mu_s}{\sigma_s}\,ds + W_t - \frac{1}{2}t\right), t \in [0,1].$$

This implies

$$\begin{aligned}\frac{dT_t}{T_t} &= \frac{\mu_t}{\sigma_t}dt + dW_t \\ &= \frac{dR_t}{\sigma_t}\end{aligned}$$

Assume the tame strategy is

$$\pi_t = \mathbf{1}_{[0,\tau]}(t)\frac{T_t}{\sigma_t}\mathbf{1}_{\{L>k+\frac{1}{2}\}}, t \in [0,1]. \tag{7.2.3}$$

Then the gains process

$$\begin{aligned}G_t &= \int_0^t \pi_s \, dR_s \\ &= \int_0^{t\wedge\tau} \frac{T_s}{\sigma_s}\mathbf{1}_{L>k+\frac{1}{2}}\sigma_s\frac{dT_s}{T_s} \\ &= \mathbf{1}_{L>k+\frac{1}{2}}(T_{t\wedge\tau} - T_0) \geq -1.\end{aligned}$$

Hence the maximum loss sustained at any point is the whole of capital. Furthermore,

$$\begin{aligned}G_1 &= \mathbf{1}_{L>k+\frac{1}{2}}\left[\exp\left(L + \int_0^\tau \frac{\mu_s}{\sigma_s}\,ds - \frac{1}{2}\tau\right) - 1\right] \\ &\geq \mathbf{1}_{L>k+\frac{1}{2}}\left[\exp\left(L - k - \frac{1}{2}\right) - 1\right] \\ &\geq 0.\end{aligned}$$

I.e. the terminal gain is non-negative and G_1 is strictly positive on the set $L > k + \frac{1}{2}$ which is a set with positive probability. Hence (7.2.3) is an arbitrage strategy.

The strategy in (7.2.3) involves holding the stock before it hits the maximum provided the maximum is greater than a certain level (related to k) and not holding anything (i.e. sell all holdings) after the stock price has reached the maximum. Almost akin to a buy low, sell high strategy. This is intuitively what one would do given the information. The advantage for the insider lies in the fact that the insider would be able to sell timeously as the stock reaches its maximum price.

Case where $\frac{\mu_t}{\sigma_t}$ bounded from above

A natural question to raise would be what about the case is where $\frac{\mu_t}{\sigma_t}$ is bounded from above, i.e. there exist $k > 0$ such that $\frac{\mu_t}{\sigma_t} \leq k,$ for all $t \in [0,1], \mathbb{P}$-a.s.?

Indeed, arbitrage opportunities exist provided the positive part of $\frac{\mu_t}{\sigma_t}$, i.e. $\left(\frac{\mu_t}{\sigma_t}\right)^+$ is well behaved in the sense that there exists $p > 2$ such that $\left(\frac{\mu_t}{\sigma_t}\right)^+ \in L^p([0,1])$. This includes the case where $\frac{\mu_t}{\sigma_t}$ is bounded from above.

Imkeller et al. (2001) points out that the *buy low, sell high* strategy employed in the case where $\frac{\mu_t}{\sigma_t}$ is bounded from below does not work in this case. A shorting strategy, as will be demonstrated, would be required.

If q is the conjugate exponent of p, then by Hölder's inequality

$$\int_\tau^t \left(\frac{\mu_s}{\sigma_s}\right)^+ ds \leq (t-\tau)^{\frac{1}{q}} \left(\int_\tau^t \left(\left(\frac{\mu_s}{\sigma_s}\right)^+\right)^p ds\right)^{\frac{1}{p}} < \infty. \tag{7.2.4}$$

Then given L the process $(\rho_t) = (L - W_{\tau+t})_{t\in[0,1-\tau]}$ is a BES^3 process. Furthermore, (ρ_t) has the same law as the process $(2W_t^* - W_t)_{t\in[0,1]}$ by Pitman's theorem (see Appendix). Then by the integrability properties of W_t^*, it can be shown that

$$\lim_{t\downarrow\tau} \frac{L - W_t}{(t-\tau)^{\frac{1}{q}}} = \infty, \quad \mathbb{P} - a.s. \text{ for } q < 2. \tag{7.2.5}$$

Combining (7.2.4) and (7.2.5), one can deduce that for a small time interval after τ,

$$\int_\tau^t \left(\frac{\mu_s}{\sigma_s}\right)^+ ds < L - W_t.$$

Hence

$$\varsigma = \inf\left\{t \geq \tau : \int_\tau^t \left(\frac{\mu_s}{\sigma_s}\right)^+ ds + \frac{1}{2}(t-\tau) > \frac{L - W_t}{2}\right\} \wedge 1$$

is a $\mathbb{G}$-stopping time that is strictly greater than τ if $\tau < 1$, $\mathbb{P}$-a.s.. Denote

$$T_t^* = \exp\left(-\int_0^t \frac{\mu_s}{\sigma_s} ds - W_t - \frac{1}{2}t\right), \quad t \in [0,1]. \tag{7.2.6}$$

Then the trading strategy defined by

$$\pi_t = -\mathbf{1}_{[\tau,\varsigma]}(t)\frac{T_t^*}{T_\tau^* \sigma_t}, \quad t \in [0,1], \tag{7.2.7}$$

is a tame strategy that leads to a gains process with strictly positive values at $t = 1$ on the set $\{\tau < 1\}$ which has a positive measure. Thus (7.2.7) is an arbitrage strategy.

The reason is that due to the existence of the Bessel process, the insider will not see W as a Brownian motion after it has reached τ, but instead will be able to observe (and anticipate) a downward drift in the stock price. The arbitrage strategy involves shorting the stock for a short period of time just after τ, which is consistent with intuition.

Remark 7.2.2. *A direct consequence of the two cases above is that there is arbitrage if $\frac{\mu}{\sigma}$ is continuous.*

7.2.2 Time information

The progressive enlargement of filtration techniques that can be employed to deal with time information were reviewed in Chapter 5. Recall from Section 5.2.2 that for an honest time L, the right continuous version of the supermartingale $\mathbb{P}(L > t|\mathscr{F}_t)$,$t \in [0,T]$ is denoted by Y^L. M^L denotes the martingale part of Y^L in its Doob-Meyer decomposition. Then the information drift according to (5.2.1) is:

$$\gamma_t = \mathbf{1}_{[0,L]}(s)\frac{\frac{d}{dt}\langle M^L, W\rangle_s}{Y_{s-}^L} - \mathbf{1}_{(L,1]}(s)\frac{\frac{d}{dt}\langle M^L, W\rangle_s}{1 - Y_{s-}^L}, \quad t \in [0,1]. \tag{7.2.8}$$

One can modify the condition of (NFLVR) in Theorem 7.2.1 with the following result:

Proposition 7.2.3. *[Proposition 1.1 of Imkeller (2002)] Suppose $t \in [0,T]$ and that*

$$\int_0^t \gamma_s^2 \, ds = \infty, \text{ on a set of positive measure.}$$

Then we also have

$$\int_0^t \alpha_s^2 \, ds = \infty, \text{ on a set of positive measure.}$$

Proof. Let $c_t = \frac{\mu_t}{\sigma_t}$ and therefore c_t is $\mathscr{F}_t$-adapted since μ_t and σ_t are. And suppose for simplicity $t = 1$. Also assume the opposite is true, i.e.

$$\int_0^1 \alpha_s^2 \, ds < \infty$$

$\mathbb{P}$-a.s.. Hence for $t \in [0,T]$

$$\begin{aligned} c_t &= \alpha_t - \gamma_t \\ |c_t| &\leq |\alpha_t| + |\gamma| \\ |c_t| &\leq \mathbb{E}(|\alpha_t||\mathscr{F}_t) + \mathbb{E}(|\gamma_t||\mathscr{F}_t) \end{aligned} \tag{7.2.9}$$

Now by the definition of γ in (7.2.8),

$$\mathbb{E}(|\gamma_t||\mathscr{F}_t) = \left| \frac{d}{dt} \langle M^L, W \rangle_t \right|,$$

and hence

$$\int_0^1 \left(\mathbb{E}(|\gamma_t||\mathscr{F}_t)\right)^2 \, dt \leq \langle M^L \rangle_1, \tag{7.2.10}$$

by the Kunita-Watanabe inequality. Since Y^L is bounded (it's a probability), $\langle M^L \rangle_1$ must be finite $\mathbb{P}$-a.s.. Hence the quantity on the left hand side of (7.2.10) must be finite, i.e.

$$\int_0^1 \left(\mathbb{E}(|\gamma_t||\mathscr{F}_t)\right)^2 \, dt < \infty. \tag{7.2.11}$$

Now with (7.2.9) and (7.2.11), by applying Jensen's inequality one obtains that $\int_0^1 c_t^2 \, dt < \infty$. And hence

$$\begin{aligned} \gamma_t &= \tilde{\mu}_t - c_t \\ (\gamma_t)^2 &\leq 4(\tilde{\mu}_t^2 + c_t^2) \\ \int_0^1 (\gamma_t)^2 \, dt &\leq 4 \left(\int_0^1 \tilde{\mu}_t^2 \, dt + \int_0^1 c_t^2 \, dt \right) < \infty, \mathbb{P}\text{-a.s..} \end{aligned}$$

which is a contradiction. □

Hence we have the following modified version of Theorem 7.2.1:

Theorem 7.2.4. *S satisfies the (NFLVR) condition if $\int_0^T \gamma_s^2 \, ds < \infty$.*

Remark 7.2.5. *Hence in order to establish whether arbitrage does exist, it is only necessary to establish the square integrability of γ.*

Example: Last crossing of a particular level by Brownian motion

In this case $L = \sup\{0 \leq t, W_t = K\}$. Recall from Proposition 5.2.6, the information drift is given by

$$\gamma_t = -\mathbf{1}_{[0,L]}(t)\frac{p_{T-t}(|W_t - K|)}{1 - F_{T-t}(|W_t - K|)}sgn(W_t - K) - \mathbf{1}_{]L,T]}(t)\frac{p_{1-t}(|W_t - K|)}{F_{1-t}(|W_t - K|)}sgn(W_t - K). \quad (7.2.12)$$

It can be shown that $\int_0^t \gamma_s^2\, ds = \infty$ on a set of positive measure. (See Proposition 2.3 of Imkeller (2002) for the case where $K = 0$.) Hence (NFLVR) is not satisfied.

Imkeller (2002) shows that if there exists $p > 2$ such that $\left(\frac{\mu}{\sigma}\right)^- \in L_p([0,1])$ (or if $\left(\frac{\mu}{\sigma}\right)$ is bounded below by a constant), then an arbitrage strategy can be constructed as follows for the case where $K = 0$:

If q is the conjugate exponent of p, then by Hölder's inequality

$$\int_L^t \left(\frac{\mu_s}{\sigma_s}\right)^- ds \leq (t-L)^{\frac{1}{q}} \left(\int_L^t \left(\left(\frac{\mu_s}{\sigma_s}\right)^-\right)^p ds\right)^{\frac{1}{p}} < \infty. \quad (7.2.13)$$

Let

$$A = \left\{L \leq \frac{1}{2}t, \tau < 1\right\},$$

where $\tau = \inf\{s \geq L, W_s = 1\}$. The process

$$\rho_s = W_{s+L}, \quad 0 \leq s \leq T_1 - L,$$

is a piece of a BES^3 by William's path decomposition (see Appendix). Using the same argument as in Section 7.2.1, it can be shown that on A

$$\lim_{t \downarrow L} \frac{W_t}{(t-L)^{\frac{1}{q}}} = \infty, \text{ for } q < 2. \quad (7.2.14)$$

Hence

$$\varsigma = \inf\left\{t \geq L : \int_L^t \left(\frac{\mu_s}{\sigma_s}\right)^- ds + \frac{1}{2}(t-L) > \frac{W_t}{2}\right\} \wedge 1$$

is a $\mathbb{G}$-stopping time that is strictly greater than L on A. If the investment strategy is set as

$$\pi_t = \mathbf{1}_{[L,\varsigma]}(t)\mathbf{1}_{\mathbb{R}_+}(W_t)\frac{T_t^*}{T_L \sigma_t}, t \in [0,1],$$

where T_t^* was given by (7.2.6).

Then for $t \in [0,1]$

$$G_t = \mathbf{1}_{\mathbb{R}_+}(W_\varsigma)\left[\exp\left(\int_L^{t\wedge\varsigma} \frac{\mu_s}{\sigma_s} ds + W_{t\wedge\varsigma} - \frac{1}{2}(t \wedge \varsigma - L)\right) - 1\right]$$

and

$$G_1 \geq \mathbf{1}_{\mathbb{R}_+}(W_\varsigma)\left[\exp\left(\frac{W_\varsigma}{2}\right) - 1\right]$$
$$\geq 0.$$

And G_1 is strictly positive on A.

Example: Time when the Brownian motion process attains the maximum

Recall from Chapter 5 that in this case $L = \tau$ is the time when W reaches its maximum in $[0, T]$. It can be shown that $\varsigma = \tau$, $\mathbb{P}$-a.s. where

$$\varsigma = \sup\{0 \leq t \leq T : W_t = W_t^*\}$$

is an honest time, hence so is τ.

Using the fact that $(W_t^* - W_t)_{t\geq 0}$ and $(|W_t|)_{t\geq 0}$ have identical laws, it was shown in Proposition 5.2.7 that for $t \in [0, T]$

$$\gamma_t = -\mathbf{1}_{[0,\tau]}(t)\frac{q_{T-t}(W_t^* - W_t)}{1 - G_{T-t}(W_t^* - W_t)} - \mathbf{1}_{(\tau,1]}(t)\frac{q_{T-t}(W_t^* - W_t)}{G_{T-t}(W_t^* - W_t)}, \tag{7.2.15}$$

where G_t and q_t denote the law and the density function of W_t^* respectively.

Using the fact that $\int_0^t \gamma_s^2\, ds = \infty$ on a set of positive measure for the case where the insider information is the time of the last crossing of a particular level by Brownian motion (i.e. in the previous example) and the equality of the laws of $(W_t^* - W_t)_{t\geq 0}$ and $(|W_t|)_{t\geq 0}$, one can immediately deduce that $\int_0^t \gamma_s^2\, ds = \infty$ on a set of positive measure also in this case. Hence (NFLVR) is also violated in this case.

Following the same methodology in Section 7.2.1, one can derive a trading strategy that leads to arbitrage. In fact, the same strategy as in (7.2.7) can be employed to derive arbitrage profit. Hence the arbitrage strategy is the same as in the case where the insider has the knowledge of the maximum value of the Brownian motion process (but doesn't know the time when it will attain the maximum). Intuitively, the insider will be able to see a downward drift in the short period after it has attained the maximum, the insider will then be able to exploit this by shorting the asset as it has reached the maximum.

Remark 7.2.6. *In both cases, it is impossible to effect a change to an equivalent martingale measure. This is due to the appearance of the 3-dimensional Bessel process (which has a drift) in the Brownian path after it has reached the random time, which the insider has knowledge of, by William's path decomposition (see Appendix).*

Note that the filtration in the case where the insider has information regarding the maximum price is essentially richer, since it is obtained by initial enlargement, i.e. the information is in $\mathscr{G}_0$, instead of being obtained by progressive enlargement as in this case. Whether one would be able to derive an arbitrage strategy that is capable of generating larger profits with the filtration incorporating the maximum price (as opposed to merely the timing of it) remains unclear.

7.2.3 Terminal information distorted by vanishing noise

So we have seen that insider information such as maximum value or the terminal value of the underlying Brownian motion process leads to infinite utility and arbitrage opportunities for the insider. But would it make a difference if this information is blurred by noise, i.e. if a degree of uncertainty is introduced for the insider?

The results obtained by Corcuera et al. (2004) regarding insider information distorted by noise that vanishes through time show that if the rate at which the additional noise in the insider's information vanishes is sufficiently slow then (NA) is still satisfied and the insider's additional utility is finite.

A brief review of the findings of Corcuera et al. (2004) is presented below.

Recall from Section 5.1 that L is parameterized with a time variable t representing information that evolves through time and have the following general formulation:

$$L_t = f(X, Y_t),$$

where X is $\mathscr{F}_T$-measurable, $Y = (Y_t)_{t\in[0,T]}$ is independent of $\mathscr{F}_T$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a given measurable function.

$\mathbb{G} = (\mathscr{G}_t)_{t\in[0,T]}$ is the "usual" augmentation of the filtration

$$(\mathscr{F}_t \vee \sigma(L_s, s \le t))_{t\in[0,T]}.$$

Y represents the (independent) noise that distort X from the insider, $Y_T = 0$ and the variance of Y should decrease to zero as $t \to T$.

Terminal value of Brownian motion with vanishing noise

Recall that in this case, the information drift is

$$\gamma_t = \frac{1}{g(T-t)}\mathbb{E}\left(\hat{W}_{g(T-t)} f'(W_T) \middle| W_t, L_t\right), \tag{7.2.16}$$

where $g : [0,T] \to [0,+\infty]$ be a strictly increasing bounded function with $g(0) = 0$ and $\hat{W}$ is a Brownian motion independent of W.

Now by the Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbb{E}((\gamma_t)^2) \le \frac{1}{g(T-t)^2}\mathbb{E}(\hat{W}^2_{g(T-t)} f'(W_T)^2) &= \frac{1}{g(T-t)^2}\mathbb{E}(\hat{W}^2_{g(T-t)})\mathbb{E}(f'(W_T)^2) \\ &= \frac{1}{g(T-t)}\mathbb{E}(f'(W_T)^2)\end{aligned}$$

Hence

$$\mathbb{E}\left(\int_0^T (\gamma_t)^2\, dt\right) \le \mathbb{E}(f'(W_T)^2)\int_0^T \frac{1}{g(t)}\, dt,$$

and therefore $\mathbb{E}\left(\int_0^T (\gamma_t)^2\, dt\right) < \infty$ if and only if $\int_0^T \frac{1}{g(t)}\, dt < \infty$. I.e. $g(t)$ must tends to 0 slow enough as t approaches 0, this means that the noise $Y_t = \hat{W}_{g(T-t)}$ must vanish sufficiently slow as well. It was pointed out in Corcuera et al. (2004) that the condition is satisfied if $g(s) = Ks^p$ with $0 < p < 1, K > 0$.

Maximum value of Brownian motion with vanishing noise

Recall that the information drift is given by

$$\gamma_t = \frac{1}{g(T-t)}\mathbb{E}(Y_t \mathbf{1}_{\{W^* > W_t^*\}} | \mathscr{F}_t \vee \sigma(L_t)). \tag{7.2.17}$$

information vanishes is sufficiently slow then (NA) is still satisfied and the insider's additional utility is finite.

A brief review of the findings of Corcuera et al. (2004) is presented below.

Recall from Section 5.1 that L is parameterized with a time variable t representing information that evolves through time and have the following general formulation:

$$L_t = f(X, Y_t),$$

where X is $\mathscr{F}_T$-measurable, $Y = (Y_t)_{t\in[0,T]}$ is independent of $\mathscr{F}_T$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a given measurable function.

$\mathbb{G} = (\mathscr{G}_t)_{t\in[0,T]}$ is the "usual" augmentation of the filtration

$$(\mathscr{F}_t \vee \sigma(L_s, s \leq t))_{t\in[0,T]}.$$

Y represents the (independent) noise that distort X from the insider, $Y_T = 0$ and the variance of Y should decrease to zero as $t \to T$.

Terminal value of Brownian motion with vanishing noise

Recall that in this case, the information drift is

$$\gamma_t = \frac{1}{g(T-t)}\mathbb{E}\left(\hat{W}_{g(T-t)} f'(W_T)\Big| W_t, L_t\right), \tag{7.2.16}$$

where $g : [0, T] \to [0, +\infty]$ be a strictly increasing bounded function with $g(0) = 0$ and $\hat{W}$ is a Brownian motion independent of W.

Now by the Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbb{E}((\gamma_t)^2) \leq \frac{1}{g(T-t)^2}\mathbb{E}(\hat{W}^2_{g(T-t)} f'(W_T)^2) &= \frac{1}{g(T-t)^2}\mathbb{E}(\hat{W}^2_{g(T-t)})\mathbb{E}(f'(W_T)^2) \\ &= \frac{1}{g(T-t)}\mathbb{E}(f'(W_T)^2)\end{aligned}$$

Hence

$$\mathbb{E}\left(\int_0^T (\gamma_t)^2\, dt\right) \leq \mathbb{E}(f'(W_T)^2)\int_0^T \frac{1}{g(t)}\, dt,$$

and therefore $\mathbb{E}\left(\int_0^T (\gamma_t)^2\, dt\right) < \infty$ if and only if $\int_0^T \frac{1}{g(t)}\, dt < \infty$. I.e. $g(t)$ must tends to 0 slow enough as t approaches 0, this means that the noise $Y_t = \hat{W}_{g(T-t)}$ must vanish sufficiently slow as well. It was pointed out in Corcuera et al. (2004) that the condition is satisfied if $g(s) = Ks^p$ with $0 < p < 1, K > 0$.

Maximum value of Brownian motion with vanishing noise

Recall that the information drift is given by

$$\gamma_t = \frac{1}{g(T-t)}\mathbb{E}(Y_t 1_{\{W^* > W^*_t\}}|\mathscr{F}_t \vee \sigma(L_t)). \tag{7.2.17}$$

Hence by applying the Cauchy-Schwarz inequality

$$\mathbb{E}\left(\int_0^T (\gamma_t)^2\,dt\right) \le \int_0^T \frac{1}{g(T-t)^2}\mathbb{E}(Y_t^2 1_{\{W^* > W_t^*\}})\,dt \le \int_0^T \frac{1}{g(T-t)^2}\mathbb{E}(Y_t^2)\,dt$$
$$= \int_0^T \frac{1}{g(T-t)}\,dt,$$

which is finite if $\int_0^T \frac{1}{g(t)}\,dt$ is. Again, this is the case if $g(s) = Ks^p, 0 < p < 1, K > 0$.

Remark 7.2.7. *For g satisfying such conditions it was shown in Corcuera et al. (2004) that* γ_t *also satisfies the Novikov condition:*

$$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^T (\gamma_t)^2\,dt\right)\right) < \infty.$$

This guarantees that

$$\mathscr{E}\left(\int_0^T \gamma_s\,dW_s\right)$$

is a martingale (by Theorem 5.16 of Hunt & Kennedy (2004)) and hence is a sufficient condition for the existence an equivalent martingale measure and hence there is no arbitrage.

Hence by applying the Cauchy-Schwarz inequality

$$\mathbb{E}\left(\int_0^T (\gamma_t)^2\,dt\right) \leq \int_0^T \frac{1}{g(T-t)^2}\mathbb{E}(Y_t^2 1_{\{W^* > W_t^*\}})\,dt \leq \int_0^T \frac{1}{g(T-t)^2}\mathbb{E}(Y_t^2)\,dt$$
$$= \int_0^T \frac{1}{g(T-t)}\,dt,$$

which is finite if $\int_0^T \frac{1}{g(t)}\,dt$ is. Again, this is the case if $g(s) = Ks^p, 0 < p < 1, K > 0$.

Remark 7.2.7. *For g satisfying such conditions it was shown in Corcuera et al. (2004) that γ_t also satisfies the Novikov condition:*

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This guarantees that

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is a martingale (by Theorem 5.16 of Hunt & Kennedy (2004)) and hence is a sufficient condition for the existence an equivalent martingale measure and hence there is no arbitrage.

Chapter 8

Conclusion

The basic mechanics of the mathematics of insider trading was reviewed in this dissertation. In particular, under the martingale theory paradigm, the *grossissement de filtrations* approach was described in detail. The power of Malliavin calculus techniques in incorporating a wider variety of information and its connection to the forward integral approach were also reviewed.

As was mentioned in the introduction, there are several research areas that were not covered in this dissertation. Furthermore, there are questions that current research has yet to answer. Most of these questions are related to how this theory of incorporating extra information is to fit within the general model of markets. For example, all the research concentrated on the mathematics of incorporating *theoretical information* such as the future value (or distribution) of the underlying Brownian motion process or the asset prices themselves. But how would a real insider translate the actual information he possesses into this type of theoretical information, when the type of information in his/her possession might be of the type - "the company is about to run out of liquidity". Obviously this is a finance topic, as oppose to a mathematical research problem, but it is nevertheless another crucial piece of the puzzle in the practical world.

Another example relates to the question of how would an insider practically price derivatives that is considered unattainable for the honest trader? How would these in turn affect related asset prices?

Furthermore, strides were made in the martingale theory paradigm since the 1990s. However, its focus is rather "long term", i.e. the effect of the information drift filters through to the asset prices is considered in a relatively large time frame. From the point of view of a day trader, any extra information in his possession would be "traded" within hours, if not minutes or seconds (processed information, rather than raw information is being referred to here). How would the results reconcile with those from the court of microstructure theory in economics? The dynamics of trading behaviour between that of a trader and a dealer (market maker) is essentially that of an insider and an honest trader. Except for the liquidity traders and passive traders (who trade on behalf of index or mutual funds), a trader would trade with a dealer simply because he believes that he possesses superior information about the asset and is trying to profit form this information at the expense of the dealer.

With the techniques that were covered in this monograph, the research effort has developed the tools necessary for further development of the theory's practical application, at least in the Brownian world. Certainly, this is only the beginning of more exciting researches on the practical aspect of the topic.

Appendix A

A.1 Bessel processes

The following results regarding Bessel processes were utilised in the main body of the dissertation. They are listed below for quick references. Most of the results were taken directly from Revuz & Yor (1991). Please refer to Revuz & Yor (1991) for the formal construction of a Bessel process and more details regarding the results below.

Definition A.1.1 ($BESQ^{\eta}$). *For every $\eta > 0$ and $x \geq 0$ the unique strong solution to the equation*

$$X_t = x + \eta t + 2\int_0^t \sqrt{X_s}\, dW_s$$

is called the square of a η-dimensional Bessel process started at x *and is denoted by* $BESQ^{\eta}(x)$.

X_t can be written as the square of the Euclidean norm of a η-dimensional Brownian motion, i.e. $X_t = \|W_t\|^2$. Then Bessel processes can be defined as follows:

Definition A.1.2 (BES^{η}). *The square root of $BESQ^{\eta}(x^2)$ where $\eta \geq 0$, $x \geq 0$ is called* the Bessel process of dimension η started at x *and is denoted by* $BES^{\eta}(x)$

For a 1-dimensional Bessel process X_t, one can show that $X_t = |W_t|$.

The 3-dimensional Bessel process is of particular interest in this dissertation due to its connection with a stopped Brownian motion process and arbitrage opportunities. The relationship between a 3-dimensional Bessel process starts at x, $BES^3(x)$, and a Brownian motion is given by the following:

Proposition A.1.3. *If (X_t) is a $BES^3(x)$ with $x \geq 0$, there is a Brownian motion W such that*

$$X_t = x + W_t + \int_0^t \frac{1}{X_s}\, ds.$$

Moreover, $\frac{1}{X_t}$ is a local martingale.

The following results are important for the analyses in the main body of the dissertation.

Theorem A.1.4 (Pitman's Theorem, Theorem 3.5, Chapter VI of Revuz & Yor (1991)). *The process $X_t = 2W_t^* - W_t$ is a $BES^3(0)$. Moreover, if X_t is a $BES^3(0)$ and $J_t = \int_{s \geq t} X_s\, ds$, then the processes $(2W_T^* - W_t, W_t^*)$ and (X_t, J_t) have the same law.*

Theorem A.1.5 (Williams' Brownian Path Decomposition, Theorem 4.9, Chapter VII of Revuz & Yor (1991)). *Fix $c > 0$ and the following four independent elements*

1. *a random variable δ uniformly distributed on $[0, c]$;*

2. *a Brownian motion W where $W_0 = c$;*

3. *two $BES^3(0)$ called X and $\tilde{X}$,*

and define

$$T_\delta = \inf\{t : W_t = \delta\},$$

$$g_{T_c} = T_\delta + \sup\{t : \delta - X_t = 0\},$$

$$T_c = g_{T_c} + \inf\left\{t : \tilde{X}_t = b\right\},$$

then the process defined by

$$X_t = \begin{cases} W_t, & \text{if } t < T_\delta; \\ \delta - X_{t-T_\delta}, & \text{if } T_\delta \le t \le g_{T_c}; \\ \tilde{X}_{t-g_{T-c}} & \text{if } g_{T_c} \le t \le T_c. \end{cases}$$

is a Brownian motion null at 0 killed when it first hits c

Remark A.1.6. *Therefore the theorem is stating that a Brownian motion process can be decomposed into three parts:*

1. *Before the Brownian motion hit δ for the first time, it remains a Brownian motion;*

2. *After the Brownian motion hit δ for the first time but before it hits 0 for the last time before it hits the level c, it can be seen as a (negative) Bessel process;*

3. *After the Brownian motion hits 0 for the last time before it hits the level c until the time when it hits the level c, it can be seen as a $BES^3(0)$.*

I.e. the effect of a stopping time of the type T_δ is that the Brownian motion can be seen as a BES^3 process after the it hits the stopping time.

A.2 Tanaka's formula

Theorem A.2.1 (Tanaka's formula, Theorem 1.2, Chapter VI of Revuz & Yor (1991)). *Let X be a continuous semimartingale. For any real number a, there exists an increasing continuous process L^a call the* local time of of X in *a such that,*

$$|X_t - a| = |X_0 - a| + \int_0^t sgn(X_s - a)\, dX_s + L_t^a,$$

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t \mathbf{1}_{(X_s > a)}\, dX_s + \frac{1}{2} L_t^a,$$

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t \mathbf{1}_{(X_s \le a)}\, dX_s + \frac{1}{2} L_t^a.$$

In particular, $|X - a|$,$(X - a)^+$ and $(X - a)^-$ are semimartingales.

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