LOCAL CONNECTEDNESS OF FRAMES

by

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To my parents
meme Ndashilwa na tate Kamati

Contents

Historical introduction

Local connectedness is one of the fundamental topological concepts. Historical facts confirm that this concept and the closely related local arcwise connectedness are older than the concept of a topological space. A short account of the history of these ideas in their classical setting may be found in [30].

One of the great stimulants in the study of local connectedness of frames is the classical result of Henriksen and Isbell[26] that the Stone-Čech compactification βX of a completely regular space X is locally connected iff X is locally connected and pseudocompact. In 1991, Baboolal and Banaschewski[1] showed that this result transfers to frames. Among other goals in the study of frame theory, the transfer of results from spaces to frames was bound to continue. In his article, *The point of pointless topology*, Johnstone[33] gives an informative historical survey of the subject. The next to be considered was the fact that any completely regular space Y containing X as a dense subspace is locally connected for locally connected and pseudocompact X ([26]). Again, Baboolal and Banaschewski[1] realized that under the assumption of the Boolean Ultrafilter Theorem (BUT), a frame counterpart of this result also holds. In 1991, through the study of congruences of frames, Chen[16] provided the frame counterpart, without assuming the BUT.

In 1996, Li[37] defined weak locale quotient morphisms and proved that local connectedness is preserved under such morphisms, a generalization of Whyburn[46]'s result that local connectedness is preserved under quotient maps of spaces. Banaschewski[14] introduced the notion of a fully complement closed subframe, and established that various familiar kinds of subframes are of this type. Moreover, he showed that any such subframe of a locally connected frame is locally connected. Baboolal[3] takes a uniform approach to local connectedness. He constructs a uniformly locally connected reflection for a locally connected uniform frame. One of its applications is that if a completely regular frame is locally connected and pseudocompact then every compactification of it is locally connected. This was then a strong temptation, so Baboolal's question whether the necessity part of the Baboolal-Banaschewski's local connectedness theorem can also be obtained using the same methods is considered here.

Synopsis

In this thesis, we undertake a systematic study of local connectedness of frames. Among other central ideas in this study is that of a connected congruence on a frame. We show that the two definitions of a connected congruence in literature (section 2.2) are not equivalent, and hence introduce a new term for one of them. We also prove that, using Baboolal's methods, if the Stone-Čech compactification βL is locally connected then L need not be locally connected for completely regular frame L. This happens in chapter 5.

Outline of the thesis

Chapter 0: This chapter contains definitions and some results which a reader is assumed to be familiar with.

Chapter 1: The notion of a connected frame is defined and basic results pertaining to it are proved. The dual concept, coconnectedness, is also discussed. Furthermore, a relatively new idea of relative local connectedness, which is due to Siweya[40] turns up.

Chapter 2: Closed homomorphisms and congruences are the crucial tools in this chapter. Two definitions of a connected congruence are presented, and it is established that they are not equivalent and hence a new term, Chen-connectedness, is introduced for the one due to Chen. The latter is characterized by a connected quotient frame L/Θ for a congruence Θ on a frame L. Then a specific look at open congruences and local connectedness occurs in the last section, where Li's work on weak locale quotient homomorphisms is presented differently.

Chapter 3: This chapter concerns compactification and local connectedness of frames and the relation between the two ideas. The compact (completely) regular coreflection of a frame obtained by Banaschewski and Mulvey[7] as a frame counterpart of the Stone-Čech compactification is briefly discussed. Most results in this chapter can be found in [1].

Chapter 4: This is devoted to fully complement closed subframes. They are useful because any fully complement closed subframe of a locally connected frame is again locally connected and because many familiar subframes such as open, closed and equalizer subframes are of this form.

Chapter 5: Here we finally turn our minds to uniform local connectedness. Through a close investigation of property S and property S^* among other concepts, the sufficiency part of the Baboolal-Banaschewski's local connectedness theorem is obtained differently. A proof that the necessity part of the theorem cannot be obtained in the same approach is also given here.

Chapter 0

Preliminaries

0.1 Introduction

In this chapter we provide definitions and some basic material that are necessary in order to read this thesis. For general introductions to frame theory, the reader is referred to [5], [32] or [43]. Further details and proofs of results regarding congruences on a frame can be found in [5], chapter 6.

0.2 Frames

A *complete lattice* is a partially ordered set with arbitrary meets and joins. A *frame* L is a complete lattice in which $x \wedge \bigvee x_i = \bigvee x \wedge x_i$ for all x and $\{x_i\}_I$ in L, where $\wedge$ and $\bigvee$ denote binary meet and arbitrary join respectively.
The bottom (zero) element is denoted by 0 and the top (unit) element by e. For an element $a \in L$, its *pseudocomplement* denoted by a^* is given by $a^* = \bigvee\{x \in L \mid x \wedge a = 0\}$.

For any topological space X, the lattice $\mathfrak{O}X$ of all open subsets of X is an example of a frame, where finite intersections and arbitrary unions correspond to finite joins and arbitrary unions respectively. A subset M of a frame L is called a *subframe* of L iff M is closed under finite meets and arbitrary joins in L. For frames M and L, a map $h : M \to L$ is a *frame homomorphism* iff h preserves finite meets and arbitrary joins (including the top and bottom elements). A frame homomorphism $h : M \to L$ is said to be *dense* iff $h(x) = 0$ implies $x = 0$ for all $x \in M$. Dually, h is called *codense* iff $h(x) = e$ implies $x = e$. Any frame homomorphism $h : M \to L$ has a *right adjoint*

$h_* : L \to M$, given by $h_*(y) = \bigvee\{x \in M \,|\, h(x) \leq y\}$. The category of frames and frame homomorphisms is denoted by **Frm**. Herrlich and Strecker's book *Category Theory*[28] provides the necessary background on category theory.

0.3 Frames and Topological spaces

We denote the category of topological spaces and continuous maps by **Top**. Remember that, for any topological space X, the lattice $\mathfrak{O}X$ is a frame. Also, any continuous map $f : X \to Y$ induces the frame homomorphism $\mathfrak{O}f : \mathfrak{O}Y \to \mathfrak{O}X$ defined by $\mathfrak{O}f(O) = f^{-1}(O)$. It is easily seen that this correspondence defines a functor $\mathfrak{O}:$ **Top** $\to$ **Frm**.

In the opposite direction we have a contravariant functor $\Sigma :$ **Frm** $\to$ **Top**. This works as follows: for each frame L, ΣL is the space of all frame homomorphisms $\xi : L \to \mathbf{2}$ with open sets $\Sigma_a = \{\xi \in \Sigma L \,|\, \xi(a) = 1\}$, and for each frame homomorphism $h : M \to L$, the associated map $\Sigma h : \Sigma L \to \Sigma M$ is given by $\Sigma h(\xi) = \xi \circ h$ for each $\xi \in \Sigma L$. The identity $(\Sigma h)^{-1}(\Sigma_a) = \Sigma_{h(a)}$ shows that Σh sends open sets to open sets hence is continuous. ΣL is called the *spectrum* of L, and Σ is called the *spectrum functor*. There exist at least two more ways of describing the spectrum of L, but here we shall use the above.

The functors Σ and $\mathfrak{O}$ are adjoint on the right ([5], proposition 3.2) with adjunctions $\eta_L : L \to \mathfrak{O}\Sigma L$ and $\varepsilon_X : X \to \Sigma\mathfrak{O}X$, where η_L is the frame homomorphism given by $\eta_L(a) = \Sigma_a$ and ε_X is the continuous map defined by $\varepsilon_X(x) = \tilde{x}$ where

$$\tilde{x}(O) = \begin{cases} 1 & if\; x \in O \\ 0 & if\; x \notin O. \end{cases}$$

A frame L is called *spatial* iff the homomorphism η_L is an isomorphism while a space X is said to be *sober* iff the continuous map ε_X is a homeomorphism. Then we have a well-established fact ([5], proposition 3.5) that the functors $\mathfrak{O}$ and Σ induce a dual equivalence between the full subcategories **SpFrm** and **Sob**, of spatial frames and sober spaces, of **Frm** and **Top** respectively.

0.4 Congruences, Nuclei and Quotients

A congruence Θ on a frame L is an equivalence relation which is a subframe of $L \times L$. The poset (under set inclusion) of all congruences on L, denoted by $\mathfrak{C}L$ is a frame with bottom $\Delta^L = \{(x,x) \mid x \in L\}$ and top $\nabla^L = L \times L$. In this congruence frame meet is intersection and join is given by the congruence generated by union of congruences on the frame L. This frame has been studied by many others (Johnstone[32], Frith[24], Banaschewski[5]).
For any frame homomorphism $h : M \to L$, its kernel defined by

$$\ker(h)=\{(x,y) \in M \times M \mid h(x) = h(y)\}$$

is a congruence ($\ker(h)=(h \times h)^{-1}[\Delta^L]$, and $(h \times h)^{-1}$ preserves equivalence relations and subalgebras) and any congruence is a kernel of a homomorphism.
A *nucleus* k on a frame L is a map $k : L \to L$ which satisfies the following conditions for any $x, y \in L$:

(i) $x \leq k(x)$
(ii) $x \leq y \Longrightarrow k(x) \leq k(y)$
(iii) $k(x \wedge y) = k(x) \wedge k(y)$
(iv) $k(k(x)) = k(x)$.

These maps feature in Banaschewski's[11] look at Tychonoff Theorem. Any nucleus k on L is associated with a frame $\mathrm{Fix}(k)=\{x \in L \mid k(x) = x\}$ and then an onto frame homomorphism $k : L \to \mathrm{Fix}(k)$. Also, any congruence $\Theta \in \mathfrak{C}L$ determines a nucleus k_Θ defined by $k_\Theta(a) = \bigvee\{x \in L \mid (a,x) \in \Theta\}$, and gives rise to a quotient frame L/Θ in a natural way. Furthermore, L/Θ is isomorphic to $\mathrm{Fix}(\Theta_k)$.
The following congruences are very important as will be seen in chapter 2. For any $a \in L$, $\nabla_a = \{(x,y) \mid x \vee a = y \vee a\}$ is the least congruence containing $(0,a)$. Any congruence of this form is said to be *closed*. The corresponding nucleus is $\sigma_a : L \to L$ defined by $\sigma_a(x) = x \vee a$, and the associated quotient map is $\tau_a : L \to \uparrow a$, given by $\tau_a(x) = a \vee x$. Nuclei and quotients of this form are also said to be *closed*. Then we also have the least congruence containing the pair (e,a), given by $\Delta_a = \{(x,y) \mid x \wedge a = y \wedge a\}$. The corresponding nucleus $\alpha_a : L \to L$ is defined by $\alpha_a(x) = a \wedge x$, and the associated quotient map $\lambda_a : L \to \downarrow a$ by $\lambda_a(x) = a \wedge x$. Congruences, nuclei and quotient maps of this form are said to be *open*.
For any $a, b \in L$, we let $\Theta_{ab} = \bigcap\{\Theta \mid (a,b) \in \Theta\}$, that is, the smallest congruence on L containing (a,b) called the *congruence generated by* (a,b). For

any congruence Θ on L we define $\Theta^+ = \{(a,b) \mid (a,b) \in \Theta,\ a \leq b\}$.

0.4.1 Lemma

For any $a, b \in L$,
(i) $\Theta_{ab} = \Delta_a \cap \nabla_b$ iff $a \leq b$
(ii) $\Delta_a = \Theta_{ae}$ and $\nabla_a = \Theta_{0a}$
(iii) $\Delta_a \wedge \nabla_a = \Delta^L$ and $\Delta_a \vee \nabla_a = \nabla^L$, that is, Δ_a and ∇_a are complementary.

As a consequence of the above, we have the following fact:
For any congruence Θ on L, $\Theta = \bigvee\{\Delta_a \cap \nabla_b \mid (a,b) \in \Theta^+\}$, that is, any congruence can be expressed in terms of these complements.

0.4.2 Proposition

(i) The map $\nabla : L \to \mathfrak{C}L$ defined by $\nabla(a) = \nabla_a$ for any $a \in L$ is a partial order embedding which preserves $\wedge$ and $\bigvee$. In fact ∇ is a frame homomorphism.

(ii) The map $\Delta : L \to \mathfrak{C}L$ defined by $\Delta(a) = \Delta_a$ for any $a \in L$ is a dual partial order embedding which takes $\wedge$ to $\vee$ and $\bigvee$ to $\bigwedge$.

(iii) The correspondence $L \rightsquigarrow \mathfrak{C}L$ is functorial such that, for any $h : M \to L$, the frame homomorphism $\mathfrak{C}h : \mathfrak{C}M \to \mathfrak{C}L$ takes a congruence on L to a congruence generated by the image of $h \times h$. Also, $\mathfrak{C}h(\Phi) \leq \Theta$ iff $\Phi \leq (h \times h)^{-1}(\Theta)$ iff $(h \times h)(\Phi) \leq \Theta$, meaning $(h \times h)^{-1}$ is the right adjoint of $\mathfrak{C}h$.

Now recall that a frame is *Boolean* if each of its elements has a complement. Then we have the following characterization:

0.4.3 Lemma

The frame homomorphism $\nabla : L \to \mathfrak{C}L$ is an isomorphism iff L is Boolean.

Proof. ($\Rightarrow$) Suppose ∇ is an isomorphism, and for any $a \in L$ consider the congruence Δ_a. Then there exists $s \in L$ such that $\nabla_s = \Delta_a$. Since $(a,e) \in \Delta_a$ then $(a,e) \in \nabla_s$ which means $a \vee s = s \vee e = e$, that is, $a \vee c = e$. Also, $(0,s) \in \nabla_s$ implies $(0,s) \in \Delta_a$ so that $s \wedge a = 0 \wedge a = 0$. This implies that s is a complement of a. Hence L is Boolean.

($\Leftarrow$) Suppose L is Boolean, and consider any $\Theta \in \mathfrak{C}L$. Then by the remark just before the above proposition, $\Theta = \bigvee\{\Delta_a \cap \nabla_b | (a,b) \in \Theta^+\}$. Since L is Boolean then a has a complement a^c such that $\Delta_a = \nabla_{a^c}$. Hence $\Theta = \bigvee\{\nabla_{a^c \wedge b} \,|\, (a,b) \in \Theta^+\} = \nabla_{\vee a^c \wedge b} = \nabla_t$ where $t = \bigvee\{a^c \wedge b \,|\, (a,b) \in \Theta^+\}$. Thus ∇ is onto. Being a partial order embedding, ∇ is one-one and therefore an isomorphism. ☐

0.4.4 Lemma

For any $a \in L$,
(i) $\Delta_a \vee \Theta = \{(x,y) \,|\, (x \wedge a, y \wedge a) \in \Theta\}$
(ii) $\nabla_a \vee \Theta = \{(x,y) \,|\, (x \vee a, y \vee a) \in \Theta\}$.

This leads to the following:

0.4.5 Lemma

(i) $\Delta_a \vee \bigcap \Theta_i = \bigcap(\Delta_a \vee \Theta_i)$
(ii) $\nabla_a \vee \bigcap \Theta_i = \bigcap(\nabla_a \vee \Theta_i)$.

Proof. (i) $$\begin{aligned}\bigcap(\Delta_a \vee \Theta_i) &= \bigcap\{(x,y) \,|\, (x \wedge a, y \wedge a) \in \Theta_i\} \\ &= \{(x,y) \,|\, (x \wedge a, y \wedge a) \in \bigcap \Theta_i\} \\ &= \Delta_a \vee \bigcap \Theta_i\end{aligned}$$
(ii) similar to the above. ☐

The following familiar construction shall be employed in chapter two.

0.4.6 Lemma

Let L and K be frames, and $h : L \to K$ be an onto frame homomorphism. Then the map $\bar{h} : L/Ker(h) \to K$ given by $\bar{h}([x]) = h(x)$ for all $[x] \in L/Ker(h)$ is well defined, that is, $[x] = [y] \Longrightarrow \bar{h}([x]) = \bar{h}([y])$. Moreover $\bar{h}$ is an isomorphism. Furthermore, if q denotes the quotient map, then Ker(q)=Ker(h) and $\bar{h} \circ q = h$. ☐

Chapter 1

Basic concepts of connectedness

1.1 Introduction to connectedness and local connectedness

We study the notion of connectedness and, indeed of particular interest, local connectedness of a frame. These concepts are defined and accompanied by some basic results. Křiž and Pultr[36] study some aspects of connectedness in coproducts of frames. Our terminology and notation is that of Baboolal and Banaschewski[1].

1.1.1 Definition

(i) For any frame L, $c \in L$ is called *connected* iff $c = a \vee b$ and $a \wedge b = 0$ implies that $c = a$ or $c = b$ for any $a, b \in L$.

(ii) A frame L is called *connected* iff its top element e is connected.

For a better appreciation of the idea, we look at:

1.1.2 Examples

(i) Any complete chain (totally ordered set) is a connected frame.

(ii) The frame of all open subsets of any connected topological space X is connected. Moreover, X is connected as a topological space iff $\mathfrak{O}X$ is connected as a frame.

(iii) The four-element Boolean frame **4** is not connected.

Remark. The bottom element of any frame is always connected.

1.1.3 Lemma

In any frame L, an element a is connected iff the top and the bottom of $\downarrow a$ are its only complemented elements.

Proof.($\Rightarrow$) Suppose $a \in L$ is connected and let u be any complemented element of $\downarrow a$. Then $u \vee u^c = a$ and $u \wedge u^c = 0$. Since a is connected it follows that either $u = a$ or $u = 0$.
($\Leftarrow$) Given that a and 0 are the only complemented elements of $\downarrow a$, let $a = u \vee v$ where $u \wedge v = 0$. Then by assumption, $u = a(v = 0)$ or $u = 0(v = a)$. Hence a is connected as asserted. ☐

As an immediate consequence of the definitions of a connected element and a connected frame, we have:

1.1.4 Lemma

In any frame L, a is connected iff the frame $\downarrow a$ is connected. ☐

1.1.5 Definition

(i) A frame L is called *locally connected* iff each of its elements is a join of connected elements.

(ii) For any frame L and $a \in L$, a *(connected)component* of a is a maximal connected $c \leq a$.

(iii) A subset S of a frame L is called *chained* iff, for any $a, b \in S$ there exist $t_0, t_1, \ldots, t_n \in S$ such that $a = t_0, t_k \wedge t_{k+1} \neq 0$ for all $k = 0, 1, \ldots, n-1$, and $b = t_n$.

Remark. In general, local connectedness does not imply connectedness. Consider the four-element Boolean frame **4**. It is locally connected but not connected. In fact, it is known that none of the two concepts imply the other. See Willard[48], p.198 for a counterexample in a space setting.

1.1.6 Lemma

In any frame L, the join of any chained set S of connected elements is connected.

Proof. Let $c = \bigvee S$ and consider any $a, b \in L$ such that $c = a \vee b$ and $a \wedge b = 0$. Then, for any $t \in S, t \leq a$ or $t \leq b$ since t is connected. Now, suppose there exist $u, v \in S$ with $u \leq a$ and $v \leq b$. Then by assumption, there exists $t_0, t_1, \ldots, t_n \in S$ for which $u = t_0$, $t_k \wedge t_{k+1} \neq 0$ for all $k = 0, 1, \ldots, n-1$, and $v = t_n$. Here if i is the first index such that $t_i \leq b$ then $0 < i$ and $0 < t_{i-1} \wedge t_i \leq a \wedge b = 0$, a contradiction. It follows that $t \leq a$ for all $t \in S$ or $t \leq b$ for all $t \in S$, and therefore $c=a$ or $c=b$, as required. □

Remark. In **Top**, connected components of elements of a space form a partition. In particular, for any two distinct elements of a given space the corresponding connected components are either the same or disjoint. This also motivates the following:

1.1.7 Corollary

In any locally connected frame L, different components of any element of L are disjoint.

Proof. Let b and c be components of any $a \in L$. If $b \wedge c \neq 0$ then $b \vee c$ is connected by Lemma 1.1.6 and hence, by maximality of b and c, $b = b \vee c = c$. □

1.1.8 Corollary

In any locally connected frame L, each element is the join of its components.

Proof. For any $a \in L$, consider the set $C = \{z \in L \mid 0 \neq z \leq a, z \text{ connected}\}$ and put

$$x \sim y \text{ iff there exists a finite sequence } z_0, z_1, \ldots, z_n \text{ in } C \text{ such that } x = z_0, z_k \wedge z_{k+1} \neq 0 \text{ for all } k = 0, 1, \ldots, n-1 \text{ and } y = z_n.$$

It is painless to check that this defines an equivalence relation on C.
Let B be an equivalence class of C. Then $B = [z] = \{y \in C \mid y \sim z\,,\ z \in C\}$. For any $y, y' \in B$, $y \sim y'$ hence $B \subseteq C$ is chained, and $\bigvee B \in C$. For any

such equivalence class B, let $x \in C$ and $y \in B$ such that $x \wedge y \neq 0$. Then $x \sim y$ and hence $x \in B$. Since $y \leq a$ and connected for each $y \in B$, now $\bigvee B \leq a$ and connected hence maximal in C. Thus $\bigvee B$ is a component of a. Since L is locally connected, $a = \bigvee C$ and therefore a is a join of its components. ☐

The following lemma gives a characterization of connectedness of frames in a corresponding manner to the result that a topological space X is connected iff each continuous function from X into a discrete space is constant. Here, the relevant comparison frame is the Boolean algebra **4**.

1.1.9 Lemma

A frame L is connected iff each homomorphism $\mathbf{4} \to L$ factors through the unique homomorphism $\mathbf{2} \to L$.

Proof.($\Rightarrow$) Let $a, b \in \mathbf{4}$ such that $a, b \neq 0$ or e. Then for any $h : \mathbf{4} \to L$, $h(a) \vee h(b) = e$ and $h(a) \wedge h(b) = 0$, hence $h(a){=}e$ or $h(b){=}e$ so that defining $\bar{h} : \mathbf{4} \to \mathbf{2}$ by $\bar{h}(a) = 1$ or $\bar{h}(a) = 0$, gives the desired factorization.
($\Leftarrow$) If $u \vee v = e$ and $u \wedge v = 0$ we obtain $h : \mathbf{4} \to L$ by letting $h(a) = u$ and $h(b) = v$, and since this factors through $\mathbf{2} \to L$ we must have $u = e$ or $v = e$ so that L is connected. ☐

Though simple, the next lemma is quite useful in this study, and its utility will be exhibited in several important subsequent results.

1.1.10 Lemma

Any dense frame homomorphism $h : M \to L$ reflects connected elements.

Proof. Let $c \in M$ with $h(c)$ connected. If $c = a \vee b$ and $a \wedge b = 0$ then $h(c) = h(a) \vee h(b)$ and $h(a) \wedge h(b) = 0$, hence $h(c) = h(a)$ or $h(c) = h(b)$, that is, $h(a) = 0$ or $h(b) = 0$. Since h is dense, $a = 0$ or $b = 0$ so that $c{=}a$ or $c{=}b$. Therefore c is connected. ☐

Remark. It is then evident that any dense frame homomorphism with a connected codomain has a connected domain. However, not every homomorphism that reflects connected elements is dense:

Counterexample. Define $h : \mathbf{3} \to \mathbf{2}$ by $h(x) = e$ if $x = e$ or $h(x) = 0$ if $x = a$ or $x = 0$. Then h is a connectedness-reflecting frame homomorphism which is not dense ($h(a) = 0$ but $a \neq 0$).

1.1.11 Lemma

Any dense onto homomorphism $h : M \to L$ whose right adjoint preserves disjoint binary joins preserves connected elements.

Proof. Given any connected $c \in M$, let $h(c) = a \vee b$ and $a \wedge b = 0$. Then by hypothesis $h_*h(c) = h_*(a) \vee h_*(b)$ and $h_*(a) \wedge h_*(b) = h_*(0) = 0$. Now since $c \leq h_*h(c)$ we have $c = (c \wedge h_*(a)) \vee (c \wedge h_*(b))$ and hence $c = c \wedge h_*(a)$ or $c = c \wedge h_*(b)$. Then it follows that $h(c) = a$ or $h(c) = b$. ☐

We now turn to the dual notion.

1.2 Coconnectedness

In this section, we define coconnectedness and prove some basic facts concerning it. Some of the facts obtained here will be used in chapter 4 to establish results on complement closed subframes.

1.2.1 Definition

In any complete lattice L, an element a is called *coconnected* iff $a = b \wedge c$ and $b \vee c = e$ implies $a = b$ or $a = c$.

Remark. In any frame, an element a is coconnected iff $\uparrow a$ is connected.

By dualizing Lemma 1.1.3, we obtain the following:

1.2.2 Lemma

In any frame L, an element a is coconnected iff the top and the bottom of $\uparrow a$ are its only complemented elements. ☐

1.2.3 Lemma

In any frame L, the pseudocomplement of any connected element is coconnected.

Proof. Suppose $a \in L$ is connected and let a^* be its pseudocomplement. Consider the frame homomorphism $h : \uparrow a^* \rightarrow \downarrow a$ given by $h(x) = x \wedge a$ for all $x \geq a^*$. If $h(x) = 0$, that is, $x \wedge a = 0$ then by definition of a^* we have $x \leq a^*$. Already $x \geq a^*$, thus $x = a^*$ which implies that h is dense. Since $a = h(e)$ is connected then by Lemma 1.1.10 e is connected, thus $\uparrow a^*$ is connected and therefore a^* is coconnected. ☐

Motivated by the above lemma, we also have:

1.2.4 Lemma

In any frame L, the pseudocomplement of a coconnected element is connected. ☐

Next, we see that more is true:

1.2.5 Lemma

The complement of any complemented coconnected element of a frame is connected.

Proof. Let a be a complemented coconnected element of a frame L. We show that a^c is connected. Consider the frame homomorphism $h : \uparrow a \rightarrow \downarrow a^c$ defined by $h(x) = x \wedge a^c$ for all $x \in \uparrow a$. It is readily seen that h is one-one. For h onto: Take any $u \in \downarrow a^c$ and put $y = u \vee a$. Then $y \in \uparrow a$ and $u = u \vee 0 = (u \wedge a^c) \vee (a \wedge a^c) = (u \vee a) \wedge a^c = y \wedge a^c = h(y)$. In all, h is an isomorphism. Since a is coconnected, $\uparrow a (\cong \downarrow a^c)$ is connected and therefore a^c is connected. ☐

Remark. It should also be noted that the complement of any complemented connected element of a frame is coconnected (this is simply because the frame homomorphism $h : \downarrow a \rightarrow \uparrow a^c$ given by $h(x) = x \vee a^c$ is an isomorphism).

1.2.6 Lemma

The right adjoint of any frame homomorphism preserves coconnected elements.

Proof. Let $h : M \to L$ be a frame homomorphism and $a \in L$ a coconnected element. Then define $g : \uparrow h_*(a) \to \uparrow a$ by $g(x) = a \vee h(x)$ for each $x \geq h_*(a)$. That g is a frame homomorphism rests on the fact that h is.
We show that g is dense: Suppose $g(x) = a$ for some $x \geq h_*(a)$. Then $a \vee h(x) = a$ which implies $h(x) \leq a$, and thus, by the properties of h_*, we have $x \leq h_* h(x) \leq h_*(a)$. Hence $x = h_*(a)$, and therefore g is dense. Now, since $\uparrow a$ is connected, then, by the density of g, $\uparrow h_*(a)$ is also connected. Therefore $h_*(a)$ is coconnected. □

1.2.7 Definition

(i) For any subset P of a frame L, a and b are called *cochained* in P iff there exists $t_1, t_2, \ldots, t_n$ in P such that $t_1 = a$, $t_i \vee t_{i+1} \neq e$ for all $i = 1, 2, \ldots, n-1$ and $t_n = b$.

(ii) P is called *cochained* iff any two of its elements are cochained.

Note that being cochained is an equivalence relation.

(iii) The corresponding equivalence classes are called *cochain components.*

1.2.8 Lemma

The meet of any cochained set of coconnected elements of a frame is coconnected.

Proof. Let P be a cochained set of coconnected elements of a frame L, and put $a = \bigwedge P$. Take any $b, c \in L$ with $b \wedge c = a$ and $b \vee c = e$. Then $t \geq a$ for any $t \in P$. Since $t \in P$, t is coconnected so $b \leq t$ or $c \leq t$. Now suppose there exists $u, v \in P$ such that $b \leq u$ and $c \leq v$. Since P is cochained, u and v are cochained in P, so there exists $t_0, t_1, \ldots, t_n \in S$ such that $u = t_0$, $t_k \vee t_{k+1} \neq e$ for all $k = 0, 1, \ldots, n-1$ and $v = t_n$. So we have $b \leq t_0$ and $c \leq t_n$. Let i be the first index such that $c \leq t_i$. Then $i > 0$ and $e > t_{i-1} \vee t_i \geq b \vee c = e$, which says $e > e$, a contradiction. Then $t \geq b$ for all $t \in P$ or $t \geq c$ for all $t \in P$. Since $a = \bigwedge P$, it follows that $a = b$ or $a = c$. Therefore a is coconnected. □

1.3 Relative local connectedness

A topological space X is called locally connected in another topological space Y iff there is a basis of Y such that $O \cap X$ is connected for every basic open set O. In his paper [23], Fox encompasses these spaces. Here we briefly study the notion of relative local connectedness of a frame which was first introduced by Siweya[40] (he calls it "local connectedness with respect to along") as a point-free version of that of Fox.

1.3.1 Definition

Let $h : M \to L$ be a frame homomorphism. Then L is said to be *locally connected relative to* h iff there is a basis B of M such that $h(b)$ is connected for each $b \in B$.

1.3.2 Lemma

Any frame which is locally connected relative to an onto frame homomorphism is locally connected.

Proof. Let $h : M \to L$ be an onto frame homomorphism with L locally connected relative to h. Take any $a \in L$. Then $a = h(x)$ for some $x \in M$. Also, $x = \bigvee_{b \in B} b$ where B is the basis of M for which $h(b)$ is connected for all $b \in B$. Hence $a = h(\bigvee_{b \in B} b) = \bigvee_{b \in B} h(b)$ with $h(b)$ connected since L is locally connected relative to h. Therefore L is locally connected. □

1.3.3 Lemma

Any domain of a dense frame homomorphism whose codomain is locally connected relative to the homomorphism is locally connected.

Proof. Let $h : M \to L$ be a dense frame homomorphism with L locally connected relative to h. Consider any $x \in M$. Then $x = \bigvee_{b \in B} b$ for the given basis B of M. Since L is locally connected relative to $h, h(b)$ is connected. Being dense, h reflects connectedness and hence b is connected. Therefore M is locally connected. □

1.3.4 Proposition

Let $h : M \to L$ *be a dense, onto frame homomorphism and* L *be locally connected relative to* h. *Then* h *preserves connected elements.*

Proof. Let $h : M \to L$ be a dense frame homomorphism with L locally connected relative to h, and $a \in M$ a connected element. We show that $h(a)$ is connected. Suppose $h(a) = x \vee y$ and $x \wedge y = 0$. Then there is a basis B of M such that $a = \bigvee_{b \in B} b$ with $h(b)$ connected. Since h is dense, by Lemma 1.1.10, it reflects connected elements so b is connected. Then we have $h(a) = x \vee y = \bigvee_{b \in B} h(b)$ which implies $h(b) \leq x \vee y$. Since $x \wedge y = 0$ and $h(b)$ is connected, we deduce that $h(b) \leq x$ or $h(b) \leq y$ hence $b \leq h_*(x)$ or $b \leq h_*(y)$. Taking joins over all such b yields $a = \bigvee_{b \in B} b \leq h_*(x) \vee h_*(y)$ and because h_* preserves meets $h_*(x) \wedge h_*(y) = 0$. Since a is connected, $a \leq h_*(x)$ or $a \leq h_*(y)$. Suppose $a \leq h_*(x)$. Then $h(a) \leq x$ and we already have $x \leq h(a)$ thus $h(a) = x$, and $y = y \wedge (x \vee y) = y \wedge h(a) = y \wedge x = 0$. Hence $h(a)$ is connected. ☐

Remark. The above result is a partial converse of Lemma 1.1.10.

Under the same hypothesis the following result was proved in [40].

1.3.5 Theorem

(i) h_* preserves disjoint binary joins.
(ii) h_* preserves and reflects connected components. ☐

By combining Lemmas 1.3.2, 1.3.3 and Theorem 1.3.5 we get the following:

1.3.6 Proposition

Let $h : M \to L$ *be a dense, onto frame homomorphism. Then* L *is locally connected relative to* h *iff* M *and* L *are locally connected and the right adjoint of* h *preserves disjoint binary joins.* ☐

Compactifications will be discussed in chapter 3, but for now recall that a compactification $h : M \to L$ of a frame L, with the right adjoint h_*, is *perfect with respect to* $a \in L$ if $h_*(a \vee a^*) = h_*(a) \vee h_*(a^*)$, and (M, h) is a *perfect* compactification of L if it is perfect with respect to every element of L ([2]).

As an example of the above proposition, a compactification of a frame is relatively locally connected iff it is perfect and its domain and codomain are locally connected.

We observe that the equivalence in the above proposition still holds when local connectedness of the codomain is dispensed.

1.3.7 Proposition

Let $h : M \to L$ be a dense onto frame homomorphism. Then L is locally connected relative to h iff M is locally connected and the right adjoint of h preserves disjoint binary joins. $\square$

The above characterization of a relatively locally connected frame can be expressed without mentioning the right adjoint of the homomorphism as follows:

1.3.8 Theorem

Let $h : M \to L$ be a dense, onto frame homomorphism. Then L is locally connected relative to h iff M is locally connected and h preserves connected elements.

Proof. ($\Rightarrow$) By Lemma 1.1.11 and Theorem 1.3.5.

($\Leftarrow$) Suppose M is locally connected and h preserves connected elements. According to the above proposition it suffices to show that $h_*(x \vee y) = h_*(x) \vee h_*(y)$ for any $x, y \in L$ with $x \wedge y = 0$. Always $h_*(x) \vee h_*(y) \leq h_*(x \vee y)$. Since M is locally connected, we can write $h_*(x \vee y) = \bigvee c$ for connected elements $c \in M$. Then $hh_*(x \vee y) = \bigvee h(c)$ so that $x \vee y = \bigvee h(c)$, a join of connected elements.

Let d be one of the the c's above. Then $h(d) = (h(d) \wedge x) \vee (h(d) \wedge y)$ and $(h(d) \wedge x) \wedge (h(d) \wedge y) = 0$, which implies $h(d) = h(d) \wedge x$ or $h(d) = h(d) \wedge y$ $\Longleftrightarrow h(d) \leq x$ or $h(d) \leq y \Longleftrightarrow d \leq h_*(x)$ or $d \leq h_*(y) \Longrightarrow d \leq h_*(x) \vee h_*(y)$. Thus $h_*(x \vee y) \leq h_*(x) \vee h_*(y)$ and hence $h_*(x \vee y) = h_*(x) \vee h_*(y)$. Therefore L is locally connected relative to h. $\square$

We close this section by mentioning that, as Siweya[40] remarks, relative local connectedness is also useful in the construction of uniform spread completions studied in [41].

Chapter 2

Closed homomorphisms and Chen-connected congruences

2.1 Introduction

In this chapter we study closed homomorphisms and Chen-connected congruences. As frame counterparts of closed continuous functions between spaces, closed frame homomorphisms turn up in works of Dowker-Papert[21], Pultr-Tozzi[39] and Chen[16]. In[18], Chen studies a stronger version of these, namely stably closed homomorphisms. Here, we define and characterize closed homomorphisms in terms of closed congruences and look at some of their general properties. In addition, we study basic properties of congruences needed later in the chapter. Moreover, we show that connectedness of a congruence as an element of a congruence frame is not equivalent to connectedness of a congruence in the sense of Chen [16, 17]. Thus, we introduce the term Chen-connected for the latter and investigate the idea. The emphasis in the last section lies on open congruences and weak locale quotient morphisms, where Li[37]'s work is presented in a slightly different way.

2.1.1 Definition

A frame homomorphism $h : M \to L$ is *closed* iff $h_*(h(x) \vee z) = x \vee h_*(z)$ for any $x \in M, z \in L$.

Remark. In the above definition, if h is the inclusion map then M is called a *closed* subframe of L.

2.1.2 Lemma

A frame homomorphism $h : M \to L$ *is closed iff* $h(x) \leq h(y) \vee z$ *implies* $x \leq y \vee h_*(z)$ *for* $x, y \in M$ *and* $z \in L$.

Proof.(⇒) Suppose h is closed and $h(x) \leq h(y) \vee z$ for $x, y \in M$ and $z \in L$. Then $x \leq h_*h(x) \leq h_*(h(y) \vee z) = y \vee h_*(z)$, that is, $x \leq y \vee h_*(z)$.

(⇐) Suppose that $h(x) \leq h(y) \vee z$ implies $x \leq y \vee h_*(z)$ for $x, y \in M$ and $z \in L$. Then, for any $u \in M$ with $u \leq x \vee h_*(z)$, we get $h(u) \leq hh_*(z) \vee h(x)$ hence $u \leq h_*h(u) \leq h_*(h(x) \vee hh_*(z))$. Since h_* preserves order and $hh_*(z) \leq z$, then $u \leq h_*(h(x) \vee hh_*(z)) \leq h_*(h(x) \vee z)$. Thus $x \vee h_*(z) \leq h_*(h(x) \vee z)$. Conversely, if $h(x) \leq h(y) \vee z$ implies $x \leq y \vee h_*(z)$ for $x, y \in M$ and $z \in L$ then the set $\{a \in M \,|\, h(a) \leq h(x) \vee z\} \subseteq \{a \in M \,|\, a \leq h_*(z) \vee x\}$. Thus $\bigvee\{a \in M \,|\, h(a) \leq h(x) \vee z\} \leq \bigvee\{a \in M \,|\, a \leq h_*(z) \vee x\} \leq h_*(z) \vee x$, that is $h_*(h(x) \vee z) \leq h_*(z) \vee x$. Therefore $h_*(h(x) \vee z) = h_*(z) \vee x$, meaning h is closed. □

2.1.3 Theorem

A frame homomorphism $h : M \to L$ *is closed iff* $(h \times h)^{-1}(\nabla_a) = \nabla_{h_*(a)}$ *for each* $\nabla_a \in \mathfrak{C}L$.

Proof. First we show that it is generally true that $(h \times h)^{-1}(\nabla_a) \geq \nabla_{h_*(a)}$. For any $(x, y) \in \nabla_{h_*(a)}$, we have $x \vee h_*(a) = y \vee h_*(a)$ hence $h(x) \vee a = h(y) \vee a$ which implies $(x, y) \in (h \times h)^{-1}(\nabla_a)$. Thus $\nabla_{h_*(a)} \leq (h \times h)^{-1}(\nabla_a)$.

(⇒) Suppose h is closed and let $(x, y) \in (h \times h)^{-1}(\nabla_a)$. Then $h(x) \vee a = h(y) \vee a$, and hence, by the closedness of h, $x \vee h_*(a) = y \vee h_*(a)$, which implies that $(x,\, y) \in \nabla_{h_*(a)}$. Hence $(h \times h)^{-1}(\nabla_a) = \nabla_{h_*(a)}$ for each $\nabla_a \in \mathfrak{C}L$.

(⇐) Suppose $(h \times h)^{-1}(\nabla_a) = \nabla_{h_*(a)}$ for each $\nabla_a \in \mathfrak{C}L$. We show that h is closed. Since $h(h_*(h(x) \vee y)) \vee y = h(x) \vee y$ for any $x \in M, y \in L$, we have $(h_*(h(x) \vee y),\, x) \in (h \times h)^{-1}(\nabla_y) = \nabla_{h_*(y)}$. Hence $h_*(h(x) \vee y) = x \vee h_*(y)$. Therefore h is closed, as desired. □

2.1.4 Lemma

For any frame homomorphism $h : M \to L$,

(i) $\mathfrak{C}h(\Delta_a^M) = \Delta_{h(a)}^L$ *for each* $\Delta_a^M \in \mathfrak{C}M$.

(ii) $\mathfrak{C}h(\nabla_a^M) = \nabla_{h(a)}^L$ *for each* $\nabla_a^M \in \mathfrak{C}M$.

Proof. (i) Recall that $\mathfrak{C}h(\Delta_a^M) = \bigcap\{\Theta \in \mathfrak{C}L \mid (h \times h)(\Delta_a^M) \subseteq \Theta\}$.
First we show that $(h \times h)(\Delta_a^M) \subseteq \Delta_{h(a)}^L$: Let $(x,y) \in (h \times h)(\Delta_a^M)$, then $(x,y) = (h(s), h(t))$ for some $(s,t) \in \Delta_a^M$. For such (s,t), $s \wedge a = t \wedge a$ and hence $h(s) \wedge h(a) = h(t) \wedge h(a)$. Therefore $(x,y) = (h(s), h(t)) \in \Delta_{h(a)}^L$.
Secondly, we show that $\Phi \supseteq \Delta_{h(a)}^L$ for any $\Phi \in \mathfrak{C}L$ with $(h \times h)(\Delta_a^M) \subseteq \Phi$.
It is sufficient to show that $(h(a), e) \in \Phi$. Trivially, $(h(a), e) = (h \times h)(a, e)$ and $(a, e) \in \Delta_a^M$, thus $(h(a), e) \in (h \times h)(\Delta_a^M)$. Hence $(h(a), e) \in \Phi$.
Note that these two steps show that $\Delta_{h(a)}^L$ is the smallest congruence on L containing $(h \times h)(\Delta_a^M)$. Hence $\mathfrak{C}h(\Delta_a^M) = \Delta_{h(a)}^L$ for each $\Delta_a^M \in \mathfrak{C}M$.
(ii) Similar to (i). ☐

2.1.5 Lemma

If $h : M \to L$ *is an onto frame homomorphism then so is* $\mathfrak{C}h : \mathfrak{C}M \to \mathfrak{C}L$.

Proof. Suppose $h : M \to L$ is onto and let $\Phi \in \mathfrak{C}L$. Then $\Phi = \bigvee(\Delta_a \cap \nabla_b)$ for some $(a,b) \in \Phi^+$. Recall that $(a,b) \in \Phi^+$ means $(a,b) \in \Phi$ with $a \leq b$ in L. Since h_* preserves order, $h_*(a) \leq h_*(b)$ and thus $(h_*(a), h_*(b)) \in \Delta_{h_*(a)} \cap \nabla_{h_*(b)}$. Now define $\Theta = \bigvee(\Delta_{h_*(a)} \cap \nabla_{h_*(b)})$ in $\mathfrak{C}M$ for such a and b.

$$
\begin{aligned}
\text{Then } \mathfrak{C}h(\Theta) &= \mathfrak{C}h(\textstyle\bigvee(\Delta_{h_*(a)} \cap \nabla_{h_*(b)})) \\
&= \textstyle\bigvee \mathfrak{C}h(\Delta_{h_*(a)} \cap \nabla_{h_*(b)}) \\
&= \textstyle\bigvee(\mathfrak{C}h(\Delta_{h_*(a)}) \cap \mathfrak{C}h(\nabla_{h_*(b)})) \\
&= \textstyle\bigvee(\Delta_{hh_*(a)} \cap \nabla_{hh_*(b)}) \\
&= \textstyle\bigvee_{(a,b)\in\Phi^+}(\Delta_a \cap \nabla_b) \qquad (h \text{ is onto}) \\
&= \Phi.
\end{aligned}
$$

Therefore $\mathfrak{C}h$ is onto as asserted. ☐

2.1.6 Theorem

Let $h : M \to L$ and $g : L \to K$ be frame homomorphisms.
(i) *If h and g are closed, then $g \circ h$ is closed.*
(ii) *If $g \circ h$ is closed and g is one-one, then h is closed.*
(iii) *If $g \circ h$ is closed and h is onto, then g is closed.*

Proof. (i) Suppose g and h are closed and let $(a, b) \in (g \circ h \times g \circ h)^{-1}(\nabla_x)$ for any $\nabla_x \in \mathfrak{C}K$. Then, $((g \circ h)(a), (g \circ h)(b)) \in \nabla_x$

$$\begin{aligned}
&\Longrightarrow g(h(a)) \vee x = g(h(b)) \vee x \\
&\Longrightarrow g_*(g(h(a)) \vee x) = g_*(g(h(b)) \vee x) \\
&\Longrightarrow h(a) \vee g_*(x) = h(b) \vee g_*(x) \\
&\Longrightarrow h_*(h(a) \vee g_*(x)) = h_*(h(b) \vee g_*(x)) \\
&\Longrightarrow a \vee h_*g_*(x) = b \vee h_*g_*(x) \\
&\Longrightarrow (a, b) \in \nabla_{h_*g_*(x)}.
\end{aligned}$$

Thus $(g \circ h \times g \circ h)^{-1}(\nabla_x) \subseteq \nabla_{h_*g_*(x)}$. The other inclusion always holds, and therefore $g \circ h$ is closed.

(ii) Suppose $g \circ h$ is closed and g is one-one. Then the latter implies that $(g \times g)^{-1}(\nabla_{g(y)}) \subseteq \nabla_y$. Conversely, for any $(c, d) \in \nabla_y$, we have $c \vee y = d \vee y$, and thus $g(c) \vee g(y) = g(d) \vee g(y)$ which implies that $(g(c), g(d)) \in \nabla_{g(y)}$. Hence $(c, d) \in (g \times g)^{-1}(\nabla_{g(y)})$, and therefore $\nabla_y \subseteq (g \times g)^{-1}(\nabla_{g(y)})$.
This establishes that $\nabla_y = (g \times g)^{-1}(\nabla_{g(y)})$.
Now,

$$\begin{aligned}
(h \times h)^{-1}(\nabla_y) &= (h \times h)^{-1}(g \times g)^{-1}(\nabla_{g(y)}) \\
&= (g \circ h \times g \circ h)^{-1}(\nabla_{g(y)}) \\
&= \nabla_{h_*g_*(g(y))} \\
&= \nabla_{h_*(y)}.
\end{aligned}$$

The last step since $y \leq g_*g(y)$ by definition of right adjoint and $g_*g(y) \leq y$ again by definition of g_* and the assumption that g is one-one. Thus h is closed.

(iii) Suppose $g \circ h$ is closed and h is onto. Then, by the above lemma, $\mathfrak{C}h$ is onto, thus $(\mathfrak{C}h) \circ (h \times h)^{-1} = id_{\mathfrak{C}L}$. Also $(h \times h)^{-1} \circ (g \times g)^{-1}(\nabla_x) = \nabla_{h_*g_*(x)}$ for each $\nabla_x \in \mathfrak{C}K$ since $g \circ h$ is closed and then $(\mathfrak{C}h) \circ (h \times h)^{-1} \circ (g \times g)^{-1}(\nabla_x) = (\mathfrak{C}h) \circ (\nabla_{h_*g_*(x)})$. Hence $(g \times g)^{-1}(\nabla_x) = \nabla_{h_*g_*(x)} = \nabla_{g_*(x)}$ for each $\nabla_x \in \mathfrak{C}K$. Therefore g is closed. ☐

2.1.7 Theorem

A quotient frame homomorphism $q : L \to L/\Theta$ is closed iff Θ is closed.

Proof.($\Rightarrow$) Suppose q is closed. Then $q_*q(a) = a \vee q_*(0)$ for any $a \in L$. Hence $\Theta = ker(q) = \{(x,y) | q_*q(x) = q_*q(y)\} = \{(x,y) | x \vee q_*(0) = y \vee q_*(0)\} = \nabla_{q_*(0)}$, where $x, y \in L$. Therefore Θ is closed.

($\Leftarrow$) Given that Θ is closed, then $\Theta = \nabla_x$ for some $x \in L$. Let $q = (-) \vee x : L \to \uparrow x$. Then, by definition of right adjoint, q_* is the inclusion map $\uparrow x \hookrightarrow L$. For any $a \in L$, such that $a \in \uparrow x$, one gets $q_*(q(a) \vee b) = q_*(a \vee x \vee b) = a \vee b = a \vee q_*(b)$. Therefore q is closed. ☐

2.1.8 Lemma

Let L be a frame. Then for any a, $b \in L$,
(i) $\nabla_a \leq \Delta_b$ iff $a \wedge b = 0$
(ii) $\Delta_a \leq \nabla_b$ iff $a \vee b = e$.

Proof. (i) ($\Rightarrow$) Suppose $\nabla_a \leq \Delta_b$ and take $(0, a) \in \nabla_a$. Then $(0, a) \in \Delta_b$, which implies $0 = 0 \wedge b = a \wedge b$.

($\Leftarrow$) Suppose $a \wedge b = 0$ and let $(x, y) \in \nabla_a$ then $x \vee a = y \vee a$ so that $(x \vee a) \wedge b = (y \vee a) \wedge b$ and consequently $x \wedge b = y \wedge b$. Therefore $\nabla_a \leq \Delta_b$.

(ii) ($\Rightarrow$) Suppose $\Delta_a \leq \nabla_b$ and take $(a, e) \in \Delta_a$. Then $(a, e) \in \nabla_b$, which means $a \vee b = e \vee b = e$.

($\Leftarrow$) Given that $a \vee b = e$, let $(x, y) \in \Delta_a$ then $x \wedge a = y \wedge a$ so that $(x \wedge a) \vee b = (y \wedge a) \vee b$ and hence $(x \vee b) \wedge e = (y \vee b) \wedge e$, that is, $x \vee b = y \vee b$. Therefore $\Delta_a \leq \nabla_b$. ☐

2.1.9 Lemma

For any family $\{\Delta_a \,|\, a \in I\}$ of open congruences on a frame L and an arbitrary congruence Θ on L, $\Theta \cap \bigcup_{a \in I} \Delta_a = \bigcup_{a \in I}(\Theta \cap \Delta_a)$.

Proof. This simply holds by set theoretic properties of intersection and union. ☐

Replacing intersection with join and union with meet in the above lemma yields the following result which appears in [3]:

2.1.10 Lemma

For any family $\{\Delta_a \mid a \in I\}$ *of open congruences on a frame* L *and an arbitrary congruence* Θ *on* L, $\Theta \vee \bigwedge_{a\in I} \Delta_a = \bigwedge_{a\in I}(\Theta \vee \Delta_a)$.

Proof. For any $(x, y) \in \bigwedge_{a\in I}(\Theta \vee \Delta_a)$, $(x, y) \in \Theta \vee \Delta_a$ for each $a \in I$ and hence by an earlier result, we get $(x \wedge a, y \wedge a) \in \Theta$ for all $a \in I$

$$\begin{aligned}
&\Longrightarrow (\bigvee x \wedge a, \bigvee y \wedge a) \in \Theta \\
&\Longrightarrow (x \wedge \bigvee a, y \wedge \bigvee a) \in \Theta \\
&\Longrightarrow (x, y) \in \Theta \vee \Delta_{\vee a} = \Theta \vee \bigwedge \Delta_a \\
&\Longrightarrow (x, y) \in \Theta \vee \bigwedge \Delta_a.
\end{aligned}$$

Hence $\bigwedge_{a\in I}(\Theta \vee \Delta_a) \leq \Theta \vee \bigwedge_{a\in I} \Delta_a$.

Conversely, $\Theta \vee \bigwedge_{a\in I} \Delta_a \leq \Theta \vee \Delta_a$ for each $a \in I$ which implies that $\Theta \vee \bigwedge_{a\in I} \Delta_a \leq \bigwedge(\Theta \vee \Delta_a)$. Therefore $\Theta \vee \bigwedge_{a\in I} \Delta_a = \bigwedge_{a\in I}(\Theta \vee \Delta_a)$. ☐

2.1.11 Proposition

If two families $\{\Theta_i \mid i \in I\}$ *and* $\{\Delta_{a_i} | i \in I\}$ *of congruences on a frame* L *satisfy the following conditions:*

(i) $\bigwedge\{\Theta_i \mid i \in I\} = \bigwedge\{\Delta_{a_i} \mid i \in I\}$,
(ii) $\Theta_i \vee \Theta_j = \nabla \ (i \neq j)$,
(iii) $\Theta_i \leq \Delta_{a_i}$ *for all* $i \in I$,

then $\Theta_i = \Delta_{a_i}$ *for all* $i \in I$.

Proof. For any $j \in I$,
$$\begin{aligned}
\Theta_j &= (\bigwedge\{\Theta_i \mid i \in I\}) \vee \Theta_j && \text{(by the above lemma)} \\
&= (\bigwedge\{\Delta_{a_i} | i \in I\}) \vee \Theta_j && (\mathit{by}(i)) \\
&= \bigwedge_{i\in I}(\Delta_{a_i} \vee \Theta_j) \\
&= \Delta_{a_j} \vee \Theta_j. && \text{(by (ii) and (iii) combined)}
\end{aligned}$$

This implies $\Delta_{a_j} \leq \Theta_j$. By (iii), $\Theta_j \leq \Delta_{a_j}$. Thus $\Delta_{a_i} = \Theta_i$ for all $i \in I$. ☐

2.2 Chen-connected congruences

First we give the two known definitions of a connected congruence, and then immediately establish that the two definitions are not equivalent.

2.2.1 Definition

(i) As an element of a congruence frame $\mathfrak{C}L$, a congruence Θ is *connected* iff $\Theta = \Phi \vee \Psi$ and $\Phi \wedge \Psi = \Delta$ implies $\Theta = \Phi$ or $\Theta = \Psi$ for $\Phi, \Psi \in \mathfrak{C}L$.

(ii) A congruence Θ on a frame L is called *connected* iff L/Θ is a connected frame (Chen[16]).

Remark. The first definition above corresponds directly to that of a connected element of a frame (Definition 1.1.1) and hence seems to be more natural than the second one. The later is due to Chen and involves a quotient frame, however it has the following quotient-free equivalence which is often used (see for example Chen[16] and Baboolal[3]).

2.2.2 Lemma

A congruence Θ on a frame L is connected (in the sense of Chen) iff $\Delta_{x\vee y} \leq \Theta, \nabla_{x\wedge y} \leq \Theta$ imply $\Delta_x \leq \Theta$ or $\nabla_x \leq \Theta$, for any $x, y \in L$.

Proof. A congruence Θ is connected iff L/Θ is connected iff $\Theta[e] = \Theta[a] \vee \Theta[b]$ and $\Theta[a] \wedge \Theta[b] = \Theta[0]$ imply $\Theta[a] = \Theta[0]$ or $\Theta[b] = \Theta[0]$ iff $\Theta[e] = \Theta[a \vee b]$ and $\Theta[a \wedge b] = \Theta[0]$ imply $\Theta[a] = \Theta[0]$ or $\Theta[b] = \Theta[0]$ iff $(a \vee b, e) \in \Theta$ and $(a \wedge b, 0) \in \Theta$ imply $(0, a) \in \Theta$ or $(a, e) \in \Theta$, iff $\Delta_{a\vee b} \leq \Theta, \nabla_{a\wedge b} \leq \Theta$ imply $\Delta_a \leq \Theta$ or $\nabla_a \leq \Theta$, for any $a, b \in L$. ☐

Observation. The above two definitions of a connected congruence are not equivalent:

Consider the four-element Boolean frame **4** and recall the fact that the frame homomorphism $\nabla : L \to \mathfrak{C}L$ is an isomorphism iff L is Boolean. Then **4** is isomorphic to $\mathfrak{C}\mathbf{4}$. Since $0 \in \mathbf{4}$ is connected then so is ∇_0 in the sense of Definition 2.2.1(i). We claim that ∇_0 is not connected in the sense of Chen: Let x and y be the non-trivial elements of **4**. Then clearly $\nabla_{x\wedge y} \leq \nabla_0$ and $\Delta_{x\vee y} \leq \nabla_0$ but $\nabla_x \not\leq \nabla_0$ and $\Delta_y \not\leq \nabla_0$. Hence ∇_0 is not connected in the sense of Chen. ☐

Remark. It is now necessary to rename one of the above concepts of connectedness of a congruence. Hereafter,

2.2.3 Definition

A congruence Θ is *Chen-connected* iff L/Θ is a connected frame.

2.2.4 Definition

Let Θ be a congruence on a frame L. Then ∇_a is called the *closure of* Θ (denoted by $\bar{\Theta}$) iff a is the largest element such that $(a, 0) \in \Theta$.
Note that $\bar{\Theta} = \nabla_a \leq \Theta$.

Remark. (i) Recall that every congruence Θ on a frame L determines a nucleus on L defined by $k_\Theta(x) = \bigvee\{y \in L \,|\, (x, y) \in \Theta\}$. Then, in the above definition, $a = k_\Theta(0)$ for the associated nucleus k_Θ.
(ii) As an immediate consequence of the definition of a pseudocomplement a^* of any $a \in L$ and that of a closure of a congruence, we get $\bar{\Delta}_a = \nabla_{a^*}$.
(iii) The frame homomorphism $g : L/\bar{\Theta} \to L/\Theta$ defined by $g(\bar{\Theta}[x]) = \Theta[x]$ for any $x \in L$, $\Theta \in \mathfrak{C}L$ is dense: Suppose $g(\bar{\Theta}[x]) = \Theta[0]$. Then $\Theta[x] = \Theta[0]$. It follows that $(x, 0) \in \Theta$ hence $x \leq a$, where $\bar{\Theta} = \nabla_a$, and thus $x \in \bar{\Theta}[0]$. Since $\bar{\Theta}$ is an equivalence relation, it follows that $\bar{\Theta}[x] = \bar{\Theta}[0]$. □

2.2.5 Proposition

Let $(\Theta_i)_{i \in I}$ be a family of Chen-connected congruences on a frame L. If there exists an $i_0 \in I$ such that $\bar{\Theta}_i \vee \Theta_{i_0} \neq \nabla$ or $\Theta_i \vee \bar{\Theta}_{i_0} \neq \nabla$ for all $i \neq i_0$ then $\bigwedge \Theta_i$ is Chen-connected.

Proof. Suppose that $\Delta_{x \vee y} \leq \bigwedge \Theta_i$ and $\nabla_{x \wedge y} \leq \bigwedge \Theta_i$. Then $\Delta_{x \vee y} \leq \Theta_{i_0}$ and $\nabla_{x \wedge y} \leq \Theta_{i_0}$ and hence, by Chen-connectedness of Θ_{i_0}, we have either $\Delta_x \leq \Theta_{i_0}$ or $\nabla_x \leq \Theta_{i_0}$. Assume $\Delta_x \leq \Theta_{i_0}$ and take any $i \neq i_0$. If $\nabla_x \leq \Theta_i$ then $\nabla_x \leq \bar{\Theta}_i$: To see this, let $\bar{\Theta}_i = \nabla_a$ where $a = \bigvee\{x \in L | (0, x) \in \Theta_i\}$. Since $(0, x) \in \nabla_x \leq \Theta_i$, $(0, x) \in \Theta_i$ hence $x \leq a$ and thus $\nabla_x \leq \nabla_a$.
Now we have $\nabla = \Delta_x \vee \nabla_x \leq \Theta_{i_0} \vee \nabla_a = \Theta_{i_0} \vee \bar{\Theta}_i$, therefore $\nabla = \Theta_{i_0} \vee \bar{\Theta}_i$ contradicting $\bar{\Theta}_i \vee \Theta_{i_0} \neq \nabla$. Thus $\Delta_x \leq \Theta_i$ for all $i \neq i_0$ which implies $\Delta_x \leq \bigwedge \Theta_i$. Similarly, if $\nabla_x \leq \Theta_{i_0}$ then $\nabla_x \leq \Theta_i$ for all $i \neq i_0$, otherwise $\Delta_x \leq \Theta_i$ which leads to the same contradiction. Thus we get $\nabla_x \leq \bigwedge \Theta_i$ and

therefore $\bigwedge \Theta_i$ is Chen-connected. $\square$

Remark. The above proposition corresponds to the result in general topology which says that the union of a family $(A_i)_I$ of connected subsets of a topological space is connected provided that there exists an index $i_0 \in I$ such that A_{i_0} is not separated from A_i for all $i \neq i_0$. (Two subsets A and B of a space X are (mutually) separated iff $\bar{A} \cap B = \varnothing = \bar{B} \cap A$). For more on this concept see Kelly[34], p.54 or Willard[48], p.192 .

2.2.6 Lemma

For any frame L, L/Δ_a is isomorphic to $\downarrow a$ for each $a \in L$.

Proof. Define the frame homomorphism $h : L \to \downarrow a$ by $h(x) = x \wedge a$ for all $x \in L$. Then for any $x \in \downarrow a$, $x = x \wedge a$ so h is onto. Moreover, $\ker(h) = \{(u,v) \in L \times L \mid h(u) = h(v)\} = \{(u,v) \in L \times L \mid u \wedge a = v \wedge a\} = \Delta_a$. Consider the following factorization of h:

$$\begin{tikzcd} L \arrow[r, "q"] & L/ker(h) \arrow[r, "\bar{h}"] & \downarrow a \end{tikzcd}$$

where $\bar{h}([x]) = h(x)$ for $x \in L$. By Lemma 0.4.6, it follows that $\bar{h}$ is an isomorphism and therefore $L/\Delta_a \cong \downarrow a$. $\square$

Remark. Dually, $L/\nabla_a \cong \uparrow a$.

2.2.7 Proposition

For any frame L,
(i) *Δ_a is Chen-connected iff $a \in L$ is connected.*
(ii) *If $\Theta \in \mathfrak{C}L$ is Chen-connected then $\bar{\Theta}$ is Chen-connected.*
(iii) *∇_a is Chen-connected iff $a \in L$ is coconnected.*

Proof. (i) We employ the above lemma. Note that $a \in L$ is connected iff $\downarrow a$ is connected iff L/Δ_a is connected iff Δ_a is Chen-connected.

(ii) Suppose Θ is Chen-connected. Then by its denseness, the frame homomorphism $L/\bar{\Theta} \to L/\Theta$ reflects connectedness. Therefore $\bar{\Theta}$ is Chen-connected.

(iii) similar to (i). $\square$

2.2.8 Lemma

(i) *If* $a \in L$ *is connected, then* ∇_{a^*} *is Chen-connected.*

(ii) *For any frame homomorphism* $h : M \to L$, *if* $\Theta \in \mathfrak{C}L$ *is Chen-connected then* $(h \times h)^{-1}(\Theta)$ *is also Chen-connected.*

Proof. (i) Suppose a is connected. Then by the above proposition Δ_a is Chen-connected, and hence so is $\nabla_{a^*} = \bar{\Delta}_a$.

(ii) Suppose $\nabla_{x \wedge y} \leq (h \times h)^{-1}\Theta$ and $\Delta_{x \vee y} \leq (h \times h)^{-1}\Theta$ for any $x, y \in M$. Then $\mathfrak{C}h(\Delta_{x \wedge y}) \leq \mathfrak{C}h(h \times h)^{-1}\Theta$ and $\mathfrak{C}h(\Delta_{x \vee y}) \leq \mathfrak{C}h(h \times h)^{-1}\Theta$. By lemma 2.1.4, $\nabla_{h(x) \wedge h(y)} \leq \mathfrak{C}h(h \times h)^{-1}\Theta \leq \Theta$ and $\Delta_{h(x) \vee h(y)} \leq \mathfrak{C}h(h \times h)^{-1}\Theta \leq \Theta$. Since Θ is Chen-connected, it follows that $\nabla_{h(x)} \leq \Theta$ or $\Delta_{h(x)} \leq \Theta$ in $\mathfrak{C}L$. Applying $(h \times h)^{-1}$ to these inequalities yields: $(h \times h)^{-1}(\nabla_{h(x)}) \leq (h \times h)^{-1}\Theta$ or $(h \times h)^{-1}(\Delta_{h(x)}) \leq (h \times h)^{-1}\Theta$. It follows directly from the definitions of these congruences that $\nabla_x \leq (h \times h)^{-1}(\nabla_{h(x)})$ or $\Delta_x \leq (h \times h)^{-1}(\Delta_{h(x)})$ and hence $\nabla_x \leq (h \times h)^{-1}\Theta$ or $\Delta_x \leq (h \times h)^{-1}\Theta$. Therefore $(h \times h)^{-1}\Theta$ is Chen-connected. ☐

Remark. The above proof of part (ii) is different from that of Chen[16]. Ours involves the characterization of a Chen-connected congruence, and Chen's argument rests on the fact that $M/(h \times h)^{-1}(\Theta) \to L/\Theta$ is a dense embedding.

Recall the definition (1.2.7) of cochained elements of a subset of a frame. Now we apply this definition to the specific case of a congruence frame as follows:

Let K be any set of congruences on L. Then, we define an equivalence relation $\sim$ on K such that, for $\alpha, \beta \in$ K, $\alpha \sim \beta$ iff $\alpha = \beta$ or there exists $\mu_0, \ldots, \mu_n \in$ K for which $\alpha = \mu_0$, $\mu_k \vee \mu_{k+1} \neq \nabla$ for all $k = 0, 1, \ldots, n-1$, and $\mu_n = \beta$. If $\alpha \sim \beta$, we say that α is *co-chained to* β in K. Moreover, K is called *co-chained* iff for any $\alpha, \beta \in$ K, α is co-chained to β in K.

2.2.9 Lemma

In any frame L, the meet of any co-chained set K *of Chen-connected congruences is Chen-connected.*

Proof. Suppose $\gamma = \bigwedge$K with $\Delta_{x \vee y} \leq \gamma, \nabla_{x \wedge y} \leq \gamma$. Then for each $\mu \in$ K, $\gamma \leq \mu$ hence $\Delta_x \leq \mu$ or $\nabla_x \leq \mu$. Suppose there exists $\alpha, \beta \in$ K such that $\Delta_x \leq \alpha$, $\nabla_x \leq \beta$. Since K is co-chained, there exists $\mu_0, \mu_1, \ldots, \mu_n \in$ K such that $\mu_0 = \alpha$, $\mu_k \vee \mu_{k+1} \neq \nabla$ for $k = 0, 1, \ldots, n-1$, and $\mu_n = \beta$. If i is the first index such that $\nabla_x \leq \mu_i$, then $0 < i$ and $\nabla = \Delta_x \vee \nabla_x \leq \mu_{i-1} \vee \mu_i$, a contradiction. It follows that $\Delta_x \leq \mu$ for all $\mu \in$ K or $\nabla_x \leq \mu$ for all $\mu \in$ K, and therefore $\Delta_x \leq \gamma$ or $\nabla_x \leq \gamma$ so that γ is Chen-connected. ☐

2.2.10 Definition

Let L be a frame. Then for $a, b \in L$ define $b \lhd a$ iff there exists a Chen-connected congruence Θ on L such that $\Delta_a \leq \Theta \leq \Delta_b$.

2.2.11 Lemma

For any connected $b \leq a$ *in a frame* L, $b \lhd a$.

Proof. Any connected $b \leq a$ implies $\Delta_a \leq \Delta_b$ with Δ_b Chen-connected. ☐

2.2.12 Lemma

Let L *be a frame and* $a \in L$. *If* $a = \bigvee\{b \in L \mid b \lhd a\}$ *then* a *is a join of connected elements.*

Proof. Let $A_a = \{b \in L \mid b \lhd a\}$ and

$$\mathcal{K} = \{\Theta \in \mathfrak{C}L \mid \Theta \,\text{Chen-connected},\ \Delta_a \leq \Theta \leq \Delta_b,\ b \in A_a\}.$$

On $\mathcal{K}$, we apply the equivalence relation $\sim$ described above, and let $\mathbf{B}$ be the set of all equivalence classes associated with the relation. Then by its definition, each equivalence class $\Phi \in \mathbf{B}$ is co-chained, and hence, by Lemma 2.2.9, $\bigwedge \Phi$ is Chen-connected. Also, for any $\Phi, \Phi' \in \mathbf{B}$ such that $\Phi \neq \Phi'$, we get $(\bigwedge \Phi) \vee (\bigwedge \Phi') = \nabla$ because of the definition of $\sim$ and the fact that $\bigwedge \Phi \in \Phi$ (Φ is closed under meet). Now, for $\Phi \in \mathbf{B}$, let $k(\Phi) = \bigvee\{b \in A_a \mid \bigwedge \Phi \leq \Delta_b\}$. Take $(c, d) \in \bigwedge \Phi$. Since $\bigwedge \Phi \leq \Delta_b$ for all b, then $c \wedge b = d \wedge b$ for all b and hence $c \wedge \bigvee b = d \wedge \bigvee b$ so that $(c, d) \in \Delta_{\bigvee b} = \Delta_{k(\Phi)}$. Thus $\bigwedge \Phi \leq \Delta_{k(\Phi)}$. From the definitions of a and $k(\Phi)$ above, we get $a = \bigvee\{k(\Phi) \mid \Phi \in \mathbf{B}\}$. Then it follows that $\Delta_a = \bigwedge\{\bigwedge \Phi \mid \Phi \in \mathbf{B}\} = \bigwedge\{\Delta_{k(\Phi)} \mid \Phi \in \mathbf{B}\}$ since, as a map, Δ sends joins to meets. Then by Proposition 2.1.11 we have $\bigwedge \Phi = \Delta_{k(\Phi)}$ for each $\Phi \in \mathbf{B}$, and hence $k(\Phi)$ is connected. Thus, from $a = \bigvee\{k(\Phi) \mid \Phi \in \mathbf{B}\}$ we deduce that a is a join of connected elements. ☐

2.2.13 Corollary

If $a = \bigvee\{b \in L \mid b \lhd a\}$ *for any* $a \in L$ *then* L *is locally connected.*

Proof. Suppose $a = \bigvee\{b \in L \mid b \lhd a\}$ for any $a \in L$. Then by the above lemma, a is a join of connected elements. Thus L is locally connected. □

2.3 Open congruences and local connectedness

As elements of a congruence frame, open congruences are special elements. As seen in the above two sections, they feature in the general study of connectedness. Here they turn up in our brief look at weak locale quotient subframes which correspond to weak locale quotient morphisms studied by Li[37]. We also give a characterization of a locally connected frame in terms of connected open congruences.

2.3.1 Definition

A subframe M of a frame L is called a *weak locale quotient subframe* iff any complemented congruence $\Theta \in \mathfrak{C}M$ is open whenever the congruence generated by Θ on L, $\mathfrak{C}i(\Theta)$, is open. Here i is the inclusion map $M \hookrightarrow L$.

2.3.2 Lemma

For any frame homomorphism $h, \mathfrak{C}h$ *preserves complemented congruences.*

Proof. $\mathfrak{C}h$ is a frame homomorphism. □

When adapted to the corresponding one-one frame homomorphism
$h : M \to L$, this concept of a weak locale quotient can be characterized in terms of closed congruences as follows:

2.3.3 Lemma

A one-one frame homomorphism $h : M \to L$ *is a weak locale quotient homomorphism iff each congruence* Θ *on* M *which is complemented in* $\mathfrak{C}M$ *is closed in* $\mathfrak{C}M$ *provided that* $\mathfrak{C}h(\Theta)$ *is closed in* $\mathfrak{C}L$.

Proof. ($\Rightarrow$) Suppose h is a weak locale quotient homomorphism and let $\mathfrak{C}h(\Theta)$ be closed in $\mathfrak{C}L$ for any $\Theta \in \mathfrak{C}M$ with $\Theta^c \in \mathfrak{C}M$. Then $(\mathfrak{C}h(\Theta))^c = \mathfrak{C}h(\Theta^c)$ is open, so that by the property of h, Θ^c is open. Hence Θ is closed. ($\Leftarrow$) Suppose each congruence Θ on M which is complemented in $\mathfrak{C}M$ is closed whenever $\mathfrak{C}h(\Theta)$ is closed in $\mathfrak{C}L$. Consider the one-one frame homomorphism $h : M \to L$. If $\mathfrak{C}h(\Theta)$ is open then $(\mathfrak{C}h(\Theta))^c = \mathfrak{C}h(\Theta^c)$ is closed. By hypothesis, Θ^c is closed hence Θ is open. Therefore h is a weak locale quotient homomorphism. □

2.3.4 Definition

A subframe M of a frame L is called an *equalizer* in L iff
$M = \{x \in L \mid g(x) = h(x)\}$ for some frame homomorphisms $g, h : L \to N$.

2.3.5 Proposition

Any equalizer subframe is a weak locale quotient subframe.

Proof. Let M be an equalizer subframe in L for some frame homomorphisms $g, h : L \to N$ and Θ a complemented congruence on M with $\Theta^c \in \mathfrak{C}M$ such that $\mathfrak{C}i(\Theta) = \Delta_a$ for some $a \in L$ and the inclusion map $i : M \hookrightarrow L$. Then $g \circ i = h \circ i$. By Lemma 2.1.4 we have $\Delta_{g(a)} = \mathfrak{C}g(\Delta_a)$, for $a \in L$. Since Θ is complemented, it suffices to show that $a \in M$.

$$\begin{aligned}
\text{Now, } \Delta_{g(a)} &= \mathfrak{C}g(\Delta_a) = \mathfrak{C}g(\mathfrak{C}i(\Theta)) && \text{(by hypothesis)}\\
&= \mathfrak{C}(g \circ i)(\Theta) && (\mathfrak{C} \text{ is functorial})\\
&= \mathfrak{C}(h \circ i)(\Theta) && (g \circ i = h \circ i)\\
&= \mathfrak{C}h(\mathfrak{C}i(\Theta)) && (\mathfrak{C} \text{ is functorial})\\
&= \mathfrak{C}h(\Delta_a) && \text{(again by hypothesis)}\\
&= \Delta_{h(a)} && \text{(by Lemma 2.1.4)}
\end{aligned}$$

that is, $\Delta_{g(a)} = \Delta_{h(a)}$ which implies $g(a) = h(a)$ since $\Delta : N \to \mathfrak{C}N$ is one-one. Thus $a \in M$. Consequently, Θ is open in $\mathfrak{C}M$ and therefore M is a weak locale quotient subframe. □

We now appeal to open frame homomorphisms.

2.3.6 Definition

A frame homomorphism $h : M \to L$ is *open* iff $h^*(h(x) \wedge y) = x \wedge h^*(y)$ for its left adjoint $h^* : L \to M$, $x \in M$ and $y \in L$.

2.3.7 Lemma

A frame homomorphism $h : M \to L$ is open iff one of the following equivalent conditions is satisfied:
(i) $h(x \longrightarrow y) = h(x) \longrightarrow h(y)$ *for* $x, y \in M$.
(ii) $h_*(y \longrightarrow h(x)) = h^*(y) \longrightarrow x$ *for* $x \in M$ *and* $y \in L$. □

Joyal and Tierney[31] proved the above result, and it was later used to establish the next theorem by Chen[16], where like with closed ones (Theorem 2.1.3), open homomorphisms are related to open congruences in a nice way:

2.3.8 Theorem

A frame homomorphism $h : M \to L$ is open iff $(h \times h)^{-1}(\Delta_a) = \Delta_{h^*(a)}$ for each $a \in L$.

Proof. First, we show that in general $\Delta_{h^*(a)} \leq (h \times h)^{-1}(\Delta_a)$: For any $(x, y) \in \Delta_{h^*(a)}$ we have $x \wedge h^*(a) = y \wedge h^*(a)$

$$\begin{aligned}
&\Longrightarrow h(x) \wedge hh^*(a) = h(y) \wedge hh^*(a) \\
&\Longrightarrow h(x) \wedge a = h(y) \wedge a \qquad\qquad (\text{since } a \leq hh^*(a)) \\
&\Longrightarrow (h(x), h(y)) \in \Delta_a \\
&\Longrightarrow (x, y) \in (h \times h)^{-1}(\Delta_a), \text{ and hence } \Delta_{h^*(a)} \leq (h \times h)^{-1}(\Delta_a).
\end{aligned}$$

($\Rightarrow$) Suppose $h : L \to M$ is open. We show that $\Delta_{h^*(a)} \geq (h \times h)^{-1}(\Delta_a)$. Take any $(x, y) \in (h \times h)^{-1}(\Delta_a)$. Then $(h(x), h(y)) \in \Delta_a$ which implies $h(x) \wedge a = h(y) \wedge a$ so that $h^*(h(x) \wedge a) = h^*(h(y) \wedge a)$. Then $x \wedge h^*(a) = y \wedge h^*(a)$, which implies $(x, y) \in \Delta_{h^*(a)}$. Thus $(h \times h)^{-1}(\Delta_a) \leq \Delta_{h^*(a)}$ and hence equality.

($\Leftarrow$) Suppose $(h \times h)^{-1}(\Delta_a) = \Delta_{h^*(a)}$ for each $a \in L$. We show that h is open. Since $h(h^*(h(x) \wedge y)) \wedge y = h(x) \wedge y$ then $(h^*(h(x) \wedge y), x) \in (h \times h)^{-1}(\Delta_y) = \Delta_{h^*(y)}$ which implies $h^*(h(x) \wedge y) \wedge h^*(y) = x \wedge h^*(y)$ and hence $h^*(h(x) \wedge y) = x \wedge h^*(y)$. Therefore h is open. □

Remark. Note that the above argument is different from the one given by Chen[16], but similar to that of a closed version in Theorem 2.1.3.

2.3.9 Theorem

A quotient frame homomorphism $q : L \to L/\Theta$ *is open iff* Θ *is open.*

Proof. ($\Rightarrow$) Suppose q is open. Then for any $x \in L$, we have $q^*q(x) = q^*(q(x) \wedge e) = x \wedge q^*(e)$. Hence $\Theta = ker(q) = \{(u, v) \in L \times L \mid q^*q(u) = q^*q(v)\}$

$$\begin{aligned} &= \{(u, v) \in L \times L \mid u \wedge q^*(e) = v \wedge q^*(e)\} \\ &= \Delta_{q^*(e)}. \end{aligned}$$

Thus Θ is open.
($\Leftarrow$) If Θ is open then $\Theta = \Delta_a$ for some $a \in L$. Define $q : L \to \downarrow a$ by $q(x) = x \wedge a$. At this point, note that we employ the fact L/Δ_a is isomorphic to $\downarrow a$. Then q^* is just the inclusion $\downarrow a \hookrightarrow L$. Now, for any $x \in L$ and $y \in \downarrow a$, $q^*(q(x) \wedge y) = q^*(x \wedge a \wedge y) = q^*(x \wedge y) = x \wedge y = x \wedge q^*(y)$. Therefore q is open. ☐

For the next proposition we require, among other facts, the following result which is due to Vermeulen[42]. We state it without a proof.

2.3.10 Lemma

Let $h : M \to L$ *be a one-one frame homomorphism, and* Θ *a complemented congruence on* M*. Then in the pushout square*

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[d, "i"'] & L \arrow[d, "j"] \\ M/\Theta \arrow[r, "h_\Theta"] & L/\mathfrak{C}h(\Theta) \end{tikzcd}$$

h_Θ *is also one-one.* ☐

2.3.11 Proposition

Any closed subframe is a weak locale quotient subframe.

Proof. Let M be a closed subframe of L, Θ a complemented congruence on M and $\mathfrak{C}i(\Theta)$ closed for the inclusion map $i : M \hookrightarrow L$. Consider the following pushout square:

$$\begin{tikzcd}
M \arrow[r, "i"] \arrow[d, "\mu_M"'] & L \arrow[d, "\mu_L"'] \\
M/\Theta \arrow[r, "i_\Theta"] & L/\mathfrak{C}i(\Theta)
\end{tikzcd}$$

Then by Theorem 2.1.7, μ_L is closed and hence so is $\mu_L \circ i$. Since $\mu_L \circ i = i_\Theta \circ \mu_M$, we deduce that $i_\Theta \circ \mu_M$ is also closed. By the above lemma, i_Θ is one-one so that by Theorem 2.1.6(ii) μ_M is closed and again by Theorem 2.1.7, Θ is closed. Therefore M is a weak locale quotient subframe. $\square$

Remark. In the above Proposition, if closed is replaced by open then a new result emerges similarly. Moreover, the later can be obtained differently: According to Banaschewski[13], any open subframe is an equalizer and hence, by Proposition 2.3.5, a weak locale quotient subframe. In addition, we shall prove, in chapter 4, that any weak locale quotient subframe of a locally connected frame is locally connected.

2.3.12 Definition

For any frame L, a congruence Θ on L is called a *Chen-connected component* of L iff Θ is Chen-connected and for any Chen-connected $\Phi \in \mathfrak{C}L$ with $\Theta \vee \Phi \neq \nabla^L$, we have $\Theta \subseteq \Phi$.

2.3.13 Lemma

Any Chen-connected component of a frame is a closed congruence.

Proof. Let $\Theta \in \mathfrak{C}L$ be a Chen-connected component of L. Then Θ is Chen-connected and hence so is its closure $\bar{\Theta}$. By definition of a closure, $\bar{\Theta} \leq \Theta$ and by that of a Chen-connected component $\Theta \leq \bar{\Theta}$. Hence $\Theta = \bar{\Theta}$. Therefore Θ is closed. $\square$

2.3.14 Lemma

Open congruences on any locally connected frame are meets of Chen-connected open congruences.

Proof. Let L be a locally connected frame and Θ an open congruence on L. Then $\Theta = \Delta_a$ for some $a \in L$. Since L is locally connected $a = \bigvee x$, for connected $x \leq a$. By Proposition 2.2.6, Δ_x is Chen-connected. Therefore $\Theta = \Delta_a = \Delta_{\vee x} = \bigwedge \Delta_x$. $\square$

2.3.15 Proposition

A frame L is locally connected iff each open congruence is a meet of Chen-connected open congruences.

Proof. ($\Rightarrow$) Clear from the above lemma.
($\Leftarrow$) Suppose $\Delta_a = \bigwedge \Delta_x$ for any $a \in L$ and Δ_x Chen-connected. Then x is connected. Clearly $\Delta_a \leq \Delta_x$ and so $x \leq a$. Since $\bigwedge \Delta_x = \Delta_{\vee x}$ we have $\Delta_a = \Delta_{\vee x}$ and hence $a = \bigvee x$. Therefore L is locally connected. $\square$

2.3.16 Lemma

Let $h : M \to L$ be a one-one frame homomorphism. If the bottom of $\mathfrak{C}L$ can be expressed as a meet of Chen-connected congruences then the same can be done with the bottom of $\mathfrak{C}M$.

Proof. Suppose $\Delta^L = \bigwedge \Theta_i$ for Chen-connected $\Theta_i \in \mathfrak{C}L$. Then by Lemma 2.2.8, $(h \times h)^{-1}(\Theta_i)$ is Chen-connected for each i. Hence

$\Delta^M = (h \times h)^{-1}(\Delta^L) = (h \times h)^{-1}(\bigwedge \Theta_i) = \bigwedge (h \times h)^{-1}(\Theta_i)$, where the first equality holds because h is one-one. $\square$

Chapter 3

Compactifications and local connectedness of frames

3.1 Introduction

Henriksen and Isbell[26] particularly investigate local connectedness in the Stone-Čech compactification. In his book *Stone Spaces*, Johnstone[32] gives a brief history and the actual construction of the Stone-Čech compactification of frames. In 1980, Banaschewski and Mulvey[7] studied this compactification in a different way, and put together constructive arguments showing that compact regular frames and compact completely regular frames are coreflective in the category of frames. These results also appear in [5]. A comprehensive view of all compactifications of a frame occurs in [12]. In a uniform setting, Banaschewski and Pultr[9] construct the compact regular coreflection of uniform frames giving some application on the completion on such frames. As in [1], here we briefly discuss compactifications and their relations with local connectedness of frames.

3.2 Compactifications of frames

We recall a few concepts.

3.2.1 Definition

(i) A frame L is said to be *regular* iff $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$, where $x \prec a$ (x *is rather below* a) means there exists some $y \in L$ such that $x \wedge y = 0$ and $a \vee y = e$.

(ii) In any lattice L, a subset $J \subseteq L$ is called an *ideal* iff, for any finite $E \subseteq J$ one has $\bigvee E \in J$, and $x \in J$ whenever $x \leq z$ and $z \in J$.

(iii) An ideal J of a frame L is called a *regular ideal* iff, for each $x \in J$ there exists $y \in J$ for which $x \prec y$.

(iv) An element $a \in L$ is called *compact* iff $a \leq \bigvee E$ implies $a \leq \bigvee F$ for some finite $F \subseteq E$ for all $E \subseteq L$. L is called *compact* iff the top e is compact.

Recall that a *compactification* of a frame L is a compact regular frame M with a dense onto homomorphism $h : M \rightarrow L$. Any frame which has a compactification is said to be *compactifiable.* Banaschewski and Mulvey[7] show that, for any frame L there is a compact regular coreflection of L given by the largest regular subframe $\mathcal{R}L$ of the frame $\mathcal{I}L$ of all ideals of L and the join map $\bigvee : \mathcal{R}L \rightarrow L$. It is then clear that a frame L has a compactification iff the coreflection map $\bigvee : \mathcal{R}L \rightarrow L$ is onto. We point out here that the coreflection map is always dense, a fact to be used in the proof of Proposition 3.2.5.

Now we need the following:

3.2.2 Definition

(i) A frame L is called *completely regular* iff $a = \bigvee\{x \in L \mid x \prec\!\!\prec a\}$ for each $a \in L$, where $x \prec\!\!\prec a$ (x is *completely below* a) means there exists a doubly indexed sequence $(x_{nk})_{n=0,1\ldots;\ k=0,1,\ldots,2^n}$ such that $x = x_{n0}$, $x_{nk} \prec x_{n\,k+1}$, $x_{n2^n} = a$, and $x_{nk} = x_{n+1\,2k}$ for all $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^n$.

(ii) An ideal J of a frame L is a *completely regular ideal* iff, for any $a \in J$ there exists $b \in J$ such that $a \prec\!\!\prec b$.

Remark. Banaschewski and Mulvey[7] establish that, given the Axiom of Countable Dependent Choice (CDC), any compact regular frame is compact completely regular. In addition, for any frame L, its compact completely regular coreflection is given by the join map $\bigvee : \mathcal{CRI}L \rightarrow L$ where $\mathcal{CRI}L$ is the frame of all completely regular ideals $J \subseteq L$. The characterization that a frame has completely regular compactifications iff it is completely regular rests on the familiar fact that the dense homomorphism $\bigvee : \mathcal{CRIL} \to L$ is onto iff L is completely regular. It is well-known that a T_0-space X is completely regular iff the frame $\mathfrak{O}X$ of its open sets is completely regular. Also, X is compact iff $\mathfrak{O}X$ is compact([5]). Thus $\mathcal{CRIL}$ is the frame analogue of the Stone-Čech compactification βX of a Tychonoff space X. Hence it is herein denoted by βL as usual. Furthermore, it has been proven in [2] that βL is a perfect compactification as defined therein.

3.2.3 Lemma

In any frame L, if $x \prec\!\prec a \vee b$ and $a \wedge b = 0$, then $x \wedge a \prec\!\prec a$.

Proof. Suppose $x \prec\!\prec a \vee b$ and $a \wedge b = 0$. Then $x \prec a \vee b$ and $a \wedge b = 0$. So, there exists $y \in L$ such that $y \wedge x = 0$ and $(a \vee b) \vee y = e$ which implies $(x \wedge a) \wedge (b \vee y) = 0$ and $a \vee (b \vee y) = e$. Thus $x \wedge a \prec a$, and by symmetry $x \wedge b \prec b$. Then note that $x_1 \prec x_2 \prec \ldots \prec x_n \prec a \vee b$ and $a \wedge b = 0$ implies the condition $x_1 \wedge a \prec x_2 \wedge a \prec \ldots \prec x_n \wedge a \prec a$. Therefore, by the definition of the relation $\prec\!\prec$, the desired result follows. ☐

We then have:

3.2.4 Lemma

For any frame L, the right adjoint of $\bigvee : \beta L \to L$ preserves disjoint binary joins.

Proof. Consider the map $k : L \to \beta L$ defined by $k(a) = \{x \in L | \, x \prec\!\prec a\}$ for all $a \in L$. We first show that k is the right adjoint of $\bigvee$. Note that $k(a)$ is an ideal of L by familiar properties of $\prec\!\prec$. That this ideal is completely regular comes from the interpolating properties of $\prec\!\prec$. For $a \leq b \in L, x \in k(a)$ implies $x \prec\!\prec a \leq b$ so that $x \prec\!\prec b$. Thus $x \in k(b)$. This says k preserves order. If $\bigvee J \leq a$ for $J \in \beta L$ and $a \in L$ then for any $x \in J$ we have $x \leq \bigvee J$ so that $x \leq a$. Since J is completely regular, there exists $b \in J$ such that $x \prec\!\prec b$. But $b \in J$ implies $b \leq a$ so $x \prec\!\prec a$ meaning that $x \in k(a)$. Hence $J \subseteq k(a)$. Conversely, if $J \subseteq k(a)$ then $\bigvee J \leq \bigvee k(a) \leq a$. Hence $\bigvee J \leq a$ iff $J \subseteq k(a)$ for any $a \in L$ and $J \in \beta L$. Being the right adjoint of $\bigvee$, the map k preserves meets.

Now we show that k preserves disjoint binary joins. Let $x \in k(a \vee b)$ for any $a, b \in L$ with $a \wedge b = 0$. Then $x \prec\!\prec a \vee b$ and by the above lemma, $x \wedge a \prec\!\prec a$ and $x \wedge b \prec\!\prec b$ therefore x can be expressed in the desired way as $x = (x \wedge a) \vee (x \wedge b)$. Hence $k(a \vee b) \subseteq k(a) \vee k(b)$. The other inclusion follows because k preserves order. In all, $k(a) \vee k(b) = k(a \vee b)$. $\square$

The next result says connectedness of a frame is a necessary and sufficient condition for its compact(completely) regular coreflection to be connected.

3.2.5 Proposition

A frame is connected iff its compact (completely) regular coreflection is connected.

Proof.($\Rightarrow$) Given a connected frame L, let $g : K \to L$ be its compact (completely) regular coreflection. Then g is dense, and hence, by Lemma 1.1.10, it follows that K is connected.
($\Leftarrow$) Suppose that K is connected and consider any $h : \mathbf{4} \to L$. Since $\mathbf{4}$ is compact and completely regular, there exists $f : \mathbf{4} \to K$ such that $gf = h$. By Lemma 1.1.9, $f : \mathbf{4} \to K$ factors through $\mathbf{2} \to K$ and hence $h : \mathbf{4} \to L$ factors through $\mathbf{2} \to L$. Therefore L is connected. $\square$

3.2.6 Theorem

For any completely regular frame L, $\bigvee : \beta L \to L$ preserves and reflects connected elements.

Proof. Recall that L has compactifications iff the frame homomorphism $\bigvee : \beta L \to L$ is onto. Then by Lemmas 1.1.11 and 3.2.4, we conclude that $\bigvee$ preserves connected elements. Note that $\bigvee : \beta L \to L$ is dense. Then by Lemma 1.1.10, it also reflects connected elements. $\square$

A similar result on normal regular frames comes up later (Proposition 3.4.5).

3.3 Local connectedness of βL

In this section we present a characterization, due to Baboolal and Banaschewski, of local connectedness of βL, for any completely regular frame L.

3.3.1 Definition

A frame is called *pseudocompact* iff for any sequence $a_0 \prec\prec a_1 \prec\prec a_2 \dots$ in L with $\bigvee a_n = e$, there exists a $k \in \mathbb{N}$ such that $a_k = e$.

Remark. An alternative definition of a pseudocompact frame which involves real numbers has been used by many authors, for example Banaschewski[15] and Markus[38]. The latter gives several characterizations of a pseudocompact completely regular frame. The above definition is due to Gilmour as quoted in [1], and is more appropriate here because it is closely related to that of a completely regular frame. It is known that a space X is pseudocompact iff $\mathfrak{O}X$ is pseudocompact ([6]).

3.3.2 Lemma

In any locally connected pseudocompact frame L, if $a \prec\!\prec b$, then only finitely many components of b meet a. ☐

Remark. Baboolal and Banaschewski[1] give a technical but choice-free proof of this lemma, which is used in proving the following:

3.3.3 Theorem

For any completely regular frame L, βL is locally connected iff L is locally connected and pseudocompact. ☐

Remark. The above theorem is the frame counterpart of the classical result that, for any Tychonoff space X, its Stone-Čech compactification βX is locally connected iff X is locally connected and pseudocompact(Henriksen-Isbell[26]). Furthermore, it is the main result of [1]. We omit its proof due to its length.

3.4 Normal regular frames, property *WS* and local connectedness

We recall the following:

3.4.1 Definition

A frame L is *normal* iff, for any $a, b \in L$ with $a \vee b = e$, there exist $u, v \in L$ such that $u \vee a = e = v \vee b$ and $u \wedge v = 0$.

It is a well-known fact that any normal regular frame has a compactification. We refer the reader to [12], p.109 where a characterization of compactifiable frames is given. For any normal regular frame L, we consider the compact regular coreflection $\bigvee : \mathcal{R}L \to L$, where $\mathcal{R}L$ is the frame of all regular ideals of L. The following three lemmas serve to show that the map $r : L \to \mathcal{R}L$ defined by $r(a) = \{x \in L \mid x \prec a\}$ for all $a \in L$ is the right adjoint of the join map $\bigvee : \mathcal{R}L \to L$.

3.4.2 Lemma

For any normal regular frame L, $r(a)$ is a regular ideal.

Proof. Let $x \in r(a)$ and $y \leq x$. Then $y \leq x \prec a$, and by the property of the relation $\prec$, $y \prec a$, which means $y \in r(a)$. This shows that $r(a)$ is a downset. Now, let $x, y \in r(a)$. Then $x \prec a$ and $y \prec a$, so that by a general property of the relation $\prec$, we get $x \vee y \in r(a)$. So, $r(a)$ is also closed under finite joins, hence it is an ideal. For regularity: Since L is normal, $\prec$ interpolates. So, for each $x \in r(a)$ there exists some $y \in L$ such that $x \prec y \prec a$ which implies $y \in r(a)$. Therefore $r(a)$ is a regular ideal. $\square$

3.4.3 Lemma

For any regular ideal J and $a \in L$, $\bigvee J \leq a$ iff $J \subseteq r(a)$.

Proof.($\Rightarrow$) Suppose $\bigvee J \leq a$ and let $x \in J$. Since J is regular, there exists $y \in J$ such that $x \prec y$. But $y \leq \bigvee J \leq a$, so $x \prec y \leq a$ which implies that $x \prec a$ and hence $x \in r(a)$. Thus $J \subseteq r(a)$.
($\Leftarrow$) Suppose $J \subseteq r(a)$. Then $\bigvee J \leq \bigvee r(a) = a$ and hence $\bigvee J \leq a$. $\square$

3.4.4 Lemma

The map $r : L \to \mathcal{R}L$ is a lattice homomorphism.

Proof. It is easy to see that r preserves order, zero, and arbitrary meets including the top. So, we only show that it preserves all binary joins. First note that $x \prec a \vee b$ implies $x \prec c \vee b$ for some $c \prec a$: if y is such that $x \wedge y = 0$ and $a \vee b \vee y = e$, then by normality of L, there exists u and v with $u \wedge v = 0$ such that $u \vee b \vee y = e = a \vee v$ which implies that $x \prec u \vee b$ and $u \prec a$. Now it follows that $x \prec a \vee b$ implies $x \leq c \vee d$ for suitable $c \prec a$ and $d \prec b$, proving that $r(a \vee b) \subseteq r(a) \vee r(b)$. The converse inclusion follows from r being order-preserving. Hence $r(a \vee b) = r(a) \vee r(b)$, and therefore r is a lattice homomorphism. $\square$

Remark. It is now clear from the above lemmas that the map r is the right adjoint of the homomorphism $\bigvee : \mathcal{R}L \to L$. In view of Lemmas 1.1.10, 1.1.11 together with Lemma 3.4.4 we obtain the following counterpart of Theorem 3.2.6.

3.4.5 Proposition

For any normal regular frame L , $\bigvee : \mathcal{R}L \to L$ preserves and reflects connectedness. $\square$

Of remarkable strength in this context, is the notion which was first translated into the frame setting with a slight change in terminology by Baboolal and Banaschewski[1], the idea of property WS.

Now the reader may need to recall the definitions of a cover and a refinement discussed later in chapter 5. Then we have the following:

3.4.6 Definition

A regular frame is said to *have property WS* iff every finite cover of L has a finite refinement consisting of connected elements.

Remark. In its original term, as property S, this notion was first considered for metric spaces by Sierpinski in 1920. Later on, Wilder[47] introduced the topological and relative version which motivated Wallace[44]'s definition that a topological space X has property S iff every finite cover of X has a refinement consisting of connected elements. More on the term later.

3.4.7 Lemma

Any regular frame L with property WS is locally connected and pseudocompact.

Proof. Let L be a regular frame with property WS and consider $x \prec a$ in L. Then, $a \vee x^* = e$ so that $\{a, x^*\}$ is a cover of L. Now, by property WS, there exists a finite refinement $C \subseteq L$ consisting of connected elements so that $c \leq a$ for any $c \in C$ provided that $c \nleq x^*$, that is, $x \wedge c \neq 0$. Then we get the following inequality:

$$\begin{aligned} x &= \bigvee\{x \wedge c \,|\, c \in C\} \\ &= \bigvee\{x \wedge c \,|\, x \wedge c = 0,\ c \in C\} \vee \bigvee\{x \wedge c \neq 0,\ c \in C\} \\ &\leq \bigvee\{x \wedge c \,|\, c \leq a, c \in C\} \\ &\leq \bigvee\{c \,|\, c \leq a, c \in C\} \\ &\leq a. \end{aligned}$$

So x lies below a join of connected elements, which in turn lies below a. Since L is regular, a is the join of all elements rather below it, and hence is a join of connected elements. Therefore L is locally connected. It remains to show that L is pseudocompact.

Suppose $a_0 \prec\!\!\prec a_1 \prec\!\!\prec a_2 \prec\!\!\prec \dots$ is any sequence in L with $\bigvee a_n = e$, and put $p := a_1 \vee (a_3 \wedge a_1^*) \vee (a_5 \wedge a_3^*) \vee \dots$ and $q := (a_2 \wedge a_0^*) \vee (a_4 \wedge a_2^*) \vee \dots$.

Note that in the join of p and q, one has $a_1 \vee (a_2 \wedge a_0^*) = a_2$ and $a_1 \vee (a_3 \wedge a_1^*) \vee (a_2 \wedge a_0^*) \vee (a_4 \wedge a_2^*) = a_4$ with $a_{n-1}^* \vee a_n = e$ for each n. Then $a_n \leq p \vee q$ for each n. Since $\bigvee a_n = e$, it follows that $p \vee q = e$. Now if C is a finite refinement of the cover $K = \{p,\, q\}$ guaranteed by property WS then each $c \in C$ is less that or equal to one of the elements

$$a_1,\ a_3 \wedge a_1^*,\ a_5 \wedge a_3^*, \dots \text{ or } a_2 \wedge a_0^*,\ a_4 \wedge a_2^*, \dots$$

because $c \leq p \vee q$ and $p \wedge q = 0$ and thus by the connectedness of c we have $c \leq p$ or $c \leq q$, and the elements in either sequence are disjoint. Since C is finite and refines K, there exists some $n \in \mathbb{N}$ such that $a_1 \vee (a_2 \wedge a_0^*) \vee \dots \vee (a_{n+2} \wedge a_n^*) = e$ and hence $a_{n+2} = e$. Therefore L is pseudocompact. $\square$

Next, we characterize property WS in terms of $\mathcal{R}L$ for normal regular frames.

3.4.8 Proposition

A normal regular frame L has property WS iff $\mathcal{R}L$ is locally connected.

Proof.($\Rightarrow$) Suppose L has property WS, and let $a \in J$ for $J \in \mathcal{R}L$. Then there exists $b \in J$ such that $a \prec b$ which implies that $a^* \vee b = e$. Hence $\{a^*,\, b\}$ is a cover of L. By property WS, this cover has a finite refinement of connected elements say $C \subseteq L$. Then $\bigvee C = e$ implies $\bigvee\{r(c) \,|\, c \in C\} = L$

by Lemma 3.4.4 and hence there exists $x_c \in r(c)$ for each $c \in C$ such that $\bigvee\{x_c \,|\, c \in C\} = e$. For any $c \in C$, $a \wedge x_c \neq 0$ implies $a \wedge c \neq 0$, hence $c \nleq a^*$ and therefore $c \leq b$. This means $\bigvee\{r(c) \,|\, a \wedge x_c \neq 0\} \subseteq J$, and since a is the join of the $a \wedge x_c \neq 0$ then a belongs to $\bigvee\{r(c) \,|\, a \wedge x_c \neq 0\}$. It follows that $J = \bigvee\{r(x) \subseteq J \,|\, x \,\text{connected}\}$. Note that these $r(x)$ are connected by Proposition 3.4.5. Therefore $\mathcal{R}L$ is locally connected.

($\Leftarrow$) Suppose $\mathcal{R}L$ is locally connected. Then by Lemma 3.4.4, $a_1 \vee \ldots \vee a_n = e$ in L implies $r(a_1) \vee \ldots \vee r(a_n) = L$ in $\mathcal{R}L$. So, $\mathcal{C} = \{r(a_1), \ldots, r(a_n)\}$ is a finite cover of $\mathcal{R}L$. By local connectedness of $\mathcal{R}L$, each $r(a_i)$ is an arbitrary join of connected elements. Since $\mathcal{R}L$ is compact, each such join can be reduced to a finite join of connected elements. Then $\mathcal{K} = \{\bigvee J \,|\, J \in \mathcal{C}\}$ is a finite cover of L. By Proposition 3.4.5, the elements of $\mathcal{K}$ are connected. Since $J \subseteq r(a_i)$ implies $\bigvee J \leq a_i$, $\mathcal{K}$ refines $\{a_1, \ldots, a_n\}$, and therefore L has property WS. ☐

With Countable Dependent Choice, the necessary condition of Lemma 3.4.7 suffices to imply property WS for normal regular frames. This results in the following characterization of property WS:

3.4.9 Proposition

Assuming CDC, a normal regular frame L has property WS iff L is locally connected and pseudocompact.

Proof. ($\Rightarrow$) This follows directly from Lemma 3.4.7.
($\Leftarrow$) Suppose a normal regular frame L is locally connected and pseudocompact. Then by the axiom of CDC, L is completely regular and $\mathcal{R}L = \beta L$. Now by Theorem 3.3.3, $\mathcal{R}L$ is locally connected and then Proposition 3.4.8 guarantees that L has property WS as desired. ☐

Remark. We point out that, as remarked in [1], in 1956, Henriksen and Isbell[26] proved that a regular space X has property S in the sense of Wallace[44] iff every finite open cover of X has a finite refinement consisting of connected open sets. In this context, the latter amounts to saying that the frame of open sets of X has property WS. This term attempts to partly emphasize that, as a topological concept, it originates with Wilder. We reserve the original term property S for uniform frames, in chapter 5, which will be the direct frame counterpart of Sierpinski's version. Collins[20] investigates this idea for uniform spaces.

Chapter 4

Fully complement closed subframes

4.1 Introduction

The idea of a fully complement closed subframe was first introduced by Banaschewski[14] in 1994. In the same paper, he shows that any such subframe of a locally connected frame is locally connected. As in[14], among other results, a point-free version of Whyburn's result (see [46]) that quotient maps preserve local connectedness is presented here. Moreover, we provide detailed proofs of several facts stated in[14] and relate the kinds of subframes involved.

4.2 The Basics

4.2.1 Definition

Let M be a subframe of a frame L. Then,

(i) M is said to be *complement closed* in L iff, for $a \in M$ and $b \in L$, $a \wedge b = 0$ and $a \vee b = e$ implies $b \in M$.

(ii) M is said to be *fully complement closed* in L iff the frame $M_c = \{x \in M \mid x \leq c\}$ is complement closed in L_c for each $c \in M$.

Remark. M is fully complement closed in L iff $a \wedge b = 0$ and $a \vee b = c$ for $a, c \in M$, $b \in L$ implies $b \in M$.

4.2.2 Lemma

Any fully complement closed subframe is complement closed.

Proof. Suppose M is a fully complement closed subframe of a frame L. Then for each $c \in M$, M_c is complement closed in L_c. In particular, for $c = e$, $M_e = M$ is complement closed in $L_e = L$. $\square$

4.2.3 Examples

(1) Any Boolean subframe of a frame is fully complement closed.
(2) Any subframe of a totally ordered frame is fully complement closed.
(3) For any topology $\mathfrak{O}$ on a set X and any equivalence relation R on X, $M = \{U \in \mathfrak{O} \,|\, x \in U, (x, y) \in R \text{ implies } y \in U\}$ is a fully complement closed subframe of $\mathfrak{O}$. The elements of M are said to be R-saturated.
(4) Any intersection of fully complement closed subframes of a given frame is also fully complement closed. More examples will emerge from later results.

Remark. Note that the first two examples are of different natures in a sense that a subframe of a totally ordered frame has fewer complemented elements than a Boolean subframe of a frame.

Regarding example (3):

Let $M = \{U \in \mathfrak{O} \,|\, x \in U, (x, y) \in R \text{ implies } y \in U\}$. For any $U \in M$, let U^c be its complement in X. If $x \in U^c$ and $(x, y) \in R$ then $x \in X$ and $x \notin U$ with $(x, y) \in R$, which implies that $y \notin U$ otherwise $x \in U$, contradicting $U \in M$. Thus $y \in U^c$. This implies $U^c \in M$. Hence M is complement closed in $\mathfrak{O}$.

Now, let $M_U = \{O \in M \,|\, O \subseteq U\}$ for each $U \in M$. Suppose $O \cap Q = \emptyset$ and $O \cup Q = U$ for $O \in M$ and $Q \in \mathfrak{O}$. For any $x \in Q, (x, y) \in R$, $x \in U$ since $Q \subseteq U$ and $U \in M$ hence $y \in U = O \cup Q$. Then, either $y \in O$ or $y \in Q$. If $y \in O$ then $x \in O$ because O is R-saturated, and this says $O \cap Q \neq \emptyset$, a contradiction. So, $y \notin O$ but $y \in Q$. Hence $Q \in M$. This shows that M_U is complement closed in $\mathfrak{O}_U$ for each $U \in M$. Therefore M is fully complement closed in $\mathfrak{O}$. $\square$

4.2.4 Definition

Let M be a subframe of a frame L. Then,

(i) M is called *nearly open* iff, for any $x \in M$ with $x^* \in L$, it follows that $x^* \in M$.

(ii) M is called *closed* iff the identical embedding $i : M \hookrightarrow L$ is a closed homomorphism, that is, $i_*(i(x) \vee y) = x \vee i_*(y)$ for all $x \in M, y \in L$.

4.2.5 Theorem

The following types of subframes of a frame L are fully complement closed:

(i) *Any nearly open subframe;*

(ii) *Any closed subframe;*

(iii) *Any equalizer;*

(iv) *Any weak locale quotient subframe;*

(v) *Any open subframe.*

Proof. (i) Suppose M is a nearly open subframe of L and let $a \vee b = c$ and $a \wedge b = 0$ for any $a, c \in M$ and $b \in L$. We show that $b \in M$. By the definition of the pseudocomplement a^* of a, $b \leq a^*$ hence $b = a^* \wedge b$. Note that $a^* \wedge b = a^* \wedge c$. Since M is nearly open, $a^* \in M$, and therefore $b \in M$. Thus M_c is complement closed in L_c, meaning M is fully complement closed in L.

(ii) Let M be a closed subframe of L. Then, the inclusion map $i : M \hookrightarrow L$ is a closed homomorphism. Consider its restriction $i_{|M_c}$ to the subframe M_c of L_c. Since $M_c \subseteq M$ and $(i_{|M_c})(x) = i(x)$ for all $x \in M_c$ then $i_{|M_c}$ is also closed so that M_c is a closed subframe of L_c. If $a \vee b = c$ and $a \wedge b = 0$ for $a, c \in M_c$ and $b \in L_c$ then $a \vee (i_{|M_c})_*(b) = (i_{|M_c})_*((i_{|M_c})(a) \vee b) = c$ and $a \wedge (i_{|M_c})_*(b) = 0$ hence $b = (i_{|M_c})_*(b)$. This implies $b \in M_c$. Thus M_c is complement closed in L_c. Hence the result.

(iii) Let M be an equalizer subframe of L. Then $M = \{x \in L \,|\, g(x) = h(x)\}$ for some frame homomorphisms $g, h : L \to N$. For each $c \in M$, consider the subframe $M_c = \{x \in M \,|\, x \leq c\}$ of the corresponding L_c. Clearly, M_c is an equalizer subframe of L_c for the restrictions of g and h to L_c, $g_{|L_c}$ and $h_{|L_c}$ respectively. Let $a \vee b = c$ and $a \wedge b = 0$ for $a, c \in M_c, b \in L_c$. Then for such a, $g_{|M_c}(a) = g(a) = h(a) = h_{|M_c(a)}$. The same holds for c.

Also $g_{|M_c}(a) \vee g(b) = g(c)$ and $g_{|M_c}(a) \wedge g(b) = 0$. The same identities hold for h and $h_{|M_c}$. Hence $g_{|M_c}(a) \vee h(b) = h(c)$ and $g_{|M_c}(a) \wedge h(b) = 0$ which implies that $h(b) = (g_{|M_c}(a))^c$. Similarly, $g(b) = (g_{|M_c}(a))^c$ in N. Since N is a bounded distributive lattice, complements in N must be unique. Thus $g(b) = h(b)$, and so $b \in M$. Since $b \leq c$ it follows that $b \in M_c$. Hence M_c is complement closed in L_c. Therefore M is fully complement closed in L.

(iv) Suppose M is a weak locale quotient subframe of L. First we show that M is complement closed in L. Secondly, we prove that M_c is a weak locale quotient subframe of L_c and hence deduce the asserted result.
Let $a \vee b = e$ and $a \wedge b = 0$ for $a \in M, b \in L$. Then the complemented congruence $\nabla_a^M = \{(x,y) \in M \times M \,|\, x \vee a = y \vee a\}$ on M generates the corresponding ∇_a^L on L. Since b is the complement of a in L, it follows that ∇_a^L is the complement of ∇_b^L and consequently $\nabla_a^L = \Delta_b^L$. Because M is a weak locale quotient subframe, $\nabla_a^M = \Delta_s^M$ for some $s \in M$ hence $\Delta_b^L = \Delta_s^L$, but then $s = b$, therefore M is complement closed in L.

Note that for any $\Phi \in \mathfrak{C}M$, $\Phi \cap (M_c \times M_c) \in \mathfrak{C}M_c$ and for $\Theta \in \mathfrak{C}M_c$, if $\tilde{\Theta} = \{(x,y) \in M \times M \,|\, (x \wedge c, y \wedge c) \in \Theta\}$ then $\tilde{\Theta} \cap (M_c \times M_c) = \Theta$. Let $\Theta \in \mathfrak{C}M_c$ have a complement $\Psi \in \mathfrak{C}M_c$. Then $\tilde{\Theta}$ has complement $\tilde{\Psi} \cap \nabla_c^M$: $\tilde{\Theta} \cap \tilde{\Psi} = (\widetilde{\Theta \cap \Psi}) = \tilde{\Delta} = \Delta_c^M$, thus $\tilde{\Theta} \cap \tilde{\Psi} \cap \nabla_c^M = \Delta^M$. Since $\Delta_c^M \subseteq \tilde{\Theta}$, $\tilde{\Theta} \vee \nabla_c^M = \nabla^M = \tilde{\Theta} \vee \tilde{\Psi}$. The second part since $(\tilde{\Theta} \vee \tilde{\Psi}) \cap (M_c \times M_c) \supseteq \Theta \vee \Psi = \nabla^{M_c}$ and the later contains $(0,c)$. On the other hand, $(c,e) \in \tilde{\Theta} \vee \tilde{\Psi}$ and hence by transitivity $(0,e) \in \tilde{\Theta} \vee \tilde{\Psi}$. So we have $\tilde{\Theta} \vee (\tilde{\Psi} \cap \nabla_c^M) = (\tilde{\Theta} \vee \tilde{\Psi}) \cap (\tilde{\Theta} \vee \nabla_c^M) = \nabla^M \cap \nabla^M = \nabla^M$.
Now, let Θ generate $\Delta_s^{L_c}$. For simplicity, indicate by ˆ the congruence generated in the larger frame. Since $\Theta \subseteq \hat{\Theta}$ we have $\tilde{\Theta} \subseteq \tilde{\hat{\Theta}}$ and since the latter is in $\mathfrak{C}L$, it follows that $\hat{\tilde{\Theta}} \subseteq \tilde{\hat{\Theta}}$. In addition, since $s \leq c$ and $\hat{\Theta} = \Delta_s^{L_c}$, we have

$$\begin{aligned}\tilde{\hat{\Theta}} &= \{(x,y) \in L \times L \,|\, (x \wedge c, y \wedge c) \in \hat{\Theta}\} \\ &= \{(x,y) \in L \times L \,|\, x \wedge c \wedge s = y \wedge c \wedge s\} \\ &= \Delta_s^L.\end{aligned}$$

Hence $\hat{\tilde{\Theta}} \subseteq \Delta_s^L$. On the other hand $\hat{\tilde{\Theta}} \supseteq \Theta$ and therefore $\hat{\tilde{\Theta}} \supseteq \hat{\Theta}$. Since $(s,c) \in \hat{\Theta}$ and $(c,e) \in \hat{\tilde{\Theta}}$, we have $(s,c), (c,e) \in \hat{\tilde{\Theta}}$ so that $(s,e) \in \hat{\tilde{\Theta}}$. This implies $\Delta_s^L \subseteq \hat{\tilde{\Theta}}$. Thus $\Delta_s^L = \hat{\tilde{\Theta}}$. Then it follows that $\tilde{\Theta}$ is open, but this implies $\tilde{\Theta} = \Delta_s^M$ and $s \in M$. Consequently, $s \in M_c$ and hence $\Theta = \Delta_s^{M_c}$. This shows that M_c is a weak locale quotient subframe of L_c. Now, by the first step, M_c is complement closed in L_c. Therefore M is a fully complement closed subframe of L.

(v) Suppose M is an open subframe of L. Then in view of the remark after Proposition 2.3.11 and part (iv) above, M is fully complement closed in L.□

Remark. As proved earlier(Lemma 2.3.3 and Proposition 2.3.5), any closed or equalizer subframe is a weak locale quotient, so facts (ii) and (iii) in the above theorem are consequences of (iv). Also, it seems to be more natural to establish (v) by using the fact that every open subframe is nearly open ($a \to 0 = a^*$ so $i(a \to 0) = i(a) \to i(0) = 0 \Longleftrightarrow i(a^*) = (i(a))^*$).

4.3 Fully complement closed subframes and local connectedness

4.3.1 Definition

A subframe M of L is called *equationally closed* iff any $x \in L$ satisfying the equations $a \vee x = b$ and $a \wedge x = c$ for some $a, b, c \in M$ belongs to M.

Remark. In [39], Pultr and Tozzi investigate the connection between these subframes and quotient spaces. They also define equationally closed homomorphisms and show that open and closed homomorphisms are of this kind. Consequently, open subframes and closed subframes are equationally closed.

4.3.2 Lemma

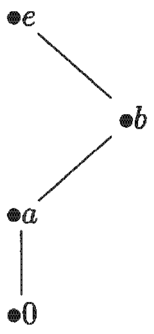
Any equationally closed subframe is fully complement closed.

Proof. Let M be an equationally closed subframe of a frame L. For each $c \in M$, suppose $a \vee b = c$ and $a \wedge b = 0$ for $a \in M$, $b \in L$. Since M is equationally closed, it follows that $b \in M$. Thus M is fully complement closed. □

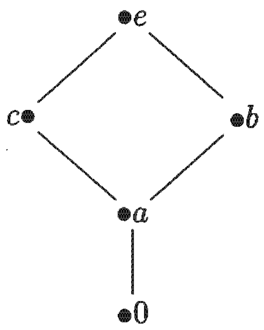
Remark. The converse of the above result does not hold in general as indicated by the following:

Example(Banaschewski[14])

$M =$



$L =$



M is a fully complement closed subframe of L which is not equationally closed.

4.3.3 Lemma

Any equalizer subframe is an equationally closed subframe.

Proof. Let M be an equalizer subframe of L and let $a \vee x = b$ and $a \wedge x = c$ for $a, b, c \in M$ and any $x \in L$. Then, there exist frame homomorphisms $g, h : L \to N$ such that $M = \{x \in L \mid g(x) = h(x)\}$. For such g and h, we get

$$g(a) \vee g(x) = g(b), \quad g(a) \wedge g(x) = g(c) \text{ and}$$
$$h(a) \vee h(x) = h(b), \quad h(a) \wedge h(x) = h(c).$$

Since $a, b, c \in M$ we obtain the following equations:

$$h(a) \vee g(x) = h(b) = h(a) \vee h(x) \text{ and } h(a) \wedge g(x) = h(c) = h(a) \wedge h(x).$$

$$\begin{aligned}\text{Then } g(x) &= g(x) \wedge (h(a) \vee g(x))\\ &= g(x) \wedge (h(a) \vee h(x))\\ &= (g(x) \wedge h(a)) \vee (g(x) \wedge h(x))\\ &= (h(a) \wedge h(x)) \vee (g(x) \wedge h(x))\\ &\leq h(x),\end{aligned}$$

that is, $g(x) \leq h(x)$. By symmetry, $h(x) \leq g(x)$. Hence $g(x) = h(x)$, and this implies $x \in M$. Therefore M is equationally closed. □

Remark. This lemma strengthens part (iii) of Theorem 4.2.5. It should also be noted that the latter, alternatively, follows from a combination of the above two lemmas.

4.3.4 Proposition

Any complement closed subframe of a locally connected frame has a cover of connected elements.

Proof. Let M be a complement closed subframe of a locally connected frame L and for the set $\mathfrak{K} = \{x \in M \,|\, x \text{ coconnected}\}$ define the subset B of M by $B = \{\bigwedge S \,|\, S \text{ a cochain component of } \mathfrak{K}\}$. By Lemma 1.2.8, each $a \in B$ is coconnected and for any $c \in \mathfrak{K}, a \vee c \neq e$ implies that a and c are cochained hence belong to the same cochain component. Since a is the meet of the latter, $a \leq c$. Let $C = \{x \in L \,|\, x \text{ connected }\}$. For each $s \in C$, put $\tilde{s} = \bigvee\{x \in M \,|\, x \leq s^*\}$. Then by Lemma 1.2.3, s^* is coconnected and hence so is $\tilde{s}$ by Lemma 1.2.6.

For any $a \in B$ and $s \in C, s \nleq a$ implies $a \vee \tilde{s} \neq e$ thus, $a \leq \tilde{s}$. It follows that $a \wedge s \leq \tilde{s} \wedge s \leq s^* \wedge s = 0$, that is, $a \wedge s = 0$. Let $\bar{a} = \bigvee\{s \in C \,|\, s \nleq a\}$ then $a \wedge \bar{a} = a \wedge \bigvee\{s \in C \,|\, s \nleq a\} = \bigvee\{a \wedge s \,|\, s \in C, s \nleq a\} = 0$. Moreover, since L is locally connected $a = \bigvee\{t \in C \,|\, t \leq a\}$ and $e = \bigvee C$ so that $a \vee \bar{a} = e$.

By definition of B, for any $s \in C$ there exists $a \leq \tilde{s}$ in B and $s \neq 0$ clearly implies $s \nleq a$ and hence $s \leq \bar{a}$. This shows $\bigvee\{\bar{a} \,|\, a \in B\} = e$. Since M is complement closed, $\bar{a} \in M$ for each $a \in B$ and by Lemma 1.2.5 $\bar{a}$ is connected. Hence M has a cover of connected elements. □

4.3.5 Theorem

Any fully complement closed subframe of a locally connected frame is locally connected.

Proof. Let M be a fully complement closed subframe of a locally connected frame L. Then for any $c \in M$, M_c is a complement closed subframe of L_c. In addition, L_c is locally connected. Hence by the above proposition, M_c has a cover of connected elements, that is, $c = \bigvee x$ for connected $x \leq c$ in M. Therefore M is locally connected. ☐

To appreciate the beauty of the generality of this theorem, we list some nice immediate consequences.

4.3.6 Corollary

Each of the following types of subframes of a locally connected frame is locally connected.
(i) *Any nearly open subframe;*
(ii) *Any weak locale quotient subframe;*
(iii) *Any equationally closed subframe;*
(iv) *Any open (closed) subframe;*
(v) *Any equalizer subframe.*

Proof. Each of these types of subframes of a locally connected frame is fully complement closed and hence by Theorem 4.3.5 locally connected. ☐

Remark. Part (ii) was first established by Li[37] in terms of locales as a point-free generalization of the classical result by Whyburn[46].

For the next theorem we need the following two lemmas:

4.3.7 Lemma

A frame homomorphism $h : M \to L$ with M regular is closed if $h(a) \vee b = e$ implies $a \vee h_*(b) = e$.

Proof. Let $h(x) \vee y \in L$ for $x \in M$ and $y \in L$. Then $h_*(h(x) \vee y) \in M$. Since M is regular, $h_*(h(x) \vee y) = \bigvee\{s \in M \mid s \prec h_*(h(x) \vee y)\}$. Next we observe that $s \prec h_*(h(x) \vee y)$ implies $h_*(h(x) \vee y) \vee s^* = e$ and then $h(h_*(h(x) \vee y)) \vee h(s^*) = h(e) = e$, which implies $h(x) \vee y \vee h(s^*) = e$ so that $h(x \vee s^*) \vee y = e$. Hence by assumption we get $x \vee s^* \vee h_*(y) = e$, that is, $s^* \vee (x \vee h_*(y)) = e$ which means $s \prec x \vee h_*(y)$. Thus $h_*(h(x) \vee y) \leq x \vee h_*(y)$. On the other hand, $hh_*(y) \leq y$ so $h(x) \vee hh_*(y) \leq h(x) \vee y$ and hence by definition of h_*, we have $x \vee h_*(y) \leq h_*(h(x) \vee y)$. In all, $h_*(h(x) \vee y) = x \vee h_*(y)$. Therefore h is closed. □

4.3.8 Lemma

Any frame homomorphism with a regular domain and compact codomain is closed.

Proof. Let $h : M \to L$ be a frame homomorphism with M regular and L compact. Also suppose $h(a) \vee b = e$ for $a \in M$ and $b \in L$. Since M is regular, $a = \bigvee\{t \in M \mid t \prec a\}$ so that $\bigvee\{h(t) \mid t \in M, t \prec a\} \vee b = e$. Since L is compact $h(t_0) \vee b = e$ for some $t_0 \in M$ with $t_0 \prec a$ and thus $t_0^* \vee a = e$. Now $b = (h(t_0^*) \wedge h(t_0)) \vee b = (h(t_0^*) \vee b) \wedge (h(t_0) \vee b) = (h(t_0^*) \vee b) \wedge e = h(t_0^*) \vee b$. Hence $h(t_0^*) \leq b$. Applying h_* to this inequality we obtain $h_*h(t_0^*) \leq h_*(b)$, so that $t_0^* \leq h_*(b)$. Then $a \vee h_*(b) \geq a \vee t_0^* = e$ and thus $a \vee h_*(b) = e$. Therefore by the above lemma, we conclude that h is closed. □

Finally, we are ready for the following:

4.3.9 Theorem

Any regular subframe of a compact locally connected frame is locally connected.

Proof. Let M be a regular subframe of a compact locally connected frame L. Then by Lemma 4.3.8, the identical embedding $i : M \hookrightarrow L$ is closed. In view of the remark following Definition 2.1.1, M is closed. Therefore, by Corollary 4.3.6(iv), M is locally connected. □

Remark. This result was first established via a study of connected congruences by Chen[16] as a solution to the problem posed by Baboolal and Banaschewski[1]. Li[37] obtains the same result through a study of weak locale quotient morphisms, and observes that the regularity of L(in the original conjecture) may not be required. Here our proof simply capitalizes on the fact that M is closed. Moreover, this theorem is a frame version of a central result in the theory of Tychonoff spaces (see [26]) which says any Tychonoff space containing X as a dense subspace is locally connected if X is locally connected and pseudocompact.

Chapter 5

Uniform Local Connectedness of Frames

5.1 Introduction

Local connectedness of a uniform space has been studied by many authors like Baboolal[4] and Collins[20]. In this chapter we look at the notion of uniform local connectedness of a uniform frame introduced by Baboolal[3]. The concept of property S briefly discussed in chapter 3 and property S^* also comes into the picture. An analogue of Gleason's uniformly locally connected coreflection of an arbitrary uniform space(see [25]), the uniformly locally connected reflection for a locally connected uniform frame is given. As in [3], we shall see that this approach leads to the sufficiency part of the Baboolal-Banaschewski's local connectedness theorem. The proof given here is different from the original one in [1] and it is credited to Baboolal. As our main result, we show that the necessity part of the local connectedness theorem in question cannot be obtained similarly.

As a matter of notation and terminology, we fix some basic essentials from the theory of structured frames.

5.1.1 Definition

A subset A of a frame L is called a *cover* of L iff the join of A is the top of L. We denote the set of all covers of L by $\mathrm{Cov}(L)$.

5.1.2 Some Notation

For $A, B \in \mathrm{Cov}(L)$ and $x, y \in L$,
(i) $Ax = \bigvee\{y \in A \mid y \wedge x \neq 0\}$.
(ii) $AB = \{Ax \mid x \in B\}$.
(iii) $A \wedge B = \{x \wedge y \mid x \in A, y \in B\}$.

Remark. By calculating the join, one can see that AB and $A \wedge B$ are covers of L for any $A, B \in \mathrm{Cov}(L)$.

5.1.3 Definition

For any $A, B \in \mathrm{Cov}(L)$ and $x, y \in L$,
(i) We say that A *refines* B (written $A \leq B$) iff for each $x \in A$ there exists $y \in B$ such that $x \leq y$.
(ii) B *star-refines* A (written $B \leq^* A$) iff $BB \leq A$.

5.1.4 Definition

Let $\mathfrak{U}L$ be a set of covers of a frame L and $a, b \in L$. Then,
(i) $a \in L$ is said to be *$\mathfrak{U}L$-strongly below b* or *a is uniformly below b* written $a \lhd_{\mathfrak{U}L} b$ iff $Ca \leq b$ for some $C \in \mathfrak{U}L$.
(ii) $\mathfrak{U}L$ is said to be *admissible* iff for each $a \in L$, $a = \bigvee\{x \in L \mid x \lhd_{\mathfrak{U}L} a\}$.
(iii) $\mathfrak{U}L$ is called a *uniformity* on L iff it is admissible, a filter relative to the relation $\leq$ and satisfies the star-refinement condition, that is, for each $A \in \mathfrak{U}L$ there exists $B \in \mathfrak{U}L$ such that $B \leq^* A$.

Remark. A uniformity without the star-refinement condition is called a *nearness*. The latter was introduced by Banaschewski and Pultr[8] and later revisited by Banaschewski[10]. Herrlich[27] investigates the concept of a nearness in a topological setting. Our study heavily involves uniformities.

5.1.5 Definition

A *uniform frame* is a frame L together with a specified uniformity on it. We denote the given uniformity by $\mathfrak{U}L$. The elements of $\mathfrak{U}L$ are called *uniform covers* of L.

5.1.6 Examples

Let $L = \mathbf{2}$. Then $\operatorname{Cov}(\mathbf{2}) = \{\{e\}, \{0, e\}\}$ and one can easily verify that:

(1) $\{\{0, e\}\}$ and $\operatorname{Cov}(\mathbf{2})$ are uniformities on $\mathbf{2}$.
(2) $\{\{e\}\}$ is not a filter hence not a uniformity on $\mathbf{2}$.

The following fact will be used several times in later results.

5.1.7 Lemma

In any uniform frame L, if $x \lhd_{\mathfrak{U}L} y$ then $x \prec y$.

Proof. Suppose $x \lhd_{\mathfrak{U}L} y$ in a uniform frame L. By definition of the relation $\lhd_{\mathfrak{U}L}$, there exists $A \in \mathfrak{U}L$ such that $Ax \leq y$. Let $z = \bigvee\{a \in A \mid a \wedge x = 0\}$. By definition of z, $x \wedge z = 0$ and $Ax \vee z = \bigvee A = e$. Since $Ax \leq y$, it follows that $z \vee y = e$. Therefore $x \prec y$. □

Recall that a frame L is regular iff $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$, then we have the following:

5.1.8 Lemma

Every uniform frame is regular.

Proof. Let L be a uniform frame. Then $a = \bigvee\{x \in L \mid x \lhd_{\mathfrak{U}L} a\}$ for each $a \in L$ by admissibility of the uniformity. Now, in view of the above lemma we have $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$. Hence L is regular. □

5.1.9 Lemma

Let $A, B \in \operatorname{Cov}(L)$.
(i) *If $A \subseteq B$ then $A \leq B$.*
(ii) *$A \leq B$ does not imply $A \subseteq B$.*

Proof. (i) Suppose $A \subseteq B$. Then $a \in B$ for any $a \in A$. Since $a \leq a$, $A \leq B$.
(ii) In $\operatorname{Cov}(\mathbf{2})$, $\{0, e\} \leq \{e\}$ but $\{0, e\} \nsubseteq \{e\}$. □

Remark. It is immediate from its definition that refinement is transitive.

5.2 Property S and Uniform Local connectedness

5.2.1 Definition

Let L be a uniform frame. Then,

(i) L is said to *have property* S iff, for any uniform cover A of L there exists a finite cover B of L consisting of connected elements such that $B \leq A$.

(ii) L is said to be *uniformly locally connected* iff, for each uniform cover A of L there exists a uniform cover B consisting of connected elements such that $B \leq A$.

(iii) L is said to be *totally bounded* iff the finite uniform covers form a basis for the uniformity on L.

5.2.2 Proposition

In any totally bounded uniformly locally connected uniform frame, each uniform cover is refined by a finite uniform cover consisting of connected elements.

Proof. Let L be a totally bounded uniformly locally connected uniform frame and let $A \in \mathfrak{U}L$. Since L is uniformly locally connected, there exists a uniform cover B consisting of connected elements such that $B \leq A$. By total boundedness there exists a finite uniform cover $C \leq B$. Then $c_i \leq b_i$ for each i. Since C is finite, $C \leq B_n = \{b_1, ..., b_n\} \subseteq B$. Then B_n is the required finite uniform cover consisting of connected elements with $B_n \leq A$. $\square$

5.2.3 Corollary

Any uniformly locally connected uniform frame which is totally bounded has property S.

Proof. Let L be a totally bounded uniformly locally connected uniform frame and let A be a uniform cover of L. Then by the above proposition, there exists a finite uniform cover C consisting of connected elements such that $C \leq A$. Therefore L has property S. $\square$

Remark. Note that the above argument establishes a stronger fact than our assertion since the finite cover required for property S need not be uniform. In what follows, by local connectedness of a uniform frame we mean local connectedness of the underlying frame.

5.2.4 Lemma

Any uniform frame having property S is locally connected.

Proof. Let L be a uniform frame which has property S and let $a \in L$. Then by admissibility, $a = \bigvee\{b \in L \mid b \lhd_{\mathfrak{U}L} a\}$. Recall that for each $b \in L$, $b \lhd_{\mathfrak{U}L} a$ implies the existence of a uniform cover C such that $Cb \leq a$. By property S, there exists a finite cover D of L consisting of connected elements such that $D \leq C$. Let $u = \bigvee\{d \in D \mid d \leq a\}$ and $v = \bigvee\{d \in D \mid d \nleq a\}$. Consider any $d \in D$ with $d \nleq a$. Since $D \leq C$, there exists $c \in C$ such that $d \leq c$. Hence $c \nleq a$. Also, $Cb \leq a$ implies that any element of C whose meet with b is not the bottom lies below a. Since $c \in C$ and $c \nleq a$ it follows that $c \wedge b = 0$. Hence $d \wedge b \leq c \wedge b = 0$ so $d \wedge b = 0$. Thus $v \wedge b = 0$. Since $u \vee v = e$ we then get $b \leq u$. Consequently, $b \leq u \leq a$ for any $b \lhd_{\mathfrak{U}L} a$. Now we have $a = \bigvee\{b \in L \mid b \lhd_{\mathfrak{U}L} a\} \leq \bigvee\{u \mid b \lhd_{\mathfrak{U}L} a\} \leq a$ which implies $a = \bigvee\{u \mid b \lhd_{\mathfrak{U}L} a\}$, a join of connected elements. Therefore L is locally connected. ☐

5.2.5 Lemma

Any uniformly locally connected uniform frame is locally connected.

Proof. The argument is similar to that of Lemma 5.2.4. The only little adjustment is: instead of the finite cover D consisting of connected elements given by property S, we have, due to uniform local connectedness, a uniform cover U of L consisting of connected elements such that $U \leq C$. ☐

Next we construct the uniformly locally connected reflection of a locally connected uniform frame.

5.2.6 Proposition

In any locally connected uniform frame L,
let $\tilde{C} = \{d \in L \mid d$ is a component of some $c \in C\}$ for each $C \in \mathfrak{U}L$.
Then $\mathfrak{B} = \{\tilde{C} \mid C \in \mathfrak{U}L\}$ is a basis for a uniformity $\tilde{\mathfrak{U}}L$ on L.

Proof. We first show that $\tilde{C} \in \mathrm{Cov}(L)$ for each $C \in \mathfrak{U}L$.
Obviously, $C \in \mathrm{Cov}(L)$ which means $e = \bigvee\{c \,|\, c \in C\}$. Since L is locally connected, each c is a join of its connected components, so $c = \bigvee\{d \,|\, d \in \tilde{C}\}$ and thus $e = \bigvee\{c \,|\, c \in C\} = \bigvee\{\bigvee d \,|\, d \in \tilde{C}\} = \bigvee\{d \,|\, d \in \tilde{C}\} = \bigvee \tilde{C}$. Hence $\tilde{C} \in \mathrm{Cov}(L)$.
Next we show that if $E = C \wedge D$ for some $C, D \in \mathfrak{U}L$ then $\tilde{E} \leq \tilde{C}$ and $\tilde{E} \leq \tilde{D}$: Let $u \in \tilde{E}$. Then u is a connected component of some $m \in E$, where $m = c \wedge d$ for some $c \in C$ and $d \in D$. So $u \leq m$ and thus $u \leq c$ and $u \leq d$. Since u is connected, $u \leq p \leq c$ and $u \leq q \leq d$ where p and q are connected components of c and d respectively. Thus $\tilde{E} \leq \tilde{C}$ and $\tilde{E} \leq \tilde{D}$ as asserted.

By the star refinement property, for each $C \in \mathfrak{U}L$ there exists $F \in \mathfrak{U}L$ such that $F \leq^* C$. We show that $\tilde{F} \leq^* \tilde{C}$: Let $t \in \tilde{F}$. Then t is a connected component of some $s \in F$. Since $F \leq^* C$, there exists $c \in C$ such that $Fs \leq c$. Hence $t \leq s \leq Fs \leq c$. Let v be a connected component of c such that $v \geq t$. Then $t \leq v \leq c$. Moreover, we claim that $\tilde{F}t \leq v$: Let $z \in \tilde{F}$ such that $z \wedge t \neq 0$. Then z is a connected component of some $s' \in F$, in particular, $z \leq s'$ so that $t \wedge s' \neq 0$ and hence $s \wedge s' \neq 0$ because $s \wedge s' \geq t \wedge s'$. Since $Fs \leq c$, we have $s' \leq c$ and from $z \leq s' \leq c$ we obtain $z \leq c$. Now $z \wedge t \neq 0$ and $t \leq v$ imply $z \wedge v \neq 0$ and hence $z \leq v$ since v is a connected component of c and $z \leq c$ is connected. This says that any element of $\tilde{F}$ whose meet with t is not the bottom lies below v. Hence $\tilde{F}t \leq v$.

By admissibility of $\mathfrak{U}L$, $a = \bigvee\{b \in L \,|\, b \lhd_{\mathfrak{U}L} a\}$ for any $a \in L$, where $b \lhd_{\mathfrak{U}L} a$ implies the existence of $C \in \mathfrak{U}L$ such that $Cb \leq a$. Let $w \in \tilde{C}$ with $w \wedge b \neq 0$. Then w is a connected component of some $c \in C$ and $c \wedge b \geq w \wedge b \neq 0$ implies $c \wedge b \neq 0$. Hence $c \leq Cb \leq a$. Thus $w \leq a$ for any such w so that $\tilde{C}b \leq a$. Hence $b \lhd_{\tilde{\mathfrak{U}}L} a$ and then $a = \bigvee\{x \in L \,|\, x \lhd_{\tilde{\mathfrak{U}}L} a\}$.
Therefore $\mathfrak{B} = \{\tilde{C} \,|\, C \in \mathfrak{U}L\}$ is a basis for a uniformity $\tilde{\mathfrak{U}}L$ on L. ☐

5.2.7 Proposition

On any locally connected uniform frame L, the uniformity $\tilde{\mathfrak{U}}L$ defined above is finer than $\mathfrak{U}L$ and L is uniformly locally connected with respect to $\tilde{\mathfrak{U}}L$.

Proof. For any $C \in \mathfrak{U}L$, take any $z \in \tilde{C}$ where $\tilde{C} \in \tilde{\mathfrak{U}}L$. Then z is a connected component of some $c \in C$. This implies $z \leq c$ for such $c \in C$. Thus $\tilde{C} \leq C$. Since $\tilde{\mathfrak{U}}L$ is a filter relative to $\leq$, it follows that $C \in \tilde{\mathfrak{U}}L$. Hence $\tilde{\mathfrak{U}}L$ is finer than $\mathfrak{U}L$. Also, for any $D \in \tilde{\mathfrak{U}}L$ there exists $C \in \mathfrak{U}L$ such that $\tilde{C} \leq D$.

Already by its definition, $\tilde{C}$ consists of connected elements. Therefore L is uniformly locally connected with respect to $\tilde{\mathfrak{U}}L$ as asserted. □

5.2.8 Definition

A frame homomorphism $h : M \to L$ between uniform frames is said to be *uniform* iff $h[C] \in \mathfrak{U}L$ for each $C \in \mathfrak{U}M$.

We briefly speak categorically.

The category of uniform frames and uniform frame homomorphisms is denoted by **UniFrm**. In addition, the categories **ULCUniFrm** and **LCUniFrm**, of uniformly locally connected uniform frames and locally connected uniform frames respectively, are full subcategories of **UniFrm**.

Next we recall the following:

5.2.9 Definition

Let $\mathcal{A}$ be a subcategory of $\mathcal{B}$ and $B \in \mathcal{O}b(\mathcal{B})$.

(i) An *$\mathcal{A}$-reflection map* for B is a $\mathcal{B}$-morphism $r : B \to A$, where $A \in \mathcal{O}b(\mathcal{A})$ with the following universal property: For any $\mathcal{B}$-morphism $f : B \to A'$ for some $A' \in \mathcal{O}b(\mathcal{A})$ there exists a unique $\mathcal{A}$-morphism $f' : A \to A'$ such that the triangle

$$\begin{tikzcd} B \arrow[r, "r"] \arrow[dr, "f"'] & A \arrow[d, "f'"] \\ & A' \end{tikzcd}$$

commutes.

(ii) The category $\mathcal{A}$ is called a *reflective subcategory* of $\mathcal{B}$ if and only if each $B \in \mathcal{O}b(\mathcal{B})$ has an $\mathcal{A}$-reflection.

5.2.10 Theorem

ULCUniFrm *is a reflective subcategory of* **LCUniFrm**.

Proof. By Lemma 5.2.5, every uniformly locally connected uniform frame is locally connected. Hence **ULCUniFrm** is a subcategory of **LCUniFrm**. It remains to show that any uniformly locally connected uniform frame has a reflection map. Let $(L, \mathfrak{U}L) \in$ **LCUniFrm** and consider $\tilde{\mathfrak{U}}L$ as described earlier. Then, by Proposition 5.2.7, $(L, \tilde{\mathfrak{U}}L)$ is a uniformly locally connected uniform frame. Since $\mathfrak{U}L \subseteq \tilde{\mathfrak{U}}L$ the identity map $id_L : (L, \mathfrak{U}L) \to (L, \tilde{\mathfrak{U}}L)$ is a uniform frame homomorphism. In the following diagram:

$$\begin{tikzcd}
(L, \mathfrak{U}L) \arrow[r, "g"] \arrow[d, "id_L"'] & (M, \mathfrak{U}M) \arrow[d, "id_M"'] \\
(L, \tilde{\mathfrak{U}}L) \arrow[r, "h"] & (M, \mathfrak{U}M)
\end{tikzcd}$$

Let $g : (L, \mathfrak{U}L) \to (M, \mathfrak{U}M)$ be any uniform frame homomorphism into a uniformly locally connected frame M and let $h : (L, \tilde{\mathfrak{U}}L) \to (M, \mathfrak{U}M)$ be defined by $h(x) = g(x)$ for all $x \in L$. It suffices to show that h is a uniform frame homomorphism. It is evident , from its definition, that h is a frame homomorphism. Let $\tilde{C} \in \tilde{\mathfrak{U}}L$ (a basic element) for $C \in \mathfrak{U}L$. Since g is uniform, $g[C] \in \mathfrak{U}M$. By uniform local connectedness of M, there exists $D \in \mathfrak{U}M$ consisting of connected elements such that $D \leq g[C]$.

Next, we show that $D \leq h[\tilde{C}]$: Let $d \in D$. Since $D \leq g[C]$, there exists $c \in C$ such that $d \leq g(c)$. Since L is locally connected, $c = \bigvee x_i$, where the x_i are connected components of c. We now have $d \leq g(c) = g(\bigvee x_i) = \bigvee g(x_i)$ and hence there exists some j such that $d \wedge g(x_j) \neq 0$. Let $u = g(x_j)$ and $v = \bigvee\{g(x_i) \mid i \neq j\}$. Since $d \leq \bigvee g(x_i)$, we have $d \leq u \vee v$ and $u \wedge v = g(x_j) \wedge \bigvee\{g(x_i) \mid i \neq j\} = \bigvee\{g(x_j) \wedge g(x_i) \mid i \neq j\} = 0$, the last step since distinct components are disjoint and hence so are their images under g. Since d is connected, we have $d \leq u$ or $d \leq v$. Since $d \wedge u \neq 0$ we have $d \leq u = g(x_i) = h(x_i)$. Thus $D \leq h[\tilde{C}]$. Since $\mathfrak{U}M$ is a filter, in particular, an upset and $D \in \mathfrak{U}M$, it follows that $h[\tilde{C}] \in \mathfrak{U}M$. Hence h is a uniform frame homomorphism. Therefore **ULCUniFrm** is a reflective subcategory of **LCUniFrm**. $\square$

Next we characterize totally bounded uniform frames.

5.2.11 Lemma

A uniform frame is totally bounded iff each of its uniform covers is refined by a finite cover.

Proof.($\Leftarrow$) Suppose L is a uniform frame such that each of its uniform covers is refined by a finite cover. Let $A \in \mathfrak{U}L$. By the star-refinement property, there exists some $B \in \mathfrak{U}L$ such that $B \leq^* A$. Take a finite, not necessarily uniform cover $F = \{f_1, f_2, ..., f_n\}$ with $F \leq B$. Then, for each $f_i \in F$ there exists some $b_i \in B$ such that $f_i \leq b_i$ and thus $e = f_i \vee ... \vee f_n \leq b_1 \vee ... \vee b_n$ which implies that $b_1 \vee ... \vee b_n = e$. Since $\{Bb \,|\, b \in B\} \leq A$, then for each $i, Bb_i = \bigvee\{x \in B \,|\, x \wedge b_i \neq 0\} \leq a_i$ in A. Let $D = \{Bb_1, ..., Bb_n\}$. Then for each $c \in B$, $c \wedge b_j \neq 0$ for some j,(otherwise $c \wedge (b_1 \vee ... \vee b_n) = 0$ and hence $c = 0$) which implies $c \leq Bb_j$. Thus $B \leq D$ and then $D \leq A$, where D is a finite uniform cover of L. Therefore L is totally bounded.

($\Rightarrow$) Suppose L is totally bounded. Then, by definition of total boundedness, each uniform cover of L is refined by a finite uniform cover. ☐

5.2.12 Proposition

(i) *Any uniform frame having property S is totally bounded.*

(ii) *Any uniform frame having property S, has property S with respect to each of its coarser uniformities.*

Proof. (i) Let L be a uniform frame such that L has property S and let $A \in \mathfrak{U}L$. By property S, there exists a finite cover B of L consisting of connected elements such that $B \leq A$. Therefore, by the above remark, L is totally bounded.

(ii) Suppose a uniform frame L has property S with respect to $\mathfrak{U}L$ and let $\mathfrak{U}'L$ be any coarser uniformity on L. Take any $A \in \mathfrak{U}'L$. Then $A \in \mathfrak{U}L$. Hence by property S relative to $\mathfrak{U}L$, there exists a finite cover B of L consisting of connected elements such that $B \leq A$. Therefore L has property S with respect to $\mathfrak{U}'L$. ☐

Remark. It would be interesting to know whether a result similar to the second part of the above proposition holds for uniform local connectedness.

5.2.13 Proposition

A locally connected uniform frame has property S iff its uniformly locally connected reflection has property S.

Proof. ($\Rightarrow$) Let $(L, \mathfrak{U}L)$ be a locally connected uniform frame having property S and let $(L, \tilde{\mathfrak{U}}L)$ be the uniformly locally connected reflection of $(L, \mathfrak{U}L)$. Then, for any basic element $\tilde{A} \in \tilde{\mathfrak{U}}L$, we have $A \in \mathfrak{U}L$. Since $(L, \mathfrak{U}L)$ has property S, there exists a finite cover B of L consisting of connected elements such that $B \leq A$. We show that $B \leq \tilde{A}$: Since $B \leq A$ then $b \in B$ implies $b \leq a$ for some $a \in A$. Since b is connected, clearly $b \leq t$ for some connected component t of a. Hence $B \leq \tilde{A}$, which means $(L, \tilde{\mathfrak{U}}L)$ has property S.

($\Leftarrow$) Suppose $(L, \tilde{\mathfrak{U}}L)$ has property S. Since $\mathfrak{U}L \subseteq \tilde{\mathfrak{U}}L$, it follows from Proposition 5.2.12 that $(L, \mathfrak{U}L)$ has property S. □

5.2.14 Definition

A cover A of L is called *normal* iff there exists a sequence $(A_k)_k$ of covers of L such that $A = A_1$ and $A_{k+1} \leq^* A_k$ for all $k = 1, 2, 3 \ldots$.

Remark. According to Baboolal[3], any uniform frame $(L, \mathfrak{U}L)$ has a finest uniformity $\mathfrak{U}_F L$ on L consisting of normal covers of L. In 1957, Henriksen and Isbell[26] already demonstrated the importance of normal covers of a space where a locally connected space is characterized in terms of such covers.

The next result, which is mentioned in Banaschewski[13], connects uniformities and compactifications.

5.2.15 Lemma

A frame is uniformizable iff it is compactifiable. □

5.2.16 Proposition

Let L be a uniformizable frame. Then L is locally connected iff L is uniformly locally connected with respect to its finest uniformity.

Proof. For a uniformizable frame L, let $\mathfrak{U}_F L$ be the finest uniformity on L.

($\Rightarrow$) Suppose L is locally connected. Then by Proposition 5.2.7, $(L, \tilde{\mathfrak{U}}_F L)$ is uniformly locally connected and $\mathfrak{U}_F L \subseteq \tilde{\mathfrak{U}}_F L$. Since $\mathfrak{U}_F L$ is the finest uniformity on L, it follows that $\tilde{\mathfrak{U}}_F L \subseteq \mathfrak{U}_F L$ and hence $\tilde{\mathfrak{U}}_F L = \mathfrak{U}_F L$. Therefore $(L, \mathfrak{U}_F L)$ is uniformly locally connected.

($\Leftarrow$) Suppose $(L, \mathfrak{U}_F L)$ is uniformly locally connected. Then by Lemma 5.2.5, L is locally connected. ☐

5.2.17 Lemma

Let $h : M \to L$ be a dense onto frame homomorphism with $h_* : L \to M$ its right adjoint. Then $a \prec b$ iff $h_*h(a) \prec b$ for $a, b \in M$.

Proof.($\Rightarrow$) Suppose $a \prec b$ in L. Then, there exists $c \in L$ such that $a \wedge c = 0$ and $b \vee c = e$. Applying h to the latter yields $h(h_*h(a) \wedge c) = hh_*h(a) \wedge h(c) = h(a) \wedge h(c) = 0$, the second last equality because h is onto. By denseness of h, we deduce $h_*h(a) \wedge c = 0$. We already have $b \vee c = e$. Therefore $h_*h(a) \prec b$.

($\Leftarrow$) Suppose $h_*h(a) \prec b$. Then by definition of h_*, we have $a \leq h_*h(a) \prec b$. Hence by a general property of the relation $\prec$, it follows that $a \prec b$. ☐

5.2.18 Definition

A uniform frame homomorphism $h : M \to L$ is called a *surjection* iff h is onto and the uniform covers $h[A]$ with $A \in \mathfrak{U}M$, generate $\mathfrak{U}L$.

5.2.19 Theorem

Any dense surjection between uniform frames reflects uniform local connectedness.

Proof. Let $h : M \to L$ be a dense surjection with L uniformly locally connected and let $A \in \mathfrak{U}M$. Then by the star-refinement property of $\mathfrak{U}M$, there exists $B \in \mathfrak{U}M$ such that $B \leq^* A$. Since h is uniform, $h[B] \in \mathfrak{U}L$. By uniform local connectedness of L, there exists $C \in \mathfrak{U}L$ consisting of connected elements with $C \leq h[B]$ and since h is surjective there exists some $D \in \mathfrak{U}M$ such that $h[D] \leq C$. Now consider $h_*[C] = \{h_*(c) \mid c \in C\}$. Since each c is connected and h is onto we have $hh_*(c) = c$ connected. By denseness of h, $h_*(c)$ is connected. We show that $D \leq h_*[C]$ which would imply

$h_*[C] \in \mathfrak{U}M$: Let $d \in D$. Then $h(d) \leq c$ for some $c \in C$. Thus, by the property of h_*, we get $d \leq h_*(c)$. Now, we claim that $h_*[C] \leq A$: Let $c \in C$. Since $C \leq h[B]$ there exists $b \in B$ such that $c \leq h(b)$. In addition, $Bb \leq a$ for some $a \in A$ so that by Lemma 5.1.7 we have $b \prec a$ and hence $h_*h(b) \prec a$ by Lemma 5.2.17. Thus $h_*(c) \leq h_*h(b) \leq a$.
In summary, for any $A \in \mathfrak{U}M$, $h_*[C]$ is a uniform cover of M consisting of connected elements such that $h_*[C] \leq A$. Therefore M is uniformly locally connected. ☐

The above result has an application in the study of a completion of a uniform frame. In their article, *Samuel compactification and completion of uniform frames*[9], Banaschewski and Pultr describe the completion of a uniform frame using compact regular coreflection of uniform frames. A systematic study of the completion is done in [10].

Before we apply the above theorem, we recall some concepts.

5.2.20 Definition

A frame L is *complete* iff any dense surjection $h : M \to L$ is an isomorphism.

5.2.21 Definition

A *completion* of a uniform frame L is a complete uniform frame M together with a dense surjection $h : M \to L$.

According to Banaschewski and Pultr[9], any uniform frame has a unique completion $\gamma : CL \to L$. The reader is referred to [10] for the description of CL. Also, Kriz[35] gives a description of uniform completion in locales. Banaschewski[15] characterizes Cauchy complete uniform frames in terms of this completion. In a classical setting, a brief discussion of the completion of a uniform space appears in [26]. Also see Isbell[29].

We are now ready to give the promised application of Theorem 5.2.19.

5.2.22 Corollary

The uniform completion of any uniformly locally connected uniform frame is uniformly locally connected.

Proof. Let L be a uniform frame which is uniformly locally connected. Consider the uniform completion $\gamma : CL \to L$ of L. By definition of a completion, γ is a dense surjection. Since L is uniformly locally connected, by Theorem 5.2.19, CL is also uniformly locally connected. ☐

Remark. In her PhD thesis[45], Walters-Wayland shows that a completely regular frame L is pseudocompact iff L is totally bounded with respect to each of its uniformities. We employ this result in proving the following fact.

5.2.23 Proposition

Let L be a completely regular frame. Then, L is locally connected and pseudocompact iff L has property S with respect to each of its uniformities.

Proof. ($\Rightarrow$) Let L be a locally connected and pseudocompact completely regular frame. By the above noted Walters-Wayland's result, L is totally bounded with respect to each of its uniformities. Since L is locally connected, by Proposition 5.2.16, L is uniformly locally connected with respect to its fine uniformity. Hence by Proposition 5.2.13, L has property S with respect to its fine uniformity and hence with respect to each of its uniformities by Proposition 5.2.12.
($\Leftarrow$) Suppose L has property S with respect to each of its uniformities. Then by Proposition 5.2.12, L is totally bounded with respect to each of its uniformities. Again by the result of Walters-Wayland, L is pseudocompact and by Lemma 5.2.4 L is locally connected. ☐

5.3 Property S^* and (Uniform) Local Connectedness

In this section, we define property S^* introduced by Baboolal[3]. We aim to show that if $h : M \to L$ is a dense onto frame homomorphism between completely regular frames and L is locally connected and pseudocompact then

M is locally connected. This will yield the sufficiency part of the Baboolal-Banaschewski local connectedness theorem. Note that, in view of Proposition 5.2.23, our hypothesis implies that L has property S. Proving that such an h reflects property S would complete the proof. However, we do not have such a direct proof at the moment. We will appeal to the fact that property S is equivalent to property S^* which is reflected by dense uniform frame homomorphisms.

Recall the definition of a Chen-connected congruence in chapter 2.

5.3.1 Definition

A uniform frame L *has property* S^* iff given any $A \in \mathfrak{U}L$ there exists a finite number of Chen-connected congruences $\Theta_1, \Theta_2, ..., \Theta_n$ such that $\bigwedge \Theta_i = \Delta^L$ and for each i there exists $a_i \in A$ such that $\Delta^L_{a_i} \leq \Theta_i$.

Remark. Note the vague similarity between this definition and that of property S. The latter requires connected elements of the uniform frame whereas the former involves Chen-connected congruences on the uniform frame.

5.3.2 Proposition

If the codomain of any dense uniform frame homomorphism has property S^* then so does its domain.

Proof. Let $h : M \to L$ be a dense uniform frame homomorphism and suppose L has property S^*. Let $A \in \mathfrak{U}M$. Then, by star-refinement property, there exists $B \in \mathfrak{U}M$ such that $B \leq^* A$. Since h is uniform, $h[B] \in \mathfrak{U}L$. By property S^* of L, there exists Chen-connected congruences $\Theta_1, \Theta_2, ..., \Theta_n$ on L such that $\bigwedge \Theta_i = \Delta^L$ and for each i, there exists $b_i \in B$ such that $\Delta_{h(b_i)} \leq \Theta_i$. Now, put $\Phi_i = (h \times h)^{-1}(\Theta_i)$. Then, by Lemma 2.2.8, Φ_i is Chen-connected and so is $\bar{\Phi}_i$ by Proposition 2.2.7. Since $\bar{\Phi}_i \leq \Phi_i$ for each i and $\bigwedge \Theta_i = \Delta^L$ it follows that $\bar{\Phi}_1 \wedge ... \wedge \bar{\Phi}_n \leq \Phi_1 \wedge ... \wedge \Phi_n = \Delta^M$, the last equality by Lemma 2.3.16. This implies $\bar{\Phi} \wedge ... \wedge \bar{\Phi}_n = \Delta^M$. Since $B \leq^* A$, there exists $a_i \in A$ such that $Bb_i \leq a_i$ for each $i = 1, 2, ..., n$ and $b_i \in B$. By Lemma 5.1.7, $b_i \prec a_i$ and hence $b_i^* \vee a_i = e$. Next, we show that $\Delta_{a_i} \leq \bar{\Phi}_i = \nabla^M_{x_i}$: Since Δ_{a_i} is the largest congruence containing (a_i, e), it suffices to show that $(a_i, e) \in \nabla^M_{x_i}$ or $a_i \vee x_i = x_i \vee e = e$. Since $b_i^* \vee a_i = e$,

it is sufficient to show that $b_i^* \leq x_i$. For any $(s,t) \in \Delta_{b_i}$, we have
$$\begin{aligned} s \wedge b_i = t \wedge b_i &\Longrightarrow h(s) \wedge h(b_i) = h(t) \wedge h(b_i) \\ &\Longrightarrow (h(s), h(t)) \in \Delta_{h(b_i)} \leq \Theta_i \\ &\Longrightarrow (s,t) \in (h \times h)^{-1}(\Theta_i) = \Phi_i. \end{aligned}$$

Thus $\Delta_{b_i} \leq \Phi_i$. Since $(b_i, e) \in \Delta_{b_i}$ we have $(b_i, e) \in \Phi_i$ and because $b_i^* \vee a_i = e$ it follows that $(b_i, b_i^* \vee a_i) \in \Phi_i$. Now we have $(b_i^* \wedge b_i, b_i^* \wedge (b_i^* \vee a_i)) = (0, b_i^*) \in \Phi_i$. But x_i is the largest element of M such that $(0, x_i) \in \Phi_i$, hence $b_i^* \leq x_i$. Thus $x_i \vee a_i = e$ so that $(a_i, e) \in \nabla_{x_i} = \bar{\Phi}_i$ and hence $\Delta_{a_i} \leq \bar{\Phi}_i$.
In summary, for the $A \in \mathfrak{U}M$ we began with, $\bar{\Phi}_1, \ldots, \bar{\Phi}_n$ are the disjoint Chen-connected congruences and for each i we found $a_i \in A$ such that $\Delta_{a_i} \leq \Phi_i$. Therefore M has property S^*. $\square$

5.3.3 Lemma

If $a \prec b$ in any frame and $\Delta_a \leq \Theta$ in the congruence frame then $\Delta_b \leq \bar{\Theta}$.

Proof. Suppose $a \prec b$ in a frame L and $\Delta_a \leq \Theta$ in $\mathfrak{C}L$. Then there exists $x \in L$ such that $a \wedge x = 0$ and $x \vee b = e$. Let $\bar{\Theta} = \nabla_c$, where c is the largest element of L such that $(0, c) \in \Theta$. To prove the assertion, as in the above proof, it suffices to show that $c \vee b = e$. We note that $\Delta_a \leq \Theta$ implies $(a, e) \in \Theta$ so that $(a \wedge x, e \wedge x) = (0, x) \in \Theta$. By the property of c, we deduce that $x \leq c$. Thus $b \vee c = e$. Hence $(b, e) \in \nabla_c$. Finally, since Δ_b is the smallest congruence containing (b, e), and $\nabla_c = \bar{\Theta}$ we have $\Delta_b \leq \bar{\Theta}$. $\square$

5.3.4 Proposition

For any uniform frame having property S^*, each of its open congruences is a meet of Chen-connected closed congruences.

Proof. Let L be a frame which has property S^* and let Δ_a be any open congruence on L. By admissibility, $a = \bigvee\{x \in L \mid x \triangleleft_{\mathfrak{U}L} a\}$. By definition of $\triangleleft_{\mathfrak{U}}$, for such x, there exists $A \in \mathfrak{U}L$ such that $Ax \leq a$. By star-refinement, there exists $B \in \mathfrak{U}L$ such that $B \leq^* A$. Since L has property S^*, there is a finite number of Chen-connected $\Theta_1, \ldots, \Theta_n$ such that $\bigwedge \Theta_i = \Delta$ and for each $i = 1, 2, \ldots, n$ there exists $b_i \in B$ such that $\Delta_{b_i} \leq \Theta_i$. Also, $B \leq^* A$ implies that there exists some $a_i \in A$ such that $Bb_i \leq a_i$ so that $b_i \prec a_i$ for $i = 1, 2, \ldots, n$ and hence $\Delta_{a_i} \leq \bar{\Theta}_i$ by Lemma 5.3.3. Since $\bar{\Theta}_i \leq \Theta_i$ for each

i and $\bigwedge \Theta_i = \Delta$ we have $\bigwedge\{\bar{\Theta}_i \,|\, i = 1, 2, ..., n\} = \Delta$.
Assume $\Delta_x \vee \bar{\Theta}_i = \nabla$ for all i. Then $\Delta_x \vee \bigwedge\{\bar{\Theta}_i \,|\, i = 1, 2, ..., n\} = \Delta_x \vee \Delta = \bigwedge\{\Delta_x \vee \bar{\Theta}_i \,|\, i = 1, 2, ..., n\} = \nabla$. Hence $\Delta_x \vee \bigwedge\{\bar{\Theta}_i \,|\, i = 1, 2, ..., n\} = \Delta_x$ and thus $\Delta_x = \nabla$. The latter implies $(0, e) \in \Delta_x$ so that $0 = x \wedge e = x$, contradicting the hypothesis $(x \neq 0)$ in the admissibility condition. Hence there exists at least one j for which $\Delta_x \vee \bar{\Theta}_j \neq \nabla$. Consider $\bar{\Theta}_1, ..., \bar{\Theta}_n$ such that $\Delta_x \vee \bar{\Theta}_i \neq \nabla$. We show that $\Delta_a \leq \bigwedge\{\bar{\Theta}_i \,|\, \Delta_x \vee \bar{\Theta}_i \neq \nabla\} \leq \Delta_x$:
For the first inequality, recall that $b_i \prec a_i$ and $\Delta_{a_i} \leq \bar{\Theta}_i$ for all i. Then $a_i \wedge x \neq 0$ otherwise we get $\Delta_{a_i} \vee \Delta_x = \Delta_{a_i \wedge x} = \Delta_0 = \nabla$ so that $\bar{\Theta}_i \vee \Delta_x = \nabla$, a contradiction. Now, $a_i \wedge x \neq 0$ together with $Ax \leq a$ implies $a_i \leq a$. Hence $\Delta_a \leq \Delta_{a_i} \leq \bar{\Theta}_i$. Therefore $\Delta_a \leq \bigwedge\{\bar{\Theta}_i \,|\, \Delta_x \vee \bar{\Theta}_i \neq \nabla\}$.
For the second inequality, note that $\bigwedge\{\bar{\Theta}_i \,|\, i = 1, ..., n\} = \Delta \leq \Delta_x$ and then

$$\bigwedge\{\bar{\Theta}_i \,|\, \Delta_x \vee \bar{\Theta}_i \neq \nabla\} \wedge \bigwedge\{\bar{\Theta}_i \,|\, \bar{\Theta}_i \vee \Delta_x = \nabla\} \leq \Delta_x$$

which implies $\bigwedge\{\bar{\Theta}_i \,|\, \Delta_x \vee \bar{\Theta}_i \neq \nabla\} \wedge \bigwedge\{\bar{\Theta}_i \vee \Delta_x \,|\, \bar{\Theta}_i \vee \Delta_x = \nabla\} \leq \Delta_x$.

Then clearly, $\bigwedge\{\bar{\Theta}_i \,|\, \Delta_x \vee \bar{\Theta}_i \neq \nabla\} \wedge \nabla = \bigwedge\{\bar{\Theta}_i \,|\, \Delta_x \vee \bar{\Theta}_i \neq \nabla\} \leq \Delta_x$. This verifies the second inequality. Hence $\Delta_a \leq \bigwedge\{\bar{\Theta}_i \,|\, \Delta_x \vee \bar{\Theta}_i \neq \nabla\} \leq \Delta_x$.
Since $a = \bigvee\{x \in L \,|\, x \triangleleft_{\mathfrak{U}L} a\}$, it follows that $\Delta_a = \Delta_{\bigvee\{x \in L \,|\, x \triangleleft_{\mathfrak{U}L} a\}} = \bigwedge \Delta_x$. This says Δ_a is a greatest lower bound of the Δ_x. Since $\Delta_a \leq \bigwedge\{\bar{\Theta}_i \,|\, \Delta_x \vee \bar{\Theta}_i \neq \nabla\}$ and $\bigwedge\{\bar{\Theta}_i \,|\, \Delta_x \vee \bar{\Theta}_i \neq \nabla\}$ is also a lower bound of the Δ_x we get $\Delta_a = \bigwedge\{\bar{\Theta}_i \,|\, \Delta_x \vee \bar{\Theta}_i \neq \nabla\}$. Since Θ_i is Chen-connected then so is $\bar{\Theta}_i$ for each i. Then we have expressed Δ_a as a meet of closed Chen-connected congruences, which completes the proof. □

5.3.5 Proposition

Any uniform frame having property S^* is locally connected.

Due to its length and technicality, the proof is omitted. It appears in [3].

5.3.6 Lemma

In any locally connected frame L, if $\Delta_a \leq \Theta \neq \nabla$, where Θ is Chen-connected, then there exists a Chen-connected Δ_b such that $\Delta_a \leq \Delta_b \leq \Theta$.

Proof. Let L be a locally connected frame and let $\Delta_a \leq \Theta \neq \nabla$ with Θ Chen-connected. Since L is locally connected, $a = \bigvee\{x \in L \,|\, x \,\text{connected}\}$.

We claim that for some of these x, $\Theta \vee \Delta_x \neq \nabla$: Assume the contrary, that is, $\Delta_x \vee \Theta = \nabla$ for all such x. Then,

$$\nabla_x = \nabla_x \wedge \nabla = \nabla_x \wedge (\Delta_x \vee \Theta) = (\nabla_x \wedge \Delta_x) \vee (\nabla_x \wedge \Theta) = \Delta \vee (\nabla_x \wedge \Theta) = \nabla_x \wedge \Theta.$$

Hence $\nabla_x \leq \Theta$. This would imply $\nabla_a = \bigvee \nabla_x \leq \Theta$ for connected $x \leq a$ and hence $\Delta_a \vee \nabla_a \leq \Theta$. Thus $\nabla = \Theta$, contradicting the hypothesis $\Theta \neq \nabla$. Hence there exists some connected $x \leq a$ such that $\Delta_x \vee \Theta \neq \nabla$.
Let $b = \bigvee\{x \in L \mid x \,\text{connected},\ x \leq a,\ \Delta_x \vee \Theta \neq \nabla\}$. Then, by properties of Δ as a map, it follows that $\Delta_b = \bigwedge\{\Delta_x \mid x \,\text{connected},\ x \leq a, \Delta_x \vee \Theta \neq \nabla\}$. Since $\Delta_a \leq \Theta$, we have $\Delta_a \vee \Theta = \Theta$, which implies $\bigwedge \Delta_x \vee \Theta = \Theta$. By Lemma 2.1.10, we deduce that $\bigwedge(\Theta \vee \Delta_x) = \Theta$ for connected $x \leq a$. Now, this implies $\bigcap(\Theta \vee \Delta_x) = \Theta$ for connected $x \leq a$ with $\Theta \vee \Delta_x \neq \nabla$. The latter implies that $\Theta \vee \bigwedge \{ \Delta_x \mid x \,\text{connected},\ x \leq a, \Delta_x \vee \Theta \neq \nabla\} = \Theta$, by Lemma 2.1.10 again. Then $\Theta \vee \Delta_b = \Theta$. Hence $\Delta_b \leq \Theta$.
It remains to show that Δ_b, as defined above, is Chen-connected. Since Δ_x is Chen-connected and $\Delta_x \vee \Theta \neq \nabla$, it follows that $\Delta_x \vee \bar{\Theta} \neq \nabla$ and hence by Proposition 2.2.5, $\Delta_x \wedge \Theta$ is Chen-connected for all such x. For a fixed x_1 and any x_2 we get

$$(\Delta_{x_1} \wedge \Theta) \vee (\Delta_{x_2} \wedge \Theta) = \Theta \wedge (\Delta_{x_1} \vee \Delta_{x_2}) \leq \Theta \neq \nabla.$$

Again by Proposition 2.2.5, $\bigwedge\{\Delta_x \wedge \Theta \mid x \,\text{connected},\ x \leq a, \Delta_x \vee \Theta \neq \nabla\}$ is Chen-connected. This says $\Theta \wedge \Delta_b$ is Chen-connected. Since $\Delta_b \leq \Theta$, $\Theta \wedge \Delta_b = \Delta_b$. Therefore Δ_b is Chen-connected and $\Delta_a \leq \Delta_b \leq \Theta$. ☐

5.3.7 Theorem

A uniform frame has property S^* iff it has property S.

Proof.($\Rightarrow$) Let L be a uniform frame having property S^* and let $A \in \mathfrak{U}L$. By definition of property S^*, there exists Chen-connected congruences $\Theta_1, ..., \Theta_n$ such that $\bigwedge \Theta_i = \Delta$ and for each $i = 1, ..., n$ there exists $a_i \in A$ such that $\Delta_{a_i} \leq \Theta_i$. Without loss of generality, we assume that $\Theta_i \neq \nabla$ for all i, otherwise any $\Theta_j = \nabla$ could be deleted and the above condition would still hold. Again, since L has property S^*, it follows from Proposition 5.3.5 that L is locally connected and by Lemma 5.3.6, there exists connected b_i such that $\Delta_{a_i} \leq \Delta_{b_i} \leq \Theta_i$ for all i. Thus $b_i \leq a_i$ for each i. Also, $\Delta_{b_1} \wedge ... \wedge \Delta_{b_n} \leq \bigwedge \Theta_i = \Delta = \Delta_e$, that is, $\Delta_{b_1} \wedge ... \wedge \Delta_{b_n} \leq \Delta_e$. Since Δ sends joins to meets,

$\Delta_{b_1 \vee ... \vee b_n} = \Delta_e$. Since Δ is one-one $b_1 \vee ... \vee b_n = e$. Hence $\{b_1, ..., b_n\}$ is a finite cover of L consisting of connected elements and $\{b_1, ..., b_n\} \leq A$. Therefore L has property S.

($\Leftarrow$) Suppose L has property S and let $A \in \mathfrak{U}L$. By definition of property S, there exists a finite cover $\{b_1, ..., b_n\}$ of L with each b_i connected such that $\{b_1, ..., b_n\} \leq A$. The latter implies that for each b_i there is some $a_i \in A$ such that $b_i \leq a_i$. Since Δ reverses order, it follows that $\Delta_{a_i} \leq \Delta_{b_i}$. Since b_i is connected, Δ_{b_i} is Chen-connected for each i. Finally, because Δ sends joins to meets, we have

$$\bigwedge \Delta_{b_i} = \Delta_{b_i} \wedge ... \wedge \Delta_{b_n} = \Delta_{\bigvee b_i} = \Delta_e = \Delta.$$

Therefore L has property S^*. ☐

We list some facts which correspond to early results involving property S:

5.3.8 Proposition

(i) *Any totally bounded uniformly locally connected uniform frame has property S^*.*

(ii) *Any uniform frame having property S^* is totally bounded.*

(iii) *Any uniform frame having property S^* has property S^* with respect to each of its coarser uniformities.*

(iv) *Let L be a completely regular frame. Then L is locally connected and pseudocompact iff L has property S^* relative to each of its uniformities.*

Proof. (i) This holds by Theorem 5.3.7 and Corollary 5.2.3.
(ii) This holds by Proposition 5.2.12 and Theorem 5.3.7 .
(iii) By Proposition 5.2.12 and Theorem 5.3.7.
(iv) This follows from Proposition 5.2.23 and Theorem 5.3.7. ☐

5.3.9 Theorem

Let M and L be completely regular frames. If $h : M \to L$ is a dense onto frame homomorphism and L is locally connected and pseudocompact then M is locally connected.

Proof. Since M and L are completely regular, they are compactifiable and hence uniformizable. Let $\mathfrak{U}M$ be any uniformity on M. Then by the above lemma, $\{h[C] \mid C \in \mathfrak{U}M\}$ generates a uniformity on L. Hence h is a dense surjection. Since L is locally connected and pseudocompact it follows, by Proposition 5.2.23, that L has property S. By Theorem 5.3.7, L has property S^*. Since any surjection is uniform, h is a dense uniform frame homomorphism. Thus, by Proposition 5.3.2, M has property S^* and hence locally connected by Proposition 5.3.5. $\square$

As a corollary, we obtain the promised sufficiency part of the Baboolal-Banaschewski local connectedness theorem.

5.3.10 Corollary

Let L be a completely regular frame. If L is locally connected and pseudocompact then its Stone-Čech compactification βL is locally connected.

Proof. Let $\bigvee : \beta L \to L$ be the Stone-Čech compactification of a completely regular frame L. Suppose L is locally connected and pseudocompact. Then $\bigvee$ is a dense onto frame homomorphism. Since L is locally connected and pseudocompact, by the above theorem, βL is locally connected. $\square$

To close this section, we show that the necessity part of the local connectedness theorem cannot be obtained in a similar fashion. We verify that for any dense onto frame homomorphism $h : M \to L$ between completely regular frames, if M is locally connected then, in general, L is not pseudocompact. We do this by providing a counter-example. Recall the following familiar facts:

fact 1. A T_0-space X is completely regular iff $\mathfrak{O}X$ is completely regular.
fact 2. A space X is pseudocompact iff $\mathfrak{O}X$ is pseudocompact.

5.3.11 Lemma

A topological space X is locally connected iff $\mathfrak{O}X$ is locally connected.

Proof. ($\Rightarrow$) Suppose X is locally connected and let $O \in \mathfrak{O}X$. Then, by a familiar characterization of open sets, O contains a neighbourhood of each of its points, that is, $O \supseteq N_x$ for each $x \in O$. Since X is locally connected, each $x \in X$ has a neighbourhood base of open connected sets. In particular, each $x \in O$ satisfies the latter condition since $O \subseteq X$. So we have $x \in C_x \subseteq N_x \subseteq O$ and hence $O = \bigcup\{C_x \mid C_x \,\text{open and connected}\}$. Therefore $\mathfrak{O}X$ is locally connected.

($\Leftarrow$) Suppose $\mathfrak{O}X$ is locally connected and let $x \in X$. Then for each $O \in \mathfrak{O}X$, $O = \bigcup\{C \in \mathfrak{O}X \mid C \subseteq O \,\text{connected}\}$. In particular, since $X \in \mathfrak{O}X$, we have $X = \bigcup\{C \in \mathfrak{O}X \mid C \,\text{connected}\}$. Now, $x \in X$ implies $x \in C$ for some open and connected $C \subseteq X$. Thus each x has a neighbourhood base of open and connected sets. Therefore X is locally connected. $\square$

Now, consider the dense embedding $i : (0,1) \hookrightarrow [0,1]$. Both $(0,1)$ and $[0,1]$ are completely regular since they are subspaces of the real space $\mathbb{R}$ which is completely regular (see for example Willard[48] p.95, Example 14.9 and Theorem 14.10 or [22]). It can be shown that $[0,1]$ is locally connected. Recall that for metric spaces, compactness is equivalent to pseudocompactness([19]). Then, being not compact, $(0,1)$ is not pseudocompact.

Counterexample. We transfer the above into a frame setting by using the open set functor $\mathfrak{O}$, and get:

$$\mathfrak{O}i = i^{-1} : \mathfrak{O}([0,1]) \to \mathfrak{O}((0,1)).$$

By fact 1, $\mathfrak{O}([0,1])$ and $\mathfrak{O}((0,1))$ are completely regular and by fact 2, $\mathfrak{O}((0,1))$ is not pseudocompact. Since $[0,1]$ is locally connected, by Lemma 5.3.11, we deduce that $\mathfrak{O}([0,1])$ is locally connected. That $\mathfrak{O}i$ is onto rests on the fact that $(0,1) \subseteq [0,1]$. We show that $\mathfrak{O}i$ is dense: Suppose $\mathfrak{O}i(U) = \emptyset$. Then $i^{-1}(U) = \emptyset$ and hence $U = ii^{-1}(U) = i(\emptyset)$, that is, $U = \emptyset$. Thus $\mathfrak{O}i$ is dense. Therefore $i^{-1} : \mathfrak{O}([0,1]) \to \mathfrak{O}((0,1))$ is a dense onto frame homomorphism between completely regular frames, where $\mathfrak{O}([0,1])$ is locally connected but $\mathfrak{O}((0,1))$ is not pseudocompact.

The space version of this counterexample was hinted to me by Professor Baboolal. Many thanks.

Concluding Remarks.

To avoid the danger of confusion, we remind the reader that the converse of Corollary 5.3.10 holds in general (Theorem 3.3.3). The point here is that one cannot establish it using the methods that led to the corollary.

We have not been able to answer the following question: Given a uniformly locally connected uniform frame $(L, \mathfrak{U}L)$ and another uniformity $\mathfrak{U}'L$ on L such that $\mathfrak{U}'L \subseteq \mathfrak{U}L$, is $(L, \mathfrak{U}'L)$ also uniformly locally connected?

Bibliography

[1] D. Baboolal and B. Banaschewski, *Compactification and local connectedness of frames*, J. Pure Appl. Algebra 70 (1991) 3-16.

[2] D. Baboolal and B. Banaschewski, *Perfect compactification of frames*, Manuscript, 2000.

[3] D. Baboolal, *Local connectedness Made Uniform,* Appl. Cat. Structures 8 (2000), 377-390.

[4] D. Baboolal, *On local connectedness*, Math. Japonica 37, 5(1992), 865-869.

[5] B. Banaschewski, *Lectures on Frames*, Univ. of Cape Town, 1988.

[6] B. Banaschewski and C. Gilmour, *Pseudocompactness and the cozero part of a frame*, Comment. Math. Univ. Carolinae 37, 3(1996) 577-587.

[7] B. Banaschewski and C.J. Mulvey, *Stone-Čech compactification of locales I*, Houston J of Math. Vol 6, 3(1980), 301-312.

[8] B. Banaschewski and A. Pultr, *Cauchy points of uniform and nearness*, Quaestiones Mathematicae, 19. 1-2(1996), 101-127.

[9] B. Banaschewski and A. Pultr, *Samuel compactification and completion of uniform frames*, Math. Proc. Camb. Phil. Soc. 108 (1990), 63-78.

[10] B. Banaschewski, *Completion in pointfree topology*, Lecture Notes in Mathematics and Applied Mathematics, No.2/96 Univ. of Cape Town, 1996.

[11] B. Banaschewski, *Another look at the localic Tychonoff Theorem. Comment.* Math. Univ. Carolinae 29, 4(1988), 647-656.

[12] B. Banaschewski, *Compactification of frames*, Math. Nachr. 149 (1990) 105-116.

[13] B. Banaschewski, *Equalizers in frames*, Research seminar notes, Univ. of Cape Town, 1992.

[14] B. Banaschewski, *On locally connected frames*, Socat 94, Univ. of Cape Town, 1994.

[15] B. Banaschewski, *The real numbers in pointfree topology*, Textos de Matemática, Série B, Departamento de Matemática da Universidade de Coimbra, 1997.

[16] X. Chen, *Closed frame homomorphism*, PhD Thesis, McMaster University, 1991.

[17] X. Chen, *On the local connectedness of frames*, J. Pure Appl. Algebra 79(1992) 35-43.

[18] X. Chen, *Stably closed frame homomorphisms*, Cahiers de Top. et. Geom. Diff. Cat. XXXVII-2 (1996).

[19] C.O Christenson and W.L Voxman, *Aspects of Topology.*

[20] P.J. Collins, *On uniform connection properties*, Amer. Math. Monthly 78, 4(1971), 372-374.

[21] C.H Dowker and D. Papert, *Quotient frames and subspaces*, Proc. London. Math. Soc. 16 (1966), 257-296.

[22] R. Engelking, *General Topology*, PWN, Warsaw, 1977.

[23] R.G. Fox, *Covering spaces with singularities*, Algebraic Geometry and topology: A symposium in honour of S. Lefschetz, edited by R.H. Fox et al, Princeton University Press, Princeton, New Jersey, 1957, 243-257.

[24] J. Frith, *Structured frames*, PhD thesis, Univ. of Cape Town, 1987.

[25] A.M Gleason, *Universal locally connected refinements*, Illinois J. Math. 7(1963), 521-531

[26] M. Henriksen and J.R. Isbell, *Local connectedness in the Stone-Čech compactification*, Illinois J. Math. 1 (1957), 574-582.

[27] H. Herrlich, *A concept of nearness*, Gen. Top. Appl. 4(1974). 191-212.

[28] H. Herrlich and G.E Strecker, *Category theory,* Heldermann Verlag Berlin 1979.

[29] J.R. Isbell, *Atomless parts of spaces*. Math. Scand. 31 (1972) 5-32.

[30] I.M. James, *History of Topology*, First Edition, Amsterdam; New York: Elsevier Science B.V., 1999.

[31] A. Joyal and M. Tierney, *An extension of Galois Theory of Grothendieck*, Amer. Math. Soc. Memoirs 309. Vol.51 (1984).

[32] P.T. Johnstone, *Stone spaces*, Cambridge Univ. Press, 1982.

[33] P.T. Johnstone, *The point of pointless topology*, Bull. A.M.S (New series), 8-1 (1983), 41-53.

[34] J.L Kelley, *General Topology*, Springer-Verlag, 1961.

[35] I. Křiž, *A direct description of uniform completion in locales and a characterization of LT groups*. Cahiers de Top. et Géom. Diff. Cat. 27(1986), 19-34.

[36] I. Křiž and A. Pultr, *Peculiar behavior of connected locales*, Cahiers de Top. et. Geom. Diff. Cat. XXX-I(1989), 25-43.

[37] Y. Li, *Weak locale quotient morphisms and locally connected frames*, J. Pure Appl. Algebra 110 (1996) 101-107.

[38] N. Markus, *Realcompactifications of frames*, MSc thesis, Univ. of Cape Town. 1994.

[39] A. Pultr and A. Tozzi, *Equationally closed subframes and representation of quotient spaces*, Cahiers de Top. et. Géom. Diff. Cat. XXXIV-3 (1993).

[40] H. J. Siweya, *Local connectedness of a strange kind*, Manuscript, 2001.

[41] H. J Siweya, *Spreads in locales and uniform locales,* PhD thesis, Univ. of Durdan-Westville, 2000.

[42] J.J.C Vermeulen, *Proper maps of locales*, J. Pure Appl. Algebra 92(1994) 79-107.

[43] S. Vickers, *Topology via Logic*, Cambridge Tracts in Theor. Comp. Sci. 5, Cambridge University Press, Cambridge, 1985.

[44] A.D. Wallace, *Extension invariance*, Trans. Amer. Math. Soc., Vol.70(1951), 97-102.

[45] J.L Walters-Wayland, *Completeness and nearly fine uniform frames*, PhD Thesis, Univ. Catholique de Louvain, 1996.

[46] G.T. Whyburn, *On quasi-compact mappings*, Duke J. Math. 19(1952), 445-446.

[47] R.L Wilder, *Topology of manifolds*, Amer. Math. Soc. colloq. Publ. 32(1949).

[48] S. Willard, *General Topology*, Addison-Wesley Series in Mathematics, 1970.