A study of circuit Complexity for Coherent States

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Abstract

Computational complexity is a popular quantity in quantum information theory. It has made huge strides in recent years in the study of black hole dynamics. A brief definition of complexity is the measure of how difficult it is to implement a task. For a quantum system, complexity evaluates the difficulty of preparing a quantum state from a given reference state by unitary transformations. However, in the dual gravity theory complexity has a geometric meaning. In some black hole context, Leonard Susskind and collaborators proposed two holographic conjectures. The Complexity=Volume (CV) states that complexity of the boundary field theory is dual to the volume of a co dimension one maximal surface that extends to the boundary of the Ads space. Complexity=Action (CA) posits that complexity of the boundary is the same as the action evaluated as an action on patch in the bulk defined as the Wheeler De Witt patch. In recent years, these two conjectures have initiated an extensive study of complexity.

This thesis is also motivated by these conjectures and will investigate complexity in the field theory side of the story. Specifically, we will explore the complexity for coherent states. We will start with a review of different methods of computing complexity. Finally, we then investigate the complexity for coherent states by using the methods of circuit complexity and operator complexity.

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Contents

Chapter 1

Introduction

1.1 Motivation & Background

Information about quantum gravity that is found in the volume of the region is encoded on the surface of the volume [3]. The best example to show how this holographic principle works, is the Bekenstein-Hawking entropy.

$$S_{BH} = \frac{A}{4G_N}, \tag{1.1}$$

which states the proportionality of the area of horizon and the black hole entropy instead of what is expected to be proportional to the volume.

Another example of holographic principle that has been making big strides in theoretical physics is known as AdS-CFT correspondence, which was proposed by Juan Maldacena [28]. The term AdS means the Anti de Sitter space time and CFT stands for Conformal field theory. The AdS-CFT correspondence works by comparing calculations of observables on both theories. This holographic principle is a type of duality. Duality is defined as the equivalence of different formulations. It is a one to one map of the observables and the dynamics on both sides. This is commonly used in situations where certain observables are easier to digest from the other. AdS-CFT correspondence is a particular type of duality which is also known as Weak/Strong duality.

AdS/CFT correspondence has been extensively used to try and solve quite a few open problems. It is particularly useful in dealing with entangled system. Entanglement is phenomena that is able to differentiate classical and quantum systems. Two non-interacting systems can be correlated with each other. This means if one has access to information of one system, the already has information of the other system.

Entanglement Entropy measures the amount of entanglement between 2 subsystems of a quantum system. In the context of quantum mechanics it is not difficult to solve the entanglement entropy. However, applying it to the realm of quantum field theory the problem becomes complicated and we end up using methods which also work with certain cases. One such method is called the replica trick, which we will discuss extensively in the next chapter. The entanglement entropy becomes even more interesting by the Ryu-Takayanagi proposal[38]. They proposed a holographic derivation of the entanglement in QFT(conformal) by using AdS/CFT. Theirs provides a geometrical meaning to entanglement entropy. They proposed that entanglement entropy in (n+1) dimensional conformal field theories can be obtained from the area of n-dimensional minimal surfaces in AdS in (n+2). This falls under what is known as holographic entanglement entropy. This allowed us to understand another interesting conjecture, namely the ER=EPR by Maldacena and Susskind [27]. This states briefly that two entangled black holes are equivalent to the Eistein Rosen-Bridge.

Black holes are really great theoretical labs to test out formulations and observables, due to the fact that the outside observers don't have access to information inside. It is difficult to learn about the dynamics that occur inside a black hole. Holographic entanglement entropy is very useful in this context. Since it provides us a probe into the interior. We realised that the time evolution of it, can provide information of what is occurring inside of the black hole. Susskind and collaborators [41] noticed that the holographic entanglement reaches a thermal equilibrium at a certain time dubbed the scrambling time, whereas the Einstein Rosen bridge evolves for longer time. Susskind and collaborators proposed holographic complexity as equivalent observable in the field theory to be dual to the volume of Einstein Rosen bridge at longer time scales [40, 42].

This introduction of computational complexity from the quantum information theory gives rise to two new gravitational observables. The first is the Complexity=Volume conjecture, which states complexity of the boundary state is proportional to the volume of maximally codimension-one bulk surface that extends to the AdS boundary [39, 20, 26]. The second gravitational observable is the Complexity= Action conjecture, which posits that the action evaluated on the bulk region known as the Wheeler De Witt patch [7], is related to complexity of the boundary field. The study of both the Complexity= Volume and Complexity=Action conjunctures have been studied extensively in various circumstances [37, 1, 4, 10, 11, 25]. Complexity in quantum field theory on the other hand it is still in its early stages. This is very important to study as it gives us a powerful tool to probe into the black hole. Moreover, this is very useful in studying certain aspects of quantum theories, it could give us more information about the Hamiltonian complexity [15], an efficient way to describe many-body wave functions [34] and study of quantum chaos [13].

In this thesis we will be focusing on complexity for coherent states. The thesis will be divided in the following sections.

Chapter 2 we will give a thorough depth into what entanglement entropy is from the context of quantum mechanics to quantum field theories and methods of how to compute it in both contexts and necessary ingredients needed to solve them. Give a brief description of what is AdS spacetime and different coordinate systems and their implications. Mention Ryu-Takayanagi by generalising it via. the Bekernstein-Hawking entropy and its amazing implications. Discuss about the time evolution of a thermofield double state and what we learn from it especially in black hole dynamics point of view.

Chapter 3 We begin defining necessary tools to make sense of the definition of computational complexity which originates from theoretical computer science. We give a brief but needed discussion on classical information and logical operations that allow classical computers. We then discuss analogous concepts from a quantum information perspective and define complexity from the context of of quantum information. Discuss the geometry of states and compare complexity and Fubini-Study and which one is the best distance indicator in such spaces. We then mention four methods of computing complexity in QFT which are the Nielsen approach. The covariance matrix approach. The Fubini-Study method and lastly the Krylov complexity. We provide simple examples and review some from past papers to show how they work.

Chapter 4 We then delve in and start computing the complexity for coherent states using the Nielsen approach.

Chapter 5 This Chapter reviews [19], which we consider as an alternative method in computing complexity. We go through the results of Heisenberg group complexity and compare with other papers results that have computed complexity for coherent states and displacement operator.

Chapter 6 We discuss the results from both chapter 4 and chapter 5 and discuss future work that might come out of the thesis.

We have the appendix, that includes a extensive calculation in parametrisation of SU(1,1) in coherent form and diagonalization of Hamiltonian for chapter 4.

Chapter 2

AdS-CFT correspondence

2.1 Motivation

From AdS-CFT duality we know that a strongly coupled field theory (CFT) on the boundary in n dimensions of a weakly coupled gravity in n+1 AdS space [30]. This type of duality is also known to be called the Strong/Weak duality.

2.2 Entanglement Entropy

As stated in the introduction, entanglement is a very distinct phenomena that differentiates classical to quantum systems. Entanglement entropy is an important concept, to understand complexity. Which is the main topic of this thesis. Actually complexity as sort of an extension to the entanglement entropy in the external black hole settings. In this chapter we will give an overview of entanglement entropy and AdS space. Then discuss the holographic entanglement entropy and conclude with the time evolution of thermofield double state and its holographical equivalence. The next section we will discuss how it is diagnosed in quantum mechanics and in quantum field theory.

2.2.1 Entanglement Entropy in Quantum mechanics

Pure states, Mixed states and Density operator

Let us consider a Hilbert space $\mathcal{H}$[1].We deconstruct the Hilbert space such that,

$$\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}. \tag{2.1}$$

We define a pure state ψ as a definite state in a Hilbert space that is not a result of statistical distribution of other states. Mixed state is a probabilistic distribution of corresponding pure states.

Let us consider a quantum system and define a state vector $|\psi\rangle \in \mathcal{H}$. Suppose that we have a subsystem A, that has $\mathcal{H}_{\mathcal{A}}$. We define subsystem B, which is a

[1]We will assume Hilbert space is finite dimensional.

compliment of A. We define $|\psi\rangle$ to be a pure state and $|\psi\rangle \in \mathcal{H}$ is separable if and only if $|\phi\rangle \in \mathcal{H_A}$ and $|\chi\rangle \in \mathcal{H_B}$ such that

$$|\psi\rangle = |\phi\rangle \otimes |\chi\rangle, \tag{2.2}$$

otherwise $|\psi\rangle$ is entangled.

The state of a system and the description of a quantum system can be seen as the same concept in QM [22]. The density operator ρ is an operator that describes the quantum system by computing the statistical outcomes of measurements in the system. We will ascribe state in a quantum system using density operator $\rho = |\psi\rangle\langle\psi|$. The density operator has certain properties it follows such as

- ρ is Hermitian.
- Tr(ρ)=1.
- $\rho \geq 0$.

The expectation value $\langle \hat{A} \rangle_\rho = Tr(A\rho)$ where $\hat{A}$ is a an operator. For a pure state the density operator $\rho^2 = \rho$, otherwise it is defined as a mixed state.

We can also diagonalize the density operator ρ such that

$$\rho = \sum_{i=1}^{n} p^i \rho^i, \tag{2.3}$$

where p^i are the eigenvalues which form the probability distribution, $\rho^i = |i\rangle\langle i|$ and $|i\rangle$ is an orthonormal basis. We can also define the density operator such that

$$\rho = \frac{1}{Z} e^{-H_A}, \tag{2.4}$$

where H_A is a formal modular Hamiltonian and partition function Z. In this instance we assume the temperature T is 1.

Reduced Density operator

Let us consider a density operator ρ^{AB} which is found in a composite system AB. We can define a partial trace,

$$Tr_A(\rho^{AB}) = Tr(\rho^A)\rho^B \text{ and } Tr_B(\rho^{AB}) = Tr(\rho^B)\rho^A. \tag{2.5}$$

Since we know Tr(ρ)=1. We define a reduce matrix density as,

$$\rho^A = tr_B(\rho^{AB}). \tag{2.6}$$

The reduced matrix density is important if we want to know information about a certain system if we have density matrix of the full system. Taking the trace of the compliment allows us to be able to focus on the system we are interested in.

Von Neumann Entropy

In classical information theory, the Shannon entropy is the measure of the amount of information contained in a message [45]. The quantum analogous of this is called the Von Neumann entropy. Let ρ be a density operator of the system, the Von Neumann entropy of the system is defined as,

$$S(\rho) = -Tr(\rho \log(\rho)). \tag{2.7}$$

Applying the spectral decomposition the Von Neumann entropy is then

$$S(\rho) = -\sum_{i=1}^{n} \lambda_i \log \lambda_i, \tag{2.8}$$

where λ_i is the probability distribution. The Von Neumann entropy can be defined as the measure of how mixed a quantum system is. It satisfies the condition S(ρ)≥ 0 and S(ρ)$= 0$ if and only if ρ is pure. An example that shows entanglement is the Bell's pair.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B), \tag{2.9}$$

where $\uparrow, \downarrow$ represent spin up and spin down. The reduced matrix density

$$\rho_A = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) = \frac{I_2}{2}, \tag{2.10}$$

where I_2 is two dimensional identity matrix[2]. The entanglement entropy is then,

$$S(A) = -Tr(\rho_A \log \rho_A) = -Tr(\frac{1}{2}\log\frac{1}{2}) = -Tr(\frac{1}{2}\log 2) = \log 2. \tag{2.11}$$

When a density matrix corresponds to an identity matrix, this tells us it is maximally entangled. Maximally entangled states are of the form $S = n \log 2$ where n is the number of qubits in each subsystem.

The Von Neumann entropy has very interesting properties. The Von Neumann entropy is invariant to unitarity ,

$$S(U\rho U^\dagger) = S(\rho). \tag{2.12}$$

This means that the time evolution has no influence in converting mixed state to pure state or vice versa.
Suppose we have a quantum system that is constructed of three subsystems such that the total Hilbert space is $\mathcal{H} = \mathcal{H_A} \otimes \mathcal{H_B} \otimes \mathcal{H_C}$. Strong subadditivity holds,

$$S(AB) + S(BC) \geq S(B) + S(ABC). \tag{2.13}$$

Using this we are able to define the concept of mutual information,

$$I(AB) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \geq 0. \tag{2.14}$$

The mutual information is a quantitative concept to correlate information between two systems. Lastly the inequality that holds with the Von Neumann entropy is the **Araki-Lieb inequality**.

$$S(AB) \geq |S(A) - S(B)| \tag{2.15}$$

This inequality tells us that the composite system can have entropy that is less than of the subsystem it contains.

[2]Note that, $Tr(I_2) = 2$

Rényi entropy

The Rényi entropy is a one-parameter generalisation of the Von Neumann entropy which is defined as,

$$S_n(\rho) = \frac{1}{1-n}(\sum_{i=1}^{n} p_i^n) = \frac{1}{1-n}\ln Tr(\rho^n). \tag{2.16}$$

When $n \to 1$ this give us $S_{n\to 1}(\rho) = S(\rho)$. The Rényi entropy gives us information on the eigenvalues than the Von Nuemann entropy. We will show in the next section that the analytical continuation will be useful in quantum field theory computing entanglement entropy .

Purification

Suppose we have system A with a mixed state ρ_A. We can extend the Hilbert space AB such that a mixed state is contained in a pure state. This procedure is called purification. We can perform this by applying the Schimdt decomposition. Suppose we have a mixed state $\rho_A \in \mathcal{H}_\mathcal{A}$. We extend the Hilbert space such that $\mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_{\mathcal{A}'}$. Where dim($\mathcal{H}_\mathcal{A}$)= dim($\mathcal{H}_{\mathcal{A}'}$) [44]. We then have a pure state such that,

$$|\Psi\rangle = \sum_i \sqrt{\lambda_i}|\chi\rangle \otimes |\phi\rangle. \tag{2.17}$$

λ_i are called a Schimdt coefficients and satisfy $\sum_i \lambda_i = 1$. The othonormal states $|\chi\rangle$ and $|\phi\rangle$ are found in $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_{\mathcal{A}'}$ respectively. The Schimdt coefficients basically classify a state as either $\lambda_i = 1$ or $\lambda_i = 0$ for all i. This tells us if our state is separable or if it is entangled. This also allows us to think of entanglement and entropy interchangeably.

Let us look at an example of purification. Let us consider consider ρ_A in a form of Eq. 2.4. This means we have

$$\rho_A = \frac{1}{Z}e^{-\beta H_A}, \tag{2.18}$$

where $\beta = \frac{1}{T}$. We can extend the system with the exact copy of H_A and call it H_B. Through the process of Schimdt decomposition we have

$$|\Psi\rangle = \frac{1}{Z}\sum_i e^{-\beta\frac{E_i}{2}}|E_i\rangle \otimes |E_i\rangle. \tag{2.19}$$

This is called the Thermofield double state.

2.2.2 Entanglement Entropy in QFT

Subsystem

In quantum field theory computing entanglement entropy is more difficult to solve as compared to quantum system where we had finite dimensional Hilbert space and we were able to localise states. The best way to think of this is to firstly think of the quantum field as spatial lattice. The question then arises what is a subsystem? From a spatial lattice stand point the Hilbert space of a local site x is $\mathcal{H}_x$. The full Hilbert space of the spatial lattice is the tensor product of all the local site x, such that $\mathcal{H} = \otimes_x \mathcal{H}_x$. Suppose we can deconstruct the full Hilbert space of the spatial lattice into a region A and its compliment region B. Which looks like $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$, where $\mathcal{H}_{\mathcal{A}} \coloneqq \otimes_{x\in A} \mathcal{H}_x$ and $\mathcal{H}_{\mathcal{B}} \coloneqq \otimes_{x\in B} \mathcal{H}_x$.

When the Spatial lattice has a continuum limit, we can still define our region A and its compliment B. The decomposition of the Hilbert space fails at the boundary that separates region A and region B, this is called the **entangling surface**. Even though we have such a problem, we can still essentially find the physics in the continuum, such as the S(A), ρ, Rényi entropy, mutual information and so on.

Density operator in path integral form

The problem we face in trying to find entanglement entropy in QFT is tracing out the degrees of freedom of the compliment. When the space is infinitely dimensional. To be able to calculate the Rényi entropy, mutual information and the Von Neumann entropy we propose to try and express the ρ in path integral form.
The propagator of two field configuration ψ_i and ψ_j can be represented using path integral.

$$\langle \psi_i | e^{iHt} | \psi_j \rangle = \int_{\psi_i}^{\psi_j} \mathcal{D}\psi e^{-iS[\psi]}, \tag{2.20}$$

where $S[\psi]$ is the action. From Eq. 2.4 , we can define $\rho(\tau) = e^{-H\tau}$. τ comes from applying the Wick's rotation where $t \to -i\tau$. This converts the path integral into a Euclidean path integral,

$$\rho(\psi_i, \psi_j, \tau) = \int_{\psi_i}^{\psi_j} \mathcal{D}\psi e^{-S_E[\psi]}. \tag{2.21}$$

The Replica Trick

It is quite difficult to calculate the entropy $S(\rho)$ in quantum field theory. A method that we will use to calculate entropy is called the **Replica Trick**. Let us consider the Rényi Entropy $S_n = -\frac{1}{n-1} \log Tr(\rho^n)$. What does a $Tr(\rho)$ look like in Euclidean path integral form?

$$Tr(\rho^n) = \int [\mathcal{D}\phi_1 \mathcal{D}\phi_2 ....\mathcal{D}\phi_n]_B \rho[\phi_1, \phi_2]\rho[\phi_2, \phi_3]...\rho[\phi_{n-1}, \phi_n]. \tag{2.22}$$

We define the trace of the density matrix due to convolution that traces the degrees of freedom of B and "glues" degrees of freedom in A.

Let us consider a wave function of ground state Ψ. The path integral with boundaries $\phi(x, 0) = \phi_0$ and $\phi(x, t \to -\infty)$ [29],

$$\Psi(\phi(x, t)) = \mathcal{N} \int_{\phi(x,t\to-\infty)}^{\phi(x,0)=\phi_0} \mathcal{D}\phi e^{-S_E[\phi]}. \tag{2.23}$$

The conjugate of the wave function with boundaries $\phi(x, 0) = \phi_1$ and $\phi(x, t \to \infty)$,

$$\Psi(\phi(x, t))^* = \mathcal{N}^* \int_{\phi(x,0)=\phi_0'}^{\phi(x,t\to\infty)} \mathcal{D}\phi e^{-S_E[\phi]}. \tag{2.24}$$

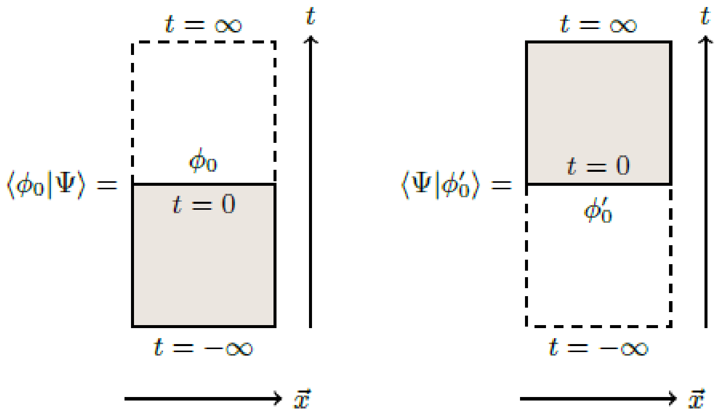

Figure 2.1: The wave function and its conjugate in a geometric form [33].

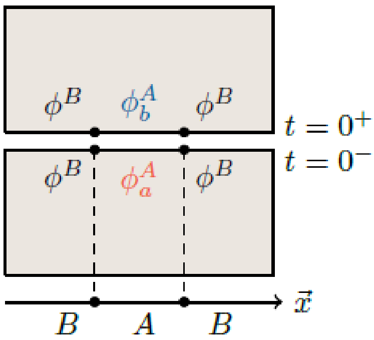

Figure 2.2: This image describes the partial trace over the compliment B, in geometrical form [33]. the $t = 0^+$ and $t = 0^-$ tells us about the the direction where the wave functions are coming from. The $\rho_A = \int_{\phi(x,0^-)}^{\phi(x,0^+})\mathcal{D}\phi e^{-S_E[\phi]}$.

This results in the matrix density being $\rho = |\Psi(\phi(x,t))\rangle\langle\Psi(\phi(x,t))|$. We can imagine Figure 2.1 the lower half plane that is shaded at the wave function and the upper half plane that is shaded as the conjugate of the wave function. The matrix density geometrically is sewing both planes together on the edges. So how do we represent the reduced matrix density geometrically? Applying the partial trace over the compliment B would be the same as attaching edges of the wave function and its conjugate along B [22]. This means that the n^{th} power of a reduced matrix density is given by attaching n copies of the sheet along a branch cut A [33].

$$\rho_A^n = \int \mathcal{D}\rho_A^{n-1}[\phi_{n-2}, \phi_{n-1}]\rho_A^n[\phi_{n-1}, \phi_n] \equiv Z_n(A). \tag{2.25}$$

This method of sewing the reduced matrix densities on top of each other is called the **replica trick**. The sewed on n-sheets are called replicated manifold $\mathcal{R}_n$ [22]. We also notice a singularity on the replicated manifold which would imply an entangled surface on ∂A. Performing a free path integral on this manifold is the same as calculating the reduced matrix density n-times ρ_A^n. The trace of the reduced density operator

$$Tr(\rho_A^n) = \frac{Z_n(A)}{Z^n}, \tag{2.26}$$

where $\frac{1}{Z^n}$ is the normalisation factor to be able to satisfy $Tr(\rho)$=1. Using the Rényi entropy we have,

$$S_n = \frac{1}{1-n}(\ln \frac{Z_n(A)}{Z^n}) \tag{2.27}$$

$$S_n = \frac{1}{1-n}(\ln (Z_n(A)) - n \ln (Z)). \tag{2.28}$$

By analytically continuing S_n so the limit of n is smooth and allowing $n \to 1$ we get ,

$$S(A) = \lim_{n \to 1} \partial_n (\ln (Z_n(A)) - n \ln (Z)). \tag{2.29}$$

2.2.3 Entanglement Entropy in Relativistic quantum field theory

Subsystem in Space time Manifold

Let us consider calculating the entanglement entropy(EE) in a relativistic quantum field theory. Before we are able to do that, we firstly have to define certain objects needed for this to be possible. We will consider a space time manifold provided with causal structure. We define a causal domain $D(S)$ in S$\subseteq$ M as a set of point $p \in M$ where every non-stretching curve that goes through point p intersects subset S [33]. We also define acausal set S as a set that has no two distinct points that lie on an identical causal curve. Cauchy slice Σ is an acausal set that is the domain of the entire manifold M [33]. Manifold that contains a Σ is globally hyperbolic. Generally globally hyperbolic manifolds have many Cauchy slices.

Observables that are governed by the Heisenberg equations that evolve with time should respect the causal structure of the manifold M. With this, we can evolve any observable on the Cauchy slice Σ [33]. This tells us that the observables found in Σ are due to them being complete, in a sense we can use equation of motion in Σ. Remembering one of the important axioms of relativistic quantum theory is that space-like observables commute. Hence Σ plays the role of a spatial manifold.

Given ρ_A in Σ there exists a unitary transformation localised on Σ such that it evolves within its causal domain D(A). The entanglement entropy remains invariant under this transformation. Let us consider region A' with different Σ' such that D(A')=D(A). This associates $\mathcal{H}_{A'} = \mathcal{H}_A$ and $\rho_{A'} = \rho_A$. This means that entanglement entropy is only dependent on the causal domain instead of the Cauchy slice. We have learnt that to find reduced matrix density ρ we have to trace out the degrees of freedom of the region that is compliment to our region of interest. The question now is how do we trace out the degrees of freedom outside the causal domain D(A)?

Let us consider a D=2 Minkowski space, with the semi-infinite region $A = \{x \geq 0\}$ on some Σ. In this system we notice that the density operator is defined in the Gibbs state form $\rho_A = e^{-H_A}$, where H_A is a modular Hamiltonian. This is a problem because modular Hamiltonians are non-local, whereas we need ours to be local. We express the semi-infinite region in Rindler coordinates. We do that by firstly describing the semi-infinite region in polar coordinates, such that the metric is

$$ds^2 = r^2 d\theta^2 + dr^2. \tag{2.30}$$

In our situation we consider θ to behave as time coordinate. Applying Wick's theorem

$$\theta = i\chi. \tag{2.31}$$

Going back to the canonical coordinates of our Minkowski space, it becomes

$$x = r\cosh\chi \ \ , t = r\sinh\chi, \tag{2.32}$$

(r, χ) only cover the semi-infinite region instead of all of Minkowski space. This in turn becomes a Rindler Wedge. The benefits of this coordinate transformation is that the Hamiltonian becomes local. We also have an temperature $T = \frac{1}{2\pi}$ which is known as the **Unruh temperature**. With the defined temperature our Hamiltonian becomes

$$H = 2\pi K, \tag{2.33}$$

where K is a boost generator. The boost generator comes from the Lorentzian picture of the generator of χ translations. The conserved quantity corresponds to a killing vector k^μ, which can expressed as $\int_\Sigma \sqrt{h} k^\mu n^\nu T_{\mu\nu}$[22]. Here h is an induced metric and n an unit normal vector. The boosts has the killing vector,

$$\frac{\partial}{\partial\chi} = x\frac{\partial}{\partial t} + \frac{\partial}{\partial x}. \tag{2.34}$$

Choosing $\chi = 0$ as our Cauchy slice for the Rindler wedge, the boosts generator becomes

$$K = \int_0^\infty xT_{00}dx, \tag{2.35}$$

where T_{00} is the stress energy tensor for the temporal component. This shows us that the boosts generator is associated with energy. This can also be derived using the Bisognino-Whichmann theorem [6].

This interesting discovery tells us that finding the reduced matrix density in zero temperature is the same as finding the thermal reduced matrix density in Rindler space. This also tells us that the thermal entropy is the same as the entropy in the semi-infinite region. The thermal entropy has interesting significant findings. We realise a high spike in temperature in the entangling surface between regions A and its compliment A^c. It becomes hotter and hotter as it gets closer to the entangling surface. We want to emphasize that the entanglement entropy is a physical phenomenon. This means the thermal entropy is also a physical phenomenon not just a mathematical abstraction.

For a field of mass m, we begin by looking at a case where the temperature is below m. We notice that s(T) which we define as entropy density vanishes and the field is frozen out. At the distance past $r = \zeta$, where ($\zeta := \frac{1}{m}$ is the correlation length), the field does not have significant influence to the entropy. This means that the field is entangled within the neighbourhood of $r \leq \zeta$ of the entangling surface. The case where the temperature is above m. We realise that s(T) is proportional to T. When r is very small the temperature diverges and we get UV divergence. To be able not have these divergences we introduce UV cutoff. The ϵ is the UV cutoff and the entropy is regulated to avoid such divergences.

$$S(A) \approx \int_0^\infty drs(T_{\text{phys}}(r)) \int_\epsilon^\zeta dr\frac{1}{r} = \ln\frac{\zeta}{\epsilon} \tag{2.36}$$

In considering an entropy density of a CFT with a central charge c. Temperature that is dependent on constant proper acceleration $a = \frac{1}{r}$. This tells us that the temperature is spatially dependent. With this information the entropy of region A is,

$$S(A) = \int_{\epsilon}^{\infty} drs(T(r)) = \frac{c}{6}\ln\frac{\zeta}{\epsilon}. \tag{2.37}$$

The next section will give us an idea of what is an AdS space. We are providing a background to this space as it is vital in us introducing holographic entanglement entropy.

2.3 Anti de Sitter Space

Suppose we have an Einstein gravity that has a cosmological constant Λ which is provided by the Einstein-Hilbert action.

$$\mathcal{S}_{EH} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^{n+1}x\sqrt{-g}(R-2\Lambda) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^n x\sqrt{-h}K. \tag{2.38}$$

Where G is the Newtons constant, g is the metric tensor of the manifold, R is the Ricci scalar and Λ is the cosmological constant in the first term of the action. The second term of the action is known as the Gibbson-York boundary term. This is included if the space time we are dealing with has a boundary. The h term is defined as the induced metric tensor and K is the extrinsic curve.

We take the variation of the action with respect to the metric $g_{\mu\nu}$ produces the Einstein field equations.

$$\frac{\delta\mathcal{S}_{EH}}{\delta g_{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.39}$$

where the momentum-energy tensor $T_{\mu\nu}$ comes from having matter in the space time. When $T_{\mu\nu} = 0$ our solutions become vacuum, this makes it maximally symmetric. The killing vector k^I vectors provide a smooth map onto the manifold that preserves the distance onto points. This is known as isometries. These killing vectors have to satisfy the condition that the Lie derivative of the metric is $\pounds_k g_{\mu\nu} = 0$. To be able to have maximum number of isometries in n+1 dimensional manifold. We must have $\frac{n(n+1)}{2}$ independent killing vectors. Such spacetimes are known as maximally symmetric spacetimes. The one we are interested in has Ricci scalar that is negative($R < 0$).

The one that is very fundamnetal to this thesis is called the Anti-de Sitter space. Anti-de Sitter space is defined as a solution for the Einstein field equation that has a negative cosmological constant Λ.

The Anti- de Sitter space that is (d+1) dimensional is embedded in a (d+2) dimensional Minkowski space. The metric is defined such that $\eta = \mathrm{diag}(-1, +1, +1, ..., -1)$. The line element is defined as,

$$ds^2 = \eta_{ab}dx^a dx^b. \tag{2.40}$$

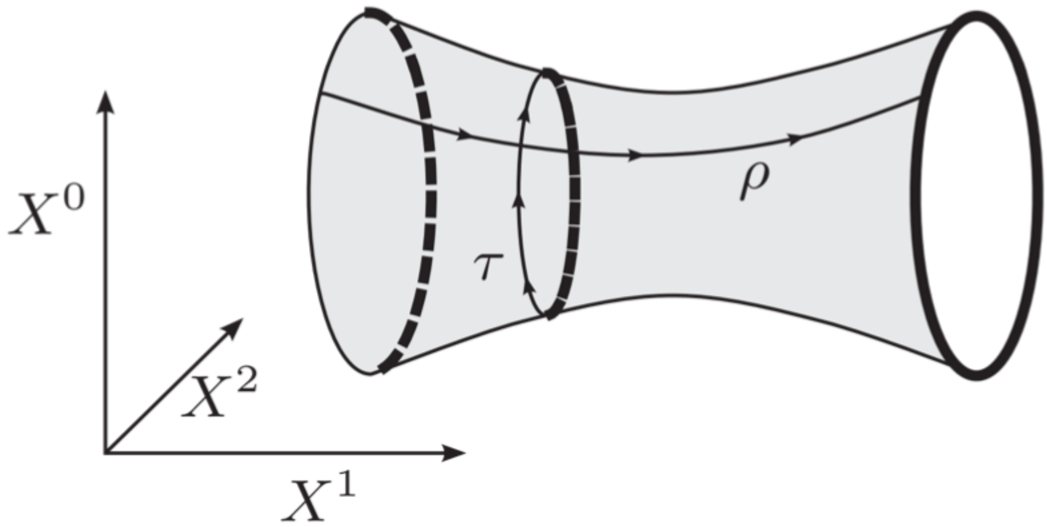

Figure 2.3: Provided is AdS_2 in global coordinates embedded in a Minkowski spacetime and has coordiantes X^0,X^1,X^2 [2].

AdS is defined as hyper surface as ,

$$\eta_{ab}dx^a dx^b = -(x^0)^2 + \sum_{i=1}^{d}(x^i)^2 - (x^{d+1}) = -L^2. \tag{2.41}$$

Where $-L^2$ is the radius of the curvature. The AdS is invariant under the group SO(d,2). Maximally symmetric space can be expressed as as a coset space. This means the $AdS_{d+1} = \frac{SO(d,2)}{SO(d,1)}$.

2.3.1 AdS space in various coordinate systems

Global coordinates

In the global coordinates. We parameterise the following to get

$$x^0 = L\cosh\rho\cos\tau \tag{2.42}$$

$$x^{d+1} = L\cosh\rho\sin\tau \tag{2.43}$$

$$x^i = L\Omega_i\sinh\rho \text{ for i =1,....,d,} \tag{2.44}$$

where $\tau \in [0, 2\pi]$, $\rho \in R_+$ and $\sum_{i=1}\Omega_i^2 = 1$. These coordinates cover all points in the hyperbloid exactly once. The line element becomes,

$$ds^2 = L^2(-\cosh\rho^2 d\tau^2 + d\rho^2 + \sinh\rho^2 d\Omega_{d-1}^2). \tag{2.45}$$

Let us consider $\tan\theta = \sinh\rho$, then our line element becomes

$$ds^2 = \frac{L^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2), \tag{2.46}$$

where $0 \leq \theta < \frac{\pi}{2}$. Suppose we neglect the conformal factor and add the the point $\theta = \frac{\pi}{2}$. The $\frac{\pi}{2}$ is related to the spatial infinity. This results into the line element being

$$ds^2 = (-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2), \tag{2.47}$$

where $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \tau \leq 2\pi$. The boundary has the same features as a Einstein static universe.

Poincare coordinates

In the Poincare coordinates. The full parametrisation becomes

$$X^0 = \frac{L^2}{2}(1 + \frac{r^2}{L^4}(\vec{x}^2 - t^2 + L^2)) \tag{2.48}$$

$$X^i = \frac{rx^i}{L} \tag{2.49}$$

$$X^d = \frac{L^2}{2r}(1 + \frac{r^2}{L^4}(\vec{x}^2 - t^2 - L^2)) \tag{2.50}$$

$$X^{d+1} = \frac{r\vec{x}}{L}, \tag{2.51}$$

where $i \in 1, ..., d-1$. The line element becomes

$$ds^2 = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}(-dt^2 + dx^2), \tag{2.52}$$

where $r > 0$ this means it only covers half of the AdS space.
The AdS_{n+1} space-time in Poincare coordinates has a form of a flat space which is given by (t,x) with a warped direction r. The metric tells us that we have two singularities. When $r \to 0$ we have a coordinate system caused singularity. When $r \to \infty$ a divergence on the flat part. This can be seen as the conformal boundary. For our convenience let us define $z = \frac{L^2}{r}$. This will allow us to locate the conformal boundary at z=0 as seen from Figure 2.4. The metric with new representation is then

$$ds^2 = \frac{L^2}{z^2}(dz^2 - dt^2 + d\vec{x}^2). \tag{2.53}$$

Euclidean AdS

In most cases of formulating AdS/CFT we tend to Euclideanise AdS. This results in the Isometry group of Euclidean to become SO(d+1,1). We define the euclidean time $\tau_E = i\tau$, hence

$$ds^2 = L^2(\cosh\rho^2 d\tau_E + d\rho^2 + \sinh\rho^2 + d\Omega_{d-1}^2). \tag{2.54}$$

2.3.2 Penrose Diagram of AdS

The Penrose diagram is one of the best ways to try to draw and provide information of entire space times in finite portions. Expectantly in such diagrams to find useful information we will have to compromise a number of issues. One important feature

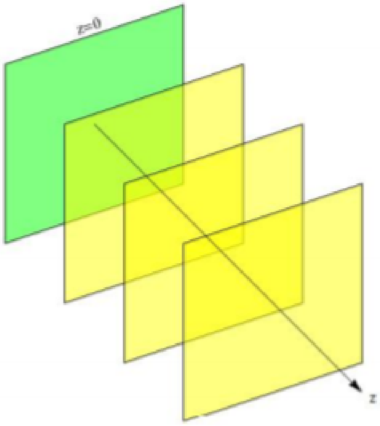

Figure 2.4: This is the AdS spacetime in Poincare coordinates. The green square at z=0 represents the boundary.

that will not be compromised or is of principle when drawing Penrose diagrams is null geodesics that run at slopes ± 1 in the diagram.
Let us consider a 2-dimensional Minkowski space. The line element is described as

$$ds^2 = -dt^2 + dx^2, \tag{2.55}$$

where the $-\infty < t, x < \infty$. We convert the non-compact coordinates to non-compact null coordinates,

$$u_{\pm} = t \pm x. \tag{2.56}$$

We compactify the non-compactifying null coordinates such that

$$\tilde{u}_{\pm} = \tan^{-1}(u_{\pm}), \tag{2.57}$$

where $|\tilde{u}_{\pm}| \leq \frac{\pi}{2}$. We provide another coordinate transformation

$$\tau = \tilde{u}_{+} + \tilde{u}_{-} \tag{2.58}$$

$$\theta = \tilde{u}_{+} - \tilde{u}_{-}, \tag{2.59}$$

where $|\tau \pm \theta| \leq \pi$.

Since we know that Minkoswki space is similar to AdS, we can convert Eq. 2.55 to polar coordinates such that,

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2. \tag{2.60}$$

When we drop the angular dependence and $r > 0$ this results to Figure 2.5.

2.4 Holographic Entanglement Entropy

From the above sections we have realised that, studying entanglement entropy in quantum field theory provides important physical insights. The problem we faced with studying entanglement entropy is that it is quite difficult to compute. We were

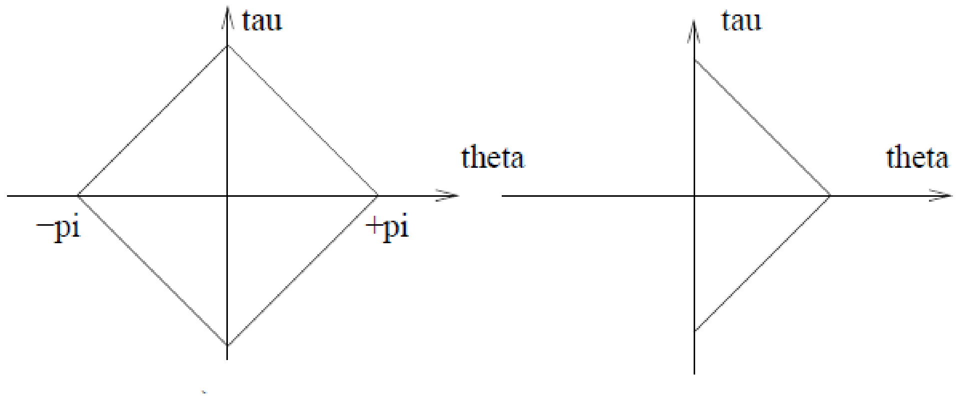

Figure 2.5: The image on the left is the 2d Minkowski space and on the left it is when $r > 0$ and the angular dependence is dropped.

only able to solve it in certain cases. With the recent developments of holographic principle. We noticed that we can solve entanglement entropy for a field theory on a boundary that is dual to a one-dimensional higher bulk geometry. This is done by finding the minimal surface area with a provided boundary. This is called the Ryu-Takayanagi (RT) formula.

2.4.1 Ryu-Takayanagi formula

Bekenstein-Hawking Entropy

Suppose we motivate the R-T formula by defining EE from the Bekenstein-Hawking entropy and generalising from there on. Let us consider a deconfined space in a holographic field. The bulk geometry is of a static, asymptotically AdS black hole. The Bekenstein-Hawking entropy is a quarter of an area of a (bipartite) horizon,

$$S_{BH} = \frac{A_{m_{hor}}}{4G_N}. \tag{2.61}$$

We have suppressed the Planck's constant and defined $k_B = 1$. We notice that the S_{BH} is thermal entropy [36], hence we have a thermal matrix density. For a thermal state is a mixed state so the purification results in a thermofield double state. Purification produces a thermofield double state that is formed from two copies of a system. In our case the system is a field theory. We will define the two copies as K and K' and these are copies found in the boundary of the space time. Symmetric two sided extended black hole space is holographically equivalent to the thermofield double state [22]. The black hole that has thermal state exists in the outer region.

We noticed that the outer regions are static but the extended black hole is not globally static[22]. This means that we cannot define standard constant time slices. We do realise that certain Cauchy slices exist, particularly those ones that have a time reflection symmetry invariance of the space time. Such an invariant σ go through the bipartite horizon m_{hor} and it is only limited to constant time slices in both outer regions. It intersects K$\cup$ K' on Σ. $\Sigma = A \cup A^c$ where A is the Cauchy

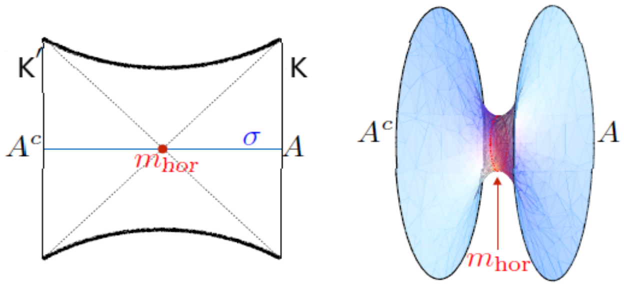

Figure 2.6: The left side of the figure we have a Penrose diagram of two sided static AdS black hole. It has a Cauchy slice σ from the bulk that is invariant from time reflection and passes through the bipartite horizon. On the right we have the induced geometry σ which is an Einstein Rosen Bridge (ERB) connecting two boundaries.

slice of K and A^c is the Cauchy slice of K'. In terms of the thermofield double state. The thermal density matrix is equivalent to of the reduced matrix density ρ_A of one of the outer regions. This implies that S(A) is then,

$$S(A) = S_{BH} = \frac{A_{m_{hor}}}{4G_N}. \tag{2.62}$$

We want to generalize Eq. 2.62. We want to think of the bifurcate horizon m_{hor} as a bulk surface. Thinking about it in that manner will deduce that σ is associated with the minimal area that divides the two regions A and A^c .

We now introduce that this minimal area feature as an ansatz and as the key aspect in connecting the surface m_{hor} with entropy S(A). For a holographic space time with σ as a time reflection invariant bulk Cauchy slice that separates A and A^c. We will assume that for a moment that the set of boundaries is in pure state. We assert that the minimal surface area m(A) that separates A and A^c gives us the entropy of A.

$$S(A) = \frac{\text{area}(m(A))}{4G_N}. \tag{2.63}$$

The essence of this, is the reduced matrix density ρ_A can be seen as the black hole that has a horizon m(A). We have to be careful with how we describe m(A) and this is done by tracking the orientation. We find that the minimal surface area is homologous to A [22], hence

$$S(A) = \frac{1}{4G_N} min_{m\sim A}\text{area}(m(A)). \tag{2.64}$$

Let us not forget that this only works due to the requirement that σ is time reflection invariant in the extended black hole space time. If it was static, the the constant time slices would automatically be time reflection invariant hence they would go through the bifurcate horizon. We also notice that the minimal surface area for A^c and A are similar just opposite orientations . The minimal surface must extend to the point of the conformal boundary. This means that it will have to touch the entangling surface. This will create UV divergence which is expected for S(A) in field theory.

To see if R-T formula works certain checks we made. We found that R-T obeys properties of EE such as strong subadditivity [23]. It is consistent with the Rényi entropy which are carried out using replica trick using the quantum gravity framework. It agrees with calculations of EE and certain divergences for certain cases. R-T formula has made so much progress as it has been applied to various space times. R-T formula has made big influence and been the center of attention in the theoretical framework of the relationship between geometry and boundary fields. It has major influence in topics such as tensor networks in holography. Entanglement related quantities and holographic error correcting codes.

2.4.2 The time evolution of Thermofield double state and its duality in black hole dynamics

In the section about Ryu-Takanayagi formula. It has opened the door to find a framework between the bulk theory and the boundary field . We are curious about the time evolution in black hole interiors. The dynamics are difficult to study due to the observer outside the horizon not having access to information. In [27] it has been noticed that a pair of entangled black holes are dual to a non transversable wormhole (Einstein Rosen-Bridge). These entangled black holes have thermal conformal field theory on their boundary. We have also stated that a two sided symmetric black hole space time is holographically equivalent to a Thermofield double state. This means if we could study the time evolution of the TFD we could have an idea of the black hole dynamics due the holographic principle.

The are two convenient methods we have for the total Hamiltonian acting on a double system.

$$H_{tot} = H_R - H_L \text{ and } H'_{tot} = H_R + H_L. \tag{2.65}$$

We realise that the eigenvalue for H_{tot} becomes zero, so the thermofield double state acted on by H_{tot} is the same as when it is time independent thermofield double state. As sated from the previous section we notice that the is a holographical duality between the thermofield double state and the eternal AdS black hole. What we notice is that, the Einstein Rosen Bridge grows for longer timescales. Whereas when we time evolve the thermofield double state it saturates as it reaches thermal equilibrium and this the time it saturates is called scrambling time. This is then where Susskind and collaborators conjectured holographic complexity to translate the long timescale of the Einstein Rosen Bridge.

The next section will define what computational complexity from its quantum and classical information background. Complexity in QFT, the four candidates of complexity in QFT.

Chapter 3

Quantum Information and Complexity

We have realised that Quantum information and complexity have made quite an influence in holography. The objective of the first few sections is to try an understand the concept of complexity in the context of quantum information. Firstly we need to make sure we understand the concepts such as bits, qubits, gates and quantum circuits. Most of this information is based on. Complexity has been studied extensively in theoretical computer science. Generally its definition has been the measure of how difficult it is to perform a task. We might think this definition is very ambiguous so it is in need of necessary ingredients. Such as space of states, concept of a simple states and concept of what an operation is.

3.1 Classical Computing

Le us consider a classical computer with n bits. We have strings that are made up of 0 and 1. Where the simplest strings is (0000..0) and (111..1). These lie in a space of states that has 2^n possible strings. We can also define a simple operation as doing an operation one bit at a time. Lastly we define a task as a system that maps one simple state to a much more complicated state. With all these concepts defined. We define complexity as the minimum number of operations needed to perform a task. This comes from the classical computing context. The next section gives an overview of how classical information and operations work in a classical computer.

3.2 Basics of classical information

Given a string with n bits, a function $f(x)$ in a classical computer map a string with n bits to a string with m bits as an output.

$$f : \{0, 1\}^n \to \{0, 1\}^m. \tag{3.1}$$

From the situation of n bit. We can also notice that m functions instead of m bit output. This allows us to express it as,

$$f : \{0, 1\}^n \to \{0, 1\}. \tag{3.2}$$

There are 2^n possible n input, which have two possible outcomes. This means that we have 2^{2n} functions taking n bit to one bit. We can express the function as elementary sequential logical operations which we define as gates.

Suppose we have n inputs $(x^1, x^2, x^3, .., x^n)$. Let us define the following logical operations.
We will firstly define a logical OR of all possible $f^{(a)}(x)$. The operation $\vee$ between two bits is expressed by

$$x \vee y = x + y - x\dot{y}, \tag{3.3}$$

that it is 0 if $x = y$ and otherwise it is 1. The next logical operation is the AND of all n bits, it is expressed as $\wedge$

$$f^{(a)}(x) = x_1 \wedge x_2 \wedge x_3 \wedge .. \wedge x_n. \tag{3.4}$$

This is true in the case where $x^{(a)} = 11111..11$. Otherwise for any other $x^{(a)}$, the function $f^{(a)}$ as the AND of n bits.
If we consider $x^{(a)} = 1001...10$. the function is,

$$f^{(a)}(x) = x_1 \wedge (\neg x_2) \wedge (\neg x_3) \wedge x_4 .. \wedge x_5 \wedge (\neg x_6). \tag{3.5}$$

The NOT operation is represented as

$$\neg x = 1 - x. \tag{3.6}$$

So we have now expressed the function with the three sequential logical operations NOT, OR and AND. There is another elementary logical operation that creates identical one bit to two bits which is called COPY.

A computer performs elementary operations on a bit or a pair of bits using AND,OR, NOT and COPY. A computation is the process of an application of a finite sequence of operations, circuits onto input bits. The result of the computation is the bits that come out after all operations have been executed.

We can perform arbitrary complex computations with just simple hardware, executing the elementary logical operations.
With this brief overview. It suffices we can begin discussing quantum information and quantum circuits.

3.3 Quantum Circuits and computational complexity

In quantum information, we consider qubits as a unit. The qubit is represented as a vector which has a basis of $|0\rangle$ and $|1\rangle$ and is found in a Hilbert space. The vector found in this space is characterised as

$$|\psi\rangle = b_1|0\rangle + b_2|1\rangle, |b_1|^2 + |b_2|^2 = 1, \tag{3.7}$$

where $b_1, b_2 \in C$. The states of these kind have unitary operators acting on them. The measurements in this space are not deterministic. The probability of finding

$|0\rangle$ is $|b_1|^2$ and $|1\rangle$ is $|b_2|^2$.

We can generalise and define quantum computation. Given a string of n qubits, let the standard initial be $|0\rangle \otimes |0\rangle \otimes |0\rangle ... \otimes |0\rangle$. We can apply the unitary transformation U, which is made up of quantum gates, that apply and action on a few qubits at a time. The result of U acting on an initial will project on $\{|0\rangle, |1\rangle\}$ basis. The output is the measurement of the computation.

Given what are elementary gates in quantum information. We can think of quantum circuit as a process where unitary transformation is broken to simpler products of unitary operation that act on one qubit at a time. Quantum circuits in an acryclic network representation can be seen as quantum gates that are connected by wires. The act of the quantum operation on some qubits is represents the quantum gates. The wires represent the action that quantum gates act on the the qubits.

Quantum circuits may have n input qubits and m output qubits. The quantum circuit prompts some quantum n input qubits to m output qubits due to an action caused by quantum gates in a certain way.

The size of a quantum circuit depends on the number of elementary operations. This gives us an idea of how hard it is to perform that task. Suppose our computer perform task in series. This tells us that the gate will perform on a qubit one time interval a time. This shows that time it takes to compute is related to the size of the circuit.

Let us consider the computer performs the task in parallel. This means that any gate simultaneously acts on a qubit. We notice that this simultaneous action is proportional to the depth of the circuit. With all this we can define Complexity as the minimum depth or size needed to to perform a circuit up to tolerance ϵ.

Let us mention a couple of logical operators in quantum information and their representation.

The NOT gate acting on a quantum state converts $0 \to 1$ and $1 \to 0$. It is characterised by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.8}$$

We also have the Hadamard gate which acts on a single qubit.

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3.9}$$

Acting on a state with the Hadamard gate twice is the equivalent to a NOT gate. There is also a controlled NOT gate (CNOT). It is a two qubit unitary gate that acts as the NOT operations on the second qubit if and only if the first qubit is $|1\rangle$.

The gate is represented by the matrix

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.10}$$

the gate act on the state $|\psi, \phi\rangle = |\psi, \psi \otimes \phi\rangle$ where $\otimes$ represents XOR $x \otimes y = x + y \mod 2$.

We also have the Toffoli gates, which maps $|\psi, \phi, \theta\rangle \rightarrow |\psi, \phi, \theta \otimes \psi\phi\rangle$. We also have the phase gate, given by

$$R_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}, \tag{3.11}$$

and the controlled phase gate

$$\Lambda(R_\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{pmatrix} \tag{3.12}$$

With these gates we defined , we can construct a universal gate. Universal gate are

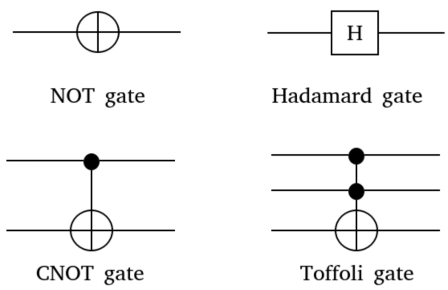

Figure 3.1: The geometric representation of the couple quantum logical operators.

gates that can approximate target state up to a tolerance in a quantum circuit.

We realise a very significant result in universal gates which comes from the Solovay-Kitaev theorem. Given a universal set G, the number of N times that you need to act the gates found in G to get to a target state of tolerance $\epsilon > 0$ is $N = O(\log^c(\frac{1}{\epsilon}))$. Where c is a constant $1 \leq c \leq 2$. This tells us that complexity has an upper bound. This becomes a problem because to reach a higher accuracy it requires circuits of exponential size in n.

3.3.1 Complexity in Quantum Field Theory

Due to the discrete number of degrees of freedom in quantum mechanics it is generally not difficult to calculate complexity. Computing complexity in quantum field theory is quite tricky since you have infinite number of degrees of freedom. It also does not have concrete definition. We will look at the four popular methods that are used to define complexity in QFT. We will begin with circuit complexity in Nielsen approach. The second will be complexity using the co variance matrix approach. The third complexity using the Fubini-Study metric. Lastly we will mention the Krylov Complexity. Recently there has been a method of complexity using path integrals [9]. This is relatively young compared to the other three it shall not be mentioned in the thesis. Before we define complexity in QFT. Let us motivate that the standard measure of distinguishing can been described better using complexity.

3.3.2 Geometry of quantum states

Fidelity and Fubini-Study metric

Let us consider a finite dimensional Hilbert space $\mathcal{H}$ where two states are described as vectors in that space. We want to know how are we going to be able to define distance between quantum states. Firstly we define a space that has no vectors that relate to the same quantum system by a complex number $a \in C$ such that $|\psi\rangle = a|\phi\rangle$. This space is known as complex projective space CP^n. It contains distinct state vectors. The distance measure in CP^n is defined as Fubini-Study metric [43]. The best way to define the Fubini-Study metric is by using the idea of Fidelity. It is the measure of the closeness of two quantum states are to each other. Incorporating Fidelity and assuming the the distinct state vectors are normalisable, we get

$$\kappa = |\langle\psi|\phi\rangle|^2. \tag{3.13}$$

Suppose we have to distinct state in CP^n such that $|\psi(x)\rangle$, $|\psi(x+\delta x) \in CP^n$. The distance between the two quantum states is,

$$\sqrt{\kappa} = |\langle\psi(x)|\psi(x+\delta x)\rangle|. \tag{3.14a}$$

We Taylor expand $\psi(x+\delta x) = \psi(x) + \partial_\mu\psi(x)dx^\mu + \frac{1}{2}\partial_\mu\partial_\nu\psi(x)dx^\mu dx^\nu + ...$

$$\sqrt{\kappa} = \langle\psi(x)|\psi(x)\rangle + \langle\psi(x)|\partial_\mu\psi(x)dx^\mu\rangle + \langle\psi(x)|\frac{1}{2}\partial_\mu\partial_\nu\psi(x)dx^\mu dx^\nu\rangle. \tag{3.14b}$$

After rearranging terms we define the Fubini-Study metric as,

$$ds_{FS} = g_{\mu\nu} = \langle\partial_\mu\psi(x)|\partial_\nu\psi(x)\rangle - \langle\psi(x)|\psi(x)\rangle\langle\psi(x)|\partial_\nu\psi(x)\rangle. \tag{3.14c}$$

With the metric of Eq. 3.14c we then can define the distance of the geometry which is $D_{FS} = \int d_{FS} = \int \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}dt$. The next section will focus on how well does Fubini-Study distance do compared to complexity as distance candidates in the geometry of quantum states.

Fidelity v.s Complexity

Lets consider reference state $|\psi_R\rangle$ and a target state $|\psi_T\rangle$. They are connected via a unitary transformation U such that $|\psi_T\rangle = U|\psi_R\rangle$. With regard to Fidelity we notice that it matters not which state the (reference or target state) the operator is acting on, this would mean to choice is very much insensitive. Suppose we have

$|\chi\rangle = |1\rangle \otimes |1\rangle \otimes |1\rangle$ and $|\tau\rangle = |1\rangle \otimes |1\rangle \otimes |1\rangle$. Using the Fidelity measure we notice that the overlap is one, this means the Fubini-study metric is zero. Let us convert one qubit in$|\chi\rangle$ into $|0\rangle$ and see it's effect. This means that $|\chi\rangle = |0\rangle \otimes |1\rangle \otimes |1\rangle$. From this change using the Fubini-Study distance this is the maximal distinction possible, even though we converted a single qubit. Now with complexity not only does it show their distinct but it also shows how close their to each other whereas with Fubini-Study metric it does not. This shows that Fubini-Study metric is not as sensitive compared to complexity. This tells us that complexity seems to be the better distance candidate in regarding space of states or space of unitaries. Let us consider a space of states of $SU(2^n)$[1]. The complexity has requirements that it must satisfy to be a distance measure such as:

- Positivity: C$\geq$ 0.
- Distinguishability: C(U,V)=0 if and only if U=V.
- Symmetry: C(U,V)=C(V,U).
- Triangular Inequality: C(U,V) $\leq$ C(U,W) + C(W,V).

We also realise that the metric on $SU(2^n)$ is local right invariant this means that,

$$C(U, V) = C(UW, VW) \tag{3.15}$$

$$C(U, V) \neq C(WU, WV). \tag{3.16}$$

Satisfying these requirements allows complexity to be the best distance candidate in a geometry of states. To even extend the definition of QFT.

3.3.3 Circuit complexity in Nielsen approach

In these following sections, we will be using differential geometry as tools to compute the minimum number of of elementary gates needed to prepare the target state given a reference state.
Let us consider a target state such that,

$$|\psi_T\rangle = U|\psi_R\rangle. \tag{3.17}$$

From the point of view of Quantum Information, we know that U can be broken down to simpler elementary discrete gates. Since we are computing complexity in quantum field theory sense. We need to describe U in a continuous manner. We consider U as a path-ordered exponential map applying the Nielsen approach [14, 31, 32]

$$U = \overleftarrow{P}[-i \int_0^1 ds H(s)], \tag{3.18}$$

where $\overleftarrow{P}$ is the path ordering of the circuit from left to right. H(s) is the time dependent Hamiltonian[2]. We then consider one-parameter family of paths in unitary space.

$$U(\phi) = \overleftarrow{P}[-i \int_0^\sigma ds H(s)] \text{ where } H(s) = \sum_I Y^I(s) O_I, \tag{3.19}$$

[1]This specific unitaries are chosen due to their ability to carry out operations for n-qubit state.
[2]These Hamiltonians are inspired by the theory of optimal control[24]

where H(s) is the Hermitian operator expressed from an expansion of Hermitian basis operators O_I. We can suppose that the operators behave in the same way as elementary gates. $Y^I(s)$ is a control function which tells us which gate should be applied at a certain time s in the circuit. In the space of unitaries it can also be thought of as a tangent vector and can be expressed as,

$$Y^I(\phi)O_I = \partial_\phi U(\phi)U^{-1}(\phi). \tag{3.20}$$

The boundary conditions associated with the unitary is $U(\phi = 0) = I$,$U(\phi = 1) = U$. With the above information we notice that we are dealing with a variational problem, where we solve for the shortest path between $U(\phi = 0) = I$ and $U(\phi = 1) = U$.

We define the distance functional that corresponds to some unitary as,

$$D[U] = \int_0^1 d\phi F(U(\phi), Y(\phi)^I), \tag{3.21}$$

where $\mathcal{F}(U, Y^I)$ is some cost function. Certain requirements should be satisfied [14, 31] which are

- Positivity: $\mathcal{F} \geq 0$ with $\mathcal{F} = 0$ if and only if $Y^I = 0$.
- Continuous and Smoothness: $\mathcal{F} \in C^\infty$.
- Homogeneity: $\mathcal{F}(U, \lambda Y^I) = \lambda\mathcal{F}(U, Y^I)$.
- Triangle Inequality: $\mathcal{F}(U, Y^I + Y^{I'}) \leq \mathcal{F}(U, Y^I) + \mathcal{F}(U, Y^{I'})$.

Even with such requirements, we still have freedom to choose which cost function we want. Different choices have different implications. Let us choose the cost function associated with Riemannian geometry. The distance functional becomes

$$D[U] = \int_0^1 d\phi \sqrt{G_{IJ}Y^I(\phi)Y^J(\phi)}, \tag{3.22}$$

where G_{IJ} chooses to favour or not specific velocities Y^I. If we choose $G_{IJ} = \delta_{IJ}$ this means every tangent vector has equal chance. Applying the theorem from [31] we notice minimizing the functional produces complexity. This is the same finding the geodesics for a specific cost function.

We want to find the most efficient circuit to prepare the target state. This means we also have to have efficient Hamiltonians which are tractable. For us to do this we have to choose operators that build elementary gates to have a closed algebra. This means for a particular O_I it has to satisfy $[O_I, O_J] = \epsilon^{ijk}O_k$. This gives us a theoretical group structure. This means we can look at circuits as trajectories in a group manifold. The shortest geodesic in the manifold is what we call complexity.

Let us consider the complexity of a coupled harmonic oscillator as done in [24] (simple Gaussian state). The applied the canonical operators x_i and p_i to define the elementary gates.

$$H = e^{ix_0p_0}, J_a = e^{ix_0p_a}, K_a = e^{ix_ap_0} \tag{3.23}$$

$$Q_{ab} = e^{ix_a p_b}, Q_{aa} = e^{\frac{i\epsilon}{2}(x_a p_a + p_a x_a)} = e^{\frac{\epsilon}{2} i x_a p_a}, \tag{3.24}$$

x_0, p_0 are defined as constants and x_a and p_b are operators that satisfy $[x_a, p_b] = i\delta_{ab}$. The quantum gates act on the Gaussian states $\psi(x_I, x_{II})$ as follows

- H creates a global phase space,

$$H\psi(x_I, x_{II}) = e^{ix_0 p_0}\psi(x_I, x_{II}) \tag{3.25}$$

- J creates a shift of x_I by ϵx_0,

$$J\psi(x_I, x_{II}) = \psi(x_I + \epsilon x_I, x_{II}) \tag{3.26}$$

- K initiates a local phase change

$$K\psi(x_I, x_{II}) = e^{ix_0 p_0}\psi(x_I, x_{II}) \tag{3.27}$$

- Q_{ab} initiates entanglement as long as $a \neq b$, where it creates a shift of x_b by ϵx_a,$Q_{(I,II)}\psi(x_I, x_{II}) = \psi(x_I, x_{II} + \epsilon x_I)$.
- $Q_{(I,I)}$ initiates re scaling of $\psi(x_I, x_{II})$ i.e. $Q_{(I,I)}\psi(x_I, x_{II}) = e^{\frac{\epsilon}{II}}\psi(e^{\epsilon} x_I, x_{II})$.

We keep ϵ very small. This keeps the changes of the gates acting on the wave function to be at the minimum. These gates are not enough to produce the target state. Knowing the target state that allows us to implement minimal set of gates to receive unitary required.

The generators $O_I = O_{ab} = ix_a p_a + \frac{1}{2}\delta_{ab}$ has a closed algebra, which in our case gl(2,R). Therefore the unitary transformation is

$$U(\phi) = \overleftarrow{P} \exp\,[i \int_0^\phi ds \sum_{a,b} Y^{ab}(s) O_{ab}]. \tag{3.28}$$

The complexity here is defined as the geodesic from $U(\phi = 0) = I$ to $U(\phi = 1)$. To measure complexity for Gaussian states we are choosing Riemaniann distance,

$$C = D_{min} = min\{\int_0^1 \sqrt{\sum_{a,b,c,d} G_{ab,cd} Y^{ab}(s) Y^{cd}(s)}\}. \tag{3.29}$$

There can be multiple such geodesics. This case the shortest geodesic is complexity.

To begin computing complexity, we firstly have to find Y^I which is known as the tangent vectors. It is expressed interms of the unitary (Eq. 3.20) and generators O.

In case of Gaussian states let $O_I \to O_{ab} \in GL(2, R)$. Express $Y^{ab}(s)$ as

$$Y^{ab}(s) = Tr(\frac{dU}{ds} U^{-1}(s) O^{ab}) \tag{3.30}$$

This works when operators are represented as matrix.

This harmonic oscillator example shows the beauty of group theoretic structure. It allows not to have to worry about the physical nature of the operators. What we have to do is re imagine the circuits as paths that lie in the group manifold G and this done by finding the Lie algebra g. We notice the convenience in using this representation. Using this trick we always choose the simplest basis in matrix form corresponding to O.

With all these ingredients, we can compute complexity by minimising the functional

$$D = \int_0^1 ds \sqrt{\delta_{ij} Tr(\frac{dU}{ds} U^{-1}(s) O^I) Tr(\frac{dU}{ds} U^{-1}(s) O^J)} = \int_0^1 ds \sqrt{g_{IJ} \dot{x}^I \dot{x}^J}. \quad (3.31)$$

With this, we can define the line element

$$ds^2 = \delta_{IJ} Tr(dUU^{-1}O^I) Tr(dUU^{-1}O^J). \quad (3.32)$$

Choosing a convenient parametrisation of U (which reflects the theoretic group structure). We can find the primary geometry, which will result in corresponding boundary conditions. This will allow us to construct geodesic, the shortest is defined complexity.

Gaussian states

Let us review [24] which they considered Gaussian state. They defined the reference and target state in terms of normal coordinates (The coupled harmonic oscillator)

$$|\psi_R\rangle = N \exp\left[\frac{-\omega_0(x_+^2 + x_-^2)}{2}\right], \; |\psi_T\rangle = P \exp\left[\frac{-\omega_+ x_+^2 + \omega_- x_-^2}{2}\right]. \quad (3.33)$$

The author re imagined this in a matrix representation. The wave functions will have the form

$$|\psi_R\rangle = N \exp\left[\frac{-(y_i Q_{ij}(s=0) y_j}{2}\right], \; |\psi_T\rangle = P \exp\left[\frac{-y_i Q_{ij}(s=1) y_j}{2}\right], \quad (3.34)$$

where $y_i = \{x_+, x_-\}$ and $Q(s=0) = \omega I$ and $Q(s=1) = diag\{\omega_+, \omega_-\}$. Q transform under unitary transformation,

$$Q(s=1) = U(s) Q(s=0) U^T(s). \quad (3.35)$$

Due to this being a coupled harmonic oscillator, [29] noticed that GL(N,R) has N^2 generators $\{O^J\}$. This is a problem as they ended up with high number of degrees of freedom. How they dealt with it is for a general element found in GL(N,R), they applied the Iwasawa decomposition [21]. This breaks down the element of GL(2,R) into an othorgonal matrices K. A positive diagonal matrices with determinant 1 called A. and N being a nilpotent group that has upper triangular matrices with 1's on the diagonal. The matrices N,K can be discarded as they increase complexity unnecessarily. The basis generators they had for GL(2,R) become

$$M_{++} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \;,\; M_{+-} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \;,\; M_{-+} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$M_{--} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Since this is defined in normal coordinates. The generators M_{+-} and M_{-+} are not considered.

$$U(s) = \exp\left(\alpha_+ M_{++} + \alpha_- M_{--}\right), \tag{3.36}$$

with $\alpha_\pm \in R$. Plugging into Eq. 3.32, it becomes

$$ds^2 = d\alpha_+^2 + d\alpha_-^2. \tag{3.37}$$

Geodesics are trivially obtained in this space time.

$$\alpha_\pm(s) = \alpha_\pm(s=0) + \alpha_\pm(s=0), \tag{3.38}$$

The above result and Eq. 3.35. It becomes

$$Q(s=1) = diag\{\omega_+, \omega_-\}. \tag{3.39}$$

The boundary conditions then become

$$\alpha_\pm(s=0) = 0, \alpha_\pm = \frac{1}{2}\log\frac{\omega_\pm}{\omega_0}. \tag{3.40}$$

The complexity of the coupled harmonic oscillator is then

$$C[U] = \frac{1}{2}\sqrt{\log\left(\frac{\omega_+}{\omega_-}\right)^2 + \log\left(\frac{\omega_-}{\omega_0}\right)^2}. \tag{3.41}$$

3.3.4 Covariance Matrix Approach

Gaussian states not only can they be represented as wave functions but also covariance matrix. For a generic quantum system we define a two point function as

$$\langle\psi|\xi^a\xi^b|\psi\rangle = \frac{1}{2}\langle\psi|\{\xi^a,\xi^b\} + [\xi^a,\xi^b]|\psi\rangle = G^{ab} + i\Omega^{ab}, \tag{3.42}$$

where ξ^a is a dimensionless canonical coordinates, which have a commutation relation that produces the symplectic form Ω^{ab} .

In a bosonic system, Ω^{ab} becomes trivial, as it is completely fixed in standard commutation relations.

$$\Omega^{ab} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \tag{3.43}$$

where G^{ab} is characterises the bosonic Gaussaian states.
For Fermionic system, $G^{ab} = \delta_{ab}$ and $i\Omega^{ab}$ characterises the fermionic Gaussian

states. For G^{ab} to be characterised as covariance matrix, G^{ab} has to be real, positive definite symmetric matrix. This demand follows from the Robert-Schrödinger uncertainty principle which states that $G^{ab} + i\Omega^{ab} \geq 0$.

For a pure bosonic and fermionic Gaussian state, we have a vanishing one point function $\langle\psi|\xi^a|\psi\rangle = 0$. We define the bosonic and fermionic states as

$$G^{ab} = \frac{1}{2}\langle\psi|\{\xi^a, \xi^b\}\psi\rangle, \quad i\Omega^{ab} = \frac{1}{2}\langle\psi|[\xi^a, \xi^b]|\psi\rangle, \tag{3.44}$$

respectively. With all this, to compute complexity , we have to consider the reference and target state in co variance matrix form. $G_T = U^\dagger(\phi)G_R U(\phi)$, $U(\phi) = \overleftarrow{P}\exp\left[-i\int_0^\phi K(s)ds\right]$ where $K(s) = \sum_I Y^I(s)K_I$. Then the complexity is given as before

$$C(U) = \int ds\sqrt{\sum_I |Y^I(s)|^2}, \tag{3.45}$$

where the $Y^I(s)$ are equally favourable due to $G_{IJ} = \delta_{IJ}$. Provided that we parametrise $U(\phi)$, we can find the boundary conditions. We will be able to utilise differential geometry as a tool to be able to compute complexity as the same as the Nielsen approach from the previous section.

We also want to see how does G^{ab} transform under the unitary transformation U. $SP(2N, R)$ has a general transformation such that

$$G^{ab}(\phi) = \frac{1}{2}\langle\psi(\phi)|\{\xi^a, \xi^b\}|\psi(\phi)\rangle \tag{3.46}$$

$$= \frac{1}{2}\langle\psi(\phi)|e^{i\phi K}\{\xi^a, \xi^b e^{-i\phi K}|\psi(\phi)\rangle \tag{3.47}$$

$$= \frac{1}{2}U(\phi)^a_c U(\phi)^b_d\langle\psi(0)\{\xi^b, \xi^d\}\psi(0)\rangle \tag{3.48}$$

$$= U(\phi)^a_c U(\phi)^b_d G^{cd}(0). \tag{3.49}$$

The covariance matrix under the unitary transformation becomes

$$G_T = UG_R U^T \tag{3.50}$$

Let us review [29] where they considered Gaussian states in normal coordinates. In covariance matrix representation of states is given as

$$G_R = \begin{pmatrix} \frac{1}{\omega_0} & 0 & 0 & 0 \\ 0 & \omega_0 & 0 & 0 \\ 0 & 0 & \frac{1}{\omega_0} & 0 \\ 0 & 0 & 0 & \omega_0 \end{pmatrix}, G_T = \begin{pmatrix} \frac{1}{\omega_+} & 0 & 0 & 0 \\ 0 & \omega_+ & 0 & 0 \\ 0 & \frac{1}{\omega_-} & 0 & 0 \\ 0 & 0 & 0 & \omega_- \end{pmatrix} \tag{3.51}$$

Compactifying it to make a 2×2 matrix

$$G_R^0 = \begin{pmatrix} \frac{1}{\omega_0} & 0 \\ 0 & \omega_0 \end{pmatrix}, G_T^{\pm} = \begin{pmatrix} \frac{1}{\omega_\pm} & 0 \\ 0 & \omega_\pm \end{pmatrix} \tag{3.52}$$

The squeezing operator, makes the the computation more digestible. So the change of basis is

$$\tilde{G^0_R} = SG^0_R S^T, \tilde{G^\pm} = SG^\pm_T S^T, \tag{3.53}$$

where

$$S = \begin{pmatrix} \sqrt{\omega_0} & 0 \\ 0 & \frac{1}{\sqrt{\omega_0}} \end{pmatrix} \tag{3.54}$$

This results into

$$\tilde{G^0_R} = I, \qquad G^0_R = \begin{pmatrix} \frac{\omega_0}{\omega_\pm} & 0 \\ 0 & \frac{\omega_\pm}{\omega_0} \end{pmatrix}. \tag{3.55}$$

For us to find the complexity. They had to compute the geodesic that follows the path from $\tilde{G^\pm_T}(\sigma) = U(\phi)^T \tilde{G^\pm_T}(\phi) U(\phi)$.

Let $U \in SP(2,R)$ with full parametrisation

$$U(\sigma,\kappa,\eta) = \begin{pmatrix} \cos\sigma_\pm \cosh\kappa_\pm - \sin\eta_\pm \sinh\kappa_\pm & -\sin\eta_\pm \cosh\kappa_\pm + \cos\eta_\pm \sinh\kappa_\pm \\ \sin\sigma_\pm \cosh\kappa_\pm + \cos\eta_\pm \sinh\kappa & \cos\sigma_\pm \cosh\kappa_\pm + \sin\eta_\pm \sinh\kappa_\pm \end{pmatrix}. \tag{3.56}$$

The author in [29] defined $\rho_\pm \equiv \sigma_\pm + \eta_\pm$

$$\begin{pmatrix} \cos(2\sigma_\pm) - \sin(\rho_\pm)\sinh(2\kappa_\pm) & \cos(\rho_\pm)\sinh(2\kappa_\pm) \\ \cos(\rho_\pm)\sinh(2\kappa) & \cosh\kappa + \sin(\rho_\pm)\sinh(2\kappa_\pm) \end{pmatrix} = \begin{pmatrix} \frac{\omega_0}{\omega_\pm} & 0 \\ 0 & \frac{\omega_\pm}{\omega_0} \end{pmatrix} \tag{3.57}$$

This results in $\kappa_\pm(\phi = 1) = \frac{1}{2}\log(\frac{\omega_\pm}{\omega_0})$, $\kappa(\phi = 1)_\pm = 0$, $\rho_\pm = \sigma_\pm + \eta_\pm = \frac{\pi}{2}$. The line element is then

$$ds_k^2 = \delta_{IJ} Tr(\partial U_k U_k^{-1} K^I) Tr(\partial U_k U_k^{-1} K^J), \tag{3.58}$$

where $k = \pm$ and

$$K^I = [\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}]^T \tag{3.59}$$

This results in

$$= d\kappa_k^2 + \cosh(2\kappa_k)\sinh^2\kappa_k d\eta_k^2 - \sinh^2(2\kappa) d\eta_k d\sigma_k + \cosh^2\kappa_k \cosh(2\kappa_k) d\sigma_k^2. \tag{3.60}$$

Since η_k, σ_k are constants, their full geometry becomes a flat space

$$ds^2 = d\kappa_+^2 + d\kappa_-^2 \tag{3.61}$$

The complexity is given by

$$C[U] = \int_0^1 d\phi \sqrt{g_{ij} \frac{d\kappa_i}{d\phi} \frac{d\kappa_j}{d\phi}} = \frac{1}{2}\sqrt{\log(\frac{\omega_+}{\omega_0})^2 + \log(\frac{\omega_-}{\omega_0})^2} \tag{3.62}$$

3.3.5 Fubini-Study Complexity

In this section, instead of considering the complexity in the space of unitaries. We are considering the space of states, this will require examining complexity using the Fubini-Study metric. In the section of geometry of states we noticed the Fubini-Study distance struggles with giving us information about the path travelled from on state to the other. This results in it being "naive" in a sense that we do not know how we reached from the reference to the target state. The complexity of the Fubini-Study metric will work well in situations we have specific set of operators $\mathcal{O}$. Which was introduced in [12]. We can also use it in situations where the curves pass through low complexity.

The difference between the previous section and the Fubini-Study metric is that our previous subsection we parametrised the unitary. The complexity of the Fubini-Study metric parametrises the state. Specifically we cast a coherent form on the target state. The coherent form has a theoretical group structure. This is explained by [16] and [35] extensively.

In this thesis, we consider the target state to be bosonic coherent states that are elements of SU(1,1). This is a review of the computation done by [29]. The representation of the group action is explained extensively in the Appendix. Due to the coherent form having a theoretical group structure, the complexity will be associated with geodesics that impose SU(1,1). The parametrisation allowed [29] to define the boundary conditions.

[29] considered path-ordered exponential unitary which is generated by $\mathcal{O} \in su(1,1)$ where,

$$|\psi_T(\phi)\rangle = \overleftarrow{P} \exp[-i \int_0^\sigma H(t)dt]|\psi_R(\phi)\rangle, H(t) = \sum_I Y^I \mathcal{O}(\phi). \tag{3.63}$$

This procedure to express reference and target state for generic SU(1,1) which is of the form of Eq. A.4. Applying the decomposition from Eq. A.6, where $\mathcal{O}_-$ annihilates $|\psi_0\rangle$ and $\mathcal{O}_0|\psi\rangle = c|\psi\rangle$ where c is a constant. So, we recast the generic SU(1,1) state as

$$|\psi(\phi)\rangle = \tilde{N(\phi)} e^{\gamma_+(\phi)\mathcal{O}^+}|\psi_0\rangle, \tag{3.64}$$

where $N(\phi) = e^{c \ln \gamma^0}$.

This is true for single modes. [29] extended to N mode states. The author defined the ground state as

$$|\psi_0\rangle = \prod_{k=0}^{N-1} |k, -k\rangle, \tag{3.65}$$

and they built the generic state.

$$|\psi(\phi)\rangle = \prod_{k=0}^{N-1} \tilde{N(\phi)} e^{\gamma_+(\phi)\mathcal{O}^+}|k, -k\rangle \tag{3.66}$$

These generators $\mathcal{O}_I$ generate simple operations. This means that the path when parametrising ϕ will reach the target state via simple states. Hence they utilised the Fubini-Study cost function Eq. 3.14c. Computing $\partial_\phi|\psi(\phi)\rangle$,

$$\partial_\phi\psi(\phi)\rangle = \frac{\gamma'^0(\phi) + 2\gamma^0(\phi)\gamma^{+\prime}(\phi)\mathcal{O}^+}{2\sqrt{\gamma^0(\phi)}} e^{\gamma^+(\phi)\mathcal{O}^+}|\psi_0\rangle \tag{3.67}$$

$$= [\frac{1}{2}\frac{\gamma'^0(\phi)}{\gamma^0(\phi)} + \gamma'^+(\phi)\mathcal{O}^+]|\psi(\phi)\rangle \tag{3.68}$$

$$\langle\psi|\partial_\phi|\psi(\phi)\rangle = \langle\psi|[\frac{1}{2}\frac{\gamma'^0(\phi)}{\gamma^0(\phi)} + \gamma'^+(\phi)\mathcal{O}^+]|\psi\rangle \tag{3.69}$$

$$= [\frac{1}{2}\frac{\gamma'^0(\phi)}{\gamma^0(\phi)} + \gamma'^+(\phi)\langle\mathcal{O}^+\rangle]. \tag{3.70}$$

The expectation value is $\langle\mathcal{O}^+\rangle = \langle\psi|\mathcal{O}^+|\psi\rangle$. The author [29] considered how $\mathcal{O}_i$ is affected by the unitary transformations. They considered a matrix representation of the form

$$U(\phi)^\dagger\mathcal{O}^i U(\phi) = M_{ij}\mathcal{O}^j \tag{3.71}$$

$$\langle\psi|\mathcal{O}^+|\psi\rangle = \langle\psi_0|M_{+j}\mathcal{O}^j|\psi_0\rangle \tag{3.72}$$

$$= \frac{1}{2}M_{+0}, \tag{3.73}$$

where they applied the identity that $\langle\psi|\mathcal{O}^+|\psi\rangle = \langle\psi|\mathcal{O}^-|\psi\rangle = 0$. This brought them to

$$\langle\partial_\phi\psi|\partial_\phi\psi\rangle = \langle\psi|[\frac{1}{2}\frac{(\gamma'^0(\phi))^*}{(\gamma^0(\phi))^*} + (\gamma'^+(\phi))^*(\mathcal{O}^+)^*][\frac{1}{2}\frac{\gamma'^0(\phi)}{\gamma^0(\phi)} + \gamma'^+(\phi)\mathcal{O}^+]|\psi\rangle. \tag{3.74}$$

Plugging the above equation in Eq. 3.14c and knowing that $\langle\mathcal{O}^+\rangle = \langle\psi|\mathcal{O}^+|\psi\rangle$ they found that

$$\mathcal{F} = [\frac{1}{2}\frac{(\gamma'^0(\phi))^*}{(\gamma^0(\phi))^*} + (\gamma'^+(\phi))^*\langle(\mathcal{O}^+)^*\rangle][\frac{1}{2}\frac{\gamma'^0(\phi)}{\gamma^0(\phi)} + \gamma'^+(\phi)\langle\mathcal{O}^+\rangle] \tag{3.75}$$

$$= |\gamma^+|^2\langle\psi(\phi)|(\mathcal{O}^+)^*\mathcal{O}^+|\psi(\phi)\rangle \tag{3.76}$$

$$= |\gamma^+|^2 L_{-j}L_{+k}\langle\psi(\phi(\mathcal{O}^j)\mathcal{O}^k|\psi(\phi)\rangle \tag{3.77}$$

$$= |\gamma^+|^2 L_{-j}L_{+k}. \tag{3.78}$$

Since they knew that $\langle\psi|\mathcal{O}^-\mathcal{O}^-|\psi\rangle = \langle\psi|\mathcal{O}^+\mathcal{O}^+|\psi\rangle = 0$. They also knew that from Eq. A.10. They got

$$L_{--}L_{++} = (1 + \frac{\gamma^+\gamma^-}{\gamma^0})^2 \tag{3.79}$$

$$L_{--}L_{++} = \frac{1}{|\gamma^0|^2}. \tag{3.80}$$

By reinstating index k, the final result of the line element of Fubini-Study is $ds^2 = \sum_k^{N-1} ds_k^2 = \sum_k^{N-1}\frac{|d\gamma_k^+|^2}{(1-|\gamma_k^+|^2)}$. They defined the distance in CP^n space as

$$D_{FS} = \sqrt{\sum_k^{N-1} s_k^2} \tag{3.81}$$

$$s_k = \int_0^1 d\phi\frac{1}{1-|\gamma_k^*(\phi)|^2}|\frac{d\gamma_{k]+(\phi)}}{d\phi}|. \tag{3.82}$$

Complexity becomes the minimal length of the curve

$$C = min_{\gamma^+(\phi)}\{D_{FS}\}. \tag{3.83}$$

They introduced a parametrisation of $\gamma(\phi)^+$. This is so they can derive complexity geodesic by computing the geometry it lies on. Knowing that $|\gamma_{k,0}| \leq 1$

$$\gamma_k^+(\phi) = |\gamma_k| \exp\left(i\phi_k(\phi)\right), |\gamma_k| \tanh\left(\frac{\theta_k(\phi)}{2}\right). \tag{3.84}$$

Plugging these in the distance for a given k, the result is

$$ds^2 = \frac{1}{4}\sum_{k=0}^{N-1}(d\theta_k^2 + \sinh\left(\theta_k\right)^2 d\sigma_k^2). \tag{3.85}$$

3.3.6 Krylov complexity

The fourth and last approach in computing complexity is defined as the Krylov complexity. This is particularly interesting in dealing with quantum chaos and many body models. The Krylov complexity has its roots from the Lanczos algorithm which is found in many-body dynamics. The Lanzcos algorithm has a recursively systemic method that constructs the space of complex operators. This is known as operator complexity.

In recent papers Krylov complexity has been suggested to be a good contender for complexity in interacting quantum field theory[8]. We realised that even when we have open definition. Computing Krylov complexity requires from us to understand the numerics and this makes it difficult to understand the universal features.

The quantum information progress that bare its fruits from holographic entanglement entropy to holographic complexity has allowed us to study observables in both the bulk and the boundary theory and has been an integral part in extensively trying to create a concrete translation in the AdS-CFT correspondence.

In this thesis we will develop Krylov complexity from a geometric approach. With this approach we will have an underlying understanding of operator complexity.

3.4 Krylov Space

Suppose that for any generic system we have Hamiltonian H in a Hilbert space $\mathcal{H}$. We define a Krylov space as minimum subspace in operator space that contains $\mathcal{O}$ at all time. The operator $\mathcal{O}$ is governed by the Heisenberg picture.

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt} = e^{i\mathcal{L}}\mathcal{O}, \tag{3.86}$$

where the super operator Liouvillian is defined as $[H, .]$. We define the Krylov space as the span of all nested commutations of H with the other operator

$$\text{span}\{\mathcal{O}, [H,\mathcal{O}], [H,[H,\mathcal{O}]], ..\}. \tag{3.87}$$

We adjust the Krylov space to the time evolution of an operator with an identified constant Hamiltonian. We would like to compose the othornormal basis for the Krylov space, provided by a thermal two point function on the operator space. The thermal two point function is

$$\langle A|B\rangle_{\beta}, \tag{3.88}$$

where $\langle A\rangle = \frac{1}{Z}Tr(e^{-\beta H}A)$ and $Z = Tr(e^{-\beta}H)$. We produce the othornormal basis by using the Lanczos algorithm, which is generated by the Gram-Schmidt othorgalization procedure. We begin by setting $b_0 \equiv 0$ and $|\mathcal{O}_{-1}\rangle \equiv 0$. Then $|\mathcal{O}_0\rangle = \frac{\mathcal{O}}{||\mathcal{O}||}$. We set

$$A_1 = [H, \mathcal{O}_0] - (\mathcal{O}_\prime, [H, \mathcal{O}_0])\mathcal{O}_0 = [H, \mathcal{O}_0]. \tag{3.89}$$

Define $b_1 = ||A_1||$, set $\mathcal{O}_1 = \frac{A_1}{b_1}$. With this we define

$$A_2 = [H, \mathcal{O}_1] - b_1\mathcal{O}_0. \tag{3.90}$$

We follow the same procedure until $b_n = 0$ and then we stop, othwerise we continue until we notice a recurring pattern from b_n. What we notice from this procedure $\{\mathcal{O}_n\}_{n=0}^{k-1}$ is orthonormal such that$(\mathcal{O}_n, \mathcal{O}_m) = \delta_{nm}$. We also notice that the Liouvillian operator can be represented as a tridiangular matrix. Time evolution on the Krylov basis. The operators time evolution can be seen as the motion along the Krylov basis.

$$\mathcal{O}(t) = \sum_{n=0}^{k-1} i^n\phi_n(t)\mathcal{O}_n \tag{3.91}$$

Using the Heisenberg equation

$$\frac{d}{dt}\mathcal{O}(t) = i[H, \mathcal{O}(t)], \tag{3.92}$$

we get

$$\dot{\phi}_n(t) = b_n\phi_{n-1}(t) - b_{n+1}\phi_{n+1}(t) \tag{3.93}$$

$$\phi_0 = \delta_{n0}. \tag{3.94}$$

As these boundary conditions ensures that $\mathcal{O}(0) = \mathcal{O}_0$.
What we realise is important is that the dynamics of the operators in the Krylov space depend on the Lanczos coefficients b_n.

Krylov Complexity

$$C_k = \sum_{n=0}^{k-1} n|\Phi_n(t)|^2 \tag{3.95}$$

Eq. 3.94 can be viewed from a physical standpoint as dynamics of a particle found on the Krylov chain[3]. With this natural measure of operator complexity arises defined as Krylov complexity. Which is defined as the average position on the Krylov basis.

[3]It is a one dimensional chain that has a one-to-one correspondence with the Krylov basis[8]

Let us consider computing the Krylov complexity of harmonic oscillator. We firstly define

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \tag{3.96}$$

$$x^\dagger = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger). \tag{3.97}$$

We set $\hbar = 1$ and $m = 1$. This results into

$$x = \sqrt{\frac{1}{2\omega}}(a^\dagger + a) \tag{3.98}$$

$$x^\dagger = \sqrt{\frac{1}{2\omega}}(a + a^\dagger). \tag{3.99}$$

The thermal two point function is

$$\langle x x^\dagger \rangle_\beta = \left(\frac{1}{2\omega}\right) \langle (a^\dagger + a)(a + a^\dagger) \rangle_\beta \tag{3.100}$$

$$= \left(\frac{1}{2\omega}\right) \langle a^\dagger a + a^\dagger a^\dagger a a + a a^\dagger \rangle_\beta \tag{3.101}$$

$$= \left(\frac{1}{2\omega}\right) \langle a^\dagger a + (1 + a^\dagger a) \rangle_\beta \tag{3.102}$$

$$= \left(\frac{1}{2\omega}\right) \langle 2n + 1 \rangle_\beta \tag{3.103}$$

$$= \left(\frac{1}{2\omega}\right) \langle 2 \left(\frac{1}{1 - e^{\lambda T}}\right) + 1 \rangle_\beta \tag{3.104}$$

$$= \left(\frac{1}{2\omega}\right). \tag{3.105}$$

The final result becomes

$$\sqrt{\langle x x^\dagger \rangle} = \left(\frac{1}{\sqrt{2\omega}}\right). \tag{3.106}$$

We set the $\mathcal{O}_0 = \frac{\mathcal{O}}{||\mathcal{O}||} = \frac{\frac{1}{\sqrt{2\omega}}(a^\dagger + a)}{\frac{1}{\sqrt{2\omega}}}$ which becomes $\mathcal{O}_0 = (a^\dagger + a)$. We compute A_1, which is

$$A_1 = [H, \mathcal{O}_0] \tag{3.107a}$$

$$= [\omega(a^\dagger a + \frac{1}{2}), a^\dagger + a] \tag{3.107b}$$

$$= \omega[a^\dagger a, a^\dagger + a] \tag{3.107c}$$

$$= \omega(a^\dagger - a) \tag{3.107d}$$

$$\sqrt{\langle A_1 A_1^\dagger} = \sqrt{\omega^2 \langle (a^\dagger - a)(a - a^{\dagger)\rangle}} \tag{3.108a}$$

$$= \sqrt{\omega^2 \langle a^\dagger a + (1 + a^\dagger a)} \tag{3.108b}$$

$$= \sqrt{\omega^2 \langle 2n + 1 \rangle} \tag{3.108c}$$

$$= \omega. \tag{3.108d}$$

We define $b_1 = \omega$, this allows us to define $\mathcal{O}_1 = \frac{A_1}{b_1} = \frac{\omega(a^\dagger - a)}{\omega} = a^\dagger - a$. With this we define

$$\begin{aligned} A_2 &= [H, \mathcal{O}_1] - b_1\mathcal{O}_0 && (3.109a) \\ &= [\omega\left(a^\dagger a + \frac{1}{2}\right), a^\dagger - a] - \omega(a^\dagger + a) && (3.109b) \\ &= \omega(a^\dagger[a, a^\dagger] + [a^\dagger, a^\dagger]a - a^\dagger[a, a] - [a^\dagger, a]a - a^\dagger - a)\omega(a^\dagger + a - a^\dagger - a) && (3.109c) \\ &= 0. && (3.109d) \end{aligned}$$

Therefore, the Lanczos algorithm for Harmonic oscillators produces $\{\mathcal{O}_0, \mathcal{O}_1\}$.

Le us compute the Krylov Complexity for harmonic oscillator. Given

$$|\tilde{\mathcal{O}}\rangle = \sum_{n=0}^{k-1} i^n\phi_n(t)\mathcal{O}_n \quad (3.110)$$

$$= i^0\phi_0(t)\mathcal{O}_0 + i^1\phi_1(t)\mathcal{O}_1 \quad (3.111)$$

$$= \phi_0(t)\mathcal{O}_0 + i\phi_1(t)\mathcal{O}_1. \quad (3.112)$$

Applying the discrete Schrodinger equation Eq. 3.94 and Let n=0

$$\begin{aligned} \dot{\phi_0}(t) &= b_0\phi_{-1}(t) - b_1\phi_1(t) && (3.113a) \\ &= -\omega\phi_1(t). && (3.113b) \end{aligned}$$

For n=1,

$$\begin{aligned} \dot{\phi_1}(t) &= b_1\phi_0(t) - b_2\phi_2(t) && (3.114a) \\ &= \omega\phi_0(t) && (3.114b) \end{aligned}$$

We differentiate Eq. 3.113 to give us

$$\begin{aligned} \ddot{\phi_0}(t) &= -\omega\dot{\phi_1}(t) && (3.115a) \\ \ddot{\phi_0}(t) &= -\omega^2\phi_0(t) && (3.115b) \end{aligned}$$

This result comes from the substitution of Eq. 3.114 into the differentiated form of Eq. 3.113. We provide the ansatz which is

$$\phi_0(t) = C_1e^{-i\omega t} + C_2e^{i\omega t}. \quad (3.116)$$

We substitute Eq. 3.116 it into Eq. 3.113 This gives us

$$\phi_1(t) = i(C_1e^{-i\omega t} - C_2e^{i\omega t}). \quad (3.117)$$

Using the boundary condition we find that $\phi_1(0) = \delta_{10} = 0$ and expressing,

$$\phi_0(t) = C_1\cos\omega t + C_2\sin\omega t. \quad (3.118)$$

Applying unitarity $\sum_{n=0}^{k-1}|\psi_n|^2 = 1$ this allows the Krylov complexity to be

$$C_K = \sum_{n=0}^{k-1} n|\phi_n(t)|^2 \tag{3.119}$$

$$= 1|\phi_1(t)|^2 \tag{3.120}$$

$$= (C_1 e^{i\omega t} - C_2 e^{-i\omega t})(C_1 e^{-i\omega t} - C_2 e^{i\omega t}) \tag{3.121}$$

$$= 1 \tag{3.122}$$

The Krylov complexity for displacement operator in quantum harmonic oscillator is examined in [8]. The result tells us that the complexity is time dependent.

In the next section we will compute the complexity of a Bosonic coherent state using the Nielsen approach.

Chapter 4

Complexity of Bosonic Gaussian state

We are interested in the complexity of the most general bosonic Gaussian state — the wave function has the form

$$\psi(x) = \left(\frac{\text{Re}\Omega}{\pi}\right)^{1/4} \exp\left(-\frac{|P|^2}{4\text{Re}\Omega}\right) \exp\left(-\frac{P^2}{4\text{Re}\Omega}\right) \exp\left(-\frac{\Omega}{2}x^2 + Px\right) \ , \tag{4.1}$$

where Ω and P are defined as in A.15c. Physically, the situation we have in mind is the ground state of a harmonic oscillator of frequency ω_0, evolved in time with a general quadratic Hamiltonian, namely a Hamiltonian of the form

$$H = \frac{1}{2}p^2 + \frac{\omega_0^2}{2}x^2 + \frac{\delta^2}{2}x^2 + Fx \ , \tag{4.2}$$

where x and p are conjugate variables, $[x, p] = i$. [See the Appendix for details.]

4.1 Gaussian states

We are interested in Gaussian states, namely states (whose wave function is) of the form Eq. 4.1. These can be obtained using the creation and annihilation operator algebra. To this end, we write

$$x = f\ a + f^* a^\dagger \ , \ p = -i(g\ a - g^* a^\dagger) \ ; \tag{4.3a}$$

in order to satisfy the appropriate commutation relations, f and g must satisfy

$$\text{Re}(fg^*) = 1/2 \ . \tag{4.3b}$$

Inverting Eq. 4.3a, the $(a, a^\dagger)$ can be written in terms of the position and momentum operators as

$$a = g^*\ x + if^*\ p \quad , \quad a^\dagger = g\ x + if\ p \ . \tag{4.4}$$

Then, wave functions with the structure of Eq. 4.1 can be obtained from the space of states generated by the $(a, a^\dagger)$ and projecting onto the position representation; in particular,

$$\underline{a|0\rangle = 0} : \psi_0(x) = \left(\frac{\omega_0}{\pi}\right)^{1/4} \exp\left(-\frac{\omega_0}{2}x^2\right) \ , \tag{4.5a}$$

$$\underline{a|z\rangle = z|z\rangle} : \psi_z(x) = \left(\frac{\mathrm{Re}\Omega}{\pi}\right)^{1/4} \exp\left(-\frac{|P|^2}{4\mathrm{Re}\Omega}\right) \exp\left(-\frac{P^2}{4\mathrm{Re}\Omega}\right) \exp\left(-\frac{\Omega}{2}x^2 + Px\right) , \tag{4.5b}$$

where we have written

$$\Omega = \frac{g^*}{f^*} \quad \text{and} \quad P = \frac{z}{f^*} . \tag{4.5c}$$

Note: Eqs. 4.5c and 4.3b tell us how fix f and g to obtain a desired Ω in Eq. 4.1 — a consistent solution is given by

$$f = \frac{1}{\sqrt{2\mathrm{Re}\Omega}} \quad , \quad g = \frac{\Omega^*}{\sqrt{2\mathrm{Re}\Omega}} . \tag{4.6}$$

4.2 Circuit complexity of Gaussian states

As stated in the previous section of complexity using Nielsen approach. Provided reference state $|\psi_R\rangle$, the target state $|\psi_T\rangle$ and $g = \exp(-i\epsilon M_I)$ as elementary gates that have M_I as group generators. The aim is to construct the structured unitary transformation $\tilde{U}(\tau)$ that begins at the reference state and ends at the target state. state:

$$\tilde{U}(\tau = 0) = I \ , \ |\psi_T\rangle = \tilde{U}(\tau = 1)|\psi_R\rangle . \tag{4.7a}$$

where the path-ordered operator $\tilde{U}(\tau)$ is

$$\tilde{U}(\tau) = \overleftarrow{\mathcal{P}} \exp\left(-i \int_0^\tau ds \ \sum_I Y^I(s) M_I\right) \tag{4.7b}$$

with the control functions defined as $\{Y^I(s)\}$and "s" in the space of unitaries parametrises the path. Then, we define a "cost functional", $\hat{\mathcal{C}}(\tilde{U})$:

$$\hat{\mathcal{C}}(\tilde{U}) = \int_0^1 ds \sqrt{\sum_I |Y^I(s)|^2} . \tag{4.8}$$

Eq. (4.8) that complexity is determined as the minimal path. This means we have to minimise w.r.t $\{Y^I(s)\}$ as a function of "s"; In space of unitaries this is known as geodesics. Hence complexity is the length of the geodesic.

In what follows, we consider a set of gates built from the most general set of operators quadratic in position and momentum operators or, equivalently, quadratic in creation and annihilation operators — we consider states obtained by evolving the system with the "Hamiltonian"

$$H = \epsilon \ a^\dagger a + \frac{1}{2}\left(\Delta aa + \Delta^* a^\dagger a^\dagger\right) + \left(\kappa a + \kappa^* a^\dagger\right) \ , \tag{4.9}$$

where the parameters $\{\epsilon, \Delta, \kappa\}$ all depend, in general, on the "circuit time" τ. Then, considering the "Heisenberg operator" $\hat{O}(\tau)$ defined by

$$\hat{O}(\tau) = \tilde{U}^\dagger(\tau) \ \hat{O} \ \tilde{U}(\tau) \ , \tag{4.10}$$

one obtains the equation of motion

$$i\partial_\tau a(\tau) = \epsilon \ a(\tau) + \Delta^* a^\dagger(\tau) + \kappa^* \ . \tag{4.11}$$

4.2.1 The Metric

To proceed, we introduce the quantities

$$\overline{a}(\tau) = \begin{pmatrix} a(\tau) \\ a^\dagger(\tau) \end{pmatrix} \quad , \quad M_2 = \begin{pmatrix} -i\epsilon & -i\Delta^* \\ +i\Delta & +i\epsilon \end{pmatrix} \quad , \quad \overline{\lambda}_2 = \begin{pmatrix} -i\kappa^* \\ +i\kappa \end{pmatrix} \; ; \tag{4.12a}$$

the equation of motion (Eq. 4.11) can be written as

$$\partial_\tau \overline{a}(\tau) = M_2 \; \overline{a}(\tau) + \overline{\lambda}_2 \; . \tag{4.12b}$$

Then, we introduce the matrix $\hat{U}(\tau)$ such that

$$\begin{pmatrix} \overline{a}(\tau) \\ 1 \end{pmatrix} = \hat{U}(\tau) \begin{pmatrix} \overline{a} \\ 1 \end{pmatrix} \; ; \tag{4.13}$$

$\hat{U}(\tau)$ satisfies the equation of motion

$$\partial_\tau \hat{U}(\tau) = M \; \hat{U}(\tau) \; , \tag{4.14a}$$

where

$$M = \begin{pmatrix} M_2 & \overline{\lambda}_2 \\ \overline{0}^T & 0 \end{pmatrix} \; . \tag{4.14b}$$

The solution of Eq. 4.14a is given by

$$\hat{U}(\tau) = \overleftarrow{\mathcal{P}} \exp \left[\int_0^\tau ds \; M(s) \right] \; ; \tag{4.15a}$$

$\hat{U}(\tau)$ has the structure

$$\hat{U}(\tau) = \begin{pmatrix} \Lambda(\tau) & \overline{\lambda}(\tau) \\ \overline{0}^T & 1 \end{pmatrix} \; , \tag{4.15b}$$

where

$$\Lambda(\tau) = \begin{pmatrix} q & p^* \\ p & q^* \end{pmatrix} \quad , \quad \overline{\lambda} = \begin{pmatrix} z \\ z^* \end{pmatrix} \tag{4.15c}$$

with $|p|^2 - |q|^2 = 1$.

With the goal of obtaining a metric, we define

$$Y^1 = \text{Im}[\Delta] \; , \; Y^2 = \text{Re}[\Delta] \; , \; Y^3 = \epsilon \; , \; Y^4 = \text{Im}[k] \; , \; Y^5 = \text{Re}[k] \; , \tag{4.16a}$$

and introduce the quantities

$$X^1 = i2 \, Y^1 \; , \; X^2 = -i2 \, Y^2 \; , \; X^3 = -i2 \, Y^3 \; , \; X^4 = -2 \, Y^4 \; , \; X^5 = -2 \, Y^5 \; ; \tag{4.16b}$$

then M can be written as

$$M = \sum_{I=1}^{5} X^I \hat{O}_I \tag{4.17}$$

where

$$\hat{O}_1 = \frac{1}{2} \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; , \; \hat{O}_2 = \frac{1}{2} \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; , \; \hat{O}_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ,$$

$$(4.18a)$$

$$\hat{O}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \ , \ \hat{O}_5 = \begin{pmatrix} 0 & 0 & +i \\ 0 & 0 & -i \\ 0 & 0 & 0 \end{pmatrix} \tag{4.18b}$$

with the $\{\hat{O}_I\}$ normalized as

$$\mathrm{Tr}[\hat{O}_I \hat{O}_J^\dagger] = \frac{1}{2}\delta_{IJ} \ . \tag{4.18c}$$

Note that $\{\hat{O}_1, \hat{O}_2, \hat{O}_3\}$ form an SU(1,1) subalgebra

$$[\hat{O}_1, \hat{O}_2] = -i\hat{O}_3 \ , \ [\hat{O}_2, \hat{O}_3] = +i\hat{O}_1 \ , \ [\hat{O}_3, \hat{O}_1] = +i\hat{O}_2 \ .$$

As per Eq. 4.8, we define the metric

$$ds^2 = \sum_I |Y^I|^2 \ . \tag{4.19}$$

To obtain this, it can be extracted (via Eqs. 4.15b and 4.18c) as

$$ds^2 = 2 \ \mathrm{Tr}\left[(\partial_\tau \hat{U})\hat{U}^{-1}\hat{O}_J^\dagger\right] 2 \ \mathrm{Tr}\left[(\partial_\tau \hat{U})\hat{U}^{-1}\hat{O}_J^\dagger\right] \ ; \tag{4.20a}$$

where we have defined

$$p(\tau) = \cosh(\theta(\tau))e^{i\phi(\tau)}, \tag{4.20b}$$
$$q(\tau) = \sinh(\theta(\tau))e^{-i\chi(\tau)}, \tag{4.20c}$$
$$p^*(\tau) = \sinh(\theta(\tau))e^{-i\phi(\tau)}, \tag{4.20d}$$
$$q^*(\tau) = \sinh(\theta(\tau))e^{i\chi(\tau)}, \tag{4.20e}$$
$$z^*(\tau) = z_1(\tau) - iz_2(\tau), \tag{4.20f}$$
$$z(\tau) = z_1(\tau) + iz_2(\tau). \tag{4.20g}$$

Thus, we obtain

$$\begin{aligned} ds^2 = {} & (1 + 4z_1^2 + 4z_2^2)d\theta^2 + (-2\sin(v)\sinh(4\theta)z_1z_2 + \cos(v)\sinh(4\theta) \\ & (-z_1^2 + z_2^2)\frac{1}{4}\cosh(4\theta)(1 + 4z_1^2 + 4z_2^2))du^2 + (\frac{1}{4} + z_1^2 + z_2^2)dv^2 \\ & + 4dz_1^2 + 4dz_2^2 + 4\cosh(2\theta)(-2\cos(v)z_1z_2 + \sin(v)(z_1^2 - z_2^2)d\theta du \\ & + (-8\cos(v)z_1z_2 + 4\sin(v)(z_1^2 - z_2^2))d\theta du - 8(\cos(v)z_1 + \sin(v)z_2)d\theta dz_1 \\ & - 8(\sin(v)z_1 + sin(v)z_2)d\theta dz_2 + (-4\sin(v)\sinh(2\theta)z_1z_2 + 2\cos(v)\sinh(2\theta) \\ & (-z_1^2 + z_2^2) + \frac{1}{2}\cosh(2\theta)(1 + 4z_1^2 + 4z_2^2))dudv + (-4\sin(v)\sinh(2\theta)z_1 \\ & + 4(\cosh(2\theta) + \cos(v)\sinh(2\theta)z_2))dudz_1 + (-4(\cosh(2\theta) - \cos(v)\sinh(2\theta)z_1) \\ & + 4\sin(v)\sinh(2\theta)z_2)dudz_2 + 4z_2dvdz_1 - 4z_1dvdz_2 \ , \end{aligned} \tag{4.21}$$

where we have defined

$$u(\tau) = \chi(\tau) + \phi(\tau)\ ,\ v(\tau) = \chi(\tau) - \phi(\tau). \tag{4.22}$$

Making the off-diagonal elements go to zero the metric becomes,

$$\begin{aligned} ds^2 = &(1 + 4z_1^2 + 4z_1^2)d\theta^2 + (2\sin(v)\sinh(4\theta)z_1 z_2 + \cos(v)\sinh(4\theta)(1 + 4z_1^2 + 4z_2^2)du^2 \\ &+ (\frac{1}{4} + z_1^2 + z_2^2)dv^2 + 4dz_1^2 + 4dz_2^2. \end{aligned} \tag{4.23}$$

This metric is included due to the fact that when we solve for complexity $G_{IJ} = \delta_{IJ}$ implies that we will only be dealing with the diagonal terms in the metric.

4.2.2 Boundary Conditions

The complexity is obtained from the geodesics of the metric Eq. 4.21; these geodesics must satisfy the appropriate boundary conditions, namely that the initial point is the reference state and the final point is the target state — the circuit $\tilde{U}(\tau)$ must satisfy Eq. 4.7a, namely

$$\tilde{U}(\tau = 0) = I\ ,\ |\psi_T\rangle = \tilde{U}(\tau = 1)|\psi_R\rangle\ .$$

As we are working with Gaussian/coherent states, the initial and final points are characterized by

$$\text{reference state}:\ (a - z_R)|z_R\rangle_a = 0 \text{ with } x = f_R\ a + f_R^* a^\dagger\ ,\ p = (g_R\ a - g_R^* a^\dagger)/i \tag{4.24a}$$

$$\text{target state}:\ (b - z_T)|z_T\rangle_b = 0 \text{ with } x = f_T\ b + f_T^* b^\dagger\ ,\ p = (g_T\ b - g_T^* b^\dagger)/i \tag{4.24b}$$

One readily finds the a and b operators are related by the Bogoliubov transformation

$$\begin{pmatrix} b \\ b^\dagger \end{pmatrix} = \begin{pmatrix} \mathcal{U} & \mathcal{V}* \\ \mathcal{V} & \mathcal{U}^* \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}\ , \tag{4.25a}$$

where

$$\mathcal{U} = g_T^* f_R + f_T^* g_R\ ,\ \mathcal{V} = g_T f_R - f_T g_R\ . \tag{4.25b}$$

To obtain the boundary conditions, we start with the definition of the target state, Eq. 4.24; we use Eqs. 4.7a and 4.25a to write

$$(b - z_T)|z_T\rangle_b = 0 :\ \left[(\mathbf{s}^T, 0)\begin{pmatrix} \overline{a} \\ 1 \end{pmatrix} - z_T\right] U(\tau = 1)|z_R\rangle_a = 0 \ \text{ where } \ \mathbf{s}^T = (\mathcal{U}, \mathcal{V}^*)\ .$$

Multiplying the above equation from the left by $U^\dagger(\tau = 1)$ and using Eq. 4.13, one obtains

$$(b - z_T)|z_T\rangle_b = 0 \quad \longrightarrow \left[(\mathbf{s}^T, 0)\hat{U}(\tau = 1)\begin{pmatrix} \overline{a} \\ 1 \end{pmatrix} - z_T\right] |z_R\rangle_a = 0\ .$$

Then, using the explicit form of $\hat{U}(\tau)$ (Eqs. 4.15b and 4.15c) and the definition of the initial state (Eq. 4.24), we arrive at the boundary conditions

$$q(\tau = 0) = 1\ ,\ \ p(\tau = 0) = 0 \ \ ,\ z(\tau = 0) = 0. \tag{4.26}$$

Using the definition of (Eq. 4.20), we result in

$$u(\tau = 0) = \chi_0 \ , \quad \theta(\tau = 0) = 0 \quad , \ v(\tau = 0) = -\chi_0, \ z(\tau = 0) = 0 \ , \tag{4.27}$$

To define the definition of the target state (Eq. 4.24)

$$q(\tau = 1) = \mathcal{U}^* \ , p(\tau = 1) = -\mathcal{V}, \ z(\tau = 1) = \mathcal{U}^*(z_R + z_T) - \mathcal{V}^*(z_R^* + z_T^*). \tag{4.28}$$

We consider the definition of (Eq. 4.20) of the target state, also using the definition of (A.21b), (A.25) which results in

$$u(\tau = 1) = \arg(\cos \Omega t - i(u_1^2 + v_1^2) \sin \Omega t) - \arg(-i2u_1 v_1 \sin \Omega t) \tag{4.29a}$$

$$\theta(\tau = 1) = \text{arcCosh}(\cos \Omega t - i(u_1^2 + v_1^2) \sin \Omega t) \tag{4.29b}$$

$$v(\tau = 1) = -(\arg(i2u_1 v_1 \sin \Omega t) + \arg(\cos \Omega t - i(u_1^2 + v_1^2) \sin \Omega t)) \tag{4.29c}$$

$$\begin{aligned} z_1(\tau = 1) = & \frac{1}{2}((z_R + z_T)(\cos \Omega t + i(u_1^2 + v_1^2) \sin \Omega t + i2u_1 v_1 \sin \Omega t) \\ & + (z_R^* + z_T^*)(\cos \Omega t - i(u_1^2 + v_1^2) \sin \Omega t + i2u_1 v_1 \sin \Omega t)) \end{aligned} \tag{4.29d}$$

$$\begin{aligned} z_2(\tau = 1) = & \frac{i}{2}((z_R^* + z_T^*)(i2u_1 v_1 \sin \Omega t - cos\Omega t + i(u_1^2 + v_1^2) \sin \Omega t) \\ & - (z_R + z_T)(cos\Omega t - i(u_1^2 + v_1^2) \sin \Omega t - i2u_1 v_1 \sin \Omega t)). \end{aligned} \tag{4.29e}$$

4.3 The circuit complexity of coherent states

Now that we have defined our metric in Eq. 4.23 and the boundary conditions in Eq. 4.27 and Eq. 4.29. With this information we can start trying to calculate the methods to find the circuit complexity which is given as,

$$C(\mathcal{U}) = \int_0^1 ds \sqrt{g_{ij} \dot{x}^i \dot{x}^j}. \tag{4.30}$$

We can firstly solve the functional from the get go. The functional has five variables this means it will not produce an analytical result hence we will have to solve the functional numerically. Let us firstly define what our metric tensor g_{ij} will be,

$$g_{ij} = \begin{pmatrix} 4z_1(\tau)^2 + 4z_2(\tau)^2 + 1 & 0 & 0 & 0 & 0 \\ 0 & 2z_1(\tau)z_2(\tau)\sinh(4\theta(\tau))\sin(v(\tau)) & 0 & 0 & 0 \\ & (z_2(\tau)^2 - z_1(\tau)^2)\sinh(4\theta(\tau)) & & & \\ \cos(v(\tau)) + \frac{1}{4}\left(4z_1(\tau)^2\right. & & & & \\ & 4z_2(\tau)^2 + 1\cosh(4\theta(\tau)) & & & \\ 0 & 0 & z_1(\tau)^2 + z_2(\tau)^2 + \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \tag{4.31}$$

With the provided metric tensor we can define the Lagrangian as ,

$$L = (du(\tau)^2 \left(2z_1(\tau)z_2(\tau)\sinh(4\theta(\tau))\sin(v(\tau)) + \left(z_2(\tau)^2 - z_1(\tau)^2\right)\sinh(4\theta(\tau))\cos(v(\tau)) + \frac{1}{4}\left(4z_1(\tau)^2 + 4z_2(\tau)^2 + 1\right)\cosh(4\theta(\tau)) + \left(z_1(\tau)^2 + z_2(\tau)^2 + \frac{1}{4}\right)dv(\tau)^2 + \left(4z_1(\tau)^2 + 4z_2(\tau)^2 + 1d\theta(\tau)^2 + 4dz_1(\tau)^2 + 4dz_2(\tau)^2\right)^{\frac{1}{2}}. \tag{4.32}$$

4.4 Killing vectors and Conserved charges

4.4.1 Killing vectors

We have from the previous section found the metric Eq. 4.21 and also found the associated boundary conditions needed to solve complexity. The next step to computing the minimal geodesic is simply using symmetries and the associated conserved charges. From the **??** one of the conditions we chose so that complexity could be a good distance candidate was for our metric to be right-invariant. From [24] it was noticed that for every generator we have a killing vector. In our sense this means we have five killing vectors. k^I.

Let us now mention how to compute them. Firstly let us consider a coordinate transformation that allows the line element to be invariant

$$x^i \to x^i + \epsilon k^i, \tag{4.33}$$

where ϵ is defined infinitesimal parameter. the purpose of this is to find coordinate shift that is invariant to the line element. For a coordinate shit for $\hat{U}(\tau)$ generally we have

$$\delta U = \partial_i U \delta x^i. \tag{4.34}$$

In this situation we consider $\hat{U} = e^{\epsilon_I \mathcal{O}^I}$, with constants ϵ_I. We then expand $\hat{U}$ and the leading term of ϵ_I reduces to

$$U\mathcal{O}_I \epsilon^I = \partial_i U \delta x^I \tag{4.35a}$$

$$\Rightarrow \epsilon_I = tr(U^{-1}\partial_i U O_I^\dagger)\delta x^i. \tag{4.35b}$$

We can invert the indices to get

$$\delta x_j = (k_I)_j \epsilon^I \tag{4.36}$$

The killing vectors are

$$(k_I)_j = \frac{1}{2}[Tr(\hat{U}_k^{-1}\partial_j U_k \mathcal{O}_I^\dagger]^{-1}. \tag{4.37}$$

We have to also verify if the killing vectors are true by plugging them into the killing equation and seeing if they satisfy the conditon.

$$0 = \nabla_i (k_I)_j + \nabla_j (k_I)_i = (g_{jl}\nabla_i + (g_{il}\nabla_j)k_J^l. \tag{4.38}$$

The killing vectors in a matrix representation are as follows

$$k_1 = (i\cos(u(\tau)), -i\cosh(2\theta(\tau))\,\mathrm{csch}(\theta(\tau))\,\mathrm{sech}(\theta(\tau))\sin(u(\tau)), 2i\,\mathrm{csch}(2\theta(\tau))\sin(u(\theta)), 0, 0) \tag{4.39a}$$

$$k_2 = (i\sin(u(\tau)), -i\cos(u(\tau)) - i\cosh(2\theta(\tau))\,\mathrm{csch}(\theta(\tau))\,\mathrm{sech}(\theta(\tau)), 2i\cos(u(\tau))\,\mathrm{csch}(2\theta(\tau)), 0, 0) \tag{4.39b}$$

$$k_3 = (0, 2i, 0, 0) \tag{4.39c}$$

$$\begin{aligned} k_4 = (0,0,0,&\frac{1}{2}(\cosh(\theta(\tau))\cos(\frac{u(\tau)+v(\tau)}{2}) + \cos(\frac{u(\tau)-v(\tau)}{2})\sinh(\theta(\tau))) \\ &,\frac{1}{2}(-\cosh(\theta(\tau))\sin(\frac{u(\tau)+v(\tau)}{2}) + \sin(\frac{u(\tau)-v(\tau)}{2})\sinh(\theta(\tau)))) \end{aligned} \tag{4.39d}$$

$$\begin{aligned} k_5 = (0,0,0,&\frac{1}{2}(\cosh(\theta(\tau))\sin(\frac{u(\tau)+v(\tau)}{2}) + \sin(\frac{u(\tau)-v(\tau)}{2})\sinh(\theta(\tau))) \\ &,\frac{1}{2}(\cosh(\theta(\tau))\cos(\frac{u(\tau)+v(\tau)}{2}) - \cos(\frac{u(\tau)-v(\tau)}{2})\sinh(\theta(\tau)))) \end{aligned} \tag{4.39e}$$

After finding the five killing vectors we naturally progress to solving the five conserved charges, $c_I = (\tilde{k}_I)^i g_{ij}\dot{x}^j$. By knowing the conserved charges we can the solve for $x^\mu = (\theta(\tau), u(\tau), v(\tau), z_1(\tau), z_2(\tau))$ which is a path in the geometry of unitaries. We realised that it is extremely difficult to find x^μ explicitly by computing the conserved charges analytically. This means that our only option that we have is to use numerical methods.

In the next section we will discuss an alternative method that c to help us find the complexity for coherent states.

Chapter 5

Operator Complexity

5.1 Operator Complexity

Recently [19] developed a alternative method of computing complexity for infinite dimensional system. Similar method was outline before in [5] for SYK models. This method of computing the length of the shortest geodesic in the group manifold is called the operator complexity. This approach is independent of the reference and target state. In this approach the desired unitary is composed of a set of basis operators which are formed from Lie algebras. Operator complexity has been studied in a variety of the quantum systems in [19] with infinite dimensional Hilbert spaces. However in [19] the authors only considered their study to subspaces of Gaussian states. They defined the basis operators using finite dimensional, non unitary, matrix representation. What is amazing about the operator approach is that the operator geometry isn't dependent to the matrix representation. So one can get choose the convenient set of matrix representation.

Let us consider reviewing the Heisenberg group complexity which is stated in [19]. Let us consider a reference state that is acted upon by a Unitary operator to produce a target state

$$|\psi_T\rangle = U_T|\psi_R\rangle, \tag{5.1}$$

where the Unitary operator is defined by path-ordered map

$$U(s) = \overleftarrow{\mathcal{P}} \exp\left(-i\int_0^\tau ds\ V^I(s)\mathcal{S}_I\right), \tag{5.2}$$

which satisfies the differential equation

$$\frac{dU(s)}{ds} = V^I U(s)\mathcal{S}_I. \tag{5.3}$$

with the boundary conditions being

$$U(s=0) = I, \ \ U(s=1) = U_T. \tag{5.4}$$

We then define the circuit depth to be Eq .3.21. The natural choice of metric we have $G_{IJ} = K_{IJ}$ which is the Cartan-Killing metric. This defined as a bi-variant metric. For our case we cannot use this metric since we are in an infinite dimensional Hilbert space. We end up using the $G_{IJ} = \delta_{IJ}$ and this means that the velocities $V^I(s)$ have equally favourable trajectories in the operator space. We define complexity as the minimisation of the depth of the circuit[1].

$$\mathcal{C}_T = min_{\{V^I\}}(\int_0^1 \sqrt{G_{IJ}V^IV^J}ds), \tag{5.5}$$

We are looking for the optimal trajectory that $V^I(s)$ needs to take to reach the desired unitary operator. Which is the solution of the Euler-Arnold equation.

$$G_{IJ}\frac{dV^J}{ds} = if^P_{IJ}V^JG_{PL}V^L, \tag{5.6}$$

where f^P_{IJ} is defined as the structural constant of the operator and it arises from $[\mathcal{S}_I, \mathcal{S}_J] = if^P_{IJ}\mathcal{S}_P$.

The next section we will be solving the operator complexity of the Heisenberg group. We will firstly describe our desired unitary operator in terms of the set of basis operators $\{\mathcal{S}_\mathcal{I}\}$ that correspond to the Heisenberg group. Find the trajectories that satisfy the Euler-Arnold equation. From that set of trajectories we will find the ones that reach our desired state through the imposition of the boundary conditions.

5.2 Heisenberg Group Complexity

The Heisenberg Group Complexity corresponds with the displacement operator in a quantum harmonic oscillator,

$$U_T = D(\alpha) = \exp{[\alpha\hat{a}^\dagger - \alpha^*\hat{a}]}. \tag{5.7}$$

The unitary operator acts upon the vacuum state which results into the coherent state $|\alpha\rangle = D(\alpha)|0\rangle$. We can generalise and think of the time dependent Displacement operator

$$U_T(t) = D(\alpha, t) = e^{iH_ft}D(\alpha)e^{-iH_ft}, \tag{5.8}$$

where there is a time evolution acting on the Displacement operator caused by Hamiltonian $H_f = \omega(a^\dagger a)$.
Having defined $U_T(t)$ as a path ordered exponential. We can generate Eq. 5.7 by using a standard set of basis operators

$$U_T(t) = \mathcal{P}\exp{[-i\int_0^1 V^I(s)\hat{e}_I ds]}, \tag{5.9}$$

where $\{\hat{e}_I\}$ represents Hermitian basis operators, which in our case are

$$\hat{e}_1 = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger), \quad \hat{e}_2 = \frac{1}{\sqrt{2}}(\hat{a} - \hat{a}^\dagger), \quad \hat{e}_3 = I. \tag{5.10}$$

[1] One can construct the quantum circuit from a given reference state to a target state in many different ways. However, complexity corresponds to the minimum circuit.

These basis operators respect the Heisenberg Lie algebra $[\hat{e}_1, \hat{e}_2] = -i\hat{e}_3$. Following from the previous statement we find that the structural constant $f_{12}^3 = -1$, whilst the other permutations are zero. The Cartan-Killing form $K_{IJ} = -\frac{1}{h^\nu} f_{IK}^L f_{JL}^K = 0$. Our desired unitary operator with respect to $\{\hat{e}_I\}$ is

$$U_T(t) = \exp[\sqrt{2}iIm(\alpha(t))\hat{e}_1 + \sqrt{2}iRe[\alpha(t)]\hat{e}_2]. \tag{5.11}$$

We parametrise the trajectories of the operator space with vectors $V^I(s)$, from s=0 to s=1. The circuit depth measures the length of the geodesic of any trajectory,on the operator space given a metric. We notice that the Cart-Killing form disappears. This means $G_{IJ} = \delta_{IJ}$ becomes our natural choice for the metric.
The trajectory from Identity to the desired unitary operator, using optimal trajectory is a a solution of the Euler-Arnold equation, which is

$$\frac{dV^1}{ds} = -V^2V^3 \tag{5.12}$$

$$\frac{dV^2}{ds} = V^1V^3 \tag{5.13}$$

$$\frac{dV^3}{ds} = 0, \tag{5.14}$$

with $V^3(s) = c_3$, the general solution for the vectors $V^I(s)$ that reach our desired state is

$$V^1(s) = c_1\cos(c_3s) + c_2\sin(c_3s), \tag{5.15}$$

$$V^2(s) = c_1\sin(c_3s) - c_2\cos(c_3s), \tag{5.16}$$

$$V^3(s) = c_3, \tag{5.17}$$

c_1,c_2,c_3 are constants which we will solve through the imposition of the boundary conditions. The complexity along the optimal trajectory is

$$\mathcal{C}_T = \int_0^1 \sqrt{G_{IJ}V^IV^J}ds \quad \sqrt{c_1^2 + c_2^2 + c_3^2}. \tag{5.18}$$

Before we implement the boundary conditions to find $\{c_I\}$. We want to simplify the composition of the unitary transformation with respect to Hermitian basis operators. $\hat{e}_3$ from its relationship with other basis operators allows us to conclude that it is a time independent overall phase space. Therefore we can set $c_3 = 0$. Hence the complexity becomes

$$\mathcal{C}_T = \sqrt{c_1^2 + c_2^2}. \tag{5.19}$$

Before finding the values, we want to have an explicit description of U_T. For this to occur we have to solve Eq. 5.3. Let us define convenient matrix representation of $\{\hat{e}_I\}$. The Lie algebra generators of Heisenberg group are

$$\hat{e}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \hat{e}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ \hat{e}_3 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A general element found in the Heisenberg group, defined by the matrix representation have a form,

$$U(s) = \begin{pmatrix} 1 & x_1(s) & x_3(s) \\ 0 & 1 & x_2(s) \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.21}$$

Since $ds^2 = Tr(dUU^1\mathcal{S}^\dagger)$, we construct the line element of the Heisenberg group space is

$$ds^2 = dx_1^2 + dx_2^2 + (x_2dx_1 + dx_3)^2. \tag{5.22}$$

Noticing that the line element has a negative curvature. This implies that the geometry of Heisenberg group is a three dimensional hyperbolic space. Using the explicit representation of Eq. 5.21 and the general solution of the vectors $V^I(s)$. We solve the differential equation via parametrisations,

$$x_1(s) = x_0 - c_1 s \tag{5.23}$$

$$x_2(s) = y_0 - c_2 s \tag{5.24}$$

$$x_3(s) = z_0 + y_0 c_1 s - \frac{1}{2}c_1 c_2 s^2, \tag{5.25}$$

where x_0, y_0, z_0 are constants. Imposing the boundary conditions where s=0, U(s=0)= I. The unitary operator is then

$$U(s) = \begin{pmatrix} 1 & -ic_1 s & \frac{1}{2}c_1c_2s^2) \\ 0 & 1 & -c_2 s \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.26}$$

We then impose s=1, U(s=1)=U_T. The solution for $c_1 = -\sqrt{2}Im(\alpha(t))$, $c_2 = -\sqrt{2}Re[\alpha(t)]$. The complexity for the displacement operator in quantum harmonic oscillator is

$$\mathcal{C}_{Heis} = \sqrt{2}|\alpha|. \tag{5.27}$$

This tells us that the complexity of a time dependent displacement operator is a time independent and corresponds to the parameter of the coherent state. We notice that there is similarity of the results with [17, 18] which are complexity for coherent states that are state dependent. Whereas a disparity with [5, 8] when studying the complexity of the displacement operator, this is due to both results being time dependent.

Chapter 6

Conclusion

We examined the circuit complexity of a time dependent generalised coherent state using the Nielsen approach. We firstly considered gates that are built from a basis operators that are quadratic to x and p. We found that the matrix representation of the basis operators that is needed to construct the unitary operator has five generators Eq. 4.18. From this and the algebra we realized that the group is an extension of SU(1,1). We computed the metric Eq. 4.21 and the diagonalized metric Eq. 4.23. This tells us that the trajectory in the extended SU(1,1) geometry is dependent on five variables $x^\mu = (\theta(\tau), u(\tau), v(\tau), z_1(\tau), z_2(\tau))$. With our implementation of coordinate transformation , this also means a change in our boundary conditions Eq. 4.27, Eq. 4.29. With the above ingredients, we are able to have a matrix form of the $\tilde{U}(\tau)$. To be able to compute complexity we have to find the variables explicitly in $x^\mu(\tau)$. We take advantage of symmetries and associated charges to try and simplify the path x^μ. We find that we have five killing vectors and this implies we have five conserved charges. Due to the complicated form of the conserved charges, we could not find any analytic or numerical solutions.

We turn our focus on reviewing the operator complexity outlined in [19]. We particularly review the complexity of displacement operator in a quantum harmonic oscillator. The system has infinite dimensional Hilbert space. The displacement operator is constructed by the Heisenberg group. the underlying geometry of the Heisenberg group is a three dimensional hyperbolic space. We find that the complexity of the Heisenberg group is proportional to the coherent state parameter Eq.(5.27) which is time independent. This result is similar to the state dependent complexity for coherent states found in [17, 18].However, it is in contrast with the complexity of displacement operators of [5, 8] as their result is time dependent whereas the result in [19] is not.

Note that, in [19] the author did not consider the generalized coherent state. The difficulty with generalized coherent state is that due to the linear terms in the Hamiltonian the group becomes longer and it is in fact the Jacobi group. The Euler Arnold equations for the Jacobi group are very difficult to solve. We leave this for a future work.

Finally we want to comment on the to operator methods the one in [19] and the Krylov complexity in [8]. We see that the Krylov complexity for displacement operator is time dependent, whereas it is not in [19]. We would like to explore this in the future.

Appendix A

The Coherent form of SU(1,1),The quadratic Hamiltonian – Diagonalization, Bogoliubov transformation, and time evolution

A.1 Coherent form of SU(1,1)

The Lie Algebra that generates the group is su(1,1). This has three generators which are $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and the commutation relations

$$[\mathcal{O}_1, \mathcal{O}_2] = -i\mathcal{O}_3, [\mathcal{O}_2, \mathcal{O}_3] = i\mathcal{O}_1, [\mathcal{O}_1, \mathcal{O}_3] = -i\mathcal{O}_2. \tag{A.1}$$

We then change the basis to a much convenient and useful form

$$\mathcal{O}_\pm = \pm i(\mathcal{O}_1 \pm i\mathcal{O}_2), \mathcal{O}_3 = \mathcal{O}_0 \tag{A.2}$$

The coherent form is defined by acting a representation T(g) of SU(1,1)[16] on a state

$$|\psi_{coherent}\rangle = T(g)|\psi_0\rangle. \tag{A.3}$$

The coherent form of SU(1,1) is

$$|\psi\rangle = .e^{\alpha_+(\sigma)\mathcal{O}^+ + \alpha_-\mathcal{O}^- + \omega\mathcal{O}^0}|\psi_0\rangle. \tag{A.4}$$

To be able to digest this we decompose the state

$$\exp\left[\alpha_+\mathcal{O}_+ + \alpha_-\mathcal{O}_- + \omega\mathcal{O}_0\right] \tag{A.5}$$

$$= \exp(\gamma_+\mathcal{O}_+)\exp(\ln\gamma_0\mathcal{O}_0)\exp(\gamma_-\mathcal{O}_-), \tag{A.6}$$

where

$$\gamma_0 = \frac{1}{(\cosh\mu - \frac{\beta}{2\mu}\sinh\mu)^2}, \gamma_\pm = \frac{2\alpha_\pm \sinh\mu}{2\mu\cosh\mu - \beta\sinh\mu}, \mu = \frac{\beta^2}{4} - \alpha_+\alpha_-. \tag{A.7}$$

This results into

$$|\gamma_+|^2 = \frac{2\alpha_- \sinh\mu - 2\alpha_- \sinh\mu}{2\mu\cosh\mu - \beta\sinh\mu 2\mu\cosh\mu + \beta\sinh\mu} \tag{A.8}$$

$$= \frac{-\alpha_+ + \alpha_- \sinh^2\mu}{\mu^2(\cosh^2\mu - \frac{\beta^2}{4\mu^2}\sinh^2\mu)} \tag{A.9}$$

With this Identity of $\alpha_+\alpha_- = \frac{\beta^2}{2} - \mu^2$ and

$$|\gamma_0| + |\gamma_+|^2 = 1 \tag{A.10}$$

A.2 Gaussian wave functions

We consider the evolution of a Gaussian state with a harmonic oscillator Hamiltonian:[1]

$$|\psi(t)\rangle = \exp(-iHt)|\psi_0\rangle \quad \text{where} \quad H = \frac{1}{2}p^2 + \frac{\omega^2}{2}x^2 \quad \text{and} \quad |\psi_0\rangle = \text{Gaussian state} \ . \tag{A.11}$$

$|\psi_0\rangle$ is readily written in position representation, $\psi_0(x) = \langle x|\psi_0\rangle$:

$$\psi_0(x) = \left(\frac{\omega_0}{\pi}\right)^{1/4} \exp\left[-\frac{\omega_0}{2}(x-x_0)^2 + i\ p_0 x\right] \ . \tag{A.12a}$$

In the momentum representation, one has

$$\phi_0(p) = \left(\frac{1}{\pi\omega_0}\right)^{1/4} \exp\left[-\frac{1}{2\omega_0}(p-p_0)^2 - i\ (p-p_0)x_0\right] \ , \tag{A.12b}$$

where $\phi_0(p) = \langle p|\psi_0\rangle$; note that $|\psi_0\rangle$ is such that

$$\langle\psi_0|\hat{x}|\psi_0\rangle = x_0 \quad , \quad \langle\psi_0|\hat{p}|\psi_0\rangle = p_0 \ .$$

Working in position representation, one has

$$|\psi(t)\rangle = \exp(-iHt)|\psi_0\rangle \longrightarrow \psi(x,t) = \int dx'\ K(x,t|x',t=0)\ \psi_0(x') \ , \tag{A.13a}$$

where $K(x,t|x',t=0)$ is the propagator

$$K(x,t|x',t=0) = \langle x|\exp(-iHt)|x'\rangle \ . \tag{A.13b}$$

For the Hamiltonian given in Eq. A.11, the propagator is[**shankar**]

$$K(x,t|x',t=0) = \left(\frac{\omega}{i2\pi\sin(\omega t)}\right)^{1/2} \exp\left\{\frac{i\omega}{2}\left[(x^2+x'^2)\cot(\omega t) - \frac{2xx'}{\sin(\omega t)}\right]\right\} \ . \tag{A.14}$$

Carrying out the (Gaussian) integral, one obtains

$$\psi(x,t) = \left(\frac{\omega_0}{\pi}\right)^{1/4} \left(\frac{\omega}{i\sin(\omega t)}\right)^{1/2} \left(\frac{1}{z}\right)^{1/2} \exp\left(-\frac{\Omega}{2}x^2 + Px + \kappa\right) \ , \tag{A.15a}$$

where

$$z = \omega_0 - i\omega\cot(\omega t) \ , \tag{A.15b}$$

and

$$\Omega = \frac{\omega}{z}[\omega - i\omega_0\cot(\omega t)] \ , \ P = \frac{\omega(p_0 - i\omega_0 x_0)}{z\sin(\omega t)} \ , \ \kappa = \frac{1}{2z}[-p_0^2 + i2\omega_0 x_0 p_0 + i\omega\omega_0 x_0^2\cot(\omega t)] \ . \tag{A.15c}$$

[1] Also of interest is the case of an inverted oscillator, namely where $H = (p^2 - \omega^2 x^2)/2$ — this is readily obtained (from the results presented here) by analytically continuing the frequency $\omega \to i\omega$ [**barton**].

A.3 Diagonalization

Upon completing the square, Eq. 4.2 is written as

$$H = \frac{1}{2}p^2 + \frac{\Omega^2}{2}(x + F/\Omega^2)^2 - F^2/2\Omega^2 \ . \tag{A.16}$$

Then, we make the canonical transformation

$$P = p \quad , \quad Q = x + F/\Omega^2 \ , \tag{A.17}$$

where $[Q, P] = i$. Introducing creation and annihilation operators in the usual way ($[b, b^\dagger] = 1$), Eq. A.16 is diagonalized

$$Q = \frac{1}{\sqrt{2\Omega}}(b + b^\dagger) \quad , \quad P = \frac{1}{i}\sqrt{\frac{\Omega}{2}}(b - b^\dagger) \ : H = \Omega(b^\dagger b + 1/2) - F^2/2\Omega^2 \ . \tag{A.18}$$

A.4 Bogoliubov transformation

We now write the Hamiltonian as $H = H_0 + H_1$ —

$$H = H_0 + H_1 : \ H_0 = \frac{1}{2}p^2 + \frac{\omega_0^2}{2}x^2 \quad , \quad H_1 = \frac{\delta^2}{2}x^2 + Fx \ ; \tag{A.19}$$

we introduce creation and annihilation operators which diagonalize H_0:

$$x = \frac{1}{\sqrt{2\omega_0}}(a + a^\dagger) \quad , \quad p = \frac{1}{i}\sqrt{\frac{\omega_0}{2}}(a - a^\dagger) \ : H_0 = \omega_0(a^\dagger a + 1/2) \ . \tag{A.20a}$$

In terms of the $\{a, a^\dagger\}$,

$$H = \left(\omega_0 + \frac{\delta^2}{2\omega_0}\right)(a^\dagger a + 1/2) + \frac{\delta^2}{4\omega_0}(a^\dagger a^\dagger + aa) + \frac{F}{\sqrt{2\omega_0}}(a + a^\dagger) \ . \tag{A.20b}$$

Using that $P = p$ and $Q = x + F/\Omega^2$, one deduces the relation between the $\{a, a^\dagger\}$ and $\{b, b^\dagger\}$ operators

$$b = u_1 a + v_1 a^\dagger + \frac{1}{2}\sqrt{\frac{\Omega}{\omega_0}}a_0 \quad , \quad b^\dagger = v_1 a + u_1 a^\dagger + \frac{1}{2}\sqrt{\frac{\Omega}{\omega_0}}a_0 \tag{A.21a}$$

where

$$\left(u_1 = \frac{\Omega + \omega_0}{2\sqrt{\Omega\omega_0}} \ , \ v_1 = \frac{\Omega - \omega_0}{2\sqrt{\Omega\omega_0}}\right) \ \text{and} \ \ a_0 = \frac{F\sqrt{2\omega_0}}{\Omega^2} \ . \tag{A.21b}$$

[Note that $u_1^2 - v_1^2 = 1$.] This Bogoliubov transformation can be expressed succinctly by defining

$$\overline{a} = \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \quad , \quad \overline{b} = \begin{pmatrix} b \\ b^\dagger \end{pmatrix} \quad , \quad \Lambda = \begin{pmatrix} u_1 & v_1 \\ v_1 & u_1 \end{pmatrix} \quad , \quad \overline{\lambda} = \frac{1}{2}\sqrt{\frac{\Omega}{\omega_0}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \ ; \tag{A.22a}$$

then, Eq. A.21a takes the form

$$\begin{pmatrix} \overline{b} \\ b_0 \end{pmatrix} = M \begin{pmatrix} \overline{a} \\ a_0 \end{pmatrix} \ \text{where} \ M = \begin{pmatrix} \Lambda & \overline{\lambda} \\ \overline{0}^T & 1 \end{pmatrix} \ . \tag{A.22b}$$

Note: We can also write $(\overline{a}, a_0)$ in terms of $(\overline{b}, b_0)$ by inverting Eq. A.22b —

$$\begin{pmatrix} \overline{b} \\ b_0 \end{pmatrix} = M^{-1} \begin{pmatrix} \overline{a} \\ a_0 \end{pmatrix} \text{ where } M^{-1} = \begin{pmatrix} \Lambda^{-1} & -\Lambda^{-1}\overline{\lambda} \\ \overline{0}^T & 1 \end{pmatrix} \tag{A.23a}$$

and

$$\Lambda^{-1} = \begin{pmatrix} u_1 & -v_1 \\ -v_1 & u_1 \end{pmatrix} \quad , \quad \Lambda^{-1}\overline{\lambda} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} . \tag{A.23b}$$

A.4.1 Time evolution

We are interested in evolving the operator a with the (full) Hamiltonian H: $\hat{U}^\dagger a \, \hat{U}$ where $\hat{U} = \exp(-iHt)$. Using Eqs. A.22b and A.23a, one obtains

$$\hat{U}^\dagger \begin{pmatrix} \overline{a} \\ a_0 \end{pmatrix} \hat{U} = M^{-1} \begin{pmatrix} \hat{U}_2 & \overline{0} \\ \overline{0}^T & 1 \end{pmatrix} M \begin{pmatrix} \overline{a} \\ a_0 \end{pmatrix} \text{ where } \hat{U}_2 = \begin{pmatrix} e^{-i\Omega t} & 0 \\ 0 & e^{+i\Omega t} \end{pmatrix} ; \tag{A.24}$$

explicitly,

$$M^{-1} \begin{pmatrix} \hat{U}_2 & \overline{0} \\ \overline{0}^T & 1 \end{pmatrix} M = \begin{pmatrix} \Lambda^{-1}\hat{U}_2\Lambda & \Lambda^{-1}(\hat{U}_2 - I)\overline{\lambda} \\ \overline{0}^T & 1 \end{pmatrix} \tag{A.25a}$$

where

$$\Lambda^{-1}\hat{U}_2\Lambda = \begin{pmatrix} \mathcal{U} & \mathcal{V}^* \\ \mathcal{V} & \mathcal{U}^* \end{pmatrix} \text{ with } \left(\mathcal{U} = \cos\Omega t - i(u_1^2 + v_1^2)\sin\Omega t \ , \ \mathcal{V} = i2u_1v_1 \ \sin\Omega t\right) , \tag{A.25b}$$

$$\Lambda^{-1}(\hat{U}_2 - I)\overline{\lambda} = \begin{pmatrix} z \\ z^* \end{pmatrix} \text{ with } z = \frac{1}{2}\left[\sqrt{\frac{\Omega}{\omega_0}}(u_1 \ e^{-i\Omega t} - v_1 \ e^{i\Omega t}) - 1\right] . \tag{A.25c}$$

Note: $|\mathcal{U}|^2 - |\mathcal{V}|^2 = 1$ — $\Lambda^{-1}\hat{U}_2\Lambda$ performs a "hyperbolic rotation".

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