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A survey of some regression-based and duality methods to value American and Bermudan options

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Abstract

We give a review of regression-based Monte Carlo methods for pricing high-dimensional American and Bermudan options for which backwards methods such as lattice and PDE methods do not work. The continuous-time pricing problem is approximated in discrete time and the problem is formulated as an optimal stopping problem. The optimal stopping time can be expressed through continuation values (the price of the option given that the option is exercised after time j conditioned on the state process at time j). Regression-based Monte Carlo methods apply regression estimates to data generated by artificial samples of the state process in order to approximate continuation values. The resulting estimate of the option price is a lower bound. We then look at a dual formation of the optimal stopping problem which is used to generate an upper bound for the option price. The upper bound can be constructed by using any approximation to the option price. By using an approximation that arises from a lower bound method we have a general method for generating valid confidence intervals for the price of the option. In this way, the upper bound allows for a better estimate of the price to be computed and it provides a way of investigating the tightness of the lower bound by indicating whether more effort is needed to improve it.

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1 Introduction

The valuation of derivative securities which early exercise features, such as American and Bermudan options, is a challenging problem in mathematical finance. Even in the simple Black-Scholes framework with one state variable, a closed form expression for the price of an American put option is not available and so it must therefore be computed numerically. Backwards methods such as lattice methods and finite-difference methods for PDEs were traditionally believed to be the most natural way to approach to the problem of early exercise. This is because early exercise decisions require knowledge of the value of the unexercised product and by working backwards this value is always available. For this reason Monte Carlo simulation, which is a forward method, has been believed until recently, to be ill-suited to this problem because knowledge of the value of the unexercised product is not readily available in a process which evolves through time.

When there is just one or two state variables, backwards methods do not present a challenge. However, where the problem is high-dimensional (in the number of state variable required to describe the state space at each exercise time), these techniques are not practically feasible. In the financial engineering practice such high-dimensional problems arise frequently, such as the LIBOR market model, and are thus of considerable interest to both researchers and practitioners. In this paper, we focus on high-dimensional American and Bermudan options, the pricing of which amount to solving an optimal stopping problem which is subject to what is commonly referred to as the "curse of dimensionality". It becomes essential to adapt the Monte Carlo simulation technique in order to cope with this problem.

More recently various methods using Monte Carlo simulation have proven successful in approximating the price of these products. The lower bound methods we consider use approximate dynamic programming (ADP) to develop a (sub-optimal) exercise strategy via linear least-squares regression. Monte Carlo is used by simulating paths of the state process, for which different exercise strategies are approximated and then used to estimate the value of the product. The supremum over all exercise strategies of the mean, in the martingale measure, of the discounted payoff process of the product is the value of the product, and the solution of the so-called primal problem, and therefore the lower bound is an unbiased estimate. In §3 we examine two such regression-based methods for finding lower bounds for the price of American and Bermudan options: The method of Tsitsiklis and Van Roy [6] and the Longstaff-Schwartz method [11] which has become particularly popular.

We also consider upper bound methods which are based on the idea of hedging against a buyer of the product who has perfect foresight. Adding a hedge of zero initial value to the discounted payoff process of the product does not affect the mean in the martingale measure, and thus the value of the product which is the supremum over all random exercise strategies. However, if the buyer uses the best possible exercise strategy amongst all random times (perfect foresight) this results in an increase in the value of the product, and adding a hedge of zero initial value gives an upper bound for the price. Rogers [13] and Haugh and Kogan [3] independently developed a dual formulation of the optimal stopping problem. They show that the infimum over all hedging portfolios of the mean, in the martingale measure, of the sum of the hedging portfolio and the discounted payoff process of the product, is the value of the product, and the solution of the dual problem. An upper bound for the price of the product can therefore be obtained by approximating the optimal hedge, which is an unbiased estimate. In §4 we see how Rogers and Haugh and Kogan use approximate value functions to estimate the optimal hedge using Monte Carlo and arrive at an upper bound for the price of a Bermudan option. We also consider Andersen and Broadie's approach [14] in arriving at an upper bound for the price, in which the product with a sub-optimal exercise strategy is chosen as an approximation to the optimal hedge. In this way Andersen and Broadie are able to construct a confidence interval for the price of the option since the exercise strategy they use is estimated using a lower bound method.

2 Pricing American and Bermudan Options

In contrast with a European option which can be exercised only at its maturity date, thus precluding exercise choices from being made on the part of the holder during the life of the option, American and Bermudan options possess early exercise features. This permits the holder to exchange the option for cash, at his discretion, at a range of dates from start until maturity, either continuous for an *American option* or discrete for a *Bermudan option*[1]. In these types of options, the holder has always the need to make an optimal decision: as each new exercise date is reached, conditional on the value of the asset, should the option be exercised or not[2]? In determining the price, we therefore take into account these extra exercise rights.

Let us assume for a moment that the value process of an American option is already known. A no-arbitrage argument (see [1] Chapter 12) shows that the option cannot be worth less than it's intrinsic value, since if it was, a rational agent simply could buy and immediately exercise the option to make a sure profit on the difference in values. Also, if the American option's value exceeds it's intrinsic value, a rational agent would choose not to exercise, as more could be gained by selling the option itself in the market. It is only in the case where the value of the American option and it's intrinsic value are equal, that it would be an error not to exercise early. An investor would be at fault if he was to ignore this exercise opportunity because as time passes, the number of these opportunities will diminish and thus the option value will decrease. Moreover for the period, during which he has chosen not to exercise, the no-arbitrage argument does not hold, thus the option value may become less than its intrinsic value. If the option is exercised he would hold cash that would grow at the risk-free rate. So although the values are equal, their rates of change are not. So in the case where the option value and the intrinsic value are equal, more money is to be made by exercising the option rather than waiting.

The assumption that the option price is already known, implies knowledge of when to exercise. This is not, however useful, because typically we need to know when to exercise in order to determine the option value. The value of an American option is the value achieved by exercising optimally. Finding this value entails finding the optimal stopping time — by solving an optimal stopping problem — and computing the expected discounted payoff of the option under this rule. The embedded optimization problem makes a difficult problem for simulation.

1. For example, the American put grants the holder the right to exchange, at any time until expiration, the underlying asset for an agreed amount cash. If the underlying asset is worth X_j at time j, the holder of the option can exchange it for cash with value K, known as the strike. In effect, the payoff upon exchange at time j is $(K - X_j)^+$. Other typical examples of such options occurring in the market are American straddles (with payoff $|X_j - K|$); Bermudan swaptions; convertible bonds; installment options, *etc.* Bermudan options are uncommon in stock markets and FX markets, but they are very common in the interest rate markets. A major application of the valuation of such options is in getting out prices and hedges for Bermudan swaptions.

2. For an American call on a non-dividend paying stock, it is clear that early exercise is never optimal (see [2] Chapter 9). This is because at time j, the owner has all the exercise opportunities open the owner of the European call whose price is bounded below by $(X_j - Ke^{-r(L-j)})^+$, where L is the maturity date. The price of the American call, c_j, must therefore be at least as much as the European call and so always exceeds its intrinsic value, i.e. $c_j > (X_j - K)^+$. In the case of the American put, however, it can be optimal to exercise early if it is sufficiently deep in the money. In this case, the profit is obtained earlier so that it can start to earn interest immediately. The European put price can be greater or equal to it's intrinsic value $(Ke^{-r(L-j)} - X_j)^+$ and the American put price, p_j, is bounded below by it's intrinsic value, i.e. $p_j \geq (K - X_j)^+$.

2.1 Problem Formulation

The holder of an American option is free to choose any exercise time τ before or at the given expiration time L. This exercise time may be stochastic but only in such a way that the decision to exercise before or at a time j only depends on the history up to time j. The problem of pricing an American option can be cast into the form of an *optimal stopping problem* — the problem of valuing an American option consists of finding an optimal exercise strategy and valuing the expected discounted payoff from this strategy under the equivalent martingale measure. It is well-known that in complete and arbitrage-free markets the price of a derivative security has this representation. In this paper we will concern ourselves with probabilistic approximation methods in the context of discrete optimal stopping, therefore we present the optimal stopping problem in discrete time rather than the original optimal stopping problem in continuous time [3, 4, 6]. The *finite-exercise Markovian formulation* is described below [3] .

Information Set Let the financial market be defined for the equally-spaced discrete times with values in $\{0, \ldots, L\}$. It is described by the complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_j)_{j=0,\ldots,L}, \mathscr{P})$, where the state-space Ω is the set of all realizations of the financial market, $\mathcal{F}$ is the σ-algebra of events at time L, and $\mathscr{P}$ is a probability measure defined on $\mathcal{F}$. The discrete time filtration $(\mathcal{F}_j)_{j=0,\ldots,L}$, is assumed to be generated by the state variables or underlying assets of the model $(X_j)_{j=0,\ldots,L}$, that evolve in a state space $\Re^d$, and are assumed to be Markov with the initial state $X_0 = x$ is deterministic. The process $(X_j)_{j=0,\ldots,L}$ records all necessary information about financial variables including the prices of the underlying assets as well as additional risk factors driving stochastic volatility or stochastic interest rates. There are other various possibilities for the choice of the process $(X_j)_{j=0,\ldots,L}$. The most simple examples are geometric Brownian motion, as for instance, in the celebrated Black-Scholes setting. More general models include stochastic volatility models, jump-diffusion processes or general Lévy processes. The model parameters are usually calibrated to observed time series data.

Exercise Dates The holder of the Bermudan option is permitted to exercise it at any one of the pre-specified exercise dates τ in $\mathscr{T}_0$, the class of all stopping times with values in $\{0, \ldots, L\}$. We require that for all $j \leq L$ we have

$$\{\omega \in \Omega;\ \ \tau(\omega) \leq j\} \in \mathcal{F}_j$$

In other words, τ must be an $\mathcal{F}_j$*-stopping time*.

Option Payoff Let the nonnegative adapted process $\{\tilde{h}_j \in \Re : j = 0, \ldots, L\}$ be the payoff of the option, where $\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_L$ are square integrable random variables such that for $j = 0, \ldots, L$

$$\tilde{h}_j = f(j, X_j),$$

for some Borel function $f(j, \cdot)$. Let $\{B_j \in \Re : j = 0, \ldots, L\}$ be the risk-free bank account process defined by $B_j = \exp\left(\int_0^j r_s ds\right)$, where r_s denotes the instantaneously risk-free rate of return at time s, which may depend on current and past state variables $X_0, \ldots, X_j$. In an arbitrage-free and complete market, using the numéraire process $(B_j)_{j=0,\ldots,L}$, there exists a risk-neutral measure, $\mathscr{Q}$, equivalent to $\mathscr{P}$, under which the price processes of all discounted

3. Here we consider a discrete-time version of the optional stopping problem, by requiring that the option be exercised at pre-specified intervals — that is we treat the option as Bermudan. Although this lowers the value of the option the difference in values is small and vanishes as the difference between allowable exercise times goes to zero.

state-contingent claims relative to the numéraire are martingales and may be determined as the expected value of their discounted payoff processes [7]. We assume that the discounted payoff process of the American option, $h_j = \left(\frac{\tilde{h}_j}{B_j}\right)_{j=0,\ldots,L}$, satisfies the following integrability condition:

$$\mathbb{E}\left[\max_{j=0,\ldots,L} |h_j|\right] < \infty,$$

where $\mathbb{E}[\cdot|\mathcal{F}_j]$ denotes the expected value under the risk-neutral probability measure $\mathcal{Q}$, conditional on the time j information $\mathcal{F}_j$.

Option Price Let $\{\tilde{V}_j \in \Re : j = 0, \ldots, L\}$ be the value process of the American option price, conditional on its not having been exercised prior to time L. We have the following optimal stopping problem characterization of the value process, $(\tilde{V}_j)_{j=0,\ldots,L}$ (See [7]), denominated in time j dollars:

$$\tilde{V}_j(x) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_j} \mathbb{E}\left[\frac{B_j \tilde{h}_\tau(X_\tau)}{B_\tau}\middle| X_j = x\right], \qquad x \in \Re^d,$$

where $\mathcal{T}_j$ is the class of all stopping times τ taking on values in $\{j, \ldots, L\}$ [4]. This can also be interpreted as the value we get in mean if we sell the option in an optimal way after time $j-1$ given $X_j = x$, i.e., the value of a newly issued option at time j starting from state x. Letting the discounted value process of the option be denoted by $V_j = \left(\frac{\tilde{V}_j}{B_j}\right)_{j=0,\ldots,L}$, we can rewrite the above denominated in time 0 dollars [5]:

$$V_j(x) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_j} \mathbb{E}[h_\tau(X_\tau)|X_j = x], \qquad x \in \Re^d. \tag{2.1}$$

Equation (2.1) defines the *Snell envelope* [6], $(V_j)_{j=0,\ldots,L}$, of the payoff process $(h_j)_{j=0,\ldots,L}$. The problem of pricing a Bermudan option, the "primal" problem, is that of computing from (2.1)

$$\boxed{V_0 = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[h_\tau(X_\tau)\right] = \mathbb{E}\left[h_{\tau^*}(X_{\tau^*})\right]} \tag{2.2}$$

Continuation Values The Q-value function is defined to be the value of the option at time j given $X_j = x$ and subject to the constraint that the option be held at time j rather than

4. We have that $\tilde{V}_j = v(j, X_j)$ for some function $v(j, \cdot)$ and $\mathbb{E}(\tilde{h}_{\tau_{j+1}}|\mathcal{F}_j) = \mathbb{E}(\tilde{h}_{\tau_{j+1}}|X_j)$. And since the initial state was assumed to be is deterministic, we have that V_0 is deterministic.

5. In suppressing explicit discounting, we have assumed that the discount factor over one period has the form $D_{j,j+1} = B_j/B_{j+1}$. But this assumption was unnecessary because the formulation in (2.1) is independent of the choice of risk-free measure. All we require is that the expectation is taken with respect to the risk-neutral measure consistent with the choice of numéraire implicit in the discount factor. For example, under the time L forward measure, we take $D_{j,j+1} = B^L_{j+1}/B^L_j$, with B^L_j denoting the time j price of a bond maturing at L.

6. Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_j)_{j=0,\ldots,L}, \mathcal{Q})$ then an adapted process $(V_j)_{j=0,\ldots,L}$ is the *Snell envelope* with respect to $\mathcal{Q}$ of the process $(h_j)_{j=0,\ldots,L}$ if
   i V is a $\mathcal{Q}$-supermartingale
   ii V dominates h, i.e. $V_j \geq h_j$ $\mathcal{Q}$-a.s. for all $j \in \{0, \ldots, L\}$
   iii If $(W_j)_{j=0,\ldots,L}$ is a $\mathcal{Q}$-supermartingale which dominates h, then W dominates V

exercising it, i.e. the *continuation value* of the option:

$$Q_j(x) = \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_{j+1}} \mathbb{E}[h_\tau(X_\tau)|X_j = x], \quad j = 0, \ldots, L-1, \tag{2.3}$$

For time L we define the corresponding continuation value as

$$Q_L(x) = 0, \qquad x \in \Re^d,$$

because the option expires at time L and hence we do not get any money if we sell it after time L. The value of the option at time 0 is Q_0.

2.2 Value Iteration and Q-Value Iteration

The *dynamic programming formulation* [10] generates a sequence of $V_L, V_{L-1}, \ldots, V_0$ of value functions, and can be written as follows:

$$\boxed{\begin{array}{l} V_L(x) = h_L(x) \\ V_j(x) = \max\{h_j(x), \mathbb{E}[V_{j+1}(X_{j+1})|X_j = x]\}, \quad j = 0, \ldots, L-1. \end{array}} \tag{2.4}$$

A primary motivation for this discretization is that it facilitates exposition of computational procedures, which typically entail discretization.

The option value, V_j, represents the maximum of exercising the Bermudan option, giving a time 0 value of h_j, or continuing at time j, giving a time 0 value of $\mathbb{E}[V_{j+1}(X_{j+1})|X_j = x]$. Note that V_j represents the time 0 value of a Bermudan option *newly issued* at time j, not the value of the option issued at time 0 which may may have been exercised prior to time j.

The optimal exercise strategy is thus fundamentally determined by the conditional expectation of the payoff from continuing to keep the option alive. The price of the option is then given by $V_0(X_0)$ where $X_0 = x$ is the initial state of the economy.

In a paper by Kohler [9], the option values in Equation (2.1) with the discrete-time Markovian formulation are shown to satisfy the value iteration above (see Proposition A.1 of the Appendix). From (A.1) of the Appendix, we have the following equivalent representation of Q-values:

$$\begin{aligned} Q_j(x) &= \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_{j+1}} \mathbb{E}[h_\tau(X_\tau)|X_j = x] \\ &= \mathbb{E}[V_{j+1}(X_{j+1})|X_j = x], \qquad j = 0, \ldots, L-1, \end{aligned} \tag{2.5}$$

so that in the value process V_{j+1} at time $j+1$ determines the continuation value, Q_j, at time j. The Q-value at time j determines the option value at time j, as in (A.2) of the Appendix:

$$V_j(x) = \max\left\{h_j(x), Q_j(x)\right\}, \quad j = 0, \ldots, L-1, \tag{2.6}$$

It also follows that $(V_j)_{j=0,\ldots,L}$ defined in (2.4) is the Snell envelope of $(h_j)_{j=0,\ldots,L}$ (see Proposition A.3 in the Appendix).

As a natural analogue to value iteration given in (2.4), we could use *Q-value iteration* instead. By substituting (2.6) into (2.5) we obtain the following representations of Q_j introduced by Tsitsiklis and Van Roy (1999) [6]:

$$\boxed{\begin{array}{l} Q_L(x) = 0 \\ Q_j(x) = \mathbb{E}[\max\{h_{j+1}(X_{j+1}), Q_{j+1}(X_{j+1})\}|X_j = x], \quad j = 0, \ldots, L-1. \end{array}} \tag{2.7}$$

The above formulation (denominated in time 0 dollars) allows a direct and recursive computation of the continuation values (and hence value functions through Equation (2.6)) by computing conditional expectations.

Comparing (2.7) with (2.4) we see that in the value iteration, the maximum occurs outside the expectation and as a consequence the value function will not be differentiable. In contrast the maximum will be smoothed by taking it's conditional expectation in the Q-value iteration. Since it is always easier to estimate smooth functions there is some reason to focus on continuation values, as in [11] and [6].

In principle, Q-value iteration can be used to price any Bermudan option. However, in applications the underlying distributions will be rather complicated and therefore it is not clear how to compute these conditional expectations in practice. Moreover, in practice the algorithm suffers from the "curse of dimensionality" – that is the computation of the conditional expectations grows exponentially in the number d of state variables. This difficulty arises because computations involve discretization of the state space, and such discretization leads to a grid whose size grows exponentially in dimension. Since one value is computed and stored for each point in the grid, the computation time exhibits exponential growth.

2.3 Stopping Times

Pricing of American options entails solving the primal problem defined by Equation (2.2) via the dynamic programming recursions discussed in the previous section. However instead of focussing on values it is also convenient to view the pricing problem through stopping times and exercise regions. In Equation (A.2) of Proposition A.1, we see that:

$$\boxed{V_j(x) = \mathbb{E}[h_{\tau_j^*}(X_{\tau_j^*})|X_j], \quad j = 0, \dots, L-1,} \tag{2.8}$$

where $\tau_j^* = \min\{k \in \{j, \dots, L\} | Q_k(X_k) \leq h_k(X_k)\}$. In particular, we have that $V_0(x) = \mathbb{E}[h_{\tau^*}(X_{\tau^*})]$, where the optimal stopping time τ^* is given by:

$$\tau^* = \tau_0^* - \min\{k \in \{0, \dots, L\} : Q_k(X_k) \leq h_k(X_k)\}. \tag{2.9}$$

Since $Q_L(x) = 0$ and $h_L(x) \geq 0$ there exists always some index where $Q_k(X_k) \leq h_k(X_k)$, so the right hand side above is indeed well defined[7]. We may interpret τ^* as follows: in order to sell the option in an optimal way, we have to sell it as soon as the value we get if we sell it immediately is at least as large as the value we get in the mean in the future, if we sell it in the future in an optimal way.

We also have the following representation of Q-values, equivalent to (2.7) (shown in Proposition A.2 in the Appendix), on which Longstaff and Schwartz (2001) [11] focus:

$$\boxed{Q_j(x) = \mathbb{E}[h_{\tau_{j+1}^*}(X_{\tau_{j+1}^*})|X_j = x], \quad j = 0, \dots, L-1,}. \tag{2.10}$$

We see that in Equation (2.10), knowledge of $Q_j(X_j)$ amounts to knowledge of an optimal stopping rule τ^*. Using (2.9) and (2.10), the dynamic programming principle can be rewritten in terms of optimal stopping times τ_j, as follows (see [4]):

$$\boxed{\begin{aligned} &\tau_L(x) = L \\ &\tau_j(x) = j\mathbb{1}_{\{h_j(X_j) \geq \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|X_j = x]\}} + \tau_{j+1}\mathbb{1}_{\{h_j(X_j) < \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|X_j = x]\}}, \quad j = 0, \dots, L-1, \end{aligned}} \tag{2.11}$$

7. From (2.6), we see that $Q_k(X_k) \leq h_k(X_k)$ is equivalent to $V_k(X_k) \leq h_k(X_k)$, so we may rewrite the optimal stopping time as $\tau^* = \tau_0^* - \min\{k \in \{0, \dots, L\} : V_k(X_k) \leq h_k(X_k)\}$.

This formulation in terms of stopping times (rather than in terms of value function) plays an essential role in the least-squares regression method of Longstaff and Schwartz [11].

Unless the dimension of the state space is small, the pricing problem becomes intractable (when traditional methods such as binomial trees are employed) and calls for the approximation of Q-value functions in (2.7) and (2.11). Several authors, especially Longstaff and Schwartz [11], and Tsitsiklis and Van Roy [6], have proposed the use of regression to estimate Q-values from simulated paths of the state process and thus to price American and Bermudan options. Each continuation value $Q_j(x)$ is the regression of the option vale $V_{j+1}(X_{j+1})$ on the current state $X_j = x$, and this suggests an approximation procedure: approximate $Q_j(x)$ by a linear combination of known functions of the current state (see §2.4) and use regression (typically least-squares) to estimate the best coefficients in this approximation (see §3).

2.4 Parametric Q-Value Functions

An *approximation architecture* (see [6]) for the Q-values is a class of parametrized functions from which we select $Q_j^{[m]}(x, \beta_j) : \Re^d \times \Re^m \mapsto \Re$, which assigns values $Q_j^{[m]}(x, \beta_j)$, for $j = 0, \ldots, L-1$, to states x, where $\beta_j = (\beta_{j1}, \ldots, \beta_{jm})'$ is a vector of free parameters. The objective becomes to choose, for each $j = 0, \ldots, L-1$, a parameter vector β_j that minimizes some approximation error so that

$$Q_j^{[m]}(x, \beta_j) \approx Q_j(x) = \mathbb{E}[V_{j+1}(X_{j+1})|X_j = x].$$

In choosing a parametrization to approximate the Q-value function, a measurable, real-valued *feature vector*, $e^m(x) = (e_1, \ldots, e_m)'$, is associated to each state $x \in \Re^d$. The feature vector is assumed to satisfy the following conditions (see [4]):

1. For $j = 1, \ldots L-1$, the sequence $(e_k(X_j))_{k \geq 1}$ is total in $L^2(\sigma(X_j))$.
2. For $j = 1, \ldots L-1$ and $m \geq 1$, if $\sum_{k=1}^m \lambda_k e_k(X_j) = 0$ a.s. then $\lambda_k = 0$ for $k = 1, \ldots, m$.

Such a feature vector is meant to represent the most salient properties of a given state[8]. In a feature based parametrization, $Q_j^{[m]}(x, \beta_j)$ depends on x only through $e(x)$, hence for some function $f : \Re^m \times \Re^m \mapsto \Re$, we have $Q_j^{[m]}(x, \beta_j) = f(e(x), \beta_j)$. The function f represents the choice of *architecture* used for the approximation.

In [11, 6] *linearly parametrized architecture* of the following form is considered:

$$\boxed{Q_j^{[m]}(x, \beta_j) = \sum_{k=1}^m \beta_{jk} e_k(x) = \beta_j' e^m(x),} \tag{2.12}$$

i.e., the Q-value function is approximated by a linear combination of feature vectors.

Tsitsiklis and Van Roy [6] go on to define an operator Φ, that maps vectors in $\Re^m$ to real-valued functions of the state, by:

$$(\Phi\beta)(x) = \sum_{k=1}^m \beta_k e_k(x).$$

8. One could use a different feature vectors at different exercise dates, but to simplify notation we suppress and dependence of on j.

Given a choice of parametrization $Q_j^{[m]}$, the computation of appropriate parameters, β_j, calls for a numerical algorithm. Approximate Q-value iteration generates a sequence of deterministic parameters $\beta_{L-1}, \beta_{L-2}, \ldots, \beta_0$ leading to approximations $Q_{L-1}^{[m]}(\cdot, \beta_{L-1}), Q_{L-1}^{[m]}(\cdot, \beta_{L-2}), \ldots, Q_0^{[m]}(\cdot, \beta_0)$ to the true Q-value functions $Q_{L-1}, \ldots, Q_0$.

2.5 Approximate Q-Value Iteration

The approximate Q-value iteration, suggested by Tsitsiklis and Van Roy [6], involves a sequence of orthogonal projection matrices (Π_j^m) for $j = 0, \ldots, L-1$ that projects any function in $L^2(\Omega)$ onto the span of $\{e_1(X_j), \ldots, e_m(X_j)\}$, with respect to a weighted quadratic norm $||V||_{\pi_j}$, defined by:

$$||V||_{\pi_j} = \left(\int_{x \in \Re^d} V^2(x) \pi_j(dx) \right)^{1/2},$$

where π_j is the probability measure on $\Re^d$ that describes the probability distribution of X_j under the risk-neutral dynamics of the process. In other words the projection operator is characterized by:

$$\Pi_j^m V = \arg\min_{\Phi b} ||V - \Phi b||_{\pi_j},$$

where $b \in \Re^m$. Note that the range of the projection is the same as that of Φ and therefore, for any function V, with $||V||_\pi \leq \infty$, there is a weight vector b such that $\Pi_j^m V(x) = \Phi b(x) = b \cdot e^m(x)$ for $X_j = x$. Working with the regression representation of Q_j in Equation (2.7), the algorithm generates iterates satisfying:

$$\boxed{\begin{array}{l} Q_L^{[m]}(x, \beta_L) = 0, \\ Q_j^{[m]}(x, \beta_j) = \Pi_j^m \mathbb{E}[\max\{h_{j+1}(X_{j+1}), Q_{j+1}^{[m]}(X_{j+1}, \beta_{j+1})\} | X_j = x], \qquad j = 0, \ldots, L-1. \end{array}} \tag{2.13}$$

The approximation algorithm offers advantages over Q-value iteration because it uses a more parsimonious representation: only m numerical values need to be stored at each stage. The algorithm generates approximate Q-value functions by mimicking Q-value iteration, while sacrificing exactness in order to maintain functions within the range of the approximator (the span of the feature vector).

Equation (2.13) can be interpreted as follows: given the state $X_{j+1} = z$ and parameters β_{j+1} at time $j+1$, the approximate Q-value at time j, is given by computing the projection of $\mathbb{E}[\max\{h_{j+1}(X_{j+1}), Q_{j+1}^{[m]}(X_{j+1}, \beta_{j+1})\} | X_j = x]$ at time $j+1$ onto the span of $e^m(x)$. In other words the approximate Q-value at time j, is that $\Phi\beta_j(x) = \Phi b(x)$ which minimizes $|| \mathbb{E}[\max\{h_{j+1}(z), \Phi\beta_{j+1}(z)\} | X_j = x] - \Phi b(x)||_{\pi_j}$. The approximate option value is then given by:

$$V_0^m = \max\{h_0(X_0), Q_0^{[m]}(X_0)\}.$$

2.6 Approximate Stopping Times

The option value achieved by following some specific exercise strategy is dominated by an optimal strategy. Any stopping time τ (for the Markov chain $X_0, X_1, \ldots, X_L$) determines a value (in general suboptimal) through:

$$V_0^{(\tau)}(X_0) = \mathbb{E}[h_\tau(X_\tau)] \leq V_0^{(\tau^*)}(X_0).$$

In other words, any algorithm that gives a sub-optimal stopping rule τ can be used to compute a lower bound on the Bermudan value V_0. To get a good lower bound, we need to find an stopping time τ that is close to some optimal exercise policy τ^*. The method of Longstaff and Schwartz [11], discussed in the proceeding section can be used to generate a candidate exercise rule that defines a lower bound price process.

Working with the representation of Q_j, as in (2.10), in terms of approximate stopping times:

$$Q_j^{[m]}(x,\beta_j) = \Pi_j^m \mathbb{E}[h_{\tau_{j+1}^{[m]}}(X_{\tau_{j+1}^{[m]}})|X_j = x],$$

the approximate Q-value iteration can be written in terms of the sub-optimal stopping times $\tau_j^{[m]}$ as follows (see [4]):

$$\boxed{\begin{aligned}\tau_L^{[m]}(x,\beta_j) &= L\\ \tau_j^{[m]}(x,\beta_j) &= j\mathbb{1}_{\{h_j(X_j)\geq \Pi_j^m \mathbb{E}[h_{\tau_{j+1}^{[m]}}(X_{\tau_{j+1}^{[m]}})|X_j=x]\}}\\ &\quad + \tau_{j+1}^{[m]}\mathbb{1}_{\{h_j(X_j)<\Pi_j^m \mathbb{E}[h_{\tau_{j+1}^{[m]}}(X_{\tau_{j+1}^{[m]}})|X_j=x]\}}, \qquad j=0,\ldots,L-1.\end{aligned}} \tag{2.14}$$

From these stopping times, we obtain the approximation of the value function:

$$V_0^m = \max\left\{h_0(X_0), h_{\tau_0^{[m]}}(X_{\tau_0^{[m]}})\right\}.$$

3 Regression-based Monte Carlo Simulation

In general, it is not always possible to compute the projections involved in the algorithms (2.13) and (2.14) and also calculating the conditional expectations poses a challenge as the state space, $\Re^d$, is potentially high-dimensional.

Regression-based Monte Carlo methods use regression estimates, generated from N artificial samples of the state process as numerical procedures to compute the above projections in (2.13) and (2.14) approximately. The algorithms in the proceeding sections construct estimates of the continuation values and (sub-optimal) estimates of the stopping time [9] .

3.1 Approximate Projection Operator

Tsitsiklis and Van Roy [6] firstly define an approximation to the projection operator. In particular they perform a Monte Carlo simulation of the underlying variable $(X_j)_{j=0,\dots,L} \in \Re^d$ i.e., for $n = 1, \dots, N$ they generate paths $(X_j^{(n)})_j = 0, \dots, L$. The data generating processes is assumed to be completely known, i.e., all parameters of the process are estimated from historical data. The sample of states $(X_j^{(1)}), \dots, (X_j^{(N)})$ are artificial independent Markov processes which are identically distributed as $(X_j)_{j=0,\dots,L}$, according to the probability distribution π_j. They then go on to define the approximate projection operator for $j = 0, \dots, L-1$:

$$\hat{\Pi}_j^m V = \arg\min_{\Phi b} \sum_{n=1}^{N} (V(X_j^{(n)}) - (\Phi b)(X_j^{(n)}))^2. \tag{3.1}$$

As N grows, this approximation becomes close to exact, in the sense that $||\hat{\Pi}_j^m V - \Pi_j^m V||_{\pi_j}$ converges to zero with probability 1.

For $j = 0, \dots, L-1$, given $\hat{\Pi}_j^m$, these so-called Monte Carlo samples are then used recursively to generate a vector of parameters $\beta_j^m = (\beta_{j1}^m, \dots, \beta_{jm}^m) \in \Re^m$ minimizing (3.1), in order to estimate Q_j by using a modified version of the approximate Q-value iteration of (2.13):

$$\begin{aligned} Q_L^{[m]}(x, \beta_L^m) =& 0, \\ Q_j^{[m]}(x, \beta_j^m) =& \hat{\Pi}_j^m \mathbb{E}[\max\{h_{j+1}(X_{j+1}), Q_{j+1}^{[m]}(X_{j+1}, \beta_{j+1}^m)\} | X_j = x], \ j = 1, \dots, L-1. \end{aligned} \tag{3.2}$$

or by using a modified version of the approximate stopping times in (2.14):

$$\begin{aligned} \tau_L^{[m]}(x, \beta_L^m) =& L \\ \tau_j^{[m]}(x, \beta_j^m) =& j \mathbb{1}_{\{h_j(X_j) \geq \hat{\Pi}_j^m \mathbb{E}[h_{\tau_{j+1}^{[m]}}(X_{\tau_{j+1}^{[m]}}) | X_j = x]\}} \\ &+ \tau_{j+1}^{[m]} \mathbb{1}_{\{h_j(X_j) < \hat{\Pi}_j^m \mathbb{E}[h_{\tau_{j+1}^{[m]}}(X_{\tau_{j+1}^{[m]}}) | X_j = x]\}}, \qquad j = 1, \dots, L-1. \end{aligned} \tag{3.3}$$

9. This kind of recursive estimation scheme was firstly proposed by Carrière (1996) [12] for the estimation of value function. In Tsitsiklis and Van Roy (1999) [5] and Longstaff and Schwartz (2001) [11] it was used to construct estimates of continuation values.

As opposed to the original version of the algorithm, in which projections posed a computational burden, this new variant involves the solution of a linear least-squares problem, with m free parameters, and admits efficient computation of projections, as long as the number of samples N is reasonable. The computation of the projections in the above algorithms entails solving a linear least-squares problem of which the m-dimensional parameter β_j^m is the solution, i.e.:

$$\beta_j^m = \arg\min_{\Phi b} \sum_{n=1}^{N} (\mathbb{E}[Y_j | X_j = X_j^{(n)}] - (\Phi b)(X_j^{(n)}))^2, \tag{3.4}$$

where Y_j is given by either of the conditional expectations in the above algorithms, i.e.:

$$Y_j = Y_j(X_{j+1}, \ldots, X_L, Q_{j+1}^{[m]}, \ldots, Q_L^{[m]}) = h_{\tau_{j+1}^{[m]}}(X_{\tau_{j+1}^{[m]}}), \tag{3.5}$$

or

$$Y_j = Y_j(X_{j+1}, Q_{j+1}^{[m]}) = \max\{h_{j+1}(X_{j+1}), Q_{j+1}^{[m]}(X_{j+1}, \beta_{j+1}^m)\}, \tag{3.6}$$

so that the approximate projection (see [4]) gives:

$$Q_j^{[m]}(X_j, \beta_j^m) = \hat{\Pi}_j^m \mathbb{E}[Y_j | X_j] = \beta_j^m \cdot e^m(X_j) = \sum_{k=1}^{m} \beta_{jk}^m e_k^m(X_j), \qquad j = 1, \ldots, L-1. \tag{3.7}$$

We remark that under the assumptions made about the vector e^m, β_j^m has the explicit solution [10]

$$\beta_j^m = (A_j^m)^{-1} \mathbb{E}[Y_j e^m(X_j)], \qquad j = 1, \ldots, L-1 \tag{3.8}$$

where A_j^m is an $m \times m$ non-singular matrix, with coefficients given by

$$(A_j^m)_{1 \le k,l \le m} = \mathbb{E}[e_k(X_j) e_l(X_j)]. \tag{3.9}$$

and then the option value is given by

$$V_0^{[m]} = \max\{h_0(X_0), \hat{\Pi}_0^m \mathbb{E}[Q_1^{[m]}(X_1)]\}.$$

or

$$V_0^{[m]} = \max\left\{h_0(X_0), \hat{\Pi}_0^m \mathbb{E}[h_{\tau_1^{[m]}}(X_{\tau_1^{[m]}})]\right\}.$$

However there is an additional obstacle that we must overcome. For each sample $X_j^{(n)}$, we must compute $\mathbb{E}[Y_j | X_j]$ in (3.7), or the expectations in (3.8) and (3.9). The variables inside the expectations have the joint distribution of the state of the underlying Markov chain and approximate continuation values at future times. This expectation is over a potentially high-dimensional space $\Re^d$ and can therefore poses a computational challenge which we deal with in the next section.

10. We want to minimize the expected squared error in this approximation with respect to the coefficients β_j^m (see [8]), so we solve for β_j^m in

$$\frac{\partial}{\partial \beta'} \mathbb{E}[(\mathbb{E}[Y_j | X_j] - \beta' e^m(X_j))^2] = 0$$

$$\implies \mathbb{E}[e^m(X_j)\mathbb{E}[Y_j | X_j]] = \mathbb{E}[e^m(X_j) e^{m'}(X_j)]\beta$$

$$\implies \beta = \mathbb{E}[e^m(X_j) e^{m'}(X_j)]^{-1} \mathbb{E}[e^m(X_j)\mathbb{E}[Y_j | X_j]]$$

Now since $e(X_j)$ is measurable with respect to X_j and by the Tower property, we have that

$$\begin{aligned} \mathbb{E}[e^m(X_j)\mathbb{E}[Y_j | X_j]] &= \mathbb{E}[\mathbb{E}[Y_j e^m(X_j) | X_j]] \\ &= \mathbb{E}[Y_j e^m(X_j)]. \end{aligned}$$

3.2 Approximate Conditional Expecation Operator

The second approximation made by Tsistsklis and Van Roy [6], is to evaluate numerically the conditional expectation $\mathbb{E}[Y_j|X_j]$ by Monte Carlo simulation and thus to find the coefficients in (3.2) and (3.3). For each sample of states $(X_j^{(n)})$, and successor state $(X_{j+1}^{(n)})$, they define the approximate conditional expectation by:

$$\hat{\mathbb{E}}[Y_j|X_j = X_j^{(n)}] = Y_j^{(n)}, \tag{3.10}$$

where $Y_j^{(n)}$ is given by either

$$Y_j^{(n)} = Y_j^{(n)}(X_{j+1}^{(n)}, \ldots, X_L^{(n)}, Q_{j+1}^{n,m,N}, \ldots, Q_L^{n,m,N}) = h^{(n)}_{\tau_{j+1}^{n,m,N}}(X^{(n)}_{\tau_{j+1}^{n,m,N}}), \tag{3.11}$$

in the case of the Longstaff-Schwartz algorithm (cf. (3.5)), or

$$Y_j^{(n)} = Y_j^{(n)}(X_{j+1}^{(n)}, Q_{j+1}^{n,m,N}) = \max\{h_{j+1}^{(n)}(X_{j+1}^{(n)}), Q_{j+1}^{n,m,N}(X_{j+1}^{(n)}, \beta_{j+1}^{(m,N)})\}, \tag{3.12}$$

in the case of the Tsitsiklis and Van Roy algorithm [11](cf. (3.6)). The computation of coefficients $\beta_j^{(m,N)} = (\beta_{j1}^{(m,N)}, \ldots, \beta_{jm}^{(m,N)}) \in \Re^m$ are described below.

The algorithms in (3.2) and (3.3) can be modified further by making this approximation to the conditional expectation so that $Q_j^{n,m,N} = \hat{\Pi}_j^m \hat{\mathbb{E}}[Y_j|X_j = X_j^{(n)}] = \hat{\Pi}_j^m Y_j^{(n)} = \Phi\beta_j^{(m,N)}(X_j^{(n)})$ [12]. This implies the application of a regression estimate to the approximative sample $\{(e^m(X_j^{(n)}))_{j=0,\ldots L}, (Y_j^{(n)})_{j=0,\ldots L}\}$ for $n = 1, \ldots, N$ to produce the coefficient estimates $\beta_j^{(m,N)}$.

The modification of the approximate $Q^{[m]}$-value iteration of (3.2) is thus given by:

$$\boxed{\begin{aligned} Q_L^{n,m,N}(x, \beta_L^{(m,N)}) &= 0 \\ Q_j^{n,m,N}(x, \beta_j^{(m,N)}) &= \hat{\Pi}_j^m \max\{h_{j+1}^{(n)}(X_{j+1}^{(n)}), Q_{j+1}^{n,m,N}(X_{j+1}^{(n)}, \beta_{j+1}^{(m,N)})\} \\ &= \hat{\Pi}_j^m \max\{h_{j+1}^{(n)}(X_{j+1}^{(n)}), \beta_{j+1}^{(m,N)} \cdot e^m(X_j^{(n)})\} \\ &= \beta_j^{(m,N)} \cdot e^m(x), \qquad j = 1, \ldots, L-1. \end{aligned}} \tag{3.13}$$

and of the recursive stopping times, $\tau^{[m]}$, of (3.3), by:

$$\boxed{\begin{aligned} \tau_L^{n,m,N} &= L \\ \tau_j^{n,m,N} &= j\mathbb{1}_{\{h_j^{(n)}(X_j^{(n)}) \geq \hat{\Pi}_j^m h^{(n)}_{\tau_{j+1}^{n,m,N}}(X^{(n)}_{\tau_{j+1}^{n,m,N}})\}} + \tau_{j+1}^{n,m,N}\mathbb{1}_{\{h_j^{(n)}(X_j^{(n)}) < \hat{\Pi}_j^m h^{(n)}_{\tau_{j+1}^{n,m,N}}(X^{(n)}_{\tau_{j+1}^{n,m,N}})\}} \\ &= j\mathbb{1}_{\{h_j^{(n)}(X_j^{(n)}) \geq \beta_j^{(m,N)} \cdot e^m(X_j^{(n)})\}} + \tau_{j+1}^{n,m,N}\mathbb{1}_{\{h_j^{(n)}(X_j^{(n)}) < \beta_j^{(m,N)} \cdot e^m(X_j^{(n)})\}}, \qquad j = 1, \ldots, L-1, \end{aligned}} \tag{3.14}$$

11. In the case of the Tsitsiklis and Van Roy algorithm, the approximation in (3.10) is:

$$\hat{\mathbb{E}}[\max\{h_{j+1}(X_{j+1}), Q_{j+1}^{[m]}(X_{j+1}, \beta_{j+1})\}|X_j = X_j^{(n)}] = \max\{h_{j+1}^{(n)}(X_{j+1}^{(n)}), Q_{j+1}^{n,m,N}(X_{j+1}^{(n)}, \beta_{j+1}^{(m,N)})\}$$

12. Because $\hat{\mathbb{E}}$ enters linearly in the approximate Q-value representation and effectively allows for the noise in the next state $X_{j+1}^{(n)}$ to be averaged out, $\hat{\Pi}_j^m Y_j^{(n)}$ is an unbiased estimator of $\hat{\Pi}_j^m \hat{\mathbb{E}}[Y_j|X_j = X_j^{(n)}]$. Such unbiasedness would not be possible with approximate value iteration, because the dependence on $\hat{\mathbb{E}}$ is nonlinear.

Both algorithms are applied in connection with linear regression. Here the estimate $(Q_j^{n,m,N})_{j=0,\ldots,L}$ is defined by (cf. (3.7)):

$$Q_j^{n,m,N}(x, \beta_j^{(m,N)}) = \hat{\Pi}_j^m Y_j^{(n)} = \beta_j^{(m,N)} \cdot e^m(x) = \sum_{k=1}^{m} \beta_{jk}^{(m,N)} e_k^m(x), \tag{3.15}$$

where for $j = 0, \ldots, L-1$, $\beta_j^{(m,N)} \in \Re^m$ is the linear least-squares estimator (cf. (3.8)):

$$\begin{aligned} \beta_j^{(m,N)} &= \arg\min_{b \in \Re^m} \sum_{n=1}^{N} \left(Y_j^{(n)} - b \cdot e^m(X_j^{(n)}) \right)^2 \\ &= (A_j^{(m,N)})^{-1} \frac{1}{N} \sum_{n=1}^{N} Y^{(n)} e^m(X_j^{(n)}), \qquad j = 1, \ldots, L-1, \end{aligned} \tag{3.16}$$

where $A_j^{(m,N)}$ is an $m \times m$ matrix, with coefficients given by (cf. (3.9)):

$$(A_j^{(m,N)})_{1 \le k,l \le m} = \frac{1}{N} \sum_{n=1}^{N} e_k(X_j^{(n)}) e_l(X_j^{(n)}). \tag{3.17}$$

Note that $\lim_{N\to\infty} A_j^{(m,N)} = A_j^m$ almost surely. Under the assumptions for the feature vector, the matrix $A_j^{(m,N)}$ is invertible for N large enough.

Finally, from $Q_1^{n,m,N}$, we have the following approximation for $V_0^{m,N}$:

$$V_0^{m,N} = \max\{h_0(X_0), \frac{1}{N} \sum_{n=1}^{N} Q_0^{n,m,N}(X_0, \beta_0^{(m,N)})\},$$

and from the variable $\tau_1^{n,m,N}$, we can derive

$$V_0^{m,N} = \max\left\{ h_0(X_0), \frac{1}{N} \sum_{n=1}^{N} h^{(n)}_{\tau_1^{n,m,N}}(X_0) \right\}.$$

3.3 Method of Tsitsiklis and Van Roy

1 *Generate N independent paths of state vector, X, conditional on initial state, X_0 (Markov chain)*
2 *At terminal nodes set $Q_L^{(m,n,N)}(X_L^{(n)}) = 0$ for all $n = 1, \ldots, N$*
3 *Apply backward induction: for $j = L-1, \ldots, 1$*

3.1 *Regress $V_{j+1}^{(m,n,N)}(X_{j+1}^{(n)})$ on $(e_1^m(X_j^{(n)}), \ldots e_m^m(X_j^{(n)}))$ where $V_{j+1}^{(m,n,N)}(X_{j+1}^{(n)}) = \max\{h^{(n)}(X_{j+1}^{(n)}), Q_{j+1}^{(n,m,N)}(X_{j+1}^{(n)})\}$*

3.2 *Set $Q_j^{(n,m,N)}(X_j^{(n)}) = \sum_{k=1}^{m} \beta_{jk}^{(m,N)} e_k^{(m)}(X_j^{(n)})$ where the $\beta_{jk}^{(m,N)}$s are the estimated regression coefficients*

End for

4 *Set $V_0^{m,N}(X_0) = (\sum_{n=1}^{N} \max\{h^{(n)}(X_1^{(n)}), Q_1^{n,m,N}(X_1^{(n)}))\}/N$.*

In full detail, a typical iteration of the algorithm in (3.14) proceeds as follows:
Given $Q^{n,m,N}(x, \beta_{j+1}^{(m,N)}) = \beta_{j+1}^{(m,N)} \cdot e^m(x)$, the vector $\beta_j^{(m,N)}$ is found by minimizing

$$\sum_{n=1}^{N} \left(\max\{h_{j+1}^{(n)}(X_{j+1}^{(n)}), \sum_{k=1}^{m} \beta_{j+1,k}^{(m,N)} e_k^m(X_{j+1}^{(n)})\} - \sum_{k=1}^{m} \beta_{jk} e_k^m(X_j^{(n)}) \right)^2,$$

with respect to β_{jk}. The exact solution is given in (3.16). In other words, we regress $\max\{h_{j+1}^{(n)}(X_{j+1}^{(n)}), \sum_{k=1}^{m} \beta_{j+1,k}^{(m,N)} e_k(X_{j+1}^{(n)})\}$, which is the option value at time $j+1$, on the span of features at time j.

3.4 Method of Longstaff and Schwartz

While the algorithm described above is that of Tsitsiklis and Van Roy [6], Longstaff and Schwartz [11] omit states $X_j^{(n)}$ where $h^{(n)}(X_j^{(n)}) < 0$ when estimating the regression coefficients, $\beta_j^{m,N}$ in (3.14), i.e. the regression involves only in the money paths, which appears to be more efficient numerically. This representation of the regression leads to a modification of (3.14) (see [4]):

$$\begin{aligned} &\hat{\tau}_L^{n,m,N} = L \\ &\hat{\tau}_j^{n,m,N} = j \mathbb{1}_{\{h_j^{(n)}(X_j^{(n)}) \geq \hat{\beta}_j^{(m,N)} \cdot e^m(X_j^{(n)})\} \cap \{h_j^{(n)}(X_j^{(n)}) > 0\}} + \tau_{j+1}^{n,m,N} \mathbb{1}_{\{h_j^{(n)}(X_j^{(n)}) < \hat{\beta}_j^{(m,N)} \cdot e^m(X_j^{(n)})\} \cup \{h_j^{(n)}(X_j^{(n)}) = 0\}}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, \ldots, L-1, \end{aligned} \tag{3.18}$$

where

$$\hat{\beta}_j^{(m,N)} = \arg\min_{b \in \Re^m} \sum_{n=1}^{N} \mathbb{1}_{\{h_j^{(n)}(X_j^{(n)}) > 0\}} \left(Y_j^{(n)} - b \cdot e^m(X_j^{(n)}) \right)^2 \tag{3.19}$$

In effect, the regression step (3.1) of the algorithm of Tsitsiklis and Van Roy, is modified in practice [13]:

(3.1*) *For* $j = L-1, \ldots, 1$ *regress* $V_{j+1}^{n,m,N}(X_{j+1}^{(n)})$ *on* $(e_1^m(X_j^{(n)}), \ldots, e_m^m(X_j^{(n)}))$, *where*

$$V_{j+1}^{n,m,N}(X_{j+1}^{(n)}) = \begin{cases} h_{j+1}^{(n)}(X_{j+1}^{(n)}), & h_{j+1}^{(n)}(X_{j+1}^{(n)}) \geq \hat{\beta}_{j+1}^{(m,N)} \cdot e^m(X_j^{(n)}) \text{ and } h_{j+1}^{(n)}(X_{j+1}^{(n)}) > 0, \\ V_{j+2}^{n,m,N}(X_{j+2}^{(n)}), & h_{j+1}^{(n)}(X_{j+1}^{(n)}) < \hat{\beta}_{j+1}^{(m,N)} \cdot e^m(X_j^{(n)}) \text{ and } h_{j+1}^{(n)}(X_{j+1}^{(n)}) = 0. \end{cases}$$

In particular, they take $V^{n,m,N}(X_{j+1}^{(n)})$ to be the *realized* discounted payoff on the nth path as determined by the exercise policy, $\hat{\tau}_k^{n,m,N}$, implicitly defined by $Q_k^{n,m,N}(X_k^{(n)})$, for $k = j+1, \ldots, L$.

13. The assumptions made for the feature vectors also need to be replaced by (see [4]):
    1. For $j = 1, \ldots, L-1$, the sequence $(e_k(X_j))_{k \geq 1}$ is total in $L^2(\sigma(X_j), \mathbb{1}_{\{h_j(X_j) > 0\}} d\mathscr{P})$,
    2. For $1 \leq j \leq L-1$ and $m \geq 1$, if $\mathbb{1}_{h_j(X_j)} \sum_{k=1}^{m} \lambda_k e_k(X_j) = 0$ a.s. then $\lambda_k = 0$ for $1 \leq k \leq m$.

3.5 The Lower Bound on the Option Price

Haugh and Kogan [3] characterize the worst-case performance of the lower bound in the following theorem:

Theorem 3.1. *The lower bound on the option price, $\underline{V}_0(X_0)$, satisfies*

$$\underline{V}_0(X_0) \geq V_0(X_0) - \mathbb{E}_0\left[\sum_{j=0}^{L} |\tilde{Q}_j(X_j) - Q_j(X_j)|\right], \tag{3.20}$$

where $\tilde{Q}_j$ is an approximation to the Q-value function.

Proof. At time j, the following six mutually exclusive events are possible: (i) $\tilde{Q}_j(X_j) \leq Q_j(X_j) \leq h_j(X_j)$, (ii) $Q_j(X_j) \leq \tilde{Q}_j(X_j) \leq h_j(X_j)$, (iii) $\tilde{Q}_j(X_j) \leq h_j(X_j) \leq Q_j(X_j)$, (iv) $Q_j(X_j) \leq h_j(X_j) \leq \tilde{Q}_j(X_j)$, (v) $h_j(X_j) \leq Q_j(X_j) \leq \tilde{Q}_j(X_j)$, (vi) $h_j(X_j) \leq \tilde{Q}_j(X_j) \leq Q_j(X_j)$. We define $\tilde{\tau}_j = \min\{s \in \{j, \ldots, L\} \cap \mathscr{T} : Q_s(X_s) \leq h_s(X_s)\}$ and

$$\underline{V}_j(X_j) = \mathbb{E}\left[h_{\tilde{\tau}_j}(X_{\tilde{\tau}_j})|X_j\right].$$

For each of the six scenarios, we establish a relation between the lower bound and the true option price.

(i), (ii) The algorithm for estimating the lower bound correctly prescribes immediate exercise of the option so that $V_j(X_j) - \underline{V}_j(X_j) = 0$.

(iii) In this case the option is exercised incorrectly. $\underline{V}_j(X_j) = h_j(X_j$ and $V_j(X_j) = Q_j(X_j)$, implying $V_j(X_j) - \underline{V}_j(X_j) \leq |\tilde{Q}_j(X_j) - Q_j(X_j)|$.

(iv) In this case, the option is not exercised though it is optimal to do so. Therefore,

$$\underline{V}_j(X_j) = \mathbb{E}\left[\underline{V}_{j+1}(X_{j+1})|X_j\right],$$

while

$$V_j(X_j) = h_j \leq Q_j(X_j) + (\tilde{Q}_j(X_j) - Q_j(X_j)) = \mathbb{E}\left[V_{j+1}(X_{j+1})|X_j\right] + (\tilde{Q}_j(X_j) - Q_j(X_j)).$$

This implies

$$V_j(X_j) - \underline{V}_j(X_j) \leq \mathbb{E}\left[V_{j+1}(X_{j+1}) - \underline{V}_{j+1}(X_{j+1})|X_j\right].$$

(v), (vi) In this case the option is correctly left unexercised so that

$$V_j(X_j) - \underline{V}_j(X_j) = \mathbb{E}\left[V_{j+1}(X_{j+1}) - \underline{V}_{j+1}(X_{j+1})|X_j\right].$$

Therefore, by considering the six possible scenarios, we find that

$$V_j(X_j) - \underline{V}_j(X_j) \leq |\tilde{Q}_j(X_j) - Q_j(X_j)| + \mathbb{E}\left[V_{j+1}(X_{j+1}) - \underline{V}_{j+1}(X_{j+1})|X_j\right].$$

Iterating and using the fact that $\underline{V}_L(X_L) = V_L(X_L)$ implies the result. □

While this theorem suggests that the performance of the lower bound may deteriorate linearly in the number of exercise periods, numerical experiments indicate that this is not the case. If the exercise strategy that defines the lower bound were to achieve the worst-case performance, then at each exercise period we erroneously would not exercise, i.e., the condition $Q_j(X_j) < h(X_j) < \tilde{Q}_j(X_j)$ would satisfied. If this were to occur, then at *each* exercise period, the state process would be close to the optimal exercise boundary. In addition, $\tilde{Q}_j$ would have to systematically overestimate the true Q-value so that the option would not exercised when it is optimal to do so. In practice, the variability of the underlying state variables, X_j, might suggest that X_j spends little time near the optimal exercise boundary. This suggests that as long as $\tilde{Q}_j$ is a good approximation to Q_j near the exercise frontier, the lower bound should be a good estimate of the true price, regardless of the number of exercise periods.

3.6 Convergence Results

Clément, Lamberton, and Protter [4] prove convergence of the Longstaff-Schwartz procedure as the number of feature vectors $m \to \infty$. The limit obtained coincides with the true price $V_0(X_0)$ if the assumptions made for the feature vectors in §2.4 hold; otherwise, the limit coincides with the value under suboptimal exercise policy and thus underestimates the true price. In practice (3.18) therefore produces low-biased estimates. The convergence of $V_0^{[m]}(X_0) = \max\left\{h_0(X_0), \hat{\Pi}_0^m \mathbb{E}[h_{\tau_1^{[m]}}(X_{\tau_1^{[m]}})]\right\}$ to V_0 is a direct consequence of the following result:

Theorem 3.2. *Assume that the feature vectors satisfy assumption 1. of §2.4, then, for $j = 1, \dots, L$, we have*

$$\lim_{m \to +\infty} \mathbb{E}[h_{\tau_j^{[m]}}(X_{\tau_j^{[m]}})|\mathcal{F}_j] = \mathbb{E}[h_{\tau_j}(X_{\tau_j})|\mathcal{F}_j],$$

in L^2.

Proof. See Theorem A.4 in the Appendix. □

In the following theorem, m is fixed, Clément, Lamberton and Protter [4] look at the convergence of $V_0^{n,m,N}(X_0)$ as N, the number of Monte-Carlo simulations, goes to infinity.

Theorem 3.3. *Assume that the feature vectors satisfy assumptions 1. and 2. of §2.4 and that for $j = 1, \dots L-1$, $\mathscr{P}(\beta_j \cdot e(X_j) = h_j(X_j)) = 0$. Then $V_0^{m,N}(X_0)$ converges almost surely to $V_0^{[m]}(X_0)$ as N goes to infinity. We also have almost sure convergence of $\frac{1}{N}\sum_{n=1}^{N} h^{(n)}_{\tau_j^{n,m,N}}(X^{(n)}_{\tau_j^{n,m,N}})$ towards $\mathbb{E}[h_{\tau_j^{[m]}}(X_{\tau_j^{[m]}})]$ as N goes to infinity, for $j = 1, \dots, L$.*

Proof. see Theorem A.5 in the Appendix. □

3.7 Final Comments

These ADP methods have performed surprisingly well on realistically high-dimensional problems (see [11] for numerical examples) and there has also been considerable theoretical work (e.g. [6, 5, 4]) justifying this. In practice, standard least-squares regression is used and because this technique is so fast the resulting Q-value iteration algorithm is also very fast. For typical problems that arise in practice, N is often taken to be on the order of 10000 to 50000 [10].

In practice it is quite common for an alternative estimate, $\underline{V}_0$, of V_0 to be obtained by simulating the exercise strategy that is defined by $\hat{\tau}^{n,m,N}$ for different paths. $\underline{V}_0$ is an unbiased lower bound on the true option value. That the estimator is a lower bound follows from the fact that $\hat{\tau}^{n,m,N}$ is

a feasible adapted exercise strategy. Typically, $\underline{V}_0$ is a much better estimator of the true price than $V_0^{n,m,N}(X_0)$ as the latter often displays significant upward bias.

Many more details are required to fully specify the algorithm in practice. In particular, model parameter values and the parametric family of approximating functions need to be chosen. The success of the method depends on the choice of basis functions of the feature vector. Polynomials (sometimes damped by functions vanishing at infinity) are a popular choice [6, 11], however in this case the number of basis functions required could grow quickly with the dimension of the underlying state vector X_j. Longstaff and Schwartz [11] use 5-20 basis functions in the examples they test. Problem-specific information is also often used when choosing basis functions. For example, if the value of the corresponding European option is available in closed form then that would typically be an ideal candidate for a basis function. Other commonly used basis functions are the intrinsic value of the option and the prices of other related derivative securities that are available in closed form. Rasmussen (2002) [17] investigates improvements based on the control variate technique

4 Duality-based Methods

While the approximate dynamic programming (ADP) methods of the previous section have been very successful, a notable weakness is their inability to determine how far the approximate solution is from optimality in any given problem. Throughout, we have formulated the Bermudan option pricing problem as one of maximizing over stopping times. Haugh and Kogan (2004) [3] and Rogers (2002) [13] have independently established dual formulations in which the price is represented through a minimization problem. Duality-based methods that can be used for constructing approximate upper bounds on the true value function $V_0(X_0)$, which are unbiased, by using Monte Carlo simulations. The Longstaff-Schwartz method in previous section yielded estimates $\tau_0^{n,m,N}$ of the optimal stopping time τ^* in order to approximate a lower bound on the value function. Haugh and Kogan showed that any approximate solution, arising from ADP could be evaluated by using it to construct an upper bound on the true value function.

4.1 Upper Bounds for Bermudan Options

In this section, we develop a method for finding upper bounds for American option prices [14] by Monte Carlo due to Rogers (2002) [13] and Haugh and Kogan [3]. The price of such options, $V_0(X_0)$, is usually written in the form of the primal problem:

$$V_0(X_0) = \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_0} \mathbb{E}[h_\tau(X_\tau)],$$

as in (2.2). The restriction to the set of stopping times, $\mathscr{T}_0$, corresponds to the fact that the holder of the option cannot see the future. As we have already seen, this supremum has a natural maximum element which is to exercise when the exercise value is greater than or equal to the continuation value [15].

If we were to increase the set of stopping times to include inadmissible stopping times, the price of the option would clearly go up and there is one stopping time that gives the highest price: the time defined by being the point of optimal exercise when foresight is allowed. Thus we have that:

$$\mathbb{E}[\max_{j=0,\dots,L} h_j(X_j)],$$

is an upper bound. However, allowing the holder to see into the future means that the price is much higher and the estimate is not particularly useful. To tighten the upper bound Rogers' took a martingale, M_j, with $M_0 = 0$ [16], that he subtracted before taking the maximum. Because M_j is a martingale it will continue to be a martingale if stopped at a bounded stopping time by the Optional Sampling theorem. Therefore, for any such hedge M_j, the value of the option is

$$V_0(X_0) = \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_0} \mathbb{E}[h_\tau(X_\tau) - M_\tau].$$

14. Both Rogers and Haugh and Kogan formulated the problem in continuous time, i.e., for American options, but the notation being used here is restricted to discrete time, because we are interested in pricing American options numerically.

15. Joshi [15] remarks that this reflects the buyer's price for the option in that the buyer chooses the exercise strategy which determines the amount he can realize. The seller on the other hand must hedge against the possibility of the buyer choosing any exercise strategy – even if the buyer chooses an exercise date at random, in which case there is a positive probability of exercising at the maximum along the path of the payoff process.

16. M_j can be viewed as the discounted price process of a portfolio of initial value zero or a hedge (a self-financing trading strategy).

Passing to exercising with maximal foresight gives still a higher number,

$$\mathbb{E}[\max_{j=0,\dots,L}\{h_j(X_j) - M_j\}],$$

which is still an upper bound for the price. It is a theorem of Rogers' that there exists a choice for M_j which makes the upper bound above equal to the price of the option:

Theorem 4.1. Rogers' Duality Relation.

$$V_0(X_0) = \inf_M \mathbb{E}[\max_{j=0,\dots,L}\{h_j(X_j) - M_j\}], \tag{4.1}$$

where $(M_j)_{j=0,1,\dots,L}$ is a martingale. The infimum is attained by by taking $M_j = M_j^*$, where M_j^* is the martingale part of the Doob-Meyer decomposition of the option price process [17]*:*

$$V_j(X_j) = M_j^* - A_j,$$

where (A_n) is a non-decreasing, predictable process, null at 0.

Proof. See Theorem A.10 of the Appendix. □

Joshi [1] expands on the original idea of Davis and Karatzas (1994) [16] to provide the following interpretion of Rogers' result for Bermudan options: it is possible for the seller of the Bermudan option to hedge perfectly by investing in its initial value at time zero and trading appropriately at each exercise date. To understand intuitively why it holds, it is useful to interpret the right hand side of (4.1) as the seller's price.

The seller of an option is subject to the exercise strategy chosen by the buyer of the option and is obligated to cover its payoff regardless of the buyer's choice of exercise date, even in the event that the buyer will use the optimal strategy or worse, in the case that the buyer exercises when the payoff process is a maximum along the path as if he is exercising with maximal foresight (which may result from choosing this date at random). Equality in (4.1) of Rogers' result says that even in these cases, the seller can hedge his exposure, by investing the buyers price, i.e., the buyer's and seller's prices are the same [18].

The seller can hedge perfectly under the assumption that the buyer is following the optimal strategy, and so "buys" (or dynamically replicates) one unit of the Bermudan option to hedge with for the buyer's price. At each exercise date, there are four possibilities according to the optimal exercise strategy and whether or not the buyer decides to exercise. In the two cases, where the buyer and seller agree then there is perfect hedging. If the buyer exercises and the seller does not when the optimal strategy says not to, then the derivative with optimal strategy – the value of the hedging portfolio – is worth more than the exercise value, so that the seller makes extra money when selling his replicating portfolio and is more than hedged. If the buyer does not exercise and the seller does when the optimal strategy says so, then the seller can exercise and buy the unexercised option with one less exercise date (worth the continuation value) for less than the exercise value and continue dynamically replicating and is ahead again. Rogers implicitly suggests that the extra cash can be used to buy numéraire bonds (or any instrument which is always of non-negative value). In this way, the seller can hedge against exercise with maximal foresight. We therefore take this optimal hedge and evaluate the expression in (4.1) to

17. That every supermartingale $(\pi_j)_{0\le j\le L}$ has this unique decomposition is proved in Proposition A.9 of the Appendix.
18. If the seller's price is not high enough to cover against any exercise strategy, he is not truly hedged – he needs to be hedged against the possibility that the buyer's ineptitude by luck imitates seeing the future. Thus a seller's price that did not allow for hedging against maximal foresight would not be sufficient. We have to be hedged if exercise occurs at a sub-optimal time.

get the seller's price.

Haugh and Kogan (2004) [3] formulate the pricing of American options, in a similar way to Rogers (2002), as a minimization problem over the class of supermartingales, π (rather than the class of martingales). They give the following upper bound on the option price for any supermartingale π,

$$\begin{aligned} V_0(X_0) &= \operatorname*{ess\,sup}_{\tau\in\mathscr{T}_0} \mathbb{E}\left[h_\tau(X_\tau)\right] = \operatorname*{ess\,sup}_{\tau\in\mathscr{T}_0} \mathbb{E}\left[h_\tau(X_\tau) - \pi_\tau + \pi_\tau\right] \\ &\leq \operatorname*{ess\,sup}_{\tau\in\mathscr{T}_0} \mathbb{E}\left[h_\tau(X_\tau) - \pi_\tau\right] + \pi_0 \\ &\leq \mathbb{E}\left[\max_{j=0,\ldots,L}\left(h_j(X_j) - \pi_j\right)\right] + \pi_0, \end{aligned} \tag{4.2}$$

where the first inequality follows from the Optional Sampling Theorem for supermartingales. They then show that there exists a choice of supermartingale π that makes the upper bound equal to the price of the option.

Theorem 4.2. Duality Relation of Haugh and Kogan.

$$V_0(X_0) = \inf_{\pi} \mathbb{E}\left[\max_{j=0,\ldots,L}\left(h_j(X_j) - \pi_j\right)\right] + \pi_0$$

The infimum is attained when $\pi_j = V_j(X_j)$, where $V_j(X_j)$ is the value process for the American option,

$$V_j(X_j) = \operatorname*{ess\,sup}_{\tau\in\mathscr{T}_j} \mathbb{E}\left[h_\tau(X_\tau)\middle| X_j\right],$$

Proof. Similar to the proof of Rogers'. Taking $\pi_j = V_j(X_j)$ we have:

$$V_0(X_0) \leq \mathbb{E}_0\left[\max_{j=0,\ldots,L}\left(h_j(X_j) - V_j(X_j)\right)\right] + V_0(X_0) \leq V_0(X_0).$$

Where the second inequality follows from the fact that $V_j(X_j) \geq h_j(X_j)$ for all j. □

Haugh and Kogan [3] define the *dual* function:

$$F(j,\pi) = \mathbb{E}[\max_{s=j,\ldots,L}\{h_s(X_s) - \pi_s\}|X_j] + \pi_j, \tag{4.3}$$

and the *dual* problem, which is to minimize the dual function at time 0 over the space of supermartingales, π_j. Theorem 4.2 therefore shows that the optimal values of the dual problem given by

$$\inf_{\pi} F(0,\pi) = \inf_{\pi} \mathbb{E}[\max_{j=0,\ldots,L}\{h_j(X_j) - \pi_j\}] + \pi_0, \tag{4.4}$$

and primal problem of (2.2)

$$\sup_{\tau\in\mathscr{T}_0} \mathbb{E}[h_\tau(X_\tau)]$$

coincide.

Theorem 4.2 implies the following characterization of the Bermudan option price:

The discounted option price, $V_j(X_j)$, is the smallest supermartingale that dominates the discounted payoff of the option, $h_j(X_j)$, at all exercise periods.

Haugh and Kogan [3] make an interesting remark, that the reverse implication holds, i.e. that the characterization above can be used to prove Theorem 4.2. Note that the dual function $F(j, \pi)$ is a supermartingale, because both terms on the right in (4.3) are supermartingales. For all j, $F(j, \pi) \geq h_j$, and so $F(0, \pi) \geq V_0$, because $V_j(X_j)$ is the smallest supermartingale that dominates $h_j(X_j)$ [19]. On the other hand, when π_j is chosen to be the option price, then $V_0(X_0) \geq F(0, \pi)$, so that the values of the primal and dual problems coincide. Theorem 4.2 therefore expresses the above characterization of the Bermudan option in a constructive form, which can be used to find an upper bound on the option price, if we can find a good approximation to the supermartingale minimizing (4.2).

4.2 A Special Case of the Optimal Martingale

Glasserman [10] specialises the continuous time optimal martingale of Rogers', of Theorem 4.1, that was specified by the Doob-Meyer decomposition in Theorem 4.1, to a discrete time version. In particular, he shows that the infimum in (4.1) can achieved by constructing an optimal martingale that he defines as follows:

$$M_j = \sum_{k=1}^{j} \Delta_k, \quad j = 1, \ldots, L, \tag{4.5}$$

with $M_0 \equiv 0$, where the martingale differences Δ_j are given by:

$$\Delta_j = V_j(X_j) - Q_{j-1}(X_{j-1}), \quad j = 1, \ldots, L. \tag{4.6}$$

That this process is indeed a martingale follows from the property of the differences

$$\mathbb{E}[\Delta_j | X_{j-1}] = \mathbb{E}[V_j(X_j) - \mathbb{E}[V_j(X_j)|X_{j-1}]|X_{j-1}] = 0.$$

He then uses induction to show that:

$$V_j(X_j) = \max\{h_j(X_j), h_{j+1}(X_{j+1}) - \Delta_{j+1}, \ldots, h_L(X_L) - \Delta_L - \ldots - \Delta_{j+1}\},$$

for all $j = 1, \ldots, L$, and that:

$$V_0(X_0) = \max_{j=1,\ldots,L} \{h_j(X_j) - M_j\},$$

thus verifying that optimality is attained in (4.1) by the martingale defined in (4.5) and (4.6) [20].

Kohler [9] uses the regression representation of the Q-value function of (2.7):

$$Q_{j-1}(X_{j-1}) = \mathbb{E}[\max\{h_j(X_j), Q_j(X_j)\}|X_{j-1}],$$

to arrive at a similar result:

19. Since an upper bound on the price of a Bermudan option can be constructed simply by evaluating $F(0, \pi)$ over an arbitrary supermartingale π_j, and in particular, if such a supermartingale satisfies $\pi_j \geq V_j(X_j)$, then the option $V_0(X_0)$ is bounded above by π_0. The standard proof of this characterization is also given in Proposition A.3 of the Appendix.

20. This holds at $j = L$ because $V_L(X_L) = h_L(X_L)$. Assuming it holds at j, then using

$$\begin{aligned} V_{j-1}(X_{j-1}) &= \max\{h_{j-1}(X_{j-1}), \mathbb{E}[V_j(X_j)|X_{j-1}]\} \\ &= \max\{h_{j-1}(X_{j-1}), V_j(X_j) - \Delta_j\} \end{aligned}$$

we see that it extends to $j - 1$. The option price at time 0 is

$$V_0(X_0) = \max\{h_0(X_0), \mathbb{E}[V_1(X_1)|X_0]\} = V_1(X_1) - \Delta_1.$$

Rewriting $V_1(X_1)$ using the induction hypothesis gives the result of Rogers' theorem.

Theorem 4.3.

$$V_0(X_0) = \inf_M \mathbb{E}[\max_{j=0,\dots L}\{h_j(X_j) - M_j\}] = \mathbb{E}[\max_{j=0,\dots L}(h_j(X_j) - M_j^*)],$$

where

$$M_j^* = \sum_{s=1}^{j}(\max\{h_s(X_s), Q_s(X_s)\} - \mathbb{E}[\max\{h_s(X_s), Q_s(X_s)\}|X_{s-1}]). \tag{4.7}$$

Proof. See Proposition A.8 in the Appendix.

□

Unfortunately, since Rogers' proof is non-constructive and depends on knowledge of the price process for the American option, we cannot apply it in practice, thus obtaining the optimal supermartingale is of course a task of a similar complexity to the original optimal stopping problem. However, there are various characterizations of the martingale associated with the optimal value function and these suggest alternative strategies for constructing approximate martingales. To this end, Andersen and Broadie [14], suggest using a price process or exercise strategy that has been found using a lower bound methodology, for example the method of Tsitsiklis and Van Roy [6] or Longstaff and Schwartz [11], respectively, in order to construct these approximate martingales. This is discussed further in §4.3 and §4.4.

4.3 Computing the Upper Bound

If $\hat{M}_j$ is a martingale that is close to the optimal martingale M_j, then we can use

$$\max_{j=0,\dots,L}\{h_j(X_j) - \hat{M}_j\} \tag{4.8}$$

to estimate an upper bound for the option price. Glasserman [10] gives a way of extracting martingale differences from approximate value functions $\hat{V}_j$, which involves computing, along a simulated path $X_0, X_1, \dots, X_L$, the differences

$$\hat{\Delta}_j = \hat{V}_j(X_j) - \mathbb{E}[\hat{V}_j(X_j)|X_{j-1}], \tag{4.9}$$

as an approximation to (4.6), such that $\hat{M}_j = \sum_{k=1}^{j}\hat{\Delta}_k$ with $\hat{M}_0 \equiv 0$. The first term on the right of (4.9) is just the value of the approximation $\hat{V}_j$ along a simulated path of the Markov chain.

We use a conditionally unbiased $\mathscr{F}_{j-1}$-measurable estimate of the conditional expectation in the second term in (4.9), in practice. Denoting this estimate by $\hat{\mathbb{E}}[\cdot|X_{j-1}]$, the $\hat{\Delta}_j$s of (4.9) are clearly martingales since

$$\mathbb{E}[\hat{V}_j - \hat{E}[\hat{V}_j(X_j)|X_{j-1}]|X_{j-1}] = 0.$$

Thus the $\hat{M}_j$s are also martingales and can be used to construct Monte Carlo estimates of $V_0(X_0)$, for which $\mathbb{E}[\max_{j=0,\dots,L}\{h_j(X_j) - \hat{M}_j\}]$, will be an upper bound [21].

21. Note that it is not true for approximate Q-value function, $\hat{Q}_j$, that

$$\hat{Q}_j(X_j) = \mathbb{E}[\hat{V}_{j+1}(X_j)|X_j], \quad j = 0, \dots, L-1.$$

Therefore by evaluating the differences $\hat{\Delta}_j = \hat{V}_j(X_j) - \hat{Q}_{j-1}(X_{j-1})$, as an approximation to (4.6), the approximation $\hat{M}_k$ will not give a valid upper bound for $V_0(X_0)$ in (4.7). This is because $\mathbb{E}[\hat{\Delta}_j|X_{j-1}] \neq 0$, i.e., the $\hat{\Delta}_j$s are not martingales.

The conditional expectation in (4.9) can be estimated using a nested Monte Carlo simulation. At each step X_{j-1} of the Markov chain, we generate N successors $X_j^{(1)}, \ldots, X_j^{(N)}$ and use

$$\hat{E}[\hat{V}_j(X_j)|X_{j-1}] = \frac{1}{N}\sum_{n=1}^{N} \hat{V}_j(X_j^{(n)}),$$

then these N successors may be discarded and a new one generated to get the next step X_j of the path of the Markov chain.

The simulated values

$$\hat{\Delta}_j = \hat{V}_j(X_j) - \frac{1}{N}\sum_{n=1}^{N} \hat{V}_j(X_j^{(n)}), \tag{4.10}$$

are martingale differences and it follows that using this $\hat{\Delta}_j$ is guaranteed to produce a high-biased estimator, even for finite N.

When the supermartingale π_j, defined by Haugh and Kogan [3] in (4.3), coincides with the discounted option value process, V_j, the upper bound $F(0,\pi)$ equals the true price of the American option. This suggests that a tight upper bound can be obtained from an accurate approximation, $\hat{V}_j$, by defining an approximate supermartingale $\hat{\pi}_j$ in such a way that when the approximate option price, $\hat{V}_j$, coincides with the exact price, V_j, $\hat{\pi}_j$ equals the discounted option price, V_j. They propose using either of the following two recursive definitions of the approximate supermartingale, $\hat{\pi}_j$, to get an upper bound on the value of the option $V_0(X_0)$ (by evaluating the dual function at time 0, $F(0,\hat{\pi})$, in (4.4)):

$$\begin{aligned} \hat{\pi}_0 &= \hat{V}_0(X_0) \\ \hat{\pi}_{j+1} &= \hat{\pi}_j + \hat{V}_{j+1}(X_{j+1}) - \hat{V}_j(X_j) - \mathbb{E}\left[(\hat{V}_{j+1}(X_{j+1}) - \hat{V}_j(X_j))^+ \middle| X_j\right], \end{aligned} \tag{4.11}$$

or

$$\hat{\pi}_{j+1} = \hat{\pi}_j \left(\frac{\hat{V}_{j+1}(X_{j+1})}{\hat{V}_j(X_j)} - \mathbb{E}\left[\left(\frac{\hat{V}_{j+1}(X_{j+1})}{\hat{V}_j(X_j)} - 1 \right)^+ \middle| X_j \right] \right). \tag{4.12}$$

The above approximate supermatingales [22], are defined in such a way that when $\hat{V}_j(X_j) = V_j(X_j)$ then $\hat{\pi}_j = V_j(X_j)$, since when $\hat{V}_j(X_j) = V_j(X_j)$ then the positive part of the expectations in (4.11) and (4.12) is zero, because of the supermartingale property of $V_j(X_j)$. It is not clear how to compare upper bounds determined by (4.11) and (4.12), but the properties of the bound obtained by (4.11) are simpler to analyze.

The first of the above definitions of the supermartingale $\hat{\pi}_j$ can be rewritten as:

$$\hat{\pi}_j = \hat{V}_j(X_j) - \sum_{s=0}^{j-1} \mathbb{E}[(\hat{V}_{s+1}(X_{s+1}) - \hat{V}_s(X_s))^+ | X_s].$$

By omitting the positive part of the expectation and taking $\hat{V}_j(X_j) = V_j(X_j)$, the above reduces to the definition of Glasserman's special case of the optimal martingale (plus a constant) in (4.6):

$$\pi_j = V_0(X_0) + \sum_{s=1}^{j} (V_s(X_s) - Q_{s-1}(X_{s-1})) = V_0(X_0) + M_j^*,$$

22. $\hat{\pi}_j$ is an adapted supermartingale for any approximation $\hat{V}_j$, since

$$\mathbb{E}_j[\hat{\pi}_{j+1} - \hat{\pi}_j] \leq 0,$$

by construction, for either definition of $\hat{\pi}_j$. Also, when $\hat{V}_j = V_j$, $\hat{\pi}_j = V_j$, because the latter process is a supermartingale and the positive part of expectations in (4.11) and (4.12) equals zero.

in the notation of Theorem 4.3. So we have that $F(0,\pi) = V_0(X_0)$ as expected, since $\hat{\pi}$ is still a supermartingale.

By omitting the positive part in the definition of $\hat{\pi}_j$, the upper bound is in fact tighter. It coincides with the supermartingale $\hat{\pi}_j$ at $j = 0$ and is always greater than $\hat{\pi}_j$ for $j > 0$. Therefore, it leads to a lower value of the upper bound defined by (4.3).

Andersen and Broadie [14] pointed out that in general tighter upper bounds can be obtained using the martingale component of the supermartingale π_j. To this end, Haugh and Kogan construct upper bounds from approximate value functions, taking π_j to be the martingale defined by

$$\begin{aligned} \pi_0 &= \hat{V}_0(X_0), \\ \pi_{j+1} &= \pi_j + \hat{V}_{j+1}(X_{j+1}) - \hat{V}_j(X_j) - \mathbb{E}\left[\hat{V}_{j+1}(X_{j+1}) - \hat{V}_j(X_j) \middle| X_j\right]. \end{aligned} \tag{4.13}$$

Rogers [13] and Haugh and Kogan [3] implement the kinds of estimates in (4.10) and (4.13) in connection with linear regression. For example, an approximation to the continuation value denoted by $\hat{Q}_j$ could result from the parametric approximation (see for example Haugh and Kogan's [3] use of a "multilayered perceptron with a single hidden layer" or the linearly parametrized architectures of Tsitsiklis and Van Roy [6]. Then the associated approximate value function would be given by $\hat{V}_j(x) = \max\{h_j(x), \hat{Q}_j(x)\}$

In particular, given estimates $\hat{Q}_j(X_j) = Q_j^{n,m,N}(X_j, \beta_j^{(m,N)})$ of the continuation values as in (3.14) we can estimate the above martingales, for each X_j and $X_j^{(1)}, \ldots, X_j^{(N)}$, by

$$\begin{aligned} \hat{M}_j &= \sum_{s=1}^{j} (\max\{h_s(X_s), Q_s^{n,m,N}(X_s, \beta_s^{(m,N)})\} - \hat{\mathbb{E}}[\max\{h_s(X_s), Q_s^{n,m,N}(X_s, \beta_s^{(m,N)})\}|X_{s-1}]) \\ &= \sum_{s=1}^{j} (\max\{h_s(X_s), Q_s^{n,m,N}(X_s, \beta_s^{(m,N)})\} - \frac{1}{N}\sum_{n=1}^{N} \max\{h_s(X_s^{(n)}), Q_s^{n,m,N}(X_s^{(n)}, \beta_s^{(m,N)})\}). \end{aligned}$$

This in turn can be used to construct Monte Carlo estimates of V_0, for which the expectation

$$\mathbb{E}[\max_{j=0,\ldots,L} (h_j(X_j) - \hat{M}_j)]$$

is greater than or equal to V_0.

As a consequence we get two kinds of estimates with expectation lower than or higher than $V_0(X_0)$ respectively, so we have available an interval in which our true price should be contained. Andersen and Broadie [14] use a suboptimal exercise policy to provide a lower bound for the price of the option. The dual value defined by extracting a martingale from the suboptimal policy gives their upper bound. We see this in the proceeding section.

4.4 Upper Bounds from Stopping Times

Andersen and Broadie [14] use the upper bound methodology developed by Rogers [13] to formulate another computational algorithm based on Monte Carlo. A distinguishing feature of their approach in constructing upper bounds, is the use of an approximate exercise policy to approximate the optimal exercise frontier, as opposed to an approximate option price, to estimate the bounds on the true price of the option.

If τ_j for $j = 1, \ldots L$ is a sequence of stopping times with the property that $\tau_j \geq j$ for all j, then τ_j has the interpretation of being the time at which the Bermudan option, issued at time j, should

be exercised (under some policy). These stopping times might, for example, be constructed from an approximation, $\hat{Q}_j$ to the Q-value function through

$$\tau_j = \min\{k \in \mathscr{T}_j : h_k(X_k) \geq \hat{Q}_k(X_k)\}, \tag{4.14}$$

for $j = 1, \ldots, L$, as in (2.9), where $\hat{Q}_i$ might be specified through the regression representation of Tsitsiklis and Van Roy [6].

Alternatively, τ_j may derive from a direct approximation to the exercise frontier as in the method of Longstaff and Schwartz [11] in (3.18), where $\tau_j = \hat{t}_j^{n,m,N}$, in which case it is not necessary to have an approximation to the value function available. Instead we can define a lower bound process given by:

$$\hat{V}_j(X_j) = \mathbb{E}[h_{\tau_j}(X_{\tau_j})|X_j], \tag{4.15}$$

as in (2.8), where $\hat{V}_j(X_j)$ is the value of a Bermudan option newly issued at time j and following the exercise strategy prescribed by (3.18). Defining the sequence of stopping times τ_j in effect allows us to the evolution of the lower bound process $\hat{V}_j(X_j)$ in terms of these newly issued options. For an given exercise strategy τ_j, Andersen and Broadie [14] define an adapted exercise indicator process $(l_j)_{j=0,\ldots,L}$ that equals 1 if exercise should take place at time j (given $\mathscr{F}_j$) and 0 otherwise. The indexed stopping time is defined as

$$\tau_j = \min\{k \in \mathscr{T}_j : l_j = 1\}. \tag{4.16}$$

If τ_j is close to the optimal strategy τ_j^*, then $\hat{V}_j(X_j)$ should be close to $V_j(X_j)$. Because $\hat{V}_j(X_j)$ can be computed for any adapted exercise strategy, specification of an exercise strategy through l_j or τ_j suffices to compute an upper bound.

Regardless of how τ_j is defined, if these stopping times define an sub-optimal exercise policy, the differences in (4.9) can be rewritten, as Glasserman [10] does, in terms of the lower bound process of (4.15):

$$\begin{aligned}
\hat{\Delta}_j &= \hat{V}_j(X_j) - \mathbb{E}[\hat{V}_j(X_j)|X_{j-1}] \\
&= \mathbb{E}[h_{\tau_j}(X_j)|X_j] - \mathbb{E}[h_{\tau_j}(X_{\tau_j})|X_{j-1}] \\
&= \begin{cases} \mathbb{E}[h_{\tau_j}(X_{\tau_j})|X_j] & - \quad \mathbb{E}[h_{\tau_j}(X_{\tau_j})|X_{j-1}] \quad \text{if } l_j = 0, \\ h_j(X_j) & - \quad \mathbb{E}[h_{\tau_j}(X_{\tau_j})|X_{j-1}] \quad \text{if } l_j = 1 \end{cases}
\end{aligned} \tag{4.17}$$

which would define a martingale for $j = 1, \ldots, L-1$, and give a valid upper bound for the option price.

Note that at time j, if continuation is prescribed, then the lower bound process $\hat{V}_j(X_j)$ is given by:

$$\hat{V}_j(X_j) = \mathbb{E}[h_{\tau_j}(X_{\tau_j})|X_j] = \mathbb{E}[\mathbb{E}[h_{\tau_j}(X_{\tau_j})|X_{j+1}]|X_j] = \mathbb{E}[\hat{V}_{j+1}(X_{j+1})|X_j],$$

but if exercise is prescribed, then $\hat{V}_j(X_j) = h_j(X_j)$, and $\mathbb{E}[\hat{V}_{j+1}(X_{j+1})|X_j]$ now represents the continuation value at time j and exercises according to τ_{j+1}, even though exercise was specified at time j. Thus, in the exercise region

$$\mathbb{E}[\hat{V}_{j+1}(X_{j+1})|X_j] = \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|X_j]$$

So in either region we need to estimate the same quantity $\mathbb{E}[\hat{V}_{j+1}(X_{j+1})|X_j]$ for $j = 1, \ldots, L-1$, with a slight difference in interpretation. Andersen and Broadie [14] suggested using sub-Monte Carlo simulations (or nested simulations) to evaluate the conditional expectations in (4.17), in their method for computing upper bounds, described presently:

1. The path $X_0, X_1, \ldots, X_L$ of the underlying Markov chain is simulated.
2. For each $j = 1, \ldots, L$ along the simulated path, either continuation or exercise is recommended by the indicator process l_j.
   (i) If continuation is indicated, use a nested Monte Carlo simulation to estimate
   $$\hat{V}_j(X_j) = \mathbb{E}[h_{\tau_j}(X_{\tau_j})|X_j].$$
   N_1 subpaths are simulated, originating from X_j and stopped according $\tau_j > j$, when $l_{\tau_j} = 1$, resulting in each subpath $X_{j+1}^{(n)}, \ldots, X_{\tau_j}^{(n)}$ terminating at the random stopping time τ_j, which is different in general for each $n = 1, \ldots, N_1$ [23]. Store the average of $h_{\tau_j}(X_{\tau_j})$ over the N_1 subpaths.
   (ii) If exercise is indicated, use a nested Monte Carlo simulation to estimate
   $$\mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|X_j],$$
   in the same way as step (i). N_2 subpaths are generated, originating from X_j and stopping at the first time, $\tau_j \geq j+1$, such that $l_{\tau_j} = 1$ [24]. Store $\hat{V}_j(X_j) = h_j(X_j)$ and the average of $h_{\tau_{j+1}}(X_{\tau_{j+1}})$ over the N_2 subpaths.
3. These quantities are then used to estimate (4.17) for $j = 1, \ldots, L$ and then setting $\hat{M}_j = \hat{\Delta}_1 + \cdots + \hat{\Delta}_j$ we can compute $\max_{j=1,\ldots,L}\{h_j(X_j) - \hat{M}_j\}$. The above procedure is repeated N_3 times (i.e. for N_3 separate simulations of the original process $X_0, \ldots, X_L$) and then averaging the results gives the desired upper bound $\mathbb{E}[\max_{j=1,\ldots,L}\{h_j(X_j) - \hat{M}_j\}]$.

In their paper [14], they apply their method in combination with stopping rules defined as in Longstaff and Schwartz [11]. They compute the upper bound in Rogers' formulation of the dual problem, by defining a martingale [25] π_j by:

$$\begin{aligned} \pi_0 &= \hat{V}_0(X_0), \\ \pi_{j+1} &= \pi_j + \hat{V}_{j+1}(X_{j+1}) - \hat{V}_j(X_j) - l_j\mathbb{E}\left[\hat{V}_{j+1}(X_{j+1}) - \hat{V}_j(X_j)|X_j\right], \qquad j = 0, \ldots, L-1 \end{aligned} \tag{4.18}$$

This representation is equivalent to (4.13), the difference here is that the exercise strategy has been specified. Of course, whether or not exercise is indicated, the term $\mathbb{E}\left[\hat{V}_{j+1}(X_{j+1}) - \hat{V}_j(X_j)\middle|\, X_j\right]$ will vanish when the stopping rule defining $\hat{V}(X_j)$ prescribes continuation. Thus we can rewrite (4.19) as

$$\pi_j = \hat{V}_j(X_j) - \sum_{k=1}^{j}\mathbb{E}[\hat{V}_k(X_k) - \hat{V}_{k-1}(X_{k-1})|X_{k-1}], \tag{4.19}$$

for $j = 1, \ldots, L$. The method for computing the upper bound defined by (4.18) is essentially the same as what was described above: in the continuation region we need to estimate $\hat{V}_j(X_j) = \mathbb{E}[h_{\tau_j}(X_{\tau_j})|X_j]$ and the conditional expectation drops away; in the exercise region the conditional

23. Observe that whereas each subpath used in (4.10) evolves for exactly one step of the Markov chain, here each subpath evolve for a random number of steps determined by the stopping rule.

24. In the exercise region, it is likely that the stopping time $\tau_{j+1} = j+1$ will indicate exercise after one step, thus these nested Monte Carlo simulations are expected to be quicker than those in the continuation region.

25. If continuation is indicated at time j then $l_j = 0$ and we have that $\hat{V}_j$ is a martingale:
$$\begin{aligned} \hat{V}_j(X_j) &= \mathbb{E}[h_{\tau_j}(X_{\tau_j})] = \mathbb{E}[\mathbb{E}[h_{\tau_j}(X_{\tau_j})|X_{j+1}]|X_j] = \mathbb{E}[\mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|X_{j+1}]|X_j] \\ &= \mathbb{E}[\hat{V}_{j+1}|X_j]. \end{aligned}$$
Thus $\mathbb{E}[\pi_{j+1}|X_j] = \pi_j$, so that π_j is a martingale in the continuation region. Also, if exercise is indicated, clearly π_j is a martingale too.

expectation can be rewritten as $h_j(X_j) + \mathbb{E}[\hat{V}_{j+1}(X_{j+1})|X_j]$ so we estimate $\mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|X_j]$. It is also worth mentioning that it is straightforward to combine the two approaches. The explicit approximation, $\hat{Q}_j$, could be used in some regions of the state space to estimate $\hat{M}_j$ while nested simulations to estimate the Δ_js could be used in other regions.

4.5 Tightness of the Upper Bound

Andersen and Broadie [14] rewrite (4.19) as:

$$\pi_j = \hat{V}_0(X_0) + \sum_{k=0}^{j-1} \left[\hat{V}_{j+1}(X_{j+1}) - \mathbb{E}[\hat{V}_{j+1}(X_{j+1})|X_j]\right],$$

in order to investigate the tightness of the bound.

Defining the difference process as $e_j = V_j(X_j) - \hat{V}_j(X_j) > 0$, and using the Doob-Meyer decomposition of $V_j(X_j)$ they then rewrite (4.19) as:

$$\pi_j = M_j + V_0(X_0) - e_0 - \sum_{k=1}^{j}(e_k - \mathbb{E}[e_k|X_{k-1}]), \quad j = 1, \ldots, L. \tag{4.20}$$

By taking $M_j = \pi_j - \hat{V}_0(X_0)$ in the dual problem in (4.1), they arrive at the following upper bound, $\bar{V}_0$, on the price of the Bermudan option:

$$\begin{aligned} V_0(X_0) \leq \bar{V}_0(X_0) &= \hat{V}_0(X_0) + \mathbb{E}[\max_{j=0,\ldots,L}\{h_j(X_j) - \pi_j\}] \\ &= \hat{V}_0(X_0) + \mathbb{E}[\max_{j=1,\ldots,L}(h_j(X_j) - M_j - V_0(X_0) + e_0 + \sum_{k=1}^{j}(e_k - \mathbb{E}[e_k|X_{k-1}])] \\ &\leq V_0(X_0) + \mathbb{E}[\max_{j=1,\ldots,L}(\sum_{k=1}^{j}(e_k - \mathbb{E}[e_k|X_{k-1}]))], \end{aligned}$$

since $h_j(X_j) \leq V_j(X_j) \leq V_0(X_0) + M_j$.

Andersen and Broadie relate the worst-case performance of the upper bound determined by the martingales in (4.18) to the accuracy of the original approximation, $\hat{V}_j(X_j)$. Because $e_k \geq 0$ and $\mathbb{E}[e_k|X_{k-1}] \geq 0$, we have:

$$\bar{V}_0 \leq V_0(X_0) + \mathbb{E}[\max_{j=1,\ldots,L}(\sum_{k=1}^{j} e_k)] \leq V_0(X_0) + \mathbb{E}[\sum_{j=1,\ldots,L} e_j]. \tag{4.21}$$

Equation (4.21) shows that if the lower bound is uniformly close to the true price, e.g., if $e_k \leq \varepsilon$, then the upper bound will differ by at most $L\varepsilon$ from the true value. Thus, the upper bound deteriorates at most linearly with the number of exercise dates. The quantity of interest is in fact, from (4.19):

$$\mathbb{E}\left[\max_{j=1,\ldots,L}\left(h_j(X_j) - \hat{V}_j(X_j) + \sum_{k=1}^{k}\mathbb{E}\left[\hat{V}_k(X_k) - \hat{V}_{k-1}(X_{k-1})\right]\right)\right].$$

In practice the upper bound appears to be much better than the worst case bound — while we would expect it to increase with the number of exercise periods, it is not clear that it should increase linearly. In fact, the quality of the upper bound should deteriorate with the number of exercise periods, but not in a linear fashion. In pricing American options when, if the horizon, L, is fixed, while the number of exercise dates in $[0, L]$ is allowed to increase, decreasing the interval

between exercise periods, Tsitsiklis and Van Roy [6] show that the approximation error, $|\hat{V}_j - V_j|$, is bounded above by a constant times $\sqrt{k}$, where k is the number of exercise periods [26].

4.6 Final Comments

Andersen and Broadie [14] go on to analyze their algorithm in (4.18) by including the noise that is introduced in the estimates for the martingale π, by the nested Monte Carlo simulations. They replace $\hat{V}_j(X_j)$ by $\hat{V}_j(X_j) + \epsilon_j$, where ϵ_j is a pure noise term. If continuation is indicated then ϵ_j is normally distributed with mean 0 and variance $1/N_1$ (the standard deviation is zero at points of exercise). They replace $\mathbb{E}[\hat{V}_{j+1}(X_{j+1})|X_j]$ by $\mathbb{E}[\hat{V}_{j+1}(X_{j+1})|X_j] + \epsilon_j^{'}$, where $\epsilon_j^{'}$ is normally distributed with mean 0 and variance $1/N_2$. Including these noise terms in (4.18) they get:

$$\hat{\pi}_j = \hat{\pi}_{j-1} + \hat{V}_j(X_j) + \epsilon_j - \hat{V}_{j-1}(X_{j-1}) - \epsilon_{j-1} - l_{j-1}\left(\mathbb{E}[\hat{V}_j(X_j) - \hat{V}_{j-1}(X_{j-1})|X_j] + \epsilon_j^{'} - \epsilon_{j-1}^{'}\right). \tag{4.22}$$

Then by induction, they write

$$\hat{\pi}_j = \pi_j + \tilde{\epsilon},$$

where $\tilde{\epsilon}$ is a sum of mean zero noise terms. So in effect, their algorithm generates the noisy estimate given by:

$$\max_{j=0,\ldots,L}(h_j(X_j) - \pi_j - \tilde{\epsilon}).$$

Andersen and Broadie then show that by using their algorithm's noisy estimate above, they still arrive a valid upper bound, which will be biased high for finite samples N_1 and N_2. The bias will be lower for higher values of N_1 and N_2. Letting m denote the index at which $h_j(X_j) - \pi_j$ assumes its maximum, they get

$$\begin{aligned}\mathbb{E}[\max_{j=0,\ldots,L}(h_j(X_j) - \pi_j - \tilde{\epsilon}_j)] &\geq \mathbb{E}[h_m(X_m) - \pi_m - \tilde{\epsilon}_m] \\ &= \mathbb{E}[h_m(X_m) - \pi_m] \\ &= \mathbb{E}[\max_{j=0,\ldots,L}(h_j(X_j) - \pi_j)],\end{aligned}$$

where the first equality follows from the zero mean of $\tilde{\epsilon}_m$

Based on N_3 independent simulation trials used in calculating the estimate of the upper bound above, denoted by $\hat{\rho}_0 = \mathbb{E}[\max_{j=0,\ldots,L}(h_j(X_j) - \pi_j - \tilde{\epsilon}_j)]$, and N independent simulation trial used in calculating the lower bound $\hat{V}_0$, Andersen and Broadie give an interval for which the $100(1-\alpha)\%$-probability confidence interval for the price of the Bermudan option V_0 must be tighter than:

$$\left[\hat{V}_0 - z_{1-\alpha/2}\frac{\hat{s}_V}{\sqrt{N}}, \hat{V}_0 + \hat{\rho}_0 + z_{1-\alpha/2}\sqrt{\frac{\hat{s}_V^2}{N} + \frac{\hat{s}_\rho^2}{N_3}}\right],$$

where $\hat{s}_V$ is the sample standard deviation of $\hat{V}_0$ and $\hat{s}_\rho$ is the sample standard deviation of $\hat{\rho}$. z_x denotes the xth percentile of a standard Gaussian distribution. Because $\hat{V}_0$ is biased low, i.e. $\mathbb{E}[\hat{V}_0] \leq V_0$, and $\hat{V}_0 + \hat{\rho}_0$ is biased high, this confidence interval is conservative, but the computational effort and precision associated with $\hat{V}_0$ and $\hat{\rho}_0$ can be set separately through the choices of N and N_3. Typically the time required to compute $\hat{V}_0$ is less that required to compute $\hat{\rho}_0$. Taking

26. In the case of perpetual options, when $L \to \infty$, while the interval between exercise times remains constant, Tsitsiklis and Van Roy [6] show that the approximation error, $|\hat{V}_j - V_j|$, can be bounded above by a constant, independent of the number of exercise periods.

this into account it is possible to reduce the standard errors appearing in the confidence interval given above. The spread between the estimated lower and upper bounds gives an indication of whether additional computational effort should be dedicated to improving the exercise policy – for example, by increasing the number of basis functions. Based on their computational results in [14], Andersen and Broadie suggest using the point estimate, $\hat{V}_0 + \frac{1}{2}\hat{\rho}$, as an estimate of the price of the Bermudan option, instead of using the lower or upper bound.

The method of Andersen and Broadie [14] involves for each of the N_3 outer simulations, paths with exercise opportunities at steps $1, \ldots, L$ and for each of the N_1 or N_2 inner simulations, additional paths of up to L steps each. Thus the worst-case computation time required to approximate $\hat{\rho}_0$ is proportional to:

$$d \times N_3 \times \max(N_1, N_2) \times L^2, \tag{4.23}$$

implying that the computation time is linear in d, the dimension of the problem. In practice inner simulations are often stopped very quickly, and so the actual computational time of the method is closer to linear in L. See [14] for numerical results.

The method of Andersen and Broadie [14] is a general method for generating valid confidence intervals for the price of Bermudan options. By approximating the solution of the dual problem associated with the primal problem via an approximation of a lower bound, the upper bound can be used to compute a better price estimate as well as indicating whether more effort is needed to improve the lower bound. In practice, the lower bound is determined very quickly and the upper bound is then used to demonstrate the tightness of the lower bound.

Appendix

Proposition A.1

For $j = 0, \ldots, L-1$, set

$$\tau_j^* = \inf\{j < k \leq L | Q_k(X_k) \leq h_k(X_k)\}.$$

Then the following holds:

$$V_j(x) = \mathbb{E}[h_{\tau_j^*}(X_{\tau_j^*})|X_j = x], \qquad j = 0, \ldots, L-1,$$

Furthermore, we have

$$V_0 = \mathbb{E}[h_{\tau^*}(X_{\tau^*})],$$

where $\tau^* = \tau_0^*$.

Proof

Kohler [9] proves Proposition A.1 by induction: For $j = L$ we have

$$\tau_L^* = L$$

and any $\tau \in \mathcal{T}_L$ satisfies

$$\tau = L$$

So in this case we have

$$\begin{aligned} V_L(x) &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_L} \mathbb{E}[h_\tau(X_\tau)|X_L = x] = \mathbb{E}[h_L(X_L)|X_L = x] \\ &= \mathbb{E}[h_{\tau_L^*}(X_{\tau_L^*})] \end{aligned}$$

For $j = 0, \ldots L-1$ assume that for $j < k \leq L$ we have

$$V_k(x) = \mathbb{E}[h_{\tau_k^*}(X_{\tau_k^*})|X_k = x].$$

Let $\tau \in \mathcal{T}_j$ be an arbitrary stopping time. Then

$$\begin{aligned} h_\tau(X_\tau) &= h_\tau \mathbb{1}_{\{\tau = j\}} + h_\tau(X_\tau)\mathbb{1}_{\{\tau > j\}} \\ &= h_j(X_j)\mathbb{1}_{\{\tau = j\}} + h_{\max\{\tau, t+1\}}(X_{\max\{\tau, t+1\}})\mathbb{1}_{\{\tau > j\}}. \end{aligned}$$

Since $\mathbb{1}_{\{\tau = j\}}$ and $\mathbb{1}_{\{\tau > j\}}$ are $\mathcal{F}_j$-measurable, we have

$$\begin{aligned} \mathbb{E}[h_\tau(X_\tau)|X_j] &= \mathbb{E}[h_j(X_j)\mathbb{1}_{\{\tau = j\}}|X_j] + \mathbb{E}[h_{\max\{\tau, t+1\}}(X_{\max\{\tau, t+1\}})\mathbb{1}_{\{\tau > j\}}|X_j] \\ &= h_j(X_j)\mathbb{1}_{\{\tau = j\}} + \mathbb{1}_{\{\tau > j\}}\mathbb{E}[h_{\max\{\tau, j+1\}}(X_{\max\{\tau, j+1\}})|X_j]. \end{aligned}$$

Now since $\max\{\tau, j+1\} \in \mathcal{T}_{j+1}$, we have that

$$\mathbb{E}[h_{\max\{\tau, j+1\}}(X_{\max\{\tau, j+1\}})|X_j] = \mathbb{E}[\mathbb{E}[h_{\max\{\tau, j+1\}}(X_{\max\{\tau, t+1\}})|X_{j+1}]|X_j],$$

and since $\mathbb{E}[h_{\max\{\tau, j+1\}}(X_{\max\{\tau, t+1\}})|X_{j+1}] \leq V_{j+1}(X_{j+1})$, we have that

$$\begin{aligned} \mathbb{E}[h_\tau(X_\tau)|X_j] &\leq h_j(X_j)\mathbb{1}_{\{\tau = j\}} + \mathbb{1}_{\{\tau > j\}}\mathbb{E}[V_{j+1}(X_{j+1})|X_j] \\ &= \max\{h_j(X_j), \mathbb{E}[V_{j+1}(X_{j+1})|X_j]\}. \end{aligned}$$

By definition of τ_j^*, we have that on $\{\tau_j^* > j\}$

$$\max\{\tau_j^*, j+1\} = \tau_{j+1}^*.$$

Using this and the induction hypothesis we have that

$$\begin{aligned}
\mathbb{E}[h_{\tau_j^*}(X_{\tau_j^*})|X_j] &= h_j(X_j)\mathbb{1}_{\{\tau_j^*=j\}} + \mathbb{1}_{\{\tau_j^*>j\}}\mathbb{E}[h_{\max\{\tau_j^*,j+1\}}(X_{\max\{\tau_j^*,j+1\}})|X_j] \\
&= h_j(X_j)\mathbb{1}_{\{\tau_j^*=j\}} + \mathbb{1}_{\{\tau_j^*>j\}}\mathbb{E}[\mathbb{E}[h_{\tau_{j+1}^*}(X_{\tau_{j+1}^*})|X_{j+1}]|X_j] \\
&= h_j(X_j)\mathbb{1}_{\{\tau_j^*=j\}} + \mathbb{1}_{\{\tau_j^*>j\}}\mathbb{E}[V_{j+1}(X_{j+1})|X_j].
\end{aligned}$$

We next show that

$$Q_j(X_j) = \mathbb{E}[V_{j+1}(X_{j+1})|X_j]. \tag{A.1}$$

By the induction hypothesis, and because $\tau_{j+1} \in \mathscr{T}_{j+1}$ we have

$$\begin{aligned}
\mathbb{E}[V_{j+1}(X_{j+1})|X_j] &= \mathbb{E}[\mathbb{E}[h_{\tau_{j+1}^*}(X_{\tau_j^*})|X_{j+1}]|X_j] = \mathbb{E}[h_{\tau_{j+1}^*}(X_{\tau_j^*})|X_j] \\
&\leq \operatorname*{ess\,sup}_{\tau\in\mathscr{T}_{j+1}} \mathbb{E}[h_\tau(X_\tau)|X_j] = Q_j(X_j).
\end{aligned}$$

Furthermore the definition of V_{j+1} implies that

$$\begin{aligned}
\mathbb{E}[V_{j+1}(X_{j+1})|X_j] &= \mathbb{E}[\operatorname*{ess\,sup}_{\tau\in\mathscr{T}_{j+1}} \mathbb{E}[h_\tau(X_\tau)|X_{j+1}]|X_j] \\
&\geq \operatorname*{ess\,sup}_{\tau\in\mathscr{T}_{j+1}} \mathbb{E}[\mathbb{E}[h_\tau(X_\tau)|X_{j+1}]|X_j] = Q_j(X_j).
\end{aligned}$$

Therefore we have that $\mathbb{E}[V_{j+1}(X_{j+1})|X_j] = Q_j(X_j)$. Using this we have that

$$\begin{aligned}
\mathbb{E}[h_{\tau_j^*}(X_{\tau_j^*})|X_j] &= h_j(X_j)\mathbb{1}_{\{\tau_j^*=j\}} + \mathbb{1}_{\{\tau_j^*>j\}}Q_j(X_j) \\
&= \max\{h_j(X_j), Q_j(X_j)\}.
\end{aligned}$$

Summarizing the above results, we have

$$\begin{aligned}
V_j(x) &= \operatorname*{ess\,sup}_{\tau\in\mathscr{T}_j} \mathbb{E}[h_\tau(X_\tau)|X_j] \leq \max\{h_j(x), \mathbb{E}[V_{j+1}(X_{j+1})|X_j = x]\} \\
&= \max\{h_j(x), Q_j(x)\} = \mathbb{E}[h_{\tau_j^*}(X_{\tau_j^*})|X_j = x],
\end{aligned}$$

which proves

$$V_j(x) = \max\{h_j(x), Q_j(x)\} = \mathbb{E}[h_{\tau_j^*}(X_{\tau_j^*})|X_j = x]. \tag{A.2}$$

Also, we have that

$$\begin{aligned}
V_0 &= \operatorname*{ess\,sup}_{\tau\in\mathscr{T}_0} \mathbb{E}[h_\tau(X_\tau)] \\
&= \operatorname*{ess\,sup}_{\tau\in\mathscr{T}_0} \mathbb{E}[h_0(X_0)\mathbb{1}_{\{\tau=0\}} + h_{\max\{\tau,1\}}(X_{\max\{\tau,1\}})\mathbb{1}_{\{\tau>0\}} \\
&= \mathbb{E}[h_0(X_0)\mathbb{1}_{\{h_0(X_0)\geq Q_0(X_0)\}} + h_{\tau_1^*}(X_{\tau_1^*})\mathbb{1}_{\{h_0(X_0)<Q_0(X_0)\}}] \\
&= \mathbb{E}[h_0(X_0)\mathbb{1}_{\{h_0(X_0)\geq Q_0(X_0)\}} + \mathbb{E}[V_1(X_1)|X_0]\mathbb{1}_{\{h_0(X_0)<Q_0(X_0)\}}] \\
&= \mathbb{E}[h_0(X_0)\mathbb{1}_{\{h_0(X_0)\geq Q_0(X_0)\}} + Q_0(X_0)\mathbb{1}_{\{h_0(X_0)<Q_0(X_0)\}}] \\
&= \mathbb{E}[\max\{h_0(X_0), Q_0(X_0)\}] = \mathbb{E}[h_{\tau^*}(X_{\tau^*})]. \quad \square
\end{aligned}$$

Proposition A.2

For $j = 0, \ldots, L-1$ the following relations hold:

$$Q_j(x) = \mathbb{E}[h_{\tau^*_{j+1}}(X_{\tau^*_{j+1}})|X_j = x]$$

$$Q_j(x) = \mathbb{E}[\max\{h_{j+1}(X_{j+1}), Q_{j+1}(X_{j+1})|X_j = x]$$

Proof

For the first relation, we have from Equation A.1 that $Q_j(x) = \mathbb{E}[V_{j+1}(X_{j+1})|X_j = x]$, so

$$\begin{aligned} Q_j(X_j) &= \mathbb{E}[\mathbb{E}[h_{\tau^*_{j+1}}(X_{\tau^*_{j+1}})|X_{j+1}]|X_j] \\ &= \mathbb{E}[h_{\tau^*_{j+1}}(X_{\tau^*_{j+1}})|X_j] \end{aligned}$$

For the second relation, because

$$\begin{aligned} h_{\tau^*_{j+1}}(X_{\tau^*_{j+1}}) &= h_{j+1}(X_{j+1})\mathbb{1}_{\{\tau^*_{j+1}=j+1\}} + h_{\tau^*_{j+1}}(X_{\tau^*_{j+1}})\mathbb{1}_{\{\tau^*_{j+1}>j+1\}} \\ &= h_{j+1}(X_{j+1})\mathbb{1}_{\{h_{j+1}(X_{j+1})\geq Q_{j+1}(X_{j+1})\}} + h_{\tau^*_{j+1}}(X_{\tau^*_{j+1}})\mathbb{1}_{\{h_{j+1}(X_{j+1})<Q_{j+1}(X_{j+1})\}}. \end{aligned}$$

Using the first relation we can conclude that

$$\begin{aligned} Q_j(X_j) &= \mathbb{E}[h_{j+1}(X_{j+1})\mathbb{1}_{\{h_{j+1}(X_{j+1})\geq Q_{j+1}(X_{j+1})\}} + h_{\tau^*_{j+1}}(X_{\tau^*_{j+1}})\mathbb{1}_{\{h_{j+1}(X_{j+1})<Q_{j+1}(X_{j+1})\}}|X_j] \\ &= \mathbb{E}[\mathbb{E}[h_{j+1}(X_{j+1})\mathbb{1}_{\{h_{j+1}(X_{j+1})\geq Q_{j+1}(X_{j+1})\}} + h_{\tau^*_{j+1}}(X_{\tau^*_{j+1}})\mathbb{1}_{\{h_{j+1}(X_{j+1})<Q_{j+1}(X_{j+1})\}}|X_{j+1}|X_j] \\ &= \mathbb{E}[h_{j+1}(X_{j+1})\mathbb{1}_{\{h_{j+1}(X_{j+1})\geq Q_{j+1}(X_{j+1})\}} + \mathbb{E}[h_{\tau^*_{j+1}}(X_{\tau^*_{j+1}})|X_{j+1}]\mathbb{1}_{\{h_{j+1}(X_{j+1})<Q_{j+1}(X_{j+1})\}}|X_j] \\ &= \mathbb{E}[h_{j+1}(X_{j+1})\mathbb{1}_{\{h_{j+1}(X_{j+1})\geq Q_{j+1}(X_{j+1})\}} + Q_{j+1}(X_{j+1})\mathbb{1}_{\{h_{j+1}(X_{j+1})<Q_{j+1}(X_{j+1})\}}|X_j] \\ &= \mathbb{E}[\max\{h_{j+1}(X_{j+1}), Q_{j+1}(X_{j+1})\}|X_j]. \end{aligned}$$

Proposition A.3

The sequence $(V_j)_{j=0,\ldots,L}$ is a $\mathcal{Q}$-supermartingale. It is the smallest $\mathcal{Q}$-supermartingale that dominates the sequence $(h_j)_{j=0\ldots,L}$.

Proof

We have:

$$V_{j-1}(x) = \max\{h_{j-1}(x), \mathbb{E}[V_j(X_j)|\mathcal{F}_{j-1}]\}.$$

It follows that $(V_j)_{j=0,\ldots,L}$ is a $\mathcal{Q}$-supermartingale dominating $(h_j)_{j=0\ldots,L}$. Let us now consider a supermartingale $(U_j)_{j=0,\ldots,L}$ that dominates $(h_j)_{j=0,\ldots,L}$. Then $U_L \geq V_L$ and if $U_j \geq V_j$ we have

$$U_{j-1} \geq \mathbb{E}[U_j|\mathcal{F}_{j-1}] \geq \mathbb{E}[V_j|\mathcal{F}_{j-1}]$$

whence

$$U_{j-1} \geq \max(h_{j-1}, \mathbb{E}[V_j|\mathcal{F}_{j-1}]) = V_{j-1}$$

A backward induction proves the assertion that (U_j) dominates (V_j). □

THEOREM A.4

Assume that $e_m(X_j)$ is total in $L^2(\sigma(X_j))$ for $j = 1, \ldots, L$, then we have

$$\lim_{m \to +\infty} \mathbb{E}[h_{\tau_j^{[m]}}(X_{\tau_j^{[m]}})|\mathcal{F}_j] = \mathbb{E}[h_{\tau_j}(X_{\tau_j})|\mathcal{F}_j],$$

for $j = 1, \ldots, L$ in $L^2(\sigma(X_j))$.

PROOF

We proceed by induction on j. The result is true for $j = L$. Let us prove that if it holds for $j+1$, it is true for j ($j \leq L-1$). Since

$$h_{\tau_j^{[m]}}(X_{\tau_j^{[m]}}) = h_j(X_j)\mathbb{1}_{\{h_j(X_j) \geq \beta_j^m \cdot e^m(X_j)\}} + h_{\tau_{j+1}^{[m]}}(X_{\tau_{j+1}^{[m]}})\mathbb{1}_{\{h_j(X_j) < \beta_j^m \cdot e^m(X_j)\}}$$

for $j < L-1$, we have

$$\begin{aligned}\mathbb{E}[h_{\tau_j^{[m]}}(X_{\tau_j^{[m]}}) - h_{\tau_j}(X_{\tau_j})|\mathcal{F}_j] &= (h_j(X_j) - \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j])(\mathbb{1}_{\{h_j(X_j) \geq \beta_j^m \cdot e^m(X_j)\}} - \mathbb{1}_{\{h_j(X_j) \geq \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j]\}}) \\ &\quad + \mathbb{E}[h_{\tau_{j+1}^{[m]}}(X_{\tau_{j+1}^{[m]}}) - h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j]\mathbb{1}_{\{h_j(X_j) < \beta_j^m \cdot e^m(X_j)\}}.\end{aligned}$$

By assumption, the second term on the right side of the equality converges to 0 and we just have to prove that B_j^m defined by:

$$B_j^m = (h_j(X_j) - \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j])(\mathbb{1}_{\{h_j(X_j) \geq \beta_j^m \cdot e^m(X_j)\}} - \mathbb{1}_{\{h_j(X_j) \geq \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j]\}}),$$

converges to 0 in $L^2(\sigma(X_j))$. Observe that

$$\begin{aligned}||B_j^m||_2 &= ||h_j(X_j) - \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j]||\mathbb{1}_{\{\mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j] > h_j(X_j) \geq \beta_j^m \cdot e^m(X_j)\}} - \mathbb{1}_{\{\beta_j^m \cdot e^m(X_j) > h_j(X_j) \geq \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j]\}} \\ &\leq ||h_j(X_j) - \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j]||\mathbb{1}_{\{|h_j(X_j) - \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j)]| \leq |\beta_j^m \cdot e^m(X_j) - \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j]|\}} \\ &\leq ||\beta_j^m \cdot e^m(X_j) - \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j]|| \\ &\leq ||\beta_j^m \cdot e^m(X_j) - \Pi_j^m(\mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j])|| + ||\Pi_j^m(\mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j]) - \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j]||.\end{aligned}$$

But

$$\beta_j^m \cdot e^m(X_j) = \Pi_j^m h_{\tau_{j+1}^{[m]}}(X\tau_{j+1}^{[m]}) = \Pi_j^{[m]}\mathbb{E}[h_{\tau_{j+1}^{[m]}}(X_{\tau_{j+1}^{[m]}})|\mathcal{F}_j],$$

since Π_j^m is the orthogonal projection on a subspace of the space of $\mathcal{F}_j$-measurable random variables. Consequently

$$||B_j^m||_2 \leq= ||\mathbb{E}[h_{\tau_{j+1}^{[m]}}(X_{\tau_{j+1}^{[m]}})|\mathcal{F}_j] - \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j]||_2 + ||\Pi_j^m\mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j] - \mathbb{E}[h_{\tau_{j+1}}(X_{\tau_{j+1}})|\mathcal{F}_j]||_2.$$

The first term of the right side of the inequality tends to 0 by the induction hypothesis and the second one by the assumption that for $j = 1, \ldots, L-1$, the sequence $(e_k(X_j))_{k \geq 1}$ is total in $L^2(\sigma(X_j))$.

THEOREM A.5

Assume that the feature vectors satisfy assumptions 1. and 2. of §2.4 and that for $j = 1, \ldots L-1$, $\mathscr{P}(\beta_j \cdot e(X_j) = h_j(X_j)) = 0$. Then $V_0^{m,N}(X_0)$ converges almost surely to $V_0^{[m]}(X_0)$ as N goes to infinity. We also have almost sure convergence of $\frac{1}{N}\sum_{n=1}^{N} h_{\tau_j^{n,m,N}}^{(n)}(X_{\tau_j^{n,m,N}}^{(n)})$ towards $\mathbb{E}[h_{\tau_j^{[m]}}(X_{\tau_j^{[m]}})]$ as N goes to infinity, for $j = 1, \ldots, L$.

Proof

The proof is based on Lemma A.6 and Lemma A.7 and is similar to the proof of Lemma Lemma A.7, therefore, we omit it. The lemmas make use of the following notation:

Notation

Given a parameter $b^m = (b_1^m, \dots, b_{L-1}^m)$ in $\Re^m \times \dots \times \Re^m$ and deterministic vectors $k = (k_1, \dots, k_L) \in \Re^L$ and $x = (x_1, \dots, x_L) \in \Re^d \times \dots \times \Re^d$, we define a vector $F = (F_1, \dots, F_L)$ by:

$$\begin{aligned} F_L(b^m, k, x) &= k_L \\ F_j(b^m, k, x) &= k_j \mathbb{1}_{\{k_j \geq b_j^m \cdot e^m(x_j)\}} + F_{j+1}(b^m, k, x) \mathbb{1}_{\{k_j < b_j^m \cdot e^m(x_j)\}}, \\ \text{for} \quad j &= 1, \dots, L-1. \end{aligned}$$

We have

$$F_j(b^m, k, x) = k_j \mathbb{1}_{B_j^c} + \sum_{i=j+1}^{L-1} k_i \mathbb{1}_{B_j \dots B_{i-1} B_i^c} + k_L \mathbb{1}_{B_j \dots B_{L-1}},$$

with

$$B_j = \{k_j < b_j^m \cdot e^m(X_j)\}.$$

We remark that $F_j(b^m, k, x)$ does not depend on $(b_1^m, \dots, b_{j-1}^m)$ and that we have

$$\begin{aligned} F_j(b^m, h_j, X_j) &= h_{\tau_j^{[m]}}(X_{\tau_j^{[m]}}) \\ F_j(b^{(m,N)}, h_j^{(n)}, X_j^{(n)}) &= h^{(n)}_{\tau_j^{n,m,N}}(X^{(n)}_{\tau_j^{n,m,N}}). \end{aligned}$$

For $j = 2, \dots, L$ we denote by G_j the vector valued function

$$G_j(b^m, k, x) = F_j(b^m, k, x) e^m(x_{j-1}),$$

and we define the functions ϕ_j and ψ_j by

$$\begin{aligned} \phi_j(a^m) &= \mathbb{E} F_j(b^m, h_j(X_j), X_j) \\ \psi_j(a^m) &= \mathbb{E} G_j(b^m, h_j(X_j), X_j) \end{aligned}$$

Observe that with this notation, we have

$$\beta_j^m = (A_j^m)^{-1} \psi_{j+1}(\beta^m),$$

where $\beta^m = (\beta_1^m, \dots, \beta_{L-1}^m)$, and similarly, for $j = 1, \dots L-1$,

$$\beta_j^{(m,N)} = (A_j^{(m,N)})^{-1} \frac{1}{N} \sum_{n=1}^{N} G_{j+1}(\beta^{(m,N)}, h_j^{(n)}, X_j^{(n)}),$$

where $\beta^m = (\beta_1^{(m,N)}, \dots, \beta_{L-1}^{(m,N)})$.

Lemma A.6

For $j = 1, \dots, L-1$, we have:

$$|F_j(b, h_j(X_j), X_j) - F_j(c, h_j(X_j), X_j)| \leq \left(\sum_{i=j}^{L} |h_i(X_i)| \right) \left(\sum_{i=j}^{L-1} \mathbb{1}_{\{|h_i(X_i) - c_i \cdot e(X_i)| \leq |b_i - c_i||e(X_i)|\}} \right),$$

using the notion above.

PROOF

Let $B_j = \{h_j(X_j) < b_j \cdot e(X_j)\}$ and $\tilde{B}_j = \{h_j(X_j) < c_j \cdot e(X_j)\}$. We have:

$$F_j(b, h_j, X_j) - F_j(c, h_j(X_j, X_j) = h_j(X_j)(\mathbb{1}_{B_j} - \mathbb{1}_{\tilde{B}_j}) + \sum_{i=j+1}^{L-1} h_i(X_i(\mathbb{1}_{B_j \dots B_{i-1} B_i^c} - \mathbb{1}_{\tilde{B}_j \dots \tilde{B}_{i-1} \tilde{B}_{L-1}^c}) + h_L(X_L)(\mathbb{1}_{B_j^c \dots B_{L-1}^c} - \mathbb{1}_{\tilde{B}_j^c \dots \tilde{B}_{L-1}^c}).$$

But

$$\begin{aligned} |\mathbb{1}_{B_j} - \mathbb{1}_{\tilde{B}_j}| &= \mathbb{1}_{\{b_j \cdot e(X_j) \le h_j(X_j) < c_j \cdot e(X_j)\}} + \mathbb{1}_{\{c_j \cdot e(X_j) \le h_j(X_j) < b_j \cdot e(X_j)\}} \\ &\le \mathbb{1}_{\{|h_j(X_j) - c_j \cdot e(X_j)| \le |b_j - c_j||e(X_j)|\}} \end{aligned}$$

Moreover

$$\begin{aligned} |\mathbb{1}_{B_j \dots B_{i-1} B_i^c} - \mathbb{1}_{\tilde{B}_j \dots \tilde{B}_{i-1} \tilde{B}_i^c}| &\le \sum_{k=1}^{i-1} |\mathbb{1}_{B_k} - \mathbb{1}_{\tilde{B}_k}| + |\mathbb{1}_{B_i^c} - \mathbb{1}_{\tilde{B}_i^c}| \\ &= \sum_{k=j}^{i} |\mathbb{1}_{B_k} - \mathbb{1}_{\tilde{B}_k}|, \end{aligned}$$

this gives

$$|F_j(b, h_j(X_j), X_j) - F_j(c, h_j(X_j), X_j)| \le \sum_{i=j}^{L} |h_i(X_i)| \sum_{i=j}^{L-1} |\mathbb{1}_{B_i} - \mathbb{1}_{\tilde{B}_i}|.$$

Combining these two inequalities, we obtain the result.

LEMMA A.7

Assume that for $j = 1, \dots, L-1$, $\mathscr{P}(\beta_j^m \cdot e(X_j) = h_j(X_j)) = 0$ them $\beta_j^{(m,N)}$ converges almost surely to β_j^m.

PROOF

We proceed by induction on j. For $j = L-1$, the result is a direct consequence of the law of large numbers. Now, assume that the result is true for $i = j, \dots, L-1$. We want to prove that it is true for $j-1$. We have

$$\beta_{j-1}^{(m,N)} = (A_{j-1}^{(m,N)})^{-1} \frac{1}{N} \sum_{n=1}^{N} G_j(\beta^{(m,N)}, h_j^{(n)}(X_j^{(n)}), X_j^{(n)}).$$

By the law of large numbers, we know that $A_{j-1}^{(m,N)}$ converges almost surely to A_{j-1}^m and it remains to prove that $\frac{1}{N} \sum_{n=1}^{N} G_j(\beta^{(m,N)}, h_j^{(n)}(X_j^{(n)}), X_j^{(n)})$ converges to $\psi_j(\beta^m)$. From the law of large numbers, we have the convergence of $\frac{1}{N} \sum_{n=1}^{N} G_j(\beta^m, h_j^{(n)}(X_j^{(n)}), X_j^{(n)})$ to $\psi(\beta^m)$ and it suffices to prove that:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left(G_j(\beta^{(m,N)}, h_j^{(n)}(X_j^{(n)}), X_j^{(n)}) - G_j(\beta^m, h_j^{(n)}(X_j^{(n)}), X_j^{(n)}) \right) = 0.$$

We note $G_N = \frac{1}{N}\sum_{n=1}^{N}\Big(G_j(\beta^{(m,N)}, h_j^{(n)}(X_j^{(n)}), X_j^{(n)}) - G_j(\beta^m, h_j^{(n)}(X_j^{(n)}), X_j^{(n)})\Big)$. We have:

$$\begin{aligned}|G_N| &\leq \tfrac{1}{N}\sum_{n=1}^{N}|e(X_{j-1}^{(n)})||F_j(\beta^{(m,N)}, h_j^{(n)}(X_j^{(n)}), X_j^{(n)}) - G_j(\beta^m, h_j^{(n)}(X_j^{(n)}), X_j^{(n)})| \\ &\leq \tfrac{1}{N}\sum_{n=1}^{N}|e(X_{j-1}^{(n)})|\sum_{i=j}^{L}|h_i^{(n)}(X_i^{(n)})|\sum_{i=j}^{L-1}\mathbb{1}_{\{|h_i^{(n)}(X_i^{(n)})-\beta_i^m\cdot e(X_i^{(n)})|\leq|\beta_i^{(m,N)}-\beta_i^m||e(X_i^{(n)})|\}}.\end{aligned}$$

Since, for $i = j, \ldots, L-1$, $\beta_i^{(m,N)}$ converges almost surely to β_i^m, we have for each $\epsilon > 0$:

$$\begin{aligned}\limsup|G_N| &\leq \limsup\tfrac{1}{N}\sum_{n=1}^{N}|e(X_{j-1}^{(n)})|\sum_{i=j}^{L}|h_i^{(n)}(X_i^{(n)})|\sum_{i=j}^{L-1}\mathbb{1}_{\{|h_i(X_i)-\beta_i^m\cdot e(X_i)|\leq\epsilon|e(X_i)|\}} \\ &= \mathbb{E}|e(X_{j-1})|\sum_{i=j}^{L}|h_i(X_i)|\mathbb{1}_{\{|h_i(X_i)-\beta_i\cdot e(X_i)|\leq\epsilon|e(X_i)|\}},\end{aligned}$$

where the last equality follows for the law of large numbers. Letting ϵ go to 0, we obtain the convergence to 0, since for $j = 1, \ldots, L$, $\mathscr{P}(\beta_i^m \cdot e(X_j) = h_j(X_j)) = 0$.

Proposition A.8

Let $\mathscr{M}$ be the set of all martingales $M_0, \ldots, M_L$ with $M_0 = 0$. Then

$$V_0 = \inf_{M\in\mathscr{M}}\mathbb{E}[\max_{j=0,\ldots L}(h_j(X_j) - M_j)] = \mathbb{E}[\max_{j=0,\ldots L}(h_j(X_j) - M_j^*)],$$

where

$$M_j^* = \sum_{s=1}^{j}(\max\{h_s(X_s), Q_s(X_s)\} - \mathbb{E}[\max\{h_s(X_s), Q_s(X_s)\}|X_{s-1}].$$

Proof

We first prove:

$$\max_{j=0,\ldots,L}\{h_j(X_j) - \sum_{s=1}^{j}(\max\{h_s(X_s), Q_s(X_s)\} - \mathbb{E}[\max\{h_s(X_s), Q_s(X_s)\}|X_{s-1}])\} = \max\{h_0(X_0), Q_0(X_0)\}. \tag{A.3}$$

To do this we observe that we have by (A.1) and (A.2), we have:

$$\begin{aligned}&\max_{j=0,\ldots,L}\{h_j(X_j) - \sum_{s=1}^{j}(\max\{h_s(X_s), Q_s(X_s)\} - \mathbb{E}[\max\{h_s(X_s), Q_s(X_s)\}|X_{s-1}])\} \\ = \;&\max_{j=0,\ldots,L}\{h_j(X_j) - \sum_{s=1}^{j}(\max\{h_s(X_s), Q_s(X_s)\} - Q_{s-1}(X_{s-1})\}.\end{aligned}$$

For $j = 1, \ldots, L$, we have that:

$$\begin{aligned} & h_j(X_j) - \sum_{s=1}^{j} (\max\{h_s(X_s), Q_s(X_s)\} - Q_{s-1}(X_{s-1})) \\ \leq \; & h_j(X_j) - \sum_{s=1}^{j-1} (Q_s(X_s) - Q_{s-1}(X_{s-1})) - (h_j(X_j) - Q_{j-1}(X_{j-1})) \\ = \; & Q_0(X_0). \end{aligned}$$

Furthermore for $j = 0$ we get:

$$h_j(X_j) - \sum_{s=1}^{j} (\max\{h_s(X_s), Q_s(X_s)\} - Q_{s-1}(X_{s-1})) = h_0(X_0),$$

which shows that

$$\max_{j=0,\ldots,L} \{h_j(X_j) - \sum_{s=1}^{j} (\max\{h_s(X_s), Q_s(X_s)\} - Q_{s-1}(X_{s-1}))\} \leq \max\{h_0(X_0), Q_0(X_0)\}.$$

But for $j = \tau^*$ we get in the case of $Q_0(X_0) > h_0(X_0)$ by definition of τ^*:

$$\begin{aligned} & h_{\tau^*}(X_{\tau^*}) - \sum_{s=1}^{\tau^*} (\max\{h_s(X_s), Q_s(X_s)\} - Q_{s-1}(X_{s-1})) \\ = \; & h_{\tau^*}(X_{\tau^*}) - \sum_{s=1}^{\tau^*-1} (Q_s(X_s) - Q_{s-1}(X_{s-1})) - (h_{\tau^*}(X_{\tau^*}) - Q_{\tau^*-1}(X_{\tau^*-1})) \\ = \; & Q_0(X_0), \end{aligned}$$

and in the case of $Q_0(X_0) \leq h_0(X_0)$ (which implies $\tau^* = 0$) we have:

$$h_{\tau^*}(X_{\tau^*}) - \sum_{s=1}^{\tau^*} (\max\{h_s(X_s), Q_s(X_s)\} - Q_{s-1}(X_{s-1})) = h_0(X_0).$$

This proves (A.3).
As shows at the end of Proposition 1:

$$V_0 = \mathbb{E}[\max\{h_0(X_0), Q_0(X_0)\}].$$

Thus we have, using this and (A.3), that:

$$\mathbb{E}[\max_{j=0,\ldots L} (h_j(X_j) - M_j^*)] = \mathbb{E}[\max\{h_0(X_0), Q_0(X_0)\}] = V_0.$$

Thus it suffices to show: For any martingale $M_0, \ldots, M_L$ with $M_0 = 0$ we have:

$$\mathbb{E}[\max_{j=0,\ldots L} (h_j(X_j) - M_j)] \geq \sup_{\tau \in \mathscr{T}} \mathbb{E}[h_\tau(X_\tau)] = V_0.$$

But this follows from the Optional Sampling Theorem, because $M_0, \ldots, M_L$ is a martingale with $M_0 = 0$ and τ is a stopping time we know:

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0] = 0,$$

and hence

$$\mathbb{E}[h_\tau(X_\tau)] = \mathbb{E}[h_\tau(X_\tau) - M_\tau] \leq \mathbb{E}[\max_{j=0,\ldots,L} \{h_j(X_j) - M_j\}].$$

Proposition A.9

Every supermartingale $(\pi_n)_{0\leq n\leq N}$ has the unique following decomposition:

$$\pi_n = M_n - A_n$$

where (M_n) is a martingale and (A_n) is a non-decreasing, predictable process, null at 0.

Proof

It is clearly seen that the only solution for $n = 0$ is $M_0 = \pi_0$ and $A_0 = 0$. Then we must have

$$\pi_{j+1} - \pi_j = M_{j+1} - M_j - (A_{j+1} - A_m).$$

so that, conditioning both sides with respect to $\mathcal{F}_j$ and using the properties of M and A

$$-(A_{j+1} - A_j) = \mathbb{E}[\pi_{j+1}|\mathcal{F}_j] - \pi_j$$

and

$$M_{j+1} - M_j = \pi_{j+1} - \mathbb{E}[\pi_{j+1}|\mathcal{F}_j].$$

(M_j) and (A_j) are entirely determined using the previous equations and we see the (M_j) is a martingale and (A_j) is a predictable non-decreasing process (because (π_j) is a supermartingale).

□

Proposition A.10

$$V_0(X_0) = \inf_M \mathbb{E}[\max_{j=0,\ldots,L}\{h_j(X_j) - M_j\}], \tag{A.4}$$

where $(M_j)_{j=0,1,\ldots,L}$ is a martingale. The infimum is attained by by taking $M_j = M_j^*$, where M_j^* is the martingale part of the Doob-Meyer decomposition of the option price process:

$$V_j(X_j) = M_j^* - A_j$$

Proof

Since the *Snell envelope* process, $V_j(X_j) = \operatorname{ess\,sup}_{\tau\in\mathcal{T}_j} \mathbb{E}\left[h_\tau(X_\tau)\middle|X_j\right]$, is a supermartingale it has a Doob-Meyer decomposition given in proposition . We have from the definition that

$$\begin{aligned} V_0(X_0) &= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0} \mathbb{E}[h_\tau(X_\tau)] \\ &= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0} \mathbb{E}[h_\tau(X_\tau) - M_\tau] \\ &\leq \mathbb{E}[\max_{j=0,\ldots,L}(h_j(X_j) - M_j)]. \end{aligned}$$

The inequality follows from the fact that V_j is dominated by the martingale
$z_j = \mathbb{E}[\max_{k=0,\ldots,j}|h_k(X_k)||X_j]$. Taking the infimum over all M proves that $V_0(X_0)$ is bounded

above by the right-hand side of (A.4). On the other hand, since $h_j(X_j) \leq V_j(X_j) = M_j^* - A_j$,

$$\begin{aligned}
\inf_M \mathbb{E}[\max_{j=0,\ldots,L}\{h_j(X_j) - M_j\}] &\leq \mathbb{E}[\max_{j=0,\ldots,L}\{h_j(X_j) - M_j^*\}] + V_0(X_0) \\
&\leq \mathbb{E}[\max_{j=0,\ldots,L}\{V_j(X_j) - M_j^*\}] + V_0(X_0) \\
&= \mathbb{E}[\max_{j=0,\ldots,L}\{-A_j\}] + V_0(X_0) \\
&= V_0(X_0)
\end{aligned}$$

as claimed □

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