Symplectic Frölicher Spaces of Constant Dimension

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ABSTRACT

Alfred Frölicher introduced the concept of a smooth structure which does not use norms. The spaces with such a structure, named Frölicher spaces or smooth spaces are being studied in recent years.

Frölicher and Kriegl showed that these spaces are objects in a cartesian closed category, which Cherenack denoted by **FRL**. The latter proved that **FRL** is topological over sets and intensively compared Frölicher spaces with differential spaces in the sense of Sikorski. Other comparisons of these spaces with smooth manifolds, diffeological spaces and convenient spaces can be found in Frölicher, Kriegl, Michor, Cap, Teichman and Tore's works.

In this thesis, we introduce the concept of dimension on Frölicher spaces. We define the concepts of Frölicher spaces of constant dimension, Frölicher spaces of class **DS**, pre-Frölicher spaces, pseudomanifolds and then present new results for the symplectic geometry in the constant dimensional case. Examples and applications to mechanics are provided.

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DECLARATION

I declare that this thesis is my own unaided work. It is being submitted for the degree of Doctor of Philosophy in the University of Cape Town. It has not been submitted for any degree or examination in any other university.

Tshidibi Augustin Batubenge.
25th day of July 2004

A

Mon regretté Père
Isidore Malengu Mwabi Mukulu
dont la parole prophétique prend effet ce jour, contre vents et marrées.

Et à

Ma femme bien-aimée
Mélanie-Esther Ngambamunyi wa Kasengele
pour tant de patience, de privations et de prières.

Contents

Chapter 1

Introduction

An intuitive abstract representation of a geometric object, considered together with a differential structure, constitutes a topological construct which is referred to in studying the motion of various phenomena. A differentiable manifold (we will say manifold for short) is known to be used as a classic model for this purpose, that is, a Hausdorff topological space which is locally Euclidean. In Mechanics for instance, which we mainly focus on throughout this thesis, the motion of a particle represented by a physical system is determined at each instant according to its phase, momentum, velocity, etc. which, in turn, are completely described by differentiation.

Note that the progress of research in science have often required that a manifold be locally homeomorphic to a space richer than the Euclidean space, for instance, a Banach space or a locally convex topological space. In fact, the differential structure obtained on a topological manifold by constructing local charts and atlases unfortunately breaks down in many cases. For instance, material related to the gravity regime of the very early universe [34], or generalized models of space-time which should be used in Cosmology possess a lot of singularities and should be modelled on other spaces than manifolds.

Therefore, many generalizations of the concept of manifold were proposed as about four decades ago. Such are for instance the concept of a manifold with boundary, a manifold with corner, a Banach manifold, and in the very early eighties the concept of a manifold modelled on convenient space was introduced by A. Frölicher, A. Kriegl and P. Michor [26], [39]. Nevertheless, it is worth observing that the point in these innovations should not be the modelling space for the manifold structure, whether it was Euclidean, Banach or convenient, but rather the idea of working out a differential geometry on the topological space without thinking of its local similarity with a modelling space. The topological construct together with such an additional differential structure generated by a ring of functions is generally called differential space. In order to face earlier developments in Differential Geometry, a category of modelling objects would be required to be cartesian closed. A recent comparative study by P. Cherenack [17] proved the non-cartesian closedness of the category of differential spaces and the closedness between this category and the category **FRL** of Frölicher spaces. Cherenack also presented the two concepts of tangent structures on a Frölicher space before he devoted his works to the Riemannian structure on these spaces, with applications to Cosmology. (see [16], [17], [18]).

This thesis is devoted to the Symplectic Geometry on Frölicher spaces. Since the whole area is still a matter of investigation, we present the very first steps towards the concepts of tangent structures and of dimensionalization in **FRL**, before we discuss our results about the Symplectic Geometry of particular classes in **FRL**. These are objects of **FRL** whose geometry coincides either with that of differential spaces of constant dimension, or with smooth manifolds. This work is organized as follows.

The first chapter gathers the existing concepts so far defined in the category **FRL**, which is briefly compared to that of diffeological spaces in the sense of J.M. Souriau [66] and convenient spaces in the sense of A. Frölicher, A. Kriegl and P. Michor (see [26], [39]. Finally it emphasizes the link with differential spaces in the sense of Sikorski [54], [34], [60]. The chapter ends with a characterization of the concept of diffeomorphism which will play a key role in the whole work.

The second chapter introduces the concept of tangent structures: the kinematic tangent structure provided by structure curves and the operational tangent structure defined via structure functions.

The third chapter presents the dimensionalization process for Frölicher spaces, following the concept of $\mathcal{M}$-independence and $\mathcal{M}$-basis. It is shown that a local diffeomorphism built from $\mathcal{M}$-independent smooth functions determines the dimension n of the operational tangent space at a given point. Such an integer n is called an $\mathcal{M}$-dimension provided that each $\mathcal{M}$-basis in the germ of smooth functions at this point has a common cardinality equal to n. Then follows a discussion of the concept of Frölicher spaces **of constant dimension**. These are defined as those Frölicher spaces with the same $\mathcal{M}$-dimension at each point. The definition of the concept of **constant dimension** in terms of the so-called **local basis of smooth vector fields** will be very useful as this basis stands for a local chart used to express many geometric concepts in local coordinates when the construct under consideration is a manifold.

In the fourth chapter we deal with the so-called **pre-Frölicher spaces** which are differential spaces whose differential structure generates the Frölicher structure on the same underlying set. The latter will be called the **DS-Frölicher spaces**. Furthermore, we briefly prove some results from our investigations about Frölicher spaces generated by the differential structure, with an additional property of being locally diffeomorphic to Frölicher subspaces of $\mathbb{R}^n$ of constant dimension equal to n. We call them **DS-pseudomanifolds**.

The fifth chapter presents the concept of a symplectic structure on linear Frölicher spaces and in the general case. We shall show that the Poisson formalism of mechanics may be introduced from the symplectic structure or in a more general setting. We finally obtain the usual canonical presentation of symplectic forms on Frölicher spaces.

In order to achieve the main goal of this investigation, we devote the last chapter of this work to the presentation of a geometrical setting for classical mechanics in the language of Frölicher spaces. We deal with Lagrangian and Hamiltonian finite-dimensional systems on **DS**-Frölicher spaces and the transition between them by the Legendre transformation. We shall finally investigate the symplectic geometry and mechanics of the Frölicher space resulting from the gluing of two pseudomanifolds.

Chapter 2

Preliminaries

In this chapter we present basic concepts and the terminology Frölicher spaces. The definition of the concepts will be followed by examples of Frölicher spaces generated by sets of maps. It shall be indicated that the smooth structure due to Alfred Frölicher, called **Frölicher structure**, has some connection with other well-known differential structures such as the differential structure introduced by Sikorski, the convenient structure due to Frölicher, Kriegl and Michor, and the diffeological structure due to Souriau. From the fact that the category **<u>FRL</u>** of Frölicher spaces is cartesian closed, most objects constructed are either initial or final. We then show that this kind of construction of objects leads to a characterization of diffeomorphisms which will play an important role in the rest of the thesis. We compare the Frölicher structure with the smooth structure of manifolds in later chapters.

2.1 Concept of a Frölicher space

In his paper titled *Catégories cartesiennement fermées engendrées par les monoides*, A. Frölicher (see [24, pp. 373-374]) showed that the submonoid $\mathcal{M} := C^{\infty}(\mathbb{R},\mathbb{R})$ of C^{∞}(or smooth) real functions of the monoid $Ens(\mathbb{R},\mathbb{R})$ of all real functions generates a cartesian closed category. Furthermore, he presented together with A. Kriegl, in a more general setting, the definition of the objects of this category which they called **smooth spaces** (see [26]). In subsequent work these spaces were called **Frölicher spaces** [39], [17].

Definition 2.1.1 *A* **Frölicher structure** *on a set M is a pair $(\mathcal{C}_M, \mathcal{F}_M)$ where $\mathcal{C}_M$ and $\mathcal{F}_M$ generate each other by setting*

$$\mathcal{C}_M := \Gamma\mathcal{F}_M = \{c : \mathbb{R} \longrightarrow M;\ f \circ c \in C^{\infty}(\mathbb{R},\mathbb{R}) \ \text{ for all } \ f \in \mathcal{F}_M\} \quad \text{and}$$
$$\mathcal{F}_M := \Phi\mathcal{C}_M = \{f : M \longrightarrow \mathbb{R};\ f \circ c \in C^{\infty}(\mathbb{R},\mathbb{R}) \ \text{ for all } \ c \in \mathcal{C}_M\}.$$

Then $\mathcal{C}_M$ is the set of **structure curves**, *and $\mathcal{F}_M$ is the set of* **structure functions**. *The triple $(M, \mathcal{C}_M, \mathcal{F}_M)$ is called a* **Frölicher space**. *A Frölicher structure $(\Gamma\mathcal{F}_0, \Phi\Gamma\mathcal{F}_0)$ is said to be generated by a set $\mathcal{F}_0 \subset \mathbb{R}^M$ of functions, while $(\Gamma\Phi\mathcal{C}_0, \Phi\mathcal{C}_0)$ is said to be a Frölicher structure generated by a set $\mathcal{C}_0 \subset M^{\mathbb{R}}$ of curves.*

Note that the Frölicher structure was defined in the most general case in the sense that the source of curves is any set S and the range of functions is any set R so that $\mathcal{C}_M \subseteq M^S$ and $\mathcal{F}_M \subseteq R^M$. In [24], only the case $S = R$ was considered by Frölicher. In this work, we deal with the case $S = R = \mathbb{R}$. Hence, from Definition 2.1.1 above we have $\mathcal{F}_M \circ \mathcal{C}_M = \mathcal{M}$.

We shall denote by $\mathcal{C}$ the set $\mathcal{C}_{\mathbb{R}}$ and $\mathcal{F}$ the set $\mathcal{F}_{\mathbb{R}}$. The triple $(\mathbb{R}, \mathcal{C}, \mathcal{F})$ is called the canonical Frölicher space on the real numbers.

2.2 Topology underlying a Frölicher space

From the definition of a Frölicher space, the initial topology is naturally induced on M by all real-valued functions $f : M \longrightarrow \mathbb{R}$. This is the weakest topology, denoted by $\tau_{\mathcal{F}}$, under which these functions are continuous. We observe that the generated Frölicher structure has another topology induced by the structure curves. We shall denote this topology by $\tau_{\mathcal{C}}$.

Note that in their book [26], A. Frölicher and A. Kriegl showed that the subbasis for $\tau_{\mathcal{F}}$ is $\mathcal{U} = \{f^{-1}(0,1)\}_{f \in \mathcal{F}}$. In his doctoral thesis, Dugmore [21, p.10] noticed that the basis of $\tau_{\mathcal{F}}$ is $\mathcal{B} = \{f^{-1}(0,\infty)\}_{f \in \mathcal{F}_M}$. In [17], Cherenack remarked that the given topologies $\tau_{\mathcal{F}}$ and $\tau_{\mathcal{C}}$ coincide when the Frölicher space under consideration is a smooth manifold. We note finally that from the study by A. Cap [13, pp. 3-4], a Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$ whose initial topology coincides with the final topology is called a **balanced space**, that the space is Hausdorff if the two topologies are both Hausdorff and that a compact Hausdorff balanced Frölicher space is called a **base space**.

Unless otherwise indicated we shall restrict ourselves to the class of base spaces. In fact any Frölicher space can be associated with one which is Hausdorff up to an equivalence relation.

2.3 Smooth maps between Frölicher spaces

Definition 2.3.1 *A map $\varphi : (M, \mathcal{C}_M, \mathcal{F}_M) \longrightarrow (N, \mathcal{C}_N, \mathcal{F}_N)$ between Frölicher spaces is called a* **smooth map** *(or* **map of Frölicher spaces***), and denoted by $\mathcal{M}$-map, if*

$$\varphi^* \mathcal{G}_0 \subseteq \mathcal{F}_M.$$

That is, $g \circ \varphi \in \mathcal{F}_M$ whenever g lies in the set $\mathcal{G}_0 \subseteq \mathbb{R}^M$, generating the Frölicher structure $(\mathcal{C}_M, \mathcal{F}_M)$ on M.

Note that in this work, $\mathcal{C}_0$, $\mathcal{F}_0$, or $\mathcal{G}_0$ generally stands for a set generating the Frölicher structure.

Proposition 2.3.1 *A map $\varphi : (M, \mathcal{C}_M, \mathcal{F}_M) \longrightarrow (N, \mathcal{C}_N, \mathcal{F}_N)$ between Frölicher spaces is smooth if and only if*

$$\varphi_* \mathcal{C}_0 \subseteq \mathcal{C}_N.$$

That is, $\varphi \circ c \in \mathcal{C}_N$, whenever c lies in $\mathcal{C}_0 \subseteq M^{\mathbb{R}}$ which is the set of all the curves generating $(\mathcal{C}_M, \mathcal{F}_M)$.

Proof. Assume that $\varphi : (M, \mathcal{C}_M, \mathcal{F}_M) \longrightarrow (N, \mathcal{C}_N, \mathcal{F}_N)$ is a smooth map. Suppose that $(\mathcal{C}_M, \mathcal{F}_M)$ is generated by $\mathcal{C}_0$ and $(\mathcal{C}_N, \mathcal{F}_N)$ generated by $\mathcal{F}_0$. That is, $g \circ \varphi \in \mathcal{F}_M$ for all $g \in \mathcal{F}_0$ by assumption. Let $c \in \varphi_* \mathcal{C}_0$. Then $c = \varphi \circ d$ for some $d \in \mathcal{C}_0$. Now, for any $g \in \mathcal{F}_0$, we have

$$g \circ c = g \circ (\varphi \circ d) = (g \circ \varphi) \circ d \in C^\infty(\mathbb{R}, \mathbb{R}).$$

Hence, $c \in \mathcal{C}_N$.
Conversely assume that $\varphi_* \mathcal{C}_0 \subseteq \mathcal{C}_N$. That is, $\varphi \circ c \in \mathcal{C}_N$ whenever $c \in \mathcal{C}_0$. Then $g \circ (\varphi \circ c) = (g \circ \varphi) \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $g \in \mathcal{F}_N$, which makes $g \circ \varphi$ into a structure function on M. ☐

Corollary 2.3.1 *$\varphi : (M, \mathcal{C}_M, \mathcal{F}_M) \longrightarrow (N, \mathcal{C}_N, \mathcal{F}_N)$ is a smooth map if and only if $f \circ \varphi \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for each $f \in \mathcal{F}_N$ and each $c \in \mathcal{C}_M$.*

Corollary 2.3.2 *Given two smooth maps $\varphi : (M, \mathcal{C}_M, \mathcal{F}_M) \longrightarrow (N, \mathcal{C}_N, \mathcal{F}_N)$ and $\eta : (N, \mathcal{C}_N, \mathcal{F}_N) \longrightarrow (P, \mathcal{C}_P, \mathcal{F}_P)$. Then $\eta \circ \varphi$ is a smooth map.*

It is proved in [26, p.4] that Frölicher spaces are **objects** and smooth maps are **morphisms** in the so-called **category of Frölicher spaces**. We denote this category by **<u>FRL</u>**. It follows that for a given Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$ the set $C^\infty(\mathbb{R}, \mathbb{R}) := \mathcal{M}$ containing all constant functions has a natural Frölicher stucture in which structure curves are all maps $c : \mathbb{R} \longrightarrow \mathcal{M}$ such that the associated map $\tilde{c} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\tilde{c}(s,t) := c(s)(t)$ is smooth together with c. Likewise, the sets $C_M = C^\infty(\mathbb{R}, M)$ and $\mathcal{F}_M = C^\infty(M, \mathbb{R})$ become Frölicher spaces. Thus $\Gamma : \mathcal{F}_M \longrightarrow \mathcal{C}_M$ and $\Phi : \mathcal{C}_M \longrightarrow \mathcal{F}_M$ are functors in the category $\mathcal{M}$, where

$$\begin{aligned} \Gamma \mathcal{F}_0 &= \{c : \mathbb{R} \longrightarrow M \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in \mathcal{F}_0\}, & (2.1) \\ \Phi \mathcal{C}_0 &= \{f : M \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in \mathcal{C}_0\} & (2.2) \end{aligned}$$

for $\mathcal{F}_0 \subseteq R^M$ and $\mathcal{C}_0 \subseteq M^S$.

Lemma 2.3.1 *The functors* Γ *and* Φ *are order reversing; that is*

$$\mathcal{C}_1 \subseteq \mathcal{C}_2 \Longrightarrow \Phi\mathcal{C}_1 \supset \Phi\mathcal{C}_2, \tag{2.3}$$

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \Longrightarrow \Gamma\mathcal{F}_1 \supset \Gamma\mathcal{F}_2 \tag{2.4}$$

Proof. Let $f \in \Phi\mathcal{C}_2$ and $c \in \mathcal{C}_1$. Since $\mathcal{C}_1 \subseteq \mathcal{C}_2$, it follows that $c \in \mathcal{C}_2$ as well. Then $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ and $f \in \Phi\mathcal{C}_1$ which proves the first inclusion. The second assertion is proved in the same way. ☐

Corollary 2.3.3 *If* $\mathcal{F}_0 \subseteq \mathcal{M}(M, \mathbb{R})$, *then* $\mathcal{F}_0 \subseteq \Phi\Gamma\mathcal{F}_0$. *Similarly, if* $\mathcal{C}_0 \subseteq \mathcal{M}(\mathbb{R}, M)$, *then* $\mathcal{C}_0 \subseteq \Gamma\Phi\mathcal{C}_0$.

Proof. The proof is a straightforward consequence of Lemma 2.3.1 above. ☐

The following interpretation of this corollary is very useful. That is to say that generators are smooth maps in the generated structure. In fact, note that $\mathcal{F}_0 \subseteq \mathcal{F}_M \equiv \Phi\Gamma\mathcal{F}_0$ (resp. $\mathcal{C}_0 \subseteq \Gamma\Phi\mathcal{C}_0$) has the initial Frölicher structure induced by $(\mathcal{C}_{\mathcal{F}_M}, \mathcal{F}_{\mathcal{F}_M})$(resp. by $(\mathcal{C}_{\mathcal{C}_M}, \mathcal{F}_{\mathcal{C}_M})$) which corroborates the interpretation.

Definition 2.3.2 *Let* $\mathcal{F}_0$ *(or* $\mathcal{C}_0$*) be the set generating a Frölicher structure* $(\mathcal{C}, \mathcal{F})$. *The structure is said to be* **finitely generated** *(resp.* **countably generated***) if* $\mathcal{F}_0$ *(or* $\mathcal{C}_0$*) is a finite set (resp. countable set). Otherwise the structure is said to be* **infinitely generated**. *If* M *is a vector space and* $\mathcal{F}_0$ *is a set of linear functions then the Frölicher structure is said to be* **linearly generated**.

Examples. 1. Let E be a vector space and E^* its algebraic dual. Then the Frölicher structure $(\Gamma F, \Phi\Gamma F)$, where $F \subseteq E^*$ separates points in E, is linearly generated and one has $F \subseteq \Phi\Gamma F$.

2. Let M be the truncated real line $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. The set $\mathcal{F}_0 = \{\mathrm{id}_{\mathbb{R}^*}, |.|\}$ yields structure curves that are all smooth real functions whose graphs do not cross the x-axis. One can see that the composition of such functions with $id_{\mathbb{R}^*}$ and $|.|$ lies in $C^\infty(\mathbb{R}, \mathbb{R})$, making the generating functions into structure functions.

3. The set of real numbers together with all C^∞ real functions shall be denoted by $(\mathbb{R}, \mathcal{C}, \mathcal{F})$ and called the **canonical Frölicher space**.

Definition 2.3.3 *A Frölicher structure* $(\mathcal{C}_M, \mathcal{F}_M)$ *on a set* M *such that* $\mathcal{F}_M \equiv \underline{M}(M, \mathbb{R})$ *is said to be* **discrete**.

Definition 2.3.4 *A Frölicher structure* $(\mathcal{C}, \mathcal{F})$ *is said to be* **finer** *than another* $(\mathcal{C}', \mathcal{F}')$ *on the same underlying set* M *if* $\mathcal{F}' \subseteq \mathcal{F}$. *The structure* $(\mathcal{C}, \mathcal{F})$ *is said to be* **coarser** *than* $(\mathcal{C}', \mathcal{F}')$ *if* $\mathcal{C}' \subseteq \mathcal{C}$.

Examples. 1. A discrete Frölicher structure on M is one that is finer than any other Frölicher structure on M. This is a straightforward consequence of the definition above. In fact, it follows from the definition that a discrete Frölicher structure is the Frölicher structure generated by an empty set of curves and in which all functions are smooth. Hence the topology underlying a discrete Frölicher space is discrete. For instance, if $\mathcal{F}_0 = \mathbb{R}^{\mathbb{R}}$ then $\mathcal{C}_{\mathbb{R}} = \mathbb{R}$ and $\mathcal{F}_{\mathbb{R}} = \mathcal{F}_0$. So the discrete Frölicher structure on $\mathbb{R}$ is infinitely generated by $\mathcal{F}_0$.

2. Consider the Frölicher structure generated on $\mathbb{R}$ by the set $\mathcal{F}_0 = \{id_{\mathbb{R}}, |.|\}$, where $|.|$ is the absolute value. Then structure curves are given by

$$\mathcal{C}_{\mathbb{R}} = \Gamma\mathcal{F}_0 = \{c : \mathbb{R} \longrightarrow \mathbb{R} \mid id_{\mathbb{R}} \circ c \text{ and } |.| \circ c \text{ are } C^\infty(\mathbb{R}, \mathbb{R})\}.$$

These structure curves are all real functions $c : \mathbb{R} \longrightarrow \mathbb{R}$ whose absolute values are C^∞. Their graphs are flatter in a neighborhood of any point where they cross the x-axis and a tangent line at such a point is horizontal. Furthermore, $\mathcal{C}_{\mathbb{R}} \subseteq \mathcal{C}$, where $\mathcal{C} = C^\infty(\mathbb{R}, \mathbb{R})$ is the set of structure curves in the canonical Frölicher structure $(\mathcal{C}, \mathcal{F})$ on $\mathbb{R}$. Hence, the structure $(\mathcal{C}, \mathcal{F})$ is coarser than the structure generated by $\{id_{\mathbb{R}}, |.|\}$.

Proposition 2.3.2 *Let $(\mathcal{C}, \mathcal{F})$ and $(\mathcal{C}', \mathcal{F}')$ be two Frölicher structures with underlying set M. Then $(\mathcal{C}, \mathcal{F})$ is finer than $(\mathcal{C}', \mathcal{F}')$ if the identity map*

$$id_M : (M, \mathcal{C}, \mathcal{F}) \longrightarrow (M, \mathcal{C}', \mathcal{F}')$$

is a smooth map.

It follows from this proposition that $(\Gamma\mathcal{F}_0, \Phi\Gamma\mathcal{F}_0)$ is the coarsest Frölicher structure generated by the set $\mathcal{F}_0$ and in which all the generating functions are structure functions, and $(\Gamma\Phi\mathcal{C}_0, \Phi\mathcal{C}_0)$ is the finest Frölicher structure generated by the set $\mathcal{C}_0$ and in which all the generating curves are structure curves.

Proposition 2.3.3 *The following identities hold for the functors Γ and Φ:*

$$\Gamma\Phi\Gamma = \Gamma, \tag{2.5}$$

$$\Phi\Gamma\Phi = \Phi. \tag{2.6}$$

Proof. Let $\mathcal{F}_0 \subseteq \mathcal{M}(M, \mathbb{R})$. One has that $\mathcal{F}_0 \subseteq \Phi\Gamma\mathcal{F}_0$ by Corollary 2.3.3 above. From Lemma 2.3.1, the functor Γ reverses order and one obtains the inclusion $\Gamma\mathcal{F}_0 \supseteq \Gamma\Phi\Gamma\mathcal{F}_0$. But $\Gamma\mathcal{F}_0 \subseteq \Gamma\Phi\Gamma\mathcal{F}_0$ as stated in the second assertion of Corollary 2.3.3, which gives the second inclusion and proves the first equality. The proof of the second equality is similar. ☐

Example. Let $M = \mathbb{R}$ and $\mathcal{F}_0 = \{id_\mathbb{R}\}$. The generated Frölicher structure $(\mathcal{C}, \mathcal{F})$ is the canonical structure on $\mathbb{R}$, where $\mathcal{C} = \mathcal{F} = C^\infty(\mathbb{R}, \mathbb{R})$. Clearly, $id_\mathbb{R}$ is smooth with respect to $(\mathcal{C}, \mathcal{F})$. In this case, the two structures $(\Gamma id_\mathbb{R}, \Phi\Gamma id_\mathbb{R})$ and $(\Gamma\Phi id_\mathbb{R}, \Phi id_\mathbb{R})$, generated by $\{id_\mathbb{R}\} := \mathcal{F}_0$ and $\{id_\mathbb{R}\} := \mathcal{C}_0$ respectively, coincide.

One can see that for $\mathcal{F}_0 = \{id_\mathbb{R}\}$, $\Gamma\mathcal{F}_0$ yields $C^\infty(\mathbb{R}, \mathbb{R})$. To show the equality $\Phi\Gamma\mathcal{F}_0 = C^\infty(\mathbb{R}, \mathbb{R})$ reduces to proving the following assertion:

$$f \circ g \in C^\infty(\mathbb{R}, \mathbb{R}) \;\text{ for all }\; g \in C^\infty(\mathbb{R}, \mathbb{R}) \;\text{ implies }\; f \in C^\infty(\mathbb{R}, \mathbb{R}).$$

Observe that if we assumed that f is not smooth but $f \circ g \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $g \in C^\infty(\mathbb{R}, \mathbb{R})$, then the particular case $g = id_\mathbb{R}$ would lead to a contradiction. This ends the proof.

The following section provides some typical Frölicher spaces considered as either initial or final objects in the category **<u>FRL</u>**. It is proved that this category has initial and final objects. Also, products and coproducts, limits and colimits exist. (see for instance [26], [39]).

2.4 Frölicher structure on cartesian products and subsets

Proposition 2.4.1 *Let* $\{\mathcal{F}_i\}_{i\in J} = \{(M_i, \mathcal{C}_{M_i}, \mathcal{F}_{M_i})\}_{i\in J}$ *be a countable family of Frölicher spaces. There exists an initial Frölicher structure induced by the family* $\{\mathcal{F}_i\}_{i\in J}$ *such that* $P := \prod_{i\in J} M_i$ *is the underlying set.*

Proof. It suffices that we obtain a characterization for the structure curves on the product. We use natural maps between the set P and the structured sets M_i. Obviously these maps are the natural projections of P onto each member of the family $\pi_j : P \longrightarrow M_j$. Let $\mathcal{F}_{0j}$ be the generating sets on M_j for all $j \in J$. Then $\pi_j^* \mathcal{F}_{0j} : P \longrightarrow \mathbb{R}$ form the generating set $\mathcal{F}_0$ for the structure on P. Consider the set

$$\mathcal{F}_0 = \bigcup_{i\in J} \pi_i^* \mathcal{F}_{0i} = \cup_{i\in J}\{f_i \circ \pi_i \mid f_i \in \mathcal{F}_{0i}\}.$$

It follows that the structure curves are maps $c = (c_i)_{i\in J} : \mathbb{R} \longrightarrow P$ such that $(f_i \circ \pi_i \circ c_i) \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $i \in J$. The latter holds if and only if $c_i \in \mathcal{C}_{M_i}$ for all $i \in J$. Hence the generated Frölicher product space is $(P, \Gamma\mathcal{F}_0, \Phi\Gamma\mathcal{F}_0)$, where the structure curves in $\Gamma\mathcal{F}_0$ are maps whose components are structure curves in the factors M_i of P. $\square$

Corollary 2.4.1 *The space* $(\mathbb{R}^n, \mathcal{C}, \mathcal{F})$ *is an initial object in* **<u>FRL</u>** *generated by the natural projections* $\pi_i : \mathbb{R}^n \longrightarrow \mathbb{R} \quad (i = 1, ..., n)$.

Proof. We know from the preceding section that $id_{\mathbb{R}}$ generates $(\mathbb{R}, \mathcal{C}, \mathcal{F})$. Then $\pi_i \circ id_{\mathbb{R}} \quad (i = 1, ..., n)$ generate the product Frölicher structure on

$\mathbb{R}^n$. It follows that the structure curves for $(\mathbb{R}^n, \mathcal{C}, \mathcal{F})$ are the usual smooth curves $c : \mathbb{R} \longrightarrow \mathbb{R}^n$; $c = (c_1, ..., c_n)$ where each c_i is a C^∞ function on $\mathbb{R}$. □

This result was proved by Jan Boman in 1967 (see [11], [9]) and is one of the most important first steps toward the construction of smooth structures which need not involve norms in their definition.

Proposition 2.4.2 *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space. Let $A \subseteq M$ be a nonempty subset. There exists on A an initial Frölicher structure induced by the structure $(\mathcal{C}_M, \mathcal{F}_M)$ such that $(A, \mathcal{C}_A, \mathcal{F}_A)$ is a* **Frölicher subspace** *of $(M, \mathcal{C}_M, \mathcal{F}_M)$.*

Proof. Note that a natural map between A and the structured set M is the inclusion map $\iota : A \hookrightarrow M$. Let $\mathcal{F}_{0M} \subseteq \mathcal{F}_M$ be the set generating $(\mathcal{C}_M, \mathcal{F}_M)$. Clearly $\iota^* \mathcal{F}_{0M} = \mathcal{F}_{0M}|A$ yields the set generating the Frölicher structure on A. The resulting object is $(A, \Gamma(\mathcal{F}_{0M|A}), \Phi\Gamma\mathcal{F}_{0M|A})$, making the map ι into a smooth map. □

Corollary 2.4.2 *There exists on the set $\mathbb{Q}$, of all rational numbers, a discrete Frölicher structure induced by the canonical structure defined on $\mathbb{R}$.*

Proof. The Frölicher structure on $\mathbb{Q}$ is obviously generated by $id_{\mathbb{Q}} = id_{\mathbb{R}}|\mathbb{Q}$. It follows that $\{id_{\mathbb{Q}}\}$ yields only constant maps $c : \mathbb{R} \longrightarrow \mathbb{Q}$ as structure curves, as $id_{\mathbb{Q}} \circ c$ must lie in $C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in \Gamma\{id_{\mathbb{Q}}\}$. Hence all functions $f : \mathbb{Q} \longrightarrow \mathbb{R}$ shall be structure functions, turning the corresponding topology into a discrete topology. □

2.5 Frölicher structure on coproducts and quotient sets

Proposition 2.5.1 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a Frölicher space and* ρ *an equivalence relation in* M *consistent with a smooth function* $f : M \longrightarrow Z$*, that is,* $a \; \rho \; b$ *if and only if* $f(a) = f(b)$*. Then there exists a Frölicher structure on the quotient set* M/ρ*, making the natural projection map*

$$q : M \longrightarrow M/\rho$$

a smooth map.

Proof. Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ and $(Z, \mathcal{C}_Z, \mathcal{F}_Z)$ be Frölicher spaces. Since ρ is consistent with f, then it is easy to observe that there exists a unique map

$$\varphi : M/\rho \longrightarrow Z$$

such that $\varphi \circ q = f$. We shall only show that this map φ is smooth, making M/ρ into a Frölicher space. In fact, $\varphi([a]) = f(a)$, where $[a]$ is the equivalence class of $a \in M$. Now let

$$\mathcal{C}_0 := q_* \mathcal{C}_{0M},$$

where $\mathcal{C}_{0M}$ is the set generating the structure $(\mathcal{C}_M, \mathcal{F}_M)$.

Consider the curves $\bar{c} \in \mathcal{C}_0$, where $\bar{c} = q \circ c$ for $c \in \mathcal{C}_{0M}$ as generating Frölicher structure on M/ρ. Hence, for all $c \in \mathcal{C}_M$, $\varphi \circ q \circ c$ is smooth by construction. It follows that $\varphi \in \Phi \mathcal{C}_0$. Thus $(M/\rho, \Gamma\Phi\mathcal{C}_0, \Phi\mathcal{C}_0)$ is the quotient Frölicher space. Finally, note that since $q_* \mathcal{C}_{0M}$ yields structure curves on M/ρ for all $c \in \mathcal{C}_{0M}$, q is a smooth map. This ends the proof. $\square$

Proposition 2.5.2 *Let* $(M_i, \mathcal{C}_{M_i}, \mathcal{F}_{M_i})_{i\in J}$ *be a family of Frölicher spaces and* $\sqcup_{i\in J}M_i$ *their coproduct. There exists a Frölicher structure on* $\sqcup_{i\in J}M_i$ *in which the injection maps* $j_i : M_i \hookrightarrow \sqcup_{i\in J}M_i = \overline{M}$ *are smooth maps.*

Proof. There are obviously the injections maps j_i of M_i into $\sqcup_{i\in J}M_i$. Let

$$\cup_{i\in J}\{j_i \circ c_i \mid \ c_i \in \mathcal{C}_{0M_i}\} = \overline{\mathcal{C}_0}$$

be the set of curves generating the structure. Then $(\overline{M}, \Gamma\Phi\overline{\mathcal{C}_0}, \Phi\overline{\mathcal{C}_0})$, which is a final object in <u>**FRL**</u>, is a Frölicher space whose underlying set is the coproduct. ☐

2.6 Diffeomorphisms on Frölicher spaces

Definition 2.6.1 *A* **diffeomorphism** *of a Frölicher space* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *onto a Frölicher space* $(N, \mathcal{C}_N, \mathcal{F}_N)$ *is a smooth bijective map* $\varphi : M \longrightarrow N$ *such that* φ^{-1} *is smooth.*

Proposition 2.6.1 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a Frölicher space. Consider a set* N *and assume that* $\varphi : (M, \mathcal{C}_M, \mathcal{F}_M) \longrightarrow N$ *is an injective map. Then there exists on the image* $\varphi(M) \subseteq N$ *a Frölicher structure making* φ *a diffeomorphism of* M *onto* $\varphi(M)$.

Proof. Let $\mathcal{C}_{0M}$ be a generating set for the Frölicher structure $(\mathcal{C}_M, \mathcal{F}_M)$ on M. Then the set $\mathcal{C}_0 = \{\widetilde{c} : \mathbb{R} \longrightarrow \varphi(M) \mid \ \ \widetilde{c} = \varphi \circ c, \ c \in \mathcal{C}_{0M}\}$ yields a final structure on $\varphi(M)$. The structure functions are given by

$$\Phi\mathcal{C}_0 = \{f : \varphi(M) \longrightarrow \mathbb{R} \mid \ \ f \circ \widetilde{c} \in C^\infty(\mathbb{R}, \mathbb{R}) \ \text{ for } \ \widetilde{c} \in \mathcal{C}_0\}.$$

It follows that the structure functions are all maps $f : \varphi(M) \longrightarrow \mathbb{R}$ such that $f \circ \varphi \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in \mathcal{C}_{0M}$. In turn, the set $\Gamma\Phi\mathcal{C}_0$ yields structure curves as follows:

$$\Gamma\Phi\mathcal{C}_0 = \{\gamma : \mathbb{R} \longrightarrow \varphi(M) \mid \quad f \circ \gamma \in C^\infty(\mathbb{R}, \mathbb{R}) \ \text{ for all } \ f \in \Phi\mathcal{C}_0\}.$$

Then $(\varphi(M), \Gamma\Phi\mathcal{C}_0, \Phi\mathcal{C}_0)$ is a final object in **<u>FRL</u>**. Hence the map

$$\varphi : (M, \mathcal{C}_M, \mathcal{F}_M) \longrightarrow (\varphi(M), \Gamma\Phi\mathcal{C}_0, \Phi\mathcal{C}_0)$$

is injective by assumption, surjective by construction and smooth since one can see that $\varphi_*\mathcal{C}_{0M} \subseteq \Gamma\Phi\mathcal{C}_0$, also by construction. It remains to show that φ^{-1} is a smooth map. Note that $\gamma = \varphi \circ c$, where $c \in \mathcal{C}_{0M}$ are structure curves on $\varphi(M)$. Then the smoothness of φ^{-1} results from the fact that the map

$$\varphi^{-1} \circ \gamma = \varphi^{-1} \circ \varphi \circ c = c$$

lies in $\mathcal{C}_M$ for all $\gamma \in \mathcal{C}_{\varphi(M)}$. Thus φ is a diffeomorphism. □

It can be remarked that if $\mathcal{F}_{0M}$ is the generating set for the initial structure on M, then the structure $(\Gamma\Phi\mathcal{C}_0, \Phi\mathcal{C}_0)$ where $\mathcal{C}_0 = \varphi_*\mathcal{C}_{0M}$ coincides with the structure $(\varphi_*\mathcal{C}_{0M}, (\varphi^{-1})^*\mathcal{F}_{0M})$. Indeed, for each $f \in \mathcal{F}_{0M}$ and each $c \in \mathcal{C}_{0M}$, we have that

$$f \circ \varphi^{-1} \circ \varphi \circ c = f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}).$$

Remark. We can see that the final structure obtained above does not mix with the existing Frölicher structure on the set N which is the range of the map φ. It will not be the case in the following propositions.

Proposition 2.6.2 *Let M be a set and $(N, \mathcal{C}_N, \mathcal{F}_N)$ a Frölicher space with the smooth structure $(\mathcal{C}_N, \mathcal{F}_N)$ generated by a set $\mathcal{G}_0 \subseteq \mathbb{R}^N$. Let φ be an injective set map of M into $(N, \mathcal{C}_N, \mathcal{F}_N)$ such that $\mathcal{G}_{0|\varphi(M)} \circ \varphi$ generates a Frölicher structure $(\mathcal{C}_M, \mathcal{F}_M)$ on M. Then the map φ between $(M, \mathcal{C}_M, \mathcal{F}_M)$ and $(N, \mathcal{C}_N, \mathcal{F}_N)$ is a diffeomorphism of M onto its image $\varphi(M)$, where*

$$\mathcal{C}_M = \Gamma(\varphi^* \mathcal{G}_{0|\varphi(M)}), \qquad \mathcal{F}_M = \Phi\Gamma(\varphi^* \mathcal{G}_{0|\varphi(M)}).$$

Proof. Since $\varphi(M) \subseteq N$, let us consider the initial Frölicher structure induced on $\varphi(M)$ by $(\mathcal{C}_N, \mathcal{F}_N)$. The generating functions of the induced structure are the restrictions $g_{t|\varphi(M)}$ of the generating functions of $(\mathcal{C}_N, \mathcal{F}_N)$, where $t \in T$ for a countable set of indices T. It follows that the set of structure curves on $\varphi(M)$ is

$$\mathcal{C}_{\varphi(M)} = \{\widetilde{c} : \mathbb{R} \longrightarrow \varphi(M) \mid \ g_{t|\varphi(M)} \circ \widetilde{c} \in C^\infty(\mathbb{R}, \mathbb{R}) \ \text{ for all } t \in T\},$$

and the resulting set of structure functions is

$$\mathcal{F}_{\varphi(M)} = \{\widetilde{g} : \varphi(M) \longrightarrow \mathbb{R} \mid \ \widetilde{g} \circ \widetilde{c} \in C^\infty(\mathbb{R}, \mathbb{R}) \ \text{ for all } \ \widetilde{c} \in \mathcal{C}_{\varphi(M)}\}.$$

Then $\varphi : (M, \mathcal{C}_M, \mathcal{F}_M) \longrightarrow (\varphi(M), \mathcal{C}_{\varphi(M)}, \mathcal{F}_{\varphi(M)})$ is smooth if and only if

$$\widetilde{g} \circ \varphi \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$$

for all $\widetilde{g} \in \mathcal{G}_{0|\varphi(M)}$, $c \in \mathcal{C}_{0M}$. Equivalently, φ is smooth if and only if

$$g_{t|\varphi(M)} \circ \varphi \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$$

for all $g_t \in \mathcal{G}_0$ and $c \in \mathcal{G}_{0M}$. But $g_{t|\varphi(M)} \circ \varphi \in \mathcal{F}_M$ by assumption. Therefore the proposition holds. □

Proposition 2.6.3 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a Frölicher space. Consider the map* $\varphi : M \longrightarrow \mathbb{R}^T$ *defined by the formula*

$$\varphi(p) := (f_t(p))_{t\in T}$$

for all $p \in M$*, where* T *is a countable set of indices and* f_t *are generating functions for* $(\mathcal{C}_M, \mathcal{F}_M)$*. If* φ *is injective on* M*, then* φ *is a diffeomorphism of* M *onto its image* $\varphi(M)$*, where this image is endowed with the initial Frölicher structure induced by the canonical Frölicher structure of* $\mathbb{R}^T$*.*

Proof. Consider on $\mathbb{R}^T$ the canonical Frölicher structure generated by smooth natural projections π_t. Then $(\varphi(M), \mathcal{C}_{\varphi(M)}, \mathcal{F}_{\varphi(M)})$ is an initial object in **<u>FRL</u>** as a subspace of $\mathbb{R}^T$ and the structure is generated by the restrictions $\pi_{t|\varphi(M)}$. It follows that

$$\mathcal{C}_{\varphi(M)} = \{\widetilde{c} : \mathbb{R} \longrightarrow \varphi(M) \mid \quad \pi_{t|\varphi(M)} \circ \widetilde{c} \in C^\infty(\mathbb{R}, \mathbb{R}), \quad \text{for all} \quad t \in T\}$$

yields $\widetilde{c} = (c_t)_{t\in T}$ where each $c_t \in \mathcal{C}_{\mathbb{R}} = C^\infty(\mathbb{R}, \mathbb{R})$ by the definition of the Frölicher product structure given in Section 2.4. Then the condition

$$\widetilde{c} \in \mathcal{C}_{\varphi(M)} \;\; \text{if and only if} \;\; \pi_{t|\varphi(M)} \circ (c_t)_{t\in T} \in C^\infty(\mathbb{R}, \mathbb{R})$$

also reduces to

$$\widetilde{c} \in \mathcal{C}_{\varphi(M)} \;\; \text{if and only if} \;\; c_t \in C^\infty(\mathbb{R}, \mathbb{R}). \qquad (*)$$

Hence φ is smooth if and only if $\varphi \circ c \in \mathcal{C}_{\varphi(M)}$. That is, $(f_t)_{t\in T} \circ c$ lies in $\mathcal{C}_{\varphi(M)}$. But $(f_t)_{t\in T} \circ c = (f_t \circ c)_{t\in T}$, where $f_t \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$. Then according to $(*)$ one has that $\varphi \circ c \in \mathcal{C}_{\varphi(M)}$. Now we use the fact that φ is smooth to show that $\varphi^{-1} : \varphi(M) \longrightarrow M$ is smooth. It suffices to show that $\mathcal{F}_M \circ \varphi^{-1} \subseteq \mathcal{F}_{\varphi(M)}$. Observe that $\mathcal{F}_M = \mathcal{F}_M \circ \varphi^{-1} \circ \varphi$. This ends the proof. □

2.7 Comparison with some other smooth structures

In this paragraph we are interested in some connection between the Frölicher structure and other differential structures that can be defined on the same point set. For basic concepts we refer the reader to [66],[39], [41], [36],[9] and [17]. Note that $\underline{\mathcal{M}}$ is used by certain authors for **FRL**.

Carlos Tore [66] notes that every Frölicher space is a diffeological space denoted by $P_1(M)$, where $M \in \underline{\mathcal{M}}$. Structure curves form the set $P_1(M)$ and are identified with 1-plaques (also called **plots** by Iglesias (see [36])), while structure functions are diffeological morphisms from $(M, P_1(M))$ into $(\mathbb{R}, P(\mathbb{R}))$. It follows that there exists a functor $\varphi : \underline{\mathcal{M}} \longrightarrow \underline{P\mathcal{M}}$ between the two categories, which is one-to-one on objects, and is an embedding. Also one can read in [39] that convenient spaces are linear Frölicher spaces with smooth curves and associated functionals. Finally, we note that Frölicher spaces and differential spaces in the sense of Sikorski are closely related. The following proposition is due to Cherenack's comparative study of the two structures in [17].

Definition 2.7.1 *The pair* $(M, \mathcal{F})$ *is called a (Sikorski) differential locally subcartesian space generated by functions* $f_1, f_2, \ldots, f_n : M \longrightarrow \mathbb{R}$ *if*
(1) for all open coverings $\{U_i\}_{i \in I}$ *on* M *and for all functions* $g : M \longrightarrow \mathbb{R}$*, the following sheaf theoretic property holds:*

$$\text{If } \ g_{|U_i} = f_{j|U_i} \text{ for each } \ i \in I \ \text{ and each } \ j = 1, 2, \ldots, n, \quad \text{then } \ g \in \mathcal{F}.$$

(2) $\omega \circ (f_1, f_2, ..., f_n) \in \mathcal{F}$ *for all* $\omega \in C^\infty(\mathbb{R}^n)$.

Theorem 2.7.1 *A Frölicher structure* $(\mathcal{C}_M, \mathcal{F}_M)$ *on a set* M *induces a differential structure in the sense of Sikorski on the set* $\mathcal{F}_M$ *of structure functions.*

Proof. Let $c : \mathbb{R} \longrightarrow M$ be a structure curve on M. Let $\{f_j\}_j$ $(j = 1, \ldots, n)$ be a collection of functions generating the structure $(\mathcal{C}_M, \mathcal{F}_M)$. Assume that $\{U_i\}_{i\in I}$ is a covering of M and $g : M \longrightarrow \mathbb{R}$ is a function on M such that $g_{|U_i} = f_{j|U_i}$ for each $i \in I$ and each $j = 1,\ 2,\ \ldots.\ n$. It follows that for all structure curves $c \in \mathcal{C}_M$ one has $f_{j|U_i} \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for each $i \in I$ and each $j = 1,\ 2,\ \ldots.\ n$; i.e., $g_{|U_i} \circ c_{|c^{-1}(U_i)} \in C^\infty(\mathbb{R}, \mathbb{R})$. Since the sets $c^{-1}(U_i)$ cover $\mathbb{R}$ totally, it turns out that $g \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$. Then $g \in \mathcal{F}_M$.

Furthermore, assume that $\omega : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a C^∞ function and $\{f_1, \ldots, f_n\}$ is a collection of structure functions. Then $f_i \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ $(i = 1, \ldots, n)$. Hence

$$(f_1 \circ c, \ldots, f_n \circ c) = (f_1, \ldots, f_n) \circ c$$

is a structure curve on the product $\mathbb{R}^n$, and from the Boman Theorem [11] one has $\omega \circ ((f_1, \ldots, f_n) \circ c)$ is in $C^\infty(\mathbb{R}, \mathbb{R})$, turning $\omega \circ (f_1, \ldots, f_n)$ into a structure function on M. □

Definition 2.7.2 *Let* $(M, \mathcal{F}_M)$ *and* $(N, \mathcal{F}_N)$ *be differential spaces. A map* $f : (M, \mathcal{F}_M) \longrightarrow (N, \mathcal{F}_N)$ *is called* **differentiable map** *if* f *satisfies the condition* $f^*\mathcal{F}_N \subset \mathcal{F}_M$.

Lemma 2.7.1 *Let* (M, C) *and* (N, D) *be differential spaces. Let* $(\mathcal{C}_M, \mathcal{F}_M)$ *and* $(\mathcal{C}_N, \mathcal{F}_N)$ *be the associated Frölicher structures on* M *and* N *respectively generated by* C *and* D*. If* $\varphi : (M, C) \longrightarrow (N, D)$ *is a smooth map then*

$$\varphi : (M, \Gamma C, \Phi\Gamma C) \longrightarrow (N, \Gamma D, \Phi\Gamma D)$$

is a smooth map.

Proof. Assume that $\varphi : (M, C) \longrightarrow (N, D)$ is a smooth map between differential spaces. That is,

$$g \circ \varphi \in C \quad \text{for all g} \in D \qquad (\dagger).$$

Observe that since C generates the Frölicher structure $(\Gamma C, \Phi\Gamma C)$, one has that $C \subseteq \Phi\Gamma C$. Then $(\dagger)$ becomes

$$g \circ \varphi \in \Phi\Gamma C \quad \text{for all g} \in D. \qquad (\ddagger)$$

Moreover, D generates $(\Gamma D, \Phi\Gamma D)$ so that $(\ddagger)$, which is equivalent to $\varphi^* D \subseteq \Phi\Gamma C$, means that φ is smooth as a map between Frölicher spaces. $\square$

Lemma 2.7.2 *Let* (M_1, C_1) *and* (M_2, C_2) *be differential spaces and* φ *a diffeomorphism of* (M_1, C_1) *onto* (M_2, C_2)*. Then*

$$\varphi : (M_1, \Gamma C_1, \Phi\Gamma C_1) \longrightarrow (M_2, \Gamma C_2, \Phi\Gamma C_2)$$

is a diffeomorphism of the associated Frölicher spaces.

Proof. From Lemma 2.7.1, it turns out that φ is smooth as a map of differential spaces and smooth as a map of Frölicher spaces.

Moreover, φ is a diffeomorphism as a map of differential spaces. Then φ^{-1} exists and is smooth in both settings. Hence it follows that φ, considered as a diffeomorphism of differential spaces, sends structure functions of (M_1, C_1) onto those of (M_2, C_2) and vice-versa. That is, φ^{-1} maps C_2 onto C_1 as follows:

$$(\varphi^{-1})^* C_1 = C_2, \qquad \varphi^* C_2 = C_1.$$

We need only show that φ, considered as a diffeomorphism of Frölicher spaces, (i) maps structure curves of the Frölicher space $(M_1, \mathcal{C}_{M_1}, \mathcal{F}_{M_1})$ onto structure curves of the Frölicher space $(M_2, \mathcal{C}_{M_2}, \mathcal{F}_{M_2})$ with respect to the structures generated by the differential structures C_1 and C_2 respectively. That is,

$$\varphi_* \Gamma C_1 = \Gamma C_2, \qquad (\varphi^{-1})_* \Gamma C_2 = \Gamma C_1; \qquad \text{and}$$

(ii) sends $\Phi\Gamma C_2$ onto $\Phi\Gamma C_1$, and $\Phi\Gamma C_1$ onto $\Phi\Gamma C_2$ such that

$$\varphi^*(\Phi\Gamma C_2) = \Phi\Gamma C_1, \qquad (\varphi^{-1})^*(\Phi\Gamma C_1) = \Phi\Gamma C_2.$$

We show the identity

$$(\varphi^{-1})^*(\Phi\Gamma C_1) = \Phi\Gamma C_2.$$

The other identity may be proved in a similar way. Let us prove first that

$$\Phi\Gamma C_2 \subseteq (\varphi^{-1})^*(\Phi\Gamma C_1).$$

Let $f \in \Phi\Gamma C_2$. That is, f is a structure function on M_2. Then $f \circ \gamma \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $\gamma \in \Gamma C_2$. Note that $\gamma \in \Gamma C_2$ is of the form $\gamma = \varphi \circ c$ where $c \in \Gamma C_1$. Therefore

$$f \circ \gamma = f \circ (\varphi \circ c) = (f \circ \varphi) \circ c \in C^\infty(\mathbb{R}, \mathbb{R}).$$

It follows that $f \circ \varphi \in \Phi\Gamma C_1$. Hence $f \in (\varphi^{-1})^* \Phi\Gamma C_1$. This proves the first inclusion $\Phi\Gamma C_2 \subseteq (\varphi^{-1})^*(\Phi\Gamma C_1)$. Now we prove the reverse inclusion. Let $g \in (\varphi^{-1})^* \Phi\Gamma C_1$. Then, for all $\gamma \in \Gamma C_1$, one has that $g \circ (\varphi \circ \gamma) \in C^\infty(\mathbb{R}, \mathbb{R})$ according to the structure on M_2. Then $\varphi \circ \gamma \in \Gamma C_2$. Hence $g \in \Phi\Gamma C_2$ which proves the inclusion $(\varphi^{-1})^*(\Phi\Gamma C_1) \subseteq \Phi\Gamma C_2$. Thus the desired equality follows. $\square$

Chapter 3

Tangent structures

In this chapter, we define basic notions related to the tangent structure on a Frölicher space. Note that there are two existing approaches to the notion of tangent vectors. On a Frölicher space, a tangent vector may be defined either from smooth functions or from smooth curves. Adapting the terminology used by A. Kriegl and P. Michor [39] on convenient spaces, we shall call **operational tangent vectors** those defined from smooth functions since their sets are closed under both operations of vector addition and scalar multiplication. We call **kinematic tangent vectors** those defined from smooth curves. It is known that curves describe the motion of a body in the study of a mechanical system. We remark that the set of operational tangent vectors is a linear space while that of kinematic tangent vectors does not, in general, have a linear structure. The tangent bundles are defined and the existence of a natural Frölicher structure on them is proved. This chapter also introduces the notion of smoothness for tangent vector fields. It should be noted that certain tangent vector fields on Frölicher spaces are not locally representable by vectors. That is, some tangent vectors on a Frölicher space may fail being extendable to a vector field. This work does not deal with these cases.

3.1 Operational tangent vectors

3.1.1 Concepts and properties

Lemma 3.1.1 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a Frölicher space. Then the set* $\mathcal{F}$ *of structure functions is a real algebra.*

Proof. The proof is immediate. We may emphasize the fact that $\mathcal{F}_M$ is closed under pointwise multiplication. In fact, any structure functions f and g define on M a map

$$\varphi : M \longrightarrow \mathbb{R}^2; \quad p \mapsto \varphi(p) := ev_p(f, g).$$

φ is smooth as the evaluation map ev is smooth ([26],[39]) and defines in turn a smooth function

$$\omega : \mathbb{R}^2 \longrightarrow \mathbb{R}; \quad \omega \circ ev_p(f, g) = \omega(f(p), g(p)).$$

Let $\omega \circ (f, g) := f \cdot g$. It turns out that $f \cdot g \in \mathcal{F}_M$ since, from Theorem 2.7.1, $\mathcal{F}_M$ has the differential structure associated with the Frölicher structure. □

Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space. The smooth derivations on M are smooth linear operators that satisfy the Leibniz rule on $\mathcal{F}_M$. Denote their set by $Der(M)$. Then at each point $p \in M$ such an operator $d : \mathcal{F}_M \longrightarrow \mathcal{F}_M$ induces a map

$$d_p : \mathcal{F}_M \longrightarrow \mathbb{R}; \quad d_p f := (df)_p \quad \text{for all} \quad f \in \mathcal{F}_M.$$

One can see that $(df)_p = ev_p(df)$. Since d is a smooth derivation and the evaluation map is smooth, it follows that $(df)_p$ is smooth.

Proposition 3.1.1 *Let d be a smooth derivation on a Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$. The map $d_p = ev_p \circ d$ is a smooth linear operator and a derivation on $\mathcal{F}_M$. That is,*

(i) $d_p(f + \lambda g) = d_p(f) + \lambda d_p(g)$ *and*
(ii) $d_p(f \cdot g) = f(p) d_p(g) + g(p) d_p(f)$

for all $p \in M$, $f, g \in \mathcal{F}_M$, $\lambda \in \mathbb{R}$.

Proof. We have just shown that d_p is a smooth map. It remains to show that d_p is linear and satisfies the Leibniz rule. In fact, since d is a derivation by assumption and $\mathcal{F}_M$ is an algebra one has

$$\begin{aligned} d_p(f + \lambda g) &= (d(f + \lambda g))_p \\ &= ev_p \circ d(f + \lambda g) \\ &= ev_p(df + \lambda dg) \\ &= (df)_p + \lambda ev_p(dg) \\ &= d_p f + \lambda d_p g. \end{aligned}$$

Moreover,

$$\begin{aligned} ev_p \circ d(f \cdot g) &= ev_p(fdg + gdf) \\ &= ev_p(fdg) + ev_p(gdf) \\ &= ev_p(f)d_pg + ev_p(g)d_pf \\ &= f(p)d_pg + g(p)d_pf. \end{aligned}$$

□

In the following we shall denote the map d_p by v.

Definition 3.1.1 *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space. An* **operational tangent vector** *v at the point $p \in M$ is a smooth derivation on the algebra $\mathcal{F}_M$. That is, for all $f, g \in \mathcal{F}_M$ and $\lambda \in \mathbb{R}$,*

(a) $v(f + \lambda g) = v(f) + \lambda v(g)$ and
(b) $v(f \cdot g) = f(p)v(g) + g(p)v(f)$.

T_pM denotes the set of all operational tangent vectors to M. The set T_pM is called the **operational tangent space** *to M at $p \in M$.*

Lemma 3.1.2 *The operational tangent space T_pM to M at the point $p \in M$ is a linear space.*

Proof. Let us show that the sum $\sigma(v_1, v_2)$ of operational tangent vectors at a point of M is an internal operation. For all $f, g \in \mathcal{F}_M$, one has

$$\begin{aligned} \sigma(v_1, v_2) &= \sigma(d_pf, d_pg) \\ &= (df)_p + (dg)_p \\ &= ev_p \circ (df + dg) \\ &= (df + dg)_p. \end{aligned}$$

Since d is a smooth derivation, $df + dg \in \mathcal{F}_M$. Hence, $\sigma(v_1, v_2) \in T_pM$. Let $\mu(\lambda, v)$ be the scalar multiplication of $v \in T_pM$ by a real λ. Similar to the first part of the proof one has that

$$\mu(\lambda, v) = \lambda(d_pf) = \lambda(df)_p = (\lambda(df))_p = (d(\lambda f))_p \in T_pM. \qquad \square$$

Proposition 3.1.2 *The operational tangent space T_pM at the point p of a Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$ is a linear Frölicher space.*

Proof. From Lemma 3.1.2 above T_pM has a linear structure. Moreover we observe that for all $f \in \mathcal{F}_M$ and $d \in Der(M)$, $(df)_p$ yields an obvious map from T_pM into $\mathbb{R}$ by setting

$$(df)_p(v) := v(f). \tag{3.1}$$

The Frölicher structure on T_PM is the one which is generated by these functionals. $\square$

Definition 3.1.2 *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space. The* **operational cotangent space** *at $p \in M$, denoted by T_p^*M, is the algebraic dual of the operational tangent space T_pM at $p \in M$. The points of T_p^*M are called* **covariant vectors** *or* **covectors**.

It follows from this definition that the operational cotangent space T_p^*M is a linear Frölicher space if T_pM is finite dimensional, then one has that $\dim T_pM = \dim T_p^*M$.

Proposition 3.1.3 *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space and $p \in M$. A linear map $v : \mathcal{F}_M \longrightarrow \mathbb{R}$ is an operational tangent vector on M at p if and* *only if* v *satisfies the following conditions:*

$$\begin{aligned} v(f) &= 0 \quad \textit{if } f \textit{ is constant;} \\ v_{|a_p^2} &= 0 \end{aligned}$$

where $a_p^2 := \{(f - f(p)) \cdot (g - g(p));\ f, g \in \mathcal{F}_M\}$.

Proof. ($\Longrightarrow$) Assume that v is an operational tangent vector. That is, v is a linear derivation.

(i) Then for any constant $c \in \mathbb{R}$ and the function $f : M \longrightarrow \mathbb{R}$ such that $f(x) = c$ for all $x \in M$, one has that $v(f) = v(c) = v(1 \cdot c) = cv(1)$. But $v(1) = v(1 \cdot 1) = 1v(1) + 1v(1) = 2v(1)$. Hence $v(1) = 0$. Thus $v(f) = 0$.

(ii) Furthermore, as the derivation at $p \in M$, v satisfies the Leibniz property $v(f \cdot g) = g(p)v(f) + f(p)v(g)$. It follows that

$$\begin{aligned} 0 &= v(f \cdot g) - f(p)v(g) - g(p)v(f) \\ &= v(f \cdot g - f(p)g - g(p)f + f(p)g(p)) \\ &= v[(f - f(p)) \cdot (g - g(p))]. \end{aligned}$$

($\Longleftarrow$) Conversely, assume that the linear map v vanishes on constants and on the set $a_p^2 := \{(f - f(p)) \cdot (g - g(p));\ f, g \in \mathcal{F}_M\}$. One has to show that v has the Leibniz property. Now,

$$\begin{aligned} 0 &= v[(f - f(p)) \cdot (g - g(p))] \\ &= v(f \cdot g - f(p)g - g(p)f + f(p)g(p)) \\ &= v(f \cdot f) - f(p)v(g) - g(p)v(f) + v(f(p)g(p)) \\ &= v(f \cdot g) - f(p)v(g) - g(p)v(f). \end{aligned}$$

Thus, $v(f \cdot g) = g(p)v(f) + f(p)v(g)$. ☐

Note that the notion of a higher order tangent vector agrees with that of the usual higher order derivative in a Euclidean space.

Definition 3.1.3 *Let M be a Frölicher space, $p \in M$ and $k \in \mathbb{N}$. A linear map $v^{(k)} : \mathcal{F}_M \longrightarrow \mathbb{R}$ is called a* **k-th order tangent vector** *on M at p if the following conditions hold:*

$$\begin{aligned} v^{(k)}(f) &= 0 \quad \text{if } f \text{ is constant} \\ v^{(k)}_{|a_p^{k+1}} &= 0, \end{aligned}$$

where $a_p^{k+1} := \{(f_1 - f_1(p)) \cdots (f_{k+1} - f_{k+1}(p)) \mid f_1, \ldots, f_{k+1} \in \mathcal{F}_M\}$. The set $T_p^{(k)}M$ of all k-th order operational tangent vectors to M at $p \in M$ for a $k \in \mathbb{N}$ is merely a linear space called **the k-th order operational tangent space on M at p.**

3.1.2 Tangent map

Lemma 3.1.3 *Let $\varphi : (M, \mathcal{C}_M, \mathcal{F}_M) \longrightarrow (N, \mathcal{C}_N, \mathcal{F}_N)$ be a smooth map between Frölicher spaces and $v \in T_pM$. Then the pair (v, φ) induces on N an operational tangent vector in a neighborhood of $\varphi(p)$.*

Proof. For all $g \in \mathcal{F}_N$, $g \circ \varphi$ is a structure function on M since φ is a smooth map. In the neighborhood of $\varphi(p) \in N$, define a linear map $\varphi_{*p}v$ by

$$\varphi_{*p}v : \mathcal{F}_N \longrightarrow \mathbb{R}; \quad \varphi_{*p}v(g) = v(g \circ \varphi).$$

Note that the linear map $\varphi_{*p}v$ inherits the properties of derivation from those of v. Hence, $\varphi_{*p}v$ is the induced tangent vector at $\varphi(p)$. $\square$

Definition 3.1.4 *Let φ be a smooth map between two Frölicher spaces M and N. Let $p \in M$ and $v \in T_pM$. The smooth linear map $\varphi_{*p} : T_pM \longrightarrow T_{\varphi(p)}N$ defined by $\varphi_{*p}(v) = v(g \circ \varphi)$ for all $g \in \mathcal{F}_N$ is called the* **tangent map** *associated with φ at p.*

3.1.3 Properties of operational tangent vectors

Lemma 3.1.4 *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space. The following conditions are equivalent:*

(i) n tangent vectors $v_1, \ \ldots \ , v_n \in \ T_pM$ are linearly independent;

(ii) For all smooth functions $f \in \mathcal{F}_M$, the map

$$\theta : \mathcal{F}_M \longrightarrow \mathbb{R}^n; \ f \mapsto \ \theta(f) = (v_1(f), \ \ldots, \ v_n(f))$$

is a surjection;

(iii) There exist n smooth functions $f_1, \ \ldots, \ f_n \in \mathcal{F}_M$ such that $v_i(f_j) = \ \delta_{ij}$, where δ_{ij} is the Kronecker symbol;

(iv) There exist n smooth functions $f_1, \ \ldots, \ f_n \in \mathcal{F}_M$ such that $det(v_i(f_j)) \neq 0$.

Proof.

(i) $\Longrightarrow$ *(ii)* Suppose that $v_1, \ \ldots, \ v_n$ are linearly independent tangent vectors at $p \in M$ and θ, which is a linear map by construction, is not a surjection. Then $\theta(\mathcal{F}_M) \subsetneq \mathbb{R}^n$ is a linear subspace of $\mathbb{R}^n$. According to the Euclidean structure on $\mathbb{R}^n$, there exists a non-zero vector $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ such that

$$\alpha_1 v_1(f) + \ \ldots \ + \alpha_n v_n(f) = (\alpha_1 v_1 + \ \ldots \ + \alpha_n v_n)(f) = 0$$

for all $f \in \mathcal{F}_M$. Then

$$\alpha_1 v_1 + \ \dots \ + \alpha_n v_n = 0$$

with $\alpha_1, \ \dots \ , \alpha_n$ not all zero. Equivalently, $v_1, \ \dots \ , v_n$ are linearly dependent, contradicting linear independence of $\{v_1, v_2, \dots, v_n\}$.

(ii) $\Longrightarrow$ (iii) Let $\{e_j \mid \ 1 \leq j \leq n\}$, where $e_j = (\delta_{ij})_{1\leq i,j\leq n}$ is the canonical basis for $\mathbb{R}^n$. Since θ is surjective, there exists $f_j \in \mathcal{F}_M$ such that

$$\theta(f_j) \ = \ e_i \quad \text{for all} \ \ i = 1, 2, \dots, n$$

which turns out to be the rule $v_i(f_j) = \delta_{ij} \ \ (i, j = 1, \dots, n)$.

(iii) $\Longrightarrow$ (iv) The rule $v_i(f_j) = \delta_{ij} \ \ (i, j = 1, \dots, n)$ above induces an $n \times n$-lower and -upper matrix whose determinant is equal to 1. Therefore, $det(v_i(f_j)) \neq 0$.

(iv) $\Longrightarrow$ (i) Let $v_i(f_j) = \delta_{ij}$ define an $n \times n$-matrix associated with the homogeneous linear system

$$\sum_{i=1}^{n} v_i(f_j)\lambda_i = 0$$

of j equations in i unknowns. The zero vector is obviously the unique solution. Hence the vectors $v_1, \dots, v_n$ are linearly independent. ☐

Examples of operational tangent vectors.

(1) Let G be the Frölicher subspace of $\mathbb{R}^2$ defined by all pairs $(x, y) \in \mathbb{R}^2$ such that $xy = 0$. A structure curve on G is a curve $c : \mathbb{R} \longrightarrow B_1 \cup B_2$, where $B_1 = \{(x, 0) \mid \ x \in \mathbb{R}\}$ and $B_2 = \{(0, y) \mid \ y \in \mathbb{R}\}$, while a structure function

on G is a real-valued function $f : G \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that f is smooth on the x-axis and the y-axis. Note that $f : G \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$; where

$$f(x,y) = f_1(x) \text{ along the } x-\text{axis},$$

$$f(x,y) = f_2(y) \text{ along the } y-\text{axis};$$

for f_1 and f_2 are smooth real functions that intersect at the origin. It follows that at $p = (0,0)$, $df = v(f)$ provides two linearly independent operational tangent vectors which are partial derivatives with respect to x and y. Thus $T_{(0,0)}G = \mathbb{R}^2$.

Contrary to this, for all $p \neq (0,0)$, one has $p = (x,0)$, where $x \neq 0$ or $p = (0,y)$, where $y \neq 0$. Then, as the derivative in the direction of the x-axis is equal to 1, the derivative in the direction of the y-axis shall be equal to 0 and vice-versa. Thus $\dim T_pG = 1$ for $p \neq (0,0)$.

(2) On the set $\mathbb{Q}$ of rational numbers considered as a Frölicher subspace of $\mathbb{R}$, only constant functions are structure curves. The resulting topology generated by the structure curves is discrete. Then every singleton $\{p\}$ is an open neighborhood of $p \in \mathbb{Q}$. Consequently the only operational tangent vector is the zero vector. That is, $T_p\mathbb{Q} = \{0\}$.

3.2 Tangent bundle, vector field, Lie Bracket

Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space. Note that as p runs through M, the map $(df)_p$ associated with $f \in \mathcal{F}_M$ globally determines a smooth map denoted by *df* on the disjoint union

$$\cup_{p\in M} T_pM = \cup_{p\in M}\{p\} \times T_pM.$$

But the right-hand side is a subset of $M \times Der(M)$ which, in turn, is a product of Frölicher spaces. So *df* is a map defined on $\cup_{p\in M} T_pM$ such that $(df)_{|T_pM} = (df)_p$. It follows that there is a natural map

$$\pi : \cup_{p\in M} T_pM \longrightarrow M$$

sending each tangent vector $v \in T_pM$ to the attachment point $p \in M$. That is, $\cup_{p\in M} T_pM$ is a bundle, which we shall call the tangent bundle. Then π is the **natural projection** of the tangent bundle.

3.2.1 Operational tangent bundle

Definition 3.2.1 *The disjoint union of all operational tangent spaces on a Frölicher space* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *is said to be the* **operational tangent bundle** *on* M *and denoted by* TM. *The tangent spaces* T_pM *at each point* $p \in M$ *are the fibers of the tangent bundle* TM.

Proposition 3.2.1 *There exists on the operational tangent bundle* TM *a Frölicher structure in which the natural projection is a smooth map.*

Proof. Observe that the set of obvious functions on TM which generates the smooth structure is

$$\mathcal{F}_T = \{df \mid f \in \mathcal{F}_M\} \cup \{f \circ \pi \mid f \in \mathcal{F}_M\}.$$

Hence df and $f \circ \pi$ are structure functions in the generated structure according to Corollary 2.3.3 in Chapter 2. The smoothness of π follows. ☐

Definition 3.2.2 *A map* $X : (M, \mathcal{C}_M, \mathcal{F}_M) \longrightarrow (\cup_{p \in M} T_pM, \Gamma\mathcal{F}_T, \Phi\Gamma\mathcal{F}_T)$ *sending* $p \in M$ *to* $v \in T_pM$ *such that* $\pi \circ X = id_M$ *is called the* **tangent vector field**.

Proposition 3.2.2 *A tangent vector field* X *on* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *is smooth if and only if* X *transforms a structure function into a structure function. That is,* $X(f) \in \mathcal{F}_M$ *whenever* $f \in \mathcal{F}_M$.

Proof. From the definition of smooth maps we know that for X to be smooth it is enough to check that its composition with the functions generating the structure is a structure function. That is,

$$X \text{ is smooth iff } (f \circ \pi) \circ X \in \mathcal{F}_M \text{ and } df \circ X \in \mathcal{F}_M.$$

The first statement is trivial since $(f \circ \pi) \circ X = f \circ (\pi \circ X) = f \circ id_M$. Note that since $X(p)$ is a tangent vector at $p \in M$, the second statement gives, at each point p,

$$(df \circ X)_p = (df)_p X(p) = (Xf)(p).$$

Then as p runs through M, X is smooth if and only if $Xf \equiv X(f) \in \mathcal{F}_M$, which ends the proof. ☐

We shall denote by $\mathfrak{X}(M)$ the set of all smooth tangent vector fields on M.

Definition 3.2.3 *Let* φ *be a diffeomorphism on a Frölicher space* M. *A smooth vector field* X *on* M *is said to be* φ**-invariant** *if*

$$d\varphi(X) = X.$$

Definition 3.2.4 *Let M be a Frölicher space. The algebraic dual of TM, denoted by T^*M, is called the* **cotangent bundle** *on M. That is,*

$$T^*M := \cup_{p\in M} T_p^* M.$$

It follows that

(1) $T^*M = \{(p,\theta) \mid \theta : T_pM \longrightarrow \mathbb{R} \text{ is linear}\}$. Then there is a natural smooth projection $\tau : T^*M \longrightarrow M$;

(2) to any smooth tangent vector field X on M there corresponds a smooth function $X^* : T^*M \longrightarrow \mathbb{R}$ defined by

$$X^*(\theta) = \theta(X(\tau(\theta))); \;\; \theta \in T^*M. \tag{3.2}$$

Proposition 3.2.3 *The cotangent bundle T^*M on a Frölicher space M carries a natural Frölicher structure generated by the set*

$$\{X^* \mid X \in \mathfrak{X}(M)\} \cup \{f \circ \tau \mid f \in \mathcal{F}_M\}.$$

Proof. The proof is similar to that of Proposition 3.2.1 above. □

3.2.2 Lie bracket

Definition 3.2.5 *Let M be a Frölicher space. The operator*

$$[\,,\,] : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

defined by

$$[X,Y] := XY - YX$$

is a smooth derivation called the **Lie bracket**.

It is easy to show that this bracket is bilinear, skew-symmetric and satisfies the Jacobi identity. Then $\mathfrak{X}(M)$ together with the bracket $[\ ,\ ]$ is the **Lie algebra of vector fields on M**.

Proposition 3.2.4 *The set of all φ-invariant smooth vector fields on a Frölicher space M is a subalgebra of the Lie algebra $\mathfrak{X}(M)$ of smooth vector fields on M.*

Proof. The proof is immediate. □

Proposition 3.2.5 *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ and $(N, \mathcal{C}_N, \mathcal{F}_N)$ be two Frölicher spaces and φ a diffeomorphism of $(M, \mathcal{C}_M, \mathcal{F}_M)$ onto $(N, \mathcal{C}_N, \mathcal{F}_N)$. Then φ induces a map $d\varphi$ from $\mathfrak{X}(M)$ to $\mathfrak{X}(N)$ such that $d\varphi$ is an isomorphism of Lie algebras.*

Proof. For all $X \in \mathfrak{X}(M)$, $g \in \mathcal{F}_N$, define the image $Y \in \mathfrak{X}(N)$ of X under the map $d\varphi$ by setting

$$Y(g) = (X(g \circ \varphi)) \circ \varphi^{-1}.$$

One has $d\varphi : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(N)$. It is easy to show that $d\varphi$ is linear, invertible and preserves the Lie bracket. □

3.3 Kinematic tangent vectors

In Physics, curves usually describe the motion of a body. The velocity is then determined by the rate of change which fixes the direction of a tangent at any point of the trajectory. Following this approach, we shall call those tangent vectors involving smooth curves kinematic in order to distinguish them from operational tangent vectors that we defined on smooth functions.

Definition 3.3.1 *Let M be a Frölicher space, $a \in \mathbb{R}$ and $x \in M$. Denote by $C_M^{a,x}$ the set of all smooth curves $c \in \mathcal{C}_M$ such that $c(a) = x$. By a* **kinematic tangent vector** *to the space M with foot point a we mean*

$$X_{c,a}(f) := \frac{d}{dt}(f \circ c)_{|t=a} = df(c(a)), \tag{3.3}$$

where $c \in C_M^{a,x}$.

Definition 3.3.2 *A* **tangent cone space** *at a point p of a Frölicher space M, denoted by T_pCM, is the set of all kinematic tangent vectors at p.*

Definition 3.3.3 *The* **cotangent cone space** *of a linear tangent cone space, denoted by T_p^*CM, is its algebraic dual. The direct sum of tangent cone spaces is called the* **tangent cone bundle** *and denoted by TCM. The algebraic dual of TCM is called the* **cotangent cone bundle** *and denoted by T^*CM.*

Examples of kinematic tangent vectors

1. $T_pC\mathbb{Q} = \{0\}$ for all $p \in \mathbb{Q}$ since structure curves on $\mathbb{Q}$, for the Frölicher structure are only the constant functions $c : \mathbb{R} \longrightarrow \mathbb{Q}$. Thus $\mathcal{C}_\mathbb{Q} = \mathbb{Q}$ and $f \circ c$ has a constant value such that (3.3) is always zero.

2. Let G be the Frölicher subspace of $\mathbb{R}^2$ defined by all pairs $(x, y) \in \mathbb{R}^2$ such that $xy = 0$. It follows that for all structure curves c one has

$$f \circ c : \mathbb{R} \longrightarrow \mathbb{R}; \quad f \circ c(t) = f(x, 0) \quad \text{or} \quad f \circ c(t) = f(0, y) \quad \forall t \in \mathbb{R}.$$

It turns out that along the x-axis there exists at least one non-zero tangent vector v_c. The resulting kinematic tangent space T_pCG is then a straight line for all $p \neq 0$, similarly on the y-axis.

At $p = (0,0)$ all the kinematic tangent vectors lie on $B_1 = \{(x,0) \mid x \in \mathbb{R}\}$ or on $B_2 = \{(0,y) \mid y \in \mathbb{R}\}$. Then the kinematic tangent space at $p = 0$ yields the whole of G.

Chapter 4

Dimension in Frölicher spaces

Before we discuss the concept of dimension, we first define the concept of $\mathcal{M}$-independence of structure functions in a neighborhood of a fixed point of a Frölicher space. Then we come to the concepts of smooth basis and that of $\mathcal{M}$-dimension. It turns out that if the considered Frölicher space is a smooth manifold, then $\mathcal{M}$-dimension coincides with the topological dimension. A Frölicher space whose $\mathcal{M}$-dimension is the same at each point is said to be of **constant dimension**.

4.1 Concept of $\mathcal{M}$-independence

Definition 4.1.1 *Let M be a Frölicher space, $p \in M$, U an open neighborhood of p and $\mathcal{G}_p(U)$ the germ of smooth functions at p in U. Let $f, g_1, \dots, g_n$ be in $\mathcal{G}_p(U)$. Then* *(a) f is said to be* $\mathcal{M}$**-dependent** *on $g_1, \dots, g_n$ if there exists $\omega \in C^\infty(\mathbb{R}^n)$ such that*

$$f = \omega \circ (g_1, \dots, g_n). \tag{4.1}$$

Otherwise, f is said to be $\mathcal{M}$**-independent** *of $g_1, \dots, g_n$.*
(b) A set $\mathcal{F} = \{f_1, \dots, f_m\} \subseteq \mathcal{F}_M$ is said to be $\mathcal{M}$**-independent** *at p if none of the functions f_i (for $i = 1, \dots, m$) in $\mathcal{F}$ is $\mathcal{M}$-dependent on the remaining others.*

It follows that every finite subset of $\mathcal{F}$ is $\mathcal{M}$-independent at p.

Example. Let $(\mathbb{R}^n, \mathcal{C}, \mathcal{F})$ be a Frölicher space with the canonical Frölicher structure defined by all real-valued smooth functions and all smooth curves. Then the set $P = \{\pi_1, \dots, \pi_n\}$ of natural projections is $\mathcal{M}$-independent.

Lemma 4.1.1 *Let A be an open subspace of a Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$. If $\mathcal{F} = \{f_1, \dots, f_n\}$ is $\mathcal{M}$-dependent at $p \in M$, then the restriction of $\mathcal{F}$ to A denoted by $\mathcal{F}_{|A} = \{f_{1|A}, \dots, f_{n|A}\}$ is also $\mathcal{M}$-dependent at p.*

Proof. The proof is immediate. □

Definition 4.1.2 *Let M be a Frölicher space, p a point in M and U a neighborhood of p. A smooth function f is said to be $\mathcal{M}$-independent on smooth functions $g_1, \dots, g_n$ at p if f is not $\mathcal{M}$-dependent on $g_1, \dots, g_n$.*

Lemma 4.1.2 *Let M be a Frölicher space, $p \in M$ and U an open neighborhood of p. Then T_pU is isomorphic to T_pM.*

Proof. The inclusion map $\iota_U : U \hookrightarrow M$ is an embedding. That is, the concept of tangent vector is a local concept. ☐

Lemma 4.1.3 *Let M be a Frölicher space, $p \in M$ and U be an open neighborhood of p. Let $\varphi := (f_1, .., f_n)$ be such that $f_1, .., f_n \in \mathcal{G}_p(U)$ and $\omega \in C^\infty(\mathbb{R}^n)$. For all $v \in T_pM$,*

$$v(\omega \circ \varphi) = \sum_{i=1}^{n} D_i\omega(\varphi(p)) \cdot v(f_i),$$

where $D_i = \dfrac{\partial}{\partial x_i}$ is the i-th partial derivative of ω.

Proof. Let $v \in T_pM$. According to Lemma 4.1.2 above there exists $\hat{v} \in T_pU$ such that $v = \iota_{*p}\hat{v}$. If $\varphi := (f_1, \ldots, f_n)$, then one has

$$\varphi : (U, \mathcal{C}_U, \mathcal{F}_U) \longrightarrow (\mathbb{R}^n, \mathcal{C}, \mathcal{F}).$$

Then φ is clearly smooth and the pair $(p, \hat{v})$ induces a vector $\varphi_{*p}\hat{v} \in T_{\varphi(p)}\mathbb{R}^n$. It follows that for all $\omega \in C^\infty(\mathbb{R}^n)$,

$$\varphi_{*p}\hat{v}(\omega) = \hat{v}(\varphi^*\omega) = \hat{v}(\omega(\hat{f}_1, \ldots, \hat{f}_n)).$$

Moreover, since $\hat{v}$ is a derivation on the set of all smooth functions on $\mathbb{R}^n$ at p, one has

$$\hat{v}(\omega \circ (\hat{f}_1, \ldots, \hat{f}_n)) = \sum_{i=1}^{n} D_i\omega(\varphi(p)) \cdot \hat{v}(f_i).$$

☐

4.2 Concept of $\mathcal{M}$-basis and $\mathcal{M}$-dimension

Definition 4.2.1 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a Frölicher space. A set* $\mathcal{B} \subseteq \mathcal{F}_M$ *is said to be a* $\mathcal{M}$-**basis** *of* $\mathcal{F}_M$ *at* $p \in M$ *if* $\mathcal{B}$ *is* $\mathcal{M}$*-independent in an open neighborhood* U *of* p *and for any* $f \in \mathcal{G}_p(U)$ *one has* $f = \omega \circ (g_1, g_2, \ldots, g_n)$, *where* $n = card(\mathcal{B})$ *and* $\omega \in C^\infty(\mathbb{R}, \mathbb{R})$.

Lemma 4.2.1 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a Frölicher space and* $\{f_1, \ldots, f_n\}$ *be an* $\mathcal{M}$*-basis in an open neighbourhood* U *of* $p \in M$*. Then the map*

$$\psi := (f_1, \ldots, f_n)$$

is a diffeomorphism of $(U, \mathcal{C}_U, \mathcal{F}_U)$ *onto* $(\psi(U), \mathcal{C}_{\psi(U)}, \mathcal{F}_{\psi(U)})$.

Proof. Let ψ be the map $\psi : U \longrightarrow \mathbb{R}^n$ defined by

$$\psi(p) := (f_1(p), \ldots, f_n(p)) \quad \forall p \in M.$$

Note that since $\{f_1, \ldots, f_n\}$ is an $\mathcal{M}$-basis, the smooth functions $f_1, \ldots, f_n$ separate points of U. Then the map ψ is injective. Also from Proposition 4.1.1, the subset $\psi(U) \subseteq \mathbb{R}^n$ carries a Frölicher structure. Thus ψ is a diffeomorphism and the lemma holds according to Section 2.6. ☐

Lemma 4.2.2 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a Frölicher space. Consider the map given by*

$$\psi(p) = (f_1(p), \ldots, f_n(p)) \quad \textit{for all} \quad p \in M,$$

where $\{f_1, \ldots, f_n\}$ *is an* $\mathcal{M}$*-basis in an open neighbourhood* U *of* p*, then the associated tangent map*

$$\psi_{*p} : T_pM \longrightarrow T_{\psi(p)}\psi(M)$$

is an isomorphism of linear spaces.

Proof. The map ψ_{*p} is linear. By Lemma 4.2.1 $\psi_{|U}$ is a diffeomorphism. Then ψ^{-1} exists and $(\psi^{-1})_{*\psi(p)} = (\psi_{*p})^{-1}$. This means that ψ_{*p} is an isomorphism.

Now assume that X is a smooth vector field on M and Y is a vector field on $\psi(M)$. It follows that for all $w \in T_{\psi(p)}\psi(M)$,

$$w = Y(\psi(p)) = \psi_{*p}(X(\psi^{-1}(\psi(p)))) = \psi_{*p}X(p).$$

Thus there exists a vector $v := X(p) \in T_pM$ such that $w = \psi_{*p}(v)$. □

Lemma 4.2.3 *Let M be a k-dimensional variety in $\mathbb{R}^n$ and $p \in M$. Let $\{\pi_1, \ldots, \pi_n\}$ be the set of n smooth projections on $\mathbb{R}^n$ and $\{\hat{\pi}_1, \ldots, \hat{\pi}_n\}$ their restrictions to M. Then $\{\hat{\pi}_1, \ldots, \hat{\pi}_n\}$ is $\mathcal{M}$-independent at p if and only if dim $M = n$.*

Proof. ($\Longrightarrow$) Suppose that $\{\hat{\pi}_1, \ldots, \hat{\pi}_n\}$ is $\mathcal{M}$-independent at p and assume that dim $M = k < n$. Let U be an open neighborhood of p in $\mathbb{R}^n$. Any $a \in U$ can be written $a = (a_1, \ldots, a_k, a_{k+1}, \ldots, a_n)$. Define $\psi = (\hat{\pi}_1, \ldots, \hat{\pi}_k)$. According to Lemma 4.2.1 the map ψ is a diffeomorphism of M onto $\psi(M) \subseteq \mathbb{R}^k$ and $\psi(a) = (a_1, \ldots, a_k)$ for all $a \in U$. Now let $b = (a_1, \ldots, a_k)$ and G be a neighborhood of b in $\mathbb{R}^k$. One can define a smooth map

$$\eta = (\eta_1, \ldots, \eta_{n-k}) : G \subseteq \mathbb{R}^k \longrightarrow \mathbb{R}^{n-k}$$

where $\eta_l(b) = a_{k+l}$ for $l = 1, \ldots, n-k$ such that $U = \{(b, \eta(b)) \mid b \in G\}$. It follows that $a_{k+l} = \eta_l \circ \psi(a) = \hat{\pi}_{k+l}(a)$. Consequently,

$$\hat{\pi}_{k+l} = \eta_l \circ (\hat{\pi}_1, \ldots, \hat{\pi}_k) \quad (l = 1, \ldots, n-k).$$

This is in contradiction to the assumption above. Therefore, $k = n$.

($\Longleftarrow$) Conversely, assume that dim $M = n$. Then M may be considered as an open neighborhood of each point $p \in M$. Let $\{\hat{\pi}_1, \ldots, \hat{\pi}_n\}$ be the set of restrictions to M of natural smooth projections $\pi_1, \ldots, \pi_n$. One asserts, according to Lemma 4.1.1, that $\hat{\pi}_1, \ldots, \hat{\pi}_n$ are $\mathcal{M}$-independent since $\pi_1, \ldots, \pi_n$ are $\mathcal{M}$-independent at any $p \in M$. □

Proposition 4.2.1 *Let M be a non-empty subset of $\mathbb{R}^n$, where $\mathbb{R}^n$ is endowed with the canonical Frölicher structure. The restrictions $\hat{\pi}_1, \ldots, \hat{\pi}_n$ to M of natural projections $\pi_1, \ldots, \pi_n$ are $\mathcal{M}$-independent if and only if $n =$ dim T_pM for every $p \in M$.*

Proof. ($\Longrightarrow$) Assume that $\{\pi_{1|M}, \ldots, \pi_{n|M}\}$ is $\mathcal{M}$-independent at $p \in M$ and suppose that dim $T_pM = k < n$. According to [47], there exists a k-dimensional variety $S \subseteq \mathbb{R}^n$ such that $U = S \cap M$ is a neighborhood of p in M and $C^\infty(S)_{|U} = C^\infty(\mathbb{R}^n)_{|U}$. Then the set $\{\pi_{1|S}, \ldots, \pi_{n|S}\}$ is an $\mathcal{M}$-dependent set at $p \in U$ according to Lemma 4.2.3. This is in contradiction to our assumption. Thus, $k = n$.

($\Longleftarrow$) Conversely, suppose that dim $T_pM = n$ and $\{\hat{\pi}_1, \ldots, \hat{\pi}_n\}$ is $\mathcal{M}$-dependent at p, where each $\hat{\pi}_i$ is the restriction of the projection π_i to M. That is, for at least one π_{j_0} in the list, there exist an open neighborhood W of p and a map $\omega \in C^\infty(\mathbb{R}^{n-1})$ satisfying

$$\pi_{j_0} = \omega \circ (\pi_1, \ldots, \pi_{i_0-1}, \pi_{i_0+1}, \ldots, \pi_n)$$

where both sides are functions restricted to W. Consider the homomorphism of linear spaces defined by

$$\varphi : T_pM \longrightarrow \mathbb{R}^n;\ v \mapsto \varphi(v) = v(\hat{\pi}_i).$$

It is known from Linear Algebra that such a φ is an isomorphism, a property which fails in the present case. Thus $\dim T_pM < n$. This is in contradiction to the assumption. Thus, the maps $\hat{\pi}_1, \dots, \hat{\pi}_n$ are $\mathcal{M}$-independent at p. $\square$

Proposition 4.2.2 *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space and $p \in M$. The set $\mathcal{F}_0 = \{f_1, \dots, f_n\} \subseteq \mathcal{F}_M$ is $\mathcal{M}$-independent at p if and only if $\dim T_pM = n$.*

Proof. Let ψ be defined as in Lemma 4.2.1 above. It is easy to see that

$$f_i = (\pi_{i|\psi(M)}) \circ \psi \ \ \forall i \in \{1, \dots, n\}$$

and since ψ is a diffeomorphism according to Lemma 4.2.1, the maps $f_1, \dots, f_n$ are $\mathcal{M}$-independent at p if and only if the restrictions $\hat{\pi}_1, \dots, \hat{\pi}_n$ of $\pi_1, \dots, \pi_n$ to $\psi(M)$ are $\mathcal{M}$-independent at $\psi(p)$, meaning that $\dim \psi(M) = n$. Furthermore, since $\psi(M)$ is a nonempty set in $\mathbb{R}^n$, the conditions above on the restrictions of the projections imply that $\dim T_{\psi(p)}\psi(M) = n$ according to Lemma 4.2.1. Thus from Lemma 4.2.2 one concludes that $\dim T_pM = n$.$\square$

Proposition 4.2.3 *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space and $p \in M$. A subset $\mathcal{F}_0 = \{f_1, \dots, f_n\} \subseteq \mathcal{F}_M$ is $\mathcal{M}$-independent at p if and only if for each open neighborhood U of the point p and for each $\omega \in C^\infty(\mathbb{R}^n)$ we have*

$$\nabla\omega(f_1(p), \dots, f_n(p)) = 0 \quad \textit{whenever} \quad \omega \circ (f_1, \dots, f_n)_{|U} = 0,$$

where ∇ is the gradient vector for the $C^\infty(\mathbb{R}^n)$ function ω evaluated at the point $(f_1(p), \dots, f_n(p))$.

Proof. ($\Longrightarrow$) Assume first that $\{f_1, \ldots, f_n\}$ is $\mathcal{M}$-independent at p in the neighborhood U in which, for $\omega \in C^\infty(\mathbb{R}^n)$,

$$\omega \circ (f_1, \ldots, f_n) = 0.$$

Consider the map $\hat{\psi} : (U, C_U, \mathcal{F}_U) \longrightarrow \mathbb{R}^n$ such that

$$\hat{\psi}(p) = (\hat{f}_1(p), \ldots, \hat{f}_n(p))$$

for all $p \in U$, where each $\hat{f}_i$ is the restriction to U of maps $f_1, \ldots, f_n$. From Proposition 4.2.2 it follows that dim $T_{\hat{\psi}(p)}\hat{\psi}(U) = n$. Now, assume that $\hat{\pi} := (\hat{\pi}_1, \ldots, \hat{\pi}_n)$, using natural projections onto $\hat{\psi}(U)$. Therefore

$$\omega \circ (f_1, \ldots, f_n)_{|U} = 0.$$

Furthermore, let $\hat{p} = \hat{\psi}(p)$, $\hat{v}_i = D_{i\hat{p}}$. Then $\hat{v}_i \in T_{\hat{\psi}(\hat{p})}\hat{\psi}(U)$ such that for all $\hat{\omega} \in C^\infty(\hat{\psi}(U))$, $D_{i\hat{p}}\hat{\omega}$ designates the i-th partial derivative of $\hat{\omega}$ evaluated at $\hat{p}$. Then $\hat{v}_1, \ldots, \hat{v}_n$ are linearly independent and $\hat{v}_i(\hat{\pi}_j) = \delta_{ij} \quad (i, j = 1, \ldots, n)$ since dim $T_{\hat{\psi}(\hat{p})}\hat{\psi}(U) = n$. It follows that

$$\begin{aligned} 0 &= \hat{v}_i(\omega \circ \hat{\pi}) \\ &= \hat{v}_i(\omega \circ (\hat{\pi}_1, \ldots, \hat{\pi}_n)) \\ &= \textstyle\sum_{j=1}^n D_j\omega(\hat{\pi}_1(\hat{p}), \ldots, \hat{\pi}_n(\hat{p})) \cdot \hat{v}_i(\hat{\pi}_j). \end{aligned}$$

In other words,

$$\sum_{j=1}^n D_j\omega(\hat{\pi}(p)) \cdot \hat{v}_i(\hat{\pi}_j) = 0$$

which leads to $D_j\omega(f_1(p), \ldots, f_n(p)) = 0$. Equivalently, one has the required identity

$$\nabla\omega(f_1(p), \ldots, f_n(p)) = 0.$$

($\Longleftarrow$) Conversely, let $\mathcal{F} = \{f_1, \ldots, f_n\} \subseteq \mathcal{F}_M$ be given such that

$$\omega \circ (f_1, \ldots, f_n)_{|U} = 0 \quad \text{implies that} \quad \nabla\omega(f_1(p), \ldots, f_n(p)) = 0. \tag{4.2}$$

We need to show that $\mathcal{F}_0$ is $\mathcal{M}$-independent at $p \in M$. Suppose that $\mathcal{F}_0$ is not $\mathcal{M}$-independent at p. Then there exist an open neighborhood U of p, an $i_0 \in \{1, \ldots, n\}$ and a smooth function $\theta \in R^{n-1}$ such that

$$f_{i_0|U} = \theta \circ (f_2, \ldots, f_{i_0-1}, f_{i_0+1}, \ldots, f_n)_{|U}.$$

Note that the proof holds for any index $i \in \{1, \ldots, n\}$. We may consider $i_0 = 1$, for instance, then

$$f_{1|U} = \theta \circ (f_2, \ldots, f_n)_{|U}$$

which is equivalent to

$$(f_1 - \theta(f_2, \ldots, f_n))_{|U} = 0. \tag{4.3}$$

This equation induces at each point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ a function

$$\omega : \mathbb{R}^n \longrightarrow \mathbb{R}$$

such that

$$(x_1, \ldots, x_n) \quad \mapsto \omega(x_1, \ldots, x_n) = x_1 - \theta(x_2, \ldots, x_n)$$

whose partial derivative with respect to x_1 is not equal to zero. (Note that in this case $D_1\omega = 1$). This is the negation of Equation 4.2. Thus $f_1, \ldots, f_n$ are $\mathcal{M}$-independent at p. $\square$

Proposition 4.2.4 *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space, $p \in M$ and $v_1, \ldots, v_k$ a set of linearly independent tangent vectors at p. Then any set $\{f_1, \ldots, f_k\}$ of structure functions satisfying the condition*

$$v_i(f_j) \; = \; \delta_{ij}, \qquad \text{where} \;\; i, j = 1, \ldots, k \tag{4.4}$$

is $\mathcal{M}$-independent at p.

Proof. We need to show that if $\{f_1, \ldots, f_k\}$ satisfies Equation 4.4, then the condition in Equation 4.2 is also satisfied. Let U be an open neighborhood of p and $\omega \in C^\infty(\mathbb{R}^k)$ such that $\omega \circ (f_1, \ldots, f_k)_{|U} = 0$. Then

$$\forall j = 1, \ldots, k, \qquad v_j(\omega \circ (f_1, \ldots, f_k)) = 0.$$

That is,

$$\forall j = 1, \ldots, k, \qquad \sum_{i=1}^{k} D_i\omega(f_1(p), \ldots, f_k(p)) \cdot v_j(f_i) = 0.$$

According to Equation (4.4) satisfied by $\{f_1, \ldots, f_k\}$, we shall have

$$(D_j\omega(f_1(p), \ldots, f_k(p)))_{j=1,\ldots,n} \; = \; \nabla\omega(f_1(p), \ldots, f_k(p)) = 0.$$

Then from Proposition 4.2.3, the set $\{f_1, \ldots, f_k\}$ is $\mathcal{M}$-independent. $\square$

Lemma 4.2.4 *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space and $\mathcal{B}$ an $\mathcal{M}$-basis at $p \in M$ with respect to $\mathcal{F}_M$. Then every smooth function defined on $\mathcal{B}$ is a restriction of a unique tangent vector at p.*

Proof. Let $\mathcal{B} \subseteq \mathcal{F}_M$ be an $\mathcal{M}$-basis at $p \in M$, $v_0 : \mathcal{B} \longrightarrow \mathbb{R}$ a smooth function on $\mathcal{B}$ and $\alpha_1, \ldots, \alpha_n \in \mathcal{B}$. Let $\omega \in C^\infty(\mathbb{R}^n)$ and $\alpha \in \mathcal{F}_M$ be such that there exists a neighborhood $U \in \tau_{\mathcal{F}}$ of p satisfying $\alpha_{|U} = \omega \circ (\alpha_1, \ldots, \alpha_n)_{|U}$. Consider now a map v defined such that

$$\sum_{i=1}^{n} D_i\omega(\alpha_1(p), \ldots, \alpha_n(p)) \cdot v_0(\alpha_i) \;=\; v(\alpha). \tag{4.5}$$

We show that the map v is well defined as a functional on $\mathcal{F}_M$. That is, v is independent of the choice of any ω in $C^\infty(\mathbb{R}^n)$. For, if we assume that another map, $\theta \in C^\infty(\mathbb{R}^n)$, also satisfies the condition

$$\alpha_{|U} = \theta \circ (\alpha_1, \ldots, \alpha_n)_{|U}.$$

It follows that

$$\omega \circ (\alpha_1, \ldots, \alpha_n)_{|U} = \theta \circ (\alpha_1, \ldots, \alpha_n)_{|U}. \tag{4.6}$$

Then

$$(\omega - \theta) \circ (\alpha_1, \ldots, \alpha_n)_{|U} = 0. \tag{4.7}$$

Since $\alpha_1, \ldots, \alpha_n \in \mathcal{B}$, they satisfy the condition in Equation (4.2). Hence

$$\text{for all } \; j = 1, \ldots, n, \qquad D_j(\omega - \theta)(\alpha_1(p), \ldots, \alpha_n(p)) = 0.$$

That is,

$$\forall j = 1, \ldots, n, \qquad D_j\omega(\alpha_1(p), \ldots, \alpha_n(p)) = D_j\theta(\alpha_1(p), \ldots, \alpha_n(p)).$$

Then we may write

$$\sum_{i=1}^{n} D_i\omega(\alpha_1(p), \ldots, \alpha_n(p)) \cdot v_0(\alpha_i) = \sum_{i=1}^{n} D_i\theta(\alpha_1(p), \ldots, \alpha_n(p)) \cdot v_0(\alpha_i).$$

The defined function has the property mentioned in Lemma 4.1.3 satisfied by all tangent vectors defined on the basis $\mathcal{B}$. This vector v is unique. □

Proposition 4.2.5 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a Frölicher space,* $p \in M$ *and* $\mathcal{B}$ *an* $\mathcal{M}$*-basis with respect to the structure* $(\mathcal{C}_M, \mathcal{F}_M)$*. Then the map* η *of* T_pM *into* $\mathbb{R}^{\mathcal{B}}$ *defined by*

$$(\forall v \in T_pM), \quad \eta(v) = v_{|\mathcal{B}}$$

is an isomorphism of linear spaces.

Proof. Note that the map η is linear, by definition, surjective according to Lemma 4.2.4 above and injective since one can see that its kernel contains only the null vector. □

Corollary 4.2.1 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a Frölicher space,* $p \in M$ *and* $\mathcal{B}$ *any* $\mathcal{M}$*-basis at* p *with respect to* $(\mathcal{C}_M, \mathcal{F}_M)$*. Then the following holds:*
(a) If $Card\ \mathcal{B} < \infty$ *then* $\dim T_pM = Card\ \mathcal{B}$*,*
(b) If $Card\ \mathcal{B} = \infty$ *then* $\dim T_pM = 2^{Card\ \mathcal{B}}$*.*

4.3 Concept of dimension

Definition 4.3.1 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a Frölicher space,* $p \in M$ *and* U *an open neighborhood of* p*. If* $Card\ \mathcal{B} = n$ *for each* $\mathcal{M}$*-basis at* p*, then* n *is called the* **$\mathcal{M}$-dimension** *of* M *at* p *(or* **differential dimension** *at* p*).*

Note that the so-called $\mathcal{M}$-dimension at a point is a local concept. A Frölicher space may have distinct $\mathcal{M}$-dimensions at distinct points. Later we shall make this concept more precise.

Chapter 5

Particular classes of Frölicher spaces

5.1 Frölicher spaces of constant dimension

Definition 5.1.1 *A Frölicher space is said to be of* **constant dimension** *if either*
(i) dim $T_pM =$ *dim* T_qM *for any* $p, q \in M, p \neq q$ *and for all* $v \in T_pM$*, there exists a vector field* X *on* M *such that* $X(p) = v$*; or*
(ii) for each point $p \in M$ *there exists an open neighborhood* U *of* p *in* M *and a local basis of vector fields over* U *making* $\mathfrak{X}(U)$ *a free module.*

Examples.

(1) The Euclidean space $\mathbb{R}^n$ is a Frölicher space of constant dimension n

whose canonical $\mathcal{M}$-basis is the set of all n smooth natural projections. Also $\dfrac{\partial}{\partial x^1}, \ldots, \dfrac{\partial}{\partial x^n}$ form a basis for the module of vector fields over an open neighborhood of a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. □

(2) An n-dimensional smooth manifold M is a Frölicher space of constant $\mathcal{M}$-dimension n. On the one hand, all tangent spaces on manifolds are vector spaces of a same dimension. On the other hand, an $\mathcal{M}$-basis for M is described as follows. For each $x \in M$, recall that in any local chart, (U, φ), $(x_1, \ldots, x_n) \in \varphi(U) \subseteq \mathbb{R}^n$ are called **local coordinates** for x. If $\hat{\pi}_1, \ldots, \hat{\pi}_n$ are natural projections on $\mathbb{R}^n$ restricted to $\varphi(U)$, then

$$\begin{aligned} x_i &= \hat{\pi}_i(x_1, \ldots, x_n) \\ &= \hat{\pi}_i(\varphi(x)) \\ &= (\hat{\pi}_i \circ \varphi)(x) \qquad \forall i = 1, \ldots, n. \end{aligned}$$

Then $\hat{\pi}_1, \ldots, \hat{\pi}_n$ and φ induce smooth functions denoted by $\zeta_1, \ldots, \zeta_n$ in the neighborhood U of x such that

$$\zeta_i : U \subseteq M \longrightarrow \mathbb{R};\ \zeta_i := \hat{\pi}_i \circ \varphi \ \ \forall i = 1, \ldots, n.$$

Since φ is a homeomorphism of U onto $\varphi(U) \subseteq \mathbb{R}^n$ and $\hat{\pi}_1, \ldots, \hat{\pi}_n$ are $\mathcal{M}$-independent at $\varphi(x)$, $\zeta_1, \ldots, \zeta_n$ define an $\mathcal{M}$-basis at each $x \in M$. Recall that functions $f : M \longrightarrow \mathbb{R}$ on smooth manifolds are smooth if and only if $f \circ \varphi^{-1}$ is smooth for each chart φ [61, p. 41]. It follows that, locally smooth functions on a smooth manifold M are smooth on the whole of M, as it is required for a Frölicher structure.

□

5.2 Pre-Frölicher and DS-Frölicher spaces

In the second chapter, we noted that for every Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$ there is a natural differential structure in the Sikorski sense turning $(M, \mathcal{F}_M)$ into a differential space (d-space). The converse is clearly not true since, for $(M, \mathcal{F})$ to be a d-space, we need no structure curves. But in becoming a Frölicher space it happens that the generated structure $(\Gamma\mathcal{F}, \Phi\Gamma\mathcal{F})$ is finer than the differential structure. That is, $\mathcal{F} \subseteq \Phi\Gamma\mathcal{F}$ with the same structure curves. The geometry on a differential space should be similar to that of its generated Frölicher space if the following equality holds:

$$\Phi\Gamma\mathcal{F} = \mathcal{F}.$$

Observe that the class of such spaces is nonempty. In fact it obviously contains the Euclidean space $\mathbb{R}^n$ and all smooth manifolds. Note that the canonically induced Frölicher space $\mathbb{Q}$ of rational numbers is not a member of this class. We emphasize the fact that the geometry on a pre-Frölicher space coincides with that of the generated Frölicher space, on the so-called of class **DS**.

Definition 5.2.1 *A differential space* $(M, \mathcal{F})$ *is said to be a* **pre-Frölicher space** *if its differential structure coincides with the set* $\mathcal{F}_M$ *of structure functions in the generated Frölicher structure* $(\Gamma\mathcal{F}, \Phi\Gamma\mathcal{F})$ *on* M. *A Frölicher space* $(M, \mathcal{C}_M, \mathcal{F}_M)$, *where the smooth structure* $(\mathcal{C}_M, \mathcal{F}_M)$ *is generated by that of the pre-Frölicher space is called Frölicher space of class* **DS**, *or* **DS**-*Frölicher space, for short.*

Theorem 5.2.1 *Let* (M_1, C_1) *and* (M_2, C_2) *be differential spaces. If* (M_1, C_1) *is a pre-Frölicher space and* $\varphi : (M_1, C_1) \longrightarrow (M_2, C_2)$ *is a diffeomorphism of differential space, then* (M_2, C_2) *is a pre-Frölicher space.*

Proof. From Lemma 2.7.2, the diffeomorphism $\varphi : (M_1, C_1) \longrightarrow (M_2, C_2)$ between differential spaces is a diffeomorphism of the corresponding Frölicher spaces $(M_1, \Gamma C_1, \Phi\Gamma C_1)$ and $(M_2, \Gamma C_2, \Phi\Gamma C_2)$. Assume that (M_1, C_1) is a pre-Frölicher space. That is, $\Phi\Gamma C_1 = C_1$. We need to show that $\Phi\Gamma C_2 = C_2$. From the assumption, we have

$$(\varphi^{-1})^*(\Phi\Gamma C_1) = (\varphi^{-1})^* C_1.$$

Then $(\varphi^{-1})^*(\Phi\Gamma C_1) = C_2$ according to the first equality in Lemma 2.7.2. Also, one has that $(\varphi^{-1})^*(\Phi\Gamma C_1) = \Phi\Gamma C_2$ as shown in the last identity of Lemma 2.7.2. Hence, $\Phi\Gamma C_2 = C_2$. ☐

Proposition 5.2.1 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a* **DS***-Frölicher space of constant dimension* n*. Then for an open neighborhood* U *of* p*, there exist a local basis* $\{W_1, \ldots, W_n\}$ *of the module* $\mathfrak{X}(U)$ *at any point* $p \in M$ *and* n *smooth functions* $\alpha_1, \ldots, \alpha_n \in \mathcal{G}_p(U)$ *such that* $W_i(\alpha_j) = \delta_{ij}$*, where* δ_{ij} *is the Kronecker delta function.*

Proof. The proof is based on a straightforward interpretation of Definition 5.1.1. Let $p \in M$ and U an open neighborhood of p. Then $W_1, \ldots, W_n$ are linearly independent tangent vector fields in $\mathfrak{X}(U)$. Therefore, $W_1(p), \ldots, W_n(p)$ are linearly independent tangent vectors in T_pM. Thus, from Lemma 3.1.4, we have that $W_i(p)(\alpha_j) = \delta_{ij}$. Since it holds for all $p \in M$, we conclude that $W_i(\alpha_j) = \delta_{ij}$. ☐

Lemma 5.2.1 *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a* **DS***-Frölicher space of constant dimension n. If $v^{1*}, \dots v^{n*} \in T_p^*M$ is a basis at the point $p \in M$ then there exist a neighborhood U of p in M and smooth functions $\alpha_1, \dots, \alpha_n$ in $\mathcal{G}_p(U)$ such that $v^{*i} = (d\alpha_i)_p$.*

Proof. Assume that $v^{1*}, \dots, v^{n*}$ form a basis B on T_p^*M. Then there exists a basis of vectors $v_1, \dots, v_n$ which is dual to B. It follows from Lemma 3.1.4 that there exist smooth functions $\alpha_1, \dots, \alpha_n \in \mathcal{G}_p(U)$ such that $v_i(\alpha_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function. Hence by Equation (3.1), the latter can be written as $(d\alpha_j)_p(v_i) = \delta_{ij}$. That is $(d\alpha_j)_p = v^{i*}$. □

Corollary 5.2.1 *If $(M, \mathcal{C}_M, \mathcal{F}_M)$ is a* **DS***-Frölicher space of constant dimension n, then for every point $p \in M$ there exist n smooth functions $\alpha_1, \dots, \alpha_n \in \mathcal{G}_p(U)$ such that $\{(d\alpha_1)_p, \dots, (d\alpha_n)_p\}$ is a basis on T_p^*M corresponding to a basis $\{v_1, \dots, v_n\}$ of the tangent space T_pM.*

5.3 Pseudomanifolds

In this section, we investigate the class of Frölicher spaces which are locally diffeomorphic to non-open subsets of the Euclidean space $\mathbb{R}^n$.

Definition 5.3.1 *A Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$ is called a* **pseudomanifold of dimension** *n, where $n \in \mathbb{N}$, if for every point $p \in M$ there exist an open neighborhood V of p in M and a diffeomorphism φ of $(V, \mathcal{C}_V, \mathcal{F}_V)$ onto $\varphi(V)$ which is a closed n-dimensional subspace $(N, \mathcal{C}_N, \mathcal{F}_N)$ of $(\mathbb{R}^n, \mathcal{C}, \mathcal{F})$.*

It follows from the definition above that for all $q \in M$, there exist $V \in \tau_{\mathcal{F}}$, $q \in V$ and a diffeomorphism $\varphi : V \longrightarrow \varphi(V) := N$; $\varphi(V) \subseteq \mathbb{R}^n$ such that

$$\overline{int(N)} = N \text{ and } \dim T_qN = \dim T_q\mathbb{R}^n.$$

Definition 5.3.2 *Let* $(M, C_M, \mathcal{F}_M)$ *be a pseudomanifold of dimension* n *and* x *a point in* M. *We say that a structure curve* c **passes through** x *if* $c(a) = x$, *where* $a \in \mathbb{R}$. *The set of curves passing through* x *with foot point* $0 \in \mathbb{R}$ *will be denoted by* $C_M^{0,x}$.

Definition 5.3.3 *Let* $(M, C_M, \mathcal{F}_M)$ *be a pseudomanifold of dimension* n *and for* $x \in M$, *let* $\{f_1, \ldots, f_n\}$ *be an* $\mathcal{M}$*-basis at* x. *In a neighborhood of* x, *let* φ *be a diffeomorphism defined by* $\varphi(x) := (f_1(x), \ldots, f_n(x))$. *Two smooth curves* c_1 *and* c_2 *in* $C_M^{0,x}$ *are said to be* **tangent at** x *if*

$$(d(\varphi \circ c_1))_0 = (d(\varphi \circ c_2))_0. \tag{5.1}$$

Lemma 5.3.1 *Let* $(M, C_M, \mathcal{F}_M)$ *be a pseudomanifold of dimension* n, $x \in M$ *and* $c \in C_M^{0,x}$. *Let* $\varphi : (M, C_M, \mathcal{F}_M) \longrightarrow (\mathbb{R}^n, C, \mathcal{F})$ *be a map defined by* $\varphi(x) := (f_1(x), \ldots, f_n(x))$ *in a neighborhood of each point* $x \in M$, *where* $\{f_1, \ldots, f_n\}$ *is an* $\mathcal{M}$*-basis at* x. *Then*

$$c_1 \sim c_2 \Longleftrightarrow (d(\varphi \circ c_1))_0 = (d(\varphi \circ c_2))$$

defines an equivalence relation on $C_M^{0,x}$. *Also, Equation (5.1) above only depends on the basis* $\{f_1, \ldots, f_n\}$ *at* x *and not on the choice of the smooth functions.*

Proof. It is obvious that $\sim$ is an equivalence relation. Let us show that Equation 5.1 does not depend on the choice of the smooth functions $f_1, \ldots, f_n$. Suppose that $\{g_1, \ldots, g_n\}$ is another $\mathcal{M}$-basis. Let

$$\psi := (g_1, \ldots, g_n) : U \longrightarrow \psi(U) \subseteq \mathbb{R}^n.$$

We know that $\psi(U)$ is a Frölicher subspace of $\mathbb{R}^n$ and both φ, ψ are diffeomorphisms of U onto $\varphi(U)$ and $\psi(U)$ respectively, according to Lemma 4.2.1. Then

$$\begin{aligned}
(d(\varphi \circ c_1))_0 &= (d(\varphi \circ \psi^{-1} \circ \psi \circ c_1))_0 \\
&= (d(\varphi \circ \psi^{-1}))_{\psi(x)}(d(\psi \circ c_1))_0 \\
&= (d(\phi \circ \psi^{-1}))_{\psi(x)}(d(\psi \circ c_2))_0 \\
&= (d(\varphi \circ \psi^{-1} \circ \psi \circ c_2))_0 \\
&= (d(\varphi \circ c_2))_0.
\end{aligned}$$

□

We shall denote by $\hat{c}$ an equivalence class of curves so obtained and use the construction above in the following lemma.

Lemma 5.3.2 *Let M be a pseudomanifold of dimension n and $p \in M$. Let U be a neighborhood of p in M and $\hat{c}$ the equivalence class of all the curves tangent at p in the sense of Lemma 5.3.1. Then the map $\eta : \mathbb{R}^n \longrightarrow T_pCM$ given by*

$$v \mapsto \eta(v) = \hat{c},$$

is a bijection.

Proof.

(1) Let us note first that if M is a Frölicher space, U a neighborhood of a

point $p \in M$ and $\varphi = (f_1, \dots, f_n)$ where $f_1, \dots, f_n$ form an $\mathcal{M}$-basis at p. Then the formula

$$c_v(t) := \varphi^{-1}(\varphi(x) + tv),$$

where $t \in \mathbb{R}$ and $v \in \mathbb{R}^n$, clearly defines a smooth curve which passes through $p \in M$.

(2) For all v and v' in $\mathbb{R}^n$, the identity $\eta(v) = \eta(v')$ means that $\hat{c}_v$ is tangent to $\hat{c}_{v'}$ at p. It follows from Definition 5.3.3 and from (1) above that

$$\begin{aligned} (d(\varphi \circ c_v))_0 &= (d(\varphi \circ \varphi^{-1})(\varphi(p) + tv))_0 \\ &= (d(\varphi(x) + tv))_0 \\ &= v. \end{aligned}$$

Similarly, one shows that $(d(\varphi \circ c_{v'}))_0 = v'$. Hence $v = v'$. Therefore, η is injective.

(3) Let $\beta \in T_pCM$ and c its representative. Let $v = (d(\varphi \circ c))_0$ be a vector in $\mathbb{R}^n$. It follows from (2) above that

$$(d(\varphi \circ c_v))_0 = (d(\varphi \circ c))_0.$$

Then c_v is tangent to c at p. Hence $\beta = \hat{c} = \hat{c}_v = \eta(v)$. ☐

Theorem 5.3.1 *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a pseudomanifold. Then at any point $p \in M$, the tangent space coincides with the tangent cone space.*

Proof. Since M is a space of constant dimension n, then in a neighborhood of every point $p \in M$ there exists a diffeomorphism of U onto a closed subset of $\mathbb{R}^n$. Moreover, T_pCM is a vector space and there exists an isomorphism of $\mathbb{R}^n$ onto T_pCM. Then dim $T_pCM = n$ and since dim $T_pM = n$, it follows that T_pCM is isomorphic to T_pM for all $p \in M$. $\square$

Chapter 6

Symplectic Frölicher Spaces

In studying dynamical systems with applications to quantum theory, a physicist expects the phase space of the system to be modelled on a differential construct equipped with a Poisson structure or with a symplectic structure. The geometry of this space is usually that of smooth manifolds. This chapter is aimed at introducing a model of phase space more general than the one which is usually considered in differential geometry. We show that although there are no local charts in the definition of Frölicher spaces, a symplectic structure can be defined. This structure plays a major role in building a generalized symplectic framework for analytic dynamics.

It is shown that the fundamental Darboux form holds for a symplectic form ω on a symplectic Frölicher space of constant dimension. We also deal with Hamiltonian systems as well as Lagrangian systems, using the Legendre transformation which links them. These usual concepts are defined in the Frölicher space language.

6.1 Basic concepts

Recall that the Frölicher spaces we deal with in the remaining part of this work are of class **DS**. As stated and proved in Section 5.2 above, geometric concepts in this class of Frölicher spaces coincide with those of differential spaces which generate them. Therefore, the main reference is W. Sasin ([54], [55]).

Definition 6.1.1 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a* **DS**-*Frölicher space and* $\mathfrak{X}(M)$ *the* $\mathcal{F}_M$-*module of all smooth derivations on* $\mathcal{F}_M$. *Then a* **global** k**-differential form** *on* M *is a* $\mathcal{F}_M$-k-*linear mapping*

$$\omega : \mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M) \longrightarrow \mathcal{F}_M,$$

where $k = 1, 2, \ldots$.

Note that the sum $\Omega(M) = \bigoplus_{k \geq 0} \Omega^k(M)$, where $\Omega^0(M) := \mathcal{F}_M$, is a graded algebra over $\mathbb{R}$ under the canonical operations of addition and scalar multiplication. In this algebra we define the operation of exterior differentiation by the usual formulas as follows.

Definition 6.1.2 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a* **DS**-*Frölicher space.*

(1) If $\alpha \in \Omega^0(M)$ *then* $(\underline{d}\alpha)(X) = X(\alpha)$ *for any* $X \in \mathfrak{X}(M)$;

(2) If $k \geq 1$ *and* $\omega \in \Omega^k(M)$ *then*

$$(\underline{d}\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \ldots, \widetilde{X}_i, \ldots, X_{k+1}))$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widetilde{X}_i, \ldots, \widetilde{X}_j, \ldots, X_{k+1}),$$

for any $X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$.

Clearly, this operator $\underline{d} : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ satisfies the axioms of the usual exterior derivative.

Definition 6.1.3 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a* **DS**-*Frölicher space and* TM *its operational tangent bundle. Let* T^kM *be the smooth cartesian product given by*

$$T^kM = \{(v_1, \ldots, v_k) \in TM \times \ldots \times TM; \quad \pi(v_1) = \ldots = \pi(v_k)\},$$

for $k = 1, 2, \ldots$ *and* π *the projection of the bundle* TM. *Then we call* **pointwise differential form** *a smooth mapping* $\omega : T^kM \longrightarrow \mathbb{R}$ *such that* $\omega_{|T_pM \times \ldots \times T_pM}$ *is skew-symmetric* $\mathbb{R}$-k-*linear is said to be on* M *for each* $p \in M$.

We shall denote by $A^k(M)$, for $k = 1, 2, \ldots$ the set of all pointwise differential forms on M. Note that the direct sum $A(M) = \bigoplus_{k \geq 0} A^k(M)$, where $A^0 := \mathcal{F}_M$ is made into a graded algebra over $\mathbb{R}$. As stated in [54, p.3], it is proved that the graded algebras $A(M)$ and $\Omega(M)$ are isomorphic on a differential space of constant dimension. This holds for **DS**-Frölicher spaces as well.

Let E be a linear Frölicher space of dimension n. That is, E has both Frölicher structure and linear structure with smooth addition and smooth scalar multiplication. Note that a tangent space at any point of E coincides with E. Then the algebraic dimension n of E is its $\mathcal{M}$-dimension.

Definition 6.1.4 *A* **symplectic structure** *on* E *is a bilinear smooth form* ω *which is skew-symmetric and nondegenrate.*

Definition 6.1.5 *A symplectic structure on a linear Frölicher space E is a skew-symmetric and nondegenerate bilinear smooth form ω on E.*

Definition 6.1.6 *A* **symplectic linear Frölicher space** *is a linear Frölicher space E which carries a smooth symplectic structure ω.*

Lemma 6.1.1 *Let E be a vector space of finite dimension n and E^* its dual. Let $\wedge^2(E^*)$ denote the space of antisymmetric covariant tensors on E. For any $\omega \in \wedge^2(E^*)$ there exists a basis $\{f^{*1}, \ldots, f^{*n}\}$ of E^* such that*

$$\omega^* = f^{1*} \wedge f^{2*} + \ldots + f^{2(p-1)*} \wedge f^{2p*}.$$

The integer p depends only on ω. That is, p is independent of the choice of the basis.

Proof. We refer the reader to the proof in [30, pp.152-153]). ☐

Corollary 6.1.1 *If ω is a symplectic form on a finite dimensional vector space E then the space is even dimensional, that is, dim $E = 2p$, where p is a nonzero integer.*

Example: Consider the skew-symmetric and nondegenrate bilinear form ω_0 on $(\mathbb{R}^{2n}, \mathcal{C}, \mathcal{F})$, given by

$$\omega_0 : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}; \quad (\overline{x}, \overline{y}) \mapsto \omega_0(\overline{x}, \overline{y}) = \sum_{i=1}^{n} x_{i+n}y_i - x_i y_{i+n},$$

where $\overline{x}, \overline{y} \in \mathbb{R}^{2n}$. Then ω_0 defines a symplectic canonical structure on $\mathbb{R}^{2n}$. The pair $((\mathbb{R}^{2n}, \mathcal{C}, \mathcal{F}), \omega_0)$, where $\mathbb{R}^{2n}$ is considered with its canonical Frölicher structure, is a symplectic linear Frölicher space.

Definition 6.1.7 *A map* $\varphi : (E,\omega) \longrightarrow (E',\omega')$ *between linear Frölicher spaces is called* **symplectic smooth map** *if*

$$\varphi^*\omega' = \omega, \quad i.e.$$

$$\omega(x,y) = \omega'(\varphi(x),\varphi(y))$$

for all $x, y \in M$. *A smooth symplectic isomorphism on a symplectic linear Frölicher space* M *is called either a* **smooth symplectic transformation,** *or an* $\mathcal{M}$**-symplectic transformation** *or* $\mathcal{M}$**-symplectomorphism**.

Observe that if dim M = dim M', such a map φ always exists. In this case, M and M' are said to be **symplectically equivalent**.

Proposition 6.1.1 *Let* E *be a linear symplectic Frölicher space of dimension* n. *The group of all smooth symplectic transformations on* E, *denoted by* $Sp_{\mathcal{M}}(n,\mathbb{R})$, *is a Frölicher-Lie group with respect to the composition of maps and*

$$A \in Sp_{\mathcal{M}}(n,\mathbb{R}) \Longleftrightarrow A^T J A = J,$$

where J *is a* $2n \times 2n$*-matrix such that* $J^2 = -I$, $J^T = -J$.

6.2 Symplectic Frölicher spaces

Definition 6.2.1 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a Frölicher space of constant dimension. We call* $\mathcal{M}$*-symplectic form* ω *(or symplectic form, for short) on* M *an exterior form which is* $\mathcal{M}$*-closed and* $\mathcal{M}$*-nondegenerate.*

Recall that the $\mathcal{M}$-nondegeneracy of the 2-form ω is equivalent to one of the following statements:

(1) For all $X, Y \in \mathfrak{X}(M)$, $\omega(X,Y) = 0$ implies that $X = 0$;

(2) For all $x \in M$ and $Y_x \in T_xM$, if $\omega_x(X_x, Y_x) = 0$ then $X_x = 0$ where ω_x is a skew-symmetric smooth bilinear form associated with the exterior form ω at the point x;

(3) The map $\omega^\flat : TM \longrightarrow T^*M$ is an injective morphism of vector bundles if ω is weakly $\mathcal{M}$-nondegenerate or an isomorphism if ω is strongly $\mathcal{M}$-nondegenerate;

(4) The map $\omega^\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ is an injective morphism of modules if ω is weakly nondegenerate or an isomorphism of modules if ω is strongly nondegenerate. In the latter case, the inverse of $\omega^\flat$ is denoted by $\omega^\sharp$ and satisfies

$$\omega^\flat(X) = i_X\omega = \alpha \ \text{ if and only if } \ (\omega^\flat)^{-1} := \omega^\sharp(\alpha) = X = X_\alpha.$$

Furthermore, a $\mathcal{M}$-closed exterior differential form ω on a Frölicher space M is the one which vanishes under the exterior differentiation, that is $\underline{d}\omega = 0$. Later on, we shall omit $\mathcal{M}$ and simply say that ω is closed and nondegenerate. Also we shall denote by the usual d the operator $\underline{d}$.

Definition 6.2.2 *Let M be a Frölicher space. The pair (M, ω) is called a* **symplectic Frölicher space** *provided ω is a closed and nondegenerate 2-form on M.*

Let (M, ω) be a symplectic Frölicher space. Then it is worth considering each of the concepts of a vector field, exterior form and flow respectively in a restricted set that is called **domain**.

Definition 6.2.3 *A* **domain** *of a vector field X on a Frölicher space M is a set denoted by D_X which is a Frölicher subspace dense and smoothly included in M such that $X : D_X \longrightarrow TD_X$ is also a smooth vector field, where $TD_X \subseteq TM$.*

Proposition 6.2.1 *Let (M, ω) be a symplectic* **DS**-*Frölicher space of constant dimension n. Then n is an even integer.*

Proof. For all $p \in M$, one has that $\dim T_pM = n$ as M is a Frölicher space of constant dimension n. Let ω be a symplectic structure on M. It follows that $\omega_p := \omega(p)$ is a symplectic structure on T_pM. Then from Lemma 6.1.1 it follows that $\dim T_pM = 2m$, with $2m = n$, $m \in \mathbb{N}$, $m \neq 0$. Hence $\dim M = 2m$. ☐

6.2.1 Normal form of symplectic forms

Theorem 6.2.1 *Let $((M, \mathcal{C}_M, \mathcal{F}_M), \omega)$ be a symplectic* **DS**-*Frölicher space of constant dimension $2n$. For every point $x \in M$ there exist an open neighborhood U of x in M and $2n$ smooth functions $q^1, \ldots, q^n, p_1, \ldots, p_n \in \mathcal{G}_x(U)$ such that*

$$\omega_{|U} = \sum_{i=1}^{n} dq^i \wedge dp_i.$$

The latter form is called the **canonical (-normal or Darboux-) form** *of* ω.

Proof. Since $(M, \mathcal{C}_M, \mathcal{F}_M)$ is a **DS**-Frölicher space of constant dimension $2n$, it turns out that for any $x \in M$ there exists an open neighborhood U of x, $W_1, \ldots, W_n, V_1, \ldots, V_n$ in $\mathfrak{X}(U)$ forming a local basis over U and smooth functions $e^1, \ldots, e^n, e^{n+1}, \ldots, e^{2n}$ in the space of germs $\mathcal{G}_x(U)$ such that

$$W_i(e^j) = \delta_{ij}, \quad W_i(e^{n+j}) = 0, \quad V_i(e^j) = 0, \quad V_i(e^{n+j}) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta symbol. From Lemma 5.2.1 it follows that

$$de^1, \ldots, de^n, de^{n+1}, \ldots, de^{2n}$$

form the basis dual to $W_1, \ldots, W_n, V_1, \ldots, V_n$ in the dual space. Then from Lemma 6.1.1 one can choose functions $q^1, \ldots, q^n, p^1, \ldots, p^n$ in $\mathcal{G}_x(U)$ such that $\{dq^1, \ldots, dq^n, dp^1, \ldots, dp^n\}$ form the basis in which $\omega_{|U}$ has the normal form

$$\omega^* := \omega_{|U} = \sum_{i=1}^{n} dq^i \wedge dp_i. \qquad \square$$

Theorem 6.2.2 *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space of constant dimension n and $N \subseteq M$. If $(N, \mathcal{C}_N, \mathcal{F}_N)$ is a Frölicher subspace of (maximal) constant dimension, that is, dim N = dim $M = n$ then every local basis of smooth vector fields $\{W_1, \ldots, W_n\}$ on M induces a local basis of smooth vector fields $\{V_1, \ldots, V_n\}$ on N.*

Proof. Let $p \in M$ and U be an open neighborhood of p in M such that $\{W_1, \ldots, W_n\}$ is a local basis over U. Observe that in this case, the inclusion map $\iota_N : N \hookrightarrow M$ is smooth and $(\iota_N)_{*p} : T_pN \longrightarrow T_pM$ ($\iota(p) = p$ for all $p \in N$) is an isomorphism so that ι_N is an embedding. Consider $V_1, \ldots, V_n$ in $\mathfrak{X}(U \cap N)$ as candidates for the local basis on N. We note that

$$V_i(p) = (\iota_N)^{-1}_{*p} W_i(p).$$

Hence, using Equation 3.1 we have

$$\begin{aligned} V_i(p)(f_{|U\cap N}) &= d(f_{|U\cap N})(V_i(p)) \\ &= d(f_{|U\cap N})(\iota_N)^{-1}_{*p} W_i(p) \\ &= (\iota_{N\cap U})^* d(f)(\iota_N^{-1})_{*p} W_i(p) \\ &= (W_i f)_{|N\cap U}. \end{aligned}$$

This proves that V_i are smooth tangent vector fields. Thus, $V_1, \ldots, V_n$ form a local basis on N. □

Lemma 6.2.1 *Let $((M, \mathcal{C}_M, \mathcal{F}_M), \omega)$ be a pseudomanifold of dimension $2n$ endowed with symplectic structure ω and $N \subseteq M$. If $(N, \mathcal{C}_N, \mathcal{F}_N)$ is a Frölicher subspace of constant maximal dimension then there exists on N a symplectic structure induced by ω.*

Proof. Let $\iota : N \hookrightarrow M$ be the smooth inclusion map. That is, ι is the identity map of M restricted to N and for all $p \in N$ the equality $\dim T_pN = \dim T_pM$ holds and $\iota_{*p} : T_pN \longrightarrow T_pM$ is an isomorphism of vector spaces. Hence,

$$\dim T_pM = 2n = \dim T_pN$$

for all $p \in N$. Then dim $N = 2n$.
Furthermore, for all $v_1, v_2 \in T_pM$ one has

$$\begin{aligned} \iota^*\omega(v_1, v_2) &= \omega(\iota_{*p}v_1, \iota_{*p}v_2) \\ &= \omega(v_1, v_2). \end{aligned}$$

Hence, the pullback $\iota^*\omega$ is a nondegenerate 2-form on N. One concludes that N together with this pullback is a symplectic Frölicher space, turning ι into a symplectic transformation on M. ☐

Definition 6.2.4 *Let (M_1, ω_1) and (M_2, ω_2) be symplectic pseudomanifolds. A smooth map $\varphi : M_1 \longrightarrow M_2$ is said to be* **symplectic or canonical** *if*

$$\varphi^*\omega_2 = \omega_1.$$

That is, for all $x \in M_1$ and all $v, w \in T_xM_1$ one has the following identity

$$\omega_{1x}(v, w) = \omega_{2\varphi(x)}(\varphi_{*x}v, \varphi_{*x}w),$$

where ω_{1x} is the evaluation of ω_1 at the point x, $\omega_{2\varphi(x)}$ is the evaluation of ω_2 at the point $\varphi(x)$ and φ_{*x} is the tangent (or derivative) of φ at x.

It follows from the above definition that

$$\varphi^*\omega_{2|\varphi(U)} = \omega_{1|U}$$

always holds for symplectic pseudomanifolds and turns the chart φ into a canonical diffeomorphism.

Note that the set of all canonical diffeomorphisms of a symplectic pseudomanifold M forms a Frölicher-Lie subgroup of the Frölicher-Lie group Diff(M) of all diffeomorphisms of M with respect to the composition of maps [26]. This subgroup is denoted by $\mathbf{Diff}_{can}(M)$ which is used in plasma dynamics.

6.2.2 Examples of symplectic Frölicher spaces

1. The Euclidean space $\mathbb{R}^{2n}$ is a symplectic Frölicher space of constant dimension with respect to the canonical symplectic structure

$$\omega_0 = \sum_{i=1}^{n} dx^i \wedge dy^i,$$

where x_i and y_i are co-ordinate functions if $\mathbb{R}^{2n}$ is considered as a smooth manifold.

The two following theorems provide our second example of a symplectic linear Frölicher space.

2. Consider the Euclidean Frölicher space $\mathbb{R}^{2n}$ together with its canonical symplectic structure ω_0.

Theorem 6.2.3 *There is a canonical symplectic structure on the linear Frölicher space $C^\infty_K(\mathbb{R}^{2n})$ of smooth functions on $\mathbb{R}^{2n}$ with compact support $K \subseteq \mathbb{R}^{2n}$.*

Proof. The exterior differential k-form $\omega : T^2C^\infty_K(\mathbb{R}^{2n}) \longrightarrow \mathbb{R}$, defined by

$$\omega(v_f, v_g) := \int_K \{f, g\}_0 \, d\mu, \qquad (*)$$

is a symplectic structure on $C^\infty_K(\mathbb{R}^{2n})$. Recall that in $(*)$, μ is the Lebesgue measure on $\mathbb{R}^{2n}$, v_f is a derivation in a direction f and v_g is a derivation in a direction g. $\{f, g\}_0 = \omega_0(X_f, X_g)$ is the Poisson bracket on the Lie algebra of smooth functions on $\mathbb{R}^{2n}$ considered as a symplectic manifold such that X_f (resp. X_g) is the vector field associated with the function f (resp. g).

Then $\int_K \{f, g\}_0 \, d\mu$ is a bilinear and nondegenerate 2-form. Moreover, it can be observed from the definition of ω that $\omega(v_f, v_g)$ in the formula (*) does not depend on a point $h \in C^\infty_K(\mathbb{R}^{2n})$ such that $v_f, v_g \in T_h C^\infty_K(\mathbb{R}^{2n})$. Note that any such h is also a vector since the space under consideration is linear (see [39, 276]). Then from this invariance of ω with respect to vectors in $C^\infty_K(\mathbb{R}^{2n})$, it follows immediately that the exterior derivative of ω is zero. That is, ω is closed. □

3. Let M be a **DS**-Frölicher space of constant dimension n.

Theorem 6.2.4 *There is a canonical symplectic structure on the cotangent bundle* T^*M*, where* M *is Frölicher space of constant dimension* n.

Proof. To obtain this canonical symplectic structure on T^*M, and on TM, we construct local bases of TU and T^*U by the process of **lifting a local basis** over an open neighborhood U of a point $p \in M$ as described below.

Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be an n-dimensional Frölicher space of class **DS** and TM its tangent bundle. Since the associated pre-Frölicher space $(M, \mathcal{F}_M)$ is a differential space of constant dimension n, it is known [60] that TM is $2n$-dimensional. Let $W_1, \ldots, W_n \in \mathfrak{X}(U)$ be a local basis on $(M, \mathcal{C}_M, \mathcal{F}_M)$. Let $\phi : U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U)$ be given by

$$\phi(p, r_1, \ldots, r_n) = \sum_{i=1}^{n} r_i W_i(p)$$

where $p \in U$, $r_i \in \mathbb{R}$ for all $i = 1, \ldots, n$. One can show that the map ϕ is a diffeomorphism. We only show that the composition with the functions generating the Frölicher structure on TM, which is induced on $\pi^{-1}(U)$, are structure functions on $U \times \mathbb{R}^n$. That is,

$$(\alpha \circ \pi \circ \phi)(p, r_1, \ldots, r_n) = \alpha(\pi(\sum_{i=1}^{n} r_i W_i(p))) = \alpha(p),$$

which proves that

$$\alpha \circ \pi \circ \phi = \alpha \circ pr_1, \quad \text{with} \;\; pr_1 : U \times \mathbb{R}^n \longrightarrow U.$$

Similarly, one gets for $d\alpha$ the following

$$\begin{aligned}(d\alpha \circ \phi)(p, r_1, \ldots, r_n) &= d\alpha(\sum_{i=1}^{n} r_i W_i(p)) \\ &= \sum_{i=1}^{n} r_i W_i(p)(\alpha) \\ &= \sum_{i=1}^{n} r_i (W_i \alpha)(p).\end{aligned}$$

Now, fix $r \in \mathbb{R}^n$ and let $\phi_r : U \longrightarrow \pi^{-1}(U)$ be given by

$$\phi_r(p) = \phi(p, r).$$

Then define the lifts $\overline{W}_1, \ldots, \overline{W}_n$ of vector fields $W_1, \ldots, W_n$ by setting

$$\overline{W}_i(v) := (\phi_r)_{*p} W_i(p),$$

where $v \in \pi^{-1}(U)$, $p = \pi(v)$ and $\Phi_r(p) = v$.

In a similar way, fix $p \in M$ and define $\phi_p : \mathbb{R}^n \longrightarrow \pi^{-1}(U)$ by

$$\phi_p(r) = \phi(p, r).$$

Then let $\frac{\partial}{\partial \dot{x}_1}, \ldots, \frac{\partial}{\partial \dot{x}_n} \in \mathfrak{X}(\pi^{-1}(U))$ be the lifts of $\frac{\partial}{\partial r^1}, \ldots, \frac{\partial}{\partial r^n} \in \mathfrak{X}(\mathbb{R}^n)$ defined by

$$\frac{\partial}{\partial \dot{x}_i}(v) = (\phi_p)_{*r} \frac{\partial}{\partial r_{k}}_{|r},$$

where $v \in \pi^{-1}(U)$, with $v = \sum_{i=1}^{n} r_i W_i(p)$. Of course $v \in T_pM$ and

$$\rho : \pi^{-1}(U) \longrightarrow \mathbb{R}^n; \; \rho(v) = r, \quad \text{where} \;\; \mathrm{r} = (\mathrm{r}_1, \ldots, \mathrm{r}_n).$$

It follows that $\{\overline{W}_1, \dots, \overline{W}_n, \ \frac{\partial}{\partial \dot{x}_1}, \dots, \frac{\partial}{\partial \dot{x}_n}\}$ is a local vector basis on $\pi^{-1}(U)$. In the similar way, we construct a local vector basis on the cotangent bundle as follows. Consider the canonical projection $\tau : T^*M \longrightarrow M$ sending each covector onto its attachment point and let $\psi : U \times \mathbb{R}^n \longrightarrow \tau^{-1}(U)$ be given by the formula

$$\phi(p,r) = \sum_{i=1}^{n} r_i W_i^*, \tag{6.1}$$

where $W_i^*(W_j) = \delta_{ij}$ and U an open neighborhood at the attachment point. The map ψ is a diffeomorphism and satisfies the identity

$$\widetilde{W_i}(\omega) = (\psi_r)_{*p} W_i(p)$$

for $\omega \in \tau^{-1}(U), \quad \tau(\omega) = p, \quad r = (\omega(W_1(p)), \dots, \omega(W_n(p)))$.
Define the lifts $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n} \in \mathfrak{X}(\tau^{-1}(U))$ by

$$\frac{\partial}{\partial y^i}(\omega) = (\psi_p)_{*r} \frac{\partial}{\partial r^i}|r,$$

Note that $\{\widetilde{W}_1, \dots, \widetilde{W}_n, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\}$ is a local vector basis on $\tau^{-1}(U) \subseteq T^*M$ considered as a Frölicher space.

Now, for $\alpha \in T^*M$, let $T_\alpha T^*M$ be the tangent space on the cotangent bundle at the covector α. Define a 1-form on T^*M by the formula

$$\theta_\alpha(v) = \alpha(\tau_{*\alpha} v)$$

for any $v \in T_\alpha(T^*M)$. We shall be interested in the **normal** expression of the 1-form θ. Let $\varphi_1, \dots, \varphi_n \in \mathcal{F}_M$ be the dual functions for the local basis $\{W_1, \dots, W_n\}$ on an open neighborhood U of a point $q \in M$. So,

$W_i(\varphi_j) = \delta_{ij}$, where $i, j = 1, \ldots, n$. That is, $(d\varphi_j)(W_i) = \delta_{ij}$, where $d\varphi_1, \ldots, d\varphi_n$ form the dual basis on T^*M. Observe that the lifts $\widetilde{W}_1, \ldots, \widetilde{W}_n$ of $W_1, \ldots, W_n$ to T^*M satisfy the condition

$$\tau_{*\alpha}\widetilde{W}_i = W_i(\tau(\alpha)),$$

for any covector $\alpha \in \tau^{-1}(U)$. It follows that each i-th coordinate of the covector α in the dual basis $\{d\varphi_1, \ldots, d\varphi_n\}$ is a smooth function of $p_1, \ldots, p_n$ given on $\tau^{-1}(U)$ by

$$p_i(\alpha) = \alpha(W_i(\tau(\alpha))).$$

Moreover, we obtain n other smooth functions $q_1, \ldots, q_n$ given by

$$q^i = \varphi_i \circ \tau, \quad \text{for } i = 1, \ldots, n$$

such that the following identities

$$\widetilde{W}_i(q^j) = \delta_i^j,$$
$$\widetilde{W}_i(p_j) = 0$$

hold for $i, j = 1, \ldots, n$. In fact, for all $\alpha \in \tau^{-1}(U)$ one has

$$\begin{aligned} \tau_{*\alpha}\widetilde{W}_j q^i(\alpha) &= W_j(\varphi_i \circ \tau)(\alpha) \\ &= ((W_j \circ \varphi_i) \circ \tau)(\alpha) \\ &= W_j \circ \varphi_i \\ &= \delta_{ij}, \end{aligned}$$

where δ_{ij} is the Kronecker delta function, since $\tau \circ \alpha = id_M$.

Now we have the following:
$\{\widetilde{W}_1, \ldots, \widetilde{W}_n, \dfrac{\partial}{\partial \dot{x}_1}, \ldots, \dfrac{\partial}{\partial \dot{x}_n}\}$ is local basis on $\mathfrak{X}(\pi^{-1}(U))$, which is a free module over smooth functions defined on $\tau^{-1}(U) \subseteq T^*M$. Then

$$\theta : T_\alpha(T^*M) \longrightarrow \mathbb{R}; \qquad v \mapsto \theta(v) = \alpha(\tau_{*\alpha}v)$$

is such that

$$\begin{aligned}\theta(\widetilde{W}_i\alpha) &= \alpha(\tau_{*\alpha}\widetilde{W}_i(\alpha)) \\ &= \alpha(W_i(\tau(\alpha))) \\ &= p_i(\alpha)\end{aligned}$$

and

$$\begin{aligned}\theta(\frac{\partial}{\partial\dot{x}}(\alpha)) &= \alpha(\tau_{*\alpha}\frac{\partial}{\partial\dot{x}_i}(\alpha)) \\ &= \alpha(\frac{\partial}{\partial\dot{x}_i}(\tau(\alpha))) \\ &= 0.\end{aligned}$$

Then $p_1, \ldots, p_n, q_1, \ldots, q_n$ are $\mathcal{M}$-independent smooth functions such that $\{dp_1, \ldots, dp_n, dq_1, \ldots, dq_n\}$ is a basis of $T^*_\alpha(T^*M)$ in which one can write θ in the following form

$$\theta_{|\tau^{-1}(U)} = \sum_{i=1}^{n} p_i dq^i.$$

Then, taking the exterior derivative of both sides yields the 2-form

$$\omega_0 = \sum_{i=1}^{n} dq^i \wedge dp_i.$$

Observe also that ω_0 is obviously symplectic, making the cotangent bundle a symplectic **DS**-Frölicher space of constant dimension. ☐

Definition 6.2.5 *The exterior form* ω_0 *is called the canonical* 2*-form on the cotangent bundle* T^*M*. The* 1*-form* θ *is called the* **Liouville** *form.*

6.3 Poisson Frölicher spaces

We introduce the Poisson structure on Frölicher spaces in a more general setting which need not be induced by the symplectic structure. We shall show that the two structures are closely related in the sense that every symplectic Frölicher space is Poisson. The converse is not true since the zero bracket turns any Frölicher space into a Poisson space. Observe that we deal with two structures since for most of the configuration spaces of mechanical studies one needs both the Poisson structure on the structure functions as well as the symplectic structure on the phase space.

Definition 6.3.1 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a* **DS**-*Frölicher space. An* **almost Poisson structure** *on* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *is a smooth map*

$$\{\ ,\ \} : \mathcal{F}_M \times \mathcal{F}_M \longrightarrow \mathcal{F}_M$$

sending (f_1, f_2) *to* $\{f_1, f_2\}$ *and satisfying*

(1) $\{\ ,\ \}$ *is real bilinear in* f_1 *and* f_2*,*

(2) $\{f_1, f_2\} = -\{f_2, f_1\}$ *antisymmetry,*

(3) $\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + f_2\{f_1, f_3\}$ *Leibniz identity*

whenever $f_1, f_2, f_3 \in \mathcal{F}_M$. *Note that* $\{\ ,\ \}$ *is also called the* **almost Poisson bracket** *and the pair* $((M, \mathcal{C}_M, \mathcal{F}_M), \{\ ,\ \})$ *is said to be an* **almost Poisson Frölicher space**.

Interpretation. Observe that for a fixed $h \in \mathcal{F}_M$, the almost Poisson bracket $\{\ ,\ \}$ induces a smooth map

$$\{h, \cdot\} : \mathcal{F}_M \longrightarrow \mathcal{F}_M.$$

Of course for all $f \in \mathcal{F}_M$, one has $\{h, \cdot\}(f) : M \longrightarrow \mathbb{R}$. We denote it by

$$p : \mathcal{F}_M \longrightarrow \underline{\mathcal{M}}(M, \mathbb{R}),$$

where $\underline{\mathcal{M}}(M, \mathbb{R})$ denotes the underlying set of the Frölicher space of all functions defined on M. Then from cartesian closedness the map p is smooth if and only if the associated map $\widetilde{p} : \mathcal{F}_M \times M \longrightarrow \mathbb{R}$ is smooth. Observe that p is linear and has the Leibniz property. Let

$$\widetilde{p}(f, x) := (df)_x.$$

It turns out that globally p is a smooth derivation on $\mathcal{F}_M$. Hence $\{h, \cdot\}$ is a smooth vector field induced by h. We shall denote it by X_h.

Furthermore from the skew-symmetry of the bracket $\{\ ,\ \}$ the identity

$$X_h(f) = -X_f(h)$$

follows.

Proposition 6.3.1 *The set of all almost Poisson structures on $(M, \mathcal{C}_M, \mathcal{F}_M)$ is a module over the ring $\mathcal{F}_M$.*

Proof. The proof is a straightforward consequence of the properties of the bracket $\{\ ,\ \}$. ☐

Definition 6.3.2 *A* **Poisson structure** *on a* DS*-Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$ is an almost Poisson structure on $(M, \mathcal{C}_M, \mathcal{F}_M)$ which satisfies the Jacobi identity:*

$$\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0.$$

The pair $((M, \mathcal{C}_M, \mathcal{F}_M), \{\ ,\ \})$ is called a **Poisson DS-Frölicher space.**

Observe that the derivation { , }, together with the commutative multiplication and the addition of functions turn the $\mathcal{F}_M$ into a Lie algebra.

Example. Consider the Abelian Lie algebra of the $\mathcal{F}_M$-module of vector fields $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ on a **DS**-Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$, such that

$$[X_i, X_j] = [X_i, Y_j] = [Y_i, Y_j] = 0.$$

For all $f, g \in \mathcal{F}_M$ define the map { , } by

$$\{f, g\} = \sum_{i=1}^{n} (X_i(f)Y_i(g) - X_i(g)Y_i(f)).$$

Then { , } is a Poisson structure.

Proposition 6.3.2 *Let* $\{\ ,\ \}_1, \{\ ,\ \}_2$ *be Poisson structures on a* **DS**-*Frölicher space* $(M, \mathcal{C}_M, \mathcal{F}_M)$. *Then the difference* $\{\ ,\ \}_1 - \{\ ,\ \}_2$ *is an almost Poisson structure on* $(M, \mathcal{C}_M, \mathcal{F}_M)$.

Proof. $\{\ ,\ \}_1 - \{\ ,\ \}_2$ is obviously bilinear and antisymmetric. One can only check the Leibniz property. In fact, for all $f, g, h \in \mathcal{F}_M$ one has

$$\begin{aligned} h[\{f,g\}_1 - \{f,g\}_2] &= \{f, hg\}_1 - \{f, hg\}_2 \\ &= \{f, k\}_1 g + h\{f, g\}_1 - \{f, h\}_2 g - h\{f, g\}_2 \\ &= (\{f, h\}_1 - \{f, h\}_2)g + (\{f, g\}_1 - \{f, g\}_2)h. \end{aligned}$$ ☐

Chapter 7

Geometrical formalism of mechanics

In this chapter we show that classical mechanical systems can be formulated on a configuration space endowed with the Frölicher smooth structure. Therefore, we shall study both Hamiltonian and Lagrangian systems. Then we observe that the Legendre transformation which links these two different but closely related formulations of mechanics is a smooth map. The chapter ends with a model of gluing symplectic structures from two different configuration spaces whose maps, functions, exterior forms and vector fields agree along a transversal intersection. It will be observed that integral curves can exist on the points of the boundary of pseudomanifolds although they are singularity points.

7.1 Hamiltonian systems in Frölicher spaces

Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a **DS**-Frölicher space of constant dimension $2n$ and ω a symplectic structure on $(M, \mathcal{C}_M, \mathcal{F}_M)$. Let

$$\sigma : (\mathcal{F}_M, \{\ ,\ \}) \longrightarrow (\mathfrak{X}(M), [\ ,\ ])$$

be a smooth map sending $H \in \mathcal{F}_M$ to a vector field $X_H := \sigma(H)$ and satisfying

$$X_H = \omega_\flat^{-1}(dH).$$

So this distinguished vector field X_H which is generated by ω and H is uniquely determined by the equation

$$\omega(X_H, \cdot) = dH(\cdot).$$

The field X_H surely exists if the associated map $\omega_\flat$ is bijective, or on a restricted domain in the general case. The uniqueness follows from the non-degeneracy of ω.

Definition 7.1.1 *A vector field X_H attached to a function $H \in \mathcal{F}_M$ such that $i_{X_H}\omega = dH$ is called the* **global Hamiltonian vector field** *and H is the* **energy function.**

In fact, note that in mechanics X_H is considered as the Hamiltonian system and H carries the total energy of the system, while $((M, \mathcal{C}_M, \mathcal{F}_M), \omega, H)$ is said to be a dynamical (-Hamiltonian in this case) system. As a consequence of this definition, note that the 1-form generated by the symplectic form ω

and the function H is dH. That is,

$$i_{X_H\omega} = \sum_{i=1}^{n}(\partial_{v_j}H.dp_j + \partial_{W_j}H.dq^j)$$

if $X_H = \sum_{i=1}^{n}(r_iW_i + s_iV_i)$ in the basis $\{W_1, \dots, W_n, V_1, \dots, V_n\}$.

Proposition 7.1.1 *Let $((M, \mathcal{C}_M, \mathcal{F}_M), \omega)$ be a symplectic* **DS**-*Frölicher space of constant dimension $2n$. The Hamiltonian vector field X_H associated with the Hamiltonian function $H : M \longrightarrow \mathbb{R}$ can be written as*

$$X_H = \sum_{i=1}^{n}(-\partial_{W_i}H \cdot V_i + \partial_{V_i}H \cdot W_i)$$

with respect to a local basis $\{W_1, \dots, W_n, V_1, \dots, V_n\}$ on $\mathfrak{X}(U)$, where U is an open neighborhood of a point $p \in M$. Hence $X_H = (\partial_{V_i}H, -\partial_{W_i}H)$ in this basis.

Proof. In the basis $\{W_1, \dots, W_n, V_1, \dots, V_n\}$ the vector field X_H can be written as

$$X_H = \sum_{i=1}^{n}(r_iW_i + s_iV_i).$$

Since

$$i_{X_H}\omega = \sum_{j=1}^{n}(\partial_{V_j}H \cdot dp_j + \partial_{W_j}H \cdot dq^j),$$

one has the identification

$$(dq^j \wedge dp_j)(r_jW_j + s_jV_j, \cdot) = \partial_{V_j}H \cdot dp_j + \partial_{W_j}H \cdot dq^j.$$

Expand the left-hand side and use the duality between W_j and V_j, dq^j and dp_j. Hence

$$\begin{aligned}(r_jW_j+s_jV_j)(dq^j\wedge dp_j) &= (r_jW_j(dq_j))\wedge dp_j-(r_jW_j(dp_j))\wedge dq^j\\ &+ (s_jV_j(dq^j))\wedge dp_j-(s_jV_j(dp_j))\wedge dq^j\\ &= r_jdp_j-s_jdq^j.\end{aligned}$$

Identifying again both the sides of the equation

$$r_jdp_j-s_jdq^j=\partial_{V_j}H\cdot dp_j+\partial_{W_j}H\cdot dq^j$$

one obtains

$$r_j=\partial_{V_j}H,\qquad s_j=-\partial_{W_j}H$$

which proves the result. □

It follows that every integral curve c of the Hamiltonian vector field X_H should have $2n$ components $q^1(t),\dots,q^n(t),p_1(t),\dots,p_n(t)$ satisfying the identities

$$\dot{q}_j=\partial_{V_j}H\circ c\qquad\text{and}\qquad\dot{p}_j=-\partial_{W_j}H\circ c$$

with respect to the local basis $\{W_1,\dots,W_n,V_1,\dots,V_n\}$.

Definition 7.1.2 *A vector field on a symplectic* **DS**-*Frölicher space* (M,ω) *is said to be* **locally Hamiltonian** *if at every point* p *of* M *there is an open neighborhood* $U\ni p$ *such that* X *restricted to* U *is Hamiltonian. Hence* $X=X_H$ *and* H *is the* **Hamiltonian function** *associated with* X_H. *That is,*

$$i_{X|U}\omega=dH_{|U}.$$

We shall denote by L_s the set of all locally Hamiltonian vector fields.

Proposition 7.1.2 *A vector field X on a symplectic* **DS**-*Frölicher space (M, ω) is locally Hamiltonian if and only if $L_X\omega = 0$.*

Proof. Note that $X = X_H$ satisfies $i_{X_H} = dH$. Thus $L_X\omega = 0$ follows from Cartan's identity $L_X = i_X d + di_X$, which ends the proof. □

Definition 7.1.3 *A smooth function f on a* **DS**-*Frölicher space is said to be a* **first integral** *of a vector field $X_h = \{h, \cdot\}$ if $\{h, f\} = 0$.*

The following material is standard in the setting of **DS**-Frölicher spaces as well as in the setting of smooth manifolds. It emphasizes the conservative properties of Hamiltonian vector fields in the **DS**-Frölicher spaces setting.

Proposition 7.1.3 *Let $((M, \mathcal{C}_M, \mathcal{F}_M), \omega)$ be a symplectic* **DS**-*Frölicher space and $H \in \mathcal{F}_M$ a Hamiltonian function for the vector field $X = X_H$. Then H is constant on the trajectories of the flow of X and the energy is conserved in the system. That is,*

$$X_H(H) = 0.$$

Proof. We have $X_H(H) = L_{X_H}(H) = \{H, H\} = 0$. □

This shows that H is a first integral of X_H.

Proposition 7.1.4 *Let $((M, \mathcal{C}_M, \mathcal{F}_M), \omega)$ be a symplectic* **DS**-*Frölicher space and $X = X_H$ a locally Hamiltonian vector field with Hamiltonian function H. Then $H \circ c$ is constant if c is an integral curve for X_H. That is, the integral trajectories of a Hamiltonian system lie on the energy surfaces $H = Const.$*

Proof. Note that c is an integral curve for X if and only if $\dot{c}(t) = X(c(t))$ for all $t \in \mathbb{R}$. By the chain rule we show as follows that the derivative of $H \circ c$ vanishes. In fact,

$$\begin{aligned} \frac{d}{dt} H \circ c(t) &= dH(c(t)) \cdot \dot{c}(t) \\ &= dH(c(t))(X_H(c(t))) \\ &= \omega(X_H(c(t)), X_H(c(t))) \\ &= 0 \end{aligned}$$

since ω is skew-symmetric. □

Proposition 7.1.5 *Let $((M, C_M, \mathcal{F}_M), \omega)$ be a symplectic* **DS**-*Frölicher space. The set $L_s = \{X \in \mathfrak{X}(M) \mid L_X\omega = 0\}$ of all locally Hamiltonian vector fields is a real Lie subalgebra of $\mathfrak{X}(M)$.*

Proposition 7.1.6 *If X is a globally Hamiltonian vector field and $Y \in L_s$ then $[X, Y]$ is a globally Hamiltonian vector field.*

Proof. Let $X, Y \in L_s$. Using Cartan's identity $[L_X, i_Y] = i_{[X,Y]}$ one has

$$\begin{aligned} i_{[X,Y]}\omega &= [L_X, i_Y](\omega) \\ &= L_X i_Y \omega - i_Y L_X \omega \\ &= L_X i_Y \omega \\ &= L_X(dh) \\ &= d(L_X h). \end{aligned}$$

where $h = H_{|U}$ is the Hamiltonian H of Y restricted to an open set U. Hence $[X, Y]$ is globally Hamiltonian. □

Observe from Proposition 7.1.6 that L_s^0 is an ideal of L_s and L_s/L_s^0 is an Abelian Lie algebra.

Definition 7.1.4 *A smooth function on a symplectic Frölicher space of class* **DS** *is called a* **Casimir function** *if* $\{f, g\} = 0$ *for any* $g \in \mathcal{F}_M$.

Note that a Casimir function is naturally associated with the zero vector field $\{f, \cdot\}$.

7.2 Lagrangian systems-Legendre transformation

Let $L : TM \longrightarrow \mathbb{R}$ be a smooth function on the tangent bundle considered as a **DS**-Frölicher space associated with the pre-Frölicher space $(M, \mathcal{F}_M)$. Recall that when TM is considered as a velocity phase space of a mechanical system the real-valued function L is called the **Lagrangian** of the system. It is proved that the Legendre transformation $\mathcal{L} : TM \longrightarrow T^*M$ given in term of L by the formula

$$< \mathcal{L}(u), v >= \frac{d}{dt}(L(u + tv))_{|t=0}, \qquad (7.1)$$

for $u, v \in T_pM$, $p \in M$, $t \in \mathbb{R}$ is a smooth map as a map of differential spaces ([58, 38-39]). According to Lemma 2.7.1 it follows that the map $\mathcal{L}$ is smooth as a map of Frölicher spaces.

Definition 7.2.1 *The Lagrangian function* L *is said to be* **regular** *if the*

Legendre transformation $\mathcal{L}$ *is a local diffeomorphism, and* **hyperregular** *if* $\mathcal{L}$ *is a diffeomorphism.*

Definition 7.2.2 *[2, p.214] If* $c : I \subseteq \mathbb{R} \longrightarrow TM$ *is a smooth integral curve of a vector field* X *on* TM*, then the smooth curve* $\pi \circ c : I \longrightarrow M$ *is called a* **base integral curve** *of* X*. Similarly a base integral curve of a covector field* X^* *for a smooth curve* $c : I \subseteq \mathbb{R} \longrightarrow T^*M$ *is* $\tau \circ c : I \longrightarrow M$*.*

Theorem 7.2.1 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be a* **DS**-*Frölicher space provided with a symplectic form* ω *and* L *be the Lagrangian function. If* L *is hyperregular then the Legendre transformation is a symplectomorphism. In this case every Hamiltonian vector field* X_H *is associated with its corresponding Lagrangian vector field* X_E*. The integral curves of* X_E *are mapped by* $\mathcal{L}$ *onto integral curves of* X_H*. Furthermore, the vector fields* X_E *and* X_H *have the same base integral curves.*

Proof. Assume that L is hyperregular. Then we define a form ω_L by

$$\omega_L = \mathcal{L}^*\omega. \tag{7.2}$$

We can see that ω_L is the pullback under $\mathcal{L}$ of the canonical symplectic 2-form on T^*M is a symplectic form on TM. It is nondegenerate. Furthermore one can see that it is closed because

$$d\omega_L = d(\mathcal{L}^*\omega_0) = \mathcal{L}^*d\omega_0 = 0,$$

as d commutes with the pullback map. Thus ω_L turns TM into a symplectic Frölicher space. It follows that

$$\mathcal{L} : (TM, \omega_L) \longrightarrow (T^*M, \omega_0)$$

is a symplectomorphism. In the sequel, let $E : TM \longrightarrow \mathbb{R}$ be the energy function given by

$$E(v) =< \mathcal{L}(v), v > -L(v), \tag{7.3}$$

where $v \in TM$. Then, taking

$$H = E \circ \mathcal{L}^{-1}, \tag{7.4}$$

one has a Hamiltonian function $H : T^*M \longrightarrow \mathbb{R}$. Hence one obtains the Lagrangian tangent vector field X_E on TM and the Hamiltonian tangent vector field X_H on T^*M respectively as follows:

$$i_{X_E}\omega_L = dE, \quad \text{and} \quad i_{X_H}\omega_0 = dH. \tag{7.5}$$

Also observe that

$$\mathcal{L}_* X_E = X_H.$$

That is, for any tangent vector field $Y \in \mathfrak{X}(T^*M)$ there exists a tangent vector field $X \in \mathfrak{X}(TM)$ such that $\mathcal{L}_* X = Y$. Then the identities in (7.5) can be written as

$$\omega_L(X_E, X) = (dE)X, \quad \text{and} \quad \omega_0(X_H, Y) = (dH)Y. \tag{7.6}$$

Now let $v \in TM$ and $w \in T_v(TM)$. We know that $E : TM \longrightarrow \mathbb{R}$ so that

$$X_E : TM \longrightarrow T(TM); \quad v \mapsto X_E(v).$$

Moreover,

$$\mathcal{L}_{*v} : T_v(TM) \longrightarrow T_{\mathcal{L}(v)}(T^*M); \quad w \mapsto \mathcal{L}_{*v}(w) := v^*$$

is an isomorphism of linear Frölicher space since $\mathcal{L}$ is a diffeomorphism. It follows that

$$\begin{aligned}\omega_0(\mathcal{L}_{*v}(X_E(v)), v^*) &= \omega_L(X_E(v), w)\\ &= dE(v)\cdot w\\ &= d(H\circ\mathcal{L})(v)\cdot w\\ &= dH(\mathcal{L}(v))\cdot\mathcal{L}_{*v}(w)\\ &= dH(\mathcal{L}(v))\cdot v^*\\ &= \omega(X_H(\mathcal{L}(v)), v^*).\end{aligned}$$

Recall that ω is nondegenerate and v^* is arbitrary. Therefore,

$$\mathcal{L}_* X_E = X_H.$$

Furthermore we know that $\mathcal{L}$ is a diffeomorphism which maps the fiber T_pM of TM onto the fiber T_p^*M of T^*M over the same point $p \in M$. That is, $\mathcal{L}$ satisfies

$$\tau\circ\mathcal{L} = \pi,$$

where $\tau : T^*M \longrightarrow M$ and $\pi : TM \longrightarrow M$ are the two natural projections.

Let C_E be an integral curve of X_E. As $E = \mathcal{L}^*H$, it follows that $C_H = \mathcal{L}\circ c_E$ is an integral curve of X_H. Hence

$$\pi\circ c_E = \tau\circ\mathcal{L}\circ c_E = \tau\circ c_H.$$

Therefore, the base integral curves $\pi\circ c_E$ and $\tau\circ c_H$ of X_E and X_H respectively, coincide. ☐

7.3 Classical mechanical systems on Frölicher spaces

7.3.1 System without potential

Definition 7.3.1 *A* **DS**-*Frölicher space* $(M, \mathcal{C}_M, \mathcal{F}_M)$, *together with a pseudo-Riemannian smooth form* g *is called a* **pseudo-Riemannian DS-Frölicher space.**

Theorem 7.3.1 *Let* $((M, \mathcal{C}_M, \mathcal{F}_M), g)$ *be a pseudo-Riemannian* **DS**-*Frölicher space of constant dimension* n *and* $L : TM \longrightarrow \mathbb{R}$ *the Lagrangian given by*

$$L(v) = \frac{1}{2} g(v, v),$$

where $v \in TM$. *Then* L *is hyperregular and the associated Legendre transformation satisfies*

$$\mathcal{L}(v)(w) = g(v, w)$$

for all $v, w \in T_pM$, $p \in M$. *The energy function*

$$E = L = \frac{1}{2} g(v, v)$$

and the associated Hamiltonian is given by

$$H(\alpha) = \frac{1}{2} < \mathcal{L}^{-1}(\alpha), \mathcal{L}^{-1}(\alpha) > .$$

Proof. Define a function $\Phi_{v,w} : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\Phi_{v,w}(t) = L(v + tw); \quad \text{for} \quad t \in \mathbb{R}.$$

Then

$$\begin{aligned}\Phi_{v,w}(t) &= \frac{1}{2}g(v+tw,v+tw)\\ &= \frac{1}{2}g(v,v)+tg(v,w)+\frac{1}{2}t^2g(w,w).\end{aligned}$$

Hence

$$\frac{d}{dt}\Phi_{v,w}(t)|_{t=0} = g(v,w) = \mathcal{L}(v)(w) \tag{7.7}$$

is the Legendre transformation $\mathcal{L}$ associated with L which is its fiber derivative. This proves the first part of the theorem.

Now we note that $\mathcal{L}$ is bijective according to the properties of g. For L to be hyperregular we need to show that $\mathcal{L}$ is a diffeomorphism of Frölicher space. It is enough to show that $\mathcal{L}^{-1}$ is smooth since $\mathcal{L}$ was assumed to be smooth. That is, we compose $\mathcal{L}^{-1}$ with generating functions of the structure on TM as follows. For an arbitrary smooth function $\alpha \in \mathcal{F}_M$,

(i) $(\alpha\circ\pi)\circ\mathcal{L}^{-1}$ is a structure function on T^*M. Indeed,

$$(\alpha\circ\pi)\circ\mathcal{L}^{-1} = \alpha\circ(\pi\circ\mathcal{L}^{-1}) = \alpha\circ\tau$$

is a structure function on T^*M.

(ii) Now we show that $d\alpha\circ\mathcal{L}^{-1}$ is also a structure function on T^*M. Let $p\in M$ and U be an open neighborhood of $p\in M$. Let $\{W_1,\ldots,W_n\}$ be a local basis over U. That is, there exist $W_1^*,\ldots,W_n^*$ smooth 1-forms such that $W_i^*(p)(W_j(p)) = \delta_{ij}$, where $i,j=1,\ldots,n$ according to Lemma 3.1.4. When considering the diffeomorphism $\psi: U\times\mathbb{R}^n \longrightarrow T^*U$ such that

$$\psi(p,r_1,\ldots,r_n) = \sum_{i=1}^{n} r_iW_i^*(p)$$

as constructed in Equation (6.1) above, one shall have to work out the smoothness of $d\alpha \circ \mathcal{L}^{-1}$ from that of $d\alpha \circ \mathcal{L}^{-1} \circ \psi$. According to the properties of the 2-form g which is a nondegenerate function, there exist unique vector fields $A_1, \ldots, A_n \in \mathfrak{X}(U)$ such that

$$W_i^*(X) = g(A_i, X)$$

where $X \in \mathfrak{X}(U)$, $i = 1, \ldots, n$. Note that $\mathcal{L}(u) = i_u(g) = g(u, \cdot)$. Then solving Equation (7.7) gives the identity

$$\mathcal{L}^{-1}(W_i^*(p)) = A_i(p),$$

where $p \in U$. It follows that for all $(p, r) \in U \times \mathbb{R}^n$ Equation (3.1) in Chapter 2 leads to

$$\begin{aligned} d\alpha(\mathcal{L}^{-1} \circ \psi)(p, r) &= \mathcal{L}^{-1} \circ \psi(p, r)(\alpha) \\ &= \mathcal{L}^{-1}(\sum_{i=1}^{n} r_i W_i^*(p))(\alpha) \\ &= \sum_{i=1}^{n} r_i \mathcal{L}^{-1}(W_i^*(p))(\alpha) \\ &= \sum_{i=1}^{n} r_i A_i(p)(\alpha). \end{aligned}$$

Observe that since A_i (for $i = 1, \ldots, n$) are smooth vector fields and α is a structure function on U we have proved that $d\alpha \circ \mathcal{L}^{-1} \circ \psi$ is a structure function on $U \times \mathbb{R}^n$. Therefore

$$d\alpha \circ \mathcal{L}^{-1} \in \mathcal{F}_{T^*U}.$$

But since $d\alpha \in \mathcal{F}_{TU}$ is a generating function, then one concludes that $\mathcal{L}^{-1}$ is a smooth map.

Recall finally that $H = E \circ \mathcal{L}^{-1}$, and $E(v) =< \mathcal{L}(v), v > -L(v)$. Hence

$$\begin{aligned} E(v) &= g(v,v) - \frac{1}{2}g(v,v) \\ &= \frac{1}{2}g(v,v) \\ &= L. \end{aligned}$$

So

$$\begin{aligned} H(\alpha) &= (E \circ \mathcal{L}^{-1})(\alpha) \\ &= E(\mathcal{L}^{-1}(\alpha)) \\ &= \frac{1}{2}g(\mathcal{L}^{-1}(\alpha), \mathcal{L}^{-1}(\alpha)) \end{aligned}$$

□

Note that in this case the Lagrangian vector field X_E is called the **geodesic spray** [2, p.225] and $((M, \mathcal{C}_M, \mathcal{F}_M), L)$ is said to be a **classical mechanical system without potential**.

7.3.2 System with potential

Theorem 7.3.2 *Let $((M, \mathcal{C}_M, \mathcal{F}_M), g)$ be a pseudo-Riemannian DS-Frölicher space of constant dimension n. Let $V : M \longrightarrow \mathbb{R}$ be a smooth function and $L : TM \longrightarrow \mathbb{R}$ be the Lagrangian defined by*

$$L(v) = \frac{1}{2}g(v,v) - (V \circ \pi)(v), \tag{7.8}$$

where $v \in TM$. Then L is hyperregular, the Legendre transformation is

$$\mathcal{L}(v)(w) = g(v,w)$$

and the energy function $E : TM \longrightarrow \mathbb{R}$ is given by

$$E(v) = \frac{1}{2}g(v,v) + (V \circ \pi)(v). \tag{7.9}$$

Furthermore, the associated Hamiltonian $H : T^*M \longrightarrow \mathbb{R}$ *is given by*

$$H = L \circ \mathcal{L}^{-1} + 2V \circ \tau. \tag{7.10}$$

Proof. The construction is similar to that in Theorem 7.3.1. A straightforward computation yields $\mathcal{L}(v)(w) = g(v, w)$, turning $\mathcal{L}$ into a diffeomorphism. So, L is hyperregular. Now computing the identity $E =< \mathcal{L}(v), v > -L(v)$ we have

$$E(v) = \frac{1}{2}g(v, v) + (V \circ \pi)(v).$$

Let $v = \mathcal{L}^{-1}(\alpha)$, for some $\alpha \in T^*M$. Hence from Equation (7.8) it follows that

$$\begin{aligned}
H(\alpha) &= E(\mathcal{L}^{-1}(\alpha)) \\
&= \frac{1}{2}g(\mathcal{L}^{-1}(\alpha), \mathcal{L}^{-1}(\alpha)) + (V \circ \pi)(\mathcal{L}^{-1}(\alpha)) \\
&= \frac{1}{2}g(v, v) + (V \circ \pi)(v) \\
&= (\frac{1}{2}g(v, v) - L(v)) + L(v) + (V \circ \pi)(v) \\
&= L(v) + 2(V \circ \pi)(v) \\
&= L \circ \mathcal{L}^{-1}(\alpha) + 2(V \circ \pi)(\mathcal{L}^{-1}(\alpha)) \\
&= L \circ \mathcal{L}^{-1}(\alpha) + 2(V \circ \tau)(\alpha)
\end{aligned}$$

□

The pair (M, L) where M is a **DS**-Frölicher space of constant dimension and L the Lagrangian defined by Equation (7.8) is called a **classical mechanical system with potential** V.

7.4 Transversal intersection of symplectic pseudomanifolds

This application aims to glue geometrical objects from two pseudomanifolds of the class **DS** which have a transversal nonempty intersection in order to obtain new objects for the geometry of the glued space. In case that they are symplectic pseudomanifolds one obtains the resulting symplectic structure. Note that geometrical objects are the underlying sets as objects in the category **FRL**. We shall glue smooth functions, vector fields, exterior forms, exterior derivative and integral curves.

7.4.1 Gluing pseudomanifolds

Definition 7.4.1 *A* **transversal intersection** *of two connected* **DS***-pseudomanifolds* (M_1, ω_1) *and* (M_2, ω_2) *is a nonempty set* $\Delta = M_1 \cap M_2$ *satisfying the following conditions with the smooth coproduct* $M_1 \sqcup M_2$ *:*

(1) $(M_1 \sqcup M_2, \Gamma\Phi\overline{C}_0, \Phi\overline{C}_0)$ *is a Frölicher space whose structure is generated by the set of curves*

$$\overline{C}_0 = \{\iota_1 \circ c_1 \mid c_1 \in \mathcal{C}_{M_1}\} \sqcup \{\iota_2 \circ c_2 \mid \ c_2 \in \mathcal{C}_{M_2}\},$$

where $\iota_1 \hookrightarrow M_1 \sqcup M_2$, $\iota_2 \hookrightarrow M_1 \sqcup M_2$ *are the insertion maps.*

(2) $(\Delta, \Gamma(\mathcal{F}_{M_1|\Delta} \ \cap \ \mathcal{F}_{M_2|\Delta}), \Phi\Gamma(\mathcal{F}_{M_1|\Delta} \ \cap \ \mathcal{F}_{M_2|\Delta}))$ *is a Frölicher space with the induced structure from that of* M_1 *and* M_2.

(3) $f \in \mathcal{F}_M$ *if and only if* $f_{|M_1} \in \mathcal{F}_{M_1}$ *or* $f_{|M_2} \in \mathcal{F}_{M_2}$.

(4) $T_pM = (\iota_1)_{*p}T_pM_1 \oplus (\iota_2)_{*p}T_pM_2$ *for* $p \in \Delta$.

(5) $(\iota_\Delta)_{*p}T_p\Delta = (\iota_1)_{*p}T_pM_1 \cap (\iota_2)_{*p}T_pM_2$, *where* $\iota_\Delta : \Delta \hookrightarrow M_1 \sqcup M_2$ *is the inclusion map.*

Definition 7.4.2 *Let* $f_1 \in \mathcal{F}_{M_1}$ *and* $f_2 \in \mathcal{F}_{M_2}$ *such that* $f_{1|\Delta} = f_{2|\Delta}$, *the smooth map* $f_1 \sqcup f_2 : M_1 \sqcup M_2 \longrightarrow \mathbb{R}$ *defined by*

$$f_1 \sqcup f_{2|M_1} = f_1 \tag{7.11}$$

$$f_1 \sqcup f_{2|M_2} = f_2 \tag{7.12}$$

is called a **conjunction map**.

Clearly, $f_1 \sqcup f_2$ is smooth by construction as f_1 and f_2 are smooth by assumption. Also, it is easy to observe that $f_1 \sqcup f_2 \in \Phi\overline{\mathcal{C}}_0 = \mathcal{F}_M$.

Definition 7.4.3 *A vector field* $X \in \mathfrak{X}(M)$ *is said to be* **tangent to the subspace** Δ *if for any point* $p \in \Delta$ *there is a tangent vector* $v \in T_p\Delta$ *such that*

$$X(p) = (\iota_\Delta)_{*p}v.$$

We shall denote by $\mathfrak{X}_\Delta(M)$ *the set of all smooth vector fields tangent to* M *which are also tangent to* Δ.

Lemma 7.4.1 *Let* (M, ω) *be a symplectic pseudomanifold and* $X \in \mathfrak{X}_\Delta(M)$. *Let* $Y : \Delta \longrightarrow T\Delta$ *be a vector field defined by*

$$(\iota_\Delta)_{*p}Y(p) = X(p), \quad p \in \Delta. \tag{7.13}$$

Then Y *is smooth as a tangent vector field on* Δ *and* Y *is unique.*

Proof. Observe that since X is smooth, it follows by construction that Y is also smooth. Moreover, one can see that ι_Δ is an embedding, so Y is unique since $(\iota_\Delta)_{*p}$ is an isomorphism of linear Frölicher spaces. ☐

Definition 7.4.4 *The vector field Y defined in Equation (7.13) above is called the* **restriction** *of $X \in \mathfrak{X}_\Delta(M)$ to the subspace Δ and is denoted by $X_{|\Delta}$.*

Proposition 7.4.1 *If $(M = M_1 \sqcup M_2, \mathcal{C}_M, \mathcal{F}_M)$ is a* **DS***-pseudomanifold, then*

$$\mathfrak{X}(M) = \mathfrak{X}_\Delta(M). \tag{7.14}$$

Proof. The inclusion $\mathfrak{X}_\Delta(M) \subseteq \mathfrak{X}(M)$ is obvious. We need only to show the reverse inclusion. Let $X \in \mathfrak{X}(M)$. Then $X \in \mathfrak{X}_{M_j \backslash \Delta}(M)$, for $j = 1, 2$. This follows from the assumption that Δ is closed as boundary, making $M_j \backslash \Delta$ an open set. Hence $X \in \mathfrak{X}_{cl(M_j \backslash \Delta)}(M)$. That is, $X \in \mathfrak{X}_{M_j}(M)$. It follows that $X(p) \in (\iota_j)_{*p} T_p M_j$ whenever $p \in \Delta$, $j = 1, 2$. So

$$X(p) \in (\iota_1)_{*p} T_p M_1 \cap (\iota_2)_{*p} T_p M_2,$$

which is equivalent to

$$X(p) \in (\iota_\Delta)_{*p} T_p \Delta.$$

Thus $X \in \mathfrak{X}_\Delta(M)$, which proves the reverse inclusion. ☐

Definition 7.4.5 *A pair (X_1, X_2) of vector fields $X_1 \in \mathfrak{X}_\Delta(M_1)$ and $X_2 \in \mathfrak{X}_\Delta(M_2)$ is said to be* **consistent on** *Δ if $X_{1|\Delta} = X_{2|\Delta}$. The unique vector field denoted by $X_1 \sqcup X_2$ such that*

$$X_1 \sqcup X_{2|M_i} = X_i, \qquad i = 1, 2$$

is called the **conjunction of vector fields** X_1 *and* X_2.

Proposition 7.4.2 *Let* ${}^{1}\mathfrak{X}^{2}{}_{\Delta}(M_{1,2}) = \{(X_1, X_2) \in \mathfrak{X}_\Delta(M_1) \times \mathfrak{X}_\Delta(M_2)\}$ *be the set of all pairs of vector fields* $X_1 \in \mathfrak{X}_\Delta(M_1)$ *and* $X_2 \in \mathfrak{X}_\Delta(M_2)$ *which are consistent on* Δ. *Then the correspondence*

$$\mathfrak{X}(M) \longrightarrow {}^{1}\mathfrak{X}^{2}{}_{\Delta}(M_{1,2})$$

is bijective.

Proof. The proof is a straightforward consequence of Proposition 7.4.1 and Definition 7.4.3 above. ☐

Proposition 7.4.3 *Let* $(M = M_1 \sqcup M_2, \mathcal{C}_M, \mathcal{F}_M)$ *be the pseudomanifold following a transversal intersection along* Δ. *Let* $c : \mathbb{R} \longrightarrow M$ *be a smooth curve on* M *such that* $c(t)$ *lies in* M_1 *for* $t < 0$, $c(t)$ *lies in* M_2 *for* $t > 0$ *and* $c(0) \in \Delta$. *Then*

$$c'(0) \in (\iota_\Delta)_{*c(0)}(T_{c(0)}\Delta).$$

Proof. Let c_- denote the restriction of c to $(-\infty, 0]$ and c_+ the restriction of c to $[0, +\infty)$. Since c is assumed smooth, it turns out that

$$c'(0) = (\iota_1)_{*c(0)}c'_-(0) = (\iota_2)_{*c(0)}c'_+(0).$$

It follows that

$$c'(0) \in (\iota_1)_{*c(0)}T_{c(0)}M_1 \ \cap \ (\iota_2)_{*c(0)}T_{c(0)}M_2 = (\iota_\Delta)_{*c(0)}(T_{c(0)}\Delta). \tag{7.15}$$

☐

Corollary 7.4.1 *For every smooth vector field $X \in \mathfrak{X}(M_1 \sqcup M_2)$ there is an integral curve at singular points.*

Proof. We only observe that a piecewise curve defined by

$$c(t) = \begin{cases} c_1(t) & \text{for} \quad t \in (-\infty, t_0] \\ c_2(t) & \text{for} \quad t \in [t_0, +\infty) \end{cases}$$

such that $c_1 : [t_0, +\infty) \longrightarrow M_1$ is an integral curve for $X_1 \in {}^{1}\mathfrak{X}^{2}{}_{\Delta}(M_{1,2})$. On the other hand $c_2 : (-\infty, t_0] \longrightarrow M_2$ is an integral curve for $X_2 \in {}^{1}\mathfrak{X}^{2}{}_{\Delta}(M_{1,2})$. Then $c_1(t_0) = c_2(t_0) \in \Delta$ is a smooth integral curve for $X \in \mathfrak{X}(M)$. ☐

Definition 7.4.6 *Two k-forms $\omega_1 \in \Omega^k(M_1)$ and $\omega_2 \in \Omega^k(M_2)$ are said to be* **consistent on Δ** *if*

$$\iota_{1\Delta}^{*}\omega_1 = \iota_{2\Delta}^{*}\omega_2, \tag{7.16}$$

where $\iota_{1\Delta}$ (resp. $\iota_{2\Delta}$) is the inclusion map of Δ into M_1 (resp. M_2).

Let $\overline{\Omega}^k(M) = \{\omega_1 \sqcup \omega_2 : \omega_1 \in \Omega^k(M_1),\ \omega_2 \in \Omega^k(M_2);\ \iota_{1\Delta}^{*}\omega_1 = \iota_{2\Delta}^{*}\omega_2\}$ denote the set of all **Δ-consistent k-forms** on M.

Definition 7.4.7 *The* **conjunction of Δ-consistent k-forms ω_1 and ω_2** *is the k-form defined by*

$$\omega_1 \sqcup \omega_2 : \mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M) \longrightarrow \mathcal{F}_M, \quad (k\text{-copies of } \mathfrak{X}(M))$$

such that

$$(\omega_1 \sqcup \omega_2)(X_1 \sqcup Y_1, \ldots, X_k \sqcup Y_k) := \omega_1(X_1, \ldots, X_k) \sqcup \omega_2(Y_1, \ldots, Y_k) \tag{7.17}$$

where $X_1 \sqcup Y_1, \ldots, X_k \sqcup Y_k \in \mathfrak{X}(M)$.

Proposition 7.4.4 *If ω_1 and ω_2 are Δ-consistent k-forms on $M = M_1 \sqcup M_2$ then $d\omega_1$ and $d\omega_2$ are Δ-consistent $(k+1)$-forms on M.*

Proof. Since the pullback map commutes with the operator of exterior differentiation d, $\iota_{1\Delta}^* d\omega_1 = \iota_{2\Delta}^* d\omega_2$ follows from $\iota_{1\Delta}^* \omega_1 = \iota_{2\Delta}^* \omega_2$. ☐

Definition 7.4.8 *The $(k+1)$-form $d\omega = d\omega_1 \sqcup d\omega_2$ obtained from $\omega = \omega_1 \sqcup \omega_2$ is called the* **exterior derivative** *of the k-form ω.*

Definition 7.4.9 *The set $\overline{\Omega}(M) = \oplus_{k=0}^{n} \overline{\Omega}^k(M)$ together with the natural operations of addition and multiplication and the operator of exterior differentiation d is a graded algebra called the* **de Rham algebra** *on the pseudomanifold $(M, \mathcal{C}_M, \mathcal{F}_M)$.*

Theorem 7.4.1 *Let (M_1, ω_1) and (M_2, ω_2) be symplectic* **DS***-pseudomanifolds. Then $\omega_1 \sqcup \omega_2$ is a symplectic form on $M = M_1 \sqcup M_2$ and $\omega_1 \sqcup \omega_2$ assigns to any tangent vector field $X = X_1 \sqcup X_2$ a unique 1-form $\alpha = \alpha_1 \sqcup \alpha_2$ on M, where α_1, α_2 are Δ-consistent 1-forms on (M_1, ω_1) and (M, ω_2) respectively.*

Proof. (i) Observe that $\omega_1 \sqcup \omega_2$ is symplectic by construction. (ii) We need to show that $\omega_1 \sqcup \omega_2$ maps $X_1 \sqcup X_2$ into $\alpha_1 \sqcup \alpha_2$, where $X_i \in \mathfrak{X}(M_i)$, $\alpha_i \in \Omega^1(M_i)$, $i = 1, 2$. In fact, in the usual way we know that

$$\omega : X \mapsto i_X \omega = \omega(X, \cdot) = \alpha$$

uniquely yields a 1-form since ω is nondegenerate, and

$$\omega(X, \cdot) = \omega_1 \sqcup \omega_2(X_1 \sqcup X_2, \cdot)$$

by definition. It follows that

$$\omega_1(X_1, \cdot) \sqcup \omega_2(X_2, \cdot) = \alpha_1 \sqcup \alpha_2. \tag{7.18}$$

(iii) It remains to show the correctness of the definition of $\alpha_1 \sqcup \alpha_2$ with respect to ω_1 and ω_2. That is, we show that α_1 and α_2, images of ω_1 and ω_2, are Δ-consistent. We have

$$\begin{aligned} (\iota_{1\Delta}^* \alpha_1)(v) &= \alpha_1((\iota_{1\Delta})_{*p} v) \\ &= \omega_1(X_1(p), (\iota_{1\Delta})_{*p} v) \\ &= \omega_1((\iota_{1\Delta})_{*p} u, (\iota_{1\Delta})_{*p} v) \\ &= (\iota_{1\Delta}^* \omega_1)(u, v) \\ &= (\iota_{1\Delta}^* \omega_2)(u, v) \\ &= \omega_2(X_2(p), (\iota_{2\Delta})_{*p} v) \\ &= (\iota_{2\Delta}^* \alpha_2)(v), \end{aligned}$$

where $v \in T_pM$, $p \in M$ and $u \in T_p\Delta$ satisfies

$$(\iota_{1\Delta})_{*p} u = X_1(p)$$

$$(\iota_{2\Delta})_{*p} u = X_2(p)$$

according to Lemma 7.4.1 above. □

Corollary 7.4.2 *There is a bijective correspondence between the module of smooth tangent vector fields and the module of smooth* 1*-forms on* M. *Two Hamiltonian vector fields* X_{H_1} *and* X_{H_2} *associated with the Hamiltonian functions* H_1 *and* H_2 *yield a Hamiltonian vector field* X_H *given by*

$$X_H = X_{H_1} \sqcup X_{H_2},$$

where $H = H_1 \sqcup H_2$ *and*

$$i_{(X_{H_1} \sqcup X_{H_2})}(\omega_1 \sqcup \omega_2) = d\ (H_1 \sqcup H_2). \tag{7.19}$$

7.4.2 Gluing Hamiltonian systems: an example

Consider the canonical pseudomanifold $(\mathbb{R}^{12}, \mathcal{C}, \mathcal{F})$ and

$$M_1 = \{(q^1, q^2, q^3, p_1, p_2, p_3, 0, 0, 0, 0, 0, 0); q^i, p_i \in \mathbb{R},\ i = 1, 2, 3\} \subseteq \mathbb{R}^{12}$$

$$M_2 = \{(0, 0, 0, 0, 0, 0, \overline{q}^1, \overline{q}^2, \overline{q}^3, \overline{p}_1, \overline{p}_2, \overline{p}_3); \overline{q}^i, \overline{p}_i \in \mathbb{R},\ i = 1, 2, 3\} \subseteq \mathbb{R}^{12}$$

two pseudomanifolds. Then M_1 and M_2 are obviously Frölicher subspaces of $\mathbb{R}^{12}$ which are considered as configuration spaces for two mechanical systems with 6 degrees of freedom. The configuration coordinates (q^1, q^2, q^3) or $(\overline{q}^1, \overline{q}^2, \overline{q}^3)$ and the momenta (p_1, p_2, p_3) or $(\overline{p}_1, \overline{p}_2, \overline{p}_3)$ determine together the instantaneous states. Then $\mathbb{R}^{12}$ can be considered as the phase space of the system.

In the Hamiltonian formulation the equations of the motion for such a classical system are written in term of first order differential equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i};\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \qquad i = 1, 2, 3. \tag{7.20}$$

A Hamiltonian function $H(q, p)$ defining the system in case of absence of constraining forces and of time dependence is the total energy of the system, that is, the kinetic plus the potential energies. Similarly to some observation made by Eledrisi [22] on structured spaces, we note that on a pseudomanifold, the set of singular points lying in the transversal intersection is

$\Delta = M_1 \cap M_2 = \{0\}$. That is, $p_0 \in \Delta$ if and only if $p_0 = (0, \ldots, 0) \in \mathbb{R}^{12}$.

Assume that (M_1, ω_1) and (M_2, ω_2) are symplectic with symplectic Δ-consistent forms given by

$$\omega_1 = \sum_{i=1}^{3} dq^i \wedge dp_i, \qquad \omega_2 = \sum_{i=1}^{3} d\overline{q}^i \wedge d\overline{p}_i.$$

Consider two potential functions $V_1 : M_1 \longrightarrow \mathbb{R}$, $V_2 : M_2 \longrightarrow \mathbb{R}$ such that $V_1(p_0) = V_2(p_0)$ and two Hamiltonian functions given by

$$H_1 = T_1 + V_1 = \frac{1}{2}\sum_{i=1}^{3}\frac{p_i^2}{m} + V_1$$

$$H_2 = T_2 + V_2 = \frac{1}{2}\sum_{i=1}^{3}\frac{\overline{p}_i^2}{m} + V_2$$

where m designates the mass of material points. We need to calculate the Hamiltonian vector field in each case. Then we shall obtain the corresponding integral curves in M_1 and in M_2.

From the identity

$$X_H = (\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q_i})$$

we obtain

$$X_{H_1} = \sum_{i=1}^{3}\left[\frac{p_i}{m} + \frac{\partial V_1}{\partial q_i}\right]\frac{\partial}{\partial q^i} - \frac{\partial V_1}{\partial q^i}\frac{\partial}{\partial p_i} \tag{7.21}$$

$$X_{H_2} = \sum_{i=1}^{3}\left[\frac{\overline{p}_i}{m} + \frac{\partial V_2}{\partial q_i}\right]\frac{\partial}{\partial \overline{q}^i} - \frac{\partial V_2}{\partial \overline{q}^i}\frac{\partial}{\partial \overline{p}_i}. \tag{7.22}$$

The integral curves for X_{H_j} $(j = 1, 2)$ are those which satisfy the (Hamilton-Jacobi) equations

$$\frac{dq^i}{dt} = \frac{\partial H_j}{\partial p_i}, \qquad \frac{p_i}{dt} = -\frac{\partial H}{\partial q_i}. \tag{7.23}$$

That is, the integral curves are solutions of the system

$$\frac{dq^i}{dt} = \frac{p_i}{m} + \frac{\partial V_j}{\partial q_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial V_j}{\partial q^i}. \tag{7.24}$$

which can be glued for the mechanics on $M = M_1 \sqcup M_2$.

Note that the potential functions are arbitrary. Hence, assuming without loss of generality that $\frac{\partial V_j}{\partial q^i}$ is a constant we have

$$\gamma(t) = (\sum_{i=1}^{3} \frac{1}{m} p_i t, -\sum_{i=1}^{3} \frac{\partial V_1}{\partial q^i} t).$$

Similarly for X_{H_2} one has

$$\overline{\gamma}(t) = (\sum_{i=1}^{3} \frac{1}{m} \overline{p}_i t, -\sum_{i=1}^{3} \frac{\partial V_2}{\partial \overline{q}^i} t).$$

Chapter 8

Conclusion

Our investigation achieved the task of showing that a set endowed with a Frölicher structure can carry an additional structure turning it into a configuration space for the geometric formalism of a mechanical system. The additional structure may be a Poisson structure in its general setting or a symplectic structure inducing a Poisson one in a natural way.

In dealing with this material, we noticed that two different tangent structures can be defined on a Frölicher space. We proved that the kinematic and the operational tangent structures coincide when the space under consideration is of constant differential dimension.

The study revealed that the categories **FRL** of Frölicher spaces and **DSP** of differential spaces are closely related. It was observed that in working out the geometry on subspaces, the category **DSP** presents good behavior compared with **FRL**, where the poorness in structure curves yields lot of structure functions. The consequence of this behavior is that all functions defined on the subspace become smooth, with the effect of turning the resulting topology discrete. Clearly, all singletons are open sets in this case. It follows that only the zero vector is a smooth derivation at every point, that is, the geometry collapses.

In order to avoid this imperfection possibly not needed, we worked out a class of Frölicher spaces whose structure is generated by a differential structure in the Sikorski sense. We proved that smooth maps in a differential space are also smooth in the generated Frölicher space, that a diffeomorphism in the generating structure conserves its properties in the Frölicher generated structure and that the structure curves generated on the differential space are those of the Frölicher space. Hence, the whole set of Frölicher structure functions coincides with the differential structure, making their geometry look alike. We called such differential spaces **pre-Frölicher** while the generated spaces are said to be Frölicher spaces of class **DS**. We ensured that this class is nonempty as it contains smooth manifolds and the Euclidean spaces $\mathbb{R}^n$. We also ensured that certain Frölicher spaces do not lie in this class. An example is the space $\mathbb{Q}$ of rational numbers. Therefore, all the tools related to the exterior algebra used in this work are those which already exist in the theory of differential geometry on differential spaces.

Our study focused on those **DS**-Frölicher spaces which are locally diffeomorphic to some closed subsets of $\mathbb{R}^n$ of constant maximal dimension.

We have worked out the canonical 1-form and the resulting symplectic 2-form on the cotangent bundle to a **DS**-Frölicher space of constant dimension by the technique of using a local basis. This allowed us to obtain the canonical expressions of symplectic forms, symplectic gradients, the Hamilton-Jacobi equations, and to a possible geometrical formalism of mechanics in this class. The link between the Lagrangian and the Hamiltonian systems was made possible by a smooth Legendre transformation. We checked that the conservative properties in the Hamiltonian formalism of mechanics also hold in this setting.

The last application showed that, up to an equivalence relation, two symplectic pseudomanifolds which have a transversal intersection may be glued in order to obtain a space which is a symplectic **DS**-pseudomanifold.

This study has opened the way to many questions of differential, symplectic or Riemannian geometry. Deep investigations should be made for the general setting of the exterior algebra on Frölicher spaces. It is also worth noting that not all Frölicher spaces satisfy the inverse function theorem. A class of those which have this property should be worked out, opening the way to a study of flows and smooth actions of Frölicher-Lie groups on Frölicher spaces.

Note finally that this study can be extended to investigating the behavior of the symplectic structure on a Frölicher quotient space resulting from an equivalence relation on a symplectic Frölicher space.

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