THE GRAVITY OSCILLATIONS OF WHITE DWARF STARS

by

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Preface

The numerical investigation of the oscillations of white dwarfs which is discussed below was begun at the suggestion of Professor Warner soon after the discovery of periocdic variations in several cataclysmic variables. Its completion within a reasonable time was greatly helped by the availability of a computer program for the solution of the equations governing the structure of white dwarf stars provided by Dr. W.B. Hubbard. Although the program could not be used in the original form for the present investigation, where the accuracy of the outer layers is critical, it provided an invaluable starting point and the coding of the equation of state of the core was used in an almost unaltered form.

Results of the investigation which are of astronomical interest are presented below in the form of a paper which has been accepted for publication in the Monthly Notices of the Royal Astronomical Society. Details of the computer programs used are discussed in appendices 2 and 3.

I am very grateful to Professor Warner for his help at all stages of this work and to Dr. Hubbard for making available the white dwarf model program. I am indebted to the Council for Scientific and Industrial Research for a bursary.

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Summary

The second order differential equations governing the non-radial adiabatic oscillations of stars have been solved for a variety of white dwarf models. It is found that the inclusion of convection in the envelopes of the models with surface temperatures corresponding to those of HL Tau 76 and G44-32 does not alter the eigenvalues significantly and the periods of the lowest order modes are shorter than the observed periods by a factor of about 2. The eigenfunctions of hot white dwarf models which include a hydrogen burning shell source in equilibrium show that the oscillations have a large amplitude only in the hydrogen envelope. The relevance of these results to the observation of periodic variations in cataclysmic variables is discussed.

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1. Introduction

The observational material on short period light variations in white dwarfs and cataclysmic variables has been reviewed by Warner and Robinson (1972a). They find strong support for an interpretation in terms of non-radial oscillations. In particular they find it necessary to invoke gravity mode oscillations to explain the existence of (multiple) periods longer than would be expected for the radial oscillations of white dwarfs. More recent observations have done little to change this view, and the numerical investigation of Osaki and Hansen (OH, 1973) has tended to give it further support.

OH find that they are able to explain the observed periods of the cataclysmic variables (15 to 70-s) by g-modes provided that the mass of the white dwarf exceeds about .5 $M_\odot$.

The requirement on the mass is in agreement with a determination of the masses of cataclysmic variables by Warner (1973) who finds that they appear to cluster near 1.2 $M_\odot$. Moreover the period luminosity relation which OH derive could be used to explain the variation of the period of Z Cam (Warner and Brickhill 1974). However Bath et al (1974) have pointed out that such a comparison is not valid. In terms of the derivation given by OH the period is determined essentially by the core temperature. The observed 20 percent change in the period of Z Cam would therefore imply an unrealistic change in the thermal energy of the white dwarf. Thus if g-mode pulsations are indeed the cause of the observed periodicities, they must be confined to a region very near the surface.

We have investigated the oscillations of a variety of hot white dwarf models of 1 and 1.2 $M_\odot$. The results of these calculations are discussed in Section 5 of this paper.

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OH find some difficulty in understanding the long-period white dwarfs. Outstanding in this group is HL Tau 76 (Landolt 1968), whose white dwarf character is well established and which has a complex spectrum of periods around 750 s. The periods of the coolest model used by OH are 210 and 291 seconds for the first and second gravity modes of the second harmonic. This model is already very much cooler than the observed colours of HL Tau indicate. However, OH suggest that convection may play an important part in increasing the periods. We have investigated the oscillations of white dwarf models with effective temperatures corresponding to those of the long-period group. The models include envelope convection and an improved treatment of the ion pressure in the core. Although the eigenperiods are longer than those found by OH it is still not possible to explain the 750-s period in terms of the g-modes of lowest order.

2. The Gravity Oscillations of White Dwarfs

It is convenient to reproduce here Ledoux's discussion of the g-modes of white dwarf stars*. If we suppose that the time dependence of the perturbations has the form exp ($i\sigma t$) and that they can be developed in a series of spherical harmonics, Y_ℓ^m (Θ, ϕ), then the equations obtained by the neglect of the perturbation to the gravitational potential may be written

$$\frac{dv}{dr} = \left[\frac{\ell(\ell+1)}{\sigma^2} - \frac{\rho r^2}{\Gamma_1 p}\right] \frac{p^{2/\Gamma_1}}{\rho} w \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

and

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* Lectures given at the South African Astronomical Observatory, March 1972.

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$$\frac{dw}{dr} = \frac{1}{r^2} (\sigma^2 + Ag) \frac{\rho}{p^{2/\Gamma_1}} v \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

where

$$A = \frac{1}{\rho} \frac{d\rho}{dr} - \frac{1}{\Gamma_1 p} \frac{dp}{dr} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

The boundary conditions are

$$\left.\begin{array}{llll} \text{at} & r = 0 & v = 0, & w = 0 \\ \text{at} & r = R & v = 0, & w = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

(Ledoux and Walraven 1958). The components of the displacement of an element with the equilibrium position (r, θ, ϕ) are given by

$$\delta r = \left(\frac{1}{\sigma^2 + Ag}\right) \frac{p^{1/\Gamma_1}}{\rho} \frac{dw(r)}{dr} Y_\ell^m(\theta,\phi) = a(r)\, Y_\ell^m(\theta,\phi) \quad (5)$$

$$r\,\delta\theta = \frac{1}{\sigma^2} \frac{p^{1/\Gamma_1}}{\rho} \frac{w(r)}{r} \frac{\partial}{\partial\theta} Y_\ell^m(\theta,\phi) = b(r) \frac{\partial}{\partial\phi} Y_\ell^m(\theta,\phi) \quad (6)$$

$$r \sin\theta\,\delta\phi = \frac{1}{\sigma^2} \frac{p^{1/\Gamma_1}}{\rho} \frac{w(r)}{r} \frac{1}{\sin\theta} \frac{\partial}{\partial\phi} Y_\ell^m(\theta,\phi) = \frac{b(r)}{\sin\theta} \frac{\partial}{\partial\phi} Y_\ell^m(\theta,\phi) \quad (7)$$

For the gravity modes of a white dwarf the frequency is small and $\rho r^2/\Gamma_1 p \ll \ell(\ell+1)/\sigma^2$ except very near the surface. In the degenerate core p varies with ρ^{Γ_1} so that $A \to 0$ but $|Ag|$ increases very rapidly in the transition zone where $\sigma^2 \ll |Ag|$. Equations (1) and (2) then show that v and w must vary very slowly in the interior and for sufficiently high degeneracy the oscillations may be largely excluded from the core. This situation is illustrated in Fig. 1 (p. 11). As σ^2 decreases with increasing order of the mode $(\sigma^2 + Ag)$ becomes negative in a large

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part of the core and oscillations of long vertical wavelength are possible (Fig.1(c)). For modes of low order the coefficients of w and v in equations (1) and (2) are both positive in the interior and the amplitude decreases more or less exponentially. The energy of the oscillation is contained almost entirely in the envelope and the excitation energy is low. By contrast the excitation energies of the radial modes of a white dwarf are high and the growth times are consequently very long ($\sim 10^9$ yr. see e.g. Vauclair 1971). Clearly the high amplitude of the g-modes in the envelope favours their excitation in white dwarf stars. At the same time it is apparent that the g-modes may fulfil the requirement imposed by the rapid changes in period which have been observed in dwarf novae if it can be shown that the oscillations are confined to the surface layers even in these relatively hot stars.

3. Input Physics, Techniques

The equation of state used in the models is that of Hubbard and Wagner (1970). It includes the ideal electron gas pressure, valid for any range of density and temperature, the Coulomb correction to the ion pressure obtained by Brush, Sahlin and Teller (1966) in the analytic approximation of Van Horn (1971), and the exchange pressure, using the zero temperature form of Zapolsky (Salpeter 1961) at high densities and the finite temperature non-relativistic form of Hubbard (1969) at lower densities. Where nuclear energy generation is included we use the result of Bahcall and May (1968) for the p-p chain, neglecting factors due to electron screening and precise chemical composition.

In some of the models the inward integration is started at the base of the model atmospheres of Wickramsinghe (1972). Since, for the purposes of this paper we are concerned mostly with effective temperatures which lie outside the range of

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Wickramsinghe's atmospheres, we have used the surface conditions

$$T_s = T_e, \quad P_s = \frac{2}{3\kappa_s} \frac{GM}{R^2} \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

(Schwarzschild 1958, p 89). This treatment is certainly adequate for the hotter stars since the solutions converge very rapidly in the radiative envelopes. For cooler objects the envelope becomes unstable to convection and the solution becomes sensitive to boundary condition. Pressure ionisation is treated by the method suggested by Eggleton, Faulkner and Flannery (1973) which is probably sufficiently accurate for the envelopes considered here. However we assume that temperature stratification in the convective zone is adiabatic and we have used the crude boundary condition (8). Suprisingly, temperatures and pressures at the base of the convective zone are in good agreement with the more accurate calculations of Böhm and Cassinelli (1971) and of Baglin and Vauclair (1973) for envelopes of similar composition. This agreement is sufficient, as is discussed below, to ensure that the periods are unaffected by the approximate treatment, and may indicate that the atmosphere condition is less important than has been previously supposed (Van Horn 1971).

The solution is followed inward using the opacity tables of Cox and Stewart (1970) for the radiative opacity. In the case of the hydrogen rich envelopes we have used the tables of Hubbard and Lampe (1969) for the conductive opacity and the analytic approximation of Sweigart (1973) for other compositions. The inward integration is terminated when the value of $\frac{r}{T}\frac{dT}{dr}$ falls below about 0.5. The temperature at this point is found to be insensitive to the limiting value chosen provided it is not so small that the analytic approximation to the conductivity tables begins to diverge ($\rho \gtrsim 10^6$).

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In general the integration terminates at a point where the degeneracy parameter has a value of about 20. The core temperatures thus found agree well with the results of Lacis (Van Horn 1971) where his calculations incorporated the same opacity tables.

The core solution is integrated from the centre, assuming an isothermal distribution, to the fitting point. The central density can then be varied to fit the surface gravity or the mass. If the final radius differs from the initially guessed value it is necessary to repeat the process. This procedure produced sufficiently rapid convergence for most of the models considered here.

The linear second order differential equations governing adiabatic non-radial oscillations were integrated by a Runge-Kutta technique. The neglect of the perturbation to the gravitational potential has been shown to be unimportant for $\ell = 2$ in the case of polytropic stars (Robe 1968). For the white dwarfs considered here the approximation is in general excellent because the oscillations have a very low amplitude in the interior.

In what follows we denote the kth lowest order gravity mode of the ℓ th harmonic by $g_{k\ell}$. Thus, for example the lowest order gravity mode of the second harmonic will be denoted by g_{12} and the corresponding period by P_{12}. The white dwarf models are labelled according to their mass and their luminosity or effective temperature. A model of .6 $M_\odot$ and 10^{-3} $L_\odot$ with an effective temperature of 12000 K is called .6M-3L or .6M 12T.

4. The Long Period White Dwarfs

The variable white dwarfs HL Tau 76, R 548, and G44-32 occupy a small part of the two-colour diagram (Lasker and Hesser 1971). This may indicate that

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they form a class separate from other periodic white dwarfs, all of which are apparently members of close binary systems undergoing mass exchange. Periods and effective temperatures are listed in Table I. The temperatures have been derived from the calibration given by Shipman (1972) for DA stars using the UBV colours from Eggen and Greenstein (1965) and Greenstein (1970).

4.1 Results of model calculations

Parameters of some of the models which we have used are summarised in Table II and some properties of the g-mode oscillations in Table III. The following points are relevant to the discussion.

(1) In most models with surface temperatures between 6000 K and 12 000 K the energy of the oscillation is contained largely in the outer layers. The energy of oscillation contained in a shell of unit thickness (dE/dr) reaches a maximum at 0.996R in model .8M8T and has fallen to 1% of this value at 0.95R. The corresponding points in model .4M8T are at 0.97R and 0.82R. If a hydrogen rich composition is allowed to extend beyond the base of the convective zone the concentration of energy towards the surface can be considerably increased (compare the excitation energies of models .8M8T(a) and .8M8T(b) in Table III).

In the more degenerate models the amplitude in the interior is sufficiently low to justify the neglect of the perturbation to the gravitational potential even in the case of $\ell = 1$ modes where the displacement at the centre is finite (Fig. 1(a)).

An interesting consequence of the concentration of the energy of oscillation in the transition zone is that the eigenfunctions corresponding to different modes $g_{k\ell}$ and $g_{k\ell'}$ are similar (Fig. 1(a) and (b)) and the periods may be related by an equation of the form

$$\frac{P_{k\ell}}{P_{k\ell'}} = \left[\frac{\ell'(\ell'+1)}{\ell(\ell+1)} \right]^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots \quad (9)$$

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TABLE I

The Long Period White Dwarfs

Name	T_e	Period(s)	Reference
R548	9800	212.9 273.0	Lasker & Hesser (1971)
HL Tau 76	8700	746.2 494.2	Fitch (1973)
G44-32	8600	600 1638) orbital ? 822)	Lasker & Hesser (1969)

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TABLE II

Parameters of cooling white dwarf models

Model	$M/M_\odot$	$L/L_\odot$	T_e (K)	P_T (dyn cm^{-2})	P_R (dyn cm^{-2})	T_R (10^6 K)	T_c (10^6 K)	$\log \rho_c$ gm cm^{-3}
.2M8T(a)	.180	2.60(-3)	8370	1.0(14)	2.6(13)	0.9	14.0	5.25
.4M6T(a)	.388	2.50(-4)	5940	1.0(15)	5.1(13)	0.6	4.36	6.05
.4M8T(a)	.388	1.00(-3)	8350	2.0(13)	2.0(13)	0.6	7.58	6.05
.4M8T(b)	.388	3.97(-3)	8350	-	6.4(16)	3.1	7.48	6.05
.6M13T(c)	.575	4.12(-3)	13000	1.2(6)	2.2(14)	1.3	11.6	6.53
.6M13T(c[1])	.575	1.92(-4)	6800	1.2(6)	2.2(14)	1.1	11.7	6.53
.8M7T(b)	.758	4.35(-4)	8340	-	8.9(15)	1.2	2.77	6.97
.8M8T(a)	.758	4.35(-4)	8340	1.0(15)	1.9(13)	0.5	4.23	6.97
.8M8T(b)	.758	1.00(-2)	18000	-	1.8(16)	1.7	4.03	6.97
.8M18T(b)	.758	7.00(-2)	28900	-	1.5(15)	1.9	14.0	6.96
.8M29T(b)	.758			-	1.5(11)	0.4	31.4	6.95

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(1) In models labelled (a) or (c) we have used the model atmospheres of Wickramsinghe or the surface condition (7) with hydrogen opacity tables (Iben mixture XII). In the former the hydrogen composition was extended at least to the base of the convective zone. The pressure at the transition between the hydrogen and the underlying helium is indicated in column 5. In the models labelled (b) the entire envelope was assumed to be helium.

(2) The core composition of model .6M13T (c^1) is helim. The others are oxygen.

(3) The temperatures and pressures at the base of the convective zone are given in columns 6 and 7.

TABLE III

g-modes of cooling white dwarf models

Model	ℓ	P_1(s)	P_2(s)	P_3(s)	E_1^* (10^{46} ergs)
.2M8T(a)	2	204	289	361	3.90
.4M6T(a)	1	417	504	631	0.12
	2	241	293	371	0.11
.4M8T(a)	1	340	439	531	1.81
	2	198	258	322	1.83
.4M8T(b)	2	207	285	366	17.80
.6M13T(c)	2	136	178	218	21.00
.6M13T(c^1)	2	121	145	172	-
.8M7T(b)	1	408	511	557	6.10
	2	236	299	368	6.08
	3	167	213	275	6.08
	4	130	165	254	5.94
.8M8T(a)	2	166	200		0.06
.8M8T(b)	2	194	248	304	9.53
.8M18T(b)	2	101	130	157	11.10
.8M29T(b)	2	70	94	113	31.30

* The excitation energy of the g_1-mode normalised at the surface: $\delta r/R = 1$

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Figure 1. The radial and horizontal displacements as a function of radius in model .8M7T(b). The amplitudes are normalised independently. At the surface the horizontal displacements (which decrease very rapidly with decreasing r/R) exceed the vertical displacements by a factor $\sim 10^3$ $(b(R)/a(R) = g/R\sigma^2)$. In general the kinetic energy associated with the horizontal displacements dominates so that the energy of the oscillation is contained largely in the surface layers. (a) The lowest order mode (k = 1) of the first harmonic ($\ell = 1$), (b) k = 1, $\ell = 2$, (c) k = 2, $\ell = 2$, (d) k = 3, $\ell = 2$. The energies (in units of 10^{44} ergs) associated with the vertical and horizontal motions are respectively (a) 0.11, 61.29, (b) 0.18, 60.34, (c) 1.03, 70.71, (d) 76.63, 252.86.

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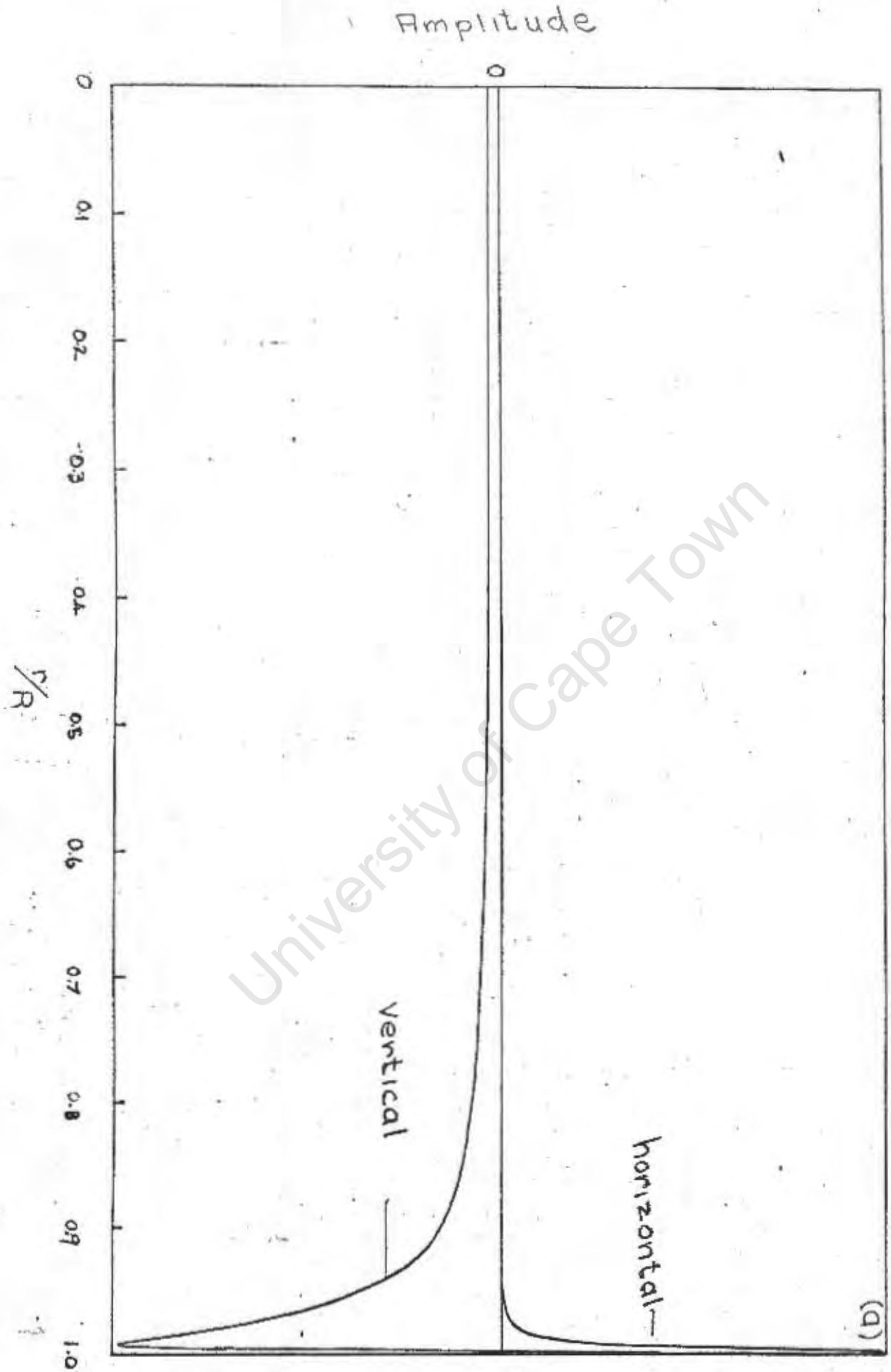

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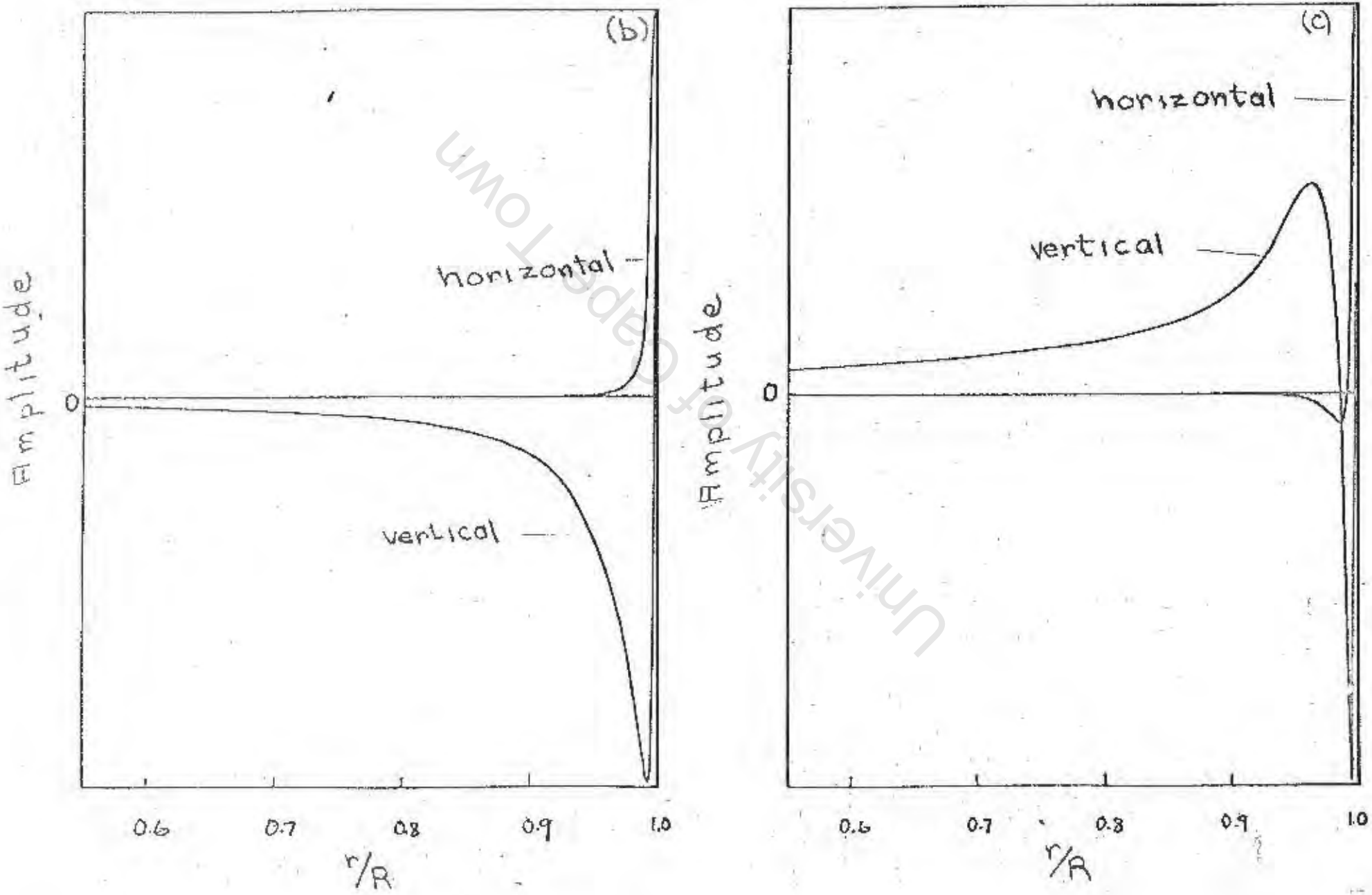

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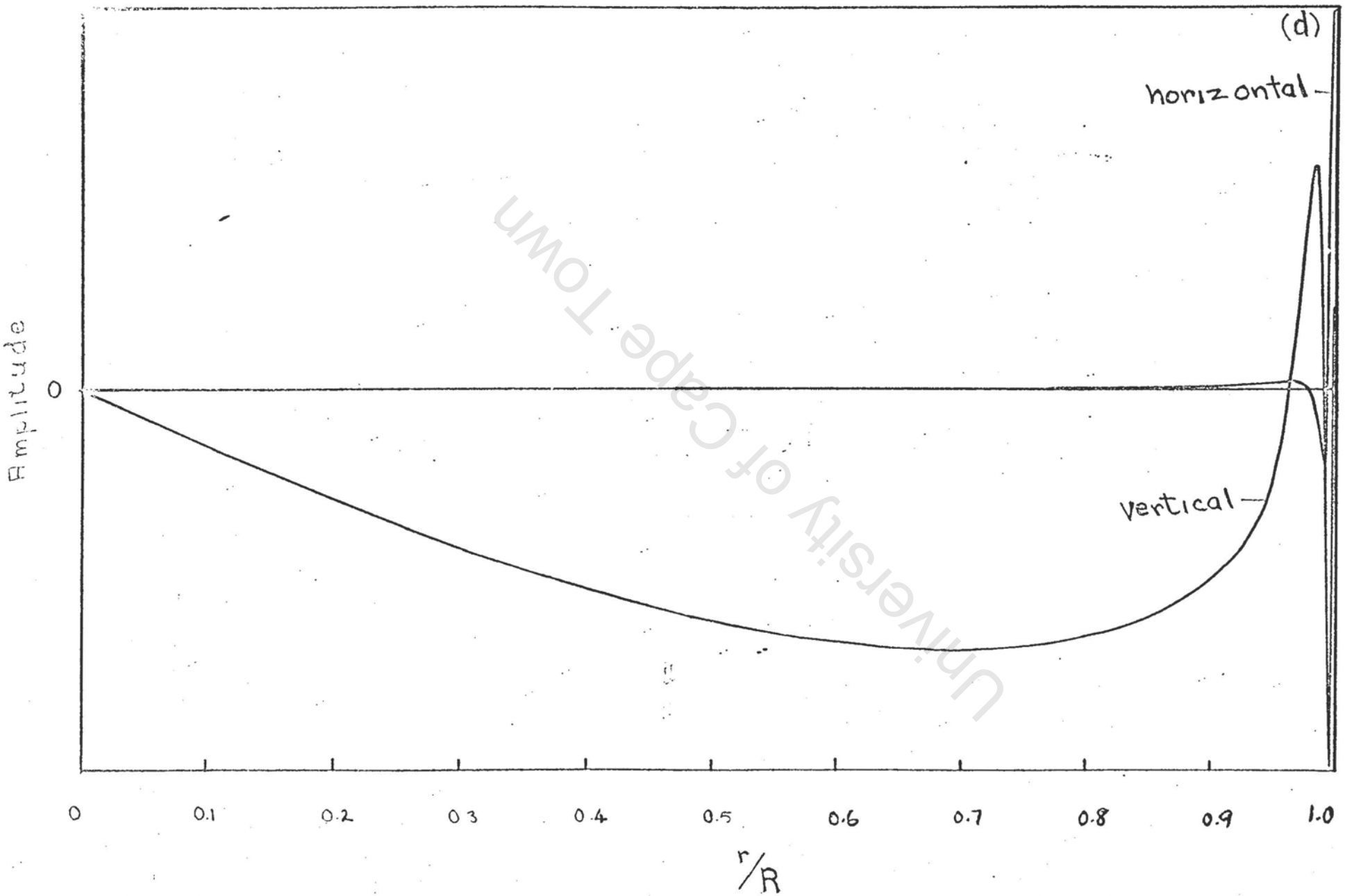

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This relation follows immediately from equations (1) and (2) if we note that in the transition zone, $-Ag \gg \sigma^2$ and $\ell(\ell+1)/\sigma^2 \gg \rho r^2/(\Gamma_1 p)$.

(2) For surface temperatures greater than ~ 8000 K the inclusion of convection in the models does not affect the eigenvalues significantly. To illustrate we compare the periods of models of $.4M_\odot$ with an effective temperature of 8350 K (Tables II and III). In model .4M8T(a) the hydrogen rich composition of the atmosphere is extended to the lower boundary of the convective zone. (This is consistent with the DA spectrum of HL Tau and R548; Baglin and Vauclair 1973). In model .4M8T(b) the entire envelope, including the atmosphere, is assumed to be helium, thus greatly increasing the extent of the convective zone. However the core temperature is only slightly altered and the period is increased by less than 5%.

The depth of the convective zone of model .4M6T(a) indicates that convection is unimportant even at 5900 K in stars where the hydrogen rich composition extends sufficiently far inwards.

Our results are in agreement with the work of Koester (1973) and of Chin and Stothers (1971) who find their core temperatures unaffected by convection for $T_e \gtrsim 6000$ K. Convection does modify the eigenfunctions near the surface, resulting in a reduction of the surface amplitude. This effect is responsible for the difference in the excitation energies of the oscillations (normalised at the surface) of the models mentioned above. The inertia of this region is however insufficient to alter the periods significantly. For the same reasons we feel that the eigenfrequencies are unlikely to be strongly affected by uncertainties in the conductive opacity in the electron-coupling zone (Hubbard and Wagner 1970).

(3) The composition of the core is not critical. A change from Z = 2 to Z = 8 causes the period to increase by a few percent and results in an increased

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amplitude in the interior of the model (c.f. models .6M13T(c) and .6M13T(c^1) but is unlikely to affect our conclusions. If a hydrogen-rich composition is extended significantly beyond the base of the convective zone the frequencies are increased by a factor of up to ~ 2 as a result of the decrease in density. We have not used any models in which the value of Z in the envelope is greater than 2. Unless convection is important in these models at pressures $\gtrsim 10^{17}$ dynes cm^{-2} the periods are unlikely to be significantly longer than those found here.

(4) At a given surface temperature the periods do not depend strongly on the mass. We are able to describe this dependence with sufficient accuracy for our purposes by the equation

$$P_{k\ell} = P^{o}_{k\ell} + 50\ M/M_{\odot} \quad s \quad \ldots\ldots\ldots\ldots\ (10)$$

The dependence of the period P_{12} on temperature is illustrated in Fig. 2. Results on models which do not include hydrogen in the envelope ((b) and (c) models) and which have oxygen cores have been combined using the relation (10) to form the curve in Fig. 2 which represents P^{o}_{12}. The masses of the models lie between .2 $M_{\odot}$ and .8 $M_{\odot}$. There is no indication that more massive models would not fit onto the curve, on the other hand it is clear that significant differences will appear at lower masses as electron degeneracy decreases (OH 1973, Warner and Brickhill 1974). The fact that a model with a surface temperature of 12000K of .2 $M_{\odot}$ fits rather well onto the curve in Fig. 2 indicates that it would be necessary to consider models of less than .1$M_{\odot}$ to reduce the discrepancy between the observed and calculated periods which we discuss below.

In addition to the relations (9) and (10) we find the following approximate relation between periods of different modes of the same harmonic:

$$P_{k\ell} = P_{k\ell}(1 + k/h) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

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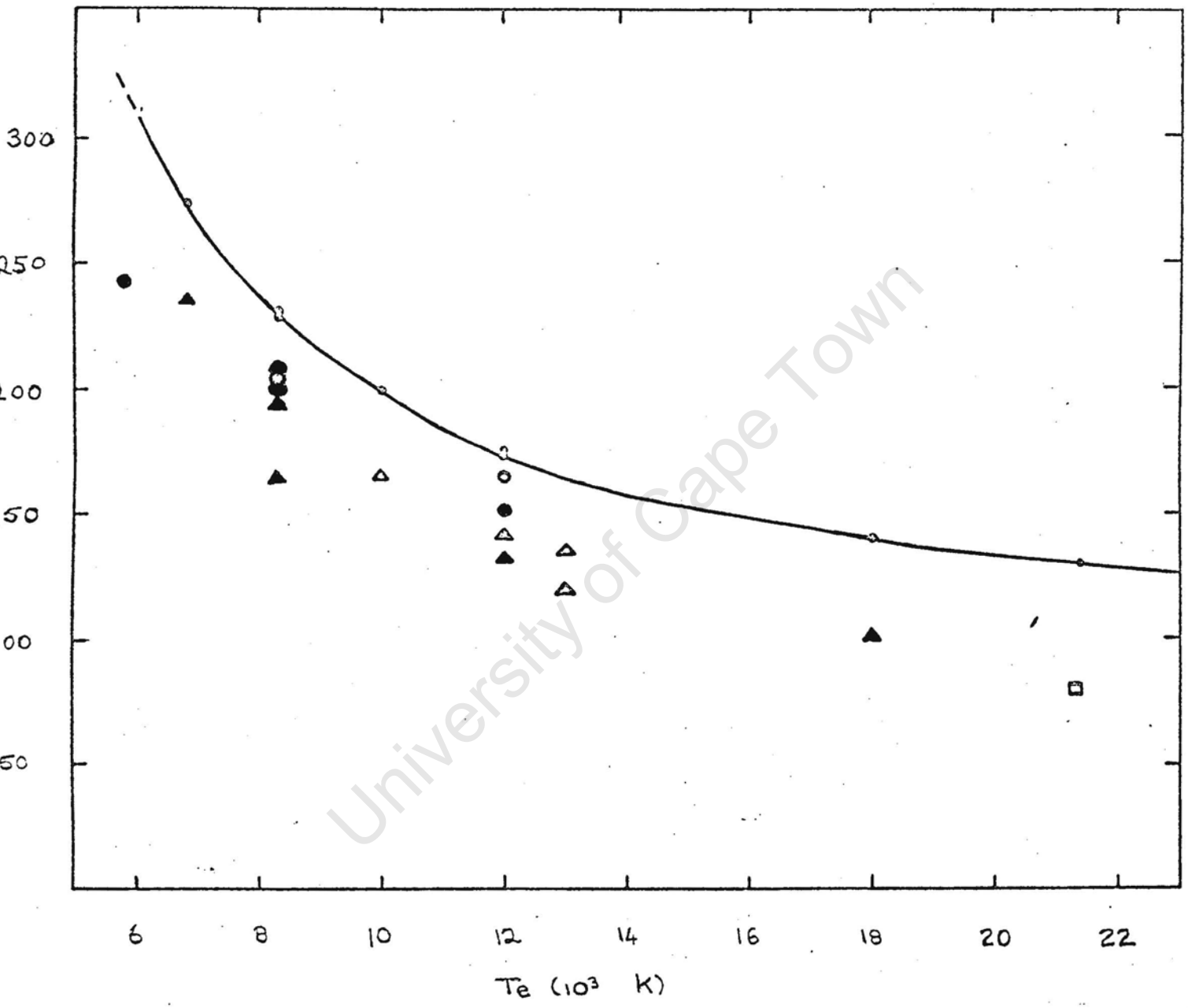

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Figure 2. Periods of cooling white dwarf models: circles $\sim .2\ M_{\odot}$, filled circles $\sim .4 M_{\odot}$, triangles $\sim .6\ M_{\odot}$, filled triangles $\sim .8 M_{\odot}$, squares $\sim 1\ M_{\odot}$, small filled c ircles $\sim P^{o}_{12}$.

h varies between 3 and 4 but is constant to within ~5% for a particular model and given value of ℓ. The periods are somewhat longer than those found by Osaki and Hansen; the increase is presumably a result of the improved treatment of the ion pressure in the cores of our models.

4.2 The Effects of Rotation

The $(2\ell + 1)$-fold degeneracy of the periods corresponding to the kth mode of the ℓ th harmonic is removed by rotation. For slow rotation the frequencies become, relative to axes rotating with the angular velocity of the star,

$$\sigma_{k\ell m} = \sigma_{k\ell} + m C_{k\ell} \Omega \quad (m = 0, \pm 1, \ldots \pm \ell) \quad \ldots\ldots\ldots\ldots \quad (12)$$

where $\sigma_{k\ell}$ is the frequency in the non-rotating star, Ω is the rotation frequency and

$$C_{k\ell} = \frac{\int_0^R \rho\,(2ab + b^2)\, r^2\, dr}{\int_0^R \rho\,(a^2 + \ell(\ell+1)b^2)\, r^2\, dr}$$

where a(r) and b(r) are given in equations (5) and (6)

(Ledoux and Walraven 1958).

We have calculated values of $C_{k\ell}$ for a number of white dwarf models. As k increases the value of b dominates over a in the outer part of the star and $C_{k\ell}$ approaches $\frac{1}{\ell(\ell+1)}$. For the low order modes ($k \lesssim 4$) the values are scattered between $\frac{1}{\ell(\ell+1)}$ and zero.

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Modes with a non-zero value of m correspond physically to waves travelling with angular velocity $-\sigma_{k\ell m}/m$ in the rotating frame. The absolute angular velocities are then $-\sigma_{k\ell}/m + \Omega(1 - C_{k\ell})$ and the observed frequencies are given by

$$\sigma_{k\ell m} = \left|\sigma_{k\ell} - m(1 - C_{k\ell})\Omega\right| \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

For $\sigma_{k\ell} \gg \Omega$ the frequencies are evenly spaced and lie close to the non-rotating frequency $\sigma_{k\ell}$. If σ is comparable with Ω (which is not unlikely for $P \sim 500$s) arbitrarily long periods are possible.

Resonance between low order g-modes and the tidal perturbation of a binary partner is therefore somewhat more probable (in the case of a white dwarf) than was argued by Cowling (1941). However there is no convincing evidence for such a resonance in the existing data on periodic white dwarfs.

4.3 Discussion

Comparing the observed periods with our theoretical results it is immediately clear that the pulsations of R548 are easily understood in terms of the g-modes of low order while the periods of the other two stars correspond to $P_{4\ell}$ or higher orders. It is interesting that the ratio of the two periods of R548, $1.283 \pm .003$, is very close to the values found in model calculations (1.28 - 1.29) and predicted by equation 4 for the ratio of periods of the third and fourth harmonics. This identification would imply an $\ell = 1$ mode at 670s.

The white dwarf character of HL Tau 76 is well established (Landolt 1968, Greenstein 1970), and photometric data on the rapid-variability is fairly extensive (Warner and Nather 1972, Warner and Robinson 1972a, Fitch 1973). Power spectra show a strong peak at 115.79 c/d (f_0) with additional peaks at

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frequencies which differ from f_0 by integral multiples of 7.41 c/d (F). The fact that the spacing of the peaks near the harmonics 2 f_0 and 3 f_0 is the same indicates that some sort of modulation process is involved. The additional peaks could be explained in terms of non-linear effects between the g-modes of a rotating star; however Fitch's suggestion that the modulation frequency F is the orbital frequency of a faint binary companion is appealing. Fitch has shown that the spectrum can be understood in terms of a pulsational period of 746s, which is non-sinusoidally modulated at the frequency F, and one other period at 494s.

The similarity of the light curves of doubly periodic cepheids led Fitch to suggest that the 746 and 494-s periods should be associated with the fundamental and first overtone radial modes. The large discrepancy between the periods and theoretical periods for the radial oscillations of a white dwarf makes the identification very improbable.

From Fig. 2 and using the relations (9) and (11) we predict the following approximate periods for the $\ell = 1$ g-modes of a white dwarf of low mass: 375, 500, 625, 750, 875, etc. If one is prepared to admit that the modes g_2 and g_4 for example can be highly excited while the amplitude of the other modes remains low then the periods can be fairly easily explained. However the comparison would certainly be more satisfactory if the theoretical periods were longer by a factor of ~ 2.

Although at this stage it is probably unwise to attach too much significance to the discrepancy we feel that there is some evidence to suggest that HL Tau is in fact related to the stars which are discussed in the following section. The binary period of $3\frac{1}{4}$ h. suggested by Fitch is typical of these objects whose pulsations are apparently associated with the accretion of mass onto the white dwarf component.

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It has already been suggested that G44-32 is related to AM CVn, a binary in which mass is being transferred to the white dwarf component and which exhibits a 120-s periodicity (Warner and Robinson 1972a and 1972b). Mass accretion would raise the apparent temperature of the white dwarf and could easily account for the discrepancy between the observed and calculated periods of HL Tau.

5. The pulsations of the Cataclysmic Variables

The stars which we consider in this section are short period binary systems in which mass is being transferred to a partner of small dimensions which is probably a white dwarf. The periodic light variations at ~30s which concern us are consistent with this view and it would be very difficult to account for the general properties of the cataclysmic variables using any other single model.

Excluding, for the present, the old nova DQ Herculis, and AM CVn in which the mass transferred is helium, the stars are either dwarf novae or nova-like variables. In the dwarf novae the periodic variations have been observed only during outbursts, at maximum luminosity or immediately following.* In cases where the luminosity decreased significantly while the pulsations were visible the period has increased on a similar timescale and the correlation with luminosity is apparently the same from one outburst to another (Warner and Robinson 1972a, Warner 1974b).

The observed properties of the ~30-s pulsations in the nova-like variables are similar. The amplitude is low (~.01 mag) and variable but in this case the period (and luminosity) remains within fairly narrow limits. Other observational

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* Recent observations of rapid variability in dwarf novae have been summarised by Warner (1974a).

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properties of the nova-like variables also indicate that they are closely related to the dwarf novae. In particular their structure is probably similar to that of the Z Cam stars at standstill. (Warner and Van Citters 1974, Nather and Robinson 1974 (NR), Osaki 1974).

It is clear that the observed periods of the dwarf novae cannot be associated with the rotation of the condensed object as in the model of Bath, Evans and Pringle for DQ Her (1974) since it would not be possible to account for the observed period changes. As Bath et al (1974) have pointed out, these observations also place a severe constraint on models based on oscillations of the star. However because the g-modes of a white dwarf whose core is sufficiently degenerate do in fact have a large amplitude only in the surface layers such models are not immediately excluded. Moreover it seems unlikely that the alternative proposed by Bath (1973) (in which the period is associated with the Keplerian motion of hot spots in the disc) is applicable to the pulsations of the nova-like variable UX UMa (NR 1974). The $\sim$30-s period of this object varies slowly from night to night but is present for $\sim 10^6$ s which is considerably longer than would be expected on the basis of the hot spot model ($\sim 2 \times 10^3$ s; Bath, Evans and Papaloizou 1974). Several other objections to this model in the case of UX UMa have been raised by NR.

5.1 The Structure of the Condensed Objects

Warner's (1973) determination of the masses of the cataclysmic variables indicates that the most probable mass for the white dwarf component is 1.2 $M_\odot$. His method gives a result which agrees, to within observational error, with an independent determination of the mass of EM Cyg by Robinson (1974). These

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authors find values of 0.70 $M_\odot$ and 0.93 $M_\odot$ respectively with an error of about 0.2 $M_\odot$ in each.

The luminosity of the nova-like variables and of dwarf novae at maximum are probably between 1 and 10^2 $L_\odot$. In the case of the dwarf novae the optical (and ultra-violet) luminosity falls through 2 to 4 magnitudes on a timescale of $\lesssim 10^6$ s indicating that the emission originates in the accretion disc. This conclusion is in agreement with the apparent size of the object, deduced from the observed colours during outburst and has been confirmed by Warner's (1974a) observations of the eclipses of Z Cha. Determination of the luminosity of the white dwarf is difficult because it is possible that a very small fraction of the total radiation is emitted at observable wavelengths. For example the surface temperature of a model of 10 $L_\odot$ is 130 000 K. The corresponding black body curve reaches a maximum at ~200 Å and the ratio F_λ / F_{max} at 3500 Å is less than 5×10^{-4}. However observations from outside the atmosphere (Rappaport et al 1974, Holm and Gallagher 1974) indicate that the luminosity of the white dwarf does not greatly exceed that of the disc. It is possible that the intrinsic luminosity of the star is considerably less than 10 $L_\odot$ but the surface temperature would still be very high as a result of the process of mass accretion. (Lynden Bell and Pringle 1974).

We assume that the envelope of the white dwarf is not greatly expanded since it would not otherwise be possible to explain the short period of the pulsations. On the other hand the small surface area of the white dwarf of ~1 $M_\odot$ then implies that it would contribute a very small fraction of the luminosity at optical wavelengths regardless of the surface temperature. Observations of the phase changes in UX UMa during eclipse by NR indicates that the pulsating object is

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about the same size as the disc. We suggest that the pulsations are observed as a result of a reflection effect off the disc or surrounding material.

Since the relevant layers of the white dwarf are necessarily in a highly perturbed state it is not possible, at this stage, to calculate accurate periods for the g-modes. Nonetheless we feel that our results concerning the location and excitation energies of the oscillations in equilibrium models do provide some insight into the probable nature of the g-mode oscillations in the cataclysmic variables.

5.2 The g-modes of Models in Thermal Equilibrium

The models we have used have masses of 1.0 $M_\odot$ or 1.2 $M_\odot$ and the luminosities range from 10^{-2} to 10 $L_\odot$. In general the envelopes were assumed to be of solar composition and the inward integration was continued to the point where the luminosity L(r) was reduced to zero as a result of the inclusion of a nuclear energy source in thermal equilibrium. The core temperatures of these models are consequently lower than those of cooling white dwarf models of the same luminosity.

The results of our calculations are summarised in Tables IV and V. The excitation energies listed in Table V are normalised at the surface so that $\delta r(R) = R$. Since the first node in δr is less than about 0.01 R from the surface the amplitude of the light variations would probably be large if $\delta r(R)/R \gtrsim 10^{-3}$. The values listed in Table V should therefore be multiplied by a factor of $\sim 10^{-5}$.

The amplitude of the oscillations in the core is very small in all the models in which the composition of the envelope is rich in hydrogen (Fig. 5). In the more luminous models the low amplitude in the interior is partly due to the

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TABLE IV

Parameters of hot white dwarf models

Model	$L/L_\odot$	R (10^8 cm)	T_c (K)	P_T* (dyn cm^{-2})	(10^6 K)	log ρ_c (gm cm^{-3})
1.0M-2L (a)	1.1(-2)	5.51	21300	1.4(19)	11.6	7.56
1.0M-2L (b)	1.1(-2)	5.51	21300	1.4(19)	17.0	7.56
1.0M-2L (c)	1.0(-2)	5.51	20600	-	13.3	7.56
1.2M-2L	7.0(-3)	3.80	22700	3.2(19)	10.4	8.25
1.2M-1L	1.0(-1)	3.86	43800	3.6(19)	20.6	8.25
1.2M0L	1.0	3.96	76900	5.7(19)	34.3	8.25
1.2M1L	1.0(1)	4.16	133300	1.1(20)	60.1	8.23

* The pressure at the base of the hydrogen rich envelope.

TABLE V

Periods and excitation energies of hot white dwarf models

Model	ℓ	P_1(s)	P_2(s)	P_3(s)	E_1(10^{45} ergs)	E_2(10^{45} ergs)	E_3 (10^{45} ergs)
1.0M-2L(a)	2	56.4	78.4	101.1	3.0	0.22	0.60
1.0M-2L(b)	2	56.2	78.2	89.9	3.0	0.23	2.10
1.0M-2L(c)	2	80.8	97.2	107.8	77.0	24.00	21.00
1.2M-2L	2	41.1	56.9	-	1.9	0.13	-
1.2M-1L	2	29.8	43.1	51.5	4.0	0.20	3.90
1.2M 0L	1	37.4	55.1	57.8	8.1	34.00	0.05
	2	22.0	33.3	39.1	8.0	0.44	6.90
	3	15.7	23.7	30.5	8.0	0.43	0.12
	4	12.2	18.4	23.8	8.0	0.44	0.08
1.2M1L	2	16.0	26.1	33.8	17.0	1.10	14.00

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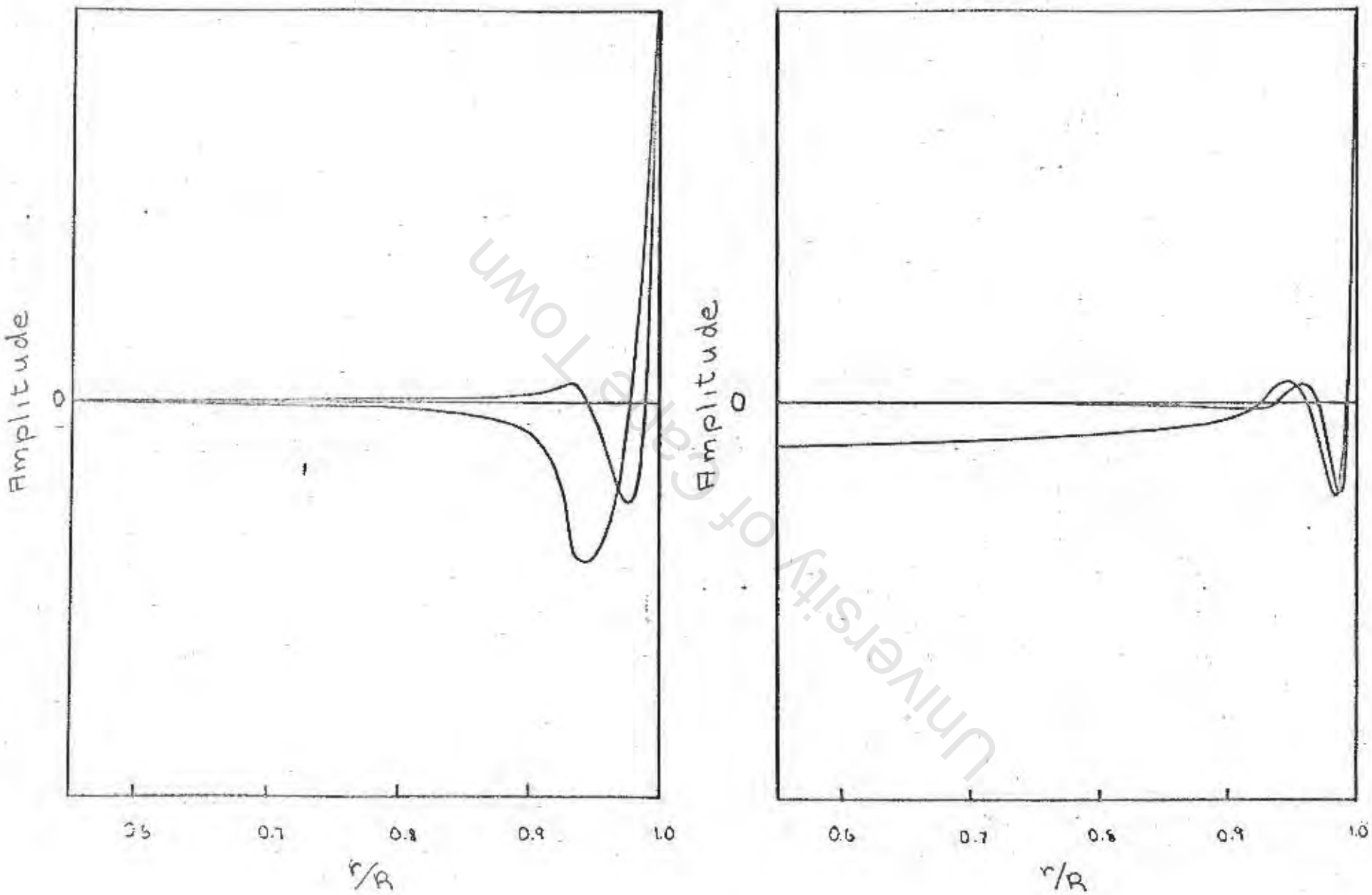

Figure 3. The radial displacement δr, of the $\ell = 2$ modes in model 1.2M0L.

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assumption that the luminosity is derived from the hydrogen burning shell source. The effect of this assumption in reducing the core temperature is not important for the cooler models, but the hydrogen rich composition of the envelope is still critical. To illustrate this point we compare the models 1M-2L (a), (b) and (c). Models (a) and (b) have hydrogen rich envelopes of the same mass which extend inwards to the point where the energy release due to the p-p chain is equal to the luminosity at the surface, but in model (b) it was assumed that the thermal structure was not affected by nuclear reactions and the integration was continued inward keeping L(r) constant and using a helium composition. The inward integration was then terminated when the temperature gradient became small (Section 3). Model (c) has the same mass and luminosity as models (a) and (b) but the entire envelope was assumed to be helium. The inward integration was stopped on the same condition as in model (b).

The periods and eigenfunctions of the g-modes of model (b) were found to be almost identical to those of model (a) despite the fact that the core temperature of the latter is increased by a factor of ~ 1.5. The comparatively low opacity of the helium envelope of model (c) results in a core temperature which is considerably lower than that of (b) but the excitation energies of the oscillations are nevertheless increased by a factor of about 10, (Table V). The difference arises because the low inertia of the hydrogen increases the natural frequency of the envelope with the result that its oscillations become largely uncoupled from the core. Increased temperature in the envelope would strengthen this effect (see Appendix).

As the order of the mode increases $(\sigma^2 + Ag)$ becomes negative throughout a large part of the interior and a coupling between long wavelength oscillations

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in the core with short wavelength oscillations of the envelope becomes possible (Fig. 5(c)). For models with helium envelopes this is the case even for the modes of lowest order.

Suppose r_o is a point such that

$$\int_{r_o}^{R} dE/dr \quad dr = \tfrac{1}{2}E$$

where E is the excitation energy of the oscillation. We may use the mass contained above this point as a quantitative measure of the extent to which the oscillations are confined to the surface layers. For the mode g_{12} of models of $1L_\odot$ and 10^{-2} $L_\odot$ this mass is 2×10^{28} gms and 3×10^{27} gms respectively. The thermal energies of the corresponding regions are $\sim 4 \times 10^{43}$ ergs and $\sim 1 \times 10^{42}$ ergs.

5.3 Discussion

Following the suggestion by Kraft (1963) it has frequently been supposed that the dwarf nova outburst is caused by a thermal instability in the hydrogen rich envelope of the white dwarf. Starrfield et al (1974) find that their fully implicit calculations do not reproduce the observed light curves of dwarf novae. However they suggest that the kinetic energy of the expanding envelope of the white dwarf may be transferred to the accretion disc which is heated by shock waves. The disc is the source of the optical luminosity which explains the observed structure of the objects and the rapid decline from maximum.

Although the thermal structure of the white dwarf immediately following the runaway is probably not very similar to that of the models it is clear that the

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temperature above the shell source will be high and given that convection is not very extensive it seems highly probable that the gravity oscillations will be confined to the envelope. Comparing the observed energy of the outburst ($\sim 10^{40}$ ergs) with the excitation energies listed in Table V, it is apparent that g-mode oscillations may well be excited. The observation of short-period variations of the right sort of frequencies may be regarded as evidence in favour of models of the dwarf novae based on a thermal instability. However we cannot confirm the suggestion that the observed increase in period is caused by cooling of the white dwarf (Warner and Robinson 1972a, Warner and Brickhill 1974). It is unlikely that the thermal energy of the layers above the site of the instability is less than 10^{43} ergs. A significant change on the timescale of the observed period change would therefore imply a luminosity of $\sim 10^{6}$ $L_{\odot}$ which is not consistent with observations.

The timescale on which the period changes is in fact more closely related to that of the changing luminosity of the disc. We are therefore led to consider alternative models of the dwarf nova outburst in which the increased luminosity is associated with a sudden increase in the rate of mass accretion, (e.g. Bath et al, 1974; Osaki 1974). In these models the <u>outer</u> layers of the envelope would be most highly perturbed by the outburst. The temperature would be increased by the dissipation of the kinetic energy of the infalling material at the surface and by the adiabatic heating of the envelope <u>above</u> the transition zone (the increase in pressure is more or less constant in the envelope $\Delta P = (\Delta M/4MR^2)g$ so that $\frac{\Delta T}{T}$ changes as $\frac{1}{P}$). The excited g-modes would have a high amplitude in the outermost layers which cool fairly rapidly, accounting for the observed rapid change in period.

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The perturbation in the envelope would probably be greatest in the equatorial region indicating that the modes likely to be excited are those having a high value of m, i.e. the travelling waves. This raises the interesting possibility that the excitation of gravity oscillations in white dwarfs accreting mass is associated with the transfer of the kinetic energy and momentum of the infalling material to the outer layers of the star, in a similar way to that in which gravity waves are excited on the surface of the sea. The observational identification of the 30-s period in UX UMa with an m = 2 mode travelling in the direction of rotation of the disc (NR 1974) is in agreement with this suggestion. We note that coherent pulsations have been observed in AM CVn (Warner and Robinson 1972b). It is highly improbable that these could be excited by nuclear reactions.

The pulsations of the old nova DQ Herculis differ from those of the dwarf novae. The period is longer (71-s) and constant to a fairly high degree. It has persisted at about the same amplitude for 20 years (e.g. Nather and Warner 1969). Observations of polarisation changes during the 71-s cycle indicate that the pulsations may originate in the strong magnetic field of a rotating object (Nather et al 1974). Oscillation theories are not excluded by these observations however (cf. Angel 1969) and the similarity of the observed phase shifts to those of UX UMa (NR, 1974), and the sinusoidal nature of the oscillations are in favour of such a model. In particular we note that the phase shift begins as the disc goes into eclipse. If the pulses originate near the white dwarf it therefore seems probable that they are partially reflected off the surface of the disc in which case the observed radiation would be polarised by electron scattering. Observations of the polarisation of light from UX UMa would test this suggestion.

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The constant nature of the oscillation suggests that the amplitude in the interior is relatively high. This may be caused by the type of coupling discussed previously or because of a comparatively high temperature in the outer part of the core. Such an oscillation would probably not be excited by the process of mass accretion. On the other hand it is clear that g-modes could have been excited by the 1934 nova explosion. The slow decrease in amplitude of the 71-s pulsation suggests that this is in fact the case (Warner et al., 1972).

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APPENDIX 1

To compare envelopes of different density we write equations (1) and (2) in the form

$$\frac{dv}{dM_r} = \left[\frac{\ell(\ell+1)}{\rho^2\sigma^2} - \frac{R^2}{\rho\Gamma_1 p} \right] \frac{p^{2/\Gamma_1}}{4\pi R^2} w \qquad (1.1)$$

$$\frac{dw}{dM_r} = \left[Ag + \sigma^2 \right] \frac{1}{4\pi R^2 p^{2/\Gamma_1}} v \qquad (1.2)$$

where we may regard p as a function of the mass of the overlying layers only. If we neglect the terms $R^2/(\rho\Gamma_1 p)$ and σ^2, and assume that the value of A (at a given pressure) remains constant, these equations show that σ changes as $1/\rho$. Accordingly, if the density of the envelope is reduced, the point at which $(Ag + \sigma^2)$ changes sign moves outwards and the amplitude in the interior is decreased.

Alternatively, sufficiently far from the surface $\left(\frac{\ell(\ell+1)}{\sigma^2} \gg \frac{\rho r^2}{\Gamma_1 p}\right)$ elimination of w between equations (1) and (2) yields

$$\frac{d^2v}{dr^2} = \frac{\ell(\ell+1)}{\sigma^2} \frac{1}{r^2} (\sigma^2 + Ag) v + \frac{\rho}{p^{2/\Gamma_1}} \frac{d}{dr}\left(\frac{p^{2/\Gamma_1}}{\rho}\right) \frac{dv}{dr} \qquad (1.3)$$

$$\frac{d^2v}{dM_r^2} = \frac{\ell(\ell+1)}{\sigma^2} \frac{1}{r^2} \frac{(\sigma^2 + Ag)}{(4\pi R^2 \rho)^2} v + \frac{\rho}{p^{2/\Gamma_1}} \frac{d}{dr}\left(\frac{p^{2/\Gamma_1}}{\rho}\right) \frac{1}{4\pi r^2 \rho} \frac{dv}{dM_r} \qquad (1.4)$$

comparison with the equation of a damped harmonic oscillation shows that in the transition zone where the solution oscillates the vertical wavelength (with respect to M_r) is proportional to $\sigma\rho/[\ell(\ell+1)(Ag+\sigma^2)]^{1/2}$. A reduction in ρ in the envelope would therefore necessitate a corresponding increase in σ if the average wavelength (mode) is to remain unchanged. Moreover it is clear that

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the oscillations would be shifted toward the surface.

If we suppose that in the envelope p varies with ρ^{γ}, we may write

$$A = \frac{-g\rho}{p}\left(\frac{1}{\gamma} - \frac{1}{\Gamma_1}\right) \tag{1.5}$$

If γ and g remain constant at a given pressure then A is proportional to ρ. However, in the changes in the white dwarf envelopes which concern us, the reduced degeneracy corresponding to a decrease in density results in an increase in γ so that the assumption that A is constant is reasonable.

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APPENDIX 2

The White Dwarf Models

The equations governing the structure of non-rotating stars in equilibrium may be written in the form

$$\frac{dp}{dr} = -\frac{GM_r}{r^2}\rho \tag{2.1}$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho \tag{2.2}$$

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon \tag{2.3}$$

$$\begin{cases} \dfrac{dT}{dr} = \dfrac{3}{4ac}\dfrac{L_r}{4\pi r^2}\dfrac{\rho}{T^3}\kappa & (2.4a) \\[2ex] \dfrac{dT}{dr} = \dfrac{\Gamma_2 - 1}{\Gamma^2}\dfrac{T}{P}\dfrac{dP}{dr} & (2.4b) \end{cases}$$

where (2.4a) holds provided the temperature gradient thus found is not greater (in absolute magnitude) than the adiabatic gradient given by equation (2.4b); in which case the temperature gradient is assumed to be given by (2.4b). In addition the pressure, density and temperature are related by the equation of state, so that, given any two, the third can be found from a relationship of the form

$$\rho = \rho(p,T). \tag{2.5}$$

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The opacity κ and the rate of energy release ε are uniquely determined (for a given composition) by the state of the gas. We may write

$$\kappa = \kappa(\rho,T) \tag{2.6}$$

$$\varepsilon = \varepsilon(\rho,T) \tag{2.7}$$

Thus, given the values of P, M_r, L_r and T at any point (r) in the star, it is possible, using equations 2.1 to 2.4, to find the values of ρ, κ, ε and hence to find P, T, M_r and L_r at a neighbouring point $r + \Delta r$.

To solve the equations for the models used above the following procedure was adopted. The effective temperature and surface gravity were specified together with an assumed value for the mass. The radius and luminosity were calculated and the surface temperature was found from the condition $T_s = T_e$ or was taken from the model atmosphere. The pressure was likewise given by conditon (8) or by the model atmosphere. The numerical integration then proceeded inwards to the point where the temperature gradient became very small (section 3) as a result of the reduced luminosity or because of the low opacity. For a white dwarf the depth at this point is generally small (<.1R). The conditions at the base of the envelope depend only on the surface temperature and gravity and are not altered by the assumed mass and radius of the star.

At the centre L_r and M_r are zero and the temperature was taken to have the value found at the base of the envelope.

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The central density was guessed and the solution followed outwards, assuming constant temperature, to the point where the pressure had fallen to the value of the base of the envelope. The radius at this point was compared with the expected radius at the base of the envelope (for the new mass) given by

$$r_e = \left[\frac{g}{(M_C+M_E)G}\right]^{-\frac{1}{2}} - D \qquad (2.8)$$

where M_C is the mass of the core, M_E is the mass of the envelope and D the depth of the envelope. If r_e differed from the core radius the outward integration was repeated with a new central density given by

$$\rho_o = \rho_o' + (r_e - r_c) \Big/ \left[\frac{dr_c}{d\rho_o} - \frac{dM_C}{d\rho_o}\,\frac{dr_e}{dM_C}\right] \qquad (2.9)$$

where ρ_o' is the new central density, ρ_o the previous central density and r_c the core radius. The derivatives $\frac{dr_c}{d\rho_o}$ and $\frac{dM_C}{d\rho_o}$ are initially a crude estimate based on the relations for cold white dwarfs. After the second outward integration they were derived from the previous results.

The process was repeated until the discrepancy was reduced. If the final value of r_e differed from the radius at the base of the envelope it was necessary to repeat the whole procedure (by rerunning the program with the new value of the mass). In general this was only necessary once because the state of the envelope at any given depth is largely independent of the mass chosen. In practice it was found that it was also possible to

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specify the mass and luminosity at the surface and to guess the radius. The central density was then varied to fit the Mass. In this case the repeated integration from the surface with a new value of R would in general result in a slightly different core temperature. However convergence was still sufficiently rapid because the structure of the core was only weakly dependent on the temperature and because it was usually possible to guess the radius fairly accurately on the basis of the table given by Hubbard and Wagner (1970) or using previous results.

The algorithm used in all the numerical integrations can be represented

$$p(r+\Delta r) = p(r) + (Dp_1 + 2Dp_2 + 2Dp_3 + Dp_4)/6 \qquad (2.10)$$

where

$$Dp_1 = \Delta r p'(r,M,p,T) \quad \left[= \Delta r \left[- \frac{GM(r)}{r^2} (\rho(p(r),T(r))\right]\right] \qquad (2.11a)$$

$$Dp_2 = \Delta r p'(r+\Delta r/2, M+DM_1/2, p + Dp_1/2, T + DT_1/2) \qquad (2.11b)$$

$$Dp_3 = \Delta r p'(r+\Delta r/2, M+DM_2/2, p + Dp_2/2, T+DT_2/2) \qquad (2.11c)$$

and

$$Dp_4 = \Delta r p'(r+\Delta r, M+DM_3, p + Dp_3, T+DT_3) \qquad (2.11d)$$

The advantage of this method over the finite difference techniques is that the step size can be readily altered. The

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increment Δr was varied so that $\frac{\Delta p}{p}$ remained within specified limits. Since the error in the integration is of fourth order in Δr it could easily be checked by reducing the step size (i.e. by changing the requirement on $\frac{\Delta p}{p}$).

The coding of the equation of state was taken directly from the program made available by Dr. Hubbard with some small changes to improve the accuracy of the thermodynamic functions. In this program the ion pressure was found using the analytic form given by Van Horn (1971). The results were checked by direct comparison with the Monte Carlo data (Brush Sahlin and Teller 1966). The electron pressure is given by an analytic approximation at low density ($\rho \lesssim 10^3$) and by integration of the Fermi-Dirac Integrals at higher densities. The subroutine was checked against the results of Cox (1968, p. 1165).

At a given point in the star the pressure and temperature were generally known and the density was required (see above). Since the subroutine provided by Dr. Hubbard could only be used to find the pressure, given the temperature and density, it was necessary to solve equation (2.5) iteratively. At each step the increment in the density was found by requiring that

$$\Delta P = \left.\frac{\partial P}{\partial T}\right|_{\rho} \Delta T + \left.\frac{\partial P}{\partial \rho}\right|_{T} \Delta\rho \qquad (2.12)$$

where ΔP and ΔT are the corresponding increments in P and T. The partial derivatives included the effects of changes in the electron molecular weight ω_e . Using this new value of the density ω_e was found (see below) and hence the electron density and the electron pressure, the ion pressure and the

radiation pressure. If the total pressure differed from the required value more than a factor of 10^{-5} the process was repeated using a new value of the density based on the latest value of the partial derivative $\left.\frac{\partial P}{\partial \rho}\right|_T$. As soon as the discrepancy in P was sufficiently small ε and κ were evaluated using the final value of ρ and the next increments in P,M,L and T calculated.

In the treatment of ionisation suggested by Eggleton, Faulkner and Flannery the ratio of atoms in the (i+1)th state of ionisation to those in the ith is given by

$$\frac{n_i^+}{n} = \frac{\omega^+}{\omega} e^{\eta+\chi_i/kT+\Delta\mu} \tag{2.13}$$

where ω and ω^+ are the statistical weights of two adjacent states of ionisation, η is the degeneracy parameter, χ the ionisation potential and $\Delta\mu$ is a correction term which describes the effects of pressure ionisation. They suggest the form

$$\Delta\mu = 2\, a_o^3 (kT + 20\chi_o)\, n_e/kT \tag{2.14}$$

where $a_o = 0.523/\bar{Z}$ and $\chi_o = 13.5\,\bar{Z}$. The correction depends only on the electron density and in their formulation the ratios need only be evaluated once at each point in the star. In our case it was still necessary to solve the equations iteratively. Each time the subroutine was called the temperature and density were given together with an approximate value for ω_e based on the values of ω_e, $\frac{\partial \omega_e}{\partial \rho}$ and $\frac{\partial \omega_e}{\partial T}$ found at the previous step.

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This value was then used to find the electron density and hence a new value of ω_e and the processes repeated until the difference between successive values became sufficiently small. The degeneracy parameter η was assumed to be given by

$$\eta = \log \omega_e - 1.5 \log kT - 90.8925 \qquad (2.15)$$

which is strictly only valid in regions of low degeneracy. Since partial ionisation was only included for Helium and Hydrogen the degeneracy parameter did not reach values greater than about 2 and the errors in η were less than about 1, corresponding to an error of a factor of e in the ratio $\frac{n_{i+1}}{n}$. However it was found that use of the correct value of η (found in the electron pressure subroutine) made very little difference to the models and was very time consuming.

Since nuclear energy generation was assumed to be due to the p-p chain only, and factors due to electron screening and precise chemical composition were neglected, the evaluation of ε was trivial. From Bachall and May (1968)

$$\varepsilon_{p-p} = 4.72\times10^{6}\ \rho\ T_6^{-2/3} e^{-\tau} X^2 (1+0.417\tau^{-1} + 12.6\tau^{-2} + 36.6\tau^{-3})$$

where

$$\tau = 33.8\ T_6^{-1/3}$$

and T_6 is the temperature in 10^6 K.

The evaluation of the radiative opacity from the tables of Cox and Stewart presented some difficulties as it was necessary

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to extrapolate their results into regions of higher density and lower temperature. The procedure adopted was as follows. For points which were within the range of their results the opacity was found by linear interpolation in the tables using the three closest grid points. If the opacity was not given at any one of these points the fourth neighbouring grid point was used. Thus if the density at the required point fell between the values ρ_i and ρ_{i+1} used in the tables and the temperature between T_j and T_{j+1} the opacity was found on the assumption that it would lie on the plane defined by the opacity at three of the points (i,j), (i,j+1), (i+1,j+1), (i+1,j). If the opacity was known at less than three of the points the required value was given by

$$\kappa = k \rho / T^3$$

(c.f. Schwarzschild 1958, p.68,70).

The constant k was found from the requirement that the formula gave the correct value of the opacity at one of the four closest grid points (provided it was known at least one of these points) or to have a value that had been found previously. The opacity found by this technique varied smoothly and k changed slowly through the envelope. To get some idea of possible errors arising from the extrapolation of the tables k could be artificially varied. It was found that the resulting changes in the models were generally small.

In order to solve the equations governing the non-radial oscillations the functions Γ_1 and A were required at each point

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in the equilibrium model and to facilitate the treatment of convection Γ_2 was also required (equation 2.4b).

The gammas were found using the expressions

$$\Gamma_3 = 1 + \frac{P\chi_T}{\rho T C_V}$$

$$\Gamma_1 = \chi_\rho + \chi_T (\Gamma_3 - 1)$$

$$\Gamma_2 = 1/(1 - 1/\chi_\rho/(\Gamma_3 - 1) + \chi_T)$$

where $\chi_\rho = \frac{\rho}{P} \left.\frac{\partial P}{\partial T}\right|_T \qquad \chi_T = \frac{T}{P} \left.\frac{\partial P}{\partial T}\right|_\rho$

and C_V is the specific heat at constant density, including the contribution due to ionisation. In general the partial derivatives were found from the derivatives of the expressions used in the coding. They could be checked against the values found by calculating the pressure and internal energy at neighbouring points in the T,ρ plane. The partial derivatives $\frac{\partial U_e}{\partial T}$ and $\frac{\partial P_e}{\partial T}$ (where the U_e and P_e represent the energy density and pressure due to the electron gas) were found from analytic expressions even at high density. Although the absolute error in these functions was fairly large (up to ~50 percent), the resulting changes in the thermodynamic functions were negligible.

A was found from the expression

$$A = \frac{1}{\rho}\frac{d\rho}{dr} - \frac{1}{\Gamma_1 p}\frac{dp}{dr}$$

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where $\frac{dp}{dr}$ was given at each point by equation (2.1) and $\frac{d\rho}{dr}$ was given by the condition

$$\frac{dp}{dr} = \left.\frac{\partial p}{\partial T}\right|_{\rho} \frac{dT}{dr} + \left.\frac{\partial p}{\partial \rho}\right|_{T} \frac{d\rho}{dr}$$

Since $\frac{dp}{dr}$, $\frac{dT}{dr}$ $\left.\frac{\partial p}{\partial T}\right|_{\rho}$ and $\left.\frac{\partial p}{\partial r}\right|_{T}$ were known at each point. Alternatively the gradients could be found by numerical differentiation of p and ρ as functions of radius in the star. The two methods were found to be in satisfactory agreement.

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Appendix 3

Calculation of Non-radial Eigenfunctions

The second order equations governing the non-radial oscillations of stars can be written in the form

$$\frac{du}{dr} = \frac{\rho g}{\Gamma_1 p} u + \left\{\frac{\ell(\ell+1)}{\sigma^2} - \frac{\rho r^2}{\Gamma_1 p}\right\} y \qquad (3.1)$$

$$\frac{dy}{dr} = -Ay + \frac{1}{r^2} (Ag + \sigma^2) u \qquad (3.2)$$

(Ledoux and Walraven 1958, p. 520)

where $u = r^2 \delta r$ and $y = p'/\rho$, p is the Sulerian perturbation to the pressure. The boundary conditions at $r = R$ follow directly from the requirement that the Lagrangian perturbation to the pressure at the surface be zero;

$$u(R) = R^2 (\delta r(R)), \quad y(R) = g(R)(\delta r(R)) \qquad (3.3)$$

For an assumed value of σ^2 these equations can be integrated directly from the surface and the divergence of u and y near the centre can be used to find the eigenvalues σ_R^2 after a few trial integrations.

To understand the divergence of the inward integrations near the centre it is best to consider the equations again in the form (1) and (2). Sufficiently close to the centre these equations may be written

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$$\frac{dv}{dr} = \frac{\ell(\ell+1)}{\sigma^2} w \tag{3.4}$$

$$\frac{dw}{dr} = \frac{\sigma^2}{r^2} v \tag{3.5}$$

At each point in the star v and w are directly related to u and y;

$$v = u\rho^{1/\Gamma}, \quad w = y\rho\rho^{-1/\Gamma} \tag{3.6}$$

so that their divergence is also simply related.

The inward integration becomes divergent as soon as the coefficients of w and v are both positive since if v for example is slightly too large relative to w the calculated increments in v and w will tend to increase the error. To illustrate suppose that v and w are both positive. If now v is relatively too large the increment in w, $(\sigma^2 v/r^2)dr$ (where dr is negative), will tend to reduce w further so that near the centre the functions v and w (and u and y) will tend towards infinity with opposite sign regardless of the precision of the numerical technique used.

It was found that if u, for example, took large positive values near the centre when σ^2 was slightly greater than an eigenvalue, σ_k^2, it would tend towards $-\infty$ if σ^2 was smaller than σ_k^2. This change of sign as σ^2 was varied through an eigenvalue could be used to find the eigenvalue to the desired degree of accuracy. The procedure adopted was as follows. The equations were integrated from the surface with a trial value of σ^2 using the numerical technique described in Appendix 2. In this case the step size was not varied automatically but it was necessary to make the increments in r very much smaller near the surface than in the interior. The values of the functions A, g and

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$\rho/\Gamma_1 p$ at each point were found by interpolation in tables generated by the white dwarf model program. The integration was continued to a point fairly close to the centre ($\sim$ .05 R) where the functions had already begun to diverge considerably. The value of u at this point, u_o say, was stored in the machine and the integration was repeated with decreasing values of σ^2 until the sign of u_o changed. The value of σ^2 at which u_o would be zero was determined by assuming that u_o varied linearly with σ^2 and the integration was repeated using this value until successive values of σ^2 were sufficiently close. Convergence was usually very rapid, the change in σ between the third and the fourth integrations was usually less than $\sim$.01 per cent. Once the eigenvalue had been thus established the equations were integrated from the surface and from the centre (at r = o, u = o and y = o so that it was necessary to use a power series development) to a point near 0.5 R using a reduced step size. The closeness of the fit at this point provided a test of the accuracy of the procedure used to find the eigenvalues.

The program was also used to find the eigenfrequencies of polytropes by interpolation in the British Association tables. The values found were in excellent agreement with the results of Robe (1968).

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