UNIVERSITY OF CAPE TOWN
DEPARTMENT OF MATHEMATICS

The Δ - Nielsen number in Products

by

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INTRODUCTION

In 1967 Robert F. Brown derived a formula which relates the Nielsen number $N(f)$ of a fibre map f to the Nielsen numbers $N(\bar{f})$,$N(f_b)$, where $\bar{f}$,f_b are induced by f. This work is concerned to prove an analogous result for the Δ-Nielsen number, $N(f,g,\Delta)$.

In Chapter I we introduce the set of coincidences of two maps $f,g\colon X \to Y, \Gamma(f,g) = \{x \varepsilon X\colon f(x)=g(x)\}$. We partition this set into equivalence classes by means of the equivalence relation of fixed end-point homotopy and then study some of the geometry of the equivalence classes. We then proceed to introduce the Δ-Nielsen number $N(f,g,\Delta)$ by means of an index, which we show satisfies the axioms of Brooks [1969] for a coincidence index. Thereafter we show $N(f,g,\Delta)$ to be a homotopy invariant.

In Chapter II we introduce the class of fibre spaces. By restricting ourselves to fibre spaces which are products of closed, finitely triangulable manifolds, we derive an analogous formula for coincidences as Brown has for fixed points. Some suggestions for a complete analogue conclude the work.

I wish to thank Dr. H. Schlagbauer who supervised this work throughout, guiding me through my first aquaintance with this field.

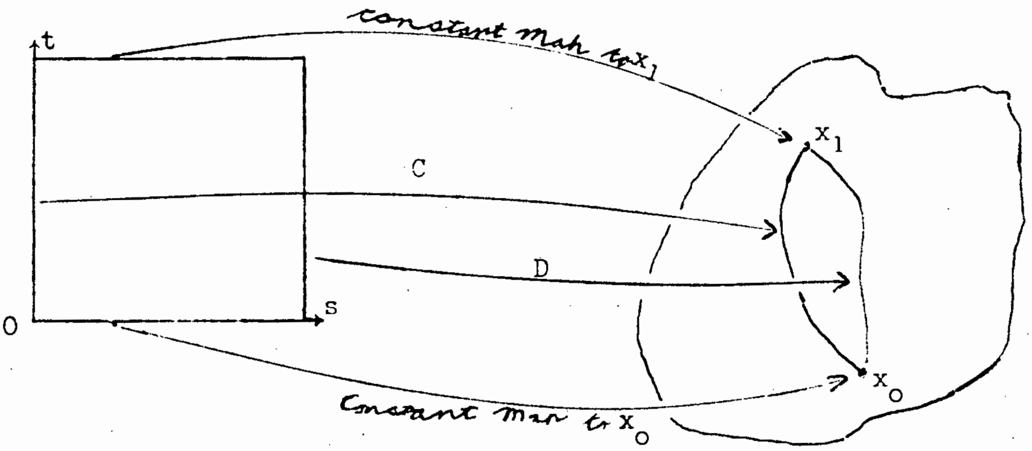
The beautiful layout and typing of Miss N. Tweedie as well as the highly competent printing of Mr. P. Gabriels are plain for all to see.

CHAPTER ONE

In this chapter we define an equivalence relation on the set $\Gamma(f,g)$, thereby obtaining equivalence classes, called coincidence classes. Some of the equivalence classes are said to be 'essential' and all maps f', g', f' homotopic to f and g' homotopic to g will be shown to have the same number $N(f,g,\Delta)$ of these 'essential' classes, a concept which implies the classes being non-empty. Thus $N(f,g,\Delta)$ is a lower bound on the number of coincidence points of any pair of maps (f',g') which are pairwise homotopic to (f,g). Restricting our spaces to compact ANRs, results in there being only a finite number of coincidence classes of f and g, thus $N(f,g,\Delta)$ is finite.

DEFINITION 1.1 Two paths, C, D, I → X are said to be fixed end-point homotopic if there is a map

$$
\begin{aligned}
\psi(s,0) &= C(0) = D(0) = x_0 && \text{for all } s \in I \\
\psi(s,1) &= C(1) = D(1) = x_1 && \text{for all } s \in I \\
\psi(0,t) &= C(t) && \text{for all } t \in I \\
\psi(1,t) &= D(t) && \text{for all } t \in I
\end{aligned}
$$



If C is a path then let $[C]$ denote the set of all paths that are fixed end-point homotopic to C. Our next theorem shows fixed end-point homotopy to be an equivalence relation, written $\simeq$. We have

THEOREM 1.2 Let C, D, E be three paths in space X. Then $C \simeq C$, $C \simeq D$ implies $D \simeq C$, and $C \simeq D$ and $D \simeq E$ together imply $C \simeq E$.

Proof. Since C is fixed end-point homotopic to D and D is fixed end-point homotopic to E there are maps ψ, Φ, : $I \times I \to X$ such that

$\psi(s,0) = C(0) = D(0) = x_0$ all $s \in I$

$\psi(s,1) = C(1) = D(1) = x_1$ all $s \in I$

$\psi(0,t) = C(t)$

$\psi(1,t) = D(t)$

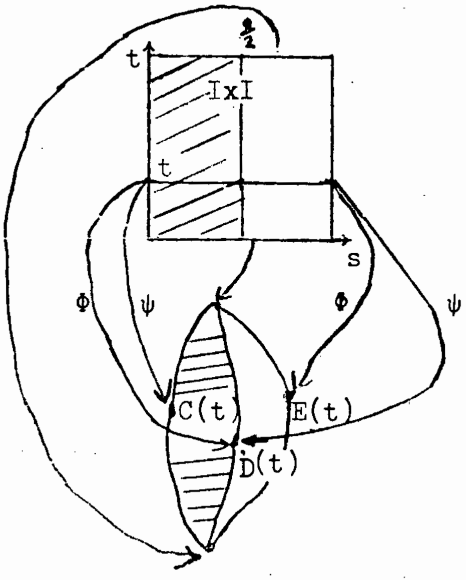
and

$\Phi(s,0) = D(0) = E(0) = x_0$ all $s \in I$

$\Phi(s,1) = D(1) = E(1) = x_1$ all $s \in I$

$\Phi(0,t) = D(t)$

$\Phi(1,t) = E(t)$



Reflexivity. Let $\Lambda(s,t) : I \times I \to X$ be defined by $\Lambda(s,t) = C(t)$, so C is fixed end-point homotopic to itself.

Symmetry. $\Lambda(s,t) = \psi(1-s,t)$ has the properties that $\Lambda(s,0) = C(0) = D(0) = x_0$, $\Lambda(s,1) = C(1) = D(1) = x_1$, $\Lambda(0,t) = \psi(1,t) = D(t)$ and $\Lambda(1,t) = \psi(0,t) = C(t)$ so D is fixed end-point homotopic to C.

Transitivity. Define $\Lambda(s,t)$ by letting $\Lambda(s,t) = \psi(2s,t)$ $0 \leq s \leq 1/2$

$= \Phi(2s-1,t)$ $1/2 \leq s \leq 1$

Then $\Lambda(s,0) = C(0) = E(0) = x_o$ $\Lambda(s,1) = C(1) = D(1) = E(1) = x_1$, $\Lambda(o,t) = \psi(o,t) = C(t)$, $\Lambda(1,t) = \Phi(1,t) = E(t)$. Since $\psi(1,t) = \Phi(o,t) = D(t)$, Λ is a continuous function from $I \times I \to X$, so C is fixed end-point homotopic to E.

DEFINITION 1.3 Let x_o, $x_1 \in \Gamma(f,g)$. x_o is said to be f,g equivalent to x_1 if there is a path $\alpha : I \to X$ such that $f\alpha$ is fixed end-point homotopic to $g\alpha$.

By the immediately preceding theorem, this relation is an equivalence relation. $\Gamma(f,g)$ is thus partitioned into disjoint equivalence classes called coincidence classes the set of all of which being denoted by $\tilde{\Gamma}(f,g)$.

DEFINITION 1.4 A compact ANR is a compact space which imbeds as a neighbourhood retract of the Hilbert cube, I^∞. A compact ANR is thus a separable metric space [Brown, 1971].

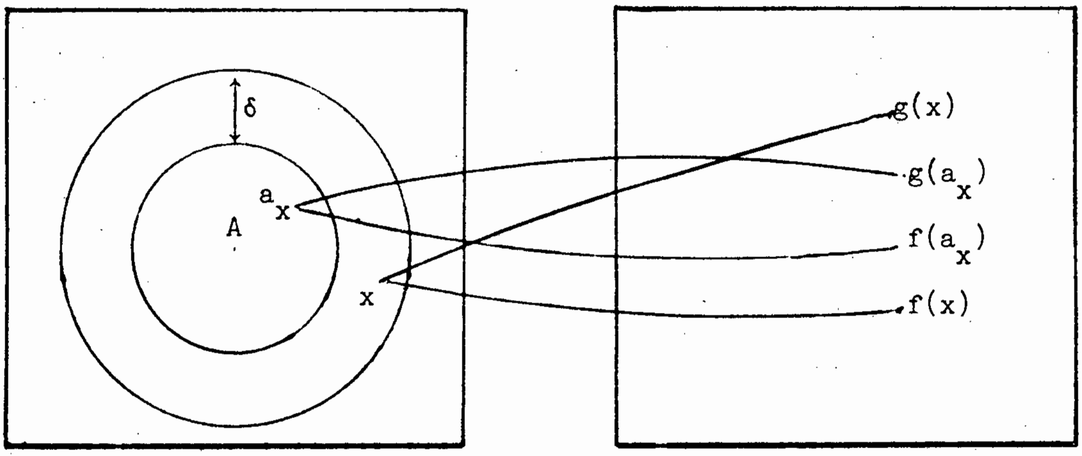
In compact ANRs we can say a fair amount about the geometry of $\tilde{\Gamma}(f,g)$.

LEMMA 1.5 Let f, g, be any two maps from metric space X to metric space Y. If $b \in X - \Gamma(f,g)$ there exists a neighbourhood $N(b)$ containing no coincidence points of f and g.

Proof. Suppose that each neighbourhood $N(b,\frac{1}{n})$, $n \in N$ contained at least one x_n of $\Gamma(f,g)$. Then sequence $\{x_n\}$ converges to b so $f(b) = g(b)$ since $\{f(x_n)\} = \{g(x_n)\}$. Hence $b \in \Gamma(f,g)$, a contradiction.

Hence $X - \Gamma(f,g)$ is open so $\Gamma(f,g)$ is a closed subset of X.

LEMMA 1.6 [Brown, 1971]. Let (Xd) and (Yp) be compact metric spaces and let A be a closed subset of X such that $A \cap \Gamma(f,g) = \phi$. Then there exists a $\delta > 0$ such that $d(x,A) < \delta$ implies $p(f(x), g(x)) > \delta$; so $x \notin \Gamma(f,g)$.



Proof. The function $h(x) = p(f(x), g(x)) : X \to R$ is a map and $A \subset X$ is compact so set $h(A) \subset (0,\infty)$ is bounded below by $z > 0$. But X and Y are compact so f and g are uniformly continuous thus there exist $\delta_1, \delta_2, > 0$ such that $d(x_1,x_2) < \delta_1$ implies $p(f(x_1),f(x_2)) < \frac{z}{3}$ all $x_1, x_2 \in X$, and $d(x_1,x_2) < \delta_2$ implies $p(g(x_1),g(x_2)) < \frac{z}{3}$. Let $\delta = \min(\delta_1,\delta_2,\frac{z}{3})$, and let $x \in X$ such that $d(x,A) < \delta$. Since A is compact there exists $a_x \in A$ such that $d(x,a_x) = d(x,A)$. Then $z \leq p(f(a_x),g(a_x)) \leq p(f(a_x),f(x)) + p(f(x),g(x) + p(g(a_x),g(x))$ which sum is less than $\frac{z}{3} + p(f(x),g(x)) + \frac{z}{3}$ so $p(f(x),g(x)) > \frac{z}{3} > \delta$ as required.

DEFINITION 1.7 A compact metric space (X,d) is said to be uniformly locally contractible (ULC) if, given $\varepsilon > 0$ there exists $\delta > 0$ such that if

$W = \{(x,x') \in X \times X : d(x,x') < \delta\}$ then there exists a map Ω : $X \times I \to X$ such that

$$\Omega(x,x',0) = x,\ \Omega(x,x',1) = x'$$

$$\Omega(x,x,t) = x \text{ for all } t \in I$$

and $\operatorname{diam}(\Omega((x,x')\times I)) < \varepsilon$ for all $(x,x') \in W$.

THEOREM 1.8 [Brown, 1971]. A compact ANR is ULC.

DEFINITION 1.9 Two maps $f,g : X \to Y$ are said to be ε- homotopic for $\varepsilon > 0$ if there exists a map $H : X \times I \to Y$ such that

$$H(x,0) = f(x) \text{ for all } x \in X$$

$$H(x,1) = g(x) \text{ for all } x \in X$$

$$\operatorname{diam}(H(x \times I)) < \varepsilon \text{ for any } x \in X.$$

Letting $H(x,t) = \Omega(f(x),g(x),t)$ we obtain the following theorem

THEOREM 1.10 Let (X,d) and (Y,p) be any compact ANRs and let $\varepsilon > 0$ be given. Then by theorem 1.8 above, there is a $\delta > 0$ such that if $p(f(x),g(x)) < \delta$ for all $x \in X$, then f and g are ε - homotopic.

DEFINITION 1.11 Let C be a path from coincidence point e to e' of $\Gamma(f,g)$. Then paths fC and gC are said to be ε - fixed end-point homotopic if there is a map $H : I \times I \to Y$ such that

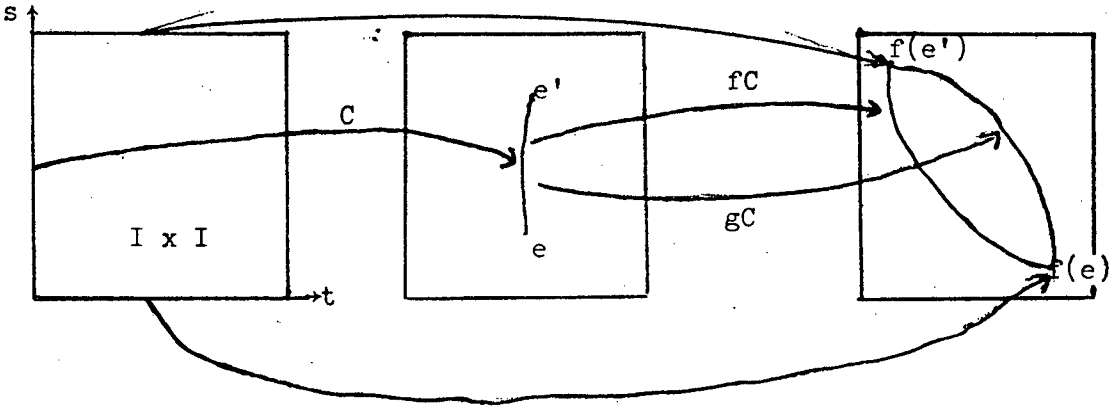
$H(0,t) = f(e) = f(C(0)) = g(C(0)) = g(e)$, all $t \in I$

$H(1,t) = f(e') = f(C(1)) = g(C(1)) = g(e')$ all $t \in I$,

$H(s,0) = f(C(s))$ all $s \in I$

$H(s,1) = g(C(s))$ all $s \in I$

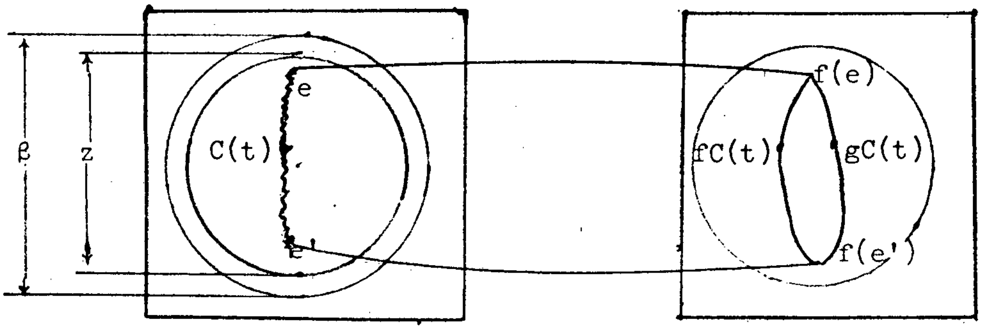
diam $H(I \times I) < E$ all $(s,t) \in I \times I$.



Clearly any two such coincidence points are f, g equivalent. Our next theorem tells us that in compact ANRs any two sufficiently close coincidence points are f, g equivalent.

THEOREM 1.12 There is a number $z > 0$ such that if e and e' are any two coincidence points in $\Gamma(f,g)$ and $d(e,e') < z$, then e and e' are f, g equivalent.

Proof.



By theorem 1.8 above there exists a $\delta > 0$ such that if $W_1 =$

$\{(y,y') \in Y \times Y : p(y,y') < \delta\}$ then there exists a map $\Omega_1 : W_1 \times I \to Y$ such that $\Omega_1(y,y',0) = y$, $\Omega_1(y,y',1) = y'$ and $\Omega_1(y,y,t) = y$ all $t \in I$. Since f and g are both uniformly continuous given $\delta > 0$ there is a $\beta > 0$ such that for all x, $x' \in X$ with $d(x,x') < \beta$, $p(f(x),f(x') < \frac{\delta}{2}$, $p(g(x),g(x')) < \frac{\delta}{2}$. Let e, $e' \in \Gamma(f,g)$ such that $d(e,e') < z$, z being chosen by theorem 1.8 to give set $W_o = \{(x,x') \in X \times X : d(x,x') < z\}$ and map $\Omega_o : W_o \times I \to X$ such that $\Omega_o(x,x',0) = x$, $\Omega_o(x,x',1) = x'$ and $\operatorname{diam}(\Omega_o((x,x') \times I)) < \beta$. Let $C(t) = \Omega_o(e,e',t)$ so C is a path from e to e' such that $\operatorname{diam}(f(C(I))) < \frac{\delta}{2}$ and $\operatorname{diam}(g(C(I))) < \frac{\delta}{2}$. Hence $p(f(C(s)), g(C(s))) < \delta$ for each $s \in I$. Define $H(s,t) = \Omega_1(f(C(s)),g(C(s)),t)$. Then $H(0,t) = \Omega_1(f(C(0)), g(C(0)),t) = \Omega_1(f(e),g(e),t) = f(e) = g(e)$ and similarly $H(1,t) = f(e') = g(e')$. We see then by theorem 1.10 that H is an ε-fixed end-point homotopy so e and e' are f, g equivalent coincidence points. They are thus in the same element K of $\tilde{\Gamma}(f,g)$.

This results in, amongst other things,

<u>THEOREM 1.13</u> If X, Y, are any two compact ANRs and f, g : X → Y any two maps then $|\tilde{\Gamma}(f,g)|$ is finite.

<u>Proof.</u> Any two distinct coincidence classes K, L of $\tilde{\Gamma}(f,g)$ are by theorem 1.12 at least a distance z apart, i.e. for any $a \in K$ and $a' \in L$, $\inf\{d(a,a')\} \geq z$. Around each e in K put an open $\frac{z}{3}$ ball, hence $K \subset \bigcup_{e \in K} \{B_{\frac{z}{3}}(e)\}$ and $K = \Gamma(f,g) \cap \{\bigcup_{e \in K} B_{\frac{z}{3}}(e)\}$ so K is open in $\Gamma(f,g)$ the latter having the subspace topology. Since $\Gamma(f,g)$

is closed in X, $\Gamma(f,g)$ is compact so only finitely many coincidence classes exist.

<u>THEOREM 1.14</u> Let (X,d) and (Y,p) be any two metric spaces and maps $f, g : X \to Y$. Then K, a coincidence class of f and g is closed in X (and hence in $\Gamma(f,g)$ with the subspace topology).

<u>Proof.</u> Let a be an accumulation point of K and let $\{x_n\}$ be a Cauchy sequence in K tending to a. Then $\{f(x_n)\} = \{g(x_n)\}$ so their continuity implies that $f(a) = g(a)$ so $a \in \Gamma(f,g)$. But theorem 1.12 above implies that all other coincidence classes are at least a distance $z > 0$ away from K so a could not have been in the closure of any other class, so $a \in K$ so K is closed.

<u>DEFINITION 1.15</u> In the function space Y^X the compact-open topology is defined as follows: For each pair of sets $A \subset X$, $B \subset Y$ let $(A,B) = \{f \in Y^X : f(A) \subset B\}$. Then the compact-open topology on Y^X is that having as subbasis all sets (A,V) where $A \subset X$ is compact and $V \subset Y$ is open.

We have

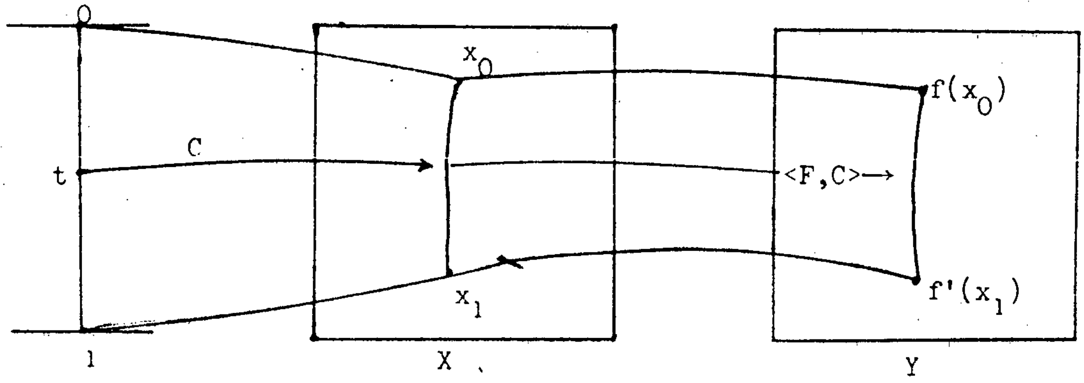
<u>THEOREM 1.16</u> [HU]. If Y^X has the compact-open topology and if X is locally compact and regular, then we may regard homotopies of maps from X into Y as paths in Y^X, and every such path may be regarded as a homotopy.

If X is path connected it is connected and locally connected, and if compact too, then it is locally compact and regular, hence the above theorem applies to path connected compact ANRs provided the relevant function spaces have the compact-open topology. Henceforth in this work, all will be assumed thus topologised.

In what follows we shall write the composition of two functions $f : X \to Y$ and $h : Y \to Z$ as $hf : X \to Z$, and the multiplication of two paths F and F' by FF' where $FF'(t) = F(2t),\ 0 \le t \le 1/2$ and

$F(2t-1),\ 1/2 \le t \le 1$.

DEFINITION 1.17 For paths F in Y^X and C in X let $\langle F, C\rangle$ be the path in Y defined by $\langle F, C\rangle(t) = F(t)(C(t))$, $F(0) = f \in Y^X$, $F(1) = f'$ say, $f' \in Y^X$.



LEMMA 1.18 Let F and F' be paths in Y^X such that $F(1) = F'(0)$ and let C and C' be paths in X such that $C(1) = C'(0)$; then

(1) $\langle F^{-1}, C^{-1}\rangle = \langle F, C\rangle^{-1}$

(2) $\langle FF', CC'\rangle = \langle F, C\rangle\langle F', C'\rangle$.

Proof. By definition, for $t \in I$, $\langle F^{-1}, C^{-1}\rangle(t) = F(1-t)(C(1-t)$ $= \langle F, C\rangle^{-1}(t)$. The definitions state that

$\langle FF', CC'\rangle(t) = FF'(t)(CC'(t))$

$$= \begin{cases} FF'(t)(C(2t)) & \text{if } 0 \leq t \leq 1/2 \\ FF'(t)(C(2t-1)) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

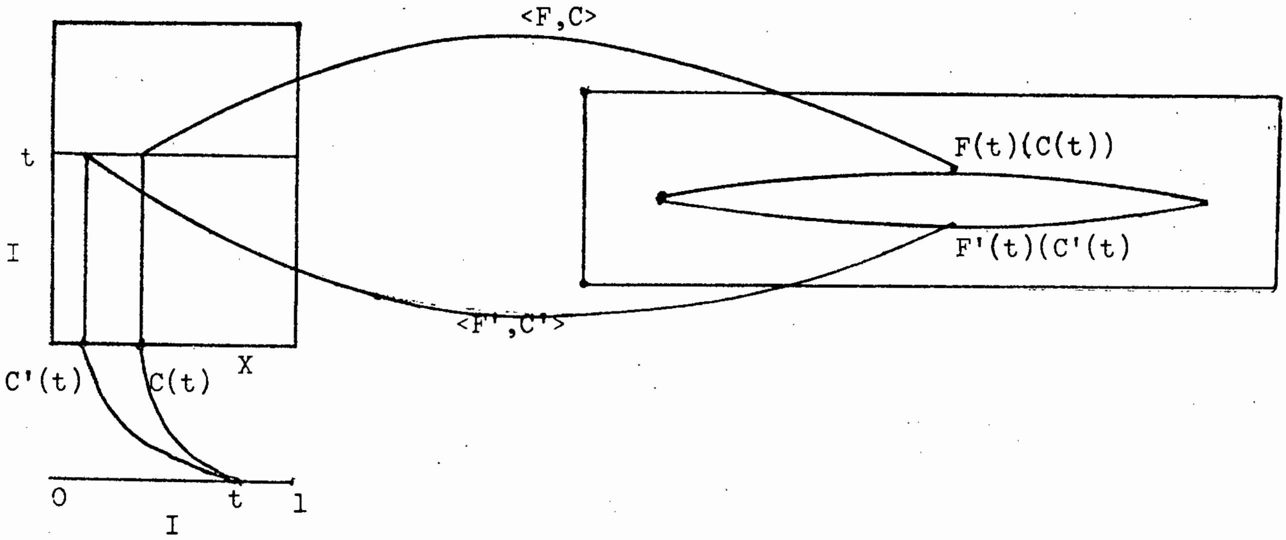
$$= \begin{cases} F(2t)(C(2t)) & \text{if } 0 \leq t \leq 1/2 \\ F'(2t-1)(C(2t-1)) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

$$= \begin{cases} \langle F,C\rangle(2t) & \text{if } 0 \leq t \leq 1/2 \\ \langle F',C'\rangle(2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

$$= \langle F,C\rangle\langle F',C'\rangle(t) \text{ as required.}$$

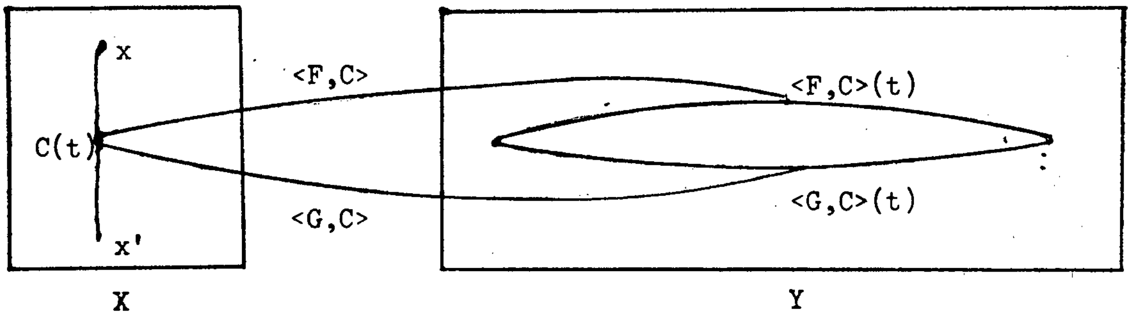
The next result shows that the operation $\langle .,.\rangle$ is also well behaved in respect of fixed end-point homotopy. Note that in path-connected compact ANRs we may, in view of HU's theorem, rewrite our definition of fixed end-point homotopy by requiring that there exist a map $\Omega : I \to X^I$ such that $\Omega(0) = C$, $\Omega(1) = C'$, $\Omega(t)(0) = C(0) = C'(0)$, $\Omega(t)(1) = C(1) = C'(1)$ for all $t \in I$.

LEMMA 1.19 If X, Y are path connected compact ANRs with F and F' paths in Y^X such that $[F] = [F']$ and C and C' are paths in X such that $[C] = [C']$ then $[\langle F,C\rangle] = [\langle F',C\rangle]$.



Proof. By hypothesis, there is a path Φ in the function space Y^{X^I} with $\Phi(0) = F$, $\Phi(1) = F'$, $\Phi(t)(0) = F(0) = F'(0)$, and $\Phi(t)(1) = F(1) = F'(1)$; and Ω in X^I with $\Omega(0) = C$, $\Omega(1) = C'$, $\Omega(t)(0) = C(0) = C'(0)$, and $\beta(t)(1) = C(1) = C'(1)$; for all $t \varepsilon I$. Define a path δ in Y^I by $\delta(t) = <\Phi(t),\Omega(t)>$ for $t \varepsilon I$. Then $\delta(0) = <F,C>$, $\delta(1) = <F'C'>$ and δ is a fixed end-point homotopy between $<F,C>$ and $<F'C'>$.

DEFINITION 1.20 If F and G are paths in Y^X then $x \varepsilon X$ is said to be F,G related to $x' \varepsilon X$, written $x - FG \rightarrow x'$ if there is a path C in X from x to x' with $[<F,C>] = [<G,C>]$.



Coincidence points of pairwise homotopic maps are associated to each other by being F,G-related. Since this concept operates between two different sets, it cannot be an equivalence relation. However the next three results show that the F,G relation is as near as it could be to being an equivalence relation on $\Gamma(f,g)$.

LEMMA 1.21 Let $f, g : X \rightarrow Y$ be maps and$(F, G)$the constant paths(f, g) in Y^X i.e. $(F(t), G(t)) = (f, g)$ all $t \varepsilon I$.
If $x_o \varepsilon \Gamma(f,g)$ then $x_o - F, G \rightarrow x_o$.

Proof. Given the constant path x_o, $<F,x_o>(t) = f(x_o) = g(x_o) = <G,x_o>(t)$, all $t \varepsilon I$ so $<F,x_o> = <G,x_o>$ as required.

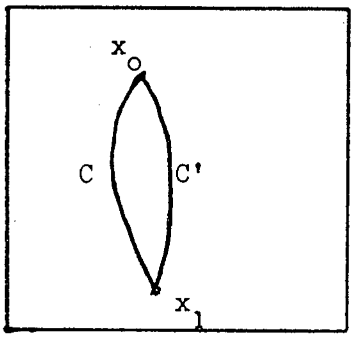
LEMMA 1.22 If F is a path in Y^X from f to f' and G a path from g to g' such that $x_0 \in \Gamma(f,g)$ and $x_1 \in \Gamma(f',g')$; then $x_0 - F, G \to x_1$ implies $x_1 - F^{-1}, G^{-1} \to x_0$.

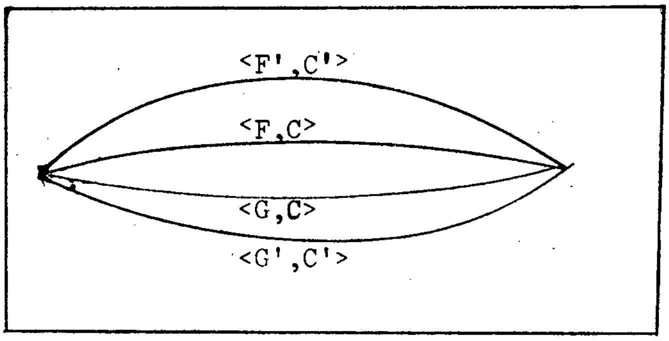
Proof. We have a path C from x_0 to x_1 such that $[\langle F,C\rangle] = [\langle G,C\rangle]$. The path C^{-1} goes from x_1 to x_0 so by part (1) of Lemma 1.18 $[\langle F^{-1},C^{-1}\rangle] = [\langle F,C\rangle^{-1}] = [\langle G,C\rangle^{-1}] = [\langle G^{-1},C^{-1}\rangle]$ as required.

LEMMA 1.23 Let F, G, F', G' be paths in Y^X such that $(F(0), G(0)) = (f, g,)$, $F(1) = F'(0) = f'$, $G(1) = G'(0) = g'$ and $(F'(1), G'(1)) = (f'', g'')$. If $x_0 \in \Gamma(f,g)$, $x_1 \in \Gamma(f'g')$ and $x_2 \in \Gamma(f'',g'')$ such that $x_0 - F,G \to x_1$ and $x_1 - F',G' \to x_2$ then $x_0 - FF', GG' \to x_2$.

Proof. There is a path C from x_0 to x_1 with $[\langle F,C\rangle] = [\langle G,C\rangle]$. Now FF' is a path from f to f'' and GG' a path from g to g'' whilst CC' is a path from x_0 to x_2. We apply part (2) of Lemma 1.18 to prove that $[\langle FF', CC'\rangle] = [\langle FC\rangle\langle F'C'\rangle] = [\langle G,C\rangle\langle G',C'\rangle] = [\langle GG', CC'\rangle]$ as required.

LEMMA 1.24 Let F, G, F', G' be paths in Y^X such that $F(0) = F'(0) = f$, $F(1) = F'(1) = f'$, $G(0) = G'(0) = g$ and $G(1) = G'(1) = g'$ with $[F] = [F']$ and $[G] = [G']$. If $x_0 - FG \to x_1$ then $x_0 - F',G' \to x_1$.

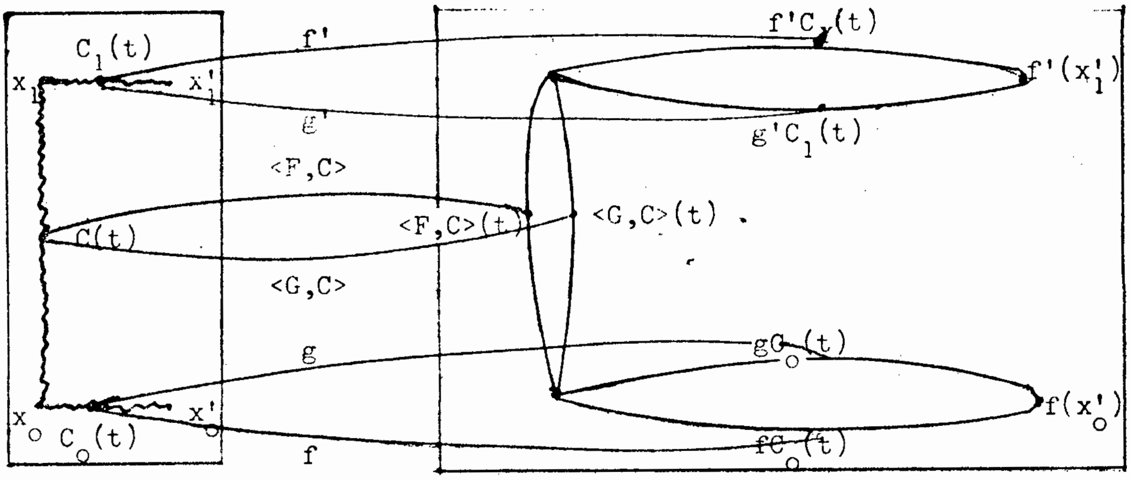




Proof. By hypothesis there is a path C, with $[<F,C>] = [<G,C>]$. By Lemma 1.19 $[F] = [F']$ and $[C] = [C']$ implies $[<F',C'>] = [<F,C>]$ which by hypothesis equals $[<G,C>] = [<G',C'>]$ by Lemma 1.19 again. So $[<F',C'>] = [<G',C'>]$ as required.

The next result shows that if $x \varepsilon K$, $K \varepsilon \tilde{\Gamma}(f,g)$ and $x_1 \varepsilon L$, $L \varepsilon \tilde{\Gamma}(f',g')$ and x_0 is F,G related to x_1 then we may make a well defined definition of coincidence class K being F,G related to coincidence class L, this relatedness being independent of which representatives x_0 and x_1 are F,G related.

THEOREM 1.25 Let F,G be paths in Y^X such that $(F(0), G(0)) = (f,g)$. and $(F(1), G(1)) = (f'g')$. Let $x_0 \varepsilon \Gamma(f,g)$ be contained in coincidence class K of $\tilde{\Gamma}(f,g)$ and let $x_1 \varepsilon L$, $L \varepsilon \tilde{\Gamma}(f'g')$. If $x_0 - FG \rightarrow x_1$ then $x_0' - F,G \rightarrow x_1'$, for any $x_0' \varepsilon K$ and any x_1', εL.



Proof. Let H be the constant path in Y^X at f and J the constant path in Y^X at g. Now since x_0 and x_0' are in the same coincidence class K there is a path C_0 in X from x_0 to x_0' such that $[fC_0] = [gC_0]$. But for each $t \in I$ $\langle H,C_0\rangle(t) = H(t)(C_0(t)) = f(C_0(t))$ so $fC_0 = \langle H,C_0\rangle$. Similarly $gC_0 = \langle J,C_0\rangle$ so we have $[\langle H,C_0\rangle] = [\langle J,C_0\rangle]$ so $x_0 - H,J \to x_0'$. Similarly $[\langle H',C_1\rangle] = [\langle J',C_1\rangle]$ so $x_1 - H',J' \to x_1'$, where H' is the constant path in Y^X at f' and J' the constant path in Y^X at g'. By Lemma 1.22, $x_0' - H^{-1},J^{-1} \to x_0$ and $H = H^{-1}$, $J = J^{-1}$. Thus by Lemma 1.23 since $x_0 - F,G \to x_1$, $x_0 - FH', GJ' \to x_1'$ applying Lemma 1.23 again it follows that $x_0' - HFH', JGJ' \to x_1'$. Define a path $\psi : I \to Y^X$ by

$$\psi(t)(s) = \begin{cases} H\left(\frac{2s}{1-t}\right) & \text{if } 0 \le s \le \frac{1-t}{2},\ t \neq 1 \\ F\left(\frac{4s+2(t-1)}{3t+1}\right) & \text{if } \frac{1-t}{2} \le s \le \frac{t+3}{4} \\ H'\left(\frac{4s-t-3}{1-t}\right) & \text{if } \frac{t+3}{4} \le s \le 1 \end{cases}$$

Then ψ shows $[HFH'] = [F]$. Similarly $[JGJ'] = [G]$ so by Lemma 1.24 $x_0' - F,G - x_1'$.

Coincidence class K is said to be F,G-related to L if there is $x \in K$, $x' \in L$ and $x - F,G \to x'$.

Our next four results show that F,G relatedness between coincidence classes of different pairs of maps, behaves analogously to that for individual members of those classes.

<u>LEMMA 1.26</u> Let $f,g : X \to Y$ be maps and F the constant path in Y^X at f and G the constant path in Y^X at g. Coincidence classes K,K' $\in \tilde{\Gamma}(f,g)$ are F,G related iff they are identical.

<u>Proof.</u> Suppose $K - K'$ and let $x \in K$. By Lemma 1.21, $x - FG \to x$;

so by definition, $K - F,G \to K'$. On the other hand if $K - F,G \to K'$ for some $x \in K$ and $x' \in K'$ then there is a path C in X from x to x' such that $[\langle H,C\rangle] = [\langle G,C\rangle]$. But $\langle F,C\rangle = fC$ so $[fC] = [gC]$. Thus x and x' are in the same coincidence class of $\Gamma(f,g)$ so $K \cap K' \neq \phi$. Coincidence classes are equivalence classes so $K = K'$.

<u>LEMMA 1.27</u> Let F,G be paths in Y^X with $(F(0), G(0)) = (f,g)$ and $(F(1), G(1)) = (f',g')$. Let $K \in \tilde{\Gamma}(f,g)$ and $L \in \tilde{\Gamma}(f',g')$ such that $K - F,G \to L$, then $L - F^{-1},G^{-1} \to K$.

<u>Proof.</u> By definition there is a point $x \in K$ which is F,G related to $x' \in L$. By Lemma 1.22 $x' - F^{-1},G^{-1} \to x$ so $L - F^{-1},G^{-1} \to K$ as required.

Similarly from Lemma 1.23, we have

<u>LEMMA 1.28.</u> Let F,F',G,G', be paths in Y^X such that $(F(0),G(0)) = (f,g)$, $F(1) = F'(0) = f'$, $G(1) = G'(0) = g'$ and $(F'(1),G'(1)) = (f'',g'')$. Let $K \in \tilde{\Gamma}(f,g)$, $L \in \tilde{\Gamma}(f'g')$ and $M \in \tilde{\Gamma}(f'',g'')$ such that $K - F,G \to L$ and $L \to F',G' \to M$; then $K - FF',GG' \to M$.

Lemma 1.24 yields

<u>LEMMA 1.29</u> Suppose F,F' are paths in Y^X from f to f' and G,G' paths in Y^X from g to g', such that $[F] = [F']$ and $[G] = [G']$. If $K \in \tilde{\Gamma}(f,g)$ and $L \in \tilde{\Gamma}(f',g')$ are F,G, related then $K - F'G' \to L$.

These four results then yield

<u>THEOREM 1.30</u> Let (F,G) be a path in Y^X from (f,g) to (f',g'). Let $K,K' \in \tilde{\Gamma}(f,g)$ and $L,L' \in \tilde{\Gamma}(f',g')$. If $K - F,G \to L$ and $K - F,G \to L'$ then $L = L'$ and if $K - F,G \to L$ and $K' - F,G \to L$ then $K = K'$.

<u>Proof</u> If $K - F,G \to L$ and $K - F,G \to L'$, then by Lemma 1.27, $L - F^{-1},G^{-1} \to K$; so by Lemma 1.28, $L - F^{-1}F,G^{-1}G \to L'$. Define H to be the constant path in Y^X at f' and J the constant path in Y^X at g'. Define a path ψ in Y^{X^I} by

$$\psi(t)s = \begin{cases} H(s) & \text{if } 0 \le s \le \frac{t}{2} \\ F^{-1}(2s-t) & \text{if } \frac{t}{2} \le s \le \frac{1}{2} \\ F(2s+t-1) & \text{if } \frac{1}{2} \le s \le \frac{2-t}{2} \\ H(s) & \text{if } \frac{2-t}{2} \le s \le 1. \end{cases}$$

ψ shows that $[F^{-1}F] = [H]$. Now by Lemma 1.29, L is H,J related to L' so by Lemma 1.26 $L = L'$, as required. For the second part of the theorem suppose that $K - F,G \to L$ and $K' - FG \to L$; then by Lemma 1.27, $L - F^{-1},G^{-1} \to K$ and $L - F^{-1},G^{-1} \to K'$ so first part of theorem yields $K = K'$ as required.

Theorem 1.30 states that an F,G-relation induces a one-to-one relation from $\tilde{\Gamma}(F(0),G(0))$ into $\tilde{\Gamma}(F(1),G(1))$.

We next introduce the Δ-notation, as used by Brooks. Brooks considers coincidences, roots and fixed points of mappings. These are characterised by letting Δ be one of the three following sets:

1. $\Delta = \Delta_1(X,Y)$, the class of all pairs of paths (F,G) in Y^X, used

in studying coincidences of two maps $f,g : X \to Y$.

2. $\Delta = \Delta_2(X,Y,y_o)$, the class of all pairs $(F,G) \varepsilon \Delta_1(X,Y)$ such that $G(t)(x) = y_o$ for all $t \varepsilon [0,1]$ and all $x \varepsilon X$, used for studying the roots of the equation $f(x) = y_o$, for an arbitrary map $f : X \to Y$.

3. $\Delta = \Delta_3(X)$, the class of all pairs $(F,G) \varepsilon \Delta_1(X,X)$ such that $G(t)(x) = x$ for all $t \varepsilon [0,1]$ all $x \varepsilon X$, used for studying fixed points of an arbitrary map $f : X \to X$.

Since our work is based on his axioms it holds for Δ_1, Δ_2 and Δ_3. We thus use a general Δ in what follows, though our work will be done with $\Delta = \Delta_1$. Let Δ be the class of all pairs of paths in Y^X. We have

DEFINITION 1.31 An Δ-essential coincidence class $K \varepsilon \tilde{\Gamma}(f,g)$ is one for which $(F,G) \varepsilon \Delta$ and $(F(0), G(0)) = (f,g)$ implies there being a coincidence class $K' \varepsilon \tilde{\Gamma}(f',g')$ such that $K\text{-}F,G \to K'$; otherwise K is called inessential.

The Δ-Nielsen number of f and g, denoted by $N(f,g,\Delta)$ is defined to be the number of Δ-essential classes of $\tilde{\Gamma}(f,g)$.

Theorem 1.30 tells us that an F,G relation defines a one-to-one function from the Δ-essential elements of $\tilde{\Gamma}(F(0),G(0))$ onto those of $\tilde{\Gamma}(F(1),G(1))$, whilst theorem 1.13 implies that, since $N(f,g,\Delta) \leq |\tilde{\Gamma}(f,g)|$, $N(f,g,\Delta)$ is finite where f and g are defined on a compact ANR.

INVARIANCE OF $N(f,g,\Delta)$ UNDER HOMOTOPY.

In this section we define a function $\jmath$ which assigns an element of an abelian group $\underset{\sim}{A}$ to each class in $\tilde{\Gamma}(f,g)$. We will show that j assigns the same element of $\underset{\sim}{A}$ to any two F,G related classes whilst if a class K is inessential $j(k)$ will be shown to equal zero. Thus the Δ-Nielsen number will be shown to be a homotopy invariant.

DEFINITION 1.32 [Brooks]. A triple (f,g,A) is Δ-admissible if $f,g : X \to Y$ are maps, $(f,g) \varepsilon \Delta$, $A \subsetneq X$ and a closed set $N \subseteq X$ such that $\bar{A} \subsetneq \text{Int}(N)$ and $\Gamma(f,g) \cap (N-A) = \phi$.

DEFINITION 1.33 An Δ-admissible index j is a function from the Δ-admissible triples into an abelian group $\underset{\sim}{A}$ such that

1. (Additivity) If $A \subset X$ and $\{A_i\}$ is a finite indexed class of subsets of A such that

   (a), (f,g,A) is Δ-admissible and (f,g,A_i) is admissible for each i, and

   (b), $(A - \cup_i A_i) \cap \Gamma(f,g) = \phi$ then $j(f,g,A) = \Sigma_i\, j(f,g,A_i)$.

2. (Homotopy) If $(F,G) \varepsilon \Delta$ and A an open subset of X such that $(F(t),G(t),A)$ is Δ-admissible for each $t \varepsilon [0,1]$ then $j(F(0),G(0)A) = j(F(1),G(1),A)$.

The Δ_1 index, which is the one we shall be using, has values in $\text{Hom}(H^*(YxY,YxY-D), H^*(X))$ where $H^*(Z)$ is the total cohomology ring of Z (with arbitrary coefficients) and D is the diagonal of Y x Y. This index $j(f,g,A)$ is the cohomology homomorphism induced by the map $h : X \to (YxY,YxY-D)$ defined by $h(x) = (f(x),g(x))$.

The Index.

Throughout this section $H^*(X,A)$ will mean $H^*(X,A,Q)$.

Recall that we think of R^n as a subset $S^n - P$ of S^n. For a point $x \in R^n$ let $|x|$ denote its distance to the origin. Observe that if $f,g : R^n \to R^n$ are maps having the property that, for any sequence $x_1, x_2, \ldots.$ in R^n such that

$\lim\limits_{n\to\infty} |x_n| = \infty$ implies that

$\lim\limits_{n\to\infty} |f(x_n)| = \lim\limits_{n\to\infty} |g(x_n)| = \infty$ we can extend $f,g : S^n \to S^n$

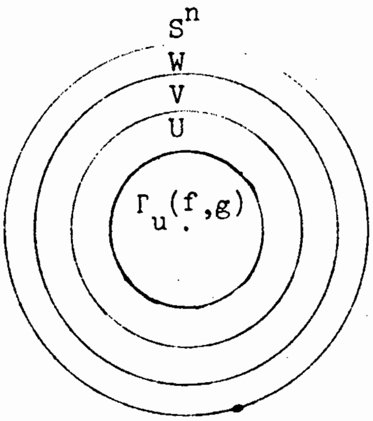
by letting $f(P) = P = g(P)$.

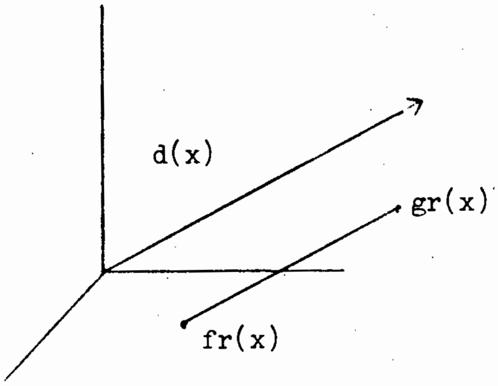
Choose a generator $\mu_1 \in H^1(R^1,R^1-0) \cong Q$. We define generators $\mu_n \in H^n(R^n,R^n-0)$ inductively, as on pg 54 of Brown as follows: Having obtained generator μ_{n-1} we have the Künneth isomorphism

$\alpha: H^{n-1}(R^{n-1},R^{n-1}-0) \otimes H^1(R^1,R^1-0) \to H^n(R^n,R^n-0)$ and we define $\mu_n = \alpha(\mu_{n-1} \otimes \mu_1)$

Let $x_0 \in R^n$ and define $d:(R^n,R^n-x_0) \to (R^n,R^n-0)$ by $d(x) = x - x_0$. Let $l:(R^n,R^n-x_0) \to (S^n,S^n-x_0)$ and $k: S^n \to (S^n,S^n-x_0)$ be inclusions. Then d^* is an isomorphism as d is a homeomorphism, $l^*: H^n(S^n,S^n-x_0) \to H^n(R^n,R^n-x_0)$ is an isomorphism by the excision theorem since $S^n = R^n - P$ and $cl(P) = P \subseteq Int(S^n-x_0)$ whilst $k^*: H^n(S^n,S^n-x_0) \to H^n(S^n)$ is also an isomorphism since $S^n - x_0$ is contractible.

Let X,Y be any finite connected polyhedra and let $T = (K,\tau)$ be a triangulation of X. Recall that we can imbed $|K|$ in R^n for n equal to the number of vertices of K, so we can imbed X,Y in $R^n \subset S^n$. Since X is an ANR and R^n is separable metric, there is an open subset W of R^n containing X and a retraction $r\colon W \to X$. If $f,g\colon X \to Y$ and (f,g,U) is a Δ-admissible triple let $r\colon W \to X$ be as above and define $V = r^{-1}(U)$, which is an open subset of W and hence of S^n.





Let $\Gamma_u(f,g)$ be the set of coincidence points of f and g that lie in U. Define $d\colon V \to R^n$ by $d(x) = gr(x) - fr(x)$. If $d(x) = 0$ then $fr(x) = gr(x)$, so $r(x) \in \Gamma_u(f,g)$. Hence $r(x) \notin \Gamma_u(f,g)$ implies $d(x) \neq 0$, hence $d(V - r^{-1}(\Gamma_u(f,g)) \subseteq R^n - 0$. We write F for $r^{-1}(\Gamma_u(f,g))$ for the rest of this section.

Let $1\colon (V,V-F) \to S^n,S^n-F)$ be inclusion; we claim that 1^* is an isomorphism by excision. By lemma 1.5 $\Gamma_u(f,g)$ is a closed subset of X. Since f and g have no coincidence points on the boundary of U, $\Gamma_u(f,g)$ is the intersection of $\Gamma(f,g)$ and the closure of U. Therefore F is closed in the compact space X, hence F is compact, hence F is closed in S^n. Since $S^n - V$ is a closed set in the open set $S^n - F$ the excision can be performed.

We have the composition

$$H^n(R^n,R^n\text{-}0) \xrightarrow{d^*} H^n(V,V\text{-}F) \xrightarrow{l^{*-1}} H^n(S^n,S^n\text{-}F) \xrightarrow{k^*} H^n(S^n)$$

where $k\colon S^n \to (S^n,S^n\text{-}F)$ is inclusion; then $k^* l^{*-1} d^*(U_n) = q\nu_n$ for same $q \in Q$. Define $q = j_r(f,g,U)$. We use the notation j_r to indicate that the definition appears to depend on the imbeddings of X in R^n, Y in R^n, the open set W and the retraction $r\colon W \to X$ because the definition of d and V depend on r and U. We will show in this section that the j_r defined above is a Δ-admissible index.

Consider the composition

$$H^n(R^n,R^n\text{-}0) \xrightarrow{d^*} H^n(R^n,R^n\text{-}x_o) \xrightarrow{l^{*-1}} H^n(S^n,S^n\text{-}x_o) \xrightarrow{k^*} H^n(S^n)$$

and define $\nu_n = k^* l^{*-1} d^*(\mu_n) \in H^n(S^n) \cong Q$.

Since the composition is an isomorphism ν_n is a generator of $H^n(s^n)$.

LEMMA 1.33 The definition of ν_n is independent of the choice of the point $x_o \in R^n$.

Proof Brown pg 54-5.

Note also that we have a Künneth isomorphism $\alpha, \alpha(\mu_p \otimes \mu_q) = \mu_{p+q}$, whilst if $\pi^*\colon H^{p+q}(S^{p+q}) \to H^{p+q}(S^p \times S^q)$ is an isomorphism we also have $\nu_{p+q} = \pi^{*-1}\alpha(\nu_p \otimes \nu_q)$.

<u>LEMMA 1.34</u> Let (f,g,U) be a Δ-admissible triple, let $F \subsetneq F' \subseteq V' \subseteq V$ where F' is closed, V' is open, and $S^n - V'$ is in the interior of $S^n - F'$. Let $d'\colon (V',V'-F') \to (R^n,R^n-0)$ be the restriction of d. If $l'\colon (S^n,S^n-F') \to (V',V'-F')$ and $k'\colon S^n \to (S^n,S^n-F')$ are inclusions, then

$$k'^*(l'^*)^{-1}d'^*(\mu_n) = j_r(f,g,U)\cdot\nu_r$$

<u>Proof:</u> We have the following commutative diagram where $h\colon (S^n,S^n-F') \to (S^n,S^n-F)$ is inclusion

$$\begin{tikzcd}
 & H^n(V,V-F) \arrow[r, "l^{*-1}"] \arrow[d] & H^n(S^n,S^n-F) \arrow[d] \arrow[dr, "k^*"] & \\
H^n(R^n,R^n-0) \arrow[ur, "d^*"] \arrow[dr, "d'^*"'] & & & H^n(S^n) \\
 & H^n(V,V'-F') \arrow[r, "l'^{*-1}"] & H^n(S^n,S^n-F') \arrow[ur, "k'^*"'] &
\end{tikzcd}$$

<u>THEOREM 1.35</u> If (f,g,U) is a Δ-admissible triple and $U_1,\ldots,U_s$ are disjoint open subsets of U such that (f,g,U_i) is Δ-admissible for each i and $\Gamma_u(f,g) \subset \bigcup_{i=1}^{s} U_i$

then $j_r(f,g,U) = \sum_{i=1}^{s} j_r(f,g,U_i)$.

<u>Proof:</u> Let $F_i = r^{-1}(\Gamma_u(f,g) \cap U_i)$ and $V_i = r^{-1}(U_i)$. By Lemma 1.34 for the composition

$$H^n(R^n,R^n-0) \xrightarrow{d^*} H^n[\bigcup_{i=1}^{s} V_i,(\bigcup_{i=1}^{s} V_i)-F_i]$$

$$\xrightarrow{l^{*-1}} H^n(S^n,S^n-F) \xrightarrow{k^*} H^n(S^n)$$

We know that $k^*l^{*-1}d^*(\mu_n) = j_r(f,g,U) \cdot \nu_n$.

By definition, if we consider, for $t = 1,.....,s$

$$H^n(R^n,R^n-0) \xrightarrow{d^*_t} H^n(V_t,V_t-F_t)$$

$$\xrightarrow{l^{*-1}_t} H^n(S^n,S^n-F_t) \xrightarrow{k^*_t} H^n(S^n)$$

then $k^*\ l^{*-1} d^*\ (\mu_n) = j_r(f,g,U) \cdot \nu_n$ where d_t is the restriction of d to V_t and l_t and k_t are inclusions. Since the V_i are disjoint open sets and $\Gamma u_i(f,g) \subset V_i$ then

$$H^n[\bigcup_{i=1}^{s} V_i,(\bigcup_{i=1}^{s} V_i)-F] = H^n[\bigcup_{i=1}^{s} V_i,\bigcup_{i=1}^{s}(V_i-F_i)] \cong \bigoplus_{i=1}^{s} H^n(V_i,V_i-F_i)$$

so we can write $d^*(\mu_n) = (u_1,......,u_s)$, where $u_t \in H^n(V_t,V_t-F_t)$.

Let $h_t\colon (V_t,V_t-F_t) \to (\bigcup_{i=1}^{s} V_i,\bigcup_{i=1}^{s}(V_i-F_i))$ be inclusion; then $dh_t = d_t$.

Furthermore, $h^*_t(u_1,.....,u_s) = u_t$ so

$h^*_t\ d^*(\mu_n) = h^*_t(U_1......u_s) = u_t = d^*_t(\mu_n)$, and thus $d^*(\mu_n) = d^*_1(\mu_n) ,......,d^*_s\ (\mu_n))$.

Diagrams

$$\begin{tikzcd}
H^n[\bigcup_{i=1}^{s} Vi,(\bigcup_{i=1}^{s} Vi)-F] \arrow[d, "h_i^*"] & \\
H^n(V_i,V_i-F_i) & H^n(S^n,S^n-F_t) \arrow[l, "l'^*_i"] \arrow[ul, "m'^*_t"']
\end{tikzcd}$$

commute for all $i = 1,....s$ when all homomorphisms are inclusion-induced. For $x \in H^n(S^n,S^n-F_t)$ write $m'_t(x) = (u'_1,.....u'_s)$, then $h_i^* m'^*_t(x) = u'_i = l'^*_i(x)$. When $i \neq t$ we have $V_i \subseteq S^n - F_t$ so l'_i can be written as the composition of inclusions

$$(V_i,V_i-F_i) \to (S^n-F_t,S^n-F_t) \to (S^n,S^n-F_t)$$

and l'^*_i is the zero homomorphism.

Therefore since $l'_t = l_t$,

$$m'^*_t(x) = (0,....,0,l_t^*\ (x),0,.....,0).$$

Let $m_t: (S^n,S^n-F) \to (S^n,S^n-F_t)$ be inclusion. Then diagram

$$\begin{tikzcd}
H^n[\bigcup_{i=1}^{s} V_i,(\bigcup_{i=1}^{s} V_i)-F] \arrow[d, "h_t^*"'] & H^n(S^n,S^n-F) \arrow[l, "l^*"'] \arrow[r, "k^*"] & H^n(S^n) \\
H^n(V_i,V_i-F_i) & H^n(S^n,S^n-F_t) \arrow[l, "l_t^*"] \arrow[ul, "m'^*_t"'] \arrow[u, "m_t^*"] \arrow[ur, "k_t^*"'] &
\end{tikzcd}$$

commutes,

because all homomorphisms are induced by inclusions.

Therefore, for $x \in H^n(S^n,S^n-F_t)$

$$m_t^*(x) = l^{*-1} m'^*_t(x) = l^{*-1}(0,....,0,l_t^*(x),0,....,0,)$$

and

$$k_t^*(x) = k^*m_t^*(x) = k^*l^{*-1}(0,\ldots.0,l_t^*(x),0,\ldots.,0)$$

In the first part of the proof we had

$$d^*(\mu_n) = (u_1,\ldots.,u_s) = (d_1^*(\mu_n),\ldots.,d_s^*(\mu_n)), \text{ so}$$

$$\begin{aligned} k_t^*l_t^{*-1}d_t^*(\mu_n) &= k_t^*l^{*-1}(u_t) \\ &= k^*l^{*-1}(0,\ldots.,0,l_t^*(l_t^{*-1}(u_t)),0,\ldots.,0) \\ &= k^*l^{*-1}(0,\ldots.,0,u_t,0,\ldots.,0) \end{aligned}$$

By the definition of direct sum

$$\begin{aligned} k^*l^{*-1}d^*(\mu_n) &= k^*l^{*-1}(u_1,\ldots.,u_s) \\ &= k^*l^{*-1}(u_1,0,\ldots.,0) +\ldots.+k^*l^{*-1}(0,\ldots.,0,u_s) \\ &= k_1^*l_1^{*-1}d_1^*(\mu_n) +\ldots.+ k_s^*l_s^{*-1}d_s^*(\mu_n) \end{aligned}$$

Applying the definitions of the j_r, we have

$$\begin{aligned} j_r(f,g,U)\cdot\nu_n &= \sum_{i=1}^{s}\left\{(j_r(f,g,U_i)\cdot\nu_n\right\} \\ &= \left(\sum_{i=1}^{s} j_r(f,g,U_i)\right)\cdot\nu_n \text{ as reqd.} \end{aligned}$$

<u>THEOREM 1.36</u> Let H,L: X x I → Y be homotopies and define $f_t(x) = H(x,t)$ and $g_t(x) = L(x,t)$.

If (f_t,g_t,U) is Δ-admissible for all $t \in I$,

then $j_r(f_o,g_o,U) = j_r(f_1,g_1,U)$.

<u>Proof:</u> Let $F' = \{x \in u: H(x,t) = L(x,t) \text{ some } t \in I\}$

Then F' is a closed subset of I such that $\Gamma_u(f,g) \subsetneq F'$.

In order to apply Lemma 1.34 we must prove that F' is closed in S^n, so $S^n - V$ is in the interior of $S^n - F'$. It is clearly sufficient to show that F' is closed in X. Let $\{x_j\}$ be a sequence of points in F' converging to a point x in the closure of U. For each x_j our definition of F' guarantees a $t_j \in I$ such that $H(x_j,t_j) = L(x_j,t_j)$. Since I is compact, there is no loss of generality in assuming that the sequence $\{t_j\}$ converges to a point $t \in I$. By the continuity of H,L we have $H(x,t) = L(x,t)$ so $x \in F'$ so F' is closed in X and hence in S^n.

Now we can apply Lemma 1.34 to state that, for the composition

$$H^n(R^n,R^n-0) \xrightarrow{d_t^*} H^n(V,V-F') \xrightarrow{(l'^*)^{-1}} H^n(S^n,S^n-F') \xrightarrow{k'^*} H^n(S^n)$$

where $d_t(x) = g_t r(x) - f_t r(x)$ to have $k'^* l^{*-1} d_t^*(\mu_n) = j_r(f_t,g_t,U) \cdot \nu_n$.

Define $D\colon (V,V-F') \times I \to (R^n,R^n-0)$ by $D(x,t) = g_t r(x) - f_t r(x)$. Then D is a homotopy between d_0 and d_1 so $d_0^* = d_1^*$ and $j_r(f_0,g_0 U) = j_r(f_1,g_1,U)$. as required.

Given polyhedra X,X',Y,Y', and maps $f,g\colon X \to Y, f'g'\colon X' \to Y'$, we have maps $f \times f', g \times g'\colon X \times X' \to Y \times Y'$. If, after imbedding, $X \subset R^n$ and $X' \subset R^{n'}$ then X and $X' \subset R^n \times R^{n'} = R^{n+n'}$. Furthermore we have open sets $W \subsetneq R^n$ containing X and $W' \subset R^{n'}$ containing

X' and retractions $r\colon W \to X$, $r'\colon W' \to X'$. Thus $W \times W'$ is an open subset of $R^{n+n'}$ containing $X \times X'$, and $r \times r'\colon\ W \times W' \to X \times X'$ is a retraction. If (f,g,U) and (f',g',U') are Δ-admissible triples then $f \times f', g \times g'$ have no coincidence points on $\partial(U\times U') = (U\times\partial U') \cup (\partial U\times U')$ where ∂ denotes the boundary. Note also that $(r\times r')^{-1}(U\times U') = V \times V'$ and that the set of coincidence points of $f \times f'$, $g \times g'$ on $U \times U'$ is $\Gamma_u(f,g) \times \Gamma_{u'}(f'g')$. We let $F = r^{-1}(\Gamma_u\ (f,g))$ as before and let $F' = r'^{-1}(\Gamma_{u'}(f,g))$ here. If $d_x' = d \times d'$ and l_x, k_x are inclusions then

$$\begin{tikzcd}
H^{n+n'}(R^{n+n'},R^{n+n'}-0) \arrow[d, "d_x^*"'] & H^{n+n'}(S^{n+n'}) \\
H^{n+n'}(V\times V',V\times V'-F\times F') \arrow[r, "l_x^{*-1}"] & H^{n+n'}(S^{n+n'},S^{n+n'}-F\times F') \arrow[u, "k_x^*"']
\end{tikzcd}$$

hence $k_x^* l_x^{*-1} d_x^*(\mu_{n+n'}) = j_{r+r'}(f\times f',g\times g',U\times U')$.

Our next result makes use several times of the identity $(X,X-A) \times (Y,Y-B) = (X\times Y,X\times Y-A\times B)$ which holds as both sides equal $(X\times Y,X-A\times Y-B)$.

<u>THEOREM 1.37</u> If (f,g,U) and $(f'g'U')$ are Δ-admissible triples then $j_{r\times r'}(f\times f',g\times g',U\times U') = j_r(f,g,U) \cdot j(f',g',U')$.

<u>Proof.</u> The argument depends on the diagram below. The α^s denote the appropriate Künneth homomorphisms or their restrictions to

subgroups. Note that result (ω) proved just alone is used several times. By definition $d_x = d \times d'$ and $l_x = l \times l'$. All diagrams except (1) and (2) commute by the naturality of the Künneth homomorphism. We observe that $F \subset R^n$, $F' \subset R^{n'}$, so subdiagrams (1) and (2) commute because they were induced by commutative diagrams of spaces and maps.

Since the diagram on pg 29 commutes

$$\alpha_4^{-1}\ \pi^*\ k_x^*\ l_x^{*^{-1}}\ d_x^*\ \alpha_1 = (k^*\otimes k'^*)(l^*\otimes l'^*)^{-1}\ (d^*\otimes d'^*).$$

We defined the μ's and ν's so that $\alpha_1(\mu_n \otimes x\mu_n) = \mu_{n+n'}$ and

$\pi^{*^{-1}}\alpha_4(\nu_r \otimes \nu_n') = \nu_{n+n'}$,

hence

$$j_{r\times r'}(f\times f', g\times g', U\times U') \cdot \nu_{n+n'}$$

$$= k_x^* l_x^{*^{-1}} d_x^*(\mu_{n+n'})$$

$$= \pi^{*^{-1}}\alpha_4(k^*\otimes k'^*)(l^*\otimes l'^*)^{-1}(d^*\otimes d'^*)(\mu_n \otimes \mu_{n'})$$

$$= \pi^{*^{-1}}\alpha_4[k^* l^{*^{-1}} d^*(\mu_n) \otimes k'^* l'^{*^{-1}} d'^*(\mu_{n'})]$$

$$= \pi^{*^{-1}}\alpha_4[j_r(f,g,U)\cdot\nu_n \otimes j_{r'}(f',g',U')\cdot\nu_{n'}]$$

$$= [j_r(f,g,U)\cdot j_{r'}(f',g',U')]\pi^{*^{-1}}\alpha_4(\nu_n \otimes \nu_{n'})$$

$$= [j_r(f,g,U)\cdot j_{r'}(f',g',U')]\cdot\nu_{n+n'}.$$

$$\begin{tikzcd}
H^{n+n'}(R^{n+n'},R^n-0) \arrow[d, "d^*_x"] & & H^n(R^n,R^n-0) \otimes H^{n'}(R^{n'},R^{n'}-0) \arrow[ll, "\alpha_1"'] \arrow[d, "d^*\otimes d'^*"] \\
H^{n+n'}(VxV',VxV'-FxF') & & H^n(V,V-F) \otimes H^{n'}(V',V',-F') \arrow[ll, "\alpha_2"'] \\
H^{n+n'}(S^{n+n'},S^{n+n'}-FxF') \arrow[u, "l^*_x"] \arrow[r, "\cong", "\pi^*"'] \arrow[d, "k^*_x"] & H^{n+n'}(S^n x S^{n'},S^n x S^{n'}-FxF') \arrow[ul, "\cong"', "(lxl')^*"] \arrow[d, "(kxk')^*"] & H^n(S^n,S^n-F \otimes H^{n'}(S^{n'},S^{n'}=F') \arrow[l, "\alpha_3"] \arrow[u, "l^*\otimes l'^*"] \arrow[d, "k^*\otimes k'^*"] \\
H^{n+n'}(S^{n+n'}) \arrow[r, "\cong", "\pi^*"'] & H^{n+n'}(S^n x S^{n'}) & H^n(S^n) \otimes H^{n'}(S^{n'}) \arrow[l, "\cong"', "\alpha_4"]
\end{tikzcd}$$

For C a path in X and $r, s \in I$ define $C^s_r : I \to X$ by $C^s_r(t) = C(r+t(s-r))$. Thus C^s_r is the restriction of C to the interval $[r,s] \subseteq [0,1]$, reparametised so that it is again a path.

<u>LEMMA 1.38</u> Let $C : I \to X$ be a path, then $(C^r_q)^{-1} = C^q_r$ and $[C^r_q C^s_r] = [C^s_q]$.

<u>Proof.</u> The first statement obtains as $(C^r_q)^{-1}(t) = C(q+(1-t)(r-q)) = C(r+t(q-r)) = C^q_r(t)$. For the second statement, we define the required fixed end-point homotopy $\psi : I \times I \to X$ by

$$\psi(u,t) = \begin{cases} C[u(q+t(s-q)) + (1-u)(q+2t(r-q))] \\ \qquad \text{if } 0 \leq t \leq \frac{1}{2} \\ C[u(q+t(s-q)) + (1-u)(r+(2t-1)(s-r))] \\ \qquad \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

<u>LEMMA 1.39</u> Let F and G be paths in Y^X and let $q,r,s \in I$. Suppose for $K(q) \in \tilde{\Gamma}(F(q), G(q))$, $K(r) \in \tilde{\Gamma}((F(r),G(r))$ and $K(s) \in \tilde{\Gamma}(F(s),G(s))$ that $K(q) - F^r_q, G^r_q \to K(r)$ and $K(r) - F^s_r, G^s_r \to K(s)$; then $K(q) - F^s_q, G^s_q \to K(s)$.

<u>Proof.</u> By Lemma 1.28 on pg.15 $K(q)$ is $F^r_q F^s_r, G^r_q G^s_r$ related to $K(s)$ whilst Lemma 1.38 above means $[F^r_q F^s_r, G^r_q G^s_r] = [F^q_s, G^q_s]$, so Lemma 1.29 on pg. 15 gives the desired result.

Considering the first half of the proof of theorem 1.13, letting $U_i = \bigcup_{e \in K_i}(B_{\frac{z}{3}}(e))$ we have

<u>LEMMA 1.40</u> Let X be a compact ANR and $f,g : X \to Y$. Then for each coincidence class K_i of $\tilde{\Gamma}(f,g)$ there exists an open subset U_i of X, $K_i \subseteq U_i$ such that $\bar{U}_i \cap \Gamma(f,g) = K_i$.

<u>LEMMA 1.41</u> The definition of $j(K_i)$ is independent of the choice of the open set U_i in X such that $K_i \subseteq U_i$ and $\bar{U}_i \cap \Gamma(f,g) = K_i$.

<u>Proof.</u> Let U_i, V_i be open subsets of X such that $K_i \subseteq U_i, K_i \subseteq V_i, \bar{U}_i \cap \Gamma(f,g) = K_i$, $\bar{V}_i \cap \Gamma(f,g) = K_i$. If $x \in U_i - (U_i \cap V_i)$ then $x \notin \Gamma(f,g)$. Thus the additivity axiom of the definition of j means

$$j(f,g,U_i) = j(f,g,U_i \cap V_i).$$

Similarly $\quad j(f,g,V_i) = j(f,g,U_i \cap V_i)$

so $\quad j(f,g,U_i) = j(f,g,V_i)$ as required

<u>LEMMA 1.42</u> Let F,G be paths in Y^X where X and Y are compact ANRs, and $r \in I$. Denote the coincidence classes of maps $F(r),G(r)$ by $K_1(r),\ldots\ldots,K_n(r)$. Then there exist open sets $U_1,\ldots\ldots,U_n$ and $\varepsilon > 0$ such that

(1) $K_j(r) \subseteq U_j$

(2) $U_j \cap U_k = \phi$ whenever $j \neq k$

(3) If $|r-s| \leq \varepsilon$ and $K(s) \in \tilde{\Gamma}(F(s),G(s))$, then there exists j such that $K(s) \subseteq U_j$ and $K(s)$ is F^r_s, G^r_s-related to $K_j(r)$.

(4) If $|r-s| \leq \varepsilon$ then $(F(s),G(s)U_j)$ is an Δ-admissible triple for all $j = 1,\ldots\ldots,n$.

Proof. This will proceed as follows:

We use theorem 1.8 and X being a metric space to give us open balls of sufficiently small size with which we construct $U_1,\ldots,U_n$ which not only satisfy (1) and (2) but (3) and (4) also.

Parts (1) and (2).

From the proof of Theorem 1.13 above we can construct open sets $U'_1,\ldots\ldots,U'_n$ satisfying (1) and (2). Let $\varepsilon' > 0$ be the number guaranteed by Theorem 1.8. Then there exists $\delta > 0, \delta < \frac{\varepsilon'}{4}$ such that $x,x' \varepsilon X, d(x,x') < \delta$ and $|r - s| < \delta$ together imply $d(F(r)(x),F(s)(x') < \frac{\varepsilon'}{4}$ and $d(G(r)(x),G(s)(x')) < \frac{\varepsilon'}{4}$ as F,G are both uniformly continuous. For $x \varepsilon K_j(r)$, choose $\varepsilon_x > 0$ so that $\varepsilon_x \leq \delta$ and $U(x,\varepsilon_x) \subseteq U'_j$. Since $(F(r),G(r))$ is compact and X path connected, there is a finite set of connected open sets $U(x(1),\varepsilon_{x(1)}),\ldots\ldots,U(x(m),\varepsilon_{x(m)})$, with $x(k) \varepsilon \Gamma(F(r),G(r)$, so that

$$\Gamma(F(r),G(r)) \subseteq \bigcup_{k=1}^{m} \{U(x(k)), \varepsilon_{x(k)}\}.$$

Let $U_j = \bigcup_k \{U(x(k), \varepsilon_{x(k)})\}$ where the union is taken over those k for which $x(k) \varepsilon K_j(r)$. Let $x \varepsilon K_j(r)$; then $x \varepsilon U(x(k),\varepsilon_{x(k)})$ for some k. Since $U(x(k),\varepsilon_{x(k)})$ is an open and connected subset of the ANR X which is locally path connected, there is a path C from x to x(k) in $U(x(k),\varepsilon_{(k)})$. For $t \varepsilon I, d(C(t),x) < \varepsilon_{x(k)} \leq \delta$, which implies that

$$p(F(r)(C(t)),F(r)(x)) < \frac{\varepsilon'}{4} \text{ , } \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$p(G(r)(C(t)),G(r)(x)) < \frac{\varepsilon'}{4} \quad p(G(r)(C(t)),F(r)(C(t)) < \frac{\varepsilon'}{4} \ \ldots(2)$$

and thus by Theorem 1.8 $[F(r)C] = [(G(r)C]$.

We have shown that x and $x(k)$ are in the same coincidence class M of $\tilde{\Gamma}(F(r),G(r))$. Hence $x(k) \in K_j(r)$ and $x \in U_j$, so $K_j(r) \subseteq U_j$. Further, $U(x(k), \varepsilon_{x(k)}) \subseteq U'_j$ when $x(k) \in K_j(r)$, which implies that $U_j \subseteq U'_j$. Therefore the sets $U_j, j = 1,.....,n$, satisfy properties (1) and (2).

Part (3).

Let ε_1 be the smallest of the numbers $\varepsilon_{x(1)},.....,\varepsilon_{x(m)}$. Since $\Gamma(F(r),G(r)) \subseteq \bigcup_{j=1}^{n} U_j$, there exists a $z > 0$ such that $x \in X - \bigcup_{j=1}^{n} U_j$ implies $p(F(r)(x),G(r)(x)) > z$. Again using the uniform continuity of $F(r)$ and $G(r)$, there exists $\varepsilon_2 > 0$, such that, if $|r-s| < \varepsilon_2$, then $p(F(r)(x),F(s)(x)) < z$ and $p(G(r)(x),G(s)(x)) < z$. Thus, if $|r-s| < \varepsilon_2$ and $x \in X - \bigcup_{j=1}^{n} U_j$, then $F(s)(x) \neq G(s)(x)$. Equivalently, $\Gamma(F(s),G(s)) \subseteq \bigcup_{j=1}^{n} U_j$. Let ε be the smaller of ε_1 and ε_2. We now show that the open sets $U_1,.....,U_n$ and $\varepsilon > 0$ satisfy conditions (3) and (4).

Suppose $|r-s| \leq \varepsilon$ and $x_s \in K(s), K(s) \in \tilde{\Gamma}(F(s),G(s))$. Since $\varepsilon \leq \varepsilon_2, x_s \in U_j$ for some $j = 1,.....,n$ and hence $x_s \in U(x(k),\varepsilon_{x(k)})$ for some $x(k) \in K_j(r)$. Let C be a path in $U(x(k),\varepsilon_{x(k)})$ from x_s to $x(k)$. For any $t \in I, F_s^r(t) = F(t')$ where $t' = s + t(r-s)$, so $|r-t'| = |1-t||r-s| \leq |r-s| < \varepsilon$.

Also, since $\varepsilon \leq \varepsilon_1 \leq \delta < \frac{\varepsilon'}{2}$ we have

$$\begin{aligned}
& p(F_s^r(t)C(t),G_s^r(t)) = p(F(t')C(t),G(t')C(t)) \\
& \leq p(F(r)C(t),F(t')C(t)) + p(F(r)C(t),G(r)C(t)), \\
& < \frac{\varepsilon'}{4} + \frac{\varepsilon'}{2} \text{ by (1), (2) respectively} \\
& < \varepsilon'.
\end{aligned}$$

and, by definition of ε', $[\langle F^r_s, C\rangle] = [\langle G^r_s, C\rangle]$. So $x_s - F^r_s, G^r_s \to x(k)$. By definition then, $K(s) - F^r_s, G^r_s \to K_j(r)$. If there were a point $x'_s \in K(s)$ such that $x'_s \in U_k$, then by the same reasoning, $K(s) - F^r_s, G^r_s \to F_k(r)$. Thus by Theorem 1.30 $k = j$, so $K(s) \subseteq U_j$ thus proving Part 3.

Part (4).

$\{\bigcup_{K(s)\subset U_j} K(s)\} \cap \bigcup\{_{K(s)\subset U_k} K(s)\} = \phi, j \neq k$, so since X is normal, there are V_i, V_k such that $\bigcup_{K(s)\subset U_j} K(s) \subset U_j \subseteq \bar{U}_j \subset V_j \subseteq \bar{V}_j \subset U'_j$, $\bigcup_{K(s)\subset U_k} K(s) \subset U_k \subseteq \bar{U}_k \subseteq V_k \subseteq \bar{V}_k \subseteq U'_k$ whilst $U'_j \cap U'_k = \phi, j \neq K$ implies $\Gamma(F(s), G(s)) \cap (\bar{V}_j - U_j) = \phi$ so $(F(s), G(s), U_j)$ is Δ-admissible.

We now have the very important

THEOREM 1.43 Let F,G be paths in Y^X from (f,g) to (f',g') and X,Y compact ANRs. Let $K \in \tilde{\Gamma}(f,g)$. If K is F,G-related to some $L \in \tilde{\Gamma}(f',g')$ then $j(K) = j(L)$; and if K is not F,G-related to any element of $\tilde{\Gamma}(f',g')$ then $j(K) = 0$.

Proof. Let $x \in K \in \tilde{\Gamma}(f,g)$, and let $B(s)$ be the set of points x' in $\Gamma(F,(s), G(s))$ such that x is F^s_o, G^s_o-related to x'; $(F^s_o(t), G^s_o(t)) = (F(s(1-t)), G(s((1-t)))$. By Theorem 1.30 $B(s)$ is either empty or a single coincidence class of $\tilde{\Gamma}(F(s), G(s))$. If $B(s) = \phi$ then by the additivity of j, $(F(s), G(s), \phi) =$

$j(F(s), G(s), \phi) + j(F(s), G(s), \phi) = 2j(F(s), G(s), \phi)$ so $j(F(s), G(s), \phi) = 0$

now $j(K) = j(F(o), G(o), U_j)$ and $j(L) = j(F(1), G(1), U_k)$; hence to show $j(K) = j(L)$ we must show $j(K) = j(F(1), G(1), U_k)$, this then proving both parts of the theorem at once. We claim that, given $r \varepsilon I$, there exists $\varepsilon >$ such that $|r-s| < \varepsilon$ implies $j(B(r)) = j(B(s))$. If this holds then $j(K) = j(L)$; to see this note. $B(o) = \{x' \varepsilon \Gamma(F(o),G(o)):$ $x' - F^o_o,G^o_o \rightarrow x, x \varepsilon K\}$, $=$ $\{x' \varepsilon \Gamma(f,g) : x' - f,g - x, x \varepsilon K\}$ i.e, $B(o) = K$, and

$B(1) = \{x' \varepsilon \Gamma(F(1),G(1)) : x' - F^1_o,G^1_o \rightarrow x, x \varepsilon K\}$ so

$B(1) = \{x' \varepsilon \Gamma(F(1),G(1)) : x' - f',g' \rightarrow x, x \varepsilon K\}$ so

$B(1) = L$. Hence if $|r-s| < \varepsilon$ implies $j(B(r)) = j(B(s))$ then $j(B(r)) = j(B(r+\frac{\varepsilon}{2})$ for all $r \varepsilon [0,1-\frac{\varepsilon}{2}]$.

Hence $j(K) = j(B(o)) = j(B(\frac{\varepsilon}{2})) = j(B(\varepsilon)) = \ldots\ldots = j(B(1-\frac{\varepsilon}{2})) = j(B(1)) = j(L)$

(a maximum of $\frac{2}{\varepsilon}$ steps, for the nearest integer bigger than $\frac{2}{\varepsilon}$).

so $j(K) = j(L)$.

We will let ε be the number obtained from Lemma 1.42. Suppose $B(s) \neq \phi$; then by Lemma 1.42 $B(s)$ is F^r_s,G^r_s-related to some $K_j(r) \varepsilon \tilde{\Gamma}(F(r),G(r))$. Now $B(s)$ is F^s_o,G^s_o-related to K, so by Lemma 1.39 $K_j(r)$ is F^r_o,G^r_o-related to K. Thus by Theorem 1.30 $K_j(r) = B(r)$ and $B(r) \neq \phi$. The contrapositive statement is that if $B(r) = \phi$, then $B(s) = \phi$ and $j(B(r)) = j(B(s)) = 0$, so proving our claim when $B(r) = \phi$.

Now suppose $B(r) \neq \phi$. Let U be the open set guaranteed by Lemma 1.42 containing $B(r)$ and $|r-s| \leq \varepsilon$, then

$U \cap \Gamma(F(s),G(s) = B(s))$..................................(1)

There are two possible cases :

<u>Case (1)</u> $U \cap \Gamma(F(s),G(s)) = \phi$, and

<u>Case (2)</u> $U \cap \Gamma(F(s),G(s)) \neq \phi$. Firstly,

<u>Case (1):</u> If $B(s) \neq \phi$ then, by Lemma 1.42, $B(s)$ _ U_j for same j, and

B(s) is F^r_s,G^r_s-related to $K_j(r)$. But we showed above that $K_j(r) = B(r)$, so $U_j = U$. Therefore $U \cap \Gamma(F(s),G(s)) = \phi$ implies $B(s) = \phi$ in which case (1) certainly holds.

Case (2): Suppose $U \cap \Gamma(F(s),G(s)) \neq \phi$. Take $x_s \in U \cap \Gamma(F(s),G(s))$; then x_s is in some coincidence class M of F(s),G(s). By Lemma 1.42, $M \subseteq U_j$ some j. But $x_s \in U \cap U_j$, so again by Lemma 1.42, $U = U_j$. Hence $M \subseteq U$ and therefore M is F^r_s,G^r_s-related to B(r). By definition K is F^r_o,G^r_o-related to B(r), so Lemma 1.39 implies K to be F^s_o,G^s_o-related to M which by Theorem 1.30 implies K = B(s). Thus, if $x_s \in U \cap \Gamma(F(s),G(s))$, then we have shown that $x_s \in B(s)$, so $U \cap \Gamma(F(s),G(s)) \subseteq B(s)$. However we have already seen that $B(s) \subseteq U$, so $B(s) \subseteq U \cap \Gamma(F(s),G(s))$, so $U \cap \Gamma(F(s),G(s)) = B(s)$, as required.

Now, by definition j(B(r)) = j(F(r),G(r),U) and by (1) we have j(B(s)) = j(F(s),G(s),U) since Lemma 1.42 states that $(F(s),G(s),U_j)$ is Δ-admissible when $|r-s| \leq \varepsilon$. Finally F^r_s,G^r_s is a homotopy from F(s),G(s) to F(r),G(r) and for $t \in I$, $F^r_s(t),G^r_s(t)=F(t'),G(t')$ where $|r-t'| \leq |r-s| \leq \varepsilon$. So by Lemma 1.42 (F^r_s,G^r_s,U) is Δ-admissible. Therefore by the homotopy axiom of j, j(B(r)) = j(B(s)), as required.

The following theorem is of major importance.

THEOREM 1.44 Let X,Y be compact ANRs and let F,G be paths in Y^X such that (F(o),G(o)) = (f,g) and (F(1),G(1)) = (f',g'). Then $N(f,g,\Delta) = N(f'g'\Delta)$.

Proof. Taking the contrapositive of the last half of Lemma 1.43, $j(K) \neq 0$ implies K is F,G related to some coincidence class L of (f'g'), i.e. K is Δ-essential. Lemma 1.43 above implies then that

$j(L) \neq 0$ so $N(f,g,\Delta) \leq N(f'g'\Delta)$. But $K\text{-}F,G \to L$ implies $L\text{-}F^{-1},G^{-1} \to K$, (Lemma 1.27), so $j(L) \neq 0$ implies $j(K) \neq 0$ so $N(f'g'\Delta) \leq N(f,g,\Delta)$, so $N(f,g,\Delta) = N(f'g'\Delta)$ as required.

CHAPTER II

In this chapter we define the class of fibre spaces, and note that a certain class of these are generalisations of products. Restricting ourselves to products, each factor of which is a closed, finitely triangulable manifold (and hence a compact ANR), will enable us to derive a formula for computing the Δ-Nielsen number $N(f,g,\Delta)$ for certain maps f,g.

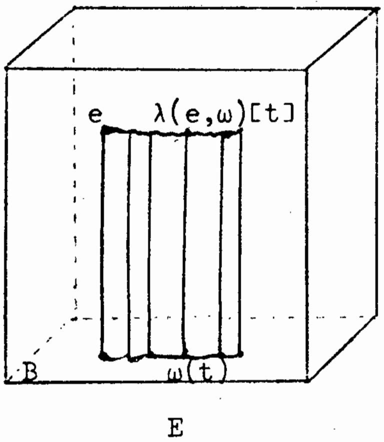
Definition 2.1 [Brown] Let E and B be any two topological spaces and p a continuous subjection from E onto B. Let $\Omega_p \subset E \times B^I$ be the subspace $\{(e,\alpha) \in E \times B\ p(e) = \alpha(0)\}$ of the cartesian product. Let $\bar{p}: E^I \to \Omega_p$ be the map $\bar{p}(\alpha) = (\alpha(0), p\alpha)$. Triple (E,p,B) is said to be a fibre space if there is a map $\lambda: \Omega_p \to E^I$ such that $\bar{p}\lambda$ is the identity on Ω_p.

Definition 2.2 [Dugundji] Let E,B be topological spaces and p a continuous subjection of E on to B and $\Omega_p = \{(e,\omega) \in E \times B^I: p(e) = \omega(0)\}$. A lifting function for (E,p,B) is a map $\lambda: \Omega_p \to E^I$ such that $\lambda(e,\omega)[0] = e$ and $p\lambda(e,\omega)[t] = [t]$ for each $(e,\omega) \in \Omega_p$ and $t \in I$.

<u>THEOREM 2.3</u> Definition 2.1 <=> Definition 2.2

<u>Proof.</u> Suppose $\bar{p}\lambda$ is the identity on Ωp. Then $\bar{p}\lambda(e,\omega) = (\lambda(e,\omega)[0], p\lambda(e,\omega)) = (e,\omega)$ so $\lambda(e,\omega)[0] = e$ and $p\lambda(e,\omega)[t] = \omega[t]$.

<=. Given $\lambda(e,\omega)[0] = e$ and $p\lambda(e,\omega)[t] = \omega[t]$, $(e,\omega) = (\lambda(e,\omega)[0], p\lambda(e,\omega)) = \bar{p}\lambda(e,\omega)$ means $\bar{p}\lambda$ is the identity on Ωp as required.



As Dugundji observes, the rôle of λ is to lift each path ω starting at $\omega(0)$ to a path in E so that the whole family of ωs is lifted 'continuously' into E, hence it is called a lifting function.

λ is said to be a regular lifting function if ω being a constant path in B implies $\lambda(e,\omega)$ is a constant path in E.

We next define maps which preserve the fibre structure.

<u>Definition 2.4</u> Let (E,p,B) and (F,q,C) be any two fibre spaces. $f: E \to F$ is called a fibre map if for any $e, e' \in E$, $p(e) = p(e')$ implies $qf(e) = qf(e')$.

We see that f sends a fibre into a fibre.

A fibre map $f: E \to F$ induces $\bar{f}: B \to C$ such that $\bar{f}p(e) = qf(e)$

$\bar{f}$ a Well Defined Map.

Let $e \in E$ and suppose $p(e) = p(e')$. As f is a fibre map $q(f(e)) = q(f(e'))$ so $\bar{f}(p(e) = \bar{f}(p(e')$ and $\bar{f}$ is well defined.

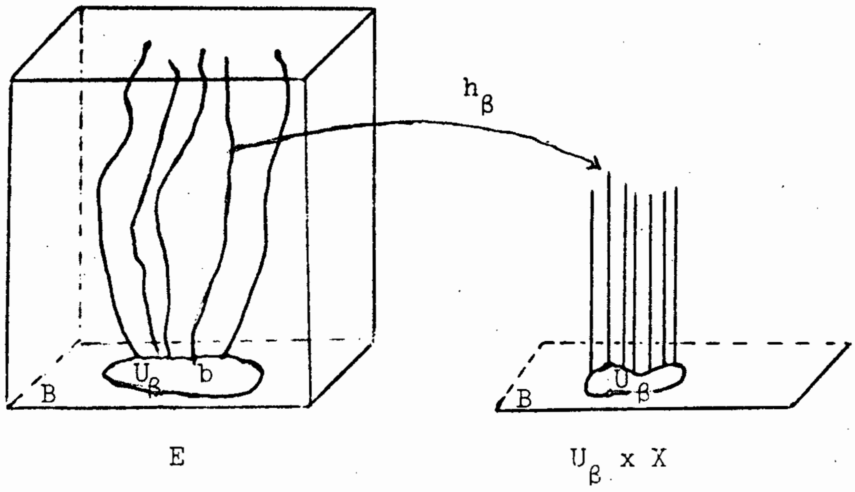
As p, q, f are maps so is $\bar{f}$ and diagram.

$$\begin{tikzcd}
E \arrow[r, "h"] \arrow[d, "f"'] & B \arrow[d, "\bar{f}"] \\
F \arrow[r, "q"'] & C
\end{tikzcd}$$

and $\bar{f}$ is unique in making diagram commute.

Definition 2.5 [Brown] Fibre space (E,p,B) is said to be locally trivial if there exists a space X, an open cover $\{U_\beta\}$ of B, homeomorphisms $h_\beta: p^{-1}(U_\beta) \to U_\beta \times X$ and a projection $\pi: B \times X \to B$ such that $\pi h_\beta = p$.

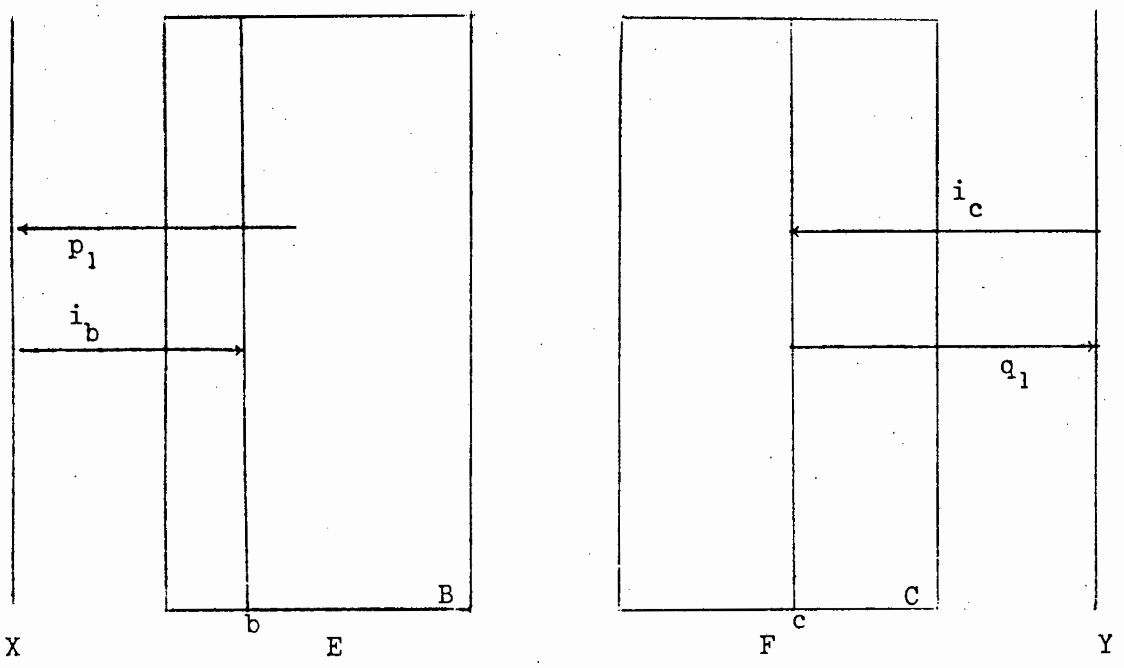
Notice that if $b \in U_\beta$ then $h_\beta(b) = (b,x)$ so $h_\beta^{-1}(b,x) = b$, and as h_β^{-1} is continuous in both variables it is continuous in each separately; in particular $X \approx p^{-1}(b)$. X is called the fibre of (E,p,B).



Definition 2.6 [Brown]. Trivial Fibre Spaces

The fibre space (E,p,B) is said to be trivial if E = B x X. Similarly fibre space (F,q,C) is trivial if F is the product C x Y, where Y is the fibre of (F,q,C).

If f, g: E → F are any two fibre maps where E = B x X and F = C x Y then they induce, in addition to $\bar{f}$, $\bar{g}$: B → C two further maps f_b, g_b: X → Y as follows: each fibre b x X of E is an imbedding of X in E by means of the homeomorphism i_b say; the family $\{i_b, b \in B\}$ is clearly a continuous family, whilst since F is also a product, there is in addition to projection q: F → C a projection q_1: F → Y. Define f_b, g_b: X → Y by $f_b(x) = q_1 f i_b(x)$ and $g_b(x) = q_1 g i_b(x)$. There is of course also projection p_1: E → X and a continuous family of imbeddings i_c: Y → c x Y (see diagram below).



Note that if $\beta: I \to B$ is a path such that $\beta(o) = b$ and $\beta(1) = b'$ we may define two continuous families of mappings, $\{f_{\beta(t)}\}$, $\{g_{\beta(t)}\}: X \to Y$ by letting $f_{\beta(t)}(x) = q_1 f i_{\beta(t)}(x)$, and $g_{\beta(t)}(x) = q_1 g i_{\beta(t)}(x)$.

THEOREM 2.7 Let B,C,X,Y be closed; finitely triangulable manifolds such that the $\dim(B) = \dim(C)$ and $\dim(X) = \dim(Y)$ and such that (E,p,B) and (F,q,C) are trivial fibre spaces, $E = B \times X$, $F = C \times Y$. If $f,g: E \to F$ are two fibre maps such that $f = \bar{f} \times f_b$ and $g = \bar{g} \times g_b$

then $N(f,g,\Delta) = N(\bar{f},\bar{g},\Delta') \cdot N(f_b,g_b,\Delta'')$ for each pair (f_b,g_b), $b \in B$.

Since every connected polyhedron is a path connected compact ANR the Nielsen number defined in Chapter I is the one used here.

The theorem above was motivated by the following theorem for fixed points in Robert F. Brown, The Nielsen Number of a Fibre Map, Annals of Mathematics (85) 1967, 484-93 and corrected by Brown and Fadell (to appear).

THEOREM Let $F = (E,p,B)$ be a locally trivial fibre space with fibre Y where E, B and Y are connected finite polyhedra and let $f: E \to E$ be a fibre map. If one of the following conditions is satisfied

(a) $\pi_1(B) = \pi_2(B) = 0$

(b) $\pi_1(Y) = 0$

(c) F is trivial and either $\pi_1(B) = 0$ or $f = \bar{f} \times f_b$

then $N(f) = N(\bar{f})\, N(f_b)$ for all f_b, $b \in R$.

As may be seen, we have restricted ourselves to giving an analogue for coincidence points for his last case only (with a further restriction) and we shall indicate what theorems would have to be proved to obtain a complete analogue

Definition 2.8 Two Coincidence Points x,x' being F,G Equivalent.

Let f_o, f_1, g_o, g_1 be maps from X to Y, $x \in \Gamma(f_o,g_o)$, $x' \in \Gamma(f_1,g_1)$. Let $\partial: I \to X$ be a path such that $\partial(o) = x$ and $\partial(1) = x'$ and let $F,G: X \times I \to Y$ be two homotopies between f_o and f_1, and g_o and g_1 respectively.

Define two paths $\delta_F, \delta_G: I \to Y$ by

$$\delta_F(t) = F(\delta(t),t)$$

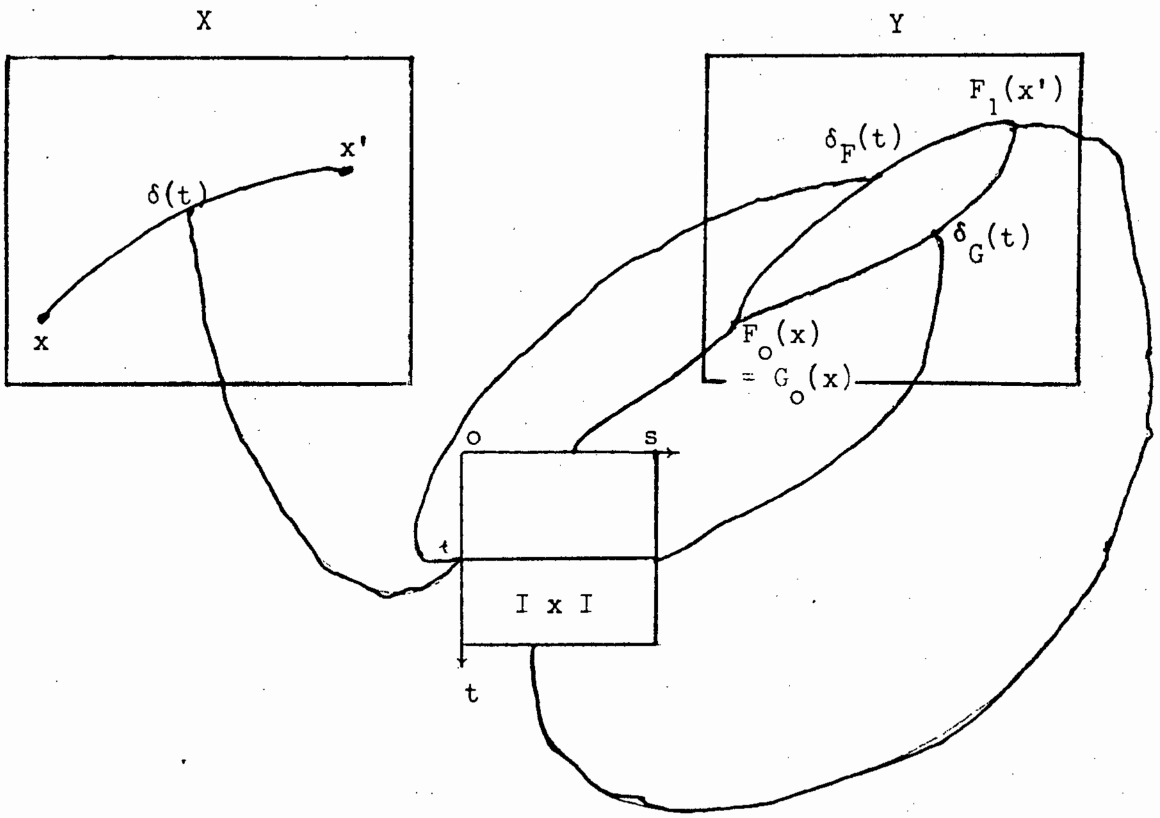
$$\delta_G(t) = G(\delta(t),t)$$

Then $\delta_F(o) = F(x,o) = f_o(x) = g_o(x)) = \delta_G(o)$

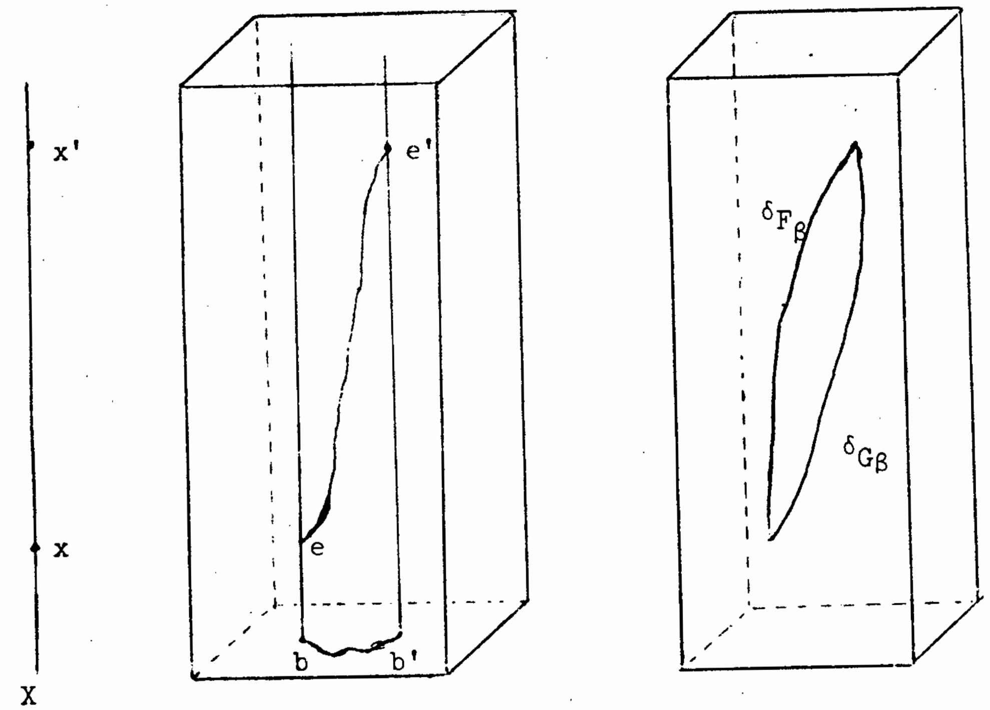
$\delta_F(1) = F(x',1) = f_1(x') = g_1(x') = \delta_G(1)$

If there exists a path δ such that δ_F and δ_G are fixed end-point homotopic, then x and x' are F,G equivalent. [This requires that there exist a map $K: I \times I \to Y$ such that $K(s,o) = F_o(x), K(s,1) = F_1(x')$

$K(o,t) = \delta_F(t), K(1,t) = \delta_G(t)$]



Definition 2.9 Two Coincidence Points being F_β, G_β - Equivalent



Let $\beta: I \to B$ be a path such that $\beta(o) = b$ and $\beta(1) = b'$

Let $x \in \Gamma(f_b, g_b)$ and $x' \in \Gamma(f_{b'}, g_{b'})$ and let $\delta: I \to X$ be a path such that $\delta(o) = x$ and $\delta(1) = x'$. Then the continuous families $\{f_{\beta(t)}\}, \{g_{\beta(t)}\}$ induce two homotopies $F_\beta, G_\beta: X \times I \to Y$ defined by letting $F_\beta(\delta(t),t) = f_{\beta(t)}(\delta(t))$ and $G_\beta(\delta(t),t) = g_{\beta(t)}(\delta(t))$.

Coincidence points x and x' are said to be F_β, G_β- equivalent if the path δ is such that δ_{F_β} and δ_{G_β} are fixed end-point homotopic.

We next have

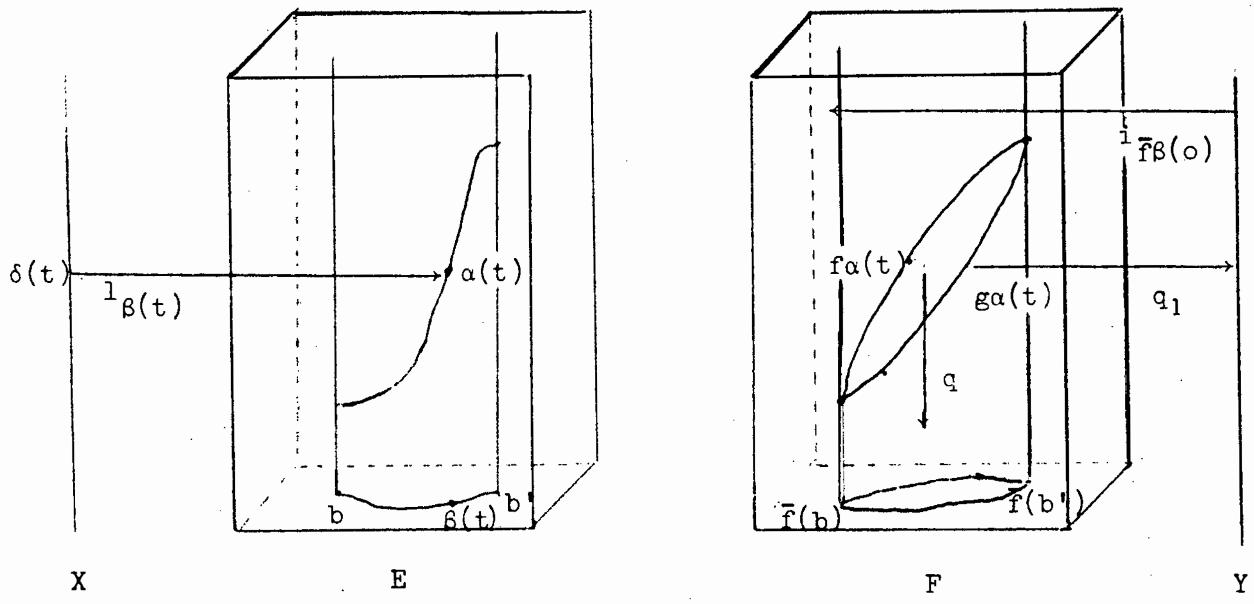
LEMMA 2.10 Let Δ'' be the class of ordered pairs of paths in Y^X. Then $N(f_b, g_b, \Delta'')$ is independant of the choice of $b \in B$.

Proof: Let b, b' be any two elements of B and let $\beta: I \to B$ be a path such that $\beta(o) = b$ and $\beta(1) = b'$. Then as in Definition 2.9 above we have continuous families $\{f_{\beta(t)}\}, \{g_{\beta(t)}\}$ inducing homotopies $F_\beta, G_\beta: X \times I \to Y$ such that $(f_b, g_b) \simeq (f_{b'}, g_{b'})$. But we showed in Chapter I that the Nielsen number is a homotopy invariant so $N(f_b, g_b, \Delta'') = N(f_{b'}, g_{b'}, \Delta'')$.

LEMMA 2.11 For $e, e' \in \Gamma(f,g)$ $e \xrightarrow{f,g} e'$ iff $p(e) \xrightarrow{\bar{f},\bar{g}} p(e')$ and $p_1(e) \xrightarrow{F_{\beta_1}, G_\beta} p_1(e')$ where the F_β, G_β relation is induced by β

Proof: Let $f, g: E \to F$ be any two fibre maps with $e, e' \in \Gamma(f,g)$. Then $\bar{f}p(e) = q(f(e) = q(g(e) = \bar{g}(f(e))$ and $\bar{f}(p(e')) = q(f(e')) = q(g(e')) = \bar{g}(p(e'))$ so $p(e), p(e') \in \Gamma(\bar{f}, \bar{g})$. Further if $e \xrightarrow{f,g} e'$ then there is a path α from e to e' and a fixed end-point homotopy $J: I \times I \to F$ such that $J(s,o) = f(e) = g(e)$, $J(s,1) = f(e') = g(e')$ $J(o,t) = f(\alpha(t))$ and $J(1,t) = g(\alpha(t))$.

Then $p(e) \xrightarrow{\bar{f},\bar{g}} p(e')$ by means of path $\beta = p\alpha: I \to B$ and homotopy $H: I \times I \to C$ where $H = qJ$ since $\bar{f}\beta = qf\alpha$ and $\bar{g}\beta = qg\alpha$.



Define path $\delta: I \to X$ by $\delta(t) = p_1(\alpha(t))$. Then $\delta(o) = p_1(e)$, $\delta(1) = p_1(e')$ and $q_1(J(s,t)$: $I \times I \to Y$ is a homotopy such that $q_1J(s,o) = f_{\beta(o)}(\delta(o)) = g_{\beta(o)}\delta(o), q_1J(s,1) = f_{\beta(1)}\delta(1) = g_{\beta(1)}\delta(1)$ $q_1J(o,t) = f_{\beta(t)}\delta(t)$ and $q_1J(1,t) = g_{\beta(t)}\delta(t)$, hence $p_1(e) \xrightarrow{F_\beta, G_\beta} p_1(e')$. Conversely suppose $e,e' \in \Gamma(f,g)$ and $p(e) \xrightarrow{\bar{f},\bar{g}} p(e')$ by means of path $\beta: I \to B$ such that $\beta(o) = p(e), \beta(1) = p(e')$ and $\bar{f}(\beta(t))$ is fixed end-point homotopic to $\bar{g}(\beta(t))$. Suppose too that $p_1(e) \xrightarrow{F_\beta, G_\beta} p_1(e')$. Thus there is a path $\delta: I \to X$ such that $\delta(o) = p_1(e)$, $\delta(1) = p_1(e')$ and a path $f_{\beta(t)}\delta(t)$ fixed end-point homotopic to $g_{\beta(t)}\delta(t)$ by means of $K: I \times I \to Y$ such that

$$K(s,o) = f_{\beta(o)}p_1(e) = g_{\beta(o)}p_1(e)$$
$$K(s,1) = f_{\beta(1)}p_1(e') = g_{\beta(1)}p_1(e')$$
$$K(o,t) = f_{\beta(t)}\delta(t)$$
$$K(1,t) = g_{\beta(t)}\delta(t)$$

Since $\{i_{\beta(t)}\}$ is a continuous family of homeomorphisms $\{i_{\beta(t)}\delta\}: I \to E$ is a path such that $i_{\beta(o)}\delta(o) = e$ and $i_{\beta(1)}(\delta(1)) = e'$. If δ is such that $pi_{\beta(t)}(\delta(t) = \beta(t)$, each $t \in I$, then the F_β,G_β-relation between $p_1(e)$ and $p_1(e')$ is said to be induced by β.

Since we are given that this is so

$$i_{\bar{f}\beta(o)}(K(s,o)) = f(e) = g(e) \text{ all } s \in I$$
$$i_{\bar{f}\beta(1)}(K(s,1)) = f(e') = g(e') \text{ all } s \in I$$
$$i_{\bar{f}\beta(t)}(K(o,t)) = fi_{\beta(t)}\delta(t) \text{ all } t \in I$$
$$i_{\bar{g}\beta(t)}(K(1,t)) = gi_{\beta(t)}\delta(t) \text{ all } t \in I$$

so $e \xrightarrow{f,g} e'$.

LEMMA 2.12 If B,C,X,Y, are closed, finitely triangulable manifolds such that $\dim(B) = \dim(C)$ and $\dim(X) = \dim(Y)$ and $\bar{f},\bar{g}: B \to C$ and $f_b, g_b,: X \to Y$, are maps, then there are maps h, j, h_b, j_b, arbitrarily close to $\bar{f},\bar{g},f_b,g_b$ respectively (and hence homotopic to them) such that $\Gamma(\bar{h}xh_b,\bar{j}xj_b)$ is finite, each coincidence point being in the interior of some maximal simplex of B x X.

Proof: Schlagbauer, Theorem 1.7.

Note that since the Nielsen number is a homotopy invariant we have that $N(\bar{f},\bar{g},\Delta') = N(\bar{h},\bar{j},\Delta')$

$$N(f_b,g_b,\Delta'') = N(h_b,j_b,\Delta'') \text{ each } b \in B$$

hence $N(\bar{f}xf_b,\bar{g}xg_b,\Delta) = N(\bar{h}xh_b,\bar{j}xj_b,\Delta)$, each $b \in B$.

Definition 2.13 Let E,F be finite polyhedra and $f,g: E \to F$ mappings. A coincidence point $\bar{e}$ of (f,g) is said to be isolated if there is an open set U in E containing $\bar{e}$ such that $\bar{U} \cap \Gamma(f,g) = \bar{e}$. For $\bar{e}$ an isolated coincidence point of f,g define $j(f,g,U) = j(f,g,\bar{e})$.

LEMMA 2.14 If $\bar{e}$ and $p(\bar{e})$ are isolated coincidence points of (f,g) and $(\bar{f},\bar{g})$ respectively and if $\bar{e}$ lies in the interior of a maximal simplex of some triangulation of E then

$$j(f,g,\bar{e}) = j(f_b,g_b,p_1(\bar{e}) \cdot j(\bar{f},\bar{g},p(\bar{e})$$

Proof: Follows from Theorem 1.37 if we let $X = B$, $X' = X$ (here the fibre) $f = \bar{f}$ and $f' = f_b$.

One result to be used shortly is

LEMMA 2.15 Let $b \varepsilon \Gamma(\bar{f},\bar{g})$. If K is a coincidence class of (f,g), then $p_1[K \cap p^{-1}(b)]$ is a single coincidence class of (f_b,g_b).

Proof: Let $e,e' \varepsilon K \cap p^{-1}(b)$ and let $\alpha: I \to E$ be a path from e to e' such that paths $f\alpha, g\alpha: I \to C \times Y$ are fixed end-point homotopic. The projection by $q_1: C \times Y \to Y$ shows that $q_1 f\alpha$ and $q_1 g\alpha$ are fixed end-point homotopic in Y, whilst $p_1\alpha(t)$ is a path in X such that $p_1\alpha(o) = p_1(e)$ and $p_1(\alpha(1)) = p_1(e')$. We may unite $\alpha(t) = (b(t),x(t)) \varepsilon B \times X$, hence $q_1 f\alpha(t) = q_1(\bar{f}(b(t)), f_b x(t)) = f_b x(t)$ whilst $p_1\alpha(t) = x(t)$ so $q_1 f\alpha(t) = f_b p_1 \alpha)t)$, and similarly $q_1 g\alpha(t) = g_b p_1 \alpha(t)$ so $p_1(e) - f_b,g_b \to p_1(e')$ as required.

We finally have then:

THEOREM. Let B,C,X,Y be closed, finitely triangulable manifolds such that dim(B) = dim(C), dim(X) = dim(Y) and such that (E,p,B) and (F,q,C) are trivial fibre spaces, $E = B \times X$ and $F = C \times Y$. If $f,g: E \to F$ are any two fibre maps such that $f = \bar{f} \times f_b$ and $g = \bar{g} \times g_b$

then $N(f,g,\Delta) = N(\bar{f},\bar{g},\Delta) \cdot N(f_b,g_b,\Delta'')$for each pair (f_b,g_b), $b \varepsilon B$.

Proof: By lemma 2.12 we may assume that $\Gamma(\bar{f}xf_b,\bar{g}xg_b)$ has only finitely many coincidences each of which lies in a maximal simplex of E.

Let K be a coincidence class of f,g, then by 2.11 p(K) is contained in a coincidence class L of $\bar{f},\bar{g}$. For $e \varepsilon K$ $p_1[K \cap p^{-1}p(e)]$ is a single coincidence class M of f_b,g_b.

We will prove that

(1) .... $j(K) = j(L) \cdot j(M)$.

Let $p(K) = (b_1,\ldots\ldots,b_n)$ (which being finite in number are isolated) and for each $u = 1,\ldots\ldots,r$ let $K \cap p^{-1}(b_u) = (e_{u_1},\ldots,e_{u,m_u})$ Let $M = p_1[K \cap p^{-1}(b_1)]$ and let b_u be any element of $p(K)$. Then as $K \cap p^{-1}(b_u) \subseteq K$ and $K \cap p^{-1}(b_1) \subseteq K$, lemma 2. tells us that $p_1[K \cap p^{-1}(b_u)] \xrightarrow{\quad F_\beta, G_\beta \quad} M$, hence

$$j[p_1(K \cap p^{-1}(b_u))] = j[M], \text{ all } u = 1,\ldots\ldots,r$$

By the additivity property of the index j

$$J(L) = \sum_{w=1}^{m_u} j(f_{b_u}, g_{b_u}, p_1(e_{u,w})), \text{ each } u = 1,\ldots\ldots,r.$$

Applying the additivity property and lemma 2.

$$\begin{aligned} j(K) &= \sum_{u=1}^{r} \sum_{w=1}^{m_u} j(\bar{f}xf\ ,\bar{g}xg\ ,e_{u,w}) \\ &= \sum_{u=11}^{r} j(\bar{f},\bar{g},b_u) \sum_{w=1}^{m_u} j(f_{bu},g_{bu},e_{uw}). \end{aligned}$$

(2)............... $= j(M) \sum_{u=1}^{r} j(\bar{f},\bar{g},b_u)$

We must now distinguish two cases. Either $p(K) = L$ or it is not. If $p(K) = L$ then

$$j(K) = \sum_{u=1}^{r} j(\bar{f},\bar{g},b_u)$$

and equation (1) is identical to equation (2). Otherwise choose $b \in L - p(K)$ and let α be a path from b to b_1 and $H: I \times I \to C$ a homotopy such that $b \xrightarrow{\bar{f},\bar{g}} b_1$. By 1.25 if $x,x' \in X$ and $x \xrightarrow{F_\beta,G_\beta} x'$

then the points of their respective coincidence classes are also F_β,G_β-equivalent. But if the points of M are F_β,G_β-related to any coincidence class N of f_b,g_b then by 2.11 $i_b(N) \subseteq K \cap p^{-1}(b)$ hence $b \in p(K)$, a contradiction. Hence M is inessential so $j(M) = 0$, so (2) implies $j(K) = 0$ and (1) is trivially satisfied too.

We have also shown that if K is essential then $p(K) = L \cap K \subseteq p^{-1}(L)$.

Let L be any essential coincidence class of $\bar{f},\bar{g}$ then by 2. and (1) $p^{-1}(L)$ contains exactly $N(f^b,g^b,\Delta)$ essential coincidence classes of f,g where by 2. $N(f_b,g_b,\Delta)$ is independent of the choice of $b \in B$. Furthermore we have proved that each essential coincidence class of $\bar{f} \times f_b$, $\bar{g} \times g_b$ lies in $p^{-1}(L)$ for some essential coincidence class L of $\bar{f},\bar{g}$.

Therefore

$$N(\bar{f}xf_b,\bar{g}xg_b,\Delta) = N(f_b,g_b,\Delta'') \cdot N(\bar{f},\bar{g},\Delta'')$$

Conclusion: We see that we have established what might be called a first case for this formula for computing the Nielsen number. To obtain such a formula for a broader class of fibre spaces, such as those defined in the Brown-Fadell paper, the following must be proved:

(1) Lemma 2. must be such that $j(f,g,\bar{e}) = j(\bar{f},\bar{g},p(e)) \cdot j(f_b,g_b,\phi_b^{-1}(\bar{e}))$ for any fibre maps f,g.

(2) The Hopf construction such as is found in Brown 1971, pg 117 must be extended to this wider class of spaces.

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