Paradoxes of Infinity and Beyond

Sets, Quantification, and Cantor's Domain Principle

By Tarryn Harding

Supervised by Professor Bernhard Weiss

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Faculty of the Humanities

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COMPULSORY DECLARATION

This work has not been previously submitted in whole, or in part, for the award of any degree. It is my own work. Each significant contribution to, and quotation in, this dissertation from the work, or works, of other people has been attributed, and has been cited and referenced.

Signature:

Signed by candidate

Date: 02 September 2025

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To Adam and Daniel, for being my midwives.

And to Bernhard, with thanks.

For everything.

Preface

Like many outstanding philosophers, Michael Dummett was occasionally prone to articulating various controversial ideas. This dissertation will be concerned with one such idea, the initial formulation of which can be found in Dummett's magnum opus, *Frege: Philosophy of Language* (1973):

> … the prime lesson of the discovery of the set-theoretical paradoxes [is] that quantification over [certain] domains cannot be regarded as yielding, in all cases, a sentence with a determinate truth-value. (p. 516)

In later work, Dummett would elaborate on this claim with recourse to the notion *of indefinite extensibility*, where indefinitely extensible concepts – which include the concepts 'set' and 'ordinal' – are such that our possessing "a definite conception" of a totality of the concept's instances in turn facilitates our "characteriz[ing] a larger totality all of whose members fall under it" (1996, p. 441).

That the set-theoretic paradoxes *do* reveal that the concept 'set' is indefinitely extensible – and that this might have consequences for classical quantification – is, at least now, a claim that enjoys run-of-the-mill status in the overall literature on absolute generality[1]. However, when viewed in light of another of Dummett's controversial ideas – namely, that indefinite extensibility is inextricably linked to our concept(s) of *infinity* – then this same claim emerges as a version of a principle Michael Hallett (1984) dubs 'the Domain Principle,' which he attributes to Georg Cantor himself:

> Each potential infinite, if it is rigorously applicable mathematically, presupposes an actual infinite. (Cantor, as quoted in Hallett, 1984, p. 25)

which, insofar as we understand 'rigorous mathematical applicability' as 'suitability for classical quantification,' transforms into the (decidedly less run-of-the-mill) idea that quantification over infinite totalities presupposes that these form a *set*[2].

Investigating this claim will be primary aim of this dissertation. We should note at the outset, however, that the Domain Principle – where it is discussed at all – is usually understood as a principle of set *existence*[3]: one testifying simply to non-emptiness of the category of the infinite Cantor dubbed the transfinite.

[1] For instance, Augustín Rayo and Gabriel Uzquiano (2006, p. 4); more generally, James Studd (2019) (p. viii).
[2] Or a "set-*like* object" Richard Cartwright (1994) (p.8).
[3] A notable exception is Ignacio Jané (2010), who remarks that the Domain Principle is, for Cantor, "not only a principle of existence, but a principle of intelligibility as well" (p. 215).

Understood thusly, the Domain Principle is not only taken (mostly) to be *true,* but also – to paraphrase W. D. Hart (1975) – to be that which "makes set theory a paradise" (p. 247); moreover, the degree to which it is *false,* is usually thought of as having less to do with quantification, and more to do with the claim – which *is* typically motivated by the paradoxes – that *not all infinite totalities form sets.*

This opposing view about the origins of the contradictions is succinctly captured by Richard Cartwright:

> It is one thing for there to be certain objects; it is another for there to be a set, or set-like object, of which those objects are the members. (1994, p. 8)

Moreover, Cartwright emphasizes that this "straightforward response" (p. 10) has the added benefit of ensuring that "the usual laws of quantificational logic are not… disturbed" (p. 12): a fact that soon becomes apparent if we consider the following version of Russell's Paradox (as formulated by its namesake in his 'Mathematical Logic as Based on the Theory of Types' (1908)):

$$(W): \forall x\ (x\ \in W\ \leftrightarrow x \notin x)$$

In that paper, Russell diagnoses his paradox as arising due to "the class W's [being] defined by reference to 'all classes,' and then turn[ing] out to be one among classes" (p. 225). However, that W 'turns out to be one among classes' can be broken down into two distinct claims: namely i) that W *exists as a class* and ii) that W *falls within the range of the quantifiers' governing its definition.*

In insisting that "the conclusion to be drawn is that there is no such class as W" (p. 11), Cartwright thus opts to reject claim (i): this in turn ensures that "quantification over all classes is permitted" (p. 12), and is, as a result, usually associated with quantifier *absolutism*: the view that absolute generality is not only *expressible,* but is (or can be) expressed by means of *ordinary quantifiers.* By contrast, rejecting claim (ii) is standardly the route taken by quantifier *relativists*, for whom the upshot of the paradoxes is precisely that our quantifiers, in falling short of ranging over all sets, thereby *fail* to range over absolutely everything.

Of course, in stipulating that our quantifiers be limited to sets (or set-like objects), the Domain Principle – if true – would itself entail a form of quantifier relativism. However, it is interesting to note that the relativist views we discuss in this dissertation – including those of James Studd and Øystein Linnebo – are typically motivated less by Domain Principle-like considerations, and more by the concern that Cartwright's 'straightforward response' threatens to render unstable our underlying *conception of set:* this, insofar as it poses, but *fails to answer*, the question of "when does it make sense to say that a collection forms 'one thing' as opposed to a mere aggregate?" (Hallett, 1984, p. 286).

In turn, to the degree that relativists enrich their languages with expressive devices *besides* quantifiers – Studd (2019), for instance, allows that we may express more wide-ranging claims about sets by means of a *modal* operator, ‘□’ – this is construed either as a means of “provid[ing] a philosophically richer story about the gears at the heart of the set-formation process” (Sutto, 2024, p. 3); or, more proximally, as a way for the relativist to diffuse the accusation that the restrictions on absolute generality she posits undermine her ability to state her view.

But, rather remarkably, all of this is usually *unaccompanied* by an account of what differentiates the sort of general claims we can formulate by means of quantifiers, from those made available by expressive devices like modal operators[4]. Consequently – besides exploring arguments that the quantifier *absolutist's* distinction between sets and ‘mere aggregates’ is at base *un-principled* – this dissertation will also grapple with the question of what it is about the form of generality expressed by the *relativist's* modal operators that allows these to transcend the (apparent) limits attendant on our quantifiers: a question that, in being asked, promises not only to illuminate the nature of these constraints, but also to shed light on whether they entail something like the Domain Principle.

A final note before we proceed: it is necessary to briefly comment on the place of naturalism (broadly construed) vis-à-vis our enquiry. Philosophers, especially of the analytic variety, have long been divided in their views on how (and whether) philosophy should interact with set theory. At the one extreme we encounter thinkers like Peter Simons, who in his paper ‘Against Set Theory’ (2005) rails against what he calls the “math envy” of those who “would assimilate philosophy to the formal sciences of mathematics and logic” (p. 143); at the other extreme is the view that, to the degree that philosophy and set theory *can* be bedfellows, this should proceed with the understanding that set theory as “a mathematical subject is beyond philosophical reproach” (Pleitz, 2019, p. 35). Studd (2019) himself adopts what he calls “a modest methodological naturalism,” where this amounts not only to the assumption that “set theory, as practised by set theorists, is broadly along the right track” but also – more importantly – that:

> Even if set theory or model-theoretic semantics are not beyond philosophical interpretation or criticism, we must, I assume*,* ultimately leave things substantially as we found them (p. viii)

This aspect of his naturalism leads Studd to “set aside responses to the set-theoretic paradoxes… which seek to radically reform mathematical practice” as well as those which “are yet to make substantial progress towards recovering classical set theory”; it also entails that his defence of quantifier relativism include arguments “show[ing] that it is consonant with mathematical and semantic practice” (p.viii).

[4] As Sutto (2024) points out: “the way and the kind of absolute generality attained by the modal operator is currently an open matter” (p. 16).

This latter consequence emerges most strongly in Studd's (2019) extensive reliance on *Mirroring Theorems*[5], which play an essential role in his demonstration that his 'potentialist' set theory (MST_P) suffices to interpret the fragment of ZFC Studd dubs s^+. However, the availability of Mirroring Theorems also means that it is possible to interpret Studd's potentialist set theory in an *entirely non-modal theory* (that is: s^+ interprets MST_P). This gives rise to an objection – standardly[6] attributed to Joel Hamkins (2018) – that potentialist theories like Studd's are, in fact "implicitly actualist" (p. 34): at the very least, such results raise the uncomfortable possibility that the shift to a modal theory has failed to "make a real difference" (Linnebo and Shapiro, 2017, p. 175).

That Studd *is* vulnerable to an objection of this sort – and that this follows from his aim of ensuring that relativism is 'consonant with mathematical and semantic practice" – raises a more general question concerning the reach, and objectives, of the philosophy of set theory: in the specific case of modal approaches, the question is "how much should a potentialist bother *not* to upset mathematicians by granting that modal set theory parallels non-modal approaches?" (Sutto, 2024, p. 12).

To circle around to my own approach to this vexed question (as well as the broader concern of what the role of set theory *is* in philosophy, and vice-versa): the utility I derive from the use of various logical formalisms means that I cannot in good faith count myself among the Peters Simons's of the world (although I *am* sympathetic to many of his claims). On the other hand, it goes without saying that I do *not* take set theory to be 'beyond philosophical interpretation or criticism'; nor am I entirely comfortable with Studd's insistence that we ought – in the wake of this interpretation and criticism – to 'ultimately leave things substantially as we found them.'

My own (tentative) view is best expressed by an observation Hallett (1984): namely that, in the case of set theory, "[a]xiomatization went hand in hand with the divorce from any attempt to understand what sets are or what conceptual role they play" (p. 303), where this now entails that "we find it hard to say why many of the key collections isolated by the axioms should be sets" (p. 304). My own version of this would involve noting – apart from the obvious difficulties accompanying the *absolutist's* 'straightforward response' – that the extant *relativist* approaches to the paradoxes, despite their technical flair, tend to leave rather basic questions unanswered.

The degree to which this *should* lead to the 'reform' of mathematical practice – or upset the work of potentialist philosophers – I leave up to them to decide.

[5] Mirroring theorems establish that the truth of a non-modal formula φ in first-order logic entails the truth of its 'modalisation,' $\varphi^\Diamond$. See below.
[6] Øystein Linnebo and Stewart Shapiro (2017) articulate their own version of the objection, see below.

Chapter 1

The Domain Principle: its Defenders and Detractors

Introduction

In his masterful exposition of the work of Georg Cantor – widely acknowledged to be the founder of modern set theory – Michael Hallett (1984) identifies the goal of the latter's philosophical work to be "provid[ing] arguments to support the legitimate mathematical employment of actual infinities, especially infinite numbers" (p. 7).

To this end, Hallett argues that Cantor adopts three principles. The second and third of these, respectively, are what Hallett calls Cantor's *finitism* – the idea that the infinite numbers Cantor's theory postulates are theoretically on par with *finite* numbers – and Cantor's doctrine of the *Absolute*: the notion that, outside of the 'actual infinities' of the transfinite, there is a class of infinite that eludes mathematical determination.

The first, and most important, principle guiding Cantor's work is what Hallett calls *the Domain Principle*; in Cantor's own words:

> Each potential infinite, if it is rigorously applicable mathematically, presupposes an actual infinite. (Cantor, as quoted in Hallett, 1984, p. 25).[7]

This prompts the question of how Cantor himself understood the terms 'potential infinite' and 'actual infinite' (which date back to Aristotle[8]). In his 'Communications on the Theory of the Transfinite,'[9] for instance, Cantor seems to cash out the potential infinite in terms of *variability* – it is "nothing more than an indeterminate… variable magnitude taking values which become either as small as we please or larger than any arbitrary finite bound" – where the actual infinite, by contrast, is "a fixed in itself, constant quantum which is larger than any finite magnitude of the same kind" (Cantor, as quoted in Jané, 1995, p. 379). Applying this reasoning to the natural numbers, he describes how:

> … a variable magnitude x successively taking the different finite whole number values 1, 2, 3, ..., ν, ... represents a potential infinite, while the set (ν) of all whole finite numbers… offers the simplest example of an actual infinite quantum. (Cantor, as quoted in Jané, 1995, p. 379)

[7] One way of understanding Cantor's Domain Principle is as a kind of indispensability argument for the inclusion of the actual infinite in mathematics: adopting it thus aligns with the overall goal for his work identified by Hallett, namely, of 'provid[ing] arguments to support the legitimate mathematical employment of actual infinities, especially infinite numbers.'
[8] For more, see Chapter 2.
[9] *Mitteilungen zur Lehre vom Transfiniten,* 1887 – 1888.

This goes some way towards explaining what the actual infinite consists in – namely, the existence of infinite sets[10]. However – besides suggesting that this is intertwined with the 'variation' of terms in a well-ordered sequence – the question of how we should understand the *potential* infinite remains unanswered (nor are we closer to an understanding of what 'rigorous mathematical applicability' consists in).

This chapter will be dedicated to unpacking these issues, beginning with the suggestion – due to Graham Priest – that the 'rigorous mathematical applicability' of a given domain consists in its being *suitable for classical quantification*. From there, we'll explore how a variety of thinkers – including Cantor, and Michael Dummett – have understood the sort of variability exhibited by infinite domains, and its implications for quantification. We'll assess whether their arguments succeed in justifying the Domain Principle, before turning our attention to a discussion of its most vocal opponent, Richard Cartwright.

In this way, we'll aim to canvass the views of both the Principle's defenders, and its detractors; we'll then conclude by charting a way forward, in light of these views, for the dissertation as a whole.

§1.1

'Rigorous mathematical applicability' and the 'variability' of infinite domains

One of Cantor's more extended justifications for the Domain Principle runs as follows:

> … we cannot do without *variable* quantities in the sense of the potential infinite; from this can be demonstrated the necessity of the actual-infinite. In order for there to be a variable quantity… the 'domain' of its variability must strictly speaking be known beforehand through a definition. However, this domain cannot itself be something variable, since otherwise each fixed support for the study would collapse. Thus, this 'domain' is a definite, actually infinite set of values. (Cantor, as quoted in Hallett, 1984, p.25)

Of crucial importance is the distinction Cantor draws between a 'domain of variability' – which, according to the Domain Principle, is *presupposed* by the rigorous mathematical study of the potential infinite – and the 'variability of a domain' which, contrastingly, leads to the 'collapse' of 'each fixed support' for this study. This gives rise to two questions:

> **Question 1:** What does the 'variability' of a 'potentially infinite' domain consist in?

[10] It is controversial whether or not Cantor always understood the actual infinite *only* in terms of infinite sets. For instance, Ignacio Jané (1995) points out that, in his earlier work, Cantor was willing to accept two varieties of the actual infinite – infinite sets, and the Absolutely infinite (which, while an actual infinite, is not an infinite *set* on pain of paradox).

> **Question 2:** How does this variability undermine the rigorous mathematical study of the potential infinite (viz., *qua* the 'variable quantities' of mathematics)?

In *Beyond the Limits of Thought* (1995), Graham Priest opts to cash out the variability of a domain in terms of its being *unspecified* or *indeterminate.* Thus, writing of Cantor's argument for the Domain Principle above, he describes it as "simple and ingenious" insofar as it is:

> … based on the equally simple observation that for a statement about some variable quantity to have determinate sense, the domain of its variability must be determinate. For example, consider the claim 'Let z be a root of the equation $ax^2 + bx + c = 0$. Then z has at least one value.' This is true if z may be complex; false if z must be real. (p. 139).

Priest's parsing of the Domain Principle in the first sentence suggests that he takes rigorous mathematical applicability to consist in the *determinacy of sense* of mathematical statements, specifically those involving *quantification* over infinite domains. Moreover, while he never makes explicit what he takes determinacy of sense to consist in, the example he provides to illustrate his point suggests that he takes this to consist, for a given sentence, in its possessing determinate (specifically, *invariant)* truth values. Consequently, Priest's (1995) response to Question 1 would likely be that 'domain variability' arises when it is indeterminate which domain is being ranged over by our quantifiers (in his example, for instance, indeterminacy arises concerning whether the quantifier ranges over complex numbers, or the reals); in turn, his response to Question 2 would probably involve the claim that this domain variability undermines the *determinate sense* of statements quantifying over these domains, by way of undermining the stability of their truth values.

Priest's (1995) decision to cash out Cantor's notion of 'rigorous mathematical applicability' in terms of the prerequisites of *quantification*, is both interesting and compelling, and we will follow him in adopting it for the purposes of this dissertation. However, if our reading of his rational reconstruction is sound, there is little besides this in his overall analysis that is groundbreaking[11]. Moreover, it is not at all clear that the sort of variability, *qua* domain indeterminacy, present in Priest's example *is* the kind of variability at play in Cantor's conception of the potential infinite. For one thing, the former seems to arise due to an underlying indeterminacy *across,* or *between,* domains: but such 'inter-domain' variability – and the indeterminacy of sense in quantified statements it engenders – plausibly arises for *finite* domains, as well as infinite ones[12].

Additionally, the kind of 'inter-domain' indeterminacy at play in Priest's example can presumably be eliminated by means of an appropriately-interpreted predicate: we might, for instance, restrict the quantifier in the original statement so that it now reads '$\forall z\ (D(z) \rightarrow \theta(z))$,' where '$\theta(z)$' abbreviates the original claim (viz., 'if z be a root of the equation

[11] As Michael Glanzberg (2004) points out, it is nothing short of a 'truism' that classical – or bivalence-preserving - quantification necessitates the specification of a domain.

$ax^2 + bx + c = 0$, then z has at least one value') and D is a predicate satisfied by either (and only) the reals, or the complex numbers. Critically, it seems that we *can* specify potentially infinite domains in this way (to use Cantor's paradigm example, let '$D(z)$' abbreviate 'z is a natural number'); however, there is reason to believe that Cantor nevertheless viewed this means of specifying infinite mathematical domains as largely *insufficient* when it comes to securing their 'rigorous mathematical applicability' – especially if our grasp of the relevant predicate (and the role this plays in securing the domain) is akin to our grasping an underlying *concept* satisfied by the elements *in* the domain. This is a point Hallett (1984) makes – specifically, he notes that, for Cantor, it is perfectly possible for us to grasp the concept 'natural number' (say, by means of the successor function) without thereby having grasped the *domain* of natural numbers: this, in Cantor's words, is because "we with our restricted being are not in a position to actually conceive the infinitely many individuals… belonging to the set ω in one intuition" (as quoted in Hallett, 1984, p. 27).

Cantor's argument here seems to be that our ability to recognize whether or not a given object is a natural number – viz., our grasp of the relevant *concept* – is not sufficient for us to secure the domain of natural numbers, for it does not entail a corresponding grasp of the *totality* of *all* natural numbers. Michael Glanzberg makes a similar point in his 'Quantification and Realism' (2003): there, he notes that we might specify the domain of natural numbers by means of an inductive definition along the lines below:

N1. 0 is a number

N2. If n is a number, so is Sn (the successor of n, i.e., n + 1)

N3. The natural numbers are the closure of this process.

That this specification is 'sharp,' as Glanzberg puts it, is a function of clauses N1 and N2: these jointly ensure that the elements of the domain are "precisely the things got from 0 by iterating the successor" (p. 547); moreover, by "sharply tell[ing] us that certain things are in the domain," these clauses *would* suffice to eliminate the kind of inter-domain variability present in Priest's example. However, Glanzberg contends that N1 and N2 nevertheless fail to specify the domain of natural numbers *determinately*, as they fail to "tell us *completely* what is in the domain" (p. 546, emphasis mine).

Put differently: while N1 and N2 provide the assurance that *every object in the domain* is a natural number (that is, a thing 'gotten from 0 by iterating the successor'), they nevertheless fail to assure us that *every object that is a natural number, is in the domain*.

Precisely something along these lines seems to be Cantor's point when he complains that our grasp of the concept 'natural number' doesn't equate to our 'actually conceiv[ing] the infinitely many individuals… belonging to the set ω in one intuition.' To use Glanzberg's terminology: the complaint is that our grasping the relevant concept, while possibly ensuring the sharpness of our specification, nevertheless fails to ensure that it is *exhaustive;* or – at the very least – that a candidate domain of the natural numbers *is* exhaustive is *not* something that 'we in our restricted being' are capable of ascertaining.

The particular kind of indeterminacy afflicting infinite domains is thus, for Cantor, less a function of whether they can be sharply delineated from other domains, and more to do with their being – for us finite beings – indeterminate in *extent.* Consequently, Cantor would likely agree with Glanzberg that clauses like N1 and N2, while necessary, are nevertheless insufficient to secure infinite domains, as they fail to guarantee that these domains are *exhaustive*.

Moreover, just as clause N3 functions to provide the requisite assurance as to these domains' extent, it seems that this is precisely the point of the *Domain Principle*, for Cantor; indeed, that the Principle succeeds where other merely 'intensional' means of specifying domains fail is, Hallett (1984) argues, a function of its profoundly *realist* implications. That is: Hallett contends that Cantor's proposed means of dealing with the kind of 'variability' afflicting potentially infinite domains – namely, the fact that the extent of these domains eludes our direct intuition – is simply to assume these domains *exist in their entirety*, and that they do so *independently* of our intuition. It is *this* assumption, Hallett argues, that is encapsulated in the Domain Principle; this, in turn, guarantees that "regardless of our limited ability to construct or generate natural numbers, there is a complete sequence of these numbers… *Our constructive frailty is irrelevant*" (p. 27, emphasis mine).

Thus, on Hallett's reading, Cantor proposes to solve what is essentially a semantic quandary with recourse to metaphysics. Specifically, the question of how to secure the 'rigorous mathematical applicability' of infinite domains, given their indeterminate extent, is answered simply with the assumption that these domains are – as Cantor puts it in a letter to Philip Jourdain – already "completed and actually existing." Critically, this assumption is intimately linked to these domains' comprising *sets;* indeed, at least in his later work, Cantor seems to take this quality of a given domain's being 'completed'(viz., determinate in extent) not only to *be guaranteed* when it is a set, but to also *imply* or *entail* that it is a set.

To sum up (and putting an intuitive spin on it): that infinite domains are exhaustive is, for Cantor, ensured by their elements together comprising 'one thing,' i.e., a set – and it is this, in turn, that secures the 'rigorous mathematical applicability' of the domains in question, despite our incapacity to survey them 'in one intuition.'

§1.2

The Defenders: Dummett and the 'haziness' of quantifier domains

We've come some way in our understanding of Cantor's Domain Principle:

> Each potential infinite, if it is rigorously applicable mathematically, pre-supposes an actual infinite.

Specifically, we've opted to explicate Cantor's notion of 'rigorous mathematical applicability' in terms of the prerequisites for quantification over infinite domains; we've also seen that Cantor's answer to Question 1 (viz., the kind of variability characterising these domains) boils down to their being *indeterminate in extent* for us finite beings.

However, the details of Cantor's response to Question 2 – namely, *how* this indeterminacy in extent undermines the 'rigorous mathematical applicability' of infinite domains – is in need of both fleshing out, and further defense. Moreover, the idea that the requisite assurance regarding the extent of these domains comes only when they comprise infinite *sets,* is in special need of justification: for while it seems clear that comprising a set would be *sufficient* to safeguard the determinate extent (or 'completion') of potentially infinite domains, it is less clear that sethood is *necessary* to ensure this.

One thinker who seems to think this *is* the case – and whose broader views have wide-ranging implications for the legitimacy of classical quantification in the mathematical realm– is Michael Dummett. Consequently, this section will be dedicated to a discussion of his views, and how they align with Cantor's; and a natural place to begin, is with the former's arguments that quantifying over infinite domains poses a *prima facie* challenge to the classical, or 'Fregean,' understanding of our first-order quantifiers.

On this picture, quantification can be thought of along broadly *truth-functional* lines[13]: for instance, in a finite or a 'surveyable' totality, the universal quantifier would be equivalent to a finite conjunction of (the truth values of) all its instances. Importantly, the sentences that result from quantification over finite domains are what Dummett calls "decidable" – which term he uses, in *Frege: Philosophy of Language* (1973), specifically to describe sentences for which there exists "some effective means which would in principle lead us to a situation in which we could determine [their] truth-value." For sentences involving quantification, this involves "a procedure for discovering every member of the totality and a criterion for determining that every member of the totality had been inspected" (p. 514); this in turn allows us to arrive at the truth-value of a classically-quantified sentence, as "the final outcome of a process which involves running through the values of all its instances" (1991, pp. 313 – 314).

Of course, 'running through the value of all its instances' is possible, at least in-principle, for a sentence quantifying over *finite* domains; that such a sentence is 'decidable' is thus a function of there being nothing (again, in principle) standing in the way of our "conducting a complete survey, establishing the truth-value of every instance of the quantified statement" (1996, p. 61). However, things look quite different when it comes to what Dummett calls 'intrinsically infinite' domains – those for which we always "have a means of finding another element… however many we have already identified" (1991, p. 318). In this case, the possibility of 'establishing the truth-value of every instance of the quantified statement' is

[13] Specifically, quantifiers are to be understood as second-level concepts,

fundamentally withheld from us, due to the in-principle *impossibility* of our "complet[ing] an infinite process" (p. 313).

Thus, given a sentence purporting to quantify over an 'intrinsically infinite' totality, Dummett (1973) declares that we are "faced with a choice" (p. 514). On the one hand, we can acknowledge that while we possess "the capacity to recognize whatever would decide conclusively in favour of the truth or of the falsity of the sentence" (p. 514), a corresponding *incapacity* to bring about such a deciding situation means that we should withhold ascribing determinate truth values to the sentence in question. Rather than assessing it as true or false, we ought instead to evaluate it as "justified or unjustified" (where a claim, for Dummett (1996), is justified "if the one who makes it can vindicate his claim" (p. 438)).

This is the choice Dummett himself opts to make – and it is, he acknowledges, of "a radical character." For the proposal is not merely that bivalence fails for sentences quantifying over intrinsically infinite domains; rather, it is that we should jettison altogether "the conception of objective truth-values determined, independently of our knowledge or means of knowledge, by a reality external to us" (p. 470).

However, Dummett also acknowledges that opting to *retain* the idea of objective truth values for sentences quantifying over infinite domains – and, that these are determined 'by a reality external to us' – *would* safeguard bivalence. This is because a realist stance would allow us to treat quantification over intrinsically infinite domains in a manner *strictly analogous* to the finite case (thus undermining the claim that quantifying over the former poses *in-principle* difficulties for classical quantification). More fully: realism about our quantifier domains would, Dummett (1996) contends, assure us that:

> … it is only a practical difficulty which impedes our determining the truth-values of sentences involving such quantification in a similar way [to the finite case]… [this] is defended by appeal to a hypothetical being who could survey infinite domains in the same manner as we survey finite ones. Thus Russell spoke of our incapacity to do this as 'a mere medical impossibility.' (p. 61)

Of course, our inability to 'survey' infinite quantifier domains just amounts to their being (for us) indeterminate in extent; and we saw in the last section that this is exactly what motivates Cantor to adopt a realist stance vis-à-vis these domains. Indeed, the idea that there exists some Being 'who could survey infinite domains in the same manner as we survey finite ones' is not, for Cantor, a hypothetical one: the existence of an omniscient Deity capable of bringing to completion operations like the successor function (and so, for Whom, all natural numbers exist as a 'completed totality') is frequently cited as part of his justification for the truth of *the Domain Principle* – the realist implications underlying which, we saw in §1.1, function primarily as a means of assuring us of the *determinate extent* of the resulting infinite domains.

That transfinite numbers and totalities – while evading our our own 'restricted, discursive comprehension' – nevertheless exist 'in and for the Divine intelligence,' is thus an integral part of Cantor's justification that 'they *are* in and for themselves'[14]; in turn, the principal function of this mathematical realism – as also enshrined in the Domain Principle – is to safeguard the 'rigorous mathematical applicability' of these transfinite numbers and totalities (which are, in the words of St Augustine, "made finite to God").[15]

Dummett would agree with this sentiment (at least in spirit): his own talk of realism's reducing our inability to survey infinite domains to merely 'a practical difficulty' expresses much the same thought. However, for Dummett, there is, another, more nuanced sense in which the truth of realism would assure us that our infinite mathematical domains are determinate in extent (which we will dedicate the remainder of this section to unpacking). This is the sense in which realism, if true, would entail that our mathematical domains domains are "completely definite," where this involves a complete absence of "any haziness about what elements [a domain] does or does not contain" (1991, p. 314).

Importantly, at first blush, talk of 'haziness about what elements a domain does or does not contain' is reminiscent more of the *inter-domain* indeterminacy characterizing Priest's example in §1.1, rather the indeterminate *extent* (viz., unsurveyable nature) of infinite domains that so worried Cantor; and admittedly, Dummett *does,* at times*,* give the impression[16] that the 'definiteness' of a domain (viz., its *lacking* in 'haziness') consists its being sharply specified, in Glanzberg's sense[17]. However, this is almost always on the assumption that *realism is true*: for instance, in *Frege: Philosophy of Mathematics* (1991) Dummett argues that, assuming realism, our grasping a domain by means of a "definite concept" (viz., a criterion of membership and one of identity) suffices to ensure that "generalisation with respect to it will yield statements with determinate truth values" – this, because:

> … We do not need to say just what objects there are which fall under the given concept: provided the concept is definite, *reality will of itself*

[14] Cantor uses this phrasing in a letter to mathematician Giuseppe Veronese,:

> . . . one must distinguish between [transfinite numbers] as they are in and for themselves, and in and for the Divine intelligence and how these same numbers appear in our restricted, discursive comprehension (Cantor, as quoted in Hallett, 1984, p. 28)

[15] See Hallett (1984), p. 35.

[16] For instance, he argues in *Frege: Philosophy of Language* (1973) that – assuming the truth of realism – a 'determinate' (and so, presumably, sharp) conception of a domain *would* suffice to render it suitable for classical quantification:

> Once we have such a conception, we have successfully conferred upon the universal quantifier a sense under which there will be associated, with every sentence formed by means of it, a determinate truth-value, true or false... Much may remain for us to do if we want to find out what the truth-value of the sentence is; but, once we have specified the domain of quantification by means of a condition of membership of that domain, no more work remains for us to do, in order to guarantee that the sentence has a definite truth-value: all the rest is, as it were, accomplished independently of us by objective reality itself. (p. 518)

[17] A specification of a domain fulfils this criterion when it "sharply tells us that certain things are in the domain" (Glanzberg, 2004, p. 546).

> *determine the truth or falsity of such statements*. On this view, reality dispels all haziness: we need to nothing further to eliminate it. (p. 31, emphasis mine)

However, we saw earlier that Dummett does *not* take sentences quantifying over infinite domains to have 'determinate truth values.' This flows from his r*ejection* of realism, which itself hinges on the claim that these domains exhibit precisely the 'haziness' we'd expect them to *lack* – given our grasp of the relevant concepts – if realism *were* true. In turn, that our grasp of the relevant 'definite concepts' is, for Dummett, beyond dispute, entails that the kind of 'haziness' at issue cannot be reduced to our (merely) lacking a sufficiently *sharp* specification of the domains in question.

This latter claim becomes especially apparent when we consider Dummett's key argument for anti-realism, from the *set-theoretic paradoxes*; wherein he contends – using Burali-Forti's paradox as an example – that "[t]he principal consequence" of the contradictions is "that even platonists were compelled to allow that there are mathematical concepts whose extensions form hazy totalities" (1991, p. 315):

> The Burali Forti Paradox ensures that no definite totality comprises everything intuitively recognizable as an ordinal number, where a definite totality is one quantification over which always yields a statement determinately true or false. For a totality to be definite in this sense, we must have a clear grasp of what it comprises: but, if we have a clear grasp of any totality of ordinals, we thereby have a conception of what is intuitively an ordinal number greater than any member of that totality. (p. 316)

Evidently, our possessing 'a clear grasp' (and so, presumably, a 'sharp conception') of some domain of ordinals not only fails to eliminate Dummett's haziness – rather, it seems to *give rise to it,* by allowing us to form 'a conception of what is intuitively an ordinal number greater than any member of that totality.' This is a function of the ordinals' being '*indefinitely extensible*,' where Dummett defines an indefinitely extensible concept as:

> … one such that, if we can form a definite conception of a totality all of whose members fall under that concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it (1996, p. 441).

Thus, to the degree that, for Dummett, the 'haziness' afflicting infinite mathematical domains is a function of their being *indefinitely extensible,* it is clear that such haziness *can* coincide with our possessing 'a definite conception' of the relevant totality. This, however, brings to the fore once again the question of the sense in which this haziness involves uncertainty 'about what elements [a domain] does or does not contain' (and, moreover, how this relates to the domain in question's being indeterminate in extent).

Some clarity on this might be gained if we track the concept of indefinite extensibility back to its origins. These, as Dummett notes[18], lie not with him but with Betrand Russell, specifically the latter's notion of 'self-reproductive processes,' which Russell first describes in his 'On Some Difficulties In The Theory of Transfinite Numbers and Order Types' (1905). There, Russell characterizes such processes as "essentially incapable of terminating, although each process is such that the class of all terms generated by it… ought to be the last term generated by that process" (p. 43). And, just as Dummett concludes that the indefinite extensibility of the ordinals ensures, on pain of contradiction, that 'no definite totality comprises everything intuitively recognizable as an ordinal number,' so too does Russell argue, in the case of 'self-reproductive' properties, that:

> …we can never collect *all* the terms having the said property into a whole; because, whenever we hope we have them all, the collection which we have immediately proceeds to generate a new term also having the said property. (p. 36)

This 'new term' is clearly the analogue of the 'intuitive ordinal' Dummett contends lies at the heart of the Burali-Forti paradox; however, one of Russell's (1905) key insights is that the phenomenon of 'self-reproductive processes' "cover[s] *all* the contradictions that have hitherto emerged in this subject" (p. 35, emphasis mine). To justify *this* idea, Russell points out that the paradoxes seem to conform to a general scheme:

> Given a [self-reproductive] property ϕ and a function f, such that, if ϕ belongs to all the members of u, $f(u)$ always exists, has the property ϕ, and is not a member of u; then the supposition that there is a class W of all terms having the property ϕ and that $f(W)$ exists leads to the conclusion that $f(W)$ both has and has not the property ϕ. (p.35)

As Hallett (1984, p. 180) points out, Russell's insight can be distilled in the following conditional:

$$(*): \forall u \, [\forall x \left(x \in u \rightarrow \phi(x)\right) \rightarrow \left(f(u) \notin u \wedge \phi\left(f(u)\right)\right)$$

where u is a set; the function f is as described above; and ϕ is the 'self-reproductive' property under consideration.

Corroborating Russell's claim, a number[19] of the set-theoretic paradoxes fit into the scheme captured by $(*)$: in each case, contradiction arises when, having assumed the existence of a set $W = \{x \mid \phi(x)\}$, we note, in Russell's word, that '$f(W)$ both has and has not the

[18] See footnote 5, p. 317 of *Frege: Philosophy of Mathematics* (1991).
[19] For instance, for Russell's Paradox, let $\phi(x) =_{df} x \notin x$; $W = \{x \mid x \notin x\}$; and let f be the identity function (such that $f(W) = W$).
Similiarly, for Mirimanoff's Paradox, let ϕ abbreviate a formula satisfied by well-founded sets; $W = \{x \mid \phi(x)\}$; and, , let f be the identity function.

property ϕ.' For instance, where ϕ is the concept 'ordinal' (and we take f to be the identity function), inconsistency emerges when we consider that the set of all ordinals, W, appears itself to be well-ordered, and thus an 'intuitive ordinal' – one that *should* be an element of the set of all ordinals, and yet which, in being greater than all these ordinals, is excluded from this set.

To link all of this back to the promised explication of Dummett's notion of 'haziness': that the indefinitely extensible (or 'self-reproductive') nature of some concept ϕ culminates in uncertainty about whether the set of all ϕs is *itself* a ϕ, *does* speak to uncertainty about what elements the domain of ϕs 'does or does not contain'; how this might relate to the indeterminate *extent* of this domain, in turn, becomes clearer when we consider this phenomenon in light of Dummett's discussion, in *Frege: Philosophy of Mathematics* (1991), of the domain of real numbers (which domain he was wont to characterize in terms of indefinite extensibility[20]). There, having proposed criteria of identity and membership (and so, provided a 'definite concept') for this domain, Dummett notes that this is:

> … quite adequate to explain what is required of a specified mathematical entity for us to recognise it as a real number; but it does not suffice as a means of circumscribing a domain of quantification, when such quantification is to yield statements with determinate truth-values… because *it fails to determine the limits of acceptable specification of something to be acknowledged as a real number*. (p. 315, emphasis mine)

This passage once again underscores how, for Dummett, eliminating the 'haziness' that undermines classical quantification over infinite domains – namely, *circumscribing* the domain in question – requires something *more* than providing a means of identifying a given object as being a real number (and so, as sharply belonging to the domain of real numbers). Instead, circumscribing this domain entails essentially that we *determine* '*the limits of acceptable specifications' of the real numbers.*

The issue, then, isn't merely whether a given object is a real number, but rather where the 'acceptable specifications' of real numbers *end*; in other words, the question is *how far the real numbers extend.*

This seems to capture the kind of uncertainty that arises when, given an indefinitely extensible ϕ, we ask whether the set of all ϕs, W, is itself a ϕ (one way of putting this question, arguably, would be to ask whether ϕ-hood *extends as far as* W). More generally, what all this appears to reveal is that the kind of 'haziness' afflicting indefinitely extensible domains, for Dummett, is akin to the sort of *indeterminacy in extent* that, for Cantor, predominantly characterizes the *potential infinite.* That is: if he were faced with Question 1 (concerning what the 'variability' of potentially infinite domains consists in), Dummett would likely respond that this is a function of the domains' in question being *indefinitely extensible*.

[20]E.g., in his (1996): "Cantor's celebrated diagonal argument to show that the totality of real numbers is not denumerable has precisely the form of a principle of extension for an indefinitely extensible concept" (p. 442).

Correspondingly, his answer to Question 2 (regarding the *impact* this has on classical quantification) would probably fall back on the 'haziness' he takes to afflict such domains, which – insofar as it indicates an indeterminacy of the 'limits of acceptable specifications' of the objects populating them – simultaneously undermines classical quantification over them.

To sum up: we have come (at least somewhat) closer to an understanding of Dummett's nebulous notion of haziness. We've also seen that this notion – and the broader phenomenon of indefinite extensibility on which it rests – furnishes us with answers to both Question 1 and 2 (and, for the purposes of this dissertation, we will indeed assume that a domain is potentially infinite just in case it is indefinitely extensible).

However, it is important to note that 'haziness about what elements a domain does or does not contain' is *not* the only kind of haziness Dummett associates with indefinite extensibility. This emerges in *Frege: Philosophy of Mathematics* (1991), when, having described 'ordinal' as one of the "mathematical concepts whose extensions form hazy totalities" (p. 315), Dummett goes on to remark that:

> Better than describing the intuitive concept of ordinal number as having a hazy extension is to describe it as having an increasing sequence of extensions: what is hazy is the length of the sequence, which vanishes in the indiscernible distance. (p. 317)

Here, haziness is a function not of the indeterminate extent of a *single* candidate extension of the concept 'ordinal,' but is rather exhibited by the 'increasing sequence of extensions' Dummett famously describes as accompanying this, and other, indefinitely extensible concepts.

Of course, this gives rise to the question of how these two forms of haziness are connected. Both, evidently, are related to the 'intuitive ordinal' accompanying our 'clear grasp' of a given totality of ordinals. However, whereas the first kind of haziness captures our oscillating between ascribing (or not) a well-order type to this 'intuitive ordinal' – and thus, an underlying uncertainty concerning the extent of the *initial* domain (viz., uncertainty about whether it *includes this ordinal*) – the second seems to entail that the latter has been definitively *excluded* from the initial domain: it now functions as the first *new* element of the next, *expanded* domain in the sequence.

This suggests that Dummett's second species of haziness may arise as a function of *eliminating the first*: indeed, just this seems to be the upshot, for him, of the Burali-Forti Paradox:

> Any definite totality of ordinals must therefore be so circumscribed as to forswear comprehensiveness, renouncing any claim to cover all that we might intuitively recognize as being an ordinal. (1991, p. 316)

Critically, we saw earlier that 'circumscribing a domain' entails nothing less than the *absence* of any 'haziness about what elements it does or does not contain.' However, that circumscribing a domain of ordinals entails 'forswear[ing] comprehensiveness… renouncing any claim to cover all that we might intuitively recognize as being an ordinal,' arguably *does* speak to the aspect of Dummett's *second* species of haziness, whereby each individual extension in the 'increasing sequence' is *essentially incomplete* (this, because each is capable of *expansion*).

If sound, this reading of Dummett's two kinds of haziness – and how they interact – has a number of important consequences for our broader discussion of quantification, and the potential infinite (or, as we've opted to cash it out, the indefinitely extensible). We've already noted Dummett's likely response to Question 2, when it comes to his first form of haziness (viz., its indicating our having failed to determine the 'limits of acceptable specifications' for the objects populating indefinitely extensible domains); however, our discussion of Dummett's second kind of haziness points to another way indefinite extensibility undermines classical quantification: specifically, it seems to call into question the idea that the resulting sentences *succeed in expressing absolutely general claims.* Dummett elaborates on this ideas in *Frege: Philosophy of Language* (1973) as follows:

> … it is not possible to suppose that, by specifying the range of some style of individual variables as being over 'all objects', or 'all sets', or 'all ordinals', we have thereby conferred a determinate truth-value on all statements containing quantifiers binding such variables …. [Any attempt] will either lead to contradiction or will prompt us to concede that we are not, after all, using the bound variables to range over absolutely everything that we could intuitively acknowledge as being an object, a set, or an ordinal number. (p. 568 – 569)

The second horn of the above – call this, *Dummett's Dilemma* – is clearly a function of the fact, when it comes to Dummett's second kind of haziness, that each individual extension in the 'increasing sequence' must 'forswear comprehensiveness': for this would indeed entail, for a quantifier ranging only over the former, that it *fails* to range over 'absolutely everything that we could intuitively acknowledge as being an object, a set, or an ordinal number.' But – critically – this in turn entails that sentences featuring these quantifiers *fail to express absolutely general claims* about all objects (or all sets, or ordinal numbers).

There is another upshot of Dummett's kind of haziness that is worth noting: for we saw earlier that this only emerges when the extensions of the 'increasing sequence' have been *appropriately circumscribed*, so as to form the 'definite totalities' *necessitated by classical quantification*. This entails, firstly, that for Dummett, our (classically-interpreted) quantifiers may *only* legitimately range over these less-than-comprehensive totalities (and thus, are permanently precluded from expressing absolutely general claims). Moreover – and perhaps, more importantly – that this process of circumscribing domains *leads* to our recognizing such objects as new 'intuitive ordinals,' strongly suggests – if we assume the Von Neumann ordinals[21] – that the kind of definiteness exhibited by these totalities is the kind specifically *associated with set*s. This seems to reveal an implicit commitment on Dummett's behalf to *the Domain Principle* – specifically, in the form of the claim that circumscribing our quantifier domains necessitates that these form *sets* – and so heralds another point of agreement between him and Cantor.

This latter point is not entirely surprising. For we saw in §1.1 that what motivates Cantor to adopt the Domain Principle is the variability *qua indeterminate extent* of potentially infinite domains (in turn, the realist implications of the Principle function primarily to secure the 'rigorous mathematical applicability' of these domains, given our own 'restricted discursive comprehension' thereof). However, in much the same way, the 'haziness' Dummett takes to afflict *indefinitely extensible* quantifier domains – in the form, specifically, of the question that arises at the *limits* of these domains, concerning 'what elements they do or do not contain' – is what lies behind Dummett's own endorsement of the Domain Principle.

Where Dummett *differs* from Cantor, is with the idea that it is the work of mathematical reality to 'dispel the haziness' characterizing our infinite domains. As he puts it in *Frege: Philosophy of Mathematics* (1991):

> In the mathematical realm, reality cannot be left to blow all haziness away: we have to remove it ourselves by contriving adequate means of laying down just what elements the domain is meant to comprise (p. 315)

Importantly, to the degree that 'contriving adequate means of laying down just what elements the domain is meant to comprise' entails determining *the limits of acceptable specifications* of these elements – in other words, *circumscribing* the domain, in Dummett's unique sense of the word – then, we've seen that this entails, for Dummett, that the domain in question form a set.

And this idea, of course, is none other than the Domain Principle.

§1.3

The Detractors: Cartwright and the 'All-in-One' Principle

Both Dummett and Cantor thus agree that classical quantification necessitates a 'definite totality' for our quantifiers to range over – where the kind of 'definiteness' both have in mind, seems specifically to be that of sets.

However, it remains unclear from either of their arguments whether *only* this kind of definiteness will suffice when it comes to the goal – to borrow Dummett's colourful turn of

[21] On Von Neumann's account, ordinals are identified with particular well-ordered sets. So the process by which we obtain new ordinals is just by forming sets of (the preceding) ordinals; that this in turn suggests, in the current context, that Dummett is committed to the *Domain Principle*, is because the fact that our clear grasp of a domain gives rise to new 'intuitive ordinals' suggests that this clear grasp entails precisely that this domain forms a set.

phrase – of 'blowing all haziness away.' For instance, in Dummett's case, he seems to *assume* that circumscribing a domain, and so eliminating the first kind of haziness accompanying indefinite extensibility (viz., that which arises when it is uncertain 'what elements a domain does or does not contain') is *only* achieved *when this domain forms a set.*

This leads to a new question for Dummett (and, indeed, for Cantor):

> **Question 3:** Why is circumscribing a domain (viz., eliminating any 'haziness about what elements it does or does not contain') established only when the domain in question comprises a set?

(a principled answer to which would, of course, simultaneously justify the Domain Principle).

One *opponent* of the idea that the only 'non-hazy' quantifier domains are sets (and who would, as a result, likely dismiss Question 3 on the whole as ill-founded) is Richard Cartwright: and indeed, in his seminal paper 'Speaking of Everything' (1994), Cartwright is widely credited with thoroughly *dis*crediting the Domain Principle (or, as he calls it there, the 'All-in-One Principle'). In line with the proposed understanding of 'definite totality,' Cartwright defines the All-in-One Principle as the idea that "we can speak of all so-and-so's only if the so-and- so's constitute some one set-like object"(p. 6); or, put in terms of quantification, that "to quantify over certain objects is to presuppose that those objects constitute a 'collection,' or a 'completed collection' – some one thing of which those objects are the members" (p. 7)[22]. However, Cartwright takes the principle to be manifestly absurd; in what may be one of the most cited paragraphs in the literature on the link between the paradoxes and quantification, he spells out this absurdity in detail:

> Consider what [the All-in-One Principle] implies: that we cannot speak of the cookies in the jar unless they constitute a set; that we cannot speak of the natural numbers unless there is a set of which they are the members; that we cannot speak of all pure sets unless there is a class having them as members. I do not mean to imply that there is no set the members of which are the cookies in the jar, nor that the natural numbers do not constitute a set, nor even that there is no class comprising the pure sets. The point is rather that the needs of quantification are already served by there being simply the cookies in the jar, the natural numbers, the pure sets; no additional objects are required. (p. 8)

Critically, that there *can* be the cookies in the jar, the natural numbers, or the pure sets, *in the absence of* the set of any one of them, depends essentially on the following idea:

[22] Cartwright goes on to note that it would "be more accurate to speak of a battery of principles, varying in strength" (p. 7).

It is one thing for there to be certain objects; it is another for there to be a set, or set-like object, of which those objects are the members. (p. 8)

Cartwright links this division to Russell, suggesting that the latter "intended to respect the distinction… in his talk of the class as many and the class as one." Indeed, we saw in the previous section that, for Russell, the existence of 'self-reproductive concepts' does ensure that, on pain of paradox, "we can never collect *all* the terms having the said property into a whole" (1905, p. 36); in his own discussion of Russell, though, Cartwright explicitly disavows the idea that the paradoxes are to do with 'self-reproductive processes' – or, more specifically, the aspect of this claim that has Russell contend, in line with Dummett's arguments from indefinite extensibility, that such 'processes' spell trouble for *classical quantification* over all sets.

This particular line of thought Cartwright finds so concerning, is one Russell pursues in his 'Mathematical Logic as Based on the Theory of Types' (1908). There, after describing a number of paradoxes, both set-theoretic and semantic, he offers the following account of their origins:

> In each contradiction *something is said about all cases of some kind*, and from what is said a new case seems to be generated, which both is and is not of the same kind as the cases of which all were concerned in what was said. (p. 224, emphasis mine)

We can make Russell's thought here – and the underlying diagnosis of the paradoxes it gestures towards – more precise, by means of the formal definition he gives of the paradox bearing his name:

$$(W)\colon \forall x\ (x\ \in W\ \leftrightarrow x \notin x)$$

Having formulated this definition, Russell notes that paradox arises due to the fact that "the class W is defined by reference to 'all classes,' and then turns out to be one among classes" (p. 225). However, one way to cash out the thought that W 'turns out to be one among classes' is as the claim that W, if it were to exist, would *fall within the range of the universal quantifier governing its definition*. Consequently, one way to dissolve the paradox – aside from denying the existence of W – would be to conclude that the quantifier in (W) in fact *fails to range over* W.

Importantly, this reasoning exactly parallels that of Dummett's Dilemma, specifically its second horn (wherein we concede that 'our bound variables fail to range over absolutely everything that we could intuitively acknowledge as… a set, or an ordinal number'). Correspondingly, that we *are* prohibited from talking at once about all sets, or all ordinals, seems precisely to be the upshot of the paradoxes for Russell:

> We can now sum up our whole discussion. After stating some of the paradoxes of logic, we found that all of them arise from the fact that an expression referring to all of some collection may itself appear to denote one of the collection… We decided that, where this appears to occur, we are dealing with a false totality, and that in fact *nothing whatever can significantly be said about all of the supposed collection*" (p. 261, emphasis mine, 1908).

It is thus clear not only that Russell views the paradoxes as forcing on us something like Dummett's Dilemma, but that his response, like Dummett's, is to opt for something like the second horn (that is, Russell concludes that 'nothing whatever significantly can be said' about all sets, all ordinals or, indeed, all *things*).

However, Dummett arrives at the same conclusion specifically as a function of his commitment to the *Domain Principle* – similiarly, as Cartwright points out in his discussion of Russell, the reasoning that leads the latter to his own version of the second horn arouses the "suspicion… that the All-in-One Principle is at work" (p. 13). Indeed, a key takeaway from the paradoxes for Russell *is* the need to establish his 'Vicious Circle Principle':

> If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total.' (1908, p. 225)

Critically, Russell seems here to be suggesting that the reason a collection might lack a total, is due to the possibility of its being *enlarged by new members*: thus, for instance, the difficulty isn't merely W's being 'defined by reference to 'all classes,'' but rather the specific consequence of this that is W's 'turn[ing] out to be one among classes.'

However – as we saw above – to reach this conclusion involves the assumption that W is *the right sort of thing* to fall in the range of the quantifiers governing its definition: in other words, that it is *a set.* This suggests that a collection's possessing a 'total' may specifically involve *sethood*; moreover, in a footnote to his definition of the 'Vicious Circle Principle' above, Russell additionally links the notion to our ability to *generalise about all members* of the relevant collection: "[w]hen I say that a collection has no total, I mean that statements about *all its members* are nonsense" (p. 225 – footnote †, emphasis mine). Contraposing this, we obtain the idea that making significant statements about all members of a collection presupposes that this collection has a 'total' – but if, as we've suggested, a collection's possessing a 'total' additionally entails that it be a set, then the result is a version of Cartwright's All-in-One Principle (equivalently: Cantor's Domain Principle).

In sum: Dummett (and possibly Russell) can be read as advancing a *reductio* argument along the lines below:

1. The Domain Principle is true.
2. Our (classically-interpreted) quantifiers range over absolutely all sets.
3. ⊥

in response to which they reject the second premise, while accepting – in confirmation of Cartwright's suspicions – the first.

Naturally, Cartwright is himself entirely unmoved by this reasoning – a point he emphasizes in his own discussion of Dummett in 'Speaking of Everything' (1994):

> Dummett's argument seems to be simply that since there is no universal [set][23], and since the All-in-One Principle is true, unrestricted quantification is illegitimate. Perhaps the first premise can reasonably be described as a lesson of the set-theoretic paradoxes… the second cannot (p. 17).

It would, however, be disingenuous to ascribe to Cartwright the more guarded claim that the non-existence of a universal set 'can reasonably be described as a lesson of the set-theoretic paradoxes'; rather that there exists no such set is *the* lesson of the set-theoretic paradoxes for Cartwright, as the following passage from 'Speaking of Everything' indicates:

> There surely are objects that do not together constitute, as its members, some one object: some objects are not members of themselves, and there is nothing the members of which are precisely the things that are not members of themselves. (p. 7)

The idea that there 'surely are objects that do not together constitute, as its members, some one object' – call this *Cartwright's Claim* – is part of what he calls the "straightforward response" to the paradoxes (which, as the name implies, is entirely in line with orthodox approaches to the contradictions, most of which involve denying that the problematic collections form a set). However, the true significance of Cartwright's Claim becomes apparent only when considered in light of his insistence that "the usual laws of quantificational logic are not… disturbed," and, more strongly, his commitment to ensuring that "quantification over all classes is permitted" (p. 12). Such statements reflect Cartwright's endorsement of *Quantifier Absolutism* – namely, the view that it is possible, at least in principle, for our classically-interpreted quantifiers to range over *absolutely everything*. Of course, a logical consequence of quantifier absolutism is the second premise of our *reductio* above; consequently, the former view is straightforwardly inconsistent with the Domain Principle. However – and importantly – it is *also* inconsistent with the more general idea that *every collection forms a set*[24]. This is why Cartwright's straightforward response consists

[23] In the original quote, Cartwright uses word 'domain' – however, he intends it to be understood synonymously with 'set.'
[24] To see this, we can return to the details of Russell's definition, (W). In that case, the paradoxical reasoning only requires, for it to go through, that W be the right kind of thing to fall within the range of the quantifiers in its definition; and while a commitment to the Domain Principle would entail this so too would a maximally liberal approach to the question of which collections form sets.

both in the denial of the Vicious Circle Principle (which he describes as "off the wall") *and* in the insistence that "the conclusion to be drawn is that there is no such class as W" (p. 10 – 11).

Similiarly, in response to Dummett's Dilemma, Cartwright quite happily accepts that our quantifiers fall short of ranging 'over absolutely everything that we could intuitively acknowledge as being… a set, or an ordinal number.' However, unlike Dummett and Russell – for whom accepting this second horn entails the falsity of quantifier absolutism –for Cartwright, the upshot is, simply, that "our intuition is just wrong"; or, as he puts it a bit later: "so much the worse for our intuitive acknowledgements" (p. 15).

§1.4

Plural Logic: Rejecting Plural Comprehension

While denying the existence of W (and of Dummett's 'intuitive' ordinal) may indeed be a 'straightforward' response to the paradoxes, it also leads to a number of questions. The first, and most pressing, of these concerns the origins of our intuitions that the relevant totalities exist: we are, in other words, left wanting for an account of *just where our intuitions go wrong.*

For Russell and Dummett, these are likely linked to an acceptance of the Domain Principle (which, for Cartwright, would also explain why the intuitions in question are mistaken). However, Dummett himself points out that accepting the existence of sets like W needn't flow from an antecedent commitment to the latter– rather, it seems simply to be "a consequence of our intuitive understanding of the concept set" (1994, p. 246)[25].

For his own part, Cartwright notes that an 'intuitive understanding' of this kind might involve a commitment to the truth of all instances of the following *Naïve Comprehension Schema*:

$$(Naïve\ Comprehension): \exists x \forall y (y \in x \leftrightarrow \varphi(y))$$

Notably, something very much like Naïve Comprehension features in one of Cantor's earliest definitions of set, offered in his 'On Infinite, Linear Point-Manifolds [Sets]'[26]:

> By a 'manifold' or 'set' I understand in general any many [Viele] which can be thought of as one [Eines], that is, every totality of definite elements which can be united to a whole through a law. (Cantor, as quoted in Hallett, 1984, p. 32)

[25] Interestingly, in this address, Dummett seems to distance himself from the Domain Principle, writing that our taking the universe of sets to form a set is "not demanded by our taking those variables as ranging over them, or by our regarding the sentences of our theory as having determinate truth-values" (p. 248)
[26] *Über unendliche, lineare Punktmannigfaltigkeiten,* (1883)

However, the idea that it is a 'law' or property that unites the element of a set 'to a whole,' is deeply paradoxical: as Cartwright points out in relation to Naïve Comprehension Schema, taking the predicate 'φ' to abbreviate '$x \notin x$' has the effect of transforming the latter schema into Russell's contradictory definition (W).

Thus, if asked to explain what lies behind our intuitions that sets like W exist, Cartwright would point to Naïve Comprehension (the intuitive appeal of which, he notes, meant that even such great thinkers as Frege were 'surprised' by Russell's Paradox); in turn, that Naïve Comprehension is invalid is, or him, what explains where our intuitions go awry.

Importantly, accepting the invalidity of Naïve Comprehension – viz., denying that all properties determine *sets* – needn't entail the additional claim that not all properties determine *some objects* possessing the property in question: denying the existence of W, for instance, needn't amount to denying the existence of the totality of *objects* with the relevant property of non-self-membership. If Cartwright rejected *this* idea, he'd have to relinquish a litany of claims he clearly endorses – one of them being the idea that 'some objects are not members of themselves'.

This suggests that Cartwright, while denying the validity of the *Naïve* Comprehension Schema, would nevertheless endorse a principle of *Plural* Comprehension; indeed, if we avail ourselves of the resources of plural logic[27] – specifically, the logic PFO[28], developed by Øystein Linnebo (2014) – we can formalize this principle as follows:

$$(Plural\ Comprehension)\text{: } \exists xx \forall y (y \prec xx \leftrightarrow \varphi(y))$$

Here, '$\exists xx$' is a *plural quantifier*, to be read 'there exists the xx' (its dual, the quantifier '$\forall xx$,' is to be read, 'for every plurality xx'); the plural variables bound by these quantifiers (in this case, the plural variable xx) range over pluralities; and, finally, '$\prec$' is a two-place 'member-plurality' predicate (such that '$u \prec xx$' reads 'u is one of the xx'). Given this translation, the Plural Comprehension Schema above states, simply enough, that there exist some object(s) – viz., the xx – all of which have the property φ; moreover, taking 'φ' to abbreviate '$y \notin y$,' we can also obtain Cartwright's assertion that 'some objects are not members of themselves.'

Indeed, armed with our plural variables and with '$\prec$,' we can define a new, two-place predicate '$\equiv$' – where '$u \equiv xx$' reads 'u is *the set* comprised of all and only the xx' – as follows:

$$u \equiv xx \leftrightarrow_{df} \forall y (y \prec xx \leftrightarrow y \in u)$$

[27] The use of plural logic was popularised by Georg Boolos in his 'To Be is to be a Value of a Variable (or to be Some Values of Some Variable)' (1984).
[28] The language of *PFO* – $\mathcal{L}_{\mathrm{PS}}$ (or $\mathcal{L}_{\mathrm{PSU}}$ with urelements) – is the language of ordinary set theory as enriched by plural variables and '$\prec$.' See Appendix A for the proof and model theory for *PFO*.

This allows us to formalize the claim that every collection, or plurality, corresponds to a *set*; following Linnebo (2010), we'll call this principle *Collapse:*

$$(Collapse): \forall xx \exists y (y \equiv xx)$$

Of course, this principle is roundly rejected by Cartwright; indeed, if we negate Collapse, we obtain a plural formulation of *Cartwright's Claim*, viz., the idea that 'there surely are objects that do not together constitute, as its members, some one object':

$$(Cartwright's\ Claim): \exists xx \sim \exists y (y \equiv xx)$$

We noted in §1.3 that the above is a natural (indeed, a logical) consequence of Cartwright's commitment to quantifier absolutism; however, there also exists a natural link between the latter view, and the Plural Comprehension Schema[29]. This is because one of the more widespread applications of plural logic is its use, in model-theoretic semantics, as a way of encoding the domain of all sets as a *plurality*. But, if Plural Comprehension is valid, then taking the predicate 'φ' in the schema to abbreviate some condition that is true of everything (say, '$x = x$'), we obtain the claim that there exists a (plurality-encoded) domain comprising *absolutely everything*:

$$(Comprehensive\ Domain): \exists xx \forall z (z \prec xx)$$

These plural formulations make vivid – by way of making formally precise – just why quantifier absolutism, when combined with the Domain Principle, engenders contradiction (specifically: if the former view entails something like Comprehensive Domain, then the truth of this and the Domain Principle swiftly leads to the existence of a *universal set* – and with this, Russell's Paradox).

We also noted in §1.3 that quantifier absolutism is jointly inconsistent with the claim that all pluralities form sets (which corresponds to the principle of Collapse formulated above) – and, just as we would expect, Collapse *is* inconsistent with Comprehensive Domain. In fact, given that Comprehensive Domain is a consequence of Plural Comprehension, it follows that the latter principle is *also* inconsistent with Collapse (the mutual inconsistency of *these* two principles soon becomes apparent, if we consider that they jointly entail *Naïve* Comprehension).

In sum: it is clear that these plural formulations extend beyond being merely rough approximations of the key positions in our debate: rather, by effectively carving up the space of logical possibilitie*s* that emerge in the wake the paradoxes, such principles arguably capture the key features – viz., the commitments and implications – of the views we've considered so far in this dissertation.

[29] Another principle to which Cartwright is implicitly committed.

Thus – granting that our use of plural logic *does* carve up logical space in this way – it soon becomes apparent that the paradoxes force on us a basic choice. Either we can join Cartwright by affirming the validity of Plural Comprehension (and thus of Comprehensive Domain) and by denying Collapse; or, we can deny Plural Comprehension, and accept the truth of Collapse (whether as a consequence of the Domain Principle – as would be the case for Dummett and Russell – or independently of any such considerations).

The remainder of this chapter will be dedicated to outlining the general shape of the views that emerge in the wake of either of these choices; the details of which– viz., their implications for quantification, in general, and the validity of the Domain Principle, in particular – will occupy the dissertation as a whole.

We'll begin with the option that would most likely be the choice made by Dummett and Russell, given their endorsement of the Domain Principle – namely, the option of *rejecting Plural Comprehension:*

$$(Plural\ Comprehension): \exists xx \forall y(y \prec xx \leftrightarrow \varphi(y))$$

We noted earlier that this guarantees the existence of a comprehensive plurality of φs by entailing, in effect, that an object is a member of the plurality in question if and only if it possesses the property φ. Consequently, the opponent of Plural Comprehension would likely couch her rejection of the latter in terms of the falsehood of one, or both, of the two conditional claims that together compose this biconditional.

The first – which I'll call *Homogenous Plurality*$_{\varphi}$ – states simply that there is a plurality whose members are *only those objects with a particular property,* φ. It can be formalized in PFO as follows:

$$(Homogenous\ Plurality_{\varphi}): \exists xx \forall z(z \prec xx \rightarrow \varphi(z))$$

Thus regimented, the truth of Homogenous Plurality$_{\varphi}$ seems self-evident (its negation would amount to denying that *any* pluralities exist containing only the instances of a single property – thus obviating the existence of such innocent pluralities as those containing, for instance, 'only ordinals less than 3').

Given this heavy cost of rejecting Homogenous Plurality$_{\varphi}$, it would almost certainly make more sense for the opponent of Plural Comprehension to reject the second conditional. This asserts, not that there exists a plurality comprising only φs, but rather that there exists a plurality containing *all* of the φs. Call this claim, *Comprehensive Plurality*$_{\varphi}$:

$$(Comprehensive\ Plurality_{\varphi}): \exists xx \forall z(\varphi(z) \rightarrow z \prec xx)$$

The above may strike the reader as familiar: for if we take 'φ' to abbreviate '$x = x$', we obtain, from Comprehensive Plurality$_{\varphi}$, the more sweeping claim that is Comprehensive

Domain (which, as we saw earlier, is an effective regimentation, in PFO, of quantifier absolutism). Moreover, if we *negate* Comprehensive Plurality$_{\varphi}$ – call the resulting claim, *No Comprehensive Plurality*$_{\varphi}$:

$$(No\ Comprehensive\ Plurality_{\varphi}): \forall xx \exists z(\varphi(z) \land \sim(z \prec xx))$$

then we obtain the claim that no plurality contains every object with the property φ – where this is because, for any plurality whatsoever, there exists a φ that *lies outside of it*.

But of course, in the case of an *indefinitely extensible* (or 'self-reproductive') φ, something along these lines is what we would expect; in particular, if the non-exhaustive pluralities to which No Comprehensive Plurality$_{\varphi}$ attests are plurality-encoded *domains,* then the latter schema perfectly encapsulates Dummett's claim – where φ is the concept 'ordinal' – that '[a]ny definite domain of ordinals must therefore be so circumscribed as to forswear comprehensiveness.'

Moreover, if we take 'φ' to abbreviate '$x = x$', we thereby secure the more general claim that is the second horn of Dummett's Dilemma: viz., that our quantifiers fail to range over an absolutely comprehensive (plurality-encoded) domain, due to there always existing additional objects that lie beyond the bounds of any such putative domain:

$$(No\ Comprehensive\ Domain): \forall xx \exists z \sim(z \prec xx)$$

Following Gabriel Uzquiano (2009), we'll call this claim *No Comprehensive Domain*. It is, of course, the negation of Comprehensive Domain: and, at least *prima facie*, it provides the opponent of Plural Comprehension with more plausible grounds to deny the latter than would a rejection of Homogenous Plurality$_{\varphi}$. For one thing, given its natural affinity with an indefinite-extensibility-based diagnosis of the paradoxes, No Comprehensive Domain is a natural choice for those who find this particular account of the paradoxes compelling; correspondingly, the view with which the principle is most associated – namely, *Quantifier Relativism* – counts among its adherents both philosophers committed to the Domain Principle (like Russell and Dummett) as well as those sympathetic to Collapse-type principles, like Charles Parsons, Øystein Linnebo, and James Studd.

However, despite this wide-ranging appeal, quantifier relativism faces an immediate and devastating objection – one we can tease out with recourse to the plural formulation of No Comprehensive Domain above. In order for this to be true, the range of its existential singular quantifier would need to be *strictly wider* than the range of the universal quantifier binding its plural variables; however, if we interpret the plural variables 'xx' as 'the zero or more objects xx,' it follows immediately that any object in the range of the singular bound variables *would fall within the range of the quantifiers governing the plural bound variables*. Put more simply: the means by which we testify to the existence of the witness that would make No Comprehensive Domain *true*, presupposes precisely the kind of quantification that, if legitimate, would render it false.

As Tim Williamson (2003) points out, the situation here is akin to the proponent of quantifier relativism attempting to express her own view, by means of the following claim:

(1) I am not quantifying over everything.

The worry, of course, is that (1) implies the obviously self-refuting sentence below:

(2) There is something I am not quantifying over.

Precisely the same difficulty in *expressing* quantifier relativism this points to, is emphasized by David Lewis (1990):

> Maybe the [relativist] replies that some mystical censor stops us from quantifying over absolutely everything without restriction. Lo, he violates his own stricture in the very act of proclaiming it! (p. 68)

James Studd (2019) dubs this cluster of worries for the quantifier relativist, the *Objection from Ineffability*; and while he is himself a committed proponent of the view, Studd concedes that the latter objection provides what are *prima facie* compelling grounds for its rejection. For what the *Objection from Ineffability* seems to reveal is that any attempt to *express* the view that quantification fails to achieve absolute generality, presupposes precisely the expressive resources (*qua* absolutely general quantification) that this view deems unacceptable.

§1.5

Plural Logic: Rejecting Collapse

It seems, then, that there are no easy options when it comes to rejecting Plural Comprehension. On the one hand, disavowing Homogenous Plurality$_{\varphi}$ amounts to denying what appears to be a self-evident truth; on the other hand, rejecting Comprehensive Domain threatens to saddle us with an unacceptable – because deeply mysterious – species of ineffability. Thus, given the joint inconsistency of Plural Comprehension and Collapse, there seems little to deter us instead from *accepting* the former and rejecting, in its place, the latter (and so, siding with Cartwright against Russell and Dummett).

Little, that is, aside from the *intuitive* appeal of Collapse – the fact that it seems, as Dummett puts it, to be 'a consequence of our intuitive understanding of the concept set.' Corroborating Dummett's claim, something very much like Collapse is suggested by another of Cantor's definitions of 'set,' this one from his *Contributions to the founding theory of transfinite numbers*[30]:

[30] *Beiträge zur Begründung der transfiniten Mengenlehre (*1895)

By a ‘set’ we understand every collection to a whole *M* of definite, well-differentiated objects *m* of our intuition or our thought. (Cantor, as quoted in Hallett, 1984, p. 33)

This definition – unlike the one we met earlier – omits any mention of these ‘definite’ elements’ being collected to a whole ‘*through a law*’ (which we’ve noted is evocative more of Naïve Comprehension, than of Collapse). However, this raises the pertinent question of whether denying the validity of Naïve Comprehension wouldn’t also give us grounds to reject Collapse (especially since both have the same paradoxical consequence, viz., securing the existence of sets like W).

That Cartwright endorses something along these lines is suggested by his arguing, in ‘Speaking of Everything’ (1994), that his straightforward response “does not involve essentially the concepts class and membership” (p.10). By way of substantiation, he notes that the definition (W), in addition to being an instance of Naïve Comprehension, is a variant of another scheme, which is also invalid:

$$(1): \exists y \forall x\ (Fxy\ \leftrightarrow \sim Fxx)$$

Here, taking the relation ‘F’ to be the ‘element of’ relation turns (1) into (W). Moreover, if we infer, from the inconsistency of (1), its negation:

$$(2): \sim\exists y \forall x\ [Fxy\ \leftrightarrow \sim Fxx]$$

and if we once more take ‘F’ to be the ‘element of’ relation, we obtain the statement that W does not exist:

$$(3): \sim\exists y \forall x\ [x\ \in y\ \leftrightarrow x\ \notin x]$$

which statement – by way of guaranteeing the existence of a plurality (specifically, of all non-self-membered sets) that does *not* comprise a set – contradicts Collapse.

However, things may not be quite as ‘straightforward’ as all this. As James Studd (2019) points out, the falsity of (W) may explain the inconsistency of *Naïve Set Theory*, which has Naïve Comprehension as an axiom; however, it leaves *unexplained* the *difference* between pluralities that form sets, and those that don’t. Consequently, Studd argues that the invalidity of Naïve Comprehension falls short as an e*xplanation* for the falsity of Collapse:

… the question of real interest is not why this instance of Naïve Comprehension yields a contradiction, but why certain sets—in this case, those that lack themselves as elements—are unable to form a set. And this cannot be explained merely by appeal to logical truths. (2013, p. 700)

Studd’s ‘question of real interest’ can fruitfully be framed in terms of Cantor’s two contrasting definitions of set. The question, specifically, is *not* whether every ‘law’ or property determines a set (which query would obviously be answered in the negative); rather, the question is *why some pluralities of 'definite, well-differentiated objects' fail to form sets.*

The importance of this line of enquiry lies in the fact that the appeal of Collapse extends, in many ways, beyond the intuitive. For instance, if we assume that any given set is 'definite' and 'well-differentiated,' then Collapse is not merely suggested by Cantor's second definition of set – rather, it seems to be a direct consequence of it. Furthermore, given precisely the same assumption, Collapse follows from what is probably the most well-known (and certainly, the most widely accepted) conception of set – the *iterative* conception – at least as it is characterized by Kurt Gödel in the following, famous paragraph:

> This concept of set ... according to which a set is something obtainable from the integers (or some other well-defined objects) by iterated application of the operation 'set of,' not something obtained by dividing the totality of all existing things into two categories, has never led to any antinomy whatsoever. (1964, p. 474–75)

However, as Jonathan Lear forcefully articulates in 'Sets and Semantics' (1977), Gödel's declaration that the iterative conception 'has never led to any antinomy whatsoever' is – on the assumption that all sets *are* well-defined objects – premature:

> There are two beliefs associated with the iterative conception of set that are apparently mutually inconsistent: (i) Given any well-determined objects, they can be collected together into a set by an application of the set of operation. (ii) There is no set of all sets. (p. 86)

Clearly, what is needed from the advocate of the iterative conception who *also* wishes to deny Collapse, is an account of just what makes an object – or, rather, a *plurality* of objects – 'well-defined': an account that, at the same time, explains why those pluralities for which Collapse fails, are *not* 'well-defined'.

One candidate account is what Charles Parsons (2005a) calls the 'genetic' explanation of the iterative conception of set, according to which "sets are taken to be 'formed' or 'constituted' from previously given objects, sets or individuals" (p. 269). Of course, this leaves open the question of just what being 'previously given' consists in; one suggestion – famously articulated by George Boolos (1971) – is to cash this out in terms of a metaphor of set 'formation' or 'construction.' Specifically, the suggestion is to think of Gödel's 'iterated applications of the 'set of' operation' as taking place according to a sequence of *stages*. In a universe without urelements (viz., a universe consisting only of pure sets) there exists, prior to the first stage, nothing at all; appropriately, then, the only set formed from what is 'previously given' at this stage (which is nothing at all) is the empty set. However, at subsequent stages, this set – now 'formed' – can be considered 'previously given': thus, at the immediately successive stage, we are free to obtain the *set* of the empty set (along with the empty set itself).

Having started thus, we go on, at each stage 'constructing' or 'forming' sets out of those 'formed' or 'constructed' at earlier stages. The resulting picture of the stages of set formation has these form a serial well-ordering (transitive, irreflexive, and linear[31]); moreover, if we are willing to countenance the existence of *limit* stages (viz., the *union* of all the preceding stages), [32] we can iterate the process of set formation into the transfinite (in fact, on the standard account of the resulting, *cumulative hierarchy*, each stage, or *rank,* is indexed by an ordinal).

This view of set formation, Boolos (1971) claims, "strikes people as entirely natural, free from artificiality, not at all ad hoc… one they might perhaps have formulated themselves" (p. 218). He further argues that it is entirely faithful to Cantor's second definition of set above:

> … when one is told that a set is a collection into a whole of definite elements of our thought, one thinks: Here are some things. *Now* we bind them up into a whole. *Now* we have a set. (p. 220)

Most important of all, Boolos's picture of set formation provides a *prima facie* plausible answer to Studd's question of why some pluralities of 'definite elements of our thought' fail to form sets. For if sets are constituted in line with the overall picture Boolos provides, this would entail, as Parsons (2005a) puts it, that "the elements of a set… [are] *prior* to the set." This relation of priority obviously precludes the possibility of a set's being an element of itself; however, if some pluralities – such as the plurality of all ordinals, or of all non-self-membered sets – *were* also to comprise sets, then these sets *would* be self-membered (and thus, *per impossible*, 'prior' to themselves).

Indeed, the claim at the heart of the iterative conception – viz., *every* element of a set must exist 'prior' to that set – provides additional reasons to reject the existence of such sets as Russell's W: for at no point in the cumulative hierarchy are all non-self-membered sets (or, indeed, all ordinals) 'previously given' in this way; and this is because, at any given rank in the hierarchy, it is always possible to go on to 'construct' more of them.

Ostensibly, then, the conceptual resources of the iterative conception – in particular, the notion of 'priority' that governs set formation – holds promise as a way of explaining why Collapse fails for certain problematic pluralities. Nevertheless (and despite this utility), precisely this notion – and specifically, the temporal metaphor underlying it – threatens to unravel the conception's own intuitive appeal. This, as Parsons (2005a) notes, is because cashing out the notion of being 'prior' (or 'previously given') in strictly temporal terms, is a non-starter (for one thing, any form of set construction in a time-like medium would, given the vast infinities of the transfinite[33], necessitate that the stages of such construction "be thought of as [taking place in] a kind of 'super-time'" (p. 273)).

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[31] See p. 223 of his (1971) for details.

[32] Each such *successor* stage is the union of preceding stage's sets, and its powerset.

[33]As we've already noted, the 'construction' or 'generation' of sets gives rise to a linear hierarchy, such that every set in the hierarchy can be assigned a *rank* that itself corresponds to an ordinal (whereby, the rank of a given set is the least ordinal

Related difficulties arise for just this metaphor of *construction* (which pervades the iterative conception). For any notion of construction presupposes a *constructor* – and if not us, then some Being (or Idealised Mind) who is responsible for carrying out the constructions in question. But, as Parsons pointedly contends, "it is hard to see what the conception of an idealized mind is that would fit here," given that:

> [Such a mind] would differ not only from finite minds but also from the divine mind as conceived in philosophical theology, for either the latter is thought of as in time, and therefore as doing things in an order with the same structure as that in which finite beings operate, or its eternity is interpreted as complete liberation from succession. (p. 273)

Parson's dilemma here is clear: either the 'Ideal Mind' posited by the temporally-inclined iterative theorist is bound by the same temporal limitations as Her non-ideal, finite counterparts (and is, consequently, equally incapable of iterating the 'set of' operation into the transfinite); or should She, on the other hand, stand in "complete liberation from succession," it seems irresistible to conclude that *all sets are 'previously given' to Her*, in just the sense required by the iterative conception – in which case, it is not clear what obstacle there should be to Her forming the set of them.

Conclusion

It is thus clear we cannot understand the iterative conception's notion of being 'previously given' in terms of *temporal* priority; however, the iterative theorist's rejection of Collapse – and so, her resolution of the paradoxes– hinges on precisely the idea that the contradictory pluralities are, at no point in the hierarchy, 'previously given' in the manner required to form a set.

Something, evidently, must be made of the notion.

Parsons (2005a), for his part, suggests that '[w]hat we need to do is to replace the language of time and activity by the more bloodless language of potentiality and actuality (p. 293) – in other words, we should replace the temporal notions predominating the iterative conception, with *modal* ones. Importantly – far from providing grounds for the rejection of Collapse – the resulting, *modal* conceptions of set instead present the possibility of *rehabilitating* it, by way of allowing for any plurality of (actual) objects to *potentially* comprise a set.

Of course, there remains the (justifiable!) worry that accepting Collapse in any form threatens to impale us on the second horn of Dummett's Dilemma (whereby we are forced to acknowledge that our bound variables fail to range simultaneously over all sets). But even

greater than the rank of all its elements). Obviously, this conception of the stages of the hierarchy presupposes the availability of the ordinals, and so threatens to devolve into circularity if the ordinals are themselves conceived of as sets – a point we'll touch on in greater detail later, in §1.2. Alternatively, if the ordinals are urelements, then it isn't clear why they shouldn't all be 'previously given' at rank 0, and so available for collection as a set: at this point, however, a version of Cantor's Paradox looms.

here, a modal account of set formation has the potential to avoid the worst excesses of the Objection from Ineffability that, as we saw in §1.3 , is the most serious objection for the proponents of Collapse. The suggestion, specifically, is that the limitations faced by our quantifiers[34] when it comes to expressing absolute generality might be overcome by the use of appropriately interpreted *modal operators* – which operators might, in turn, facilitate our generalising simultaneously over all sets.

An in-depth discussion of modal conceptions of set will, however, be one we take up only at the end of Chapter 2 (moreover, our overall assessment of this conception – and, specifically, whether it manages to answer the three key questions we've posed in our discussion so far – will have to wait until Chapter 4).

In the interim – and in view, specifically, of the choice between Plural Comprehension and Collapse we've contended is forced on us by the paradoxes – our focus will turn to the question of how the proponent of Cartwright's Claim (viz., the quantifier *absolutist*) might motivate his rejection of second principle.

Our discussion of this so far has focused on the difficulty of explaining the 'intuitive appeal' of Collapse, if indeed it is false; however, aside from calling into question our intuitions about the relationship between pluralities and sets, rejecting Collapse may also have potentially far-reaching – and, importantly, *negative* – consequences for quantification over all sets.

To a discussion of these issues – which will be the main focus of Chapter 2 – we now turn.

[34] Which limitations, we will argue in Chapter 4, are a function of their being subject to the Domain Principle.

Chapter 2

To be or not to be (a set): Collapse, and the Problem of Vagueness for Quantification Over All Sets

Introduction

In Chapter 1, we saw that the set-theoretic paradoxes essentially force on us the choice of rejecting (at least) one of two jointly inconsistent principles[35]: firstly, a principle of *Collapse*, according to which every plurality forms a set:

$$(Collapse): \forall xx \exists y (y \equiv xx)$$

and, secondly, a principle of *Plural Comprehension*, according to which every property, φ, determines a plurality of its instances:

$$(Plural\ Comprehension): \exists xx \forall y (y \prec xx \leftrightarrow \varphi(y))$$

Moreover, we saw that rejecting either one is naturally accompanied by a particular view regarding the implications of the paradoxes for *quantification*. On the one hand, a commitment to *Quantifier Absolutism* – viz., the idea that it is possible, in principle, for our classically-interpreted quantifiers to range over *absolutely everything* – is usually endorsed by those who *oppose* Collapse; for with rejecting the latter comes the freedom to affirm the following, key consequence of *Plural Comprehension*:

$$(Comprehensive\ Domain): \exists xx \forall z (z \prec xx)$$

which, in attesting to an all-encompassing plurality, simultaneously affirms the possibility of an absolutely comprehensive (plurality-encoded) *domain*.

On the other hand, endorsing the *negatio*n of Comprehensive Domain:

$$(No\ Comprehensive\ Domain): \forall xx \exists z \sim (z \prec xx)$$

emerges as a natural choice for *Quantifier Relativists* who are also sympathetic to Collapse: and indeed, for such thinkers as Dummett and Russell, this combination of relativism *and*

[35] Since together they entail the Naïve Comprehension Scheme:

$$(Naïve\ Comprehension): \exists x \forall y (y \in x \leftrightarrow \varphi(y))$$

Collapse is a straightforward consequence of their overall commitment to the Domain Principle – or, as Cartwright (1994) calls it, the All-in-One Principle:

> … to quantify over certain objects is to presuppose that those objects constitute a 'collection,' or a 'completed collection' – some one thing of which those objects are the members (p. 7)

Of course, the absolutist who opposes Collapse must also reject the Domain Principle (as, indeed, Cartwright does). However, independent of one's commitment to the latter, rejecting the former necessarily entails, in Cartwright's words, that "it is one thing for there to be certain objects," while "it is another for there to be a set… of which those objects are the members" (p. 8).

This, however, appears to undermine the idea – arguably a cornerstone of contemporary set theory – that a set *is completely characterized by its elements*; a claim Øystein Linnebo (2010) unpacks as follows:

> One aspect of this idea is that part of the nature of a set is what elements it has. In particular, two sets cannot be identical unless they have the same elements. Another aspect is the converse namely, that the nature of a set is exhausted by what elements it has (p. 149).

Evidently, the opponent of Collapse must reject the latter aspect of the overall claim; but this leads to the question of what – *in addition to its elements* – the nature of a given set consists in. In other words, the opponent of Collapse is faced with the following, key query:

> **Question 4:** What differentiates those pluralities that are subject to Collapse (i.e., form sets) from those that aren't?

Any proposed answer to Question 4 will naturally have implications for the Domain Principle, insofar as the 'actual infinities' to which the Principle attests are infinite *sets*; indeed, we will see in this chapter that the question of which potential infinities correspond to actual infinities is itself a version of Question 4 – viz., the question of *what differentiates those well-ordered sequences of ordinals that form sets from those that do not.*

This, of course, is just the question posed by the Burali-Forti Paradox – and one way around the difficulties it presents, would simply be to reject the existence of the actual infinite (*qua* infinite sets) altogether. Such (more or less) was the approach of Aristotle, whose own encounter with the paradoxes plaguing Ancient Greek thought on the infinite – notably, Zeno's Paradoxes of motion – led him to formulate the initial distinction between the potential and actual infinite, as well as the first (and most influential) arguments against the validity of Collapse-type principles.

It is therefore appropriate, given the focus of this chapter – viz., what distinguishes the pluralities that are subject to Collapse, from those that aren't – that we begin by canvassing

Aristotle's views on the matter; including a brief discussion of the paradoxes that prompted him to formulate these views. To this task, we now turn.

§2.1

Aristotle, Cantor, and the rehabilitation of the actual infinite

Perhaps the most famous of Zeno's paradoxes is that of Achilles and the tortoise. Bostock's version of the paradox runs as follows:

> Suppose we have a body (which we may call Achilles) which moves a finite distance from a starting point to a finishing point (which we may call the tortoise). Now the distance over which Achilles has to travel to reach the tortoise will contain infinitely many points… any infinite series of such points will mark out the distance into infinitely many discrete parts. Consequently Achilles has to traverse infinitely many parts of the distance, one after the other, before he has traversed the whole distance. But this is as much as to say that before he can reach the tortoise he must have completed a series of infinitely many different tasks… So Achilles cannot catch his tortoise after all. (1972 – 1973, p. 40)

At the heart of the paradox is the idea that Achilles must complete an infinite series of tasks (viz., traversing an infinite series of distances) – and, of course, that this is impossible to do. Aristotle firmly agreed with the latter intuition (a conviction that is reflected in his own, preferred definition of the infinite as "that which is impossible to traverse" (203b30), given in book III of the *Physics*). However, as Adrian Moore (2019) points out in his treatment of Aristotle, the latter was also committed to the claim that the distance Achilles has to traverse is 'infinite by division'[36]; and the obvious worry with Aristotle's endorsing *this*, alongside his preferred definition of the infinite, is that it would appear to commit him to the d claim that Achilles – in catching up to the tortoise – manages to traverse the untraversable.

Nevertheless, Moore notes that we can manage this tension by teasing out two senses of the phrase 'infinite by division':

i) There is a possible situation *s*, such that for every natural number *n*, the distance between Achilles and the tortoise is divided into more than *n* parts in s.
ii) For every natural number *n*, there is a possible situation *s*, such that the distance between Achilles and the tortoise is divided into more than *n* parts in *s*.

Endorsing the sense of 'infinite by division' accompanying (i) commits us to the existence of a possible situation where an *infinite* number of divisions of the distance between Achilles and the tortoise has been effected; and as Moore points out, if Aristotle *did* understand the 'infinite by division' in this way, this *would* amount to his believing that Achilles manages to

[36] Moore (2019) parses this the idea that "between any two points… there is a third" (p. 3).

‘traverse the untraversable’ (*qua* traversing each of the infinitely many distances in the series).

Moore (2019), however, argues that Aristotle would have found this sense of ‘infinite by division’ incoherent: this is because the series of divisions, in being present ‘all at once,’ would witness what Aristotle called the *actual infinite* – that “of which no part is outside… complete and whole.” But, as the latter famously claims in the *Physics*, the infinite is “*not* that of which *no* part is outside, but that of which some part is *always* outside” (206b33, emphasis mine).

This conception of the infinite as that for which “there is always another and another to be taken” (206a29) – which is far more aligned with the sense of ‘infinite by division’ accompanying (ii) – is what Aristotle called the *potential infinite*; and with it, comes his solution to the paradox. For, rather than committing Aristotle to the claim that Achilles traverses the untraversable, that the distance between the latter and the tortoise is *potentially* infinite (viz., ‘infinite by division’ in the sense of (ii)) simply entails – given a situation where this distance has been divided a *finite* number of times – there are always *more* divisions we could perform.

Importantly, both (i) and (ii) above presuppose the infinitude of the natural numbers; however, for Aristotle, these are themselves a paradigm of the potential infinite, in that there is *no end to the process of counting*. An important corollary is Aristotle’s rejecting the possibility of infinite *numbers*:

> … for number, or what has number, is countable, and so, if it is possible to count what is countable, it would then be possible to traverse the infinite. (204a34)

This same sentiment would be echoed centuries later in Leibniz’s insistence, in a letter to fellow mathematician Johann Bernoulli, that the notion of a ‘last number’ implies ‘a contradiction’:

> When it is said that there are infinitely many terms, it is not being said that there is some specific number of them, but that there are more than any specific number. (Leibniz, quoted in Levey, 1998, pp. 76–7)

Leibniz’s reasoning here would lead him to conclude that “an infinity of things is not one whole… there is no aggregate of them” (p. 86). To link all of this back to the question animating this chapter – and swapping out talk of ‘wholes’ for talk of *sets* – it is plausible that both Aristotle and Leibniz would conclude that what differentiates the pluralities that form sets from those that do not, is that the latter are *infinite*.

Such would be the received mathematical wisdom for nearly two centuries after Leibniz’s death; that is, until Cantor upended mathematical orthodoxy by asserting that, in addition to the *potential* infinite (viz., infinite well-ordered sequences), the *actual* infinite – specifically, infinite *sets* – are a legitimate object of mathematical study. This is a claim Cantor justifies not only by arguing for the *viability* of the actual infinite, but also by demonstrating the *mathematical fruitfulness* of marrying the latter with the potential infinite – a fact that becomes especially apparent when we consider his two principles of ordinal generation:

(1) if α is a (finite or transfinite) ordinal number, there exists a new ordinal number, $\alpha + 1$, which is the immediate successor of α.

(2) given any infinite sequence of increasing ordinal numbers, there is a new ordinal number, γ, that is the least ordinal to follow all of these – and which is their 'limit.'

Principle (1) clearly exemplifies the good-old-fashioned, Aristotelian conception of the potential infinite: for any successor ordinal obtained by applying principle (1) to some finite ordinal α will itself *only ever be finite.*[37] However, that it is possible (*contra* Aristotle) to 'traverse' the sequences generated by means of (1), is precisely what Cantor's *second* ordinal generating principle attests to: for the limit ordinal γ secured by principle (2) corresponds to the order-type (indeed, it measures the *length*) of the infinite sequence it follows.

Moreover, should we opt (in line with standard set-theoretic practice) to identify ordinals with the set of their predecessors, then securing the existence of this limit ordinal just is to secure the *infinite set* of the ordinals in the well-ordered sequence preceding it. In this way, the claim at the heart of principle (2) – viz., that any potential infinite corresponds to an actual infinite[38] – emerges as a testament to the *validity of Collapse for infinite pluralities*. Following Hallett (1984), we can make this implication clear by reparsing principle (2) along the lines below:

> (2') given any unending (i.e., potentially infinite) sequence of increasing ordinal numbers, there is a smallest infinite set (i.e., an actual infinite) containing them all – to which corresponds a unique, limit ordinal number.

The power of principle (2') lies, in Cantor's words, in its "giv[ing] us the ability to break through every barrier in the formation of the concept of real, whole numbers" (as quoted in Hallett, 1984, p. 58). For instance, (2') allows us to obtain the first ordinal *after* all the finite ordinals – the first countably infinite ordinal, ω[39] – and so, quite literally, count the countable. Furthermore, that (2') is applicable to *any* infinite sequence of ordinals is what allows us,

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[37] Indeed principle (1) accords entirely with Aristotle's characterization of the infinite as that which "is in virtue of one thing's constantly being taken after another – *each thing taken is finite*, but it is always one followed by another" (206a9, emphasis mine).

[38] Importantly, the claim that 'every potential infinite corresponds to an actual infinite' does *not* amount to the Domain Principle – for it omits any mention of this being a prerequisite of the *rigorous mathematical applicability* of the potential infinite. (This additional ingredient is also, more broadly, what differentiates the Domain Principle from Collapse).

[39] I will use ω to refer, interchangeably, to the first countably infinite ordinal, and to the set of all finite ordinals. (In contemporary set theory, these two notions are equivalent, given that an ordinal is identified with the set of its predecessors).

given the unending sequence of the countably infinite ordinals (ω, ω +1… ω + ω…), to obtain the first *uncountably*[40] infinite ordinal, ω_1 – and so on.

Indeed, the unlimited applicability of (2') not only allows us to extend the counting process arbitrarily far, by way of obtaining sets with well-order types of ever increasing, infinite *lengths*; it also facilitates our obtaining sets of ever increasing, infinite *sizes* (a fact Cantor exploited to great effect by deploying (2') in the construction of his 'number classes'[41], each of which correspond to a unique measure of infinite cardinality).

It is tempting, in view of all this, to conclude that Cantor's principal innovation lies only in his rehabilitation of the actual infinite (as encapsulated in the unrestricted applicability of (2'), and the principle of Collapse underlying it). However, the mathematical fruitfulness of the actual infinite is chiefly found *in concert* with the potential infinite[42]: for it is only by working *together* that principles (1) and (2') deliver the (actually) infinite sets that, in the words of W. D. Hart (1975), "make set theory a paradise" (247).

That these infinite sets can, moreover, be proven to stand in determinate mathematical relations[43] to one another, not only justifies Cantor's assertions that they are "just as capable of being grasped by our restricted understanding as is the finite in its way" (as quoted in Hallett, 1984, 14)[44] – it also further motivates the applicability of (2') to ever more potential infinites, viz., unending sequences of ordinals generated by means of principle (1).

§2.2

The Burali-Forti Paradox and Cantor's Absolute

There is, however, one unending, potentially infinite sequence of ordinals to which (2') does not apply, on pain of paradox. That is potentially infinite sequence is the ordinal sequence in its *entirety*; and the paradox, of course, is Burali-Forti's.

Of course, one way to *dissolve* the paradox would be to deny that the ordinal sequence as a whole is subject to principle (2'); however, as we saw in the last section, it is precisely the purportedly unlimited applicability of (2') that 'makes set theory a paradise.' Countenancing an exception to (2') thus calls into question its validity when it comes to *other* potentially infinite sequences of ordinals; indeed, since the applicability of (2') in these cases depends crucially on whether the resulting pluralities are subject to Collapse, the question of the former's validity ends up boiling down to the question we posed in the Introduction:

[40] The first 'uncountable' ordinal is the least ordinal greater than all the countable ordinals.
[41] These are the (initial) ordinals obtained by an application of (2') to a given, potentially infinite sequence of ordinals generated by means of principle (1). For details, see Hallett (1984), p. 70.
[42] Thus, Zermelo claimed that Cantor's genius lay in his marrying the 'two opposite tendencies of the thinking spirit… the idea of creative advancement and that of collective completion' (as quoted in Ebbinghaus and Kanamori, 2010, p. 431)
[43] The transfinite ordinals, for instance, can be shown to share a common arithmetic with their finite counterparts.
[44] A crucial idea at play here is the notion – which Hallett (1984) dubs Cantor's principle of 'finitism' – that the transfinite is, in an important sense, 'just like the finite.'

Question 4: What differentiates those pluralities that are subject to Collapse (i.e., form sets) from those that aren't?

Cantor's own attempt to furnish us with an answer to this – as well as the question of the validity of (2') – hinges on what Hallett (1984) calls his doctrine of the Absolute. The basic idea is that the realms of the infinite are not exhausted by the transfinite; there exists a further infinite – the Absolute – which, unlike the transfinite, is not amenable to mathematical determination.

Notably, references in Cantor's work to the Absolute can be found alongside his earliest definition of set[45], and his construction of the number classes in his '*On Infinite, Linear Point-Manifolds [Sets]*'[46]; indeed, in the extract below, his aim is precisely to distinguish the infinite *sets* of the transfinite, from the Absolute:

> I fix conceptually once and for all the various levels of the proper [i.e. actual] infinite through the number-classes... I have no doubt at all that in this way we extend ever further, never reaching an insuperable barrier, but also never reaching any even approximate comprehension of the Absolute. The Absolute can only be recognized, never known, not even approximately. (Cantor, as quoted in Hallett, 1984, p. 42)

Later on in the same passage, Cantor describes the sequence of number classes (equivalently, the ordinal sequence as a whole) as, itself, an "absolutely-infinite totality…. an absolutely-infinite sequence":

> … the absolutely infinite sequence of numbers therefore seems to me in a certain sense a suitable symbol of the Absolute. Whereas, hitherto, the infinity of the first number-class [ω] alone has served as such a symbol, for me, precisely because I regarded that infinity as a tangible or comprehensible idea, it appeared as an utterly vanishing nothing in comparison with the absolutely infinite sequence of numbers. (Cantor, as quoted in Hallett, 1984, p. 42)

That the ordinal sequence as a whole symbolizes the Absolute is crucial for Cantor's proposed dissolution of the Burali-Forti Paradox: the key assumption underlying the latter is the idea that we can *numerically determine* the sequence of ordinals in its entirety (*qua* using (2') to obtain the limit ordinal corresponding to its well-order type). But, if this sequence represents the Absolute – and if the Absolute is *not* amenable to numerical determination – it follows that the ordinal sequence as a whole is an exception to (2') (and this, naturally, suffices to dissolve the paradox).

[45] 'Any many [*Viele*] which can be thought of as one [*Eines*]… every totality of definite elements which can be united to a whole through a law' (Cantor, as quoted in Hallett, 1984, p. 32).

[46] *Über unendliche, lineare Punktmannigfaltigkeiten* (1883)

Of course, this implies that the ordinal sequence as a whole is also not subject to Collapse. Cantor, however, accepted this implication wholeheartedly: indeed, in a series of letters to Richard Dedekind– and in the time-honoured tradition of mathematicians' turning paradoxes into proofs – Cantor proposed that the Absolute be used as a *criterion* for distinguishing those pluralities that are subject to Collapse (which he calls 'consistent' multiplicities), from those that aren't (and which are, correspondingly, 'inconsistent' multiplicities):

> … a multiplicity can be such that the assumption that *all* of its elements "are together" leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as "one finished thing". Such multiplicities I call *absolutely infinite* or *inconsistent multiplicities.*
>
> If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as "being together", so that they can be gathered together into *"one* thing", I call it a *consistent multiplicity* or a "set." (Cantor, as quoted in Van Heijenoort, 1967, p. 114)

Cantor establishes the link between the Absolute and 'inconsistent multiplicities' precisely by means of the Burali-Forti Paradox: specifically, by deploying an early version of the Axiom of Replacement[47], he argues that a given multiplicity is inconsistent if and only if it has a cardinality equivalent to (or greater than) the ordinal sequence. Pleasingly, Øystein Linnebo (2010) has shown that we can mimic Cantor's reasoning here – and so, derive an analogue of his result – in PFO. For if we assume a *Principle of Cardinal Comparability* along the lines below (p. 151):

$$(Cardinal\ Comparability): \forall xx \forall yy(|xx| < |yy| \ \vee \ |xx| \geq |yy|)$$ [48]

where '$|xx| < |yy|$' abbreviates 'the xx are strictly less numerous than the yy,' (and '$|xx| \leq |yy|$,' that 'the xx are either less numerous to, or as numerous as, the yy'), then we can, by means of the relation '$\leq$,' formulate a *plural* version of the Axiom of Replacement:

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[47] The Axiom of Replacement states that any multiplicity that is the *same size* as a set, is itself a set – analogously, in his letter to Dedekind, Cantor stipulates that "[t]wo equivalent multiplicities either are both 'sets' or are both inconsistent."

[48]This notation is due to Studd (2019). As he notes, we can state that a plurality, the xx, is at least as big as another plurality, the yy ('$|xx| \geq |yy|$') when there exists a plurality-encoded function mapping the xx *onto* the yy; in turn, we can say that the xx are strictly less numerous than the yy ('$|xx| < |yy|$') when '$|yy| \geq |xx|$' holds, but not vice versa. Finally, we state that two pluralities, the xx and the yy, are the same cardinal size when both '$|xx| \geq |yy|$' and '$|yy| \geq |xx|$' are true. Given this, Linnebo's Principle of Cardinal Comparability amounts to the claim that any two pluralities – the xx and the yy – are such that the xx are either i) less numerous than the yy, or ii) at least as numerous as, or *greater* in number than, the yy.
Studd points out that, given this formulation, Cardinal Comparability amounts to a strong Choice assumption; in his own discussion, Linnebo (2013) notes that the principle follows from the assumption of Global Well-Ordering (according to which there are some ordered pairs that form a well-ordering of the universe). He notes, further, that "[w]riters on plural logic and set theory tend to be implicitly committed to GWO" (p.161), due to their commitments not only to Choice but to reflection principles (according to which any property of the set-theoretic universe is 'reflected' in some set-sized sub-universe). These commitments jointly compel GWO due to the idea that, if GWO failed, then this would 'reflect down' into the universe of sets, and produce a counterexample to the claim, which follows from Choice, that any set can be well-ordered.

$$(Plural\ Replacement): \forall xx \forall yy[(\exists u(u \equiv xx) \wedge |yy| \leq |xx|) \rightarrow \exists v(v \equiv yy)]$$

With this in place, Linnebo is able to derive the following, plural version of Cantor's claim that a multiplicity is consistent if and only if it is of a strictly smaller cardinality than the ordinals[49] (denoted by the plural variable 'oo'[50]):

$$(OO): \forall xx(\exists u(u \equiv xx) \leftrightarrow |xx| < |oo|)$$

The left-hand side of the above biconditional is none other than Collapse itself; and indeed, we can think of (OO) as capturing Cantor's answer to Question 4 (viz., what distinguishes those pluralities that are subject to Collapse from those that aren't).

That the proposed criterion specifically boils down to the relative *cardinality* of these pluralities, in turn positions Cantor's doctrine of the Absolute as the intellectual forerunner to classic limitation of size doctrines – and while these are standardly associated with the work of John Von Neumann[51], the phrase 'limitation of size' was, in fact, first coined by Russell. Indeed, it appears in the same (1905) paper where the latter introduces his notion of 'self-reproductive processes' – the underlying logical form of which Russell distils in the conditional below (where ϕ is the 'self-reproductive,' or indefinitely extensible, property under study):

$$(*): \forall u\ [\forall x\left(x \in u \rightarrow \phi(x)\right) \rightarrow \left(f(u) \notin u \wedge \phi\left(f(u)\right)\right)$$

We saw in Chapter 1 that Russell characterises the 'self-reproductive processes' profiled by $(*)$ as "essentially incapable of terminating," where this in turn makes it "natural to suppose that any aggregate embracing all the terms generated by one of these processes cannot form a class." However, this particular failure of Collapse has, for Russell, specific implications in connection with notions of *limitation of size:*

> … Consequently… there will be a certain limit of size which no class can reach; and any supposed class which reaches or surpasses this limit is an improper class, *i.e.,* is a non-entity. (1905, p. 43)

Indeed, just as the difference between Cantor's 'consistent' and 'inconsistent' multiplicities – as captured by (OO) above – boils down to whether the multiplicity in question is of a smaller cardinal size than the *ordinal* sequence, so too does Russell suggest, with reference to $(*)$, that:

> It is probable, in view of the above general form for all known contradictions, that, if ϕ is any demonstrably non-predicative property[52],

[49] See Linnebo (2010, p. 162) for a proof sketch.
[50] Linnebo (2010) points out that the assumption that there exists a plurality 'oo' of ordinals is one to which the opponents of Collapse are "firmly committed" (p. 152).
[51] Specifically, his (1925) axiomatization of set theory.
[52] 'Non-predicative,' in the context of Russell's (1905) paper, refers to those concepts that do not determine sets.

we can actually construct a series, ordinally similar to the series of all ordinals, composed entirely of terms having the property ϕ. (p. 36).

With his talk of 'constructing a series ordinally similar to the ordinals,' Rusell means that should be possible, given a putatively self-reproductive property ϕ, to construct a 1-1 mapping – call this G – from the ordinals to the ϕs; and indeed, Russell effectively[53] proceeds to construct such a mapping, by defining G as follows:

$$G(\gamma) = f(\{G(\beta): \beta < \gamma\})$$

where f is the function in $(*)$, and $G(0)$ is some arbitrarily chosen set of ϕs (which, by virtue of only containing ϕs, would itself have ϕ, and so be defined for f).

The more general upshot of all this is the clear link that exists for Russell, as for Cantor, between the ordinal sequence – viz., the Absolutely infinite – and the *failure of Collapse* accompanying the paradoxes; in a sense, what is revealed is that, for these thinkers, the contradictions are all *variations on the Burali-Forti Paradox*.

A similar connection – as well as a link between Cantor's Absolute, and Dummett's notion of *indefinite extensibility* – has been noted by contemporary philosophers Crispin Wright and Stewart Shapiro, who in 'All Things Indefinitely Extensible' (2006) argue that we should take "the ordinals to be the paradigm case of an indefinitely extensible totality, and the mechanics of the Burali–Forti paradox to be the paradigm of indefinite extension" (p. 259). In that same paper, the authors draw on none other than Russell's $(*)$ in providing their own formal account of what the indefinite extensibility of a concept P consists in, which I've summarised below:

(Δ) Given a concept P of objects of type τ, and a concept of concepts of type Π, then P is *indefinitely extensible* if and only if there is a function F, from items of the same type as P to items of type τ, such that if X is any sub-concept of P, and $\Pi(X)$, the following three conditions are fulfilled:

i. $F(X)$ falls under the concept P
ii. It is not the case that $F(X)$ falls under X
iii. $\Pi(X')$, where X' is the concept instantiated just by $F(X)$ and every item which instantiates X

For instance, where P is the concept of 'non-self-membered set, Wright and Shapiro stipulate that, for a given sub-concept X of P, $\Pi(X)$ if and only if the Xs form a set, while F is itself the 'set of' operation (correspondingly, $F(X)$ is the set of the Xs). Given this, condition (i) amounts to the claim that $F(X)$ is *itself* a non-self-membered set, while condition (ii) entails that $F(X)$ is not a member of itself; finally, condition (iii) entails that the totality (viz., the

[53] This simplified reconstruction is due to Hallett (1984) – see p. 36 of Russell's (1905), for the latter's full explanation.

instances of X') comprising $F(X)$, and all the non-self-membered sets it contains, itself forms a set.

Given this, the parallels between (Δ) and $(*)$ soon become clear:

$$(*): \forall u\ [\forall x\ \big(x\ \in u\ \rightarrow\ \phi(x)\big) \rightarrow \Big(f(u) \notin u\ \wedge\ \phi\big(f(u)\big)\Big)$$

For the concept P and the function F in (Δ) evidently correspond, respectively, to the property ϕ and the function in *f* in $(*)$. Moreover, the second half of the conjunction that is $(*)$'s consequent (viz., '$\phi\big(f(u)\big)$') is clearly equivalent to condition (i) of (Δ) – for instance, in Russell's Paradox, both entail that the relevant object is itself a non-self-membered set. Similiarly, the first half of the conjunction is evidently equivalent to condition (ii) (to use again the example of Russell's Paradox: both entail that the Russell set is not an element of itself).

Furthermore: just as, in $(*)$, the function f is defined not just for a given term in the series (so, for an arbitrary ϕ set) but for *collections* of these terms – specifically, *sets* of ϕ sets – so too, in (Δ), is the function F defined not just for a given sub-totality (the Xs) of Ps, but also for the same totality *as enlarged by the addition of* $F(X)$. This is the outcome ensured by condition (iii); and the resulting sequence generated through repeated applications of F –

$$X, \{X\}, \big\{X, \{X\}\big\} \ldots$$

– is precisely analogous to the series, itself 'ordinally similar to the ordinals,' obtained by means of Russell's mapping G.

For their part, Wright and Shapiro intend this series to capture the 'increasing sequence of extensions' that Dummett famously associated with indefinitely extensible concepts. Indeed, just as the latter (1991) diagnoses the inconsistency accompanying a given indefinitely extensible concept – like 'set' or 'ordinal' – as a function of our "assuming its extension to constitute a definite totality" (p. 318), so too does paradox arise, in the case of Wright and Shapiro's (Δ), when we assume that P *itself has the property* Π. For this ensures that P is defined for F, and thus that $F(P)$ both is and is not a P (as per conditions (i) and (ii), respectively).

Of course, where F is the 'set of' operation, this paradox-inducing assumption amounts to nothing less than the idea that the Ps *comprise a set*. This in turn uncovers another point of commonality between Dummett and Cantor. For as revealed by his letters to Dedekind, Cantor takes the origins of paradoxes to be our mistakenly treating Absolutely infinite (*qua* 'inconsistent') multiplicities as 'consistent' – that is, as subject to *Collapse.* However, if Wright and Shapiro are correct to take the ordinals as 'the paradigm… of an indefinitely extensible totality,' then this emerges as identical to Dummett's admonition not to 'assume' that the extension of an indefinitely extensible concept is 'a definite totality.'

In sum: for both Dummett and Cantor, the cost of our treating Cantor's Absolutely infinite multiplicities as 'consistent' – or Dummett's indefinitely extensible totalities as 'definite' – is paradox.

§2.3

Relative and Absolute Indefinite Extensibility

In view of their many points of convergence (which we canvassed in Chapter 1), it is not entirely surprising that Cantor would agree with Dummett in this instance too. However, what *is* surprising about this particular point of agreement, is that it re-aligns Cantor with the Aristotelian-scholastic understanding of infinity he is widely viewed as having *rejected.*

For the reader will recall, again from Chapter 1, that we opted to cash out Cantor's notion of *potential infinity* in terms of indefinite extensibility; this, in turn, means that a natural way of cashing out the idea that Cantor's Absolute is indefinitely extensible, is as the claim that the latter is *irreducibly potentially infinite*[54]. But – if *this* is correct – then Cantor's directive not to treat Absolutely infinite multiplicities as 'consistent' (or, equivalently, 'indefinitely extensible totalities' as 'definite') is revealed as precisely the Aristotelian injunction not to treat *potential* infinites as *actual.*

Importantly, Cantor's denying that there exists an actual infinite corresponding to this *particular* potential (Absolute) infinite, amounts to far less wide-ranging a rejection than Aristotle's[55]. It *does,* however, mean that Cantor is himself faced with a version of Question 4: namely, *what distinguishes the potential infinities that do correspond to actual infinities, from those that do not.*

Happily, we can get a handle on this question – and the distinction between potential infinites to which it points – by means of Wright and Shapiro's (2006) analysis of indefinite extensibility: indeed, the relevant distinction is analogous to one they themselves formulate, using the conceptual resources of (Δ), between a concept's being indefinitely extensible *proper*, and its being merely *extensible-up-to-an-ordinal-λ*.

Key to this distinction is the function F in (Δ), which Wright and Shapiro identify with "the principle of extendability" (1991, p. 317) Dummett describes as giving rise to the new instances of a given, indefinitely extensible concept. And, just as Dummett claims that our identifying these instances (as in the paradigm case of his intuitive ordinal) flows from our forming a *definite conception* of the corresponding totality, so too, as we saw in §2.2, is F

[54] Wright and Shapiro (2006, p.279) attest to the plausibility of identifying the potential infinite with the indefinitely extensible That Cantor himself should be read – at least in his later work – as advancing an understanding of *the Absolute* as irreducibly potential, is an idea that has received support from some of his commentators: e.g., Ignacio Jané (1995, p.389).
[55] As Aristotle puts it in his *Physics*: "[t]he infinite, then, *is in no other way*, but is in this way, potentially" (205b12, emphasis mine).

defined only for 'definite' sub-concepts of P (viz., those Xs that collectively possess the property Π[56]).

All this is quite in line with Russell's $(*)$, where *f* is similiarly defined only for *sets* of ϕs; where Wright and Shapiro *expand* on Russell's analysis, though, is in distinguishing those cases where *every* sub-totality of Ps is 'definite' (*qua* instantiating Π) from those cases where *not* every sub-totality of Ps is 'definite'[57].

In the former case, that every sub-totality of $Ps,$ including those obtained by means of F, is 'definite,' entails that there is no "ordinal limit on the length of the series of Π-preserving applications of F" to sub-totalities of Ps (p. 279). Put (a great deal) more simply: that every sub-totality of Ps is 'definite' entails that every such totality is subject to the principle of extendability (viz., the function F), *and* that this process of applying F can be iterated without (ordinal) limit. In the paradigm case of the ordinals, this just means that any candidate totality of ordinals can itself comprise a set, and that the resulting totality (consisting of this set together with its elements), can be well-ordered (viz., itself assigned an ordinal).

Importantly, Wright and Shapiro note that in the case of these concepts – which they call 'indefinitely extensible *proper*, but which we'll refer to as *absolutely* indefinitely extensible – the fact that *every* sub-totality of Ps is 'definite' gives us "powerful intuitive reason to regard P itself as having the property Π" (p. 269). However, just this 'intuition' is what gives rise to the paradoxes associated with the (absolutely) indefinitely extensible: for instance, where P is the concept 'non-self-membered set,' that *the* totality of Ps instantiates Π, entails that this totality *itself forms a set* – and the inconsistency this engenders is, of course, Russell's Paradox.

Consequently, concepts that are *absolutely* indefinitely extensible (or rather, the pluralities' comprising their instances) are *not* consistently subject to Collapse; things, however, are quite different for Ps that are merely extensible up-to-an-ordinal-λ (henceforth, *relatively indefinitely extensible*); as an example, Wright and Shapiro point to the finite ordinals, which they analyze, in terms of (Δ), along the lines below:

> **Finite ordinals:** $P(x) \leftrightarrow x$ is a finite ordinal; $\Pi(X) \leftrightarrow$ the Xs are finite; $F(X) =$ the successor of the largest X.

It is immediately clear why the concept 'finite ordinal,' on this analysis, is only *relatively* indefinitely extensible, viz., there being a *limit* to the Π-preserving applications of F to

[56] For instance, in the case of Russell's Paradox, F is the 'set of' function, and is defined for a sub-concept X of P if and only if $\Pi(X)$, where $\Pi(X)$ is true if and only if the Xs *form a set*).
[57] In the case of Russell's $(*)$, no such distinction exists, given that the sorts of collections it countenances are sets, which are 'definite' by default.

totalities of finite ordinals. The exception in question emerges with *the* totality of finite ordinals; and that this doesn't have Π, is because it is *not itself finite*.

This explains, for Wright and Shapiro, why the concept 'finite ordinal' – and other relatively indefinitely extensible concepts – are comparatively paradox-free; in particular, our being comparably *unwilling* to ascribe Π to the relevant pluralities is what accounts for the latter's being consistently subject to *Collapse.* To link all of this back to Question 4 (or, the version of it we are considering here, viz., which potential infinities correspond to actual infinities): Wright and Shapiro would likely reply that the potential infinites in question are just the extensions of concepts that are *relatively indefinitely extensible.*

However – and importantly – it is not entirely clear that this way of accounting for the consistency of relatively indefinitely extensible concepts *is* explanatory. For one thing, that the totality of finite ordinals is not itself finite seems, at least *prima facie,* unrelated to this totality's being consistently subject to Collapse[58]. More pressingly, Wright and Shapiro's analysis relies essentially on Π's being a "relativized notion" (p. 266): for instance, Π denotes the property of being finite in the case of the finite ordinals – but this differs not only from the case of *absolutely* indefinitely extensible concepts (where Π denotes sethood), but also from *other* relatively indefinitely extensible concepts[59].

The worry is that a relativized notion of 'definiteness' hardly seems satisfactory as part of an account of *Dummett's* notion of indefinite extensibility: as our discussion in Chapter 1 made clear, Dummett by all accounts employs a single notion– that of *sethood* – throughout his writings. Consequently, a more satisfying account of indefinite extensibility – and of the difference between concepts that are absolutely, as opposed to relatively, indefinitely extensible – would arguably employ a uniform notion of 'definiteness' throughout.

Fortunately, such an account soon suggests itself, if we attend to the details of where the principle of extendability governing a *relatively* indefinitely extensible concept fails to kick in. We've seen that this occurs when, at some point in the series of iterated applications of F, a totality of Ps is obtained that is no longer Π, and consequently, no longer subject to F. However, that F is undefined for a particular totality of Ps just means that, at that stage in the generative process, *no 'new' Ps are produced.* To use the example of the finite ordinals: at some stage in the process of obtaining greater and greater totalities of finite ordinals by means of F, we will eventually arrive at a totality that *is itself no longer finite*.[60]

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[58] However, we will see a bit later on that there *is* a link between our being unwilling to ascribe the property 'finite' to the totality of finite ordinals, and their being subject to Collapse: for our willingness to countenance the existence of ordinals *beyond* the finite (which, in turn, allows us to designate the cardinality of this totality as something other than finite) *is* what allows us to treat the totality of finite ordinals as a set.

[59] Such as the concept 'countably infinite ordinal,' where Π, on their analysis, denotes being 'countably infinite'

[60] Analogously, in the case where P is the concept 'countably infinite ordinal' (and where the principle of extendability, similarly to the case of the finite ordinals, involves obtaining the successor of the union of a totality of Ps): the claim that we will eventually arrive at a totality that no longer has the property Π really amounts to nothing more than the idea that, at some point in the repeated applications of F to totalities of Ps, we will obtain a totality that is *no longer countably infinite* – that is, no longer a P.

Notably, that the relevant principle of extendability – in that case, f – will eventually fail to generate *new* instances, is what explains why *Russell's* $(*)$ fails to characterise such concepts as 'finite ordinal':

$$(*): \forall u\, [\forall x\, (x\, \in u\, \rightarrow\, \phi(x)) \rightarrow \big(f(u) \notin u\, \wedge\, \phi\big(f(u)\big)\big)$$

This is because $(*)$ only characterizes a concept ϕ when every entity got by applying f to a given set of ϕs, is itself a ϕ. However, where ϕ is the concept 'finite ordinal' – and where f is the identity function – it is *not* the case that every set of finite ordinals, itself is a finite ordinal (the notable exception's being the set of *all* finite ordinals, which is the first transfinite ordinal).

In sum: the reason why $(*)$ fails to characterise such concepts as 'finite ordinal' is because, at some point in the process of applying f to sets of ϕs, an entity – $f(u)$ – will be obtained that is not *itself* a ϕ. But this, I contend, is also a (more) natural way to make sense of Wright and Shapiro's distinction between absolute versus relatively indefinitely extensible concepts: what differentiates the former from that latter is the fact that the relevant principle of extendability *never fails to produce new instances of the concept in question* .

This way of accounting for the distinction also holds the promise of allowing us to adopt a uniform notion of 'definiteness,' namely *sethood:* for if those totalities of ϕs that are *not* 'definite' merely comprise *pluralities* – and if, as per Wright and Shapiro's initial analysis, the relevant principle of extendability kicks in only for 'definite totalities,' *qua* sets, of ϕs – then we can incorporate the insight captured by this distinction, together with the other key features of (Δ), into the following *plural* version of Russell's $(*)$:

$$(Plural\, *): \forall xx\, [\big(\forall y(y \prec\, xx\, \rightarrow\, \phi(y))\, \wedge\, \exists u(u\, \equiv xx)\big) \rightarrow \big(f(u) \notin u\, \wedge\, \phi\big(f(u)\big)\big)]$$

In $(Plural\, *)$, the distinction between 'indefinite' and 'definite' totalities emerges in the antecedent; specifically, we see that it is only when a plurality of ϕs is also 'definite' (i.e., forms a set) that the relevant principle of extendability delivers the new object, $f(u)$, that is a further instance of ϕ.

Moreover, unlike $(*)$ – where the only assumption required to engender paradox was the existence of a set $W = \{x \mid \phi(x)\}$ – in the case of $(Plural\, *)$, *two* such assumptions are needed. The first is a claim we first met in Chapter 1, *Comprehensive Plurality*$_\phi$:

$$(Comprehensive\ Plurality_{\phi}): \exists xx \forall z(\phi(z) \rightarrow z \prec xx)$$

This secures the existence of an all-encompassing plurality of ϕs; in turn, given the second assumption – Collapse – this entails the existence of a corresponding *set* comprising all the ϕs; this, of course, leads to paradox for much the same reasons that assuming the existence of W does in the case of $(*)$.

In sum: we can think of $(Plural\ *)$, like $(*)$, as providing a logical profile of indefinitely extensible concepts. However, that *two* assumptions are required to engender paradox in the former case, facilitates a more fine-grained analysis: indeed, it transpires that we can track the distinction between absolute and relative indefinite extensibility, in terms of which of these assumptions are retained, and which discarded.

For instance, in the case of the concept 'finite ordinal,' we are typically willing to retain both Collapse, *and* the idea that there exists a comprehensive plurality of finite ordinals; in turn., what prevents the emergence of paradox[61] is the fact that the set of finite ordinals obtained by means of Collapse is *not itself a finite ordinal* (viz., not a ϕ).

However, when it comes *absolutely* indefinitely extensible concepts, this strategy seems unavailable: given the concept 'ordinal,' for instance, it seems that any set corresponding to a plurality of ordinals *must itself be an ordinal*.

Nevertheless, for those – like Richard Cartwright – who wish to countenance a comprehensive plurality of ordinals, there remains the option of *denying that this plurality forms a set*; that is, there remains the option of *rejecting Collapse.*

As we saw in Chapter 1, just this is what Cartwright opts to do when faced with the paradoxes; however, we also saw there that an alternative would be to *deny the validity of the Plural Comprehension Scheme*:

$$(Plural\ Comprehension): \exists xx \forall y(y \prec xx \leftrightarrow \phi(y))$$

The analysis afforded by $(Plural\ *)$ further buttresses this alternative response. For the left-to-right direction of Plural Comprehension is precisely Comprehensive Plurality$_\phi$; and since it is only in tandem with *this* assumption that Collapse engenders inconsistency, the possibility emerges – when faced with a concept profiled by $(Plural\ *)$ – that we might instead reject the former claim[62].

But, of course, if Comprehensive Plurality$_\phi$ *is* false – that is, if no plurality includes *all* ϕ*s* – it follows that no plurality comprises every *thing*. Thus, from the falsity of Comprehensive Plurality$_\phi$ there soon follows the negation of *Comprehensive Domain*:

$$(No\ Comprehensive\ Domain): \sim\exists xx \forall z(z \prec xx)$$

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[61] Indeed, in that case, $(Plural\ *)$ will *fail* to characterise the concept 'finite ordinal,' given that the latter will satisfy the former's antecedent, but not its consequent.

[62] Indeed, given a ϕ profiled by $(Plural\ *)$, rejecting Comprehensive Plurality$_\phi$ is the outcome of the following, quite straightforward chain of reasoning: if there corresponds to each set u of ϕs a further object, $f(u)$, that is both an instance of ϕ and which lies outside of u – but if, additionally, all pluralities of ϕs form sets – then it follows that *no plurality of* ϕs *includes all* ϕs.

But of course, the truth of this claim is, for Russell and Dummett, not only a general consequence of the paradoxes, but more specifically an upshot of the kind of indefinite extensibility (or 'self-reproductive processes') we've discussed in this section.

To sum up: it is clear that, in differentiating between relative and absolute indefinite extensibility, $(Plural\ *)$ additionally permits a more fine-grained analysis of the choices we face in our encounter with the paradoxes accompanying the latter. This it achieves, specifically, by throwing into sharp relief the relative commitments of those who oppose Collapse – and for whom the primary implications of the paradoxes concern *which pluralities comprise sets* – versus those who reject *Plural* Comprehension, and for whom the key lesson of the paradoxes is that *our quantifiers fail to range over a comprehensive domain.*

§2.4

The Objection from Vagueness

Of course, for all that this fine-grained analysis does achieve, this does not extend to providing a clear answer to the question animating this chapter:

> **Question 4:** What differentiates those pluralities that are subject to Collapse (i.e., form sets) from those that aren't?

Nevertheless, $(Plural\ *)$ does succeed in *sharpening* it: for if the pluralities that are consistently subject to Collapse are just the extensions of *relatively* indefinitely extensible concepts, then we can frame the same question in terms of what differentiates the latter from its absolutely indefinitely extensible cousin. And here, at least, the beginnings of an answer are forthcoming: as we saw in the previous section, what distinguishes concepts that are only *relatively* indefinitely extensible, is the fact that the relevant principle of extendability will eventually fail to produce new instances of the concept in question.

However, what makes this only the *beginnings* of an answer, is our lacking an account of the *mechanisms behind this failure*; until such an account is provided, any attempt to answer Question 4 in terms of the distinction between relative and absolute indefinite extensibility, will be incomplete.

This, of course, is chiefly a concern for the opponent of Collapse – however, her troubles do not end there. For in addition to sharpening our sense of Question 4, the more fine-grained analysis afforded by $(Plural\ *)$ also succeeds in making it *more pressing*. This is because the very idea that we *need* an account of the difference between pluralities that comprise sets and those that don't, assumes that the source of the paradoxes is Collapse; however, one upshot of our discussion in §2.3 was that we might instead attribute these to Comprehensive Plurality$_{\phi}$ (and, more generally, the Plural Comprehension Scheme).

This has the effect of shifting the burden of proof onto the opponents of Collapse. For the question is no longer (merely) what accounts for the fact that some pluralities are subject to

Collapse, while others aren't; rather, the issue is now whether, when faced with the paradoxes, Question 4 is *the right question to ask in the first place.*

Of course, one way of motivating an affirmative response to *this* query, would be to provide a principled answer to Question 4 (viz., providing reasons – independent of the inconsistencies – to endorse a distinction between pluralities and sets). Moreover, should the dividing line between sets and pluralities indeed boil down to the distinction between relative and absolute indefinite extensibility, the resulting account would almost certainly involve some form of *limitation of size*.

However, besides its being unclear to what extent limitation of size criteria *are* independent of considerations of paradox, such accounts face another pressing worry: they presuppose that we have, first and foremost, *some grasp of the extent of the ordinals*. This is a difficulty Russell himself (1905) picks up on, when he first introduces the notion of limitation of size: he remarks that such theories are "of very little use, until we know how far the series of ordinals goes … it is not easy to see where this series begin to be non-existent, if such a bull may be permitted" (p. 36). Later on, he elaborates:

> A great difficulty of this theory is that it does not tell us how far up the series of ordinals it is legitimate to go. It might happen that ω was already illegitimate: in that case all proper [sets] would be finite… a series ordinally similar to a segment of the series of ordinals would necessarily be a finite series. Or it might happen that ω^2 was illegitimate, or ω_1 or any other ordinal having no immediate predecessor. (p. 44).

We can sharpen our sense of Russell's concern, here, by returning to the detail of Cantor's two number generating principles, specifically principle (2'). For the worry, clearly, is that we need an account of 'how far the series of ordinals goes' if we are to employ the series as a means of demarcating the threshold cardinality for sets; but 'how far the series of ordinals goes' is a function of which ordinals *exist,* which in turn depends on which unending sequences of ordinals are subject to principle (2'). Key to determining *this,* though, is the question of whether the ordinals in the relevant sequence *form a set*.

At this point, vicious circularity looms: for what sets there are depends on the extent of the ordinals (viz., what ordinals there are); but what ordinals there are, in turn depends on which unending sequences of ordinals are subject to (2') – in other words, it depends *on which sets there are.*

Linnebo (2010) points to a similar circularity undermining the use of (OO)[63] to distinguish sets and non-sets:

[63] (OO), the reader will recall, provides a plural formulation of the basis (namely, being cardinally equivalent to the ordinals) of Cantor's original distinction between 'consistent' and 'inconsistent' multiplicities.

$$(OO): \forall xx(\exists u(u \equiv xx) \leftrightarrow |xx| < |oo|)$$

Echoing the sentiment behind Russell's 'bull' above, Linnebo contends that using (OO) to determine a threshold cardinality for sets invariably leads to the question of "*why there are not more ordinals than the* oo" (emphasis mine):

> … why cannot the plurality oo form a set, which would then be an additional ordinal, larger than any member of oo? According to the view under discussion, the explanation is that oo are too many to form a set, where being too many is defined as being as many as oo. Thus, the proposed explanation moves in a tiny circle. The threshold cardinality is what it is because of the cardinality of the plurality of all ordinals, but the cardinality of this plurality is what it is because of the threshold. (pp. 153 – 154)

Besides engendering uncertainty concerning 'why there are not more ordinals than the oo,' this circularity equally threatens to undermine our warrant to believe that the ordinals *extend as far as we standardly take them to do.* We can formulate this latter worry in an especially rigorous way, by means of $(Plural *)$:

$$(Plural *): \forall xx\,[\big(\forall y(y \prec xx \rightarrow \phi(y)) \land \exists u(u \equiv xx)\big) \rightarrow \big(f(u) \notin u \land \phi(f(u))\big)]$$

and the attendant distinction between relatively, versus absolutely, indefinitely extensible ϕs. We've already noted this analysis leaves us with the question of what, for the former species of concept, accounts for the relevant principle of extendibility's *failing* to give rise to further ϕs. More technically: the question is *why* – when the principle of extendability, f, is applied to some plurality of relatively indefinitely extensible ϕs – the object obtained is *not itself a* ϕ.

The concept 'finite ordinal' is illustrative in this regard. We've noted, in that case, that the relevant plurality is *the* plurality of finite ordinals, and that the reason the relevant object (viz., the *set* of finite ordinals) is not a finite ordinal, is because it is the first *trans*finite ordinal. However, this suggests that the reason the concept 'finite ordinal' is only relatively indefinitely extensible – that is, why there is an ordinal limit to the number of instances of the concept produced by the principle of extendability – is because there exist *further species of ordinals beyond the finite*.

More generally: what seems to be the hallmark of the sorts of ordinals that are relatively indefinitely extensible, is the fact that these are of a *comparatively limited variety*; there exist, in each case, species of ordinals beyond the kind in question. This distinguishing feature, in turn, ensures that every plurality comprising these ordinals – including, in particular, the comprehensive plurality comprising *each* such ordinal – is consistently subject to Collapse, and so, principle (2') (indeed, applying the latter is what facilitates our shift, so to speak, to the 'next species of ordinals up').

However, if positing the existence of ordinals beyond the finite allows us to apply principle (2') without fear of paradox in that case – and *mutatis mutandis*, for other relatively indefinitely extensible concepts – then we might wonder why we shouldn't employ the same strategy when it comes to the absolutely indefinitely extensible concept that is 'ordinal' *simpliciter*: why, in other words, should we insist on the answer of 'no,' when it comes to Linnebo's (2010) question of 'whether there are more ordinals than the *oo*'?

The immediate retort is summed up well by Wright and Shapiro (2006): "because we are supposed to be dealing with ALL of them—or at least trying to" (p. 272). To flesh this out: in the case of the concept of 'ordinal' *simpliciter,* it seems we lack the option of positing a different *sort* of number (which option, it seems, we *do* have, and use, in the case of the finite ordinals) – and this is because 'ordinal' is *the maximally general category of number we have*. There simply is no 'beyond' the ordinals to which we might appeal, when we apply Collapse to *that* particular all-encompassing plurality (or, equivalently: when we apply principle (2') to that particular potential infinite).

However, a reply is in the offing: it concerns what warrant we have to think that there *is* such a 'beyond' to which we might appeal, in the finite case. Put differently: what warrant do we have to believe that we are *not*, in fact, 'dealing with ALL of them,' when it comes to the finite ordinals? (Importantly: to respond to this by falling back on the claim that the finite ordinals' are consistently subject to (2'), would be to beg the question).[64]

At this point, the proponent of the transfinite might argue that we *do* possess a principled account of why Collapse – and so, (2') – is applicable to the finite ordinals (and, consequently, why there exist ordinals beyond this limited kind): his is because the existence of the relevant set, ω, is proven by our best current set theory (Zermelo Fraenkel set theory – ZF – or with the Axiom of Choice, ZFC).

However, ZFC only proves the existence of ω as a result of the Axiom of Infinity; moreover, should we drop this axiom, the sentences of the resulting theory (call this ZFC^{-}) will all still be true, even if the universe in question consists of *only hereditary finite sets*[65]. The upshot of this is that ZFC vindicates our applying Collapse to the finite ordinals *only* to the extent that we accept the Axiom of Infinity – and since this axiom is arguably *equivalent* to accepting the validity of Collapse for the finite ordinals, it seems inescapable that falling back on ZFC as a means of justifying the latter, similiarly threatens to devolve into circularity.

Of course, if we *do* lack any (non-circular) justification for applying Collapse – and so, principle (2') – to the finite ordinals, then it remains open to us to entertain the possibility that these *are*, in fact, 'all of them.' However, this would mean that the the concept 'finite ordinal,' rather than being relatively indefinitely extensible, would instead exemplify *absolute* indefinite extensibility: in particular, any set obtained from any plurality of finite

[64] This is because, as our analysis reveals, being consistently subject to (2') itself relies on the existence of further ordinals – and so, precisely the assumption that the finite ordinals 'are not all of them.'

[65] Hereditary finite sets are those that i) are finite and ii) contain as elements only finite sets, recursively down to the empty set.

ordinals would itself *only ever be finite* – including the (ostensible) set of all finite ordinals itself.

Something like this arguably forms the core of Russell's worry concerning 'how far up the series of ordinals it is legitimate to go'; Dummett himself outlines a similar concern in *Frege: Philosophy of Mathematics* (1991), in relation to the cardinals:

> .. to someone who has long been used to finite cardinals, and only to them, it seems obvious that there can only be finite cardinals. A cardinal number, for him, is arrived at by counting; and the very definition of an infinite totality is that it is impossible to count it. This is not a stupid prejudice… All the same, the prejudice is one that can be overcome: the beginner can be persuaded that it makes sense, after all, to speak of the number of natural numbers…When he has become accustomed to this idea, he is extremely likely to ask, 'How many transfinite cardinals are there?'… He is very likely to be answered by being told, 'You must not ask that question.' But why should he not? If it was, after all, all right to ask, 'How many numbers are there?', in the sense in which 'number' meant 'finite cardinal,' how can it be wrong to ask the same question when 'number' means 'finite or transfinite cardinal'?... merely to say, 'If you persist in talking about the number of all cardinal numbers, you will run into contradiction,' is to wield the big stick, not to offer an explanation. (p. 315)

Dummett's question of 'How many transfinite cardinals are there?' is the clear analogue to Linnebo's 'why are there not more ordinals than the oo'; his broader point also resonates with many aspects of our discussion of $(Plural\ *)$, and the distinction between relative versus absolute indefinite extensibility. For instance, that we can be 'persuaded that it makes sense to speak of the number of natural numbers,' reflects our willingness, where ϕ is the concept 'finite ordinal,' to endorse both Collapse and Comprehensive Plurality$_{\phi}$. But, as we've seen, this requires denying that the set corresponding to this plurality – the object $f(u)$[66] – is itself a ϕ; contrastingly, where $f(u)$ would be the set of all ordinals (and so, the first to succeed them all), we are told – much like Dummett's beginner, in response to his query about 'the number of all cardinal numbers' – that this ordinal doesn't exist (or, equivalently, that we 'must not ask that question'). However – as our analysis of $(Plural\ *)$ reveals – it transpires that the source of this prohibition is, simply, the fact that any answer we might hazard as to the number of the transfinite ordinals will *itself be a transfinite ordinal* (analogous considerations, clearly, apply to the cardinals).

Critically, the point here is not (merely) that, where 'ϕ' is absolutely indefinitely extensible – and u is (ostensibly) the set of all ϕs – any candidate $f(u)$ will itself be a ϕ. Rather, the claim is that the issue of when (or for which ϕs) $f(u)$ exists – and subsequently, the issue of whether $f(u)$ is itself a ϕ – are questions *we ourselves decide*. To borrow Dummett's turn of

[66] In the particular case where u is the set of finite ordinals.

phrase: there is no in-principle reason why – having 'been having persuaded that it makes sense to speak of the number of natural numbers' – we should *not* ask 'how many transfinite cardinals there are': for the same considerations that facilitate our countenancing numbers beyond the natural numbers (viz., those we can count), just as easily motivate our countenancing the existence of numbers 'beyond' the transfinite (conversely: to the degree that we are unwilling to countenance such numbers, our speaking of 'the number of the natural numbers' begins to make *less* sense).

To sum up: the distinction between absolute and relative indefinite extensibility is, at base, *unstable* – a fact that should significantly unsettle the opponent of Collapse, insofar as this rejection is limited to (and explained by) concepts that are absolutely indefinitely extensible. Indeed, so far as *solutions* to the paradoxes go, this strategy amounts to little more than wielding Dummett's big stick; more worryingly, it threatens to imbue our conception of set with an unacceptable species of *vagueness.*

This a point Wright and Shapiro (2006) make by means of their 'Sorites strip' thought experiment:

> We construct a long strip of paper…whose colour starts out scarlet at one end but then fades very gradually and seemingly continuously to a yellowish-orange at the other… On it are inscribed a series of randomly selected decimal numerals.. as close together as they can be consistently with their ready distinguishability one from the next. (p. 294)

With this set-up in place, Wright and Shapiro ask us to consider the following statement:

$$(A): \forall x(R(x) \ \rightarrow D_7(x))$$

where '$R(x)$' abbreviates 'x is a numeral on a red background,' and '$D_7(x)$,' the claim that x denotes a multiple of 7. They then note that the truth value of (A) is clearly *indeterminate,* since there is "no sharp cut-off between the numerals that satisfy its antecedent and those which do not" – this, because the former "are *indeterminate in extent* within a wider population of numerals on the strip" (p. 295, emphasis mine).

Interestingly, Wright and Shapiro caution that the sort of vagueness affecting their Sororites strip is *not* perfectly analogous to that afflicting the boundary between absolute versus relative indefinitely extensibility: for whereas, in the former case, "indeterminacy of extent *within a wider population* [seems] essential to the intended point" (p. 295, emphasis mine), when it comes to the ordinals, "there is… no 'larger strip' on which they peter out." However, it isn't clear that something like 'indeterminacy of extent within a wider population' *isn't* what vexes the ordinals: for the question of whether a given species of ordinal is relatively or absolutely indefinitely extensible, *just is* the question of whether there exists 'a wider population of ordinals' in which the species in question 'peters out.'

The crucial point, in any case, is that our decision to 'cap' various species of ordinals where we do – or, equivalently, our taking these species to extend *as far* as we standardly do – is at base unprincipled. And this Wright and Shapiro do concede:

> … it is conceptually open whether to regard them as confined to the finite, or the recursive, or the countable, or the accessible, or . . . , or whether we let them rip, stopping only when the apparatus buckles on Burali-Forti. (p. 295)

But to concede that the extent of the ordinals is 'conceptually open' in this way, would appear to concede nothing less than that our concept 'ordinal' is vexed by vagueness – which, insofar as the ordinals serve as a marker for the distinction between sets and non-sets, equally afflicts our concept of 'set.'

In fact, the species of vagueness at issue is strongly reminiscent of the first kind of Dummettian 'haziness' we encountered in Chapter 1: for its being "conceptually open" whether the ordinals are "confined to the finite, or the recursive, or the countable, or the accessible, or… " seems to indicate nothing less than our having *failed* to determine 'the limits of acceptable specifications' of the concept 'ordinal' – and so, our having failed, relative to the *domain* of ordinals, to determine 'what elements it does or does not contain'[67].

Indeed, that this vagueness (*qua* Dummettian 'haziness') is specifically a function of *rejecting Collapse,* additionally promises to shed light on a question we formulated – but left unanswered – back in Chapter 1:

> **Question 3:** Why is circumscribing a domain (viz., eliminating any haziness about what elements it does or does not contain') established only when the domain in question comprises a set?

For we've seen that the kind of haziness at issue arises due to the deeply *ad hoc* nature of the wedge the opponent of Collapse is forced to establish between the pluralities – or potentially infinite ordinal sequences – that (he says) comprise sets, and those that (he says) do not. As Wright and Shapiro (2006) quip: "[a]ny proposed place to stop the expansion of the actual infinite, say at 2ω, ω_1, or the ninth strong inaccessible, is artificial… Why stop *there*?" (p. 282). However, if this 'artificiality' is a function specifically of our *rejecting Collapse*, then this implies that *accepting* the latter would entail that the former (viz., any haziness about what the domain does or does not contain) is eliminated – and so, that the domain in question has (now) been effectively circumscribed.

[67] The reader will recall that, for Dummett, the paradigm of such 'haziness' (and the uncertainty about what elements a domain comprises that accompanies it) is the question of whether his intuitive ordinal *is in fact an ordinal* – but this, of course, is just a particular instance of the broader question of whether the object $f(u)$, corresponding to a set u of ϕs, is itself a ϕ.

And, of course, where we *fail* to do this, there soon follows an indeterminacy concerning what 'the limits of acceptable specifications' are, of the objects populating this domain – and this in turn, as Dummett claims, has the effect of invalidating classical quantification over it. Similarly, in the case of the statement (A) quantifying over the numerals in their 'Sorites strip,' Wright and Shapiro argue that an "indeterminacy in the range of admissible witnesses" engenders a corresponding "indeterminacy in the truth-conditions" of this statement:

> 'Numeral on a red background' exactly fails to specify any determinate set of inscribed numerals, with indeterminacy in the truth-conditions of quantifications over them the immediate result. (p. 295)

Extending this consideration to the question of quantification over the *ordinals* – given *their* indeterminate extent – the authors conclude that "if this matter is indeed conceptually open," it swiftly follows that there is:

> … going to be potential indeterminacy in the truth-conditions of quantifications over all ordinals… *There will be such indeterminacy because it is indeterminate what goes into the argument pool.* (p. 295, emphasis mine)

In this way, the absolutist's rejection of Collapse – in the name of retaining absolutely general quantification over all sets – ironically ends up undermining it.

§2.5

Potentialism and the rehabilitation of Collapse

One upshot of the Objection from Vagueness is that our attention is shifted onto *Plural Comprehension* as the culprit in the set-theoretic paradoxes. Indeed, the natural option[68] for advocates of Collapse who are also *quantifier relativists,* is to endorse the negation of Comprehensive Plurality$_\phi$:

$$(No\ Comprehensive\ Plurality_{\phi}): \forall xx \exists z(\phi(z) \wedge \sim(z \prec xx))$$

where this has, as a consequence, *No Comprehensive Domain*:

$$(No\ Comprehensive\ Domain): \forall xx \exists z \sim(z \prec xx)$$

[68] Or we might say, one that is more natural than rejecting *Homogenous Plurality*$_\phi$ (which claim is the left-to-right conditional that, together with *Comprehensive Plurality*$_\phi$, makes up the Plural Comprehension scheme). See §1.4, for a discussion.

which provides a natural formulation of relativism[69] (which view – given the worries the rejection of Collapse itself poses for absolute generality – may now seem more palatable).

However, we saw in Chapter 1 that No Comprehensive Domain leads straight to the *Objection from Ineffability,* viz., the complaint that the truth of No Comprehensive Domain would require its singular existential quantifier to range more widely than its plural counterpart (and, in so doing, to transcend the very limits on quantification that are a cornerstone of the relativist's view). Plainly put: in stating her view by means of *No Comprehensive Domain,* the quantifier relativist risks undermining it.

There is, however, a way for the quantifier relativist to respond to this worry – one with the potential of rehabilitating both No Comprehensive Domain *and* Collapse. The response is due to Charles Parsons, and forms part of his attempts to address the inadequacies[70] plaguing a temporal account of the notion of 'priority' at the heart of the *iterative conception of set:*

> … that any available objects can be formed into a set is, I believe, correct, provided that it is expressed abstractly enough, so that 'availability' has neither the force of existence at a particular *time* nor of giveness to the human mind, and formation is not thought of as an action… What we need to do is to replace the language of time and activity by the more bloodless language of potentiality and actuality. (2005a, p. 293)

In essence, Parson's proposal is to cash out the notion of 'priority' (or 'availability') in *modal* terms: that is, we are to understand the claim 'the elements of a set are prior to the set' along the following lines: "relative to that of its elements, the existence of a set is *potential*" (2005b, p. 315, emphasis mine).

That we might respond to the paradoxes by shifting to a modal account of sets – and so, become *set-theoretic potentialists* (Sutto, 2024) – is a suggestion that has since been taken up and developed in great detail by Øystein Linnebo and by James Studd. For both, the upshot of the paradoxes is an "expansion of the mathematical ontology" (Linnebo, 2010, p. 208), where this is specifically a function of expanding quantifier *domains*. Indeed, in a manner vividly reminiscent of the 'increasing sequence of extensions' Dummett associates with indefinitely extensibility, Studd (2019) describes how these expanding domains form an ascending *hierarchy*[71]*:*

$$\langle M_0, E_0, S_0\rangle, \langle M_1, E_1, S_1\rangle, \langle M_2, E_2, S_2\rangle, \langle M_3, E_3, S_3\rangle, \ldots$$

[69] Especially – as will be emphasized in the conclusion of this chapter – if the relativism in question is motivated by an indefinite extensibility-based account of the paradoxes: such that, in particular, the properties (or ϕs) for which Comprehensive Plurality$_\phi$ fails are specifically the indefinitely extensible ones.

[70] Which conception, the reader will recall from §1.5, seems to entail Collapse – or at the very least something like it – in the guise of the claim that "given any well-determined objects, they can be collected together into a set" (Lear, 1977, p. 86).

[71] Each $\langle M_i, E_i, S_i\rangle$ for each $i \in I$ (where I is an index set) is, on Studd's (2019) account, an interpretation of the language of set theory ($\mathcal{L}_{\mathrm{S}}$ or, for a theory with urelements, $\mathcal{L}_{\mathrm{SU}}$): the overall hierarchy is, in turn, the hallmark of his *interpretational expansionist* potentialist account. See chapters 3 and 4 for our discussion of his view.

where each M_i corresponds to a *single* quantifier domain for the language of set theory (and E_i and S_i to the extensions, at M_i, of the 'set' predicate, 'ß,' and the 'element of' relation, '∈,' respectively). Moreover, the length of this sequence is similiarly 'hazy' in the sense of 'vanish[ing] into the indiscernible distance[72]' – and in line with Dummett's claim that this phenomenon vitiates classical quantification over all sets, Studd and Linnebo concur that no single quantifier ranges simultaneously over all the elements comprising every domain in the hierarchy. However, the latter thinkers nevertheless endorse the possibility of generalising *across*, or *between*, these 'actualist' quantifier domains, by means of a *modal* operator, '□'[73] (the corresponding possibility operator '$\Diamond\psi$' – here, to be read as 'potentially' – is defined in the usual way, as '$\sim\Box\sim\psi$'). Specifically, a modal formula '$\Box\psi$' – which we can gloss as 'it is absolutely the case that ψ' – is true if and only if ψ is true, in *absolutely every domain of sets in the potential hierarchy.*

Key to set-theoretic potentialism is thus the idea that the language of set theory has "an implicit modal character" (Linnebo, 2010, p.155). We can make this character explicit by means of a 'modal translation' of the language of ordinary set theory – formally, a translation *function*[74], which maps each formula, ϕ, of the 'actualist' language (say[75], $\mathcal{L}_{PS}$) onto its 'modalised' counterpart, $\phi^{\Diamond}$, in the language of *modal* set theory, $\mathcal{L}_{MPS}$.

Critically, the 'modalised' quantifiers in these latter formulae are no longer restricted to a single domain in the hierarchy, but rather range over the elements of every such domain. Indeed, just this is what facilitates the potentialist's proposed rehabilitation of Collapse; or, as Parsons puts it, the attempt to "use modal concepts in order to save the idea that *any* multiplicity of objects can constitute a set… one makes only the proviso that they 'can exist together'" (2005a, p. 295)[76]. More fully: if this co-existence entails that the elements in question collectively *comprise a plurality*[77], then we can quite easily formalise Parson's

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[72] The indeterminate extent of this domain *hierarchy* makes it a paradigm of the potential infinite.

[73] Strictly speaking, Studd's modal account contains not one but two modal operators – a 'forwards-' and 'backwards-looking' one, respectively; by contrast, Linnebo's (2013) modal theory deploys a single modal operator. An advantage of the latter is thus its "leaner ideology" (Studd, 2019, p. 162); however, given the expressive power associated with Studd's theory, this dissertation will focus on his account. See Appendix B for the details of how Studd's operators are interpreted by the hierarchy of domains, as well as the proof theory for the resulting logic, $MPFO$ (which enriches PFO with the axioms and rule-schemas governing Studd's 'absolute' operator, '□').

[74] In Studd's (2019) account, the translation proceeds, given a non-modal formula ϕ of an 'actualist' language (that is, $\mathcal{L}_S$, $\mathcal{L}_{PS}$, or their analogues with urelements) by appending a '□' and a '$\Diamond$' to each universal and each existential quantifier in ϕ, respectively; additionally, each atomic formula θ of ϕ is prefixed with a '$\Diamond$,' to obtain $\Diamond\theta$ (following Studd's lead, we'll abbreviate the formulas '$\Diamond(u \prec xx)$' and '$\Diamond(u \in v)$' to '$u \prec^{\Diamond} xx$,' and '$u \in^{\Diamond} v$'). See Appendix B for details.

[75] This translation is uniform for both singular languages ($\mathcal{L}_S$, or with urelements, $\mathcal{L}_{SU}$), and plural ones ($\mathcal{L}_{PS}$, or with urelements, $\mathcal{L}_{PSU}$).

[76] Here, Parsons is drawing inspiration from Cantor's own characterizations of sets in his various letters – for instance, besides those written to Dedekind, Cantor also describes in a letter to Hilbert that a set can:

> … be thought of as *finished* [...] if it is possible without contradiction (as can be done with finite sets) to think of *all its elements as existing together…* or (in other words) if it is *possible* to imagine the set as *actually existing* with the totality of its elements. (Cantor, as quoted in Ewald, 1996, p. 927)

[77] The intuitive idea that pluralities only have, as members, objects that actually co-exist, is one we will be able to make more precise in Chapter 4 by means of the notion of *modal plural rigidity.*

suggestion that "objects that exist together *can* constitute a set" (p. 292) – call this *Parson's Suggestion* – as follows:

$$(Parson's\ Suggestion): \Box\forall xx\Diamond\exists y\Box\forall u(u \prec^{\Diamond} xx \leftrightarrow u \in^{\Diamond} y)$$

However, given the definition of the predicate '≡' (where '$u \equiv xx$' reads 'u is the set comprised of the xx'), *Parson's Suggestion* above emerges as the 'modal translation' of Collapse (call this *Collapse*$^{\Diamond}$):

$$(Collapse^{\Diamond}): \Box\forall xx\Diamond\exists y(y \equiv^{\Diamond} xx)$$

Read idiomatically, Collapse$^{\Diamond}$ asserts the existence of a *potential* set corresponding to *absolutely every plurality.* However, if pluralities comprise only objects that *actually co-exist*, then the fact that the set containing these objects is merely *potential* relative to them, entails that it is impossible for the former to fall amongst the latter. This, in turn, suffices to remove any threat of paradox – a fact Parsons (2005b) demonstrates below, specifically with reference to Russell's Paradox:

> From our point of view, the objects that *are* F can constitute a set, not the objects that *would be* F if the set were formed … Suppose 'Fx' is '$x \notin x$.' We want to assume that there is potentially a set of all objects that are not elements of themselves, but that means all objects that are *actually* not elements of themselves. (p. 315)

A key thing to note here is that the truth of Collapse$^{\Diamond}$ (viz., there being 'potentially a set of all objects that are not elements of themselves') is compatible with there being 'all objects that are *actually* not elements of themselves'. This means that Collapse$^{\Diamond}$, unlike its non-modal counterpart, is consistent with *Comprehensive Plurality*$_{\varphi}$ – and indeed, with the stronger Plural Comprehension Scheme[78].

However, (joint) inconsistency re-emerges if the potentialist additionally chooses to affirm *modalised* Plural Comprehension (call this, *Plural Comprehension*$^{\Diamond}$):

$$(Plural\ Comprehension^{\Diamond}): \Diamond\exists xx\Box\forall y(y \prec^{\Diamond} xx \leftrightarrow \varphi^{\Diamond}(y))$$

The rub, as in the non-modal case, comes with the claim that there exists a Comprehensive Plurality of φs – or rather, in the modal context, the claim that there *potentially* exists a plurality comprising *absolutely every potential* φ[79] (call this, *Comprehensive Plurality*$_{\varphi}^{\Diamond}$):

[78] Studd (2019) adopts the latter as an axiom of $MPFO$.
[79] It is useful, in the context of set-theoretic potentialism, to extend our discussion of modalised formulae to *properties* – so, for instance, where φ abbreviates a formula that captures a particular property, its modalisation (which would correspond to a 'potential' property) will be written as $\varphi^{\Diamond}$.

$$(Comprehensive\ Plurality_{\varphi}{}^{\Diamond})\Diamond\exists xx\Box\forall y(\varphi^{\Diamond}(y) \rightarrow y \prec^{\Diamond} xx)$$

For, in attesting to the *potential* existence of a plurality comprising absolutely all $\varphi^{\Diamond}$s, Comprehensive Plurality$_{\varphi}{}^{\Diamond}$ in fact affirms the *actual* existence of this plurality, at some domain M_i in the potentialist hierarchy. But this would entail that, at M_i, this plurality is 'available' for Collapse$^{\Diamond}$ – and so, that there exists a potential set, at M_{i+1}, that comprises absolutely every $\varphi^{\Diamond}$. Of course, where the $\varphi^{\Diamond}$s are relatively indefinitely extensible – say, where $\varphi^{\Diamond}$ abbreviates 'is potentially a finite ordinal' – this would not be a problem: however, where the property captured by $\varphi^{\Diamond}$ is one that it is *absolutely* indefinitely extensible – say, where $\varphi^{\Diamond}$ abbreviates 'is a *potential ordinal*'' – then the potential set of $\varphi^{\Diamond}$s guaranteed by Collapse$^{\Diamond}$ would itself be a $\varphi^{\Diamond}$.

The result, as we've seen throughout this chapter, is paradox.

Clearly, then, the same tension that we took to be the hallmark of the contradictions in the context of ordinary (plural) set theory re-emerges, at the modal level, in the joint inconsistency of Plural Comprehension$^{\Diamond}$ and Collapse$^{\Diamond}$. Given this, the opponent of Collapse may be tempted not only to declare a failure the attempt to 'use modal concepts … to save the idea that *any* multiplicity of objects can constitute a set'; he may dismiss as entirely redundant the shift to a modal account of sets in the first place.

However, this response may be too quick. For the utility of a modal account of sets extends beyond rehabilitating Collapse; in particular, the expressive resources accompanying this theory may furnish the quantifier relativist with the means to respond to the Objection from Ineffability.

We saw at the beginning of this section that one form this objection takes concerns the plural formulation of the relativist's view, No Comprehensive Domain:

$$(No\ Comprehensive\ Domain): \forall xx\exists z{\sim}(z \prec xx)$$

where the worry, specifically, was that the truth of No Comprehensive Domain would require its singular existential quantifier to range strictly more widely than its plural quantifier. However, when it comes to the *modalisation* of No Comprehensive Domain – call this, *No Absolutely Comprehensive Domain*$^{\Diamond}$:

$$(No\ Absolutely\ Comprehensive\ Domain^{\Diamond}): \Box\forall xx\Diamond\exists y{\sim}(y \prec^{\Diamond} xx)$$

no corresponding difficulty arises; moreover, this is for precisely the same reasons that Collapse$^{\Diamond}$ manages to avoid (at least some of) the worries plaguing its non-modal counterpart.

This is because the objects' comprising the pluralities ranged over by modalised quantifier, '$\Box\forall xx$' in No Absolutely Comprehensive Domain$^{\diamond}$, are only those that *actually co-exist* at some domain in the hierarchy. However, on the modal picture, it is perfectly possible for the (potential) witness corresponding to the existential quantifier, '$\Diamond\exists y$,' to *fail* to (actually) co-exist with a given plurality of existing objects. Indeed, for the potentialist *relativist,* there is, at *every* domain M_i in the hierarchy, a plurality of actual objects to which there corresponds an object that *necessarily fails to co-exist* with this plurality. The plurality in question is that comprising *every object in* M_i (in other words, it *just is* the domain M_i, plurally-encoded); and the object which necessarily fails to co-exist with this plurality is the set obtained from it by means of Collapse$^{\diamond}$ – one which, in falling outside of M_i, serves as the initial element of the subsequent, expanded domain in the sequence.

In sum: with the advent of modal resources comes the possibility, for the quantifier relativist, of *actually managing to express her view*. However, the benefits accompanying the shift to potentialism needn't accrue only to the relativist; as Studd (2019) points out, the absolutist additionally stands to profit from "the availability of the modality," as "it permits him to characterize his view *without presupposing it to be true*" (p. 163, emphasis mine).

The difficulty Studd is gesturing towards, here, is in many ways the inverse of that faced by the relativist – for where her worry was that (non-modal) formulations of relativism are trivially self-*refuting*, the converse complaint, for the absolutist, is that many attempts to express his view succeed in doing so only to the extent that they *presuppose it is true.*

To see this, we can consider the following example – due to Tim Williamson (2003) – of a statement uttered by the absolutist, intending to assert the truth of absolutism:

(1) It is possible to quantify over everything.

Clearly, (1) succeeds in expressing quantifier absolutism (and is true) *only to the degree that the quantifier 'everything' achieves absolute generality;* but of course, to assume this, is to beg the question against the relativist. For analogous reasons, the latter is unlikely to be swayed by the (non-modal) plural formulation of absolutism that is *Comprehensive Domain*:

$$(Comprehensive\ Domain)\text{: } \exists xx\forall z(z \prec xx)$$

For while every member of the plurality to which Comprehensive Domain attests may indeed be ranged over by its singular universal quantifier, this does not, on the relativist's view, preclude the existence of a (potentially) *more encompassing* plurality (over whose members the universal quantifier in Comprehensive Domain *fails* to range). Thus, the relativist may go so far as to accept both (1) and Comprehensive Domain[80] as true; however, this is a function

[80] Indeed, since Comprehensive Domain follows from Plural Comprehension –which is itself an axiom of $MPFO$ –the former emerges as a trivial truth, in that context.

of her simultaneously being able to *deny* that either claim succeeds in expressing quantifier absolutism in manner that does not beg the question.

For his own part, Williamson – himself an absolutist – concedes that an utterance of (1) is unlikely to persuade the relativist; however, he contends that "what matters much more is whether the generality-absolutist can regard generality-absolutism as expressible":

> … For the problem for generality-relativism was that by the generality-relativist's own lights it seems to be inexpressible. According to the generality-absolutist, in a context in which 'everything' is used unrestrictedly, (1) does express generality-absolutism. (2003, p. 433)

But as we've seen, the use of modal resources *does* allow the relativist to express her view (via modalised No Absolutely Comprehensive Domain$^{\Diamond}$). Moreover, we can plausibly take the modal context to be one 'in which 'everything' is used unrestrictedly' – or, at least, is *intended* to be used unrestrictedly – by the absolutist: what this means is that the absolutist, rather than formulating his view by means of (1) or Comprehensive Domain, might instead choose to express it by means of a *modal* claim, viz., *Absolutely Comprehensive Domain*$^{\Diamond}$:

$$(Absolutely\ Comprehensive\ Domain^{\Diamond}):\ \Diamond\exists xx\Box\forall y(y \prec^{\Diamond} xx)$$

Read idiomatically, Absolutely Comprehensive Domain$^{\Diamond}$ attests to the existence, at some point in the potentialist hierarchy, of a plurality (or a plurality-encoded *domain*) comprising absolutely everything. Understood thusly, it clearly succeeds in expressing absolutism in the context of set-theoretic potentialism: however, it does so *without presupposing the existence of this plurality*. Moreover (and unlike its non-modal counterpart) Absolutely Comprehensive Domain$^{\Diamond}$ is no longer a *trivial* truth in the context of $MPFO$: it may, in other words, *fail* to be true. Studd (2019) hails this as a victory for the absolutist – for unlike Comprehensive Domain, the relativist is no longer able to "brush aside Absolutely Comprehensive Domain as a trivial truth that is perfectly consonant with relativism."

Studd (2019) thus concludes that, far from being redundant[81], the shift to a modal theory "improves the dialectical situation for both sides":

[81] It is possible for the absolutist to concede the utility of the modal operators (for example, when it comes to expressing quantifier absolutism) while nevertheless maintaining the modal theory i*s* ultimately-speaking, redundant. This is due to a result Studd calls 'Flattening', whereby a slightly stronger version of Absolutely Comprehensive Domain$^{\Diamond}$ (viz., one dropping the '$\Diamond$' from the outermost quantifier) entails that the truth of any modalised formula implies the truth of its non-modal counterpart (and vice versa):

$$\exists xx\Box\forall y(y \prec^{\Diamond} xx) \vdash_{MPFO} \phi^{\Diamond} \leftrightarrow \phi$$

Studd (2019) emphasizes, in light of this result, that there is no need for the absolutist to object to the shift to a modal theory, as "modalisation gives him a harmlessly baroque way to re-express what, by his lights, he could already say with his non-modal language interpreted over the absolutely comprehensive domain" (p. 162).

> The advantage of the modal statements… is that they permit each side to frame a coherent thesis that their opponent needs, by his or her own lights, to deny. The normal standard expected of view-stating can be met: neither side need presuppose the correctness of their view in order to state it. (p. 163)

Moreover, if we *do* take the modal context to be that in which the absolutist intends his use of 'everything' to be (absolutely) unrestricted, then the availability of modal resources not only allows each side of the debate to 'frame a coherent thesis'; it also helps set the stage for what Studd (2011) calls "Williamson's Crucial Test" (p. 68) which the latter proposes in response to the relativist's complaint that (1) fails to capture the absolutist's thesis. We've seen that Williamson, by way of reply, contends that the absolutist "does not claim that (1) is intrinsically an expression of generality-absolutism, but only that it expresses generality-absolutism when used in a suitable context" (p. 433 – 444). However, he goes on to suggest that an utterance of (1), when made in such a context – and as evaluated from the standpoint of what is, potentially, a more *liberal* context – should serve as the final arbiter of the standoff between absolutism and relativism about quantifiers:

> Generality-absolutists should instead take a stand with respect to a judiciously selected original context *C*, and argue in *C** that everything was already quantified over in *C*... they should have the nerve to stake their position on *C,* and say in *C** 'If I was not quantifying in *C* over everything, then I give up'. Generality-relativists should likewise have the nerve to stake their position on the shift from *C* to *C*,* and say in *C** 'If you were quantifying in *C* over everything, then I give up'… the disputants may agree to treat the shift from *C* to *C** as a crucial test to decide the issue between them. (p. 434 – 435)

In the potentialist context, we can think of Williamson's original context *C* as the all-encompassing domain to whose potential existence Absolutely Comprehensive Domain$^{\diamond}$ attests; in turn, the shift to a more expansive domain *C** (or – in Linnebo's (2013) words – the relativist's 'expansion of mathematical ontology') is one she proposes to effect by means Collapse$^{\diamond}$, specifically, the potential set to which *it* attests.

Thus, just as Dummett's Dilemma revolves, at base, around the existence (or not) of the latter's 'intuitive ordinal', so too does Williamson's Crucial Test boil down, in part, to the question of whether the potential sets secured by Collapse$^{\diamond}$ exist. However (and this is Studd's key point), it is by means of the shift to a modal theory that these questions can coherently be posed – and that they might at last be answered.

Conclusion

Of course, while giving us an especially powerful means of formulating the claim at the heart of Dummett's Dilemma and Williamson's Crucial Test – namely, Collapse$^{\diamond}$ – shifting to a modal theory does not *force* it on us. For the opponent of Collapse – or equivalently, the proponent of Cartwright's Claim – can simply dig in his heels; indeed he may go so far as to formulate a modal version of his own view, along the lines below:

$$\left(Cartwright's\ Claim^{\diamond}\right): \diamond\exists xx{\sim}\diamond\exists y(y \equiv^{\diamond} xx)$$

This, however, amounts to a far stronger claim than its non-modal counterpart – the idea is not merely that some pluralities *do not* form sets, but rather that some pluralities *cannot* form sets.[82] But this just raises the question, as Studd (2013) puts it, of "[w]hat is it about the world that allows some sets to form a set, whilst prohibiting others from doing the same?" (p. 699). This, in turn, is just a variant of Question 4 – which, as we've seen throughout this dissertation, admits of no easy answer.

Importantly, none of this is to say that the advocate of Collapse$^{\diamond}$ is without her own problems: for in the wake of her acceptance of the latter comes the new challenge of justifying her corresponding rejection of modalised Plural Comprehension$^{\diamond}$:

$$\left(Plural\ Comprehension^{\diamond}\right): \diamond\exists xx\Box\forall y(y \prec^{\diamond} xx \leftrightarrow \varphi^{\diamond}(y))$$

The question, for advocates of Collapse$^{\diamond}$, is thus no longer what distinguishes pluralities that form sets from those that don't, but rather *which pluralities potentially exist* – or, more precisely, *which $\varphi^{\diamond}$s determine a comprehensive plurality of their instances.* The question, in other words, is what accounts for the falsity of Comprehensive Plurality$_{\varphi}^{\diamond}$:

$$\left(Comprehensive\ Plurality_{\varphi}^{\diamond}\right): \diamond\exists xx\Box\forall y(\varphi^{\diamond}(y) \rightarrow y \prec^{\diamond} xx)$$

Or, alternatively, for which $\varphi^{\diamond}$s is the following claim *true*:

$$\left(No\ Comprehensive\ Plurality_{\varphi}^{\diamond}\right): \Box\forall xx\diamond\exists y({\sim}\left(y \prec^{\diamond} xx\right) \land \varphi^{\diamond}(y))$$

Critically for Dummett and Russell – and indeed, for Studd and Linnebo – a possible answer is forthcoming: the $\varphi^{\diamond}$s which fail to determine an absolutely comprehensive plurality of their instances, are just those that are *absolutely indefinitely extensible*[83]. For if *any* collection of instances of the concept gives rise to a new instance – as is the case for concepts that are absolutely indefinitely extensible – then it is arguably impossible for all of these instances to *co-exist*, and so, to comprise a single, all-encompassing plurality.

[82] See Studd (2011, p.132)

[83] In turn, the distinction between *absolutely* and *relatively* indefinitely extensible concepts boils down to the question of whether it is *possible*, at some point, for all the instances of the concept in question to co-exist (*qua* comprise a plurality).

Moreover, where the putatively comprehensive pluralities whose bounds are exceeded by these $\varphi^{\Diamond}$s are (plurally-encoded) quantifier *domains*, then the $\varphi^{\Diamond}$s in question additionally bear witness to the truth of No Absolutely Comprehensive Domain$^{\Diamond}$:

$$(No\ Absolutely\ Comprehensive\ Domain^{\Diamond}): \Box\forall xx \Diamond \exists y \sim (y \prec^{\Diamond} xx)$$

Consequently, it would appear that the *quantifier relativist* has at her disposal at least the beginnings of a motivation for rejecting of Plural Comprehension$^{\Diamond}$ – one which aligns seamlessly with Dummett's twin sentiments concerning the upshot of the set-theoretic paradoxes (namely, that these 'reveal the existence of indefinitely extensible concepts,' where the 'prime lesson' of this, in turn, is that quantification over the corresponding domains 'cannot be regarded as yielding… a sentence with a determinate truth-value').

But, importantly, we saw in Chapter 1 that both of these claims are intimately related to Dummett's acceptance of *the Domain Principle*. By contrast, for Studd and Linnebo – both of whom are quantifier relativists – affirming an indefinite-extensibility-based account of the paradoxes is seemingly *unrelated,* for either thinker, to any corresponding endorsement of the Domain Principle (indeed, Studd (2019) goes so far as to actively distance himself from the Principle, describing it as "dubious," and further, as "discredited" (p.178)).

However, it is not entirely clear that something like the Domain Principle *doesn't* fall out of quantifier relativism – *especially in the modal context*. When it comes to the potentialist's domain hierarchy, for instance, the non-modal quantifiers are restricted to ranging over pluralities of actually existing objects: but if we take seriously the Objection from Vagueness – especially in the context of motivating Collapse$^{\Diamond}$ – then this is arguably equivalent to these quantifiers' being restricted, in their range, to the elements of (potential) *sets.* Moreover, if the relativist's diagnosis of the paradoxes boils down to our mistakenly taking our non-modal quantifiers to *range more widely*, then this seems to imply that classical quantification is *only* valid when our quantifiers range over the elements of (potential) sets; which, of course, is just to say – paraphrasing Cartwright – that we 'can quantify classically over the so-and-so's *only* if the so-and-so's constitute some one set-like object.'

The issue of whether the modal relativist is in fact committed to the Domain Principle, is one we will explore in depth in Chapter 4; in the meantime, though, there remains the question of whether the quantifier relativist *in fact succeeds in passing Williamson's Crucial Test*. For accepting that there exist (potential) sets that lie outside of any given domain is not yet to have *argued* that the latter *do,* in fact, undergo expansion: what this latter claim additionally requires, is an account of the *mechanisms* that might give rise to domain expansion.

Outlining the details of one such an account – Studd's (2019) *interpretational expansionism* – and subsequently investigating whether it sustains the quantifier relativist's rejection of Plural Comprehension$^{\Diamond}$, will be the task of Chapter 3.

Chapter 3

Interpretational Expansionism: Meaning, Use, and the Many Faces of Absolutism

Introduction

In Chapter 2, we saw that shifting to a potentialist account of sets might enable the quantifier relativist to avoid the Objection from Ineffability – rather than attempting to express her view by means of the self-undermining claim that is *No Comprehensive Domain*, the suggestion is that she should instead advance its *modalised* counterpart, *No Absolutely Comprehensive Domain*$^{\Diamond}$:

$$(No\ Absolutely\ Comprehensive\ Domain^{\Diamond})\text{: } \Box\forall xx\Diamond\exists y{\sim}(y \prec^{\Diamond} xx)$$

Moreover, we saw that a similar shift, on the *absolutist's* part, enables him to formulate his own view without *presupposing it to be true.* For where the non-modal claim of *Comprehensive Domain*, below:

$$(Comprehensive\ Domain)\text{: } \exists xx\forall z(z \prec xx)$$

succeeds in expressing quantifier absolutism only if the plurality to which it attests *already comprises absolutely everything,* the corresponding modalised claim of *Absolutely Comprehensive Domain*$^{\Diamond}$:

$$(Absolutely\ Comprehensive\ Domain^{\Diamond})\text{: } \Diamond\exists xx\Box\forall y(y \prec^{\Diamond} xx)$$

captures his core claim (in the modal context), *sans* any such presuppositions.

That modal resources enable both sides in the debate to formulate consistent – and substantial! – theses, in turn paves the way for what Studd (2011) calls Williamson's Crucial Test: namely, a scenario where the relativist – relative to a domain the absolutist declares to be absolutely comprehensive – attempts to specify one that is *more* expansive. However – as we saw at the end of Chapter 1 – even if the relativist shows that we have good reason to believe our quantifiers *do* range more inclusively in the course of the paradoxical reasoning (say, by convincing us of the validity of Collapse$^{\Diamond}$), she nevertheless faces a lingering worry. Studd (2019) calls this *the Objection from Mysteriousness*, and it involves "question[ing] whether the relativist is able to adequately explain what stands between our quantifiers and absolute generality" (p. 15); in the case of *expansionism*, one guise this query can take concerns *how it is that quantifier domains expand.*

The expansionist evidently cannot draw on the most obvious means by which this might occur, viz., a form of *set-theoretic creationism:* for while, as Jarred Warren (2017) points out in his discussion of quantifier relativism, this *would* "make meta-semantic sense" (p. 85) of how our quantifiers come to range more expansively – what did not exist before, they could not have ranged over – a constructionist account of sets faces numerous (and, for many mathematicians and philosophers, insurmountable) difficulties[84].

For just these reasons, Studd (2019) himself declares a creationist account of set-domain expansion as "a manifest non-option" (p. 103), thereby eluding Warren's (2017) "Syclla of metaphysical implausibility"; in turn, how he proposes to avoid the "Charybdis of metasemantic mystery" (p. 85) is by advancing an *interpretational account of domain expansion.* The claim is that our quantifiers come to range more expansively as a function of *an underlying shift in our interpretation of the language of set theory.*

Of course, without an account of how these shifts take place, interpretational expansionism fares little better than set-theoretic creationism: in fact, an immediate worry for the interpretational expansionist is that it seems hard to explain these shifts – especially when they entail our quantifiers' ranging over 'new sets' – in terms *other than* creationism. Studd is, however, at pains to argue that the shifts in interpretations he posits are akin to commonplace changes in the meanings of expressions that can occur over time. For instance, he points to the contemporary meaning of the word 'meat' – which now refers to the flesh of an animal– in contrast to its meaning at the time of Henry VIII, where meat (or, for ease of delineation, meat_{1509}) referred, more generally, to any edible item. However, this restriction in the meaning of 'meat' *has not been accompanied by the literal destruction of items that previously were* $\textit{meat}_{1509}$ (as Studd puts it: "[t]o attach a less liberal meaning to 'meat' is not to reduce the stock of meat … nor the stock of non-meat_{1509}" (p. 108)).

In turn, Studd argues that "[a]n exactly parallel response is open to the charge of creationism against interpretational expansionism" (p. 108). For we may imagine (perhaps as a result of developments in large cardinal theory) that the domain of the language of set theory at the time of Studd's writing *Everything, More or Less* – viz., M_{2019} – is more expansive than the domain of the same language in the year Zermelo proved his Quasi-Categoricity Theorem: namely, M_{1930}. Now, as a result of this theorem, we know that M_{1930} can itself be associated with a 'standard model' of $ZFCSU_p$[85] (plural set theory with urelements), where this has the form $V_k(U)$ (or, in this case, $V_{k_{1930}}(U)$[86]): more importantly, this standard model is itself a set. Thus, in this context, the creationist charge is that the shift between M_{1930} and M_{2019} is "impossible … since the pure set $V_{k_{1930}}$– the set comprising every pure set_{1930} – is not the kind of thing that can be brought into being" (p. 108).

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[84] See Chapter 1, §1.5, for a discussion of these difficulties in relation to the iterative conception of set.
[85] The (impure) theory $ZFCSU_p$ – that is, (plural) Zermelo–Fraenkel set theory, with Choice and the Urelement Set Axiom – is the standard theory Studd works with throughout *Everything, More or Less* (2019).
[86] Where U is the urelement base and k has the properties of a strongly inaccessible cardinal.

However – as Studd rightly points out – the objection misses the point: for while the set $V_{k_{1930}}$ indeed was not a thing$_{1930}$, but *is* a thing$_{2019}$, this does not imply that this set has 'come into being' by some mysterious act of creation:

> It was something$_{2019}$ back in 1930 and remains something$_{2019}$ now… It was nothing$_{1930}$ back in 1930 and remains nothing$_{1930}$ now. To attach a more liberal meaning to 'thing' or 'something' is not to increase the inventory of things or to bring something into being (nor is it to increase the inventory of things$_{1930}$ or to bring something$_{1930}$ into being). (p. 108)

While this may dispel the worry that interpretational expansionism presupposes set-theoretic creationism, we should note that the kinds of semantic shifts *Studd* posits are importantly different to the sort he describes above.

In fact, we will see later on that it is possible for the extension of the *'set' predicate* to systematically expand, without this necessarily being accompanied by an expansion in the range of our *quantifiers.*

The resulting view – which Studd (2019) calls *Third-Way Absolutism* – will be the subject of an in-depth discussion in §3.3 and §3.4; what will occupy us, in the interim, is *Studd's* account of the interpretational shifts that he believes underlie expansionism, which are first and foremost a function of a change in the interpretation of our quantifiers.

Studd's suggestion, specifically, is that we understand these shifts as akin to that which occurs *when an initially quantifierless linguistic community comes to adopt quantifiers.* By adapting a strategy first used by Vann McGee (2000) to show that this latter shift can plausibly be accounted for by the community's *learning,* and *following,* the rules of quantifier inference, Studd contends that coming to *use* quantifier expressions in the right sort of way, and in a systematic manner, can also lead to their domains' *expanding.*

Outlining McGee's original strategy, and Studd's adaption thereof, will be the task of the first two sections of this chapter; to these details, we now turn.

§3.1

McGee, Studd, and learning to quantify unrestrictedly

In his paper, 'Everything'(2000) – where he first presents his rule-based account of how our quantifiers come to range unrestrictedly – McGee is concerned to answer a worry that emerges in Hilary Putnam's discussion, in 'Models and Reality' (1980), of the Lowenhein-Skölem theorem. The theorem states that, for every formal language that has a model with an infinite domain, that same language has a model where the domain is (only) *countably* infinite; Putnam however, extends these concerns to *our entire language*. The thought is that if we were to restrict the quantifiers we use in our discourse to a (merely) countable subset – call this S – of the infinite domain we standardly take them to range over, *precisely the same* *sentences would nevertheless be true*. But, of course, the worry this leads to – which is the subject of McGee's (2000) paper – is that it is not possible to determine "whether a speaker is speaking a language in which the variables range over S or a language in which the variables range over everything" (p. 58). Given a child trying to learn our language, for instance. McGee complains that:

> … There is, it appears, nothing in [her] behaviour… that determines whether the variables the child employs and takes her elders to be employing range over S or over everything. This will be so even if the child shows a precocious ability for meta-linguistic reflection. (p. 58)

That "there is nothing you or I can do, say, or think to determine which mode of quantification we are using" (p. 59) is precisely the conclusion McGee aims to challenge, by demonstrating that unrestricted quantification over everything (as opposed, merely, to restricted quantification over S) is *learnable*. In particular, he argues that our correctly applying the standard rules governing the universal quantifier suffices to confer on the latter its intended (*qua* unrestricted) interpretation, in the form of its classical condition for *truth in a model,* $\mathfrak{A}$:

$(T-\mathfrak{A})$: $\forall v\varphi$ is $\text{true}_{\mathfrak{A},\sigma}$ iff every entity a with $a \in M_{\mathfrak{A}}$ is such that φ is $\text{true}_{\mathfrak{A},\sigma}[v/\,a]$.

where $M_{\mathfrak{A}}$ is the domain of the model, φ is a formula, and σ is an assignment of values to the variables of the language[87] (call this $\mathcal{L}$).

McGee's overall strategy is deceptively simple. He proceeds, firstly, by outlining the standard introduction- and elimination- rules governing the universal quantifier:

(UG): If $\Gamma \vdash \phi(c)$, then $\Gamma \vdash \forall x\phi(x)$

Where Γ is a set of formulae, and 'c' does not occur in ϕ, or in the formulae in Γ.

(UI): From $\forall x\phi(x)$, infer $\phi(c)$

He then opts to cash out the 'correctness' of these rules in terms of their being '*open-endedly valid*' – where this entails not only that the rules preserve truth in all models, but that they are, additionally, valid "not only within the language $\mathcal{L}$, but… will remain valid however the language may be enriched by the addition of new sentences" (p. 66).

With these assumptions in place, McGee proceeds simply by noting that the universal quantifier's being restricted to S threatens to violate the assumed validity of (UI):

> Suppose that our first-order variables are restricted, so that a lies outside their range, and imagine that we are permitted to name anything we like

[87] Since McGee considers a language with only constants (and no variables), his version of this clause would, strictly speaking, omit the variable assignment, σ.

> by an individual constant… Then we can take c to be a constant referring to a, and we can take F to be a predicate referring to those things that are within the range of our first- order variables. Then, $F(c)$ will be false, and $\forall x F(x)$ will be true, invalidating (UI). (p. 68)

As Shaughan Lavine (2006) points out in his discussion of McGee, there already exists a well-known predicate (first formulated by Quine (1969)) that applies to all and only those things within the range of '$\forall$': '$\exists y(y = x)$.' Thus, in line with Lavine's recommendations, we can define McGee's predicate F along the lines below:

$$F(x) =_{df} \exists y(y = x)$$

Given this, the latter's argument boils down to the claim that an object a's falling outside the range of our quantifiers threatens to render invalid the following instance of (UI):

$$(UI_C): \forall x \exists y(y = x) \vdash \exists y(y = c)$$

Critically, McGee (2000) does *not* take his argument to show that a – viz., the object denoted by the constant 'c' – *must* be in the range of '$\forall$'; he concedes, for instance, that it is possible "for there to be restrictions on what individuals it is possible to name by an individual constant or on what collections of individuals it is possible to name by a predicate" (p. 68).

Nevertheless, McGee *does* conclude that "the default value of '$\forall$', the value it takes when no occasion-specific restrictions are in effect, is quantification over everything" (p. 69); however, it is precisely here that the utility of McGee's absolutist strategy lies, for the purposes of the relativist *qua expansionist*. For the reader will recall that this brand of relativism flows not from the idea that our quantifiers are systematically *restricted*, but rests rather on the claim that it is *possible for the range of otherwise unrestricted quantifiers to increase*. But of course, to demonstrate this requires demonstrating, first, that it *is* possible for our quantifiers to range unrestrictedly: the first step of Studd's (2019) adaption of McGee consequently consists in doing just this.

However, where McGee's strategy focuses on the introduction- and elimination-rules governing '$\forall$' in the context of a standard first-order language, Studd's (2019) version takes place against the background of the Language of Generalised Quantifiers (henceforth, $\mathcal{L}_{GQ}$), where quantifiers are composed from a determiner (e.g., '$every$,' '$some$') and a predicate (e.g., '$thing$,' '$donkey$'), and denote properties of predicate extensions. Thus, where McGee takes the semantic value of '$\forall$' to consist in its conditions for truth-in-a-model, in the context of an extensional semantics for $\mathcal{L}_{GQ}$, by contrast, the extensions of quantifiers are *sets of predicate extensions*: $\mathcal{L}_{GQ}$'s universal quantifier – '$everything$' – thus denotes the property of being a predicate extension whereof the domain, M (which is itself the extension of the predicate '$thing$') is a subset. In turn, the extension (*qua* intended interpretation) of '$everything$,' is equivalent to the set of all unary predicate extensions that are co-extensional with M.

With this framework in place – and in line with McGee's (2000) proposal that "we try to identify a scenario in which creatures like ourselves acquire a pattern of usage recognizable as universal quantification" (p. 60) – Studd (2019) asks us to consider an idealized linguistic community (which he calls Community Q, or 'the Q'ers') who, at least, initially lack quantificational resources. Then, just McGee proceeds by way of assuming adherence to the introduction and elimination rules for '∀', Studd analogously assumes that the Q'ers, upon enriching their vocabulary[88] with determiners like '$every$,' learn how to use these expressions by *following particular rules*, in the form of patterns of inference formalized by means of sequents. For instance, Studd describes how a particular pattern of use might be formalized by the following sequent:

$$(U): \varphi_1, \dots, \varphi_m \Rightarrow \psi_1, \dots, \psi_n$$

the interpretation of which he glosses as follows: "members of the community find it incoherent both to accept each formula φ_p (with $p = 1, \dots, m$) and to reject each formula ψ_q (with $q = 1, \dots, n$)" (p.222). In the case of the determiner '$everything$,' Studd assumes that the Q'ers adopt the following patterns of use:

$$(U - every)^{89}: \quad every\eta\theta, \eta(\tau) \Rightarrow \theta(\tau) \qquad \frac{\eta(v), \varphi_1, \dots, \varphi_m \Rightarrow \psi_1, \dots, \psi_n, \theta(v)}{\varphi_1, \dots, \varphi_m \Rightarrow \psi_1, \dots, \psi_n, every\eta\theta}$$

Importantly, rather than cashing out the correctness of these rules, as McGee does, in terms of their *preserving truth in all models*, Studd opts to understand this in terms of the Q'ers' use-patterns' being open-endedly *materially sound*. For individual sequents like (U), this entails simply that there is no interpretation, $[\![\cdot]\!]$, and no assignment, σ, of variables over $[\![\cdot]\!]$, that makes all the formulae $\varphi_1, \dots, \varphi_m$, true, and all the formulae $\psi_1, \dots, \psi_n$, false. Similiarly, in the case of sequent *rules* (such as the right-hand sequent in $(U - every)$), material soundness entails that the soundness of the *initial* sequents of these rules under $[\![\cdot]\!]$ implies the soundness of their *end* sequents under $[\![\cdot]\!]$. Finally, that the sequents and sequent rules are materially sound in an *open-ended* way, just entails that they remain materially sound (under an interpretation, $[\![\cdot]\!]$, of Q-ish) even as the lexicon of Q-ish expands with new expressions[90].

With all this in place, Studd's proof, in the case of '$everything$,' proceeds in much the same way as McGee's does with '∀': on the assumption that the other expressions (besides the quantifiers) receive their intended interpretations, the material soundness of $(U - every)$ succeeds in conferring on '$everything$' its intended (unrestricted) interpretation. That is: Studd is able to demonstrate[91], on the basis of these assumptions, that the extension of '$everything$' – that is, $[\![everything]\!]$ – is identical to the set of all unary predicate extensions, based on the Q'ers' universe M, that are co-extensional with M:

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[88] See Appendix C for the complete lexicon of the Q'ers, as well as the sequent rules formalizing their patterns of use.
[89] This sequent is subject to the side-condition that v does not occur free in the end-sequent.
[90] See Appendix C for Studd's (2019) formal account of open-endedness.
[91] See p. 226 in Studd (2019) for his proof (which he calls *the Q-Quantification Theorem*).

$$[\![everything]\!] = \{\theta \subseteq M \mid M \subseteq \theta\}$$

In light of this result, Studd (2019) concludes that "[t]he initially quantifierless Community Q come to quantify by using quantifiers in the right way" (p. 227); more specifically – in line with McGee – he concludes that the Q'ers, in correctly using their quantifiers, come to quantify *unrestrictedly* over their universe.

However, having come this far, Studd departs from McGee –his idealized scenario with the Q'ers continues to unfold. Firstly, the latter acquire what Studd calls "rudimentary plural resources" (viz., the unary plural predicate '$things$,' the binary predicate, '$\prec$,' and plural variables). Moreover, "a philosophical divide starts to open within the imaginary community" – quantifier absolutists on one side, and relativists on the other – such that the latter group:

> … think that it remains to be seen whether they might not come to speak a more liberal language, one in which they might describe the expressive limitations of their present language…a sub-community calling themselves Community E resolve to isolate themselves from the rest of their community, and attempt to do just this. (p. 231).

The resulting lexicon of Community E (or 'the E'ers') is almost entirely identical to the Q'ers' lexicon, save for the addition of a set-term operator, '$\{\cdot\}$,' governed by the following, simple syntax: where 'xx' is a plural variable, '$\{xx\}$' is a singular term. The operator '$\{\cdot\}$' is thus a "singularizing device" in Linnebo and Florio's (2021) sense – however, to make room for the possibility that the pluralities denoted by some plural variables *fail* to form sets, the semantics of '$\{\cdot\}$' allow for it to be *undefined* for certain plural terms[92].

Moreover, in this extended scenario, Studd allows for variable assignments over *either* an interpretation of Q-ish (viz., $[\![\cdot]\!]_Q$) or one of Eish (viz., $[\![\cdot]\!]_E$). This makes room for the possibility that a (variable) expression which *fails* to denote in Q-ish, may nevertheless denote in E-ish.

Finally, the E'ers come to adopt specific, *trans*-language patterns of use[93], which Studd formalizes by means of labelled sequents, like (TU) below:

$$(TU){:}\ v_1{:}\ \varphi_1, \dots, v_m{:}\ \varphi_m \Rightarrow \zeta_1{:}\ \psi_1, \dots, \zeta_n{:}\ \psi_n$$

Like the intra-language sequent, (U), (TU) can be understood in terms of the E'ers' finding it incoherent to accept each of the formulas $\varphi_1, \dots, \varphi_m$, as uttered in language v, and to reject each of the formulas $\psi_1, \dots, \psi_n$, as uttered in language ζ[94] (in this context, the language

[92] This is necessary to avoid running afoul of a plural version of Cantor's theorem (it also avoids begging the question against the *opponent* of Collapse-type principles, in the context of Studd's expansionist theorem).

[93] See Appendix C for the rules governing these patterns of use, as well as the overall details of the E'ers' lexicon (and its interpretation).

[94] While the example of (TU) does not reflect this, it is possible to have multiple formulae of different languages occur on both sides of the sequent arrow – for an example of this, see the stability sequent rules, $(U_E{-}\in)$ and $(U_E{-}\prec)$, governing '$\in$' and '$\prec$' respectively, in §3.3.

variables 'v' and 'ζ' would be replaced with either an 'E,' or a 'Q' indexing E-ish or Q-ish respectively).

Of course, given that the lexicon of E-ish expands Q-ish only with the set-term operator, '$\{\cdot\}$,' the most important trans-language sequents are those governing its interaction with the plural variables of either language. On this front, Studd proposes that the E'ers' use-patterns be encoded by the following sequents:

$$Q\colon things(vv) \Longrightarrow E\colon thing(\{vv\})$$

$$(U_E - \{\cdot\})\colon \qquad Q\colon things(vv), Q\colon x \prec vv \Longrightarrow E\colon x \in \{vv\}$$

$$Q\colon things(vv), E\colon x \in \{vv\} \Longrightarrow Q\colon x \prec vv$$

With this architecture in place, Studd proceeds to present his expansionist argument: he argues that the E'ers' correctly using the expressions of their expanded lexicon – in line with the sequents above – not only ensures that their quantifier, '$everything$,' ranges unrestrictedly over *their* universe (M_E); it also ensures that this universe *strictly more expansive* than that of the Q'ers (M_Q).

To arrive at this conclusion, Studd again assumes that the trans-language patterns of use adopted by the E'ers are open-endedly materially sound: for those (trans-language) sequents containing formulae from both Q-ish and E-ish, this entails that the sequents be interpreted by a *pair* of interpretations, $\langle [\![\cdot]\!]_Q, [\![\cdot]\!]_E \rangle$; in turn, the *open-ended* material soundness of relevant sequents amounts to there being no variable assignment, over either $[\![\cdot]\!]_Q$ or $[\![\cdot]\!]_E$, that renders the formulae to the left of the sequent arrow true, and those on the right, false[95]. That an E-ish use-pattern is *open-endedly* materially sound under a pair of interpretations, $\langle [\![\cdot]\!]_Q, [\![\cdot]\!]_E \rangle$, just entails that this use-pattern remains sound, no matter how the respective lexicons of E-ish and Q-ish expand[96].

Thus, given the assumption that the E'ers' use-patterns are open-endedly materially sound under a pair of interpretations $\langle [\![\cdot]\!]_Q, [\![\cdot]\!]_E \rangle$, Studd is able to prove that the E-ish interpretation of the pair, $[\![\cdot]\!]_E$, is "a proper stable expansion" of the corresponding Q-ish interpretation of the pair, $[\![\cdot]\!]_Q$ (where an interpretation of a language, $[\![\cdot]\!]_v$, stably expands that of another, $[\![\cdot]\!]_\zeta$,if the domain of the first, M_v, is a subset of the domain of the second, M_ζ, and both interpretations agree on the extensions of their shared lexicon, relative to M_v).

The proof – which Studd (2019) dubs the *Expansion Theorem* – is essentially a version of Russell's Paradox: a plural term of Q-ish (call this *rr*) is taken to denote all those sets in M_Q that are not in the extension of $[\![\in]\!]_E$ – that is, those sets $a \in M_Q$ such that $\langle a, a\rangle \notin [\![\in]\!]_E$. The

[95] Material soundness for the sequent rules in this extended scenario consists, as before, in the preservation of material soundness between initial and end sequents.
[96] See Appendix C for the details of how Studd (2019) extends his technical account of open-endedness to E-ish.

soundness of the first of the trans-language sequents governing $\{\cdot\}$ then ensures that '$thing(\{rr\})$' is true as uttered in E-ish, and so that $\{rr\} \in M_E$; and, given the presence of additional axioms ensuring that $[\![\cdot]\!]_E$ and $[\![\cdot]\!]_Q$ agree on the extensions of '$\in$,' and of '$\prec$' (which axioms, in turn, ensure that $[\![\cdot]\!]_E$ *stably* expands $[\![\cdot]\!]_Q$), it follows that $\{rr\}$, on pain of paradox, lies outside of the universe M_Q.

But of course, since $\{rr\} \in M_E$, it follows that $M_Q \subset M_E$ – in other words, it must be that the domain of E-ish, as interpreted by $[\![\cdot]\!]_E$, is strictly larger than the corresponding domain of Q-ish, as interpreted by $[\![\cdot]\!]_Q$. Finally ,Studd points out that we can run a version of McGee's original proof in the context of Community E, so as to demonstrate that '$everything$' – as used in E-ish – ranges unrestrictedly over M_E[97].

In this way, we arrive at the more precise conclusion that *the range of the E'ers' quantifiers* has undergone an expansion – or, as Studd (2019) puts it: "in E-ish, '$everything$' expresses universal quantification over the new wider universe" (p. 237).

§3.2

Charity and subsentential interpretation

Importantly, that the E'ers, in shifting their *language,* have also shifted to a more *expansive domain*, is by no means an irresistible conclusion. There are a number of grounds on which absolutist might dispute it; chief amongst which is the crucial assumption that the E'ers' patterns of use – especially those governing '$\{\cdot\}$' – are *open-endedly materially sound*.

Studd (2019) motivates this in the case of both the E'ers, and the Q'ers by way of a circumscribed *Principle of Charity*. He is careful to emphasize that the idea is *not* that the Q'ers' and E'ers' sequents are to be interpreted according to a crude principle of *correctness maximization*, whereby "sentences are so interpreted as to maximize the proportion of instances of acceptance or rejection that are correct … i.e., where a truth is accepted or a falsehood rejected" (p. 229): for understanding Charity along these lines would have the unwelcome consequence that the *incorrect* use of expressions – if suitably systematic, and widespread – might lead to *a shift in the meanings* of the expressions incorrectly used. By way of example, Studd points to the possibility of "erroneous computations with hitherto-uncalculated-with integers" (p. 229): in that case, if Charity were understood in terms of correctness maximization (and if such erroneous calculations were sufficiently widespread), then our interpreting the relevant inferences charitably might have the effect of shifting the meanings of basic arithmetical terms as '+' and '×.'

Studd thus proposes instead that we understand his Principle of Charity in terms of *knowledge maximization*: *ceteris paribus*, sentences ought to be "so interpreted as to maximize the proportion of instances of acceptance or rejection that constitute knowledge"

[97] See p. 237 in Studd (2019) for his proof of the Expansion Theorem.

(p. 229)[98]. Importantly, understanding Charity along these lines is quite consistent with interpreting some of the inferences made by a community as *unsound* (for instance, interpreting as correct the problematic calculations with the as-yet-uncalculated-with integers mentioned above would arguably *decrease* the overall stock of knowledge in the mathematical community in question). At the same time, though, interpreting the inferences made by a particular community as materially sound *in general* – especially when these patterns of use are widespread and systematic (as, by hypothesis, the E'ers' and Q'ers' use patterns are) – *is* consonant with knowledge maximization.

All this leaves the absolutist about quantifiers with some wiggle room, though: for he may contend that no Principle of Charity, even one grounded in knowledge maximization, can mandate that we interpret E'ers' patterns of use for '$\{\cdot\}$' soundly – and that is because, by his lights, *no sound interpretation of these patterns is possible.* For, under the assumption that the E'ers' universe (M_E) does *not* extend the Q'ers' universe (M_Q), then the only possible way the following trans-language sequent:

$$(U_E - \{\cdot\}): Q: things(vv) \Rightarrow E: thing(\{vv\})$$

can be soundly interpreted would be if there were a member of M corresponding to the denotation of each plural term in the language. However, in the case where M is encoded as a set – and plural denotations are subsets of M – then this would clash with the well-known theorem that no set has all of its subsets as members.

Indeed, even where M is plurality-encoded – such that the plural denotations of the language are sub-*pluralities* (as opposed to sub*sets*) of M – the claim that there corresponds a member of M to each of its sub-*pluralities*, runs afoul of a *plural* version of Cantor's theorem, due to Øystein Linnebo and Salvatore Florio (2021, pp. 42 – 44). The proof is remarkably simple – we begin by defining a two-place predicate '$\preccurlyeq$,' where '$xx \preccurlyeq vv$' reads 'xx is a sub-plurality of vv':

$$xx \preccurlyeq vv \ \leftrightarrow_{df} \forall u(u \prec xx \ \rightarrow u \ \prec vv)$$

From this, we can define a plurality identity predicate, '$\approx$':

$$vv \approx xx \ \leftrightarrow_{df} (vv \preccurlyeq xx \ \land xx \preccurlyeq vv)$$

Then, the plural version of Cantor's theorem (henceforth, Plural Cantor) proceeds by showing that, given a plurality vv with at least two members[99], there is no surjective function from the members of vv to its sub-pluralities – that is, no function f that satisfies the following condition:

[98] Studd attributes this way of understanding Charity to Tim Williamson (2007).

[99] This assumption is necessary to prove, given the purported surjective function f, that there is a member of vv such that $x \nprec f(x)$. See pp. 42 – 44 of Linnebo and Florio (2021) for the full derivation of Plural Cantor.

$$\forall yy(yy \preccurlyeq vv \rightarrow \exists x(x \prec vv \wedge f(x) \approx yy))$$

For, on the assumption that the relevant function exists, it follows that there is a member of vv such that $x \nprec f(x)$. From this, and Plural Comprehension, it is possible define the following plurality, dd:

$$(DD): \forall x(x \prec dd \leftrightarrow (x \prec vv \wedge x \nprec f(x)))$$

Given that f is surjective, it follows that there is a $d \prec vv$ such that $f(d) \approx dd$. But, instantiating the universal quantifier in (DD) to d, we obtain:

$$d \prec dd \leftrightarrow d \nprec dd$$

The upshot of Plural Cantor is thus that no surjective function mapping the members of a given plurality to its various sub-pluralities is possible – in the current context, this entails that it is impossible for there to be a unique, singular-term extension corresponding to each plural term denotation of the language in question.

For the absolutist, this has the desired consequence that there *simply is no way* to interpret the E'ers' patterns of use soundly: for, certainly, these patterns are unsound over the original universe M_Q; and, since he rejects the possibility that the E'ers' universe, M_E, extends M_Q, it follows that no sound interpretation of the E'ers' use patterns is possible *tout court*. Indeed, the absolutist may even contend that the only way Studd's Expansion Theorem can go through, is on the assumption that it *is possible for domains to expand* – an assumption which, while naturally acceptable to the expansionist, the absolutist is liable to reject as close to question-begging.

In *Everything, More or Less*, Studd (2019) concedes that his expansion argument is unlikely to persuade the absolutist, for just this reason; however, in *Absolute and Relative Generality* (2011), Studd *does* attempt to answer the objection that his proof *presupposes* domain expansion:

> … the relativist will agree that the absolutist rejoinder is correct to the extent that the following is true: the patterns of use are sound only if the initial universe fails to be absolute. However, whether the relativist must *presuppose* the initial universe's non-absoluteness turns on a further substantive question in metasemantics. It depends on whether, in the cases in question, subsentential or sentential content has metasemantic priority over the other; whether, in the first instance, Charity operates at a sentential or subsentential level. (p. 208)

Studd's (2011) point seems to be this: if Charity functions as a guide for how we should interpret the expressions of a given linguistic community, this gives rise to the question of which expressions – or, rather, which *level* of expressions – are to be given priority in the interpretive process. For instance, in applying Charity to the (E-ish) pattern of use below:

$$(U_E - \{\cdot\}) \colon Q \colon things(vv) \Rightarrow E \colon thing(\{vv\})$$

we might begin at the *subsentential level* – namely, by assigning semantic values to the plural variable 'vv,' and the singular term '$\{vv\}$' – before proceeding to assess whether, on the basis of these possible valuations, it is possible to interpret the sequent $(U_E - \{\cdot\})$ soundly as a whole. However, if we were to opt for this approach, then the only way we might end up interpreting $(U_E - \{\cdot\})$ soundly would be if our *initial evaluations* of the possible denotations of '$\{vv\}$' (prior, that is, to our evaluation of '$thing(\{vv\})$') allowed for these to lie outside of the domain containing the denotation of 'vv.'

To that degree, then, according priority to the *subsentential* components of the E'ers' use-patterns in the course of interpreting the latter, might be said to presuppose expansionism; Studd (2011) himself concludes that "if a metasemantic account that… grants priority to subsentential expressions is the right one, then it seems that the absolutist is right to dig in his heels about soundness" (p. 209). However – and this seems to be Studd's (2011) point – to prioritise the subsentential components of the E'ers' expressions is not the only option available to us: for we might proceed instead by assigning semantic values, firstly, to the *complex* expressions '$things(vv)$' and '$thing(\{vv\})$' – so as to render the sequent sound – and then on *that* basis determine the denotations of 'vv,' and '$\{vv\}$.' In this case, expansionism would still follow: however, rather than our having to countenance this as a possibility at the *outset* of the interpretive process (viz., in our initial evaluations of the possible semantic values of such singular terms as '$\{vv\}$'), this would instead be the outcome of interpreting the other expressions in $(U_E - \{\cdot\})$ in line with soundness.

The need to distinguish between a sentential versus a sub-sentential approach to interpretation – specifically when it comes to sentences containing *quantifiers* – has been emphasized by other philosophers. For instance, in his 'Quantifier Variance and Indefinite Extensibility' (2017), Jarred Warren argues that those thinkers (he calls them 'bottom up theorists') who endorse the view that interpretation proceeds *firstly* via the sub-sentential components of expressions, tend to "make a quantifier's domain relevant to explaining the semantic properties of sentences containing the quantifier" (p. 88). In the context of E'ers' expressions – and, specifically, the trans-language sequents that govern how these relate to the corresponding expressions in Q-ish – a 'bottom up' approach would thus presuppose an already-given domain of candidate denotations for the singular terms of both languages. But of course, relative to a *single* domain – and whatever our attitude is towards Charity – it simply *cannot* be that there are as many denotations for such singular terms as '$\{vv\}$,' as there are possible assignments of values to the plural variable vv.

By contrast, on what Warren (2017) calls a 'top-down' approach to the interpretation, the "facts about a quantifier's domain are explained by facts about the semantic properties of whole sentences involving the quantifier, typically the truth conditions of these sentences" (p. 88). In the case of the E'ers and the Q'ers, such an approach would entail that the soundness of the patterns governing the former's use of '{·}' *is what explains the corresponding facts about the quantifier domains in question*: in particular (and as Studd claims), the soundness of these use patterns would *explain* why, accompanying the shift to E-ish, there has been a shift in – properly, an *expansion* of – the domain of Q-ish.

However – to return to the worry with which we began this section – Studd (2011) maintains that this does not *beg the question* against the absolutist. For the expansion of the initial universe indeed flows from a charitable interpretation of the E'ers' expressions that prioritises those at the sentential level – but, given independently-motivated reasons as to why our charitable interpretive practices *should* proceed in this way, it does not follow that this approach *presupposes* expansionism.

And, as Studd points out, it seems that we do possess independently-motivated reasons to prioritise sentential-level expressions in our application of Charity, when it comes to *mathematical* or *logical* expressions. To see this, we might consider the (contrasting) case where the subsentential components of our expressions, such as constants, denote concrete objects with which we can interact causally. This would pair well with an account of the truth conditions of the relevant sentences that has these depend on an initial, non-semantic link between the subsentential expression, and its referent – consequently, a 'bottom-up' approach to the interpretation of the relevant expressions would be justified.

However, the extensions of our *mathematical* expressions are, on most accounts, objects from which we are causally *isolated*. This naturally raises the question of just how these denotations are to be determined: and one idea – which goes back to Frege – is to settle this question by way of grasping the truth conditions of the *sentences* in which they occur. But precisely this seems to be what Studd (2011) is suggesting: the thought is that we "[t]hink of sentential content in an unstructured way, as giving the truth-conditions of sentences"; then, granting that use *determines* these unstructured contents, we can apply Charity, and thereby "work backwards to the contents of subsentential expressions" (2011, p. 209).

§3.3

Open-endedness and Stability

At this point, though, we do well to remember – in Studd's (2011) words – that "whether or not universe expansion occurs is a matter of *subsentential* interpretation… it is a question of whether or not new objects, objects outside the initial universe, come to be the extensions of singular terms" (p. 211)[100]; moreover, the assumption of material soundness is not sufficient

[100] This point is made quite vivid by Studd's (2019) Expansion Theorem: whether the E'ers' quantifier ranges more widely than the Q'ers', hinges on the question of whether the *singular* term $\{rr\}$ succeeds in denoting a (new) object.

by itself to secure the intended (*qua expansionist*) interpretation of the E'ers' subsentential expressions:

> … one way for [the sequents in $(U_E - \{\cdot\})$] to be [sound] is for the subsentential expressions… to receive their intended interpretations over a strictly wider universe. But there are many other ways that the subsentential expressions could be re-interpreted in order to achieve this result… granted that Soundness operates at the level of sentential content, *there is no unique backwards route from the content of a whole sentence to the content of its subsentential expressions* (p. 211 – 212, emphasis mine).

This raises the question of how it is that we *do* end up arriving at the desired (expansionist) interpretation of the E'ers' subsentential expressions – to understand this, it is useful to return to McGee's original demonstration, in 'Everything' (2000), that our first-order quantifiers range unrestrictedly. There, in relation to the worry that his strategy leaves room for uncertainty concerning the interpretation of the first-order variables, McGee notes the following:

> The rules of inference do something quite amazing. They create a uniquely defined semantic role for each of the connectives and quantifiers, where before there was none. But they do not do something magical. They do not create this uniquely determined semantic role out of nothing… they presuppose the whole apparatus of naming and predication, for *the bounds of quantification are fixed by determining to what things our names and predicates are able to refer*. (p. 68, emphasis mine)

Key to the success of McGee's strategy is thus the assumption – over and above the validity of the quantifier rules – that the other, non-logical parts of the language (viz., predicates and constants) receive *their* intended interpretation.

Importantly, Studd avails himself of precisely the same assumption in his own proof that the Q'ers – upon enriching their language with quantificational resources – come to quantify unrestrictedly. This ensures that the intended interpretation of the Q'ers' subsentential expressions is obtained by fiat; moreover, when it comes to the (more controversial) case of the *E'ers*, exactly the same outcome is obtained – viz., securing the intended (and *expansionist*) interpretation of their subsentential expressions – *to the degree that the E'ers' use of their expressions corresponds to the Q'ers.*

And, indeed, for the vast majority of the E'ers' expressions, these *do* correspond. For, save for the for the set-term operator, '$\{\cdot\}$' (and the expressions formed therefrom), the lexicons of E-ish and Q-ish are entirely indistinguishable. Over and above this, though – as we saw in §3.1 – the two interpretations $[\![\cdot]\!]_Q$ and $[\![\cdot]\!]_E$ featuring in Studd's Expansion Theorem also agree on the *interpretations* of these common expressions; at least, they do so, relative to the Q'ers universe M_Q[101].

This is a function of various 'Stability axioms' Studd adopts in the course of his proofs: in his Expansion Theorem, these take the form of additional trans-language sequents, which ensure that the binary predicates, '∈' and '≺,' as used in both Q-ish and E-ish, agree on their extensions in M_Q:

$$Q{:}\,u \in v \Rightarrow E{:}\,u \in v$$

$(U_E{-}\in)$: $$Q{:}\,thing(u), Q{:}\,thing(v), E{:}\,u \in v \Rightarrow Q{:}\,u \in v$$

$(U_E{-}\prec)$: $$Q{:}\,u \prec v \Rightarrow E{:}\,u \prec v$$

$$Q{:}\,thing(u), Q{:}\,things(vv), E{:}\,u \prec vv \Rightarrow Q{:}\,u \prec vv$$

Stability axioms play an analogous role in the *modal* context, in relation to a proof Studd (2019) provides of the modalised claim No Absolutely Comprehensive Domain$^{\diamond}$*:*

$$(No\ Absolutely\ Comprehensive\ Domain^{\diamond}){:}\ \Box\forall xx\Diamond\exists y{\sim}\big(y \prec^{\diamond} xx\big)$$

In that case, the relevant axioms concern both '∈' and the predicate 'ß'[102] (abbreviating 'is a set'):

$(STA{-}\in)$: $\Box\forall u\forall v(\Diamond(u \in v) \rightarrow u \in v)$
$(STA - ß)$: $\Box\forall u\big(ß^{\diamond}(u) \rightarrow ß(u)\big)$

The effect, however, is entirely the same: for just as the trans-language stability sequents governing '∈' and '≺' ensure that these expressions, as used in *both* E-ish and Q-ish, overlap in extension relative to M_Q, so too do their modal counterparts ensure that, *at a given domain in the potentialist hierarchy*, the predicates '∈' and '$\in^{\diamond}$'– and those of 'ß' and '$ß^{\diamond}$'– agree on their extensions.

Moreover, when combined, in each case, with the claim that the initial domain is *comprehensive* (or, in the modal case, the claim that such a domain potentially exists), the

[101] This, the reader will recall, was what made the E-ish interpretation of the pair a 'stable expansion' of its Q-ish counterpart.
[102] Studd incorporates this predicate due to his default theory's being the (impure $ZFCSU_p$. This will become useful in our discussion in §3.4 of *third-way absolutism* (the difficulties besetting which specifically emerge in the context of impure theories).

Stability axioms have the effect of collapsing any distinction *at all* between the languages under consideration (as opposed, merely, to the distinctions between the languages relative to the *initial domain* under consideration).

In the case of Studd's Expansion Theorem, for instance, the trans-language Stability sequent governing '∈' ensures that $[\![\in]\!]_Q = [\![\in]\!]_E \cap (M_Q \times M_E)$; but, when combined with the claim that M_Q is comprehensive (i.e., that $M_Q = M_E$), the result is that any difference in extension between $[\![\in]\!]_E$ and $[\![\in]\!]_Q$ falls away altogether (i..e, $[\![\in]\!]_Q = [\![\in]\!]_E$).

In a similiar way, if we assume a slightly stronger principle than the standard absolutist claim, Absolutely Comprehensive Domain$^\diamond$ – call this, *Actually Comprehensive Domain*$^\diamond$:

$$(Actually\ Comprehensive\ Domain^\diamond): \exists xx \Box \forall z (z \prec^\diamond xx)$$

then this leads to what Studd calls the 'flattening' of the potential hierarchy of interpretations[103], into a single, maximal interpretation:

$$\langle M_\infty, E_\infty, S_\infty \rangle$$

where M_∞ is an all-inclusive domain, and E_∞ and S_∞ the (maximal) extensions of 'ß' and '∈,' when respectively interpreted over this domain:

$$M_\infty = \bigcup_{i \in I} M_i \qquad S_\infty = \bigcup_{i \in I} S_i \qquad E_\infty = \bigcup_{i \in I} E_\infty$$

However, this is accompanied by the collapse of any distinction between statements of the *modal* language of plural set theory ($\mathcal{L}_{MPS}$), and those of its non-modal counterpart ($\mathcal{L}_{PS}$). As Studd demonstrates (by means of a theorem he dubs 'Flattening'[104]) this means that every modalised formula of $\mathcal{L}_{MPS}$ is provably equivalent, in the *MPFO*, to its *non-modal* counterpart:

$$(Flattening): Actually\ Comprehensive\ Domain^\diamond \vdash_{MPFO} \phi^\diamond \leftrightarrow \phi$$

Given this (extended) discussion, the reader may be wondering just what is so controversial about Studd's use of Stability axioms in securing the intended interpretations of the (subsentential) expressions he deploys in his proofs. The answer is this: by ensuring (for instance) that the languages of Q-ish and E-ish are almost entirely indistinguishable, such

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[103] We saw in §2.5 that, for the interpretational expansionist, the hierarchy of quantifier domains (and their associated interpretations) can be represented by the following sequence:

$$\langle M_0, E_0, S_0 \rangle, \langle M_1, E_1, S_1 \rangle, \langle M_2, E_2, S_2 \rangle, \langle M_3, E_3, S_3 \rangle, \ldots$$

where each M_i (for all $i \in I$, for some index set I) corresponds to a domain – and E_i and S_i to the extensions, at M_i, of 'ß' and '∈' respectively.

[104] See pp. 257 – 258 of Studd (2019) for his derivation of this theorem. The result extends to the analogues of these languages that include urelements.

axioms threatens to go against a key stipulation McGee makes in the course of *his* original demonstration; one which Studd (2019) himself purports to adopt, in his own proofs. This is the stipulation that the rules governing our quantifier expressions remain materially sound – or, in McGee's context, valid in all models – *no matter how we enrich our language with new expressions*: in other words, it is the stipulation that the rules preserve truth *open-endly.*

To get a handle on why Studd's use of Stability may undermine this assumption, it is useful, once again, to return to McGee – or, rather, to Shaughan Lavine's (2006) extensive critique of the latter's arguments. We saw in §3.1 that these are intended to answer the worry, given a child who is learning to quantify, that there is "nothing in [her] behaviour… that determines whether the variables the child employs and takes her elders to be employing range over S or over everything" (p. 58), where S is a – merely countable – subset of absolutely everything. McGee's claim, in turn, is that "S-quantification is not learnable" (p. 59), which he demonstrates – assuming that learning to quantify entails following the classical quantifier rules – by showing that our quantifiers' being *restricted* to S would violate these rules, specifically by rendering the following instance of Universal Instantiation invalid:

$$(UI_C)\colon \forall x \exists y(y = x) \vdash \exists y(y = c)$$

where c denotes an object outside the range of '$\forall$.'

Importantly, Lavine (2006) emphasizes that the choice of 'c' – besides being arbitrary – must also *be open-ended*: this, in line with the open-endedness of the *rules*, which "apply not just in the present logical language but in any extension of it, whether… envisioned or not" (p. 113). Consequently, Lavine contends that the free variable in $F(x)$ has less the character of "a referential or substitutional variable" (p.119), and is more akin to a "full schematic variable," where "what counts as an acceptable substitution instance is open ended and automatically expands as the language in use expands" (p. 118).

Given this, Lavine provides the following, *schematic* characterization of McGee's predicate $F(x)$, which Lavine calls "the everything axiom" (p. 124):

$$(E)\colon \exists y(y = s)$$

where 's' is a full schematic letter.

With (E) in hand, Lavine (p. 106) notes that if, as per McGee's set-up above, the difficulty *were* accounting for how we learn S-quantification – when construed as a form of *restricted* quantification, relative to an absolutely comprehensive universe of discourse – then the everything axiom, together with McGee's (2000) other assumptions, *would* suffice to show that 'S-quantification is not learnable.' However, Lavine argues that, in framing the debate in this manner, McGee has *fundamentally mischaracterized the issue at hand*:

> The actual challenge is… rather different. We are to envision twin users of *two languages* who differ in only one respect: One takes the quantifiers to

> range over universe of discourse S, the other takes them to range over universe of discourse *T.* (p. 110, emphasis mine).

That is: rather than considering the (potentially restricted) quantifiers of a *single* language, $\mathcal{L}$, Lavine contends that we should instead consider *two* languages – say, $\mathcal{L}_1$ and $\mathcal{L}_2$ (with respective domains, D_1 and D_2) – where each comes equipped with its own everything axiom, relativized to the domain in question:

$$(E_1): \exists y \in D_1(y = s) \qquad\qquad (E_2): \exists y \in D_2(y = t)$$

The 'actual challenge,' in *this* context, is to show that the validity of the standard quantifier rules ensures that $D_1 = D_2$. As Lavine puts it, "to be in D_1 is to exist$_1$ … and to be in D_2 is to exist$_2$, and we want to know if we can prove that existence$_1$ and existence$_2$ coincide" (p. 127); or, put in terms of the languages' respective everything axioms: for a singular term c to be a substitution instance for the schematic variable s in E_1 is for s to denote$_1$ (and, similiarly, for it to be a substitution instance for t in E_2 is for it to denote$_2$) – and we want to know if every term that denotes$_1$ also denotes$_2$ (and vice versa).

However*,* should we attempt to prove this *using McGee's original strategy*, Lavine points out that a potentially devastating difficulty soon emerges:

> … if an arbitrary element a of D_2 should happen *not* to be in D_1, that is, if a should happen not to exist$_1$, then from the perspective established by $\mathcal{L}_1$ it is *not possible to let c denote a*—a constant symbol must denote$_1$ an object$_1$, and this *c* does not. In fact, the addition of *c* to the language of the logic $\mathcal{L}_1$ will *only be acceptable if a is in* D_1. (p. 127, emphasis mine)

The key take-away, from the above, is this: for McGee's strategy to work (viz., demonstrating that our quantifiers' *failing* to range unrestrictedly would violate the validity of the rules), it must be permissible for constants of the language to denote objects which may lie beyond its domain of discourse. Lavine's claim, however, is that this *simply isn't possible*. For to assume that a constant is an appropriate substitution instance for a given language's everything axiom – or, as McGee puts it, to assume that 'we are permitted to name anything we like by an individual constant' – *just is to assume that the object denoted by that constant falls within the bounds of that language's domain*. But this, of course, is precisely what McGee's strategy *is meant to demonstrate*: as Lavine (2006) pointedly puts it, "[t]o simply assume that *a* is in D_1 is to beg the question at issue… whether there could be a universe of discourse D_2 not contained in D_1" (p. 127).

This leads Lavine to conclude that:

> McGee's argument for the learnability of unrestricted quantification turns out to be at best a self-consistency argument—it, at best, could hope to

show that, assuming that there is such a thing as unrestricted quantification, unrestricted quantification is learnable. (p. 128)

More specifically: where McGee's argument fails – or, at least, *would* fail to convince the sceptic of absolutely general quantification – is in being unable to prove that there is a *unique* domain of absolutely everything. This is a point Patrick Dieveney (2014) makes, in his discussion of Lavine's critique of McGee: referring to the speakers of $\mathcal{L}_2$ and $\mathcal{L}_1$ respectively as 'Harry' and 'Sally', he notes that "we cannot prove that Harry and Sally are, in fact, quantifying over the same domain, i.e., a domain of absolutely everything" (p. 302), because:

> To prove that the domain of Sally's quantifiers is the same as Harry's quantifiers, it must be acceptable for Sally to add the name [c] to her language. Addressing this problem requires an account of *when it is acceptable to expand a given language to include a new name* (p. 303, emphasis mine).

However – to link all this back to the central topic of this section – the question of whether it is acceptable to expand a language to include a new name is akin (if not equivalent) to the question of the extent, or nature, of its *open-endedness.* In turn, Lavine's critique of McGee can plausibly be recast as the complaint that the latter has adopted far too *liberal* an understanding of open-endedness – one whereby it is possible to enrich a given language with *any new expression whatsoever.*

In the context of *multiple* related languages – such as the case of Harry and Sally – this amounts to the (unjustified, for Lavine and Dieveney) assumption that any constant of Harry's language can be added to Sally's, and vice versa. In the context of multiple related and *evolving* languages – such as E-ish and Q-ish – the assumption (albeit, in McGee's case, tacitly made) is that, in the open-ended process of language *enrichment*, there has not also been a language *shift* (where this shift would be accompanied by a corresponding shift in the meaning – indeed, the *extent* – of 'everything').

Just this nuance is what Studd exploits, in *his* account of quantifier expansion: for it is only by virtue of the Q'ers' and the E'ers' *speaking different languages* that it remains open to him, firstly, to argue that the Q-ers' quantifiers range comprehensively over *their* universe, while *also denying* that the Q'ers' quantifiers range over the object denoted by '$\{rr\}$' (this latter fact being, of course, a result of $\{rr\}$'s failing, so to speak, to denote$_Q$ – indeed, its *not being expression of Q-ish*).

However, it is at just this point that difficulties emerge for Studd's account of domain expansion, specifically concerning the role played by open-endedness therein. For while we might concede that there *is* a difference between the open-ended *enrichment* of a single language, on the one hand, and its *shifting* to become a new language, on the other, it is not at all clear that the E'ers' split from the Q'ers *should be classed in terms of the latter, rather than the former.*

For as we've already noted, not only are the *lexicons* of E-ish and Q-ish virtually indistinguishable; so too – as a function of Studd's adoption of the Stability axioms – are the *interpretations* of their common vocabulary essentially identical. But, given this, the absolutist might plausibly contend that we have good reason to treat the terms of E-ish – including the constants formed by means of '$\{\cdot\}$' – as the expressions, not of a *new* language, but rather, of a (comparatively minor) extension of Q-ish: a version we arrive at, moreover, in the process of *open-endedly enriching the latter's lexicon*.

In turn, to the degree that Studd *denies* this claim – that is, to the degree that he maintains that Q-ish and E-ish really are *different languages* – the absolutist might argue that Studd's argument violates, if not the (quite technical) letter, then at least the *spirit* of open-endedness. In particular, he might contend that, if Studd *were* in fact being faithful to this spirit – as captured by McGee's original strategy – then the open-endedness of the patterns of use governing the Q'ers' quantifiers would ensure that these *also range over the new constants formed by means of* $\{\cdot\}$.

This way, of course, lies paradox; however, in view of the foregoing, the contradictions take on a new significance. For our discussion suggests that these emerge not only from a combination of Collapse-type principles with an absolutely comprehensive domain – rather, they require the additional ingredient of *Stability.*

This soon becomes evident if we attend to the details of Studd's (2019) Expansion Theorem: there, the relevant Collapse-type principles are encapsulated in the E'ers' use patterns for '$\{\cdot\}$', as captured by $(U_E - \{\cdot\})$. These ensure that the object denoted by the E-ish term '$\{rr\}$' exists; in turn, the assumption that M_Q is comprehensive ensures that $\{rr\} \in M_Q$, and so that $\{rr\}$ is a candidate for self-membership (we have it that $\{rr\} \in \{rr\}$ if and only if $\langle\{rr\}\ ,\{rr\}\rangle \notin [\![\in]\!]_E$).

However, *this is not quite yet a paradox*: full-fledged inconsistency only emerges if – as Stability guarantees – $[\![\in]\!]_E$ and $[\![\in]\!]_Q$ *agree on their extensions in* M_Q. Only in that case does the truth, *in Q-ish,* of $\{rr\} \in \{rr\}$ entail – via $\langle\{rr\},\{rr\}\rangle \in [\![\in]\!]_Q$ and the stability of $\in$ – that $\langle\{rr\},\{rr\}\rangle \in [\![\in]\!]_E$, and thus the paradoxical conclusion that, contrary to our assumption, $\{rr\} \notin \{rr\}$.

Stability plays a similarly paradox-generating role, in the modal context, in Studd's (2019) proof, via *reductio,* of No Absolutely Comprehensive Domain$^{\diamond}$. His own reasoning proceeds straight via the Flattening theorem we met earlier on in this section:

$$(Flattening): Actually\ Comprehensive\ Domain^{\diamond} \vdash_{MPFO} \phi^{\diamond} \leftrightarrow \phi$$

where the reader will recall that Actually Comprehensive Domain$^{\diamond}$ is a slightly stronger version of the absolutist's claim Absolutely Comprehensive Domain$^{\diamond}$:

$$(Actually\ Comprehensive\ Domain^{\diamond}): \exists xx\Box\forall z(z \prec^{\diamond} xx)$$

Moreover, in the place of Collapse$^{\diamond}$, Studd instead assumes a weaker principle which he calls ‘Unlimited Collectability for Sets’:

$$(UC\text{ß}): \Box\forall xx(\text{ß}xx \rightarrow \Diamond\exists y\left(\text{ß}^{\diamond}(y) \wedge \Box\forall u\big(u \prec^{\diamond} xx \leftrightarrow u \in^{\diamond} y\big)\right))$$

where ‘$\text{ß}xx$’ is a plural[105] predicate defined as follows:

$$\text{ß}xx =_{df} \forall u(u \prec xx \rightarrow \text{ß}(u))$$

Thus, $(UC\text{ß})$ amounts to the claim that, to absolutely any plurality of *actual sets*, there corresponds a *potential* set[106]. Of course, given Flattening, this emerges as none other than the (non-modalised) claim, Collapse: paradox is then obtained in the usual way (namely, via the instance of plural comprehension testifying to a plurality comprising all non-self-membered sets).

However, we can run a version of the argument more akin to that used in Studd’s Expansion Theorem, and so make vivid, as we did there, the crucial role played by Stability. For just as, in that context, the relevant plurality consists of all the sets $a \in M_Q$ such that $\langle a, a\rangle \notin [\![\in]\!]_E$, we can, in the modal context, obtain the following instance of plural comprehension:

$$1. \forall y[y \prec rr \leftrightarrow (\text{ß}(y) \wedge \sim\big(y \in^{\diamond} y\big))]$$

Then, given that the plurality rr consists only of actual sets, we can – assuming the truth of $(UC\text{ß})$ – secure the existence of the potential set, R, comprising all and only the non-self-membered sets among the rr:

$$2.\ \text{ß}^{\diamond}(R) \wedge \Box\forall u\big(u \prec^{\diamond} rr \leftrightarrow u \in^{\diamond} R\big)$$

For ease of exposition, let’s call the comprehensive domain to which Actually Comprehensive Domain$^{\diamond}$ testifies, mm – the result is that the latter emerges as the following claim testifying to the comprehensiveness of mm:

$$(Actually\ Comprehensive\ Domain^{\diamond}{}_{mm}): \Box\forall z(z \prec^{\diamond} mm)$$

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105 This is an example of a *distributive* predicate, where a predicate P is distributive if and only if the following equivalence holds:

$$Pxx \leftrightarrow \forall u(u \prec xx \rightarrow P(u))$$

See pp.16 – 18 of Florio and Linnebo (2021) for a discussion of distributive predicates, and their counterpart in the guise of *collective* plural predicates - which can *only* be true of pluralities, as opposed to (also) true of their members.

106 Analogously, in the context of the E’ers and the Q’ers, the trans-language sequents for ‘{·}’ entail that the denotation of any plural term in *Q-ish* corresponds to the denotation of a singular (set) term in *E-ish*).

From the comprehensiveness of mm, it follows both that $R \prec mm$, and that $rr \preccurlyeq mm$ (i.e., that the rr are a subplurality of mm); that mm is comprehensive also means that the quantifiers in (1) and (2) are effectively (if implicitly) relativized to mm. All this ensures that R falls within the range of the quantifiers in both (1) and (2) – in particular, it ensures that R is a candidate for self-membership, via its being one of the rr.

However – as we saw with Studd's Expansion Theorem – this is *not yet a paradox*. For, as per (1), the rr consist, firstly only of *actual* non-self-membered sets. Consequently, to the degree that R is *merely potential* – that is, it is the case both that $ß^{\Diamond}(R)$, and that $\sim ß(R)$ – then it cannot be a member of this plurality, and inconsistency is thus avoided.

However, paradox beckons once more if we adopt the Stability axiom governing ß:

$$(STA - ß): \Box\forall u\big(ß^{\Diamond}(u) \rightarrow ß(u)\big)$$

This guarantees that relative to any given domain in the hierarchy, $ß^{\Diamond}$ and ß are identical in extension: thus, relative to mm in particular, it ensures that R's being a potential set entails that it is *also* an actual set. This means that it is possible for R to fall amongst the rr; in turn, when combined with the Stability axiom for ∈, below:

$$(STA{-}\in): \Box\forall u\forall v(\Diamond(u \in v) \rightarrow u \in v)$$

which ensures that $\sim(R \in R)$ implies $\sim\big(R \in^{\Diamond} R\big)$[107], full-blown paradox swiftly follows, in the form of the below, now-familiar conclusion:

$$3. R \in R \leftrightarrow \sim(R \in R)$$[108]

§3.4

Third-way absolutism and the Objection from Super-Meanings

Thus far, we have taken the set-theoretic paradoxes to indicate a basic tension between *Collapse-type principles*, like $(UCß)$ above, and principles like Plural Comprehension$^{\Diamond}$. However, the foregoing discussion suggests a way out of this dilemma – or, what is now better characterised as a *trilemma* – for the absolutist: rather than rejecting either Plural Comprehension$^{\Diamond}$ (and so, quantifier absolutism) or intuitive principles like Collapse, he might instead reject the Stability axioms that govern how (say) the expressions of $\mathcal{L}_{MPS}$, and those of $\mathcal{L}_{PS}$, interact.

[107] The reverse direction – namely, $\sim\big(R \in^{\Diamond} R\big) \rightarrow \sim(R \in R)$ – follows from the semantics for, and rules governing, '□.'
[108] Proof sketch: for the left-to-right direction, we assume $R \in R$, which entails $\sim\big(R \in^{\Diamond} R\big)$, and thus $\sim(R \in R)$. For the right-to-left direction, we assume $\sim(R \in R)$: this, given the Stability axiom for ∈, ensures that $\sim\big(R \in^{\Diamond} R\big)$; then, given the Stability axiom for ß, which ensures that $ß^{\Diamond}(R)$ is true, we arrive at the conclusion that $(R \in R)$ after all.

The resulting position – which Studd (2019) calls 'Third-Way Absolutism' – has a 'best-of-both-worlds' flavour. On this view – versions of which have been proposed by Gabriel Uzquiano (2015) and Tim Williamson (1998) – the paradoxes are viewed, in line with quantifier relativism, as symptomatic the set-theoretic universe's being indefinitely extensible. However, where Uzquiano and Williamson depart from the relativist, is with the assertion that this manifests as shifts in the *meanings* of our core set-theoretic predicates – like 'ß' and '∈' – as opposed to underlying shifts in the domains of our *quantifiers.* Thus Williamson (1998), for instance, describes the upshot of the paradoxes as being that, "given any reasonable assignment of meaning to the word 'set' we can assign it a more inclusive meaning while feeling that we are going on in the same way" (p. 20). In much the same vein, Uzquiano (2015) describes how we can understand the iterative conception of set in terms of "a cumulative process of *reinterpretation of the set-theoretic vocabulary*":

> … the cumulative process of reinterpretation is indefinitely extensible: no matter how we interpret the primitive set-theoretic predicates… we can, if we like, move to a more comprehensive interpretation. (p. 151)

Importantly, Studd (2019) points out that, like the expansionist's hierarchy of quantifier domains, the ever more comprehensive interpretations of 'ß' and '∈' can themselves be taken to form something like a potential hierarchy:

$$\langle M_\infty, E_0, S_0\rangle, \langle M_\infty, E_1, S_1\rangle, \langle M_\infty, E_2, S_2\rangle, \langle M_\infty, E_3, S_3\rangle, \ldots$$

This hierarchy, however, is very much absolutist-friendly: for the shifts in the interpretations of 'ß' and '∈' – corresponding, respectively, to each S_i and E_i in the sequence – take place relative to an *all-encompassing domain*, M_∞.

A key consequence of this is that the sets that potentially exist at 'later' interpretations fall within the range of the *non-modal* quantifiers that range over this comprehensive domain. Thus, for instance, the third-way absolutist can quite happily assert the following:

$$(1): \Diamond\exists x ß(x) \rightarrow \exists x \Diamond ß(x)$$

This is an instance of the more general *Barcan Formula* (specifically, its '◇' form) :

$$\left(Barcan\ Formula^{\Diamond}\right): \Diamond\exists x \varphi(x) \rightarrow \exists x \Diamond \varphi(x)$$

$$(Barcan\ Formula^{\Box}): \forall x \Box \varphi(x) \rightarrow \Box \forall x \varphi(x$$

However, both this formula and its converse, below:

$$(Converse\ Barcan\ Formula^{\Box}): \Box \forall x \varphi(x) \rightarrow \forall x \Box \varphi(x)$$

$$\left(Converse\ Barcan\ Formula^{\Diamond}\right): \exists x \Diamond \varphi(x) \rightarrow \Diamond \exists x \varphi(x)$$

must be rejected by the expansionist – for, contrary to quantifier relativism, the formulae together ensure that the domain ranged over by the modalised quantifiers ($\Diamond\exists$' and '$\Box\forall$') is *identical to those ranged over by their non-modal counterparts* (namely, '$\exists$' and '$\forall$').

This becomes apparent if we consider the instance of the Barcan Formula that is (1) above: for this entails that the witness to the *potential* existence claim in the antecedent also falls within the range of the *non-modal* quantifier governing the consequent. The latter, of course, is unacceptable to the expansionist: consequently, while she would accept '$\Diamond\exists x ß(x)$' – claims of this sort are a cornerstone of her view – she rejects the implication, '$\exists x \Diamond ß(x)$'.

By contrast, for the third-way *absolutist*, that '$\Diamond\exists x ß(x)$' implies '$\exists x \Diamond ß(x)$' is, arguably, the cornerstone of *his* view: this is because, on his account, every possible future member of the extension of 'ß' – that is, every satisfier of $\Diamond ß(x)$ (or, as we've often written it, $ß^{\Diamond}$) – is *already an element of the overarching domain M_∞*.

However, it is precisely here that the trouble begins for the third-way absolutist: specifically, when the attempt is made to marry the latter with any set theory that includes urelements (so-called 'impure' set theories[109]). This is because the comprehensiveness of M_∞ ensures, as Studd (2011) puts it, that "every future set lurks hidden among the urelements" (p. 142) – where the '*urelemente*' predicate is defined, in the standard way, as 'non-set':

$$Ur(x) =_{df} {\sim}ß(x)$$

Studd's point is that the extension of Ur contains, at each stage in the absolutist-friendly potential hierarchy, both 'standard' urelements *and* every 'potential' set that, at the relevant stage, is yet to be reinterpreted as 'actual.' But, as Stewart Shapiro (2003) remarks, this poses a difficulty for any impure set theory that includes the so-called 'urelemente axiom,' whereby there exists a set comprising every urelement:

$$(UA): \exists y(ß(y) \land \forall z({\sim}ß(z) \rightarrow z \in y)$$

The worry is that, if all future sets are indeed 'hidden among the urelements' – and if the quantifiers in (UA) indeed range over absolutely everything – then the axiom guarantees the existence of an *actual* set that has, as members, *every potential, future set*. Indeed, as interpreted at the first stage of the absolutist-friendly potential hierarchy, the only sets that *aren't* in the extension of the predicate 'Ur' (which, by (UA), forms a set – call this 'U') are those 'actual' sets in the current extension of 'ß' (namely, those sets $s \in S_0$).

However, should the absolutist countenance principles as 'Unlimited Collectability for Sets,'– the acceptance of which was taken to be an *advantage* for this version of absolutism – then the existence of an (actual) universal set is not far off. For $(UCß)$ guarantees the

[109] Most set theories are so-called 'pure' theories – however, as Studd (2011) points out, the ability to extend these theories to encompass urelements is an important requirement when it comes to applying set theory to non-mathematical areas of enquiry (consequently, Studd's (2019) own default theory is $ZFCSU_p$).

existence, at a given stage of the reinterpretative process, of a (potential) set of all sets in the extension of 'ß' at that stage – thus, at the first stage, we obtain the potential set P, comprising every actual set $s \in S_0$. However, the potentiality of P is (so to speak) only 'temporary' – consequently, at the subsequent stage (where it is reinterpreted as 'actual') it would be possible to obtain the union of it and U, thereby securing the existence of a *truly* universal set: one comprising all the actual sets at that stage (viz., all $s \in S_1$) together with every potential set in the hierarchy.

Shapiro (2003) thus concludes that, in the context of (modal) third-way absolutism, McGee's urelemente axiom "fails and fails badly in any language which has a set theory like ZFC and a quantifier ranging over absolutely everything" (p. 477). However, the incompatibility of third-way absolutism with impure set theories extends beyond those that include (UA). Studd (2019), for instance, considers the standard Pairing axiom of (impure) set theory:

$$(P): \forall x \forall y \exists u (ß(u) \land \forall z (z \in u \leftrightarrow (z = x \lor z = y))$$

where the quantifiers, in line with third-way absolutism, range over absolutely everything, including future potential sets. This ensures that, for each such set, we can obtain its singleton (which, itself, is an 'actual' set); from this, it follows that – relative to the stage where (P) is itself interpreted – the plurality of actual sets at that stage (call this plurality, ss) is of the same, or greater, cardinality as the plurality comprising every *potential* set (call this plurality, pp) at the same stage.

However, if 'Unlimited Collectability for Sets' is also true, it follows that there corresponds, to each plurality of *actual* sets at the stage in question, a *potential* set that has these (actual) sets as elements: in other words, we can assign, to every sub-plurality $aa \preccurlyeq ss$, one of the pp. But, by Plural Cantor, this should not be possible if, in fact, the ss *are* at least as numerous as the pp. For the former ensures that there are more *sub-pluralities* of the ss than there are *member*s of ss: consequently, if every such sub-plurality corresponds to one of the pp, it follows that the pp must be of a strictly greater cardinality than the ss – contrary to what we could expect by Pairing.

Of course, all of the foregoing difficulties boil down to the need, in the context of third-way absolutism, to distinguish between urelements *qua* potential sets, and urelements *qua urelements* (this, for the purpose of restricting the range of quantifiers in axioms like (P) either to the latter, or to actual sets). Consequently – as Studd (2019) concedes – the third-way absolutist may point out that the issue lies not with the idea that the potential sets 'lurk hidden among the urelements' of an all-encompassing domain, but rather with the *formulation* of axioms like (P):

> … He may object that (these axioms) fail to respect an important distinction between 'permanent' urelements such as donkeys and spacetime points, which never enter the extension of the set predicate, and 'temporary' urelements such as the predicate's extensions themselves,

which do eventually 'become' sets when we reach a sufficiently liberal structure. (p. 200)

By way of reintroducing this distinction, the third-way absolutist might suggest that we incorporate a predicate (call this, 'A') that applies only to 'available' objects at a given stage of the reinterpretative process (in effect, either to 'actual' sets, or 'permanent' urelements):

$$A(x) =_{df} ß(x) \vee \sim ß^{\diamond}(x)$$

Thus, in the case of (P), the resulting, restricted axiom would look something like this:

$$(P_A): \forall x \forall y((A(x) \wedge A(y)) \rightarrow \exists u(ß(u) \wedge \forall z(z \in u \leftrightarrow (z = x \vee z = y)))$$

However, as Studd (2019) is quick to point out, the difficulty with this strategy is that it *severely limits the applicability of sets guaranteed by Pairing*. For instance, the (unrestricted) Pairing axiom, (P), plays a crucial role in the Kuratowski definition of ordered pairs; these, in turn, are widely used by set theorists to encode the extensions of arbitrary predicates (for instance, the extension of the identity predicate, '$=$,' can be taken to be the plurality of absolutely all ordered pairs $\langle a, a \rangle$). However, the application of (P_A) delivers, not this plurality, but rather the plurality of all *available* ordered pairs; thus, insofar as the third-way absolutist relies on ordered pairs to encode the extensions of his predicates, the strategy of restricting (P) to the available objects has the consequence that the extensions of *all* the theory's predicates – not merely 'ß' and '$\in$' – become stage-relative.

The availability of the defined predicate 'A' – and, especially, its reliance on the 'potential set' predicate, '$ß^{\diamond}$' – raises an additional, critical question for the third-way absolutist. This concerns whether he is willing to countenance what Studd calls '*super meanings*': namely, those predicates, like '$ß^{\diamond}$,' the extensions of which comprise every possible member of every possible extension of the predicate 'ß.'

Of course, in the context of *Studd's* potentialism, the operator '$\diamond$' would by itself suffice to formulate such predicates; and Uzquiano (2015) himself cashes out his own 'linguistic model of indefinite extensibility' in modal terms that similiarly permit the definition of predicates like '$ß^{\diamond}$' and '$\in^{\diamond}$.' However, with the availability of these predicates comes the renewed threat of paradox: for instance, Studd (2019) demonstrates that we might run the same modal argument we encountered in §3.3, using the following, modified version of 'Unlimited Collectability for Sets':

$$(UCß^{\diamond}): \forall xx(ß^{\diamond}xx \rightarrow \exists y\left(ß^{\diamond}(y) \wedge \forall u(u \prec xx \leftrightarrow u \in^{\diamond} y)\right))$$

where the modal operators governing the quantifiers are dropped (by the third-way absolutist's lights, these are, in any case, superfluous), but the potentiality operators in the so-called 'super-predicates,' '$ß^{\diamond}$' and '$\in^{\diamond}$,' are retained – call this principle, 'Unlimited

Collectability for *Potential* Sets'[110]. We can similiarly modify the needed instance of plural comprehension, as follows:

$$1.1\ \forall y[y \prec rr \leftrightarrow (ß^{\Diamond}(y) \wedge \sim(y \in^{\Diamond} y))]$$

As before, the relevant Collapse-type principle – in this case, $(UCß^{\Diamond})$ – secures the existence of a potential set, R, that has as elements all the (this time, potential) sets in rr:

$$2.1\ ß^{\Diamond}(R) \wedge \forall u(u \prec rr \leftrightarrow u \in^{\Diamond} R)$$

In turn, the comprehensiveness of M_{∞} ensures that R lies within the range of the quantifiers in (1.1) and (2.1), and so is a candidate for self-membership.

However, this time, there is *no need to invoke Stability to engender inconsistency*. For R is a potential set, and the availability of the predicates $ß^{\Diamond}$ and $\in^{\Diamond}$ secures the legitimacy[111] of the relevant instance of plural comprehension (viz., (1.1) above). The paradoxical conclusion soon follows:

$$2.1\ R \in^{\Diamond} R \leftrightarrow R \prec rr \leftrightarrow \sim(R \in^{\Diamond} R)$$

Call the worry this paradox poses for the third-way absolutist, the *Objection from Super-Meanings.* Importantly, it is no threat to the *expansionist* – for while the expressive resources needed to formulate '$ß^{\Diamond}$' and '$\in^{\Diamond}$' are indeed available in Studd's potentialist account, the fact that our quantifier domains might, on his view, *expand,* allows for the possibility that *R* lies beyond the range of the quantifiers in both (1.1) and (2.1), thus diffusing the threat of paradox.

By contrast, for the third-way *absolutist,* positing a less-than-inclusive domain is not an option: consequently, should he wish to retain both these expressive resources, *and* absolutely unrestricted quantification, he is left with no option other than to reject the Collapse-type principle that is $(UCß^{\Diamond})$ – and thus, to have any distinction between his view, and standard absolutism (together with any advantages that might accrue from *accepting* Collapse) fall away.

Of course, *should* the third-way absolutist wish to retain these advantages, there remains the option of eschewing these expressive resources. Such seems to be Williamson's (1998) approach: having posited that the paradoxes are symptomatic of underlying shifts in the *meaning* of 'set,' he goes on to note that "[t]he inconsistency is not in any one meaning… it

[110] Strictly speaking, rather than assuming $(UCß^{\Diamond})$ outright, Studd (2019) derives it from the version of $(UCß)$ obtained when the initial plurality comprises potential sets, using a version of his Flattening theorem ('Stability-Free Flattening'). See p. 198 for this derivation (and pp. 257 – 258 for his account of Stability-Free Flattening).

[111] On the picture of pluralities we sketched in Chapter 2 – whereby it isn't possible for these to have potential sets as members – this instance of Plural Comprehension would not be valid. However, this account of the nature of pluralities is one the third-way absolutist will have to drop, should he retain the idea that we can encode *quantifier domains* as pluralities (for M_{∞} contains both potential as well as actual sets).

is in the attempt to combine all the different meanings that we could reasonably assign it into a single super-meaning" (p. 20).

An obvious difficulty with this diagnosis (and the resulting eschewal of super-meanings), though, is that, rather than collapsing any distinction between third-way absolutism and *standard* absolutism, it threatens to erase any meaningful distinction between the former, and *non*-absolutism. For we've seen that the compatibility of third-way absolutism with impure set theories hinges on our being able to restrict (at least some of) the absolutist's quantifiers to 'available' objects: earlier, we considered how this would work in the case of Pairing; as another example, Studd (2019) considers the Power Set Axiom:

$$(\wp): \forall x(ß(x) \rightarrow \exists y(ß(y) \wedge \forall z(z \in y \leftrightarrow z \subseteq x)))$$

However, the problem with axioms like ($\wp$) is that they *fail to express the intended generalisation*. In the case of the standard Power Set axiom, this concerns the existence, for *absolutely every set*, of its powerset; however, all ($\wp$) manages to assert is that there exists a powerset corresponding to each *actual* set. And the obvious problem with this, is that actual sets – at least by the third-way absolutist's lights – simply fail to comprise all the (actual *and* potential) sets there are.

That his generalisations thus fall short of encompassing *absolutely* all sets in the universe, arguably renders the third-way absolutist's retention of an all-inclusive universe a merely pyrrhic victory: indeed, for these reasons, Williamson's strategy of eschewing super-meanings threatens to run the third-way absolutist aground a version of the *Objection from Mysteriousness*. As we saw in the Introduction, this concerns whether the quantifier relativist can give an adequate account of the mechanisms that thwart our quantifiers' achieving absolute generality. Of course, for the third-way *absolutist,* the challenge isn't to explain why our *quantifiers* fail to range over absolutely everything; rather, the question is *what accounts for our being unable to formulate predicates like '*$ß^{\diamond}$*' and '*$\in^{\diamond}$*'* (this, especially, given that every satisfier of these predicates is contained in the all-inclusive domain M_{∞}, over which the third-way absolutist's quantifiers *unrestrictedly range*).

Williamson (1998) offers a possible reply to this worry, by way of positing an underlying *unpredictability* in the shifts in meaning the relevant predicates undergo. In his discussion of the semantic paradoxes (where such super-meanings give rise tp Revenge phenomena, like the Strengthened Liar), for instance, he posits that "[w]e cannot construct a strengthened Liar… because we cannot anticipate our future understanding in our present meaning" (p. 17). In a similar vein, Uzquiano (2015) ponders whether the intuition to treat the (super) extension of '$ß^{\diamond}$' as (yet another) possible extension of 'ß,' might be weakened if we concede that the open-ended process of interpretation entails that *the very idea of a candidate interpretation is itself open-ended*. He elaborates:

> It is perhaps tempting to assume at the outset that there is a perfectly delimited range of ever more comprehensive interpretations of the set-

theoretic vocabulary that remains invariant no matter where we locate ourselves in the cumulative process of reinterpretation. But this suggestive image may in fact distort the open-ended nature of the cumulative process of reinterpretation… Maybe the range of candidate interpretations evolves as we ascend the ladder of ever more comprehensive interpretations and is as open-ended as the interpretation of the set-theoretic vocabulary itself. (p. 161).

However, Uzquiano hastens to point out that“[t]his general line of response is not without consequence” (p. 162): for if the applicability of our set-theoretic statements *is* limited to a particular stage in the process of reinterpretation – if, indeed, what constitutes a candidate interpretation for our set-theoretic vocabulary is *itself stage-relative* – then the risk is that our "set- theoretic statements might turn out to be too parochial to be of interest." This is the difficulty we marked above, in relation to the Powerset Axiom; Uzquiano himself, in the case of Pairing, notes that if the axiom "is only concerned with the current interpretation of the set-theoretic vocabulary, then it will remain an open question whether [it] will still obtain when we move to more comprehensive interpretations of the language" (p. 162).

Just here lies the greatest obstacle for the strategy of rejecting super-meanings – and, by extension, the strategy that is third-way absolutism. The difficulty is not so much that our set-theoretic statements *do* exhibit the sort of parochiality Uzquiano (2015) describes above – rather it is that, at least *prima facie*, they *fail* to do so. As Studd (2011) points out: "the admissible interpretations of 'ß' do not appear to be hard to anticipate… we have a very full account of what the more extensive extensions will be like" (p. 145m emphasis).

But, if this *is* the case – if, that is, the various extensions he attaches to 'ß' *are* fully determinate in this manner – then the third-way absolutist's corresponding *inability* to formulate the correlative super-meanings is rendered utterly mysterious: a mystery that only deepens given his retention of unrestricted quantification over the entities that exhaustively comprise every such possible extension.

To sum up: just as the quantifier relativist faces the challenge of explaining 'what stands between our quantifiers and absolute generality', so too must the third-way absolutist provide an account of what stands between us, and the super-meanings he deems beyond our reach – this, especially, given that the expressive resources needed to formulate these meanings seem to fall squarely within our grasp.

Conclusion

What the foregoing arguments demonstrate is that eschewing Stability threatens to land the third-way absolutist on the horns of a dilemma. On the one hand, rejecting the possibility of super-meanings for predicates like 'ß' saddles him with the burden of explaining why such meanings are inaccessible to us, given how markedly predictable each subsequent extension of 'ß' is. On the other hand, should he accept the possibility of such super-meanings, paradox beckons once more, this time via the modified Collapse-type principle, Unlimited Collectability for *Potential* Sets $(UCß^{\diamond})$. (And, of course, retaining the super-meanings of predicates like 'ß' while rejecting $(UCß^{\diamond})$ serves to undermine any distinction between his position, the standard absolutism).

Consequently, Studd concludes that the path of least resistance for the proponent of Collapse-type principles like $(UCß^{\diamond})$ is to retain Stability, with the consequence that the predicates 'ß' and '$ß^{\diamond}$' (and '$\in$' and '$\in^{\diamond}$') coincide in extension relative to any given domain under consideration. The result of this, in turn, is that any *additional* candidates for membership in these extensions must fall outside of the (initial) domain in question (thereby witnessing – should these additional members exist – the *expansion* of the domain in question).

In view of all – and given, specifically, the weaknesses of third-way absolutism – we might take the Objection from Mysteriousness to have been answered. However, to have shown the cost of *rejecting* a particular claim is not yet to have argued conclusively *for* it; and while the third-way absolutist's rejection of Stability saddles him with the Objection from Super-Meanings, the expansionist's retention of Stability is not itself unaccompanied by controversy.

For one thing, we saw in §3.3 that the overlap in meaning this engenders in the case of Q-ish and E-ish, threatens to undermine any real sense in which the two languages differ – and so, in particular, any reason to treat the constants formed by means of the set-term operator as constants *only of E-ish*, as opposed to (open-ended) additions made to Q-ish. But this itself appears to pose something of a dilemma for Studd: for should we end up concluding, on the one hand, that these languages *are* one and the same, then this would entail that his Expansion Theorem – and with it, the basis of his reply to the Objection from Mysteriousness – *fails to go through*. On the other hand, upholding the distinction between these languages renders him vulnerable to the accusation that he has failed to do justice to the assumption of open-endedness that played so crucial a role in McGee's (2000) original strategy (a failure made all the more egregious by Studd's first employing this strategy – *sans modifications* – in a proof that the Q'ers quantify unrestrictedly over *their* universe).

Moreover, even if we *do* conclude that Studd succeeds in elucidating the mechanisms underlying quantifier domain expansion, it is not at all clear that the Objection from Mysteriousness – viz., the *broader* question of 'what stands between our quantifiers and absolute generality' – has been exorcised in its entirety. Indeed, the expansionist may be vulnerable to a version of this objection that is the counterpart of the third-way absolutist's Objection from Super-Meanings: for just as this leaves the third-way absolutist with the challenge of articulating what stands in the way of his formulating the relevant super-predicates (this, given the availability of modal resources, and how 'well-behaved' each expansion of the extensions of 'ß' and '$\in$' is) so too, for the expansionist, is the question of 'what stands between our quantifiers and absolute generality' lent renewed force by the observation that the domain expansions *she* posits are markedly predictable – *if not wholly determinate* – in nature.

This observation forms the basis of *the Objection from Implicit Actualism*, which we'll discuss in detail in Chapter 4. At its heart is the worry, as Øystein Linnebo and Stewart Shapiro (2017) put it, that potentialist theories like Studd's are "just actualism in potentialist garb" (p. 175). But – if true – this raises the question of why our 'actualist' (*qua* non-modal) quantifiers shouldn't range over every domain in the expanded (potential) hierarchy.

This isn't the only renewed version of the Objection from Mysteriousness the expansionist faces: as we will see throughout Chapter 4, versions of it are liable to arise wherever there is a lack of clarity as to the *differing kinds of generality* expressed by the expansionist's modal operators, on the one hand, and her *non-modal* quantifiers, on the other (in fact, one way we can think about the Objection from Mysteriousness, in the modal context, is as *calling into question* the purportedly distinct nature of these two forms of generality).

That Studd's potentialist theory is especially vulnerable to this objection, will be among the key ideas we defend in this final instalment of our study. Moreover, our own pursuit of an account of the distinction between modal and quantificational generality – and the differing prerequisites underlying each – also promises, at last, to shed light on the validity of Cantor's Domain Principle.

To a discussion of these issues, we now turn.

Chapter 4

On the Origins of the Objection from Mysteriousness: Quantification, Modality, and Cantor's Domain Principle

Introduction

We concluded Chapter 3 by noting that, just as the Objection from Super Meanings boils down to the question of what stands between the third-way absolutist and the relevant super-meanings, so too is the overall Objection from Mysteriousness – viz., the question of "what stands between our quantifiers and absolute generality" (Studd, 2019, p. 15) – very much an open one, for the expansionist.

To get a sense of the broader implications of this question – and how they extend beyond (merely) a concern with the mechanisms underlying quantifier domain expansion – it is useful to consider it in light of a closely-related objection the relativist faces: the Objection from Ineffability. This, as we've seen, is the complaint that any attempt to *express* that quantification fails to achieve absolute generality, seems to *presuppose* absolute generality. However, the worry here isn't just that quantifier relativism is self-refuting – rather, it is with the fact that *knowing* our quantifiers are limited seems to involve what Tim Williamson (2003) calls "unrestrictedly general thought" (p. 452).

Something like this dialectic is captured by a series of what we might call 'local' versions of the Objection from Mysteriousness – raised by Williamson in his tour-de-force 'Everything' (2003) – which we will take as a springboard for our broader discussion in this chapter. The first of these objections – which Studd (2019) collectively dubs the *Objections from Semantic Theorising* – concerns the relativist's ability to *interpret* her limited quantifiers. For instance, in the case of the relativist who takes these limitations to be *contextually*-determined, Williamson notes that she might advance the following clause:

($\forall C$): For every context C, $\forall x\varphi$ is true in C under σ if and only if every member d of D_c is such that φ is true in C under $\sigma_{[x/d]}$.

where φ is a formula of the language (call this $\mathcal{L}$) under consideration; σ is an assignment to its variables; and D_C denotes the domain of an arbitrary context C.

Importantly, by the contextual relativist's lights, ($\forall C$) is itself proposed in a particular theoretical context (call this *CT*); consequently, the metalanguage quantifier 'every member' ranges over the domain associated with *this* context (namely, D_{CT}) rather than D_C. However, if ($\forall C$) is to accord the relativist's quantifiers with their intended interpretation, this would require not only that that D_{CT} be *at least as wide* as D_C[112] – i.e., $D_C \subseteq D_{CT}$ – but that this holds for *any arbitrary context C of $\mathcal{L}$*. That is: the intended generality of $(\forall C)$ requires that that the domain of theoretical context CT have, as subdomains, *every domain associated with every possible context of the language*.

This poses the possibility – antithetical to quantifier relativism – that the meta-language quantifier 'everything' is itself *unrestricted*; moreover, Williamson argues that the only way the relativist might *avoid* this conclusion, would be to posit the existence of objects that are not in *any* of $\mathcal{L}$'s contextually-limited domains (and which are, consequently, "utterly ineffable in $\mathcal{L}$" (p. 445)). The obvious difficulty with this is that while some languages *are* so limited[113] as to plausibly have such 'semantic pariahs,' it is not at all clear that this phenomenon is widespread (and certainly, not so widespread as would have to be the case for contextual relativism about *all* quantifiers – including the meta-language ones – to be true).

Importantly, a disparity in expressive resources between object- and meta-languages is hardly non-standard (especially, as Williamson notes, in cases where the object language contains context-dependent expressions). The peculiar nature of the worry for the quantifier relativist, though, is that unlike the *absolutist* about quantifiers – who can happily countenance the existence of absolutely unrestricted meta-language quantifiers – such expressive resources are, by her own lights, utterly illicit.

Of course, for the *potentialist* relativist, an obvious option would be to draw on *modal resources* as a means of bridging this gap – and, as we will see, Studd (2019) proposes to do just this. However, the coherence of this strategy relies essentially on the claim that the generality expressed by Studd's non-modal quantifiers, differs in kind from that expressed by hi modal operators: to the degree, then, that we lack an account of what this distinction amounts to, the claim that we are unable to express the generality in question *by means of quantifiers* remains utterly mysterious.

Indeed, Williamson contends that this worry – namely, that the relativist, in interpreting her limited quantifiers, relies on a mysterious form of absolute generality – is not limited to the latter's semantic theorising: it also emerges in her attempts to formulate generalisations about comparatively *limited* kinds (such as the totality of pebbles on Camps Bay beach, or all donkeys). Given their comparatively limited scope, we would expect such generalisations to be unproblematic even for the quantifier relativist; indeed, she might even use her *quantifiers* to formalize such claims as the (presumably true) generalisation, 'No donkey talks,' as follows:

$$(1): \forall_1 x(D(x) \to {\sim}T(x))$$

[112] For if, on the contrary, there were a member *d* of D_C that were not *also* a member of D_{CT}, it would be possible, by the lights of (∀C), for $\forall x\varphi$ to be true in C, even if not all the members of D_C were such that φ were true.
[113]As an example, Williamson points to the language of Peano Arithmetic.

In line with relativism, the domain ranged over by the quantifier (indexed by the subscript) is a limited one. However, even with this restriction in place, it is seems perfectly possible for (1) to express the proposition, 'No donkey talks,' *so long as absolutely every donkey falls within the domain of* $\forall_1$ (for, in that case, (1) is true if and only if absolutely no donkey talks).

However, despite this overlap, Williamson contends that (1), in an important sense, *fails* to express the same proposition as 'No donkey talks', for:

> … one might stipulate that the domain is to contain all and only terrestrial animals, under the mistaken impression that there are talking donkeys on Mars. By uttering the formula with that interpretation, one would not say that (absolutely) no donkey talks, even though the domain does in fact contain absolutely every donkey, since it contains absolutely every donkey on earth, and in fact absolutely every donkey is on earth (p. 437)

Williamson's point is that while (1) and 'No donkey talks' may be *extensionally* equivalent, they are not *intensionally* equivalent (for the truth of (1) – so interpreted – is compatible with the existence of talking donkeys on Mars, while the truth of 'No donkey talks' is not).

More generally, the upshot is that expressing a kind generalisation requires that the relevant domain include members of the kind associated with *different possible worlds*[114] – in particular, Williamson argues that for (1) to succeed in expressing the proposition, 'No donkey talks,' this would require that the speaker have "some sort of access to the information that absolutely every donkey is in the domain" (p.437).

In turn, to the degree that this information is *accessible*, we would also expect that it is *expressible* – say, by means of the formula below:

$$(2): \forall x(D(x) \rightarrow \exists_1 y(y = x))$$

where the quantifier $\exists_1 y$ ranges over the same domain as $\forall_1$ in (1).

However – by the relativist's lights – the outermost quantifier in (2) is *itself limited in range*; consequently, our attempt to express the idea that absolutely every donkey is in the initial domain, if we are relativists, in fact amounts to the following:

$$(2.1)\ \forall_2 x(D(x) \rightarrow \exists_1 y(y = x))$$

where $\forall_2$ itself ranges over a comparatively limited domain[115].

[114] See Studd (2011, p.103)
[115] Although one which, we may grant, includes the domain ranged over by $\exists_1$.

The worry with this, though is, that it *needn't dent the optimism of Williamson's Martian donkey whisperer*: for the truth of (2.1) is consistent with there being a talking donkey – one that is neither on earth (viz., in the domain ranged over by $\exists_1 y$) nor in the domain ranged over $\forall_2 x$. Indeed, Williamson contends that "[t]his looks dangerously like the start of a vicious regress… to have access to one piece of information in the series, we need prior access to a previous piece of information in the series" (p. 437).

The result is an objection that combines the most powerful elements of the Objection from Ineffability, and the Objection from Mysteriousness. The peculiar nature of the worry – which will recur throughout this chapter – is not only that the relativist is unable to express particular *facts:* rather, it is that these facts themselves appear to be *absolutely general* in nature. In the specific case of generalisations about all donkeys: not only is the relativist unable to express the information – should she indeed have 'prior access' to it – that absolutely all donkeys fall within a given, limited domain; such prior access itself appears essentially involve "unrestrictedly general thought" (p. 452).

Of course – much like her response to the worries we raised with $(\forall C)$ – the potentialist relativist would likely argue that such thought is what is captured by her *modal operators*: she might even argue that Williamson's objections only apply to the quantifier relativist who fails to countenance alternative forms of generality.

But – once again – this response is only effective in the presence of an account of what this modal generality consists in; and, above all, of what differentiates it from the kind expressed by non-modal quantifiers.

Indeed, for the potentialist, the force of the Objections from Semantic Theorising – and the overall Objection from Mysteriousness – arguably lies in their shining a light on precisely this distinction; and as we will see in this chapter, the degree to which the potentialist fails to justify it, is the degree to which both objections still stand.

§4.1

Semantic theorising for the potentialist

Studd (2019) deploys another strategy – aside from potentialism – in his own response to the Objections from Semantic Theorising; to get a handle on this, it is worthwhile outlining why these objections are ineffective against *absolutism.*

This, as Williamson (2003) points out, boils down to the fact that the absolutist "treats the presence or absence of contextual restrictions as a feature of the relevant semantic structures" (p. 437); this in turn means that he is free to make to make any restrictions on his quantifiers *explicit*. Thus, for instance, he might formalize the kind generalisation 'No Donkey Talks' along the lines below:

$$(3): \forall x(E(x) \rightarrow (D(x) \rightarrow \sim T(x)))$$

where 'E' abbreviates 'lives on earth,' and serves to restrict the universal quantifier. However, the key point is that the quantifier in (3) is *otherwise wholly unrestricted*; omitting the restriction thus delivers the intended kind generalisation:

$$(4): \forall x(D(x) \rightarrow \sim T(x))$$

with the universal quantifier ranging over a domain of absolutely everything.

Similiarly, a disparity between her object- and meta-language quantifiers poses no significant threat to the *absolutist's* ability to (consistently) maintain his view: should he (for instance) wish to provide a semantics for a context-sensitive language, he may put forth a clause like $(\forall C)$, while quite happily taking the meta-language quantifier 'every member' to be unrestricted (viz., having its domain include as subdomains, all the domains associated with the object-language quantifier '$\forall$').

However, it is at just this point that Studd's (2019) own response to the Objections from Semantic Theorising kicks in. Specifically, he contends that *expansionism* in fact enjoys *precisely the same benefits as absolutism* when it comes to semantic theorising – where this, in turn, is because Williamson's objections are only effective against the relativist *qua restrictionist* (viz., the relativist who thinks our quantifiers are limited as a function of being *systematically restricted*, relative to a – possibly comprehensive – background domain).

By contrast, for the relativist *qua expansionist,* it is perfectly acceptable to posit that our quantifiers are *unrestricted,* relative to a given domain – but this, Studd argues, all but disarms the myriad difficulties Williamson raises for the relativist. For instance, granted the possibility of unrestricted quantifiers, Studd contends that the expansionist can *follow the absolutist* in taking the meta-language quantifier in clauses like $(\forall C)$ to be entirely unrestricted. Moreover, when it comes to kind generalisations like 'No donkey talks' the expansionist is similiarly free to make *explicit* any restrictions on the quantifiers governing her claims about all donkeys: above all, Studd argues that she is free to *omit* these restrictions, and so forestall Williamson's vicious regress altogether.

In view of this, Studd (2019) concludes that many of the benefits enjoyed by absolutism in the area of semantic theorising carry over to expansionism – this, because the two views:

> … need not differ on what items make up the universe of a given language, or what the extensions of its expressions are. Instead they disagree over the relationship this universe bears to others: specifically, on whether the universe is open to expansion (p. 110).

However, it is not entirely clear that *this* disagreement does not itself give rise to other worries Williamson raises for the relativist in 'Everything' (2003). For as we saw in §3.4, the fact that the expansionist disagrees with the absolutist 'on whether the universe is open to expansion', means that the latter must – in the context of *potentialism*– reject the Barcan Formula (in both its '$\Box$' and '$\Diamond$' forms) below:

$$(Barcan\ Formula^{\Box}): \forall x \Box \varphi(x) \rightarrow \Box \forall x \varphi(x)$$

$$(Barcan\ Formula^{\Diamond}): \Diamond \exists x \varphi(x) \rightarrow \exists x \Diamond \varphi(x)$$

together with the Converse Barcan Formula:

$$(Converse\ Barcan\ Formula^{\Box}): \Box \forall x \varphi(x) \rightarrow \forall x \Box \varphi(x)$$

$$(Converse\ Barcan\ Formula^{\Diamond}): \exists x \Diamond \varphi(x) \rightarrow \Diamond \exists x \varphi(x)$$

However, this renders the expansionist vulnerable to an objection that Williamson (2003) levels against what he calls 'the operator strategy,' which involves the (contextual) relativist's attempting to formulate her view by means of a claim along the following lines:

(5) In some context C_1, not everything is such that, in this context, it is something.

which is, in the modal context, equivalent to idea – endorsed by the expansionist – that potentially not everything is such that, actually, it is something. As Williamson points out, both claims are inconsistent with the Barcan Formula[116] – consequently, accepting such statements as (5) (or its modal equivalent, 'potentially not everything is such that, actually, it is something') would, predictably, be accompanied by a rejection of the former. However, Williamson's worry with this strategy is that neither (5) nor its modal equivalent appear to provide *a strong enough articulation* of the variety of quantifier relativism in question. In the case of the modal expansionist's version, the worry is that:

> … [i]n asserting that there could have been something other than what there actually is, they do not assert outright that there is something other than what there actually is. That a list could have been incomplete does not imply that it is incomplete. (p. 432).

Key to the objection is the claim that an adequate statement of the expansionist's view should not merely entail that, potentially, some (additional) set exists: rather, it should entail that the set in question actually *does* exist – *just not relative to the current domain*. However, insofar as this utterance is itself interpreted relative to the current domain, it is about as convincing as the E'ers' attempts – *while speaking Q-ish* – to convince the latter speakers that their domain fails to encompass everything.

[116] For if we were (say) to take '$\varphi(x)$' to abbreviate 'is not actually something,' then 'potentially not everything is such that, actually, it is something' – together with the truth of the Barcan Formula – entails the (absurd) claim that some (actual) thing possibly is not *actually* something.

But, of course, the E'ers make no such attempt: rather, they shift to a language with a more expansive domain, and from *that* perspective, attempt to persuade the Q'ers of the (relative) restrictedness of their domain. And indeed, Studd (2011) offers a reply along these lines to Williamson's objection to the 'operator strategy': he claims that the expansionist *does* "contradict the unrestrictedness of [the initial domain] if… she can succeed in shifting to a more extensive context and truly utters [her claim] in that context" (p. 95). Consequently, we can take the interpretational expansionist's commitment to her view to consist not in her acceptance of (5), but rather, roughly-speaking, in:

> … her being disposed to believe that each object language's universe fails to be comprehensive, when she recognises herself to be speaking (or thinking) in an admissible metalanguage, relative to which this claim can be expressed. (p. 80)

Reverting to talk of 'admissible interpretations': we can take the expansionist's commitment to her view to involve her being disposed to assert that a given object language's universe fails to be comprehensive, upon finding herself speaking a language, the *interpretation of which* she recognizes as an admissible *re*interpretation of the object language.

This, however, prompts an additional question for the expansionist, viz., *what counts as an admissible reinterpretation of the object language under consideration* – and, moreover, how it is that she *recognises* it as such. The point is sharpened if we consider that the criteria of admissibility for interpretations, on Studd's account, are far more closely constrained than mere *logical* possibility[117]. Williamson (2003) raises a similar question for the quantifier relativist, in the context of her attempts to express limited kind generalisations like 'No Donkey Talks' – her first stab at which, the reader will recall, looked something like this:

$$(1)\colon \forall_1 x(D(x) \rightarrow \sim T(x))$$

Williamson notes that, in response to the concern that (1) is compatible with the existence of talking donkeys *outside* the range of $\forall_1$, the relativist might attempt to express *schematically* the idea that absolutely all donkeys are in the relevant domain, by way of a scheme like $(B[i])$ below:

$$(B[i])\colon \forall[i]x(D(x) \rightarrow \exists_1 y(x = y))$$

$(B[i])$ is not so immediately self-defeating as the restrictionist's original attempt to assert that the range of $\forall_1$ includes absolutely all donkeys; however, as Williamson proceeds to point out, the (contextual) relativist's commitment to $(B[i])$ must "involve commitments to instances in other contexts" – including those that are "semantically accessible only from other contexts." Critically, given the relative inaccessibility of these instances from our

[117] As he points out in *Everything, More or Less* (2019), "not every logically-possible interpretation need be counted admissible" (p. 148)

current context, it follows that the scheme $(B[i])$ "is given not by a list of its instances but by a general condition for being an instance"; consequently, Williamson contends that the restrictionist must, in putting forward $(B[i])$, provide "a general condition for an expression to be *of the form* $\forall[i]$" (p. 439, emphasis mine).

Studd's (2019) own reliance on the notion of an admissible interpretation itself has a notably schematic flavour; consequently, it seems that he, too, is obliged to provide a *general criterion* separating instances from non-instances. In the context of the contextual relativist's reliance on $(B[i])$, Williamson (2003) remarks that "the obvious proposal is that the condition be a *semantic* one" (p. 439, emphasis mine) – namely, an expression counts as an instance of $\forall[i]$ *if it expresses universal quantification over a domain*, which condition Williamson formalizes by means of the below (where # is a place-holder expression):

> $SECOND$: # is an instance of $\forall[i]$ if and only if for some domain D, for every formula a and variable v, $\#v\ a$ is a formula and for every assignment σ, $\#v\ a$ is true under σ if and only if a is true under $\sigma[v/d]$ for every member d of D.

However, Williamson's worry with the use of semantic conditions like *SECOND* – specifically as a means of distinguishing instances from non-instances of schemes like '$\forall[i]$' – is that our understanding of such conditions seems to "embed the conceptual resources for absolute generality" (p. 440). For instance, *SECOND* itself makes use of quantifiers ranging both over domains ('for some domain D') *and* the elements of these domains ('every member d of D'): granted these expressive resources, it is quite straightforward to define the complex – and *absolutely unrestricte*d – quantifier, 'for every domain D, and every member d of D.'

In his own response to this objection, Studd (2019) insists that the means he himself deploys to distinguish between admissible and inadmissible (re)interpretations are characterized less in terms of *semantics,* and more along *inferential* lines, in line with his use-based account of universe expansion:

> Universe-expansion is brought about by using one's expressions in the right sort of way… by adopting the right trans-language patterns of inference… The condition for [an interpretation] to be an admissible [reinterpretation] can consequently be framed in a conceptually licit way in terms of inferential relations. (p. 138)

Such inferential relations would naturally include the trans-language sequents – such as those governing the set-term operator – that co-ordinate the E'ers' use of set-theoretic expressions with that of the Q'ers; and indeed, in *Absolute and Relative Generality* (2011), Studd deploys the idealized scenario of the Q'ers and the E'ers to diffuse the further objection that our grasp of the relations' holding *between* the relevant languages, illicitly embeds absolutely general conceptual resources:

> … use provides a conceptually licit handle on semantic adequacy… describing such patterns of use does not call for absolutely general quantification, but only circumscribed generality about the linguistic and mental behaviour of a particular linguistic community. (p. 216).

However, it is not entirely clear that Studd's criteria for distinguishing admissible from inadmissible interpretations *are* straightforwardly non-semantic. When it comes to articulating what counts for the *modal* expansionist as an admissible interpretation, Studd seems to rely on what are clearly semantic criteria – most obviously, the assumption that the potentialist's domain-hierarchy is so structure as to validate the Stability axioms governing '∈' and 'ß:

$$(STA{-}\in): \quad \Box\forall u\forall v(\Diamond(u \in v) \rightarrow u \in v)$$
$$(STA - ß): \quad \Box\forall u\big(ß^{\Diamond}(u) \rightarrow ß(u)\big)$$

For this emerges as nothing less than the (obviously semantic) stipulation that, for elements j and i in the index set $\mathcal{I}$, the *extensions* S_i and S_j, and E_i and E_j – corresponding to the extensions of 'ß' and '∈,' at M_i and M_j, respectively – agree on their extensions in the common universe, $M_i \cap M_j$.

Studd would likely respond that this is less an *assumption* made by the expansionist, and more a *consequence* of the soundness of the Stability axioms. For instance, in the context of the E'ers and the Q'ers, the fact that the interpretation ($[\![\cdot]\!]_E$) of E-ish 'stably expands'[118] that ($[\![\cdot]\!]_Q$) of Q-ish, is the *conclusion* of the latter's Expansion Theorem: one he reaches based not on the (question-begging) assumption that the *domains* of the E-ish and Q-ish are related in the relevant way, but rather as a function of the stipulation that the trans-language sequents governing the E-ers' patterns of use are open-endedly materially sound.

However – as we noted in our discussion of Stability in §3.3 – that Studd does obtain the desired result of universe expansion in the course of his Expansion Theorem is a function both of his adopting Stability, *and* of the additional stipulation that the Q'ers' expressions[119] receive their *intended interpretation* over their universe: only in conjunction with this semantic assumption do the Stability sequents ensure that the interpretation of E-ish 'stably expands' the corresponding interpretation of Q-ish. Moreover, while Studd's idealized scenario with the E'ers and the Q'ers plausibly involves 'only circumscribed generality about the linguistic and mental behaviour of a particular linguistic community,' the same is clearly not true when it comes to the relations that hold between the various interpretations of $\mathcal{L}_{PS}$, which together constitute the expansionist's potential domain-hierarchy.

[118] An interpretation $[\![\cdot]\!]_E$ stably expands another, $[\![\cdot]\!]_Q$, if the domain of the first, M_E, is a subset of the domain of the second, M_Q, and both interpretations agree on the extensions of their common predicates relative to the first domain (see).

[119] Besides the determiner '$every$,' and the connectives '~' and '→' – the intended interpretations of which are the subject of Studd's proof (see §3.3).

Indeed, it is in the modal context that the most devastating version of Williamson's (2003) Objection from Semantic Theorising emerges for Studd – specifically, in relation to the Kripke-style semantic clause he takes as providing the intended interpretation of the non-modal quantifier, ∀:

(k_i-∀): For any admissible interpretation i and assignment σ: $\forall v \psi$ is true$_{i,\sigma}$ iff every item $a \in M_i$ is such that ψ is true$_{i,\sigma[v/a]}$

For just as, in the case of $SECOND$, reference to both the *domains* of arbitrary contexts and the *elements* of these domains allowed us to define the complex quantifier, 'for every element of each domain,' so too does the clause (k_i-∀) – by way of its talk both of arbitrary admissible interpretations, *and* quantification over the elements in the domains of these interpretations – threaten to 'embed the conceptual resources for absolute generality.' In fact, the challenge this poses for the expansionist is analogous to that arising for the contextual relativist, in relation to the clause, ($\forall C$):

($\forall C$): For every context C, $\forall x \varphi$ is true in C under σ if and only if every member d of D_c is such that φ is true in C under $\sigma_{[x/d]}$.

As we saw in the Introduction, the worry with ($\forall C$) is that it succeeds in according the contextual relativist's quantifier ∀ with its intended interpretation only if the domain of the theoretical context – D_{CT} – has, as subdomains*, every possible domain of every possible context for the object language*: this, in turn, raised the spectre of absolute generality for the restrictionist, at the level of her metalanguage. However – as Studd (2019) acknowledges – the clause (k_i-∀) gives rise to the same worry for the expansionist*,* at the level of *her* metalanguage:

> … (k_i-∀) needs, in effect, to function as a super-semantic condition, specifying the intended semantics for $\forall v$ under absolutely every admissible interpretation. But… (k_i-∀) fails to capture the supersemantics unless the metalanguage universe is absolutely comprehensive. (p. 172)

The clauses Studd takes to provide the truth conditions for his modal operators – like (k_i-□), below – are subject to their own version of this objection:

(k_i-□): For any admissible interpretation i and assignment σ: $\Box\psi$ is true$_{i,\sigma}$ iff every admissible interpretation j in the hierarchy is such that ψ is true$_{j,\sigma}$

The difficulty with (k_i-□) is twofold. First – and most straightforwardly – the relevant hierarchy of admissible interpretations is, on Studd's (2019) view, *set-encoded*[120]: however,

[120]In particular, each interpretation $\langle M_i, E_i, S_i \rangle$ for each stage i (for some index set *I*) in the interpretive process is encoded as a *set* (specifically, an ordered triple).

the intended interpretation of '□' has it ranging over absolutely all interpretations, a totality that systematically outstrips any set.

Secondly – and more concerningly – the clause (k_i-□) *quantifies* over admissible interpretations; moreover, it does so in a *non-modal meta-language.* But of course, if our ordinary non-modal quantifiers fail to achieve absolute generality, there seems little reason to believe that the non-modal quantifier 'every admissible interpretation' succeeds in ranging over all such interpretations.

Consequently, just the contextual relativist's clause, ($\forall C$), succeeds in interpreting ∀ only at the cost of violating quantifier restrictions at the meta-language level, so too does the success of (k_i-□) require that the range of the non-modal quantifier, 'every admissible interpretation,' extend beyond the legitimate bounds of such quantification*, as it is understood by the expansionist.* And, of course, to the degree this quantifier does *not* exceed these bounds, the clause (k_i-□) *fails* to accord '□' with its intended, absolutely general interpretation.

§4.2

Modalising the Kripke Clauses

We can think of the difficulties posed by the Objection from Semantic Theorising – which we've seen are faced by quantifier relativists of *both* expansionist and restrictionist stripes – in terms of a dilemma. Either clauses like ($\forall C$) – or, for the expansionist, (k_i-∀) and (k_i-□) – *fail* to confer on the relevant expressions their intended interpretation; or, if they *succeed* in doing so, these clauses are vulnerable to the accusation that the conceptual resources needed to interpret these expressions illicitly entail *absolutely general thought*. At the core of the Objection from Semantic Theorising is thus none other than the Objection from Mysteriousness: in particular, the question is what *could* stand in the way of our quantifiers and absolute generality, *given our ability to think absolutely general thoughts*.

By way of a response to the difficulties faced by (k_i-□), in particular, Studd notes that a version of the first worry we raised for this clause – viz., that the potentialist's hierarchy of admissible interpretations is set-encoded, while the *intended* interpretation of '□' requires that this hierarchy outstrip any set – is also faced by the *absolutist*. Specifically, no (set-based[121]) model theory can be used to provide the intended interpretation of the language of set theory with absolutist quantifiers, since the former requires that the domain of quantification be a set, while the latter range over a totality that – according to absolutism – *fails* to comprise a set (by virtue of *including* all sets). Correspondingly, Studd points out that one way *both* the

[121]One way the absolutist might attempt to elude this difficulty is by encoding the domain, M, and extensions of ß' and '∈, respectively, as *pluralities.* However, Studd (2019) points out that if we are interested specifically in interpreting the richer language of generalised quantifiers, $\mathcal{L}_{GQ}$ then this requires not merely plural, but *superplural* resources. These admit of no obvious natural language paraphrase, which has led some philosophers (such as Parsons (2006)) to question their existence. Indeed, even interpreting the standard (and simpler) plural language $\mathcal{L}_{PS}$ requires superplural resources; and if we are semantic optimists in Linnebo and Rayo's (2012) sense – "[g]iven an arbitrary language, it should be possible to articulate a generalised semantic theory for that language" (p. 276) – then interpreting this superplural metalanguage would require even richer (super *duper*) plural resources, and so on *ad infinitum.*

absolutist and relativist might diffuse this tension, is by distinguishing between a minimally *instrumental* role for set-based model theory (say, deploying it to provide an account of logical consequence for a given object language), in contrast to its use as a means of *grasping the intended interpretation of this language*. In the former instance, Studd argues that both the absolutist and her relativist counterpart can, in good conscience, avail themselves of model-theoretic resources[122].

Of course, the real rub of the Objection from Semantic Theorising arises, rather, with the *latter* use of such clauses as $(k_i\text{-}\Box)$ – namely, deploying them as a means of *coming to understand* the expressions they interpret. More pressingly, in the context of the *expansionist*'s modal operators, the worry is that, in "seek[ing] to semantically reduce the modality to quantification over admissible interpretations" (p. 172), clauses like $(k_i\text{-}\Box)$ raise the spectre of absolute generality that is *quantificational in nature.*

Studd's reply to these concerns is simply to concede that "this style of semantics fails precisely because the modal operator is not reducible in this way" (p. 172). However, rather than having the expansionist abandon the use of model theory to illuminate the intended interpretation of $\mathcal{L}_{MPS}$, Studd suggests that clauses like $(k_i\text{-}\Box)$ *may*, in fact, provide the intended interpretation of the modal operators – *if the clauses are themselves modalised.* That is, Studd proposes that the expansionist:

> … attribute tacit modal content to an ostensibly non-modal Kripke-style semantics in just the same way as in the case of ostensibly non-modal set theory… by the relativist's lights, the non-modal clause $(k_i\text{-}\Box)$ cannot hope to capture the intended generality of '$\Box$,' no similar problem faces its modalisation. (p. 173)

Thus, in the case of $(k_i\text{-}\Box)$'s worrisome metalanguage quantifier 'every admissible interpretation,' Studd's suggestion is that we understand this as '*absolutely* every admissible interpretation'; indeed, he further contends that we can make the idea of an admissible interpretation – specifically, *potential* admissible interpretation – formally precise, by means of a *modalised model theory*, viz., one grounded in a *potentialist theory of sets*.

Studd's own attempt to make good on this proposal is carried out against the backdrop of his bi-modal axiomatization of set theory[123], which he develops in a step-by-step manner reminiscent of Boolos's landmark axiomatization of ZFC in 'The Iterative Conception of Set' (1971). However, where Boolos opts to cash out the conceptions key notions in terms of *quantification* over the stages of set formation, this (as we've remarked) is clearly precluded in Studd's case: consequently, his own axiomatization (2013, 2019) draws instead on talk of preceding and succeeding *interpretations* of the language of (pure) set theory, $\mathcal{L}_s$ (where

[122] Specifically, Studd (2019) argues that when it comes to investigating the logical properties of a given language, its intended interpretation is "model-theoretically irrelevant" (p. 153).
[123] See pp. 163 – 167 of Studd (2019), as well as Studd (2013), for the details of his bi-modal axiomatization.

generalisations *about* these interpretations, as well as the overarching metatheory, are formulated in the plural modal language, $\mathcal{L}_{MPS}$).

In this context, Boolos's (1971) original formulation of the core idea of the iterative conception – "[e]very member of a set is formed *before,* i.e., at an earlier stage than, the set" (p. 223) – emerges as the claim (which Studd dubs *Priority*) that, given any set in the domain of an admissible interpretation of $\mathcal{L}_s$, there is a *preceding* admissible interpretation, the domain of which contains the elements of the set in question:

$$Priority\colon \Box\forall y \Diamond_{<} \exists xx (y \equiv^{\Diamond} xx)$$

Similiarly, the claim that "at any stage every *possible* collection (or set) of sets formed at earlier stages is formed" (1971, p. 223), as Boolos puts it, becomes the (admittedly, wordy) Collapse-type principle – which Studd (2019) dubs *Plenitude* – that any sets in the domain of *an admissible interpretation* of $\mathcal{L}_s$ are such that the set comprising them belongs to the domain of every *succeeding* interpretation of $\mathcal{L}_s$; or equivalently, the (less wordy) formulation of $\mathcal{L}_{MPS}$ below:

$$Plenitude\colon \Box\forall xx \Box_{>} \exists y (y \equiv^{\Diamond} xx)$$

Studd (2019) notes that adding these two principles to his default modal plural logic $MPFO$, together with a modalisation of the set-theoretic principle of Extensionality:

$$Extensionality^{\Diamond}\colon \Box\forall y \Box\forall z [y = z \leftrightarrow \Box\forall x (x \in^{\Diamond} y \leftrightarrow x \in^{\Diamond} z)]$$

results in a logic – which he calls MST_P – that has, as theorems, the modalisation of the Regularity axiom schema[124] together with the modalisations of the Empty Set axiom, as well as the axioms of Pairing, Union, Separation, and the Powerset axiom. MST_P also proves the modalisation of Zermelo's Rank Theorem below (where the variable α ranges over variables):

$$Rank\ Theorem^{\Diamond}\colon \Box\forall y \Diamond\exists x \Diamond\exists \alpha (R_\alpha(x) \wedge y \subseteq x)$$ [125]

With these results in hand, Studd (2019) completes his bi-modal axiomatization by demonstrating that it is possible to derive, from the aforementioned theorems of of MST_P, the *non*-modal versions of these axioms (which, together with Zermelo's Rank Theorem and Extensionality, comprise a fragment of ZFC[126] he dubs s^{+}). That is: just as Boolos (1971)

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[125] In the formulation of the Rank theorem here, $R_\alpha(x)$ abbreviates the following formula:

$$\exists f dom(f) = \alpha + 1 \wedge f(\alpha) = y \wedge (\forall \beta \leq \alpha) \forall x (x \in f(\beta) \leftrightarrow (\exists y < \beta) x \subseteq f(\gamma)))$$

For a full derivation of the rank theorem, see p. 262 of *Everything, More or Less* (2019).

[126] Moreover, in the presence of appropriate reflection principles, it is also possible to obtain the modalisations of the axiom of Infinity, and the axiom of Replacement; moreover, if additionally we assume what Studd (2019) calls the Plurality Choice Schema, then it is also possible to derive the modalisation of the Axiom of Choice (see pp. 263 - 264 of *Everything, More or Less* (2019)).

proposes to "complete the description of the iterative conception of set by showing how to derive the axioms of a theory of sets from the stage theory" (p. 224), Studd (2019) demonstrates that it is possible to derive the axioms of s^+ from their modalised counterparts in MST_P by means of the following *Mirroring* theorem[127]:

$$Mirroring\colon \Gamma \vdash_{FOL} \varphi \;\text{ iff }\; \{\gamma^{\Diamond} : \gamma \in \Gamma\} \vdash_{MFO} \varphi^{\Diamond}$$

where Γ is a set of non-modal formulae of $\mathcal{L}_S$, and MFO is the sub-logic of $MPFO$ with only singular variables. The availability of theorem[128] ensures, in the context of Studd's potentialism, that the derivability of any formula of first-order logic entails the derivability of its modalisation in MFO; in this way, Mirroring ensures that MST_P *interprets* s^+:

$Proposition\ 1\colon s^+ \vdash_{FOL} \varphi$ only if $MST_P \vdash_{MPFO} \varphi^{\Diamond}$

This completes Studd's bi-modal axiomatization of ZFC; however, while certainly impressive in its own right, our own interest in this result – in view of the Objection from Semantic Theorising – lies in Studd's use thereof as a means of providing the *intended interpretation* of his modal operators, and the expressions of $\mathcal{L}_{MPS}$ more generally. To achieve *this* end, Studd firstly defines what he calls 'a Kripkean translation' of the sentences of $\mathcal{L}_{MPS}$, which takes the form of a function, $\langle\cdot\rangle$, mapping the formulae of $\mathcal{L}_{MPS}$ to sentences of $\mathcal{L}_S$. The translation of (the clause for) the non-modal quantifier, '∀' runs as follows:

$$\langle \forall v \psi \rangle^a =_{df} (\forall v \in V_a)\langle \psi \rangle^a$$

while those for Studd's 'backward' and 'forward' looking modal operators are:

$$\langle \Box_{>} \psi \rangle^a =_{df} (\forall \beta > \alpha)\langle \psi \rangle^{\beta} \qquad\qquad \langle \Box_{<} \psi \rangle^a =_{df} (\forall \beta < \alpha)\langle \psi \rangle^{\beta}$$

where the variables 'β' and 'α' range over interpretations in the potential hierarchy; '$>$' is the ordering on admissible interpretations; and V_a abbreviates, not the α-th *rank* in the usual cumulative hierarchy, but rather the α-th *universe stage* in the overall sequence of admissible interpretations. Further, to aid in the translation of the plural formulae of MST_P – given that the range of $\langle\cdot\rangle$ consists of the expressions of a singular language (viz., those formulated in $\mathcal{L}_S$) – Studd introduces what he calls 'proxy plural variables' (x^p, y^p, z^p, ...), in addition to the standard variables (x, y, z, …) already in $\mathcal{L}_S$. Thus, the Kripkean translation of the clause for the member-plurality predicate, '$\prec$,' is as follows:

$$\langle u \prec vv \rangle^a =_{df} u \in v^p \wedge u \in V_a \wedge v^p \subseteq V_a$$

[127] See p. 255 of Studd (2019) for his derivation of this theorem.
[128] Mirroring Theorems are only available for potentialist systems that validate S4.2 (or stronger); this will become especially important in §4.3, where it forms the basis of Joel Hamkins's (2018) objection that such systems are 'implicitly actualist.'

while that for the plural quantifier, '$\forall vv$' reads:

$$\langle \forall vv\psi \rangle^{a} =_{df} (\forall v^{p} \subseteq V_{a})\langle \psi \rangle^{a}$$

With $\langle \cdot \rangle$ defined, Studd notes that we can read the Kripkean translation, $\langle \psi \rangle^{a}$, of a modal formula, ψ, as giving the truth conditions of ψ *relative to the a-th stage in the hierarchy of interpretations*. Importantly, though, this is not yet to have provided the intended, *potentialist* interpretation of the expressions of $\mathcal{L}_{MPS}$: for insofar as the hierarchy of interpretations in question is still 'actual', the Kripkean translation $\langle \psi \rangle^{a}$ of a modal formula ψ is no more effective at furnishing the latter with its intended interpretation than are the semantic clauses – like (k_i-□) – that Studd originally proposes.

But, of course, the reader will recall that Studd's professed strategy to ensure the success of these clauses in interpreting $\mathcal{L}_{MPS}$, is to *modalise them*. Consequently, given his Kripkean translation $\langle \cdot \rangle$, Studd suggests that we *modalise the formulae* that collectively comprise the values of this function; moreover, by means of a result he calls the *Kripke Normal Form Theorem*[129], Studd is able to show – on the assumption that the α-th interpretation is the current one (which claim corresponds to the MST_P formula $U_a(M)$) – that a modal formula ψ is inter-derivable, in MST_P, with the modalisation of its Kripkean translation:

$$Kripke\ Normal\ Form\colon U_a(M) \vdash_{MPFO} \ \psi \leftrightarrow (\langle \psi^{a} \rangle)^{\diamond}$$

In light of this crucial result, Studd contends that the Kripkean translation, $\langle \psi \rangle^{a}$, of a modal formula ψ, of $\mathcal{L}_{MPS}$ – or, more specifically, the *modalisation* of its Kripkean translation, $(\langle \psi^{a} \rangle)^{\diamond}$ – succeeds in providing the intended interpretation of ψ (this time, relative to the α-th stage in the *potential* hierarchy of interpretations). He sums up:

> When suitably modalised, the Kripkean truth-conditions do capture the intended interpretation of the sentences of the modal language. And in this sense a relativist-friendly MT-hierarchy provides, by the relativist's lights, a miniature representation of the potential hierarchy at large. (p. 174).

§4.3

The Objection from Implicit Actualism

We've already remarked that Studd's result is an impressive one; however, it nevertheless remains unclear that his overall approach succeeds in avoiding the worry that he is 'semantically reduc[ing] the modality to quantification over admissible interpretations.'[130] This is especially apparent when we consider the fact the Kripkean translation of his modal

[129] See pp. 173 – 174, and p. 263 of Studd's (2019) his discussion of the Kripke Normal Form Theorem.

[130] Thus Linnebo (2013), in his discussion of Studd's bi-modal axiomatization of ZFC, remarks that the "expressive power" afforded by Studd's overall approach, while advantageous, is also "dangerously strong," as it "can be seen to be equivalent to that provided by outright quantification over worlds" (p. 226)

operators – which, as we saw in §4.2, has these range *over stage-universes in the hierarchy of admissible interpretations*.

Studd (2019), for his part, disputes the idea that his interpretational modality should be so understood; relating this to his Kripke Normal Form Theorem, he remarks:

> …it bears emphasis that the Kripke Normal Form Theorem does *not* reduce the modal operators to quantifiers. Modal operators appear on *both* sides of the biconditional in the statement of the theorem. (p. 174)

Of course, this response relies essentially on the claim that the generality afforded by Studd's modal operators *is* of a fundamentally different sort to ordinary, non-modal quantification. However, precisely this is what Studd's overall approach calls into question (and which claim, as a result, requires defending); moreover, this is not a concern limited to the role played by Studd's Kripkean translation in his Kripke Normal Form theorem. For the reader will recall, from §4.2, that a key step in Studd's (bi)modal axiomatization of ZFC involved his demonstrating that the potentialist logic MST_P interprets the fragment of ZFC, s^+:

$$Proposition\ 1: s^+ \vdash_{FOL} \varphi \text{ only if } MST_P \vdash_{MPFO} \varphi^\Diamond$$

However, it is also possible – by means of Studd's Kripkean translation – to prove a converse of this result: namely, that s^+ *interprets* MST_P:

$$Proposition\ 2: MST_P \vdash_{MPFO} \psi \text{ only if } s^+ \vdash_{FOL} \langle\psi\rangle^a$$

In fact, we can strengthen Proposition 2, above, to a biconditional: for each formula of s^+ is equivalent, in first-order logic, to the Kripkean translation of its modalisation:

$$Proposition\ 3: s^+ \vdash_{FOL} \varphi \leftrightarrow \langle\varphi^\Diamond\rangle^a$$

This allows us to derive[131] a strengthened version Proposition 1:

$$Proposition\ 4: s^+ \vdash_{FOL} \varphi \text{ if and only if } MST_P \vdash_{MPFO} \varphi^\Diamond$$

Studd's Kripke Normal Form theorem in turn allows us to extend this result to the other modal formulae of MST_P (as opposed, merely, to the modalisations of the formulae of s^+); this allows us to derive the desired biconditional:

[131] Proof sketch (adapted from Studd (2019), p. 263): given that we already have the left-to-right direction in Proposition 1, it remains only to prove the right-to-left direction. Thus, assume $MST_P \vdash_{MPFO} \varphi^\Diamond$. Then, by Proposition 2, it follows that $s^+ \vdash_{FOL} \langle\varphi^\Diamond\rangle^a$, and in turn, by Proposition 3, that $s^+ \vdash_{FOL} \varphi$. Thus we have it that $s^+ \vdash_{FOL} \varphi$ if $MST_P \vdash_{MPFO} \varphi^\Diamond$, which claim, in conjunction with Proposition 1, entails Proposition 4: $s^+ \vdash_{FOL} \varphi$ if only if $MST_P \vdash_{MPFO} \varphi^\Diamond$.

$$Proposition\ 5: MST_P \vdash_{MPFO} \psi \text{ if and only if } s^+ \vdash_{FOL} \langle\psi\rangle^{a}$$ [132]

Studd (2019) himself concludes that Proposition 5 amounts to nothing less than a demonstration that "the theorems of MST_P are precisely the Kripkean translations of s^+-theorems" (p. 263). What this seems to amount to, though, is the claim that the relevant modal facts – as expressed by means of MST_P – can ultimately be cast in the *non-modal terms of ordinary set theory*. But this raises the question of *what need there is for the modal concepts in the first place*, if the facts they purportedly capture may just as easily be articulated using *a non-modal* conceptual apparatus. As Linnebo and Shapiro put it in 'Actual and Potential Infinity' (2017), such results as Proposition 5 mean that:

> … it needs to be confirmed that the modality makes a real difference…
> that the resulting theory isn't just actualism in potentialist garb. (p. 175)

Of course, that Studd's modal theory is 'just actualism in potentialist garb,' is the worry that lies at the heart of the Objection from Mysteriousness; in their paper, Linnebo and Shapiro note that it arises specifically in relation to the availability of Mirroring Theorems:

$$Mirroring: \Gamma \vdash_{FOL} \varphi \ \text{ iff } \{\gamma^{\Diamond}: \gamma \in \Gamma\} \vdash_{MFO} \varphi^{\Diamond}$$

which, as we saw in §4.2, play a crucial role in Studd's (2019) bi-modal axiomatization of ZFC. However, Linnebo and Shapiro (2017) point out that such theorems are only available given the modal axiom (G), below:

$$(G): \Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$$

which is an axiom of S4.2; that is, (G) is itself only available in what Linnebo, in another paper with philosopher Joel Hamkins (2022), calls "convergen*t* forms of potentialism" (p. 2), where the corresponding modal validities are S4.2 or higher, and the accessibility relations between the worlds (or domains) exhibit the feature of *directed convergence,* below:

$$(Directed\ Convergence): \forall w_1 \forall w_2 \exists w_3 (w_1 \leq w_3 \ \wedge w_2 \leq w_3)$$

This ensures, as Linnebo and Hamkins (2022) put it, that in the relevant potentialist systems,[133] "any two possibilities can be brought together again in a common further possibility" (p. 2). For instance, in Linnebo's (2010, 2013) own potentialist account (which validates S4.2), not all worlds (or domains) access each other; however, it is nevertheless guaranteed that, for any two worlds, there is a common, third world that *both access.* Studd's

[132] Proof sketch (adapted from Studd (2019), p. 263): assume that $s^+ \vdash_{FOL} \langle\psi\rangle^a$. Then, by Proposition 4, it follows that $MST_P \vdash_{MPFO} (\langle\psi^a\rangle)^{\Diamond}$. In turn, on the assumption that the current interpretation is the a-th, and that $\langle\psi\rangle^a$ contains no proxy plural variables, it follows that $MST_P \vdash_{MPFO} \psi$.
[133] A 'potentialist system' – as per Hamkins's (2018) definition – is a collection of model-theoretic structures ('worlds') in a common signature $\mathcal{L}$, that are equipped with a reflexive and transitive accessibility relation – whereby, in particular, the domain of a given world is contained in the domain of another, if the first world accesses the second.

own potentialist system – which validates the strongest modal logic, S5 – additionally contains axiom (5) below:

$$(5): \Diamond\Box\varphi \rightarrow \varphi$$

This ensures that Studd's theory exhibits what Linnebo and Hamkins (2022) call 'completeness' or 'maximality,' any potentially necessary truth is (already) an actual truth – or to use the authors' tense-like gloss: "any statement that could become permanent has already been made permanent" (p. 2).

The relevance of all this for our discussion – specifically, in light of the need to 'confirm that the modality makes a real difference' – soon becomes apparent if we consider how these features of convergent potentialism differ from those exhibited by systems where *only S4* is validated. Linnebo and Hamkins (2022) describe such systems as being:

> … [of] a far more radical branching character, one in which fundamental bifurcations in possibility are revealed as the universe unfolds in one manner as opposed to another incompatible manner. (p. 2)

This contrasts with the various domains – or, as Hamkins (2018) calls them, "universe fragments" – of the systems validating S4.2 or above, which "form a coherent system" (p. 34). Indeed, he and Linnebo (2022) point out that the 'coherence' exhibited by the totality of worlds (call this $\mathcal{W}$) in a convergent potentialist structure entails that "$\mathcal{W}$'s converges to a limit structure $\mathcal{M}$, which is simply the union of the structures of $\mathcal{W}$" (p. 5). This convergence is a function, specifically, of the fact that "any given world in $\mathcal{W}$ can be enlarged so as to encompass any given individual of $\mathcal{M}$": consequently, while each world in $\mathcal{W}$ is associated with its own (comparatively limited) domain, these all come together (*converge*) into a single, overarching domain – that of $\mathcal{M}$ itself.

Studd's own modal account is no exception to this: in that case the corresponding limit structure – with its single, all-encompassing domain – is none other than the maximal interpretation $\langle M_\infty, E_\infty, S_\infty \rangle$, that is the union of each of admissible interpretation $\langle M_i, E_i, S_i \rangle$, for each stage i in the potential domain hierarchy. However, given the existence of this limit structure – and specifically in view of Linnebo and Shapiro's (2017) worry that his modal system is merely 'actualism in potentialist garb' – we might wonder what distinguishes the *individual* interpretations in the hierarchy from the *maximal* interpretation, $\langle M_\infty, E_\infty, S_\infty \rangle$.

Something like this seems to be Hamkins's (2018) own critique of convergent potentialism: he complains that the limit models accompanying these systems have "an implicit existence," and that, as a result "convergent forms of potentialism are much closer to actualism than are the more radically potentialist theories" (p. 34). Indeed, Hamkins goes so far as to declare that convergent potentialisms are "*implicitly actualist*" (emphasis mine) – this, because the 'implicit existence' of the limit models in question ensures that there is:

> … very little at stake in the ontological dispute concerning the actual existence of this limit model. What does it matter if the potential objects that might come to exist do not yet actually exist, if the way that they will come to exist is unique and deterministic? To experience potentiality in convergent potentialism is simply to wait for the inevitable. (p. 34)

Importantly, Hamkins (2018) points out that the 'implicit existence' of these limit structures are precisely what enable us to "interpret the full actualist universe inside the potentialist ontology" (p. 34), by way of ensuring the availability of the Mirroring Theorems that bridge the actualist set theory to be interpreted, and the potentialist (modal) theory doing the interpreting. However – as we saw in the case of Studd's Proposition 5 – *this goes both ways:* that is, the presence of these limit structures equally ensures (at least, given something like Studd's Kripkean translation) that we can 'interpret the potentialist ontology inside the actualist universe.'

At precisely this point, a version of Hamkins' complaint – call this, the *Objection from Implicit Actualism* – emerges for Studd's expansionism: specifically, given the relations of *mutual interpretation* that we've seen hold between Studd's modal and non-modal theories, the worry is that the ontologies associated with these two (ostensibly distinct) theories, are *really not all that different.*

And *this* worry is, of course, itself a version of the Objection from Mysteriousness. In particular: given the limit structure, $\langle M_\infty, E_\infty, S_\infty \rangle$ that we've seen accompanies Studd's theory – and, above all, the overarching domain associated with this structure, M_∞ – the claim that it isn't clear what stands between our quantifiers and absolute generality morphs into the complaint that the expansionist has failed to articulate what distinguishes the *individual domains* in the potentialist hierarchy, from the *overall limit domain* to which these domains, in their 'coherence', converge. Moreover, if we *were* to collapse the boundaries between these structures, this would in turn dismantle the basis for the distinction between the two forms of *generality* – quantificational and modal – by which means (according to the potentialist) we generalise over these ostensibly divergent domains.

Put (somewhat) more briefly: in generalising over a (limit) domain that is seemingly indistinguishable from the domains ranged over by our *non-modal* quantifiers, the concern is that Studd's *modal* operators are, in fact, 'implicitly actualist.'

§4.4

The Objection from Implicit Potentialism

Worryingly for Studd, it is possible to frame a version of this complaint *from the other direction as well*. That is: in addition to the worry that his modal operators are implicitly actualist, he also faces the accusation his non-modal quantifiers may be '*implicitly potentialist*.'

We can get a handle on this idea, and the associated critique, by means of Max Cresswell's seminal paper, 'In Defence of the Barcan Formula (1991); where the Barcan Formula and its converse, the reader will recall, are modal principles that are *invalid* in Studd's potentialist system.

There, Cresswell notes that systems where the Barcan formula fails standardly interpret their quantifiers as ranging over a world-*relative* domain, D_w: consequently, the semantic clause stipulating their truth-at-a-world conditions usually take the following form:

$(V\forall')$: $\forall x\varphi(x)$ is true in w iff for all $d \in D_w$, $\varphi(d)$ is true in w[134]

The clause $(V\forall')$ is more or less[135] identical to (k_i-∀), which we saw in §4.1 provides the intended interpretation of Studd's non-modal quantifier:

(k_i-∀): For any admissible interpretation i and assignment σ: $\forall v\psi$ is true$_{i,\sigma}$ iff every item $a \in M_i$ is such that ψ is true$_{i,\sigma[v/a]}$

Correspondingly, in contrast to both clauses, Cresswell notes that modal systems that *validate* the Barcan formula, typically interpret their quantifiers (relative to a world, w) as follows:

$(V\forall)$: $\forall x\varphi(x)$ is true in w iff for all $d \in D$, $\varphi(d)$ is true in w

where D is a domain comprising absolutely all individuals (including those that don't exist – i.e., that are merely possible – relative to the world in question).

The overarching domain D in $(V\forall)$ is clearly analogous to the domain of the limit structure $\mathcal{M}$ that featured so prominently in the foregoing discussion; indeed, just as Hamkins (2018) questions whether the 'actual existence' of this limit model lowers the stakes in the 'ontological dispute' over the existence of potential sets, Cresswell (1991) similiarly queries what further barrier there could be to assuming the existence of D, once we've taken the "metaphysical plunge" (p. 274) of assuming the existence of each world-relative D_w:

> Even if each world w has its own domain D_w of the things which exist in w there is no reason why all these D_ws can't be collected into one single domain D… why must you interpret the quantifier only in the domain of things which exist in the world in question? Why not interpret it in the whole domain which is the union of the domains of each world? (p. 273)

[134] Strictly speaking, the clauses in question would require a more complex formulation, with reference to variable assignments, etc. We follow Cresswell in omitting these complications, for the same reasons he gives ("such complexity is not required for my purposes" (p. 272)).
[135] The only difference being that talk of worlds, ranged over by the variable w, is replaced by talk of interpretations, ranged over by the variable i.

Creswell does concede that a potential difficulty with this suggestion is that it appears to commit one to the view that "exactly the same things exist in all possible worlds," which in turn "looks like the view that everything is a necessary existent." Correspondingly, he notes that any semantics that validates the Barcan Formula – viz., any semantics that adopts $(\vee\forall)$, rather than $(\vee\forall')$, for its quantifiers – starts to "look not only like a special case of a more general semantics, but in fact like an implausible special case" (p. 272).

In response to this 'special case' argument, though, Cresswell argues that we in fact have good reason to treat systems that validate the Barcan Formula – and which adopt $(\vee\forall)$ – as more basic than those that adopt $(\vee\forall')$. The key point is that the quantifiers governed by $(\vee\forall')$ – which Cresswell dubs 'actualist quantifiers' – can be defined in terms of the 'possibilist' quantifiers characterized by the former clause, if the relevant logic contains an appropriate existence predicate, $E!$. This predicate can be either primitive or defined: indeed, in languages with identity, the definition standardly proposed for the existence predicate is a (non-schematic version of) Lavine's 'everything axiom':

$$E!(x) =_{df} \exists y(y = x)$$

Given the availability of $E!$, Cresswell points out that we can define an 'actualist' quantifier ranging only over the entities in existence at a world w (following Cresswell, we'll denote this with 'Π') in terms of the possibilist quantifier, '$\forall$,' which ranges over *absolutely all entities* (including the merely potential):

$$(\Pi_{def}): \Pi x\varphi(x) =_{df} \forall x(E!x \rightarrow \varphi(x))$$

But, given this, Cresswell argues that a natural conclusion would be that "systems without [the Barcan Formula][136] but which have an existence predicate emerge as subsystems of systems with [the Barcan Formula]" (p. 275); given the logical form of $\Pi x\varphi(x)$, Cresswell further contends that:

> If systems with [the Barcan formula] are taken as basic… then the quantifiers in systems without [the Barcan Formula] are best regarded as restricted quantifiers. *Restricted to the domain of the world in question*." (p. 274, emphasis mine)

This, of course, presupposes that (Π_{def}) *is* the correct logical form to ascribe to 'Π'; Cresswell, however, argues that we *also* have good reason to believe that *this* is the case, based on the logical behaviour of the actualist quantifier – in particular, based on its conforming (or not) to the standard *quantifier rules*, like Universal Instantiation (UI).

In the case of Cresswell's 'possibilist' quantifier, this rule has the following, standard form:

[136] I've used 'Barcan Formula' – written in full – where Cresswell uses the abbreviation 'BF.'

$$(\forall 1): \forall x \varphi(x) \rightarrow \varphi(c)$$

However, on the assumption that variable assignments are allowed *outside* of the domain of the world of evaluation, the version of the rule for the actualist quantifier, 'Π':

$$(\Pi 1): \Pi x \varphi(x) \rightarrow \varphi(c)$$

is invalid. Indeed, its invalidity emerges for much the same reasons that the restricted generalisation (below) of 'No Donkey Talks':

$$\forall x(D(x) \rightarrow \sim T(x))$$

does *not* imply that a particular human being designated by the constant 'c' – say, former president of South Africa, Jacob Zuma – doesn't talk:

$$\sim T(c)$$

This is because the quantifier '$\forall$' ranges more widely than the extension of the restricting predicate 'D' – and while the truth of the generalisation entails that no element of the extension of 'D' talks, it implies nothing at all about entities (like Mr Zuma), who are *not* donkeys, while nevertheless falling within the range of '$\forall$' (and who, again like Mr Zuma, may be quite loquacious,). Similarly, if we define the actualist quantifier 'Π' as a restriction of '$\forall$,' then $(\Pi 1)$ will be false whenever 'c' designates an individual outside the domain of the world of evaluation, D_w.

In fact, the situation here is entirely analogous to McGee's (2000) proposed counter-example to the relativist claim that our quantifiers may be restricted, which we encountered in Chapter 3. As we saw there, McGee's argument is that, should there indeed be an entity, *a*, that lies beyond the range of our quantifiers, it seems that we could:

> … take c to be a constant referring to *a,* and we can take F to be a predicate referring to those things that are within the range of our first-order variables. Then, $F(c)$ will be false, and $\forall x \phi(x)$ will be true, invalidating (UI). (p. 68)

However – as Lavine (2006) points out in his critique of McGee – the crucial assumption on which this counter-example rests is that we are, as McGee puts it, 'permitted to name anything we like by an individual constant.' In turn, this same assumption (or something like it) is what lies behind the invalidity of $(\Pi 1)$: in this case, being 'permitted to name anything we like' is equivalent to countenancing assignments to our variables where values assigned lie beyond the domain of the world of evaluation.

Importantly, the reader will also recall that this is an assumption that Studd himself *drops* in the context of his Expansion Theorem (specifically, by making room for the existence of entities that we are only 'permitted to name' upon shifting to a new language). However, when it comes to his *modal theory*, this assumption is instead *retained* by Studd – specifically, he allows for variable assignments *over the entire hierarchy.* This means that the usual version of the rule of Universal Instantiation is invalid for his quantifiers; however, in view of this, Studd opts to *modify* (UI), to obtain the following:

$$(FUS): \forall x \varphi(\boldsymbol{x}) \rightarrow (E!\boldsymbol{u} \rightarrow \varphi(\boldsymbol{u}))$$ [137]

where '$E!$' is defined as above.

However – and importantly – the logical form of (FUS) is entirely consonant with understanding the quantifier '$\forall$' to be *restricted*: indeed, Studd's proposed modification of (UI) is strictly equivalent to one Cresswell considers for the actualist quantifier 'Π,' so as to the restore the validity of $(\Pi 1)$:

$$(\Pi 1E): (\Pi x \varphi(x) \wedge E!c) \rightarrow \varphi(c)$$

For $(\Pi 1E)$ – like (FUS) – ensures that any entities denoted by the constant 'c' exist relative to the world (or domain) of evaluation; this is sufficient to avoid McGee-style counter-examples to the rule.

It bears emphasis, though, that the (restored) validity of these modified rules is *not* a consequence of giving up the assumption that 'we are permitted to name anything we like by means of an individual constant.' This is a point Cresswell pointedly makes in his discussion of $(\Pi 1E)$ above, which he notes is "one of the forms [Universal Instantiation] can have in 'free logic'" (p. 274). Cresswell, however, cautions against taking free logic as the right model for the modal logic under discussion, as:

> … [i]n a non-modal free logic it is tempting to think of the variable x, in those cases when $(\Pi 1)$ fails, as a 'non-denoting' term. This is because in non-modal logic we don't normally have a class of things which don't happen to exist but might have. But in a modal semantics as I have been presenting it so far there are no non-denoting terms… *The c in* $(\Pi 1)$ *denotes all right. But the thing it denotes does not exist in the world at which it is being evaluated.* (p. 276, emphasis mine)

Cresswell's point amounts to this: that a constant, c, in a modal system validating a variable-domain interpretation of the quantifiers, does *not* denote at a particular world of evaluation, does not entail that it is *non-denoting simpliciter*. In an important sense, c *does* denote – it does so, *relative to the overarching domain, D.*

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[137] The (bold) variables '$\boldsymbol{x}$' and '$\boldsymbol{u}$' are either singular or plural.

Correspondingly, to the degree that they *do* countenance variable assignments outside of the domain of a given world of evaluation, the complaint is that modal accounts (like Studd's) that advocate a variable-domain interpretation of their quantifiers not only fail to avoid presuming the existence of D – they in fact *rely* on precisely this domain, when it comes to the interpretation of their *actualist quantifiers.* In Studd's case, this commitment emerges especially strongly, due to his adopting, as an axiom, '$\Diamond E!\,v$,' where the variable 'v' ranges over M_∞ (which is, in that context, the equivalent of D). This ensures – quite in line with Cresswell's remarks above – that any constant or variable of $\mathcal{L}_{MPS}$ does indeed denote, *even if the entities denoted only exist potentially*. Indeed, Studd's adopting '$\Diamond E!\,v$' as an axiom ensures not only that we can, in a sense, name entities that do not (yet) exist – it also ensures that our variables and constants never denote an entity that, in an *absolute* sense, *fails* to exist[138].

Importantly, Cresswell's critique is not aimed at modal *systems* (like Studd's) that endorse a domain-variable interpretation of their quantifiers (and consequently, reject the Barcan Formula). Rather, his claim is that the quantifiers of such systems – in view of the modified rules they require – behave less as if they range unrestrictedly over a domain of actual existents, and more like *restricted* quantifiers: ones whose range, when these restrictions are dropped, comprises *absolutely everything*.

The argument is especially devastating in Studd's case: for the idea that the latter's non-modal quantifiers are *genuinely unrestricted* plays a key role throughout his various arguments in *Everything, More or Less* (2019). Indeed, this claim forms the basis Studd's insistence that expansionism resists many of the objections Williamson raises for quantifier relativism, which together comprise the Objection from Semantic Theorising.

Studd's choice, however, to retain variable assignments over the *entire* potential hierarchy – together with his subsequent modification of the standard quantifier rules to *accommodate* these liberal assignments – complicate the idea that his non-modal quantifiers range unrestrictedly only over what actually exists. And indeed, (UI) is not the only quantifier rule that requires amending as a result: for Studd additionally proposes to "liberalize" the rule of Universal *Generalisation* (UG), of which the standard sequent form is as follows:

$$(UG)\text{: If } \Gamma \vdash \phi(c)\text{, then } \Gamma \vdash \forall x \phi(x)$$ [139]

It is easy to see why (UG) would be unacceptable to Studd: the antecedent sequent is secured only in the class of models where the entity denoted by 'c' actually exists (which, for the expansionist, is only a subclass of all possible models). Correspondingly, he proposes the following amendment to the standard rule:

$$(FUG)\text{: If } \Gamma \cup \{E!\,c\} \vdash \phi(c)\text{, then } \Gamma \vdash \forall x \phi(x)$$

[138] Namely, by not coming to exist at some stage in the potential domain-hierarchy.
[139] Where 'c' does not occur in ϕ, or in the formulae in Γ.

To see why this constitutes a *liberalization* of (UG), it is useful to consider a modification of the predicates in (FUG) along the lines of our discussion of $(\Pi 1)$ (viz., the version of universal instantiation for Cresswell's 'actualist' quantifier) – specifically, we'll replace '$E!$' with the predicate 'D' ('is a donkey'), to obtain the following:

$$(FUG\_donkeys)\text{: If } \Gamma \cup \{D(c)\} \vdash \phi(c)\text{, then } \Gamma \vdash \forall x\phi(x)$$

Importantly, the resulting rule – $(FUG\_donkeys)$ – is *invalid*: a fact that becomes clear if we take 'ϕ' to abbreviate '$\sim T$' ('doesn't talk'). For while it is (presumably) the case – in all models and under all possible assignments to 'c' – that we *can* infer 'c doesn't talk' ('$\sim T(c)$') from 'c is a donkey' ('$D(c)$'), this does *not* entitle us to the claim that *absolutely nothing* talks (as the counter-example of Mr Zuma, amongst others, shows).

The problem with $(FUG\_donkeys)$ is thus that it allows us to infer the truth of this generalisation (viz. 'absolutely nothing talks'), even when the domain includes things besides donkeys – or, indeed, *no donkeys at all.* That is: what the modified rule allows for is the derivation of a generalisation, *absent the instances with which it is concerned*. However, we can level precisely the same complaint at Studd's adaption of (UG). In his case, the modified rules allow us infer ordinary non-modal generalisations like '$\forall x\phi(x)$' in *any domain whatsoever of the potential hierarchy* – but this, naturally, calls into question the idea that '$\forall x\phi(x)$' is concerned only with the elements of *one* particular domain *of* this hierarchy.

Of course, the cost of abandoning these modifications – as we've seen repeatedly throughout this section – is the invalidity of Studd's quantifier rules. However, we might wonder whether something like this *isn't what we would expect*, if our quantifiers *did* genuinely range only over actual sets. For instance, when it comes to the invalidity of $(FUG\_donkeys)$, this seems to be a consequence of the relevant generalisation's being, in a sense, *too strong*: we've inferred, based on a comparatively limited kind (donkeys) that a property (not talking) is true of absolutely all things. However, it isn't clear that something like this shouldn't be true of our generalisations about all *actual sets,* given that – in the modal context – these *aren't the only sets there (potentially) are*. And indeed – in contrast with this – the sorts of generalisations captured by Studd's non-modal quantifiers display a notable resilience in the face of subsequent domain expansions: the modified rules effectively ensure that these non-modal generalisations *remain true no matter how the domain expands*.

A similiar stability in the case of Studd's *modal* truths, is part of what motivates Hamkins's Objection from Implicit Actualism: specifically, the existence of the limit structure $\mathcal{M}$ ensures – as he and Linnebo (2022) put it – that "the worlds of $\mathcal{W}$ must all agree with on the atomic truths and therefore also with each other" (p. 4). By contrast, in systems where only S4 is validated – and which, in Hamkins's (2018) words, exhibit a "more radical form of potentialism" – there is:

> … truly branching possibility, statements that could become true, but might not. For this kind of potentialism, one is living in a universe

> fragment… the question of what might become true and verified in a larger universe fragment depends on precisely how the universe unfolds. (p. 34)

We can, it seems, raise a similar complaint to this, in the case of Studd's *non-modal* quantifiers: namely, that the statements formed by means thereof exhibit far less parochiality than we would expect, if their range – and instances – *were,* in fact, limited to individual domains of actual sets in the potential hierarchy.

Indeed, given Studd's variable assignments over the entirety of the hierarchy – together with his modification of the quantifier rules to accommodate these restrictions – it seems irresistible to conclude that his quantifiers *are* best construed as restricted relative to an overarching domain comprising both actual *and* potential existents: in other words, that they are, in fact, *implicitly potentialist.*

§4.5

Plural Modal Rigidity and Cantor's Domain Principle

We can think of the Objections from Implicit Actualism and Implicit Potentialism, respectively, as approaching the Objection from Mysteriousness from two distinct angles. The first objection – in highlighting the similiarity between the potential hierarchy's *individual* domains, and the *limit* domain that is their union – undermines the idea that the prerequisites of modal generality differ from the quantificational kind; the second, in turn, calls into question the claim that our quantifiers fail to range over the entire domain hierarchy, and so similiarly challenges the idea that the form of generality expressed by the latter differs from that expressed by the modal operators.

Given his vulnerability to both objections, it is evident that the thrust of the Objection from Mysteriousness – namely, the question of *what stands between our quantifiers and absolute generality* – is not only retained, but amplified for Studd. However, this is not to say that the objection cannot be answered; indeed, one response due to Øystein Linnebo – developed in extended work with Stewart Shapiro, and Salvatore Florio – holds the promise not only of confirming that the modality makes a real difference', but also of shedding light on the role of the *Domain Principle,* when it comes to securing the determinacy of *quantificational* generality.

The key innovation at the heart of Linnebo's response is the notion of the *modal rigidity* – an idea Linnebo first introduces in 'Actual and Potential Infinity' (2017) with Stewart Shapiro (which paper, the reader will recall, is also where the authors formulate their own version of the Objection from Implicit Actualism[140]). There, the authors argue that a key difference between actualist (i.e., non-modal) set theory, and potentialist set theory, lies in the principles of higher-order logic – in particular, the principles of *plural logic* – each validates; a claim they justify, in turn, with recourse to the notion of pluralities' being *modally rigid.* This

[140] Namely, the concern that potentialist systems' validating S4.2 are 'actualism in potentialist garb.'

entails that "when x is one of some objects yy, then this is necessarily so… and likewise when x is not one of yy" (p. 177) – by way of example, Linnebo and Shapiro ask us to:

> … [c]onsider Barack and Michelle. If Michelle were not one of some people, then these people would not be Barack and Michelle but some other people. Likewise, if Vladimir were one of some people, these people would not be Barack and Michelle but some other people. (p. 177)

That all pluralities exhibit this feature of modal rigidity receives a more extended treatment – as well as a detailed defence – in Linnebo's book-length treatment of plural logic (with Salvatore Florio), *The Many and the One* (2021). There, the authors begin by pointing out that a *principle of extensionality* is "widely assumed to provide a criterion of identity for pluralities" (p. 206) – and indeed, the reader will recall from Chapter 3 (§3.2) that a version of extensionality helped us to define the plural identity predicate, '$\approx$':

$$vv \approx xx \leftrightarrow_{df} (vv \preccurlyeq xx \wedge xx \preccurlyeq vv)$$

where '$\preccurlyeq$' is the sub-plurality predicate, and defined as follows:

$$xx \preccurlyeq vv \leftrightarrow_{df} \forall u(u \prec xx \rightarrow u \prec vv)$$

Of course, the import of principles of extensionality when it comes to the identity of *sets* – as encapsulated in the Axiom of Extensionality, below – is already well-appreciated:

$$Extensionality: \forall x \forall y(\forall u(u \in x \leftrightarrow u \in y) \leftrightarrow x = y)$$

However, a clear parallel between plural identity and that of sets is soon revealed if, in light of our definition of '$\preccurlyeq$' above, we render the definition of the plural identity predicate as follows:

$$vv \approx xx \leftrightarrow_{df} \forall u(u \prec xx \leftrightarrow u \prec vv)$$

That both sets and pluralities are thus extensional in nature, forms the core of Linnebo and Shapiro's arguments that these collections are *modally rigid*; in the case of sets, this entails that if something is (not) an element of a set, then it is *necessarily* (not) an element of that set:

$$(Set\_Rgd^{+}): \Box \forall x \forall y(x \in y \rightarrow \Box(E!\, y \rightarrow x \in y))$$

$$(Set\_Rgd^{-}): \Box \forall x \forall y(\sim(x \in y) \rightarrow \Box \sim(x \in y))$$

Linnebo and Florio take the conjunction of (Set_Rgd^{+}) and (Set_Rgd^{-}) to encapsulate the claim that membership in a set is not subject to 'drift': that is, a set necessarily contains the elements it in fact does, at every world where it exists. But, critically, they argue that *precisely the same principle holds true of pluralities*: that is, necessarily, any member of a plurality is necessarily a member of that plurality, whenever the plurality in question exists:

$$(Pl\_Rgd^{+}): \Box\forall x \forall yy(x \prec yy \rightarrow \Box(E!yy \rightarrow x \prec yy))$$[141]

And moreover, that, necessarily, whenever an entity is *not* a member of plurality, then it is *necessarily* not so:

$$(Pl\_Rgd^{-}): \Box\forall x \forall yy({\sim}(x \prec yy) \rightarrow \Box{\sim}(x \prec yy))$$

Linnebo and Florio thus declare, in view of plural modal rigidity, that "[e]very plurality thus exhibits extensionality in its purest form" (p. 227); and indeed, while (Pl_Rgd^{+}) and (Pl_Rgd^{-}) – like their set-theoretic counterparts – are by no means indisputable[142], both principles also enjoy a fair amount of intuitive appeal. In the modal context, though, this claim has a number of notable consequences – especially for the particular principles of *plural logic* validated by the modal (or potentialist) theory, on the one hand, and its non-modal (actualist) counterpart, on the other.

For one thing, the modal rigidity of pluralities would appear to give us good grounds to believe that not all instances of the modalised principle of Plural Comprehension$^{\Diamond}$ are *true:*

$$(Plural\ Comprehension^{\Diamond}): \Diamond\exists xx \Box\forall y(y \prec^{\Diamond} xx \leftrightarrow \varphi^{\Diamond}(y))$$

Of course, that Plural Comprehension$^{\Diamond}$ is invalid is, as we've seen throughout this dissertation, a cornerstone of *quantifier relativism* – where the relativist's reasons for this rejection hinge, specifically, on her taking the following key instance of Plural Comprehension$^{\Diamond}$ to be false:

$$(Absolutely\ Comprehensive\ Domain^{\Diamond}): \Diamond\exists xx \Box\forall y(y \prec^{\Diamond} xx)$$

However, the phenomenon of plural rigidity promises to further buttress the relativist's response. For, as Linnebo and Florio point out, what it appears to reveal is that "plurals are governed by strong extensionality principles *whose satisfaction is a non-trivial matter*" (p. 229, emphasis mine); but this in turn entails that the claim encapsulated by Plural Comprehension$^{\Diamond}$ – viz., that *any* (modalised) condition $\varphi^{\Diamond}$ determines a potential plurality of its instances – itself has decidedly non-trivial (and, indeed, fairly substantial) content.

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[141] There are a number of options available to us when it comes to the formulation of the plural existence predicate, $E!xx$ – one is to take it as primitive. Alternatively, we can opt for a defined predicate: for instance, given his bi-modal operators, Studd defines $E!xx$ as follows:

$$E!xx =_{df} {\sim}\Diamond_{>}\exists u(u \prec \Diamond xx \wedge \Box_{<}{\sim}E!u)$$

which translates (somewhat wordily) as, 'for absolutely all subsequent reinterpretations of the language of set theory, there is no element u such that u is one of the xx and in absolutely all prior interpretations, u did not exist.' Evidently, this definition is not only compatible with plural rigidity, but can be seen to entail it.

[142] See pp. 213 – 216 of Linnebo and Florio (2021) for the objections one might raise to the modal rigidity *of sets;* similar considerations carry over to the plural case.

For example, in the case of the potential set predicate, ß$^{\diamond}$ the corresponding instance of Plural Comprehension$^{\diamond}$ testifies to the existence, at some point in the domain hierarchy, of a plurality comprising absolutely every set – where this, in the *interpretational* expansionist context, in turn corresponds to an absolutely comprehensive set-theoretic domain. However, that this absolutely comprehensive (plurality-encoded) domain is *modally rigid* entails not only that other domains comprising additional sets – and so, alternative *interpretations* countenancing these sets – *do not* exist, but rather than they *cannot* exist. The relevant instance of Plural Comprehension$^{\diamond}$ thus amounts to the claim that no further admissible interpretations of our set-theoretic vocabulary where this plurality-encoded domain gains new members, *are possible.*

Contrastingly, for the potentialist *relativist,* the strong nature of this claim arguably forms part of her overall motivation for *rejecting* Plural Comprehension$^{\diamond}$[143]. Indeed, the phenomenon of plural rigidity helps at the same time to explain why the potentialist relativist can comfortably retain the former principle's *non-modal* counterpart:

$$(Plural\ Comprehension): \exists xx \forall y(y \prec xx \leftrightarrow \varphi(y))$$

any *prima facie* tension caused by the discrepancy between relativist's accepting the latter principle[144], while rejecting its modal counterpart, is soon diffused if we note that membership in this extension of φ – insofar as this extension comprises a plurality – is always *stable at* a given domain, or world (even if it is subject to growth – and so defines a new plurality – in other domains or worlds).

In sum: plural modal rigidity arguably helps the potentialist relativist justify her rejection of certain modalised principles, *despite* her acceptance of their non-modal counterparts, and vice versa[145]. However, precisely this unique distribution of acceptance/rejection is what lies behind Linnebo and Shapiro's argument that actualism and potentialism validate *different principles of higher order logic* (and, consequently, that modal accounts of set are *not* 'actualism in potentialist garb'). Indeed, in 'Actual and Potential Infinity' (2017) the authors take this idea one step further, by "deny[ing] that the non-modal language is *fully explicit*" (p.

[143] Of course, on her view, the invalidity of this principle is a natural consequence of the fact that the (plurality-encoded) extensions of some modalised predicates are liable to gain new members (and so, define new pluralities) as interpretations of our set-theoretic vocabulary – and their associated domains – shift (especially where the predicates in question correspond to *absolutely indefinitely extensible concepts*). Indeed, this consideration extends even to certain *non-modalised* conditions (such as, for instance, Studd's (2019) singular existence predicate, '$E!$,' the plurality-encoded extension of which comprises everything relative to each – crucially, *expanding* – domain in the hierarchy).

[144] Plural Comprehension is an axiom of both the plural logic PFO, and its modal counterpart $MPFO$. See Appendices A and B, respectively, for the axioms and rules of each.

[145] The relativist potentialist can justify her acceptance of Collapse$^{\diamond}$ – together with her *rejection* of (non-modal) Collapse – for much the same reasons she accepts Plural Comprehension, while rejecting Plural Comprehension$^{\diamond}$. For once a plurality (potentially) exists, modal rigidity ensures that membership in this plurality is stable, which in turn ensures that these elements can *consistently* comprise a (potential) set, as per Collapse$^{\diamond}$. By contrast, the truth of *non-modal* Collapse would entail that some pluralities – such as, notably, those encoding the extension of '$E!$' at a domain – are liable to gain *new members*, thereby contradicting the claim that the pluralities in question are (modally) rigid.

168, emphasis) – a claim they elaborate on at length in another paper, 'Predicativism as a Form of Potentialism' (2023):

> … we use the extra expressive resources afforded by the modal language to engage in reasoning that cannot take place in the corresponding non-modal language, not even when all of its quantifiers are understood as implicitly modalised. The modal language thus allows us to look at the subject matter under a finer resolution, which the mirroring theorems enable us to turn on and off, according to our needs. (p.9)

Critically, a version of this reply to the Objection from Implicit Actualism may even be available to Studd. For he indeed concedes, in *Everything, More or Less* (2019), that at least in the case of *first-order* (singular) languages, the claims of the modal language *are* equivalent, in quite a straightforward way, to those formulated in the non-modal language: specifically, satisfying a modalised formula $\varphi^{\Diamond}$ relative to the potential hierarchy of interpretations is strictly equivalent to satisfying its non-modal counterpart, φ, when this formula is interpreted relative to the single, maximal interpretation, $\langle M_{\infty}, E_{\infty}, S_{\infty}\rangle$. Importantly, this equivalence *extends to the quantifiers*: as Studd points out, satisfying the *modal* quantified statement '$\Box\forall x\varphi^{\Diamond}$,' as interpreted by his extended sequence of interpretations, is equivalent to satisfying its non-modal counterpart, '$\forall x\varphi$,' as interpreted by $\langle M_{\infty}, E_{\infty}, S_{\infty}\rangle$.

Consequently, in the context of singular languages, Hamkins's Objection from Implicit Actualism rings true; things change significantly, however, with the advent of *plural* languages. This swiftly becomes apparent, if we attend to the details of Studd's Mirroring Theorem:

$$Mirroring\text{: } \Gamma \vdash_{FOL} \varphi \quad \text{iff } \{\gamma^{\Diamond}\text{: } \gamma \in \Gamma\} \vdash_{MFO} \varphi^{\Diamond}$$

Crucially, Γ is a set of non-modal formulae of the *singular* language $\mathcal{L}_S$ – correspondingly, any candidate modalisation $\varphi^{\Diamond}$ of a formula derived from Γ will itself *only contain singular variables*.

And indeed, Mirroring *is* only valid for formulae of singular languages, in the modal context: for instance, while a (left-to-right) version of the theorem holds for some formulae of the language of plural set theory ($\mathcal{L}_{PS}$), it fails for (non-modal) Plural Comprehension (where this, as Studd points out, is simply because $MPFO$ "does not prove the modalisation of this axiom" (p. 256). Similiarly, the right-to-left direction doesn't hold for all formulae of $MPFO$ (the notable exception being Collapse$^{\Diamond}$, the non-modal counterpart of which is inconsistent in PFO).

Consequently – to paraphrase Linnebo and Shapiro – it would appear that the shift to a modal theory *does* 'make a real difference,' at least when it comes to the principles of plural logic validated. However, it is important to note that this mismatch does *not* extend to the case of other higher-order logics: for instance, when it comes to the comprehension scheme of classic *second-order* logic (where the variable 'F' ranges over concepts):

$$(2\_Comprehension): \exists F \forall x (F(x) \leftrightarrow \varphi(x))$$

and its corresponding, *modalised* counterpart:

$$(2\_Comprehension^{\Diamond}): \Diamond \exists F \Box \forall x (F(x) \leftrightarrow \varphi^{\Diamond}(x))$$

it would appear – in contrast to the case with plural logic – that the potentialist relativist has no reason (plural-rigidity-based or otherwise) to reject *either* principle. In particular, when it comes to the *modalised* second-order comprehension scheme, there seems to be no reason to expect that the extension of a concept will *not* vary from world to world (in fact, it seems perfectly coherent for a given concept to be one and the same in two different worlds, even if its extension varies *between* these worlds).

Linnebo and Florio (2021), chalk this up to the fact that "plural membership and predication have different modal profiles" (p. 118). However, while this does mean that Linnebo's claim that actualism and potentialism validate different principles of higher order logic – and so, his reply to the Objection from Implicit *Actualism* – is limited in scope, the claim that plural membership and predication exhibit 'different modal profiles' may end up furnishing us with a reply to the Objection from Implicit *Potentialism*; specifically by helping to illuminate a critical prerequisite for the generality expressed by our (so far, only ostensibly) actualist quantifiers.

The prerequisite in question (which *fails* to constrain our modal operators[146]) is a potentialist version of *Russell's Vicious Circle Principle* – which, the reader will recall, was first formulated by its namesake as a response to the idea that the paradoxes "arise from the fact that an expression referring to all of some collection may itself appear to denote one of the collection." Correspondingly, the Vicious Circle Principle stipulates that "where this appears to occur… nothing whatever can significantly be said about all of the supposed collection" (p. 261).

Practically speaking, this amounts to a prohibition on *impredicative definitions* in set theory (where this prohibition, in particular, is what prompted Richard Cartwright's (1994) assessment of Russell's diagnosis of the paradoxes as "off the wall" (p. 10)). However, in their own discussion of these issues in 'Predicativism as a Form of Potentialism' (2023), Linnebo and Shapiro call into question this over-hasty evaluation. Drawing on the work of Henri Poincaré – in particular, the latter's 'The Logic of Infinity'[147] (1908) – they argue that the Vicious Circle Principle, and predicativism more generally, can be understood primarily as a means of ensuring definitional stability' in the context of quantification over totalities to

[146] That modal operators do not conform to this will be subject of detailed discussion in the conclusion of this chapter, and overall thesis.
[147] *La Logique de l'Infini* (1908).

which, in Poincaré's words, "we can add new elements unceasingly" – where this, in turn, poses the risk our set-theoretic definitions' being "disordered [*bouleversé*] by the introduction of new elements" (p. 47). Linnebo and Shapiro (2023) subsequently extend this consideration to the case of *potentially* infinite domains – the 'potentiality' characterizing which, they argue, threatens the stability of our set-theoretic definitions in a similar way:

> The belief that certain domains are merely potential poses a threat to our attempts to define new elements of such domains: will our definitions be stable as more and more elements are added to the domain or might the definitions instead be 'disordered' by such additions? (pp. 3 – 4)

Having articulated this worry, Linnebo and Shapiro (2023) contend that "the easiest and most natural way" to safeguard 'definitional stability' is "to require that a definition of a new element of a collection restricts all its quantifiers to 'old' elements." However – as the authors proceed to note – to do this is "precisely to impose the Vicious Circle Principle" (p. 3).

In fact, we can make the inchoate notion of 'restricting quantifiers to old elements' more precise, with recourse to precisely the phenomenon of *plural modal rigidity*. This is because the modal rigidity of pluralities would rule out, in particular, 'the introduction of new elements' that, on Linnebo's and Shapiro's (and Poincaré's) analysis, risks 'disordering' our set-theoretic definitions; consequently, given the existence of domains to which we *do* 'add new elements unceasingly,' restricting our non-modal quantifiers to *pluralities* emerges as a natural means of ensuring definitional stability.[148]

The upshot of all this is that something very much like the Vicious Circle Principle – in the guise of the stipulation that the range of our quantifiers be limited to *pluralities,* or some other modally rigid totality – emerges as an obvious means of ensuring the stability of the definitions featuring these quantifiers. However – as we've seen throughout our discussion – the utility (if not the *necessity*) of Vicious Circle-like principles is especially apparent in contexts where our definitions risk being "disordered by the introduction of new elements": which contexts, as Linnebo and Shapiro (2023) point out, arguably include *potentially infinite domains.*

Correspondingly, the foregoing arguments for the Vicious Circle Principle promise also to shed light on the validity of Cantor's *Domain Principle:*

> Each potential infinite, if it is rigorously applicable mathematically, pre-supposes an actual infinite. (Cantor, as quoted in Hallett, 1984, p. 25)

[148] Linnebo and Florio (2021, p. 270), offer a response along these lines in relation to their own, web-page-based version of Russell's Paradox.

Indeed, over and above the more general link to 'potentiality' Linnebo and Shapiro (2023) note above, Poincaré's (1908) talk of domains "to which we can add elements unceasingly" (p. 47) clearly evokes the phenomenon of *indefinite extensibility*, and thus the (Dummettian) answer we ourselves opted for in response to the question – the first we posed in this dissertation – of how to understand the 'variability' of the potentially infinite domains with which the Principle is concerned:

> **Question 1:** What does the 'variability' of a given domain – specifically, a 'potentially infinite' domain – consist in?

However, the foregoing discussion – especially our account of how modal rigidity might prevent our definitions' being "disordered by the introduction of new elements" (p. 47) – promises to shed light on our further questions (and their answers!), which we raised in relation to Cantor's and Dummett's *justification* for the Principle.

For instance, in the case of Question 2 below:

> **Question 2:** How does 'variability' of domain undermine its rigorous mathematical applicability?

we argued that that the 'variable' *qua indefinitely extensible* nature of mathematical domains entails their being indeterminate, or 'hazy,' in a number of ways. On the one hand, 'haziness' is a feature of the *sequence* of domains accompanying indefinitely extensible concepts (the length of which Dummett describes as "vanish[ing] in the indiscernible distance" (1991, p. 317)). This haziness – the second kind Dummett adumbrates in relation to indefinite extensibility – is the sort instantiated by the potentialist relativist's domain hierarchy; in turn, it *undermines* classical quantification simply due to the fact that no single quantifier ranges simultaneously over the elements of *every* domain in the hierarchy.

By contrast, the first – and somewhat more nebulous – species, affects a *single* indefinitely extensible domain, and involves "haziness about what elements [this domain] does or does not contain" (p. 314). However, we saw in Chapter 1 that *eliminating* this species of haziness – a process Dummett describes in terms of *circumscribing* the relevant domain – entails something over and above *sharply specifying* it: as revealed, in particular, by Dummett's discussion of the reals in *Frege: Philosophy of Mathematics* (1991), circumscribing a domain involves not only determining "what is required of a specified mathematical entity for us to recognise it as a real number," but also "determin[ing] *the limits of acceptable specification* of something to be acknowledged as a real number" (p. 315, emphasis mine).

And indeed, we encountered a number of scenarios in Chapter 2 where precisely the 'limits of acceptable specification' of a given mathematical entity were up for debate. This began with Wright and Shapiro's (2006) worry that the instances of certain concepts – notably, the concept 'finite ordinal' – appear to be "indeterminate in extent within a wider population" of ordinals (p. 295); however, we saw in §2.4 that this worry is *not limited* to the (merely) relatively indefinitely extensible: as made vivid by Linnebo's question of "why there are not more ordinals than the *oo*" (2010 p. 153) – where '*oo*' denotes *the* plurality of *all* ordinals – the same concerns apply equally (if not primarily) to concepts that are *absolutely* indefinitely extensible.

All of this formed the basis of what we in Chapter 2 called the *Objection from Vagueness.* Moreover, to the degree that this 'vagueness' (viz., the troublesome distinction between sets and pluralities) *undermines classical quantification* – specifically, by engendering an "indeterminacy in the range of admissible witnesses" (Wright and Shapiro, 2006, p. 295) for our quantified statements – we saw that one way to *ameliorate* this, would be to accept Collapse$^{\diamond}$ (and so, essentially, eliminate the problematic distinction altogether).

This idea formed the core of our response to Question 3 below:

> **Question 3:** Why is circumscribing a domain (viz., eliminating any 'haziness about what elements it does or does not contain') established only when the domain in question comprises a set?

However – and importantly – the notion of plural rigidity promises further to buttress this initial response. For as we saw specifically in §1.2, the peculiar nature of this question about what elements an indefinitely extensible domain does or does not contain, is that it arises specifically at the *limits* of these domains (the question is *how far* the finite ordinals extend; or, in Linnebo's phrasing, whether 'there are *not more ordinals* than the *oo*').

What all this seems to entail, is that circumscribing a domain is fundamentally a matter of *determining its extent*: specifying 'the limits of acceptable specifications' for the objects in this domain, in other words, amounts to the requirement that the latter *exhaustively comprise every (potential) object in questio*n. But an intuitive – if not straightforwardly obvious – way to cash out this requirement, is as claim that this domain will *not potentially gain (or lose) members*. And this is just the requirement that the domain in question be *modally rigid*,

Of course, we should note that to claim that a domain can be exhaustively circumscribed only when it is modally rigid, is not *quite* to assert – as per the letter of Question 3 – that this is *only* assured when the domain in question comprises *a set* (modal rigidity is, after all, a feature of *pluralities,* as well as sets). However, it does come awfully close: indeed, that *both* sets and pluralities exhibit modal rigidity arguably helps to narrow even further the gap between these two species of collections[149] (in fact, all this can plausibly be taken as yet another indication that the relativist's response to Question 4 – concerning the difference

[149] Indeed, we can plausibly take modal rigidity to be a pleasing way of making formally precise Parsons's (2005a) notion of *potential co-existence.* As we saw in §2.5, Parsons contends that, where the members of a multiplicity *cannot* possibly co-exist, this constitutes an "essential obstacle" to this multiplicity's "being collected into a unity" – by contrast, "the possibility of all the elements of a multiplicity *being together* [is] the possibility of their being collected together into one thing" (p. 281). However, if the elements of a collection's 'possibly being together' is equivalent to that collection's being *modally rigid* (at least, once the collection in question exists), then we can reparse the former claim as the idea that it is the *modal rigidity* of pluralities that ensures that there corresponds, to each plurality, a (potential) set.

between sets and pluralities – is basically right: *no absolute difference between these totalities exists*).

In sum: if our overall analysis in this section is correct, then it follows that the modal rigidity (or 'set-like' nature) of a domain not only suffices to ensure that this domain can be circumscribed (viz., its elements exhaustively specified); rather – and far, far more strongly – the implication is that the degree to which a domain *admits* of being circumscribed *is the degree to which it is modally rigid.* In turn, if exhaustively circumscribing a domain is equivalent to securing the 'determinate range of admissible witnesses' Wright and Shapiro (2006) describe as necessary for classical quantification, then it follows that the only domains suited to the latter are those that, *qua* being modally rigid, comprise either a set, or set-like object.

This, of course, is none other than Cartwright's All-in-One Principle – the sentiment underlying which was first formulated by Cantor over a century before, in the guise of his Domain Principle.

Conclusion

We already noted in Chapter 2 that an implicit commitment to the Domain Principle (indeed, to the modal rigidity-inspired version of it just articulated) is already present in Studd's (2019) potentialist account of sets – despite his protestations to the contrary[150] – insofar as his non-modal quantifiers are limited to modally rigid (*qua* set- or plurality-encoded) domains. This stands in contrast to his modal operators, which range over the entire domain hierarchy (the decidedly non-rigid nature of which is captured by Dummett's (1991) descriptions of its height as "hazy… vanish[ing] in the indiscernible distance" (p. 315)).

However, if Studd's non-modal quantifiers are so limited due to the *requisites* of classical quantification – if, that is, these requisites *explain* why our quantifiers *fail* to range over the entire domain hierarchy – then this naturally gives rise to the question of what it is about the generality expressed by Studd's *modal operators* that frees them from this constraint (where this question is merely a further iteration of one we've posed a number of times in this chapter – and which forms the core of the Objection from Mysteriousness – viz., *what distinguishes modal from quantificational generality*).

Linnebo's own response to this hinges on a distinction he introduces in 'Generality Explained' (2022), between *instance*- and *non-instance-based generality.* The first species of generality – as its name implies – entails that the explanation of a given generalisation "proceed via individual instances of the generalisation" (p. 349); for instance, given a universal generalisation, Linnebo notes that this:

[150] Studd (2019) describes the Principle as "dubious" and "discredited" (p.178).

> … requires that it be possible to consider all of its instances, provide an explanation of each instance, and then conjoin all of these individual a explanations to produce the desired instance-based explanation. (p. 352)

However, something along these lines seems to be what we take the truth of a *universally quantified* statement to consist in; for instance, in Dummett's descriptions of classical quantification, the truth-value of a classically quantified sentence is "the final outcome of a process which involves running through the values of all its instances" (1991, pp. 313 – 314). Indeed, the need to "establish the truth-value of *every instance* of the quantified statement," is what lies behind the latter's insistence that classical quantification requires the (at least in-principle) possibility of "conducting a *complete survey*" of the domain (1996, p. 61, emphasis mine): that is, the need to 'consider all of [the] instances' of a quantified statement is what gives rise to the need to establish that our quantifier domains are *exhaustive.*

Consequently, where there *isn't* "a definite range of instances to consider" – as would be case if we *lacked* the assurance that the relevant domain is exhaustive – then it follows that "the instance-based approach is unavailable" (Linnebo, 2022, p. 352). However, Linnebo and Shapiro (2023) point out that this needn't entail that we "abandon any attempt at true universal generalisations over an incompletable domain" (p.6): for we might instead avail ourselves of a form of generality which proceeds not "via individual instances," but is rather "based on *general facts* about the properties or operations involved in the claim that is generalized" (Linnebo, 2022, p. 349).

In 'Actual and Potential Infinity' (2017), Linnebo and Shapiro offer a rough sketch of what this 'non-instance based' generality might look like, based on remarks by Hermann Weyl in his 'On the New Foundational Crisis of Mathematics'[151] (1921). There, Weyl questions the possibility of instance-based generalisations about all numbers, by way of declaring "a completed run through an infinite sequence [as] nonsensical"; however, he nevertheless contends that it *is* possible to successfully ascribe (say) a property to *all* numbers, on the condition that this is grounded in "the insight… that it lies in the essence of number to have the property" (p. 98). Correspondingly, Linnebo and Shapiro (2017) argue that there are certain generalisations – such as 'every red object is coloured' and 'every atom of gold consists of 79 protons' – that "seem unconcerned with individual red objects or atoms of gold," *despite including these objects in their scope*; consequently, the authors conclude that such generalisations are 'made true' not by recourse to their instances, but rather by "*what it is* to be red or coloured" (p. 181, emphasis mine).

We should note that Linnebo's (2022) distinction between instance- and non-instance-based generality is *not* how Studd (2019) himself opts to cash out the different kinds of generality expressed by his quantifiers and his modal operators. However – aside from his *stipulations* that the quantifiers are interpreted over individual domains, and the operators, over the *entire*

[151] *Über die neue Grundlagenkrise der Mathematik,* (1921).

domain hierarchy – it is not at all clear that Studd makes any attempt to distinguish the forms of generality each of these expressions captures.

This omission makes Studd especially vulnerable to the Objection from Mysteriousness; however, it also reveals the utility Linnebo's (2022) distinction might have for his account, insofar as it explains what it is about the generality expressed by his modal operators that allows these – in contrast to his non-modal quantifiers – to range over the *entire* domain hierarchy. In fact, the claim that this form of generality is of a more 'essence-based' kind may allow Studd to diffuse the version of Williamson's (2003) *SECOND* objection we levelled against the clauses – like (k_i-∀) below – governing the interpretations of his non-modal quantifiers:

(k_i-∀): For any admissible interpretation i and assignment σ: $\forall v \psi$ is true$_{i,\sigma}$ iff
every item $a \in M_i$ is such that ψ is true$_{i,\sigma[v/a]}$

As we saw in §4.1, the difficulty with these clauses comes with the requirement that the expansionist grasp the notion of 'any admissible interpretation,' where this is sufficiently constrained so as to rule out as inadmissible many interpretations that are nevertheless *logically* possible. The worry is that this grasp would appear just as easily to facilitate the expansionist's grasping *every* admissible interpretation, and so *merging* these together into an overarching, 'maximal' interpretation.

However, there may be a way out of this quandary, should the expansionist avail herself of the notion of 'essence-based generality.' As Linnebo and Shapiro (2017) point out, there plausibly exist what are specifically "essence-based constraints on any future generation of the objects studied by mathematics" (p. 181); correspondingly, Studd might rely on similar constraints relative to our concept of 'set' in the *interpretational* expansionist context, to provide the necessary limitations on what counts as an admissible future *reinterpretation* of our set-theoretic vocabulary.

That Studd's modal generality is non-instance-based might also help him explain the feature of his modal *truths* that made these an especial target of Joel Hamkins's (2018) ire, in the context of the Objection from Implicit Actualism – namely, the markedly *stable* nature of these truths in the face of expanding domains. We saw in §4.3 that this a function of Studd's potentialist account's being *convergent* (*qua* validating a modal logic at least as strong as S4.2): this contrasts with the "more radical form[s] of potentialism" Hamkins takes to be exhibited by systems' validating S4, where there is "truly branching possibility… statements that could become true, but might not" (p. 34).

However, Linnebo (2022) argues that a unique feature of the non-instance-based form of generality he describes – one that allows it to step into the gap left by its instance-based counterpart when there *isn't* 'a definite range of instances to consider' – is its being "robust enough to ensure that the generalisation will continue to hold no matter how the range of available objects expands" (p. 357): which, in the potentialist context, translates as the fact that the *modal* truths validated "enjoy a high degree of modal robustness" (p. 361). Correspondingly, to the degree that Studd's modal truths *do* reflect this kind of generality, this would explain *why* they are so 'robust' as to remain true, in the face of the domain expansions accompanying each reinterpretation of our set-theoretic vocabulary.

Importantly, no such explanation would be forthcoming, should Studd's modal truths be instance-based: for, at least on Linnebo's (2022) account, these generalisations exhibit a comparably *low* degree of modal robustness. This boils down to the (possible) existence of counter-examples, beyond the instances of the generalisation available in the current domain (as Linnebo – drawing on a temporal metaphor – puts it: instance-based generalisations are "problematic… [e]ven though all the objects available today satisfy some generalisation, a counterexample might arise tomorrow" (p. 357)).

However – and unfortunately, for Studd – a renewed version of the Objection from Implicit Potentialism rears its head at just this point: one that is *sharpened* by the distinction between instance- and non-instance-based generality (this, ironically, despite the distinction's diffusing aspects of the Objection from Implicit *Actualism*).

We first raised the former objection in §4.4: there, we contended that the *logical behaviour* of Studd's non-modal quantifiers – as a function of his modifying the standard quantifier rules – is consonant less with these quantifiers' ranging *unrestrictedly* over a less-than-comprehensive domain, and more with their being *restricted*, relative to a domain comprising absolutely everything. In the particular case of Universal Instantiation, though, Studd's proposed modification had the consequence of effectively *ruling out the possibility of counter-examples to a given, universally quantified statement.* This meant that the resulting sentences exhibited a remarkable *lack* of parochiality in the face of subsequent domain shifts – which, in the current context, is equivalent to their displaying the 'high degree of modal robustness' associated more with non-instance-based, *modal* generality.

We can extend this concern to Studd's modification of Universal *Generalisation* – the 'liberalization' of which, the reader will recall, allowed us effectively to infer the truth of a universally quantified sentence *in the absence of its instances.* However, if this generalisation *were* instance-based, then – on the foregoing analysis – *this should not even be a possibility* (by contrast, as per Linnebo's (2022) remarks above, that we can formulate a generalisation *absent the instances with which it is concerned* is a hallmark of *non*-instance-based generality).

Of course, Studd (2019) might easily reply to this enhanced version of the Objection from Implicit Potentialism by rejecting our attempt to map the different forms of generality expressed by his quantifiers and modal operators onto Linnebo's (2022) distinction between instance- and non-instance-based generality. However, an obvious downside to this strategy is the gap it leaves in the wake of Studd's own failure to account for this difference in any substantial way.

That we *shouldn't* understand the difference between Studd's quantificational and modal generality in terms of Linnebo's distinction is especially implausible, if we consider that the latter in many ways explains the differences that *are* exhibited by these divergent forms of generality. Indeed, this extends to the potentialist's diagnosis of the paradoxes: for the claim that the latter are a function of our quantifiers' failing to range simultaneously over all (potential) sets explains very little, unless accompanied by an account of why our quantifiers should be restricted to (pluralities of) actual sets. In turn, the distinction between instance and non-instance-based generality promises not only to provide such an account (and so, to explain *why* the paradoxes arise), but also points to a way they might be *dissolved* (namely, by countenancing additional forms of generality).

However, if we *are* to understand the difference between Studd's modal and quantificational generality in terms of Linnebo's distinction, then the version of the Objection from Implicit Potentialism we articulated earlier still stands: indeed, we can frame it as a dilemma. On the hand, Studd can endorse an *entirely* instance-based interpretation of the generality expressed by his quantifiers (where this risks rendering the resulting generalisations less 'modally robust'). Alternatively, he can *retain* this 'modal robustness' (which would amount to his leaving his interpretation of the quantifiers – including his modification of the quantifier *rules* – as is). However, the consequence of *this* is that the distinction between modal and non-modal generality – which forms the basis of his potentialist account – is rendered unstable.

We should note that Linnebo (2022) entertains a third option for the expansionist: specifically, he suggests that "being instance-based or generic is a matter of degree" (p. 2), elaborating on this idea as follows:

> … perfectly univocal generalisations can, when true, be explained in interestingly different ways: some in a highly instance-based way, some in a purely generic way, and some in both of these ways. (p.3)

That some generalisations admit of a *combination* of instance- and non-instance-based explanations would explain the varied features of Studd's quantified statements (for instance, that they are modally robust, while nevertheless being limited to individual domains). However, the obvious worry with this is that it is it is difficult to see how it could be maintained without collapsing into the second horn of our dilemma above, where the crucial distinction between modal and non-modal generality risks being erased altogether.

In view of all of this, it is difficult not to conclude – to paraphrase Dummett (1991) – that 'what the paradoxes revealed' was 'not the existence of concepts with inconsistent extensions,' *nor merely* 'what may be called indefinitely extensible concepts': rather, what the paradoxes revealed was that we lack a fleshed-out account of *just what quantificational generality consists in* – and so, what its underlying requisites are.

In turn, the degree to which we lack such an account, in the current context, is the degree to which the question of 'what stands between our quantifiers and absolute generality' – and so, the overall Objection from Mysteriousness – still stands.

Appendix A: PFO

An outline of the plural logic PFO can be found in Linnebo (2014). The presentation below largely mimics Studd's (2019, pp. 245 – 249), with some minor adaptions.

A.1 Language

Singular Languages

It is useful to begin with a brief overview of the language of first-order set theory, $\mathcal{L}_{\mathrm{S}}$ (or, with urelements, $\mathcal{L}_{\mathrm{SU}}$).

<u>Primitive symbols</u>

- non-logical predicates: $\in$ (and, for impure languages, β)
- logical predicate: $=$
- connectives: $\sim$ and $\rightarrow$
- countably many singular variables: $x, y, \ldots$
- singular quantifiers: $\forall x, \forall y \ldots$

<u>Recursive definitions</u>

- Whenever *u* and *v* are singular variables:
  - $u = v$ is an (atomic) formula
  - $\beta(u)$ and $u \in v$ are (atomic) formulas
- Whenever ϕ, ϕ_1 *and* ϕ_2 are formulas, $\sim\phi$ *and* $(\phi_1 \rightarrow \phi_2)$ are formulas
- Whenever ϕ is a formula and $\forall v$ a singular quantifier, $\forall v\phi$ is a formula.

Plural Languages

The plural language $\mathcal{L}_{\mathrm{PS}}$ (or, with urelements, $\mathcal{L}_{\mathrm{PSU}}$) extends the singular language $\mathcal{L}_{\mathrm{S}}$ ($\mathcal{L}_{\mathrm{SU}}$) by adding the following primitive expressions:

<u>Primitive symbols</u>

- Logical predicate: $\prec$
- Countably many plural variables: $xx, yy, \ldots$
- Plural quantifiers: $\forall xx, \forall yy, \ldots$

The following clauses are then added to the recursive definitions of $\mathcal{L}_{\mathrm{S}}$ ($\mathcal{L}_{\mathrm{SU}}$):

<u>Recursive definitions</u>

- Whenever u is a singular variable and vv a plural variable, $u \prec vv$ is an (atomic) formula.
- Whenever ϕ is a formula and $\forall vv$ is a plural quantifier, $\forall vv\phi$ is a formula.

We thus adopt the following notational conventions:

Table A.1 Notational conventions

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u, v – singular variable
uu, vv – plural variable
u, v – variable (either singular or plural)
ϕ – non-modal formula
Γ – set of non-modal formulae

We adopt the same conventions for 'decorated variants' (e.g. u and u' will also stand for singular variables); moreover, for our modal languages, we will additionally adapt the following conventions:

ψ – formula (modal or non-modal)
Δ – set of (modal or non-modal) formulae

A.2 Model theory

Given that Studd's (2019) default theory is an impure one (viz., $ZFCSU_p$), we will focus on providing a model theoretic interpretation for a language with urelements, $\mathcal{L}_{\text{PSU}}$.

It is useful to once again start with a singular language ($\mathcal{L}_{\text{SU}}$). A model-theoretic interpretation (henceforth, MT-interpretation) for $\mathcal{L}_{\text{SU}}$ is an ordered triple $\langle M_i, S_i, E_i \rangle$ where M_i is a non-empty set, $S_i \subseteq M_i$ and $E_i \subseteq M_i \times M_i$. In turn, an assignment σ over an interpretation i maps each singular variable in the lexicon to a member of M_i. Thus, relative to an MT-interpretation i and an assignment σ over it, truth$_{i,\sigma}$ is defined as follows:

(T − β) βv is true$_{i,\sigma}$ iff $\sigma(v) \in S_i$.

(T− ∈) $u \in v$ is true$_{i,\sigma}$ iff $\langle \sigma(u), \sigma(v) \rangle \in E_i$.

(T − =) $u = v$ is true$_{i,\sigma}$ iff $\langle \sigma(u), \sigma(v) \rangle \in I_i$

(T − ~) $\sim\phi$ is true$_{i,\sigma}$ iff ϕ is not true$_{i,\sigma}$.

(T − →) $\phi_1 \rightarrow \phi_2$ is true$_{i,\sigma}$ iff ϕ_1 is not true$_{i,\sigma}$ or ϕ_2 is true$_{i,\sigma}$.

(T − $\forall v$) $\forall v\phi$ is true$_{i,\sigma}$ iff every $a \in M_i$ is such that ϕ is true$_{i,\sigma[v/a]}$.

In the clauses, I_i is the intended extension of the identity predicate based on M_i:

$$I_i =_{\text{df}} \{\langle a, b\rangle \in M_i \times M_i : a = b\}$$

Moreover, $\sigma[v/a]$, in the clause for (T − $\forall v$), is the assignment which maps v to a and assigns other variables the same values as σ does.

To extend this to the plural language $\mathcal{L}_{\text{PSU}}$, we stipulate that σ also assigns, to each plural variable, a subset of M_i. Consequently, we have the following definition of the member-plurality predicate based on M_i:

$$P_i =_{\text{df}} \{\langle a, B\rangle \in M_i \times P(M_i) : a \in B\}$$

We then add two additional clauses concerning the intended interpretations of '$\prec$' and the plural quantifier:

(T − $\prec$) $u \prec vv$ is $\text{true}_{i,\sigma}$ iff $\langle \sigma(u), \sigma(vv)\rangle \in P_i$.

(T − $\forall vv$) $\forall vv\phi$ is $\text{true}_{i,\sigma}$ iff every $A \subseteq M_i$ is such that ϕ is $\text{true}_{i,\sigma[vv/A]}$.

As in the singular case, $\sigma[vv/A]$ is the assignment which maps vv to the subset A and assigns other variables the same values as σ does.

We define model-theoretic consequence and validity as usual:

- We write $\Gamma \vDash \phi$ when ϕ is $\text{true}_{i,\sigma}$ under every assignment σ over any MT-interpretation i that renders $\text{true}_{i,\sigma}$ each member of Γ.
- A formula ϕ is *valid* when $\emptyset \vDash \phi$.

A.3 Proof theory for PFO

The axioms and rules for PFO are as follows:

TAUT Each truth-functional tautology is an axiom.

MP Whenever ϕ_1 and $\phi_1 \rightarrow \phi_2$ are derivable from Γ, so is ϕ_2

US Whenever u and v are both singular or both plural variables, and u is substitutable for v in ϕ, then $\forall v\phi \rightarrow \phi(u/v)$ is an axiom

UG $\phi_1 \rightarrow \forall v\phi_2$ is derivable from Γ whenever $\phi_1 \rightarrow \phi_2(u/v)$ is derivable from Γ, if i) u and v are both singular or both plural variables, with u substitutable for v in ϕ_2, and ii) no free occurrence of u in $\phi_1 \rightarrow \forall v\phi_2$ or any formula in Γ.

REF Whenever v is a singular variabl, $v = v$ is an axiom.

SUB Whenever u and u' are singular variables that are each substitutable for v in ϕ, then $u = u' \rightarrow (\phi(u/v) \rightarrow \phi(u'/v)$ is an axiom

PC Whenever $\phi(u)$ is a formula that lacks free occurrences of vv, $\exists vv \forall u(u \prec vv \leftrightarrow \phi(u))$ is an axiom.

A formula ϕ is said to be *derivable* from a set of formulas Γ (that is, $\Gamma \vdash \phi$) if ϕ is the last member of a finite sequence, of which each member is i) an axiom, ii) a member of Γ, or iii) the result of applying one of the above rules of inference to earlier members of the sequence.

For PFO, ϕ and each member of Γ is a formula of either $\mathcal{L}_{\mathrm{PS}}$ or $\mathcal{L}_{\mathrm{PSU}}$ (while each axiom and rule is an instance, for the relevant language, of a PFO-axiom- or -rule-schema). When the relevant system may be ambiguous, we will indicate it with a subscript (e.g. $\vdash_{\mathrm{PFO}}$).

PFO is (strongly) sound with respect to its model theory (i.e. $\Gamma \vdash_{\mathrm{PFO}} \phi$ implies $\Gamma \vDash_{\mathrm{PFO}} \phi$); however, Studd (2019) does note that "as usual for plural logic… we cannot expect completeness for the full semantics" (p. 249)

Appendix B: $MPFO$

As with our account of PFO in Appendix A, our account of $MPFO$ below will be a (condensed) reproduction of Studd's (2019) presentation (see pp. 149 – 158, and pp. 250 – 253).

B.1 Language

The modal languages Studd considers in *Everything, More or Less* (2019) – viz., the singular language $\mathcal{L}_{\text{MS}}$; its plural counterpart, $\mathcal{L}_{\text{MPS}}$; and their counterparts with urelements – are obtained from their non-modal counterparts ($\mathcal{L}_{\text{S}}, \mathcal{L}_{\text{PS}}, \mathcal{L}_{\text{SU}}, \mathcal{L}_{\text{PSU}}$) by enriching the latter with forwards- and backwards-looking modal operators, $\Box_{>}$ and $\Box_{<}$. From these, we can define a *weakly-forwards-looking*, *weakly-backwards-looking*, and *absolute* operator, as follows:

$$\Box\psi =_{\text{df}} \Box_{<}\psi \wedge \psi \wedge \Box_{>}\psi$$

$$\Box_{\geq}\psi =_{\text{df}} \Box_{>}\psi \wedge \psi$$

$$\Box_{\leq}\psi =_{\text{df}} \Box_{<}\psi \wedge \psi$$

Modalisation

Given a non-modal formula ϕ of $\mathcal{L}_{\text{S}}$ ($\mathcal{L}_{\text{SU}}$), we obtain its modalisation $\phi^{\Diamond}$ by:

- appending a $\Box$ to each universal quantifier $\forall v$ in ϕ, to obtain $\Box\forall v$
- appending a $\Diamond$ to each existential quantifier $\exists v$ in ϕ, to obtain $\Diamond\exists v$
- appending a $\Diamond$ to each atomic subformula Φ in ϕ, to obtain $\Diamond\Phi$ (for readability, we may write $[\phi]^{\Diamond}$, $[\phi(x)]^{\Diamond}$ $[\beta v]^{\Diamond}$ and $[u \in v]^{\Diamond}$ as $\phi^{\Diamond}$, $\phi^{\Diamond}(x), \beta^{\Diamond}v$ and $u \in^{\Diamond} v$).

Modalisation is extended to plural languages ($\mathcal{L}_{\text{PS}}$, $\mathcal{L}_{\text{PSU}}$) in the same way: in particular, given a plural formula ϕ, we append a $\Box$ to each universal (plural) quantifier, and a $\Diamond$ to each existential (plural) quantifier as well as to each atomic sub-formula in ϕ, to obtain $\phi^{\Diamond}$.

B.2 Model Theory

Given that Studd's modality is interpretational – i.e., it corresponds to *possible admissible interpretations of the language of set theory* – it is useful briefly to return to the model theory for $\mathcal{L}_{\text{SU}}$ we outlined in A.2 (we'll start, as we did there, with the singular case, before extending the relevant considerations to the plural languages).

We saw in A.2 that a MT-interpretation for $\mathcal{L}_{\text{SU}}$ is a triple $\langle M_i, S_i, E_i \rangle$ where M_i is the universe of discourse, and S_i and E_i correspond to the extensions, at M_i, of the two non-logical predicates, β and $\in$, in the language. In the case of interpretational expansionism, the variable i indexes *stages in the interpretation process*, which collectively form an open-ended sequence, or hierarchy:

$$\langle M_0, S_0, E_0\rangle\ \langle M_1, S_1, E_1\rangle, \langle M_2, S_2, E_2\rangle, \ldots$$

The modal language $\mathcal{L}_{\mathrm{MSU}}$ is interpreted by a (set-encoded) sequence of this kind, where each term is either a MT-interpretation or the empty interpretation, $\langle\emptyset, \emptyset, \emptyset\rangle$.

Thus, in the context of the modal languages, we talk not only of MT-interpretations, but also of an overarching MT-*hierarchy,* which is an indexed-set of triples $\{\langle M_i, S_i, E_i\rangle : i \in I\}$ which meets the following constraints:

<u>Serial Well-Order</u>

The set of indices I is a non-empty set equipped with an ordering $<_I$, under which the indices form a serial well-order, and so meets the following conditions:

(a) irreflexivity: there is no $i \in I$ with $i <_I i$.
(b) transitivity: for any $i, j, k \in I$ with $i <_I j$ and $j <_I k$, we have $i <_I k$.
(c) linearity: for any distinct $i, j \in I$, $i <_I j$ or $j <_I i$.
(d) seriality: for any $i \in I$, there is some $j \in I$, with $i <_I j$.
(e) well-foundedness: every non-empty subset A of I has a minimal member (i.e. there is some $i_0 \in A$ with no $j \in A$ such that $j <_I i_0$).

<u>Monotonicity</u>

Whenever i and j are indices in I with $i <_I j$, M_i is a subuniverse of M_j (i.e. $M_i \subseteq M_j$).

<u>Stability</u>

Whenever i and j are indices in I, the extensions S_i and S_j and the extensions E_i and E_j agree on their common domain $M_i \cap M_j$:

(a) $S_j \cap \left(M_i \cap M_j\right) = S_i \cap \left(M_i \cap M_j\right)$.
(b) $E_j \cap \left(M_i \cap M_j\right)^2 = E_i \cap (M_i \cap M_j)^2$.

Interpreting the operators

When it comes to the intended interpretation of Studd's forwards- and backwards-looking operators, we can provide the following, informal English gloss:

$\Box_{>}$: 'however the lexicon is interpreted by succeeding interpretations.'	$\Box_{<}$: 'however the lexicon is interpreted by preceding interpretations.'

In the context of Studd's MT-semantics, we can put this more precisely: given a MT-hierarchy satisfying the Serial Well-order, Monoticity and Stability constraints above, then – given an assignment σ over the hierarchy mapping each variable v to a member of some universe M_i (with $i \in I$) – the conditions for a formula of the form $\Box_{>}\psi$ or $\Box_{<}\psi$ to be $\text{true}_{i,\sigma}$ (i.e., true at index i of the relevant MT-hierarchy, given the assignment σ) are as follows:

$(\mathrm{K}_i - \Box_{>})$ $\Box_{>}\psi$ is $\text{true}_{i,\sigma}$ iff every $j > i$ is such that ψ is $\text{true}_{j,\sigma}$.

$(\mathrm{K}_i - \Box_{<})$ $\Box_{<}\psi$ is $\text{true}_{i,\sigma}$ iff every $j < i$ is such that ψ is $\text{true}_{j,\sigma}$.

Moreover, we obtain the following $\text{truth}_{i,\sigma}$ conditions for the defined modal operators[152]:

$(\mathrm{K}_i - \Box)$ $\Box\psi$ is $\text{true}_{i,\sigma}$ iff every $j > I$ is such that ψ is $\text{true}_{j,\sigma}$.

$(\mathrm{K}_i - \Box_{\geq})$ $\Box_{\geq}\psi$ is $\text{true}_{i,\sigma}$ iff every $j \geq i$ is such that ψ is $\text{true}_{j,\sigma}$.

$(\mathrm{K}_i - \Box_{\leq})$ $\Box_{\leq}\psi$ is $\text{true}_{i,\sigma}$ iff every $j \leq i$ is such that ψ is $\text{true}_{j,\sigma}$.

The corresponding possibility operator is defined, in the standard way, from each necessity operator (e.g. $\Diamond\psi =_{\text{df}} {\sim}\Box{\sim}\psi$).

A modal formula ψ is true_i relative to a MT-hierarchy if it is $\text{true}_{i,\sigma}$ under every assignment over it; finally, a formula ψ i *valid* if it is true at each index under each assignment over *every* MT-hierarchy.

These considerations are extended to the *plural* modal language $\mathcal{L}_{\text{MPSU}}$ in a straightforward way. We saw in A.2 that a MT-interpretation for its non-modal counterpart ($\mathcal{L}_{\text{PSU}}$) extends an interpretation for $\mathcal{L}_{\text{SU}}$ by additionally assigning each plural variable in $\mathcal{L}_{\text{PSU}}$ to a subset of M_i (with $i \in I$). Given such an assignment σ, we can add the following two clauses to those concerning the intended interpretation of the modal formulae of $\mathcal{L}_{\text{SU}}$:

$(\mathrm{K}_i{-}{\prec})$ $u \prec$ is $\text{true}_{i,\sigma}$ iff $\langle\sigma(u), \sigma(vv)\rangle \in P_i$.

$(\mathrm{K}_i - \forall vv)$ $\forall vv\psi$ is $\text{true}_{i,\sigma}$ iff every set A with $A \subseteq M_i$ is such that ψ is $\text{true}_{i,\sigma[vv/A]}$.

where P_i, as in the non-modal case, the intended extension for the member-plurality predicate based on M_i:

$$P_i =_{\text{df}} \{\langle a, B\rangle \in M_i \times P(M_i) : a \in B\}$$

[152] Notation: $j \leq i$ abbreviates $j < i$ or $j = i$; $j \geq i =_{\text{df}} i \leq j$.

B.3 Proof Theory

The plural modal logic – $MPFO$ – validated by the model theory outlined above is free and tense-like.

In addition to the standard plural analogues of the rules TAUT, MP, SUB, and PC (see A.3), the logic contains the following amended versions of FUS, FUG, and FREF (which especially govern the quantifiers, and their interactions with the existence predicate, $E!$):

FUS $\quad \forall \boldsymbol{v}\psi \rightarrow (E!\,\boldsymbol{u}$[153] $\rightarrow \psi(\boldsymbol{u}/\boldsymbol{v}))$ is an axiom whenever u and v are both singular or both plural variables, and u is substitutable for v in ψ.

FUG $\quad \psi_1 \rightarrow \psi_2$ is derivable from Δ whenever $\psi_1 \rightarrow \psi_2(\boldsymbol{u}/\boldsymbol{v})$ is derivable from $\Delta \cup \{E!\,\boldsymbol{u}\}$, provided that u and v are both singular or both plural variables, with u substitutable for v in ψ_2, and no free occurrences of u in $\psi_1 \rightarrow \forall v\psi_2$ or any formula in Δ.

FREF $\quad \Phi(\boldsymbol{v}) \rightarrow E!\,\boldsymbol{v}$ is an axiom whenever $\boldsymbol{v}$ is a singular or plural variable and $\Phi(\boldsymbol{v})$ is an atomic formula that contains $\boldsymbol{v}$**.**

The existence predicate, $E!$ featured in these amended rules, is defined as follows in the singular and plural case:

$$E!\,u =_{\text{df}}\ u = u \qquad\qquad E!\,uu =_{\text{df}}\ {\sim}\text{EXT}_x[\Diamond(x \prec uu)]$$

For the plural existence predicate, $\text{EXT}_x[\psi(u)]$ abbreviates the following formula:

$$\text{EXT}_u[\psi(u)] =_{\text{df}}\ \Diamond_{>}\exists u(\psi(u) \land \Box_{<}{\sim}E!\,u)$$

$MPFO$ also includes all instances of the non-modal Plural Comprehension Scheme:

$$\exists xx\forall y(y \prec xx \leftrightarrow \psi(y))$$

(whereby, given a condition $\psi(y)$, there are zero or more items that comprise every item that satisfies $\psi(y)$).

The structure of the potential MT-hierarchy is then captured by the following rule- and axiom-schemes:

K $\quad L(\psi_1 \rightarrow \psi_2) \rightarrow (L\psi_1 \rightarrow L\psi_2)$ is an axiom whenever L is one of the modal operators $\Box_{>}$ or $\Box_{<}$.

NEC $\quad L\psi$ is derivable from Δ whenever ψ is a formula that is derivable from $\emptyset$) and L is one of $\Box_{>}$ or $\Box_{<}$.

[153] The (bold) variables '$\boldsymbol{x}$' and '$\boldsymbol{u}$' are either singular or plural.

CV $\psi \rightarrow LM\psi$ is an axiom whenever L and M are either respectively $\Box_<$ and $\Diamond_>$ or respectively $\Box_>$ and $\Diamond_<$.

LÖB $\Diamond_<\psi \rightarrow \Diamond_<(\psi \wedge \Box_< {\sim}\psi)$ is an axiom whenever ψ is a formula.

H $M\psi_1 \wedge M\psi_2 \rightarrow (M(\psi_1 \wedge \psi_2) \vee M(\psi_1 \wedge M\psi_2) \vee M(\psi_2 \wedge M\psi_1)$ is an axiom whenever M is one of $\Diamond_>$ or $\Diamond_<$.

D $\Box_>\psi \rightarrow \Diamond_>\psi$ is an axiom whenever ψ is a formula.

Finally, $MPFO$ contains the following axioms, which concern how the modal operators interact with the predicates and quantifiers:

CBF $\Box_>\forall \boldsymbol{v}\psi \rightarrow \forall \boldsymbol{v}\Box_>\psi$ is an axiom whenever v is a singular or plural variable.

STA-Φ $\Box\forall\boldsymbol{v}(\Diamond\Phi(\boldsymbol{v}) \rightarrow \Phi(\boldsymbol{v})$ is an axiom whenever $\Phi(\boldsymbol{v})$ is an atomic formula, v is the string of variables it contains free and $\forall\boldsymbol{v}$ is the corresponding string of quantifiers.

E_1 $\Diamond E!\,\boldsymbol{v}$ is an axiom whenever v is a singular or plural variable.

The above proof theory sustains the deduction theorem (i.e. $\Delta \cup \{\psi_1\} \vdash \psi_2$ only if $\Delta \vdash \psi_1 \rightarrow \psi_2$), and is sound with respect to the model theory presented in B.2. That is: if we define $\Delta \vdash_{\mathrm{MPFO}} \psi$ as usual – and where $\Delta \vDash_{\mathrm{MPFO}} \psi$ means ψ is $\mathrm{true}_{i,\sigma}$ under any index i of, and any assignment σ over, any MT-hierarchy which renders $\mathrm{true}_{i,\sigma}$ each member of Δ – then we have it that $\vdash_{\mathrm{MPFO}} \psi$ only if $\Delta \vDash_{\mathrm{MPFO}} \psi$.

Finally, we briefly compare Studd's bi-modal system to its uni-modal cousins:

- The absolute operator, $\Box$ (where $\Box\psi =_{\mathrm{df}} \Box_<\psi \wedge \psi \wedge \Box_>\psi$) conforms to the rules and axioms governing S5.
- The weakly-forwards-looking operator conforms to the rules and axioms of S4.3.

Appendix C: Q-ish and E-ish

Our presentation here reproduces Studd's (2019) account of E-ish and Q-ish (see pp. 218 – 237), as well as his presentation of the language of generalised quantifiers (see pp. 64 – 69), with some minor adaptions.

C.1 Q-ish

Syntax

The Q'ers (initially) speak a fragment of the language of generalised quantifiers, $\mathcal{L}_{\text{GQ}}$, that does not contain quantifiers. Their lexicon includes the following (syntactic categories of) expressions:

- Unary (e.g., $donkey$, $thing$, $pebble$, set,…) and binary predicates[154] (e.g., $element$, $loves$, …)
- Variables: *x, y, z, …*
- Predicate-abstraction operators: $\hat{x}, \hat{y}, \hat{z}$ …[155]

Upon learning to quantify, the Q'ers enrich their lexicon with connectives ('~,' and '→') and determiners ($every$) . Quantifiers are then formed from determiners and unary predicates (e.g., $everything$, $nodonkey$); combining a quantifier with another unary predicate then results in a formula (e.g., $nodonkeytalks$).

In this initial, intra-language context (and following Studd), we'll let $LEXICON_{Q0}$ denote the set of expressions of Q-ish (including the determiner '$every$' and the connectives '~,' and '→').

Semantics

Having outlined the Q'ers' lexicon in line with the above, Studd (2019) presents two key semantic assumptions that govern how these expressions are to be interpreted by a function $[\![\cdot]\!]$. The first (which he calls 'Q-Extensionality') stipulates that these expressions are accorded *extensional* semantic values (based on an universe of discourse, M) that are *appropriate to their semantic type*; the second (which he calls 'Q-Compositionality')

[154] In $\mathcal{L}_{\text{GQ}}$, formulae can be formed by combining *n*-ary predicates with *n* occurrences of singular terms or variables; by means of an *n*-ary connective (which is then combined with *n* occurrences of formulae); or by means of predicate abstraction operators (see below).

[155] In $\mathcal{L}_{\text{GQ}}$, variable binding takes place by means of predicate-abstraction operators rather than quantifiers (e.g., the operator $\hat{x}$ combines with a formula φ to form the complex unary predicate $\hat{x}.\varphi$, which binds all the free occurrences of x in φ – $\hat{v}.\varphi$ thus reads read 'is a x such that φ').

stipulates that the semantic values of complex expressions are determined *compositionally,* from the values of their sub-expressions.

We'll deal with each of these assumptions in turn.

Q-Extensionality

Let $[\![\cdot]\!]$ be a function, the domain of which is a set of expressions of $\mathcal{L}_{\text{GQ}}$ (call this $LEXICON$; then $[\![\cdot]\!]$ is what Studd calls a *Q-Extensional interpretation* if:

(a) The elements of $LEXICON$ include the unary predicate $thing$, the variable expressions of $LEXICON_{Q0}$, and zero or more additional expressions listed in clause (c) below.
(b) The universe of discourse, M, corresponds to the non-empty set $[\![thing]\!]$.
(c) For each (non-variable) expression e in $LEXICON$, $[\![e]\!]$ is an extension based on M, which is conferred on e in accordance with its syntactic category (as listed below):

(E-τ) a singular-term-extension $[\![\boldsymbol{\tau}]\!]$ is a member of M.

(E-θ) an *n*-ary-predicate-extension $[\![\theta]\!]$ is a set of *n*-tuples of singular-term-extensions based on M.

(E-ϕ) a formula-extension $[\![\varphi]\!]$ is a truth-value – T (true) or F (false).

(E-o) an *n*-ary-connective-extension $[\![\bullet]\!]$ is a set of *n*-tuples of formula-extensions.

(E-q) a quantifier-extension $[\![\boldsymbol{q}]\!]$ is a set of unary-predicate-extensions based on M.

(E-d) a determiner-extension $[\![\boldsymbol{d}]\!]$ is a function, which maps unary-predicate-extensions based on M to quantifier-extensions based on M

Q-Compositionality

Let $[\![\cdot]\!]$ be a *Q-Extensional interpretation* of $LEXICON$. Then, given an assignment σ over $[\![\cdot]\!]$ – and where $LEXICON^*$ comprises complex expressions formed from the expressions in $LEXICON$ (together with the latter expressions themselves) – then the *Q-Compositional extension* of $[\![\cdot]\!]$ relative to σ is a *Q-Extensional interpretation* $[\![\cdot]\!]^\sigma$ of $LEXICON^*$, that meets the following conditions:

(a) $[\![x]\!]^\sigma = \sigma(x)$ for each variable x;
(b) $[\![e]\!]^\sigma = [\![e]\!]$ for each non-variable expression e in $LEXICON$;
(c) otherwise, $[\![e]\!]^\sigma$ is determined accordance with the following clauses:

(c-θ) for each n-ary predicate θ, and terms $\tau_1, \ldots, \tau_n$:
$[\![\theta(\tau_1, \ldots, \tau_n)]\!]^\sigma = \text{T iff } \langle [\![\tau_1]\!]^\sigma, \ldots, [\![\tau_n]\!]^\sigma \rangle \in [\![\theta]\!]^\sigma$

(c-q) for each quantifier q and each unary predicate θ: $[\![q(\theta)]\!]^\sigma = \text{T iff } [\![\theta]\!]^\sigma \in [\![q]\!]^\sigma$

(c-o) for each n-ary connective o, and formulas $\phi_1, \ldots, \phi_n$:
$[\![\text{o}(\phi_1, \ldots, \phi_n)]\!]^\sigma = \text{T iff } \langle [\![\phi_1]\!]^\sigma, \ldots, [\![\phi_n]\!]^\sigma \rangle \in [\![\text{o}]\!]^\sigma$.

(c-d) for each determiner d and each unary predicate η: $[\![d(\eta)]\!]^\sigma = [\![d]\!]^\sigma([\![\eta]\!]^\sigma)$.

(c-$\hat{v}$) for each abstraction operator $\hat{v}$ and each formula ϕ:
$[\![\hat{v}.\phi]\!]^\sigma = \{a \in M : [\![\phi]\!]^{\sigma[v/a]}$.

Clause (c-d) ensures that the Q'ers' determiner '$every$' has the following intended interpretation:

(s-$every$) $[\![every]\!](\eta) = \{\theta \subseteq M : \eta \subseteq \theta\}$

Moreover, the function for the connectives is defined as follows:

(s-~) $[\![\sim]\!] = \{\phi : \phi = \text{F}\}$
(s-→) $[\![\rightarrow]\!] = \{\langle \phi_1, \phi_2 \rangle : \phi_1 = \text{F or } \phi_2 = \text{T}\}$

This gives us the following conditions for truth$_{i,\sigma}$ and satisfaction$_{i,\sigma}$ (where $i = [\![\cdot]\!]$, and σ is an assignment over $[\![\cdot]\!]$):

(T$_i$-~) $\sim\phi$ is true$_{i,\sigma}$ iff ϕ is not true$_{i,\sigma}$.

(T$_i$-→) $\phi_1 \rightarrow \phi_2$ is true$_{i,\sigma}$ iff ϕ_1 is not true$_{i,\sigma}$ or ϕ_2 is true$_{i,\sigma}$.

(T$_i$-every) every$\eta\theta$ is true$_{i,\sigma}$ iff every item that satisfies$_{i,\sigma}$ η satisfies$_{i,\sigma}$ θ.

Use

As we saw in §3.1, Studd (2019) encodes the Q'ers' use-patterns in terms of *sequents*, like the one below:

$$(U): \varphi_1, \ldots, \varphi_m \Longrightarrow \psi_1, \ldots, \psi_n$$

He glosses the interpretation of this as follows: "members of the community find it incoherent both to accept each formula φ_p (with $p = 1, \ldots, m$) and to reject each formula ψ_q (with $q = 1, \ldots, n$)" (p.222).

The Q'ers adopt the following, general sequent rules:

(REP) $\phi \Rightarrow \phi$

(DIL) $\dfrac{\phi_1,\ldots,\phi_m \Rightarrow \psi_1,\ldots,\psi_n}{\phi,\phi_1,\ldots,\phi_m \Rightarrow \psi_1,\ldots,\psi_n}$ $\dfrac{\phi_1,\ldots,\phi_m \Rightarrow \psi_1,\ldots,\psi_n}{\phi_1,\ldots,\phi_m \Rightarrow \psi_1,\ldots,\psi_n,\psi}$

(CUT) $\dfrac{\chi,\phi_1,\ldots,\phi_m \Rightarrow \psi_1,\ldots,\psi_n \qquad \phi_1,\ldots,\phi_m \Rightarrow \psi_1,\ldots,\psi_n,\chi}{\phi_1,\ldots,\phi_m \Rightarrow \psi_1,\ldots,\psi_n}$

Moreover, they adopt the following patterns for '~,' '→,' and '$every$':

(U- ~) $\phi, \sim\phi \Rightarrow$ $\Rightarrow \phi, \sim\phi$

(U-→) $\phi, \phi \rightarrow \psi \Rightarrow \psi$ $\dfrac{\phi, \phi_1, \ldots, \phi_m \Rightarrow \psi_1, \ldots, \psi_n, \psi}{\phi_1, \ldots, \phi_m \Rightarrow \psi_1, \ldots, \psi_n, \phi \rightarrow \psi}$

(U-$every$)[156] $every\eta\theta, \eta(\tau) \Rightarrow \theta(\tau)$ $\dfrac{\eta(v), \phi_1, \ldots, \phi_m \Rightarrow \psi_1, \ldots, \psi_n, \theta(v)}{\phi_1, \ldots, \phi_m \Rightarrow \psi_1, \ldots, \psi_n, every\eta\theta}$

Open-ended material soundness

Studd's key assumption is that the Q'ers' use-patterns – as encoded by the sequents above – are *open-endedly materially sound.*

To (briefly) recap material soundness: that an individual sequent-rule like (U) is *materially sound* means that there is no interpretation, ⟦·⟧ and no assignment of variables σ over ⟦·⟧, that makes the formulae to the left of the sequent-arrow (in (U)'s case, $\varphi_1, \ldots, \varphi_m$) true, and the formulae on the right of the arrow (in that case, $\psi_1, \ldots, \psi_n$) false. For sequent *rules*, material soundness just entails that the soundness of the initial sequents under ⟦·⟧ implies the soundness of the *end* sequents under ⟦·⟧.

Given this, Studd (2019, p. 226) then defines *open-ended* soundness as follows:

First, he stipulates that a function $⟦\cdot⟧^+$ with domain $LEXICON^+$ *extends* another, ⟦·⟧, with domain $LEXICON$ if

i) $LEXICON \subseteq LEXICON^+$, and
ii) $⟦e⟧^+ = ⟦e⟧$, for all expressions e in $LEXICON$.

Moreover, if $⟦\cdot⟧^+$ and ⟦·⟧ are both interpretations of Q-ish, then $⟦\cdot⟧^+$ is to be called a *Q-extension* of ⟦·⟧.

[156] This sequent is subject to the side-condition that v does not occur free in the end-sequent.

He then stipulates – relative to a class $\boldsymbol{E}$ of Q-extensions of $[\![\cdot]\!]$ – that a use-pattern (either a sequent-rule, or a sequent scheme) is *sound over* $\boldsymbol{E}$ if every Q-interpretation $[\![\cdot]\!]^+$ in $\boldsymbol{E}$ is such that every $[\![\cdot]\!]^+$-instance of the scheme is materially sound under $[\![\cdot]\!]^+$; moreover, $\boldsymbol{E}$ is a full class of extensions of $[\![\cdot]\!]$ if it comprises $[\![\cdot]\!]$, and every Q-extension of $[\![\cdot]\!]$.

Given this, we finally have it that a use-pattern is *open-endedly Q-sound* under $[\![\cdot]\!]$ if it is sound over the full class $\boldsymbol{E}$ of Q-extensions of $[\![\cdot]\!]$.

With this assumption in place – together with the assumption that the Q'ers' expressions (besides the determiner '$every$,' and the connectives '~' and '→') receive their intended interpretations relative to M – Studd (2019) is able to prove his *Q-Quantification Theorem*[157], below:

> Suppose that $[\![\cdot]\!]$ is a Q-interpretation[158] of the Q-ers' lexicon $\mathrm{LEXICON}_{Q0}$ with universe M. Then, under $[\![\cdot]\!]$, the Q-ers' patterns of use for the determiner (describer in U-every) are open-endedly Q-sound if and only if $[\![\cdot]\!]$ accords every its intended semantic value based on M (as per s-every). The same result holds for the connectives, $\sim$ and $\rightarrow$. (p. 226)

C.2 E-ish

Syntax

The reader will recall from Studd's idealised scenario that, prior to the split with the E'ers, the Q'ers enrich their lexicon with the following plural expressions.

- unary plural predicate ($things$) and binary singular-plural predicate ($\prec$)
- plural variables: $xx, yy, zz, \ldots$

In terms of syntax: the unary predicate $things$ combines with plural variables (e.g., xx) to form a formula (e.g., $things(xx)$), while the binary predicate combines with singular and plural variables in the usual way.

Studd calls the resulting lexicon – viz., that obtained from the addition of plural resources to $LEXICON_{Q0}$ – $LEXICON_Q$.

Then, to obtain the lexicon of E-ish ($LEXICON_E$) we add the (non-variable) set-term operator $\{\cdot\}$, the syntax governing which is as in §3.1.

Semantics

[157] See p. 226 in Studd for a proof sketch.
[158] A Q-interpretation just is a Q-Extensional interpretation, as per the Q-Extensionality clause above.

As with Q-ish, the interpretation of E-ish is governed by the key semantic assumptions of Extensionality and Compositionality. We deal with each of these in turn.

E-Extensionality

The key change to the Extensionality assumption in the case of E-ish, is that it is permissible for some singular terms – specifically, those formed by the set-term operator, $\{\cdot\}$ – to fail to denote relate to the (initial) universe of discourse, M.

To represent this in his MT-semantics, Studd deploys a dummy singular-term-extension $*$, which lies outside the universe M – in turn, when a singular term fails to denote, we can represent this in the semantics by assigning '$*$' as the extension of that term (see, specifically, (E$'$-τ) and (E$'$-f) below).

That is: let $[\![\cdot]\!]$ be a function, the domain of which is a set of expressions of $\mathcal{L}_{\mathrm{GQ}}$, $LEXICON$; then $[\![\cdot]\!]$ is what Studd calls a *E-Extensional interpretation* if:

a) $LEXICON$ includes the unary predicates $thing$ and $things$ the binary predicate $\prec$; the variable expression in $LEXICON_Q$; and zero or more additional expressions listed in clause (e) below.
b) The universe of discourse, M, corresponds to the non-empty set $[\![thing]\!]$.
c) The extension of $things$ is the set of all subsets of M (i.e., $[\![things]\!] = \{\tau\tau : \tau\tau \subseteq M\}$).
d) The extension of $\prec$ is the set of ordered pairs based on M, of which – for each pair – the second element is a subset of M, and the first is an element of that subset (i.e., $[\![\prec]\!] = \{\langle\tau, \tau\tau\rangle : \tau \in \tau\tau \text{ and } \tau\tau \subseteq M\}$).
e) For each (non-variable) expression e in $LEXICON$, $[\![e]\!]$ is an extension based on M, which is conferred on e in accordance with either one of the clauses listed above governing Q-extensional interpretations (viz., E-θ, E-ϕ, E-o, E-q and E-d), or one of the additional clauses below:

(E$'$-τ) a singular-term-extension $[\![\boldsymbol{\tau}]\!]$ based on M is either a member of M or the dummy singular-term-extension $*$

(E$'$-f) a plural-singular-term-function-symbol-extension based on M is a function that maps each plural-term-extension based on M to either a singular-term-extension based on M or $*$.

(E-$\tau\tau$) a plural-term-extension $[\![\boldsymbol{\tau\tau}]\!]$ based on M is a subset of M.

(E-Θ) a unary-plural-predicate-extension $[\![\boldsymbol{\Theta}]\!]$ based on M is a set of plural-term-extensions based on M.

(E-Θ^2) a binary-singular-plural-predicate-extension $[\![\boldsymbol{\Theta}^2]\!]$ based on M is a set of pairs whose first co-ordinate is a singular-term-extension based on M and whose second co-ordinate is a plural-term-extension based on M.

Compositionality

The key change from the Q-ish case, is that the compositionality assumption for E-ish is *liberalised* to allow for assignments over either an interpretation, $[\![\cdot]\!]_Q$, of $LEXICON_Q$, *or* an interpretation, $[\![\cdot]\!]_E$, of $LEXICON_E$ (where both $[\![\cdot]\!]_Q$ and $[\![\cdot]\!]_E$ are *E-Extensional* interpretations).

Specifically: let $[\![\cdot]\!]$ be a *E-Extensional interpretation* of $LEXICON$. Then, given an assignment σ over *either* $[\![\cdot]\!]_Q$ or $[\![\cdot]\!]_E$ – and where $LEXICON^*$ comprises complex expressions formed from the expressions in $LEXICON$ (together with the latter expressions themselves) – then the *E-Compositional extension* of $[\![\cdot]\!]$ relative to σ is a *E-Extensional interpretation* $[\![\cdot]\!]^\sigma$ of $LEXICON^*$, if it meets the following conditions:

(a) for each singular or plural variable v: $[\![v]\!]^\sigma = \sigma(v)$;
(b) for each non-variable expression e in $LEXICON$: $[\![e]\!]^\sigma = [\![e]\!]$;
(c) otherwise, $[\![e]\!]^\sigma$ is determined – in addition to C-θ, C-q, C-o, C-d, and C-$\hat{v}$ – in accordance with the following clauses,:

(C-f) for each plural-to-singular-term-function-symbol f and plural term $\tau\tau$ with $[\![\tau\tau]\!]^\sigma \subseteq M$: $[\![f(\tau\tau)]\!]^\sigma = [\![f]\!]^\sigma([\![\tau\tau]\!]^\sigma)$ and otherwise $[\![f(\tau\tau)]\!]^\sigma = *$.

(C-Θ) for each plural term $\tau\tau$ and unary plural predicate Θ: $[\![\Theta(\tau\tau)]\!]^\sigma = \mathrm{T}$ iff $[\![\tau\tau]\!]^\sigma \in [\![\Theta]\!]^\sigma$.

(C-Θ^2) for each singular term τ, plural term $\tau\tau$, and binary singular-plural predicate Θ^2: $[\![\Theta^2(\tau,\tau\tau)]\!]^\sigma = \mathrm{T}$ iff $\langle[\![\tau]\!]^\sigma, [\![\tau\tau]\!]^\sigma\rangle \in [\![\Theta^2]\!]^\sigma$.

Thus, given his proposed revisions to the Compositionality and Extensionality clauses, respectively, Studd's (2019) semantics implements "the truth-conditions distinctive of a negative free semantics for singular or plural terms that fail to denote a member or members of the universe." (p. 234).

<u>Use</u>

As we saw in §3.1, in addition to the Q'ers' patterns of use, the E'ers adopt what Studd (2019) calls 'trans-language' use-patterns like the one below:

(TU) $$v_1 : \phi_1, \ldots, v_m : \phi_m \;\Rightarrow\; \varsigma_1 : \psi_1, \ldots, \varsigma_n : \psi_n$$

where the variables v and ζ range over languages – specifically, those in which the formulae appearing in the sequent (in this case, ϕ_i and ψ_i, where $i = 1, 2, 3, \ldots$) are uttered.

As in the Q-ish case, the E'ers' accepting (TU) is to be understood in terms of their finding it incoherent to accept each of the formulae $\varphi_1, \dots, \varphi_m$, as uttered in language v, and to reject each of the formulae $\psi_1, \dots, \psi_n$, as uttered in language ζ.

The specific trans-language use-patterns adopted by the E'ers include those governing the set-term operator:

(U_E-{·}) $$\text{Q}: things(vv) \Rightarrow \text{E}: thing(\{vv\})$$

$$\text{Q}: things(vv), \text{Q}: v \prec vv \Rightarrow \text{E}: v \in \{vv\}$$

$$\text{Q}: things(vv), \text{E}: v \prec \{vv\} \Rightarrow \text{Q}: v \prec vv$$

And the stability axioms for $\prec$ [159]:

(U_E-$\prec$) $$\text{Q}: u \prec vv \Rightarrow \text{E}: u \prec vv$$

$$\text{Q}: thing(u), \text{Q}: things(vv), \text{E}: u \prec vv \Rightarrow \text{Q}: u \prec vv$$

Open-ended material soundness

Materially soundness in the case of the E'ers' use-patterns is identical to the Q-ish case, except for the fact that, rather than these patterns' being interpreted by a *single* interpretation $[\![\cdot]\!]$, they are instead interpreted by a *pair* of interpretations, $\langle [\![\cdot]\!]_{\text{Q}}, [\![\cdot]\!]_{\text{E}} \rangle$.

Correspondingly, Studd (2019, p. 236) adapts his definition of open-ended soundness to the E-ish case, as follows:

First, he stipulates that a pair of E-interpretations[160] $\langle [\![\cdot]\!]^+_{\text{Q}}, [\![\cdot]\!]^+_{\text{E}} \rangle$ – which he writes as $[\![\cdot]\!]^+$ – is an *E-extension* of the pair $\langle [\![\cdot]\!]_{\text{Q}}, [\![\cdot]\!]_{\text{E}} \rangle$ – which he writes as $[\![\cdot]\!]$ – if $[\![\cdot]\!]^+_{\text{Q}}$ extends $[\![\cdot]\!]_{\text{Q}}$ *and* $[\![\cdot]\!]^+_{\text{E}}$ extends $[\![\cdot]\!]_{\text{E}}$.

Then – given a class A of such pairs – a trans-language use-pattern is *sound over* A if each pair of interpretations $[\![\cdot]\!]^+$ in A is such that each $[\![\cdot]\!]^+$-instance of the schema is sound under $[\![\cdot]\!]^+$. Moreover, Studd stipulates that A is a *full class of E-extensions of* $[\![\cdot]\!]$ if it contains $[\![\cdot]\!]$, as well as every E-interpretation that extends $[\![\cdot]\!]_{\text{Q}}$ (as the left co-ordinate of one of its pairs),

[159] This is a specific instance of the more general stability axiom governing a given binary predicate θ:

(U_E-θ) $$\text{Q}: \theta(u, v) \Rightarrow \text{E}: \theta(u, v)$$
$$\text{Q}: thing(u), \text{Q}: things(v), \text{E}: \theta(u, v) \Rightarrow \text{Q}: \theta(u, v)$$

[160] An E-interpretation just is an E-Extensional interpretation (with a domain of either $\text{LEXICON}_{\text{Q}}$ or $\text{LEXICON}_{\text{E}}$), as per the E-Extensionality clause outlined above.

in addition to every E-interpretation that extends $[\![\cdot]\!]_{\mathrm{E}}$ (as the right co-ordinate of one of its pairs).

Finally, we have it that a use-pattern is *open-endedly E-sound* under a pair of interpretations $[\![\cdot]\!]$ if it is sound over the full class A of E-extensions of $[\![\cdot]\!]$.

With this assumption in place, Studd (2019) is able to prove what he calls his *E-Expansion theorem*[161]:

E-Expansion Theorem. Suppose that the lexicon of Q-ish and E-ish are respectively interpretated by a pair of E-interpretations $\langle [\![\cdot]\!]_{\mathrm{Q}}, [\![\cdot]\!]_{\mathrm{E}} \rangle$ under which the E-ers' patterns are open-endedly E-sound. Then $[\![\cdot]\!]_{\mathrm{E}}$ is a proper stable expansion[162] of $[\![\cdot]\!]_{\mathrm{Q}}$ and each of the E-interpretations accord every, $\sim$, and $\rightarrow$ their intended extensions based on its universe. (p. 237

[161] See p. 237 for his proof sketch.
[162] The reader will recall, from §3.1, that an E-interpretation $[\![\cdot]\!]_{\mathrm{E}}$ *stably expands* another, $[\![\cdot]\!]_{\mathrm{Q}}$, if

i) the universe M_{Q} is a sub-universe of M_{E} (i.e. $M_{\mathrm{Q}} \subseteq M_{\mathrm{E}}$), and
ii) the two interpretations agree on the extensions of their common predicates within M_{Q} (this latter fact is especially ensured by the stability sequents).

Moreover, a stable expansion $[\![\cdot]\!]_{\mathrm{E}}$ of $[\![\cdot]\!]_{\mathrm{Q}}$ is a *proper stable expansion* if M_{Q} is a proper subuniverse of M_{E} (i.e. $M_{\mathrm{Q}} \subset M_{\mathrm{E}}$).

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