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Implementing a Filtered Term Structure Model in the South African Bond Market

Angela Ririe

Supervisor: Dr Brett Dugmore

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Abstract

A key feature of the local bond market is that trade is concentrated in a few liquid government bonds. We review and implement the filtered term structure model proposed by Gombani, Jaschke and Runggaldier that defines an arbitrage free pricing system that is consistent with liquid bond prices.

The model is derived in two stages called the underlying and perturbed models. The underlying model defines the theoretical arbitrage free term structure. It is assumed to be a multi-factor, affine HJM type model where the stochastic factors satisfy a linear diffusion equation.

Gombani et al. argue that the differences between the theoretical and market prices should be interpreted as unobserved errors. The perturbed model models the prices of the observed bonds as their theoretical values distorted by noise.

Assuming that the information available at any point in time is the market prices of a finite number of liquidly traded bonds, the perturbed model is used to derive a continually updated pricing system that is arbitrage free with respect to the observed prices. The method is based on the Kalman filter.

We implement a particular three-factor version of the model and calibrate it to the South African market. We discuss the relevant data and numerical and statistical techniques including principal component analysis and yield curve construction.

We apply the formulas for pricing European options on zero-coupon and coupon bearing bonds for Gaussian HJM models to the perturbed model and present two examples to demonstrate the application of the model to bond and option pricing.

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Chapter 1

Introduction

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In this report we review and implement a filtered term structure model in the South African market. Below we provide a brief overview of some yield curve models in order to place the model that we implement in context.

Term structure models provide a description of how interest rates change through time. The classical approach models the evolution of the yield curve in terms of the behaviour of the instantaneous risk-free rate (called the short-rate). The basic idea is that in a risk-neutral world the process followed by the short-rate determines the fair value for any future cash flow which in turn allows the entire term structure of interest rates to be determined, see for example Hull (2003). The value of the short-rate at time t is denoted $r(t)$ and the general form of the process specifying the dynamics of the short-rate is assumed to be

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t), \tag{1.1}$$

where $W(t)$ is a Wiener process.

The simplest of the short-rate models, called one-factor models, assume that the process for the short-rate is driven by a single source of uncertainty in which case $W(t)$ is one-dimensional. Two of the most well known of the early one-factor short-rate models are the Vasiček (1977) model, where the short-rate dynamics are assumed to follow a mean reverting Gaussian process and the Cox, Ingersoll and Ross (1985) model (CIR hereafter) where short-rate dynamics are assumed to follow a mean reverting square-root process.[1] The major difference between these

[1] For the Vasiček model, the dynamics of the short-rate under the martingale measure are given by

$$dr(t) = (b - ar(t))dt + \sigma dW(t), \ (a > 0),$$

where a and b are constants. For the CIR model they are given by

$$dr(t) = a(b - r(t))dt + \sigma\sqrt{r}dW(t).$$

two models is that under the Vasiček model (and in fact Gaussian models in general) there is a non-zero probability of negative interest rates, while under the CIR model interest rates are nonnegative.[2]

One of the main reasons for the popularity of the Vasiček and CIR models is that under both models bonds and bond options can be priced analytically. Jamshidian (1989) derives a formula for valuing European options on zero-coupon and coupon bonds under the Vasiček model that is similar to the Black-Scholes formula for pricing equity options. For the CIR model, a formula for the valuation of options on zero-coupon bonds is provided by Cox et al. (1985); a formula for the valuation of European options on coupon bonds is derived by Longstaff (1993).[3]

There are two basic types of term structure models, equilibrium and no-arbitrage. Equilibrium models start with assumptions about economic variables. The dynamics of the term structure are then derived from these assumptions. The Vasiček and CIR models are examples of equilibrium term structure models. The main disadvantage of equilibrium models is that they do not necessarily fit the observed initial term structure. By a process called 'inverting the yield curve', the model parameters can be chosen to make the fit as close as possible but the possible shapes of the yield curve are limited. For example, the Vasiček model cannot produce an inverted curve. This can be problematic, for instance when pricing derivatives it seems unreasonable to use a model that does not price the underlying bonds correctly.

A no-arbitrage model is designed to be exactly consistent with the observed initial term structure. For short-rate models this is generally achieved by making one or more of the model parameters time dependent. The first of the no-arbitrage models was proposed by Ho and Lee (1986). Ho and Lee's model was originally formulated in terms of a binomial tree but a continuous time limit of the model that fits into the short-rate framework can be derived, see for example Jamshidian (1988). The Ho-Lee model fits the initial term structure exactly and is analytically tractable: both bonds and European bond options can be valued analytically. The disadvantage of the model is that the volatility structure is inflexible and all rates are equally variable. The model is also not mean reverting.

Some equilibrium models can be converted to no-arbitrage models by making the drift of the short-rate process a function of time. In particular Hull and White (1990a, 1990b) derive extensions of the Vasiček and CIR models that fit the observed initial term structure. The extended version of the Vasiček model is analytically tractable while in general the extended version of

[2]Under the Vasiček model the short-rate is characterised by a Gaussian distribution while under the CIR model short-rate is characterised by a non-central chi-squared distribution.

[3]Longstaff's approach is consistent with Jamshidian's and it is in fact possible to generalise Jamshidian's approach for other analytically tractable one-factor models in a natural way. See Brigo and Mercurio (2001).

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the CIR model is not.[4]

Many other short-rate models have been suggested. A notable class of model that we have not mentioned above are the lognormal models such as Black, Derman and Toy (1990) and Black, and Karasinski (1991). These models guarantee nonnegative interest rates but are not analytically tractable making calibration to the market more difficult.

The major problem with all one-factor models is that since they assume that all yield curve movement is driven by a single source of uncertainty, rates of all maturities are perfectly correlated which implies that all rates move in the same direction. For this reason one-factor models such as those described above are generally acceptable when the instrument being priced depends only on a single point on the yield curve. If however the correlation of interest rates is relevant to pricing it is necessary to use a model that allows for more realistic correlation patterns. Models that allow a wider variety of term structure movements and volatility shapes can be achieved by introducing additional factors.

Two or three factors are usually required to give a realistic description yield curve dynamics over time. This observation goes back to Litterman and Scheinkman (1991) who show that in general almost all of the variation in the returns on any fixed income security can be explained by three attributes of the yield curve which they call level, slope and curvature. These factors as well as the percentage of historical yield curve variation explained by them can be estimated by a principal component analysis of historical yield curve volatility. Principal component analysis is discussed further in Section 3.5.

Numerous two-factor short-rate models have been suggested. A particular example is the Brennan and Schwartz (1979, 1982) model where the second factor is chosen to be the yield on a perpetual bond (called a consol). The Brennan and Schwartz model therefore incorporates both the level and steepness of the yield curve. This can be seen by noting that if the short-rate is interpreted as the level of the curve then the steepness of the curve is implied by the consol yield. Other well-known examples of two-factor models are Hull and White (1994) and Longstaff and Schwartz (1992). Balduzzi, Das, Foresi and Sundaram (1996) derive a framework for three-factor affine short-rate models. In particular they consider an example where the three factors are the short-rate, the mean-reversion level and the volatility of the short-rate and show that the model describes the level, slope and curvature of the yield curve. A problem with the multi-factor short-rate models such as those discussed above is that as they become more realistic, inversion of the yield curve becomes more difficult.

Heath, Jarrow and Morton (1992) (hereafter HJM) propose a different approach. Instead of

[4] A version of the extended CIR model that is analytically tractable has been proposed by Jamshidian (1995).

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modelling the instantaneous short-rate, they propose a framework where the evolution of the yield curve is modelled in terms of the process followed by instantaneous forward rates. In this case the initial forward rate curve is an input to the model, hence HJM type models automatically fit the observed initial term structure and inversion of the yield curve is not required. A further advantage of the HJM framework is that it allows for complete freedom in the choice of volatility structure. It can also be shown that many short-rate models can be recast in terms of the HJM framework, see for example Chiarella and Kwon (2001). A key problem with general HJM models is that they are computationally intensive and often have to be implemented using Monte-Carlo simulation since trees are generally non-recombining. The HJM framework is discussed further in Section 3.4.

A commonly cited drawback of the HJM framework is that it is expressed in terms of instantaneous forward rates which are not directly observable in the market. This led to the development of the LIBOR market model (Brace, Gatarek and Musiela 1997, Jamshidian 1997, Miltersen, Sandmann and Sondermann 1997) which is expressed in terms of traded forward rates. It is however worth noting that in most markets, including the South African market, only short dated forward rates are traded. This means that to calibrate the LIBOR market model, longer dated forward rates need to be estimated via some type of curve construction procedure. This procedure often involves the construction of instantaneous forward curves.

The model that we propose to implement in the South African market, derived by Gombani, Jaschke and Runggaldier (2005) fits into the HJM framework. The model is derived in two stages called the underlying and perturbed models. The underlying model is assumed to represent the 'true' dynamics of the yield curve. These dynamics are assumed to be stable through time. The observed market prices are however assumed to be affected by 'errors' such as liquidity adjustments. The perturbed model adjusts the 'true' dynamics by incorporating an error term to account for deviations from the theoretical market prices. This model has a useful application in the South African context since the Bond Exchange of South Africa (BESA) publishes two types of empirical yield curve called 'perfect fit' and 'best decency' but do not provide any link between the two. The 'perfect fit' curves price the instruments from which they where created exactly whereas the 'best decency' curves will be smoother than the 'perfect fit' curves but do not necessarily price the input instruments exactly. Gombani et al.'s model can be used to provide a theoretical connection between the two types of curve. The theoretical arbitrage free term structure is determined by calibrating the underlying model to historical yield curve volatility via a principal component analysis of historical 'best decency' forward curves. A 'perfect fit' curve constructed from the market prices of liquid bonds is then used to determine the market prices for the perturbed model.

The next chapter gives an overview of the South African fixed income market. Some well

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known results and notation used later are set out in Chapter 3. Chapter 4 reviews Gombani et al.'s model focusing on the particular three-factor case that we propose to implement. Where necessary we provide details omitted in the original paper. We also consider the application of formulas for pricing European bond options for Gaussian HJM models to the proposed model. Implementation of the model is considered in Chapter 5 where we discuss the calibration of the model and the relevant data and numerical techniques including yield curve construction. A significant part of the calibration process is the implementation of the Kalman filter. Gombani et al. do not discuss the solution of the differential Riccati equation needed to calculate bond prices. We discuss two possible methods for solving the relevant equation. The final chapter concludes with two examples that demonstrate the application of the model to bond and option pricing in the South African market.

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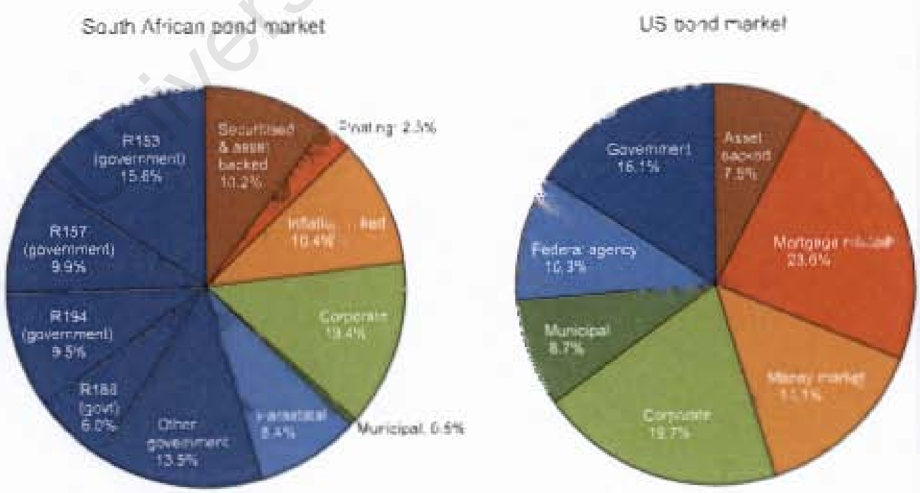
Chapter 2

Overview of the South African fixed income market

The South African bond market is amongst the most liquid and largest of the emerging markets. A key feature of the local bond market is that the market capitalisation and trade is concentrated in a few liquid government bonds. This is illustrated by Figure 2.1 below which shows a breakdown of the market capitalisation of the South African and US bond markets as of 30 June 2006.[1]

Figure 2.1: Breakdown of the South African and US bond markets by percentage of market capitalisation (30 June 2006)



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[1]RSA Data: 'Bond Data' spreadsheet, Bond Exchange of South Africa. 30 June 2006, available from www.bondexchange.co.za. US Data: Research Quarterly, Bond Market Association, 31 August 2006, available from www.bondmarkets.com.

As of 30 June 2006, the market capitalisation of the South African bond market was approximately R731.3 billion. The US market was about 258 times larger[2] than the local market with a market capitalisation of approximately $26.4 trillion.

The market capitalisation of the South African bond market has grown by about 56% over the past five years from approximately R468.8 billion as of 29 June 2001 to approximately R731.3 billion as of 30 June 2006. The extent to which the market composition has changed over the period can be seen by comparing Figure 2.1 with Figure 2.2 below which shows a similar breakdown of the South African market for 29 June 2001.[3] Specifically, although the current concentration in government bonds is particularly high, it has decreased considerably since 2001. The percentage of parastatal bonds has also decreased while the percentage of bonds issued by corporates has increased. Securitised and asset backed instruments have been introduced and make up over 10% of the market. The percentage of floating rate notes has approximately doubled and the percentage of inflation linked bonds has increased significantly.

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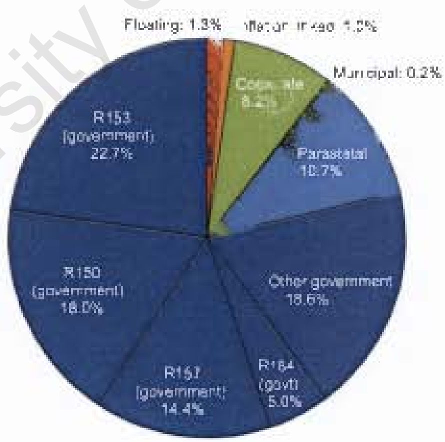
Figure 2.2: Breakdown of the South African bond market by percentage of market capitalisation (29 June 2001)



Bonds are listed on the Bond Exchange of South Africa (BESA). The details of the main South African government bonds at present are given in Table 2.1 below. Some of these are used in the yield curve construction process and form the basis of the calibrated model that we discuss

[2]Using an exchange rate of R7.15 to the dollar.

[3]Data: 'Static Data' spreadsheet, Bond Exchange of South Africa, 29 June 2001, available from www.bondexchange.co.za.

later. We also use the bonds in Table 2.1 in the examples in the final chapter.

Table 2.1: South African government bond details

Bond code	Issue date	Coupon rate	Maturity date	Coupon dates	Book close dates
R194 *	1 Apr 2001	10.00%	28 Feb 2007/08/09	28 Feb / 31 Aug	18 Feb / 21 Aug
R153 *	22 Jun 1989	13.00%	31 Aug 2009/10/11	28 Feb / 31 Aug	18 Feb / 21 Aug
R206	11 Aug 2005	7.50%	15 Jan 2014	15 Jan / 15 Jul	5 Jan / 5 Jul
R201	27 May 2003	8.75%	21 Dec 2014	21 Jun / 21 Dec	11 Jun / 11 Dec
R157 *	18 Jan 1991	13.50%	15 Sep 2014/15/16	15 Mar / 15 Sep	5 Mar / 5 Sep
R203	7 May 2004	8.25%	15 Sep 2017	15 Mar / 15 Sep	5 Mar / 5 Sep
R204	11 Aug 2004	8.00%	21 Dec 2018	21 Jun / 21 Dec	11 Jun / 11 Dec
R207	17 Jun 2005	7.25%	15 Jan 2020	15 Jan / 15 Jul	5 Jan / 5 Jul
R186 *	1 Apr 1998	10.5%	21 Dec 2025/26/27	21 Jun / 21 Dec	11 Jun / 11 Dec

* The R194, R153, R157 and R186 are 'triple-redemption' bonds, meaning that a third of the principal is redeemed at each of the dates in the maturity column of the above table. The bonds are however priced as if the entire principal was to be redeemed on the middle date using the standard BESA bond pricing formula, see BESA (2005).

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2.1 Bond indices

The first indices for the South African bond market were introduced by the Johannesburg Stock Exchange (JSE) in 1983. The indices, collectively called the JSE-Actuaries Fixed Income Index, consisted of price and yield indices for various sub-sectors of the market but did not include an overall bond index. These early indices were however not widely accepted and were replaced by a new set of indices collectively called the JSE-Actuaries Bond Performance Index in 1988.

The JSE-Actuaries Bond Performance Index consisted of values for a 'Price Index' and 'Interest Yield' for four term sub-indices and an overall All Bond Index. The Price Index and Interest Yield were simply the weighted averages of the clean prices and running yields, respectively, of the bonds in each category.[4]

Indices are primarily used as a benchmark for measuring portfolio performance and the main drawback of the JSE-Actuaries Bond Performance Index was that only approximate total returns could be calculated from the index values. This problem was addressed by the introduction of the current indices called the BEASSA Total Return Indices, launched by the Bond Exchange of South Africa (BESA) and the Actuarial Society of South Africa (ASSA) in July 2000, see BESA, ASSA (2000).

[4]The Indices were back-dated to January 1986. The Price Index values were based to a level of 100 on 1 January 1986.

The BEASSA Total Return Indices consists of three main indices:

- The All Bond Index (ALBI) which consists of the top twenty[5] listed bonds ranked according to a dual market capitalisation and liquidity ranking system.[6]
- The Government Bond Index (GOVI) which consists of the South African government bonds that are ranked in the top ten in the dual ranking.
- The Other Bond Index (OTHI) which consists of all the bonds in the ALBI that do not belong to the GOVI.

The ALBI is also split into four sub-indices by term to maturity. In particular the term splits are for bonds with terms to maturity of 1-3 years, 3-7 years, 7-12 years and greater than 12 years.

The index constituents are reselected quarterly with the reweightings taking effect on the first Thursdays of February, May, August and November.[7]

The index values as well as the index modified durations, convexities and other statistics are published daily by BESA.

The advantage of the above total return indices is the simplicity with which accurate total returns can be calculated: if the value of the index is I_0 at time $t = t_0$ and I_1 at time $t = t_1 > t_0$ then the total return for the period t_0 to t_1 is given by the formula[8]

$$\text{total return} = \frac{I_1}{I_0} - 1. \tag{2.1}$$

Also, unlike the previous indices, the current indices are replicable in the sense that the actual performance of a replicating portfolio will match the performance implied by the published index values if the portfolio manager can trade at the prices used to rebalance the index.

2.2 Yield curves

The first yield curve for the South African market, called the JSE-Actuaries Yield Curve, was introduced concurrently with the JSE-Actuaries Bond Performance Index. The curve was constructed by first using a type of cluster analysis to group a weighted set of input bonds into five clusters and determine centre points of these clusters. The curve was then constructed by using

[5]The number is subject to change at the discretion of the index selection committee.

[6]Only bonds that pay fixed semi-annual coupons are eligible for selection for the ALBI. Bonds are also required to have a term to maturity of greater than one year for the entire period for which the selection is being made.

[7]Special reweightings may occur at the discretion of the index selection committee.

[8]Coupons are assumed to be reinvested proportionately across the index.

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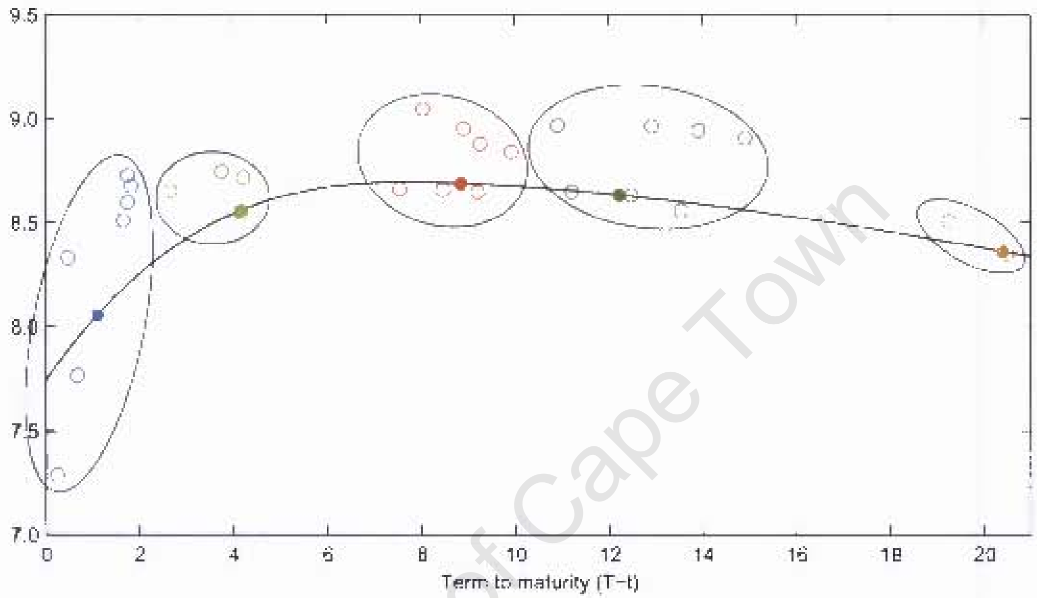
cubic splines to join these centre points, see McLeod (1990). An example of a curve for 30 June 2006 generated using this methodology is show in Figure 2.3 below.

Figure 2.3: JSE-Actuaries Yield Curve (30 June 2006)



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The JSE-Actuaries Yield Curve was however less than ideal – shortcomings included the fact that the curve was constructed from bonds with varying credit quality and the fact that the curves would not necessarily value any of the input bonds correctly. The JSE-Actuaries Yield Curve was replaced by the current curves, called the BEASSA zero-coupon curves, which have been published in their present format since December 2001, see Etheredge (2003). The BEASSA zero-coupon curves are published daily by the Bond Exchange of South Africa and consist of various curves created from instruments in the local bond and swap market.

The zero-coupon bond curves are constructed from the bonds belonging to the GOVI index, and a variety of money-market instruments. Two different types of zero-coupon bond curves are published namely 'perfect fit' and 'best decency'. The 'perfect fit' curve prices the input instruments exactly. That is to say if the cash flows of any of the input bonds are valued using the 'perfect fit' curve, the market price of the bond is obtained. Bonds that are not inputs into the curve construction process will however not necessarily be priced correctly by the curve. The 'best decency' curve aims to balance the requirements of pricing accuracy and smoothness - it will be smoother than the 'perfect fit' curve but will not necessarily value the input instruments exactly. For details of the curve construction procedure see Etheredge (2003).

Implementation of our model requires both 'perfect fit' and 'best decency' type forward and zero curves. The details of the calibration procedure and our curve construction process, which is similar to Etheredge (2003), are discussed in Chapter 5.

'Perfect fit' and 'best decency' swap curves are also published. The swap curves are based on the same construction process as the bond curves, except that that the input instruments are swap rates instead of bonds. Again the 'perfect fit' swap curves will price the input swaps exactly, while the 'best decency curves' sacrifice some degree of pricing accuracy in return for smoother curves. The model that we discuss can be calibrated to swap curves in a similar way using the procedure outlined later, and can provide a theoretical link between the 'best decency' and 'perfect fit' swap curves.

2.3 Other instruments and derivatives

Until recently the market for South African swaps has been largely an interbank market. Swaps trade over-the-counter (OTC) and are usually standard fixed vs. floating swaps linked to JIBAR[9] where payments are exchanged quarterly. The local banks mainly trade swaps with maturities up to 10 years. Longer dated swaps with maturities up to 30 years are primarily traded by foreign banks.

A market for options on South African bonds exists but is currently a lot less liquid than the underlying bond market. The most liquid derivatives during the 1990s were options on Eskom bonds. During this period asset managers were very active in the options market and there were numerous market makers (banks) trading options. At this time only OTC options traded and physical option contracts were exchanged. Many international and domestic players stopped trading the South African options market after the 1998 emerging market (Russian) crisis and option turnover decreased significantly.

In the last few years option trade has been concentrated in options on South African government bonds – very few, if any Eskom options trade. Option pricing tends to be based on the price of options on a single benchmark bond, currently either the R153 or R157. To price options on other South African government bonds, the market convention is to calculate the relevant implied volatilities from the benchmark bond volatility using a relatively naive volatility conversion formula. Options with 3, 6, 9 and 12 month expiries trade, with 3 month options being the most liquid – the longer dated options tend to exhibit wider bid-offer spreads.

[9] Johannesburg Interbank Agreed Rate.

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In Section 5.3 we discuss how the perturbed model can be calibrated to a benchmark option price. The second example in Chapter 6 then compares the prices implied by the perturbed model for options on other (non benchmark) bonds with the prices obtained using the market volatility conversion convention. The model that we implement provides a more sophisticated methodology for pricing options on less liquid bonds which is consistent with observed yield curve dynamics.

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Chapter 3

Preliminaries

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This section sets out some well known results and notation used later.

3.1 Some interest rate definitions and notation

The fundamental objects of fixed income modelling are zero-coupon bonds. A zero-coupon bond with maturity date T is a contract that guarantees to pay the holder 1 unit of currency at time T. The price of a time T maturity zero-coupon bond at time $t \leq T$ is denoted by $p(t, T)$.[1]

The time t, $T - t$ year zero-coupon rate or spot rate is denoted by $Y(t, T)$. This is the interest rate earned on money invested at time t to be returned at time T. We will assume that $Y(t, T)$ is a continuously compounded interest rate. Since buying a T-maturity zero-coupon bond at time t is equivalent to investing $p(t, T)$ at time t to receive 1 at time T it follows that

$$p(t, T)e^{Y(t,T)(T-t)} = 1. \tag{3.1}$$

The price of the zero-coupon bond $p(t, T)$ is therefore given by

$$p(t. T) = e^{-Y(t,T)(T-t)}. \tag{3.2}$$

The time t, time-weighted yield of the zero-coupon bond maturing at time T is denoted by

$$y(t, T) = Y(t, T)(T - t). \tag{3.3}$$

[1]The definitions that follow are subject to the assumptions that:

- There is a frictionless market for zero-coupon bonds of every maturity $T > 0$.
- $p(T, T) = 1$ for all $T \geq 0$.
- For every fixed t, $p(t, T)$ is differentiable with respect to T.
- There is no risk of default.

The price at time t of a coupon bearing bond with cash flows c_j occurring at times T_j for $j = 1, \ldots, m$ is given by

$$\mathcal{A}(t, c_m, T_m) = \sum_{j=1}^{k} c_j p(t, T_j). \tag{3.4}$$

For $t < S < T$, the continuously compounded forward rate $F(t, S, T)$ is the continuously compounded rate at time t for the period S to T. A simple no-arbitrage argument, see for example Bjork (1998), shows that $F(t, S, T)$ is given by

$$F(t, S, T) = -\frac{\ln p(t, T) - \ln p(t, S)}{T - S}. \tag{3.5}$$

The instantaneous forward rate, $f(t, T)$, is the rate at time t for the infinitesimal interval $[T, T + dT]$ and is given by

$$\begin{aligned} f(t, T) &= \lim_{\Delta T \to 0} F(t, T, T + \Delta T) \\ &= = - \lim_{\Delta T \to 0} \frac{\ln p(t, T + \Delta T) - \ln p(t, T)}{\Delta T} \\ &= -\frac{\partial \ln p(t, T)}{\partial T}. \end{aligned} \tag{3.6}$$

Solving the differential equation (3.6) gives the zero-coupon bond price $p(t, T)$ in terms of the instantaneous forward rates:

$$p(t, T) = \exp\left\{ - \int_t^T f(t, s) ds \right\}. \tag{3.7}$$

The instantaneous short term risk-free rate or short rate at time t is denoted by $r(t)$. The short rate is defined as the instantaneous forward rate at time t for the infinitesimal interval $[t, t + dt]$, that is

$$r(t) = f(t, t). \tag{3.8}$$

The money-market account is defined as the investment that continually accrues interest at the prevailing risk-free rate. Assuming a value of 1 at time $t = 0$, the value of the money-market account at time t is given by

$$M(t) = \exp\left\{ \int_0^t r(s) ds \right\}. \tag{3.9}$$

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3.2 Matrix functions

Both the underlying and perturbed models are formulated in terms of matrix functions. Matrix functions and the rules for differentiating and integrating them are defined as follows:[2]

The matrix function $A(t)$ defined by

$$A(t) = [a_{ij}(t)]_{n\times m} \quad i = 1,\dots,n,\ j = 1,\dots,m, \tag{3.10}$$

is an $n \times m$ matrix with the functions $a_{ij}(t)$ as entries.[3]

The derivative and integral of the matrix function $A(t)$ are defined by

$$\frac{dA(t)}{dt} = \left[\frac{d}{dt}a_{ij}(t)\right]_{n\times m} \tag{3.11}$$

and

$$\int_u^t A(s)ds = \left[\int_u^t a_{ij}(s)\,ds\right]_{n\times m}. \tag{3.12}$$

The rules for differentiating and integrating matrix functions are similar to those for ordinary functions. In particular for matrix functions $A(t)$ and $B(t)$ and constant matrix C of suitable dimensions we have

$$\frac{d}{dt}CA(t) = C\frac{d}{dt}A(t), \tag{3.13}$$

$$\frac{d}{dt}\left(A(t) + B(t)\right) = \frac{dA(t)}{dt} + \frac{dB(t)}{dt}, \tag{3.14}$$

$$\frac{d}{dt}A(t)B(t) = A(t)\frac{dB(t)}{dt} + \frac{dA(t)}{dt}B(t), \tag{3.15}$$

$$\int_u^t \left(A(s) + B(s)\right) ds = \int_u^t A(s)\,ds + \int_s^t B(s)\,ds. \tag{3.16}$$

3.3 Matrix exponential

For any constant $n \times n$ matrix, A, and scalar, τ, the matrix exponential, $e^{A\tau}$, is defined by the power series expansion

$$e^{A\tau} = \sum_{k=0}^{\infty} \frac{(A\tau)^k}{k!}. \tag{3.17}$$

We use the MATLAB function *expm* to calculate matrix exponentials. Although it can be shown that the series (3.17) converges for all values of τ, it does not converge fast enough to use as an approximation, see Grewal and Andrews (2001). Moler and Van Loan (2003) discuss the various possibilities for calculating the matrix exponential, including Padé approximation and scaling and squaring, the techniques used by the function *expm*.

[2] See Grewal and Andrews (2001), Zill and Cullen (1992).

[3] The functions $a_{ij}(t)$ are assumed to be defined on a suitable interval I.

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3.4 The Heath-Jarrow-Morton drift condition and affine HJM models

As discussed in Chapter 1, the HJM framework (Heath et al. 1992) models the evolution of the yield curve in terms of the process followed by instantaneous forward rates. The well known HJM drift condition specifies the relationship between the parameters of the forward rate process. The most commonly used version of the condition states the relationship between the drift and diffusion coefficients when the forward rate dynamics are specified under the martingale measure:[4]

HJM drift condition: *If the dynamics of the instantaneous forward rates, specified under the martingale measure* $\mathbb{Q}$*, are given by*

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t) \\ f(0,T) &= f^*(0,T), \end{aligned} \tag{3.18}$$

where $W(t)$ *is an* n*-dimensional* $\mathbb{Q}$*-Wiener process,* $\sigma(t,T)$ *is an* n*-dimensional vector valued function,* $\alpha(t,T)$ *is a scalar function and* $f^*(0,T)$ *is the observed forward rate curve at time* $t=0$ *then the relationship between the drift coefficient,* $\alpha(t,T)$*, and diffusion coefficient,* $\sigma(t,T)$*, must be*

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)'\,ds. \tag{3.19}$$

In the above formula $'$ *denotes transpose.*[5] □

Condition (3.19) implies that although the volatility structure of an HJM type model can be specified freely; once it has been chosen the drift parameters are uniquely determined.

An n-dimensional HJM model is said to be Markovian if there exists an n-dimensional Markov process $x(t)$, such that the forward rates can be expressed in the form $f(t,T;x(t))$. The notation implies that the effect of the Wiener process enters only through the variable x and any further dependence on t and T is deterministic.

Further, a Markovian HJM model is said to be affine if the forward rate process can be written in the form

$$f(t,T;x(t)) = h(t,T)x(t) + h_0(t,T), \tag{3.20}$$

[4] See for example Bjork (1998).

[5] Expanding the matrix multiplication in (3.19) gives

$$\alpha(t,T) = \sum_{i=1}^{n} \sigma_i(t,T)\int_t^T \sigma_i(t,s)\,ds.$$

The multi-factor HJM drift condition is often cited in this form, see for example Hull (2003).

where $h(t,T)$ and $h_0(t,T)$ are deterministic functions. The model derived by Gombani et al. (2005) that we propose to implement fits into the affine HJM framework. Conditions on the volatility parameters under which a general HJM model can be written in affine form are derived by Chiarella and Kwon (2001).[6]

3.5 Principal component analysis of the yield curve

Principal component analysis (hereafter PCA) is a multivariate statistical technique for simplifying a dataset by reducing its dimensionality. The discussion below is based on Johnson and Wichern (1992).

Suppose that a dataset consists of m observations of the p random variables $X_1, X_2, \ldots, X_p$. Algebraically, PCA aims to explain the variance-covariance structure of the dataset through a few linear combinations of the original variables:

Suppose the covariance matrix associated with the random vector $X = [X_1, X_2, \ldots, X_p]$ is given by $\Sigma = [\sigma_{i,j}]$, where $\sigma_{ij} = \text{cov}(X_i, X_j)$, $i, j = 1, 2, \ldots, p$. If the linear combinations $Y_1, Y_2, \ldots, Y_p$ are given by

$$\begin{array}{rcl} Y_1 &=& \ell_{11}X_1 + \ell_{21}X_2 + \cdots + \ell_{p1}X_p = \ell_1'X \\ Y_2 &=& \ell_{12}X_1 + \ell_{22}X_2 + \cdots + \ell_{p2}X_p = \ell_2'X \\ \vdots && \vdots \qquad\qquad\qquad \vdots \\ Y_p &=& \ell_{1p}X_1 + \ell_{2p}X_2 + \cdots + \ell_{pp}X_p = \ell_p'X, \end{array} \tag{3.22}$$

then it follows that

$$\begin{array}{rcll} \text{var}(Y_i) &=& \ell_i'\Sigma\ell_i, & i = 1, 2, \ldots, p \\ \text{cov}(Y_i, Y_j) &=& \ell_i'\Sigma\ell_j, & i, j = 1, 2, \ldots, p. \end{array} \tag{3.23}$$

The principal components are defined to be the uncorrelated linear combinations $Y_i = \ell_i'X$, $i = 1, 2, \ldots p$ that have maximum variances subject to the constraint that $\ell_i'\ell_i = 1$.[7]

Specifically the first principal component is the linear combination $\ell_1'X$ that maximises $\text{var}(\ell_1'X)$ subject to $\ell_1'\ell_1 = 1$.

[6] In general, a term structure model is said to be affine if zero-coupon bond prices are of the form

$$p(t,T) = e^{A(t,T)+B(t,T)r(t)}, \tag{3.21}$$

where $A(t,T)$ and $B(t,T)$ are deterministic functions. Duffie and Kan (1996) discuss the general theory of affine term structures.

[7] Since for any particular coefficient vector, ℓ_i', $\text{var}(\ell_i'X)$ can be increased by multiplying ℓ_i' by a constant, the coefficient vectors are required to be of unit length.

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The second principal component is the linear combination $\ell_2'X$ that maximises var$(\ell_2'X)$ subject to $\ell_2'\ell_2 = 1$ and cov$(\ell_1'X, \ell_2'X) = 0$.

The ith principal component is the linear combination $\ell_i'X$ that maximises var$(\ell_i'X)$ subject to $\ell_i'\ell_i = 1$ and cov$(\ell_i'X, \ell_j'X) = 0$ for $j < i$.

It can be shown, see Johnson and Wichern (1992), that the principal components are determined by the eigenvectors and eigenvalues of the covariance matrix. The covariance matrix, Σ can be diagonalised[8] as $\Sigma = Q\Lambda Q'$ where Λ is the diagonal matrix of the eigenvalues of Σ, namely $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$, and Q is the matrix with the corresponding eigenvectors, $e_1 \geq e_2 \geq \ldots \geq e_p$, as columns. The matrix Q is orthogonal, that is $QQ' = Q'Q = I$.

The ith principal component is given by

$$Y_i \;=\; e_{1i}X_1 + e_{2i}X_2 + \cdots + e_{pi}X_p \;=\; e_i'X \tag{3.24}$$

for $i = 1, 2, \ldots, p$, and

$$\begin{aligned} \text{var}\,(Y_i) &= e_i'\Sigma e_i = \lambda_i, \quad i = 1, 2, \ldots, p \\ \text{cov}\,(Y_i, Y_j) &= e_i'\Sigma e_j = 0, \quad i \neq j. \end{aligned} \tag{3.25}$$

The total variance explained by the principal components is equal to the total variance of the original variables. This follows from the fact that the sum of the diagonal entries of a matrix is equal to the sum of its eigenvalues,[9] see for example Strang (1998), hence

$$\sum_{i=1}^{p} \text{var}\,(X_i) = \sum_{i=1}^{p} \sigma_{ii} = \sum_{i=1}^{p} \lambda_i = \sum_{i=1}^{p} \text{var}\,(Y_i)\,. \tag{3.26}$$

It follows from (3.26) that the proportion of the total variance explained by the ith principal component is

$$\frac{\lambda_i}{\lambda_1 + \lambda_2 + \ldots + \lambda_p}. \tag{3.27}$$

Although p components are required to reproduce the total system variability, a large percentage of the variability can sometimes be explained by a small number of components, say k. In this case the first k components can replace the original p variables as they contain almost as much information.

The original variables can be expressed as a linear combination of the principal components: a

[8]All symmetric matrices can be diagonalised and will have real eigenvalues and orthogonal eigenvectors, see for example Strang (1998).

[9]This number is called the trace of the matrix.

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particular observation, $x = [x_1, x_2, \ldots, x_p]$, of the random vector $X = [X_1, X_2, \ldots, X_p]$ can be expressed as

$$x = \sum_{j=1}^{p} \alpha_j e_j. \tag{3.28}$$

The α's are called the principal component 'scores'.

PCA can also be interpreted geometrically. By viewing the observations of the random vector $X = [X_1, X_2, \ldots, X_p]$ as m points in p-dimensional space, PCA can be interpreted as a rotation of the original dataset to a new coordinate system where the new axes represent the orthogonal directions of maximum variability. In this context the scores are the representation of the original data in terms of this new coordinate system. Replacing the original data with the first k principal components is equivalent to projecting the original data onto the k-dimensional hyperplane that best approximates the data.

A PCA of historical yield curve data[10] can be used to determine the main types of yield curve shift that occurred over the period. Figure 3.1 below shows the coefficient vectors of the first three components of a PCA of historical South African forward curve data for the period 30 June 2003 to 30 June 2006.[11] The principal components are calculated from the covariance matrix of the forward curve shifts. Construction of the forward curves and PCA is discussed further in Chapter 5.

As mentioned in Chapter 1, most movement in the yield curve can usually be described by three main types of shifts. In other words, most of the curve volatility can be explained by the first three principal components. For the components shown in Figure 3.1, the first three components explain approximately 90.2% of the curve volatility. Usually the first three principal components can be interpreted as level shifts, slope changes and curvature changes.

The first component is called a level shift since it describes a shift that is usually close to parallel. In other words level shifts cause the curve to move in the same direction across all maturities. For the components shown in Figure 3.1, a positive level shift implies that the entire curve shifts upwards while a negative level shift implies the entire curve moves downwards.

Slope changes, which are defined by the second component, represent a steepening or flattening of the curve. Slope changes cause short-dated and long-dated rates to move in opposite directions. For the components shown in Figure 3.1, a positive slope change represents a steepening

[10]Specifically, a history of discrete approximations to the historical yield curve shifts, where for each date the curve is approximated by a vector of interest rates corresponding to a common term vector.

[11]The principal components were calculated using curves with a monthly term vector. However, to highlight the discrete nature of the coefficient vector only every third point is plotted.

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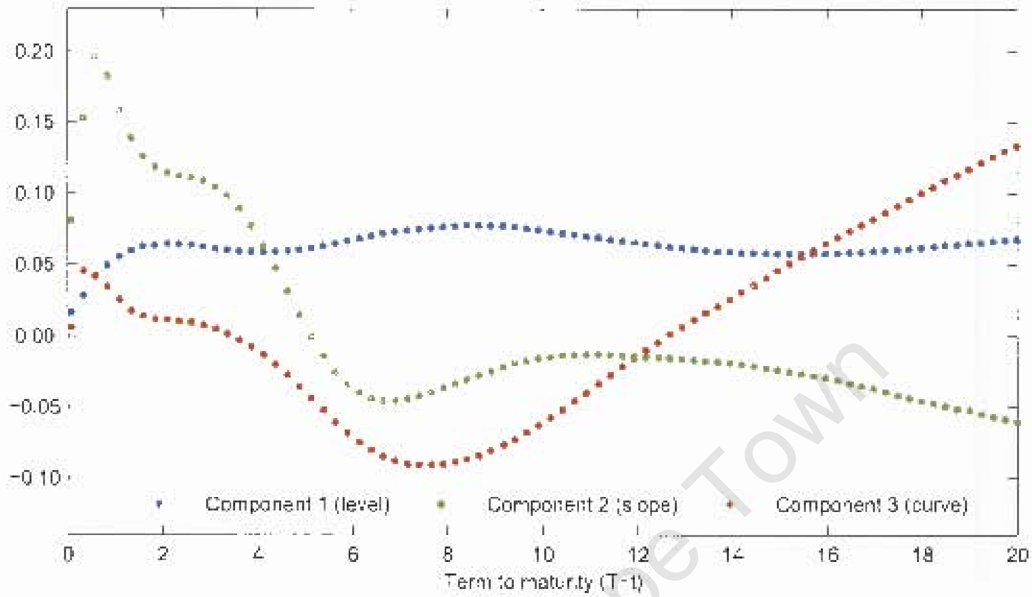
Figure 3.1: Principal components



of the curve while a negative slope change causes the curve to flatten.

The third component defines curvature changes which cause short-dated and long-dated rates to move in the opposite direction to rates in the middle of the curve. For the components shown in Figure 3.1, a positive curvature shift causes short-dated and long-dated rates to move upwards and rates in the middle of the curve to move downwards, causing the curve to become less "humped". Conversely a negative curvature change causes the curve to become more "humped".

An arbitrary shift in the yield curve can be expressed (approximately) as a linear combination of a level shift, slope change and curvature change. The coefficients of this linear combination, the scores, can be thought of as three random variables that determine the shifts in the yield curve. The random variable $W(t)$ essentially determines the values of the scores.

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Chapter 4

The model

The model that we propose to implement in the South African market was developed by Gombani et al. (2005). They derive an arbitrage free pricing system to price illiquid bonds that is consistent with the observed prices of a finite number of liquidly traded bonds. The model is derived in two stages called the underlying and perturbed models.

The underlying model defines the theoretical arbitrage free term structure. It is assumed to be a multi-factor, affine HJM type model, like (3.20), where the stochastic factors satisfy a linear diffusion equation. As a result of the affine nature of the model, the term structure at each point in time is determined by the state vector defined by the stochastic factors.

Since the initial forward rate curve is an input to an HJM type model, the theoretical prices implied by the underlying model at time $t = 0$ will correspond to the observed prices.[1] This is however generally not the case at times after $t = 0$. Given the prices of as many zero-coupon bonds as there are stochastic factors (in our case three), the value of the state vector can be reconstructed such that the prices of these zero-coupon bonds correspond with the prices implied by the underlying model. The prices implied by the underlying model for other bonds will however generally not correspond exactly to the current observed term structure.

Gombani et al. (2005) argue that the differences between the model and market prices should be interpreted as unobserved errors. They justify this setup on the basis that uncertainty is introduced by various factors such as model misspecification, liquidity constraints and bid-offer spreads as well as the fact that since the term structure is actually infinite dimensional there will always be some residual error when using a finite dimensional model, see Gombani and Runggaldier (2001).

The, second stage of the process, the perturbed model, models the prices of the observed bonds

[1] Provided that the instrument being priced is valued correctly by the initial forward rate curve.

as their theoretical values distorted by noise. The available information is assumed to be the prices of N liquid zero-coupon bonds and the perturbed model is formed by adding an extra factor of dimension N to model the extra uncertainty associated with each of the observed prices.

By defining an extended state vector, the perturbed model can be written in the same form as the underlying model but for the perturbed model the available information is insufficient to reconstruct the extended state vector. Stochastic filtering techniques, in particular the Kalman filter, are used to estimate the extended state vector.

4.1 The underlying model

The underlying model defines the theoretical arbitrage free term structure. As described in Gombani et al. (2005) the underlying model is assumed to be a multi-factor affine HJM type model where interest rates and bond prices are functions of n abstract factors.

Given a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{Q})$, the abstract factors, $x(t)$, are assumed to satisfy an n-dimensional linear diffusion equation of the form[2]

$$\begin{cases} dx(t) = A(t)x(t)dt + B(t)dW(t) \\ x(0) = 0, \end{cases} \tag{4.1}$$

where $A(t)$ and $B(t)$ are $n \times n$ matrix valued functions[3] and $W(t)$ is an n-dimensional Wiener process. The instantaneous forward rates are defined by

$$f(t,T) = C(t,T)x(t) + G(t,T), \tag{4.2}$$

where $C(t,T)$ is an n-dimensional vector valued function and $G(t,T)$ is a scalar function. Both $C(t,T)$ and $G(t,T)$ are assumed to be differentiable with respect to t.

From this point forwards, we are going to consider a three-factor model. As discussed in Section 3.5 three factors are usually sufficient to explain most of the volatility of the term structure. A pleasing feature of the perturbed model is that there is a closed form solution for bond pricing which is not the case for a general HJM model. Thus three factors do not significantly complicate pricing.

[2]Gombani et al. (2005) show that the assumption that $x(0) = 0$ is not restrictive.

[3]The matrix functions $A(t)$ and $B(t)$ are required to be locally bounded. Also $A(t)$ and $A(s)$ are required to commute for all s, $t > 0$, see Gombani et al. (2005). We will later assume that A and B are constant matrices which therefore trivially satisfy these regularity conditions.

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It is possible to write the underlying model in the usual HJM form. Differentiating (4.2) with respect to t gives

$$df(t,T) = [C_t(t,T)x(t) + C(t,T)A(t)x(t) + G_t(t,T)]\,dt + C(t,T)B(t)dW(t), \tag{4.3}$$

where $C_t(t,T)$ and $G_t(t,T)$ denote derivatives of $C(t,T)$ and $G(t,T)$ with respect to t. Comparing (4.3) with (3.18) shows that the underlying model is a three-factor HJM model with drift coefficient

$$\alpha(t,T) = C_t(t,T)x(t) + C(t,T)A(t)x(t) + G_t(t,T), \tag{4.4}$$

and diffusion coefficient

$$\sigma(t,T) = C(t,T)B(t). \tag{4.5}$$

In the general HJM framework, σ (and hence α) may depend on previous values of the forward rates. In this case however, σ and α are deterministic functions of t and T. A deterministic volatility structure results in a so called Gaussian HJM model. Under a Gaussian HJM model, forward rates are normally distributed.[4]

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The functions $C(t,T)$ and $G(t,T)$ cannot be chosen arbitrarily - using the HJM drift condition (3.19) Gombani et al. (2005) show that $C(t,T)$ and $G(t,T)$ must satisfy the following conditions:

$$C(t,T) = C(T)e^{\int_t^T A(s)ds}. \tag{4.6}$$

where $C(T) = C(T,T)$ is a locally bounded function, and

$$G(t,T) = f^*(0,T) + \frac{1}{2}\int_0^t \beta_T(s,T)ds, \tag{4.7}$$

where $f^*(0,T)$ are the observed forward rates at time zero, $\beta_T(t,T)$ denotes the derivative of $\beta(t,T)$ with respect to T and $\beta(t,T)$ is given by

$$\beta(t,T) = \left\| \int_t^T C(t,u)B(t)du \right\|^2. \tag{4.8}$$

Equations (4.6) to (4.8) show that functions chosen for f^*, $A(t)$, $B(t)$ and $C(T)$ determine the functions $C(t,T)$ and $G(t,T)$.

From here onwards we assume that $A(t) = A$, $B(t) = B$ and $C(T) = C$ are constant matrices. In this case equation (4.6), the condition on $C(t,T)$, simplifies to

$$C(t,T) = Ce^{A(T-t)}. \tag{4.9}$$

[4] General Gaussian HJM models are discussed in Section 13.3 of Musiela and Rutkowski (1997).

Choosing $A(t) = A$, $B(t) = B$ and $C(T) = C$ to be constant results in stationary volatility: Noting (4.9), the diffusion coefficient (4.5) can be written in the form[5]

$$\begin{aligned} \sigma(t,T) &= Ce^{A(T-t)}B \\ &= \left[\begin{array}{ccc} \sigma_1(T-t) & \sigma_2(T-t) & \sigma_3(T-t) \end{array} \right]. \end{aligned} \tag{4.10}$$

Thus volatility is stationary since $\sigma(t,T)$ is only a function of time to maturity, $T-t$.

If we further assume that A is invertible then there is a closed form solution for $G(t,T)$: [6]

$$\begin{aligned} G(t,T) &= f^*(0,T) + \frac{1}{2}\left\{ \left\|CA^{-1}e^{AT}B\right\|^2 - \left\|CA^{-1}e^{A(T-t)}B\right\|^2 \right\} \\ &\quad + CA^{-1}\left[e^{A(T-t)} - e^{AT}\right] BB'A'^{-1}C'. \end{aligned} \tag{4.11}$$

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4.1.1 Solving for x(t)

Assuming that at time t, we have exactly three observed bond prices, we can, in general, determine the state vector $x(t)$: If $y(t,T) = \int_t^T f(t,s)ds$ is the time t, time-weighted yield of the zero-coupon bond maturing at time T then from (3.7), (4.2) and (4.9) we have

$$y(t,T) = \int_t^T C(s)e^{A(s-t)}ds\, x(t) + \int_t^T G(t,s)ds.$$

If the observed bonds have maturities $T_1 < T_2 < T_3$, we can write

$$\begin{bmatrix} y(t,T_1) \\ y(t,T_2) \\ y(t,T_3) \end{bmatrix} = H(t) \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \int_t^{T_1} G(t,s)ds \\ \int_t^{T_2} G(t,s)ds \\ \int_t^{T_3} G(t,s)ds \end{bmatrix} \tag{4.12}$$

where $H(t)$ denotes the 3×3 matrix

$$H(t) = \begin{bmatrix} \int_t^{T_1} C(s)e^{A(s-t)}ds \\ \int_t^{T_2} C(s)e^{A(s-t)}ds \\ \int_t^{T_3} C(s)e^{A(s-t)}ds \end{bmatrix}.$$

[5] If we were considering the one-factor case, defining $\sigma = CB$ and $a = -A$ would reduce (4.10) to

$$\sigma(t,t) = \sigma e^{-a(T-t)},$$

which is the volatility structure required to recast the Hull-White (extended Vasiček) model in the HJM framework, see Bjork (1998).

[6] See Gombani et al. (2005), Bjork and Gombani (1999).

The factors can be reconstructed from the observed bond prices if (4.12) can be solved for $x(t)$ which can be done provided that $H(t)$ is invertible. The factors $x(t)$ are said to be observable as their values can be reconstructed from the available information.

4.2 The perturbed model

In practice the underlying model described above should give a reasonable description of the long-term, time series features of the term structure. However at times $t > 0$ it is likely that the underlying model will not fit all the observed prices exactly.[7] Section 4.1.1 shows that at times $t > 0$ the underlying model can only be fitted exactly to three zero-coupon bond prices.

Gombani et al. (2005) argue that the differences between the model and actual prices should be interpreted as unobserved errors and that the prices of the observed bonds should be modelled as the theoretical values implied by the underlying model perturbed by noise.

Assuming that there are N observed bonds, the underlying model is extended to the perturbed model by adding an extra factor, $\xi(t)$, of dimension N to model the extra uncertainty associated with each of the observed prices.

For the perturbed model the available information, which is assumed to be the prices of the observed liquid bonds, is insufficient to reconstruct the factors exactly. Stochastic filtering techniques, in particular the Kalman filter, can be used to estimate the factors.

The perturbed model is defined as follows:

Assume that the maturities of the observed zero-coupon bond prices are $T_1, \ldots, T_N$. Then the perturbed version of (4.2) is given by

$$dx(t) = Ax(t)dt + BdW(t) \quad (4.13)$$

$$d\xi(t) = A_\xi(t)\xi(t)dt + B_\xi(t)dW_\xi(t) \quad (4.14)$$

$$\tilde{f}(t,T) = C(t,T)x(t) + C_\xi(t,T)\xi(t) + \tilde{G}(t,T) \quad (t < T). \quad (4.15)$$

where $W_\xi(t)$ is an N-dimensional Wiener process that is independent of W, $x(0) = 0$ and $\xi(0) = 0$. For fixed T, the function $C_\xi(t,T)$ is an N-dimensional row vector of locally bounded functions. The function $\tilde{G}(t,T)$ is assumed to be differentiable with respect to t.

[7] Since the model is fitted to the initial term structure, at time $t = 0$ the prices implied by the underlying model of any instruments that are priced correctly by the initial forward curve will correspond to the observed prices.

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The components of $\xi(t)$, namely $\xi_i(t)$, $i = 1, \ldots, N$ represent the extra uncertainty associated with each of the observed bonds.

By defining an extended state vector

$$\tilde{x}(t) = \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}, \tag{4.16}$$

the perturbed system can be written in the same form as the underlying model:

$$d\tilde{x}(t) = \tilde{A}(t)\tilde{x}(t) + \tilde{B}(t)d\tilde{W}(t), \tag{4.17}$$

$$\tilde{f}(t,T) = \tilde{C}(t,T)\tilde{x}(t) + \tilde{G}(t,T), \tag{4.18}$$

with

$$\begin{aligned}
\tilde{A}(t) &= \begin{bmatrix} A & 0 \\ 0 & A_\xi(t) \end{bmatrix}, \\
\tilde{B}(t) &= \begin{bmatrix} B & 0 \\ 0 & B_\xi(t) \end{bmatrix}, \\
\tilde{C}(t,T) &= \begin{bmatrix} C(t,T) & C_\xi(t,T) \end{bmatrix}, \\
\tilde{W}(t) &= \begin{bmatrix} W(t) \\ W_\xi(t) \end{bmatrix}.
\end{aligned}$$

The functions $\tilde{C}(t,T)$ and $\tilde{G}(t,T)$ have to satisfy similar no-arbitrage restrictions to $C(t,T)$ and $G(t,T)$, see Gombani et al. (2005). In particular

$$\tilde{C}(t,T) = \tilde{C}(T)e^{\int_t^T \tilde{A}(s)ds}. \tag{4.19}$$

where $\tilde{C}(T)$ is a locally bounded function, and

$$\tilde{G}(t,T) = \tilde{f}^*(0,T) + \frac{1}{2}\int_0^t \tilde{\beta}_T(s,T)ds, \tag{4.20}$$

with

$$\tilde{\beta}(t,T) = \left\| \int_t^T \tilde{C}(t,u)\tilde{B}(t)du \right\|^2 = \beta(t,T) + \left\| \int_t^T \tilde{C}_\xi(t,u)\tilde{B}_\xi(t)du \right\|^2. \tag{4.21}$$

Typically one would recalibrate the underlying model on a daily basis to deal with the fact that, at times after the model has been calibrated, the prices implied by the model do not correspond with the prices observed in the market. The perturbed model is an improvement on the underlying model in the sense that recalibration to the market is built into the model. Essentially the extended state vector ensures that the prices of the liquid bonds given by the perturbed model will be consistent with the market prices and the prices of illiquid bonds implied by the perturbed model will be arbitrage free with respect to the market prices of the liquid bonds.

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4.2.1 Choice of parameters

From here onwards we focus on a particular choice of parameters that results in a model similar to the one considered in an earlier paper by Gombani and Runggaldier (2001). Specifically we look at the case where

$$A_\xi = 0,\ B_\xi = bI \text{ and } C_\xi^i(T) = \chi_{(T_{i-1},T_i]}(T), \tag{4.22}$$

where b is a scalar and $\chi_{(T_{i-1},T_i]}(T)$ denotes the indicator function.[8]

This choice of parameters equates to adding a perturbation driven by the Wiener process $W_\xi^i(t)$ for each of the observed bonds $p(t,T_i)$ where the ith perturbation, $\xi^i(t)$, only affects the prices of bonds with maturities less than or equal to T_i. This can be illustrated as follows:

Suppose that for the bond $p(t,T)$, T is in the interval $(T_{k-1}, T_k]$ with $1 \le k \le N$. If A_ξ, B_ξ and $C_\xi^i(T)$ are chosen according to (4.22) then (4.14) becomes

$$d\xi(t) = bdW_\xi(t). \tag{4.23}$$

and (4.19) simplifies to

$$\tilde{C}(t,T) = [C(t,T),\ C_\xi(T)] \tag{4.24}$$

and (4.20) simplifies to

$$\tilde{G}(t,T) = G(t,T) + b^2 \begin{cases} (T - T_{k-1})\,t & \text{if } \ t \le T_{k-1} \\ -\frac{1}{2}(T_{k-1})^2 + Tt - \frac{1}{2}t^2 & \text{if } \ t > T_{k-1}, \end{cases} \tag{4.25}$$

where $G(t,T)$ is given by (4.11). (See Appendix A for derivations of (4.24) and (4.25).)

From (3.7) and (4.15), the price at time t of a bond maturing at time T is given by

$$\tilde{p}(t,T) = e^{-\int_t^T C(t,s)x(t) + C_\xi(s)\xi(t) + \tilde{G}(t,s)ds}, \tag{4.26}$$

where from (4.16), (4.18) and (4.24)

$$C_\xi(s)\xi(t) = C_\xi^1(s)\xi^1(t) + \ldots + C_\xi^N(s)\xi^N(t). \tag{4.27}$$

If $T \le T_{i-1}$ then $C_\xi^i(s) = \chi_{(T_{i-1},T_i]}(s) = 0$ for all $t \le s \le T$. Therefore since $T \in (T_{k-1}, T_k]$, $C_\xi^i(s)\xi^i(t) = 0$ for $i > k$ and it follows that the values of $\xi^{k+1}(t), \ldots, \xi^N(t)$ have no effect on $\tilde{p}(t,T)$.

[8] The indicator function, $\chi_{(T_{i-1},T_i]}(T)$ is defined by

$$\chi_{(T_{i-1},T_i]}(T) = \begin{cases} 1 & \text{if } T \in (T_{i-1}, T_i] \\ 0 & \text{otherwise.} \end{cases}$$

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4.2.2 Summary of the perturbed model

For the choice of parameters discussed above, the perturbed model can be summarised as follows:

$$\begin{aligned} d\tilde{x}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}d\tilde{W}(t), \\ \tilde{f}(t,T) &= \tilde{C}(t,T)\tilde{x}(t) + \tilde{G}(t,T), \end{aligned} \tag{4.28}$$

where

$$\begin{aligned} \tilde{x}(t) &= \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}, \\ \tilde{A} &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \\ \tilde{B} &= \begin{bmatrix} B & 0 \\ 0 & bI \end{bmatrix}; \\ \tilde{C}(t,T) &= \begin{bmatrix} C(t,T) & C_\xi(T) \end{bmatrix}, \\ \tilde{W}(t) &= \begin{bmatrix} W(t) \\ W_\xi(t) \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} C(t,T) &= e^{A(T-t)}, \\ C_\xi^i(T) &= \chi_{(T_{i-1},T_i]}, \\ \tilde{G}(t,T) &= G(t,T) + b^2 \begin{cases} (T-T_{k-1})\,t & \text{if} \quad t \le T_{k-1} \\ -\frac{1}{2}(T_{k-1})^2 + Tt - \frac{1}{2}t^2 & \text{if} \quad t > T_{k-1}; \end{cases} \\ G(t,T) &= f^*(0,T) + \frac{1}{2}\left\{ \left\| CA^{-1}e^{AT}B \right\|^2 - \left\| CA^{-1}e^{A(T-t)}B \right\|^2 \right\} \\ &\quad + CA^{-1}\left[e^{A(T-t)} - e^{AT} \right] BB'A'^{-1}C'. \end{aligned} \tag{4.29}$$

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4.2.3 Estimating $\tilde{x}(t)$

For the perturbed model the dimension of the extended state vector $\tilde{x}(t)$ is $3 + N$, the sum of the dimension of $x(t)$ and the number of observed bonds. This means trying to determine $\tilde{x}(t)$ using the method of Section 4.1.1 results in a system of N equations in $3 + N$ unknowns. Consequently, the matrix $\tilde{H}(t)$ is not invertible. This implies that it is not possible to solve the equation for $\tilde{x}(t)$, or in other words the extended state vector is not observable. Gombani et al. (2005) therefore resort to Kalman filtering, a stochastic filtering technique, to estimate the state vector at each point in time. The Kalman filter is discussed in Section 4.3.2.

4.3 The 'projected price system'

The perturbed model can be used to derived a formula for zero-coupon bond prices that are arbitrage free with respect to the observed bond prices. Gombani et al. (2005) call the term structure implied by this formula the 'projected price system'.

Gombani et al. (2005) derive the 'projected price system' in three steps. Firstly they use the usual martingale measure method to define a pricing system using a particular observed bond as the numeraire. (For details on martingale measure pricing and numeraires see for example Bjork (1998).) They then show that it is possible to extend this formula to the general case where the numeraire is unobservable. Lastly they show that by choosing the unobserved money-market account as the numeraire, computing the projected system reduces to computing the conditional mean and covariances of the extended state vector $\tilde{x}(t)$ which can be done using the Kalman filter.

We follow the notation of Gombani et al. (2005) and define martingale measure pricing as follows:

Definition 4.1 *For a numeraire $N(T)$ with martingale measure $\mathbb{Q}$ on a filtration $\mathscr{F}_t$, the price system defined by the triple $(\mathbb{Q}, N, \mathscr{F})$ for the $\mathscr{F}_T$-measurable claim X is*

$$\Pi_{t,T}(X) = N(t)\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(T)}{N(T)}\right| \mathscr{F}_t\right], \tag{4.30}$$

where $\Pi_{t,T}(X)$ denotes the price at time t of the claim X that matures at time T. In particular, for the observed bond prices one then has

$$\tilde{p}(t,T) = N(t)\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{N(T)}\right| \mathscr{F}_t\right]. \tag{4.31}$$

□

Usually the money-market account is chosen as the numeraire. In this case however, Gombani et al. (2005) choose the numeraire to be the normalised price of the bond with longest maturity,[9] $\frac{\tilde{p}(t,T_N)}{\tilde{p}(0,T_N)}$. It follows from Definition 4.1 that under the corresponding martingale measure $\mathbb{Q}^N$,

$$\tilde{p}(t,T) = \tilde{p}(t,T_N)\mathbb{E}^{\mathbb{Q}^N}\left[\left.\frac{1}{\tilde{p}(T,T_N)}\right| \mathscr{F}_t\right] \text{ for } t \leq T \leq T_N.$$

As noted in Section 4.2.3 above, the factors of the perturbed model are unobservable as they cannot be reconstructed from the observed bond prices. This corresponds to only having partial information available. The partial information available from the observed bond prices corresponds to a sub-filtration, denoted $\hat{\mathscr{F}}$, of the model filtration $\mathscr{F}$. Assuming that the information

[9]This choice is for convenience; the same results are achieved if the numeraire is defined using any of the other observed bond prices as $\frac{\tilde{p}(t,T_i)}{\tilde{p}(0,T_i)}$ and the corresponding martingale measure $\mathbb{Q}^i$ is used to take expectations.

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available at time t is given by $\hat{\mathscr{F}}_t$, Gombani et al. (2005) define the 'projected price system' as follows:[10]

Definition 4.2 *The 'projected price system' is denoted by $\hat{p}(t,T)$ and is given by*

$$\hat{p}(t,T) = \tilde{p}(t,T_N)\mathbb{E}^{\mathbb{Q}^N}\left[\frac{1}{\tilde{p}(T,T_N)}\bigg|\hat{\mathscr{F}}_t\right] \text{ for } t \leq T \leq T_N. \tag{4.32}$$

□

Since $\tilde{p}(t,T_N) \in \hat{\mathscr{F}}_t$, the projected prices, $\hat{p}(t,T)$, are $\hat{\mathscr{F}}$-adapted processes. Also, since $\frac{\hat{p}(t,T)}{\hat{p}(t,T_N)}$ is a $(\mathbb{Q}^N, \hat{\mathscr{F}})$-martingale $\forall T \leq T_N$ the 'projected price system' is arbitrage free.

The above definition requires the numeraire to be observable ($\hat{\mathscr{F}}_t$–measurable). Gombani et al. (2005) prove the following proposition which extends the definition of the 'projected price system' to the general case where the numeraire is not observed.

Proposition 4.1 *If $N(t)$ is a numeraire with corresponding martingale measure $\mathbb{Q}$ on $\mathscr{F}$ such that $N(t) \notin \hat{\mathscr{F}}_t$. Then, letting*

$$\hat{N}(t) = \frac{1}{\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{N(t)}\Big|\hat{\mathscr{F}}_t\right]}$$

one has

$$\hat{p}(t,T) = \hat{N}(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\hat{N}(T)}\bigg|\hat{\mathscr{F}}_t\right].$$

□

If the unobserved money-market account $\tilde{M}$ is chosen as the numeraire, with corresponding martingale measure $\mathbb{Q}$, then it follows from Proposition (4.1) that

$$\begin{aligned}\hat{p}(t,T) &= \frac{\left[\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\tilde{M}(T)}\Big|\hat{\mathscr{F}}_T\right]\Big|\hat{\mathscr{F}}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\tilde{M}(t)}\Big|\hat{\mathscr{F}}_t\right]} \\ &= \frac{\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\tilde{M}(T)}\Big|\hat{\mathscr{F}}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\tilde{M}(t)}\Big|\hat{\mathscr{F}}_t\right]} \text{ because } \hat{\mathscr{F}}_t \subset \hat{\mathscr{F}}_T.\end{aligned} \tag{4.33}$$

From Definition 4.1

$$\frac{\tilde{p}(t,T)}{\tilde{M}(t)} = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\tilde{M}(T)}\bigg|\mathscr{F}_t\right], \tag{4.34}$$

[10] The concept of consistent price systems for sub-filtrations is derived in a general setting in Gombani, Jaschke and Runggaldier (2004).

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therefore

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{\tilde{p}(t,T)}{\tilde{M}(t)}\right|\hat{\mathscr{F}}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[\left.\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{\tilde{M}(T)}\right|\mathscr{F}_T\right]\right|\hat{\mathscr{F}}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{\tilde{M}(T)}\right|\hat{\mathscr{F}}_t\right] \text{ since } \hat{\mathscr{F}}_t \subset \mathscr{F}_t.\end{aligned} \tag{4.35}$$

Substituting the above equation into (4.33) gives

$$\hat{p}(t,T) = \frac{\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{\tilde{p}(t,T)}{\tilde{M}(t)}\right|\hat{\mathscr{F}}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{\tilde{M}(t)}\right|\hat{\mathscr{F}}_t\right]}. \tag{4.36}$$

Equation (4.36) is used as the definition of the projected prices.

4.3.1 Computation of the projected prices by Kalman filtering

This section discusses how the projected prices derived in the previous section can be computed. Gombani et al. (2005) show that if $\hat{\mathscr{F}}_t$ is defined as the filtration generated by the N observed bond prices $(\tilde{p}(t,T_i))_{i=1,\dots,N,}$ or equivalently the filtration generated by time-weighted yields, $(\tilde{y}(t.T_i))_{i=1,\dots,N,}$ where

$$\tilde{y}(t,T) = -\log(\tilde{p}(t,T)) = \int_t^T \tilde{f}(t,s)ds, \tag{4.37}$$

then the pricing formula (4.36) can be written as

$$\frac{\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{\tilde{p}(t,T)}{\tilde{M}(t)}\right|\hat{\mathscr{F}}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{\tilde{M}(t)}\right|\hat{\mathscr{F}}_t\right]} = \exp\left\{-\hat{y}(t,T) + \frac{1}{2}\Gamma_1(t,T) + \Gamma_2(t,T)\right\}, \tag{4.38}$$

with

$$\hat{y}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\tilde{y}(t,T)|\hat{\mathscr{F}}_t\right] \tag{4.39}$$

$$\Gamma_1(t,T) = \mathrm{var}^{\mathbb{Q}}\left[\tilde{y}(t,T)|\hat{\mathscr{F}}_t\right] \tag{4.40}$$

$$\Gamma_2(t,T) = \mathrm{cov}^{\mathbb{Q}}\left[\left.\tilde{y}(t,T), \int_0^t \tilde{f}(s,s)ds\right|\hat{\mathscr{F}}_t\right]. \tag{4.41}$$

From equation (4.38) it is clear that that if the conditional mean and variances (4.39) - (4.41) can be calculated then so can the projected prices defined by (4.36).

Gombani et al. (2005) show that the values of $\hat{y}(t,T)$, $\Gamma_1(t,T)$ and $\Gamma_2(t,T)$ can in fact be calculated in terms of the conditional distribution of the extended state vector $\tilde{x}(t)$ given the observed bond prices. This conditional distribution corresponds to the solution of a stochastic filtering problem where the factors $\tilde{x}(t)$ form the unobservable component of the system. Kalman filtering can be used to find this solution.

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Before discussing the Kalman filter some notation needs to be introduced. From (4.15) and (4.37) with $\tilde{C}(t) = \tilde{C}(t,t)$ and $\tilde{G}(t) = \tilde{G}(t,t)$, the yield dynamics can be written in the form

$$\begin{aligned} d\tilde{y}(t,T) &= -\tilde{f}(t,t)dt + \int_t^T d\tilde{f}(t,s)ds \\ &= -\tilde{C}(t)\tilde{x}(t)dt - \tilde{G}(t)dt + \left(\int_t^T \tilde{C}(t,s)ds\tilde{B}\right) d\tilde{W}(t) + \left(\int_t^T \tilde{G}_t(t,s)ds\right) dt. \end{aligned} \tag{4.42}$$

To simplify notation $\tilde{z}(t)$ is defined as

$$\tilde{z}(t) = \begin{bmatrix} \tilde{y}(t,T_1) - \int_t^{T_1} \tilde{G}(t,s)ds \\ \tilde{y}(t,T_2) - \int_t^{T_2} \tilde{G}(t,s)ds \\ \vdots \\ \tilde{y}(t,T_N) - \int_t^{T_N} \tilde{G}(t,s)ds \end{bmatrix}, \tag{4.43}$$

giving

$$d\tilde{z}(t) = -\begin{bmatrix} \tilde{C}(t) \\ \tilde{C}(t) \\ \vdots \\ \tilde{C}(t) \end{bmatrix} \tilde{x}(t)dt + \begin{bmatrix} \int_t^{T_1} \tilde{C}(t,s)ds\tilde{B} \\ \int_t^{T_2} \tilde{C}(t,s)ds\tilde{B} \\ \vdots \\ \int_t^{T_N} \tilde{C}(t,s)ds\tilde{B} \end{bmatrix} d\tilde{W}(t). \tag{4.44}$$

By defining $C_e(t)$ and $V(t)$ as the terms in the first and second brackets in (4.44) respectively, the partially observed system can be written in the form

$$\begin{cases} d\tilde{x}(t) &= \tilde{A}\tilde{x}(t)dt + \tilde{B}d\tilde{W}(t) \\ d\tilde{z}(t) &= C_e(t)\tilde{x}(t)dt + V(t)d\tilde{W}(t). \end{cases} \tag{4.45}$$

The system is partially observable since the value of $z(t)$ can be calculated from the observed bond prices using (4.43) but, as discussed in Section 4.2.3, the value of $\tilde{x}(t)$ cannot.

The sub-filtration generated by the observed bond prices can be defined as

$$\hat{\mathscr{F}}_t = \sigma\left\{\tilde{z}(s),\ s \leq t\right\}, \tag{4.46}$$

since the information provided by $\tilde{p}(t,T)$ and $\tilde{z}(t)$ is the same.

4.3.2 The Kalman filter

The Kalman filter is an algorithm for determining the optimal estimate of the unobservable component of a partially observable stochastic system like (4.45).[11] The Kalman filter was initially

[11] The Kalman filter can in fact be applied in a more general case where an extra component of uncertainty is introduced when measuring the observable component, see for example Grewal and Andrews (2001) (discrete time case) or Lipster and Shiryayev (1977) (continuous time case).

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derived in discrete time in an engineering setting.[12] See Kalman (1960). The continuous time version of the filter that can be used to estimate $\tilde{x}(t)$ is often referred to as the Kalman-Bucy filter.

The optimal estimate of $\tilde{x}(t)$ given by the filter is the expected value of $\tilde{x}(t)$ conditional on the filtration generated by the observed bond prices. It can be shown, see for example Lipster and Shiryayev (1977), that the estimate is optimal with respect to any quadratic function of the estimation error.

The Kalman-Bucy filter provides a complete statistical characterisation of the state of the system at any point in time in the sense that at any point in time it provides the conditional distribution of the component being estimated and continually updated this distribution based on the available information. The conditional distribution is Gaussian and thus is determined by its mean and variance. The conditional mean (required to calculate $\hat{y}(t,T)$) is propagated in feedback form by a system of linear differential equations. The corresponding covariance matrix (required to calculate $\Gamma_1(t.T)$ and $\Gamma_2(t,T)$) is propagated by a nonlinear differential equation.

The Kalman-Bucy method yields a closed system of equations for the estimate that has a unique continuous solution.

For simplicity the specific case of the Kalman-Bucy filter presented in Gombani et al. (2005) is reproduced below. The general version of this proposition is derived in Lipster and Shiryayev (1977).

Proposition 4.2 *Let the system $(\tilde{x}(t), \tilde{z}(t))$ satisfy (4.45) and $\hat{\mathscr{F}}_t$ be given by (4.46). Then the conditional distribution of $\tilde{x}(t)$, given $\hat{\mathscr{F}}_t$, is Gaussian with mean*

$$\hat{x}(t) = \mathbb{E}^{\mathbb{Q}}\left[\tilde{x}(t)|\,\hat{\mathscr{F}}_t\right], \tag{4.47}$$

and covariance matrix

$$\bar{P}(t) = var^{\mathbb{Q}}\left[\tilde{x}(t)|\,\hat{\mathscr{F}}_t\right], \tag{4.48}$$

which is deterministic

$$\bar{P}(t) = \mathbb{E}^{\mathbb{Q}}\left[\left(\tilde{x}(t) - \hat{x}(t)\right)\left(\tilde{x}(t) - \hat{x}(t)\right)'\right]. \tag{4.49}$$

Assuming that the matrix

$$D(t) = \left[V(t)V(t)'\right]^{1/2} \tag{4.50}$$

[12] The principal uses of the Kalman filter have been in been in vehicle tracking and navigation. In fact, one of the first applications of the filter was in the design of the navigation system for NASA's Apollo project, the manned mission to the moon, see McGee and Schmidt (1985). Recently the Kalman filter has been applied to various problems in finance. See Gombani and Runggaldier (2001) and references therein.

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is invertible, the conditional mean has dynamics

$$d\hat{x}(t) = \tilde{A}\hat{x}(t)dt + \hat{B}d\hat{w}(t), \tag{4.51}$$

with $\hat{x}(0) = 0$,

$$\hat{B}(t) = \left(\tilde{B}V(t)' + \bar{P}(t)C_e(t)'\right)\left[D(t)'\right]^{-1} \tag{4.52}$$

and $\hat{w}(t)$ *is the innovations process*

$$d\hat{w}(t) = D(t)^{-1}\left[d\tilde{z}(t) - C_e(t)\hat{x}(t)dt\right]. \tag{4.53}$$

Furthermore, $\bar{P}(t)$ *is the solution of the differential Riccati equation*

$$\begin{aligned}\frac{d\bar{P}(t)}{dt} &= \tilde{A}\bar{P}(t) + \bar{P}(t)\tilde{A}' - \left[\tilde{B}V'(t) + \bar{P}(t)C_e'(t)\right]\left(D(t)D'(t)\right)^{-1} \times \\ &\quad \left[\tilde{B}V(t)' + \bar{P}(t)C_e'(t)\right]' + \tilde{B}\tilde{B}'\end{aligned} \tag{4.54}$$

with initial condition $\bar{P}(0) = 0$. □

Integrating (4.54) is computationally intensive however, since (4.54) is deterministic its values can be precomputed. Two methods for integrating the DRE are given in Section 5.4.

Proposition 4.2 provides a way to calculate the conditional mean, variance and covariance of $y(t,T)$. Gombani et al. (2005) show that $\hat{y}(t,T)$, $\Gamma_1(t,T)$, and $\Gamma_2(t,T)$ are given by

$$\begin{aligned}\hat{y}(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\tilde{y}(t,T)|\hat{\mathscr{F}}_t\right] = \textstyle\int_t^T \tilde{C}(t,s)ds\hat{x}(t) + \int_t^T \tilde{G}(t,s)ds \\ \Gamma_1(t,T) &= \text{var}^{\mathbb{Q}}\left[\tilde{y}(t,T)|\hat{\mathscr{F}}_t\right] = \left(\textstyle\int_t^T \tilde{C}(t,s)ds\right)\bar{P}(t)\left(\int_t^T \tilde{C}'(t,s)ds\right) \\ \Gamma_2(t,T) &= \text{cov}^{\mathbb{Q}}\left[\textstyle\int_0^t \tilde{f}(s,s)ds, \tilde{y}(t,T)\Big|\hat{\mathscr{F}}_t\right] = \left(\int_0^t \tilde{C}(s,s)\bar{P}(s)e^{\tilde{A}(t-s)}ds\right)\left(\int_t^T \tilde{C}'(t,s)ds\right).\end{aligned} \tag{4.55}$$

Substituting $\hat{y}(t,T)$, $\Gamma_1(t,T)$ and $\Gamma_1(t,T)$ as above into (4.38) it follows that the projected prices $\hat{p}(t,T)$ are given by

$$\begin{aligned}\hat{p}(t,T) &= \exp\left\{\left[-\left(\int_t^T \tilde{C}(t,s)ds\right)\hat{x}(t) - \int_t^T \tilde{G}(t,s)ds\right]\right. \\ &\quad + \left[\frac{1}{2}\left(\int_t^T \tilde{C}(t,s)ds\right)\bar{P}(t)\left(\int_t^T \tilde{C}'(t,s)ds\right)\right] \\ &\quad \left. + \left[\left(\int_0^t \tilde{C}(s,s)\bar{P}(s)e^{\tilde{A}(t-s)}\right)\left(\int_t^T \tilde{C}'(t,s)ds\right)\right]\right\}.\end{aligned} \tag{4.56}$$

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4.4 Pricing options

Formulas for pricing European options on zero-coupon and coupon bearing bonds in Gaussian HJM models have been derived by several authors. Below we apply the formulas presented in Musiela and Rutkowski (1997) to the perturbed model. Both option pricing formulas are formulated in terms of the zero-coupon bond price volatilities, denoted by $S(t, T)$.

It can be shown, see for example Bjork (1998), that if forward rates are specified by (3.18) then the zero-coupon bond dynamics (under the martingale measure) are given by

$$dp(t,T) = p(t,T)\left\{r(t)dt + S(t,T)dW(t)\right\}, \tag{4.57}$$

where

$$S(t,T) = -\int_t^T \sigma(t,s)ds. \tag{4.58}$$

From (4.28), it follows using similar arguments to Section 4.1, that for the perturbed model the diffusion coefficient of the forward rate process is given by

$$\sigma(t,T) = \tilde{C}(t,T)\tilde{B}. \tag{4.59}$$

Therefore from (4.58) the zero-coupon bond price volatilities under the perturbed model are given by

$$S(t,T) = -\int_t^T \tilde{C}(t,s)\tilde{B}ds. \tag{4.60}$$

The value of a European option on a zero-coupon bond is given by Proposition 4.3 below.

Proposition 4.3 Options on zero-coupon bonds *The price at time $t \in [0,T]$ of a European call option with expiry T and strike price K on a zero-coupon bond which matures at time $U \geq T$ is given by*

$$C_t = p(t,U)N(h_1(p(t,U),t,T)) - Kp(t,T)N(h_2(p(t,U),t,T)). \tag{4.61}$$

where $N(\cdot)$ is the cumulative probability distribution for the standard normal distribution,

$$h_{1,2}(\cdot,t,T) = \frac{\ln(\cdot/K) - \ln p(t,T) \pm \frac{1}{2}v_U^2(t,T)}{v_U(t,T)} \tag{4.62}$$

and

$$v_U^2(t,T) = \int_t^T |S(s,U) - S(s,T)|^2\, ds. \tag{4.63}$$

□

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The value of a European call option on a coupon bearing bond with cash flows c_j occurring at times T_j for $j = 1, \ldots, m$ is given by Proposition 4.4 below.

Proposition 4.4 Options on coupon bonds *The price at time $t \leq T$ of a European call option with expiry $T \leq T_1$ and strike price K on a coupon bearing bond is given by*[13]

$$C_t = \sum_{j=1}^{m} p\left(t, T_j\right) J_1^j - Kp(t, T) J_2, \tag{4.64}$$

where

$$J_1^j = \mathbb{P}\left\{\sum_{\ell=1}^{m} c_\ell p\left(t, T_\ell\right) e^{\zeta_\ell + \upsilon_{\ell j} - \frac{1}{2}\upsilon_{\ell\ell}} > Kp(t, T)\right\} \tag{4.65}$$

for $j = 1, \ldots, m$,

$$J_2 = \mathbb{P}\left\{\sum_{\ell=1}^{m} c_\ell p\left(t, T_\ell\right) e^{\zeta_\ell - \frac{1}{2}\upsilon_{\ell\ell}} > Kp(t, T)\right\} \tag{4.66}$$

where $(\zeta_1, \ldots, \zeta_m)$ is a random variable that has a Gaussian distribution under $\mathbb{P}$ with mean zero and covariance given by

$$cov(\zeta_\ell, \zeta_m) = \upsilon_{\ell j} = \int_t^T \gamma\left(s, T_\ell, T\right) \cdot \gamma\left(s, T_j, T\right) ds \tag{4.67}$$

for $j, \ell = 1, \ldots, m$ where $\gamma\left(s, T_\ell, T\right) = S\left(s, T_\ell\right) - S\left(s, T\right)$. □

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[13] Any coupons that will be paid during the life of the option are known at the trade date and therefore will not affect the option price. Therefore in the case where the underlying bond pays coupons before T, these coupons can be excluded. This is consistent with Black's model which is formulated in terms of the forward bond price and therefore excludes any coupons paid during the life of the option. (The formula for Black's model is given in Appendix C.) The probabilities in (4.65) and (4.66) have to be estimated numerically – by vectorising the code, a Monte Carlo type simulation can be done relatively efficiently in MATLAB using the functions *mvnrnd* or *lhsnorm*.

Chapter 5

Implementation in the South African Market

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This chapter discusses the implementation of the underlying and perturbed models in the local market. In particular we discuss yield curve construction, calibration of the underlying model using PCA, calibration of the perturbed model to a benchmark option price and two numerical techniques for integrating the DRE (4.54).

In general, a model can be calibrated in two broad ways, either by using some type of statistical technique to calibrate the model parameters to historical data or by implying the models parameters from the current market prices of liquidly traded derivatives. The simplest analogy for the difference between the two approaches is the difference between using historical or implied volatilities for Black-Scholes pricing.

Given the functional form of the underlying volatilities specified by (4.10), the function parameters can either be estimated from a time series of historical forward rates or calibrated to observed market volatilities. As a result of the lack of liquidity in the option market, South African implied volatility data is either unavailable or unreliable. We therefore choose to estimate the historical volatilities and use these volatilities to determine the parameters A, B and C. Although this means that the option prices implied by the underlying model are not necessarily going to correspond with the observed market prices, the calibration of the underlying model is intended to capture the general form of the bond market volatility. This does not seem unreasonable since we will show that the perturbed model can be fitted to one option price. As mentioned in Section 2.3 the South African bond option market essentially derives prices from one liquid option price.

Since the yield curves form an input into the calibration process we will discuss their construction first. Various curves are needed to implement the two models: the underlying model is calibrated using a PCA of historical forward curves, an initial forward curve is required for both models and daily zero curves are required for updating the perturbed model. We will use 'best decency' type curves to calibrate the underlying model. We are thus assuming that 'best decency' curves are the best representation of the 'true' yield curve dynamics and that the 'perfect fit' curves are obtained after an error adjustment.

The initial forward curve and daily zero curves need to be 'perfect fit' curves. Since it makes more sense to discuss the construction of 'perfect fit' curves before discussing the construction of 'best decency' curves we will discuss the construction of 'perfect fit' curves first and then show how a constraint can be relaxed to obtain 'best decency' curves. The details of the calibration of the underlying and perturbed models are then discussed in Sections 5.2 and 5.3 respectively.

5.1 Yield curve construction

Forward and zero curves are needed but market information is obtained in the form of coupon bearing government bonds. The relevant curves therefore have to be constructed from the yields of these bonds. Although the zero curves published by BESA are derived by constructing instantaneous forward curves (Etheredge 2003), these forward curves are not published. One possibility for obtaining forward curves would be to use some sort of interpolation technique to work back to instantaneous forward curves from the published discrete zero curves, but this is not ideal since there is a loss of information due to the repeated interpolation. Also, since for the perturbed model, we want curves constructed only from the most liquid bonds, we may not necessarily want the same input set used by BESA. We therefore construct our own curves. We begin with a discussion of the selection of input instruments.

5.1.1 Instrument selection

To determine the liquid bonds we use the same liquidity measure used in the BEASSA Total Return Indices selection process, namely the twelve month average value traded, see BESA, ASSA (2000).[1] Figure 5.1 below shows twelve month average value traded for the government bonds that satisfy the Index selection requirements.[2]

[1] The trades included in the averaging process are standard turnover and option exercises. Both legs of repurchase (repo) trades are excluded. The value traded is defined to be the clean consideration of each trade. For bonds that have been in issue for less than a year, the averaging is done over the months that the bond has been in issue. The initial issuance is not included in the calculation, see BESA, ASSA (2000). (Data: 'Detailed turnover' spreadsheets published by the Bond Exchange of South Africa, available from www.bondexchange.co.za)

[2] To be eligible for selection a bond must pay fixed semi-annual coupons and is also required to have a term to maturity of greater than 1 year for the entire period that the selection is being made. The unlabeled black lines show the other government bonds that are or were in issue namely the R201, R124, R126, R133, R151, R152, R177, R184, R203, R204, R206 and R207.

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Figure 5.1: Average monthly turnover of South African government bonds

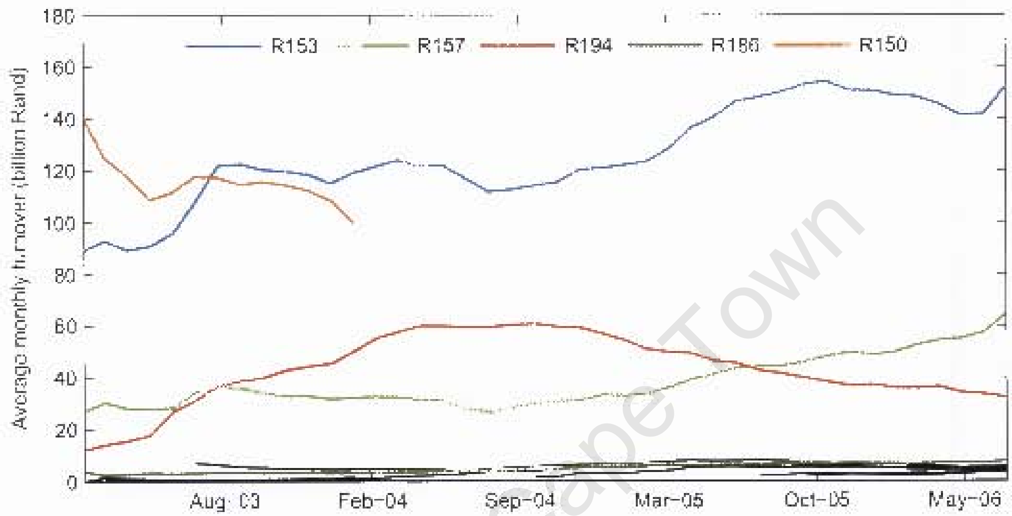

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Figure 5.1 illustrates the concentration of trade in a few liquid bonds. We used the R153, R157, R194 and R186 as our current input bonds. For the construction of the historical curves for the PCA we also included the R150 up until January 2004.[3] The R186 was the most liquid long dated bond and was included to anchor the long end of the curve. To ensure that the short end of the curve was well defined we used the 30 and 60 day treasury bills as additional input instruments.

We will later perform a PCA on the shifts in the historical forward curves, see Section 5.2. Since the set of bonds which from which the historical forward curves are constructed changes over time, we minimise the effect of changing the input instruments by constructing two curves for dates where this occurs. The shift from the previous date to the date where the input set changes is calculated using a curve constructed from the old set of input bonds. The shift from the date where the input set changes to the following date is calculated using a curve constructed from the new set of input bonds.

[3]The details of the R150 are given below, the details of the other bonds were given in Table 2.1.

Bond code	Issue date	Coupon rate	Maturity date	Coupon dates	Book close dates
R150*	2 May 1989	12.00%	28 Feb 2004/05/06	28 Feb / 31 Aug	31 Jan / 31 Jul

* The R150 was a 'triple-redemption' bond

Our curve construction methodology is similar to Etheredge (2003). Like Etheredge the construction of both the 'perfect fit' and 'best decency' type curves is done in two steps. Firstly quadratic forwards are used to generate a discrete approximation to the forward curve at the points corresponding to the cash flows of the input instruments. An interpolation technique is then used to generate a continuous instantaneous forward curve and corresponding zero curve.

5.1.2 Constructing 'perfect fit' curves

We require that the initial forward curve and the zero curves used to update the perturbed model price the input bonds exactly. The instantaneous forward curve, $f(t,T)$, is said to price a bond exactly if the present values of the bond's individual cash flows, discounted at the zero rates determined by $f(t,T)$, sum to the bond's market price.

In the South African market bonds trade on yield-to-maturity not on price. It is however possible to associate a unique market price with a given yield using the BESA bond pricing formula, see BESA (2005). We denote the market (all-in) price of a bond trading at a yield-to-maturity of y at time t by $\mathcal{A}(t,y)$.[4]

Assuming that a bond with market price $\mathcal{A}(t,y)$ has cash flows $c_1,\ldots,c_k$ occurring at times $T_1,\ldots,T_k$. The forward curve, $f(t,T)$, is said to price the bond exactly if

$$\mathcal{A}(t,y) = \sum_{j=1}^{k} c_j e^{-\int_t^{T_j} f(t,s)ds}. \tag{5.1}$$

The above 'perfect fit' condition (5.1) can be rewritten in terms of discount factors. Recalling that given an instantaneous forward curve $f(t,T)$, the corresponding discount factors $\mathcal{D}(t,T)$ are defined by

$$\mathcal{D}(t,T) = p(t,T) = e^{-\int_t^T f(t,s)ds}, \tag{5.2}$$

equation (5.1) can be written as

$$\mathcal{A}(t,y) = \sum_{j=1}^{k} c_j \mathcal{D}(t,T_j). \tag{5.3}$$

Our first aim is to estimate $\mathcal{D}(t,T)$ at discrete points corresponding to the cash flows of the input instruments.

[4]The R153, R157, R194, R186 and R150 pay coupons semi-annually. Their yields are therefore quoted as semi-annual rates. The three and six month treasury bill rates are discount rates. The time t corresponds to a settlement date. In the South African market standard settlement is 't+3', that is, trades settle three business days after they were traded.

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Let $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_p)$ denote the set of instruments used as inputs for the curve construction. Define the time vector $T = [t, T_1, \ldots, T_N]$, where t denotes the time corresponding to the forward curve date and $T_1 < T_2 < \ldots < T_N$ is an ordered list of terms corresponding to the cash flows of the full set of input instruments. Also define the $p \times (N+1)$ cash flow matrix $\mathcal{C}$ as follows:

$$\mathcal{C}_{i(j+1)} = \begin{cases} -\mathcal{A}_i(t, y) & \text{for } j = 0 \text{ and } i = 1, \ldots, p \\ c_{ij} & \text{for } j = 1, \ldots, N \text{ and } i = 1, \ldots, p. \end{cases} \tag{5.4}$$

where $\mathcal{A}_i(t, y)$ is the market price of instrument $\mathcal{G}_i$ at time t and c_{ij} is the cash flow corresponding to $\mathcal{G}_i$ that occurs at time T_j. (If $\mathcal{G}_i$ does not have a cash flow corresponding to time T_j then $c_{ij} = 0$.)

The first step of the curve construction procedure is to use an optimisation technique to search for the vector of discount factors that corresponds to the smoothest quadratic approximation to the forward curve, subject to the constraint that the input instruments are all priced exactly.

If we denote the vector that we are searching for by $\mathcal{D}$, where $\mathcal{D} = [\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_N]$, where $\mathcal{D}_0 = \mathcal{D}(t, t)$ and $\mathcal{D}_j = \mathcal{D}(t, T_j)$ for $j = 1, \ldots, N$, then the requirement that all the input instruments are priced exactly can be expressed in matrix notation by

$$\mathcal{C}\mathcal{D} = 0. \tag{5.5}$$

In addition to the above 'perfect fit' condition, for $\mathcal{D}$ to be a valid discount vector it must also satisfy the following constraints:

- $\mathcal{D}_0 = 1$.
- $\mathcal{D}_j > 0$ for $j = 0, \ldots, N$.
- $\mathcal{D}$ is strictly decreasing, that is $\mathcal{D}_j > \mathcal{D}_{j-1}$ for $j = 1, \ldots, N$. (5.6)

Quadratic approximation

For a particular discount vector $\mathcal{D} = [\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_N]$, the corresponding quadratic approximation to the instantaneous forward curve is found by fitting parabolas to the zero rates implied by consecutive sets of three discount factors and differentiating the fitted curve at the middle point.[5] (See Etheredge (2003) for details of the calculations.)

[5] The relationship between the discount factors and zero-coupon rates, Y(t,T), is given by

$$\mathcal{D}(t, T) = e^{-Y(t,T)(T-t)}.$$

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Using this procedure each discount vector determines a vector of quadratic forward rates. We denote this vector by $f^q = [f_1^q, \ldots, f_N^q]$ where f_j^q is the quadratic approximation to the instantaneous forward rate $f(t, T_j)$.

The aim of the optimisation is to find the discount vector that corresponds to the smoothest forward curve. In a similar way to Etheredge (2003) we use two measures of smoothness. The global smoothness measure defined by

$$S_1 = \sum_{j=2}^{N} \left(f_j^q - f_{j-1}^q \right)^2. \tag{5.7}$$

measures the smoothness of the general shape of the curve. The local smoothness measure defined by

$$S_2 = \sum_{j=3}^{N} \left(\frac{f_j^q - f_{j-1}^q}{T_j - T_{j-1}} - \frac{f_{j-1}^q - f_{j-2}^q}{T_{j-1} - T_{j-2}} \right)^2, \tag{5.8}$$

measures the change in the slope of the curve between consecutive points. We use the MATLAB function for constrained nonlinear optimisation problems, *fmincon*, to find the discount vector that minimises

$$S = S_1 + S_2, \tag{5.9}$$

subject to the 'perfect fit' and discount factor validity constraints discussed above. The discount vector corresponding to a bootstrapped zero curve is used as the starting point for the optimisation algorithm.

The minimalist interpolator

For the discount vector found by the optimisation the corresponding discrete forward rates $F(t, T_j - 1, T_j)$ are given by[6]

$$F(t, T_j - 1, T_j) = \frac{\ln \mathcal{D}(t, T_j) - \ln \mathcal{D}(t, T_{j-1})}{T_j - T_{j-1}}. \tag{5.10}$$

Since we need forward rates for all maturities $T - t$ we need to interpolate to construct the

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Equating the above equation with (5.2) gives

$$Y(t,T)(T-t) = \int_t^T f(t,s)ds.$$

Differentiating with respect to T gives

$$f(t,T) = Y(t,T) + (T-t)\frac{\partial Y(t,T)}{\partial T},$$

from which it follows that an approximation to the instantaneous forward curve can be found by approximating $\frac{\partial Y(t,T)}{\partial T}$.

[6]Recall that $f(t; T_1, T_2)$ is the continuously compounded forward rate at time t for the period T_1 to T_2.

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instantaneous forward curve from the discrete forwards. A comprehensive review of interpolation methods used for curve construction is presented in Hagan and West (2005). In this paper, Hagan and West note that one of the major problems with many common interpolation schemes is that they implicitly assume that the discrete forwards, $F(t, T_j - 1, T_j)$, apply only at T_j, the right end-point of the interval $[T_{j-1}, T_j]$ and not over the entire interval. The minimalist interpolator is one of two new methods introduced in this paper that correctly interpret the discrete forwards as being properties of the whole interval.

Hagan and West's minimalist interpolator models the instantaneous forward curve by the quadratic

$$f(\tau) = a_j + b_j (\tau - \tau_{j-1}) + c_j (\tau - \tau_{j-1})^2 \text{ for } \tau_{j-1} \leq \tau \leq \tau_j \tag{5.11}$$

for $j = 1, \ldots, N$, where $\tau = T - t$, $\tau_j = T_j - t$ and t $f(\tau) = f(t, T)$.

The interpolated curve must be consistent with the discrete forward rates, this is assured by requiring that

$$\frac{1}{\tau_j - \tau_{j-1}} \int_{\tau_{j-1}}^{\tau_j} f(\tau) d\tau = F_j \text{ for } j = 1, \ldots, N, \tag{5.12}$$

where $F_j = F(t, T_j - 1, T_j)$.

Integrating (5.11) and defining $h_j = \tau_j - \tau_{j-1}$ the above requirement becomes

$$F_j = a_j + \frac{b_j}{2} h_j + \frac{c_j}{3} h_j^2 \text{ for } j = 1, \ldots, N. \tag{5.13}$$

The second constraint ensures that the curve is continuous by requiring that

$$a_{j+1} = a_j + b_j h_j + c_j h_j^2 \text{ for } j = 1, \ldots, N - 1. \tag{5.14}$$

Constraints (5.13) and (5.14) define a system of $2N - 1$ equations in $3N$ unknowns. Most interpolation techniques impose a further $N + 1$ constraints by requiring the first or second derivatives of (5.11) to be continuous. Instead for this method, Hagan and West (2005) determine the forward curve parameters by minimising a penalty function which they define to be a weighted sum of:

- the sum of squares of the changes in the first derivatives at the points $\tau_1, \ldots \tau_n - 1$

$$f'(\tau) = b_j + 2c_j(\tau - \tau_{j-1}) \text{ for } \tau_{j-1} \leq \tau \leq \tau_j \tag{5.15}$$

$$J_{1,j} = b_{j+1} - b_j - 2c_j h_j \tag{5.16}$$

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- a sum of the second derivatives of $f(\tau)$ weighted by the length of the interval $[\tau, \tau_{j-1}]$

$$f''(\tau) = 2c_i \text{ for } \tau_{j-1} \leq \tau \leq \tau_j \tag{5.17}$$

$$J_{2,j} = 2c_j h_j \tag{5.18}$$

For $\omega \in (0, 1)$ the penalty function is given by[7]

$$P_\omega = \omega \sum_{j=1}^{n-1} J_{1,j}^2 + (1 - \omega) \sum_{j=1}^{n} J_{2,j}^2. \tag{5.19}$$

Hagan and West (2005) show that the above constrained minimisation problem can be transformed into a system of linear equations using Lagrange multipliers from which the coefficients a_j, b_j, c_j, $j = 1, \ldots, N$ can be determined using Gaussian elimination.

Recalling the zero-coupon rate $Y(t,T) = \frac{1}{T-t}\int_t^T f(t,s)ds$, the zero curve, $Y(\tau) = Y(t,T)$ corresponding to $f(\tau)$ is given by

$$\begin{aligned} Y(\tau) &= \frac{1}{\tau}\int_0^\tau f(s)ds \\ &= \frac{1}{\tau}\left[\sum_{k=1}^{j-1}\left(a_k h_k + \frac{b_k}{2}h_k^2 + \frac{c_k}{3}h_k^3\right) + a_j(\tau - \tau_j) + \frac{b_j}{2}(\tau - \tau_j)^2 + \frac{c_j}{3}(\tau - \tau_j)^3\right]. \end{aligned} \tag{5.20}$$

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The zero curves generated by the above methodology are similar to the 'perfect fit' curves published by BESA. Figure 5.2 below shows the forward curve and corresponding zero curve constructed for the 30 June 2006.[8] The forward curve is shown in black and the corresponding zero curve is shown in blue. The BESA 'perfect fit' zero curve is shown in green.

'Perfect fit' curves constructed in this way are used for the initial forward curves for both models and the zero curves used to update the perturbed model.

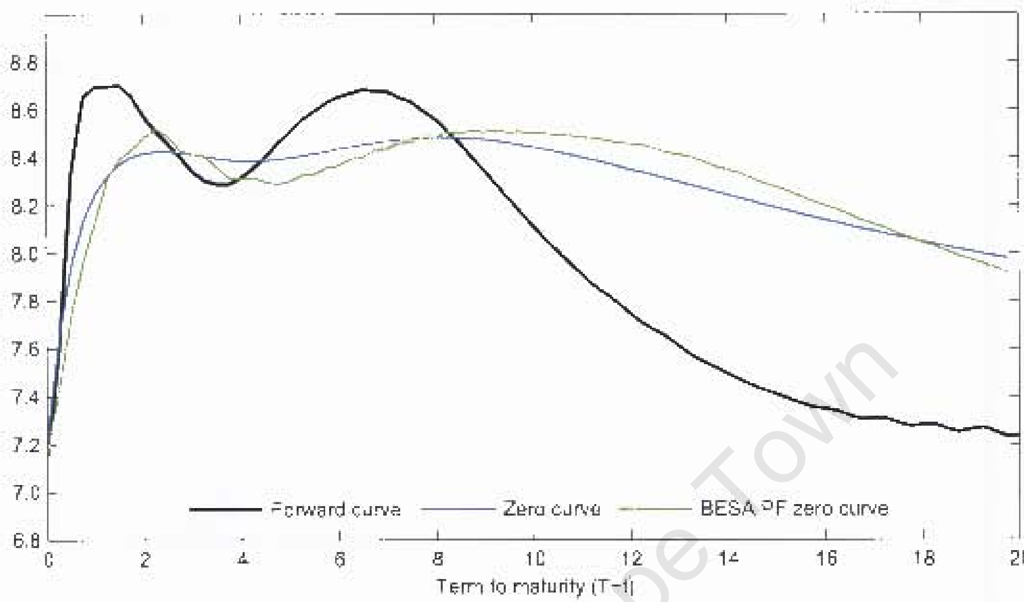
5.1.3 Constructing curves for the PCA

The underlying model is calibrated to observed historical volatility. This is achieved by estimating the matrices A, B and C from (4.10). The three factors are estimated via a PCA of historical forward curves. The curves used for the PCA are 'best decency' type curves. Since the parameterised volatility functions will in any case only approximate the volatility functions implied by

[7]The results are sensitive to the choice of ω, we use $\omega = 0.8$ as suggested by Hagan and West (2005).

[8]The inputs instruments for the curve construction were the R194, R153, R157, R186 and the 30 and 60 day treasury bills.

Figure 5.2: Forward and zero curves (30 June 2006)



the PCA, it is the general shape of the components that is the most important. The optimisation is more successful if the components are smooth. Using forward curves that sacrifice a small degree of pricing accuracy in return for increased smoothness, in particular curves that are locally smooth in the long end, results in principal components that are significantly more stable.

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The curve construction process is identical to the above except that, to generate 'best decency' curves, the perfect fit condition (5.3) is replaced by

$$\mathcal{A}(t, y+\epsilon) \leq \sum_{j=1}^{k} c_j \mathcal{D}(t, T_j) \leq \mathcal{A}(t, y-\epsilon). \tag{5.21}$$

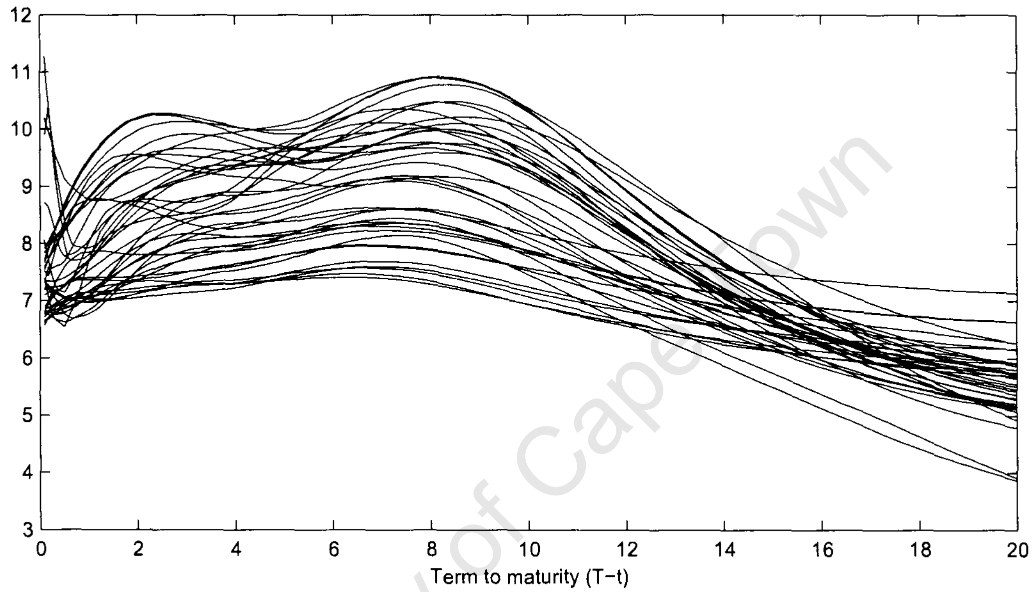
This relaxes the 'perfect fit' constraint and instead requires the curve to price the curve within a specified tolerance of the market yield-to-maturity.[9] These curves are similar to the BEASSA 'best decency' curves except that Etheredge (2003) varies the trade off between smoothness and pricing depending on the input data whereas we define a maximum pricing error and find the smoothest curve satisfying this constraint. One could apply our calibration process to 'best decency' curves as defined by Etheredge (2003).

[9] The constraint (5.5) is replaced by

$$\begin{bmatrix} \mathcal{A}(t, y_1+\epsilon) - \mathcal{A}(t, y_1) \\ \vdots \\ \mathcal{A}(t, y_p+\epsilon) - \mathcal{A}(t, y_p) \end{bmatrix} \leq CD \leq \begin{bmatrix} \mathcal{A}(t, y_1-\epsilon) - \mathcal{A}(t, y_1) \\ \vdots \\ \mathcal{A}(t, y_p-\epsilon) - \mathcal{A}(t, y_p) \end{bmatrix}.$$

For the input instruments discussed Section 5.1.1, allowing a mis-pricing of 2 basis points, that is $\epsilon = 0.0002$, results in forward curves that yield smooth principal components. Figure 5.3 shows the monthly forward curves for the period 30 June 2003 to 30 June 2006 that were used to generate the components shown in Figure 3.1.

Figure 5.3: Monthly forward curves (30 June 2003 - 30 June 2006)



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5.2 Calibrating the underlying model

We have shown that choosing the underlying model parameters A, B and C to be constant matrices results in stationary volatility. We now calibrate the underling model by estimating A, B, and C.

Given a history of forward rate curves it is possible to estimate the volatility functions $\sigma_i(T-t)$, $i = 1, 2, 3$, using principal component analysis. Assuming that we have a history of $m+1$ instantaneous forward rate curves for dates t, $t+\Delta$, $\ldots$, $t+m\Delta$, we can approximate the curves by the vectors

$$\begin{bmatrix} f(t,t) \\ f(t,t+\Delta) \\ \vdots \\ f(t,t+p\Delta) \end{bmatrix}, \begin{bmatrix} f(t+\Delta,t+\Delta) \\ f(t+\Delta,t+2\Delta) \\ \vdots \\ f(t+\Delta,t+(1+p)\Delta) \end{bmatrix}, \ldots, \begin{bmatrix} f(t+m\Delta,t+m\Delta) \\ f(t+m\Delta,t+(m+1)\Delta) \\ \vdots \\ f(t+m\Delta,t+(m+p)\Delta) \end{bmatrix}. \quad (5.22)$$

We used $\Delta = 1$ month, thus our curves were observed monthly and each curve had monthly points.

The volatility functions can be estimated from a PCA of the historical forward curve shifts defined by,

$$\begin{bmatrix} [f(t+(j+1)\Delta, t+(j+1)\Delta) - f(t+j\Delta, t+j\Delta)]/\Delta \\ [f(t+(j+1)\Delta, t+(j+2)\Delta) - f(t+j\Delta, t+(j+1)\Delta)]/\Delta \\ \vdots \\ [f(t+(j+1)\Delta, t+(j+p+1)\Delta) - f(t+j\Delta, t+(j+p)\Delta)]/\Delta \end{bmatrix} \tag{5.23}$$

for $j = 0, 1, \ldots, m-1$. Specifically the estimate for $\sigma_i(T-t)$ is given by

$$\begin{bmatrix} \sigma_i(\Delta) \\ \sigma_i(2\Delta) \\ \vdots \\ \sigma_i(p\Delta) \end{bmatrix} = \sqrt{\lambda_i} e_i, \tag{5.24}$$

where λ_i and e_i are the ith eigenvalue and eigenvector of the PCA respectively. (See for example Wilmott 2000.) The PCA can be performed in MATLAB using the function *princomp*.

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To specify the parameters, A, B and C we then use (4.10) and (5.24) and find matrices that minimise

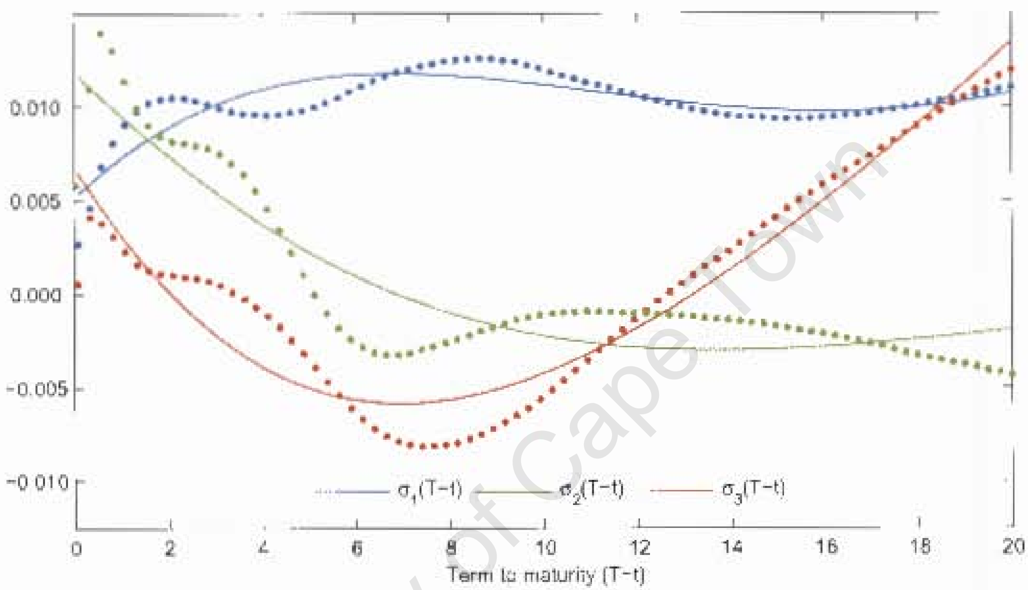
$$\sum_{j=1}^{p} \left\| \begin{bmatrix} \sigma_1(j\Delta) & \sigma_2(j\Delta) & \sigma_3(j\Delta) \end{bmatrix} - Ce^{Aj\Delta}B \right\|. \tag{5.25}$$

For the estimates of the volatility functions, as given by (5.24), from a PCA of the forward curves shown in Figure 3.1, the matrices that minimise (5.25) are:

$$\begin{aligned} A &= \begin{bmatrix} 0.1237 & -0.2435 & -0.2249 \\ -0.0184 & 0.0090 & -0.0601 \\ 0.2205 & -0.1526 & -0.2281 \end{bmatrix} \\ B &= \begin{bmatrix} 0.1079 & -0.7587 & -0.1190 \\ 0.3035 & 0.1307 & -0.0568 \\ -0.0399 & -0.5110 & 0.0657 \end{bmatrix} \\ C &= \begin{bmatrix} -0.0694 & 0.0554 & 0.1037 \end{bmatrix}. \end{aligned} \tag{5.26}$$

The estimates of the volatility functions from the PCA, as given by (5.24), are shown by the dotted lines in Figure 5.4 below.[10] The approximations implied by (4.10) using the matrices specified above are shown by the solid lines.

Figure 5.4: Volatility functions



Once the matrices A, B and C have been specified the shape of the curve is determined by the value of the state vector $x(t)$.[11] As discussed in Section 3.5 using a three-factor model allows for a flexible volatility structure. Specifically the model allows for shifts that result in upward sloping, inverted and humped curves. Figure 5.5 below shows some possible future curve shapes that can be produced by the underlying model calibrated as discussed above.[12]

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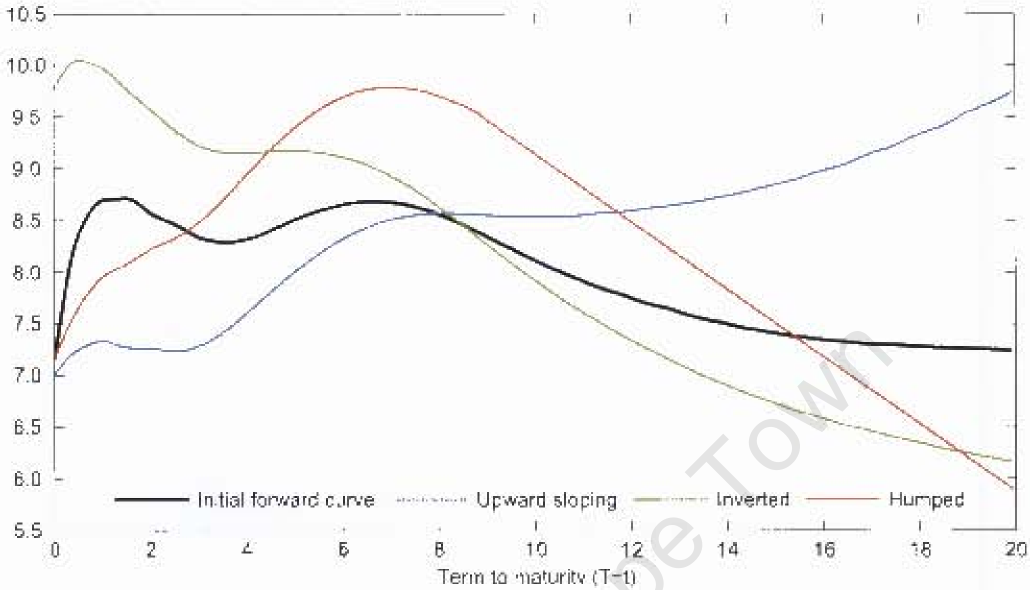
[10] Again the estimates of the volatility functions were calculated corresponding to a monthly term vector but only every third point is plotted.

[11] The value of the state vector is analogous to the principal component scores.

[12] Using the forward curve for 30 June 2006, shown in Figure 5.1.2, as the initial curve.

Figure 5.5: Possible forward curve shapes



5.3 Calibrating the perturbed model

The perturbed model is formulated in terms of a number of liquid zero-coupon bond prices, however, in the South African market the liquid bonds are all coupon bearing. To implement the perturbed model we therefore define the required zero-coupon bonds be the zero-coupon bonds corresponding to the cash flows of the most liquid coupon bearing bonds. The prices of these zero-coupon bonds are determined from a 'perfect fit' zero curve constructed from the prices of the liquid coupon bonds.

Once the underlying model has been calibrated using PCA as discussed in the previous section, the extra parameter in the perturbed model, b, can be used to calibrate the model to the market price of a benchmark option.

In the South African market bond options are traded on implied volatility, with the price being determined using Black's model.[13] Since we later compare our model prices with Black's model we give a brief overview of Black's model below.

For an option traded at time t with expiry T, the assumptions underlying Black's model are as follows:

- The distribution of underlying bond price at time T is lognormal.

[13]Black's model is used to price European options. For American options, the market uses the 'intrinsic-correction', that is the price is taken to be max(European price, intrinsic value).

- The standard deviation of the logarithm of the bond price at time T is given by $\sigma\sqrt{(T-t)}$ where σ is the volatility of the forward bond price.
- The expected value of the bond price at time T is equal to the forward price at time t. The forward bond price at time t, denoted $\mathcal{A}_F$, is given by

$$\mathcal{A}_F = \frac{\mathcal{A}(t,y) - I}{1 + \text{rfr} \times (T-t)}. \tag{5.27}$$

  where I is the present value of the coupons that will be paid during the life of the option and rfr is the (simple) risk-free rate for the period $T-t$.

The formula for Black's model is given in Appendix C.

The perturbed model is calibrated by finding the value of b for which the option price implied by Proposition 4.64 is equal to the market price of the benchmark option. Table 5.1 below shows the details of a benchmark, three month at-the-money call option on the R153 on the 30 June 2006.[14]

Table 5.1: Three month R153 at-the-money call option details

Underlying instrument:	R153
Trade date:	30 June 2006
Trade settlement date:	5 July 2006
Expiry date:	30 September 2006
Expiry settlement date:	5 October 2006
Spot yield:	8.55%
Strike yield:	8.55%
Implied volatility:	4.10%
Risk-free rate:	7.10%
Call option premium:	R 7,485 per R1 million nominal

Using the above option price to calibrate the perturbed model implies that[15]

$$b = 0.0271. \tag{5.28}$$

[14]It is important to note that South African bond options are struck on yield not on price. For European options this has no effect on option prices. Smit (2000) discusses the effect of the strike convention on American options, in particular she shows that the effect is dependent on term of the option and the shape of the underlying yield curve.

[15]Using A, B and C as given by (5.26) as the underlying model parameters and the initial forward curve shown in Figure 5.2.

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5.4 Numerical integration of the differential Riccati equation

Once we have found estimates for the parameters of the volatility functions $\sigma_1(T-t)$, $\sigma_2(T-t)$ and $\sigma_3(T-t)$ pricing essentially reduces to the calculation of $\bar{P}(t)$, the solution to the DRE (4.54). Below we consider two possible methods of direct numerical integration of the DRE.[16]

5.4.1 Vectorising the DRE

The first possibility is to vectorise the DRE and integrate the resulting system of ordinary differential equations (ODEs), see Dieci (1992). Specifically, if we denote the element in the ith row and jth column of $\bar{P}$ by $\hat{P}_{ij}$ for $i, j = 1, \ldots, n+N$ then the DRE, (4.54), can be rewritten componentwise as the $(n+N)^2$-vector differential equation

$$\frac{d\bar{p}}{dt} = f(t, \bar{p}) \tag{5.29}$$

where we have set

$$\bar{p} = \left[\bar{P}_{11}, \bar{P}_{21}, \ldots, \bar{P}_{(n+N)1}, \ldots, \bar{P}_{(n+N)(n+N)}\right]'. \tag{5.30}$$

The resulting system of ODEs can then be solved using any numerical integration scheme. The explicit Runge-Kutta schemes are the usual choice. (Dieci 1992) We use the MATLAB function *ode45* which implements the explicit Runge-Kutta(4,5) formula described in Dormand and Prince (1980).

5.4.2 Matrix formulation of the backward differentiation formulas

For large-scale problems Dieci (1992) argues that implicit numerical integration methods should be used. In particular he shows that the matrix formulation of the backward differentiation formulas can be used to take advantage of the structure of the DRE .[17] In particular the DRE reduces to an algebraic Riccati equation (ARE) after discretisation:

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[16]There are several other possible methods, see for example Benner and Mena (2004).

[17] Dieci (1992) considers DREs of the form

$$\begin{aligned} &\tfrac{d\bar{P}(t)}{dt} = A_{21}(t) + A_{22}(t)\bar{P}(t) - \bar{P}(t)A_{11}(t) - \bar{P}(t)A_{12}(t)\bar{P}(t) \\ &X(0) = 0 \end{aligned} \tag{5.31}$$

where A_{11}, A_{12}, A_{21}, A_{22} are matrix valued functions. We can write (4.54) in this form by setting

$$\begin{aligned} A_{11}(t) &= C_e'\left(D(t)D'(t)\right)^{-1}V(t)\tilde{B}' \\ A_{12}(t) &= C_e'(t)\left(D(t)D'(t)\right)^{-1}C_e(t) \\ A_{21}(t) &= \tilde{B}\tilde{B}' + \tilde{B}V(t)'\left(D(t)D'(t)\right)^{-1}V(t)\tilde{B}' \\ A_{22}(t) &= -A_{11}'(t). \end{aligned}$$

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Define

$$
\begin{array}{l} f\left(t, \bar{P}(t)\right) = \frac{d\bar{P}(t)}{dt} \\ \bar{P}(0) = 0, \end{array} \tag{5.32}
$$

where $\frac{d\bar{P}(t)}{dt}$ is given by the differential Riccati equation (DRE) (4.54).

The matrix formulation of the BDF applied to (5.32) is (Dieci 1992)

$$
\bar{P}\left(t_{k+1}\right) = \sum_{j=0}^{p-1} \alpha_j \bar{P}\left(t_{k-j}\right) + h\beta f\left(t_{k+1}, \bar{P}\left(t_{k+1}\right)\right). \tag{5.33}
$$

where h is the step size and $1 \leq p \leq 5$ is the order of the formula. The coefficients α_j and β for the various order methods are given in Table 5.2.

Table 5.2: Coefficients for the p-step BDF methods

p	β	α_0	α_1	α_2	α_3	α_4
1	1	1				
2	$\frac{2}{3}$	$\frac{4}{3}$	$-\frac{1}{3}$			
3	$\frac{6}{11}$	$\frac{18}{11}$	$-\frac{9}{11}$	$\frac{2}{11}$		
4	$\frac{12}{25}$	$\frac{48}{25}$	$-\frac{36}{25}$	$\frac{16}{25}$	$-\frac{3}{25}$	
5	$\frac{60}{137}$	$\frac{300}{137}$	$-\frac{300}{137}$	$\frac{200}{137}$	$-\frac{75}{137}$	$\frac{12}{137}$

Expanding (5.33) gives

$$
\begin{aligned} \bar{P}\left(t_{k+1}\right) &= \sum_{j=0}^{p-1} \alpha_j \bar{P}\left(t_{k-j}\right) + h\beta f\left(t_{k+1}, \bar{P}\left(t_{k+1}\right)\right) \\ &= \sum_{j=0}^{p-1} \alpha_j \bar{P}\left(t_{k-j}\right) + h\beta \Big\{ \tilde{A}\bar{P}(t_{k+1}) + \bar{P}(t_{k+1})\tilde{A} - \left[\tilde{B}V'(t_{k+1}) + \bar{P}(t_{k+1})C_e'(t_{k+1})\right] \times \\ &\quad \left(D(t_{k+1})D'(t_{k+1})\right)^{-1} \left[\tilde{B}V(t_{k+1})' + \bar{P}(t_{k+1})C_e'(t)\right]' + \tilde{B}\tilde{B}' \Big\}. \end{aligned} \tag{5.34}
$$

Rearranging terms, noting that $-\bar{P}\left(t_{k+1}\right) = -\frac{1}{2}I\bar{P}\left(t_{k+1}\right) - \bar{P}\left(t_{k+1}\right)\frac{1}{2}I'$, gives the ARE

$$
\left(h\beta\tilde{B}\tilde{B}' + \sum_{j=0}^{p-1} \alpha_j \bar{P}\left(t_{k-j}\right)\right) + \left(h\beta\tilde{A} - \frac{1}{2}I\right)\bar{P}\left(t_{k+1}\right) + \bar{P}\left(t_{k+1}\right)\left(h\beta\tilde{A} - \frac{1}{2}I\right)' -
$$

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$$
\begin{aligned}
&\left[\tilde{B}V'\left(t_{k+1}\right)+\bar{P}\left(t_{k+1}\right)C_e'\left(t_{k+1}\right)\right]h\beta\left(D\left(t_{k+1}\right)D'\left(t_{k+1}\right)\right)^{-1}\times\\
&\left[\tilde{B}V'\left(t_{k+1}\right)+\bar{P}\left(t_{k+1}\right)C_e'\left(t_{k+1}\right)\right]'=0.
\end{aligned}
\tag{5.35}
$$

The value of $\bar{P}(t_{k+1})$ is found by solving the ARE (5.35). This can be done using MATLAB's *care* function which solves AREs of the form

$$
A'XE+E'XA-\left(E'XB+S\right)R^{-1}\left(B'XE+S'\right)+Q=0. \tag{5.36}
$$

Equation (5.34) can be written in this form by setting

$$
\begin{aligned}
&A'=h\beta\tilde{A}-\frac{1}{2}I\Rightarrow A=h\beta\tilde{A}'-\frac{1}{2}I\\
&E=I\\
&B=C_e'\left(t_{k+1}\right)\\
&S=\tilde{B}V\left(t_{k+1}\right)\\
&R^{-1}=h\beta\left(D\left(t_{k+1}\right)D'\left(t_{k+1}\right)\right)^{-1}\Rightarrow R=\frac{1}{h\beta}D\left(t_{k+1}\right)D'\left(t_{k+1}\right)\\
&Q=h\beta\tilde{B}\tilde{B}'+\sum_{j=0}^{p-1}\alpha_j\bar{P}\left(t_{k-j}\right).
\end{aligned}
\tag{5.37}
$$

We implemented both of the above approaches in order to check our solutions.

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Chapter 6

Examples

This chapter focuses on two examples to demonstrate the application of the model to bond and option pricing. The first example shows bond prices derived from both the underlying and perturbed models and compares these prices with market prices. The second compares the prices implied by the perturbed model of options on various government bonds with the prices obtained using market conventions.

6.1 Example 1

In Section 4.1.1 we discussed the fact that at any point in time the underlying model can be fitted to three benchmark zero-coupon bond prices. Figure 6.1 below shows the daily market prices as well as the prices implied by the underlying model of the R153, R157 and R206 for the period 30 June 2006 to 30 September 2006 where at each date the underlying model is fitted to the 2, 5 and 10 year zero-coupon bond prices.

The market prices are calculated from the closing yields of the day using the BESA bond pricing formula.[1] The underlying model prices are calculated as follows:

The underlying model is calibrated to the market data for the 30 June 2006 as discussed in the previous chapter. Specifically A, B and C are given by (5.26) and the initial forward curve, $f^*(0,T)$, is assumed to be the 'perfect fit' curve constructed using the R194, R153, R157, R186 and the 30 and 60 day treasury bills as input instruments shown in Figure 5.2.

At each time[2] t, we use the curve construction method described in Section 5.1 to construct a zero curve using the input instruments' current market yields. Equation (4.12) is then used to calculate the current value of the underlying model state vector, $x(t)$, from the 2, 5 and 10 year

[1] Data: I-Net Bridge.

[2] For each trade date, say d, from 30 June 2006 to 30 September 2006, t is the time in years between the settlement date for d and the settlement date for 30 June 2006.

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(time weighted) yields implied by the zero curve. Given the value of $x(t)$, the forward rates for any term are determined by (4.2) and the coupon bond prices can be calculated using (3.4) and (3.7).

As expected, fitting the underlying model to the three benchmark yields gives the general direction of bond price movements but the prices of the coupon bonds implied by the underlying model are not exactly equal to the market prices.

The sharp drop in the R153 and R157 prices on 28 August and 12 September respectively are a result of the bonds going ex-coupon.[3] To eliminate this effect we can rather compare the yields implied by the underlying model price with the traded yields. The prices implied by the underlying model can be converted back to yields using the BESA bond pricing formula. These yields are shown together with the market yields in Figure 6.2 below.

The underlying model prices (or equivalently implied yields) are highly dependent on the choice of the benchmark zero-coupon bonds. As this example is purely illustrative, the 2, 5 and 10 year zero-coupon bonds were chosen arbitrarily. For this particular choice of zero-coupon bonds this choice the average errors in the R153, R157 and R206 yields were 6, 3 and 3 basis points respectively. The maximum errors were 9, 6 and 11 basis points respectively.

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[3]The settlement date for 28 August 2006 is 31 August 2006, the book close date of the R153. The settlement date for 12 September 2006 is 15 September 2006, the book close date of the R157.

Figure 6.1: R153, R157 and R206 market prices and prices implied by the underlying model

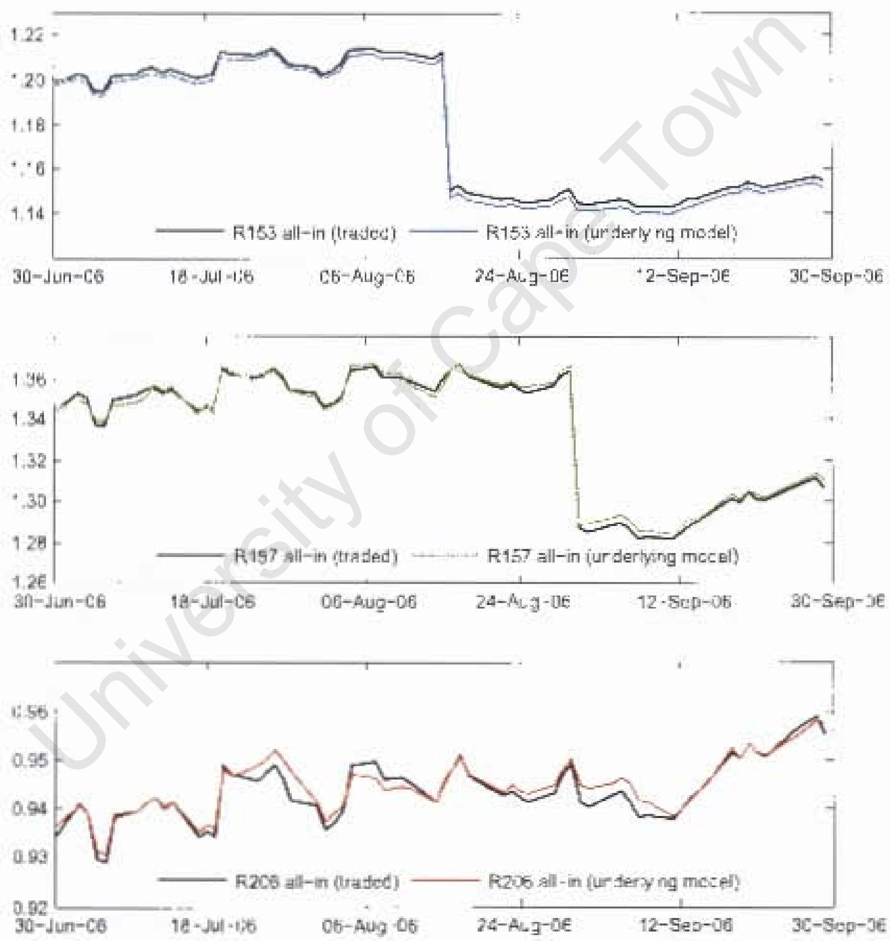
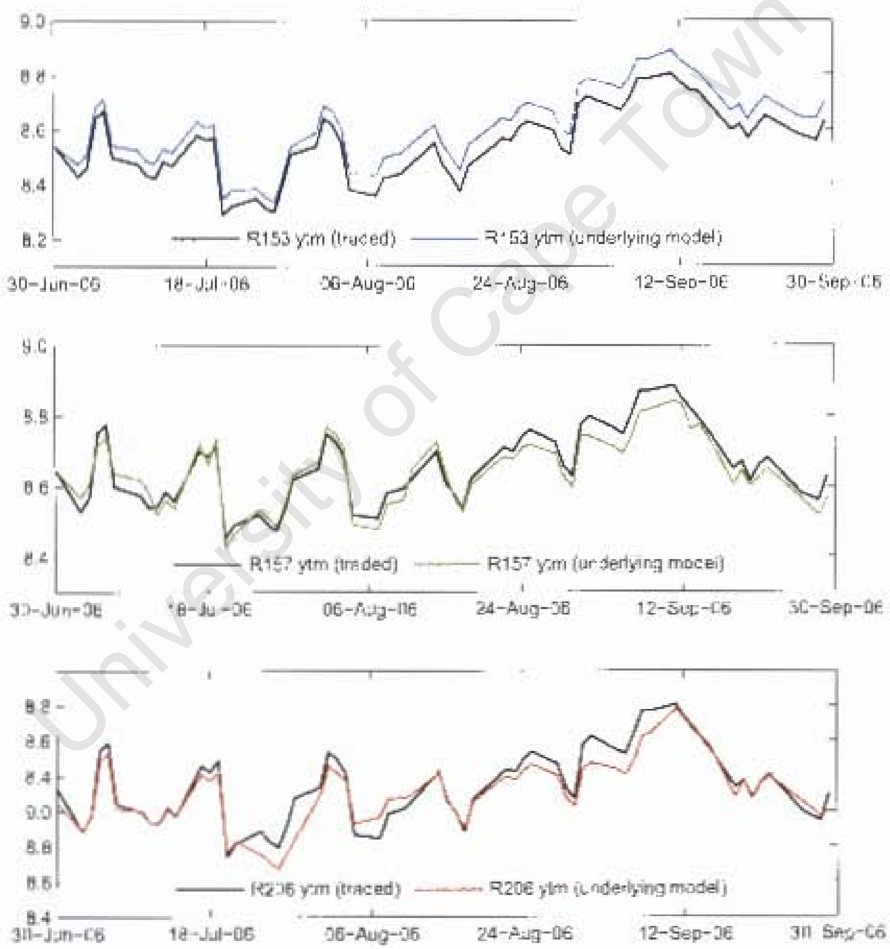


Figure 6.2: R153, R157 and R206 traded yields and yields implied by the underlying model prices



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A similar exercise using the perturbed model illustrates the advantage of using the perturbed model over the underlying model.

Figure 6.3 shows daily market prices and prices implied by the perturbed model of the R153, R157 and R206 for the period 30 June 2006 to 30 September 2006. The perturbed model prices are calculated as follows:

As for the underlying model the perturbed model is first calibrated to the market data for the 30 June 2006 as discussed in the previous section with A, B and C given by (5.26), $f^*(0,T)$ as shown in Figure 5.2 and b given by (5.28).

At each time t, the coupon bond prices are given (3.4) with the zero coupon bond prices given by the projected price formula (4.36). Essentially all that needs to be determined in order to evaluate (4.36) is the current value of $\hat{x}(t)$ since the value of $\bar{P}(t)$ and all the integrals are all deterministic. (See Appendix B for details on calculating the integral of $\tilde{G}(t,T)$.)

The current value of $\hat{x}(t)$ is approximated using Proposition 4.2. Assuming that $\hat{x}(t^-)$ denotes the previous value of $\hat{x}$ at time t^-, it follows from (4.51) that

$$\hat{x}(t) \approx x(t^-) + \tilde{A}\hat{x}(t^-)\left(t - t^-\right) + \hat{B}(t^-)\delta\hat{w}, \tag{6.1}$$

where

$$\delta\hat{w} = D(t^-)^{-1}\left[\delta z - C_e(t^-)\hat{x}(t^-)\left(t - t^-\right)\right]. \tag{6.2}$$

from (4.53) and

$$\delta\tilde{z} = \tilde{z}(t) - \tilde{z}(t^-). \tag{6.3}$$

The current value of $\tilde{z}(t)$ is calculated using (4.43) from the time weighted yields of the liquid zero-coupon bonds. These time-weighted yields are determined by a zero curve constructed using the current market yields of the input coupon bonds as discussed in Section 5.1.2.[4]

Figure 6.3 shows that the input bonds are priced correctly by the perturbed model - the prices of the R153 and R157 implied by the perturbed model are consistent with the market prices over the whole period. The prices implied by the perturbed model for bonds that were not inputs to the curve construction procedure will be arbitrage free with respect to the prices of the liquid bonds but will not necessarily correspond with market prices as can be seen from the R206 graph. The yields implied by the perturbed model are shown together with the market yields in Figure 6.4.

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[4] The R194, R153, R157 and R186.

Figure 6.3: R153, R157 and R206 market prices and prices implied by the perturbed model

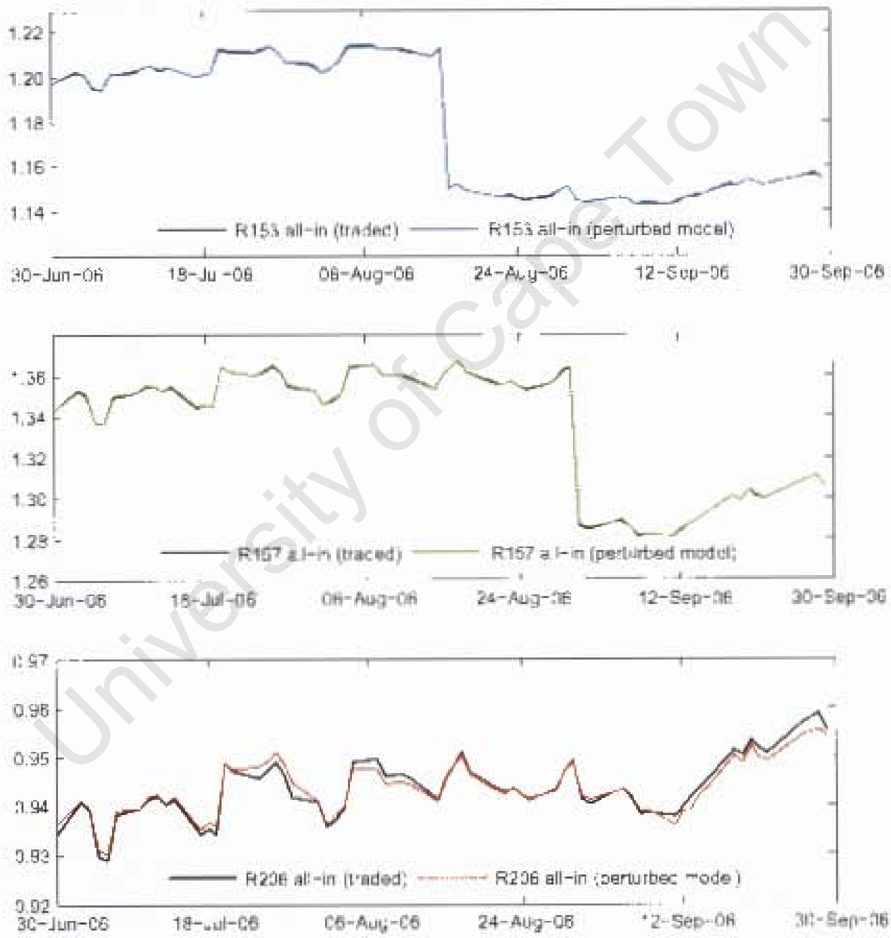

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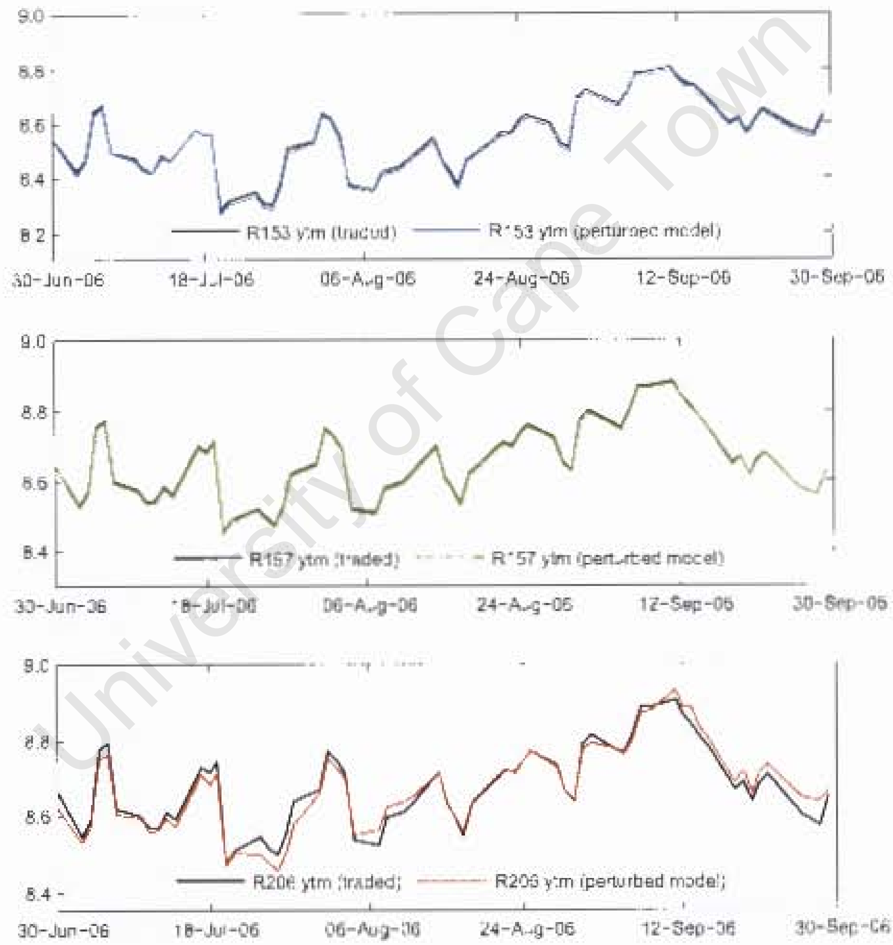
Figure 6.4: R153, R157 and R206 traded yields and yields implied by the perturbed model prices



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6.1 Example 2

We have mentioned the fact that reliable implied bond option volatilities are only available for short dated options on one or two benchmark bonds, currently the R153 or R157. To price options on other bonds, the market convention is to use a relatively naive volatility conversion formula to convert the implied volatility of the benchmark bond into implied volatilities for the other bonds.

A calibrated yield curve model can be used to calculate prices of options on any bond. This example compares the prices of options on various bonds across the curve implied by the perturbed model (that has been calibrated to the benchmark option price as discussed in Section 5.3), with prices calculated using the market volatility conversion convention.

In Section 5.3 we discussed the fact that the South African market uses Black's model to price bond options. Although the options are struck on yield, Black's model is formulated in terms of the (forward) bond price and the quoted volatilities, used as an input to the model, are price volatilities. The market convention used to determine at-the-money implied volatilities for bonds other than the benchmark bond is to assume that forward yield volatilities are constant across the curve and then to use the relationship between yield and price volatilities to convert the benchmark at-the-money implied volatility to volatilities for other bonds.[5]

The relationship between the underlying bond's forward price, $\mathcal{A}_F$, and its forward yield which we denote y_F, is given by

$$\frac{\delta \mathcal{A}_F}{\mathcal{A}_F} \approx -D\,\delta y_F, \tag{6.4}$$

or

$$\frac{\delta \mathcal{A}_F}{\mathcal{A}_F} \approx -D\,y_F \frac{\delta y_F}{y_F}, \tag{6.5}$$

where D is the modified duration of the bond at the forward yield at the option maturity date.

The terms $\frac{\delta \mathcal{A}_F}{\mathcal{A}_F}$ and $\frac{\delta y_F}{y_F}$ are percentage changes in the forward price and yield respectively. Since volatility is the measure of the standard deviation of the percentage changes in the value of a variable, Equation 6.5 implies that the relationship between the forward price volatility, σ, and forward yield volatility, σ_y, is given by[6] (Hull 2003)

$$\sigma \approx D\,y_F \sigma_y. \tag{6.6}$$

[5]For in-the-money or out-the-money options the standard market convention is to adjust the implied price volatility by 25 basis points per standard option strike where the standard strikes occur every 25 basis points.

[6]The relationship follows from the fact that if the standard deviation of the random variable X is σ_X, then the standard deviation of the random variable $Y = cX$, where c is a constant, is $\sigma_Y = |c|\sigma_X$. (Rice 1995).

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Equation (6.6) is used to convert the benchmark bond's forward price volatility to a forward yield volatility. The forward yield volatility of the non benchmark bond is assumed to be equal to the yield volatility of the benchmark bond. Equation (6.6) is then used again to convert the yield volatility back to a price volatility for the non benchmark bond, based on the non benchmark bond's modified duration.

The procedure can be summarised by the volatility conversion formula

$$\sigma = \frac{D\, y_F}{D_b\, y_{b_F}} \sigma_b, \tag{6.7}$$

where σ is the forward price volatility of a bond with modified duration, D, and forward yield y_F, and D_b, y_{b_F} and σ_b are the modified duration, forward yield and price volatility of the benchmark bond.

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Table 6.1 below shows the prices of three month, at-the-money call options on various government bonds for 30 June 2006. The implied volatilities in column 5 are calculated using (6.7) with the R153 implied volatility as the benchmark.[7] The call premiums are calculated using Black's model as discussed in Section 5.3. A risk-free rate of 7.10% was used for the calculation of the forward yields and option premiums. The premiums are quoted in Rands per R1 million nominal of the underlying bond.

Table 6.1: Market implied volatilities and at-the-money call premiums

Trade date: 30 June 2006 Trade settlement date: 5 July 2006
Expiry date: 30 September 2006 Expiry settlement date: 5 October 2006

Bond	Spot/strike yield	Strike price	Forward yield	Modified duration	Implied volatility (market conversion)	Call premium (Black's model)
R194	8.515%	1.02877	8.776%	1.28	1.70%	2,008
R153	8.550%	1.15745	8.818%	3.06	4.10%	7,485
R206	8.665%	0.95459	8.955%	5.33	7.24%	11,979
R157	8.650%	1.30513	8.937%	5.58	7.57%	17,137
R201	8.660%	1.03033	8.949%	5.61	7.62%	13,661
R203	8.650%	0.97653	8.937%	6.97	9.45%	16,423
R204	8.635%	0.97562	8.919%	7.34	9.93%	17,364
R207	8.560%	0.91322	8.829%	7.91	10.60%	17,563
R186	8.350%	1.23835	8.578%	8.96	11.66%	26,637

Figure 6.5 below shows forward price volatilities plotted against modified duration. The call

[7] The forward yields, strike prices and modified durations are calculated using the trade and maturity settlement dates.

option premiums, as a percentage of the strike prices,[5] are plotted against modified duration in Figure 6.6.

Figure 6.5: Market implied volatility conversion

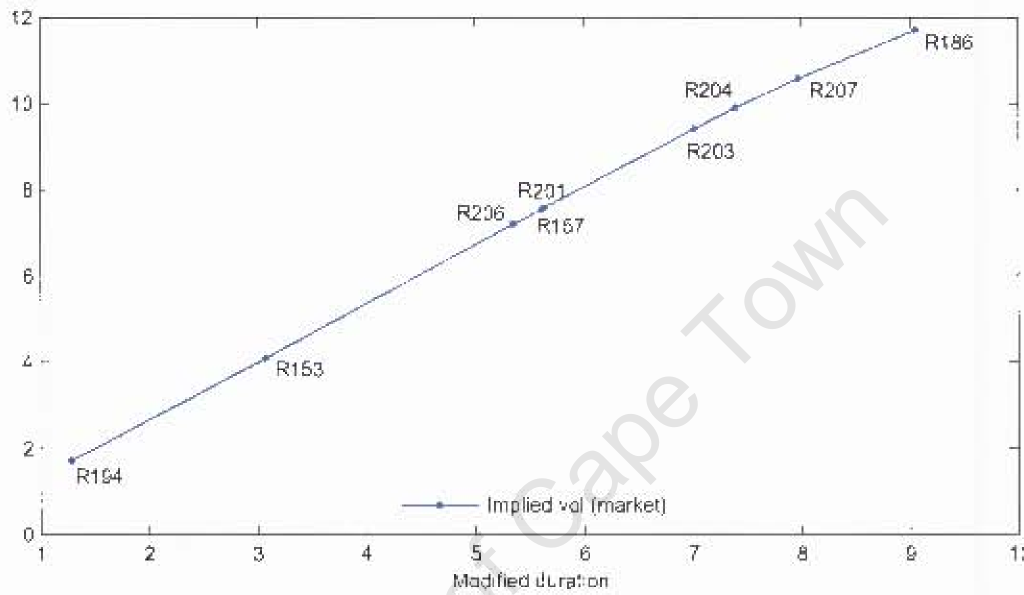

Figure 6.6 shows that volatility conversion convention used by the market implies an approximately linear relationship between the modified duration of the underlying bond and the option premium (as a percentage of price).

To compare the prices implied by the perturbed model with the prices obtained using market conventions we price the same three month call options on the various government bonds using Proposition 4.64. The resulting premiums are shown in Table 6.2 below.

It is easiest to compare the premiums graphically. Figure 6.7 below shows the option premiums implied by the perturbed model together with the premiums calculated using market conventions. Again the premiums are expressed as a percentage of price and are plotted against modified duration.

Black's formula can be used to convert the option prices implied by the perturbed model back to implied volatilities. These implied volatilities are shown together with the implied volatilities

[5] Since the option premiums are expressed in Rands per R1 million nominal of the underlying bond, the percentages are $\frac{\text{premium}/1\,000\,000}{\text{strike price}}$.

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Figure 6.6: At-the-money call premiums (Black's model)

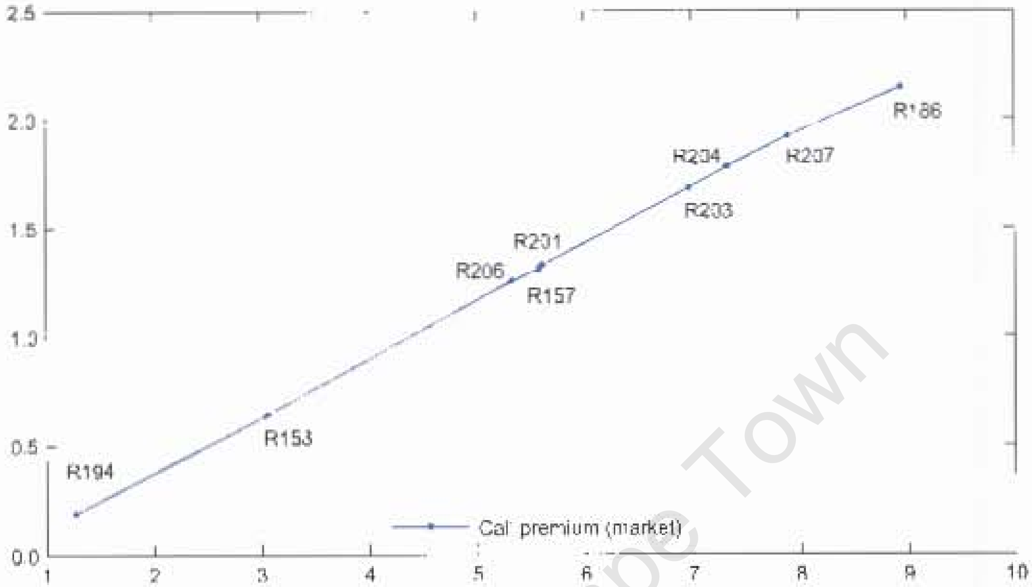

Table 6.2: Perturbed model at-the-money call premiums and implied volatilities

Trade date: 30 June 2006 Trade settlement date: 5 July 2006
Expiry date: 30 September 2006 Expiry settlement date: 5 October 2006

Bond	Spot/strike yield	Strike price	Implied volatility (perturbed model)	Call premium (perturbed model)
R194	8.515%	1.32877	2.20%	2,953
R153	8.350%	1.15745	4.10%	7,476
R206	8.663%	0.95459	7.00%	11,523
R157	8.650%	1.30513	7.13%	15,941
R201	8.660%	1.03033	7.20%	12,835
R203	8.650%	0.97653	9.49%	16,485
R204	8.635%	0.97362	10.43%	18,261
R207	8.560%	0.91322	10.32%	17,070
R186	8.330%	1.23835	10.13%	22,922

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calculated using the market convention in Figure 6.8 below.

Figures 6.7 and 6.8 suggest that yield volatilities are in fact not constant across the curve. The price of the R194 option implied by the perturbed model is higher than the price calculated using market conventions while, except for the R203 and R204 the premiums implied by the perturbed model for options on bonds that are longer dated than the R153 are less than the premiums implied by market conventions. The difference is most significant for the R186 where the pre-

Figure 6.7: At-the-money call premiums (Perturbed model)

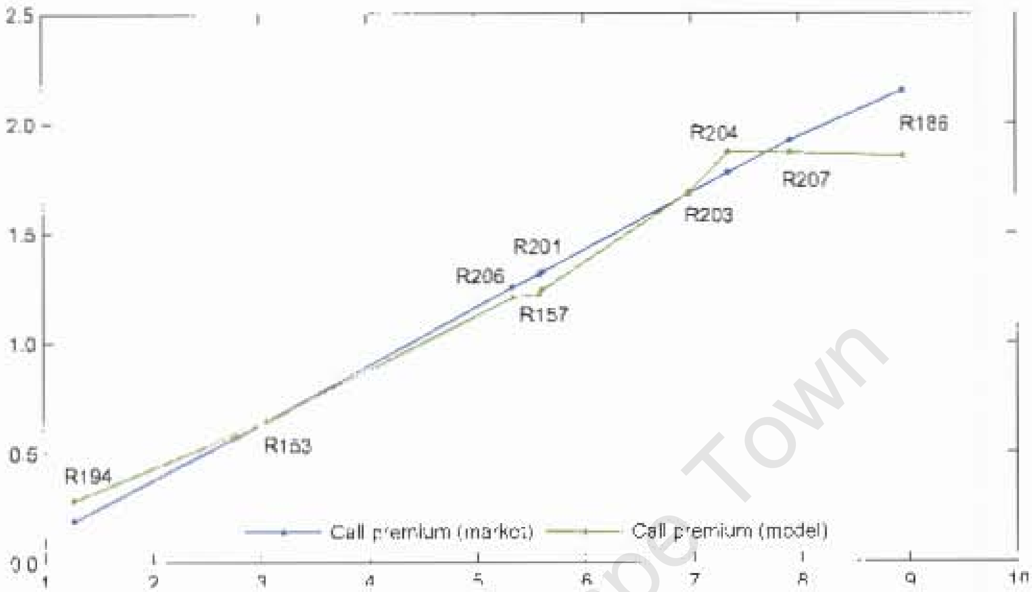
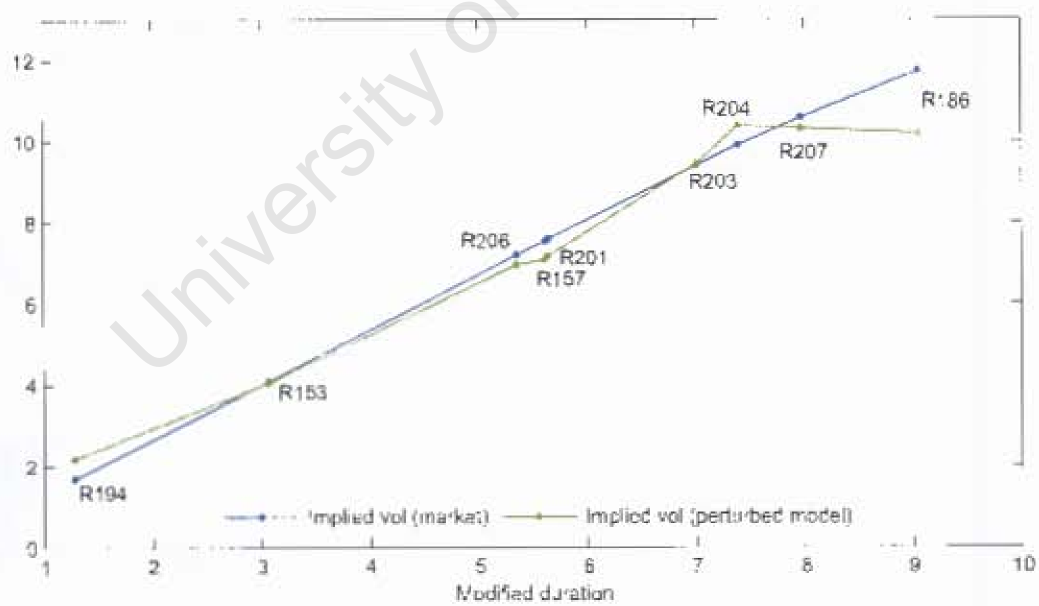


Figure 6.8: Volatilities implied for Black's model by the perturbed model prices



mium calculated using market conventions (as a percentage of price) is approximately 30 basis points higher than implied by the perturbed model. This suggests that the market method is inconsistent with the observed dynamics and thus there is theoretical mis-pricing. The theoretical mis-pricing implies that options the R153 and R194 are cheap relative to options on the R186.

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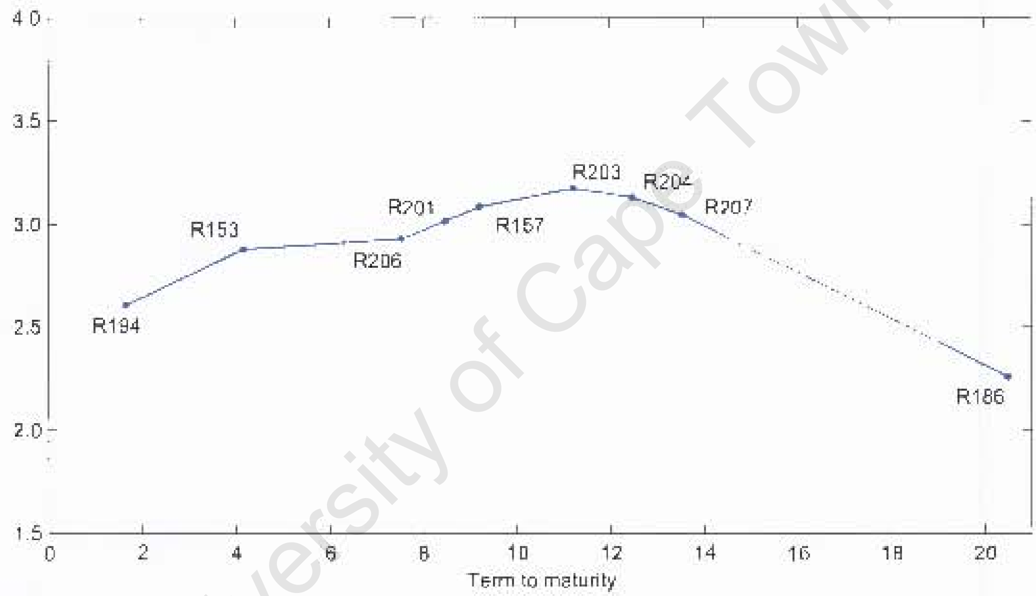
This implies that the R186 implied volatility should be lower than is assumed by the market, which suggests that yield volatilities of long-dated bonds have been lower than those of short-dated bonds. We cannot compare the historical volatilities of the particular bonds directly as their terms to maturity shorten over time, however to get some idea of whether this is in fact true, we can look at the volatilities of generic par bonds with constant maturities equal to the current maturities of the particular bonds that we are considering. Figure 6.9 below shows volatilities calculated from monthly par curves from 30 June 2003 to 30 June 2006.

Figure 6.9: Par bond volatilities



As expected the volatility of the par bond corresponding to the R186 is lower than the volatilities of the par bond corresponding to the other bonds.

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Chapter 7

Conclusions

We have discussed a recently proposed yield curve model that assumes that the yield curve dynamics are driven by a 'stable' process and that observed market prices deviate from these theoretical prices due to 'errors' (possibly caused by such factors as liquidity constraints and bid-offer spreads.) The model fits into the affine HJM framework. Features of the model include a closed form solution for bond prices and a means to update the model daily via the Kalman Filter.

The model is particularly well suited to the South African market since two sets of yield curves are published by the Bond Exchange of South Africa and the model can be used to provide a theoretical connection between the two types of curves. The underlying model can be fitted to the dynamics of the 'best decency' curves but does not necessarily price bonds exactly. The perturbed model adds an error term to achieve market prices determined by the 'perfect fit' curves.

We have illustrated the process of linking 'best decency' and 'perfect fit' curves by calibrating the model to curves that we construct using a similar methodology to BESA. As an application we discuss the relatively naive volatility conversion used by the market to price options on less liquid bonds and show that the model can be used to determine the relative value of different bond options. This allows an end user to evaluate relative cheapness of option prices and provides a market maker with a more accurate means of valuing illiquid options.

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Appendix A

Simplifications resulting from the choice of A_ξ, B_ξ and $C_\xi(T)$

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Simplification of (4.19) to (4.24):

Since $\tilde{A}$ is constant

$$\begin{aligned}\tilde{C}(t,T) &= \tilde{C}(T)e^{\int_t^T \tilde{A}(s)ds} \\ &= \tilde{C}(T)e^{\tilde{A}(T-t)}.\end{aligned} \tag{A1}$$

Also since $\tilde{A}$ is of the form

$$\tilde{A} = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right], \tag{A2}$$

we have

$$\tilde{A}^k = \left[\begin{array}{c|c} A^k & 0 \\ \hline 0 & 0 \end{array}\right]. \tag{A3}$$

This implies that

$$\begin{aligned} e^{\tilde{A}\tau} &= I + \tau\tilde{A} + \frac{\tau^2}{2!}\tilde{A}^2 + \frac{\tau^3}{3!}\tilde{A}^3 + \dots \\ &= \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right] + \tau\left[\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right] + \frac{\tau^2}{2!}\left[\begin{array}{c|c} A^2 & 0 \\ \hline 0 & 0 \end{array}\right] + \frac{\tau^3}{3!}\left[\begin{array}{c|c} A^3 & 0 \\ \hline 0 & 0 \end{array}\right] + \dots \\ &= \left[\begin{array}{c|c} I + \tau A + \frac{\tau^2}{2!}A^2 + \frac{\tau^3}{3!}A^3 + \dots & 0 \\ \hline 0 & I \end{array}\right] \\ &= \left[\begin{array}{c|c} e^{A\tau} & 0 \\ \hline 0 & I \end{array}\right]. \end{aligned} \tag{A4}$$

Substituting

$$e^{\tilde{A}(T-t)} = \begin{bmatrix} e^{A(T-t)} & 0 \\ 0 & I \end{bmatrix}$$

into (A1) and recalling that $\tilde{C}(T) = \begin{bmatrix} C, & C_\xi(T) \end{bmatrix}$ gives

$$\begin{aligned} \tilde{C} &= \left[Ce^{A(T-t)},\ C_\xi(T)\right] \\ &= \left[C(t,T),\ C_\xi(T)\right]. \end{aligned} \tag{A5}$$

□

Simplification of (4.20) to (4.25):

Since $B_\xi(t) = I$, (4.21) simplifies to

$$\tilde{\beta}(t,T) = \beta(t,T) + \left\| \int_t^T C_\xi(t,u)du \right\|^2. \tag{A6}$$

Differentiating (A6) with respect to T gives

$$\tilde{\beta}_T(t,T) = \beta_T(t,T) + \frac{\partial}{\partial T}\left\| \int_t^T C_\xi(t,u)du \right\|^2. \tag{A7}$$

Substituting (A6) into (4.20) gives

$$\begin{aligned} \tilde{G}(t,T) &= \tilde{f}^*(0,T) + \frac{1}{2}\int_0^t \left[\beta_T(s,T) + \frac{\partial}{\partial T}\left\| \int_s^T C_\xi(s,u)du \right\|^2\right] ds \\ &= \tilde{f}^*(0,T) + \frac{1}{2}\int_0^t \beta_T(s,T)ds + \frac{1}{2}\int_0^t \frac{\partial}{\partial T}\left\| \int_s^T C_\xi(s,u)du \right\|^2 ds \\ &= G(t,T) + \frac{1}{2}\int_0^t \frac{d}{dT}\left\| \int_s^T C_\xi(u)du \right\|^2 ds, \end{aligned} \tag{A8}$$

where $C_\xi(s,u) = C_\xi(u)$ follows from (A5). The first term is given by (4.11); the second term is calculated as follows:

Since $C_\xi(T)$ is defined by $C_\xi^i(T) = \chi_{(T_{i-1},T_i]}(T)$ for $i = 1,\ldots,N$, we have

$$\int_s^T C_\xi(u)du = \begin{bmatrix} \int_s^T \chi_{(T_0,T_1]}(u)du & \int_s^T \chi_{(T_1,T_2]}(u)du & \ldots & \int_s^T \chi_{(T_{N-1},T_N]}(u)du, \end{bmatrix}. \tag{A9}$$

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Since

$$\int_s^T \chi_{(T_{k-1},T_k]}(u)du = \begin{cases} 0 & \text{if} \quad T \leq T_{k-1} \\ T - T_{k-1} & \text{if} \quad T_{k-1} \leq T \leq T_k \ \& \ 0 \leq s \leq T_{k-1} \\ T - s & \text{if} \quad T_{k-1} \leq s < T \leq T_k \\ T_k - s & \text{if} \quad T_{k-1} \leq s < T_k \leq T \\ T_k - T_{k-1} & \text{if} \quad s \leq T_{k-1} \ \& \ T_k < T, \end{cases} \tag{A10}$$

it follows that

$$\left\| \int_s^T C_\xi(u)du \right\|^2 = \sum_{k=1}^N \left(\int_s^T \chi_{(T_{k-1},T_k]}(u)du \right)^2. \tag{A11}$$

Taking the derivative of the above equation with respect to T gives

$$\begin{aligned} \frac{d}{dT} \left\| \int_s^T C_\xi(u)du \right\|^2 &= \frac{d}{dT} \sum_{k=1}^N \left(\int_s^T \chi_{(T_{k-1},T_k]}(u)du \right)^2 \\ &= \sum_{k=1}^N \frac{d}{dT} \left(\int_s^T \chi_{(T_{k-1},T_k]}(u)du \right)^2 \\ &= 2\sum_{k=1}^N \left(\int_s^T \chi_{(T_{k-1},T_k]}(u)du \right) \frac{d}{dT} \int_s^T \chi_{(T_{k-1},T_k]}(u)du. \end{aligned} \tag{A12}$$

Now from (A10),

$$\frac{d}{dT} \int_s^T \chi_{(T_{k-1},T_k]}(u)du = \begin{cases} 1 & \text{if} \quad T_{k-1} \leq T \leq T_k \\ 0 & \text{if} \quad T < T_{k-1} \leq T \text{ or } T > T_k, \end{cases} \tag{A13}$$

therefore assuming $T_0 \leq T \leq T_N$, T will lie in exactly one of the intervals $(T_{k-1}, T_k]$. This implies that exactly one of the terms in (A12) will be non-zero. Therefore

$$\begin{aligned} \frac{d}{dT} \left\| \int_s^T C_\xi(u)du \right\|^2 &= 2\int_s^T \chi_{(T_{k-1},T_k]}(u)du \\ &= \begin{cases} 2\,(T - T_{k-1}) & \text{if} \quad s \leq T_{k-1} \\ 2\,(T - s) & \text{if} \quad s > T_{k-1}, \end{cases} \end{aligned} \tag{A14}$$

where $(T_{k-1}, T_k]$ is the interval containing T. From (A14) it follows that

$$\frac{1}{2}\int_0^t \frac{d}{dT} \left\| \int_s^T C_\xi(u)du \right\|^2 ds = \begin{cases} (T - T_{k-1})\,t & \text{if} \quad t \leq T_{k-1} \\ -\frac{1}{2}\,(T_{k-1})^2 + Tt - \frac{1}{2}t^2 & \text{if} \quad t > T_{k-1}. \end{cases} \tag{A15}$$

□

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Appendix B

Integral of $\tilde{G}(t, T)$

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Implementing the perturbed model requires evaluation of the integral

$$\int_t^T \tilde{G}(t, s)\, ds. \tag{B1}$$

For the choice of parameters specified in Chapter 4, $\tilde{G}(t, s)$ is defined by

$$\tilde{G}(t, T) = G(t, T) + b^2 \begin{cases} (T - T_{k-1})\, t & \text{if} \quad t \leq T_{k-1} \\ -\frac{1}{2}(T_{k-1})^2 + Tt - \frac{1}{2}t^2 & \text{if} \quad t > T_{k-1}, \end{cases} \tag{B2}$$

where $(T_{k-1}, T_k]$ is the interval containing T.

The integral of the first term of (B2), $G(t, s)$, has to be computed numerically but the integral of the second part can be calculated analytically as follows:

Let $I(t, s)$ denote

$$I(t, s) = \begin{cases} (s - T_{k-1})\, t & \text{if} \quad t \leq T_{k-1} \\ -\frac{1}{2}(T_{k-1})^2 + st - \frac{1}{2}t^2 & \text{if} \quad t > T_{k-1}, \end{cases} \tag{B3}$$

where $(T_{k-1}, T_k]$ is the interval containing s.

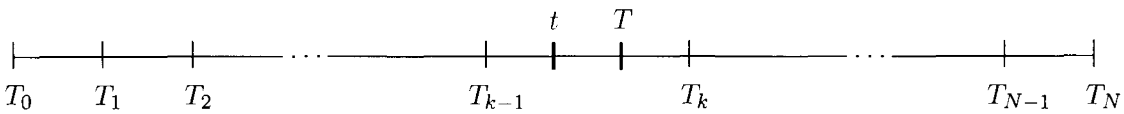
There are two cases for calculating $\int_t^T I(t, s)\, ds$. The relevant case is determined by where t and T lie with respect to the maturities of the N observed bonds:

Case 1: t lies in the same interval as T

If t lies in the same interval as T, that is $T_{k-1} \leq t \leq T \leq T_k$, as in Figure B1 then

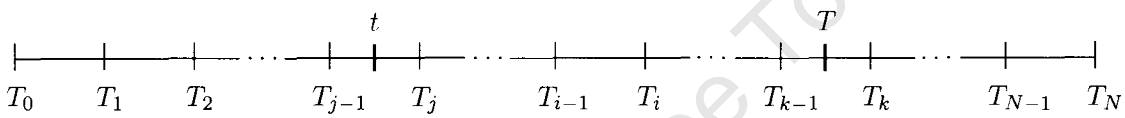
$$\begin{aligned} \int_t^T I(t, s) ds &= \int_t^T -\frac{1}{2}(T_{k-1})^2 + st - \frac{1}{2}t^2\, ds \\ &= -\frac{1}{2}\left[(T_{k-1})^2 + t^2\right](T - t) + \frac{t}{2}(T^2 - t^2). \end{aligned} \tag{B4}$$

Figure B1: t lies in the same interval as T



Case 2: t and T lie in different intervals

Figure B2: t and T lie in different intervals



If t and T lie in different intervals, that is $T_{j-1} \leq t \leq T_j$ and $T_{k-1} \leq T \leq T_k$ with $j < k$, as in Figure B2 then

$$\begin{aligned}\int_t^T I(t,s)ds &= \int_t^{T_j} -\frac{1}{2}(T_{j-1})^2 + st - \frac{1}{2}t^2\, ds + \sum_{i=j+1}^{k-1} \int_{T_{i-1}}^{T_i} (s - T_{i-1})\, t\, ds \\ &\quad + \int_{T_{k-1}}^{T} (s - T_{k-1})\, t\, ds \\ &= -\frac{1}{2}\left[(T_{j-1})^2 + t^2\right](T_j - t) + \frac{t}{2}\left[(T_j)^2 - t^2\right] + \sum_{i=j+1}^{k-1} \frac{t}{2}\,(T_i - T_{i-1})^2 \\ &\quad + \frac{t}{2}(T - T_{k-1})^2.\end{aligned} \tag{B5}$$

It follows that

$$\int_t^T \tilde{G}(t,s)\, ds = \int_t^T G(t,s)\, ds + b^2 \int_t^T I(t,s)\, ds, \tag{B6}$$

where the first term is calculated numerically using (4.11) and the second term is given by either (B4) or (B5) depending on the values of t and T. □

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Appendix C

Black's model

The value at time t of a European call option with a strike price of $\mathcal{A}(T_s, y_{\text{strike}})$ that expires at time T, on a coupon bearing bond trading at a spot price of $\mathcal{A}(t_s, y_{\text{spot}})$ is given by (Hull 2003)[1]

$$C_t = S\,N(d_1) - \frac{K\,N(d_2)}{1 + \text{rfr} \times \tau},$$

with

$$d_1 = \frac{\ln(S/K) + \ln(1 + \text{rfr} \times \tau) + \tau/2\sigma^2}{\sigma\sqrt{\tau}},$$

$$d_2 = d_1 - \sigma\sqrt{\tau},$$

$$\tau = T - t,$$

$$S = \frac{\mathcal{A}(t_s, y_{\text{spot}}) - I}{1 + \text{rfr} \times (t_s - t)},$$

$$I = \sum_i \frac{c_i}{1 + \text{rfr} \times (T_i - t_s)},$$

$$K = \frac{\mathcal{A}(T_s, y_{\text{strike}})}{1 + \text{rfr} \times (T_S - T)},$$

where t_s and T_s are the times corresponding to the settlement dates of the trade and expiry date respectively, c_i are the coupons, paid at times T_i, that occur between the trade and expiry settlement dates, σ is the implied volatility and rfr is the (simple) risk-free rate. $N(\cdot)$ is the standard normal cumulative probability distribution function.

[1]The formulas below are adjusted to take into account the fact that the BESA bond pricing formula prices bonds on a settlement date.

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