Mixed Finite Element Analysis For Arbitrarily Curved Beams

BY

K. ARUNAKIRINATHAR

SUBMITTED
TO THE UNIVERSITY OF CAPE TOWN
IN FULFILMENT OF THE REQUIREMENTS FOR THE
DEGREE OF MASTERS OF SCIENCE IN APPLIED MATHEMATICS

CAPE TOWN

SEPTEMBER 1991.

To My Parents

ACKNOWLEDGEMENT

I wish to thank my supervisor, Prof. Dayanand Reddy for his invaluable assistance and guidance during the course of this studies. I would also like to express my deepest gratitude for his willingness to take over supervision of my thesis and proof reading the script.

Many thanks to Robin Eve for his help in computation. I would also like to thank my other friends for their encouragement and support.

Further I wish to thank the Department of Applied Mathematics for the use of their facilities.

Finally I would like to thank the Foundation for Research and Development and the FRD/UCT Center for Research in Computational and Applied Mechanics for their financial support.

CONTENTS

Abstract

A convergence of a mixed finite element method for three-dimensional curved beams with arbitary geometry is investigated. First, the governing equations are derived for linear elastic curved beams with uniformly loaded based on the Timoshenko-Reissner-Mindlin hypotheses. Then, standard and mixed variational problems are formulated. A new norm, equivalent to H^1-type norm, is introduced. By making use of this norm, sufficient conditions for existence and uniqueness of the solutions of the above problems are established for both continuous and discrete cases.

The estimates of the optimal order and minimal regularity are then derived for errors in the generalised displacement vector and the internal force vector. These analytical findings are compared with numerical results, verifying the role of reduced integration and the accuracy of the methods.

1 Introduction

The beam is without doubt the simplest and most common structural element, with wide use in civil, mechanical as well as aerospace engineering structures. Thus the formulation of suitable finite elements for curved beams has been the subject of quite intensive research.

There has been a long history of attempts to develop successful finite element formulations for curved beam elements, as is clear from the papers [2, 3, 5, 15, 16, 21, 30, 35, 37, 41, 42, 43]. The analysis of curved beams by the finite element method requires that one take full account of the effects of both in-plane and out-of-plane forces and deformations. Three approaches have been used to formulate the necessary theory:

1. Intrinsic formulations [42]:
   The characteristic feature of these formulations is that the equilibrium and strain-displacement equations are derived by making use of a curvilinear coordinate system, and the techniques of one-dimensional differential geometry;
2. Shallow-beam approximations [45, 46]:
   These formulations are valid only for curved beams which can be classified as shallow, in that the radius of curvature is large relative to the length of the beam. The shallow arch is one such particular example [41].
3. Degenerate approximations [16, 49]:
   In these formulations finite element approximations are obtained by applying various appropriate physical and kinematic constraints simultaneously with the finite element discretisation, with a view to deriving a set of equations which is easier to treat.

Most work which has appeared in the literature has been based on the first and second approaches, and the bulk of this work has been confined to plane problems. We shall make use of the first approach in this thesis, and from the outset consider beams of arbitrary shape, which undergo both shear and axial deformation in addition to bending and twisting. While the Kirchhoff approximation of zero shear has been popular, the complete formulation possesses several advantages. First, the inclusion of all possible effects allows an improvement in the modelling of the behaviour of thick and anisotropic beams, while the Kirchhoff approximation is valid only for very thin beams. Secondly, the full formulation requires only the use of conventional Lagrangian basis functions, that is, it suffices to use continuous functions. The Kirchhoff theory gives rise to a set of differential equations of fourth order, which in turn requires the use of Hermite-type basis functions which are continuously differentiable.

The general formulation is not without its complications, though. It is found that the performance of elements degenerates rapidly as the beam becomes very thin. Two forms of degeneracy are observed: one is in the nature of a nonuniform convergence, in that the mesh size becomes thickness-dependent. In the second phenomenon, known as locking, the solution to the problem approaches zero as the thickness of the beam becomes very small. This is as a result of the equations becoming too stiff, roughly speaking. Locking is associated with both shear and axial terms, and when both of these are present the situation is correspondingly more complex.

Fortunately the reasons for locking, and ways in which it can be circumvented, are now reasonably well- understood, at least for plane problems. Some effective computational techniques which have emerged are:

1. the use of high order polynomials for approximating generalised displacements. In [14, 21] quintic-quintic C^1 elements are used. However, elements with higher order derivative degrees of freedom have some disadvantages in interpreting the boundary conditions, particularly in the context of shell (that is, two-

dimensional) problems. Stolarski [46] has successfully used cubic isoparametric C^0 elements.

2. the use of selective reduced integration for the transverse shear terms in the case of beams, and for the membrane term in the case of inextensional deformation. This approach has been successfully advocated by Hughes et al. [20], Zienkiewicz [50] and others. Prathap and Bhasyam [34] have introduced the notion of 'true' and 'spurious' constraints. Both these types of constraints are elininated by reduced integration.

3. the use of mixed variational formulations in which the displacements and some internal forces are retained as variables, and approximated independently.

The mixed method has become a powerful tool not only because it converges rapidly and is stable, but also because it yields results for the internal forces which are more accurate than those obtained after post-processing in the displacement problem.

There has been considerable progress in the mathematical analysis of mixed and reduced integration methods, with consequent mathematical justification for the faith placed in these methods. Arnold [3] presented a complete analysis of the beam problem, in which he showed firstly that the displacement formulation gives rise to nonuniform convergence, and secondly that the mixed formulation gives rise to convergent finite element approximations. He also showed that the reduced integration method works because it is equivalent to the mixed method. This work has subsequently been extended by Kikuchi [22] in an analysis of the circular arch problem without shear deformation, and by Reddy [41] and Reddy and Volpi [42] in analysis of the shallow arch and circular arch, respectively; in both these contributions the effects of shear and axial deformation are considered simultaneously. Kikuchi's analysis has the further merit that he examines the dependence of solutions on the thickness parameter d, and shows by means of asymptotic analysis that solutions depend analytically on this parameter. Loula et al. [27] consider the same problem as Reddy and Volpi [42], but for the case of arches which are free at one end and fixed at the other (that is, mixed Dirichlet and Neumann conditions), and with the use of the Petrov-Galerkin method.

In Reddy's [41] investigation of the shallow arch problem with two sources of deformation the existence, uniqueness and convergence of approximate solutions was shown, as was the equivalence between the mixed and reduced integration problems. The convergence results derived there are only valid, though, for the case in which the degree s of the polynomial approximation of the arch shape satisfies $1 \leq s \leq \min\{2, r\}$, r being the degree of polynomial approximation of the displacements. Reddy and Volpi [42] obtained results for the circular arch problem which are free of such restrictions. In this work, as in most investigations of this kind, one of the key sources of difficulty is the verification of the Babŭska-Brezzi conditions [11] for the approximate problem.

The aim of this thesis is to extend the investigations begun in [41, 42] to the case of beams of arbitary shape which experience full three-dimensional behaviour. The study commences in Chapter 2 with a detailed formulation of the governing differential equations for the problem. An intrinsic formulation is used, and it will be seen that this leads to a remarkably simple structure for the equations. In Chapter 3 the problem is formulated in both standard and mixed variational forms. In order to obtain results for the existence and uniqueness of solutions it is necessary to define a new norm, which is shown to be equivalent to the standard H^1-type norm. With this norm available, existence and uniqueness of solutions are shown.

In Chapter 4 we construct finite element formulations of the problems of Chapter 3. As in the case of the planar problems, the standard and mixed problems are not equivalent; rather, it can be shown that the mixed problem is equivalent to the standard problem with selective reduced integration of the non-flexural terms. Existence and uniqueness of solutions to the problems are shown, and the mixed method is shown further to be convergent.

With a view to comparing numerical solutions with closed-form solutions Chapter 5 is devoted to the derivation of an analytical solutions for two problems with constant curvature and torsion, and with constant applied force. Thereafter, in Chapter 6 numerical results are compared against the solutions obtained in Chapter 5 for two particular cases: the plane circular arch uniformly loaded out of plane, and the circular helicoidal beam with uniformly distributed load.

2 Geometry of Beam and Governing Equations

In this Chapter, we develop the equations which describe the undeformed geometry of a beam [24] of arbitrary shape. The equations which govern the behaviour of the beam under various loading conditions, are then derived. From now on, we shall often deal with vector-valued functions; these will be distinguished by means of bold face characters.

2.1 Differential Geometry of Regular Space Curves

Let $\boldsymbol{\alpha} := [0, L] \to I\!\!R^3$ be a regular curve parametrized by its arc length s. That is, if $\boldsymbol{\alpha}(s) = (x(s), y(s), z(s))$ represents a point of the curve in a rectangular cartesian coordinate system for each $s \in \mathbf{I} = [0, L]$, then $x'^2 + y'^2 + z'^2 > 0$ (Figure 2.1). The functions x, y, z are assumed to be in C^3. Here and henceforth, a superposed prime denotes the derivative with respect to s.

The vector $(x'(s), y'(s), z'(s)) = \boldsymbol{\alpha}'(s)$ is called the **tangent vector** of the curve $\boldsymbol{\alpha}$ at s and, is denoted by $\mathbf{t}$.

Since the curve is assumed to be regular, the tangent vector is defined at each point along the curve. Furthermore, it is seen that $\mathbf{t}$ is a unit vector and hence the magnitude $|\,\boldsymbol{\alpha}''(s)\,|$ of the second derivative measures the rate of change of the angle which neighboring tangents make with the tangent at s.

The length $|\,\boldsymbol{\alpha}''(s)\,|$ is called the **curvature** of $\boldsymbol{\alpha}$ at s and is denoted by $\kappa(s)$ (or $\frac{1}{\rho}(s)$ where ρ is **radius of curvature** of the curve at s).

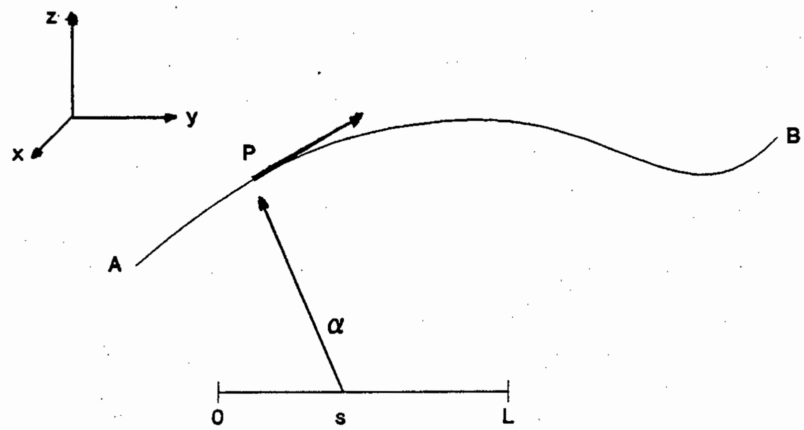


Figure 2.1: Space Curve

A unit vector $\mathbf{n}$ in the direction $\boldsymbol{\alpha}''(s)$ is chosen by requiring it to form with the tangent vector $\mathbf{t}$, a positively oriented system, and to satisfy the equation

$$\boldsymbol{\alpha}''(s) = \kappa(s)\mathbf{n} \ ; \ \kappa(s) \geq 0.$$

Conventionally, we choose the direction of $\mathbf{n}$ towards the convex-side of the curve $\boldsymbol{\alpha}$ at s. Moreover, by differentiating the equation $\mathbf{t} \cdot \mathbf{t} = 1$, we obtain $\mathbf{t}' \cdot \mathbf{t} = 0$, so that $\boldsymbol{\alpha}''(s)$ is normal to $\boldsymbol{\alpha}'(s)$. That is, $\mathbf{n} \perp \mathbf{t}$. The vector $\mathbf{n}$ is called **the normal vector** at s.

The plane determined by $\mathbf{t}$ and $\mathbf{n}$ is called the **osculating plane** at s. The unit vector $\mathbf{b} = \mathbf{t} \wedge \mathbf{n}$ is normal to the osculating plane and is known as the **binormal vector** at s (Figure 2.2).

Since $\mathbf{b}$ is a unit vector, the magnitude $\mid \mathbf{b}' \mid$ measures the rate of change of the neighboring osculating planes with the osculating plane at s . To compute $\mathbf{b}'$ we

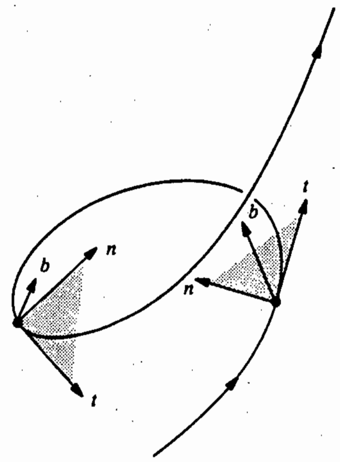


Figure 2.2: osculating plane

observe that, on the one hand, $\mathbf{b}'$ is normal to $\mathbf{b}$ (since $\mathbf{b} \cdot \mathbf{b} = 1$) and that, on the other hand,

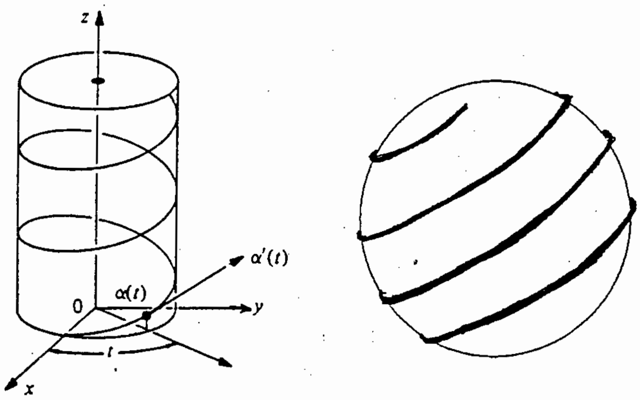
$$\begin{aligned} \mathbf{b}' &= (\mathbf{t} \wedge \mathbf{n})' \\ &= \mathbf{t}' \wedge \mathbf{n} + \mathbf{t} \wedge \mathbf{n}' \\ &= \mathbf{t} \wedge \mathbf{n}'. \end{aligned}$$

Hence $\mathbf{b}'$ is normal to $\mathbf{t}$. Therefore $\mathbf{b}'$ is parallel to $\mathbf{n}$, and we write

$$\mathbf{b}' = -\tau \mathbf{n} \quad \text{for some function } \tau(s) \ .$$

The negative sign is introduced so that τ has a particular geometrical significance which will be discussed in what follows. The function $\tau(s)$ is called the *torsion* of the curve $\boldsymbol{\alpha}$ at s .

We have thus defined three mutually orthogonal unit vectors ($\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$), which form a local basis at each point on the curve. The derivatives $\mathbf{t}' = \kappa\mathbf{n}$, $\mathbf{b}' = -\tau\mathbf{n}$ of the vectors $\mathbf{t}$ and $\mathbf{b}$, when expressed in the basis $(\mathbf{t}, \mathbf{n}, \mathbf{b})$, yield geometrical properties



(a): Helix (b) : Barrel curve

Figure 2.3: Examples of space curves

(curvature κ, torsion τ) which gives us information about the behaviour of the curve α in a neighbourhood of s .

Since $\mathrm{n} = \mathrm{b} \wedge \mathrm{t}$, we have

$$\begin{aligned} \mathrm{n}' &= \mathrm{b} \wedge \mathrm{t}' + \mathrm{b}' \wedge \mathrm{t} \\ &= -\kappa \mathrm{t} + \tau \mathrm{b}. \end{aligned}$$

We summarise the relationship between $\mathrm{t}, \mathrm{n}, \mathrm{b}$ and their derivatives in the matrix form

$$\begin{bmatrix} \mathrm{t}' \\ \mathrm{n}' \\ \mathrm{b}' \end{bmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{bmatrix} \mathrm{t} \\ \mathrm{n} \\ \mathrm{b} \end{bmatrix}$$

or more concisely,

$$\mathrm{t}_i' = \Gamma_{ij} \mathrm{t}_j \tag{2.1}$$

where $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3) \equiv (\mathbf{t}, \mathbf{n}, \mathbf{b})$. This identity is known as the **Serret-Frenet formula**; note that matrix Γ is skew-symmetric.

This completes the basic differential geometry of regular space curves needed for subsequent developments. We conclude with the following remarks.

Remark 1: According to the definition of τ we have

$$\begin{aligned}
\tau &= -\mathbf{n} \cdot \mathbf{b}' \\
&= -\mathbf{n} \cdot (\mathbf{t} \wedge \mathbf{n}') \\
&= \left[-\tfrac{\alpha''}{\kappa}, \alpha', (\tfrac{\alpha''}{\kappa})'\right] \\
&= \left[-\tfrac{\alpha''}{\kappa}, \alpha', \tfrac{\alpha'''}{\kappa}\right] - \left[-\tfrac{\alpha''}{\kappa}, \alpha', \tfrac{\alpha''\kappa'}{\kappa^2}\right] \\
&= -\tfrac{1}{\kappa^2}\left[\alpha'', \alpha', \alpha'''\right]
\end{aligned}$$

where $[\cdot,\cdot,\cdot]$ denotes the scalar triple product of vectors. This shows that $\tau > 0$ when $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ form a positively oriented basis for $\mathbb{R}^3$ and we require that $\alpha(s)$ belongs to C^3 to define τ.

Remark 2: $\mathbf{t}'$, $\mathbf{n}'$, $\mathbf{b}'$ can be written in an alternative form. If we define the angular velocity vector $\boldsymbol{\omega}$ of the orthonormal triad (t,n,b) by

$$\boldsymbol{\omega} = \tau \mathbf{t} + \kappa \mathbf{b} \qquad \text{for } \tau > 0\, ; \kappa > 0,$$

then

$$\mathbf{t}' = \boldsymbol{\omega} \wedge \mathbf{t}, \quad \mathbf{n}' = \boldsymbol{\omega} \wedge \mathbf{n}, \quad \mathbf{b}' = \boldsymbol{\omega} \wedge \mathbf{b}.$$

Remark 3: We consider curves whose curvature and the torsion are continuous functions which are assumed to be bounded above. That is,

$$0 \leq \tau(s) \leq \tau_0 \qquad \text{and} \qquad 0 \leq \kappa(s) \leq \kappa_0 \qquad \text{for } s \in \mathbf{I}. \tag{2.2}$$

Remark 4: When κ is constant, the curve lies in a cylinder. If τ and κ are both constants, then the curve is known a *helix*. If $\tau = 0$, the curve lies on a plane and if $\tau = 0$ and κ is constant, the curve becomes an arc of a circle.

2.2 Governing Equations for Beam

In this Section we derive the equations governing the behaviour of naturally curved, twisted beams undergoing small displacements and rotations. In the undeformed reference state a beam is described in terms of smooth space curve, the centre line, and normal cross sections located in the normal planes with their centroids on the line. We make the basic assumption that as the beam deforms due to extension and bending, material cross sections are constrained to remain plane, but not necessarily normal to the centre line, after deformation (see Figure 2.4).

Equilibrium Equations

Figure 2.5 illustrates a segment of beam which is in equilibrium under a distributed load. The internal forces and bending moments acting at A and P are $\mathbf{N}(0)$, $\mathbf{N}(s)$ and $\mathbf{M}(0)$, $\mathbf{M}(s)$ respectively. The distributed load per unit length acting along the beam is $\mathbf{F}(\mathrm{s})$. We assume here that there are no external couples acting on the beam.

Equilibrium of the segment AP requires firstly that the algebraic sum of the forces is equal to zero, that is,

$$\mathbf{N}(s) + \int_0^s \mathbf{F}(\xi)\, d\xi - \mathbf{N}(0) = \mathbf{0},$$

and secondly, that the net moment about any point is zero: taking moments about the origin O, we obtain

$$\mathbf{M}(s) + \mathbf{r}(s) \wedge \mathbf{N}(s) + \int_0^s \mathbf{r}(\xi) \wedge \mathbf{F}(\xi)\, d\xi - \mathbf{r}(0) \wedge \mathbf{N}(0) - \mathbf{M}(0) = \mathbf{0}.$$

These equations are true for any value of $s \in [0, L]$. It follows then that we can

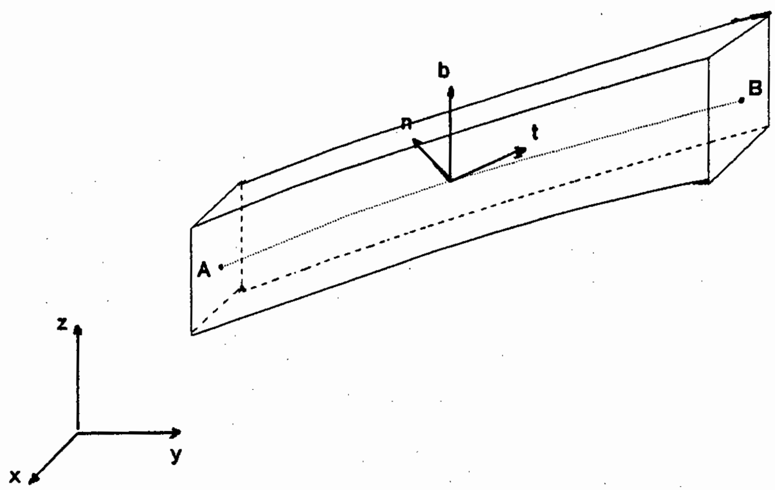


Figure 2.4: Three dimensonal beam element

differentiate each equation with respect to s, to obtain

$$\frac{d\mathbf{N}}{ds} + \mathbf{F} = \mathbf{0}$$

and

$$\frac{d\mathbf{M}}{ds} + \mathbf{t} \wedge \mathbf{N} + \mathbf{r} \wedge \frac{d\mathbf{N}}{ds} + \mathbf{r} \wedge \mathbf{F} = \mathbf{0}.$$

Finally, substituting the first equation into the second equation we obtain the equilibrium equations in the simplified form

$$\mathbf{N}' + \mathbf{F} = \mathbf{0}, \tag{2.3}$$

$$\mathbf{M}' + \mathbf{t} \wedge \mathbf{N} = \mathbf{0}. \tag{2.4}$$

Kinematics of Deformation

The initial configuration of the centroidal line of the beam, denoted by $\boldsymbol{\alpha}$: $[0, L] \rightarrow I\!R^3 \in C^3$, is a regular curve with curvature κ and torsion τ. Along the undeformed centre line, a right handed orthonormal set of base vectors $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is

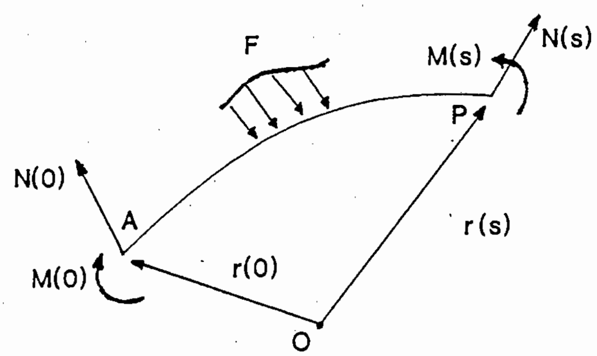


Figure 2.5: Beam element under external and internal forces

introduced, which we refer to as an intrinsic frame.

In view of the basic assumptions, the deformed configuration of the beam is completely described by the displacements of the centroidal line of the beam and the rotation of the cross section at each point. We denote the centroidal displacement vector by $\mathbf{u} : [0, L] \rightarrow \mathbb{R}^3$, and the rotation vector by $\boldsymbol{\theta}$: $[0, L] \rightarrow \mathbb{R}^3$. The components of $\mathbf{u}$ and $\boldsymbol{\theta}$ will be given with respect to the intrinsic basis.

We now derive expressions which describe the deformation of the beam. The deformation may vary from one point to another, and hence the generalised strains, which measure this deformation, will be functions of arc length s. We shall define generalised strains to be quantities which vanish in a rigid body motion.

Suppose, as shown in Fig 2.6, that a curved beam undergoes a rigid body displacement; this is completely defined by the displacement $\mathbf{u}(0)$ and the rotation $\boldsymbol{\theta}(0)$. Indeed, we then have

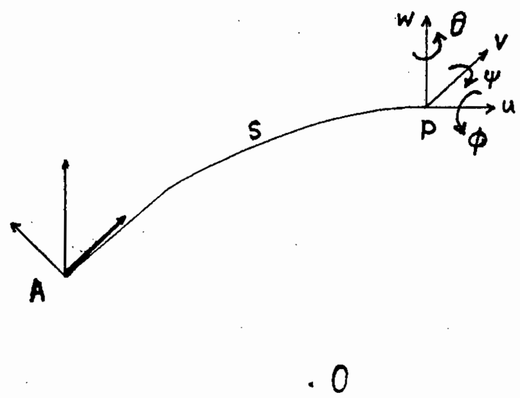


Figure 2.6: Kinematic describtion

$$\boldsymbol{\theta}(s) = \boldsymbol{\theta}(0), \tag{2.5}$$

while the displacement will be given by

$$\mathbf{u}(s) = \mathbf{u}(0) + \boldsymbol{\theta}(0) \wedge (\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(0)). \tag{2.6}$$

Differentiating the equation (2.5), we see that

$$\frac{d\boldsymbol{\theta}}{ds} = \mathbf{0} \tag{2.7}$$

is a necessary condition for a rigid body displacement. Also, from equation (2.6), a further necessary condition is

$$\begin{aligned} \frac{d\mathbf{u}}{ds} &= \boldsymbol{\theta}(0) \wedge \frac{d\boldsymbol{\alpha}}{ds} \\ &= \boldsymbol{\theta}(s) \wedge \mathbf{t} \end{aligned} \tag{2.8}$$

It is thus possible to define two vector-valued functions

$$\boldsymbol{\varepsilon}(s) = \mathbf{u}'(s) - \boldsymbol{\theta}(s) \wedge \mathbf{t} \tag{2.9}$$

and

$$\boldsymbol{\eta}(s) = \boldsymbol{\theta}'(s) \tag{2.10}$$

We recognise that the functions $\boldsymbol{\varepsilon}(s)$ and $\boldsymbol{\eta}(\mathrm{s})$ are measures of deformation of the beam, in the sense that if each function is zero for all s, the beam undergoes a rigid body displacement. Conversely if $\boldsymbol{\varepsilon}(s)$, $\boldsymbol{\eta}(\mathrm{s})$ are not zero for all s, the displacement of the beam is not a rigid body displacement and the beam has deformed. The two functions $\boldsymbol{\varepsilon}(s)$ and $\boldsymbol{\eta}(\mathrm{s})$ are the generalised strains for the beam.

Elastic Constitutive Equations

The constitutive equations are the mathematical description of the particular material response. Since we consider linear elastic materials, the stress-strain relations can be written in the form [44]

$$\mathbf{N} = \mathbf{D}\boldsymbol{\varepsilon}(s) \tag{2.11}$$

and

$$\mathbf{M} = \mathbf{E}\boldsymbol{\eta}(s), \tag{2.12}$$

where

$$\mathbf{D} = \mathrm{diag}[EA, GA_1, GA_2] \tag{2.13}$$

$$\mathbf{E} = \mathrm{diag}[GJ, EI_1, EI_2] \tag{2.14}$$

$A_1 = Ak_1$ and $A_2 = Ak_2$,

Here A the cross sectional area, k_1 and k_2 are shear correction factors, E and G are the elastic and shear moduli and I_1 and I_2 the moment of inertia of the cross section.

The quantities GA_1 and GA_2 denote the shear stiffness along the axes n and b, EI_1 and EI_2 are the principal bending stiffness relative to n and b, and EA and GJ are the axial and torsional stiffnesses of the beam. Since the beam is assumed to be homogeneous these quantities are constants.

Boundary Conditions

We consider beams which are clamped at both ends, so that we have the set of homogeneous Dirichlet boundary conditions

$$\mathbf{u}(0) = \mathbf{u}(L) = \mathbf{0}, \tag{2.15}$$

$$\boldsymbol{\theta}(0) = \boldsymbol{\theta}(L) = \mathbf{0}. \tag{2.16}$$

This completes the formulation of the mathematical model for the three-dimensional deformation of naturally curved and twisted beams. In the next Chapter we reformulate the problem in weak or variational form.

3 Standard and Mixed Variational Problems

The proper formulation of variational problems depends crucially on the selection of spaces of appropriate admissable functions. We thus begin by introducing some real function spaces (see [40, 39, 8] for references) on the open interval $(0,1) = \mathbf{I}$, which will play an important role in the variational formulations of the problem. The standard and mixed variational formulations are developed for the governing equations which were derived in the previous Chapter. The existence and uniqueness of solutions to these problems are then analysed.

3.1 Function Spaces

Let $L_2(\mathbf{I})$ be the space of real square integrable functions on $\mathbf{I}$. This is a Hilbert space with inner product and norm defined by

$$(u,v)_0 = \int_0^1 uv\, ds \quad , \quad \|u\|_0 = (u,u)_0^{\frac{1}{2}}. \tag{3.1}$$

We denote by $H^m(\mathbf{I})$, the space of all real-valued functions which together with their first m generalized derivatives belong to $L_2(\mathbf{I})$. This is a Hilbert space with inner product and norm defined by

$$(u,v)_m = \sum_{i=0}^{m} (D^i u, D^i v)_0, \quad \|u\|_m = (u,u)_m^{\frac{1}{2}}. \tag{3.2}$$

We denote by $H_0^m(\mathbf{I})$ the subspace of $H^m(\mathbf{I})$ consisting of functions whose values and whose derivatives up to order $m-1$ vanish at the ends of the interval. That is,

$$H_0^m(\mathbf{I}) := \{v \in H^m(\mathbf{I}) \,:\, D^i v(0) = D^i v(1) = 0 \quad \text{for all} \quad 0 \le i \le m-1\} \tag{3.3}$$

Here the boundary values are understood to be in the sense of traces. The space $H_0^m(\mathbf{I})$ is a closed subspace of $H^m(\mathbf{I})$, and hence a Hilbert space with the inner product $(u,v)_m$ and the norm $\|u\|_m$.

The dual space of $H_0^m(0,1)$ is denoted by $H^{-m}(0,1)$. The action of a member $f \in H^{-m}(0,1)$ on H_0^m is denoted by

$$f(u) = \langle f, u \rangle$$

and the norm on $H^{-m}(0,1)$ is defined by

$$\|f\|_{-m} = \sup_{u \in H_0^m(0,1)} \frac{\langle f, u \rangle}{\|u\|_m}; \qquad u \neq 0.$$

The following inequalities are frequently used in the sequel. For any $u, v \in L_2(\mathbf{I})$,

$$|(u,v)_0| \leq (u,u)_0^{\frac{1}{2}}(v,v)_0^{\frac{1}{2}} \quad \text{(Schwarz inequality);} \tag{3.4}$$

$$\|u\|_0 \leq \|u'\|_0 \quad \text{(Poincaré inequality).} \tag{3.5}$$

The semi-norm $|\, u \,|_1$ on $H^1(0,1)$, defined by $\|u'\|_0$, is equivalent to the H^1-norm, by Poincaré inequality, and hence is a norm on $H_0^1(0,1)$.

It is convenient to introduce product spaces $\mathbf{Q}$ and $\mathbf{V}$ defined over the interval $\mathbf{I}$. Accordingly, we set $\mathbf{Q} = \{L_2(\mathbf{I})\}^3$. This is a Hilbert space with inner product and norm defined by

$$(\mathbf{u},\mathbf{v})_0 = \int_0^1 \mathbf{u}\cdot\mathbf{v}\, ds \quad \text{and} \quad \|\mathbf{u}\|_0 = (\mathbf{u},\mathbf{u})_0^{\frac{1}{2}} \tag{3.6}$$

We also define $\mathbf{V} = \{H_0^1(\mathbf{I})\}^3$; this is a Hilbert space of vector functions with inner product and norm

$$(\mathbf{u},\mathbf{v})_1 = \int_0^1 (\mathbf{u}\cdot\mathbf{v} + \dot{\mathbf{u}}\cdot\dot{\mathbf{v}})\, ds \quad \text{and} \quad \|\mathbf{u}\|_1 = (\mathbf{u},\mathbf{u})_1^{\frac{1}{2}} \tag{3.7}$$

for $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. A superposed dot denotes the vector of derivatives of the components of a vector, that is

$$\dot{\mathbf{v}} = v_1'\mathbf{t} + v_2'\mathbf{n} + v_3'\mathbf{b}. \tag{3.8}$$

This should be distinguished from $\mathbf{v}'$, which represents the derivative of the vector $\mathbf{v}$ and whose components include contributions from the derivatives of the local basis vectors. Thus we have, from (2.1)

$$\begin{aligned} \mathbf{v}' &= \dot{\mathbf{v}} + \Gamma\mathbf{v} \\ &= (v_1' - \frac{v_2}{\rho})\mathbf{t} + (v_2' + \frac{v_1}{\rho} - \tau v_3)\mathbf{n} + (v_3' + \tau v_2)\mathbf{b}. \end{aligned} \tag{3.9}$$

At this stage we introduce a semi-norm on $\mathbf{V}$, defined by $(\mathbf{v}', \mathbf{v}')_0^{\frac{1}{2}}$, and which we denote by $\mid \mathbf{v} \mid_1$. In fact, we will show that the semi-norm $\mid \mathbf{v} \mid_1$ is a norm on $\mathbf{V}$ equivalent to the H^1- norm. that is, there are positive constants c_1 and c_2 such that

$$c_1\|\mathbf{v}\|_1 \leq \mid \mathbf{v} \mid_1 \leq c_2\|\mathbf{v}\|_1 \qquad \text{for all } \mathbf{v} \in \mathbf{V}.$$

Thus $\mathbf{V}$ is a Hilbert space with the norm $\mid \mathbf{v} \mid_1$. We present the proof of these statements in the following Lemmas. Before doing so, it is worth introducing the product space $\mathbf{W}$, defined by

$$\mathbf{W} = \mathbf{V} \times \mathbf{V};$$

this is a Hilbert space with the usual product norm

$$\mid \mathbf{z} \mid_1 = \{\mid \mathbf{v} \mid_1{}^2 + \mid \boldsymbol{\Psi} \mid_1{}^2\}^{\frac{1}{2}} \qquad \text{for } \mathbf{z} = (\mathbf{v}, \boldsymbol{\Psi}) \in \mathbf{W}.$$

We denote by $\mathbf{W}'(\mathbf{I})$, the topological dual space of $\mathbf{W}$; this is endowed with the norm

$$\|\mathcal{F}\|_{-1} = \sup_{\mathbf{z}\in\mathbf{W}} \frac{\langle \mathcal{F}, \mathbf{z}\rangle}{|\mathbf{z}|_1} \quad ; \quad \mathbf{z} \neq \mathbf{0}$$

for $\mathcal{F} \in \mathbf{V}' \times \mathbf{V}' = \mathbf{W}'$.

Lemma 3.1 *Let $\|\cdot\|_1$ be the usual norm on $\mathbf{V}$, and $|\cdot|_1$ the semi-norm defined above. Then there exists a positive constant $c_2(\alpha)$ such that*

$$|\mathbf{v}|_1 \leq c_2(\alpha)\|\mathbf{v}\|_1 \text{ for all } \quad \mathbf{v} \in \mathbf{V}.$$

Furthermore, $c_2(\alpha) = \sqrt{1+\kappa_0^2+\tau_0^2}$, where κ_0 and τ_0 are defined by (2.2).

Proof. Since $\mathbf{v} = u\mathbf{t} + v\mathbf{n} + w\mathbf{b}$, it follows from (3.9)

$$\begin{aligned} \mathbf{v}' \cdot \mathbf{v}' &= (u' - \tfrac{v}{\rho})^2 + (v' + \tfrac{u}{\rho} - \tau w)^2 + (w' + \tau v)^2 \\ &\equiv \mathbf{X}^T\mathbf{A}\mathbf{X} \end{aligned} \tag{3.10}$$

where $\mathbf{X}^T = (u, v, w, u', v', w')$ and

$$\mathbf{A} = \begin{pmatrix} \Gamma^T\Gamma & \Gamma \\ \Gamma^T & \mathbf{I} \end{pmatrix}, \qquad \Gamma \text{ being the Serret-Frenet matrix defined in (2.1)}$$

The characteristic equation of $\mathbf{A}$ is $|\mathbf{A} - \lambda\mathbf{I}| = 0$, which gives

$$\lambda^3(1-\lambda)(1+\tau^2+\kappa^2-\lambda)^2 = 0$$

The eigenvalues λ_i ; $i = 1, \cdots, 6$ of A are thus $0, 0, 0, 1, (1+\tau^2+\kappa^2)$ and $(1+\tau^2+\kappa^2)$.

Since $\mathbf{A}$ is symmetric there exists an orthogonal matrix $\mathbf{P}$ such that

$$\mathbf{P}^{\mathbf{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$$

where $\mathbf{D}$ is the diagonal matrix whose elements are the eigenvalues of $\mathbf{A}$, that is,

$$\mathbf{D} = diag\,[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6]\,.$$

We set $\mathbf{X} = \mathbf{P}\mathbf{Y}$ so that (3.10) becomes

$$\begin{aligned} \mathrm{v}' \cdot \mathrm{v}' &= \mathbf{X}^T\mathbf{A}\mathbf{X} = \mathbf{Y}^{\mathbf{T}}\mathbf{P}^{\mathbf{T}}\mathbf{A}\mathbf{P}\mathbf{Y} \\ &= \mathbf{Y}^T\mathbf{D}\mathbf{Y} \\ &= \lambda_1 y_1^2 + \cdots\cdots + \lambda_6 y_6^2. \end{aligned}$$

But

$$|\,\mathbf{X}\,|^2 = \mathbf{X}^T\mathbf{X} = \mathbf{Y}^{\mathbf{T}}\mathbf{P}^{\mathbf{T}}\mathbf{P}\mathbf{Y} = \mathbf{Y}^T\mathbf{Y} = |\,\mathbf{Y}\,|^2 \qquad (\text{since } \mathbf{P}^{\mathbf{T}}\mathbf{P} = \mathbf{I}\,)$$

as a consequence, we may deduce the inequality

$$\mathrm{v}' \cdot \mathrm{v}' \leq (1 + \tau_0^2 + \kappa_0^2)\,|\,\mathbf{X}\,|^2$$

which yields the desired result. □

Because the minimum eigenvalue of $\mathbf{A}$ is zero, we are unable to derive the reverse inequality by this method. This is obtained by adopting an alternative method.

Lemma 3.2 *The vector space* $\mathbf{V}$ *is an inner product space with inner product defined by*

$$(\mathbf{u}, \mathbf{v}) = \int_0^1 \mathbf{u}' \cdot \mathbf{v}' \, ds$$

where $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbf{V}$ *and*

$$\mathbf{u}' = (u_1' - \tfrac{u_2}{\rho}, u_2' + \tfrac{u_1}{\rho} - \tau u_3, u_3' + \tau u_2).$$

Proof. The inner product $(\cdot, \cdot)$ on the vector space should satisfy the following conditions, for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbf{V}$ and α , $\beta \in I\!R$,

1. $(\alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w}) = \alpha(\mathbf{u}, \mathbf{v}) + \beta(\mathbf{v}, \mathbf{w})$;
2. $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$;
3. $(\mathbf{u}, \mathbf{u}) \geq 0$;
4. $(\mathbf{u},\mathbf{u}) = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

According to the definition of the inner product $(\cdot, \cdot)$, the first three conditions are trivial. To verify condition 4, we see that

$$\begin{aligned} (\mathbf{u}, \mathbf{u}) = 0 \quad &\Leftrightarrow \quad \int_0^1 \mathbf{u}' \cdot \mathbf{u}' \, ds = 0 \\ &\Leftrightarrow \quad \mathbf{u}' = \mathbf{0} \qquad \text{almost everywhere on } \mathbf{I}. \end{aligned}$$

Integrating and applying the boundary conditions, this implies that

$$\mathbf{u} = 0 \qquad \text{almost everywhere on } \mathbf{I}.$$

Hence the proof is completed. □

Thus the semi-norm defined by $|\mathbf{u}|_1 = (\mathbf{u},\mathbf{u})^{\frac{1}{2}}$ is a norm on $\mathbf{V}$, since it is generated by an inner product. However we have yet to prove the norm $|\cdot|_1$ is equivalent to the norm $\|\cdot\|_1$.

Lemma 3.3 *$(\mathbf{V}, |\cdot|_1)$ is a complete inner product space. That is, $\mathbf{V}$ is Hilbert space with the norm $|\cdot|_1$.*

Proof. Suppose that $\{\mathbf{u}_n\}$ is a Cauchy sequence in $\{\mathbf{V}, |\cdot|_1\}$, then

$$\begin{aligned}\lim_{m,n\to\infty} |\mathbf{u}_m - \mathbf{u}_n|_1^2 &= \lim_{m,n\to\infty} \|\mathbf{u}'_m - \mathbf{u}'_n\|_0^2 \\ &= 0;\end{aligned}$$

In other words, the sequence $\{\mathbf{u}'_n\}$ is a Cauchy sequence in $\mathbf{Q}$. We recall that $\mathbf{Q}$ is complete. Hence the sequence $\{\mathbf{u}'_n\}$ has a definite limit, say $\mathbf{w}$, in $\mathbf{Q}$.
That is, $\mathbf{u}'_n \to \mathbf{w}$ in $\mathbf{Q}$ when $n \to \infty$.

From the inequality (Poincaré type which will be proved in the end of this section) $\|\mathbf{u}_n - \mathbf{u}_m\|_0 \leq \|\mathbf{u}'_n - \mathbf{u}'_m\|_0$ for every n,m, the sequence $\{\mathbf{u}_n\}$ is Cauchy in $\mathbf{Q}$. Since $\mathbf{Q}$ is complete, $\{\mathbf{u}_n\}$ has a limit in $\mathbf{Q}$, say $\mathbf{u}$.

We now observe that the following holds for every $n = 1, 2....$

$$\int_0^1 \mathbf{u}'_n \cdot \mathbf{v}\, ds = -\int_0^1 \mathbf{u}_n \cdot \mathbf{v}'\, ds \quad \text{for every } \mathbf{v} \in \mathbf{V}$$

Taking the limit in both sides, we obtain

$$\lim_{n\to\infty} \int_0^1 \mathbf{u}'_n \cdot \mathbf{v}\, ds = -\lim_{n\to\infty} \int_0^1 \mathbf{u}_n \cdot \mathbf{v}'\, ds$$

which yields

$$\int_0^1 \mathbf{w} \cdot \mathbf{v}\, ds = -\int_0^1 \mathbf{u} \cdot \mathbf{v}'\, ds$$

and hence $\int_0^1 (\mathbf{w} - \mathbf{u}') \cdot \mathbf{v}\, ds = 0$.
Since $\mathbf{v}$ is arbitary

$$\mathbf{u}' = \mathbf{w} \quad \text{almost everywhere.}$$

Therefore

$$\lim_{n \to \infty} | \mathbf{u}_n - \mathbf{u} |_1 = 0.$$

Therefore every Cauchy sequence in $(\mathbf{V}, |\cdot|_1)$ is convergent, and hence $(\mathbf{V}, |\cdot|_1)$ is complete. □

Theorem 3.1 *On the vector space* $\mathbf{V}$, *the norms* $|\cdot|_1$ *and* $\|\cdot\|_1$ *are equivalent.*

Proof. Define the operator $T : (\mathbf{V}, \|\cdot\|_1) \to (\mathbf{V}, |\cdot|_1)$ by $\mathbf{v} \mapsto \mathbf{v}$.
Clearly, $\mathbf{T}$ is bijective and continuous. Since we have already proved that $| \mathbf{v} |_1 \leq c_2 \|\mathbf{v}\|_1$, $\mathbf{T}^{-1}$ is continuous. By the bounded inverse theorem(see [40],page 122), there exists a positive constant c_1 such that

$$c_1 \|\mathbf{v}\|_1 \leq | \mathbf{v} |_1 \quad \text{for all } \mathbf{v} \in \mathbf{V},$$

which implies that, over the space $\mathbf{V}$, the norm $|\cdot|_1$ is equivalent to the norm $\|\cdot\|_1$. Hence the proof of the Theorem is complete. □

Hereafter, whenever we refer to the Hilbert space $\mathbf{V}$, we mean the normed space consisting of the linear vector space of functions as defined previously, together with the norm $|\cdot|_1$.

We derive a Poincaré-type inequality for the norm $|\cdot|_1$.

$$\text{Since} \quad \mathbf{u}(s) = \int_0^s \mathbf{u}'(t)\,dt,$$

we have

$$\begin{aligned}
|\,\mathbf{u}(s)\,| &= \left|\int_0^s \mathbf{u}'(t)\,dt\right| \\
&\leq \int_0^s |\,\mathbf{u}'\,|\,dt \\
&\leq \textstyle\int_0^1 |\,\mathbf{u}'\,|\,dt \\
&\leq \{\textstyle\int_0^1 1^2\,dt\}^{\frac{1}{2}}\{\textstyle\int_0^1 |\,\mathbf{u}'\,|^2\,dt\}^{\frac{1}{2}} \\
&= \|\mathbf{u}'\|_0.
\end{aligned}$$

Here we have applied the Schwarz inequality in $\mathbf{Q}$. Squaring and integrating this inequality over $\mathbf{I}$, we have

$$\textstyle\int_0^1 |\,\mathbf{u}(t)\,|^2\,dt \leq \|\mathbf{u}'\|_0{}^2 \displaystyle\int_0^1 dt$$

which leads to the Poincaré inequality

$$\|\mathbf{u}\|_0 \leq \|\mathbf{u}'\|_0 = |\,\mathbf{u}\,|_1 \qquad \square$$

3.2 Standard Variational Problem

We are interested in finding displacement and rotation vector fields $\mathbf{u}(s)$ and $\boldsymbol{\theta}(s)$ defined on $\mathbf{I}$, which satisfy the equilibrium equations derived in Section 2.2,

$$\frac{d\mathbf{N}}{ds} + \mathbf{F} = \mathbf{0}, \tag{3.11}$$

$$\frac{d\mathbf{M}}{ds} + \mathbf{t} \wedge \mathbf{N} = \mathbf{0}, \tag{3.12}$$

with boundary conditions

$$\mathbf{u}(0) = \mathbf{u}(L) = \mathbf{0}, \tag{3.13}$$

$$\boldsymbol{\theta}(0) = \boldsymbol{\theta}(L) = \mathbf{0}. \tag{3.14}$$

Here $\mathbf{N}$ and $\mathbf{M}$ are given as functions of $\mathbf{u}$ and $\boldsymbol{\theta}$, that is
$\mathbf{N} = \mathbf{D}(\mathbf{u}' - \boldsymbol{\theta} \wedge \mathbf{t})$ and $\mathbf{M} = \mathbf{E}\boldsymbol{\theta}'$.

We can recast this system of differential equations in non-dimensional form by defining

$$\bar{s} = \frac{s}{L}, \quad \bar{\mathbf{u}} = \frac{\mathbf{u}}{L},$$

$$d = \frac{I_2}{AL^2}, \quad \boldsymbol{f} = \frac{L^3}{EI_2}\mathbf{F},$$

$$\bar{\mathbf{D}} = \operatorname{diag}\left[1, \frac{Gk_1}{E}, \frac{Gk_2}{E}\right] \text{ and } \bar{\mathbf{E}} = \operatorname{diag}\left[\frac{GJ}{EI_2}, \frac{I_1}{I_2}, 1\right].$$

Substituting in (3.11),(3.12) and then dropping overbars on $\bar{s}$, $\bar{\mathbf{u}}$,$\bar{\mathbf{D}}$ and $\bar{\mathbf{E}}$ for the sake of simplicity, the above system of differential equations becomes

$$d^{-1}\mathbf{D}\boldsymbol{\varepsilon}'(\mathbf{w}) + \boldsymbol{f} = \mathbf{0}, \tag{3.15}$$

$$\mathbf{E}\boldsymbol{\theta}'' + d^{-1}\mathbf{t} \wedge \mathbf{D}\boldsymbol{\varepsilon}(\mathbf{w}) = \mathbf{0}, \tag{3.16}$$

while the boundary conditions become

$$\mathbf{u}(0) = \mathbf{u}(1) = \mathbf{0},$$

$$\boldsymbol{\theta}(0) = \boldsymbol{\theta}(1) = \mathbf{0}. \tag{3.17}$$

Here and hence forth we write $\mathbf{w} = (\mathbf{u}, \boldsymbol{\theta})$. A superimposed prime denotes the derivative with respect to non-dimensionalised arc length s.

We observe that the above equations depend explicitly on a parameter d which is, in general, proportional to the ratio of thickness to the length. For thin beams $d \ll 1$.

To obtain a weak variational form of (3.15- 3.17), we take the scalar product of equations (3.17) and (3.18) with arbitary functions $\mathbf{v}, \boldsymbol{\Psi} \in \mathbf{V}$ respectively, to obtain

$$d^{-1}\boldsymbol{D}(\mathbf{u}' - \boldsymbol{\theta} \wedge \mathbf{t})' \cdot \mathbf{v} + \boldsymbol{f} \cdot \mathbf{v} = \mathbf{0},$$

$$\boldsymbol{E}\boldsymbol{\theta}'' \cdot \boldsymbol{\Psi} + d^{-1}\mathbf{t} \wedge \boldsymbol{D}(\mathbf{u}' - \boldsymbol{\theta} \wedge \mathbf{t}) \cdot \boldsymbol{\Psi} = \mathbf{0}.$$

We now integrate these two equations over the interval [0,1] , then integrate by parts and make use of boundary conditions (3.19) to obtain

$$-\int_0^1 d^{-1}\boldsymbol{D}(\mathbf{u}' - \boldsymbol{\theta} \wedge \mathbf{t}) \cdot \mathbf{v}'\, ds + \int_0^1 \boldsymbol{f} \cdot \mathbf{v}\, ds = \mathbf{0},$$

$$-\int_0^1 \boldsymbol{E}\boldsymbol{\theta}' \cdot \boldsymbol{\Psi}'\, ds + d^{-1}\int_0^1 \mathbf{t} \wedge \boldsymbol{D}(\mathbf{u}' - \boldsymbol{\theta} \wedge \mathbf{t}) \cdot \boldsymbol{\Psi}\, ds = \mathbf{0}.$$

By adding these two equations together and rearranging, we obtain a single variational equation[1]

$$\int_0^1 \boldsymbol{E}\boldsymbol{\theta}' \cdot \boldsymbol{\Psi}'\, ds + d^{-1}\int_0^1 \boldsymbol{D}(\mathbf{u}' - \boldsymbol{\theta} \wedge \mathbf{t}) \cdot (\mathbf{v}' - \boldsymbol{\Psi} \wedge \mathbf{t})\, ds = \int_0^1 \boldsymbol{f} \cdot \mathbf{v}\, ds \qquad (3.20)$$

Since elements of $\boldsymbol{D}$ and $\boldsymbol{E}$ are diagonal with positive elements and we are interested in determining qualitative properties, without loss of generality, we take $\boldsymbol{D} = \boldsymbol{E} = \mathbf{I}$, the identity matrix, for the sake of simplicity. One can thus rewrite the

[1]When $\tau = 0$, $\kappa =$ constant and the binormal component of load is zero then we have $w = \phi = \psi = 0$: then

$$\boldsymbol{\theta}' = \vartheta'\mathbf{b} \quad \textit{and} \quad \mathbf{u}' - \boldsymbol{\theta} \wedge \mathbf{t} = (u' - \tfrac{v}{R})\mathbf{t} + (v' + \tfrac{u}{R} - \vartheta)\mathbf{n}$$

where $R = \kappa^{-1}$. Accordingly, equation (3.20) then gives the variational form for the circular arch loaded in plane.

If tangential and normal components of the load are zero, then $u = v = \vartheta = 0$, and

$$\boldsymbol{\theta}' = (\phi' - \tfrac{\psi}{R})\mathbf{t} + (\psi' + \tfrac{\phi}{R})\mathbf{n}, \qquad \mathbf{u}' - \boldsymbol{\theta} \wedge \mathbf{t} = (w' + \psi)\mathbf{b}$$

where $R = \kappa^{-1}$. With this definition, the equation (3.20) becomes the variational form for the circular arch loaded out of plane.

above variational equation (3.20) as

$$A_d(\mathbf{w},\mathbf{z}) = \langle \mathcal{F}, \mathbf{z} \rangle \qquad \text{for all } \mathbf{z} = (\mathbf{v}, \boldsymbol{\Psi}) \in \mathbf{W} \tag{3.21}$$

where the symmetric bilinear form $A_d : \mathbf{W} \times \mathbf{W} \to I\!R$ is defined by

$$A_d(\mathbf{w},\mathbf{z}) = \int_0^1 \boldsymbol{\theta}' \cdot \boldsymbol{\Psi}' \, ds + d^{-1} \int_0^1 \boldsymbol{\varepsilon}(\mathbf{w}) \cdot \boldsymbol{\varepsilon}(\mathbf{z}) \, ds,$$

and the linear functional $\mathcal{F} : \mathbf{W} \to \mathbf{W}'$ is defined by

$$\langle \mathcal{F}, \mathbf{z} \rangle = \int_0^1 \boldsymbol{f} \cdot \mathbf{v} \, ds.$$

The standard variational problem is thus:

$\mathbf{S_d}$: Given $\mathcal{F} = (\boldsymbol{f}, 0) \in \mathbf{W}'$, find $\mathbf{w} = (\mathbf{u}, \boldsymbol{\theta}) \in \mathbf{W}$ such that

$$A_d(\mathbf{w},\mathbf{z}) = \langle \mathcal{F}, \mathbf{z} \rangle \text{ for all } \mathbf{z} \in \mathbf{W}.$$

3.3 Existence and Uniqueness of Solutions

In the preceding Section, we have formulated the standard variational problem. It is necessary to ask whether or not solutions to S_d exist and, if so, whether these are unique. Sufficient conditions for the existence and uniqueness of the solutions to S_d are [11, 3] :

(S1) : the bilinear form A_d is continuous; that is, there exists a positive constant α_1 such that

$$| A_d(\mathbf{w},\mathbf{z}) | \le \alpha_1 | \mathbf{w} |_1 | \mathbf{z} |_1$$

for all $\mathbf{w}, \mathbf{z} \in \mathbf{W}$;

(S2) : A_d is $\mathbf{W}$- elliptic; that is, there exists a positive constant α_0 such that

$$A_d(\mathbf{z},\mathbf{z}) \geq \alpha_0 |\ \mathbf{z}\ |_1{}^2$$

for all $\mathbf{z} \in \mathbf{W}$;

(S3) : the linear funtional $\mathcal{F}$ is continuous on $\mathbf{W}'$.

The above conditions are verified in the following Lemma.

Lemma 3.4 *(a) For $d \in (0,\ 1]$, the bilinear form $A_d(\cdot,\cdot)$ is $\mathbf{W}$- elliptic and continuous.*
(b) The linear functional $\mathcal{F}$ is continuous.

Proof.
W-ellipticity: For any $\mathbf{z} = (\mathbf{v}, \boldsymbol{\Psi}) \in \mathbf{W}$,

$$\begin{aligned} \|\mathbf{v}'\|_0 &\leq \|\mathbf{v}' - \boldsymbol{\Psi}\wedge\mathbf{t}\|_0 + \|\boldsymbol{\Psi}\wedge\mathbf{t}\|_0 \\ &\leq \|\mathbf{v}' - \boldsymbol{\Psi}\wedge\mathbf{t}\|_0 + \|\boldsymbol{\Psi}\|_0; \quad \text{since } \|\boldsymbol{\Psi}\wedge\mathbf{t}\|_0 \leq \|\boldsymbol{\Psi}\|_0 \\ &\leq \|\mathbf{v}' - \boldsymbol{\Psi}\wedge\mathbf{t}\|_0 + \|\boldsymbol{\Psi}'\|_0 \text{ by the Poincaré inequality.} \end{aligned}$$

Making use of the inequality $(a+b)^2 \leq 2(a^2+b^2)$ for real numbers a and b and adding the term $\|\boldsymbol{\Psi}'\|_0{}^2$ in both sides, we have

$$\begin{aligned} \|\mathbf{v}'\|_0{}^2 + \|\boldsymbol{\Psi}'\|_0{}^2 &\leq \{\|\mathbf{v}' - \boldsymbol{\Psi}\wedge\mathbf{t}\|_0 + \|\boldsymbol{\Psi}'\|_0\}^2 + \|\boldsymbol{\Psi}'\|_0{}^2 \\ &\leq 2\|\mathbf{v}' - \boldsymbol{\Psi}\wedge\mathbf{t}\|_0{}^2 + 3\|\boldsymbol{\Psi}'\|_0{}^2, \end{aligned}$$

and so

$$\begin{aligned} A_d(\mathbf{z},\mathbf{z}) &= \int_0^1 \boldsymbol{\Psi}'\cdot\boldsymbol{\Psi}'\,ds + d^{-1}\int_0^1 (\mathbf{v}' - \boldsymbol{\Psi}\wedge\mathbf{t})\cdot(\mathbf{v}' - \boldsymbol{\Psi}\wedge\mathbf{t})\,ds \\ &\geq \|\boldsymbol{\Psi}'\|_0{}^2 + d^{-1}\|\mathbf{v}' - \boldsymbol{\Psi}\wedge\mathbf{t}\|_0{}^2 \\ &\geq \alpha_0\{\|\mathbf{v}'\|_0{}^2 + \|\boldsymbol{\Psi}'\|_0{}^2\} \\ &= \alpha_0 |\ \mathbf{z}\ |_1{}^2 \end{aligned} \tag{3.22}$$

where $\alpha_0 = \frac{1}{3}$.
Hence $A_d(\cdot,\cdot)$ is $\mathbf{W}$-elliptic.

Continuity: we have

$$\begin{aligned}
|\ A_d(\mathbf{w},\mathbf{z})\ | &\leq |\ \int_0^1 \boldsymbol{\theta}' \cdot \boldsymbol{\Psi}'\, ds\ | + |\ d^{-1}\int_0^1 \boldsymbol{\varepsilon}(\mathbf{w}) \cdot \boldsymbol{\varepsilon}(\mathbf{z})\, ds\ | \\
&\leq \|\boldsymbol{\theta}'\|_0\|\boldsymbol{\Psi}'\|_0 + d^{-1}\|\boldsymbol{\varepsilon}(\mathbf{w})\|_0\|\boldsymbol{\varepsilon}(\mathbf{z})\|_0 \quad \text{(Schwarz inequality).}
\end{aligned}$$

But

$$\begin{aligned}
\|\boldsymbol{\varepsilon}(\mathbf{z})\|_0 &\leq \|\mathbf{v}'\|_0 + \|\boldsymbol{\Psi} \wedge \mathbf{t}\|_0 \\
&\leq \|\mathbf{v}'\|_0 + \|\boldsymbol{\Psi}\|_0 \\
&\leq \|\mathbf{v}'\|_0 + \|\boldsymbol{\Psi}'\|_0;
\end{aligned}$$

Making use of this inequality and the inequality $a + b \leq \sqrt{2}(a^2 + b^2)^{\frac{1}{2}}$ we have

$$\begin{aligned}
|\ A_d(\mathbf{w},\mathbf{z})\ | &\leq \|\boldsymbol{\theta}'\|_0\|\boldsymbol{\Psi}'\|_0 + d^{-1}(\|\mathbf{u}'\|_0 + \|\boldsymbol{\theta}'\|_0)(\|\mathbf{v}'\|_0 + \|\boldsymbol{\Psi}'\|_0) \\
&\leq (\|\boldsymbol{\theta}'\|_0{}^2 + \|\mathbf{u}'\|_0{}^2)^{\frac{1}{2}}(\|\boldsymbol{\Psi}'\|_0{}^2 + \|\mathbf{v}'\|_0{}^2)^{\frac{1}{2}} + \\
&\quad\ d^{-1}\{\sqrt{2}(\|\boldsymbol{\theta}'\|_0{}^2 + \|\mathbf{u}'\|_0{}^2)^{\frac{1}{2}} \cdot \sqrt{2}(\|\boldsymbol{\Psi}'\|_0{}^2 + \|\mathbf{v}'\|_0{}^2)^{\frac{1}{2}}\} \\
&\leq |\ \mathbf{w}\ |_1|\ \mathbf{z}\ |_1 + 2d^{-1}|\ \mathbf{w}\ |_1|\ \mathbf{z}\ |_1 \\
&\leq \alpha_1|\ \mathbf{w}\ |_1|\ \mathbf{z}\ |_1
\end{aligned} \tag{3.21}$$

where $\alpha_1 = (1 + 2d^{-1})$ Thus $A_d(\mathbf{w},\mathbf{z})$ is bounded above and hence, is continuous.
Continuity of $\mathcal{F}$
we have

$$\begin{aligned}
|\ \mathcal{F}(\mathbf{z})\ | &= |\ \int_0^1 \boldsymbol{f} \cdot \mathbf{v}\, ds\ | \\
&\leq \|\boldsymbol{f}\|_{-1}\|\mathbf{v}\|_0 \\
&\leq \|\boldsymbol{f}\|_{-1}\|\mathbf{v}'\|_0 \qquad \text{(Poincaré inequality)} \\
&\leq \|\mathcal{F}\|_{-1}|\ \mathbf{z}\ |_1.
\end{aligned} \tag{3.22}$$

This completes the proof. □

Theorem 3.2 **(a)** *For given $\mathcal{F} \in \mathbf{W}'$ and $d \in (0,\ 1]$, there exists a unique solution $\mathbf{w} = (\mathbf{u}, \boldsymbol{\theta}) \in \mathbf{W}$ to the* problem $\boldsymbol{S}_d$.

(b) *There exists a positive constant c_0, independent of d, such that*

$$| \mathbf{w} |_1 + d^{-1}\|\varepsilon(\mathbf{w})\|_0 \leq c_0\|\mathcal{F}\|_{-1}.$$

Proof.(a): Lemma (3.4) assures the existence and uniqueness of the solution to the problem $\boldsymbol{S}_d$.

(b): For any $\mathbf{q} \in \mathbf{Q}$ a vector function $\bar{\mathbf{z}} \in \mathbf{W}$ can be constructed with the properties

$$\varepsilon(\bar{\mathbf{z}}) = \mathbf{q}, \qquad | \bar{\mathbf{z}} |_1 \leq \bar{c}\|\mathbf{q}\|_0, \tag{i}$$

where $\bar{c}$ is a positive constant (this construction is carried out later in Lemma 3.7).

Then by setting $\mathbf{q} = \varepsilon(\mathbf{w})$, $\bar{\mathbf{z}} = (\mathbf{v}, \boldsymbol{\Psi})$ we find that

$$\begin{aligned} (\boldsymbol{\theta}', \boldsymbol{\Psi}') + d^{-1}\|\varepsilon(\mathbf{w})\|_0{}^2 &= \mathcal{F}(\bar{\mathbf{z}}) \\ d^{-1}\|\varepsilon(\mathbf{w})\|_0{}^2 &\leq \|\mathcal{F}\|_{-1}| \bar{\mathbf{z}} |_1 + \|\boldsymbol{\theta}'\|_0\|\boldsymbol{\Psi}'\|_0(\text{Schwarz inequality}) \\ &\leq (\|\mathcal{F}\|_{-1} + | \mathbf{w} |_1)| \bar{\mathbf{z}} |_1 \qquad \text{(ii)} \end{aligned}$$

From the ellipticity of A_d and the continuity of $\mathcal{F}$ it follows that

$$| \mathbf{w} |_1 \leq \frac{1}{c}\|\mathcal{F}\|_{-1} \tag{iii}$$

Combining (i), (ii)and (ii) we obtain the required result.

3.4 Mixed Variational Problem

To obtain a mixed variational problem corresponding to the standard problem $\mathbf{S_d}$, we first introduce the Lagrange multiplier vector $\mathbf{q}$ by

$$\mathbf{q} = d^{-1}\varepsilon(\mathbf{w}) \tag{3.23}$$

From the definition we see that $\mathbf{q}$ represents the vector of internal forces. We thus have the following mixed variational problem:

$\mathbf{M_d}$: Given $\mathcal{F} = (\boldsymbol{f}, \mathbf{0})$ and $d \in [0,1]$, find $(\mathbf{w}, \mathbf{q}) \in \mathbf{W} \times \mathbf{Q}$ such that

$$a(\mathbf{w},\mathbf{z}) + b(\mathbf{z},\mathbf{q}) = \mathcal{F}(\mathbf{z}) \qquad \forall \mathbf{z} \in \mathbf{W}, \tag{3.24}$$

$$-d(\mathbf{q},\mathbf{r}) + b(\mathbf{w},\mathbf{r}) = 0 \qquad \forall \mathbf{r} \in \mathbf{Q}\,. \tag{3.25}$$

Here the bilinear forms $a : \mathbf{W} \times \mathbf{W} \to \mathbb{R}$ and $b : \mathbf{W} \times \mathbf{Q} \to \mathbb{R}$ are defined by

$$\begin{aligned} a(\mathbf{w},\mathbf{z}) &= \textstyle\int_0^1 \boldsymbol{\theta}' \cdot \boldsymbol{\Psi}'\, ds, \\ b(\mathbf{z},\mathbf{q}) &= \textstyle\int_0^1 \mathbf{q} \cdot \varepsilon(\mathbf{z})\, ds. \end{aligned} \tag{3.26}$$

Solving (3.25) for $\mathbf{q}$,we have

$$\mathbf{q} = d^{-1}\varepsilon(\mathbf{w}) \qquad \text{in } \mathbf{Q}. \tag{3.27}$$

By substitution of this result in equation (3.24), we recover the standard variational problem $\mathbf{S_d}$. This demonstrates the equivalence of the two formulations $\mathbf{S_d}$ and $\mathbf{M_d}$.

Existence and Uniqueness of M_d

The conditions for existence and uniqueness of solution for the above mixed problem are assured by the extension to the case $d \in [0,1]$ of Brezzi's Theorem [11] by Arnold [3].

Sufficient conditions for existence and uniqueness of a solution to the problem M_d are:

M1: the bilinear form $a(\cdot,\cdot)$ is symmetric, positive semidefinite and continuous;

M2: $a(\cdot,\cdot)$ is $\mathbf{X}$ -elliptic; that is, there exists a positive constant c_1 such that

$$a(\mathbf{z},\mathbf{z}) \geq c_1 |\, \mathbf{z}\, |_1^{\,2} \qquad \text{for all } \mathbf{z} \in \mathbf{X}$$

where

$$
\begin{aligned}
\mathbf{X} &:= \{\mathbf{z} = (\mathbf{v}, \boldsymbol{\Psi}) \in \mathbf{W} : b(\mathbf{z}, \mathbf{r}) = 0 \quad \text{for all } \mathbf{r} \in \mathbf{Q}\ \} \\
&= \{\mathbf{z} \in \mathbf{W} : \mathbf{v}' - \boldsymbol{\Psi} \wedge \mathbf{t} = 0\};
\end{aligned}
$$

M3: there exists a constant $\beta > 0$ such that

$$
\sup_{\mathbf{z} \in \mathbf{W}} \frac{b(\mathbf{z}, \mathbf{r})}{|\,\mathbf{z}\,|_1} \geq \beta \|\mathbf{r}\|_0 \text{ for all } \mathbf{r} \in \mathbf{Q} \text{ and } \mathbf{z} \neq \mathbf{0}.
$$

Lemma 3.5 *The bilinear forms* $a : \mathbf{W} \times \mathbf{W} \rightarrow I\!R$ *and* $b : \mathbf{W} \times \mathbf{Q} \rightarrow I\!R$ *are continuous.*

Proof. Continuity of $a(\cdot,\cdot)$:

we have

$$
\begin{aligned}
|\, a(\mathbf{w}, \mathbf{z})\, | &= |\, \textstyle\int_0^1 \boldsymbol{\theta}' \cdot \boldsymbol{\Psi}'\, ds\, | \\
&\leq |\, \textstyle\int_0^1 \boldsymbol{\theta}' \cdot \boldsymbol{\Psi}'\, ds\, | \\
&\leq \|\boldsymbol{\theta}'\|_0 \|\boldsymbol{\Psi}'\|_0 \qquad \text{(by Schwarz inequality)} \\
&\leq \{\|\boldsymbol{\theta}'\|_0{}^2 + \|\mathbf{u}'\|_0{}^2\}^{\frac{1}{2}} \{\|\boldsymbol{\Psi}'\|_0{}^2 + \|\mathbf{v}'\|_0{}^2\}^{\frac{1}{2}} \\
&= |\,\mathbf{w}\,|_1 |\,\mathbf{z}\,|_1.
\end{aligned}
$$

as desired.

Continuity of $b(\cdot,\cdot)$:

Since

$$
\begin{aligned}
\|\varepsilon(\mathbf{z})\|_0 &= \|\mathbf{v}' - \boldsymbol{\Psi} \wedge \mathbf{t}\|_0 \\
&\leq \|\mathbf{v}'\|_0 + \|\boldsymbol{\Psi}'\|_0 \\
&\leq \sqrt{2} |\,\mathbf{z}\,|_1,
\end{aligned}
$$

we have

$$
\begin{aligned}
|\, b(\mathbf{z}, \mathbf{q})\, | &= |\, \textstyle\int_0^1 \mathbf{q} \cdot \varepsilon(\mathbf{z})\, ds\, | \\
&\leq \sqrt{2}\, \|\mathbf{q}\|_0 |\,\mathbf{z}\,|_1,
\end{aligned}
$$

as desired. □

Lemma 3.6 *The bilinear form $a(\cdot,\cdot)$ is* $\mathbf{X}$ *- elliptic. That is, there exists a positive constant c_1, independent of d , such that*

$$a(\mathbf{z},\mathbf{z}) \geq c_1 |\, \mathbf{z} \,|_1^{\,2} \qquad \textit{for every } \mathbf{z} \in \mathbf{X}.$$

Proof. Since A_d is $\mathbf{W}$-elliptic (Lemma 3.4) we have

$$\begin{aligned} \alpha_0 |\, \mathbf{z} \,|_1^{\,2} &\leq a(\mathbf{z},\mathbf{z}) + b(\mathbf{q},\mathbf{z}) \\ &= a(\mathbf{z},\mathbf{z}) \quad \text{for all } \mathbf{z} \in \mathbf{X}. \end{aligned}$$

Hence the ellipticity of a. □

Lemma 3.7 *There exists a constant $\beta > 0$, independent of d, such that*

$$\sup_{\mathbf{z}\in\mathbf{W}} \frac{b(\mathbf{z},\mathbf{r})}{|\, \mathbf{z} \,|_1} \geq \beta \|\mathbf{r}\|_0 \quad \textit{for all } \mathbf{r} \in \mathbf{Q},\ \mathbf{z} \neq \mathbf{0}.$$

Proof. Let $\mathbf{r} = (\lambda, \mu, \xi)$ be an arbitrary element of $\mathbf{Q}$. We proceed as follows:

Step i   Construct $\mathbf{z}_1 = (\mathbf{v}_1, \boldsymbol{\Psi}_1) \in \mathbf{W}$ such that

$$(\boldsymbol{\varepsilon}(\mathbf{z}_1), \mathbf{r}) = \|\lambda\|_0^{\,2}\ ,\ |\, \mathbf{z}_1 \,|_1 \leq c_1 \|\mathbf{r}\|_0.$$

Step ii   Construct $\mathbf{z}_2 = (\mathbf{v}_2, \boldsymbol{\Psi}_2) \in \mathbf{W}$ such that

$$(\boldsymbol{\varepsilon}(\mathbf{z}_2), \mathbf{r}) = \|\mu\|_0^{\,2}\ ,\ |\, \mathbf{z}_2 \,|_1 \leq c_2 \|\mathbf{r}\|_0.$$

Step iii   Construct $\mathbf{z}_3 = (\mathbf{v}_3, \boldsymbol{\Psi}_3) \in \mathbf{W}$ such that

$$(\boldsymbol{\varepsilon}(\mathbf{z}_3), \mathbf{r}) = \|\xi\|_0^{\,2}\ ,\ |\, \mathbf{z}_3 \,|_1 \leq c_3 \|\mathbf{r}\|_0.$$

If we set $\bar{\mathbf{z}} = \mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3$ then

$$\begin{aligned} b(\bar{\mathbf{z}}, \mathbf{r}) &= b(\mathbf{z}_1, \mathbf{r}) + b(\mathbf{z}_2, \mathbf{r}) + b(\mathbf{z}_3, \mathbf{r}) \\ &= \|\lambda\|_0{}^2 + \|\mu\|_0{}^2 + \|\xi\|_0{}^2 = \|\mathbf{r}\|_0{}^2 \\ &\geq \beta\|\mathbf{r}\|_0 |\, \bar{\mathbf{z}} \,|_1 \end{aligned}$$

since

$$\begin{aligned} |\, \bar{\mathbf{z}} \,|_1 &\leq |\, \mathbf{z}_1 \,|_1 + |\, \mathbf{z}_2 \,|_1 + |\, \mathbf{z}_3 \,|_1 \\ &\leq (c_1 + c_2 + c_3)\|\mathbf{r}\|_0 \qquad \text{according to steps (i) , (ii) and (iii).} \end{aligned}$$

It follows that

$$\beta\|\mathbf{r}\|_0 \leq \frac{b(\bar{\mathbf{z}}, \mathbf{r})}{|\, \bar{\mathbf{z}} \,|_1} \leq \sup_{\mathbf{z} \in \mathbf{W}} \frac{b(\mathbf{z}, \mathbf{r})}{|\, \mathbf{z} \,|_1} \qquad (\text{for } \mathbf{z} \neq \mathbf{0}),$$

as required, where $\beta = 1/(c_1 + c_2 + c_3)$.

We next show how the vector $\mathbf{z}_1$ may be constructed; vectors $\mathbf{z}_2$ and $\mathbf{z}_3$ are constructed in a similar manner. We also consider that $\tau(s) \neq 0$ and $\kappa(s) = \frac{1}{\rho} \neq 0$; the degenerate cases may be treated in much the same as in [42].

To construct a vector function $\mathbf{z}_1$ with the desired properties, we begin by solving

$$\varepsilon(\mathbf{z}_1) \equiv \mathbf{v}_1' - \boldsymbol{\Psi}_1 \wedge \mathbf{t} = \lambda \mathbf{t} \qquad \text{for any } (\lambda, 0, 0) \in \mathbf{Q}$$

with $\mathbf{v}_1(0) = \mathbf{0}$, $\boldsymbol{\Psi}_1(0) = \mathbf{0}$ and $\mathbf{v}_1(1) = \mathbf{0}$, $\boldsymbol{\Psi}_1(1) = \mathbf{0}$.

In component form, this can be written as :

$$\begin{aligned} u' - \frac{v}{\rho} &= \lambda \qquad &\text{(i)} \\ v' + \frac{u}{\rho} - \tau w - \vartheta &= 0 \qquad &\text{(ii)} \\ w' + \tau v + \psi &= 0 \qquad &\text{(iii)} \end{aligned}$$

where $\mathbf{v}_1 = (u, v, w)$ and $\boldsymbol{\Psi}_1 = (\phi, \psi, \vartheta)$.

We set

$$\begin{aligned} v &= -\rho\{\textstyle\int_0^1 \lambda(t)\,dt\}p(s), && \text{(iv)}\\ \psi &= \tau\rho\{\textstyle\int_0^1 \lambda(t)\,dt\}p(s), && \text{(v)} \end{aligned}$$

where

$$p(s) = 30s^2(1-s)^2.$$

Note here that v and $\psi \in H_0^1(0,1)$. Then solving (i) for u by integrating and using the boundary condition $u(0) = 0$, we have

$$u(s) = \int_0^s \lambda(t)\,dt - \{\textstyle\int_0^1 \lambda\,dt\} \displaystyle\int_0^s p(t)dt. \qquad \text{(vi)}$$

It can also be shown that $u(1) = 0$ by using the fact that $\int_0^1 p(t)\,dt = 1$.

Equation (iii) gives

$$\begin{aligned} w' &= -\psi - \tau v \\ &= -\tau\rho\{\textstyle\int_0^1 \lambda\,dt\}p(s) + \tau\rho\{\textstyle\int_0^1 \lambda\,dt\}p(s) \\ &= 0. \end{aligned}$$

Integrating and using the boundary condition $w(0) = 0$, we obtain

$$w(s) = 0 \qquad \text{for all } s \in \mathbf{I},$$

Finally, substitution of u, v and w into (ii) gives

$$\begin{aligned} \vartheta(s) &= v' + \tfrac{u}{\rho} - \tau w \\ &= -\rho\{\textstyle\int_0^1 \lambda\,dt\}p'(s) - \rho'\{\textstyle\int_0^1 \lambda\,dt\}p(s) \\ &\quad + \tfrac{1}{\rho}\left\{\int_0^s \lambda dt - \{\textstyle\int_0^1 \lambda\,dt\} \displaystyle\int_0^s p(t)dt\right\} && \text{(vii)} \end{aligned}$$

Since $p'(0) = p'(1) = 0$, we see that $\vartheta(0) = \vartheta(1) = 0$.

The tangential component of the rotation vector is chosen to be zero. that is

$$\phi(s) = 0 \qquad \text{for all } s \in \mathbf{I}.$$

From (v), (vii) and the properties of ρ and τ we can find positive constants β_2 , β_3 , β_4 such that

$$\begin{aligned}
\|\phi' - \frac{\psi}{\rho}\|_0 &\leq \beta_2\|\lambda\|_0, \\
\|\psi' + \frac{\phi}{\rho} - \tau\vartheta\|_0 &\leq \beta_3\|\lambda\|_0, \\
\|\vartheta' + \tau\psi\|_0 &\leq \beta_4\|\lambda\|_0;
\end{aligned}$$

these lead to the required inequality

$$\|\boldsymbol{\Psi}_1'\|_0 \leq \beta\|\lambda\|_0, \qquad \beta > 0 \qquad \text{(viii)}$$

Similarly, we have

$$\begin{aligned}
\|\mathbf{v}_1'\|_0 &= \|\lambda\mathbf{t} + \boldsymbol{\Psi}_1 \wedge \mathbf{t}\|_0 \\
&\leq \|\lambda\|_0 + \|\boldsymbol{\Psi}_1 \wedge \mathbf{t}\|_0 \\
&\leq \|\lambda\|_0 + \|\boldsymbol{\Psi}_1\|_0 \\
&\leq (1 + \beta_1)\|\lambda\|_0 \qquad \text{(ix)}
\end{aligned}$$

where β_1 is a positive constant, depending on the geometry.

Equations (viii) and (ix) yield the result that there is a positive constant c_1 such that

$$|\ \mathbf{z}_1\ |_1 \leq c_1\|\lambda\|_0 \leq c_1\|\mathbf{r}\|_0.$$

In a similar way, we can construct $\mathbf{z}_2$ and $\mathbf{z}_3$ with the required conditions. □

Theorem 3.3 (a) *The mixed variational problem* $\mathbf{M}_\mathrm{d}$ *has a unique solution.*

(b) *For $\mathcal{F} \in \mathbf{W}'$; $d \in [0,1]$, there is a constant C, independent of d, such that*

$$| \mathbf{w} |_1 + \|\mathbf{q}\|_0 \leq C\|\mathcal{F}\|_{-1}.$$

Proof.

The proof of (a) follows from Lemmas (3.5),(3.6) and (3.7).

Part (b) follows by setting $\mathbf{q} = d^{-1}\varepsilon(\mathbf{w})$ in part (b) of the theorem 3.2.

4 Finite Element Approximations

In this Chapter we construct and analyse finite element approximations S_d^h and M_d^h of problems S_d and M_d, and show that solutions to the mixed problem M_d^h are convergent. Finally we discuss the role of selective reduced integration methods.

4.1 Preliminary Notions and Selection of Subspaces

The finite element method for one-dimensional problems is based on approximating the solution by piecewise smooth functions, specifically polynomials, on regular subintervals. In general, the degree of the polynomials is fixed. The finite element approximation of the problems S_d and M_d requires that we construct subspaces of **W** and **Q**. Let

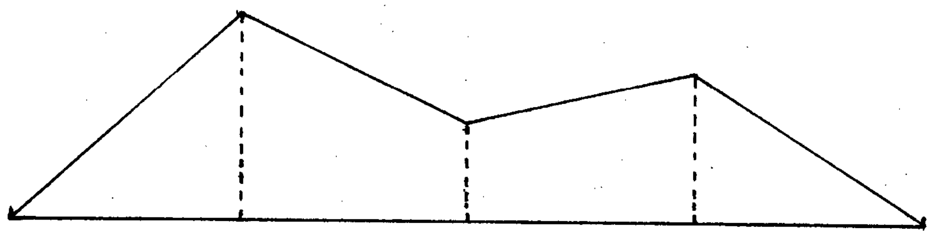
$$\Pi_n : 0 = s_0 < s_1 < \cdots < s_n = 1$$

be a partition of the interval **I**$= [0\ ,\ 1]$ with mesh parameter

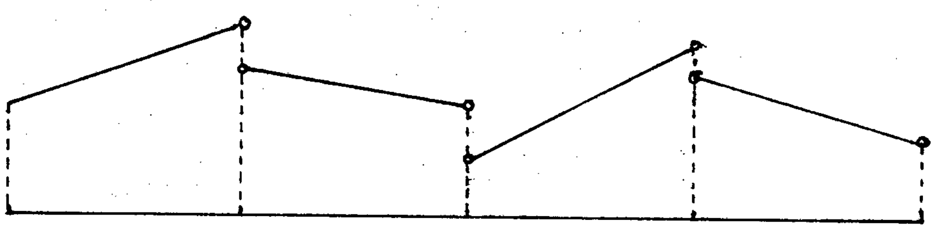
$$h = \max\{(s_i - s_{i-1}) : i = 1, 2, \cdots n\};$$

mesh refinements are assumed to be σ- quasi-uniform: that is, there exists a constant $\sigma > 0$ such that

$$\frac{\min(s_i - s_{i-1})}{\max(s_i - s_{i-1})} \geq \sigma.$$



(a)



(b)

Figure 4.1: Simple example of an element of (a) $\bar{P}^h_{1,0}$ and (b) P^h_1.

Let P^h_r denotes the set of square integrable functions whose restrictions to each subinterval $\Delta_i = [s_{i-1}, s_i]$ are polynomials of degree r; that is,

$$P^h_r := \{v \in L_2(\mathbf{I}) : v\,|_{\Delta_i} \in P_r(\Delta_i); i = 1, 2, \cdots n\}.$$

Here $P_r(\Delta_i)$ denotes the space of polynomials of degree r on Δ_i. We also set, for nonnegative integer k,

$$P^h_{r,k} := P^h_r \cap \mathrm{C}^k[0,1],$$

the subspace of k times continuously differentiable functions, and

$$\bar{P}^h_{r,0} := P^h_{r,0} \cap H^1_0[0,1].$$

Typical members of P^h_r and $\bar{P}^h_{r,0}$ are illustrated in Figure (4.1) for $r = 1$. We define finite-dimensional closed subspaces $\mathbf{V}^h$ of $\mathbf{V}$ and $\mathbf{Q}^h$ of $\mathbf{Q}$ by

$$\mathbf{V}^h = \{\bar{P}^h_{r,0}\}^3, \quad \text{and} \quad \mathbf{Q}^h = \{P^h_{r-1}\}^3.$$

We also define $\mathbf{W}^h = \mathbf{V}^h \times \mathbf{V}^h$, which is a closed subspace of $\mathbf{W}$.

We will make use of the L_2-orthogonal projections $\pi_r : L_2(\mathbf{I}) \to P^h_r$ and $\pi^0_r : L_2(\mathbf{I}) \to \bar{P}^h_{r,0}$ defined by $(q - \pi_r q, q_r) = 0 \qquad \text{for all } q \in L_2(\mathbf{I}),\ q_r \in P^h_r$ and $(q - \pi^0_r q, q_r) = 0 \qquad \text{for all } q \in L_2(\mathbf{I}),\ q_r \in \bar{P}^h_{r,0}$. L_2-orthogonal projections of $\mathbf{V}$ onto $\{P^h_r\}^3$ and $\mathbf{V}^h$ are denoted by $\pi_r\mathbf{u} = (\pi_r u, \pi_r v, \pi_r w)$ and $\pi^0_r\mathbf{u} = (\pi^0_r u, \pi^0_r v, \pi^0_r w)$.

We also frequently use the following best approximation property
$\|u - \pi_{r-1}u\|_0 \le ch^r|u|_r \qquad \text{for } u \in H^r.$

4.2 Analysis of the Discrete Problem

In this Section, we formulate the discrete problems and analyse the stability and convergence of the solutions, as well as the conditions under which the commonly employed selective reduced integration versions of the standard problem are equivalent to the mixed problem.

We define

S^h_d: Given $\mathcal{F} = (\boldsymbol{f}, \mathbf{0}) \in \mathbf{W}'$, find $\mathbf{w}_h = (\mathbf{u}_h, \boldsymbol{\theta}_h) \in \mathbf{W}^h$ such that

$$A_d(\mathbf{w}_h, \mathbf{z}_h) = \langle \mathcal{F}, \mathbf{z}_h \rangle \text{ for all } \mathbf{z}_h \in \mathbf{W}^h, \tag{4.1}$$

and

$\boldsymbol{M}_d^h$: Given $\mathcal{F} = (\boldsymbol{f}, 0) \in \mathbf{W}'$, find $(\mathbf{w}_h, \mathbf{q}_h) \in \mathbf{W}^h \times \mathbf{Q}^h$ such that

$$a(\mathbf{w}_h, \mathbf{z}_h) + b(\mathbf{z}_h, \mathbf{q}_h) = \langle \mathcal{F}, \mathbf{z}_h \rangle \qquad \text{for all } \mathbf{z}_h \in \mathbf{W}^h, \tag{4.2}$$

$$-d(\mathbf{q}_h, \mathbf{r}_h) + b(\mathbf{w}_h, \mathbf{r}_h) = 0 \qquad \text{for all } \mathbf{r}_h \in \mathbf{Q}^h. \tag{4.3}$$

From (4.3) we deduce that, for $d \neq 0$,

$$(d^{-1}\varepsilon(\mathbf{w}_h) - \mathbf{q}_h, \mathbf{r}_h) = 0 \qquad \text{for all } \mathbf{r}_h \in \mathbf{Q}^h \tag{4.4}$$

so that the discrete internal force field can be obtained as the $\mathbf{Q}$-projection of the strain field onto $\mathbf{Q}^h$. That is,

$$\mathbf{q}_h = d^{-1}\pi_{r-1}\varepsilon(\mathbf{w}_h) \in \mathbf{Q}^h. \tag{4.5}$$

Note here that the projection is only applied to components of vector not to the basis vectors.

The force vector $\mathbf{q}_h$ may be eliminated from (4.2) using (4.5). Thus the mixed variational problem $\boldsymbol{M}_d^h$ is equivalent to the problem

$(\boldsymbol{S}_d^h)_\pi$: Given $\mathcal{F} \in \mathbf{W}'$, find $\mathbf{w}_h \in \mathbf{W}^h$ such that

$$\bar{A}_d(\mathbf{w}_h, \mathbf{z}_h) = \langle \mathcal{F}, \mathbf{z}_h \rangle \qquad \text{for all } \mathbf{z}_h \in \mathbf{W}^h, \tag{4.6}$$

where the bilinear form $\bar{A}_d : \mathbf{W} \times \mathbf{W} \to I\!R$ is defined by

$$\bar{A}_d(\mathbf{w}, \mathbf{z}) = a(\mathbf{w}, \mathbf{z}) + d^{-1}(\pi_{r-1}\varepsilon(\mathbf{w}), \pi_{r-1}\varepsilon(\mathbf{z})) \quad \text{for all } \mathbf{w}, \mathbf{z} \in \mathbf{W}. \tag{4.7}$$

By comparing the discrete counterpart of the bilinear form $A_d(\cdot,\cdot)$ with the bilinear form $\bar{A}_d(\cdot,\cdot)$, we see that $\bar{A}_d(\cdot,\cdot) \neq A_d(\cdot,\cdot)$. Thus problems $\boldsymbol{S}_d^h$ and $\boldsymbol{M}_d^h$ are no longer equivalent as in the continuous case. But of course $\boldsymbol{M}_d^h$ is equivalent to $(\boldsymbol{S}_d^h)_\pi$.

Existence and Uniqueness

By Lemma 3.4, problem S_d^h has a unique solution $\tilde{\mathbf{w}}_h$ in $\mathbf{W}^h$, since $\mathbf{W}^h$ is a closed subspace of $\mathbf{W}$. Studies of special cases of S_d^h by, for example Arnold [3] (for straight beams) and Kikuchi [22] (for circular arches) make it clear that, while S_d^h does have a unique solution, this solution does not necessarily converge to the solution of S_d, due to the problem of locking. Therefore we focus attention on M_d^h. In order to show the existence, uniqueness and stablity of the solution to problem M_d^h, we require that the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, defined in (3.28), satisfy the discrete version of conditions analogous to the conditions **M1- M3** for existence and uniqueness of the solution to the continuous problem M_d. These are :

$(M1)_h$: $a(\cdot,\cdot)$ is symmetric and positive semi-definite;

$(M2)_h$: $a(\cdot,\cdot)$ is $\boldsymbol{X}^h$-elliptic; that is, there exists a positive constant c_h such that

$$a(\mathbf{z}_h, \mathbf{z}_h) \geq c_h |\, \mathbf{z}_h \,|_1^2$$

for all $\mathbf{z}_h \in X^h$, where

$$\begin{aligned} X^h &:= \{\mathbf{z}_h \in \mathbf{W}^h : b(\mathbf{z}_h, \mathbf{q}_h) = 0 \ \text{ for all } \mathbf{q}_h \in \mathbf{Q}^h\} \\ &= \{\mathbf{z}_h \in \mathbf{W}^h : \pi_{r-1}(\mathbf{v}_h' - \boldsymbol{\Psi}_h \wedge \mathbf{t}) = \mathbf{0}\}; \end{aligned}$$

$(M3)_h$: there exists a positive constant β_h, independent of h, such that

$$\sup_{\mathbf{z}_h \neq \mathbf{0}} \frac{b(\mathbf{z}_h, \mathbf{q}_h)}{|\, \mathbf{z}_h \,|_1} \geq \beta_h \|\mathbf{q}_h\|_0$$

for all $\mathbf{q}_h \in \mathbf{Q}^h$.

Before establishing these conditions, let us prove the following error estimates under the norm $|\cdot|_1$.

Lemma 4.1 *Let $\pi_{r-1} : \mathbf{V}^h \to \mathbf{Q}^h$ denote the* $\mathbf{Q}$*-projection. Then there exists a constant $c_0 > 0$ such that*

$$\|\mathbf{v}'_h - \pi_{r-1}\mathbf{v}'_h\|_0 \leq c_0 h \mid \mathbf{v}_h \mid_1$$

for every $\mathbf{v}_h \in \mathbf{V}^h$.

Proof. From the standard finite element interpolation theory and the well known best approximation property [13, 40] of orthogonal projections,
for every $u \in H^1$,

$$\|u - \pi_{r-1}u\|_0 \leq k_0 h |u|_1 \qquad \text{(i)}$$

where $k_0 > 0$. Now let $\mathbf{v}_h = (u_h, v_h, w_h) \in \mathbf{V}^h$: then

$$\begin{aligned}
\|\mathbf{v}'_h - \pi_{r-1}\mathbf{v}'_h\|_0 &\leq \|(u'_h - \kappa v_h) - \pi_{r-1}(u'_h - \kappa v_h)\|_0 \\
&\quad + \|(v'_h + \kappa u_h - \tau w_h) - \pi_{r-1}(v'_h + \kappa u_h - \tau w_h)\|_0 \\
&\quad + \|(w'_h + \tau v_h) - \pi_{r-1}(w'_h + \tau v_h)\|_0 \\
&\leq \{\|\kappa v_h - \pi_{r-1}(\kappa v_h)\|_0 + \|\kappa u_h - \pi_{r-1}(\kappa u_h)\|_0\} \\
&\quad + \{\|\tau w_h - \pi_{r-1}(\tau w_h)\|_0 + \|\tau v_h - \pi_{r-1}(\tau v_h)\|_0\} \\
&\quad (\text{since } (u'_h, v'_h, w'_h) \in \mathbf{Q}^h) \\
&\leq c_0 h \mid \mathbf{v}_h \mid_1.
\end{aligned}$$

using (i), since $\kappa,\tau \in \mathrm{C}(\mathbf{I})$. Here c_0 depends on the geometry of the curve. □

Lemma 4.2 *For sufficiently small h there exists a positive constant α_h, independent of h, such that*

$$a(\mathbf{z}_h, \mathbf{z}_h) \geq \alpha_h \mid \mathbf{z}_h \mid_1{}^2 \qquad \textit{for every } \mathbf{z}_h \in \mathbf{X}^h$$

where $\mathbf{X}^h := \{\mathbf{z}_h \in \mathbf{W}^h : \pi_{r-1}(\mathbf{v}'_h - \boldsymbol{\Psi}_h \wedge \mathbf{t}) = \mathbf{0}\}$.

Proof. For any $\mathbf{z}_h \in \mathbf{X}^h$,

$$\pi_{r-1}(\mathbf{v}_h' - \Psi_h \wedge \mathbf{t}) = \mathbf{0}$$

so that $\pi_{r-1}\mathbf{v}_h' = \pi_{r-1}(\Psi_h \wedge \mathbf{t})$. As a consequence

$$\begin{aligned} \|\pi_{r-1}\mathbf{v}_h'\|_0 &= \|\pi_{r-1}(\Psi_h \wedge \mathbf{t})\|_0 \\ &\le \|\Psi_h \wedge \mathbf{t}\|_0 \le \|\Psi_h\|_0. \end{aligned}$$

Making use of this inequality, we have

$$\begin{aligned} a(\mathbf{z}_h, \mathbf{z}_h) &= \int_0^1 \Psi_h' \cdot \Psi_h' \, ds \\ &= \|\Psi_h'\|_0^2 \\ &\ge \tfrac{1}{2}\{\|\Psi_h'\|_0^2 + \|\Psi_h\|_0^2\} \text{ (by the Poincaré inequality)} \\ &\ge \tfrac{1}{2}\{\|\Psi_h'\|_0^2 + \|\pi_{r-1}\mathbf{v}_h'\|_0^2\}. \end{aligned}$$

Now consider

$$\begin{aligned} \|\mathbf{v}_h'\|_0 &\le \|\mathbf{v}_h' - \pi_{r-1}\mathbf{v}_h'\|_0 + \|\pi_{r-1}\mathbf{v}_h'\|_0 \\ &\le c_0 h \|\mathbf{v}_h'\|_0 + \|\pi_{r-1}\mathbf{v}_h'\|_0; \end{aligned}$$

by Lemma 4.1. Thus

$$(1 - c_0 h)\|\mathbf{v}_h'\|_0 \le \|\pi_{r-1}\mathbf{v}_h'\|_0,$$

and so

$$a(\mathbf{z}_h, \mathbf{z}_h) \ge \tfrac{1}{2}\{\|\Psi_h'\|_0^2 + (1 - c_0 h)^2 \|\mathbf{v}_h'\|_0^2\}.$$

For $h \le h_0$ where $h_0 = \frac{1}{c_0}$, there exists $\alpha_h > 0$, independent of h, such that

$$a(\mathbf{z}_h, \mathbf{z}_h) \ge \alpha_h |\, \mathbf{z}_h \,|_1^2,$$

with $\alpha_h = \frac{1}{2} \min \{1, (1 - c_0 h)^2\}$.

Lemma 4.3 *For sufficiently small h there exists a positive constant β_h, independent of h, such that*

$$\sup_{0 \neq \mathbf{z}_h \in \mathbf{W}^h} \frac{b(\mathbf{z}_h, \mathbf{q}_h)}{|\mathbf{z}_h|_1} \geq \beta_h \|\mathbf{q}_h\|_0 \quad \textit{for all } \mathbf{z}_h \in \mathbf{W}^h \ .$$

Proof. We follow roughly the same procedure of the corresponding proof in Lemma 3.7. We begin by choosing two functions $m(s)$ and $n(s) \in \bar{P}^h_{r,0}$, $r \geq 1$, with the following properties:
if functions $M(s)$ and $N(s)$ are defined by

$$M(s) = \int_0^s \pi_{r-1} m(t)\, dt \quad \text{and} N(s) = \int_0^s \pi_{r-1} n(t)\, dt$$

then we require that $M(1) \neq 0$, $N(1) \neq 0$ and

$$\delta \equiv \frac{M(1)}{N(1)} \int_0^1 \pi_{r-1}(\kappa N(t))\, dt - \int_0^1 \pi_{r-1}(\kappa M(t))\, dt \neq 0.$$

For sufficiently large number of elements this is always possible[42]. We now set $f_h = am(s) + bn(s) \in \bar{P}^h_{r,0}$, where a and b are arbitary constants to be determined, and solve the equation

$$v_h' = \pi_{r-1} f_h(s) \qquad \text{(i)}$$

with the boundary condition $v_h(0) = 0$, to obtain

$$v_h(s) = aM(s) + bN(s). \qquad \text{(ii)}$$

The boundary condition $v_h(1) = 0$ yields

$$aM(1) + bN(1) = 0. \qquad \text{(iii)}$$

Now solve for u_h the equation

$$u_h' - \pi_{r-1}(\kappa v_h) = \lambda_h, \qquad u_h(0) = 0,$$

to obtain

$$\begin{aligned} u_h(s) &= \int_0^s \lambda_h\, dt + \int_0^s \pi_{r-1}(\kappa v_h)\, dt \\ &= \int_0^s \lambda_h\, dt + a \int_0^s \pi_{r-1}(\kappa M(t))\, dt + b \int_0^s \pi_{r-1}(\kappa N(t))\, dt. \end{aligned}$$

The requirement $u_h(1) = 0$ yields

$$a \int_0^1 \pi_{r-1}(\kappa M(t))\, dt + b \int_0^1 \pi_{r-1}(\kappa N(t))\, dt = -\int_0^1 \lambda_h(t)\, dt \qquad \text{(iv)}$$

From equations (iii) and (iv) we solve for a and b, to obtain

$$a = \frac{1}{\delta}\int_0^1 \lambda_h(t)\, dt \quad \text{and } b = -\frac{M(1)}{N(1)}\, a.$$

Next we choose $g_h(s) \in \bar{P}^h_{r,0}$ such that $\int_0^1 \pi_{r-1} g_h(t)\, dt \neq 0$ and set

$$\psi_h(s) = \alpha\, g_h(s)$$

where α is a constant to be determined. Solving the equation

$$w_h' + \pi_{r-1}(\tau v_h) + \pi_{r-1}\psi_h = 0 \qquad \text{with } w_h(0) = 0$$

we have

$$\begin{aligned} w_h(s) &= -\int_0^s \pi_{r-1}(\tau v_h + \psi_h)\, dt \\ &= -\int_0^s \pi_{r-1}(\tau v_h)\, dt - \alpha \int_0^s \pi_{r-1} g_h\, dt \end{aligned}$$

where $\mathbf{v}_h$ is given by (ii). By using the condition $w_h(1) = 0$ we have

$$\alpha = -\frac{\int_0^1 \pi_{r-1}(\tau v_h)\,dt}{\int_0^1 \pi_{r-1} g_h\,dt}.$$

Finally we set

$$\vartheta_h = f_h + \pi_r^0(\kappa u_h - \tau w_h);$$

clearly ϑ_h belongs to $\bar{P}_{r,0}^h$.

we choose the tangential component of the rotation vector is to be zero. That is

$$\phi_h(s) = 0 \quad \text{for all } s \in [0,1].$$

Next we see that

$$\begin{aligned} v_h' + \pi_{r-1}(\kappa u_h - \tau w_h - \vartheta_h) &= \pi_{r-1} f_h + \pi_{r-1}(\kappa u_h - \tau w_h) \\ &\quad - \{\pi_{r-1} f_h + \pi_{r-1}\pi_r^0(\kappa u_h - \tau w_h)\} \\ &= \pi_{r-1}\{(\kappa u_h - \tau w_h) - \pi_r^0(\kappa u_h - \tau w_h)\} \\ &\neq 0. \end{aligned}$$

Thus we have constructed $\mathbf{z}_{h1} = (u_h, v_h, w_h, \phi_h, \psi_h, \vartheta_h) \in \mathbf{W}^h$ such that

$$\begin{array}{lll} \boldsymbol{\varepsilon}(\mathbf{z}_{h1}) \cdot \mathbf{t} & \equiv u_h' - \pi_{r-1}(\kappa v_h) & = \lambda_h, \\ \boldsymbol{\varepsilon}(\mathbf{z}_{h1}) \cdot \mathbf{n} & \equiv v_h' + \pi_{r-1}(\kappa u_h) - \pi_{r-1}(\tau w_h) - \pi_{r-1}(\vartheta_h) & = \pi_{r-1}\{(\kappa u_h - \tau w_h) \\ & & \quad -\pi_r^0(\kappa u_h - \tau w_h)\}, \\ \boldsymbol{\varepsilon}(\mathbf{z}_{h1}) \cdot \mathbf{b} & \equiv w_h' + \pi_{r-1}(\tau v_h) + \pi_{r-1}(\psi_h) & = 0. \end{array}$$

Hence if $\mathbf{r}_h = (\lambda_h, \mu_h, \xi_h) \in \mathbf{Q}^h$ then

$$\begin{aligned} b(\mathbf{z}_{h1}, \mathbf{r}_h) &= (\varepsilon(\mathbf{z}_{h1}), \mathbf{r}_h) \\ &= (\pi_{r-1}\varepsilon(\mathbf{z}_{h1}), \mathbf{r}_h) \\ &= \|\lambda_h\|_0{}^2 + (\mu_h, \pi_{r-1}\{(\kappa u_h - \tau w_h) - \pi_r^0(\kappa u_h - \tau w_h)\}). \end{aligned}$$

Now

$$\begin{aligned} &(\mu_h, \pi_{r-1}\{(\kappa u_h - \tau w_h) - \pi_r^0(\kappa u_h - \tau w_h)\}) \\ &\geq -|(\mu_h, \pi_{r-1}\{(\kappa u_h - \tau w_h) - \pi_r^0(\kappa u_h - \tau w_h)\})| \\ &\geq -\|\mu_h\|_0 \, \|\pi_{r-1}\{(\kappa u_h - \tau w_h) - \pi_r^0(\kappa u_h - \tau w_h)\}\|_0 \\ &\geq -\|\mu_h\|_0 \, \|(\kappa u_h - \tau w_h) - \pi_r^0(\kappa u_h - \tau w_h)\|_0 \\ &\geq -C\, h\|\mu_h\|_0 |\, \mathbf{z}_{h1} \,|_1 \end{aligned}$$

by standard interpolation results, equivalence of $|\cdot|_1$ to H^1-type norm and the continuity of κ and τ.

Thus

$$b(\mathbf{z}_{h1}, \mathbf{r}_h) \geq \|\lambda_h\|_0{}^2 - C\, h\|\mu_h\|_0 |\, \mathbf{z}_{h1} \,|_1.$$

We first show that $\|\mathbf{v}_{h1}\|_0 \leq \bar{\beta}_1 \|\lambda_h\|_0$. It is worth noting that, in view of the constructed functions

$|a| \leq c_1\|\lambda_h\|_0$, $|b| \leq c_2\|\lambda_h\|_0$, $\|\pi_{r-1} f_h\|_0 \leq c_3\|\lambda_h\|_0$ and $|\alpha| \leq c_4\|\lambda_h\|_0$.

Now we observe that

$u_h' - \kappa v_h = \lambda_h + \pi_{r-1}(\kappa v_h) - \kappa v_h$

so that

$$\begin{aligned} \|u_h' - \kappa v_h\|_0 &\leq \|\lambda_h\|_0 + \|\pi_{r-1}(\kappa v_h) - \kappa v_h\|_0 \\ &\leq \|\lambda_h\|_0 + A_0\|v_h\|_0 \qquad (A_0 \text{ is a constant}) \\ &\leq c_2\|\lambda_h\|_0; \qquad\qquad \text{(a)} \end{aligned}$$

similarly

$$\begin{aligned} v_h' + \kappa u_h - \tau w_h &= \pi_{r-1} f_h + \kappa \int_0^s \lambda_h(t)\,dt \\ &\quad + a\kappa \int_0^s \pi_{r-1}(\kappa M(t))\,dt + b\kappa \int_0^s \pi_{r-1}(\kappa N(t))\,dt \\ &\quad + \tau \int_0^s \pi_{r-1}(\tau v_h)\,dt + \alpha\tau \int_0^s \pi_{r-1} g_h\,dt \end{aligned}$$

so that

$$\|v_h' + \kappa u_h - \tau w_h\|_0 \le c_3 \|\lambda_h\|_0. \qquad \text{(b)}$$

Finally,

$$\begin{aligned} w_h' + \tau v_h &= -\pi_{r-1}(\tau v_h + \psi_h) + \tau v_h \\ &= -\alpha\, \pi_{r-1} g_h + \tau v_h - \pi_{r-1}(\tau v_h) \end{aligned}$$

and hence

$$\|w_h' + \tau v_h\|_0 \le c_4 \|\lambda_h\|_0. \qquad \text{(c)}$$

From (a),(b) and (c), we can thus find a positive constant $\bar{\beta}_1$, independent of h, such that

$$\|\mathbf{v}_{h1}'\|_0 \le \bar{\beta}_1 \|\lambda_h\|_0. \qquad \text{(A)}$$

Next we would like to show that

$$\|\boldsymbol{\Psi}_{h1}'\|_0 \le \bar{\beta}_2 \|\lambda_h\|_0$$

where $\bar{\beta}_2 > 0$.

From constructed functions $\phi_h' - \kappa\psi_h = -\kappa\alpha g_h$, so that

$$\|\phi_h' - \kappa\psi_h\|_0 \leq c_5\|\lambda_h\|_0. \tag{d}$$

Similarly,
$\psi_h' + \kappa\phi_h - \tau\vartheta_h = -\alpha g_h' - \tau\{f_h + \pi_\tau^0(\kappa u_h) - \pi_\tau^0(\tau w_h)\};$
this leads to the inequality

$$\|\psi_h' + \kappa\phi_h - \tau\vartheta_h\|_0 \leq c_6\|\lambda_h\|_0 \tag{e}$$

where $c_6 > 0$.
Finally, $\vartheta_h' + \tau\psi_h = \{f_h + \pi_\tau^0(\kappa u_h) - \pi_\tau^0(\tau w_h\}' + \tau\,\alpha\, g_h,$
so that we can find a constant $c_7 > 0$ such that

$$\|\vartheta_h' + \tau\psi_h\|_0 \leq c_7\|\lambda_h\|_0. \tag{f}$$

Inequalities (d), (e) and (f) yield

$$\|{\Psi_{h_1}}'\|_0 \leq \bar{\beta}_2\|\lambda_h\|_0. \tag{B}$$

From (A) and (B), we obtain the function $\mathbf{z}_{h1}$ with the required condition
$|\ \mathbf{z}_{h1}\ |_1 \leq \bar{c}_1\|\lambda_h\|_0 \leq \bar{c}_1\|\mathbf{r}_h\|_0.$ $\qquad (*)$.

Now the bound for $b(\mathbf{z}_{h1}, \mathbf{r}_h)$ becomes

$$\begin{aligned} b(\mathbf{z}_{h1}, \mathbf{r}_h) &\geq \|\lambda_h\|_0^{\,2} - c\,h\,\|\lambda_h\|_0\|\mu_h\|_0 \ \ (\text{from } (*)) \\ &\geq (1 - B\,h)\|\lambda_h\|_0^{\,2} - B\,h\|\mu_h\|_0^{\,2} \end{aligned}$$

where $B = \frac{-c}{2}$.

In a similar way we can construct functions $\mathbf{z}_{h2}$ and $\mathbf{z}_{h3}$ and, we will obtain the bound like above. Therefore finally we obtain

$$\begin{aligned} b(\bar{\mathbf{z}}_h, \mathbf{r}_h) &\geq (1-\nu_1 h)\|\lambda_h\|_0{}^2 + (1-\nu_2 h)\|\mu_h\|_0{}^2 + (1-\nu_3 h)\|\xi_h\|_0{}^2 \\ &\geq (1-\nu_h h)\|\mathbf{r}_h\|_0{}^2 \end{aligned}$$

where $\nu_h = \max\{\nu_1, \nu_2, \nu_3\}$.

For $h \leq h_0$, where $h_0 = \frac{1}{\nu_h}$, there exists a constant $\bar{\nu} > 0$, independent of h, such that

$$b(\bar{\mathbf{z}}_h, \mathbf{r}_h) \geq \bar{\nu}\|\mathbf{r}_h\|_0{}^2$$

This completes the proof of the Lemma. □

Theorem 4.1 *(a) For given $\mathcal{F} = (\boldsymbol{f}, 0) \in \mathbf{W}'$ and $d \in [0,1]$, there exists a unique solution $\mathbf{w}_h = (\mathbf{u}_h, \boldsymbol{\theta}_h) \in \mathbf{W}^h$, $\mathbf{q}_h \in \mathbf{Q}^h$ to the problem M_d^h with the estimate*

$$|\,\mathbf{w}_h\,|_1 + \|\mathbf{q}_h\|_0 \leq C\,\|\mathcal{F}\|_{-1} \tag{4.8}$$

holds where $C > 0$, independent of h and d.
(b) If $(\mathbf{w}, \mathbf{q}) \in \mathbf{W} \times \mathbf{Q}$ is the solution of the problem M_d and $(\mathbf{w}_h, \mathbf{q}_h) \in \mathbf{W}^h \times \mathbf{Q}^h$ is the solution of M_d^h then for $\mathcal{F} \in \{H^{r-1}(0,1)\}^6$, there is a constant c_1, independent of h and d, such that

$$|\,\mathbf{w} - \mathbf{w}_h\,|_1 + \|\mathbf{q} - \mathbf{q}_h\|_0 \leq c_1\,h^r. \tag{4.9}$$

Proof. The proof of the theorem follows from the above Lemma 4.2 and 4.3. Part(b) follows from the Lemma 4.1, the standard approximation theory and the part(a).

4.3 Mixed Methods and Selective Reduced Integration

In this Section we show how exact numerical integration rule leads to locking in the standard problem. We then show how employing a reduced integration rule on the term involving d^{-1} in A_d renders the problem equivalent to M_d^h.

Numerical Integration

From a computational point of view, numerical integration is the most convenient way of carrying out the integration. It has become common practice to use Gaussian quadrature for this purpose. Let $I_r(\cdot)$ denote the quadrature rule (for a continuous function f) then

$$I_r(f) = \sum_{e=1}^{E} I_e(f) \quad , \quad I_e(f) = \sum_{j=1}^{G} W_j^e f(\xi_j)$$

where E is the number of elements, W_j^e the quadrature weights, and ξ_j the quadrature points in element $\Omega_e \subset [0,1]$. Suppose that $f \in P_{2r}^h$: then we need $r+1$ quadrature points to integrate f exactly. In other words, r quadrature points can integrate exactly a function whose restriction to Ω_e is a polynomial of degree $\leq 2r-1$.

In the case of selective reduced integration we use a numerical quadrature scheme for evaluating the "penalty" integral $\int_e \varepsilon(\mathbf{w}_h)\varepsilon(\mathbf{z}_h)\, ds$ which is of one order lower than that required to integrate this term exactly.

We formulate the following reduced integration version of the discrete standard problem S_d^h.

Reduced standard problem

$S_d^{h,red}$ Given $\mathcal{F} \in \mathbf{W}'$ and $d \in (0,1]$, find $\hat{\mathbf{w}}_h \in \mathbf{W}^h$ such that

$$(\hat{\theta}_h', \Psi_h') + d^{-1} I_r\{(\varepsilon(\hat{\mathbf{w}}_h), \varepsilon(\mathbf{z}_h))\} = \langle \mathcal{F}, \mathbf{z}_h \rangle \tag{4.10}$$

for all $\mathbf{z}_h \in \mathbf{W}^h$.

Constraints

We shall now proceed to give a heuristic argument of why exact integration rule causes problem, for the case of a thin beam.

When the beam is thin, parameter d is very small in which case the states of inextensional and shearless deformations are enforced by the presence of large multiplier (d^{-1}). The resulting constraint equations produce excessively stiff solutions. This has led many investigators to use 'Reduced integration' for standard finite element method. Such devices have been advocated by Zienkiewicz [49] and Hughes [20] and others. Prathap and Bhasyam [34] showed reduced integration produces properly coupled constraints equations and eliminates some or all spurious constaraints. This can be seen more precisely by looking at the stiffness contributions of the axial and shear terms.

We now show how exact integration of the element stiffness matrix leads to an overly stiff element. Consider the 'linear curved beam element' of length h with constant curvature where we use linear displacements and rotations shape functions. In this case we require we require two-point Gaussian quadrature to integrate exactly.

For linear local basis functions N_1 and N_2
$I_2(N_1N_1) = I_2(N_2N_2) = \frac{h}{3}$,
$I_2(N_1N_2) = \frac{h}{6}$,
$I_2(N_1'N_1') = I_2(N_2'N_2') = -I_2(N_1'N_2') = \frac{1}{h}$,
$I_2(N_1N_1') = I_2(N_1'N_2) = I_2(N_1N_2') = I_2(N_2N_2') = -\frac{1}{2}$.

Thus the penalty term will be

$$\|\varepsilon(\mathbf{w}_h{}^e)\|_0{}^2 = \int_e \left\{(u_h' - \frac{v_h}{\rho})^2 + (v_h' + \frac{u_h}{\rho} - \tau w_h - \vartheta_h)^2 + (w_h' + \tau v_h + \psi_h)^2\right\} ds$$

The first term of the above integral is the axial response. The other two terms are shear responses. When we apply the two-point quadrature the axial response becomes

$$\int_e (u_h' - \frac{v_h}{\rho})^2\, ds = \left\{\frac{u_2 - u_1}{h} - \frac{1}{\rho}\frac{v_2 + v_1}{2}\right\}^2 h + \frac{1}{12\rho^2}\left\{\frac{v_2 - v_1}{h}\right\}^2 h^3 \tag{4.11}$$

The transverse shear responses become

$$\begin{aligned}\int_e (v_h' + \frac{u_h}{\rho} - \tau w_h - \vartheta_h)^2\, ds = \\ \left\{\frac{v_2 - v_1}{h} + \frac{1}{\rho}\frac{u_2 + u_1}{2} - \tau\frac{w_2 + w_1}{2} - \frac{\vartheta_2 + \vartheta_1}{2}\right\}^2 h \\ + \frac{1}{12}\left\{\frac{1}{\rho}\frac{u_2 - u_1}{h} - \tau\frac{w_2 - w_1}{h} - \frac{\vartheta_2 - \vartheta_1}{h}\right\}^2 h^3\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}\int_e (w_h' + \tau v_h + \psi_h)^2\, ds &= \left\{\frac{w_2 - w_1}{h} + \tau\frac{v_2 + v_1}{2} + \frac{\psi_2 + \psi_1}{2}\right\}^2 h \\ &\quad + \frac{1}{12}\left\{\tau\frac{v_2 - v_1}{h} + \frac{\psi_2 - \psi_1}{h}\right\}^2 h^3\end{aligned} \tag{4.13}$$

When parameter d tends to zero the following constraints are enforced:

$\frac{u_2 - u_1}{h} - \frac{1}{\rho}\frac{v_2 + v_1}{2} = 0;\quad \frac{v_2 - v_1}{h} = 0,$

$\frac{v_2 - v_1}{h} + \frac{1}{\rho}\frac{u_2 + u_1}{2} - \tau\frac{w_2 + w_1}{2} - \frac{\vartheta_2 + \vartheta_1}{2} = 0;\quad \frac{u_2 - u_1}{h} - \tau\frac{w_2 - w_1}{h} - \frac{\vartheta_2 - \vartheta_1}{h} = 0,$

$\frac{w_2 - w_1}{h} + \tau\frac{v_2 + v_1}{2} + \frac{\psi_2 + \psi_1}{2} = 0;\quad \frac{v_2 - v_1}{h} + \frac{\psi_2 - \psi_1}{h} = 0.$

The continuous equivalents of these discretized constraints are

$u' - \frac{v}{\rho} = 0$ and $v' = 0$,

$v' + \frac{u}{\rho} - \tau w - \vartheta = 0$ and $\frac{1}{\rho}u' - \tau w' - \vartheta' = 0$,

$w' + \tau v + \psi = 0$ and $\tau v' + \psi' = 0$.

Here $u' - \frac{v}{\rho} = 0$, $v' + \frac{u}{\rho} - \tau w - \vartheta = 0$ and $w' + \tau v + \psi = 0$ are the true Kirchhoff constraints to be enforced in the limit. The remaining equations are the spurious and undesirable constraints which cause locking.

From equations (4.11), (4.12) and (4.13) we can expect that spurious constraints can be relaxed by increasing the number of elements as h^3 tends to zero faster than h. This does not allow one to obtain the optimal solution.

We now apply reduced integration rule (one-point quadrature) to the penalty term of the stiffness matrix, over an element of length h. For one-point quadrature
$I_1(N_1N_1) = I_1(N_2N_2) = I_1(N_1N_2) = \frac{h}{4}$,
$I_1(N_1'N_1') = I_1(N_2'N_2') = I_1(N_1'N_2') = \frac{1}{h}$,
$I_1(N_1'N_1) = I_1(N_1'N_2) = -I_1(N_1N_2') = -I_1(N_1N_2') = -\frac{1}{2}$.

Thus the axial term will be

$$\begin{aligned} I_1\{(u_h' - \tfrac{v_h}{\rho})^2\} &= I_1\{(u_h')^2\} - \tfrac{2}{\rho} I_1\{u_h' v_h\} + \tfrac{1}{\rho^2} I_1\{(v_h)^2\} \\ &= \left\{\frac{u_2 - u_1}{h} - \frac{1}{\rho}\frac{v_2 + v_1}{2}\right\}^2 h \end{aligned}$$

When we use the reduced integration there is no appearence of axial related spurious constraint due to discretization as d tends to zero, and it produces only the true Kirchhoff constraint $u_h' - \frac{v_h}{\rho} = 0$

The shear terms become

$$\begin{aligned} I_1\{(v_h' + \tfrac{u_h}{\rho} - \tau w_h - \vartheta_h)^2\} &= h\left\{\frac{v_2 - v_1}{h} + \frac{1}{\rho}\frac{u_2 + u_1}{2} - \tau\frac{w_2 + w_1}{2} - \frac{\vartheta_2 + \vartheta_1}{2}\right\}^2, \\ I_1\{(w_h' + \tau v_h + \psi_h)^2\} &= h\left\{\frac{w_2 - w_1}{h} + \frac{v_2 + v_1}{2} + \frac{\psi_2 + \psi_1}{2}\right\}^2. \end{aligned}$$

One can see that these terms lead to the true Kirchhoff constraints in the limit.

The stablity and the convergence of the approximation for the problem $S_d^{h,red}$ can be identified by showing the equivalence of the problem $S_d^{h,red}$ and discrete mixed problem $\boldsymbol{M}_d^h$. In order to show the equivalence we propose the following

Theorem 4.2 (a) *The reduced standard problem $S_d^{h,red}$ is equivalent to the discrete mixed problem M_d^h of finding $\hat{\mathbf{w}}_h \in \mathbf{W}^h$ and $\hat{\mathbf{q}}_h \in \mathbf{Q}^h$ which satisfy*

$$(\hat{\boldsymbol{\theta}}'_h, \boldsymbol{\Psi}'_h) + (\hat{\mathbf{q}}_h, \boldsymbol{\varepsilon}(\mathbf{z}_h)) = \mathcal{F}(\mathbf{z}_h) \tag{4.14}$$

$$(d^{-1}\, \boldsymbol{\varepsilon}(\hat{\mathbf{w}}_h) - \hat{\mathbf{q}}_h, \mathbf{r}_h) = 0 \tag{4.15}$$

(b) *Problem $S_d^{h,red}$ or M_d^h has a unique solution $\hat{\mathbf{w}}_h \in \mathbf{W}^h$, $\hat{\mathbf{q}}_h \in \mathbf{Q}^h$ with*

$$\hat{\mathbf{q}}_h = d^{-1}\, \pi_{r-1} \boldsymbol{\varepsilon}(\hat{\mathbf{w}}_h)$$

Furthermore, if $\mathbf{w} \in \{H^r(0,1)\}^6$, then there is a positive constant c, independent of d and h, such that

$$|\, \hat{\mathbf{w}}_h - \mathbf{w}\, |_1 \leq c\, h^r\, |\mathbf{w}|_r$$

Proof. Comparing the equation (4.10) with the equation (4.7) we see that

$$(\pi_{r-1}\boldsymbol{\varepsilon}(\mathbf{w}_h), \pi_{r-1}\boldsymbol{\varepsilon}(\mathbf{z}_h)) = I_r\{(\boldsymbol{\varepsilon}(\mathbf{w}_h), \boldsymbol{\varepsilon}(\mathbf{z}_h))\}$$

thus to establish equivalence it suffices to show that

$$(\pi_{r-1} u_h, v_h) = I_r(u_h, v_h) \qquad \text{for all } u_h, v_h \in P_r^h.$$

Let $J(f_h) \in P_{r-1}^h$ be the interpolation of $f_h \in P_r^h$ at the Gaussian points. Then by making use of Gaussian quadrature scheme and definition of the interpolation we have, for any $g_h \in P_{r-1}^h$,

$$(J(f_h), g_h) = I_r\{(J(f_h), g_h)\} = I_r\{(f_h, g_h)\} = (f_h, g_h),$$

since the degree of polynomial $f_h g_h$ is $2r-1$

$$I_r\{(f_h, g_h)\} = (f_h, g_h)$$

.

By the definition of projection operators

$$J(f_h) = \pi_{r-1} f_h \qquad \text{for any } f_h \in P_r^h.$$

Accordingly

$$\begin{aligned}(\pi_{r-1} f_h, g_h) &= (J(f_h), g_h) \\ &= I_r\{J(f_h), g_h\} \\ &= I_r\{(f_h, g_h)\}\end{aligned}$$

Hence the proof. □

5 Analytical Solutions

This Chapter is devoted to derive analytical solutions to problems which we developed in the chapter 2. As it is very difficult to find solutions for all the cases we confine our attention to problems having constant curvature and constant force components in tangential, normal and binormal directions. These solutions will be used in Chapter 6, where a comparison will be made with numerical results.

The non-dimensionalised form of the governing differential equations for the symmetric cross sectional beam are, from (3.17) and (3.18),

$$d^{-1}\boldsymbol{D}\boldsymbol{\varepsilon}' + \boldsymbol{f} = \mathbf{0}, \tag{5.1}$$

$$\boldsymbol{E}\boldsymbol{\theta}'' + d^{-1}\mathbf{t} \wedge \boldsymbol{D}\boldsymbol{\varepsilon} = \mathbf{0}, \tag{5.2}$$

where

$\boldsymbol{D} = [1, \frac{Gk_1}{E}, \frac{Gk_2}{E}] = [1, d_1, d_2]$, and
$\boldsymbol{E} = [\frac{GJ}{EI}, 1, 1] = [e_1, 1, 1]$ (since $I_1 = I_2 = I$).

We recall from Chapter 3 that

$$\boldsymbol{\varepsilon} = \mathbf{u}' - \boldsymbol{\theta} \wedge \mathbf{t} \quad \text{and that } d^{-1} = \frac{AL^2}{I}$$

Integrating equation (5.1), we have

$$\boldsymbol{D}\boldsymbol{\varepsilon} = -d\int \boldsymbol{f} = -\mathbf{g}(s) \qquad \text{(say)}, \tag{5.3}$$

and substitution into (5.2), yields

$$\boldsymbol{E}\boldsymbol{\theta}'' - d^{-1}\mathbf{t} \wedge \mathbf{g} = \mathbf{0}.$$

This implies that

$$\boldsymbol{\theta}'' = d^{-1}\boldsymbol{E}^{-1}(\mathbf{t} \wedge \mathbf{g}).$$

In component form, this equation reads, with $\boldsymbol{\theta} = \phi\mathbf{t} + \psi\mathbf{n} + \vartheta\mathbf{b}$,

$$(D^2 - \frac{1}{\rho^2})\phi - \frac{2}{\rho}D\psi + \frac{\tau}{\rho}\vartheta = 0, \tag{5.4}$$

$$\frac{2}{\rho}D\phi + (D^2 - \gamma^2)\psi - 2\tau D\vartheta = -d^{-1}g_b, \tag{5.5}$$

$$\frac{\tau}{\rho}\phi + 2\tau D\psi + (D^2 - \tau^2)\vartheta = d^{-1}g_n, \tag{5.6}$$

where $\gamma^2 = \tau^2 + \frac{1}{\rho^2}$ and $D(\cdot) = \frac{d(\cdot)}{ds}$.

If we write $\mathbf{X} = (\phi, \psi, \vartheta)^T$, then the above set of equations becomes

$$\ddot{\mathbf{X}} + 2P\dot{\mathbf{X}} + P^2\mathbf{X} = \mathbf{k}$$

where $\dot{\mathbf{X}} = (\phi', \psi', \vartheta')^T$, $\ddot{\mathbf{X}}$ is similarly defined, $\mathbf{k} = (0, -d^{-1}g_b, d^{-1}g_n)^T$, and

$$P = \begin{pmatrix} 0 & -\frac{1}{\rho} & 0 \\ \frac{1}{\rho} & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

Transforming the above matrix equation by

$$\mathbf{X}(s) = e^{-sP}\mathbf{Y}(s), \tag{5.7}$$

we obtain $e^{-sP}\ddot{\mathbf{Y}} = \mathbf{k}$

consequently,

$$\ddot{\mathbf{Y}} = e^{sP}\mathbf{k}. \tag{5.8}$$

We now find an expression for e^{sP}. According to the Cayley-Hamilton Theorem , every square matrix satisfies its own characteristic equation.

therefore

$$P^3 + \gamma^2 P = 0.$$

By making use of this result and Taylor series, the exponential matrix e^{sP} can be expressed in the form

$$\begin{aligned} e^{sP} &= I + sP + \frac{s^2}{2!}P^2 + \frac{s^3}{3!}P^3 + \frac{s^4}{4!}P^4 + \frac{s^5}{5!}P^5 + \cdots \\ &= I + sP + \frac{s^2}{2!}P^2 - \frac{\gamma^2 s^3}{3!}P - \frac{\gamma^2 s^4}{4!}P^2 + \frac{\gamma^4 s^5}{5!}P + \cdots \\ &= I + (\gamma s - \frac{\gamma^3 s^3}{3!} + \frac{\gamma^5 s^5}{5!} - \cdots)\frac{P}{\gamma} - (1 - \frac{\gamma^2 s^2}{2!} + \frac{\gamma^4 s^4}{4!}\cdots)\frac{P^2}{\gamma^2} + \frac{P^2}{\gamma^2} \\ &= (I + \frac{P^2}{\gamma^2}) + \frac{1}{\gamma}sin\gamma s P - \frac{1}{\gamma^2}cos\gamma s P^2 \end{aligned}$$

Consequently

$$e^{sP} = \begin{pmatrix} \frac{1}{\gamma^2}(\tau^2 + \frac{1}{\rho^2}cos\gamma s) & -\frac{1}{\rho\gamma}sin\gamma s & \frac{\tau}{\rho\gamma^2}(1 - cos\gamma s) \\ \frac{1}{\rho\gamma}sin\gamma s & cos\gamma s & -\frac{\tau}{\gamma}sin\gamma s \\ \frac{\tau}{\rho\gamma^2}(1 - cos\gamma s) & \frac{\tau}{\gamma}sin\gamma s & \frac{1}{\gamma^2}(\frac{1}{\rho^2} + \tau^2 cos\gamma s) \end{pmatrix}.$$

Replacing s by $-s$ into the above matrix , we obtain

$$e^{-sP} = \begin{pmatrix} \frac{1}{\gamma^2}(\tau^2 + \frac{1}{\rho^2}cos\gamma s) & \frac{1}{\rho\gamma}sin\gamma s & \frac{\tau}{\rho\gamma^2}(1 - cos\gamma s) \\ -\frac{1}{\rho\gamma}sin\gamma s & cos\gamma s & \frac{\tau}{\gamma}sin\gamma s \\ \frac{\tau}{\rho\gamma^2}(1 - cos\gamma s) & -\frac{\tau}{\gamma}sin\gamma s & \frac{1}{\gamma^2}(\frac{1}{\rho^2} + \tau^2 cos\gamma s) \end{pmatrix}.$$

so that equation becomes, with $\mathbf{g} = g_t\mathbf{t} + g_n\mathbf{n} + g_b\mathbf{b}$,

$$\ddot{\mathbf{Y}} = \begin{pmatrix} \frac{1}{\gamma^2}(\tau^2 + \frac{1}{\rho^2}cos\gamma s) & -\frac{1}{\rho\gamma}sin\gamma s & \frac{\tau}{\rho\gamma^2}(1 - cos\gamma s) \\ \frac{1}{\rho\gamma}sin\gamma s & cos\gamma s & -\frac{\tau}{\gamma}sin\gamma s \\ \frac{\tau}{\rho\gamma^2}(1 - cos\gamma s) & \frac{\tau}{\gamma}sin\gamma s & \frac{1}{\gamma^2}(\frac{1}{\rho^2} + \tau^2 cos\gamma s) \end{pmatrix} \begin{bmatrix} 0 \\ -d^{-1}g_b \\ d^{-1}g_n \end{bmatrix}$$

or

$$\ddot{\mathbf{Y}} = d^{-1}\begin{bmatrix} \frac{g_b}{\rho\gamma} sin\gamma s + \frac{\tau g_n}{\rho\gamma^2}(1 - cos\gamma s) \\ -g_b cos\gamma s - \frac{\tau g_n}{\gamma} sin\gamma s \\ -\frac{\tau g_b}{\gamma} sin\gamma s + \frac{g_n}{\gamma^2}(\frac{1}{\rho^2} + \tau^2 cos\gamma s) \end{bmatrix}$$

Next, we have to find an expression for $\mathbf{g}$ in terms of $\boldsymbol{f}$. From 5.3 we have

$$d\boldsymbol{f} = \mathbf{g}'.$$

In other words, with $\mathbf{g} = g_t\mathbf{t} + g_n\mathbf{n} + g_b\mathbf{b}$ and $\boldsymbol{f} = F_t\mathbf{t} + F_n\mathbf{n} + F_b\mathbf{b}$ this equation reads in component form,

$$g_t' - \frac{g_n}{\rho} = dF_t = f_t, \tag{5.9}$$

$$g_n' + \frac{g_t}{\rho} - \tau g_b = dF_n = f_n, \tag{5.10}$$

$$g_b' + \tau g_n = dF_b = f_b. \tag{5.11}$$

τ (5.9) + $\frac{1}{\rho}$ (5.11) gives

$$\tau g_t' + \frac{1}{\rho} g_b' = (\tau f_t + \frac{1}{\rho} f_b) \equiv M,$$

so it follows that

$$\tau g_t + \frac{1}{\rho} g_b = Ms + C \tag{5.12}$$

where c is an arbitary constant.

Differentiating (5.9) with respect to s and substitution of (5.10) and (5.12) yields

$$\begin{aligned} g_t'' - \frac{1}{\rho}(f_n - \frac{g_t}{\rho} + \tau g_b) &= 0, \\ g_t'' + \frac{1}{\rho^2} g_t - \tau(Ms + C - \tau g_t) &= \frac{1}{\rho} f_n, \\ g_t'' + \gamma^2 g_t &= \tau(Ms + C) + \frac{1}{\rho} f_n. \end{aligned}$$

The general solution of g_t is

$$g_t(s) = Acos\gamma s + Bsin\gamma s + \frac{\tau}{\gamma^2}(Ms + C) + \frac{1}{\rho\gamma^2}f_n,$$

where A and B are arbitary constants. Similarly, we can find solutions for the other components. These are

$$
\begin{aligned}
g_t &= Acos\gamma s + Bsin\gamma s + \frac{\tau}{\gamma^2}(Ms + C) + \frac{1}{\rho\gamma^2}f_n, && (5.13)\\
g_n &= -\rho\gamma Asin\gamma s + \rho\gamma Bcos\gamma s + \frac{N}{\gamma^2}, && (5.14)\\
g_b &= -\tau\rho(Acos\gamma s + Bsin\gamma s) + \frac{1}{\rho\gamma^2}(Ms + C) - \frac{\tau}{\gamma^2}f_n, && (5.15)
\end{aligned}
$$

where $M = (\tau f_t + \frac{1}{\rho}f_b)$ and $N = (\tau f_b - \frac{1}{\rho}f_t)$.

Substituting for g_n and g_b and integrating twice with respect to s, we have

$$
\mathbf{Y} = d^{-1}\begin{bmatrix}
(\frac{\tau}{\rho\gamma^4}N - \frac{B\tau}{\gamma})(\frac{s^2}{2} + \frac{1}{\gamma^2}cos\gamma s) - \{\frac{C}{\rho^2\gamma^3} - \frac{A\tau}{\gamma} - \frac{\tau}{\rho\gamma^3}f_n\}\frac{1}{\gamma^2}sin\gamma s \\
+\frac{M}{\rho^2\gamma^3}\{-\frac{s}{\gamma^2}sin\gamma s - \frac{2}{\gamma^3}cos\gamma s\} + a_1 s + b_1 \\
\frac{\tau\rho A}{2}s^2 - \{\frac{\tau}{\gamma^2}f_n - \frac{C}{\rho\gamma^2}\}\frac{1}{\gamma^2}cos\gamma s - \frac{M}{\rho\gamma^2}\{-\frac{s}{\gamma^2}cos\gamma s + \frac{2}{\gamma^3}sin\gamma s\} \\
+\frac{\tau N}{\gamma^5}sin\gamma s + a_2 s + b_2 \\
(\frac{B\tau^2\rho}{\gamma} + \frac{N}{\rho^2\gamma^4})\frac{s^2}{2} - (\frac{B}{\rho\gamma} + \frac{\tau^2 N}{\gamma^4})\frac{1}{\gamma^2}cos\gamma s \\
+\{\frac{A}{\rho\gamma} + \frac{\tau C}{\rho\gamma^3} - \frac{\tau^2 f_n}{\gamma^3}\}\frac{1}{\gamma^2}sin\gamma s \\
-\frac{\tau M}{\rho\gamma^3}\{-\frac{s}{\gamma^2}sin\gamma s - \frac{2}{\gamma^3}cos\gamma s\} + a_3 s + b_3
\end{bmatrix}
$$

where a_i, b_i $(i = 1,2,3)$ are arbitary integral constants, to be determined by the boundary conditions.

The boundary conditions give

$$
\begin{aligned}
\mathbf{X}(0) &= e^0\mathbf{Y}(0) = \mathbf{0},\\
\mathbf{X}(1) &= e^{-P}\mathbf{Y}(1) = \mathbf{0},
\end{aligned}
$$

from which it follows that

$$\mathbf{Y}(0) = \mathbf{0},$$
$$\mathbf{Y}(1) = \mathbf{0}. \tag{5.16}$$

Equation 5.16 implies that

$$\mathbf{0} = \begin{bmatrix} (\frac{\tau N}{\rho\gamma^4} - \frac{B\tau}{\gamma})(\frac{1}{\gamma^2}) + \frac{M}{\rho^2\gamma^3}\{-\frac{2}{\gamma^3}\} + b_1 \\ -\{\frac{\tau}{\gamma^2}f_n - \frac{C}{\rho\gamma^2}\}(\frac{1}{\gamma^2}) + b_2 \\ -\{\frac{B}{\rho\gamma} + \frac{\tau^2 N}{\gamma^4}\}(\frac{1}{\gamma^2}) - \frac{\tau M}{\rho\gamma^3}\{-\frac{2}{\gamma^3}\} + b_3 \end{bmatrix}$$

and

$$\mathbf{0} = \begin{bmatrix} (\frac{\tau}{\rho\gamma^4}N - \frac{B\tau}{\gamma})(\frac{1}{2} + \frac{1}{\gamma^2}cos\gamma) - \{\frac{C}{\rho^2\gamma^3} - \frac{A\tau}{\gamma} - \frac{\tau}{\rho\gamma^3}f_n\}\frac{1}{\gamma^2}sin\gamma \\ +\frac{M}{\rho^2\gamma^3}\{-\frac{1}{\gamma^2}sin\gamma - \frac{2}{\gamma^3}cos\gamma\} + a_1 + b_1 \\ \frac{\tau\rho A}{2} - \{\frac{\tau f_n}{\gamma^2} - \frac{C}{\rho\gamma^2}\}\frac{1}{\gamma^2}cos\gamma - \frac{M}{\rho\gamma^2}\{-\frac{1}{\gamma^2}cos\gamma + \frac{2}{\gamma^3}sin\gamma\} \\ +\frac{\tau N}{\gamma^5}sin\gamma + a_2 + b_2 \\ (\frac{B\tau^2\rho}{\gamma} + \frac{N}{\rho^2\gamma^4})\frac{1}{2} - (\frac{B}{\rho\gamma} + \frac{\tau^2 N}{\gamma^4})\frac{1}{\gamma^2}cos\gamma \\ +\{\frac{A}{\rho\gamma} + \frac{\tau C}{\rho\gamma^3} - \frac{\tau^2 f_n}{\gamma^3}\}\frac{1}{\gamma^2}sin\gamma \\ -\frac{\tau M}{\rho\gamma^3}\{-\frac{1}{\gamma^2}sin\gamma - \frac{2}{\gamma^3}cos\gamma\} + a_3 + b_3 \end{bmatrix}.$$

Eliminating the constants a_1 , a_2 , a_3 , b_1 , b_2 and b_3 , using the above equations, we obtain

$$\mathbf{Y} = d^{-1}\begin{bmatrix} \frac{\tau}{2\gamma}(\frac{N}{\rho\gamma^3} - B)(s^2-s) \\ +\frac{1}{\gamma^3}\{\frac{\tau N}{\rho\gamma^3} - B\tau - \frac{2M}{\rho^2\gamma^3}\}(cos\gamma s + (1-cos\gamma)s - 1) \\ -\frac{1}{\gamma^3}\{\frac{C}{\rho^2\gamma^2} - A\tau - \frac{\tau f_n}{\rho\gamma^2}\}(sin\gamma s - s\,sin\gamma) - \frac{Ms}{\rho^2\gamma^5}(sin\gamma s - sin\gamma) \\ \frac{\tau\rho A}{2}(s^2-s) - \frac{1}{\gamma^4}\{\tau f_n - \frac{C}{\rho}\}(cos\gamma s + (1-cos\gamma)s - 1) + \\ \frac{1}{\gamma^5}\{\tau N - \frac{2M}{\rho}\}(sin\gamma s - s\,sin\gamma) + \frac{Ms}{\rho\gamma^4}(cos\gamma s - cos\gamma) \\ \frac{1}{2\gamma}\{B\tau^2\rho + \frac{N}{\rho^2\gamma^3}\}(s^2-s) \\ -\frac{1}{\gamma^3}\{\frac{B}{\rho} + \frac{\tau^2 N}{\gamma^3} - \frac{2\tau M}{\rho\gamma^3}\}(cos\gamma s + (1-cos\gamma)s - 1) \\ +\frac{1}{\gamma^3}\{\frac{A}{\rho} + \frac{\tau C}{\rho\gamma^2} - \frac{\tau^2 f_n}{\gamma^2}\}(sin\gamma s - s\,sin\gamma) + \frac{\tau Ms}{\rho\gamma^5}(sin\gamma s - sin\gamma) \end{bmatrix}$$

It follows that

$$\mathbf{X} = e^{-sP}\mathbf{Y}$$

or

$$\begin{bmatrix} \phi \\ \psi \\ \vartheta \end{bmatrix} = d^{-1}\begin{bmatrix} \frac{\tau}{2\gamma}\{\frac{N}{\rho\gamma^3} - Bcos\gamma s + Asin\gamma s\}(s^2-s) \\ +\frac{1}{\gamma^3}\{-B\tau + \frac{R}{\rho\gamma^2}sin\gamma s + \frac{Q}{\rho\gamma^3}cos\gamma s\}(cos\gamma s + (1-cos\gamma)s - 1) \\ +\frac{1}{\gamma^3}\{A\tau + \frac{Q}{\rho\gamma^3}sin\gamma s - \frac{R}{\rho\gamma^2}cos\gamma s+\}(sin\gamma s - s\,sin\gamma) \\ +\frac{Ms}{\rho^2\gamma^5}sin\gamma(1-s) \\ \frac{\tau\rho}{2}\{Bsin\gamma s + Acos\gamma s\}(s^2-s) \\ +\frac{1}{\gamma^4}\{Rcos\gamma s - \frac{Q}{\gamma}sin\gamma s\}(cos\gamma s + (1-cos\gamma)s - 1) \\ +\frac{1}{\gamma^4}\{\frac{Q}{\gamma}cos\gamma s + Rsin\gamma s\}(sin\gamma s - s\,sin\gamma) \\ +\frac{Ms}{\rho\gamma^4}(1 - cos\gamma(1-s)) \\ \frac{1}{2\gamma}\{\frac{N}{\rho^2\gamma^3} + \tau^2\rho Bcos\gamma s - \tau^2\rho Asin\gamma s\}(s^2-s) \\ -\frac{1}{\gamma^3}\{\frac{B}{\rho} + \frac{\tau R}{\gamma^2}sin\gamma s + \frac{\tau Q}{\gamma^3}cos\gamma s\}(cos\gamma s + (1-cos\gamma)s - 1) \\ +\frac{1}{\gamma^3}\{\frac{A}{\rho} - \frac{\tau Q}{\gamma^3}sin\gamma s + \frac{\tau R}{\gamma^2}cos\gamma s\}(sin\gamma s - s\,sin\gamma) \\ -\frac{\tau Ms}{\rho\gamma^5}sin\gamma(1-s) \end{bmatrix}$$

where $Q = \tau N - \frac{2M}{\rho}$ and $R = \frac{C}{\rho} - \tau f_n$.

This completes the solution for the components of $\boldsymbol{\theta}$. To solve for $\mathbf{u}$ we return to (5.3), which in component form is

$$\begin{aligned} u' - \frac{v}{\rho} &= -g_t, \\ v' + \frac{u}{\rho} - \tau w &= -\frac{g_n}{d_1} + \vartheta, \\ w' + \tau v &= -\frac{g_b}{d_2} - \psi. \end{aligned}$$

This system of differential equations may be written in matrix notation, if we set $\mathbf{X} = (u, v, w)^T$; the we have

$$\dot{\mathbf{X}} - \Gamma\mathbf{X} = (-g_t, -\frac{g_n}{d_1} + \vartheta, -\frac{g_b}{d_2} - \psi)^T \tag{5.17}$$

Where Γ is Serret-Frenet matrix and $\dot{\mathbf{X}} = (u', v', w')^T$.

As we have seen, the nature of the solutions to the system will depend on the eigenvalues of the matrix Γ, that is, on the roots of the characteristic equation

$$0 = |\Gamma - \lambda I| = \begin{vmatrix} -\lambda & \frac{1}{\rho} & 0 \\ -\frac{1}{\rho} & -\lambda & \tau \\ 0 & -\tau & -\lambda \end{vmatrix}.$$

These roots are $\lambda_i = 0$, $i\gamma$ and $-i\gamma$, where $i = \sqrt{-1}$.

The eigenvector corresponding to the eigenvalue 0 is $(\tau, 0, \frac{1}{\rho})^T$, while the eigenvectors corresponding to $i\gamma$ is $(\frac{1}{\rho}, i\gamma, -\tau)^T$, and that corresponding to $-i\gamma$ is $(\frac{1}{\rho}, -i\gamma, -\tau)^T$.

We can therefore define the diagonalising matrix of Γ

$$B = \begin{pmatrix} \tau & \frac{1}{\rho} & \frac{1}{\rho} \\ 0 & i\gamma & -i\gamma \\ \frac{1}{\rho} & -\tau & -\tau \end{pmatrix}$$

that is B satisfies

$B^{-1}\Gamma B = \text{diag}[0, i\gamma, -i\gamma]$.

Setting $\mathbf{X} = B\mathbf{Y}$ in (5.17), we have

$$B\dot{\mathbf{Y}} - \Gamma B\mathbf{Y} = (-g_t, -\frac{g_n}{d_1} + \vartheta, -\frac{g_b}{d_2} - \psi)^T$$

and since B^{-1} exists, the above equation becomes

$$\dot{\mathbf{Y}} - B^{-1}\Gamma B\mathbf{Y} = B^{-1}(-g_t, -\frac{g_n}{d_1} + \vartheta, -\frac{g_b}{d_2} - \psi)^T$$

or

$$\dot{\mathbf{Y}} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\gamma & 0 \\ 0 & 0 & -i\gamma \end{pmatrix} \mathbf{Y} = \begin{pmatrix} \frac{\tau}{\gamma^2} & 0 & \frac{1}{\rho\gamma^2} \\ \frac{1}{2\rho\gamma^2} & -\frac{i}{2\gamma} & -\frac{\tau}{2\gamma^2} \\ \frac{1}{2\rho\gamma^2} & \frac{i}{2\gamma} & -\frac{\tau}{2\gamma^2} \end{pmatrix} \begin{bmatrix} -g_t \\ -\frac{g_n}{d_1} + \vartheta \\ -\frac{g_b}{d_2} - \psi \end{bmatrix}$$

If $\mathbf{Y} = (y_1, y_2, y_3)^T$, the above system reduces to the simple system of non-homogeneous first order differential equations

$$y_1' = -\frac{\tau}{\gamma^2}g_t - \frac{1}{\rho\gamma^2}(\frac{g_b}{d_2} + \psi) \tag{5.18}$$

$$y_2' - i\gamma y_2 = -\frac{1}{2\rho\gamma^2}g_t - \frac{i}{2\gamma}(-\frac{g_n}{d_1} + \vartheta) + \frac{\tau}{2\gamma^2}(\frac{g_b}{d_2} + \psi) \tag{5.19}$$

$$y_3' + i\gamma y_3 = -\frac{1}{2\rho\gamma^2}g_t + \frac{i}{2\gamma}(-\frac{g_n}{d_1} + \vartheta) + \frac{\tau}{2\gamma^2}(\frac{g_b}{d_2} + \psi) \tag{5.20}$$

We also observe that the conjugate of y_2 is equal to y_3, that is,

$$\bar{y}_2 = y_3$$

Equation (5.18) becomes, upon substituitions,

$$\begin{aligned} y_1' &= -\frac{\tau}{\gamma^2}\{Acos\gamma s + Bsin\gamma s + \frac{\tau}{\gamma^2}(Ms + C) + \frac{1}{\rho\gamma^2}f_n\} \\ &\quad -\frac{1}{\rho\gamma^2 d_2}\{-\tau\rho(Acos\gamma s + Bsin\gamma s) + \frac{1}{\rho\gamma^2}(Ms + C) - \frac{\tau}{\gamma^2}f_n\} - \frac{1}{\rho\gamma^2}\psi \\ &= -\frac{\tau}{\gamma^2}(Acos\gamma s + Bsin\gamma s)\{1 - \frac{1}{d_2} + \frac{d^{-1}}{2}(s^2 - s)\} - \frac{1}{\gamma^4}(Ms + C)(\tau^2 + \frac{1}{\rho^2 d_2}) \\ &\quad -\frac{\tau f_n}{\rho\gamma^4}(1 - \frac{1}{d_2}) - \frac{d^{-1}R}{\rho\gamma^6}\{1 - (1 - s)cos\gamma s - s\,cos\gamma(1 - s)\} \end{aligned}$$

$$-\frac{d^{-1}Q}{\rho\gamma^7}\{s\,sin\gamma(1-s)-(1-s)sin\gamma s\}-\frac{Md^{-1}}{\rho^2\gamma^6}s(1-cos\gamma(1-s)).$$

Integrating with respect to s, we obtain

$$\begin{aligned} y_1 \;=\; & -\frac{\tau A}{\gamma^2}\{(1-\frac{1}{d_2}+\frac{d^{-1}}{2}(s^2-s))\frac{sin\gamma s}{\gamma}+\frac{d^{-1}}{2\gamma^2}(2s-1)cos\gamma s-\frac{d^{-1}}{\gamma^3}sin\gamma s\} \\ & -\frac{\tau B}{\gamma^2}\{-(1-\frac{1}{d_2}+\frac{d^{-1}}{2}(s^2-s))\frac{cos\gamma s}{\gamma}+\frac{d^{-1}}{2\gamma^2}(2s-1)sin\gamma s-\frac{d^{-1}}{\gamma^3}cos\gamma s\} \\ & -\frac{1}{\gamma^4}(M\frac{s^2}{2}+Cs)(\tau^2+\frac{1}{\rho^2 d_2})-\frac{\tau f_n}{\rho\gamma^4}(1-\frac{1}{d_2})s \\ & -\frac{d^{-1}R}{\rho\gamma^6}\{s-\frac{(1-s)}{\gamma}sin\gamma s+\gamma^2 cos\gamma s+\frac{s}{\gamma}sin\gamma(1-s)-\gamma^2 cos\gamma(1-s)\} \\ & -\frac{d^{-1}Q}{\rho\gamma^7}\{-\frac{(1-s)}{\gamma}cos\gamma s-\gamma^2 sin\gamma s-\frac{s}{\gamma}cos\gamma(1-s)-\gamma^2(1-s)\} \\ & -\frac{d^{-1}M}{\rho^2\gamma^6}\{\frac{s^2}{2}+\frac{s}{\gamma}sin\gamma(1-s)-\gamma^2 cos\gamma(1-s)\}+c_1 \qquad (5.21) \end{aligned}$$

From the equation (5.19)

$$
\begin{aligned}
\frac{d}{ds}(y_2e^{-i\gamma s}) &= e^{-i\gamma s}\{-\frac{1}{2\rho\gamma^2}g_t - \frac{i}{2\gamma}(-\frac{g_n}{d_1}+\vartheta)+\frac{\tau}{2\gamma^2}(\frac{g_b}{d_2}+\psi)\} \\
&= \left[\{-\frac{1}{2\rho\gamma^2}g_t+\frac{\tau}{2\gamma^2}(\frac{g_b}{d_2}+\psi)\}cos\gamma s-\frac{1}{2\gamma}(\vartheta-\frac{g_n}{d_1})sin\gamma s\right] \\
&\quad -i\left[\{-\frac{1}{2\rho\gamma^2}g_t+\frac{\tau}{2\gamma^2}(\frac{g_b}{d_2}+\psi)\}sin\gamma s+\frac{1}{2\gamma}(\vartheta-\frac{g_n}{d_1})sin\gamma s\right] \\
&= \{-\frac{\rho}{2\gamma^2}(Acos\gamma s+Bsin\gamma s)(\tau^2+\frac{1}{\rho^2d_2})cos\gamma s \\
&\quad -\frac{\tau}{2\rho\gamma^4}(1-\frac{1}{d_2})(Ms+C)cos\gamma s-\frac{f_n}{2\gamma^4}(\tau^2+\frac{1}{\rho^2d_2})cos\gamma s \\
&\quad +\frac{d^{-1}}{4\gamma^2}(\tau^2\rho A-\frac{N}{\rho^2\gamma^3}sin\gamma s)(s^2-s) \\
&\quad +\frac{d^{-1}}{2\gamma^4}\{\frac{\tau R}{\gamma^2}+\frac{B}{\rho}sin\gamma s\}(cos\gamma s+(1-cos\gamma)s-1) \\
&\quad +\frac{d^{-1}}{2\gamma^4}\{\frac{\tau Q}{\gamma^3}-\frac{A}{\rho}sin\gamma s\}(sin\gamma s-s\,sin\gamma)+\frac{\tau Msd^{-1}}{2\rho\gamma^6}(cos\gamma s-cos\gamma) \\
&\quad -\frac{\rho}{2d_1}Asin^2\gamma s+-\frac{\rho}{2d_1}Bsin\gamma scos\gamma s+\frac{N}{2\gamma^3d_1}sin\gamma s\} \\
&\quad i\{-\frac{\rho}{2\gamma^2}(Acos\gamma s+Bsin\gamma s)(\tau^2+\frac{1}{\rho^2d_2})sin\gamma s \\
&\quad -\frac{\tau}{2\rho\gamma^4}(1-\frac{1}{d_2})(Ms+C)sin\gamma s-\frac{f_n}{2\gamma^4}(\tau^2+\frac{1}{\rho^2d_2})sin\gamma s \\
&\quad +\frac{d^{-1}}{4\gamma^2}(\tau^2\rho B+\frac{N}{\rho^2\gamma^3}cos\gamma s)(s^2-s) \\
&\quad +\frac{d^{-1}}{2\gamma^4}\{\frac{\tau R}{\gamma^2}+\frac{A}{\rho}cos\gamma s\}(sin\gamma s-s\,sin\gamma) \\
&\quad -\frac{d^{-1}}{2\gamma^4}\{\frac{\tau Q}{\gamma^3}+\frac{B}{\rho}cos\gamma s\}(cos\gamma s+(1-cos\gamma)s-1) \\
&\quad +\frac{\tau Msd^{-1}}{2\rho\gamma^6}(sin\gamma s-sin\gamma)-\frac{\rho}{2d_1}Bcos^2\gamma s \\
&\quad +\frac{\rho}{2d_1}Asin\gamma scos\gamma s-\frac{N}{2\gamma^3d_1}cos\gamma s\}
\end{aligned}
$$

so that, integrating with respect to s, we have

$$
\begin{aligned}
y_2 e^{-i\gamma s} = & \Big[\frac{\rho A}{2\gamma^2}(\tau^2+\frac{1}{\rho^2 d_2})\{\frac{sin2\gamma s}{4\gamma}+\frac{1}{2}s\}+\frac{\rho B}{2\gamma^2}(\tau^2+\frac{1}{\rho^2 d_2})\frac{cos2\gamma s}{4\gamma} \\
& -\frac{\tau M}{2\rho\gamma^4}(1-\frac{1}{d_2})\{\frac{s}{\gamma}sin\gamma s+\frac{cos\gamma s}{\gamma^2}\}-\frac{\tau C}{2\rho\gamma^4}(1-\frac{1}{d_2})\frac{sin\gamma s}{\gamma} \\
& -\frac{f_n}{2\gamma^4}(\tau^2+\frac{1}{\rho^2 d_2})\frac{sin\gamma s}{\gamma}+\frac{\tau^2\rho d^{-1}A}{4\gamma^2}(\frac{s^3}{3}-\frac{s^2}{2}) \\
& -\frac{Nd^{-1}}{4\rho^2\gamma^5}\{-(s^2-s)\frac{cos\gamma s}{\gamma}+(2s-1)\frac{sin\gamma s}{\gamma^2}+\frac{2}{\gamma^3}cos\gamma s\} \\
& +\frac{\tau d^{-1}R}{2\gamma^6}\{\frac{sin\gamma s}{\gamma}+(1-cos\gamma)\frac{s^2}{2}-s\} \\
& +\frac{Bd^{-1}}{2\rho\gamma^4}\{-\frac{cos2\gamma s}{4\gamma}+(1-cos\gamma)(-s\frac{cos\gamma s}{\gamma}+\frac{sin\gamma s}{\gamma^2})+\frac{cos\gamma s}{\gamma}\} \\
& -\frac{Ad^{-1}}{2\rho\gamma^4}\{\frac{s}{2}-\frac{sin2\gamma s}{4\gamma}-sin\gamma\,(-s\frac{cos\gamma s}{\gamma}+\frac{sin\gamma s}{\gamma^2})\} \\
& +\frac{\tau d^{-1}Q}{2\gamma^7}\{-\frac{cos\gamma s}{\gamma}-\frac{s^2}{2}sin\gamma\}+\frac{\tau M d^{-1}}{2\rho\gamma^6}\{s\frac{sin\gamma s}{\gamma}+\frac{cos\gamma s}{\gamma^2}-\frac{s^2}{2}cos\gamma\} \\
& -\frac{\rho A}{2d_1}\{\frac{s}{2}-\frac{sin2\gamma s}{4\gamma}\}-\frac{\rho B}{2d_1}\frac{cos2\gamma s}{4\gamma}-\frac{N}{2\gamma^3 d_1}\frac{cos\gamma s}{\gamma}+c_2\Big] \\
& -i\Big[-\frac{\rho A}{2\gamma^2}(\tau^2+\frac{1}{\rho^2 d_2})(-\frac{cos2\gamma s}{4\gamma})-\frac{\rho B}{2\gamma^2}(\tau^2+\frac{1}{\rho^2 d_2})(\frac{s}{2}-\frac{sin2\gamma s}{4\gamma}) \\
& -\frac{\tau M}{2\rho\gamma^4}(1-\frac{1}{d_2})\{-\frac{s}{\gamma}cos\gamma s+\frac{sin\gamma s}{\gamma^2}\}-\frac{\tau C}{2\rho\gamma^4}(1-\frac{1}{d_2})\frac{cos\gamma s}{\gamma} \\
& +\frac{f_n}{2\gamma^4}(\tau^2+\frac{1}{\rho^2 d_2})\frac{cos\gamma s}{\gamma}+\frac{\tau^2\rho d^{-1}B}{4\gamma^2}(\frac{s^3}{3}-\frac{s^2}{2}) \\
& +\frac{Nd^{-1}}{4\rho^2\gamma^5}\{(s^2-s)\frac{sin\gamma s}{\gamma}+(2s-1)\frac{cos\gamma s}{\gamma^2}-\frac{2}{\gamma^3}sin\gamma s\} \\
& +\frac{\tau d^{-1}R}{2\gamma^6}\{-\frac{cos\gamma s}{\gamma}-\frac{s^2}{2}sin\gamma\} \\
& -\frac{Bd^{-1}}{2\rho\gamma^4}\{\frac{s}{2}+\frac{sin2\gamma s}{4\gamma}+(1-cos\gamma)(s\frac{sin\gamma s}{\gamma}+\frac{cos\gamma s}{\gamma^2})-\frac{sin\gamma s}{\gamma}\} \\
& +\frac{Ad^{-1}}{2\rho\gamma^4}\{-\frac{cos2\gamma s}{4\gamma}-sin\gamma(s\frac{sin\gamma s}{\gamma}+\frac{cos\gamma s}{\gamma^2})\} \\
& -\frac{\tau d^{-1}Q}{2\gamma^7}\{\frac{sin\gamma s}{\gamma}+(1-cos\gamma)\frac{s^2}{2}-s\} \\
& +\frac{\tau M d^{-1}}{2\rho\gamma^6}\{-s\frac{cos\gamma s}{\gamma}+\frac{sin\gamma s}{\gamma^2}-\frac{s^2}{2}sin\gamma\} \\
& -\frac{\rho B}{2d_1}\{\frac{s}{2}+\frac{sin2\gamma s}{4\gamma}\}-\frac{\rho A}{2d_1}\frac{cos2\gamma s}{4\gamma}-\frac{N}{2\gamma^3 d_1}\frac{sin\gamma s}{\gamma}+c_3\Big] \\
& = p-iq \qquad \text{(say)}.
\end{aligned}
$$

then

$$\begin{aligned} y_2 &= e^{i\gamma s}\{p - iq\} \\ &= \{pcos\gamma s + qsin\gamma s\} + i\{psin\gamma s - qcos\gamma s\}. \end{aligned}$$

The simplified form of the real part and the imaginary part of the solution are

$$\begin{aligned} &pcos\gamma s + qsin\gamma s = \\ &-\frac{\rho A}{2\gamma^2}(\tau^2 + \frac{1}{\rho^2 d_2})\{\frac{sin\gamma s}{4\gamma} + \frac{s\,cos\gamma s}{2}\} + \frac{\rho B}{2\gamma^2}(\tau^2 + \frac{1}{\rho^2 d_2})\{\frac{cos\gamma s}{4\gamma} - \frac{s\,sin\gamma s}{2}\} \\ &-\frac{\tau M}{2\rho\gamma^6}(1 - \frac{1}{d_2}) + \frac{\tau^2 \rho d^{-1}}{4\gamma^2}(\frac{s^3}{3} - \frac{s^2}{2})(Acos\gamma s + Bsin\gamma s) - \frac{Nd^{-1}}{4\rho^2\gamma^5}\{-\frac{(s^2 - s)}{\gamma} + \frac{2}{\gamma^2}\} \\ &+\frac{\tau Rd^{-1}}{2\gamma^6}\{(\frac{s^2}{2} - s)cos\gamma s - \frac{s^2}{2}cos\gamma(1 - s)\} \\ &+\frac{Bd^{-1}}{2\rho\gamma^4}\{\frac{1}{\gamma} - \frac{s}{2}sin\gamma s - \frac{cos\gamma s}{4\gamma} - \frac{s}{\gamma}(1 - cos\gamma)\} \\ &-\frac{\tau Qd^{-1}}{2\gamma^7}\{\frac{1}{\gamma} + (\frac{s^2}{2} - s)sin\gamma s + \frac{s^2}{2}sin\gamma(1 - s)\} \\ &-\frac{Ad^{-1}}{2\rho\gamma^4}\{\frac{s}{2}cos\gamma s - \frac{sin\gamma s}{4\gamma} + \frac{s}{\gamma}sin\gamma\} \\ &\frac{\tau Md^{-1}}{2\rho\gamma^6}\{\frac{1}{\gamma^2} - \frac{s^2}{2}cos\gamma(1 - s)\} - \frac{\rho A}{2d_1}\{\frac{s}{2}cos\gamma s - \frac{sin\gamma s}{4\gamma}\} \\ &-\frac{\rho B}{2d_1}\{\frac{s}{2}sin\gamma s + \frac{cos\gamma s}{4\gamma}\} - \frac{N}{2\gamma^4 d_1} + c_2 cos\gamma s + c_3 sin\gamma s \end{aligned}$$

and

$$\begin{aligned} &psin\gamma s - qcos\gamma s = \\ &-\frac{\rho A}{2\gamma^2}(\tau^2 + \frac{1}{\rho^2 d_2})\{\frac{cos\gamma s}{4\gamma} + \frac{s\,sin\gamma s}{2}\} + \frac{\rho B}{2\gamma^2}(\tau^2 + \frac{1}{\rho^2 d_2})\{-\frac{sin\gamma s}{4\gamma} + \frac{s\,cos\gamma s}{2}\} \\ &-\frac{\tau Ms}{2\rho\gamma^5}(1 - \frac{1}{d_2}) - \frac{\tau C}{2\rho\gamma^5}(1 - \frac{1}{d_2}) - \frac{f_n}{2\gamma^5}(\tau^2 + \frac{1}{\rho^2 d_2}) \\ &+\frac{\tau^2\rho d^{-1}}{4\gamma^2}(\frac{s^3}{3} - \frac{s^2}{2})(Asin\gamma s - Bcos\gamma s) - \frac{Nd^{-1}}{4\rho^2\gamma^7}(2s - 1) \\ &+\frac{\tau Rd^{-1}}{2\gamma^6}\{\frac{1}{\gamma} + (\frac{s^2}{2} - s)sin\gamma s + \frac{s^2}{2}sin\gamma(1 - s)\} \\ &+\frac{Bd^{-1}}{2\rho\gamma^4}\{\frac{sin\gamma s}{4\gamma} + \frac{s}{2}cos\gamma s + \gamma^2(1 - cos\gamma)\} \\ &+\frac{\tau Qd^{-1}}{2\gamma^7}\{(\frac{s^2}{2} - s)cos\gamma s - \frac{s^2}{2}cos\gamma(1 - s)\} \\ &-\frac{Ad^{-1}}{2\rho\gamma^4}\{\frac{s}{2}sin\gamma s - \frac{cos\gamma s}{4\gamma} + \frac{sin\gamma}{\gamma^2}\} \\ &+\frac{\tau Md^{-1}s}{2\rho\gamma^6}\{\frac{1}{\gamma} + \frac{s}{2}sin\gamma(1 - s)\} - \frac{\rho A}{2d_1}\{\frac{s}{2}sin\gamma s - \frac{cos\gamma s}{4\gamma}\} \\ &+\frac{\rho B}{2d_1}\{\frac{s}{2}cos\gamma s + \frac{sin\gamma s}{4\gamma}\} + c_2 sin\gamma s - c_3 cos\gamma s \end{aligned}$$

Finally, we have

$$\begin{aligned}
\mathbf{X} &= PY \\
&= \begin{bmatrix} \tau y_1 + \frac{1}{\rho}(y_2 + y_3) \\ i\gamma(y_2 - y_3) \\ \frac{1}{\rho}y_1 - \tau(y_2 + y_3) \end{bmatrix}
\end{aligned}$$

So by using the fact that $y_3 = \bar{y}_2$, we obtain

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \tau y_1 + \frac{2}{\rho}Re(y_2) \\ -2\gamma Im(y_2) \\ \frac{1}{\rho}y_1 - 2\tau Re(y_2) \end{bmatrix}.$$

The boundary conditions give

$$\begin{bmatrix} \tau y_1(0) + \frac{2}{\rho}Re(y_2)(0) \\ -2\gamma Im(y_2)(0) \\ \frac{1}{\rho}y_1(0) - 2\tau Re(y_2)(0) \end{bmatrix} = \mathbf{0} \tag{5.22}$$

and

$$\begin{bmatrix} \tau y_1(1) + \frac{2}{\rho}Re(y_2)(1) \\ -2\gamma Im(y_2)(1) \\ \frac{1}{\rho}y_1(1) - 2\tau Re(y_2)(1) \end{bmatrix} = \mathbf{0}. \tag{5.23}$$

From (5.22) and (5.23) we have six equations from which we can find the unknowns A,B,C, a_1, a_2 and a_3. These six equations are

$$y_1(0) = y_1(1) = 0 \tag{5.24}$$

$$Re(y_2)(0) = Re(y_2)(1) = 0 \tag{5.25}$$

$$Im(y_2)(0) = Im(y_2)(1) = 0 \tag{5.26}$$

Elimination of these constants is not as easy as in the rotational components, so they are calculated by computer, using Gaussion elimination.

6 Numerical Results

In this Chapter we compare the results for some examples using the standard finite element formulation with and without reduced integration (recall that the mixed formulation is equivalent to the reduced integration version of the standard formulation). We plot graphs of Log| error $|_1$ against Log(mesh size) to obtain actual rates of convergence; these are compared against the estimates of Lemma 4.1.

6.1 Finite Element Model

The strength of the finite element method lies in its ability to construct global basis fuctions in a systematic and consistent manner. The unit interval [0,1] is partitioned into NELT finite elements of equal length. The basis functions or shape functions are defined locally on each element; these are then combined to obtain the global basis functions. Suppose that there are NNT nodes in the mesh; then a member $\mathbf{w}_h$ of $\mathbf{W}^h$ has the form, in terms of the nodal values $\mathbf{w}_i$,

$$\mathbf{w}_h = \sum_{i=1}^{NNT} \mathbf{w}_i \Phi_i,$$

where $\mathbf{w}_i = (u_i, v_i, w_i, \phi_i, \vartheta_i, \psi_i)^T$ are the nodal values and $\Phi_i(s)$ are the global basis functions constructed from local basis functions $N_i^e(s)$, and satisfying the condition

$$\Phi_i(s_j) = \delta_{ij}; \qquad i, j = 0, 1, \cdots, NNT.$$

As a consequence, $\mathbf{w}_h$ has the form

$$\mathbf{w}_h^e(s) = \sum_{i=1}^{NNEL} \mathbf{w}_i^e N_i^e(s)$$

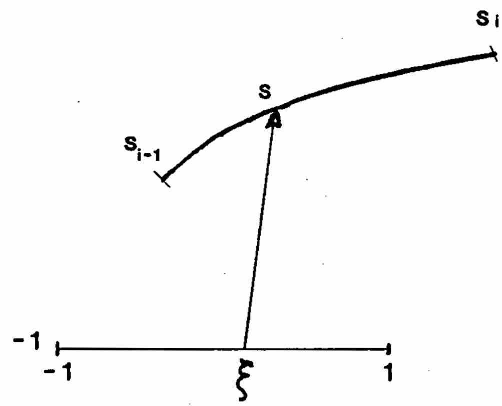


Figure 6.1: Isoparametric Map

at element level, where $N_i^e = \Phi_i|_{\Delta_e}$ and NNEL is the number of nodes per element. Note that the numbering at local and global levels does not necessarily coincide.

We now introduce an isoparametric map from reference element $\bar{\Delta}_e$ to a master element $\hat{\Delta}_e = [-1, 1]$, defined by (see figure (6.1))

$$s = \sum_{i=1}^{NNEL} s_i \hat{N}_i^e(\xi)$$

where s_i is the i^{th} nodal coordinate of the element.

At element level, the isoparametric interpolations allow us to write

$$\mathbf{w}_h^e(\xi) = \sum_{i=1}^{NNEL} \mathbf{w}_i^e \hat{N}_i^e(\xi) \tag{6.1}$$

$\hat{N}_i(\xi) = N_i^e(\xi(s))$.

Accordingly, the strain vector $\boldsymbol{\eta}^e$ on element e becomes

$$\begin{aligned}
\boldsymbol{\eta}^e &= \boldsymbol{\theta}_h^{e\prime} \\
&= \begin{bmatrix} \phi' - \dfrac{\psi}{\rho} \\ \psi' + \dfrac{\phi}{\rho} + \tau\vartheta \\ \vartheta' + \tau\psi \end{bmatrix}_h^e \\
&= A^e \mathbf{a}^e
\end{aligned}$$

where $\mathbf{a}^e$ denotes the column vector for the degrees of freedom of $\mathbf{w}_h$ to the element e and A^e is a matrix of the form $(A_1^e, \cdots, A_{NNEL}^e)$, where

$$A_i^e = \begin{pmatrix} 0 & 0 & 0 & \dfrac{N_i'}{|J|} & -\dfrac{N_i}{\rho} & 0 \\ 0 & 0 & 0 & \dfrac{N_i}{\rho} & \dfrac{N_i'}{|J|} & -\tau N_i \\ 0 & 0 & 0 & 0 & \tau N_i & \dfrac{N_i'}{|J|} \end{pmatrix}.$$

Similarly, the strain $\boldsymbol{\varepsilon}^e$ on element e becomes

$$\begin{aligned}
\boldsymbol{\varepsilon}^e &= (\mathbf{u}' - \boldsymbol{\theta} \wedge \mathbf{t})^e \\
&= \begin{bmatrix} u' - \dfrac{v}{\rho} \\ v' + \dfrac{u}{\rho} - \tau w + \vartheta \\ w' + \tau v - \psi \end{bmatrix}_h^e \\
&= B^e \mathbf{a}^e.
\end{aligned}$$

Here $B^e = (B_1^e, \cdots, B_{NNEL}^e)$, $\mathbf{a}^e = (\mathbf{w}_1^e, \cdots, \mathbf{w}_{NNEL}^e)$ and

$$B_i^e = \begin{pmatrix} \dfrac{N_i'}{|J|} & -\dfrac{N_i}{\rho} & 0 & 0 & 0 & 0 \\ \dfrac{N_i}{\rho} & \dfrac{N_i'}{|J|} & -\tau N_i & 0 & 0 & N_i \\ 0 & \tau N_i & \dfrac{N_i'}{|J|} & 0 & -N_i & 0 \end{pmatrix}.$$

Substituting these equations into equation (3.20) we obtain

$$A_d(\mathbf{w}_h, \mathbf{z}_h) = \sum_{e=1}^{NELT} A_d^e(\mathbf{w}_h^e, \mathbf{z}_h^e)$$

and

$$\langle \mathcal{F}, \mathbf{z}_h \rangle = \sum_{e=1}^{NELT} \langle \mathcal{F}, \mathbf{z}_h^e \rangle_e.$$

The element stiffness matrix $(K^e + d^{-1}G^e)$ is defined by

$$\begin{aligned} A_d^e(\mathbf{w}_h^e, \mathbf{z}_h^e) &= \int_e \boldsymbol{\Psi}_h'^T \boldsymbol{E} \boldsymbol{\theta}_h' |J|\, d\xi + d^{-1} \int_e \boldsymbol{\varepsilon}(\mathbf{z}_h)^T \boldsymbol{D} \boldsymbol{\varepsilon}(\mathbf{w}_h) |J|\, d\xi, \\ &= \mathbf{b}^T \{K^e + d^{-1}G^e\} \mathbf{a}^e; \end{aligned} \tag{6.2}$$

thus

$$\begin{aligned} K^e &= \int_e A^{eT} \boldsymbol{E} A^e |J|\, d\xi, \\ G^e &= \int_e B^{eT} \boldsymbol{D} B^e |J|\, d\xi. \end{aligned}$$

The generalised force vector $\{F^e\}$ is defined by

$$\mathbf{b}^T\{F^e\} = \int_e \mathcal{F}^e \cdot \mathbf{z}_h |J| d\xi;$$

It can be easily shown that K^e and G^e are symmetric, since the bilinear form A_d is symmetric.

6.2 Computer Programme

Based on the concepts outlined in the preceding Sections, we have developed a computer program to analyse the accuracy and convergence of the finite element solution. The Isoparametric linear and quadratic elements are used to calculate the stiffness matrix and vector. The special feature of this programme is that the global matrix is stored in a one dimensional array taking advantage of the banded symmetric nature of the stiffness equations. This minimises the computational cost and core storage requirements. (see Hinton and Owen [17], Chapter 11, for further details).

This programme is currently limited to 200 linear and 100 quadratic elements, but the number of elements may be increased by changing the appropriate dimensions and common statements. The calculation were carried out on a Vax 6000-330 in double precision arithmetic.

The main structure of the program contains six blocks which can be described in the following flowchart.

Reading, verification and organization of input quantities

Read Calculate and Print:

- number of nodes per element
- nodal coordinates
- element connectivities
- material properties
- bandwidth of the global matrix

↓

Construction of global matrix and vector

For each element:

- extract element informations
- construct element vectors(F^e) and element matrices K^e, G^e
- assemble element matrices and vectors into global matrix and vector
- store in one dimensional array

↓

1

1

Solution of the system of equations

- modify global matrix and vector to satisfy boundary conditions
- triangularization
- compute solutions

Exact solution for the governing equations

- evaluate the exact nodal values

Error norm

- evaluate the error under the norm $|\cdot|_1$
- calculate logarithm of error norm and mesh size

output of results

- print exact and approximated nodal values
- print logarithm of error norm for displacement vector rotation vector and mesh size

The following gives a brief description of the main subroutine we have used in the computer programme VBEAM.

Block I Subroutines:

INPUT The purpose of this subroutine is to read number of nodes per element and to store prescribed values in global arrays.

GRILLE This calculates the coordinates and element connectivity.

ELPROP Input and output of material properties and geometry of the beam and calculate the parameter d.

DMAT,EMAT This calculates elastic compliant matrices D and E.

CURVE Calculate the curvature and the torsion.

SYSBAN The function of the subroutine is to determine the system upper half band width.

Block II Subroutines:

ENODES This is to extract nodal coordinates and element connectivities for each element.

ELEB This routine formulates stiffness matrix $[K^e]$ for each element in turn according to (6.2).

ELES This routine formulates stiffness matrix $[G^e]$ for each element in turn according to (6.2).

LOAD This subroutine performs the evaluation of each element column according to 6.3

vector.

STORVEC The purpose of this routine is to assemble element matrices into global matrix which is stored in one dimentional array and column vectors.

Block III Subroutines:

GREDUC This subroutine adjusts the system equations with prescribed values and performs the equation elimination process for equation solution by Gaussian reduction. This is only applicable to banded symmetric equations.

BAKSUB The object of this routine is to undertake the basic substitution process required after equation elimination by Gaussian reduction. This results the all nodal values of generalised displacements.

Block IV Subroutines:

EXACTV To calculate the exact nodal solution derived in Chapter 5.

Block V Subroutines:

DISERO The function of this subroutine is to evaluate the finite element solutions for generalized displacements and then to calculate and output the logarithm of the error error of displacement and rotation vectors under the norm $|\cdot|_1$.

Block VI Subroutines:

RESOL This routine is to output all the nodal values of the generalized displacements.

6.3 Error Estimates

In this Section we consider some numerical examples that illustrate the performance of the formulations described above. Attention is focused on two examples: the balcony beam, and the helicoidal beam; in both examples uniformly distributed load are considered. Two sets of computations are presented: The first set of computations is obtained by using linear elements and the second with quadratic elements.

Our motivation is to assess the element on the basis of their dependence on thickness parameter d and the effect of selective reduced integration. We therefore obtain the solution error norms of the displacement and rotation for $d = 10^{-1}$ and $d = 10^{-6}$ and plot graphs of $\log|\ \mathbf{U} - \mathbf{U}_h\ |_1$ against log h (where $\mathbf{U}$ represents either $\mathbf{u}$ or $\boldsymbol{\theta}$) to test the asymptotic behaviour predicted by Lemma 4.1. According to Lemma 4.1, one expects the results to lie on straight lines with slope r where r is the degree of the approximating polynomial. .

Example 1.

We consider a circular helicoidal beam as illustrated in figure 6.2. The parametric equation of the centroidal line of such a beam is given by

$$\boldsymbol{\alpha}(s) = (A\cos\tfrac{s}{n}, A\sin\tfrac{s}{n}, C\,\tfrac{s}{n})$$

where A and B are constants which depends on the geometry of the beam, $n = \sqrt{A^2 + C^2}$ and s is the arc length. For the purposes of our numerical experiments we set $A = 1m$, $C = 1m$ and $n = \sqrt{A^2 + C^2} = \sqrt{2}$.
Hence the curvature $\kappa = \dfrac{A}{A^2 + C^2} = 0.5$ and the torsion $\tau = \dfrac{C}{A^2 + C^2} = 0.5$. The length of the beam $= \pi\, n = \sqrt{2}\pi$. The cross section of the beam is considered as circles of radius l therefore area of the cross section is πl^2 and moment of inertia $\frac{\pi l^4}{4}$. Thus the parameter d is equal to $(\frac{l}{2L})^2$.

Force components are: $F_t = -\frac{C}{n}P$, $F_n = 0$ and $F_b = -\frac{A}{n}P$ for the vertical helicoidal beam with uniform load P.

Errors for the standard finite element model $\boldsymbol{S}_d^h$ are shown in Figures 6.4-6.9. Graphs 6.4 and 6.5 show the results for a rather thick beam with $d = 10^{-1}$ in which the convergence is at the predicted rates in both exact and reduced integration; the results for reduced integration are not shown here since this case produces the same results.

Results for the case $d = 10^{-6}$ are shown in figures 6.6-6.9. In the linear case we do not obtain convergence with exact integration, except possibly for a very large number of elements. In the quadratic case the behaviour with exact integration appears to be satisfactory. Convergence is at the predicted rates in both cases with reduced integration. The uniform optimal order of convergence is evident from the figures 6.7 and 6.9 .

Example 2.

The numerical calculations are also performed for a circular arch with circular cross section, under the influence of a uniform load applied perpendicular to the plane of the beam (Figure 6.3). The parametric equation is given by the same as that of the beam in example 1 with $C = 0$. The constant A then represents the radius of the arch. We choose A = 1 m, and therefore curvature = 1 (radius of the arch = 1) torsion = 0. Length of the beam is π and radious of the cross section is l. Force components are: $F_t = 0$, $F_n = 0$ and $F_b = -P$.

This example produces similar results to those obtained for example 1.

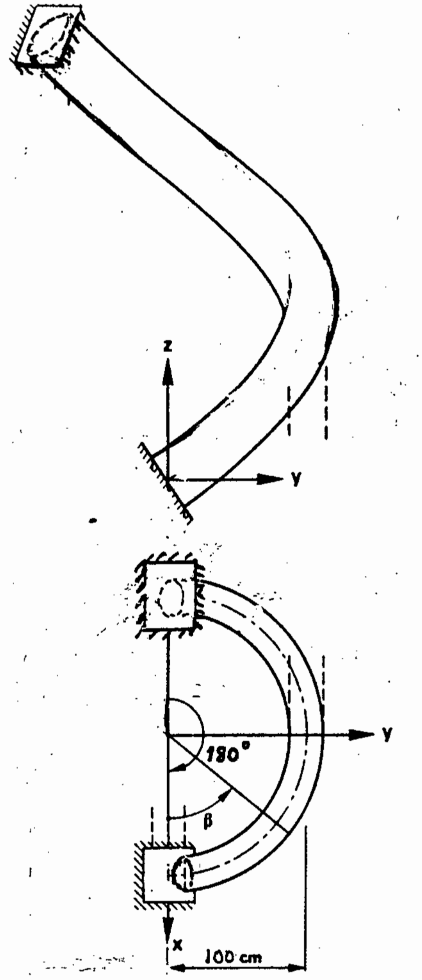


Figure 6.2: Helicoidal beam subjected to uniform load

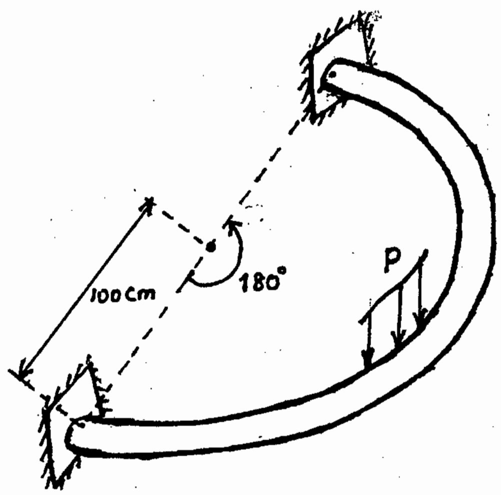


Figure 6.3: Balcony beam subjected to uniform load

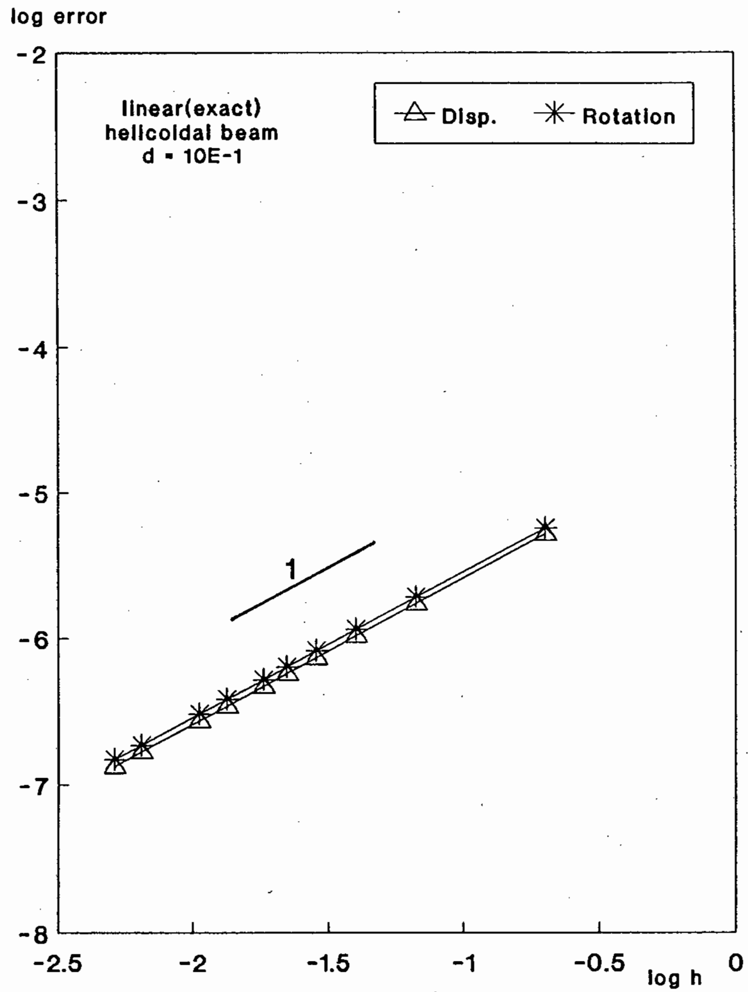


Figure 6.4: Log $|error|_1$ vs Log h for linear element with exact integration

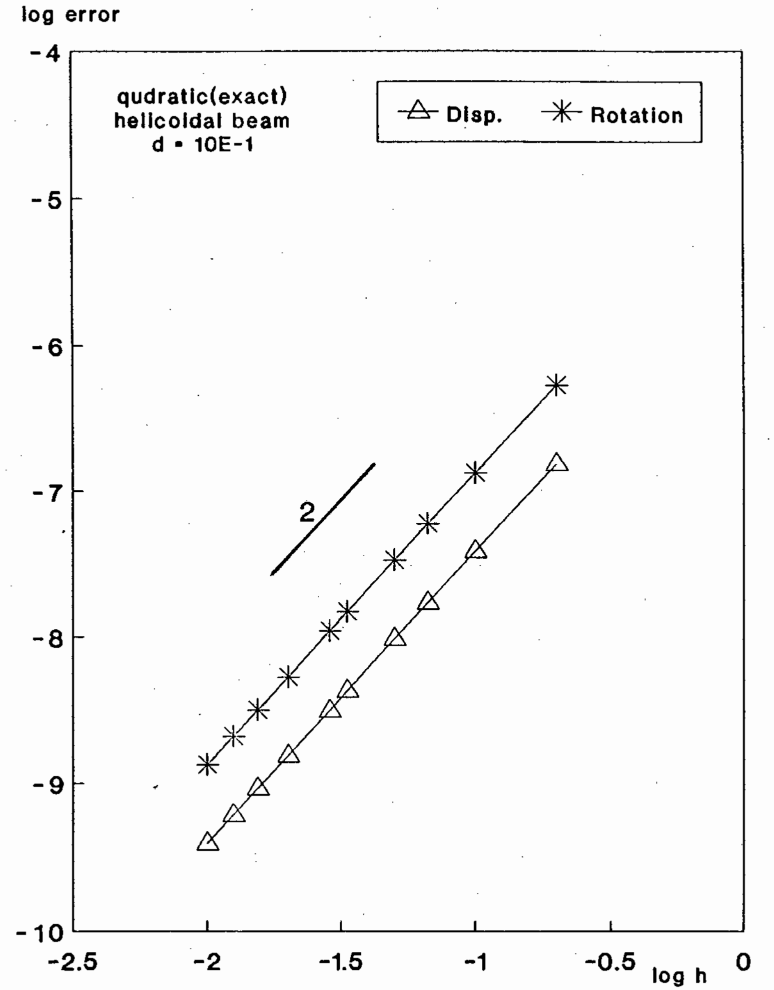


Figure 6.5: Log $|error|_1$ vs Log h for quadratic element with exact integration

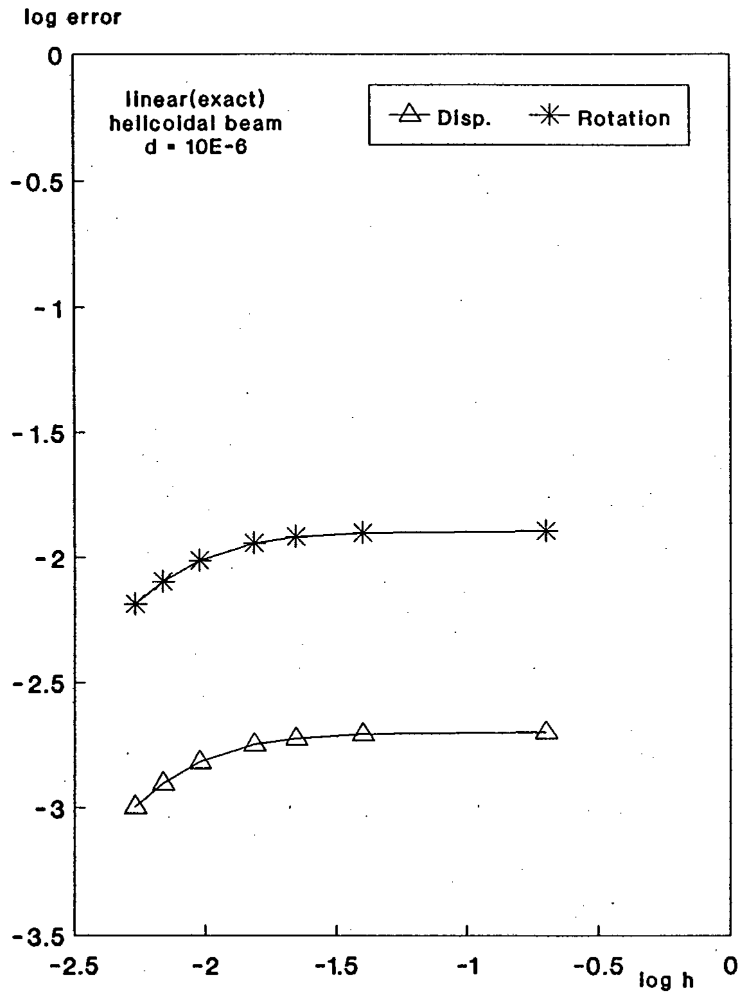


Figure 6.6: Log $|error|_1$ vs Log h for linear element with exact integration

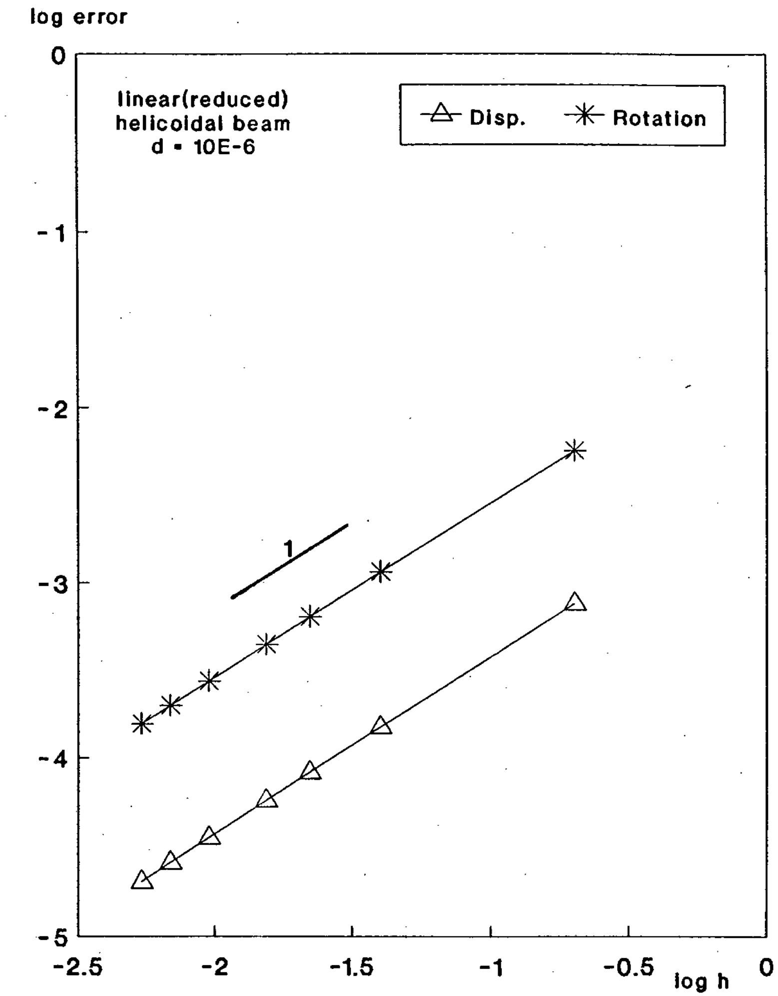


Figure 6.7: Log $|error|_1$ vs Log h for linear element with reduced integration

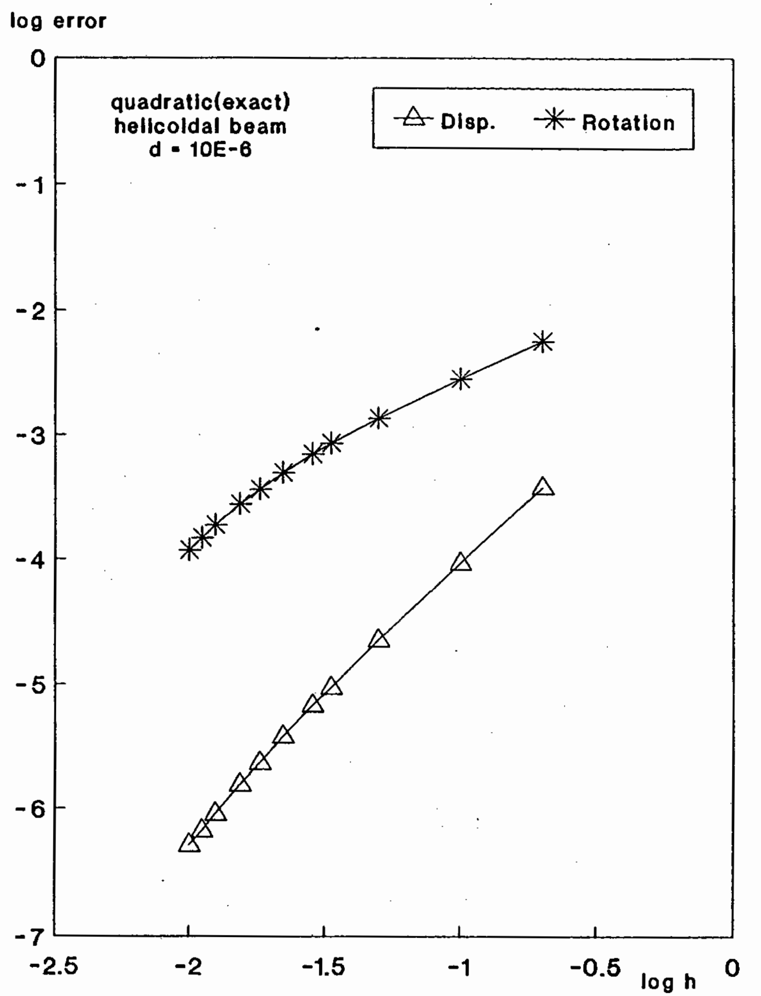


Figure 6.8: Log $|error|_1$ vs Log h for quadratic element with exact integration

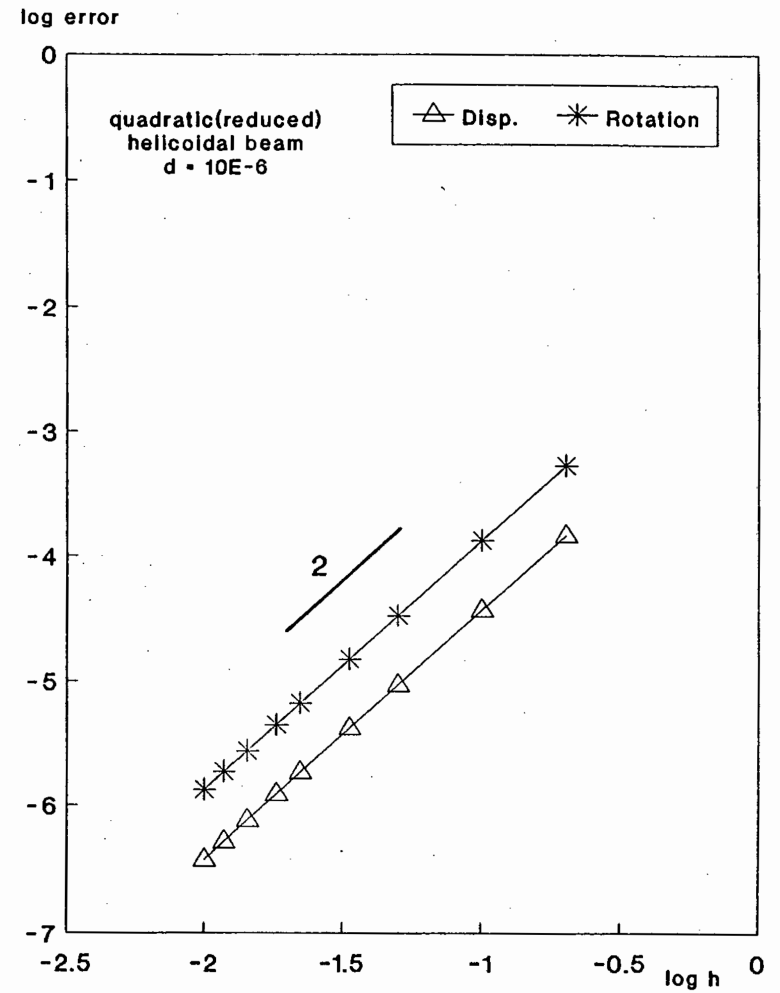


Figure 6.9: Log $|error|_1$ vs Log h for quadratic element with reduced integration

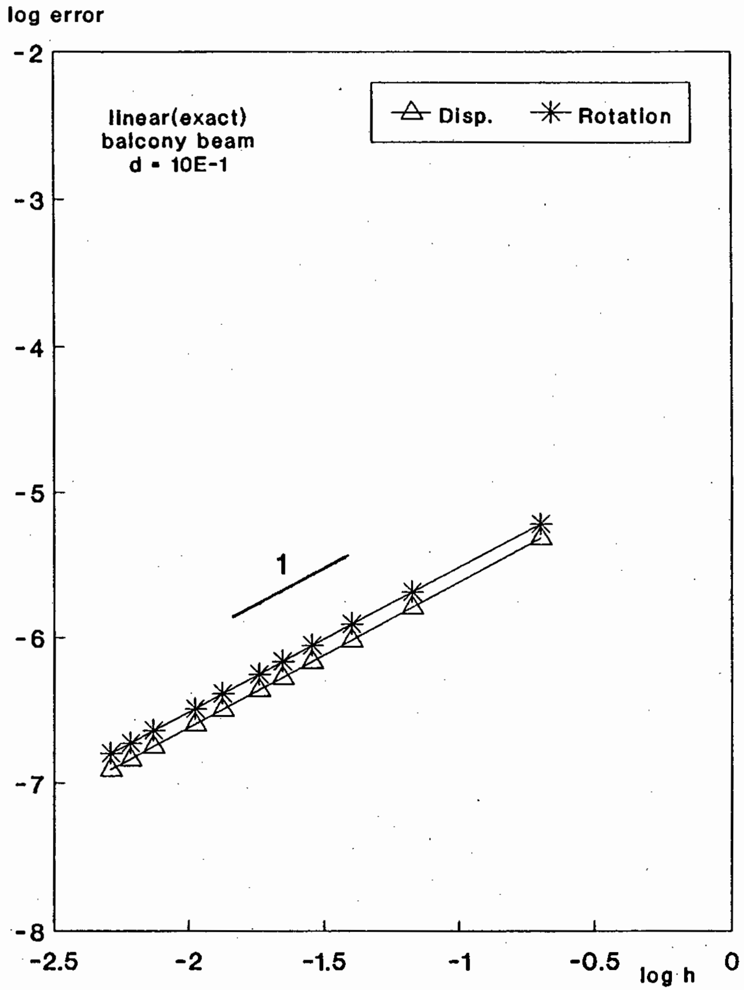


Figure 6.10: Log $|error|_1$ vs Log h for linear element with exact integration

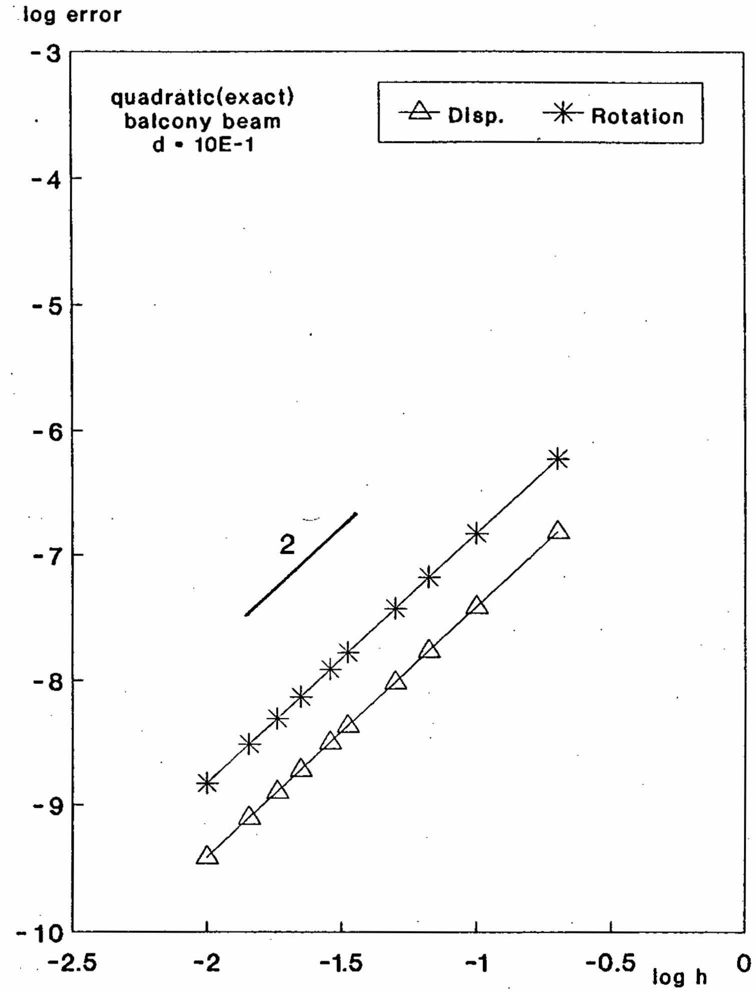


Figure 6.11: Log $|error|_1$ vs Log h for quadratic element with exact integration

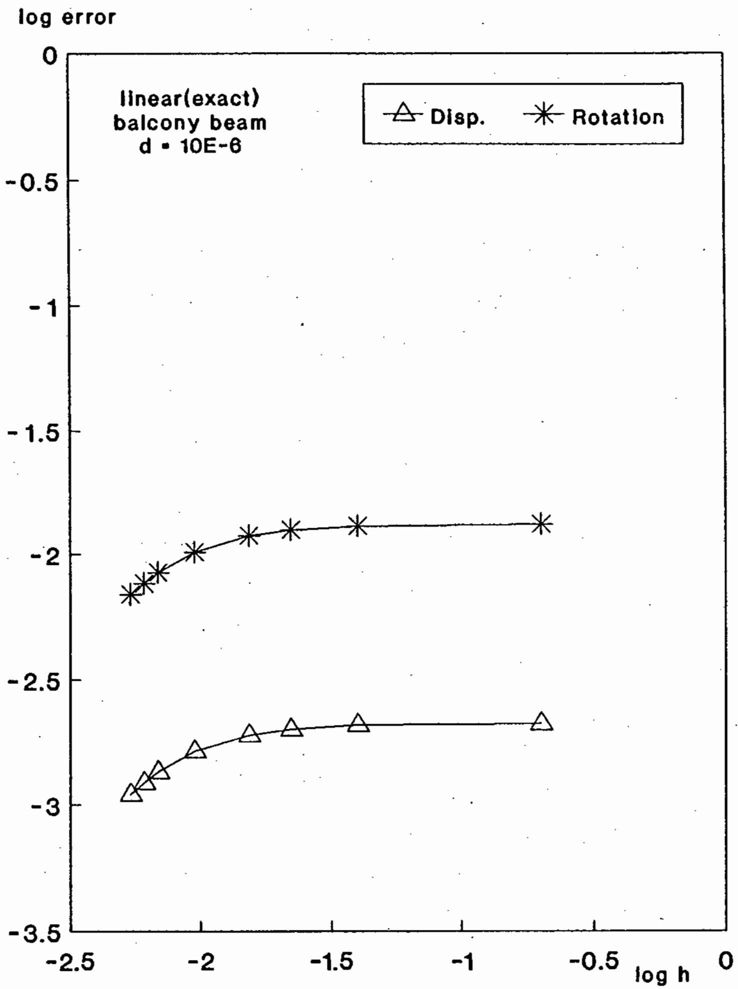


Figure 6.12: Log $|error|_1$ vs Log h for linear element with exact integration

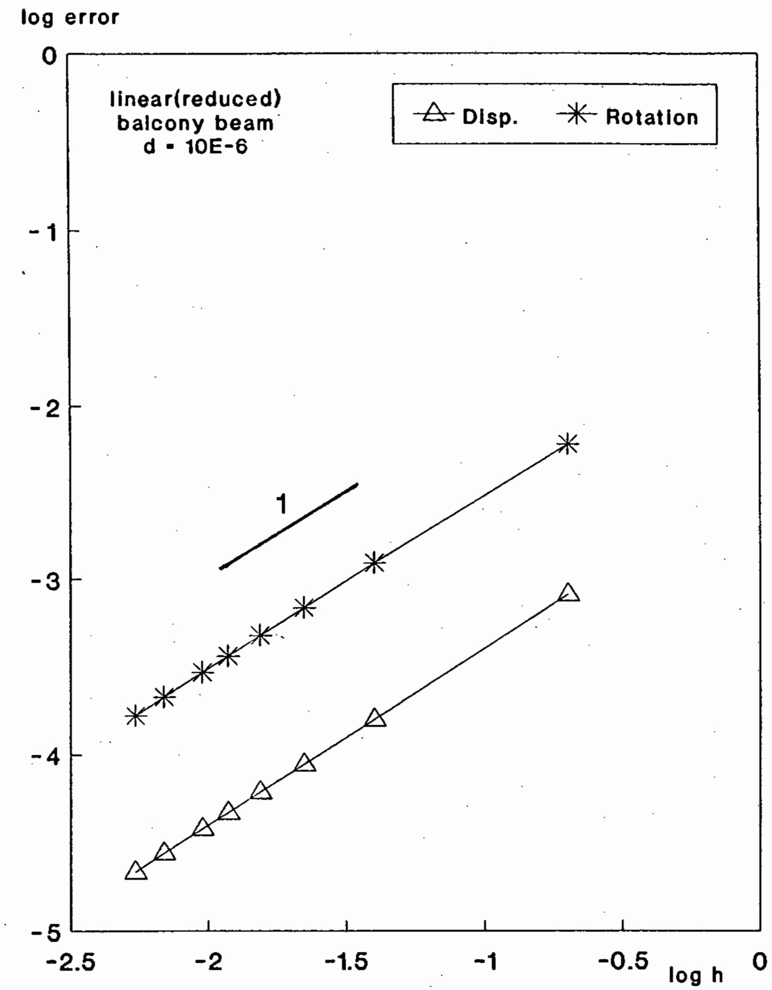


Figure 6.13: Log $|error|_1$ vs Log h for linear element with reduced integration

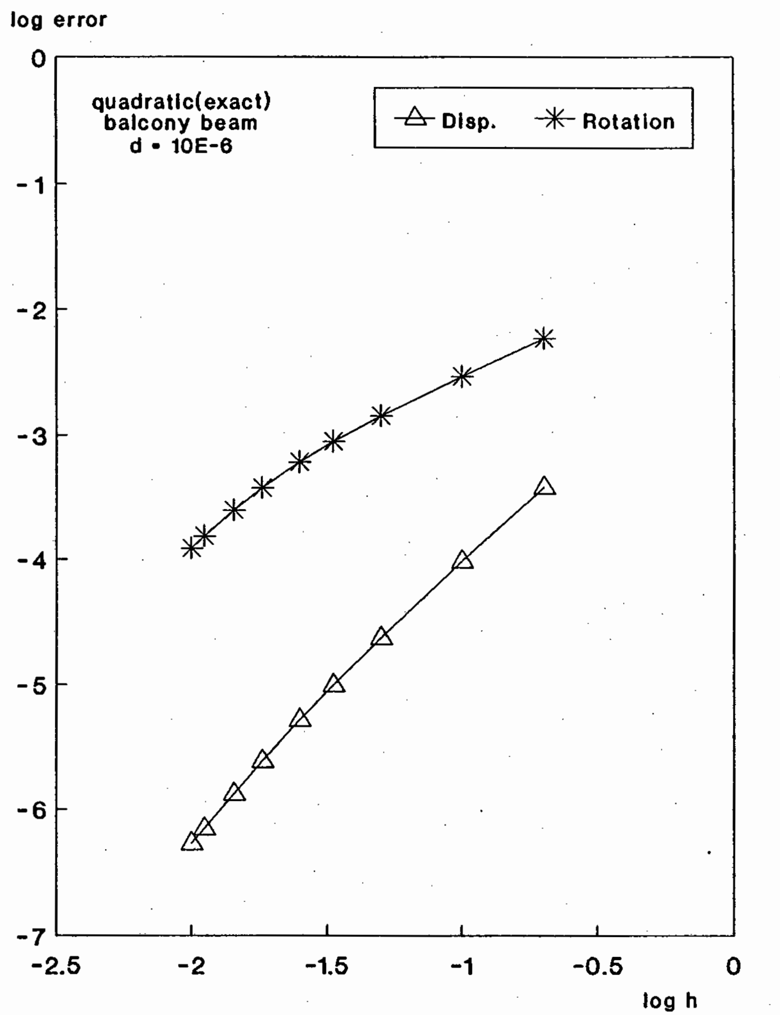


Figure 6.14: Log $|error|_1$ vs Log h for quadratic element with exact integration

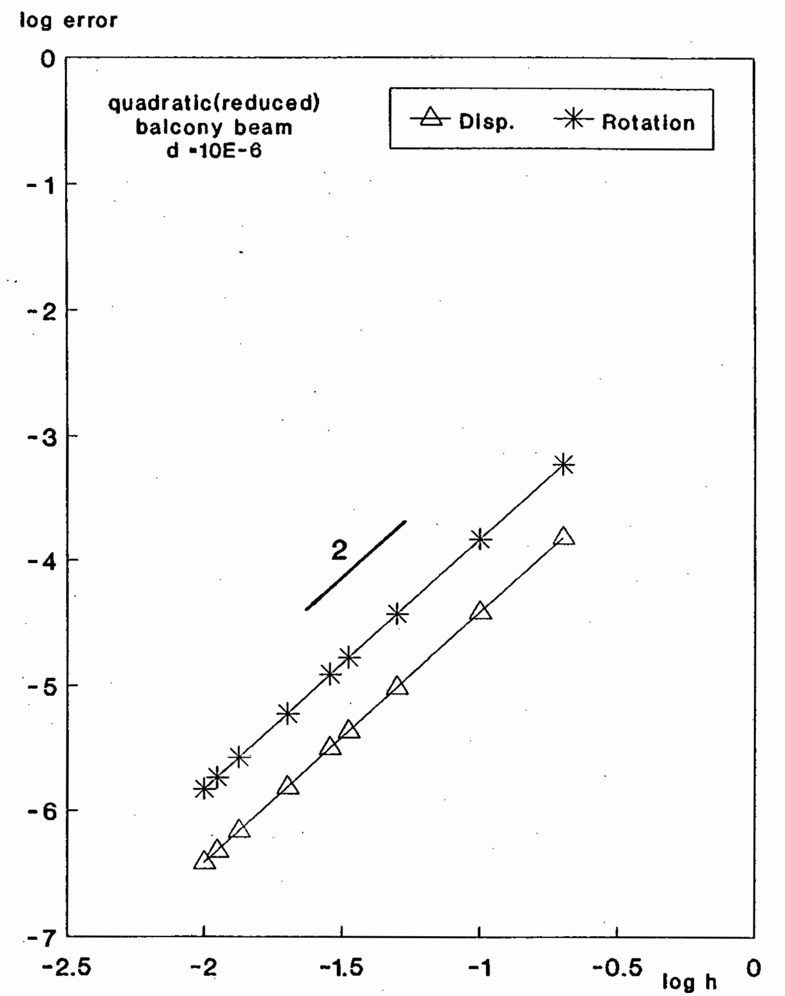


Figure 6.15: Log $|error|_1$ vs Log h for quadratic element with reduced integration

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