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Hyperconvexity and Endpoints of T_0-Quasi-Metric Spaces

by

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Abstract

Over the last decades much progress has been made in the investigation of hyperconvexity in metric spaces (see for instance [1, 2, 4, 9, 11]). Recently Kemajou et al [3] have published an article concerning hyperconvexity in T_0-quasi-metric spaces. In 1964 Isbell [2] introduced and studied the concept of an endpoint of a metric space. The aim of this dissertation is to begin an investigation into hyperconvexity and endpoints of T_0-quasi-metric spaces. It starts off with basic definitions and some well-known properties of quasi-pseudometric spaces. We conclude by commencing an investigation into hyperconvexity and endpoints of T_0-quasi-metric spaces. In this dissertation several results obtained for hyperconvexity and endpoints in metric spaces are generalized to T_0-quasi-metric spaces, and some original results for hyperconvexity and endpoints of T_0-quasi-metric spaces are presented. We also discuss for a partially ordered set the connection between its Dedekind-MacNeille completion and the q-hyperconvex hull of its natural T_0-quasi-metric space.

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To my family and friends thank you for all the support and encouragement.

Dedication

To my special woman Ingayamwena.

Declaration

I, PAULUS HAIHAMBO

hereby declare that this thesis is my own unaided work is being submitted for the degree of Master of Science at the University of Cape Town. It has not been submitted for any degree or examination in any other university.

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Contents

Introduction

The term hyperconvex space appeared first with N. Aronszajn and P. Panitchpakdi in their famous paper [1] to refer to those metric spaces that have the extension property for nonexpansive mappings. Hyperconvex spaces are also called injective spaces. In [2] Isbell has shown that every metric space X has an injective hull T_X, which is compact provided that X is totally bounded. He also showed that all injective hulls of X are isometric. In particular a metric space and its completion have exactly the same injective hull. It is well known that a metric space is hyperconvex if and only if it is injective in the category of metric spaces and nonexpansive maps (compare [4, 11, 9]).

Chapter 1 is dedicated to well-known results about quasi-pseudometric spaces and it follows the outline of [3] and [8]. We have given a complete proof of Example 1.1.1, because most of our main results do make use of this T_0-quasi-metric u and in [3] its proof was not given.

In Chapter 2 we have [3] as our main reference and most of its results are due to these authors. It is devoted to the concepts of convexity in T_0-quasi-metric spaces. Analogously to Isbell [2], Kemajou et al have shown that every T_0-quasi-metric space X has a q-hyperconvex hull Q_X, which is joincompact provided that X is totally bounded [3, Proposition 8] . They also showed that all q-hyperconvex hulls of X are isometric and a T_0-quasi-metric space X is q-hyperconvex if and only if it is injective in the category of T_0-quasi-metric spaces and nonexpansive maps [3, Theorem 1].

In Chapter 3 we present some of our main results concerning extensions of T_0-quasi-metric spaces and [4] is our main reference in this chapter. Most of the results in this chapter can be found in [5], but the proofs in this chapter and [5] respectively may sometimes differ. In [4] Dress introduced and studied the concept of the tight span of a metric space and proved among other things that every metric space X has a tight span T_X, which is compact if and only if $\bar{X}$ is compact, where $\bar{X}$ denotes the completion of X. It is known that Dress's theory is equivalent to the theory of the injective hull of a metric space due to Isbell. In this chapter we show that large parts of the theory developed by Dress do not use the symmetry condition and when appropriately modified hold essentially unchanged in the quasi-metric setting.

Chapter 4 contains our other main results pertaining to endpoints (resp. startpoints) in T_0-quasi-metric spaces and most of our results are similar to those of Isbell [2] and they are essentially contained in [6] and [7], in which we have modified most of the results in [2] to hold essentially in the quasi-metric setting. During his investigation on the hyperconvex hull of a metric space Isbell [2] introduced the concept of an endpoint of a (compact) metric space and proved among other things that the hyperconvex hull of a compact metric space is equal to the hyperconvex hull of its subspace of endpoints (compare also [9]). We present the concept of an endpoint (resp. a startpoint) in partially ordered sets and among other things give their characterizations. We also discuss for a partially ordered set X equipped with its natural T_0-quasi-metric d the connection between its Dedekind-MacNeille completion $DM(X)$ and its q-hyperconvex hull Q_X.

Finally in Chapter 5 we present our conclusions, which contain most of our summarized major results. In this chapter we also present some open problems encountered during the investigations of the materials in this dissertation.

Chapter 1

Quasi-pseudometric spaces

In this chapter we are going to discuss the basic definitions and some properties of quasi-pseudometric spaces.

1.1 Basic definitions

1.1.1 Definition. *([3, Definition 1]) Let X be a set and let $d : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of nonnegative reals.*
Then d is called a quasi-pseudometric *on X if*
(a) $d(x, x) = 0$ for all $x \in X$,
(b) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
If further,
(c) $d(x, y) = 0 = d(y, x)$ implies that $x = y$ for all $x, y \in X$, we shall say that d is a T_0-quasi-metric.

The ordered pair (X, d) is called a *quasi-pseudometric* (*resp.* T_0*-quasi-metric) space.*

If d is a quasi-pseudometric on a set X, then d^{-1} defined by $d^{-1}(x,y) = d(y,x)$ whenever $x, y \in X$ is also a quasi-pseudometric on X called the *conjugate* of d. As usual a quasi-pseudometric d on X such that $d = d^{-1}$ is called a *pseudometric*.
The function mapping $d^s(x,y) = \max\{d(x,y), d^{-1}(x,y)\} = d(x,y) \vee d^{-1}(x,y)$ whenever $x, y \in X$ is a pseudometric on X. (Here $\vee$ is used to denote a supremum, i.e. for any $x, y \in X, d(x,y) \vee d^{-1}(x,y)$ means $\sup\{d(x,y), d^{-1}(x,y)\}$).
Observe that d^s is a metric on X, whenever d is a T_0-quasi-metric on X.
One must note that the following inequalities hold for any quasi-pseudometric d on X.

$$d(x,y) \leq d^s(x,y) \text{ and } d^{-1}(x,y) \leq d^s(x,y) \text{ whenever } x, y \in X.$$

1.1.1 Example. *For any $x, y \in \mathbb{R}$, define $u(x,y) = \max\{x-y, 0\}$. Then d is a T_0-quasi-metric on $\mathbb{R}$. We shall write $x \dot{-} y$ to represent this T_0-quasi-metric, (i.e. $x \dot{-} y = \max\{x-y, 0\}$ whenever $x, y \in \mathbb{R}$).*

Proof. Observe that $u^s(x,y) = \mid x - y \mid$ whenever $x, y \in \mathbb{R}$.
Let $x, y, z \in \mathbb{R}$. Then $u(x,x) = \max\{x-x, 0\} = 0$ and $u(x,z) + u(z,y) = \max\{x-z, 0\} + \max\{z-y, 0\} \geq \max\{x-y, 0\} = u(x,y)$. If $u(x,y) = 0 = u(y,x)$, then $u^s(x,y) = 0$. Therefore $x = y$ and u is a T_0-quasi-metric as required. □

1.1.2 Definition. *([3, Definition 3]) Let (X,d) be a quasi-pseudometric space. Then X is called* metrically convex *if for any points $x, y \in X$ and non-negative reals r, s such that $d(x,y) \leq r + s$, then there exists $z \in X$ such that $d(x,z) \leq r$ and $d(z,y) \leq s$.*

1.1.2 Example. *([3, Example 4]) Consider the so-called Sorgenfrey T_0-quasi-metric on $\mathbb{R}$ which is defined for each $x, y \in \mathbb{R}$ as follows: $d(x,y) = x - y$ if $x \geq y$ and $d(x,y) = 1$ otherwise. Then d is not metrically convex.*

1.1.3 Definition. *A map* $f : (X, d_X) \to (Y, d_Y)$ *between two quasi-pseudometric spaces* (X, d_X) *and* (Y, d_Y) *is called* nonexpansive *provided that* $d_Y(f(x), f(y)) \leq d_X(x, y)$ *whenever* $x, y \in X$.

A map $f : (X, d_X) \to (Y, d_Y)$ *between two quasi-pseudometric spaces* (X, d_X) *and* (Y, d_Y) *is called an* isometric map *provided that* $d_Y(f(x), f(y)) = d_X(x, y)$ *whenever* $x, y \in X$.

Two quasi-pseudometric spaces (X, d_X) *and* (Y, d_Y) *will be called* isometric *provided that there exists a bijective isometric map between them.*

1.2 The topology of a quasi-pseudometric space

1.2.1 Definition. *([8, 1.1.2]) Let* (X, d) *be a quasi-pseudometric space. For each* $x \in X$ *and* $\epsilon > 0$, $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ *denotes the open* ϵ*-ball at* x. *The collection of all open* ϵ*-balls yields a base for a topology* $\tau(d)$ *on* X, *(i.e. a subset* G *of* X *belongs to* $\tau(d)$ *if and only if for each* $x \in G$ *there exists* $\epsilon = \epsilon_x > 0$ *such that* $B_d(x, \epsilon) \subseteq G$*). It is called the topology induced by* d *on* X.

If τ is a topology on X, such that $\tau = \tau(d)$ for some quasi-pseudometric d on X, then X is said to be *quasi-pseudometrizable.*
Observe that if d is a T_0-quasi-metric on X, then $\tau(d)$ is a T_0-topology on X.
For an arbitrary quasi-pseudometric d on X an interesting simple connection between the topologies $\tau(d)$ and $\tau(d^{-1})$ is given by the following result.

1.2.1 Remark. *([12, Remark 2.1.5]) Let* d *be a quasi-pseudometric on a set* X. *If* $\tau(d)$ *has a base* $\mathcal{B}$ *of (infinite) cardinality* κ, *then* $\tau(d^{-1})$ *has a base of cardinality* κ. *This means that* $\tau(d)$ *is second countable if and only if* $\tau(d^{-1})$ *is* *second countable.*

Proof. See [12, 2.1.5]. □

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a quasi-pseudometric space (X, d). The convergence of $(x_n)_{n\in\mathbb{N}}$ to $x \in X$ with respect to $\tau(d)$, called d-convergence and denoted by $x_n \xrightarrow{d} x$, can be characterized in the following way:

$$x_n \xrightarrow{d} x \iff d(x, x_n) \to 0.$$

Also

$$x_n \xrightarrow{d^{-1}} x \iff d^{-1}(x, x_n) \to 0 \iff d(x_n, x) \to 0.$$

The following proposition contains some simple properties of convergent sequences in quasi-pseudometric spaces.

1.2.1 Proposition. *([8, Proposition 1.1.2]) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a quasi-pseudometric space (X, d).*
1. *If $(x_n)_{n\in\mathbb{N}}$ is d-convergent to x and d^{-1}-convergent to y, then $d(x, y) = 0$.*
2. *If $(x_n)_{n\in\mathbb{N}}$ is d-convergent to x and $d(y, x) = 0$, then $(x_n)_{n\in\mathbb{N}}$ is also d-convergent to y.*

Proof. See [8, Proposition 1.1.2]. □

1.2.2 Proposition. *([8, Proposition 1.1.6]) Let (X, d) be a quasi-pseudometric space, then*
1. *Given $\epsilon > 0$, any ball $B_d(x, \epsilon)$ is $\tau(d)$-open and a ball $C_d(x, \varepsilon)$ is $\tau(d^{-1})$-closed, where $C_d(x, \varepsilon) = \{y \in X : d(x, y) \le \varepsilon\}$ whenever $\varepsilon \ge 0$. The ball $C_d(x, \varepsilon)$ need not be $\tau(d)$-closed. Also, the following inclusions hold*

$$B_{d^s}(x,\epsilon) \subseteq B_d(x,\epsilon) \text{ and } B_{d^s}(x,\epsilon) \subseteq B_{d^{-1}}(x,\epsilon),$$

with similar inclusions for the closed balls.
2. *The topology $\tau(d^s)$ is finer than the topologies $\tau(d)$ and $\tau(d^{-1})$, indeed $\tau(d^s) = \tau(d) \vee \tau(d^{-1})$ (see [14]): This means that:*

- *any $\tau(d)$-open (closed) set is $\tau(d^s)$-open (closed), similar results hold for the topology $\tau(d^{-1})$,*
- *the identity mappings from $(X, \tau(d^s))$ to $(X, \tau(d))$ and to $(X, \tau(d^{-1}))$ are continuous,*
- *a sequence $(x_n)_{n\in\mathbb{N}}$ in X is $\tau(d^s)$-convergent if and only if it is $\tau(d)$-convergent and $\tau(d^{-1})$-convergent.*

3. *For every fixed $x \in X$, the mapping $d(x,\cdot) : X \to (\mathbb{R}, |\cdot|)$ is $\tau(d)$-upper semicontinuous and $\tau(d^{-1})$-lower semicontinuous.*
For every fixed $y \in X$, the mapping $d(\cdot,y) : X \to (\mathbb{R}, |\cdot|)$ is $\tau(d)$-lower semicontinuous and $\tau(d^{-1})$-upper semicontinuous.
4. *The mapping $d(x,\cdot) : X \to (\mathbb{R}, |\cdot|)$ is $\tau(d)$-continuous at $x \in X$ if and only if $cl_{\tau(d)}(B_d(x,\epsilon)) \subseteq C_d(x,\epsilon)$ whenever $\epsilon \geq 0$.*

Proof. The proof of this proposition can be found in [8, Proposition 1.1.6]. Let us give a proof of 1 as in [8], there is a minor misprint in its proof.
For $y \in B_d(x,\epsilon)$ we have $B_d(y,\epsilon_y) \subseteq B_d(x,\epsilon)$, where $\epsilon_y = \epsilon - d(x,y) > 0$. Also if $y \in X - C_d(x,\varepsilon)$ and $\varepsilon_y = d(x,y) - \varepsilon > 0$, then $B_{d^{-1}}(y,\varepsilon_y) \cap C_d(x,\varepsilon) = \emptyset$ or equivalently $B_{d^{-1}}(y,\varepsilon_y) \subseteq X - C_d(x,\varepsilon)$. Indeed if $z \in B_{d^{-1}}(y,\varepsilon_y) \cap C_d(x,\varepsilon)$, then

$$d(x,y) \leq d(x,z) + d^{-1}(y,z) < \varepsilon + \varepsilon_y = d(x,y).$$

a contradiction.

Since $d(x, y) \leq d^s(x, y)$ and $d^{-1}(x, y) \leq d^s(x, y)$ for all $x, y \in X$, then the given inclusions hold.

$\square$

Chapter 2

Convexity in T_0-quasi-metric spaces

The theory of convexity in metric spaces is well developed by several authors (see for instance [1, 2, 4, 9, 10, 11]). A metric space (X, m) is called *hyperconvex* (see [9, Definition 2.5]) if $\bigcap_{i \in I} C_m(x_i, r_i) \neq \emptyset$ for each family $(x_i)_{i \in I}$ of points in X and a family of nonnegative real numbers $(r_i)_{i \in I}$ for which $d(x_i, x_j) \leq r_i + r_j$ whenever $i, j \in I$, where $C_m(x, r)$ denotes the closed ball of (nonnegative) radius r at x. In 1964 Isbell [2] and independently, in 1984 Dress [4] constructed the hyperconvex hull of a metric space. A metric space (X, m) is called *injective* (see [2, 9, 11]) provided that for any metric space (Y, ρ), any subspace A of (Y, ρ) and any nonexpansive map $f : A \to (X, m)$, f can be extended to a nonexpansive map $g : (Y, \rho) \to (X, m)$. It is well-known that a metric space is hyperconvex if and only if it is injective (see for example [2, 9, 11]) in the category of metric spaces and nonexpansive maps. In this chapter we are going to discuss some recent known results pertaining concepts of convexity in T_0-quasi-metric spaces due to Kemajou et al [3], which they call *Isbell-convex*, but we are going to call this concept *q-hyperconvex* in this dissertation. Kemajou et al have shown that a

T_0-quasi-metric space is q-hyperconvex if and only if it is injective in the category of T_0-quasi-metric spaces and nonexpansive maps [3, Theorem 1]. Most of the results in this chapter will be needed later in proving our main results in Chapter 3 and 4, respectively.

2.1 Hyperconvexity in T_0-quasi-metric spaces

2.1.1 Definition. *([3, Definition 2]) A T_0-quasi-metric space (X,d) will be called* q-hyperconvex *if $\bigcap_{i\in I}(C_d(x_i,r_i)\cap C_{d^{-1}}(x_i,s_i))\neq\emptyset$ for each family $(x_i)_{i\in I}$ of points in X and families of nonnegative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ for which $d(x_i,x_j)\leq r_i+s_j$ whenever $i,j\in I$.*

2.1.1 Example. *([3, Example 1]) The set $\mathbb{R}$ of the reals equipped with the T_0-quasi-metric $u(x,y)=x\dot{-}y$ whenever $x,y\in\mathbb{R}$ is q-hyperconvex.*

2.1.2 Definition. *([3, Definition 5]) If (X,d) is a T_0-quasi-metric space, a family of closed balls $(C_d(x_i,r_i),C_{d^{-1}}(x_i,s_i))_{i\in I}$ with $r_i,s_i\in[0,\infty)$ and $x_i\in X$ whenever $i\in I$ is said to have a* mixed binary intersection property *(*MIB *in short) if for all indices $i,j\in I$, $C_d(x_i,r_i)\cap C_{d^{-1}}(x_j,s_j)\neq\emptyset$.*

If (X,m) is a metric space, a family of closed balls $(C_m(x_i,r_i))_{i\in I}$ with $r_i\in[0,\infty)$ and $x_i\in X$ whenever $i\in I$ is said to have a binary intersection property *(*BIP *in short) if $C_m(x_i,r_i)\cap C_m(x_j,r_j)\neq\emptyset$ whenever $i,j\in I$.*

2.1.3 Definition. *([3, Definition 6]) A T_0-quasi-metric space (X,d) is called* Isbell-complete *if every family $(C_d(x_i,r_i),C_{d^{-1}}(x_i,s_i))_{i\in I}$ of closed balls, where $r_i,s_i\in[0,\infty)$ and $x_i\in X$ whenever $i\in I$, having a mixed binary intersection property satisfies $\bigcap_{i\in I}(C_d(x_i,r_i)\cap C_{d^{-1}}(x_i,s_i))\neq\emptyset$.*

A metric space (X, m) is called hypercomplete *if every family of closed balls $(C_m(x_i, r_i))_{i \in I}$ with $r_i \in [0, \infty)$ and $x_i \in X$ whenever $i \in I$, having a binary intersection property satisfies $\bigcap_{i \in I} C_m(x_i, r_i) \neq \emptyset$.*

2.1.1 Proposition. *([3, Proposition 1]) A T_0-quasi-metric space (X, d) is q-hyperconvex if and only if it is metrically convex and Isbell-complete.*

Proof. See [3, Proposition 1]. □

2.1.2 Proposition. *([3, Proposition 2]) (i) A T_0-quasi-metric space (X, d) is q-hyperconvex (resp. Isbell-complete, metrically convex) if and only if (X, d^{-1}) is a q-hyperconvex (resp. Isbell-complete, metrically convex) T_0-quasi-metric space.*
(ii) If (X, d) is a q-hyperconvex (resp. Isbell-complete) T_0-quasi-metric space, then the metric space (X, d^s) is hyperconvex (resp. hypercomplete). The corresponding statement about "metrically convex" does not hold as the following example shows.

Proof. See [3, Proposition 2]. □

2.1.2 Example. *([3, Example 5]) Consider the points of the unit circle in the Euclidean plane $\mathcal{C}$. The distance $d(p_1, p_2)$ from $p_1 \in \mathcal{C}$ to $p_2 \in \mathcal{C}$ is set equal to the arc length measured counterclockwise from p_1 to p_2. Then d is a T_0-quasi-metric on $\mathcal{C}$, which is clearly metrically convex, but d^s is not metrically convex, since d^s does not admit any values between 0 and π.*

2.1.1 Corollary. *([3, Corollary 2]) Each metric space (X, m) that is q-hyperconvex (resp. Isbell-complete) is hyperconvex (resp. hypercomplete).*

The above corollary shows that considering metric spaces, the class of hyperconvex (resp. hypercomplete) spaces is contained in the class of q-hyperconvex (resp. Isbell-complete) spaces.
One must note that each metric space is a quasi-pseudometric space.

2.1.4 Definition. *(compare [12]) A T_0-quasi-metric space (X, d) will be called* bicomplete *provided that the metric space (X, d^s) is complete or, equivalently a T_0-quasi-metric space (X, d) will be called* bicomplete *provided that the uniformity $\mathcal{U}_{d^s}$ induced by d^s is complete.*

It is well known that every hyperconvex metric space is complete, similarly in the case of T_0-quasi-metric spaces we have the following result.

2.1.2 Corollary. *([3, Corollary 3]) Every q-hyperconvex T_0-quasi-metric space is bicomplete.*

Proof. See [3, Corollary 3]. □

2.2 q-Hyperconvex hulls of T_0-quasi-metric spaces

Isbell [2] called a mapping of metric spaces $e : X \to E$ an injective hull of X if E is injective, e is an isometric embedding, and no injective proper subspace of E contains $e(X)$ metrically. He shows that every metric space has an injective hull and all of its injective hulls are equivalent, where two injective hulls $e : X \to E$ and $f : X \to F$ are said to be equivalent if they related by an isometry $i : E \to F$, such that $i \circ e = f$.

2.2.1 Definition. *(compare [2, page 71]) A mapping of T_0-quasi-metric spaces $e : (X, d) \to (E, \rho)$ is called a* q-hyperconvex hull *of X if E is q-hyperconvex, e is an isometric embedding, and no q-hyperconvex proper subspace of E contains $e(X)$ quasi-metrically. In such case we will simply call (E, ρ) the q-hyperconvex hull of X.*

2.2.2 Definition. *([3, Definition 7]) Let (X, d) be a T_0-quasi-metric space and let $\mathcal{FP}(X, d)$ be the set of all function pairs $f = (f_1, f_2)$ where $f_i : X \to [0, \infty)$*

whenever $i \in \{1,2\}$.
We define a T_0*-quasi-metric* D *on* $\mathcal{FP}(X,d)$ *as follows:*
$D(f,g) = \sup\{f_1(x) \dot{-} g_1(x) : x \in X\} \vee \sup\{g_2(x) \dot{-} f_2(x) : x \in X\}$ *whenever* $f,g \in \mathcal{FP}(X,d)$.

2.2.1 Remark. *([3, Remark 3]) One must note that* $D^s(f,g) = \sup\{|f_1(x) - g_1(x)| : x \in X\} \vee \sup\{|f_2(x) - g_2(x)| : x \in X\}$ *whenever* $f,g \in \mathcal{FP}(X,d)$ *is a metric on* $\mathcal{FP}(X,d)$*. Furthermore* $(\mathcal{FP}(X,d),D)$ *is q-hyperconvex. Therefore by Corollary 2.1.2 we have that* $(\mathcal{FP}(X,d),D)$ *is bicomplete.*

Proof. See [3, Remark 3]. □

2.2.1 Lemma. *([3, Lemma 1]) Let* (X,d) *be a* T_0*-quasi-metric space. For any* $a \in X$*, set* $f_a(x) = (d(a,x), d(x,a))$ *whenever* $x \in X$*. For any* $a,b \in X$ *we have* $d(a,b) = D(f_a,f_b)$*. Therefore,* $e_X : (X,d) \to (\mathcal{FP}(X,d),D)$ *defined by* $e_X(x) = f_x$ *is an injective isometric map.*

Proof. See [3, Lemma 1]. □

We shall say that a function pair $f \in \mathcal{FP}(X,d)$ is *ample* if for all $x,y \in X$, we have $d(x,y) \leq f_2(x) + f_1(y)$. The set of all ample function pairs on a T_0-quasi-metric space (X,d) will be denoted by P_X. (In such situations we may also write $P_{(X,d)}$ in cases where d is not obvious).
We say that $f \in P_X$ is *minimal* (*extremal*) among the ample pairs if for each $g \in P_X$ such that $g_1(x) \leq f_1(x)$ and $g_2(x) \leq f_2(x)$, then $g = f$.
By Q_X we shall denote the set of all minimal ample pairs on (X,d) with the restriction of D to $Q_X \times Q_X$, which for convenience we shall also denote by D. It is known that D is indeed a (real-valued) T_0-quasi-metric on $Q_X \times Q_X$ [3, Remark 6].

2.2.2 Lemma. *([3, Lemma 3]) Let* (X,d) *be a* T_0*-quasi-metric space. Each* $f \in Q_X$ *satisfies* $f_1(x) - f_1(y) \leq d^{-1}(x,y)$ *and* $f_2(x) - f_2(y) \leq d(x,y)$ *when-*

ever $x, y \in X$. Hence $f_1 : (X, d^{-1}) \to ([0,\infty), u)$ and $f_2 : (X, d) \to ([0,\infty), u)$ are nonexpansive maps, where u denotes the restriction of the T_0-quasi-metric defined in Example 1.1.1.

Proof. See [3, Lemma 3]. $\square$

2.2.1 Proposition. *([3, Proposition 3]) Let (X, d) be a T_0-quasi-metric space and $f \in P_X$ be such that $f_1 : (X, d^{-1}) \to ([0,\infty), u)$ and $f_2 : (X, d) \to ([0,\infty), u)$ are nonexpansive maps, where u denotes the restriction of the T_0-quasi-metric defined in Example 1.1.1. Furthermore suppose that there is a sequence $(a_n)_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} f_1(a_n) = 0 = \lim_{n\to\infty} f_2(a_n)$. Then $f \in Q_X$.*

Proof. See [3, Proposition 3]. $\square$

2.2.3 Lemma. *([3, Lemmas 4, 5, 6 and 8]) Let (X, d) be a T_0-quasi-metric space.*
(i) For each $a \in X$, f_a is a minimal ample pair.
(ii) If $f \in Q_X$ such that $f_1(a) = 0 = f_2(a)$ for some $a \in X$. Then $f = e_X(a)$.
(iii) A function pair f belongs to Q_X if and only if

$$f_1(x) = \sup\{d(y, x) \dot{-} f_2(y) : y \in X\} = \sup\{(f_x)_2(y) \dot{-} f_2(y) : y \in X\}$$

and

$$f_2(x) = \sup\{d(x, y) \dot{-} f_1(y) : y \in X\} = \sup\{(f_x)_1(y) \dot{-} f_1(y) : y \in X\}.$$

(iv) $D(f, g) = \sup\{f_1(x) \dot{-} g_1(x) : x \in X\} = \sup\{g_2(x) \dot{-} f_2(x) : x \in X\}$ whenever $f, g \in Q_X$.
(v) $D(f, f_a) = f_1(a)$ and $D(f_a, f) = f_2(a)$ whenever $a \in X$ and $f \in Q_X$.

Proof. See [3, Lemmas 4, 5, 6 and 8]. □

2.2.2 Proposition. *(compare [4, Section 1.3]) Let (X, d) be a T_0-quasi-metric space. Then Q_X consists of all functions pairs which are minimal in P_X.*

Proof. Obvious from the way Q_X is defined. □

2.2.3 Definition. *Let (X, d) be a T_0-quasi-metric space. Then (X, d) is called* joincompact *provided that the metric space (X, d^s) is compact.*
A T_0-quasi-metric space (X, d) is called totally bounded *provided that the metric space (X, d^s) is totally bounded.*

2.2.3 Proposition. *([3, Proposition 8]) If (X, d) is a totally bounded T_0-quasi-metric space, then (Q_X, D) is a totally bounded T_0-quasi-metric space, too, (i.e. the q-hyperconvex hull Q_X of a totally bounded T_0-quasi-metric space X is always totally bounded).*

Proof. See [3, Proposition 8]. □

2.2.1 Corollary. *(compare [3, Corollary 5]) If (X, d) is a joincompact T_0-quasi-metric space, then (Q_X, D) is also a joincompact T_0-quasi-metric space.*

Proof. Suppose (X, d) is joincompact, then (X, d) is totally bounded as (X, d^s) is totally bounded by the compactness of (X, d^s). By Proposition 2.2.3 (Q_X, D) is totally bounded, too and hence (Q_X, D^s) is totally bounded. Since (Q_X, D) is q-hyperconvex, then (Q_X, D^s) is complete by Corollary 2.1.2. Therefore (Q_X, D) is joincompact as (Q_X, D^s) is compact. □

2.2.4 Definition. *([3, Definition 8]) Let (X, d) be a T_0-quasi-metric space. Then it is called* injective *provided that for any T_0-quasi-metric space (Y, ρ), any subspace A of (Y, ρ) and any nonexpansive map $f : A \to (X, d)$, f can be extended to a nonexpansive map $g : (Y, \rho) \to (X, d)$.*

2.2.1 Theorem. *([3, Theorem 1]) A T_0-quasi-metric space (X, d) is q-hyperconvex if and only if it is injective.*

Proof. See [3, Theorem 1]. $\square$

2.2.4 Proposition. *([3, Proposition 7]) (i) (Q_X, D) is q-hyperconvex.*
(ii) $e_X : (X, d) \to (Q_X, D)$ is a q-hyperconvex hull of (X, d).
(iii) (X, d) is q-hyperconvex if and only if for each $f \in Q_X$ there is $x \in X$ such that $f = f_x$, (i.e. (X, d) is q-hyperconvex if and only if $Q_X = e_X(X)$).

Proof. See [3, Proposition 7]. $\square$

Chapter 3

A concept of an extension in T_0-quasi-metric spaces

Let X be a subspace of a metric space (Y, d). Then Y is called a tight extension of X provided that for any metric ρ on Y such that $\rho \leq d$ and agrees with d on $X \times X$ we have that $\rho = d$. In 1984 Dress [4] gave an independent, equivalent approach to Isbell's theory of the injective hull of a metric space that is based on the concept of a tight extension. He showed that every metric space X has a tight extension T_X which is compact if and only if $\bar{X}$ is compact, where $\bar{X}$ is the completion of X. He also showed that T_X is a maximal tight extension of X, (i.e. if Y is a tight extension of X, then there exists an isometric embedding $Y \to T_X$ that keeps X pointwise fixed [4, Section 1.14]). In this chapter we are going to show that most of the results in Dress [4] can be modified to hold essentially unchanged in the quasi-metric setting. Indeed we manage to show that every T_0-quasi-metric space X has a maximal T_0-quasi-metric tight extension, namely its q-hyperconvex hull Q_X.

3.1 T_0-quasi-metric tight extensions

In this section we are going to generalize some important results about tight extensions of metric spaces from Dress [4] to our quasi-metric setting and most of the results in this chapter can essentially be found in [5]. The following proposition shows that if (X, d) is a T_0-quasi-metric space, then for each $g \in P_X$, there exists $f \in Q_X$ such that $f \leq g$, (i.e. below each ample function pair there is a minimal ample function pair).

3.1.1 Proposition. *(compare [4, Section 1.9]) For every T_0-quasi-metric space (X, d) there exists a retraction map $p : P_X \to Q_X$, (i.e. a map that satisfies the following conditions):*
1. $p(f) \leq f$ for all $f \in P_X$. (In particular $p(f) = f$ for all $f \in Q_X$, since each $f \in Q_X$ is minimal).
2. $D(p(f), p(g)) \leq D(f, g)$ whenever $f, g \in P_X$.

Proof. Let $f \in P_X$, define a function pair f^* as follows,
$f_1^*(x) := \sup\{d(y, x) \dot{-} f_2(y) : y \in X\}$ and $f_2^*(x) := \sup\{d(x, y) \dot{-} f_1(y) : y \in X\}$ whenever $x \in X$.
The proof will proceed by showing the following claims first.

3.1.1 Claim. *For each $f \in P_X, f^* \leq f$, with equality if and only if $f \in Q_X$.*

Proof. Since $d(y, x) \leq f_2(y) + f_1(x)$ whenever $x, y \in X$ and thus $d(y, x) - f_2(y) \leq f_1(x)$. Therefore $f_1^*(x) = \sup\{d(y, x) \dot{-} f_2(y) : y \in X\} \leq f_1(x)$ whenever $x \in X$. In a similar manner, we can show that $f_2^*(x) \leq f_2(x)$ whenever $x \in X$. Thus $f^* \leq f$.
$f^* = f \Leftrightarrow$ for any $x \in X, f_1^*(x) = \sup\{d(y, x) \dot{-} f_2(y) : y \in X\} = f_1(x)$ and $f_2^*(x) = \sup\{d(x, y) \dot{-} f_1(y) : y \in X\} = f_2(x) \Leftrightarrow f \in Q_X$ by Lemma 2.2.3 (iii).
□

Given $x, z \in X$, the definition of f^* gives $f_2(x) + f_1^*(z) \geq d(x,z)$ and $f_2^*(x) + f_1(z) \geq d(x,z)$. This is so because

$$\begin{aligned} f_2^*(x) + f_1(z) &= \sup\{d(x,y) \dot{-} f_1(y) : y \in X\} + f_1(z) \\ &\geq d(x,z) - f_1(z) + f_1(z) \\ &= d(x,z) \end{aligned}$$

whenever $x, z \in X$. Similarly we have

$$\begin{aligned} f_2(x) + f_1^*(z) &= f_2(x) + \sup\{d(y,z) \dot{-} f_2(y) : y \in X\} \\ &\geq f_2(x) + d(x,z) - f_2(x) \\ &= d(x,z) \end{aligned}$$

whenever $x, z \in X$.
Define $q : P_X \to P_X$ by $f \mapsto q(f)$, where

$$q(f) := \tfrac{1}{2}(f + f^*) = \tfrac{1}{2}(f_1 + f_1^*, f_2 + f_2^*) = (\tfrac{1}{2}(f_1 + f_1^*), \tfrac{1}{2}(f_2 + f_2^*))$$

whenever $f \in P_X$.

3.1.2 Claim. *For each $f \in P_X, q(f)$ belongs to P_X and $q(f) \leq f$, with equality if and only if $f \in Q_X$.*

Proof. Let $f \in P_X$ and $x, y \in X$ we have that

$$
\begin{aligned}
(q(f))_2(x) + (q(f))_1(y) &= \frac{1}{2}(f_2(x) + f_2^*(x)) + \frac{1}{2}(f_1(y) + f_1^*(y)) \\
&= \frac{1}{2}[(f_2(x) + f_1^*(y)) + (f_2^*(x) + f_1(y))] \\
&\geq \frac{1}{2}[2d(x,y)] \\
&= d(x,y).
\end{aligned}
$$

This shows that $q(f)$ is ample, (i.e. $q(f) \in P_X$).
Since $f^* \leq f$, then $q(f) := \frac{1}{2}(f + f^*) \leq \frac{1}{2}(f + f) = f$.

$$
q(f) = f \Leftrightarrow \tfrac{1}{2}(f + f^*) = f \Leftrightarrow f + f^* = 2f \Leftrightarrow f^* = f \Leftrightarrow f \in Q_X.
$$

$\square$

3.1.3 Claim. *If $f, g \in P_X, D(q(f), q(g)) \leq D(f,g)$.*

Proof. Let $f, g \in P_X$ and $x \in X$. Then

$$
\begin{aligned}
f_1^*(x) &= \sup\{d(y,x) \dot{-} f_2(y) : y \in X\} \\
&\leq \sup\{d(y,x) \dot{-} g_2(y) + g_2(y) \dot{-} f_2(y) : y \in X\} \\
&\leq \sup\{d(y,x) \dot{-} g_2(y) : y \in X\} + \sup\{g_2(y) \dot{-} f_2 y : y \in X\} \\
&\leq g_1^*(x) + D(f,g).
\end{aligned}
$$

Therefore $f_1^*(x) - g_1^*(x) \leq D(f,g)$ and hence
$\sup\{(q(f))_1(x) \dot{-} (q(g))_1(x) : x \in X\} \leq \frac{1}{2}\sup\{f_1(x) \dot{-} g_1(x) : x \in X\} + \frac{1}{2}\sup\{f_1^*(x) \dot{-} g_1^*(x) : x \in X\} \leq \frac{1}{2}D(f,g) + \frac{1}{2}D(f,g) = D(f,g)$.
Similarly we can show that $\sup\{(q(g))_2(x) \dot{-} (q(f))_2(x) : x \in X\} \leq D(f,g)$ whenever $f, g \in P_X$. Thus $D(q(f), q(g)) \leq D(f,g)$ whenever $f, g \in P_X$.

$\square$

3.1.4 Claim. *If $f \in P_X, n \in \mathbb{N}$, then $q^n(f)$ is ample, (i.e. $q^n(f) \in P_X$ whenever $n \in \mathbb{N}$).*

Proof. Let us show this by induction on n. The case $n = 1$ is obvious by Claim 3.1.2.
Let $n \in \mathbb{N}, f \in P_X$ and assume that $q^n(f) \in P_X$. Now we have that $q^{n+1}(f) = q(q^n(f)) \in P_X$, because $q^n(f) \in P_X$ by our assumption and $q(f) \in P_X$ whenever $f \in P_X$ by Claim 3.1.2. □

Iterating the map q one obtains for every $f \in P_X$ a sequence of function pairs $f \geq q(f) \geq q^2(f) \geq .... \geq q^n(f) \geq .... \geq 0$ in P_X. Hence the map $p(f) := \lim_{n\to\infty} q^n(f)$ exists, where the pointwise limit pair is taken with respect to the usual topology $\tau(u^s)$ on $[0, \infty)$, where u is the T_0-quasi-metric in Example 1.1.1. Since for each $n \in \mathbb{N}, q^n(f) \leq f$ and $D(q^n(f), q^n(g)) = D(q(q^{n-1})(f), q(q^{n-1})(g)) \leq D(q^{n-1}(f), q^{n-1}(g)) \leq ..... \leq D(q(f), q(g)) \leq D(f, g)$. Then for each $n \in \mathbb{N}, q^n$ satisfies conditions 1 and 2, too.
Hence $p(f) \in P_X$ and satisfies condition 1 and 2, as $q^n(f) \in P_X$ and satisfies condition 1 and 2 whenever $n \in \mathbb{N}$ and $f \in P_X$.

3.1.5 Claim. *If $f, g \in P_X$ such that $f \leq g$, then $g^* \leq f^*$.*

Proof. In fact

$$f_1^*(x) = \sup\{d(y,x) \dot{-} f_2(y) : y \in X\} \geq \sup\{d(y,x) \dot{-} g_2(y) : y \in X\} = g_1^*(x)$$

and

$$f_2^*(x) = \sup\{d(x,y) \dot{-} f_1(y) : y \in X\} \geq \sup\{d(x,y) \dot{-} g_1(y) : y \in X\} = g_2^*(x)$$

whenever $x \in X$. This shows that $g^* \leq f^*$ whenever $f, g \in P_X$ such that $f \leq g$. □

To complete the proof we need to show that $p(f) \in Q_X$ whenever $f \in P_X$.
Let $f \in P_X$. For all $n \in \mathbb{N}$ we have $p(f) \leq q^n(f)$ and hence by Claim 3.1.5 we have $p(f)^* \geq q^n(f)^*$.
So $0 \leq p(f) - p(f)^* \leq q^n(f) - q^n(f)^* = 2(q^n(f) - q^{n+1}(f))$ (compare [10, Proof of Proposition 3.1]), as $q^{n+1}(f) = \frac{1}{2}(q^n(f) + q^n(f)^*)$. Since $(q^n(f) - q^{n+1}(f)) \to 0$ as $n \to \infty$, then $p(f) = p(f)^*$. Therefore by Claim 3.1.1 we have that $p(f) \in Q_X$. □

3.1.1 Lemma. *(compare [4, Section 1.3]) Let (X,d) be a T_0-quasi-metric space and let $f \in P_X$. For each $x \in X$, set $p_x(f) := ((p_x(f))_1, (p_x(f))_2)$ where*

$$(p_x(f))_1(z) = f_1(z) \text{ if } z \neq x \text{ and}$$
$$(p_x(f))_1(x) = \sup\{d(y,x) \dot{-} f_2(y) : y \in X\}.$$

and

$$(p_x(f))_2(z) = f_2(z) \text{ if } z \neq x \text{ and}$$
$$(p_x(f))_2(x) = \sup\{d(x,y) \dot{-} f_1(y) : y \in X\}.$$

Then for each $x \in X$ we have,
(a) $p_x(f) \in P_X$ whenever $f \in P_X$.
(b) p_x satisfies conditions 1 and 2 of Proposition 3.1.1.

Proof. Let $f \in P_X$ and $x \in X$.
(a) To show that $p_x(f) \in P_X$ we shall consider the following four cases.
Case 1: If $z = x = y$, then the assertion holds as $d(x,x) = 0$.
Case 2: If $z \neq x$ and $y = x$, then

$$
\begin{aligned}
(p_x(f))_2(z) + (p_x(f))_1(y) &= f_2(z) + \sup\{d(m,y) - f_2(m) : m \in X\} \vee 0 \\
&\geq f_2(z) + (d(z,y) - f_2(z)) \vee 0 \\
&= d(z,y).
\end{aligned}
$$

Case 3: If $z = x$ and $y \neq x$, this can be shown in a similar manner as in Case 2.
Case 4: If $z \neq x$ and $y \neq x$, then

$$(p_x(f))_2(z) + (p_x(f))_1(y) = f_2(z) + f_1(y) \geq d(z,y).$$

Hence $p_x(f) \in P_X$ whenever $f \in P_X$ and $x \in X$.
(b) By the way p_x was constructed we have $(p_x(f))_1(z) \leq f_1(z)$ and $(p_x(f))_2(z) \leq f_2(z)$ whenever $z \in X$ and $f \in P_X$. Therefore $p_x(f) \leq f$ whenever $f \in P_X$ and $x \in X$.
Also

$$
\begin{aligned}
(p_x(f))_1(x) &= \sup\{d(y,x) \dot{-} f_2(y) : y \in X\} \\
&\leq \sup\{d(y,x) \dot{-} g_2(y) + g_2(y) \dot{-} f_2(y) : y \in X\} \\
&\leq \sup\{d(y,x) \dot{-} g_2(y) : y \in X\} + \sup\{g_2(y) \dot{-} f_2(y) : y \in X\} \\
&\leq (p_x(g))_1(x) + D(f,g).
\end{aligned}
$$

Hence $(p_x(f))_1(x) - (p_x(g))_1(x) \leq D(f,g)$.
Similarly we can show that $(p_x(g))_2(x) - (p_x(f))_2(x) \leq D(f,g)$ whenever $x \in X$.

This implies that

$$
\begin{aligned}
D(p_x(f), p_x(g)) &= \sup\{(p_x(f))_1(y) \dot{-} (p_x(g))_1(y) : y \in X\} \\
&\vee \sup\{(p_x(g))_2(y) \dot{-} (p_x(f))_2(y) : y \in X\} \\
&= \sup\{f_1(y) \dot{-} g_1(y), (p_x(f))_1(x) \dot{-} (p_x(g))_1(x) : y \in X - \{x\}\} \\
&\vee \sup\{g_2(y) \dot{-} f_2(y), (p_x(g))_2(x) \dot{-} (p_x(f))_2(x) : y \in X - \{x\}\} \\
&\leq \sup\{f_1(y) \dot{-} g_1(y), D(f,g) : y \in X - \{x\}\} \\
&\vee \sup\{g_2(y) \dot{-} f_2(y), D(f,g) : y \in X - \{x\}\} \\
&= D(f,g).
\end{aligned}
$$

Hence $D(p_x(f), p_x(g)) \leq D(f,g)$ whenever $f \in P_X$ and $x \in X$. □

3.1.1 Remark. *We next note that Proposition 3.1.1 can also be proved by Zorn's Lemma (compare [4, Section 1.9]).*

Proof. Let $\mathcal{P}$ be the set of all maps from P_X to P_X satisfying conditions 1 and 2 of Proposition 3.1.1. Define an order on $\mathcal{P}$ as follows:

$$
p \preceq q \iff (p(f))_1 \leq (q(f))_1, (p(f))_2 \leq (q(f))_2
$$
and
$$
D(p(f), p(g)) \leq D(q(f), q(g))
$$

whenever $f, g \in P_X$ and $p, q \in \mathcal{P}$. Then $\mathcal{P} \neq \emptyset$ as $p_x \in \mathcal{P}$ whenever $x \in X$, where p_x is the map defined in Lemma 3.1.1.

Now let us show that $\preceq$ is a partial order on $\mathcal{P}$.

1. Reflexivity is obvious as every map equals itself.

2. Let $p, q \in \mathcal{P}$ be such that $p \preceq q$ and $q \preceq p$. Then we have that for any $f, g \in P_X$ and $x \in X$,

$$
p \preceq q \Rightarrow (p(f))_1(x) \leq (q(f))_1(x), \quad (p(f))_2(x) \leq (q(f))_2(x)
$$
and

$$D(p(f), p(g)) \leq D(q(f), q(g)).$$

Moreover

$$q \preceq p \Rightarrow (q(f))_1(x) \leq (p(f))_1(x), \quad (q(f))_2(x) \leq (p(f))_2(x)$$
$$\text{and}$$
$$D(q(f), q(g)) \leq D(p(f), p(g)).$$

This implies that

$$(p(f))_1(x) \leq (q(f))_1(x) \leq (p(f))_1(x)$$

and

$$(p(f))_2(x) \leq (q(f))_2(x) \leq (p(f))_2(x).$$

Therefore by the antisymmetry of $\leq$ on $[0, \infty)$ we have that

$$(p(f))_1(x) = (q(f))_1(x)$$

and

$$(p(f))_2(x) = (q(f))_2(x).$$

Therefore $p = q$ and $\preceq$ is antisymmetric.

3. Let $p, q, s \in \mathcal{P}$ be such that $p \preceq q$ and $q \preceq s$. Then we have that for any $f, g \in P_X$ and $x \in X$,

$$p \preceq q \Rightarrow (p(f))_1(x) \leq (q(f))_1(x), \quad (p(f))_2(x) \leq (q(f))_2(x)$$
$$\text{and}$$
$$D(p(f), p(g)) \leq D(q(f), q(g)).$$

Similarly

$$q \preceq s \Rightarrow (q(f))_1(x) \leq (s(f))_1(x), \quad (q(f))_2(x) \leq (s(f))_2(x)$$
$$\text{and}$$
$$D(q(f), q(g)) \leq D(s(f), s(g)).$$

Therefore by the transitivity of $\leq$ on $[0, \infty)$ we have that $(p(f))_1(x) \leq (s(f))_1(x)$, $(p(f))_2(x) \leq (s(f))_2(x)$ and $D(p(f), p(g)) \leq D(s(f), s(g))$. Thus $p \preceq s$ and $\preceq$ is transitive. Therefore $(\mathcal{P}, \preceq)$ is a partially ordered set.
Let $\mathcal{K} \subseteq \mathcal{P}$ be a nonempty chain and define a map $s : P_X \to P_X$ as follows

$$s(f)(x) := (\inf_{k \in \mathcal{K}}(k(f))_1(x), \inf_{k \in \mathcal{K}}(k(f))_2(x))$$

whenever $x \in X$, where the infima are taken pointwise in $[0, \infty)$.
It is easily seen that $s(f) \in P_X$ whenever $f \in P_X$, since $\mathcal{K}$ is a chain (compare [3, Proof of Theorem 1]). Then for each $f \in P_X$ and each $k \in \mathcal{K}$, we get $s(f) \leq k(f) \leq f$.
Furthermore for any $f, g \in P_X$ and $k \in \mathcal{K}$ we have

$$\begin{aligned} D(f,g) &\geq D(k(f), k(g)) \\ &\geq \sup\{(t(f))_1(x) \dot{-} (t(g))_1(x) : x \in X\} \\ &\vee \sup\{(t(g))_2(x) \dot{-} (t(f))_2(x) : x \in X\} \end{aligned}$$

whenever $t \in \mathcal{K}$ and $t \preceq k$.
Fix $x \in X$. In the following u is the T_0-quasi-metric defined in Example 1.1.1. For each $n \in \mathbb{N}$ choose $k'_n, k''_n \in \mathcal{K}$ such that $k'_n, k''_n \preceq k$ and such that the decreasing sequence $(k'_n(f))_1(x))$ converges to $\inf_{k \in \mathcal{K}}(k(f))_1(x)$ and the decreasing sequence $(k''_n(g))_1(x))$ converges to $\inf_{k \in \mathcal{K}}(k(g))_1(x)$ with respect to the usual topology

$\tau(u^s)$ on $[0,\infty)$.
For each $n \in \mathbb{N}$ set $t_n = \min\{k'_n, k''_n\}$. Then the sequence $((t_n(f))_1(x))$ converges to

$$\inf_{k\in\mathcal{K}}(k(f))_1(x)$$

and the sequence $((t_n(g))_1(x))$ converges to

$$\inf_{k\in\mathcal{K}}(k(g))_1(x)$$

with respect to the usual topology $\tau(u^s)$ on $[0,\infty)$.
Therefore

$$\begin{aligned} D(f,g) &\geq D(k(f),k(g)) \\ &\geq \inf_{k\in\mathcal{K}}(k(f))_1(x) \dot{-} \inf_{k\in\mathcal{K}}(k(g))_1(x). \end{aligned}$$

A similar argument works for the second coordinates.
Hence

$$\begin{aligned} D(f,g) &\geq D(k(f),k(g)) \\ &\geq \sup\{\inf_{k\in\mathcal{K}}(k(f))_1(x) \dot{-} \inf_{k\in\mathcal{K}}(k(g))_1(x) : x \in X\} \\ &\vee \sup\{\inf_{k\in\mathcal{K}}(k(g))_2(x) \dot{-} \inf_{k\in\mathcal{K}}(k(f))_2(x) : x \in X\} = D(s(f),s(g)). \end{aligned}$$

We have shown that $\mathcal{K}$ has a lower bound s in $\mathcal{P}$. By Zorn's Lemma $\mathcal{P}$ has a minimal element, say m.
To show that $m(f) \in Q_X$ whenever $f \in P_X$ we make use of p_x $(x \in X)$ defined

in Lemma 3.1.1. For each $x \in X$ we obviously have that $p_x \in \mathcal{P}$ and $p_x \circ m \preceq m$. Hence by minimality of $m, p_x \circ m = m$ whenever $x \in X$.

It follows that for each $x \in X, p_x(m(f)) = m(f)$ whenever $f \in P_X$. Thus by the definition of the elements of Q_X we conclude that $m(f) \in Q_X$ whenever $f \in P_X$ by [13, Remark 2]. $\square$

3.1.2 Proposition. *(compare [4, Section* 1.11*]) Let A be a (nonempty) subspace of a T_0-quasi-metric space (X, d). Then there exists an isometric embedding $\tau : Q_A \to Q_X$ such that $\tau(f)|_A = f$ whenever $f \in Q_A$.*

Proof. Fix $a_0 \in A$ and choose a retraction $p : P_X \to Q_X$ satisfying the conditions 1 and 2 of Proposition 3.1.1. Furthermore let $s : Q_A \to P_X$ be defined as $s(f) = \bar{f}$ whenever $f \in Q_A$, where

$$\bar{f}_1(x) = f_1(x) \text{ whenever } x \in A \text{ and } \bar{f}_1(x) = f_1(a_0) + d(a_0, x) \text{ whenever } x \in X - A$$

and

$$\bar{f}_2(x) = f_2(x) \text{ whenever } x \in A \text{ and } \bar{f}_2(x) = f_2(a_0) + d(x, a_0) \text{ whenever } x \in X - A.$$

We prove that $\bar{f} \in P_X$ by considering the following four cases:

Case 1: Suppose $x \in A$ and $y \in A$, then $\bar{f}_2(x) + \bar{f}_1(y) = f_2(x) + f_1(y) \geq d(x, y)$.

Case 2: Suppose $x \in X - A$ and $y \in X - A$, then $\bar{f}_2(x) + \bar{f}_1(y) = (f_2(a_0) + d(x, a_0)) + (f_1(a_0) + d(a_0, y)) \geq d(x, a_0) + d(a_0, y) \geq d(x, y)$.

Case 3: Suppose $x \in X - A$ and $y \in A$, then $\bar{f}_2(x) + \bar{f}_1(y) = (d(x, a_0) + f_2(a_0)) + f_1(y) \geq d(x, a_0) + d(a_0, y) \geq d(x, y)$.

Case 4: Suppose $x \in A$ and $y \in X - A$, then $\bar{f}_2(x) + \bar{f}_1(y) = f_2(x) + (f_1(a_0) + d(a_0, y)) \geq d(x, a_0) + d(a_0, y) \geq d(x, y)$. Thus $\bar{f} \in P_X$.

Define $\tau = p \circ s : Q_A \to Q_X$. Since for each $f \in Q_A$, $p(\bar{f}) \leq \bar{f}$, then $p(\bar{f})|_A \leq \bar{f}|_A = f$. Hence $\tau(f)|_A = p(s(f))|_A = p(\bar{f})|_A \leq f$. Therefore by minimality of f

on A we have $\tau(f)|_A = f$ whenever $f \in Q_A$.
Given $f, g \in Q_A$ we have

$$\begin{aligned} D(f,g) &= D(\tau(f)|_A, \tau(g)|_A) \\ &\leq D(\tau(f), \tau(g)) \\ &= D(p(\bar{f}), p(\bar{g})) \\ &\leq D(\bar{f}, \bar{g}) \\ &= D(f,g), \end{aligned}$$

where the last equality follows from the definition of $\bar{f}$ and $\bar{g}$, because $D(\bar{f}, \bar{g}) = D(f,g)$ on A and if $x \in X - A$ we have that

$$\begin{aligned} \bar{f}_1(x) \dot{-} \bar{g}_1(x) &= \max\{f_1(a_0) - g_1(a_0), 0\} \\ &= f_1(a_0) \dot{-} g_1(a_0) \end{aligned}$$

and

$$\begin{aligned} \bar{g}_2(x) \dot{-} \bar{f}_2(x) &= \max\{g_2(a_0) - f_2(a_0), 0\} \\ &= g_2(a_0) \dot{-} f_2(a_0). \end{aligned}$$

Therefore $D(\tau(f), \tau(g)) = D(p(\bar{f}), p(\bar{g})) = D(\bar{f}, \bar{g}) = D(f,g)$ whenever $f, g \in Q_A$. Therefore τ is an isometric embedding as required.

$\square$

The following definition can also be found in [3].
Let us recall that given two T_0-quasi-metric spaces (Y, e) and (X, d), then (Y, e) is called an *extension* of (X, d) provided that (X, d) can be embedded into (Y, e) quasi-metrically.

3.1.1 Definition. *(compare [3, Remark 7]) Let X be a subspace of a T_0-quasi-metric space (Y, d). Then Y is called a tight extension of X if for any quasi-pseudometric e on Y such that $e \leq d$ and agrees with d on $X \times X$ we have that $e = d$.*

In [3, Remark 7] it was shown that for any T_0-quasi-metric space (X, d) we have that the isometric embedding $e_X : X \to Q_X$ is tight, (i.e. Q_X is a tight extension of $e_X(X)$).

3.1.2 Remark. *(compare [4, Section 1.12]) For any T_0-quasi-metric tight extension Y_1 of X, any T_0-quasi-metric extension (Y_2, d) of X and any nonexpansive map $\psi : Y_1 \to Y_2$ satisfying $\psi(x) = x$ whenever $x \in X$, ψ is necessarily an isometric map.*

Proof. Otherwise the quasi-pseudometric $\rho : Y_1 \times Y_1 \to [0, \infty)$ defined by $\rho(x, y) = d(\psi(x), \psi(y))$ would contradict the tightness of the extension Y_1 of X. $\square$

3.1.3 Proposition. *(compare [4, Section 1.13]) Let (Y, d) be a T_0-quasi-metric tight extension of X. Then the restriction map defined by $f \mapsto f|_X$ whenever $f \in Q_Y$ is a bijective isometric map from Q_Y to Q_X.*

Proof. Choose a retraction $p : P_X \to Q_X$ satisfying the conditions 1 and 2 of Proposition 3.1.1 and let $\psi : Q_Y \to Q_X : f \mapsto p(f|_X)$ denote the composition of p with the restriction map $Q_Y \to P_X : f \mapsto f|_X$. Then ψ is nonexpansive and Q_Y and Q_X are T_0-quasi-metric extensions of X. According to Remark 3.1.2 ψ must be an isometric map, since Q_Y is a tight extension of X, because Q_Y is a tight extension of Y and Y is a tight extension of X.

By Proposition 3.1.2 there is an isometric embedding $\tau : Q_X \to Q_Y$ satisfying $\tau(f)|_X = f$ whenever $f \in Q_X$. Then we have $\psi(\tau(f)) = p(\tau(f)|_X) = p(f) = f$ for all $f \in Q_X$ and thus ψ is necessarily surjective. But a surjective isometric map on a T_0-quasi-metric domain is necessarily bijective. So $\tau : Q_X \to Q_Y$ has to be the map inverse to ψ and thus for any $f \in Q_Y$ we necessarily have the formula $f|_X = \tau(\psi(f))|_X = \psi(f) \in Q_X$, that is, the restriction map $Q_Y \to P_X : f \mapsto f|_X$ maps Q_Y already onto Q_X, without having to be composed with the retraction map p. Hence we see that for any T_0-quasi-metric tight extension Y of X the restriction map $Q_Y \to Q_X : f \mapsto f|_X$ yields a bijective isometric map between Q_Y and Q_X. □

3.1.3 Remark. *By Proposition 3.1.3 we see that if (Y,d) is a T_0-quasi-metric tight extension of X, then the q-hyperconvex hull Q_Y of Y is isometric to the q-hyperconvex hull Q_X of X, (i.e. in the case that X is a (nonempty) subspace of (Y,d), where (Y,d) is a T_0-quasi-metric tight extension of X, then Q_X and Q_Y are indistinguishable up to isometries).*

3.1.4 Proposition. *compare [9, Theorem 2]) Let X be a subspace of a T_0-quasi-metric space (Y,d). Then the following conditions are equivalent:*

(a) Y is a tight extension of X.

(b) $d(y_1,y_2) = \sup\{(d(x_1,x_2) - d(x_1,y_1) - d(y_2,x_2)) \vee 0 : x_1, x_2 \in X\}$ whenever $y_1, y_2 \in Y$.

(c) $f_y|_X(x) = (d(y,x), d(x,y))$ with $x \in X$ is minimal on X whenever $y \in Y$ and the map $\phi : (Y,d) \to (Q_X, D) : y \mapsto f_y|_X$ is an isometric embedding that keeps X pointwise fixed, (i.e. Q_X is a maximal tight extension of X in the sense of Dress [4]).

Proof. $(a) \to (b)$: Let Y be a T_0-quasi-metric tight extension of X. By Proposition 3.1.3 the map $Q_Y \to Q_X$ defined by $f \mapsto f|_X$ defines a bijective isometric map between Q_Y and Q_X. By [3, Remark 7] the extension Q_X of X fulfils condition (b) of Proposition 3.1.4. Hence the extension Y of X, as a subspace of Q_Y, fulfils condition (b), too.

$(b) \to (c)$: For any $x_1, x_2 \in X$ and $y_1 \in Y$ we have that

$$d(x_1, x_2) \leq d(x_1, y_1) + d(y_1, x_2).$$

Therefore for any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ we see that

$$d(x_1, x_2) - d(x_1, y_1) - d(y_2, x_2) \leq d(y_1, x_2) - d(y_2, x_2).$$

Consequently for any $y_1, y_2 \in Y$ we have by (b) that

$$\begin{aligned} d(y_1, y_2) &= \sup\{(d(x_1, x_2) - d(x_1, y_1) - d(y_2, x_2)) \vee 0 : x_1, x_2 \in X\} \\ &\leq \sup\{(d(y_1, x_2) \dot{-} d(y_2, x_2)) : x_2 \in X\} \\ &\leq d(y_1, y_2). \end{aligned}$$

Similarly

$$d(x_1, x_2) \leq d(x_1, y_2) + d(y_2, x_2)$$

whenever $x_1, x_2 \in X$ and $y_2 \in Y$.

It follows that for each $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ we have that

$$d(x_1, x_2) - d(y_2, x_2) - d(x_1, y_1) \leq d(x_1, y_2) - d(x_1, y_1).$$

Thus for any $y_1, y_2 \in Y$ we see by (b) that

$$
\begin{aligned}
d(y_1, y_2) &= \sup\{(d(x_1, x_2) - d(y_2, x_2) - d(x_1, y_1)) \vee 0 : x_1, x_2 \in X\} \\
&\leq \sup\{(d(x_1, y_2) \dot{-} d(x_1, y_1)) : x_1 \in X\} \\
&\leq d(y_1, y_2).
\end{aligned}
$$

Hence we conclude that $d(y_1, y_2) = D(f_{y_1}|_X, f_{y_2}|_X)$ whenever $y_1, y_2 \in Y$.
As we have just shown, for any $y_1, y_2 \in Y$ we have that

$$d(y_1, y_2) = \sup\{d(x_1, y_2) \dot{-} d(x_1, y_1) : x_1 \in X\}$$

and

$$d(y_1, y_2) = \sup\{(d(y_1, x_2) \dot{-} d(y_2, x_2) : x_2 \in X\}.$$

Substituting $x_2 \in X$ for y_2 and $x_1 \in X$ for y_1, respectively, we obtain the two equations

$$f_{y_1}|_X(x_2) = d(y_1, x_2) = \sup\{(d(x_1, x_2) \dot{-} d(x_1, y_1) : x_1 \in X\}$$

whenever $y_1 \in Y$ and $x_2 \in X$,
and

$$(f_{y_2}|_X)_2(x_1) = d(x_1, y_2) = \sup\{(d(x_1, x_2) \dot{-} d(y_2, x_2) : x_2 \in X\}$$

whenever $y_2 \in Y$ and $x_1 \in X$.
By [13, Remark 2] the restriction $f_y|_X$ is minimal on X whenever $y \in Y$.
$(c) \to (a)$: Let $q : Y \times Y \to [0, \infty)$ be a quasi-pseudometric on Y such that $q \leq d$ and $q|_{X \times X} = d|_{X \times X}$. By (c) and since $f_y|_X$ is minimal whenever $y \in Y$ we have

$$\begin{aligned} d(y_1, y_2) &= D(f_{y_1}|_X, f_{y_2}|_X) \\ &= \sup\{d(y_1, x) \dot{-} d(y_2, x) : x \in X\} \\ &= \sup\{d(x, y_2) \dot{-} d(x, y_1) : x \in X\} \end{aligned}$$

whenever $y_1, y_2 \in Y$ by [3, Lemma 7].
Then by substituting

$$d(x_1, y_2) = \sup\{d(x_1, x_2) \dot{-} d(y_2, x_2) : x_2 \in X\}$$

into the formula

$$d(y_1, y_2) = \sup\{d(x_1, y_2) \dot{-} d(x_1, y_1) : x_1 \in X\},$$

we obtain

$$\begin{aligned} d(y_1, y_2) &= \sup\{\sup\{(d(x_1, x_2) - d(x_1, y_1) - d(y_2, x_2)) \vee 0 : x_2 \in X\} : x_1 \in X\} \\ &\leq \sup\{(q(x_1, x_2) - q(x_1, y_1) - q(y_2, x_2)) \vee 0 : x_1, x_2 \in X\} \leq q(y_1, y_2) \end{aligned}$$

whenever $y_1, y_2 \in Y$ by our assumption. Consequently $q = d$. Therefore condition (a) is satisfied. $\square$

Chapter 4

Endpoints in T_0-quasi-metric spaces

In this chapter we intend to generalize results due to Isbell [2] to the category of T_0-quasi-metric spaces and nonexpansive maps. During his investigation on the hyperconvex hull of a metric space Isbell [2] introduced the concept of an endpoint of a (compact) metric space and proved among other things that the hyperconvex hull of a compact metric space is equal to the hyperconvex hull of its subspace consisting of its endpoints (see also [9]). It turns out that in the quasi-metric setting it is natural to consider also the dual concept of an endpoint, which we shall call a startpoint. In this way we succeed in generalizing several results on endpoints of metric spaces to the quasi-metric setting. Indeed we manage to show that the q-hyperconvex hull of a joincompact T_0-quasi-metric space is isometric to the q-hyperconvex hull of the subspace of its endpoints and startpoints.

4.1 Collinearity in T_0-quasi-metric spaces

4.1.1 Definition. *(compare [9, page 184]) Let (X,d) be a T_0-quasi-metric space.*

1. *A finite sequence $(x_1, x_2, ..., x_n)$ in X is called* collinear *in (X,d) provided that $d(x_i, x_k) = d(x_i, x_j) + d(x_j, x_k)$ whenever $i < j < k \leq n$.*

2. *An element $x \in X$ is called an* endpoint *provided that there exists $y \in X$ such that $d(y,x) > 0$ and such that for any $z \in X$ collinearity of (y,x,z) implies that $x = z$. In such a case we say that y witnesses that x is an endpoint of (X,d).*
The set of endpoints of a T_0-quasi-metric space (X,d) will be denoted by E_d.

3. *An element $x \in X$ is called a* startpoint *of (X,d) if it is an endpoint of (X, d^{-1}).*
The set of startpoints of a T_0-quasi-metric space (X,d) will be denoted by S_d.

Since for a point $x \in X$ to be an endpoint (resp. a startpoint), one needs a $y \in X$ such that $d(y,x) > 0$ (resp. $d(x,y) > 0$), then $y \neq x$. Hence a T_0-quasi-metric space possessing only one point cannot have an endpoint (resp. a startpoint).

4.1.1 Remark. *Note that (x_1, x_2, x_3) is collinear in the T_0-quasi-metric space (X,d) if and only if (x_3, x_2, x_1) is collinear in (X, d^{-1}).*

In the following example we are going to find the endpoints and startpoints of a special two point space.

4.1.1 Example. *Let $X = \{0,1\}$ be equipped with a T_0-quasi-metric d defined as follows:*
Set $d(1,0) = a, d(0,1) = b$ and $d(1,1) = 0 = d(0,0)$, where $a, b \in [0, \infty)$ such that $a + b \neq 0$. We shall denote this space by (X_{ab}, d).
Case 1: If $a = 0$, then 0 is the only startpoint in (X_{ab}, d) and 1 is the only endpoint in (X_{ab}, d).
Case 2: If $b = 0$, then 0 is the only endpoint in (X_{ab}, d) and 1 is the only startpoint in (X_{ab}, d).

Case 3: If $a, b \in (0, \infty)$, then both 0 and 1 are endpoints and startpoints in (X_{ab}, d).

Proof. Case 1: If $a = 0$, then $d^{-1}(0,1) = d(1,0) = a = 0$, thus 0 and 1 are not endpoints and startpoints in (X_{ab}, d), respectively. Since $d^{-1}(1,0) = d(0,1) = b > 0$ and $(x, 0, 1)$ is collinear in (X_{ab}, d) if and only if $x = 0$, then 1 witnesses that 0 is a startpoint in (X_{ab}, d). Also $d(0,1) = b > 0$ and $(0, 1, x)$ is collinear in (X_{ab}, d) if and only if $x = 1$, then 0 witnesses that 1 is an endpoint in (X_{ab}, d).
Case 2: If $b = 0$, then $d^{-1}(1,0) = d(0,1) = b = 0$, thus 0 and 1 are not startpoints and endpoints in (X_{ab}, d), respectively. Since $d^{-1}(0,1) = d(1,0) = a > 0$ and $(x, 1, 0)$ is collinear in (X_{ab}, d) if and only if $x = 1$, then 0 witnesses that 1 is a startpoint in (X_{ab}, d). Also $d(1,0) = a > 0$ and $(1, 0, x)$ is collinear in (X_{ab}, d) if and only if $x = 0$, then 1 witnesses that 0 is an endpoint in (X_{ab}, d).
Case 3: This case can be obtained by combining Case 1 and Case 2 to conclude that 0 and 1 are both endpoints and startpoints in (X_{ab}, d). $\square$

4.1.1 Lemma. *(compare [9, page 184]) If (x_1, x_2, x_3) and (x_1, x_3, x_4) are collinear in a T_0-quasi-metric space (X, d), then so are (x_1, x_2, x_4) and (x_2, x_3, x_4). (Hence (x_1, x_2, x_3, x_4) is collinear.)*

Proof. Since,

$$\begin{aligned} d(x_1, x_4) &\leq d(x_1, x_2) + d(x_2, x_4) \\ &\leq d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) \\ &= d(x_1, x_3) + d(x_3, x_4) \\ &= d(x_1, x_4), \end{aligned}$$

then $d(x_1, x_4) = d(x_1, x_2) + d(x_2, x_4)$ and $d(x_2, x_4) = d(x_2, x_3) + d(x_3, x_4)$. Hence $d(x_1, x_4) = d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4)$. Therefore $(x_1, x_2, x_3, x_4), (x_1, x_2, x_4)$ and (x_2, x_3, x_4) are collinear in (X, d) as required. $\square$

4.1.2 Lemma. *(compare [2, 2.6]) Let (X, d) be a joincompact T_0-quasi-metric space and $f \in Q_X$. Given $x \in X$ such that $f_2(x) > 0$ there is $y \in X$ such that (f_x, f, f_y) is collinear in (Q_X, D), (i.e. given an $x \in X$ such that $f_2(x) > 0$ there is $y \in X$ such that $d(x, y) = f_2(x) + f_1(y)$).*
Similarly for each $x \in X$ such that $f_1(x) > 0$ there is $y' \in X$ such that $(f_{y'}, f, f_x)$ is collinear in (Q_X, D), (i.e. given an $x \in X$ such that $f_1(x) > 0$ there is $y' \in X$ such that $d(y', x) = f_2(y') + f_1(x)$).

Proof. Assume that $m \in \mathbb{N}$ is such that $\frac{1}{m} \leq f_2(x)$. (This is possible by the nonarchimedean property of $\mathbb{R}$). Consider $n \in \mathbb{N}$ such that $n \geq m$. Suppose for all $y \in X$ we have $d(x, y) + \frac{1}{n} \leq f_2(x) + f_1(y)$. Then set $h_2(z) = f_2(x) - \frac{1}{n}$ if $z = x$ and $h_2(z) = f_2(z)$ whenever $z \in X - \{x\}$. Then $(0, 0) \leq (f_1, h_2) < (f_1, f_2)$, but (f_1, h_2) is an ample function pair on (X, d), so that f is not minimal– a contradiction.
Therefore for each $n \in \mathbb{N}$ such that $n \geq m$, there is $y_n \in X$ such that $f_2(x) + f_1(y_n) < d(x, y_n) + \frac{1}{n}$.
By joincompactness of (X, d) there are $y \in X$ and a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\geq m}$ such that $d^s(y_{n_k}, y) \to 0$. Then we get that $f_2(x) + f_1(y) \leq d(x, y) \leq f_2(x) + f_1(y)$, because we have that $|f_1(y_{n_k}) - f_1(y)| \leq d^s(y_{n_k}, y)$ and $|d(x, y_{n_k}) - d(x, y)| \leq d^s(y_n, y)$ whenever $k \in \mathbb{N}$, because f is a minimal ample pair (see Lemma 2.2.2). The second statement is proved analogously. $\square$

With a closer look at the first part of the preceding proof, we obtain the following corollary.

4.1.1 Corollary. *(compare [2, 2.6]) Let (X, d) be a T_0-quasi-metric space, $f \in Q_X, x \in X$ and $\epsilon > 0$. Suppose that $f_2(x) > 0$. Then there is $y \in X$ such* *that $d(x, y) + \epsilon > f_2(x) + f_1(y)$. Similarly suppose that $f_1(x) > 0$. Then there is $y' \in X$ such that $d(y', x) + \epsilon > f_2(y') + f_1(x)$.*

4.1.3 Lemma. *(compare [2, 2.6]) Let (X, d) be a T_0-quasi-metric space, f be an ample function pair on (X, d) such that for all $x \in X$ there is $y \in X$ such that $d(x, y) = f_2(x) + f_1(y)$ and for all $x \in X$ there is $y' \in X$ such that $d(y', x) = f_2(y') + f_1(x)$. Then f is a minimal function pair on (X, d).*

Proof. Suppose that h is a function pair on (X, d) such that $h < f$, say $h_2(x_0)+\epsilon < f_2(x_0)$ for some $x_0 \in X$ and $\epsilon > 0$. Then by Corollary 4.1.1 there is $y \in X$ such that $d(x_0, y) > f_2(x_0) - \epsilon + f_1(y) > h_2(x_0) + h_1(y)$. Hence h is not an ample function pair on (X, d) and f is therefore a minimal function pair on (X, d). The second case is dealt with analogously. □

4.1.1 Proposition. *Let (X, d) be a T_0-quasi-metric space. Fix $y \in X$ and set $a_1 \leq_y a_2$ if and only if (y, a_1, a_2) is collinear in (X, d). Then $\leq_y$ is a partial order on X.*

Proof. For each $a \in X, a \leq_y a$ as (y, a, a) is collinear (X, d). Therefore $\leq_y$ is reflexive.

Let $b, c \in X$ be such that $b \leq_y c$ and $c \leq_y b$, then (y, b, c) and (y, c, b) are collinear in (X, d). Hence $d(y, b)+d(b, c) = d(y, c)$ and $d(y, c)+d(c, b) = d(y, b)$. Therefore $(d(y, c) + d(c, b)) + d(b, c) = d(y, c)$, thus $d(c, b) + d(b, c) = 0$. By the T_0-property of X we have that $b = c$. Therefore $\leq_y$ is antisymmetric.

Let $b, c, e \in X$ be such that $b \leq_y c$ and $c \leq_y e$. Then (y, b, c) and (y, c, e) are collinear in (X, d). Therefore (y, b, e) is collinear in (X, d) by Lemma 4.1.1. Thus $b \leq_y e$, which implies that $\leq_y$ is transitive. Hence $\leq_y$ is a partial order on X as required. □

4.2 Endpoints in q-hyperconvex hulls of T_0-quasi-metric spaces

Given an arbitrary T_0-quasi-metric space (X,d), it is natural to ask the following questions.

1. When does it possess endpoints (resp. startpoints)?

2. Do (Q_X,D) and (X,d) have the same endpoints (resp. startpoints)?

3. How is the q-hyperconvex hull (Q_X,D) of (X,d) related to the q-hyperconvex hull (Q_B,D) of its subspace B, where $B=E_d\cup S_d$?

The results in this section will try to answer the above questions.

4.2.1 Proposition. *Let (X,d) be a joincompact T_0-quasi-metric space. Moreover let two distinct points $y,a\in X$ with $d(y,a)>0$ be given. Then there is a startpoint s of (X,d) and an endpoint e of (X,d) such that (y,a,e) and (s,y,a) are collinear in (X,d).*

Proof. Consider $\mathcal{M}_{y,a}=\{a'\in X:(y,a,a')$ is collinear in $(X,d)\}$. It is nonempty as $a\in\mathcal{M}_{y,a}$. Then $\mathcal{M}_{y,a}$ equipped with the restriction of the partial order $\leq_y$ in Proposition 4.1.1 is a poset. Let $\mathcal{K}\subseteq\mathcal{M}_{y,a}$ be a nonempty chain. Let us consider a net $x_k=k$ where $k\in\mathcal{K}$, which is directed by the linear order of the chain $\mathcal{K}$. Since $(X,\tau(d^s))$ is compact, we know that there is a subnet $(x_{k_e})_{e\in E}$ of $(x_k)_{k\in\mathcal{K}}$ converging to some point x in (X,d^s). Let us show that x is an upper bound of $\mathcal{K}$ in $\mathcal{M}_{y,a}$. Indeed for each $e\in E$ we have that $d(y,a)+d(a,x_{k_e})=d(y,x_{k_e})$ as $x_{k_e}\in\mathcal{M}_{y,a}$. Taking limits in $\mathbb{R}$ equipped with its usual topology we have $d(y,a)+d(a,x)=d(y,x)$, since $|d(a,x_{k_e})-d(a,x)|\leq d^s(x_{k_e},x)$ and $|d(y,x_{k_e})-d(y,x)|\leq d^s(x_{k_e},x)$ whenever $e\in E$.

Thus (y,a,x) is collinear in (X,d) and $x\in\mathcal{M}_{y,a}$. Since $(x_{k_e})_{e\in E}$ is a subnet of $(x_k)_{k\in\mathcal{K}}$, given $k\in\mathcal{K}$ we see that $k\leq_y k_e$ eventually. By definition of $\leq_y$, if $k\leq_y k_e$ then we have that $d(y,x_k)+d(x_k,x_{k_e})=d(y,x_{k_e})$. Taking as above limits, we get that $d(y,x_k)+d(x_k,x)=d(y,x)$. Consequently (y,x_k,x) is collinear in

(X, d) and $k = x_k \leq_y x$. Thus x is an upper bound of $\mathcal{K}$ in $\mathcal{M}_{y,a}$.
Hence by Zorn's Lemma, $\mathcal{M}_{y,a}$ has a maximal element, say m. Of course $m \neq y$, as $y \neq a$. We show that m is an endpoint of (X, d): Indeed let $x \in X$ such that (y, m, x) collinear in (X, d). Since (y, a, m) is collinear in (X, d), we see that $a \leq_y m \leq_y x$. Thus (y, a, x) is collinear in (X, d) and therefore $x \in \mathcal{M}_{y,a}$ and $m = x$ by maximality of m in $\mathcal{M}_{y,a}$. Hence m is an endpoint of (X, d) and (y, a, m) is collinear in (X, d) as required. Applying the same argument to the space (X, d^{-1}) we obtain the second part of the statement about startpoints. □

The above proposition yields the following corollary.

4.2.1 Corollary. *Let (X, d) be a joincompact T_0-quasi-metric space. Moreover let two distinct points $y, a \in X$ with $d(y, a) > 0$ be given. Then there is an endpoint e and a startpoint s in (X, d) such that (s, y, a, e) is collinear in (X, d).*

Proof. By Proposition 4.2.1 there is a startpoint $s \in X$ such that (s, y, a) is collinear in (X, d). In particular $d(s, a) = d(s, y) + d(y, a) > 0$. Therefore applying Proposition 4.2.1 again we get an endpoint $e \in X$ such that (s, a, e) is collinear. Then (s, y, a, e) is collinear by Lemma 4.1.1. □

4.2.1 Lemma. *Let (X, d) be a T_0-quasi-metric space. If x is a startpoint with witness y and (x, y, z) is collinear in (X, d), then z is also a witness that x is a startpoint in (X, d). Similarly if x is an endpoint with witness y and (z, y, x) is collinear in (X, d), then z is also a witness that x is an endpoint in (X, d)*

Proof. Since y witnesses that x is a startpoint in (X, d), then $d(x, y) > 0$. Note that the collinearity of (x, y, z) in (X, d) implies that $d(x, z) = d(x, y) + d(y, z) \geq d(x, y) > 0$. Let $a \in X$ be such that (a, x, z) is collinear in (X, d). Since (x, y, z) and (a, x, z) are collinear in (X, d), then (z, y, x) and (z, x, a) are collinear in (X, d^{-1}) by Remark 4.1.1. Applying Lemma 4.1.1 we have that (y, x, a) is collinear in (X, d^{-1}) and again by Remark 4.1.1 we have that (a, x, y) is collinear in (X, d)

and thus $x = a$, as y witnesses that x is a startpoint in (X, d). Therefore z witnesses that x is a startpoint of (X, d). The result about endpoints is proved analogously. □

4.2.2 Proposition. *(compare [2, 2.12]) Let (X, d) be a joincompact T_0-quasi-metric space. Then x is an endpoint of (X, d) if and only if x is an endpoint of its q-hyperconvex hull (Q_X, D).*

Proof. Suppose that x is an endpoint of X. Then there exists $y \in X$ such that $d(y, x) > 0$ and the collinearity of (y, x, a) in (X, d) implies that $x = a \quad (\star)$ whenever $a \in X$. Let $w \in Q_X$ be such that (y, x, w) is collinear in (Q_X, D). (Take note that here we are considering X as a subspace of Q_X). Then $0 < D(y, x) \leq D(y, w)$. It suffices to show that $x = w$ so that we can conclude that x is an endpoint of (Q_X, D). Since (X, d) is joincompact and $w_2(y) = D(y, w) > 0$, then by Lemma 4.1.2 there is $u \in X$ such that (y, w, u) is collinear in (Q_X, D).
Since (y, x, w) and (y, w, u) are collinear in (Q_X, D), then by Lemma 4.1.1 we have that (y, x, u) is collinear in (Q_X, D). So $x = u$ by assumption $(\star)$. Consequently $D(y, x) + D(x, w) = D(y, w)$ and $D(y, w) + D(w, u) = D(y, u)$ by the two collinearity assumptions. Since $x = u$, the second equation becomes $D(y, w) + D(w, x) = D(y, x)$. Thus the first equation becomes $(D(y, w) + D(w, x)) + D(x, w) = D(y, w)$, which implies that $D(w, x) + D(x, w) = 0$. Since D is a T_0-quasi-metric on Q_X, then $x = w$. Therefore x is an endpoint of Q_X as required.
Now let us show that each endpoint of (Q_X, D) is an endpoint of (X, d). To this suppose $g \in Q_X$ is an endpoint in (Q_X, D). Then there exists $f \in Q_X$ such that $D(f, g) > 0$ and such that for any $h \in Q_X$ the collinearity of (f, g, h) in (Q_X, D) implies that $g = h \quad (\star\star)$. Since $D(f, g) > 0$ and Q_X is a T_0-quasi-metric tight extension of X (where X is identified with $e_X(X)$), then by Proposition 3.1.4 (b) there are sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in X such that the increasing sequence $(D(x_n, y_n) - D(x_n, f) - D(g, y_n))_{n\in\mathbb{N}}$ converges to $D(f, g)$ with respect to the usual topology on $\mathbb{R}$. (Since here and in the following X is identified with $e_X(X)$,

for each $x \in X$, we shall identify x with $e_X(x) = f_x$).
By joincompactness of (X, d) there are a subsequence $(n_k)_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ and $x, y \in X$ such that $d^s(x_{n_k}, x) \to 0$ and $d^s(y_{n_k}, y) \to 0$. Consequently $|D(x_{n_k}, f) - D(x, f)| \leq d^s(x_{n_k}, x)$ and $|D(g, y_{n_k}) - D(g, y)| \leq d^s(y_{n_k}, y)$ whenever $k \in \mathbb{N}$. Therefore $D(f, g) = D(x, y) - D(x, f) - D(g, y)$ and thus $D(x, y) = D(x, f) + D(f, g) + D(g, y)$. Since $D(x, f) + D(f, g) + D(g, y) = D(x, y) \leq D(x, f) + D(f, y) \leq D(x, f) + D(f, g) + D(g, y)$, then it follows that $D(f, y) = D(f, g) + D(g, y)$ and $D(x, y) = D(x, f) + D(f, y)$. Thus (f, g, y) and (x, f, y) are collinear in (Q_X, D). By assumption $(\star\star)$ we conclude that $g = y$. Since (x, f, y) is collinear in (Q_X, D), then by Lemma 4.2.1 x is a witness in X that y is an endpoint in (Q_X, D).
Let us show that $g = y$ is an endpoint in (X, d). To this let $a \in X$ be such that (x, y, a) is collinear in (X, d). Then (x, y, a) is also collinear in (Q_X, D) and since x witnesses that $y = g$ is an endpoint in (Q_X, D), then $y = a$ and we can conclude that $y = g$ is an endpoint in (X, d). $\square$

4.2.2 Corollary. *Let (X, d) be a joincompact T_0-quasi-metric space. Then x is startpoint of (X, d) if and only if x is a startpoint of its q-hyperconvex hull (Q_X, D).*

Proof. By Proposition 4.2.2 each endpoint of (X, d^{-1}) is an endpoint of $(Q_{(X,d^{-1})}, D)$. Thus each startpoint of (X, d) is a startpoint of $(Q_{(X,d^{-1})}, D^{-1})$. By [3, Proposition 4] $s : (Q_{(X,d^{-1})}, D^{-1}) \to (Q_{(X,d)}, D)$ is an isometric bijection where $s(f_1, f_2) = (f_2, f_1)$ whenever $f \in Q_{(X,d^{-1})}$. Thus each startpoint of $(Q_{(X,d^{-1})}, D^{-1})$ is a startpoint of $(Q_{(X,d)}, D)$. Hence each startpoint of (X, d) is a startpoint of $(Q_{(X,d)}, D)$. Similarly by [3, Proposition 4] $s : (Q_{(X,d)}, D^{-1}) \to (Q_{(X,d^{-1})}, D)$ is an isometric bijection where $s(f_1, f_2) = (f_2, f_1)$ whenever $f \in Q_{(X,d)}$. Thus each endpoint of $(Q_{(X,d)}, D^{-1})$ is an endpoint of $(Q_{(X,d^{-1})}, D)$ and by Proposition 4.2.2 each endpoint of $(Q_{(X,d^{-1})}, D)$ is an endpoint of (X, d^{-1}). Thus each startpoint of $(Q_{(X,d)}, D)$ is a startpoint of (X, d). Take note that the copy of (X, d) in $(Q_{(X,d)}, D)$ is isometric to the conjugate copy of (X, d^{-1}) in $(Q_{(X,d^{-1})}, D^{-1})$, (i.e.

$e_{(X,d)}(X)$ is isometric to the conjugate of $e_{(X,d^{-1})}(X)$). $\square$

4.2.1 Example. *Consider the space (X_{ab}, d) of Example 4.1.1, where $a, b \in (0, \infty)$. In Case 3 of Example 4.1.1 it was shown that 0 and 1 are both startpoints and endpoints in (X_{ab}, d).*
We can identify the q-hyperconvex hull $Q_{X_{ab}}$ of (X_{ab}, d) with the rectangle $[0, a] \times [0, b]$ equipped with the T_0-quasi-metric

$$\rho((x_1, y_1), (x_2, y_2)) = \max\{x_1 \dot{-} x_2, y_1 \dot{-} y_2\}$$

whenever $(x_1, y_1), (x_2, y_2) \in [0, a] \times [0, b]$.
It is readily checked that we obtain exactly the following minimal function pairs (f_1, f_2) on (X_{ab}, d): Namely $(f_1(0), f_1(1)) = (x, y)$ and $(f_2(0), f_2(1)) = (b - y, a - x)$ whenever $(x, y) \in [0, a] \times [0, b]$.
Hence we can identify the points of $Q_{X_{ab}}$ with the points of $[0, a] \times [0, b]$. Obviously in this way via the embedding $e_{X_{ab}} : (X_{ab}, d) \to (Q_{X_{ab}}, D)$ the point 0 of (X_{ab}, d) is identified with the function $(f_0)_1 = (0, b)$ on X_{ab} and the point 1 of (X_{ab}, d) is identified with the function $(f_1)_1 = (a, 0)$ on X_{ab}.
Let us show that $(0, b)$ and $(a, 0)$ are both startpoints and endpoints in $(Q_{X_{ab}}, D)$. Let $(x, y) \in [0, a] \times [0, b]$ be such that $((x, y), (a, 0), (0, b))$ is collinear in $([0, a] \times [0, b], \rho)$, then $a \geq x = y + a \geq a$. Hence $x = a$ and $y = 0$. Therefore $(0, b)$ witnesses that $(a, 0)$ is a startpoint in $([0, a] \times [0, b], \rho)$, as $\rho((a, 0), (0, b)) = a > 0$. Similarly collinearity of $((0, b), (a, 0), (x, y))$ in $([0, a] \times [0, b], \rho)$ whenever $(x, y) \in [0, a] \times [0, b]$ will imply that $x = a$ and $y = 0$, hence $(0, b)$ witnesses that $(a, 0)$ is an endpoint in $([0, a] \times [0, b], \rho)$ as $\rho((0, b), (a, 0)) = b > 0$. Analogously we can show that $(a, 0)$ witnesses that $(0, b)$ is both an endpoint and a startpoint in $([0, a] \times [0, b], \rho)$. Observe that $\rho((0, 0), (x, y)) = 0 = \rho((x, y), (a, b))$ whenever $(x, y) \in [0, a] \times [0, b]$. So there is no point $(x, y) \in [0, a] \times [0, b]$ which can witness that $(0, 0)$ and (a, b) are startpoints and endpoints in $([0, a] \times [0, b], \rho)$, respectively.
Let us show that $(0, 0)$ is not an endpoint in $([0, a] \times [0, b], \rho)$, to this assume

that $(x,y) \in [0,a] \times [0,b]$ witnesses that $(0,0)$ is an endpoint in $([0,a] \times [0,b], \rho)$. Note that $\rho((x,y),(0,0)) > 0$ and thus $(x,y) \neq (0,0)$. Let $\bar{y} = \max\{x,y\}$. Then $((x,y),(0,0),(0,\bar{y}))$ is collinear in $([0,a]\times[0,b],\rho)$ if $\bar{y} = x$ and $((x,y),(0,0),(\bar{y},0))$ is collinear in $([0,a] \times [0,b], \rho)$ if $\bar{y} = y$. Thus no element of $[0,a] \times [0,b]$ can witness that $(0,0)$ is an endpoint in $([0,a] \times [0,b], \rho)$. Analogously we can show that there is no element in $[0,a] \times [0,b]$ that can witness that (a,b) is a startpoint in $([0,a] \times [0,b], \rho)$. In fact it will follow from Proposition 4.2.2 and Corollary 4.2.2 that $(0,b)$ and $(a,0)$ are the only startpoints and endpoints of $([0,a] \times [0,b], \rho)$. Therefore $(0,b)$ and $(a,0)$ are the only startpoints and endpoints of $(Q_{X_{ab}}, D)$. Thus we have shown that $(Q_{X_{ab}}, D)$ and (X_{ab}, d) have exactly the same endpoints (resp. startpoints).

The following result shows that given a joincompact T_0-quasi-metric space (X,d), any point $f \in Q_X - e_X(X)$ lies between some points of X.

4.2.3 Proposition. *Let (X,d) be a joincompact T_0-quasi-metric space and $e_X : (X,d) \to (Q_X, D)$ be the standard isometric embedding of (X,d) into its q-hyperconvex hull (Q_X, D). Then for each $f \in Q_X - e_X(X)$, there exists a startpoint s and an endpoint e in (X,d) such that (s,f,e) is collinear in (Q_X, D).*

Proof. Since X is identified with $e_X(X)$ in Q_X, then given $x \in X$ we shall identify f_x with x.
Fix $y \in X$. Since (Q_X, D) is a T_0-space, we have that $D(f,y) > 0$ or $D(y,f) > 0$ whenever $f \in Q_X - e_X(X)$. Otherwise if $D(f,y) = 0$ and $D(y,f) = 0$, then $f = f_y \in e_X(X)$– a contradiction.
We consider only the first case, (i.e. $f_1(y) = D(f,y) > 0$). The second one is analogous. By Lemma 4.1.2 there is $y' \in X$ such that (y', f, y) is collinear in (Q_X, D). Then $D(y', y) = D(y', f) + D(f, y) > 0$. Therefore by Corollary 4.2.1 there are a startpoint s and an endpoint e in (X,d) such that (s, y', y, e) is

collinear in (Q_X, D). It follows that

$$\begin{aligned} D(s,e) &\leq D(s,f) + D(f,e) \\ &\leq (D(s,y') + D(y',f)) + (D(f,y) + D(y,e)) \\ &= D(s,y') + D(y',y) + D(y,e) \\ &= D(s,y) + D(y,e) \\ &= D(s,e). \end{aligned}$$

Therefore $D(s,e) = D(s,f) + D(f,e)$ and hence (s,f,e) is collinear in (Q_X, D) as required. $\square$

4.2.4 Proposition. *(compare [2, Remark 3.2]) A bicomplete T_0-quasi-metric space (X,d) is q-hyperconvex if for every $\epsilon > 0$ there exists a q-hyperconvex T_0-quasi-metric subspace S of X such that every point of X is within ϵ of some point s in S with respect to the metric d^s, (i.e. for each $x \in X$ we find an $s \in S$ such that $d^s(x,s) < \epsilon$).*

Proof. Let f be a minimal ample pair of functions on (X,d). By our assumption for each $n \in \mathbb{N}$ we can find a subspace S_n of X such that S_n is q-hyperconvex and given any $y \in X$ there is $s_n \in S_n$ such that $d^s(y, s_n) < 2^{-n}$. Consider the restriction $f|_{S_n}$ of f to S_n. That restriction is ample on S_n, hence by Proposition 3.1.1 has a minimal function pair on S_n below it, so that there is $p_n \in S_n$ with $(d(p_n,.), d(., p_n)) \leq f|_{S_n}$, since S_n is q-hyperconvex. We shall show that $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d^s). Let $\epsilon > 0$ and $m \in \mathbb{N}$ such that $2^{-m} < \epsilon$. Suppose first that $f_2(p_m) > 0$. Then by Corollary 4.1.1 there is $z_m \in X$ such that $\epsilon + d(p_m, z_m) \geq f_2(p_m) + f_1(z_m)$, since f is minimal on (X,d). Furthermore there is $a_m \in S_m$ such that $d^s(z_m, a_m) < \epsilon$. Hence $f_2(p_m) \leq \epsilon + (d(p_m, z_m) - d(p_m, a_m)) + (d(p_m, a_m) - f_1(a_m)) + (f_1(a_m) - f_1(z_m)) \leq \epsilon + d^s(z_m, a_m) + 0 + d^s(a_m, z_m) \leq 3\epsilon$ whenever $m \in \mathbb{N}$ such that $2^{-m} < \epsilon$.
Similarly one shows that $f_1(p_m) \leq 3\epsilon$ whenever $m \in \mathbb{N}$ and $2^{-m} < \epsilon$.
Hence for any $n, m \in \mathbb{N}$ such that $\max\{2^{-n}, 2^{-m}\} \leq \epsilon$ we have $d(p_m, p_n) \leq$

$f_2(p_m) + f_1(p_n) < 3(\epsilon + \epsilon)$ and thus $d^s(p_m, p_n) \leq 3\epsilon$.
Therefore $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d^s). By bicompleteness of (X, d) there is $p \in X$ such that $d^s(p_n, p) \to 0$. We are going to show that $f = f_p$ on (X, d). Consider any $\epsilon > 0$ and any $y \in X$. There is $m \in \mathbb{N}$ such that $2^{-m} < \epsilon$, and also there is $k \in \mathbb{N}$ such that $k \geq m$ and $d^s(p, p_k) < \epsilon$.
Furthermore there is $b \in S_k$ such that $d^s(y, b) < \epsilon$.
Therefore $d(p, y) - f_1(y) \leq (d(p, y) - d(p_k, b)) + (d(p_k, b) - f_1(b)) + (f_1(b) - f_1(y)) \leq d^s(p, p_k) + d^s(y, b) + 0 + d^s(y, b) < \epsilon + \epsilon + 0 + \epsilon = 3\epsilon$.
Since $\epsilon > 0$ was arbitrary, we have $d(p, y) \leq f_1(y)$ whenever $y \in X$. Analogously one shows that for any $y \in X$ we have $d(y, p) \leq f_2(y)$. By minimality of f we conclude that $f_1(y) = d(p, y)$ and $f_2(y) = d(y, p)$ whenever $y \in X$. We have shown that for each $f \in Q_X$, there exists $p \in X$ such that $f = f_p$. Therefore (X, d) is q-hyperconvex by Proposition 2.2.4.

□

4.2.5 Proposition. *[2, page 73] Let (X, d) be a joincompact T_0-quasi-metric space possessing more than one point with a d^s-closed set K not containing some point of $B = E_d \cup S_d$. Then X is not a T_0-quasi-metric tight extension of K.*

Proof. Suppose that x is an endpoint of (X, d) and K is a d^s-closed subset of X such that $x \in X - K$. Then the definition of an endpoint in (X, d) gives $y \in X$ such that $d(y, x) > 0$ which witnesses that x is an endpoint of (X, d). Furthermore $H = K \cup \{y\}$ is d^s-closed and thus joincompact. In the light of condition (b) of Proposition 3.1.4 it suffices to show that $f_x|_H$ is not minimal on H whenever $f \in Q_X$ where $f_x|_H := (d(x, .), d(., x))$. Then X is not a tight extension of H, so certainly not of K. Of course, we have that $f_x|_H \in P_H$. If $f_x|_H \in Q_H$, then by joincompactness of H and Lemma 4.1.2 there is $k \in K$ such that $d(y, k) = (f_x|_H)_2(y) + (f_x|_H)_1(k) = d(y, x) + d(x, k)$. Since y witnesses that x is an endpoint of (X, d), then $x = k$—a contradiction. We conclude that $f_x|_H \notin Q_H$. Obviously the case that x is a startpoint of (X, d) and K is a d^s-closed subset of X such that $x \in X - K$ can be treated similarly.

$\square$

4.2.6 Proposition. *(compare [9, Theorem 4]) Let (X, d) be a joincompact T_0-quasi-metric space. Then there is a bijective isometric map $Q_X \to Q_B$ where $B = E_d \cup S_d$.*

Proof. In light of Proposition 3.1.4 we must show that X is a T_0-quasi-metric tight extension of B. Indeed we can show that $d(x_1, x_2) = \sup\{d(b_1, b_2) - d(x_1, b_1) - d(x_2, b_2) \vee 0 : b_1, b_2 \in B\}$ whenever $x_1, x_2 \in X$ in the sense of Proposition 3.1.4 condition (b), so that we can conclude that X is a T_0-quasi-metric tight extension of B. Let $x_1, x_2 \in X$. Suppose first that $d(x_1, x_2) = 0$. By the triangle inequality for any $b_1, b_2 \in B$ we have that $d(b_1, b_2) \leq d(b_1, x_1) + d(x_1, x_2) + d(x_2, b_2) = d(b_1, x_1) + d(x_2, b_2)$. Therefore $d(x_1, x_2) = 0 = \sup\{d(b_1, b_2) - d(x_1, b_1) - d(x_2, b_2) \vee 0 : b_1, b_2 \in B\}$. Let us consider now the case that $d(x_1, x_2) > 0$. By Corollary 4.2.1 there are a startpoint s and an endpoint e in (X, d) such that (s, x_1, x_2, e) is collinear in (X, d). Thus $d(s, e) = d(s, x_2) + d(x_2, e)$ and therefore $d(s, x_1) + d(x_1, x_2) + d(x_2, e) = d(s, x_2) + d(x_2, e) = d(s, e)$. Consequently $d(x_1, x_2) = d(s, e) - d(s, x_1) - d(x_2, e)$. Hence the pair (B, X) satisfies the condition (b) of Proposition 3.1.4. Thus X is a T_0-quasi-metric tight extension of B. Hence by Proposition 3.1.4 there is a bijective isometric map between Q_X and Q_B, namely $f \mapsto f|_B$, where $f \in Q_X$. $\square$

4.3 T_0-quasi-metrics induced by partial orders

In this section we will illustrate the concepts of a startpoint (resp. an endpoint) in the case of T_0-quasi-metrics induced by partial orders. Given a partially ordered set $(X, \leq)$.

Define a function mapping $d : X \times X \to [0, \infty)$ as follows:

$$d(x, y) = 0 \text{ if and only if } x \leq y \text{ and}$$

$$d(x, y) = 1 \text{ if and only if } x \nleq y.$$

Then it is easily seen that d is a T_0-quasi-metric on X. We shall call d the *natural T_0-quasi-metric* d of $\leq$.

Proof. Let $x, y, z \in X$, then $d(x, x) = 0$ as $x = x$. To show the triangle $(d(x, z) + d(z, y) \geq d(x, y))$ inequality we consider four cases.
Case 1: $x \leq z$ and $z \leq y$, then by transitivity of $\leq$ we have $x \leq y$ and $d(x, z) + d(z, y) = d(x, y)$.
Case 2: $x \leq z$ and $z \nleq y$, then $d(x, z) + d(z, y) = 1 \geq d(x, y)$.
Case 3: $x \nleq z$ and $z \leq y$ is similar to Case 2.
Case 4: $x \nleq z$ and $z \nleq y$, then $d(x, z) + d(z, y) = 2 \geq 1 \geq d(x, y)$.
If $d(x, y) = 0 = d(y, x)$, then $x \leq y \leq x$ and by antisymmetry of $\leq$ we have that $x = y$.

$\square$

Observe that if d is the natural T_0-quasi-metric induced by a partial order $\leq$, then d^{-1} is the natural T_0-quasi-metric induced by $\geq$.

Given a partially ordered set $(X, \leq)$. Let us first note that Proposition 4.2.1 is useless for an infinite partially ordered set X with its natural T_0-quasi-metric d, since d^s is the discrete metric and therefore $\tau(d^s)$ is compact if and only if X is finite.

In the following let us give some characterizations of endpoints (resp. startpoints) in partially ordered sets equipped with their natural T_0-quasi-metrics.
Let $(X, \leq)$ be a partially ordered set and $y \in X$. Set $\uparrow y := \{x \in X : y \leq x\}$ and $\downarrow y := \{x \in X : x \leq y\}$.

4.3.1 Lemma. *Let $(X, \leq)$ be a partially ordered set, d its natural T_0-quasi-metric and $x \in X$. Then x is an endpoint of (X, d) if and only if there is $y \in X$ such that x is a maximal element in $X - \uparrow y$.*

Proof. Suppose that x is an endpoint of (X, d). There is a witness $y \in X$ with $d(y, x) > 0$, thus $y \not\leq x$, such that for all $a \in X$, collinearity of (y, x, a) in (X, d) implies that $x = a$. Let $a \in X - \uparrow y$ be such that $x \leq a$. Then (y, x, a) is collinear as $d(y, a) = 1, d(y, x) = 1$ and $d(x, a) = 0$, hence $x = a$ as y witnesses that x is an endpoint in (X, d). Thus x is maximal in $X - \uparrow y$.
Suppose x is maximal in $X - \uparrow y$. Since $y \not\leq x$, then $d(y, x) = 1 > 0$. Let $a \in X$ be such that (y, x, a) is collinear in (X, d). Then $1 \geq d(y, a) = d(y, x) + d(x, a) = 1 + d(x, a) \geq 1$, hence $d(y, a) = 1$ and $d(x, a) = 0$, thus $y \not\leq a$, $x \leq a$. Therefore $x = a$ by the maximality of x in $X - \uparrow y$. Hence x is an endpoint in (X, d) witnessed by y. $\square$

4.3.1 Corollary. *Let $(X, \leq)$ be a partially ordered set, d its natural T_0-quasi-metric and $x \in X$. Then x is a startpoint of (X, d) if and only if there is $y \in X$ such that x is a minimal element in $X - \downarrow y$.*

The following definition can essentially be found in [15, Definition I-4.21].

4.3.1 Definition. *Let $(X, \leq)$ be a partially ordered set. An element $x \in X$ is called* completely j-irreducible *if either x is minimal in X, but different from the smallest element, or the set $\downarrow x - \{x\}$ has a largest element, which will be denoted by x^+. Dually one defines* completely m-irreducible *elements in X.*

4.3.1 Proposition. (a) *Each completely j-irreducible element x in a partially ordered set $(X, \leq)$ is a startpoint in (X, d) where d is the natural T_0-quasi-metric of X.*
(b) *(compare [15, Remark I-4.23]) Let X be a complete lattice equipped with its natural T_0-quasi-metric d and $x \in X$. If x is a startpoint in (X, d), then x is completely j-irreducible.*

Proof. (a) Let $x \in X$ be j-irreducible. Suppose first that x is minimal in X, but not the smallest element. Then there is $y \in X$ such that $x \not\leq y$. Therefore x is minimal in $X - \downarrow y$ and x is a startpoint in (X, d) by Corollary 4.3.1.
Suppose now that $\downarrow x - \{x\}$ has a largest element x^+. By Corollary 4.3.1 it suffices to show that x is minimal in $X - \downarrow x^+$. So let $a \in X$ with $a < x$. Then $a \in \downarrow x - \{x\}$ and consequently $a \leq x^+$. Hence $a \notin X - \downarrow x^+$ and x is minimal in $X - \downarrow x^+$. By Corollary 4.3.1 x is a startpoint in (X, d).
(b) Let $y \in X$ witness that x is a startpoint in (X, d). If x is minimal, then x is j-irreducible, as it cannot be the smallest element in X, because $x \not\leq y$ as $d(x, y) = 1$.
If x is not minimal in X, then $\emptyset \neq \downarrow x - \{x\} \subseteq \downarrow y$, since x is minimal in $X - \downarrow y$ by Corollary 4.3.1. Since X is a complete lattice, then $x^+ := \bigvee(\downarrow x - \{x\})$ exists and $x < x^+$, as $x^+ \leq y$. Thus $\downarrow x - \{x\}$ has a largest element, namely x^+. We have shown that x is completely j-irreducible in either case. $\square$

4.3.2 Corollary. (a) *Each completely m-irreducible element x in a partially ordered set $(X, \leq)$ is an endpoint in (X, d) where d is the natural T_0-quasi-metric of X.*
(b) *(compare [15, Remark I-4.23]) Let X be a complete lattice equipped with its natural T_0-quasi-metric d and $x \in X$. If x is an endpoint in (X, d), then x is completely m-irreducible.*

We recall that an element x in a complete lattice X is called *completely join-irreducible* if for each subset E of X, $x = \bigvee E$ implies that there is $e \in E$ such that $x = e$. *Completely meet-irreducible* elements are defined dually (see [16, Definition 10.26]).
One must take note that in a complete lattice the completely j-irreducible elements are exactly the completely join-irreducible elements and the completely m-irreducible elements are exactly the completely meet-irreducible elements.

4.3.3 Corollary. *Let X be a complete lattice and d its natural T_0-quasi-metric. Then $x \in X$ is a startpoint in (X, d) if and only if x is completely* *join-irreducible. Similarly $x \in X$ is an endpoint in (X,d) if and only if x is completely meet-irreducible.*

Let us recall that a subset E of a poset $(X,\leq)$ is called *join-dense* provided that for each $x \in X$ there exists $E' \subseteq E$ such that $x = \bigvee_X E'$ (see [16, p. 53]). Dually one defines the concept of a *meet-dense* subset of a poset $(X,\leq)$.

4.3.2 Proposition. *Let X be a poset equipped with its natural T_0-quasi-metric d.*
(a) *If E is a meet-dense subset of X. Then all endpoints of (X,d) belong to E.*
(b) *If E is a meet- and join-dense subset of X, then all endpoints of X are endpoints of E.*

Proof. (a) Suppose that x is an endpoint of (X,d). Then by Lemma 4.3.1 there is $y \in X$ such that x is maximal in $X - \uparrow y$. Since $y \not\leq x$, by our assumption on E there must be $e \in E$ such that $x \leq e$ and $y \not\leq e$. Hence by maximality of x we obtain $x = e$. Therefore $x \in E$.
(b) Let us continue the proof of part (a) dealing with endpoints: By our assumption on E, in the proof of part (a) we can find $y' \in E$ such that $y' \leq y$ and $y' \not\leq e = x$, then x is maximal in $X - \uparrow y'$. We conclude that $y' \in E$ witnesses that $x \in E$ is an endpoint in E. $\square$

4.3.4 Corollary. *Let X be a poset equipped with its natural T_0-quasi-metric d.*
(a) *If E is a join-dense subset of X. Then all startpoints of (X,d) belong to E.*
(b) *If E is a join- and meet-dense subset of X, then all startpoints of X are startpoints of E.*

Let $(X,\leq)$ be a linearly ordered set and let $a, b \in X$ be such that $a < b$, but that there does not exists a $z \in X$ such that $a < z < b$. As usual, then the pair (a,b) is called a *jump* in X.

4.3.3 Proposition. *Let $(X, \leq)$ be a linearly ordered set equipped with its natural T_0-quasi-metric d. The first elements of jumps in X are exactly the endpoints of (X, d) and the second elements of jumps in X are exactly the startpoints of (X, d).*

Proof. Suppose that (a, b) is a jump in X. Then a is a maximal element in $X- \uparrow b$. Hence by Lemma 4.3.1 a is an endpoint of (X, d).
To prove the converse assume that a is an endpoint of (X, d). Then by Lemma 4.3.1 there is $y \in X$ such that a is a maximal element in $X- \uparrow y$. Since $\leq$ is a linear order, we have that $a < y$ and (a, y) is a jump in X. $\square$

4.3.1 Example. *In $\mathbb{Z}$ equipped with its usual linear order and its natural T_0-quasi-metric d, each point is an endpoint and a startpoint, while in the subset $\mathbb{N}$ of $\mathbb{Z}$ each element is an endpoint and all elements except 0 are startpoints.*
The closed unit interval of rational numbers equipped with its usual linear order and the T_0-quasi-metric induced by that order does not have any endpoints nor startpoints.

4.3.5 Corollary. *Let $(X, \leq)$ be a partially ordered set equipped with its natural T_0-quasi-metric d. If for some $c \in X, \downarrow c$ consists of two elements, then c is a startpoint in (X, d).*
Similarly if for some $c \in X, \uparrow c$ consists of two elements, then c is an endpoint in (X, d).

Proof. Suppose that $\downarrow c = \{c, m\}$ with $m \neq c$. Obviously c is minimal in $X- \downarrow m$, hence a startpoint by Lemma 4.3.1.
Analogously one shows the second statement about endpoints. $\square$

4.3.6 Corollary. *Let $(X, \leq)$ be a partially ordered set equipped with its natural T_0-quasi-metric d. Each atom in a poset X with a smallest element is a startpoint of (X, d). Similarly each co-atom in a poset X with a largest element is an endpoint of (X, d).*

4.3.4 Proposition. *Let X be a partially ordered set equipped with its natural T_0-quasi-metric d. A largest element 1 cannot be an endpoint of (X, d). Similarly a smallest element 0 cannot be a startpoint of (X, d).*

Proof. Since for each $x \in X, x \leq 1$, then $d(x, 1) = 0$. Thus there cannot be an element in X witnessing that 1 is an endpoint in (X, d). The statement about startpoints is proved similarly. $\square$

4.4 The Dedekind-MacNeille completion of a partially ordered set versus the q-hyperconvex hull of its natural T_0-quasi-metric space

The following definition can be found in [16].

Let $(X, \leq)$ be a partially ordered set and $A \subseteq X$. Then we define the *set of upper bounds* of A, that is, $A^u = \{x \in X : a \leq x$ whenever $a \in A\}$ and the *set of lower bounds* of A, that is, $A^l = \{x \in X : x \leq a$ whenever $a \in A\}$. Let $DM(X) = \{A \subseteq X : A^{ul} = A\}$. Then the partially ordered set $(DM(X), \subseteq)$ is a complete lattice and it is known as the *Dedekind-MacNeille completion* of X. Furthermore $\phi : X \to DM(X)$ defined by $\phi(x) = \downarrow x$ is an order-embedding such that $\phi(X)$ is both join-dense and meet-dense in $DM(X)$. In light of [16, Theorem 7.41] this is indeed the characteristic property of the Dedekind-MacNeille completion of a partially ordered set $(X, \leq)$.

4.4.1 Proposition. *Let $(X, \leq)$ be a partially ordered set and d its natural T_0-quasi-metric. Let D be the natural T_0-quasi-metric of $(DM(X), \subseteq)$. Then (X, d) and $(DM(X), D)$ have exactly the same endpoints.*

Proof. For the proof we consider X as a subset of $DM(X)$ where it is identified with $\phi(X)$. By Proposition 4.3.2 each endpoint of $(DM(X), D)$ is an endpoint of (X, d), since X is both meet-dense and join-dense in $DM(X)$. Suppose now that x is an endpoint of (X, d) with a witness $y \in X$. Let $f \in DM(X)$ be such that (y, x, f) is collinear in $(DM(X), D)$. Thus $y \not\leq x, x \leq f$ and $y \not\leq f$. Since X is meet-dense in $DM(X)$ there is $f' \in X$ such that $f \leq f'$ and $y \not\leq f'$. Thus $x \leq f \leq f'$ and (y, x, f') is collinear in (X, d). Hence $f' = x$ and thus $f = x$, too. Therefore y witnesses that x is an endpoint of $(DM(X), D)$. $\square$

4.4.1 Corollary. *Let $(X, \leq)$ be a partially ordered set and d its natural T_0-quasi-metric. Let D be the natural T_0-quasi-metric of $(DM(X), \subseteq)$. Then (X, d) and $(DM(X), D)$ have exactly the same startpoints.*

4.4.1 Example. *(see [16, p. 169]) Considering P_4 as a subset of $DM(P_4)$. By Proposition 4.4.1 and according to Corollaries 4.4.1 and 4.3.3 in the complete lattice $DM(P_4)$ the set of endpoints of P_4 becomes the set of the (completely) meet-irreducible elements of $DM(P_4)$ and the set of the startpoints of P_4 becomes the set of the (completely) join-irreducible elements of $DM(P_4)$.*

Let $(X, \leq)$ be a partially ordered set equipped with its natural T_0-quasi-metric d.

We shall say that X is *o-hyperconvex* if for any $A \subseteq X$ and families $(r_x)_{x\in A}, (s_x)_{x\in A}$ in $\{0,1\}$ such that $d(x, y) \leq r_x + s_y$ whenever $x, y \in A$, it follows that $\bigcap_{x\in A}(C_d(x, r_x) \cap C_{d^{-1}}(x, s_x)) \neq \emptyset$. Take note that if (X, d) is q-hyperconvex then $(X, \leq)$ is o-hyperconvex. Hence given a partially ordered set $(X, \leq)$ with its natural T_0-quasi-metric d the concept of o-hyperconvexity is weaker than the concept of q-hyperconvexity.

4.4.2 Proposition. *Let $(X, \leq)$ be a complete lattice and d its natural T_0-quasi-metric. Then (X, d) is o-hyperconvex.*

Proof. Let $A \subseteq X$ and $(r_x)_{x\in A}, (s_x)_{x\in A}$ be families in $\{0,1\}$ such that $d(x, y) \leq r_x + s_y$ whenever $x, y \in A$. Set $R = \{x \in A : r_x = 0\}$ and $S = \{x \in A : s_x = 0\}$.

Note that $R \cap S$ contains at most one element, since (X, d) is a T_0-space. This is so because if $x, y \in R \cap S$ we have $d(x, y) \leq r_x + s_y = 0$ and $d(y, x) \leq r_y + s_x = 0$, thus $d(x, y) = 0 = d(y, x)$, hence $x = y$. Since $d(r, s) \leq r_r + s_s = 0$ whenever $r \in R$ and $s \in S$, then $R \subseteq S^l$. Since X is a complete lattice, $\bigvee R$ exists. Then

$$\bigvee R \in (\bigcap_{x \in R} \uparrow x) \cap (\bigcap_{x \in S} \downarrow x) = (\bigcap_{x \in A} C_d(x, r_x)) \cap (\bigcap_{x \in A} C_{d^{-1}}(x, s_x)),$$

because $C_d(x, r_x) = X$ if $x \in A - R$, $C_d(x, r_x) = \uparrow x$ if $x \in R$ and $C_{d^{-1}}(x, s_x) = X$ if $x \in A - S$, $C_{d^{-1}}(x, s_x) = \downarrow x$ if $x \in S$. Thus (X, d) is o-hyperconvex. $\square$

The following example shows that o-hyperconvexity does not imply q-hyperconvexity in a partially ordered set when equipped with its natural T_0-quasi-metric d.

4.4.2 Example. *Let $X = \{0, 1\}$ be equipped with its usual order $\leq$ and its natural T_0-quasi-metric d. Then (Q_X, D) can be identified with $([0, 1], u)$ (see [3, Example 8]) where u is the T_0-quasi-metric defined in Example 1.1.1. It is known that (X, d) is not q-hyperconvex, although $(X, \leq)$ is a complete lattice and hence o-hyperconvex by Proposition 4.4.2.*

The following result shows that given a partially ordered set $(X, \leq)$ equipped with its natural T_0-quasi-metric d, $(Q_{(X,d)}, D)$ contains the Dedekind-MacNeille completion $DM(X)$ of X.

4.4.1 Lemma. *Let $(X, \leq)$ be a partially ordered set and d its natural T_0-quasi-metric. Let F_X be the set of all those minimal ample functions pairs (f_1, f_2) on (X, d) such that $f_i : X \to \{0, 1\}$ $(i = 1, 2)$.*
Consider any arbitrary function pair $(f_1, f_2) : X \to \{0, 1\}$. Then the following conditions are equivalent:
(a) $(f_1, f_2) \in F_X$.
(b) $f_1^{-1}\{0\} = (f_2^{-1}\{0\})^u$ *and* $f_2^{-1}\{0\} = (f_1^{-1}\{0\})^l$.
(c) $(f_2^{-1}\{0\})^{ul} = f_2^{-1}\{0\}$ *and* $f_1(x) = \sup\{d(y, x) \dot{-} f_2(y) : y \in X\}$ *whenever* $x \in X$.

Proof. $(a) \rightarrow (b)$: Let $(f_1, f_2) \in F_X$, given $x \in X$, consider the equation

$$f_1(x) = \sup\{d(y, x) \dot{-} f_2(y) : y \in X\}.$$

Since the functions f_1, f_2 and d attain only the values 0 and 1, we see that these equations are equivalent to the equation $f_1^{-1}\{0\} = (f_2^{-1}\{0\})^u$. This is so because given $x \in X$, we have that $f_1(x) = 0$ if and only if (for any $y \in X, d(y, x) = 1$ implies that $f_2(y) = 1$) if and only if (for any $y \in X, f_2(y) = 0$ implies that $d(y, x) = 0$) if and only if $x \in (f_2^{-1}\{0\})^u$.
Similarly one verifies that the equation $f_2^{-1}\{0\} = (f_1^{-1}\{0\})^l$ is equivalent to the equations $f_2(x) = \sup\{d(x, y) \dot{-} f_1(y) : y \in X\}$ whenever $x \in X$ and condition (b) holds.
$(b) \rightarrow (c)$: By (b) we conclude that $(f_2^{-1}\{0\})^{ul} = f_2^{-1}\{0\}$. Furthermore the second part of (c) is equivalent to $f_1^{-1}\{0\} = (f_2^{-1}\{0\})^u$, as we have just noted above. Hence condition (c) is satisfied.
$(c) \rightarrow (a)$: Observe that $(f_2^{-1}\{0\})^{ul} = f_2^{-1}\{0\}$ and $f_1^{-1}\{0\} = (f_2^{-1}\{0\})^u$ together imply that $f_2^{-1}\{0\} = (f_1^{-1}\{0\})^l$. But the latter equality is equivalent to $f_2(x) = \sup\{d(x, y) \dot{-} f_1(y) : y \in X\}$ whenever $x \in X$, as we have observed above. Thus $(f_1, f_2) \in F_X$ and condition (a) holds. $\square$

4.4.3 Proposition. *Let $(X, \leq)$ be a partially ordered set with its natural T_0-quasi-metric d and F_X be defined as in Lemma 4.4.1.*
The map $\psi : (F_X, \leq_D) \rightarrow (DM(X), \subseteq)$ defined by $(f_1, f_2) \mapsto f_2^{-1}\{0\}$ is an order-isomorphism between F_X (endowed with the specialization order $\leq_D$ induced by D) and the Dedekind-MacNeille completion $(DM(X), \subseteq)$ of X. Furthermore for each $x \in X$, $\psi(f_x) = \downarrow x$.

Proof. By Lemma 4.4.1 each set $f_2^{-1}\{0\} \in DM(X)$, as $f \in F_X$. For each $x \in X$, $f_x \in F_X$. Also for each $x \in X$, we have that $(f_x)_2^{-1}\{0\} = \downarrow x$. The specialization order $\leq_D$ induced on F_X by the T_0-quasi-metric D is defined by $f \preceq g$ if and only if $D(f, g) = 0$ whenever $f, g \in F_X$. Hence we have for any $f, g \in F_X$,

$$D(f,g) = \sup\{g_2(x) \dot{-} f_2(x) : x \in X\} = 0 \text{ if and only if}$$
$$\psi(f) = f_2^{-1}\{0\} \subseteq g_2^{-1}\{0\} = \psi(g).$$

This implies exactly that $g_2 \leq f_2$ with respect the usual pointwise order on the real-valued functions. In particular ψ is injective. Furthermore for $A \subseteq X$ with $A^{ul} = A$ we define $f_2 : X \to \{0,1\}$ so that $A = f_2^{-1}\{0\}$ and $f_1(x) = \sup\{d(y,x) \dot{-} f_2(y) : y \in X\}$ whenever $x \in X$. Then $f = (f_1, f_2) \in F_X$ according to Lemma 4.4.1 and $\psi(f) = A$ and hence ψ is surjective. We conclude that the set F_X can be identified with the ground-set of the Dedekind-MacNeille completion of X and $\psi : (F_X, \leq_D) \to (DM(X), \subseteq)$ is an order-isomorphism satisfying $\psi(f_x) = \downarrow x$.

□

Chapter 5

Conclusions

In Chapter 3 we have shown that if X is a subspace of a T_0-quasi-metric space (Y, d), then the q-hyperconvex hull Q_X of X is quasi-metrically contained in the q-hyperconvex hull Q_Y of Y. In the case Y is a T_0-quasi-metric tight extension of X, then Q_Y is contained in Q_X quasi-metrically. Hence we have that Q_X is a maximal T_0-quasi-metric tight extension of X. In Proposition 3.1.1 it has been shown that via Zorn's Lemma, below each ample function pair there is a minimal ample function pair.

In Chapter 4 we have shown that the q-hyperconvex hull Q_X of a joincompact T_0-quasi-metric space X is isometric to the q-hyperconvex hull Q_B of B, where B is the subspace of X consisting of its endpoints and startpoints. We also showed that every joincompact T_0-quasi-metric space X having at least two elements with a nonzero distance between them has at least one endpoint (resp. startpoint). In the case of a joincompact T_0-quasi-metric space X, every point in $Q_X - e_X(X)$ lies between some points of X in (Q_X, D). We showed that for any joincompact T_0-quasi-metric space (X, d), its q-hyperconvex hull (Q_X, D) has exactly the same endpoints (resp. startpoints) as (X, d). Therefore in constructing the hull of a joincompact T_0-quasi-metric space one does not get new endpoints (resp. startpoints). In Lemma 4.3.1 and Corollary 4.3.1 we have given a characterization of endpoints (resp. startpoints) in partially ordered sets, when equipped with their natural T_0-quasi-metric d, namely that a point $x \in X$ is an endpoint in (X, d) if and only if there is $y \in X$ such that x is a maximal element in $X - \uparrow y$. Similarly $x \in X$ is a startpoint in (X, d) if and only if there is $y \in X$ such that x is a minimal element in $X - \downarrow y$. We also showed that in a partially ordered set X equipped with its natural T_0-quasi-metric d its Dedekind-MacNeille completion $DM(X)$ is contained in its q-hyperconvex hull Q_X. In Proposition 4.4.1 and Corollary 4.4.1 it has been shown that for a partially ordered set $(X, \leq)$, (X, d) and $(DM(X), D)$ have exactly the same endpoints (resp. startpoints), where d and D are the natural T_0-quasi-metrics for X and $DM(X)$, respectively. We have observed that in a complete lattice the endpoints (resp. startpoints) in our sense are exactly the completely meet-irreducible (resp. join-irreducible) elements.

During our investigations of the materials in this dissertation we encountered the following open problems. We hope to study these questions in future work.

5.0.1 Problem. *Given a T_0-quasi-metric space (X, d) compare the set of endpoints (resp. startpoints) of (X, d) with the set of endpoints (resp. startpoints) of (Q_X, D).*

In this dissertation we established that these two sets coincide if $\tau(d^s)$ is compact.

5.0.2 Problem. *Given a T_0-quasi-metric space (X, d) when does it possess endpoints (resp. startpoints)?*

In this dissertation we established that a T_0-quasi-metric space (X, d) possess endpoints (resp. startpoints) if $\tau(d^s)$ is compact and X has at least two elements with a nonzero distance between them.

5.0.3 Problem. *Given a T_0-quasi-metric space (X, d) compare the q-hyperconvex hull Q_B of its subspace B consisting of its endpoints and startpoints with the q-hyperconvex hull Q_X of X.*

In this dissertation we established that the two sets are isometric if $\tau(d^s)$ is compact.

5.0.4 Problem. *Given a partially ordered set $(X, \leq)$ equipped with its natural T_0-quasi-metric d compare the j-irreducible (resp. m-irreducible) elements in $(X, \leq)$ with the startpoints (resp. endpoints) in (X, d).*

In this dissertation we established that the j-irreducible (resp. m-irreducible) elements in $(X, \leq)$ are exactly the startpoints (resp. endpoints) in (X, d) provided that $(X, \leq)$ is a complete lattice.

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