THE HISTORY OF MATHEMATICS AND ITS IMPLICATIONS FOR TEACHING

Thesis submitted for the Degree of Master of Education

by

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The current crisis in mathematics teaching shown by the general lack of achievement and interest in mathematics at all levels arises from fundamental misconceptions of the nature of mathematics and mathematical activity. Poor attitudes are generated largely by views of mathematics as entirely utilitarian, or as a collection of abstract structures, and ignorance of the way in which it arises from and influences human situations. A considerable part of the activity of the mathematician is concerned with the consolidation and systematisation of structures so that the original problem-situations are lost, and the essential dialectic removed. It is suggested that changes in attitude can be brought about by a deliberate attempt to introduce the historical-evolutionary dialectic presently omitted from mathematics teaching.

To this end, a number of versions of the nature of history and historical explanation are explored, and four approaches to the history of mathematics are defined. Views of the nature of mathematics and methods of teaching are examined, and some models useful for teaching and examining the nature and development of mathematics are offered for consideration.

Examples of developing mathematics show the dialectic at work, and four 'case studies' are given to demonstrate the relevance of history for the teacher at different levels in school and college mathematics.

A brief survey of reports on the teaching of mathematics containing remarks on the place of history of mathematics in the curriculum leads into a discussion of some past and current courses and examinations, the availability of sources, and proposals for courses for non-specialist and specialist mathematics teachers.

Finally, statements of how and what we learn from the history of mathematics are brought together in a series of Philosophical, Pedagogical and Methodological implications for the mathematics teacher at all levels.

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Note:

(i) Citations appear as:

Author Title Publisher/City Date for books

and

Author Title Journal Volume (Number) Year (Pages)

for articles.

(ii) Abbreviations where used are as follows:

Archive for the History of the Exact Sciences	Arch H.E.S.
American Mathematical Monthly	A.M.M.
Bulletin of the Institute of Mathematics and its Applications.	Bull.I.M.A.
Historia Mathematica.	H.M.
International Journal of Mathematics Education in Science and Technology	Int.J.Math. Ed.Sci. Technol.
Mathematical Gazzette.	Math. Gaz.
Mathematics Teacher. (NCTM)	Math.Tchr.
Mathematics Teaching (ATM)	Maths.Tchg.

(iii) Footnotes are numbered consecutively within each section except for Section 4 where each case study is given its own set of footnotes.

SECTION 1

Introduction

(a) Motivation

The motivation for this study arises from two main sources. First, the common experience of teachers and teacher-trainers relating to the majority of students in secondary and college education, namely poor achievement in mathematics. Whatever criticisms there are of the particular tests applied, there is general agreement in the results that a great many students at all levels are less mathematically able than we should expect. Considering the proportion of time given to the teaching of mathematics in schools, in these terms, we are clearly being much less efficient than we would hope. Further investigations suggest that behind this lack of ability lie many complex factors like examination systems, role expectations, teaching methods, aptitudes and attitudes.[1] My particular concern is, that while we can attempt to remedy lack of achievement by improving methods intended to teach content, this is only an 'ad hoc' move[2] and we require to look deeper into the nature of mathematics and the attitudes of teachers and students towards mathematics if we are to arrive at any more substantial answers.

The second source is within mathematics itself. It would be foolish to attempt a once-for-all definition

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1. Manchester University (1968) is an example of such a study.
2. This includes many 'new mathematics' syllabuses as well as suggestions for improving particular topics and classroom methods.

of mathematics, for we know that different ages, cultures and individuals have regarded it in different ways. This disparity is not necessarily a disadvantage, for in the very recognition of variety, we have a powerful tool for the widening of peoples' appreciations of mathematics. The communication of the ways in which this variety has evolved and relates to the needs of individuals and societies is a factor generally omitted from mathematics teaching. In fact, mathematics teaching has traditionally been more concerned with the communication of mathematics as a set of tools for the improvement of the intellect and the application to the physical world.[3] The omitted part of mathematics, the dialectic concerned with the communication of its variety, is largely a historical-evolutionary study and as such has not yet found a place either in the mathematics syllabus which deals with 'facts', or the history syllabus which continues to ignore a significant proportion of our scientific and mathematical culture.

In a way, mathematics is itself to blame for its own failure to communicate a large part of its essential nature, for the main activity of the mathematics communicator - the consolidation and systematisation of existing structures - deletes the original activity of the mathematics creator from the record. Briefly, mathematics

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3. ATM (1967) Chapter 1 contains a philosophy of mathematics teaching based on the nature of mathematics as an activity, which typifies some more recent thinking and is radically different from traditional views.

is a subject which defines away its past, and mathematics creators and communicators unwittingly subscribe to the continuation of this situation.[4] . . .

The hypotheses I propose, therefore, is that changes in attitudes (and consequently achievement)[5] can be brought about by a deliberate attempt to introduce the historical - evolutionary dialectic hitherto omitted from mathematics teaching.[6] Appreciation of the history of mathematics - the evolution of mathematics in our culture, and the development of mathematical concepts - show this dialectic at work, and can provide a means for teachers at all levels to develop attitudes towards mathematics which will help to make it more readily accessible to themselves and their students.

(b) Some general problems

Consider a description of some popular views of mathematics, necessarily crude, but designed to introduce

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4. I do not mean to imply that the roles of creator and communicator are mutually exclusive.
5. Naturally changes in attitudes might affect the manner and the content of the mathematics taught, so that any attempt to quantify a comparison of achievement may be fundamentally impossible, see, for example ATCDE (1965).
6. This dialectic is not the same as teaching the history of mathematics, although this latter can play an important part in its presentation. See below, Section 3 and Section 5 in particular.

some of the problems to be considered.[7]

Two popular views of mathematics are what I shall call the practical and the technical.[8] These distinguish the majority of users of mathematics in the following way. The practical view considers mathematics as a fixed body of knowledge which can be learnt and applied to various problems. Generally this requires enough arithmetic to enable ordinary computations to be done; simple monetary transactions, weights and measures and simple proportions and estimations; no algebra, and some rudimentary practical geometry of the type used in home carpentry, for example[9]. In addition to this unskilled use, we have the semi-skilled use of mathematical formulae in trades and crafts which often requires fairly sophisticated specialist knowledge of some areas of algebra, trigonometry and calculus, together with the ability to use particular calculating devices.

At this low level of mathematical culture parents might teach their children to count (in the most elementary sense),

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7. What follows is a development of the views expressed in Wilder (1965), p. 283.
8. I use the word practical to suggest the practitioners; the craftsmen, tradesmen, technicians and others using elementary mathematics and mathematical instruments. Technical implies that mathematics in one form or other is used for problem-solving in specific fields.
9. Arithmetic, Algebra and Geometry are used in their traditional meaning here.

and in some cases attempt to help with homework. Their general inability to do either adequately leads to distress and frustration for the children, parents and teachers, reinforcing misconceptions of the nature of mathematics and the intellectual capacities required to understand it.[10]

The technical view considers mathematics not only as a body of knowledge, but also as a collection of methods or models which can sometimes be improved and often adapted to solve specific problems. This view includes the former, practical approach, but has its specialist knowledge at a higher level, that of the technician, engineer or scientist, still using mathematics as a tool, with little appreciation of the originality or creativity of the mathematics itself. There is, however, an added dimension here, that of mathematical method, or problem-solving by mathematical modelling which is at least implicit in the ways that such higher-level mathematics is used.[11]

At this level of culture we expect most mathematics to be available for transmission but these seem no more successful than others in transmitting even the most

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10. A common experience of mathematics teachers is the adult who expresses some guilt at failure in mathematics at school.
11. Only in very few cases are pure mathematical models constructed and manipulated here. Polanyi (1964) and Davis (1967) make it clear that a great deal of 'knowledge of the art' is required for successful problem-solving in many fields.

elementary mathematics to their children;[12] higher mathematics is equated with harder mathematics and the same misconceptions are inherited. The practical and technical views are held by the majority of the population.

In contrast to these two popular views is another, the intellectual, which recognises that a well-established body of knowledge exists but is also aware of the growth of that knowledge, and the creation of new mathematics.[13] This view recognises the existence of particular schools of mathematical thought to which mathematicians may subscribe at different times, and concerns itself not only with the evolution, but also the origin of mathematics and the nature of mathematical activity. This view is held only by a minority, most of whom are specialist mathematics teachers. This group represents the highest level of mathematical culture and while it does not necessarily follow that the mathematics transmitted by them (outside their professional capacity) is any wider in content, there is a chance that

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12. The technical view, generally associated with the professional middle-class means that on the whole, children are generally better motivated to learn and parents to instruct them. The largest pressures on teachers comes from this social group since professional parents seek to preserve the social structure in which they have succeeded; but a full discussion of this is beyond the scope of this work.

13. Intellectuals here are intended to be those who think about mathematics and the mental processes involved in doing mathematics. This is in contrast to research in 'foundations' where the attempts to formalise mental processes have evolved into technical discussions.

the attitudes conveyed will be more sympathetic. Those engaged in mathematical research, form what might be called the body of <u>working mathematicians</u> who usually act like the technician, applying, developing and transmitting specialised sections of mathematics, but can adopt the attitudes of the intellectual. On such occasions, considerations may concern the methods, or the nature of the mathematics used, but usually only occur at critical points in the solution of a problem, or the development of a theory.[14]

Most original contributions to the body of mathematics derive from the working mathematician, but some can also be suggested by the technician who, on occasion, has invented procedures or even theories to solve his problems.[15] Little of this, however, finds its way into the general culture nowadays except in the rather special forms of 'modern'

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14. These critical points can occur at all levels, from the contemplation of a child solving a problem to the development of subtle changes in concepts or the bases of proofs (which generate arguments as to what is or is not mathematics, and what is or is not allowable procedure). For further discussion see Section 3.
15. For example the re-appraisal of the meaning of function due to Fourier's work, Dirac's specially defined functions for Quantum Mechanics, and the use of logical forms derived from Computer Science by Spencer Brown (1969).

mathematics taught in schools.[16]

The kinds of contributions made fall into two broad categories (i) those adding to the body of mathematical techniques and theories and (ii) those influencing mathematical philosophy as a consequence of these new techniques and theories.

The transmission of both of these kinds of contributions to the general culture is practically zero. This is not surprising since few mathematicians see this kind of communication as part of their function, indeed academic popularisers are regarded with some suspicion by pure scientists.[17]

Thus the image of mathematics conveyed to the majority of the population is largely practical and technical, and the popular image of the mathematician is one of cold precision with intellectual powers not given to the common man.

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16. It might be argued that the use of electronic calculators is an example of modern mathematics being transmitted to the general culture, but this is only a technological improvement which increases the power of the individual in mathematics already transmitted. Such devices do not necessarily increase understanding or appreciation, and can often increase the mystery.

17. The image of the mathematician has been a cause for concern recently, primarily because the low recruitment to mathematics courses and to mathematics teaching threatens the security of the academic. See Bondi (1975), Newsom (1972).

Investigation of attitudes towards mathematics in groups of students entering teaching by Manchester University School of Education shows that the majority consider it to be the study of numerical calculations with practical applications and at least half the sample view mathematics with disfavour.[18] We might say this implies the practical view is held with some idea of technical applications. Considering preferred age-groups and sex differences, it is no surprise to find that women who are potential primary teachers have the poorest attitude of all.[19] There is no reason to suggest that the situation has significantly improved.[20]

In a report by members of the ATCDE, one of the main factors investigated is 'second-order thinking'; described as 'The act of considering the child's reply in terms of the possible range of the child's thinking...'[21] The realisation

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18. Manchester University (1968)(37-52)
19. If this is any indication of the general position, the general view of mathematics (compared with other school subjects) conveyed by mothers to their children must be pitiable. See Kerslake (1974) Lumb (1974) and Rogers (1974) for a general discussion of female entrants to teacher education.
20. Recent reduction in teacher entry to colleges may raise the general standard, but that is no guarantee that mathematics will be any better off.
21. ATCDE (1965) P41.

that this type of thinking is possible and acceptable, in terms of one's own attempts at solving problems, as well as regarding other people, comes as a surprise to many. It is suggested that this second order thinking correlates with mathematical ability, originality and teaching skill.[22] If this is even approximately true, then the realisation that second-order thinking is possible in mathematics is a vital contribution to a positive attitude. One of the ways this might be encouraged is by emphasising the evolutionary process, and the cultural connections of mathematics.

The majority of teachers maintain and transmit the practical or the technical view either implicitly or explicitly according to their use of mathematics.[23] This, together with its abstract nature, reinforces the dehumanisation of mathematics and the belief in the unchallengeable nature of mathematical truth. Teachers of mathematics with the practical view are rarely able to engage in second-order thinking in mathematics, while the intellectual approach is beyond them. Those with the technical view are often in a similar position, though both groups may be capable of a high degree of second-order thinking within their own subject area.

Added to this, mathematics is a compulsory subject for a very large proportion of the time a child spends in

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22. Loc. cit.
23. This means teachers of all subjects; the historian by ignoring the history of mathematics, the engineer by conveying only rules.

school and is included in syllabuses for motives which are mainly utilitarian. Because of this utilitarian emphasis there is a great deal of technique to be transmitted and little time given to ways in which children and students can find out about mathematics.[24] Having to endure this situation can hardly encourage positive attitudes.

Balancing the formality of mathematics with our appreciation of its human and cultural aspects is one way in which we can introduce a broader view to teachers and children. This is advocated by a number of educators[25] and their proposals suggest three areas for consideration.

i) The development of mathematics as a body of knowledge.

This can be regarded as covering the history of mathematics, the evolution of mathematical ideas and the problems of what topics to choose and how to teach them at various levels. This might seem fairly straight-forward, the availability of reliable sources for the history of various topics has improved considerably over recent years, and more general interest is being shown by mathematicians and students in the history of their

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24. This does not necessarily mean that the teaching of techniques should be abandoned (see Section 3).
25. For example various points of view are found in Kline (1970) May (1971) Newsom (1972) and Aiton (1972).

subject.[26] However, while the situation in colleges and universities may be improving the position of the teacher in school is much more difficult. Not only is it hard to find time to include any form of history in the mathematics curriculum, but while the present school examination systems prevail, pupils are less likely to be interested in non-examinable or less immediately rewarding topics.[27] Within such a system action of official bodies is often required to give sanction to changes, and this is even more the case where radical changes might have to be made to introduce another subject into the curriculum.[28]

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26. I cite as examples the recent increase in frequency of articles on the history of mathematics and the evolution of subject areas published in journals and the willingness of the mathematical community to support Journals such as Archive for the History of the Exact Sciences and Historia Mathematica, and the Open University Course in History of Mathematics.
27. There have been examinations in history of mathematics at school level (see Section 5), but there is, to my knowledge, no syllabus in history of mathematics offered by any of the national school examining boards at the present moment.
28. For example, an official report supporting the teaching of the history of mathematics from a teachers organisation; or a revised syllabus, as now seems possible with the introduction of N and F level examinations. These are discussed in Section 5.

On the other hand there is a growing interest shown by practising teachers in the history of mathematics, and their major problem appears to be the availability of easily usable source of material.[29]

ii) The evolution of mathematical curricula.

This concerns the influence of the body of mathematical knowledge on the mathematics taught, and also the philosophical, social and economic influences that shape the demands and expectations that society makes of the educational system. This is usually a significant portion of a specialist mathematics teachers course but emphasis is given mainly to the structural aspect of mathematics and the recent changes in syllabus content, while the other aspects are usually neglected.[30] It is also less likely that the college teacher will have available the kinds of sources that give accounts of these latter aspects[31] particularly with regard to the large 'popular tradition' of mathematics education.[32]

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29. Knowing of its existence is the first step, finding it is quite another. Hallerberg (1969) and Rogers (1975) are two attempts to assist. (See Section 5).
30. One sees a large number of studies and comparisons of the content of various syllabuses but rarely one on their origins.
31. Two recent publications are welcome here: Griffiths and Howson (1974) and Frendenthal (1974) help to provide much needed background.
32. See Wallis P. and Wallis R. (1975).

iii) <u>The formation of mathematical ideas on the individual.</u>

The individual is here concerned both as learner and creator of mathematics. Again courses for teachers include some such study but the main emphasis lies in the area of psychology and not in epistemology or a study of the <u>origins</u> of mathematics.[33] Investigating the origins of mathematics we can be interested in both the evolution of mathematical forms[34] and the inception of ideas in individuals.[35]

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33. I wish to make the distinction between the history of mathematics and the origins of mathematics. The former consists of the study of problems, both practical and theoretical, and their solutions; the latter is the investigation of the fundamental forms of mathematics and the ways in which we can recognise their functioning in individuals. The origins of mathematics has an historical aspect in that we can be interested in both the evolution of mathematical forms and the inception of ideas in individuals. (For further discussion see below, Section 3.).
34. For example, the identification of the origins or the evolution of a mathematical concept as in Boyer (1946).
35. Both Hadamard (1945) and Koestler (1959) investigate this aspect, but neither seem to give a satisfactory account. Hanson (1958) on the other hand, produces some interesting ideas from the point of view of the philosophy of science. Popper's approach is discussed in Section 3.

Questions to be asked in this context include:
Does a knowledge of the history of mathematics suggest any general activities for the classroom which are likely to encourage fruitful mathematical experience ?
Does the history of mathematics help us to recognise the mental constructs or operations that constitute mathematical experience and are we thereby able to assist our pupils to appreciate the nature of mathematics without necessarily requiring of them a great deal of technical expertise ?

These three areas have raised a number of problems. The history of mathematics course might be better given when students have a sufficiently wide knowledge of mathematics to see the significance of the developments: On the other hand, is it possible to convey the spirit of the developments in a satisfactory way to those who are not mathematics specialists ? Is it legitimate to tell historical stories to children, and if so, can this be done in both an interesting and honest way ? How has the mathematics curriculum evolved, and what can we learn from a study of the internal and external influences that have caused this evolution ? Is it possible, for example, to isolate any fundamental factors which can be of use to teachers in planning curricula ? And lastly, can the investigation of history provide any data for the study of the origins of mathematics ?

None of these questions can finally be answered here. The best that can be done in the following pages is to survey what has already been achieved and from this suggest a number of lines of investigation which may be worthy of further study.

(c) History of mathematics and the Methodology of Mathematics Teaching.

Having defined the general problem areas we now turn to some of the available material to see which questions have been tackled and how the history of mathematics has influenced the teaching of mathematics.

The simple view of history, as a 'study of the past', gives the impression that past mathematics has little to do with the present. The reasons for this are as much to do with a basically chronological view of history[36] as with the belief that only present mathematics is correct. Intrinsic interest in the biography of individuals, or the history of mathematical topics is not sufficient to persuade holders of the practical or technical view of mathematics that there is any long-term advantage in spending time in the study of the history of mathematics.

Aspects of mathematical education that appear in contemporary courses include the learning of algorithms, practice in problem solving, appreciation of structures and possibly some opportunities to create original mathematics. Rarely, however, do we find opportunities for discussing the activities of mathematics, or finding out <u>about</u> mathematics.[37]

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36. Further discussion of the nature of history appears in Section 2.

37. I do not deny that this occurs in some classrooms but there is little official sanction given to this kind of activity at school level, largely due to the problems of organising and examining such activities.

It seems that in general the exclusion of history of mathematics from school syllabuses[38] is due not only to the apparent irrelevance of past mathematics for present topics, but also to the fact that a study of its history can lead to the discussion and investigation of the nature of mathematics.[39]

If we consider the range of aspects of history available we have a two-level structure, the first dealing with the 'facts' of history, and the second with the theories of historical interpretation.[40] On the first level we can identify biography, and the reference to the first appearance and content of proofs, papers, books and theories, a kind of 'who discovered what' approach, the main emphasis being chronological. On the second level we have a comprehensive discussion of the development of particular topics and the general evolution of mathematical ideas where the emphasis lies more on philosophy.[41]

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38. There have been school examinations in history of mathematics. These will be discussed further in Section 5(a) below.

39. Time is another factor but if any subject is agreed to be relevant time will be found to teach it.

40. If teachers have only the practical or technical views or are unable to engage in second level thinking their confidence in discussing such matters will be impaired.

41. Crudely, chronology investigates historical ordering while philosophy investigates conceptual ordering.

Clearly to be in a favourable position to study the history of mathematics the student (at whatever level) must be able to appreciate some philosophical aspects sufficiently well. This means that either history of mathematics should be excluded from courses because students are incapable of philosophical activity, or that it could be available in some form in order to encourage the development of such awareness and widen the student's experience of the nature of mathematics.[42]

For the teacher an added dimension appears. Since much of the teacher's interest lies in the manner and order in which mathematical concepts are developed in the individual, any information which helps to place contemporary mathematics in a wider context, whereby the relative importance of concepts and areas of mathematics may be judged, can be most useful. While psychology may suggest a pedagogical ordering for certain mathematical concepts, this can be tempered with a knowledge of the origins and nature of the mathematics itself so that alternative interpretations may be possible.[43] Teachers need to be aware of the changes in mathematics and the ways in which these have influenced both the content of syllabuses and the views of psychologists. The interest in the development of mathematical concepts for teaching purposes is relatively

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42. Suggestions for suitable forms at school and college level are to be found in Section 5 below.

43. For example, see Section 4(b).

recent[44] and signifies an important phase in the evolution of mathematics teaching.

The change from private and piecemeal mathematics instruction to nationally organised mathematics teaching began in the latter part of the nineteenth century. Commercially based interests[45] gave way to more intellectual ideals as mathematics became part of the public school curriculum, to be accepted, with classics, as an instrument of mental discipline. This mathematics included arithmetic, algebra, and the geometry of Euclid; the latter a requirement for university entrance which dominated school mathematics and the teaching of geometry in particular, until the early part of the present century.[46] The fight against Euclid, initiated by the Association for the Improvement

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44. 'Formation of the Association for Improvement of Geometrical Teaching in 1871 marks the beginning of collective awareness by teachers.

45. In the seventeenth and eighteenth centuries the popular tradition was strong and mathematics instruction widespread and in the hands of many 'practitioners' who taught the arts of surveying, navigation, accounting etc. See Taylor E.G.R. (1954) and Wallis (1975).

46. Significant developments in school mathematics in the nineteenth and early twentieth centuries are sketched by Siddons (1936) and (1956) and Wilson (1921) while Hollingdale (1974) describes some changes in University mathematics of the time.

of Geometrical Teaching, (A.I.G.T.)[47] appeared to be motivated by a wider appreciation of mathematics as well as by the interests and abilities of the learner.[48] Gradual introduction of new content and new approaches into a systematic course aimed to show mathematics as a connected whole, and a number of texts were written for schools by well-known mathematicians in the spirit of this reform.[49]

About this time advocates for the inclusion of history of mathematics in school syllabuses appear but their arguments are not very forceful.[50] Later, a more fruitful

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47. Founded 1871. Rawdon Levett's letter to Nature of 26th May, 1870 appears to be the first public statement of the aims of such an association. See Griffiths and Howson (1974) p.128.
48. Wilson (1921) (243-344)
49. Wilson's Geometrical Texts were first, but Lamb's Calculus, Lamb (1897) (Reviewed in Math. Gaz.1 (13) 1898) is perhaps the most enduring of these, the last reprint being 1949.
50. The main thrust of the early reforms were directed to mathematical content and organisation. When history was suggested to occupy only an 'outside illustrative position' (Heppel (1893)) it stood little chance of being taken seriously.

approach was initiated by Cajori[51] who produced a text intended for teachers where he claims that teachers should not only have an acquaintance with the history of mathematics, but that it provides material relevant to the classroom, both in the explanation of content and in the application of the 'Biogenetic Law'.[52]

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51. Cajori (1896) 'A History of Elementary Mathematics with <u>Hints on Methods of Teaching</u>'(underlining mine). The 'hints' are general remarks on the importance of practical work, observation and experiment, the logical and psychological difficulties of some texts (especially geometry), and the legitimate use of algebra or arithmetic to solve a geometrical problem.

52. The 'Biogenetic Law' claims that 'the genesis of knowledge in the individual follows the same course as the genesis of knowledge in the race' or 'Ontogeny recapitulates Phylogeny'. Cajori (1896) in his preface quotes Spencer (1894) p.122. who attributes the first enunciation of this principle to Comte. See Cavenagh (1932) (78-87). Lakatos (1963/4) p.6. attributes the idea to Haeckel. Mathematicians of considerable standing at the time use this principle to justify the use of history of mathematics, for example Poincare (1908) p.437, and Klein (1932) Vol. 1. p. 268, to suggest that mathematics instruction should in general be modelled on historical developments.

The work of D.E. Smith on the Teaching of Elementary Mathematics[53] has a strong historical bias. He intends to help teachers ".. to know something of these great questions of teaching, - Whence came this subject ? Why am I teaching it ? How has it been taught ? What should I read to prepare for my work ? The subject is thus considered as in a state of evolution, while comparative method rather than dogmatic statement is the keynote."[54] The growth of arithmetic, algebra and geometry is considered in turn, so that some statement of the nature of these subjects and reasons for teaching them may be made. Surveys of teaching methods then suggest necessary revisions of the syllabus and the approaches that might be used. The final chapter "The Teachers Bookshelf"[55] contains references to books and periodicals in the History of Mathematics.

While Cajori's work was clearly a version of the elementary part of his larger History of Mathematics[56] with some added sections on school texts and teachers' organisations, and a largely implicit philosophy in the 'hints', Smith's is much more down to earth - a concise

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53. Smith D.E. (1900) The Teaching of Elementary Mathematics the first American textbook on the teaching of mathematics (Bidwell & Clason (1970) p.120). His better known History of Mathematics (1925) and Source Book in Mathematics (1929) were largely motivated by a desire to make both the history of mathematics and some of the original work of mathematics more widely accessible.
54. Underlining mine. Smith (1900) p.vii
55. Smith (1900) (297-305)
56. Cajori (1894)

argument from a trainer of teachers. The general philosophy of Smith's work is indicated by the following remarks from the Editor's introduction, "Mathematics ... studies an aspect of all knowing and reveals to us the universe as it presents itself, in one form, to mind. To apprehend this and to be conversant with the higher developments of mathematical reasoning, are to have at hand the means of vitalising all teaching of elementary mathematics. In the present book, the purpose of which is to present in simple and succinct form to teachers the results of mathematical scholarship, to be absorbed by them and applied in their classroom teaching, the author has wisely combined the genetic and the analytic methods. He shows how elementary mathematics has developed in history, how it has been used in education and what its inner nature really is."[57]

Smith's use of history, to investigate the 'inner nature' of mathematics and then to see how best this can be communicated, has much to commend it.[58]

The influence of new attitudes to learning is clearly seen in the works of Benchara Branford.[59] Here the central preoccupation is the balancing of theoretical and practical work where "... all the branches of elementary mathematics, pure and applied, theoretical and experimental,

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57. Smith (1900) (xi-xii)
58. The method of investigation is fundamentally independent of any particular philosophy of mathematics or of teaching. It is the data from which these are constructed. See below. Sections 2 and 3.
59. Branford (1908)

are co-mingled at appropriate times, so that the mind sees and uses its mathematical conceptions and processes as a beautiful, well-ordered and powerful whole, instead of a thing of shreds and patches."[60] Intending to give practical and theoretical advice to all teachers of mathematics, he suggests a realistic approach and states that, "All educational principles are, in effect, ideals; and the degree in which they are realisable must depend upon actual circumstances... The realisable is ultimately the resultant of two forces - the strength of the ideal and the resistance of the actual."[61] He is of the opinion that mathematical history has had little influence on teachers, and "... has rarely been interpreted as an integral part of the historic movement of racial experience.."[62] In view of his advocacy of the biogenetic law, a knowledge of history is essential to teachers. The influence of current theories of child development[63] and the recent struggles of the A.I.G.T. are clear in his statement of some of the fundamental conditions of teaching:

"I. The particular mathematical experience which forms the material of the educational process must, at every stage, both in quantity and quality, be appropriate to the

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60. ibid p. (vii-viii)

61. ibid p. (ix) Branford was a school inspector at this time, in a good position to know the practical and beauracratic difficulties facing a zealous teacher.

62. ibid p. (x) We might share this opinion even today.

63. For example the works of Pestalozzi and Froebel were known at this time.

to the present capacity of the individual who is expected to assimilate it.

II. The correlation between the different branches of past mathematics themselves, and between these latter and the manifold applications of mathematics must be natural, closely interdependent, interesting and continuous throughout."[64]

These conditions are achieved in the application of the biogenetic principle (called culture-epoch principle by Branford): "The path of most effective development of knowledge and power in the individual coincides, in broad outline, with the path historically traversed by the race in developing that particular kind of knowledge and power."[65]

Obvious and simple as it may seem, this principle can be attacked on psychological, historical and mathematical grounds. Likening the infant's mind to an animal's and judging past mathematics in terms of contemporary theories for example, are views unacceptable today. However, whether such theories are acceptable or not, the appearance of this book, and others of a similar nature shows that the teaching of mathematics is becoming a significant field of study.

The use of history in teaching mathematics is further developed by Toeplitz[66] who advocated the 'genetic' approach in his textbook on calculus. The genetic approach was an attempt to answer questions concerning the motivations for theorems, definitions and techniques in mathematics textbooks and syllabuses by showing their historical derivation.

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64. Branford (1908) p.243

65. ibid p.244. He claims the validity of this principle is borne out by teachers' experience as well as the opinions of a number of philosophers, scientists and educators.

66. Toeplitz (1963)

Toeplitz takes the important mathematical ideas in the context of contemporary development as concepts to be explained by examination of a number of aspects of a particular concept as it developed in the past. His aim is "... to select and utilize from mathematical history only the origins of those ideas which came to prove their value... It is not history for its own sake in which I am interested, but the genesis, at its cardinal points, of problems, facts and proofs."[67] Thus the content, structure and level of rigor is decided by contemporary standards, and History is used to provide explanations of the 'why' and 'how' questions that arise.

For example, the first chapter on "The Nature of the Infinite Process"[68] begins with Greek infinitesimal ideas, the appearance of irrationals, and an intuitive theory of infinite processes. The theory of proportion is then introduced to motivate a continuity axiom which makes the method of exhaustion rigorous. He claims the modern concept of number has its origins in Greek mathematics, and Archimedes' measurement of the circle is seen as the first step in the development of trigonometric formulae and infinite series for the calculation of functions of angles. All this is a preface to the real purpose of the chapter, a discussion of infinite series and modern definitions of limits and convergence. Toeplitz' aim is a clear exposition of what he considers to be the basic concepts and hence does not always either follow the historical development or use historical ideas.

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67. ibid. p.(v). The outline of his idea was first given in 1926.
68. ibid (1-42)

While the advantages of this approach encourage students to realise that mathematics has a history, the disadvantages include the strong implication that past mathematics was only an imperfect form of present theory, and the deletion of a great deal of material, both mathematical and cultural, due to the prevailing fashions in theory and intentions of the author.[69]

The pedagogical lectures of Felix Klein, translated as "Elementary Mathematics from an Advanced Standpoint"[70] first appeared in 1908 and contained a large number of references to the history and development of mathematics. In considering the content of mathematical curricula, Klein identifies two approaches: "Plan A is based on a more particularistic conception of science which divides the total field into a series of mutually separated parts and attempts to develop each part for itself ... its ideal is to crystalise out each of the partial fields into a logically closed system ... Plan B lays the chief stress upon the organic combination of the partial fields, and upon the stimulation which these exert one upon another". The ideal here is "the comprehension of the sum total of mathematical science as a great connected whole."[71] He compares the current school syllabus to 'plan A' and advocates the adoption of 'plan B' as a basis for reorganising the content and methods of mathematics teaching.[72]

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69. For further remarks on fashions in mathematics see Section 2(b).

70. Klein (1932/9)

71. Klein (1932/9) Vol I p.78. Underlining in these quotations is italics in Klein's text.

72. Klein is, of course, referring to German school syllabuses but his remarks could well have been echoed in England.

In order to gain a 'complete understanding of the development of mathematics', Klein introduces a third 'plan C' which is the algorithmic process, described as "a quasi-independent, onward-driving force, inherent in the formulas, operating apart from the intention and insight of the mathematician, at the time, often indeed in opposition to them."[73]

Using these three plans, he then proceeds to analyse mathematical history, stating the dominance of plans 'A', 'B', or 'C' at different times in the past.[74] His closing remarks in this section suggest that since school mathematics has been for so long under the influence of 'plan A' any reform must come from a swing to the most universal 'plan B' where: "In this connection I am thinking, above all, of an impregnation with the genetic method of teaching, of a stronger emphasis upon space perception, as such, and, particularly, of giving prominence to the notion of function, under fusion of space perception and number perception !"[75]

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73. ibid p.79.
74. ibid (79-85). This particular kind of exercise can be regarded as a 'mathematical' analysis of history from Klein's own viewpoint. A similar analysis appears in Bontroux (1955) who appears to have had some influence on Piaget. See Section 4(c).
75. ibid p.85. The translator of Toeplitz (1963) connects Klein's plan with Toeplitz' intentions in the description 'genetic'.

Not only do we have here a belief in the relevance of history for mathematics teaching in general, but also a clear indication of the importance of the structural viewpoints for the improvement of curricula derived from a wide knowledge of the developments in nineteenth century mathematics.

While Klein's suggestions found acceptance because of his stature as a mathematician, other influences from outside mathematics were already at work contributing to the cultural background and easing the task of the innovator. The theories of Froebel and the work of Pestalozzi have already been mentioned,[76] but apart from educational theory some purely political and economic decisions had been made by governments to reorganise their education systems and provide the means to supply their nations with mathematicians, engineers, scientists and soldiers.

The foundation of the Ecole Polytechnique after the French Revolution[77] and the reorganisation of Prussian education in the early nineteenth century[78] are examples

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76. See Note 63.

77. Napoleon officially recognised the importance of mathematics to military training, particularly in artillery and engineering. We associate many leading French mathematicians with this establishment, both as teachers and as former pupils.

78. Gerstell (1975) suggests a list of some fifty mathematicians from 1716 to 1875 who could have been encouraged in some way by such a move.

of the encouragement of the development of mathematics for political ends.[79] Although a large amount of pure research derives from such situations the view of the politician is largely practical or technical, in desiring the results of the applications of mathematics. Thus, while the status of mathematics in the culture may rise, the general view of the nature of mathematics for the majority remains a practical one.[80]

What teachers need to realise is that mathematics teaching has a history and that a study of the influences on the curriculum and methods of teaching both from within and without mathematics can provide insights which may be useful in the wider professional sense.[81]

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79. It would be rash to suggest that the work of particular mathematicians is the result of such political decisions, the most outstanding would probably have flourished in any case. On the other hand, official encouragement has an obvious effect upon an individual's circumstances, and the general status of mathematics in the culture.

80. The social history of mathematics cannot be investigated in detail here. Wilder (1965) and (1968), Curtiss (1937), Dresden (1942) and Fisher (1966) contribute in various ways to the idea that mathematics does not arise only from internal problems or from problems associated with science, industry or commerce. Further remarks on this subject can be found in section 2.(b)

81. Some fundamental arguments will be sketched in Sections 2 and 3, while Griffiths & Howson (1974) contains much material of a more general nature.

It is easy to collect recommendations for the value of the history of mathematics for teachers and pupils,[82] and a summary of these may be made under the headings of Chronological, Logical and Pedagogical reasons.[83]

Chronological Reasons

These deal mainly with historical facts about definitions and names and the general sequence and timing of

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82. An early exhortation is Heppel's (1893) address to the A.I.G.T., and a number of reports of the Mathematical Association contain reference to the usefulness of historical material in specific contexts, for example in the Secondary School (1959) or in Calculus (1951). These uses tend to be either as biographical anecdotes and general background, or fairly specialised subject history. Notable exceptions are the M.A. Report on Mechanics (1965) where some of the wider issues are suggested, and the I.A.A.M. (1957) where the sections on history and philosophy of mathematics, emphasise history and questions of the nature of mathematics as a part of mathematics teaching. The Ministry of Education (1958) pamphlet contains and elaborates many of the previous arguments and concludes (p.154) that mathematics can only be taught and understood properly against a background of its own history. These points will be reviewed in Section 5.

83. These headings are taken from Jones (1969) and the following summary includes the reasons given in Ministry of Education (1958) (134-154).

discoveries. Here we find reasons for the psychological basis of mathematical systems, the necessity of definitions and the arbitrariness of undefined terms. For example, the evolution of the meaning of the word 'number' can be traced, showing the different ways it has been interpreted or understood.

Logical Reasons

Here we consider the derivation of axiomatic systems and mathematical structure in general, the development of forms of proof[84] and the mistakes, paradoxes and controversies that have arisen in mathematics. Illustrations here include many techniques used to solve problems long before the methods employed were supplied with logical foundations.

Pedagogical Reasons

The reasons given here fall into three areas concerned with content and methods, cultural connections and the nature of mathematics.

Content and Methods include the ideas that historical knowledge may help in the selection, presentation and connection of mathematical topics in the curriculum, and that knowledge of patterns of discovery may improve a teacher's heuristic method. Knowledge of history also provides illustrations of the development of mathematical models and the applications of ideas unforseen at the time of their inception.

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83. These headings are taken from Jones (1969) and the following summary includes the reasons given in Ministry of Education (1958) (134-154).

84. An outstanding example from a largely logical point of view is given by Lakatos (1962/3).

<u>Cultural</u> connections can be made by discussing the relations between mathematics and other subject areas and the various stimuli for mathematical ideas. The way in which mathematics has influenced the culture, both in its applications and modes of thought are also to be included here.

The mathematicians views of the <u>nature of mathematics</u> change with time and the realisation of this fosters intellectual curiosity and dispels fear by suggesting that mathematics can be questioned.

These claims all raise further questions:

i) What is the nature of history that it is claimed to be such a significant yet absent factor in mathematics education, and what general approaches are possible ?

ii) How does mathematics relate to our culture and how are we able to describe the evolution of mathematical concepts in cultural terms ?

iii) Does a critical philosophy of mathematics, which includes historical data, provide models useful for the teaching of mathematics ?

iv) Is it possible to isolate any fundamental ideas, activities or processes from mathematical history which may be used as a basis for the development of the mathematical curriculum ?

These questions will be examined in more detail in the following sections.

SECTION 2

From History to History of Mathematics

a) 1. Understanding and Explanation.[1]

The process of understanding and explaining events is an activity common to history and to science. Since the growth of the interest of scientists in History of Science and the Philosophy of History, attempts have been made to formulate a philosophy of history which is seen to be 'scientific' by employing scientific method to formulate generalisations about historical events.

It seems to me that attempts to produce a scientific method for history (at least in the traditional sense) arise from misunderstandings about the nature of the activities of understanding and explanation in history and in science. Most of the study of history of science is carried out by scientists who carry over much of the methodology from science to history. The criteria for judging 'good' from 'bad' history of science are often more scientific than historical.

In order to produce a scientific method for history it is argued that the nature and purpose of understanding and explanation are the same both in history and science. I claim that not only are they different, requiring different modes of thought, but that in the traditional sense

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1. The notes in the first parts of this section on the nature of the study of History and of History of Science follow the different types of historical explanation to be found in Dray (1957), Walsh (1958), Carr (1964) and Hempel (1965). The extensions of these ideas into History of Mathematics are my own.

'scientific method' produces a restricted view of the nature of scientific enquiry.

This discussion leads into one concerning the nature of the history of mathematics, suggesting reasons why one might study it, four styles of interpretation, and what one might learn from such a study.

2. Events and explanations.

History is concerned with the study of the human past and is interested in the activities of individuals, but not totally or exclusively in their activities. These activities are of interest to the historian only insofar as they have what one might call significant consequences. It is, of course, part of the historians task to decide what the significant consequences are and which activities led to them.[2] It is said, then, that the task of history is to establish and understand the facts. This involves the historian in giving explanations.

Because of the apparent similarity with science, it is held by some that we are able to use historical explanation, the explanation of past events, as a guide to future action. (i.e. that in some way we can learn from history.)

We may be able to learn from history, (what we learn and how will be discussed later) but this first crude generalisation is generally considered to be unjustified.

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2. It is worth noting that since history can be said to deal with unique events, the significance of events and their consequences may change radically due either to new evidence being discovered or a change in the culture, attitudes, philosophy etc. of the historian. In this sense history is less permanent than science. (i.e. historical explanations can be more easily changed than scientific ones.)

3. Explanations and Laws.

Scientific explanation concerns events which are predictable since they are considered to occur as instances of general laws. These general laws are considered to be universal empirical statements, and their construction conforms to a standard deductive pattern. Having formulated a law in science we are then in a position to say when a particular series of events may lead to a certain outcome. This at least is the standard picture of scientific explanation and the model to which scientific history should be considered to conform.[3]

Historical explanation aims (in some respects at least) at showing that an event was not merely a matter of chance; that in some sense it had to happen, or that it was not surprising that the event did happen.[4] If this is true historical explanation has at least a deductive aspect, even if some would not admit it to be wholly deductive. This kind of explanation gives reasons (a reasonable explanation) for the particular course of events turning out as it did. Clearly, one can argue that a certain amount of hindsight is necessary to furnish explanations at this level.

It is important to note that most historians claim that prediction is not attempted in an historical context - what predictions occur are really in some other area. A large part of economics, for example, can be thought of

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3. It might be argued that we also make generalisations when we are unable to produce a law.
4. Other ways of considering historical explanation will be discussed later in this section.

as history in this predictive sense.

This kind of explanation I will call retrospective-deductive, because it is concerned with demonstrating why a particular event or series of events had certain consequences. This is only possible after the events have occurred, by the very nature of historical enquiry. Historical explanation does not deal in generalisations or universal statements.

4. The Uniqueness of Historical Events.

The laws of science apply to kinds of events, and aspects of the events so classified are essentially regular and repeatable.

It is argued that classifying events in this way leads to a destruction of history. Historical events may be classified (for example, a 'revolution') but the kind of classification involves a general conceptual framework, or a structural procedure (the French revolution, Industrial revolution, Copernican revolution etc.), rather than a precisely identifiable sequence of events.

With the emphasis on the uniqueness of historical events one mode of explanation has been to try to give more complete details of the events. (This, taken as an ideal, is the basis of the Inductive theory of history. The more complete detail we give, the more inevitable or obvious the result.) This position may at first appear attractive but is philosophically untenable if only because of the problems it poses concerning the nature and status of what may be admitted as detail or evidence.

The historian wishes to achieve a reasonable explanation (retrospective-deductive) on the strength of what in the scientists' terms is minimal evidence (i.e. only one event). In giving this explanation all kinds of scientific laws are implied. The historian however, is not making laws, but using them in a rather special way. It is the intention with which a law is used that decides the status of the result of its application, and the laws of science, as applied in history (e.g. in sociology, economics, psychology, physics, astronomy, meteorology etc.,) are generally used in a secondary - even unimportant - role to make the explanation reason - able. In many cases the 'laws' invoked are more like commonsense principles than scientific facts.

It may be argued that such a conglomeration of scientific laws will in fact produce a logical conjunction which may be regarded as a new law, manufactured, as it were, from the others. Apart from the logical difficulties involved, we should bear in mind that it is not the historians intention to manufacture such laws, least of all to test them in anything approaching a scientific way.[5]

The scientist is not interested in events as unique events but as prospective members of a class of events to illustrate a universal statement. In a rather special sense, also, the universal statement may be anticipated before many events are investigated. Scientists admit to concentrating on particular aspects of complex events in order to obtain repeatable observations.

The historian, interested in the unique event, is

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5. There is also the possibility of obtaining an explanation reasonable to an historian, but which might use laws deriving from logically incompatible scientific theories.

also interested in the complex detail surrounding them, as a means to giving, ideally, a complete explanation, or more realistically, a plausible story, but the ideal of historical explanation suggested is never reached.

5. The Flow or Continuity of Events.

Another aspect of historical explanation is to consider the flow or the continuity of events. (The word pattern is often used in this context, but obviously in a very different way from the use of pattern in establishing a scientific law.) Another way of suggesting the structure sought here is to use ideas like the interweaving or the interrelations of events. The historians intention here is to show how one event follows from another, or how one follows (i.e. is the result of) many others. While the historian may be able to give what he considers to be a reasonable explanation, it is difficult to define precisely how events are related to each other, and so any scientific explanation meets with difficulties.[6]

As before, a complete explanation is impossible in principle, not only are the precise connections between the events impossible to define precisely, but much of the (historical) evidence is lost.

6. The Rationality of Actions.

This kind of historical explanation consists in the historian somehow putting himself in the mind of the person (or persons) involved, or knowing someone else's thinking. The historian here must re-think the agent's thoughts to

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6. This is another aspect of retrospective-deductive explanation where the historian is using general laws to give a reasonable explanation.

see - from the agent's point of view - what was the thing to do at the time. If this is possible to any degree, much of the historian's activity consists in absorbing the culture of the period in question and getting to know the problem in a rather special way.

In general this will not do. Few historical actions were the results of reasons consciously entertained by the agent, and some are well-known to be the results of quite irrational actions in the normally accepted sense. Another difficulty here is that historians talk of nations, institutions, movements etc., and would have to account for this collective thought in terms of the thoughts of the individuals. It seems that this mode of explanation poses many problems for the historian.

On the other hand, it looks as though this is what history of science tries to do. History of science attempts to show that the theories of the scientists in history were the result of rational thought (by the nature of the activity we call science) and tries to discover what their considerations were.

The rational action to be considered here consists not only of consciously entertained ideas, cultural influences, deductive logic etc., but also the mental functions of induction, intuition, etc. which help the scientist to make discoveries.[7]

A naive assumption often wrongly taken as a basic principle of history of science is that it is possible to know what an individual was thinking about. (i.e. exactly how a scientist arrived at an argument or made a discovery.) This is all the more plausible when we consider the nature of science. Scientific events are well-defined (at least in principle), and limited to particular aspects of the world. Furthermore, they are defined in such a way as to be repeatable in terms of specified observations. All this might suggest that the activities of past scientists are the same (in principle) as those of present practising scientists. Because science is apparently so limited, it is assumed that the science (or scientific activity) of the past is the same as contemporary science. This leads to a situation where the <u>past activity</u> and its results is judged by the <u>present criteria</u>.

In considering the history of mathematics we may regard it as an investigation into mathematical thinking, and in view of the discussion above, trying to find out what an

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7. Kepler's discovery of eliptical orbits is an interesting example. We see in his accounts how influenced he was by the Greek-Copernican circles, and of his struggles to fit the evidence to the theory. It was only after many years that he was able to break out of the circles and 'discover' the ellipse. His own account of his mental struggles is unique and is the kind of ideal explanation that is often suggested here. An account is given in Koestler (1959)(313-315).

individual mathematician was thinking about.

This interpretation of the history of mathematics encounters the same difficulties as those of history of science. Judging past mathematics by contemporary standards makes even the most significant breakthrough look trivial. Moreoever, there is also the question of the status of mathematical thought, activity and results.[8] The kind of evidence we are dealing with in the history of mathematics are the results of a mental activity. The results written down are the end of the story: if we are lucky we have something of the starting points., the original problems, but what can we say of the path between ? Attempting to answer this question poses enormous difficulties when we have contemporary mathematicians to talk to; with historical figures it is impossible.

If we are able to demonstrate the rationality of a scientist's or mathematician's actions in an historical context completely, we will then have solved the induction problem. Since this demonstration of rationality of action can be related to the whole question of providing explanations in history, it is clear that another rationale for the study of history must be found.

7. Non-Universal Generalisations.

The following are other attempts to describe situations where historians may be engaged in forming weaker versions of the scientific law.[9]

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8. These are discussed in Section 3.

9. These attempts, like the ones above, try to cast the historian in the ideal mould of the scientist, to make history respectable, as it were.

(i) A statistical law is an explanatory generalisation which is not invalidated by the occurrence of one or more counter-examples or exceptions to the law. (An interesting argument arises here concerning the status of scientific laws within science itself. It can be argued that all scientific laws are statistical laws, and that their certainty is only deduced by the fact that no counter-examples have yet occurred. In fact, in many scientific laws counter-examples do occur and they are either ignored, or the law is changed to accommodate them. The fact that a counter-example may be found does not invalidate the law in any case; it only means, logically, that the exceptions have turned up sooner rather than later.)

(ii) Laws of normal circumstances. These can be considered 'common sense' laws, or working generalisations from immediate experience, past or present. These might conform to an accepted standard of behaviour. If so, it is difficult to see them as different from rational explanations considered above.

(iii) Limited or restricted generalisation. These are intended to be laws based on a deep knowledge of the period being studied, and applicable only to that period. What may be true of one period may not be so of another.

These weaker versions of law-like statements seem to be rather 'ad hoc' procedures to patch up an inadequate theory, but it is claimed that whatever we may think of them as inadequate, they have a certain counterfactual force.[10] I think this is taking logical justification too far. How do we know, for example, that a limited general-

10. i.e. they imply something about events that might have occurred but did not.

isation and its implications are true ? The truth of the generalisation is supposed to follow from the conditions of the period. Testing the truth of this brings us back to the application of supposed laws.

(iv) Generalisation applied to a named individual. This idea is based on the characteristics of the more important agents. Leaving aside the problem of deciding who the most important agents were, we have a situation where the explanations of events are based on the explanations of the dispositions of individual persons. Simple dispositions, (for example Newton's avoidance of controversy may be cited as a reason for his not publishing the Principia until 1687), may be suggested, but even simple dispositions are more complex than they seem at first, and to attribute an individual's action to a particular tendency means we also need to check this action against all other possible reasons to make sure this was the most reasonable. This procedure is unlikely, even if theoretically possible and so these so-called dispositions commit the investigator to further assumptions.

8. The kinds of questions answered by History.

It was stated earlier that the historian is interested in the significant consequences of actions of individuals. If this is so, the task of history is seen to be deciding what the significant consequences are, establishing and understanding the events leading up to them and furnishing rational explanations for their occurrence.[11]

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11. That the actions of an individual in history might not be rational does not mean that the explanation of these actions is not rational. 'That he was mad' is a perfectly rational explanation.

The significant consequences chosen for investigation by the historian can often depend on the historian's cultural background and attitudes, particularly so if the object of study is the growth of an institution or an idea rather than some more well-defined event.[12] In these cases, the significance of the consequences of an individual's action or thought is seen in the light of the contemporary state and importance of the concept.[13] This is particularly true of the history of science: although we may be careful to try to judge the value of an individual's contribution by standards of his own time, the choice of what we examine is largely determined by our own ideas of what was (or is) seen to be important. The meaning of understanding in this context is that we suggest a chain of events or pattern of thoughts from one significant stage to another. It is often the case that different historians form different chains, in fact it is, in a sense, their business to do so and offer these different explanations to the public. These explanations cannot be tested in the ordinary sense because the experiments are unrepeatable.

This is a very different activity from the scientist who hopes for uniformity of explanation by repeatable

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12. A 'well-defined' event could be the outcome of a battle for example.

13. The calculus, and all that has sprung from it, forms a large part of contemporary mathematics. It would generally be admitted that these significant consequences can be regarded as the result of the actions of men like Newton and Leibniz.

experiments in order to predict the outcome of similar circumstances. In his activity, the scientist is asking a 'Why ?' question. 'Why does this happen ?' 'It happens because...' The historian on the other hand, is asking what I call a 'How ?' question. 'How did this happen ?' 'It happened like this ...' The intention of the scientist is to explain classes of events, the intention of the historian is to explain unique events. The unique events are seen to be reasonable in that they conform in general to the known laws of science, and so they can be deduced by hindsight as it were.[14]

The 'How ?' question was first used above in discussing the flow or continuity of events. The 'How ?' question of history is answered not by showing that an event <u>had</u> to happen, but by showing that there was <u>no reason to suppose that it should not have happened</u>. The whole emphasis has changed.

This interpretation of understanding and explanation poses many fewer problems than the 'scientific method' approach. We are able to consider the growth of ideas, arguments of individuals, the formation and disollution of societies, the publication of journals etc., in a much more fruitful way. Since the problem here is not to demonstrate the necessity and inevitability of events, there is no need to subsume the occurrence of events under a law of any kind. Here we demonstrate the <u>possibility</u> of the event by removing the basis for the expectation that it would not happen.

The historian may then go on to ask 'why ?' questions but there is no necessity to do so. The answers to these

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14. This is why I have called historical explanation retrospective-deductive.

questions lie more within the fields of sociology, economics and so on, where scientific method is more likely to be applicable.

9. The Interpretation of Historical Events.

We see now that in view of the preceeding discussion and the present idea of the 'how ?' questions, the interpretation of historical events is not so much discovering necessary and sufficient conditions for the occurrence of an event, but more likely to be the activity of relating parts to a whole. In fact, the very nature of history is the collecting, compounding and relating of unique events into a unique whole. The kind of explanation offered is in terms of the synthesis of the parts into a new whole.[15] This is a particularly useful idea when the subject of our study is something which is still continuing, for example, many branches of mathematics had their origins centuries ago, and are still flourishing today. In the study of the history of mathematics we are relating the ideas of individuals of the past to the ideas of present mathematicians. This continuity of idea is an important factor in the building of mathematical structures. However, although the structure we may build is unique, it may not be the only possible structure.

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15. This is very near to the synthesis of a kind of epistemological structure. Epistemology tries to answer the question 'how we come to know', history, at least part of the time, tries to answer the question 'how this came about'. The methodology is similar, while the subject matter is different.

10. <u>The Reliability of Historical Explanation</u>.

As we have seen, the 'why ?' question of science is secondary to the 'how ?' question and the relating of parts to the whole in historical explanation. Although complete explanation may theoretically depend ultimately on the discovery and use of universal laws, the main emphasis in historical explanation is the relating of events one to another so that the 'how ?' questions may be answered. Generalisations thus made are not intended to be universal laws (they are not generalisations at all, in the scientific sense), but to be plausible working principles by which the parts are related to the whole.[16]

Since such generalisations are not intended as laws there is no reason to attribute the same kind of reliability to historical explanation as to scientific explanation. The consequence of this is that it is much easier to change historical explanation without sacrificing much in the way of rationality or plausibility.

11. <u>The Nature of the Subjects of Historical Enquiry</u>.

It is important to clarify the nature of the subjects of historical enquiry. An historical event may be considered as something that happens instantaneously or over a fairly short period of time (like the outcome of a battle), as something that happens over a long period of time with a recognisable beginning and end (the first world war), or as something which began in the past and continues to occur and develop now (democratic society). All these are the subjects of historical enquiry. However, one might equally

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16. There are 'maxims' of historical investigation as there are in other subjects. See Polanyi (1964) and Section 3 note (38).

argue that these kinds of events too are the subject matter of science. We get nowhere by trying to specify types of events as subject matter for one discipline or another, the difference lies in what use is made of them. For example, the comet of 1682 was, to Halley, the astronomer, a scientific event. He was interested in classes of similar events and with the aid of Newton's gravitational theory was able to predict its return in 1758. To the historian of science the comet was a significant event in Halley's career and in the scientific community at the time, because it brought to the fore a class of phenomena which were seen to be susceptible to mathematical analysis and which had hitherto been considered to be unreachable in the accepted sense of scientific experiment. To the historian the comet's significance lies in the subsequent changes of attitude towards science and the role of scientific thought in influencing the culture.

The most important thing to recognise here is that the historian of science, while describing the interrelations of ideas in the historical sense, has also to carry in some way the classifications of the scientist, and the purpose of scientific activity both then and now. It is not surprising that historians of science carry over scientific methodology into history and look for laws to govern historical events.[17]

It is often forgotten that because the major part of the study of history, and history of science particularly, concerns the interrelations of concepts and ideas, that

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17. A contemporary source of discussion along these lines can be found in the work surrounding the 'structure of scientific revolutions'.Kuhn (1962);Lakatos and

they are not at all susceptible to the traditional methodology of science. It is impossible to state necessary and sufficient conditions for the emergence of a democratic society or a scientific revolution because our concepts of democracy and science are so general and subject to constant development.

When our subject of historical study is mathematics we have to be exceedingly careful. Mathematics itself is, in a sense, the most reliable and the most certain of the sciences. So far as we can discover, there is some way in which, throughout history, this has always been so. Even though contemporary mathematics limits this reliability and puts conditions on consistency; the results of mathematical activity are mental constructs and as such have no necessary connection with the real world. (Even though starting points for these ideas may be found in practical problems.)[18] It is just because this reliability is independent of the real world that any attempts to explain mathematical invention in terms of scientific laws are doomed to failure. Mathematics as a body of knowledge can be explained in terms of its own logic, but mathematical invention cannot be explained in terms of mathematical logic. Similarly, it is impossible to analyse scientific activity totally in scientific terms.

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18. Contrast this with science whose reliability is measured by how well it matches with the real world.

(b) The Four Histories of Mathematics.

1. Introduction

When we talk of 'History of Mathematics', there is often an idea that it has something to do with 'who invented what' or how a particular theorem or theory came to be like it is. This is fairly representative of the popular but naive idea that history of mathematics has little to do with the real mathematics taught today and is, to some extent, a luxury afforded only by research departments. The fundamental pupose of History, the general study of change through time, is highly relevant not only to the mathematics we study today, but also, more importantly, to the communication of mathematics at all levels.

In the study of the history of mathematics we are concerned with two general aspects which are of equal importance. First, the 'past' itself - the documents, records, events, the 'facts' because they form the basis of mathematical theories from which today's theories and techniques are derived. In reading mathematical papers and other documents we attempt to discover the state (or nature) of concepts involved, so the importance of the records in mathematics concerns not only arranging them to determine a sequence of events, but studying them to discover the empirical facts about the problems, and the theoretical facts about the solutions to those problems. In this sense, concepts form part of the basic data of the history of mathematics. Secondly, from the data we then have attempts to reconstruct and interpret the past. The importance of a particular event or concept, the raising of its status from a mere fact about the past to a fact of history,[19] depends entirely on the interpretation we might want to put upon that fact. These interpretations range from the conscious accounts of events and attempts at reconstruction by historians of mathematics, through to the unconscious interpretations by the working mathematician or teacher communicating mathematics. We might deplore the ignorance of history and lack of sensitivity in the impressions given of mathematics springing ready-made into existence, but at least as bad are the uncritical statements still found in abundance like 'Newton invented the calculus', or 'Galois invented group theory', which are reinforced by our habit in mathematics of naming a theorem or technique after its so called inventor, often contrary to even the most obvious historical evidence.

The axiomatic view of mathematics[20] is a major contributing factor to the lack of regard for the history of mathematics; we are so anxious to show that mathematics is about <u>anything</u> that we forget the <u>something</u> that gave

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19. Carr (1964) (10-13) Facts about the past are all the undifferentiated pieces of evidence we may have available; facts of history are facts about the past which have relevance and meaning when they are used to describe or explain other events. For example, the writing of a book by a mathematician may be a fact about the past, while the influence of that book can be a fact of history.

20. This is epitomised in Russell's famous quotation ".. mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true." (Newmann (1956) p.1577).

rise to it; and for many purposes mathematics is regarded to have begun in the middle of the nineteenth century. The relevance of the history of mathematics for the development and communication of mathematics has been argued elsewhere,[21] what I wish to offer here is an outline for classifying, and hence criticising, different types of approach to the history of mathematics.

Amongst all the writing, research, discussion and popular exposition, it is possible to distinguish four hostories of mathematics, each dealing with recognisably different aspects of mathematics and its communication, and each deserving recognition. These four aspects I am calling Empirical Reconstruction, Conceptual Reorganisation, Socio-economic Development, and Patterns of Discovery.

Empirical Reconstruction is perhaps what most people understand to be the history of mathematics; it consists of the attempts to reconstruct past mathematics by the examination of documents, etc. and the motivations for this mathematics by discovering the relevant problems of the time. This kind of approach shows the development of mathematics in history, the physical and mathematical problems tackled, the new mathematics resulting from research, and the application of this new mathematics to both physical and theoretical problems.

Conceptual Reorganisation concerns both current mathematics and the interpretation put on the past. Contemporary mathematics influences value-judgements about past mathematics by describing the past in terms of current concepts, and by deciding, consciously or unconsciously,

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21. See Wilder 1972.

whether a particular piece of mathematics has relevance or merit, or whether a particular theorem is proved rigorously or not. Judging the past in terms of the present is a danger common to all aspects of history, not only mathematics, but it is particularly dangerous in mathematics and difficult to avoid because of the concept of mathematical structure. The structures of mathematics raise deep philosophical and psychological questions, only seriously tackled by writers like Beth and Piaget,[22] and relevant to mathematics history not only because the group was probably the first recognisable abstract structure, but also because we are concerned with the central concepts by which structures are described, how they came into being and to what extent they may be complete or still evolving, and the contingency or inevitability of their rules of operation.[23]

Socio-economic Development looks at history from the general standpoint of forces external to the theory and structures of mathematics. It examines how social changes can determine the centres of mathematical development, how various kinds of patronage encourage the free development, the priorities and the fashions of mathematics, the influence of individuals on research programmes, the technical and social problems considered ameanable to mathematics, the demands of investors and the restrictions imposed by economic conditions. These influences are important, for while they may not decide the detail of mathematical theory, they often determine its general

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22. See Beth and Piaget (1966).
23. The Influence of Axiomatics on the Structuralist philosophy is seen in Piaget (1971).

direction and its rate of development.[24]

Patterns of Discovery concerns the attempts to build a Philosophy of Mathematics[25] and investigates the creative intellectual processes of individuals, and the contributions history can make in the formulation of a logic of discovery in mathematics, and a psychology of invention. This area tends not to be taken seriously by mathematicians in general, which is surprising when we consider that the greatest single problem mathematics has is communicating its relevance to the culture at large, and making itself intellectually accessible at all kinds of levels. This cannot be done without a good philosophy of mathematics, which, in turn, must draw on history for much of its data.[26.27.]

More historians of mathematics are becoming aware of these four histories of mathematics and are either attempting to encompass all of these aspects, or making their position clear at the outset. The most difficult task, that of

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24. Many examples come to mind, like the strategic necessity for operations research, or the investment in the computer industry.
25. This is contrasted with both research in 'foundations' and what are commonly called philosopies of mathematics; logicism, formalism and intuitionism, because none of these pay much attention to the history of mathematics.
26. The contributions of Hadamard (1945); Polya (1945) (1954) (1962); Meschkowski (1964), Lakatos (1963/4); and Davis (1967) are all attempts to tackle this field.
27. A strong plea for a relevant philosophy of mathematics and mathematics teaching is made by Thomas (1972).

encompassing all, has been begun by Wilder[28] where the concept of culture has been adapted to form a fruitful context for the discussion of the general evolution of mathematics.

Each of these four aspects of history will now be considered in turn, to form a basis for further discussion.

2. Empirical Reconstruction

The example taken here will be the history of a particular problem; the determination of the modes of vibration of a stretched string, over a period of about fifty years at the beginning of the eighteenth century. The period itself is well documented[29], the mathematicians well-known, and the following outline is drawn mainly from readily available histories.[30]

The mathematics of music, begun by Pythagoreans, continued to be studied by a number of notable men in the seventeenth century. Their approach was mainly experimental and no significant new results were obtained from the mathematical point of view. By about 1700 it was well known that a string could vibrate in a number of modes, and that the tone produced by a string vibrating in k parts is the k^{th} harmonic. Brook Taylor, using the method of fluxions derived the fundamental frequency relation and solved the fluxional equation:

28. Wilder (1965) (1968) (1972) (1974)

29. Learned societies and their journals have been founded and the convention of publishing papers is beginning.

30. Kline (1972), Grattan-Guinness (1970a), Manheim (1964), Truesdell (1968).

$$a^2 \ddot{x} = s\dot{y}\ddot{y}$$

where $s = \sqrt{\dot{x}^2 + \dot{y}^2}$ and the fluxions are with respect to time, which gave a result of the form:

$$y = A \sin \frac{\pi x}{l}$$

for the steady state, t constant.

This breakthrough began an attack on the general problem which was to have far reaching results.

Mathematicians on the Continent, eager to prove the power of the differential techniques, took up this, and various related problems. John Bernoulli considered the motion of a weightless elastic string loaded with a number of equally speced masses. He recognised that the force on each mass was -k times its displacement, and solved the simple harmonic motion equation:

$$\frac{d^2x}{dt^2} = -kx$$

he then tackled the continuous string, showed, like Taylor that it must have the shape of a sine curve, and solved the general equation:

$$\frac{d^2y}{dx^2} = -ky$$

Both Taylor and Bernoulli treated only the fundamental frequency at this stage.

About this time we have evidence of the systematic study of second order differential equations in the work of Euler, showing the techniques of substitution and transformation, and the introduction of the exponential function, vital to the solution of second order equations.

Daniel Bernoulli investigated the hanging chain problem and suggested his solution was applicable to the vibrating string. He had tackled the problem of the massless string loaded with two weights, calculated the frequencies of the two modes and showed that when there are n weights, there are n different modes. He then extended the theory to the continuous heavy cord and showed there were infinitely many modes of vibration. While he suggested the application and showed he knew the theory for the stretched string, he published nothing on it till much later.

Euler's research in the theory of differential equations began to bear fruit in a series of works on the theory of music, the hanging loaded string, and the harmonic oscillator in particular; he popularised the use of partial differential equations and obtained the results:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \quad \text{..... (i)}$$

and

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 x}{\partial y \partial x} \quad \text{.... (ii)}$$

for a function of two variables, x and y. His techniques included ways of integrating (i) for various forms of the coefficients dx and dy.[31]

Partial Differential equations had already been used by D'Alembert in his Traité de Dynamique of 1743, when he

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31. Euler's notation was that of 'differentials', the relations appearing thus:

$$dz = \frac{dz}{dx} dx + \frac{dz}{dy} dy \quad \text{and} \quad \frac{ddz}{dxdy} = \frac{ddz}{dydx}$$

Concepts and techniques have altered since that time and the question arises whether our 'modern' partial differential notation is really 'equivalent' to Euler's. In fact the basic concepts and techniques that Euler was employing were very different from ours. For an interesting discussion of this point see Bos (1972) and (1974) later applied them to the string problem and obtained the equation:

$$\frac{\partial^2 y}{\partial t^2}(t,x) = a^2 \frac{\partial^2 y}{\partial x^2}(t,x)$$

(where $a^2 = \frac{T}{\sigma}$, and σ = the mass per unit length).

For the string fixed at $x = 0$ and $x = \ell$, where and $y = 0$, and zero velocity at $t = 0$, he showed that every solution of the partial differential equation above is the sum of a function of $(at + x)$ and a function $(at - x)$. He derived the solution:

$$y(t,x) = F(at + x) + S(at - x)$$

The reasoning we use today to obtain this solution, is familiar and the forms of general functions are used in many contexts, so that we accpet this expression without protest. But to the eighteenth century mathematicians these 'general' functions were suspect, not sufficiently justified by the fact that they 'worked' when they were differentiated and substituted in the original equation.

What was the nature of the functions F and G ? To D'Alembert, all the functions used in the solution were 'continuous', they obeyed the 'law of continuity'. This 'law of continuity' was a principle, basically geometric, which was invoked by mathematicians to justify proofs, to explain techniques and to assist intuition frequently in the eighteenth and nineteenth century.[32] In our terms, continuous in this context means 'differentiable'. D'Alembert's functions were analytic expressions formed by the operations of algebra and the calculus (i.e. infinite

32. We still have this about today as an intuitive description of situations in analysis. For further discussion see sections 4c and 4d.

algebraic processes) on algebraic formulae. Regarded in this way, as a calculus of operators, all algebraic expressions gave a definite result after the operations. Since the calculus operations could be described, and at least in principle carried out in detail (the results of the infinitesimal processes being intuitively obvious), and since algebra was in effect, a kind of generalised arithmetic, mathematicians were generally happy to develop their techniques, believing that the obvious consistency of arithmetic lay behind it all. Thus, for D'Alembert's analysis of the wave equation, if two functions agreed in one interval, they must agree in all intervals, and this was sufficient to show that they were the same function.

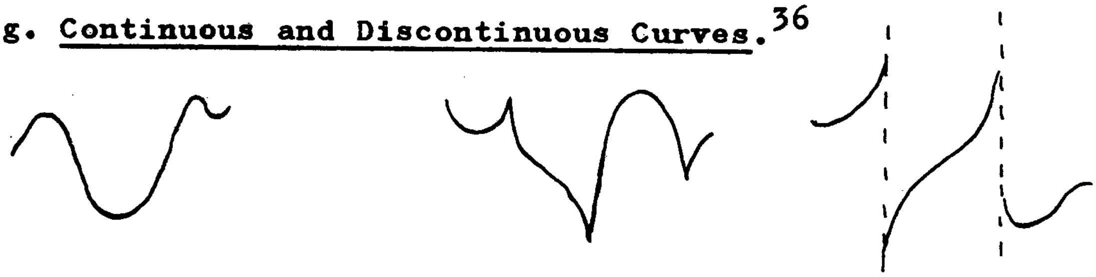
On the other hand, Euler, on seeing D'Alembert's solution tackled the problem by a similar method, but had a totally different idea as to what functions could be admitted as initial curves, and therefore as solutions of partial differential equations. Well before this time, Euler had allowed 'functions' formed by piecing different parts of well-known curves together,[33] and he used the same idea here, arguing that the initial condition - and hence possibly any number of subsequent positions - must be a function with a 'corner' to allow for the initial plucking of the string. Euler realised that "considering such functions as are subject to no law of continuity opens to us a wholly new range of analysis."[34]

33. At least by 1734.

34. Euler to D'Alembert, 1763 Dec.20. Quoted by Truesdell, 1960 p.276 = Opera 2,11 sect. 1,2.

Euler's 'discontinuous' or 'mixed' curves (see below) were criticised by D'Alembert, who was uneasy about what happened at the corners. In modern terms the second derivative is not defined at that point. Euler's use of infinitesimals[35] enabled him to find a solution and obscured the difficulty raised by D'Alembert. Euler claimed that anyway, since the vibrations were small, the corner angles would be small, and the curve would be only infinitesimally different from a continuous one.

Fig. Continuous and Discontinuous Curves.[36]



(i) continuous (Euler, D'Alembert) (ii) discontinuous (Euler, D'Alembert) (iii) contiguous (later)[37]

Now these are called:

(i) differentiable (ii) continuous non-differentiable (iii) discontinuous

Euler differs from D'Alembert in admitting all kinds

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35. Euler had derived infinitesimal relations like: $a+dx = a$, $\sqrt{dx} + dx = \sqrt{dx}$, $dx + (dx)^2 = dx$, for different powers of the infinitesimal dx. These appear in the <u>Institutiones calculi differentialis</u>, 1755, St.Petersburg = Opera 1,10.

36. After Grattan - Guiness (1970a)p.7

37. 'Contiguous' is a nineteenth century label, dating from at least the time of Fourier. Neither Euler nor D'Alembert would have had cause to use a function like this in their analysis of the problem. The confusion is still posing quite a lot of difficulty a century later, see De Morgan 1852 (40-45), Section 4(d) below. p.219.

of initial curves - the individual modes whose periods are fractions of the fundamental, but D'Alembert allowed only analytic initial curves and their solutions and insisted that a solution was possible only "... for the cases where different shapes of the vibrating string can be included in one and the same equation. In all the other cases it seems to me to be impossible to give y in a general form."[38]

Soon after, Daniel Bernoulli, continuing ideas he had expressed earlier, claimed that "... all sonorous bodies contain potentially an infinity of sounds and an infinity of corresponding ways of making their regular vibrations..."[39] By physical argument based on the idea of the superposition of frequencies of vibration he obtained an infinite series:

$$y(t,x) = \alpha \sin \frac{\pi x}{\ell} \cos \frac{\pi c t}{\ell} + \beta \sin \frac{2\pi x}{\ell} \cos \frac{2\pi c t}{\ell} + \dots\dots$$

where each fundamental mode generated "... multiplies an infinite number of times to accord to each interval an infinite number of curves..."[39] He gave no <u>mathematical</u> proof of the generality of the function y(t,x), and suggested that the constants etc., could be chosen carefully so that the series would agree with any given function at an infinite number of points. However, he gave no method for actually calculating these coefficients.

There ensued a great deal of discussion between Euler, Bernoulli and D'Alembert over the next few years in letters and journals as to the nature of the solution, and the

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38. Mem. Acad. Sci. Berlin 6 1750 (355-360) Publ.1752
39. Hist de L'Acad.de Berlin 9, 1753, (147-172),(173-195) publ. 1755.

justification of their approaches. Later, Lagrange and Laplace offered solutions, Lagrange's being perhaps nearest to what we might call 'Fourier coefficients'.[40,41] The justification of the general solution, however, seemed at this stage to be so intractable that apart from attempts at problems of non-uniform heavy strings, the vibration of drums, and other musical instruments, the general problem itself fell out of fashion because at the time it seemed that the mathematicians lacked sufficiently sophisticated techniques. The fact that much of this was due to conceptual difficulties and differences, as to technical problems is a story we can only tell by hindsight.

The routine solution we teach now derives from Fourier[42] and was discovered by him on experimental investigations into the conduction of heat, the fashionable problem at the beginning of the nineteenth century.

This one problem and the attempts at its solution in the first half of the eighteenth century highlight a number of significant developments which were to have major importance in the subsequent evolution of mathematics, not only in terms of the solution of the problem itself, but also in their wide relevance for other, at first apparently unrelated, fields.

The differential and infinitesimal techniques yielded

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40. Kline 1972 (510-514)

41. Grattan-Guinness 1970 (13-21) claims that Lagrange could not have spotted the terms as 'Fourier coefficients' because of his conceptual position over the problems of periodicity.

42. Grattan-Guinness and Ravetz (1972)

significant results, but increasingly from this time mathematicians were being pressed, and themselves pressing for greater rigor of the operations of the calculus.[43] This brought into question the whole class of operations of the calculus, and their justification in terms of a kind of 'algebra of infinities and infinitesimals'.

The question of periodicity, a local problem in terms of the wave equation, had two basic aspects, the geometric problem - the obvious continuous nature of the string, the forms it takes, and the question of whether there existed an algebraic expression to represent its shape; and the algebraic or analytic problem - that of the formulae representing possible and sensible shapes, the possibility of compounding these in analytic terms, and the question of what we now call oddness and evenness of functions. The conflict of geometric intuition and analytic precision is still with us today.

More obviously general is the notion of the definition of function. During this period we are moving from the idea that 'function' is a label applied to single algebraic formulae which completely describe a curve, to the idea that a function can be defined by piecing formulae together, which eventually led to the more analytic idea that one variable depends upon another, and that the nature of this dependence is a matter of arbitrary definition.

The principle of continuity, another geometric intuition, was taken seriously well into the nineteenth century.[44] Here, the continuity intuition applied to

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43. See Section 4 d.

44. See Section 4 c.

analysis begins to call into question infiniteness of series and their convergence, the calculation of coefficients, the comparative size of terms, and the anomalies of their use.[45]

It is not the case that the easiest problems were tackled first, though we can only say this in the light of history, and in a perverse kind of way, because the difficult problems tackled did not yield to the techniques available, this was to the advantage of mathematics as a whole, where further deep questions of the existence of mathematical entities, the role of intuition and the nature of proof, the definition of concepts and the nature of operations, came to the surface and have remained important to this day.

It is interesting to comment on the possible interpretations of the fifty years we have examined in the context of the different historical accounts that have been given, an that are possible. Giving a general history of the period, or even of the problems, in inductivist terms, is impossible.[46] One has to be selective and the available accounts show the respective interests and interpretations of the writers. Kline,[47] in the context of the general development of mathematics in the eighteenth and nineteenth centuries, shows the story as aspects of the development

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45. Examples of Euler's style of reasoning, with contemporary commentary is given in Polya (1954) (17-22) and (90-107) (Vol.I)
46. Inductive history of science or the inductivist programme, consists in giving an impartial account of 'all' the facts.
47. Kline (1972) (502-522), (468-501)

of differential equations: Grattan-Guinness[48] is much more interested in the conceptual problems in analysis and their implications for future developments; Manheim[49] sees it as a stage in the development of the concepts of point sets and the arithmetization of analysis; and Truesdell[50] as a passage in the history of rational mechanics. The relative importance of each particular event in this period depends on the interpretation we want to put on it; for example, we could write the period in terms of the history of notation and the concepts carried with and implied by the notations invented at the time;[51] if we were to write a history of the theory of functions, the beginning of the eighteenth century would have significance. This was also a great period in the debate about the nature of proof, the reliance of geometric and algebraic arguments, and also of a deep change in the philosophy of mathematics, questioning the nature of mathematical objects and their relation to the physical world.

Social changes also influence the development of mathematics here; national economies are flourishing to the extent that it is desirable and prestigious to fund the individuals conducting research, to found learned

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48. Grattan-Guinness (1970), (2-21)

49. Manheim (1964), (35-65)

50. Truesdell (1968), (107-110),(112-114)

51. Cajori (1923) and (192[illegible]) are hardly histories, more catalogues or chronologies. A good history of mathematical symbolism has yet to be written. This is also the view of Dr. Cecily Tanner, in a paper given at the International Congress on Mathematics Education, 1972.

academies, to finance and publish journals; and many of those taking part also found themselves involved in public and military duties. With these changes the relevance of mathematics to physical problems, and the consequent need to communicate this mathematics to many different people became important, and so it is also a period of relevance to the teacher of mathematics, in terms of the development of mathematical education.[52]

Finally, taking an overall view, these factors and others could be examined in terms of the general evolution of mathematics, and the place of mathematics in the culture.

3. Conceptual Reorganisation

A sad comment given on the general impression of university mathematics courses reflects the difficulty of communicating the nature of mathematical concepts at different levels of sophistication: "Mathematical concepts cannot be rationally criticised, either because they are considered arbitrary or because they are considered to directly reflect reality. Thus the concept is there because it's there. All there is in the history is successive elimination of deviations from today's rigour."[53] A much more detailed critique from the historical point of view[54] reflects the tendency of text books to write out the history of mathematics completely. This, often unconscious, elimination of the background of mathematical concepts, might make mathematics harder to learn, not easier.

If the remarks are in the context of a history of mathematics there is a tendency to judge past mathematics

52. This was first mentioned in Section 1.p.29

53. Thomas (1972) p.188

54. Grattan-Guinness (1973) in particular (442-450)

by contemporary standards of rigour. Historians of mathematics tend to be much more careful today, but the danger still exists.[55] It may be an interesting exercise to identify the origins of a particular concept,[56] but the implications of even the title may be misleading. For example, Manheim's book, 'The Genesis of Point Set Topology' suggests that he has identified the conceptual origins of a branch of contemporary mathematics, and since the general idea being investigated is 'nearness',[57] he is able to connect the infinitesimal problems and limit definitions of the seventeenth century through to the notion of an abstract space, and the axioms of Hausdorff.

A recent paper by Marie Goldstein is entitled 'The historical development of group theoretical ideas in connection with Euclid's axiom of congruence' and is intended to show that "group theoretic ideas have been present in the minds of men since ancient times".[58] The problem with this kind of approach is that as the structures of mathematics themselves become more abstract and more general, they become more easily applicable to the past. It seems much easier now to identify powerful concepts of, say, abstract algebra in past mathematics, than it was fifty or

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55. See, for example, Cajori (1919) p.238, the remarks on Euler's use of infinite series, and Bell (1945), (462-463) on the principle of continuity in nineteenth century geometry, while Scott (1958) abounds with similar remarks.

56. Boyer (1949) is a classic example of this kind of writing.

57. Manheim (1964),p.1.

58. Goldstein (1972)

even twenty years ago. This problem was touched on above,[59] and it seems very difficult to avoid getting into a situation where we may claim that certain mathematical structures are innate,[60] which may give us licence to analyse the past in terms of the present.

A careful analysis by Boyer, 'Proportion, equation, function; three steps in the development of a concept'[61] suggests that the changes occur when algebraic expression of relations becomes possible and the proportion, previously verbally expressed, becomes identifiable in an algebraic principle or 'law'; the function concept appears when the 'law' is generalised into an analytic definition motivated by both the current problems in physics,[62] and the inconsistencies in the algebra. Thus, rather than emphasising the unity of the conceptual pedigree, Boyer is attempting to point out periods when the concept which we now identify as 'function' underwent its evolutionary changes. This enables us to criticise the account in terms of contemporary mathematics, if we wish, while having regard for the mathematical concepts and procedures of the time.

The classic description of conceptual change over time is given in Lakatos,[63] where the versions of 'Euler's theorem' for polyhedra are examined over a period of some three hundred years. Lakatos suggests different motivations

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59. Above, p.54 in comments on structuration, and note (23)
60. Chomsky (1968) suggests that some of the structures of language may be innate.
61. Boyer (1946)
62. See, for example, the discussion on the stretched string problem above.
63. Lakatos (1963/4)

for the changes in concepts, internal to mathematics, and to do with the distinctions between the procedures of proof (i.e. 'demonstration' or articulation of a thought-experiment) and the concept of proof-analysis (the logical criticism of the proof), arising in the early nineteenth century.[64] While the distinction between proof and proof-analysis and the rigour of proof and the rigour of proof-analysis is clear from about this time, any mathematician following the argument becomes aware of the existence of changing standards of proof applicable at all times in mathematical history. The relevance of this for the present discussion is that it is possible, and important, for mathematicians at all levels to be aware of changing standards of rigour, and thus to avoid both judging the past in terms of the present, and imagining that rigorous mathematics is only a hundred or so years old.[65]

The outcome of the unthinking application of modern standards of rigour to history is the complete writing out of history from the contemporary standard textbooks. This, in itself, can be taken as a writing of mathematical history by implication. Not only does the textbook tend to reverse the historical order, be beginning at the currently fashionable definition and then attempting to 'criticise' these by raising unmotivated and often incomprehensible objections,

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64. For example, Lakatos (1963/4) note p. (59-60).
65. Grattan-Guinness (1973) (442-444) remarks on analysis give examples of the impressions and also the incorrect and misleading information given cheerfully in many texts.

it also removes the background of mathematics, the important 'memory' of the mathematical culture, which today is available only to a few.[66]

With every phase in mathematical history comes a reformulation, a consolidation, a new beginning. This is often considered necessary purely from the practical point of view, the proliferation of writing is such that for sheer efficiency of communication choices have to be made.[67] The latest textbook begins at the new beginning, and often has difficulty in relating to the past, in fact this is hardly ever attempted.

Students are thus cut adrift from the background of mathematics - particularly so at the transition from school to university where often unmotivated rigor is suddenly introduced as the proper way of doing mathematics, and the more intuitive operations and proofs of school mathematics are no longer respectable.[68] [69] It is important that teachers and students should be aware of these dangers.

Another important point that follows from a realisation

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66. For the 'memory' argument see Marwick (1970)(12-19).
67. The two important facts, that mathematics develops itself by generating abstract structures, which are then applied to select mathematics necessary for learning; and the questionable pedagogical value of this procedure are investigated in Section 3.
68. It is important to stress that it is not an argument for <u>banning rigor from school</u> mathematics, rather for introducing <u>intuition and heuristic into university courses</u>. Good heuristic can motivate rigor.
69. See Thomas 1972, p.188., and Grattan-Guinness 1973,p.443.

of the dangers of a modern interpretation of the past, is that since mathematics now has many specialities within itself, a mathematician working in one area may have a very different view of what is important or significant from one working in a different area. This point is made by Fisher (1966) and his example will be discussed next.

4. Socio-Economic Development

A number of examples can be given of particular developments in mathematics being arrested or encouraged by factors largely external to the structures of mathematics. In terms of Wilder's analysis[70] the factors I want to identify have come under the forces listed as Environmental Stress, Cultural Lag, and Cultural Resistance.

Environmental stress, in its most general form, can be seen, in the first place, as the creation or lack of conditions which foster the activity of mathematics. For example, mathematics, as with other academic studies, was kept alive in Europe during the Black Death and the Hundred Years War, largely by the monastic tradition which preserved the only conditions in which learning could survive. It is largely when individuals are able to rely on others for support that creative work is possible. Active encouragement of individuals in the form of the patronage of a ruler or state, of groups in the form of the founding of a society or a journal, or of institutions or universities for motives academic, practical, political and prestigious, is only possible when there is a section of the community with enough economic and social independence to provide both the means to achieve this encouragement,

70. Wilder, 1968 p.169; 1972 p.488.

the capital, buildings, etc., and the people to take advantage of it. For example, only towards the end of the nineteenth century did the United States begin to make any significant contributions to pure mathematics. The picture drawn by Hogan[71] of the pioneers in the 'howling wilderness' shows that only the obvious and most important applications of mathematics; simple computation, calendar calculations, surveying, map-making and navigation, were at all widely practiced, and even the well-known scientists like Benjamin Franklin were principally concerned with the practical applications of mathematics and considered 'theoretical' pursuits a waste of time.

Cultural lag and cultural resistance can both be exemplified in the introduction of the written Arabic numerals to Europe. For practical purposes the abacus worked well enough, and the fact that merchants were slow to change was due largely to the established tradition in calculation, the lack of obvious advantage in the written mode, and to the scarcity and expense of writing materials. Disputes between 'algorists' and 'abacists' were common, the practical advantage often going to the abacists, while the advantages of the algorists' methods were often outweighed by the feeling that symbols were suspicious objects; (they were strongly connected with magic), they disguised the <u>reality</u> of the counting, and were easily altered. Only the standardisation of the written numerals through the medium of the printing press, and the increase in the supply of cheap paper overcame both the natural lag and the active resistance to the new symbols and the methods of

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71. Hogan 1974 p.155.

calculation.[72]

Mathematical theories are made by individuals, but the acceptance or otherwise of these theories depends on social groups and their psychological condition. Precedents and preferences are often as much to do with group psychology as with mathematical rigor. It is not necessarily the case that the currently accepted theory is any more correct or rigorous than its rivals - nor even may it account for the facts any better.[73] The crucial factor is social acceptance by other mathematicians. "Therefore, for the observer, both at a particular instance and over time, a theory is not a fixed object but a social category which changes with the changing perspectives of mathematicians."[74]

There are many examples of acceptance and rejection of a theory by individual mathematicians. Sometimes the positive encouragement of a protégé is notable, as that shown to Pascal by mathematicians of the 'Mersenne School',[75] but sadly, and more often we seem to record the cases of individual dislike or discouragement, of Galois by the French educational system and the Paris Academy,[76] of Cantor by Kronecker and Mittag-Leffler,[77] or even the casual remark like the words of Gauss over the work of the young Bolyai.

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72. Menninger 1969, in particular, (431-445)
73. This thesis has been discussed variously by Hanson (1958), Popper, (1963), Kuhn (1962), Lakatos and Musgrave (1970).
74. Fisher (1966) p.137
75. Boyer (1963).
76. Sarton (1937).
77. Grattan-Guinness (1971).

On the whole there are few cases where individual mathematicians have delayed or diverted the development of mathematics for long. Much more often this has been the result of the work of a group, though we may identify particular individuals as (spokesmen or) leaders of a group (at different times). A good example of this is the theory of invariants which flourished in the latter half of the nineteenth century, only to be 'killed off' by Hilbert in 1896.[78]

Fisher defines the social existence of a theory in terms of two categories of mathematicians: those who are known practitioners of the theory and who actively contribute to its development; and those who do not contribute directly, but who recognise that the theory has a valuable contribution to make to their own specialism, or to mathematics in general.

If numbers of the former group decline, the theory is only kept alive by the latter group. Since none of these actually practice the theory, they may soon find that they can do without it, forget about it, and it dies. (That is, it passes out of social existence.) (Inhabits the third world.)

In the example of invariant theory, it seems to have been called into existence in 1841 by Boole, and over the next fifty years mathematicians such as Cayley, Sylvester,[79] Hesse, Eisenstein, Clebsch, Gordan,[80] Lindemann, Hermite, Brioschi, Peano, Klein, Lie, Hilbert and Noether all made contributions to the field. The fact that all of these are well-known mathematicians who made important contributions

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78. Fisher (1966) p.145.
79. Called by Bell (1937) the 'invariant twins'.
80. Student of Clebsch and tutor to Emmy Noether.

to other fields, most of which still survive, seems to indicate that, at the time, it was considered an important and respectable part of mathematical theory.

Cayley's definition of the general problem "to find all derivatives of any number of functions, (called quantics) which have the property of preserving their form unaltered after any linear transformation of the variables."[81] set the scene for investigations which, starting with two-variable functions and originally geometrical in character, evolved quickly into abstract algebra as the problem was generalised to functions of any number of variables.

As often happens, crucial theorems are proved by newcomers to the field and this was the case when Hilbert produced his proof that a finite system of independent invariants existed for quantics in any number of variables.[82] Hilbert did not find any systems of invariants, nor did he give any procedures for finding them, he just proved there were finite systems of invariants for a given quantic.

We may remark, in terms of the development of mathematics, that what Hilbert had done was to avoid one problem (the finding of the invariants) by re-defining it (monster-adjustment) and solving the new problem.

Gordan's reported remark: "That is not mathematics, that is theology." seems to be open to different interpretations,[83] but whatever is actually correct, it shows he recognised a major conceptual change, whether he was ready to accept it or not.

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81. Cayley, 1846, quoted in Fisher 1966,p.142.
82. See Reid (1970)(247-253) for a summary of Hilbert's contributions.
83. Fisher, p.145.

Hilbert's breakthrough within mathematics - the application of a piece of mathematical technique from one area (the formal logic of the existence proof) to a difficult problem in another area (which originally required geometric imagery) also had consequences for the existence of invariant theory as an identifiably seperate piece of mathematics.[84]

Hilbert later terminated the social existence of invariant theory by first suggesting that there were three stages in the development of a mathematical theory: naive, formal and critical, and that whilst all other contributions belonged to the first two stages, only his belonged to the third;[85] and secondly, in another paper he publicly pronounced invariant theory dead: "with this, I believe, are attained the most general goals of functional fields of invariants."[86]

Hilbert's three stages are an over-easy generalisation, disputed at the time[87] but the public pronouncement and private shift of Hilbert's interest were enough to persuade most mathematicians that there was no point in pursuing the problem. Some of the old guard of course remained and still trained research students. Gordan was one of these, who tutored Emmy Noether, and she later produced a series of papers in invariant theory dealing with aspects Hilbert had not considered. These aspects, however, became increasingly more general and abstract, and coming under Hilbert's influence she generalised invariant theory out of existence

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84. Reported in White (1899)

85. Hilbert (1896)

86. Hilbert 1893, Fisher (1966) p.145.

87. Meyer, 1897.

as a standard result in the theory of rings.[88]

It is difficult, if not impossible to play a 'what might have been' game with history, for we are set with our own conceptual framework. It might be true to say that any contemporary mathematician, brought up on abstract algebra, might not be able to 'see' the problem that confronted people like Cayley, and so it may not be possible to carry on mathematics as it were, say, from where Gordan left off. Be that as it may, we have here an example of the extinction of an area of mathematics which, because of the conceptual reorganisation involved, has been completely written out of the text books. This occurred largely through the efforts of a single mathematician, David Hilbert, whose prestige and influence was enough to put an end

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88. Fisher, p.148. An ironic remark here is to notice that Emmy Noether was assigned to the 'unfashionable' Gordan who was on the edge of retirement. This was a natural, but perhaps administratively convenient selection, because he was a friend of the family. Her thesis, a formidable piece of invariant theory, is in strong contrast to her later work in abstract algebra. Gordan dug the grave, Hilbert pronounced sentence, and Noether performed the rites ! A biographical account of Noether's training, work and influence is given in Kimberling (1972).

to it.[90]

We can consider not only the social destruction but also the social creation of areas of mathematics by prestigious individuals or groups. This may raise an interesting point for philosophy: if a theory is suggested, it can be mathematically existent. If it is not accepted by the mathematical community, it is not socially existent, and therefore has no influence on the subsequent development of mathematics, though it is possible that mathematics may develop in reaction to the new theory. The theories may inhabit Popper's 'third world', but only influence the development of mathematics insofar as they overflow into the 'second world' of the belief-systems of mathematicians and their dispositions to act.[91]

Some theories may apparently return to life. The social rebirth of an area of mathematics is exemplified in the theory of infinitesimals. If we consider the period from the seventeenth century to today, the theory of infinitesimals was socially assassinated by respectable mathematics

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90. Invariant theory as distinct from the mention of 'invariants' appears to survive in a number of different areas of specialisation: the theory of the general conic (Barton 1958) in advanced level geometry; Algebraic geometry, Continuum mechanics, Quantum field theory. (Fisher 152-3). None of these (except perhaps the first two) seem to know of the existence of the others, and their individual techniques are very different.
91. See Section 3.

in the nineteenth century, kept alive as a heuristic in nineteenth and twentieth century textbooks, and revived in an acceptable form in the twentieth century in non-standard analysis.[92]

We can consider the migration of mathematicians as a factor in determining the course of mathematics, and have an interesting example in the movement of mathematicians from Germany to the United States in the 1930's.[93] This occurs principally as the result of political acts and racial persecution of non-Nazis, particularly the Jews, in those years.

Felix Klein's teaching at Gottingen, the Erlanger Program unification of geometry, and the careful planning and administration of the Mathematics faculty, prodeced a unique situation at the time. Klein's idea of Gottingen as the centre of the scientific world found support, and the university was gradually ringed with a series of scientific and technical institutes which became "the model for scientific-technological complexes which were later to grow

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92. See case study: Calculus: Metaphysics and Practice. Sect.4(d) Historically speaking, it is not the same theory of infinitesimals in spite of the fact that Robinson sees himself as the successor of the seventeenth century mathematicians. (Robinson 1966 Chp.10).

93. Of course, not all went directly or even finally to the United States, the pattern is exceedingly complex and requires much deeper analysis than is possible here.

up around various universities in America."[94]

By about 1895 there was an American colony of mathematicians in Gottingen, drawn by advertisements of Klein's courses in the Bulletin of the American Mathematical Society. The fact that both Gottingen grew in prestige and that many mathematicians from all over the world were able to attend courses there was due as much to the socio-economic factors in Europe and America as to the brilliance and popularity of the teachers there. The American contacts made in the last years of the nineteenth and the early years of the twentieth century were to have significant pay off about thirty years later.

Attracted by the general intellectual atmosphere, and by Klein in particular, a number of outstanding mathematicians came to Gottingen, and when Hilbert began to play the major role, this tendency continued. Thus Klein, and later Hilbert, were able to influence a large number of mathematicians and students who were already, or who later became, teachers and professors in universities throughout the industrialised west.[95]

It is gênerally considered that Hilbert founded the formalist school of mathematical philosophy, and it is inevitable that the methods implied by this view, if not the view itself, would have a strong influence on his

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94. Reid 1970, p.95.

95. An interesting comparison can be made here with the famous artistic centre Bauhaus, and the teachers and students there.

students and collaborators.[96]

To make a list of all the mathematicians influenced by the formalist school would be a formidable task, but it may be possible to describe a trend, and list a few of the outstanding mathematicians who might be said to have had the most influence on contemporary mathematical practice and philosophy.

In the 1930's Nazi Germany began the process of banishing Jews and other academics with Jewish and radical sympathies. Fortunately, the ground was already prepared for their ready acceptance in other parts of the world, particularly in the United States. Not only were the American universities able to support them with facilities for work, money for projects, and publication outlets, but they were on the whole favourable to the general underlying mathematical philosophy that they held.[97] These refugees also formed formalist 'schools', and began to prepare a second generation for formalism. This second

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96. Klein's views were very different, more in the Kronecker constructivist tradition, while there were a few who opposed Hilbert like Brouwer and, for a time, Weyl, on the intuitionist side; and Russell, who began the ambitious logicist programme, was an occasional visitor.

97. This sympathy may also have been assisted by what one might crudely call the general pioneering-opportunist attitude of American society. Enterprise is rewarded by gain in mathematical as well as commerical fields.

generation of teachers and professors has had a strong influence on U.S. university and school mathematics programmes.

As a particular example, Emil Artin left Germany in the early 1930's and stayed in America during the second world war, to return to Germany some twenty years later. His name appears heading a list of "Working group experts to draw up a synopsis of a modern curriculum for school mathematics."[98] [99]

Due to the economic growth of the United States, and the various aid and advisory services, formalism in school mathematics has now spread to many countries. England and other countries where the possibility of 'internal' reforms exist are not immune; it is almost inevitable that any official programme embraces something of the formalist philosophy.[100]

More obviously, political and economic causes give rise to new areas of mathematics through direct encouragement by government or industry. For example, the use of mathematicians during the second world war in various arms of the services to solve apparently unrelated problems like

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98. O.E.E.C. report 1961 p.307. Texts mentioned in the appendix to this report include Modern Algebra by van der Waerdan, who was a pupil of Emmy Noether.

99. Dresden (1942) lists the refugees, their origins and American appointments from 1933 through to early 1942.

100. This is perhaps underlined by the famous 'declaration' by a number of eminent teachers that formalism and abstract axiomatics had had too great an influence on the teaching of mathematics. See Griffiths and Howson (1975) (235-236) and individual contributions like Kline (1970).

those of the protection of the atlantic convoys, or the bomber raids in Europe gave rise to operations research;[101] the contemporary funding of the computer industry by governments and business has produced a vast increase in software methods; research interests in logic, particularly model theory; and an application of computer technique to many problems previously thought intractable because of the sheer volume of calculation. It also raises interesting questions about proof methods and the nature of mathematical truth.[102]

Mathematical education has much to do with social movements, the founding of the Association for the Improvement of Geometrical Teaching, its evolution into the Mathematical Association; the breaking away of the Association for Teaching Aids in Mathematics which in its turn became the Association of Teachers of Mathematics, through their journals influence the styles of future teaching, and the mathematical and pedagogical content of many school programmes. The student protest movement of recent years is represented by the paper by Thomas referred to earlier,[103] which is a comment on the teaching of university mathematics, and shows how young people are objecting to the insensitive and unmotivated catechization to which many of their number are subjected. This contains an economic element, for in order to maintain department staff, student quotas must be maintained, and so courses must be made attractive. This not only means developing new courses, more 'relevant' to today's needs (though this is questionable)[104] but also

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101. Waddington (1973), was originally written as the official monograph for Costal Command, R.A.F. in 1946.

102. See Davis (1972).

103. Thomas (1972)

104. Bondi (1975) discusses the supply & demands of University

examining the methodology of mathematics teaching, heuristic techniques, traditional standards of attainment, the content of syllabuses, and so on.

There is a sociological character about some controversies in the historiography of mathematics. An example here is the dispute between Freudenthal and Grattan-Guinness[105] over the possibility that Cauchy might have drawn on some of Bolzano's results involving the definition of convergence and the concept of continuity without acknowledgement.

This is not the place to go into the mathematical details of the dispute, but only to point out that the argument not only depends on conceptual and interpretational differences, but also on the admittance or non-admittance of sociological and psychological evidence: namely the circumstantial and inferential details of the relationships existing between mathematicians of the French Academy at the time. That such evidence might or might not be admitted to an historical account is a sign of the changing awareness of historians of mathematics to what Wilder calls 'cultural-environmental stress'.[106] We see it occurs not only in the period in question, (the eighteenth century) but in the historical accounts and interpretations of that period as well.

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105. Freudenthal (1971) in criticism of Grattan-Guinness (1970b). The former not only attacks the latter's interpretation, but also his conceptual and mathematical analysis of the situation. This is the kind of personal attack one imagined last occurred in the nineteenth century, a tremendous piece of polemic !

106. Wilder (1968)(169-170)

There are a number of possible aspects of the history of mathematics that could be studied under the general heading of 'fashions in mathematics'.

Is it the case, for example, that national characteristics broadly determine the style of mathematics which predominates, or the aspects of mathematics which flourish in different countries ? It is often remarked, for example, without much evidence I feel, that Indians are much better at calculation and worse at geometry than the English.[107] However much such remarks may be due to a degree of nationalist bias,[108] or sweeping over-generalisation, while we may claim that they are less true today with our opportunities for communication and cross-fertilisation, they may have been true to some extent in the past, because the mathematics produced by a country or civilisation is an aspect of the total culture, and would consequently be expected to contain evidence of the evolution of that culture.

Communication in one form or other is an important vehicle for the fashions of mathematics. At different times it has been fashionable to travel to different places to learn techniques and skills from the masters. In the early

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107. This kind of generalisation is often made by teachers. Perhaps one might cite the story of Ramanujan; here it is clear, for example, that he did not have the Cambridge concept of mathematical proof.

108. For example Duehm (1954) sees 'deep' and 'ample' minds as national characteristics of the French and the English. (Part I Ch. 4).

seventeenth century Durer travelled to Italy to be instructed in the mysteries of perspective construction;[109] at the turn of the last century young American mathematicians tended to take their PhD degrees in Germany.[110] Travelling to centres of mathematical learning is easier and even more frequent today.

Journals too, make their contribution to the fashions in mathematics. The obvious contribution is the nature and content of the mathematical papers they print, but there are hidden influences they may have on the style and practice of mathematics. First, it has now become a requirement for mathematicians to publish research - not necessarily because the research contains any startling new discoveries but because they must be seen to be 'useful' by the administrators. Many institutions operate an 'efficiency bar'. The pressure to publish produces an enormous backlog of papers for the more prestigious journals. This has a number of consequences. New journals are founded to deal with new specialist areas, or even because the authors cannot get them published in existing journals. 'Preprints' are now becoming more common, where authors circulate their papers privately before official publication; and those more fortunate with time and finance can bypass the blocked official channels by attending small gatherings, symposia, colloquia and conferences.

We can talk of fashions in notation. A good example is the use of $\int_0^\pi \sin x dx$: the "$\int$" and the "dx" are from

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109. Panofsky (1956). See also case study on perspective. Sect.4(c)

110. Curtiss (1937) p.559

the infinitesimal techniques of Leibniz, the definite integral $\int_0^{"\pi"}$ from Fourier a century later; the reasons for the introduction of these particular symbols are often not given, even if they are known, the concepts have changed, yet we happily continue with $\int_0^{\pi}\sin x dx$ because by evolving convention, we somehow 'know' what it means. Because of its widespread use, and particularly because it appears so much <u>in print</u>, there is tremendous 'cultural resistance'[111] to Menger's attempt at the rationalisation of the calculus notation where this particular expression would appear as $\int_0^{\pi}\sin$.[112] Menger regards the calculus as a calculus of differential and integral operatios on functions in the modern sense,[113] and wants to change the notation accordingly.

Analysis of the quantity and type of published mathematical literature can show fashions in types of problems tackled; even suggesting reasons for basic changes in direction of mathematical activity, as after Hilbert's 1899 paper on the foundations of geometry,[114] or reasons why a field died out as a major field of interest, like the theory of determinants.[115]

It is even conceivable that eminent mathematicians writing 'survey' articles might suggest future developments and fashions. This was so in the case of Hilbert's famous twenty-three problems, and could also be the result

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111. Wilder (1968)(176-179)

112. Menger (1952/3/5) typescript preface p.4.

113. Menger (1966) summary.

114. Curtiss (1937)p. 563.

115. May (1968/9).

of Dieudonne's "open problems" at the end of his paper on the history of algebraic geometry[116] or Birkhoff's "Current Trends in Algebra".[117]

It would be interesting to discover to what extent these people were able to identify the significant problems and to what extent they determined them.

5. Patterns of Discovery.

This approach contrasts directly with Conceptual Reorganisation in that it is concerned, in the historical context, with past creative acts. These creative acts themselves can be regarded as the conceptual reorganisations of mathematicians in history: the acts of abstraction, generalisation, symboling, consolidation and selection that produce recognisable (and often spectacular) breakthroughs in problem-situations. The problems can be empirical, physical situations - in which casetthe mathematics is usually 'applied', in the sense that it is used as a tool for discovery after the creative act of modelling the physical situation; the majority of accounts available come from the areas of astronomy and physics, a prime example being the investigations into the law of free fall,[118] or they can be conceptual problems, like the problem of continuity, a problem in 'pure' mathematics. The distinction, if any, between 'pure' and 'applied' mathematics is entirely one of function, for both are concerned with the world of reality, and a piece of mathe-

116. Dieudonne (1972)

117. Birkhoff (1973).

118. Koyre (1955) examines the documentary evidence for the gradual conceptual changes from Kepler to Newton in great detail.

matics can be 'pure' or 'applied according to the function it has in a given problem situation.[119]

Mathematical techniques in science serve as techniques for inference-drawing. They serve to draw a picture or model of the physical problem which emphasises certain aspects that the investigator feels are significant. Thus, in many cases, the creation of mathematics is bound to the investigation of problems in the physical world.[120]

We can distinguish here two kinds of mathematical creation, each to do with the function of the mathematics involved. Pure mathematics, regarded as an abstract axiomatic system, has its creative aspect in the formation of concepts, axioms and rules of inference. This is followed by the discovery of theorems which are completely determined by the axioms and rules of inference. A special case of the creative aspect is the application of concepts and procedures from one field of mathematics to another - requiring a degree of conceptual reorganisation.[121] Applied mathematics can be regarded as the development and use of abstract systems to describe, explain, and predict events in the physical world. Thus creation here consists of the formation of theoretical concepts which describe aspects of the physical system, followed by the formulation of relationships which are described by mathematics. The mathematics here is either taken over from standard sources

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119. For elaboration of this view, see discussion of Popper's "three worlds" in Section 3.

120. A previously quoted example is Fourier, note 42 above.

121. The revolutionary effect of the work of Emmy Noether is given in Kimberling (1972).

or adapted by the experimenter.[122] Discovery then consists in working through the mathematical model to obtain further relationships which are subsequently re-interpreted as physical laws.[123]

The application of mathematics to the physical world in the seventeenth century gave rise to a new methodology in science which, in principle, enables us exactly to describe, explain and predict physical phenomena.

The concept of exactness lies in the spectacular nature of the predictions involved. Astonomical phenomena, way out of direct reach of the individual, were predicted with impressive increasing accuracy. It was a natural development that the principle that the application and development of more powerful (and meant more exact) mathematics would provide more precise prediction. This lies at the root of our modern concept of the 'exact' sciences, and the habit of mind we have developed (ably assisted by scientific philosophy) that the practical intellectual pursuits where exact mathematics is widely applicable are

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122. Dirac's investigations into the wave functions of quantum mechanics produced functions which 'worked' (they described the physical phenomena), but whose 'existence' was disputed by mathematicians. It was largely through Dirac's personal prestige that the techniques were accepted (similar remarks apply to infinitesimals, See Section 4d). Fourier also, is another case in point.

123. The basic distinction between creation and discovery and pure and applied mathematics comes from Wilder (1972(b)) p.37.

more respectable, more deserving of the classification 'science'.

Investigation of patterns of discovery is the domain of the philosopher as well as the historian, and in a sense the fact that historians may try to find out, as far as is possible, exactly what theorem a mathematician discovered, or what concept he was struggling with, implies the possession of an underlying philosophy, namely some belief in a logic of discovery.[124] The difference between philosopher and historian lies in the emphasis of interest: the philosopher is primarily concerned with the analysis of problem situations for the understanding of theoretical systems and critical arguments: the historian's first concern is the reconstruction of problem-situations. The philosopher uses the historical facts as his data, while the historian establishes those facts. Obviously neither activity excludes the other.

There is also an aspect of such situations which may be regarded as the concern of psychology, for the reasons why a person was disposed to act in such a way, or the mental mechanisms involved, suggest a study of the psychology of discovery and creation.[125]

In investigating an individual the historian of mathematics cannot help using both a logic and a psychology of discovery, though these may be more on the level of

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124. See Collingwood (1946), (282-302) 'history as re-enactment of past experience'; see also above: 'Rationality of Actions'. pp(39-42) above.

125. See Hadamard (1945).

working principles or 'maxims'[126] rather than well-developed theoretical tools.

Both creation and discovery are the concerns of pedagogy. Heuristic is used to assist the creation of mathematical concepts by placing the person in a position where they may be able to achieve conceptual reorganisations, and the discovery of mathematics by providing a set of maxims for the procedural analysis of analagous situations.[127]

The historian of mathematics has a particularly difficult task, for while on the surface he is engaged in establishing the facts of history (and this means deciding by some criteria which of the facts about the past deserve raising to that status), he is also engaged in a conceptual reconstruction of elements of the third world; the past problems, theoretical systems and critical arguments.[128]

Critical arguments, (the schema of conjectures and refutations) encompass problem-situations and theoretical systems, and the history of mathematics may be regarded not as a history of mathematical theories, but as a history of mathematical problem-situations and their modifications, for every attempt to understand a theory opens up an historical investigation into the theory and its problem-situation, thus "... the main aim of historical understanding is the hypothetical reconstruction of a historical

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126. Polani (1962) (30-31), (49-50) and also Davis (1967).

127. Polya (1945) was the first contemporary attempt to bring these rules to the surface, but there are many other examples in the history of pedagogy like Recorde's 'Whettesone of Witte', and De Morgan's 'Study and Difficulties of Mathematics' in the Penny Cyclopaedia.

128. Popper (1972) (106-190) See also Section 3.

problem-situation."[129] It is important to distinguish between the hypothetical reconstruction of the problem-situation, which is a conjecture about the actual problem of the mathematician at the time, and the problem of understanding the reconstruction. Confusion of the meta-problems and meta-theories of the historians of mathematics and the problems and theories of the mathematicians in history can lead to a great deal of argument.[130] [131]

Both the historian of mathematics and the pedagogue are keenly interested in failures as well as successes, for failures give clues to conceptual organisations which on examination, become unacceptable to either the individual suggesting them, or to the mathematical community at large. Failures give perhaps more clues than successes to the processes of problem-solving employed, for the "... schema of problem-solving by conjecture and refutation or a similar schema may be used as an explanation theory of human actions, since we can interpret an action as an attempt to solve a problem."[132] Galileo's theory of the tides, which was a 'failure', shows the importance of the reconstruction of Galileo's problem-situation for the understanding of Galileo's theory.[133]

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129. Popper (1972)p.170.

130. The dispute referred to above (105) contains these confusions. Freudenthal accepts neither Grattan-Guinness' reconstruction of the Cauchy-Bolzano problem situation, nor his understanding of it.

131. It may be suggested that most of the discussions in 'mathematical education' are metaproblems and metatheories about understanding problem-situations.

132. Popper (1972) (underlining mine).

133. Discussed in Popper (1972) (170-180).

The reconstruction of past acts of creation and discovery are critically reliant on variations in documentary evidence. A recent example of a major change in viewpoint comes in the accounts of the discovery of the law of free fall by Galileo. The account in Two New Sciences gives no clue to Galileo's actual process of discovery. In fact it is misleading, so much so that from a diagram[134] said to be copied from Oresme, and other textual evidence, it was inferred that the medieval mean speed theorem was the basis of Galileo's theory.[135] New documentary evidence suggests a different story.[136]

Stillman Drake identifies two types of discovery in mathematical physics; D_1, which are systematic deductions or implications from the basic theory, unforseen by the original investigator, or unsuspected at the beginning of an investigation; this is equivalent to what I have called 'discovery' above; and D_2, the "... perception that a certain mathematical relationship holds for physical phenomena considered in a certain way."[137] This contains a certain amount of conceptual reorganisation, and corresponds to what I have called 'creation' above. He considers that history consists mainly of 'discoveries', D_1, with the acts of 'creation' D_2 as the rarer but most important facts.

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134. Galileo (1638) Dover, p.173.

135. Wartofsky (1968). The summary in appendix (419-473) goes into great detail about Galileo's discovery process based on this assumption.

136. Drake (1973), (1974).

137. Drake (1974)p.130.

According to Drake's investigations, the rediscovery of the Eudoxian theory of proportion (in the context of Euclid Book V), encouraged Galileo to treat the growth of speed as continuous, it provided a new mathematical model for a phsical phenomenon. Galileo's procedures were thus mathematical, and mathematical reasoning replaced the traditional Aristotalian form of argument. This reconstruction of Galileo's argument, principally from manuscript notes and letters where he was trying to explain his ideas to a colleague, shows how, in the process of the attempted explanation, the concept of acceleration emerged.

We see here Drake's attempt to reconstruct Galileo's <u>problem-situation</u> in the light of new historical evidence.[138] Application of mathematics derives <u>theorems</u> which are then interpreted in terms of the problem-situation.[139] The reconstruction of Galileo's procedure depends on a number of factors, amongst which are: a knowledge of the cultural context in which Galileo was working, in this respect his change in mode of reasoning was all the more important; familiarity with the mean speed theorem as understood in the late sixteenth and early seventeenth century, and the conventions of Aristotelian reasoning as applied to it;

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138. The bound, unordered volume 72 of the Galilean manuscripts, notes and letters, contains facts about the past. Those mentioned in Drake's paper are now facts of history for they are vital to the reconstruction of Galileo's theory.

139. Galileo, in fact, derived some correct results from false premises, but that is not relevant to the present argument.

fact that previous versions of the story have shown how Galileo might have obtained the law of free fall from the mean speed theorem shows how there was enough data in Galileo's context to have been able to pursue the traditional lines of reasoning and come up with the same answer;[140] a familiarity with Galileo's previous work shows how the Eudoxian theory of proportion, attractive as a mathematical theory, sheds light on the vexed question of time and space continuity.

It is important also to realised that Drake's 1974 paper is a correction of the earlier 1973, where the basic idea in the explanation of Galileo's problem-situation is the same, (the importance of the theory of proportion), but where a piece of evidence was omitted.[141]

We have here an example of how the discovery of new evidence changes the picture of <u>how</u> Galileo discovered the law of free fall, but does not alter the previously established fact that he <u>did</u> discover it. Thus empirical reconstruction is furnished with more data (the identification of the problem-situation) by means of an investigation into the <u>actual process of discovery</u>.

A reconstruction on a number of different levels of a discovery by Euler is given in Polya's book on <u>Induction and Analogy in Mathematics</u>.[142] He considers Euler's own

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140. For example, Wartofsky (1968) 454-465)

141. Drake (1974), p.139.

142. Polya (1954) Vol 1, Chp. VI is a translation and discussion of: <u>Discovery of a most Extraordinary Law of the Numbers Concerning the Sum of their Divisors</u>. Euler: Opera Omnia ser 1 vol 12 (241-253) Polya (1954) (91-98).

account is worth studying because he "... made important discoveries .... by induction, that is, by observation, daring guess, and shrewd verification."[143] It becomes clear that the kind of induction meant is not mathematical induction as we know it, but scientific induction which would hopefully lead to a rigorous mathematical proof.[144]

Euler's own account of his discovery as translated by Polya, shows two aspects of the investigation. The terms and notations used are first explained, and the law for the sum of the divisors is given. In this, Euler shows some anxiety that he is unable to provide what he regards as a rigorous proof, and regrets that the best he can do in this respect is to "... present evidence .. almost equivalent to a rigorous demonstration."[145]

He is virutally advertising for someone to come up with a more satisfactory proof, and to assist them, he retraces the path he claimed he followed in discovering the law. This consists in his own reconstruction of his attempts to find a proof for the equivalence of a continued infinite product of separate factors, and the infinite power series which results when the factors are multiplied out. Euler says: "I proceeded as follow. Being given that the two

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143. Polya (1954) p.90 vol 1

144. Polya (1954) vol 1 p.3 quotes some of Euler's remarks on his methodology from Opera Omnia ser.1., vol.2, p.459 where he carefully distinguishes between the results of induction and mathematical truth.

145. Polya (1954) p.91 underlining mine.

expressions:

$$\text{I.}\quad s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\ldots$$

$$\text{II.}\quad s = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+\ldots$$

are equal, I got rid of the factors in the first by taking logarithms:

$$\log s = \log(1-x)+\log(1-x^2)+\log(1-x^3)+\log(1-x^4)+\ldots$$

In order to get rid of the logarithms, I differentiate and obtain the equation:

$$\frac{1}{s}\frac{ds}{dx} = -\frac{1}{1-x}-\frac{2x}{1-x^2}-\frac{3x^2}{1-x^3}-\frac{4x^3}{1-x^4}-\frac{5x^4}{1-x^5}\ldots$$

or

$$-\frac{x}{s}\frac{ds}{dx} = \frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\frac{4x^4}{1-x^4}+\frac{5x^5}{1-x^5}+\ldots$$

From the second expression for s, and infinite series, we obtain another value for the same quantity:

$$-\frac{x}{s}\frac{ds}{dx} = \frac{x+2x^2-5x^5-7x^7+12x^{12}+15x^{15}-22x^{22}-26x^{26}+\ldots}{1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-\ldots}$$

11. Let us put

$$-\frac{x}{s}\frac{ds}{dx} = t$$

We have above two expressions for the quantity t. In the first expression I expand each term into a geometric series and obtain:

$$\begin{array}{lccccccccl}
t = & x & +x^2 & +x^3 & +x^4 & +x^5 & +x^6 & +x^7 & +x^8 & +\ldots \\
 & & +2x^2 & & +2x^4 & & +2x^6 & & +2x^8 & +\ldots \\
 & & & +3x^3 & & & +3x^6 & & & +\ldots \\
 & & & & +4x^4 & & & & +4x^8 & +\ldots \\
 & & & & & +5x^5 & & & & +\ldots \\
 & & & & & & +6x^6 & & & +\ldots \\
 & & & & & & & +7x^7 & & +\ldots \\
 & & & & & & & & +8x^8 & +\ldots
\end{array}$$

Here we easily see that each power of x arises as many

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146. Euler's notation was $\int n$, Polya footnote, p.91.

times as its exponent has divisors, and that each divisor arises as a coefficient of the same power of x. Therefore, if we collect the terms with like powers, the coefficient of each power will be the sum of the divisors of its exponent. And, therefore, using the above notation $\sigma(m)$ for the sum of the divisors of n, (137) I obtain:

$$t = \sigma(1)x + \sigma(2)x^2 + \sigma(3)x^3 + \sigma(4)x^4 + \sigma(5)x^5 + \ldots$$

The law of the series is manifest. And although it might appear that some induction was involved in the determination of the coefficients, we can easily satisfy ourselves that this law is a necessary consequence."[147]

This connection with the divisor law is made by hindsight. Sometime after, the results of the investigation into divisors brought up the number pattern which was then connected with the unproven algebraic identity.

The fact that he can find a rigorous proof for neither of these relations brings out his method of demonstration, of convincing himself and others by rational argument that they are correct.

But what, in fact, he demonstrates is <u>not so much his method of arriving at a law</u>, but his <u>search for a proof of a different</u> (but logically equivalent) <u>law</u>.

The only way in which we can interpolate <u>how</u> Euler discovered the <u>law of divisors</u>, is to say that at one time he was investigating particular products and infinite series and trying to find a proof of their equivalence, and <u>that later</u> he happened to remember the pattern of powers and coefficients when he was engaged on what was originally

147. Polya (1954) (97-98).

a very different problem, that of the divisors of numbers.

Polya's account is something else again; it is a contemporary interpretation of the logic of Euler's discovery according to Euler. Polya is rationalising Euler's reasoning and justifying his proof-methods as a means of obtaining discoveries in mathematics. Polya's account is a metatheory, an attempt at understanding Euler's problem-situation, which he then uses to support his case for employing heuristic reasoning in teaching mathematical discovery. Polya's work is interesting and valuable, for he uses accounts of historical problem-situations to suggest a methodology which encourages creation by analogy and conjecture, and discovery by heuristic and plausible reasoning.

Perhaps it is even more difficult when we have the written account of a mathematician of one of his own discoveries, for the habit of the polished presentation dies hard, and time clouds the memory, so that even the personal account is something that may be viewed with caution.[148]

6. <u>Summary</u>

An early critique of different types of mathematical history points out some of the pitfalls of historical writing, "It is the impression conveyed by a number of sentences or by whole pages which is the most important element of a history of mathematics, and if such a history is largely made up of statements taken in substance from various sources without being fully digested it is very apt to convey more false impressions than the actual

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148. Some such accounts are given in Hadamand (1945).

inaccuracies in individual sentences would seem to indicate. In view of the great variety of subjects covered in a history of modern mathematics, it seems almost inevitable to introduce to some degree false impressions into such a history even when each statement taken by itself is practically correct. The reader who realises this difficulty can use to great advantage a work which otherwise might be harmful to him."[149]

We have looked at four approaches to the history of mathematics, and though they have been broadly classified according to the conscious (or unconscious) purpose for which they were written, we see that it is both impossible for the writer to exclude other aspects when taking a particular point of view, and important for the reader to be aware what the main purpose of the writing is; empirical reconstruction, a straightforward account of the main problems and developments; conceptual reorganisation, a largely unconscious (and perhaps unintended) account of the past in contemporary terms; socio-economic development, the examination of the general reasons for changes, and patterns of discovery, the attempt to reconstruct individual mathematicians problem situations.

There is no way in which we can achieve Dickson's ideal, "... what is generally wanted is a full and correct statement of the facts, not an historian's personal explanation of these facts. The more completely the historian remains in the background or the less conscious the reader is of the historian's personality, the better the history."[150] Better honest bias than unthinking error.

149. Miller (1921)p.10.

150. Dickson (1920) preface. This typical inductivist history of mathematics has a methodology and style which derives from Ranke and Whewell.

Section 3

The Philosophy[1] of Mathematics and the Methodology of Mathematics Teaching.

1. Argument

The teaching of mathematics is biased by an individual teacher's 'belief-system'[2] which contains an uncritical ideology[3] of mathematics. A knowledge of the history of mathematics helps to turn uncritical ideology into critical philosophy by providing instances of different levels of mathematical activity from heuristics to proofs, thus exhibiting the method of proofs and refutations in action, thereby suggesting some critical methodological approaches to the teaching of mathematics.

2. Some Problems in the Training of Teachers.

During a given course the teacher-trainer has to try to impart two main areas of skill; first: methods of teaching which can range from the general strategies of curriculum design to the particular tactics of classroom techniques, and second: aspects of learning which covers the growth and communication of knowledge in the individual and in the group. In the context of the teaching and learning of mathematics, students study mathematics as a subject

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1. This discussion is concerned with epistemology and the origins of mathematics rather than with foundations of mathematics which has latterly been interpreted as the totality of philosophy.
2. "Belief-system" roughly a generally under-developed cognitive schema, or set of ideas, for details see below p.115
3. Ideology. Def = science of ideas (Here used more in the sense of ideé fixeé or uncritical assumption).

where particular techniques and ideas contribute towards, the building of an overall structure, and in doing so, are helped to become aware of their own learning of mathematics.[4] Such a wide range of activity is difficult to encompass even with the most highly motivated students with good mathematical backgrounds; for the majority of students undergoing a 'method' course the problems are more intract able, not only is the general mathematical background poor,[5] but for the majority of non-specialists (and these are mainly primary teachers) the attitude towards mathematics is at best indifference and at worst antagonism and fear.[6]

3. General Attitudes

The training of mathematics teachers at all levels comes hard up against popular conceptions of science and mathematics, conceptions which are included in the student's belief-system. The popular conceptions run approximately thus: Theories of science are derived from facts, or facts from theories, and as a corollary, theories are accepted or rejected on the basis of factual (i.e. experimental) evidence. The simple conclusion from these ideas is that the scientist readily abandons his old theory in favour of the new, given sufficient evidence. The simplistic approach contains an obvious contradiction. On the one hand, science is right - it 'proves' things about the world, while on the

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4. I am attempting to generalise here, to include the training of non-specialist primary teachers as well as specialists for secondary schools and colleges.
5. It is not necessary to have even O-level mathematics to gain entry to a teacher's college.
6. Manchester University Report (1968). See also Section 1 note 19.

other, old or rejected theories are wrong.[7] Few, if any, contradictory situations are seriously studied in general science education, it is more likely that the old theories are ridiculed and their history and supporting arguments forgotten.[8]

The situations in mathematics education is much worse. All mathematics is right. 'Wrong' mathematics simply does not exist, or if it does, it is only possible by contradiction of right mathematics, and is transient and unacceptable. So far as the relation between teacher and pupil goes, mathematics is transmitted, but not discussed.[9] Can we wonder that the majority of students, unable to cope with transmission are bewildered or even antagonistic. The survivors, so-called 'mathematicians' have their problems, for efficient absorption of mathematics technique may be necessary, but is certainly not a sufficient condition for later creativity.

The actual picture, we know, is much more complex. In science, new theories are derived from others by extension:

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7. While this extreme popular view may be thought to be an exaggeration, I would remind the reader of the many instances where a scientist is reported as proving something right or wrong.
8. For example, Phlogiston Theory. Briefly disposed of on Sherwood Taylor's popular chemistry text, we can see in Agassi (1963) (41-48) how Priestly held on in spite of Lavoisier's alternative.
9. Again, as in the case of science education, there will be many notable exceptions but the old joke about mathematics going from the blackboard to the student's pad without going through the mind of either teacher or pupil still has some currency.

by modification and re-interpretation. Theories shape facts, some are emphasised, some suppressed and theories determine the acceptance or rejection of facts.

From the practice and the history of science we see that the scientist often attempts to 'explain away' anomalous facts, and theories are defended - often fiercely and against all comers by their supporters. This state of affairs is possible because scientific theories appeal to the intellect. They are attempts to provide all-embracing explanations or interpretations of the real world. In this sense, they are symbolic and objective.

Pure mathematics with its heavy current emphasis on structure deals exclusively with symbolic-objective forms. The bases of the constructions of these structures are heuristic, and their evolution is a philosophical historical process.

While for both science and mathematics there is consensus over a large body of knowledge regarded as 'true', the actual picture also contains a number of rival theories or other possible interpretations where science and mathematics are discussed and degrees of truth or acceptability are matters of opinion.

4. Training in mathematical techniques: Discovery methods and Investigation.

The successful mathematics teacher imparts not only knowledge but skill and is able to encourage mathematical intuition and train problem-solvers.[10] Using his belief-

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10. Again, this is not only the case with gifted mathematics specialists at higher levels, but also with 'ordinary' teachers and children in primary schools.

system the individual teacher consciously attempts to train students in particular techniques, and more often unconsciously, gives them some insights into the heuristic procedures necessary for successful problem-solving.

This two-fold aspect of mathematics teaching has been recognised bor a long time by teachers such as Polya,[11] and more recently by Biggs, where attempts to develop children's problem-solving capabilities were separated from the routines of mathematical technique.[12] This latter, an example of the 'discovery method' owes at least as much to the developing methodology of primary education as it does to a critical mathematical philosophy. A continuing theme of such an approach is that children should have had relevant 'experience' before particular mathematical ideas or techniques are introduced.

Discovery methods are not new,[13] the traditional idea being to draw out from the student the answer supposedly lying within. However, the answer is also known to the teacher as the correct result, and the path from ignorance to enlightenment is often carefully contrived so that the student is allowed to discover only what the teacher knows. With the kind of mathematical ideology held by most primary teachers, it has not been possible to improve on a situation which differs in no real way from the past; the body of mathematical knowledge is fixed and children <u>have</u> to discover it. Primary school discovery methods

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11. Polya (1945),(1954) and (1962/5).
12. Biggs (1965) and Hartung and Biggs (1971).
13. The Geometrical Experiment with Meno's slave Plato (1956) is taken as the first example in the discussion of the evolution of Discovery Teaching in Jones (1970)

cannot evolve while the teacher's view of the nature of mathematics is restricted. Because of this, they lack the confidence to accept as respectable mathematics many of the valuable and genuine discoveries of their pupils. The main difficulty here is the attempt to teach heuristic (problem-solving) within the context of mathematics. Polya's attempts speak mainly to the trained mathematician and are difficult to generalise.[14] Rules for problem-solving are like the inductive rules in science - applicable often only by hindsight. A recent development has produced a scheme for suggesting a variety of conjectures or problem-situations,[15] and might be usefully applied to help break through the students inhibitions about the inability to generalise and solve problems.[16] Problem generation and solution is a fundamental mathematical skill and while the problems may not be very interesting to a sophisticated mathematician the practice offered to the student is invaluable.

The place where the teaching of technique and heuristic seemed to be equally successful appears to be in the work of R.L. Moore, where accounts describe the student as having no formal lectures; being set problems but not being allowed to consult texts or papers; being encouraged to

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14. One of the major difficulties is that the communication of heuristic is often personal and charismatic.
15. Walter, and Brown,.(1969) also Brown and Walter (1970) describe "What if Not ?" a strategy for seeking problem generalisations.
16. The encyclopaediaic possibilities of the what-if-not? procedures may be daunting. Choice and Motivation also figure highly - see ATM(1966).

engage in public discussion of the conjectures and proofs of himself and his fellows, and as a result, often finding himself in a position to solve challenging research problems and make original contributions to mathematics.[17]

A number of college teachers were sympathetic to these views and developed them in various ways to use as 'Investigations' in mathematics courses for teachers.[18] Investigations are intended to be original pieces of research at the student's own level and are often suggested by a tutor to a particular student after careful discussion. Progress is monitored in regular tutorials and work is assessed in relation to the student's own mathematical background.[19]

With such an aspect of a course, while students mathematical knowledge and technical expertise may not be so great, their independence, self-reliance and intuition are encouraged, and with this training they will hopefully be

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17. Whyburn (1970). References to the method are also found in Moise (1965) and Wilder (1967).
18. Notably Bell, Morley, and Sturgess at Nottingham, and Caldwell, Leaton and others in the colleges offering the 'Alternative Syllabus' in the London Institute.
19. This introduces problems of the subjectivity of assessment, especially when comparison with most traditional courses is attempted. An account is given in ATM (1966) and it is an obvious theme in ATM(1967). Banwell, Saunders and Tahta (1972) continues to encourage similar activities.

able to develop the skills they may require by themselves.[20] Attempts to justify this kind of procedure claim that mathematics is as much an activity as a set of results, and that traditional mathematics teaching does not do enough towards encouraging the other essential skills a student requires if he is to operate as a mathematician.[21] These skills, it seems, have been acquired by the gifted almost by accident after a long period of initiation.

Here is the dilemma. Do we solve problems by application of acquired skills, or by a knowledge of the kind of skills to develop with the means at our disposal ? The rapidly changing content of mathematics makes the ways of thought a much more useful proposition, if we can find means to classify and communicate them. It is not necessarily true that insights into mathematical heuristic are acquired only in the context of practice in the techniques of mathematics; examinations of the ways in which mathematics arises in the individual and in the social context are also valid means.

5. Descriptions of Teaching

Descriptions of teaching range from teaching as a science to teaching as an 'art', suggesting a wide range of

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20. As with all courses there are successes and failures. 'Investigations' work against a long established tradition and it is not easy to convince students of the value of such an activity. It is also vastly time-consuming if done at all conscientiously, particularly with weaker students.

21. See Morley (1973).

approaches. Methods of teaching and improving teaching question the nature of science, art, learning, knowledge, etc., and in this context, the nature of mathematics. If we accept that theories of science and mathematics have philosophical bases, then theories of mathematics teaching imply philosophical standpoints. Epistemology, the discussion of the origins of mathematics and the grounds of mathematical knowledge can be found in the writings of many mathematical educators.[22] Those views have been carried into teaching usually by implication, though more recently, explicit statements about the nature of mathematics have appeared in books written for teachers at both primary and secondary level.[23] Over the last ten years or so we have seen a much greater emphasis on the production of material for teachers - to interpret that produced for children.[24] The bases of these interpretations lie in a number of different areas, for primary mathematics the major influence appears to come from developmental psychology, while for the secondary school, the influence is more clearly derived from the structures of mathematics itself.

If we examine some theories which may contribute to the interpretation of mathematics, we find that theories in different fields have similar general forms. For example,

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22. For example, those mentioned in Section 1.
23. In both ATM (1967) and Banwell, Saunders and Tahta (1972) teachers are positively encouraged to explore the nature of mathematics and mathematics teaching and learning.
24. For example many secondary textbooksseries have 'teachers books' associated with them while the intention of the Nuffield Project was to provide guides for primary teachers to methods & material already available

Bruner's description of the intellectual growth of the child[25] is similar to Polanyi's description of the historical development of scientific theory,[26] and these both have elements in common with Wilder's ideas of the evolution of mathematical concepts.[27]

While the details of the theories differ, there are similarities in broad outline, and the object of this brief comparative exploration is to suggest that in each we become involved in the gradual objectivisation of

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25. Bruner (1968)(5-6)
26. Polanyi (1964)(160-171)
27. Wilder (1968) (169-181),(207-209).

experience via the use and development of symbols.[28]

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28. For comparison the theories can be briefly described thus:

Bruner

Intellectual development depends on systematic and contingent interaction between teacher and learner via symbols. This growth is characterised by: (i) Increasing independence of response from the immediate nature of the stimulas; (ii) Ability to internalise events into a pattern or system that corresponds to the environment; (iii) Increasing capacity to use symbols to bring order into the environment and to deal with several alternatives simultaneously.

Polanyi

Stages of scientific belief have their own values and vision of reality. Science embraces a consistent pursuit of gradually changing, ever more enlightened elevated intellectual aspirations. (i) Bases of belief lie in numbers and geometrical figures, as, for example, in Pythagorean and Aristotelian science; (ii) Mechanically constrained masses form the basis of gravitation theory and corpuscular theory in the seventeenth century; (iii) More recently, the bases of scientific theories have been in systems of mathematical invariances, as in wave theory, quantum mechanics and continuum mechanics.

Wilder

There exist historical laws which govern the evolution of mathematical concepts. There are two main influences; internal and external stresses which manifest themselves

Footnote 28 continued....

in a number of ways acting at all times, to a greater or lesser degree. Phases of development can be distinguished in individual mathematical topics, for example in the evolution of number concepts we have :

(i) Distinction of units, tallying, ideograms;

(ii) Numerical systems and operations;

(iii) New number types and logical analysis of structures.

From these examples it may be gathered that there are theories of science and the development of knowledge that may be useful - though not originally intended - for teaching. These theories concern the descriptions of intellectual processes via critical dialectic arguments concerning the growth of knowledge.[29]

6. Models useful for teaching mathematics and examining the nature and development of mathematics.

a) The Teacher-Belief-System

The idea of a teacher 'belief-system' is briefly as follows:[30]

i) A teacher's perception and interpretation of sensory data and decision making is shaped by a belief system;[31]

ii) We can influence a teacher by understanding and modifying his belief-system;

iii) Modifications of belief-systems are allowed only reluctantly. Experts in modification are counsellors, psychiatrists, psychologists etc.;

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29. For example the work of Kuhn (1962), Lakatos (1963), and Popper (1963).

30. Davis (1967), p.12.

31. An alternative here might be to use 'cognitive schema'; where a schema is a mental structure which (i) integrates existing knowledge, and (ii) is a tool for the aquisition of new knowledge. However, I think it is important to include affective aspects like awareness, control of attention, measure of response, attitude and valuing besides the more usual cognitive factors like knowledge, comprehension, analysis, synthesis and evaluation. See Bloom (1970) for details of these aspects.

iv) We must find a rhetoric[32] which makes use of the sensory data available to the teacher to improve discourse - on the level of 'practitioners maxims';[33]

v) Consideration of belief-systems allows contact with relevant data wherever it may be found.[34]

vi) Education is the modification of belief-systems.

The idea that we should be able to borrow the skills already available in other fields, develop a media for communication, and widen our awareness of what data may be relevant is powerfully argued in Davis' monograph. An obvious modification in this context is to suggest that a mathematical belief-system may be worth considering. The use of the word belief suggests that it may be largely uncritical, at least initially, and contains, among other things, an ideology of mathematics which is a priori with regard to facts, utilitarian with regard to processes and infallible with regard to truth.[35] The commonly-found mathematical belief-system contains little or nothing about the development of mathematics[36] and no way in which the teacher or student

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32. Rhetoric def = art of persuasive or impressive speaking or writing.

33. Polanyi (1964) see below p.118

34. Relevant data includes the history of mathematics as discussed in Section 2.

35. This amplifies the crude classifications of practical and intellectual attitude in Section 1.

36. Except insofar as this is used to provide data for theories of mental development. Correspondences attempted between the development of mathematics and the development of mental structures are often suspect as we have seen.

can easily obtain an orientation in or understanding of the general mathematical environment.[37]

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37. A brief summary of the justifications of historical study in providing social orientation and contextual understanding from Marwick, (1970), (14-18) is as follows:

(a) "... history is necessary: it meets a basic instinct and need of man living in society."(p.14).

(b) History has two aspects:

(i) Functional. It meets the need which society has to know itself and understand its relationship with the past and with other societies and cultures.

(ii) Poetic. It satisfies the individual's innate curiosity about the past.

(c) Justifications related to these aspects are:

(i) A strong social element. History is the necessary recollection of the past which enables the individual and society to orientate themselves.

(ii) The study of history is part of man's attempt to understand his environment, physical, temporal and social.

(d) History is useful in new situations.

" ... not because it provides a basis for prediction but because a full understanding of human behaviour in the past makes it possible to find familiar elements in present problems and thus makes it possible to solve them most intelligently." (p.18).

b) Practitioners Maxims[38]

Descriptions of the act of teaching mathematics are at this level. Many situations are at present only incompletely described by sets of rules, directions and procedures[39] which can only have full meaning from an inside

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38. Polanyi (1964).

(a) "Maxims are rules, the correct application of which is part of the art which they govern ... maxims cannot be understood, still less applied by anyone not already possessing a good practical knowledge of the art. They derive their interest from our appreciation of the art and cannot themselves either replace or establish that appreciation." (p.31).

(b) "Rules of art can be useful, but they do not determine the practice of an art; they are maxims, which can serve as a guide to an art only if they can be integrated into the practical knowledge of the art. They cannot replace this knowledge." (p.50).

(c) "To learn by example is to submit to authority. You follow your master because you trust his manner of doing things even when you cannot analyse and account in detail for its effectiveness. By watching the master and emulating his efforts in the presence of his example, the apprentice unconsciously picks up the rules of the art including those which are not explicitly known to the master himself."

(p.53 underlining mine)

39. Notwithstanding the most careful and well-intentioned analysis by psychologiests and other researchers.

knowledge of the art. The classic example here is the fact that descriptions of the mechanics of bicycle riding do not help someone to learn to ride a bicycle. In this context, learning a particular mathematical technique does not necessarily enable us to apply it to problem situations. Motivations, appreciation of classes of problem - situations, and an ability to select and modify appropriate techniques are also important. The student's difficulties in applying techniques to solve problems is a well-known experience of teachers at all levels. The application of mathematical techniques is often learnt by example from practitioners or teachers. The method of fluxions in England and the methods of differentials on the Continent in the seventeenth and eighteenth century were learnt by following the 'masters'.[40] There is a wealth of historical evidence to show that the transmission of mathematical techniques by the use of practitions maxims is the way in which heuristic or metaphor is taught.[41]

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40. Technical misunderstanding by certain individuals exaggerated philosophical and logical dilemmas. See Section 4(d).

41. The heuristic or metaphor is the underlying intuitive idea or theory. Practitioners maxims assist the student with examples of the application of the heuristic, and link new heuristic with established data.

(c) The Three Worlds[42]

In relation to the present discussion, Popper's

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42. Three Worlds. (Popper (19)(106-152) A brief outline of the argument is as follows:

a) Three Worlds are:

(i) Physical objects or physical states.

(ii) States of consciousness or mental states or behavioural dispositions to act; (belief-systems, including practitioners maxim).

(iii) Objective contents of thought. This third world is man-made and changing and exists only as a consequence of (i) and (ii). It contains; theoretical systems, problems, problem situations, conjectures, refutations and critical arguments (i.e. the state of a discussion, demonstrations, justifications and proof-processes).

b) Knowledge related to the three worlds is as follows:

(i) "I exist" objects exist.

(ii) "I know" - subjects know. This is knowledge or thought in the subjective sense. (Heuristic knowledge.)

(iii) Knowledge or thought in an objective sense. Problems, theories, arguments - independent of anyone's claim to know, belief or disposition to assent, assert or act. It is knowledge without a knowing subject. This knowledge exists in the same way that the contents of a library exist independently of anyone knowing them.

Footnote 42 contd/....

Footnote 42 contd....

(c) The existence of mathematical structures - produces two classes of problems:

(i) Structure problems.

The study of the structures themselves, the third world axiomatic theories and their relations to other structures.

(ii) Production problems.

The study of the methods used in their construction, and the problems concerned with the acts of production which are in the second world.

(d) The study of the third world can throw light on the second world of subjective consciousness, especially on the subjective thought processes of mathematicians.

three worlds concern:

i) Physical objects, states and potential problem-situations of mathematics. In a sense these are neutral and can only be recognised and acted upon by individuals.

ii) Creative processes of mathematicians, their belief-systems and the transformations possible upon them. Here lie the practitioners maxims which enable the transmission of heuristic, the whole a subjective phenomenon.[43]

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43. The belief that maxims can serve as objective rules is at the root of many problems of teaching mathematics: "We can learn more about the heuristics and the methodology and even about the psychology of research by studying theories, and the arguments offered for or against them than by any direct behaviouristic or psychological or sociological approach. In general, we may learn a great deal about behaviour and psychology from a study of the products." (Popper (1972) p.14.) No theory is ever 'complete' but we examine the quasi-complete stages and probe the logical and methodological difficulties, and at each stage we can make discoveries about the creative processes of mathematicians. Psychology and sociology help to give the background to these developments.

iii) Mathematical structures and the formal processes for the production and practice of mathematics. These structures are often largely autonomous. For example, once the integers exist, we also have a large part of number theory waiting to be 'discovered'.

Much of mathematical teaching has been traditionally concerned with structure problems (merely communicating structures). Mathematical learning, however, is concerned with production problems (problem solving is a subjective situation). Understanding, in this context, is the individual subjectivisation of objective knowledge.

d) Proofs and Refutations.[44]

The method of proofs and refutations concerns the discussion of mathematical conjectures and the improvement and modification of theorems. This method exposes the evolutionary dynamic dialectic of mathematics in contrast to the inhibiting traditional view. Popper's idea that

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44. Lakatos (1963/4) proposes the method of proofs and refutations as a development of Popper's philosophy of science and Polya's heuristic methodology: "... mathematics does not grow through a monotonous increase of the number of indubitably established theorems, but through the incessant improvement of guesses by speculation and criticism .." The example he uses to demonstrate his thesis is the development of Euler's theorem for polyhedra, $F + V = E + 2$, known to Descarte in 1639, and stated by Euler in 1730 to apply to 'all' polyhedra. A typical contemporary modification of the statement

Footnote 44 contd...

of the theorem appears in Hilbert and Cohn-Vossn Geometry and the Imagination, Chelsea 1952, (290-295) as "For a simple polyhedron, simply-connected, F + V = E + 2 Lakatos exposes the two hundred or so years of 'guesses, speculation and criticism' involved in the words simple and simply-connected. The theorem is still called Euler's theorem when generalised in many dimensions as in Coxeter, HSM Regular Polytopes MacMillan (1963).

mathematics grows through the criticism of guesses and bold informal proofs encourages a search behind the accepted statement of a theorem, to discover what may have been the steps in its development.[45]

Traditional teaching of structures often begins with a contemporary theorem where the historical process ends. Motivations for and insights into the nature of the mathematics involved can often be furnished by beginning with an early statement of a theorem, at a convenient point in the historical process, and encouraging a parallel, 'imitative' development.[46] For example, the idea of a limit can begin as being the loose idea of 'near to' and can then be taken through $\delta-\epsilon$ techniques to neighbourhoods.[47]

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45. Popper's critical philosophy of mathematics (Popper 1972 p.136) presupposes the (third world) linguistic-symbolic formulation of the guesses and proofs. Here language is more than a means of communication, it is an indispensable medium of critical discussion, where the objectivity of the statement rests on the criticisability of the argument. The processes whereby these formulations are made are second world inhabitants and the concern of the teacher.
46. One need not go back to the 'beginning', only far enough to make the point about dialectic being involved. For a similar view see Grattan-Guinness, (1973).
47. A typical formal progression can be found in Smith W.K. (1964).

The idea that an examination of third world proofs can throw light on second world heuristic is used by Lakatos in an attempt to communicate that heuristic, by critical examination of facts, theorems and structures.[48] In contrast, Polya's heuristic method is too simple and not very helpful. Its main use is in teaching problem-solving strategies, but for students (and others) there lingers a strong feeling that in order to develop the method one needs to know the result at the beginning. The attempt (if any) to place these strategies in the third world and thus produce general problem-solving <u>methods</u> has not been successful. As such they are commentaries on episodes in mathematics,[49] and are really maxims, inhabitants of the second world which can only be fully communicated by individuals.[50]

7. <u>Historical Studies and Mathematics</u>

We are now in a position to make some further comments on the relevance of history to the study of mathematics. If we accept the general principle of the 'three worlds' we find that we have a useful metaphor for the study of mathematics, for they help to separate for individual attention the formal processes, the heuristic devices, and the problem-situations. The construction of mathematics as a body of knowledge concerns the structures of mathe-

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48. Lakatos is concerned with the growth of the second world arguments while Popper (1972) maintains that the third world is the only respectable study. (P.114)
49. See Thomas (1972)
50. Both third and second worlds are susceptible to analysis. The <u>third</u> world to traditional <u>logic</u> and the <u>second</u> to a philosophy of <u>personal knowledge</u>.

matics as logico-mathematical entities themselves. The study of structures and proofs at this formal objective level is the official activity of the research mathematician, but besides advancing mathematical knowledge qua mathematics, it can also indicate the path to this advancement, the stages that the proofs went through, in order to reach their currently accepted validity. Old theorems and incorrect proofs are still objective, and so the evolution of mathematical structures is potentially, if not actually available when we study current theory. The proof-forms and theorems are data for the historian and the mathematician as the objective products of current cultural and philosophical climates. The structure of mathematical culture lies in the third world.

The creative processes of individual mathematical activity lie in the subjective second world. While the third world provides the formal data; the philosophical standpoints, the belief-systems and the heuristic processes all contribute the means whereby the mathematical culture evolves. This provides the functional activity whereby the student of mathematics can begin to understand its relationship with the past, with other mathematics, and with other subject areas.

To assist this understanding, we need data from the first world; knowledge about problem-situations, the motivations for and generation of techniques, algorithms, etc., from physical problems; the general mathematical climate created by the societies, journals, communications, institutions and the general socio-economic background.

An important aspect of this description is the realisation that the status of a particular piece of mathematics may vary according to the part it plays in a theory. For example, infinitesimals for seventeenth century mathematicians were formal constructs or inhabitants of the third world; later, as philosophical climates and proof-forms changed, they became relegated to second world heuristics.[51] The 'reinstated' infinitesimals of non-standard analysis are new and different inhabitants of the third world. Objects called by the same name are not necessarily the same entity, but this lesson is often not easily learnt in the history of mathematics.

The status of a piece of mathematics varies according to the use we wish to make of it: the formal theory of arithmetic lies clearly in the third world, but the algorithms required in a particular calculation are first world inhabitants. It is the distinction of the use and hence the status of particular areas of mathematics that make the life of the teacher both difficult and challenging, for he needs constantly to be aware of the fluctuating situation applied to individuals.

8. Maxims for Mathematics Teaching and the History of Mathematics.

There are three general groups of maxims for the teaching of mathematics that can be derived from the foregoing discussion; these concern:

a) The teacher's belief-system which is altered and developed by involving the teacher in the criticisms of mathematics;

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51. Note that the Leibnitz infinitesimals remain third world inhabitants. The evolved and less believable infinitesimals of eighteenth and nineteenth century mathematics lie in the second world.

b) The teaching of mathematics which is improved and developed by active participation in the method of proofs and refutations;

c) The practice of mathematics which is the application of incomplete or inconsistent theories and their critical development.

The history of mathematics can provide data for all three classes of maxims, and if a theory of teaching can be derived, such that a critical-evolutionary approach to the learning of mathematics is encouraged, we will have an in-built instrument for curriculum development arising from the nature of the activity of mathematics itself.

Section 4

Case Studies from the History of Mathematics

(a) The Roots of Modern Mathematics

1. Introduction

'Modern mathematics' is an ambiguous label. It can be understood in a number of different ways, depending on the level and context of teaching, and the individual preferences of the teacher. Before attempting to define a common denominator for discussion, it may be useful to distinguish four general interpretations that are common today.

First, the content of a course may be modified by the introduction of new material, which could be historically new, or could recently have acquired new significance.
Second, the re-thinking of traditional material is a modernisation both with respect to the place of that material in the context of mathematics and the current changes in teaching methods.
Third, emphasis on the immediate relevance of mathematics taught, sometimes called the 'utilitarian' attitude, aims to show students that mathematics contains applicable tools as well as abstract ideas.
Fourth, the development of psychological theory and methods influences mathematics teaching not only by suggesting shifts of emphasis, like 'learning' or 'teaching', but also by focussing our attention on the acquisition of particular concepts at different levels.

Since the teaching of mathematics needs to take into account all of these aspects it can be seen that something like syllabus revision plays only a small part in the [illegible]. [illegible] as a teacher to [illegible] subject [illegible] is necessary renewal of mathematics for teaching, for if we replace one list of requirements by another, we are making only a marginal and temporary change, and to argue that this may be a permanent advantage is questionable.

The modernisation of mathematics is a continuous process, and continuous renewal requires an attitude of mind that accepts the idea of a developing, lively subject, relevant not only to sophisticated applications, but also to the individual's way of regarding and structuring the world. It is relatively easy to say that deep concepts are most important, and it does not matter much how these are put across, but it is often difficult to decide what the really significant concepts are, and by what means they should be communicated.

One can make a case for the inclusion of almost any mathematical topic or subject area at different levels in mathematics teaching. To engage in such arguments, on the whole, seems rather pointless since championing a particular topic often depends on the individual's personal taste rather than any substantial pedagogical argument.

Deciding what is important or significant in contemporary mathematics and choosing the content of a teaching programme appropriately depends on the teacher's sensitivity to the four interpretations given above, and besides being as well-informed as possible of contemporary developments, an essential ingredient of the teacher's repertoire is the maturity and hindsight that history can provide.

We have seen, over recent years, the introduction of a wide variety of topics, techniques, methods and projects covering the whole range of school and college mathematics. How is a teacher to make a choice, which subject area is more important, which technique essential, when faced with the whole re-structuring of a syllabus, or the demands of the examination at the end of a term ?

With the depth of view given to us by a knowledge of the historical development of mathematics parallelling our knowledge of the contemporary developments, we may be better able to judge the importance of different aspects of mathematics to our pupils.

For the sake of the present discussion, I take the emphasis on the general concepts of algebraic structure to typify what is generally understood to be 'modern mathematics' so far as this label applies to the content of contemporary mathematics teaching in schools and colleges. The emphasis here is on the first and second interpretations given above; the introduction of structural concepts requiring the reworking of traditional material, being a major fact of the history of mathematical education. An examination of the origins of structural concepts will hopefully assist us in our attempts to answer the 'why' and 'how' questions that may be around when we look at modern mathematics today.

2. General Background.

It is possible to discern many of the most general and powerful ideas in contemporary mathematics emerging in the first half of the nineteenth century. I make this claim because, not only do we have a number of significant breakthroughs occurring in the separate fields of geometry, algebra, analysis and logic, but we have a deep change in social and philosophical attitude which recognises and accepts both the need for unifying concepts and their discussion by the mathematical community at large. We may speculate that this general attitude was motivated by social movements like the French Revolution, but the fact that it began and gained momentum through the nineteenth century is important in the development of western culture, a signficant part of which is mathematics.

A consequence of the application of mathematics in the eighteenth century to more and more complicated physical problems was the emergence of the idea of the mathematical discipline as and end in itself.[1] Foundational problems emerged, the existence and status of infinitesimals being probably the most outstanding, and the more mathematics that was done, the more it created problems for itself. Gradually the connection between the frontiers of mathematical research and the physical world become more and more tenuous, so that by the beginning of the nineteenth century, mathematicians were ready to admit that in some sense at least, parts of their discipline had no necessary connection with reality. These changes in attitude allowed the bolder creation of objects and operations, the discovery of principles or laws, and the hitherto unsuspected relationships between different parts of mathematics. The nineteenth century is dominated by an emerging need to simplify and consolidate the wealth of mathematics discovered earlier; we may even consider it a psychological necessity for the mathematical culture to search for some kind of unifying approach to help

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1. Formation of societies, foundation of journals, and instruction in the applications of mathematics encourage this view.

them come to terms with the mathematics they had created.[2]

Probably the most well-known events of mathematical history of the nineteenth century are the discovery of non-euclidean geometry, the arithmetisation of analysis, and the foundation of the theory of sets. As achievements they are outstanding, but equally important, and generally less well-known (apart from passing references to Abel and Galois) is the emergence of the idea of algebraic structure. Because of their abstraction and generality, structural concepts underlie all areas of mathematics, and consequently are difficult to trace in detail, while the consequences of their application are manifest.

In the first half of the nineteenth century we have a chance to see these structural concepts emerging, initially as vague guiding principles, or as fruitful methods, to be examined and refined into the concepts familiar today. By way of preparation, I should make some remarks about those well-known events mentioned earlier.

3. Non-Euclidean Geometry

By the end of the eighteenth century it was fairly well-known among mathematicians that the long history of investigation into the independence of the parallel postulate had produced a situation where alternative geometries were logically possible, and the only way of justifying

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2. Wilder (1968,1974) carefully distinguishes between cultural, psychological and social influences on the development of mathematics. His (1974) examines 'Hereditary stress' as a force in the evolution of mathematics, but I know of no work that specifically deals with the social psychology of mathematical history.

the parallel postulate was to suggest that its truth was derivable from experience.[3] Euclidean geometry was generally considered to be an idealisation of experience, even an innate property of the world,[4] and the acceptance of a logically consistent but anti-commonsensical geometry as a valid piece of mathematics took a long time. Lobachevsky and Bolyai were the first to publish complete systems and claim them as a new geometry, but it was the change in atmosphere that allowed them to make public what Gauss preferred to keep untold.[5]

The two greatest general consequences of the publicity of the new geometry for the early nineteenth century were the gradual acceptance of the idea that mathematics was not necessarily about objects in the real world, and that it was possible to construct logically consistent systems

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3. Lambert (1766) 'Theorie der Parallellinien' see Kline (1972)p.868.
4. Kant (1781) 'Critique of Pure Reason'. The section on 'The metaphysical exposition of space' claims the principles of Euclidean geometry as 'a priori synthetic' truths.
5. There are a number of studies of the discovery of non-Euclidean geometry; as a preface to a concise work on modern geometry in Blumenthal (1961), a popular exposition in Reid (1963), a detailed technical discussion in Eves and Newsom (1966), historical accounts in Boyer (1968) and Kline (1972), and translations and commentary on Bolyai and Lobachevsky in Bonola (1955).

from an arbitrary collection of axioms.

4. The growth of Rigor in the calculus

The spectacular developments in the calculus in the eighteenth century, particularly in applications of the new techniques to physical problems, had emphasised the need for a logically rigorous basis for the fundamental processes. The belief that some basis would eventually be found contributed to the development of the 'metaphysics of the calculus' which attempted to deal with such problems as the existence and status of the 'infinitesimals' that the new subject contained. Justification of the infinitesimal processes had been a serious problem right from the beginning, but it was not until the nineteenth century that new difinitions for the processes appear that begin to do away with metaphysics and introduce logic in its stead. Most outstanding in this period is the work of Bolzano, Cauchy, Dirichlet, Abel, Riemann and Weierstrass. In general the period is fairly well documented, but from the historical point of view there are still a number of controversial and unsettled issues.[6]

In contrast to the appearance of a new geometry, developments in the calculus were initially much more of a tidying up process, new definitions giving rise to clearer concepts and preparing the way for the next set of problems. In broad terms, developments in the calculus were probably more influenced by the change in atmosphere, allowing mathematicians to experiment with more abstract definitions, rather than having a great influence outside

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6. Simple interpretations of the story can be found in any standard history book. Kline (1972) gives most general detail, while the studies of Boyer (1959), Manheim (1964) Pesin(1970) & Grattan-Guinness(1970) begin to show up

analysis at this time.

5. The Theory of Sets.

The development of set theory can generally be regarded as a consequence of the detailed investigations in analysis, algebra and geometry, and as such falls outside the period under discussion. Cantor's work on the theory of sets can be said to have been motivated by his investigations into the representation of functions by means of trigonometric series, a problem originating with the work of Fourier some sixty years before.[7] While a consistent set theory did not appear until the latter part of the century, we can say that mathematicians were beginning to talk about what we today would call sets, relations and operations - using set language - in the earlier part of the century, particularly in relation to the developments in algebra and logic.[8] Again, it is in the early developments that we see the increasing willingness to consider abstract definitions and arbitrary laws as the basis of mathematical theory.

6. Projective Geometry

The subject of projective geometry appears now to have fallen out of fashion as a part of courses in

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7. Cantor's first paper on the theory of sets appeared in 1874. For a detailed investigation into the background of this development see Dauben (1971) and Grattan-Guinness (1971).

8. Bolzano's work on the arithmetical theory of real numbers dates from the 1830's, and his 'Paradoxes of the Infinite' appeared in 1851. See van Roostellar (1962).

mathematics, surviving only occasionally.[9] To my view, this is unfortunate since it developed as a distinct branch of mathematics in the nineteenth century, providing many of the concepts and methods alive in modern mathematics. The fact that these concepts and methods have largely been taken over by algebra, is part of the evolutionary story of mathematics, but the first occurrence of these ideas in a geometrical context provides an interesting and relatively simple introduction to these general and powerful ideas.

The story of the emergence of projective geometry in the seventeenth century from the theory of perspective appears later,[10] and while the tradition of Desargues and Pascal had been kept alive, mainly by the work of La Hire and Bosse[11] and the applications in technology, the texts themselves were lost, and the mathematicians of the nineteenth century re-created most of projective geometry before the original work was recovered.

It is considered that the man who provided the main motivation for the revival of projective geometry was Monge, a great proponent of synthetic methods in geometry in opposition to the analytical applications of Cartesian

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9. Projective geometry survives in some courses for teachers. It might be interesting to discover the motivations for the inclusion of this topic in such courses.
10. See Section 4(c).
11. Bosse (1648), La Hire (1685). Pascal's short (1640) was recovered in 1779, and Desargues' (1639) was found in a manuscript version made by La Hire by Chasles in 1845. Kline (1972) (288-299).

geometry of the time. Monge became a military engineer, and the kind of calculations necessary in that context required simple, practical procedures. He developed what is known as 'descriptive geometry', and might be called the originator of the techniques of engineering drawing. At the turn of the century he was teaching at l'École Polytechnique in Paris and had a direct influence on the next generation of French geometers who were to revive and develop the new branch of mathematics. While Monge provided the inspiration, many of the results and theorems derived from La Hire, who had developed the work of Desargues and Pascal. I think it is appropriate to give a brief summary of the achievements of the seventeenth century in order to indicate the main ideas that were developed later.

Desargues' basic ideas are as follows:

1. Every family of parallel lines meets at a <u>point at infinity</u>: points at infinity define a <u>line at infinity</u>.[12]
2. A relation between points on a line, involution, was defined which was invariant under the operation of projection.[13]
3. Developing this, <u>conjugate points</u> were defined, a real point on a line being the conjugate of the point at infinity.
4. He then proves that an involution <u>projects into an</u> involution.
5. A <u>harmonic set</u> is defined, and the proof supplied that a <u>harmonic set projects into a harmonic set</u>.
6. The property of <u>pole and polar</u> is developed to all

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12. This idea seems to originate with Kepler in his 'Astronomiae Pars Optica' of 1604. Kline (1972) p.290.
13. Pappus Book VII prop. 130 defines a relation called cross-ratio for plane figures which Desargues developed.

conics, and he shows that the diameter of a conic is the polar of the point at infinity.[14]

7. In all this, he establishes the method of projection and section as a proof-method.

The surviving work of Pascal is much shorter, but it is clear that his declared aim was to reduce the properties of conic sections to be derivable from the smallest number of basic properties. Like Desargues, he had the idea of certain properties of figures remaining substantially unchanged after a projective transformation,[15] and in particular projective properties were invariant under linear transformations. He also exploited the generalised proof-methods, and from their work emerged the non-metric aspects of the new geometry.

It can be seen that while Pascal and Desargues built on the work of their predecessors, what they built was an entirely new geometry, defining a new kind of space where parallel lines met at infinity. Their own declared aims, however, were to produce a new approach to Euclidean geometry which was considered to be a more thorough and more simple description of the real world. The fact that their geometry was not pursued at the time probably had something to do with this attitude, but was also largely overshadowed by the concurrent developments in analytical geometry and

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14. Apollonius Book III gives harmonic properties of pole and polar applied to plane circles.
15. Kepler's work of 1604 contains the idea that the conic sections can be continuously derivable one from another.

analysis.[16]

The underlying ideas coming through to the nineteenth century were the principle of continuity - a general working principle that a property attributable to one figure could also be found in another figure derivable from the first by projection,[17] the idea of a property being invariant under some transformation, and the beginnings of a distinction between invariant properties or results, and invariant operations, a realisation that it was generally impossible and inappropriate to attempt to preserve any metric properties of figures under the transformation of projection, and the use of the method of projection and section along with the principle of continuity as an acceptable method of proving theorems.

Poncelet, a pupil of Monge, is considered to have made the greatest single nineteenth century contribution to the rebirth and establishment of projective geometry as an important branch of mathematics. He wrote a major textbook[18] which reaffirmed and developed the ideas of his

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16. It is often not recognised that Cartesian geometry brings a powerful new technique to the old, Euclidean, geometry while projective geometry is a new geometry. This seemed to be too much for mathematicians to take at the time. This was also the period of the great 'pre-calculus' developments of Cavalieri, Torricelli, Fermat, Roberval, Wallis, Barrow and others.
17. For example, we can regard a circle as a special case of an ellipse where the two foci are coincident.
18. Poncelet, '*Traite des proprietes projectives des figures*' Paris, 1822.

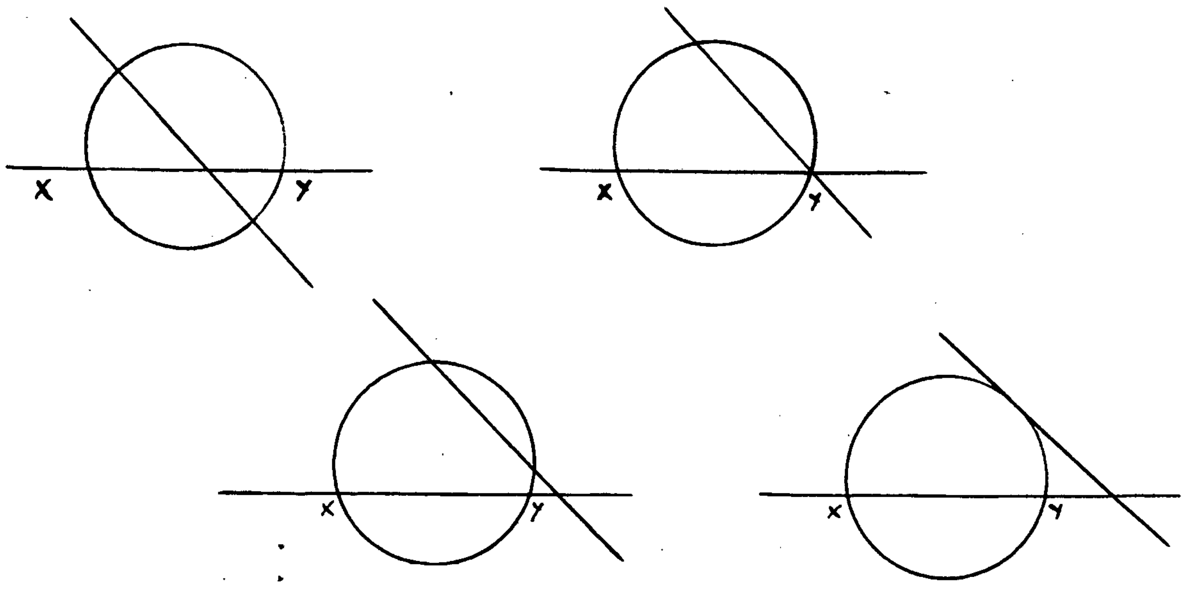
predecessors bringing out new concepts and exploiting the old ideas in a new way. Here we find defined projectivity, perspectivity, and the concept of homology, where two figures are homologous if one can be derived from another by the method of projection and section; the principle of continuity is adopted as an 'absolute truth' of the methodology, and applied to discover new theorems from old;[19] the pole and polar properties are examined, and the remarkable principle of duality is first established where it is noticed that the words 'point' and 'line' are interchangeable in theorems that apply to non-metric properties of figures.[18a] In all this, Poncelet provided the mathematicians of the nineteenth century with a collection of ideas and methods that were to prove fruitful and controversial for the rest of the century.

18a. It was Gergonne who introduced the term 'duality' and generalised Poncelet's idea to apply to non-metric situations. Kline (1972), p. 845.

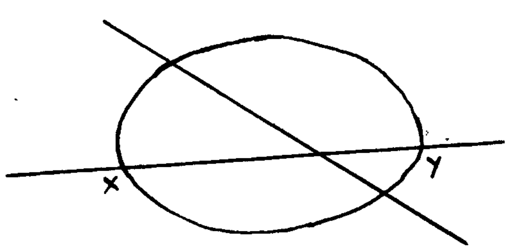
19. Please see following page.

Footnote 19

An example of the application of the principle of continuity appears in Fletcher, T. Some Lessons in Mathematics 1964 (266-269) where angle properties of the circle are demonstrated and discussed by moving a pair of intersecting straight lines drawn on a piece of tracing paper or acetate over a fixed circle drawn on another piece of paper.



By moving the line XY in the horizontal direction we may also get dynamic variations on the intersecting chords theorem, and by projection and section, we can turn the circle into an ellipse, and examine whether similar theorems hold in this case.



Another French geometer, Chasles, wrote a historical study of geometric methods,[20] and re-discovered a work of Desargues.[21] He developed the idea of homography, the general linear transformation of a plane in space, and supported Poncelet in the defence of the principle of continuity as a logical truth against those who regarded it only as an heuristic working principle. Chasles put forward principles for developing proofs, suggesting that special theorems should be generalised to obtain the most general result, which would be 'simple' and 'natural',[22] and that a proof should contain a 'principal truth' that would be immediately recognised because other theorems would result from a 'simple transformation'. The belief that great truths of geometry were simple and intuitive supported the idea that although they could be demonstrated by algebraic methods, the principles and proofs themselves did not depend on algebraic proofs. Ideas like the principle of continuity were accepted at this time as intuitively clear and had much the same status as an axiom.[23]

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20. Chasles, M. Apercu historique sur l'origine et le development des methodes en geometrie. Paris, 1837. In which he admits ignoring German writers because he did not know the language.
21. See above (11).
22. This principle of generalisation was not new at the time, we find it implicitly or explicitly throughout mathematical history. Developed as heuristic, we find it most recently in the work of Polya (1965), and Lakatos (1963-64). (Chasles, (1837), Kline (1972) p.835)
23. Another famous principle was to appear in algebra about the same time. See below. p.151

The next major contributions in this period come from two Germans, Steiner and von Staudt. Steiner produced a systematic development based on three principles:[24]

1. Points, lines and planes were the essential data of geometry, and all other objects, figures or configurations must be deduced from these before we are allowed to discuss them.
2. The principle of duality was extensively used to further this aim, and double column printing was employed to emphasise the methodology.[25]
3. The points, lines and planes were assembled into a fundamental concept, projective forms; ranges of collinear points, pencils of concurrent lines and pencils of concurrent planes.

From Steiner we have the now standard method of defining conics in projective geometry as a set of points of intersection of all pairs of corresponding lines of two projective pencils.

Von Staudt perceived two basic weaknesses of projective geometry which he was able to resolve. The first concerned the definition of cross-ratio from which other projective relations were derived. This was still dependent on length and he sought a projective relation independent of a metric concept. The second was the definition of the imaginary

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24. Steiner (1832); Kline (1972) (846-848).
25. Double column printing was to become a standard layout for texts on projective geometry through to the twentieth century. See Cremona (1885), and later editions, for example.

entities used freely for example, when two circles intersect in two points, the points define a real line, while two non-intersecting circles were said to intersect in an imaginary line. Knowing that it was possible to define imaginary numbers in terms of real numbers in algebra,[26] von Staudt asked if there was some analagous way of defining the imaginary entities of geometry. He succeeded in defining a projective relation[27] to be when two fundamental one-dimensional forms (ranges of points, pencils of lines etc.), have their members in one-one correspondence, and a harmonic set of one corresponds to a harmonic set of the other. To tackle the imaginary elements von Staudt decided to define an imaginary line as all points common to two planes with no common real line, with corresponding definitions for plane and point.

A further development was his 'algebra of throws'. This was an outcome of a search for another meaning for cross-ratio independent of distance, and provided a projective algorithm for cross-ratio and imaginary entities. The algebra uses coordinates as identification symbols free from any connection with length, and defines a harmonic throw to have a value of -1. All rational throws can be built from this and it can be shown that they correspond to cross-ratios in the Euclidean plane. The algorithm is also able to distinguish between a complex entity and its conjugate. In order to achieve this, von Staudt had to use geometric constructions that defined the operations with

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26. Hamilton (1837) see below, p.154
27. Called collineation.

his coordinate symbols so that they obeyed the commutative, associative and distributive laws. The coordinate symbols were, in fact, numbers, and thus he was able to use the ordinary laws of arithmetic to operate in his geometry.[28]

Von Staudt not only freed projective geometry from its dependence on metric concepts, but showed that it was more fundamental - that euclidean geometry could be derived from it.

The first fifty years of the nineteenth century saw some of the most fundamental ideas of projective geometry being developed. The intellectual atmosphere, the freedom from necessary connection with reality exploiting the emerging axiomatic method, the concurrent advance in algebraic description of projective concepts of Mobius and Plucker and others,[29] and the developments in abstract algebra, led to the systematic description of geometry from the transformation viewpoint in the famous 'Erlanger

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28. More detailed accounts of von Staudt's geometry are given in Coolidge (1940) (99-101) and Kline (1972) (850-851).
29. Mobius (1827), Plucker (1828-31), where homogeneous and trilinear coordinates are introduced respectively. Plucker also introduces the concept of line coordinates and extended projective concepts to the algebra of curves of the third and higher degree. Kline (1972) 852-855

Programm' of Felix Klein.[30]

7. The Laws of Algebra.

The algebra of the early nineteenth century was very different in style from that we know today. Although many important technical developments had been made in the application of algebraic methods to geometry, in the solution of equations, in the algorithms and processes arising in the calculus, and in the development of notations and techniques arising from them, the general belief was that algebra was some kind of 'literal arithmetic' where if its truth was in doubt, numbers could be substituted to check the calculations.

At this time there was no precise definition of real or complex numbers and no logical justification for the operations on them, although they were freely used; letters were manipulated as if they were integers and the results were assumed valid when any number was substituted. The algebra of literal expressions was thought to possess a logic of its own which accounted for the effectiveness and correctness of its results. The justification of operations with literal or symbolic expressions was a major problem that emerged at the beginning of the nineteenth century.

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30. Klein, (1872) in N.Y. Math.Soc.Bulletin, 2,1893, (215-249). A good exposition of these views can be found in Klein (1939). Here we have topology first defined (in 1872) as the study of all properties of a space that are invariant under one-one bicontinuous mappings. (1939) (105-108).

While not justifying operations, a kind of classification or hierarchy existed which described the type of numbers the algebras dealt with, however, this was not at all clear, since the numbers themselves were not well defined.

The different algebras were generally described as follows:

Universal Arithmetic was the algebra of integers and other positive real numbers.

Single Algebra was the algebra that allowed the introduction of negative quantities in addition to positive quantities, what would today correspond to the algebra of positive and negative reals.[31]

Double Algebra was the recently established algebra of complex numbers, known only at this time in their geometrical forms given by Argand and Wessel.[32]

Triple Algebra, not yet achieved, was the name given to the attempted extensions of the double algebra into three dimensions.[33]

Across this, we have in the 1830's Peacock's classification into arithmetical and symbolical

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31. While negative quantities had been used in algebra on and off for some time, there was still some debate as to the 'reality' or status of negative quantities. How to interpret the negative root of an equation was a perennial problem.
32. Wessel (1797), Argand (1806), see below, p. 153 note 41.
33. This was attempted by a number of mathematicians. See Crowe (1967) for details.

algebra.[34] Arithmetical algebra was where the symbols represented positive integers, and only operations leading to positive integers were permissible. There was no need to justify this kind of algebra since it was derived from the operations of arithmetic. Symbolical algebra used the same rules as arithmetical algebra, but its operations did not necessarily lead to positive integers alone. The justification for results here seemed to rely partly on a belief in the consistency of the rules,[35] and partly on some vague idea that the internal logic of arithmetic somehow carried over into negative, irrational and complex quantities. Results in arithmetical algebra were said to be general in form, but specific in value, while those in symbolical algebra were general in form and general in value. For example, the quadratic equivalence:

$$ax^2 + bx + c \equiv (x-x_1)(x-x_2)$$

is general in form. It is specific in value when x_1 and x_2 are positive integers, but general in value when x_1 and x_2 are allowed to take any value, positive, negative, real or complex. 'General in form, specific in value' in a sense

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34. Peacock's ideas can be seen forming in his (1830) and (1833), to be stated more completely in the second edition of his (1842-45).
35. The rules identified at this stage were the commutative and distributative laws, identified by Servois in 1814/15, and described in terms of the operations of + and x.

defines the rule, while 'general in form, general in value' extends the rule to other classes of numbers.

The basic principles of symbolical algebra were that the symbols should be unlimited in both value and representation, the operations on them should be possible in all cases, and that the laws of combination of the symbols should correspond with those of arithmetical algebra and the operations should go by the same names as in arithmetical algebra.

This appears to be a first step in defining operations in a new domain to correspond with operations in an already existing domain. The insistence on correspondence of operations can have two general results, new numbers can be invented (negative and complex numbers had already appeared as a result), and the search for extensions of number systems with analagous properties can be made more difficult, as the hoped-for 'triple algebra' was to show.

Peacock's <u>Principle of Permanence of Equivalent Forms</u> enunciated in 1833,[36] was 'whatever algebraical forms are equivalent when the symbols are general in form but specific in value will be equivalent likewise when the symbols are general in value as well as in form.' This became both a metaphysical guarantee of the logic of the algebra, and a heuristic device in the search for extensions

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36. Peacock's 'Report on Recent Progress and the Present State of Certain Branches of Analysis' to the British Association, 1833.

of algebraic operations.[37]

The second edition of Peacock's 'Treatise on Algebra' (1842) claimed that algebra was a deductive science where the processes of algebra were based on a complete statement of the laws that dictate the operations used, and the symbols of algebra have no meaning except that given to them by the laws. The laws given here were the associative and commutative laws for multiplication and addition, the distribution of multiplication over addition, and a cancellation law, $ac \Leftrightarrow bc = a=b, c \neq 0$.(38). The principle of permanence of equivalent forms justified the extension of these laws from the original arithmetical domain to the more general algebraic situation.

De Morgan also held that algebra was a collection of meaningless symbols and a set of arbitrary laws obeyed by them.[39] It was in the attempts to minimise the number of symbols and laws that the fundamental logical processes were exposed.

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37. The principle was used earlier by Woodhouse (1803) 'The Principles of Analytical Calculation' p.3, where in the equation: $\dfrac{1}{1-r} = 1+r+r^2 + \ldots$

    the equality sign has a 'more extended signification' than simple arithmetical equality. For discussion of this see Kline (1972) (974-977).
38. The term 'associative law' for the property of the operation such that a (b c)=(a b) c, was first used by Hamilton. See Crowe (1967) p.16.
39. De Morgan (1849) 'Trigonometry and Double Algebra' see note 72. p.164 below.

The principle of permanence of equivalent forms was to survive, mainly as a heuristic rule, for some considerable time. In spite of the fact that it begged questions about the properties of numbers, its justification was, to say the least, highly suspiscious, and if maintained, rigorously destroyed the possibilities of generality in algebra.

Hamilton's definition of complex numbers as ordered couples began to erode the principle, because it opened the way to consideration of more general 'numbers' and the operations possible on them.[40]

At that time there were two general ways of <u>representing</u> complex numbers, or imaginaries, the directed line segments of Wessel, where geometrical representation of vectors defined operations with complex numbers, and the rotation operation of Argand extending the line with positive and negative direction into the plane by a quarter turn representing $\pm\sqrt{-1}$.[41]

While most regarded these geometrical representations as a basis, both Gauss and Hamilton were suspicious and thought of geometrical representation as an <u>aid to</u>

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40. Hamilton (1837). Gauss in 1831 also describes complex numbers as ordered couples and describes operations of addition and multiplication for them. See Crowe (1967)(8-9)

41. For a summary of the differences and details see Crowe (1967) (5-11).

<u>intuition</u> but not a satisfactory justification.[42]

Hamilton's algebra of ordered couples incorporated the 'imaginary' quantity 'i' in the definition of the operations and founded complex numbers on the reals. Permanence of form and the concept of double algebra became redundant, but this was realised only after some time for Hamilton's ideas were generally slow to be taken up, and the terminology survived, Hamilton himself talking of a 'triple algebra' as a possible three-dimensional extension of his idea.[43]

The search for the three-dimensional complex number was the next step to be attempted, and we know that Gauss had made some investigations in 1819 proceeding from the idea that if a+bi represented perpendicular displacements in the plane, there should be a third component perpendicular to the plane, and had developed a non-commutative algebra. Abandoning the commutative law, thought to be a basic logical principle, was unacceptable, at least to the general mathematical community, and Gauss did not publish his results. Twenty-five years later the discoveries in mathematics and the changes in the intellectual climate

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42. Crowe p.9., considers Hamilton heard of Gauss' work only after he had developed the main ideas of his number couples and quaternions.

43. Hamilton's definitions have become standard. For two complex numbers a+bi, c+di, $(a,b) \pm (c,d) = (a \pm c, b \pm d)$

$$(a,b).(c,d) = (ac-bd,\ ad+bc) \qquad \text{and}$$

$$\frac{(a,b)}{(c,d)} = \left(\frac{ac+bd}{c^2+d^2},\ \frac{bc-ad}{c^2+d^2}\right)$$

allowed Hamilton, not without a struggle, to abandon his own insistence on a commutative rule.

Hamilton's hoped-for properties for a 'triple algebra' were as follows:[44]

The operations of addition and multiplication should be associative and commutative; multiplication should be distributive over addition; division should be unambiguous; NX = N' should be unique and X should be of the same form as N and N'; the law of moduli should hold;[45] and the new numbers must be interpretable in terms of three-dimensional space. All these conditions are satisfied in two dimensions by complex numbers.

The nearest Hamilton could get to his specified properties were the quaternions, four-term hypercomplex numbers of the form w + ix + jy + kz. The basic relations between the complex factors were $i^2=j^2=k^2=ijk = -1$, and the commutative property was replaced by anticommutativity where, in general, for two quaternions q, and q', qq' = -q'q. Hamilton talked of the quaternion Q as being Q = scalarQ + Vector Q and interpreted the imaginary part as a three-dimensional vector.

Being unable to find an algebra of number triples that conformed to his requirements,[46] Hamilton developed

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44. Crowe (1967) p.28

45. The law of moduli for multiplication of two number triples is if $(a_1+b_1i+c_1j)(a_2+b_2i+c_2j) = (a_3+b_3i+c_3j)$ then

$$(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2) = (a_3^2+b_3^2+c_3^2)$$

46. C.S. Peirce showed a number-triple algebra to Hamilton's requirements was not possible in 1881.

an algebra of number quadruples. It has been suggested[47] that Hamilton's development of an algebra of number-couples made it more likely that he would entertain the more general idea of an extension to number triples, number quadruples, etc. Also, reliance on geometrical representation as a justification of the new algebra would have tied him conceptually to number triples as the next stage and any four-dimensional or higher algebra would have been hard to conceive. In any case, the four-dimensional algebra with its sacrifice of the commutative law, even though replaced by anti-commutativity was hard for the mathematical community to take. Hamilton's discovery was the first well-known, consistent and significant number system that did not obey the laws of ordinary arithmetic. The fact that a meaningful algebra was allowed to violate the laws of arithmetic stimulated mathematicians to search not only for other algebras, but also for the logical justifications, and to provide another growing point for what has become the 'foundations' of mathematics.

Hamilton worked on a further generalisation and Grassmann's calculus of extension was an n-dimensional geometry and a generalisation of complex numbers. Grassmann's work was difficult[48] and was not translated into English until much later, but he has been suggested as the real creator of the geometry of R^n.[49]

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47. Crowe (1967) (26-27)
48. Grassmann (1844) 'The theory of linear extension, a new branch of analysis'. The revised edition of 1862 was more acceptable.
49. See Klein (1939) p.61.

8. The Solution of Equations and Galois Theory.[50]

The fundamental theorem of algebra had been stated by Girard in the early seventeenth century and an adequate proof of the proposition that every polynomial equation of degree n has n roots had been sought since that time.[51]

An interesting recent discovery shows that D'Alembert provided the first proof in 1746, and Gauss, Argand, Legendre and Cauchy followed his line with a series of proofs using the properties of polynomials as analytic functions, thus making the theorem part of complex analysis.[52]

In the latter part of the eighteenth century Lagrange analysed the known methods for solving second, third and fourth degree equations in the hope of providing clues for the solution of equations of higher degree.[53] He devised a general method for solving second, third and fourth degree equations but failed at a solution of equations of the fifth degree. After a great deal of careful work, Lagrange was led to suspect that a general solution of a polynomial equation of degree n was likely to be impossible for $n>4$.

The type of solution sought was in terms of radicals. This meant that the expression obtained for a root of an equation was composed of the operations of addition, subtraction, multiplication, division and the extraction

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50. Kline (1972) (752-771)
51. Girard, A. 'L'Invention nouvelle en l'algebre', 1629 gave the first general statement. A large number of mathematicians in the seventeenth and eighteenth centuries gave some time to the contemplation of this problem. See Kline (1972).
52. Petrova (1974)
53. Lagrange (1770/1), Kline (1972)(600-606)

of roots; an example is Fontana's solution of $x^3 + px + q = 0$ where:

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} \quad (54)$$

Lagrange's suspicion that the solution of the general quintic and higher degree equations was impossible in these terms came not so much because he couldn't find a solution, but because his investigations furnished his intuition and gave insight into some reasons why his methods were successful for $n \leq 4$ and not for $n > 4$. His technique showed that solutions were possible where certain functions of the roots of an equation remained unchanged after certain permutations.[55] Lagrange had derived the theorem that the order of a sub-group must be a divisor of the order of a group in these terms. These methods later led to the development of the theory of substitution and permutation groups, and the techniques for elimination of unknowns provided material for the theory of determinants and the theory of matrices.[56]

Gauss too became convinced that the general solution of the quintic was impossible, and showed in one of his

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54. Stewart (1973)p.xiv. This was published by Cardan in the Ars Magna of 1545.

55. For example, in $x^2+bx+c=0$, the functions $x_1+x_2 = -b$ and $x_1x_2 = c$ are symmetric functions because their value does not change when x_1 and x_2 are interchanged.

56. Vandermonde (1772) was the first to give a theory of determinants, [illegible] HCS on [illegible], and Ruffini made several attempts to show the solution of the fifth degree equation impossible at the turn of the century. Kline (1972)(605-606).

proofs of the fundamental theorem of algebra that when the roots of the cyclotomic equation, $x^p - 1 = 0$ for p a prime can be expressed in radicals, the corresponding polygon with p sides can be constructed with ruler and compass.[57]

The proof was finally furnished by Abel who was acquainted with the work of Lagrange, Gauss and Cauchy, and who after first thinking he had found a solution of the general quintic, proved the impossibility of solving by radicals an equation of degree greater than four in 1824.[58]

One might think that this was the end of the story, but as in so many instances of mathematical discovery, it was the beginning of a vast new field. It was well known that some equations of the fifth degree and higher were soluble by radicals, and so the next part of the research programme was to attempt a classification of all polynomial equations to sort out the soluble from the insoluble.

This is in fact what Galois initiated. He began by giving a simpler proof of Abel's result and then extending the argument to obtain necessary and sufficient conditions for equations of all degrees to be algebraically solvable.[59]

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57. Gauss (1801). The p^{th} roots of unity in this case are given by: $x = \cos \frac{2\pi k\alpha}{p} + i \sin \frac{2\pi k\alpha}{p}$ where k = 1,2,3...p and p is a Fermat prime. Kline (1972)(752-754).

58. Abel(1826) Kline (1972)(754-755)

59. Galois' life was tragic and rebellious. His biography and mathematical achievements have been extensively discussed. See for example, Sarton (1937),Birkhoff(1937) and Infeld (1948).

An important new idea that he introduced was that of a normal subgroup. If we can express a given polynomial in the roots of an equation as an algebraic function of the coefficients of the equation, then we can express its conjugates - obtained by permuting the roots - similarly. Galois also showed that the permutations leaving any polynomial and all its conjugates invariant were a special kind of subgroup of the symmetric group, the normal subgroup. Galois was the first to know the exact relations between group theory and the theory of equations and begin to understand them as we know them today, but he was not the first to see that there was a relation, as the previous fifty years had shown.

It was not until 1870 that Galois' work became fully public, but many of his ideas had infiltrated other areas of mathematics to be used with great effect.[60] The concept of the group organised algebra, and algebraic geometry to inspire Klein's reorganisation of geometry from the transformation viewpoint in 1872.[61] Here the different branches of geometry are characterised by groups of transformations leaving fundamental relations invariant. Further investigation into the nature of invariance, together with the group concept and axiomatic procedure has produced the

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60. Liouville edited and published part of Galois' work in 1846, Serret used some of his ideas in a textbook in 1866, but the first full and clear exposition of Galois appears in Jordan, 'Traite des substitutions et des equations algebriques' (1870).

61. This was the 'Erlanger Programm' of Klein (1872).

structure of contemporary modern algebra.[62]

A deep and pervasive influence of group theory appears in the philosophical school of structuralism. The view that a structure is a system of transformations is elaborated in a number of disciplines that can be said to exhibit a system together with laws of composition.[63] The largest class of these structures are not strictly logical or mathematical for their transformations evolve over time, and are governed by laws which are not strictly 'operations' in the mathematical sense. This meaning of transformation laws depends on the idea of 'feedback', and is said to operate in the areas of linguistics, sociology, psychology, anthropology, etc. An early application of simple mathematical structure in economics[64] was synchronous with the structural studies of mathematicians themselves, but the first abstract structures to be isolated and identified for what they were, were necessarily mathematical. Today, "... the structural models of Levi-Strauss, the acknowledged master of present-day social and cultural anthropology, are a direct adaptation of general algebra."[65] Because of the abstract and general nature of the group concept, Piaget sees in them "... a kind of prototype of structures in general, and since they are defined and used in a domain

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62. Van der Waerden's 'Moderne Algebre' is often considered to be the major text that marks the beginning of 'modern' mathematics.
63. This view is expressed in Piaget (1971)p.5-16.
64. Cournot, 'Researches into the mathematical principles of wealth' (1838).
65. Piaget (1971)p.17.

where every assertion is subject to demonstration, we must look to them to ground our hope for the future of structuralism. ... the group concept or property is obtained ... by a mode of thought characteristic of modern mathematics and logic - 'reflective abstraction' - which does not derive properties from things but from our ways of acting on things, the operations we perform on them perhaps, rather, from the various fundamental ways of coordinating such acts or operations - 'uniting', 'ordering', 'placing in one-one correspondence', and so on ... Group structure and transformation go together. But when we speak of transformations we mean an intelligible change, which does not transform things beyond recognition at one stroke, and which always preserves invariance in certain respects... It is because the group concept combines transformation and conservation that it has become the basic constructivist tool. Groups are systems of transformations; but more important, groups are so defined that transformation can, so to say, be administered in small doses, for any group can be divided into subgroups and the avenues of approach from any one to any other can be marked out."[66]

Further discussion along these lines suggests that the basic structures of mathematics (as suggested by Bourbaki) correspond to those necessary for all intellectual activity, a conclusion apparently reached independently by mathematics and psychology.[67] This latest version of the biogenetic

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66. Piaget (1971) (19-21)

67. Piaget (1971) p.27 also to be found in Lamon (1972) (117-136).

law[68] need not be argued here, but it shows the profound influence of abstract mathematics not only on philosophical thought, but also on practical mathematical education.

9. Mathematical Logic.

Formal analysis of Aristotelian logic appeared with the work of De Morgan and Boole in 1847.[69] The general motivations for these works came from the developments in algebra, analysis and geometry seeking the fundamental laws of mathematics. Hamilton's quaternions disobeying the laws of arithmetic, Liouville's proof of the existence of transcendental numbers,[70] and Bolzano's attempt at an

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68. The biogenetic law appears in various contexts and guises. Its contemporary enunciation 'ontegeny recapitulates phylogeny', or 'the path of mental development follows that of historical development' appears to be due to the German philosopher Haeckel. To subscribe to this requires a structuralist view of the history of mathematics. (See above Introduction p.   Section 1.)
69. De Morgan (1847) 'Formal Logic', Boole (1847) 'Mathematical Analysis of Logic'1 Leibniz, in his 'De Arte Combinatoria' of 1666, (published 1690) developed the ideas of logical operations and abstract relations. The depth of his ideas were not appreciated at the time, and were only realised when his work was edited at the beginning of this century. So far as we know, he had no influence on the early nineteenth century.
70. Liouville (1844) Kline (1972) p.981.

arithmetical theory of real numbers[71] showed both the possibilities and the problems ahead. De Morgan's work on the laws of algebra and mathematical logic is closely related,[72] and in popular writing he claims the first principles of mathematics to be ".. self-evident, and though derived from observation do not require more of it than has been made by children in general."[73]

Boole's approach was more subtle, going deeper than pure observational experience to establish fundamental ideas, he claimed that "The laws of thought, in all its processes of conception and of reasoning, in all those operations of which language is the expression or the instrument, are of the same kind as the laws of the acknowledge processes of mathematics."[74] His example of the general structural identity of logic and algebra is developed by Piaget and others into the concept of the Logical Group which forms a basis for the description

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71. Bolzano (1830-35) See van Roostellar (1962).
72. De Morgan's Trigonometry and Double Algebra appeared in 1849. Here he suggested algebra was a collection of meaningless symbols, (0,1,+,-,x,-,(), and letters), and a set of arbitrary laws, the associative, commutative, distributive and some others he considered essential.
73. De Morgan (1831) 'The Study and Difficulties of Mathematics' from the 'Library of Useful Knowledge'.
74. Boole (1854) 'An Investigation into the Laws of Thought' quoted in Tahta (1972)p.71.

of thought processes in evolution and operation.[75]

Later in the century mathematical logic was to emerge as a pure science of symbols for the foundation of arithmetic, and hence of all mathematics. This general aim has now been abandoned, but the influence of mathematical logic both in its practical applications and its philosophical implications is profound.[76]

10. Conclusion

The period we have considered was one of profound and remarkable change; social changes questioned the traditional order of life, and philosophical changes questioned the traditional attitudes and beliefs of science

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75. Piaget (1953) 'Logic and Psychology' and Piaget (1972) 'The Principles of Genetic Epistemology' (19-51) in particular.

76. The logical foundation of mathematics was begun by Frege (1879), and furthered by the work of Russell and Whitehead. Russell claimed that Boole invented pure mathematics, see Newmann (1956)(1576-1590). In Russell's sense, logic and pure mathematics are synonymous. The claim, however, is over-rated, for what Boole did was to contribute a symbolic system to an algebra of logic. While the general philosophical programme failed, the techniques of logic and logical algebras combined with those of modern engineering have produced a vast field of applications, and the influence of the idea that mathematics is a 'logical' subject - it inevitably obeys arbitrary rules - is still strong with the non-mathematician.

and mathematics.[77] There was a dynamic interaction between expectation of change and acceptance of change, particularly in the social and intellectual fields.

The application of mathematics to physical problems had made tremendous advances in the previous century, encouraging attempts at more subtle and difficult questions, which in their turn raised mathematical problems demanding a deeper analysis of the mathematics involved.

Here we see emerging views on the objectivity of mathematics, beliefs in the arbitrary and abstract nature of the fundamental processes, encouraged by the creation of new numbers and geometries, and the exploration of their relation with established, well-known systems. Classification problems arose to deal with the new entities, and hierarchies of algebra and geometry were derived.

The interaction of the generalisation of entities and operations was carried forward by heuristic principles, of continuity, of permanence of form, which were slowly abandoned as the definitions of the entities and operations became clearer. The crystallisation of the idea of something being the same but different after an operation, the recognition of what had changed and how, and the possibility or otherwise of reversing such a change, gave rise to the

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77. The century began with Volta's invention of the battery. Dalton's atomic theory, the battle of Waterloo, the synthesis of organic chemicals by Wohler, Lyell's 'Principles of Geology', Comte's 'Cours de Philosophie Positive', Faraday's electromagnetic induction, the Communist Manifesto and the Voyage of the Beagle; all occurred in the first fifty years.

generality of the concepts of transformation and invariance.

The appearance of duality, the interchangeability of entities giving an alternate consistent system suggested a deeper truth, that of a structure behind a structure. Axiom systems were emerging, meaningless symbols manipulated by arbitrary rules, which gave rise to problems of interpretation and truth on the one hand, and on the other to hopes for the description of logical structures that would provide secure foundations for the whole of mathematics.

The increase in abstraction and generality had a gradual but profound effect on proof methods, and there arose an increasing division between intuition and logic as the unofficial and official media of communication.

All this can be seen happening in the first fifty years of the ninettenth century, and a historical study while the ideas are still young - and simple enough to be able to see the basic outlines without requiring too much background - shows the interconnection of mathematical ideas with each other and the general culture.

It is important for teachers to have some idea of the origins of modern mathematics. The mathematics we teach has arrived by a process of evolution, and it is vital to have an understanding of that process, for it helps to identify the central ideas and to isolate the growing points which on the whole are deeper and more abstract than particular topics or techniques taught.

There are logical and psychological reasons for teaching particular topics or techniques at certain stages in the individual's mathematical development, the ways and means of teaching and learning are discussed in heuristic, and our knowledge of history helps us to make judgements about the overall relative importance of a topic, and to explain how it came to be.

b) <u>Some Aspects of Elementary Mathematics</u>

The psychological, developmental and linguistic bases of childrens learning were first investigated in a systematic way by Piaget[1] and Vygotsky[2] and have continued to be studied notably by Bruner.[3] These studies have what is generally recognised to be a scientific/philosophical foundation, in contrast to the earlier, more 'craftsman-like' approach of many teachers which culminated in the work of Froebel[4] and Montessori.[5] The contrast is between theories obtained from established and developing theories in experimental psychology of a hopefully objective nature, and the intuitive theories based on a teacher's belief-system and observations of children in the classroom.

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1. Piaget (1926). Piaget's first paper (on verbal reasoning in children) appeared in 1921.
2. Vygotsky (1965), first published posthumously in 1934. (Vygotsky began this work in 1924). Chapter 6 deals with 'The Development of Scientific Concepts in childhood.'
3. Bruner J.S. Goodnow J.J. and Austin G.A. (1956)
4. For example: Froebel F. <u>Friedrich Froebel's Pedagogics of the Kindergarten</u>. (Tr.Josephine Jarvis) London 1906.
5. For example: Montessori Maria, <u>The Montessori Method</u> N.Y. 1912.

From these investigations a number of contrasting and contradicting ideas emerge, but the more recent 'scientific' theories have been interpreted in Biggs[6] to produce the following 'Deductions from Research':

"1. Children learn mathematical concepts more slowly than we realised. They learn by their own activities.

2. Although children think and reason in different ways they all pass through certain stages depending on their chronological and mental ages and their experience.

3. We can accelerate their learning by providing suitable experiences, particularly if we introduce the appropriate language simultaneously.

4. Practice is necessary to fix a concept once it has been understood, therefore practice should follow, and not precede, discovery."

These 'deductions' and others of a similar nature, form the basis of the child-centered approach to the learning of mathematics and are to be found, explicitly or implicitly in practically every contemporary text.[7]

I find these deductions dangerous. Since they are stating the obvious (any thoughtful teacher might arrive at them given time and opportunity for discussion), they are so simplified as to be open to all kinds of interpretations.

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6. Biggs (1965)p.9.

7. The fact that they are produced as deductions from scientific theories as opposed to deductions that a teacher with a good knowledge of the nature and processes of mathematics might have made on a basis of classroom experience, I find a remarkable example of the authority of science.

They are truisms. Their negations are statements which are obviously wrong to anyone with experience of young children, and as such, it is doubtful if they provide any real help.[8] In particular, the mathematics which children are now encouraged to learn at the elementary level does not necessarily reflect their interests or modes of thought, nor is it necessarily relevant to their stage of development. Yet teachers are persuaded to act as though it were.

How has this situation come about ? Most of the elementary mathematics taught in primary school is still called 'number work' and as such reflects the apparently simple and obvious number facts and relations which (it is said), first became apparent to our ancestors. The mathematical-theoretical basis of this work now lies in what is called the 'Foundations of Mathematics'.

Without going into too much detail[9] we can sketch the basic ideas of the foundations of mathematics which give us that area called the structure of the integers. We are familiar with the notion of 'set' which, in some way is interpreted as a collection of objects classified according to some agreed criteria, and the 'equality of sets' which is interpreted as the ability to put any two 'like' sets into one-one correspondence. 'Cardinal number'

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8. Of course, Biggs goes on to detail how these deductions may be put into practice on the communicating and organising level. Little or nothing of real significance is said about the nature of mathematics, or mathematics <u>learning</u>.

9. This outline is taken from Wilder (1965) Chapters

is a label attached to a set of a particular 'size', and two sets have the same number if they can be put into one-one correspondence with each other, there being no 'elements' unmatched. An essential interpretation of the mathematical idea of 'cardinal number' is that the 'number' of a given set is constant and independent of any change in configuration of the discrete elements of the set. In its interpretation in the object-world, one certain **thing** two arbitrary sets can have in common is their 'number'.

Having established the idea of cardinal number we then proceed to relate this with the idea of 'size' as a relative attribute, and a relation '$<$' is defined such that if α and β are two cardinal numbers, the statement '$\alpha < \beta$' corresponds to the statement '$A \subset B$' for sets. Thus we obtain an 'ordering' of numbers. This can be looked on as the interpretation of a kind of 'one-more-ness' in the size of sets. From here we proceed to the idea of an ordered pair (a,b), and ordered triple (a,b,c), etc., and the idea of 'order type' where the sets $\{a,b,\}$ and $\{b,a,\}$; and $\{a,b,c,\}$ $\{b,a,c,\}$ are of the same 'order type': (by convention 'ordered pair' is synonymous with 'order type 2', etc.) Thus a 'numeral' say, '5' can be considered as a cardinal number (some indication of the 'size' of a set), and also as an 'ordinal number' which designates an 'order type', and is interpreted in some way as a concept of 'next in size', or 'next in line'.

These ideas have arisen from the development,[10]

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10. See 9. and also Eves and Newsom (1965). The axiomatic method, as understood today, was initiated by Pasch in Vorlesungen uber neueve Geometric Leipzig, 1882.

particularly in the last hundred years, of a rigorous axiomatic structure, the theory of sets and of its interpretations as the structure of the integers. The main point to note here is that this is fundamentally a logical-mathematical development which is purely abstract, conforming to arbitrary but consistent rules, with no necessary relation to the real world. This, at least, is what the logician claims.[11]

My contention is that because this is taken to be the 'true' development of the number system, it is also used as a set of criteria by which correct teaching should be judged. But this sequence interpreted from axiomatic structure is not necessarily the way in which we learn about 'number.'

The teaching of number in primary schools is based on the practical interpretation of such theories and lists of the important 'concepts' are found to take the form:

"Here is a summary of these concepts, processes and facts.

i. Sorting and classifying objects into sets. Comparing sizes of two sets (the number of objects in each set) by matching or one-to-one correspondence; learning the language, and later the symbols of inequality; is greater than $>$, is less than $<$.

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11. A rather tongue-in-cheek version of this claim can be found in Russell's 'Mathematics and the Metaphysicians' which contains the passage about mathematics being "... the subject in which we never know what we are talking about, nor whether what we are saying is true." See Newman J.R. (Ed) 1956 (1576-77)

ii. Counting the number of objects in a set (cardinal number). This, in effect, involves putting each object into one-to-one correspondence with one in the series of number names. Conservation of number. .....

iii. The number line. Numbers in sequence or in order up to 100 (ordinal numbers). ....."[12]

Of course, it may be argued that research into childrens learning of mathematical concepts tells us that this is so. But this is a dangerous assumption. We must be careful to distinguish <u>how children learn</u> something we choose to teach them from <u>what they know</u>. The examples given by such experiments may tell us something about how children learn (in the behavioural sense), but little, if anything about what children know (in the epistemelogical sense). In fact, while Piaget is aware of many of the difficulties involved, some of his interpreters, especially those read by teachers in training, are apparently unaware of this distinction.[13]

Classic examples of this may be found in two recent books where the learning experiments described are carefully designed to bring out the fundamental ideas of the

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12. Biggs (Ed) (1965) p.11

13. For example, I can find no mention of this distinction in Isaacs (1960) or Beard (1969). They are concerned with the description of Piagetian theory, the classification of stages of mental development, and the appropriate mathematical concepts to teach at those stages.

axiomatic structures described above.[14]

Such is the influence of the abstract axiomatic structure on the design of the learning experiments, and their results; we find that the early history of mathematics is being rewritten to explain this learning. This is surely a travesty of the meaning of history.

Smeltzer[15] is a typical example. 'Primitive' man learnt to count by a process of set recognition, one-one correspondence and ordering. While we may feel that, at a certain stage of development, this may be a plausible interpretation of the available evidence, we have no means of being sure that this was the only way in which man learnt about numbers, nor have we any evidence to suggest that the stage of primitive written records indicated the beginning of number (or any other mathematical) concepts. The axiomatic myth is being further perpetrated in books which can be read by the children themselves.[16]

This is not the first time that assumptions about the nature of number have prescribed both history and teaching.

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14. See Copeland (1970) "First experiences with number" (57-85) and Lovell (1971) gives a summary of suggested activities for children (36-40).
15. Smeltzer (1953) (2-19) in particular.
16. Lerch (1966). This is the story of a (base five) number system invented by the inhabitants of a make-believe island. The stages of invention are: one-one correspondence, tallying, and a coded base-five system.

Kroenecker "... asserted that 'God made the integers, all the rest is the work of man'. By 1910, some of the most wary mathematicians were inclined to regard the natural numbers as the most effective net ever invented by the devil to snarl unsuspecting man. Others, of a yet more mystical sect, maintained that the natural numbers have nothing super-natural of either kind about them, asserting that the 'unending sequence' 1,2,3,... is the one trustworthy 'intuition' vouchsafed to Rousseau's natural man. The tribes of the Amazon Basin were not consulted."[17]

Much more realistic remarks about the early history of mathematics are to be found in Boyer:[18] "Statements about the origins of mathematics, whether of arithmetic or geometry, are of necessity hazardous, for the beginnings of the subject are older than the art of writing. It is only during the last half-dozen millenia that man has been able to put his records and thoughts in written form. For data about the prehistoric age we must depend on interpretations based on the few surviving artefacts, on evidence provided by current anthropology, and on a conjectural backward extrapolation from surviving documents." Neugebauer[19] in a similar context remarks: " The common belief that we gain 'historical perspective' with increasing distance seems to me to utterly misrepresent the actual situation. What we gain is merely confidence in generalisations which we would never dare to make if we had access to the real wealth of contemporary evidence."

Bell (1945) p.170

Boyer (1968), (5-6)

Neugebauer (1957), viii

Both these authors are acutely aware of the dangers of generalising from little evidence, interpreting past mathematics in terms of current ideas, and of the 'context' which mathematics develops within a culture. While recognising a development which may be similar to the current 'axiomatic approach' as one interpretation,[20] we must also be on the lookout for other evidence (both 'mathematical' and 'non-mathematical') for different versions of the early development of mathematics. This other evidence indicates that:

i. 'Primitive' arithmetic is not as simple as it appears in the current popularly accepted versions of that period of history. Nor are the people themselves as 'primitive' as once believed.

ii. The origins of mathematics pre date written records and thus may be based upon very different ideas from those at the time when writing became apparent.

iii. Since we are today ready to admit many more situations as 'mathematical' than hitherto, (relational aspects of objects, or the formation of language, for example) there are many more potential interpretations of the little evidence we do have.

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20. Concerning interpretations, there is circumstantial evidence to suggest that Piaget's view of children's logical development derives from Russell's formulation of logic. Aiton's remark (1972) that it is instructive to read Russell's 'Introduction to Mathematical Philosophy' alongside Piaget's developmental psychology reinforces this view. Also, Piaget subscribes (at least in principle) to the bio-genetic principle. (Ginsburg and Opper (1969)p.10). Considering the evolutionary basis of his epistemology, this is hardly surprising.

A common unthinking criticism of early man and of so-called 'primitive' cultures existing today is that they are in some way 'backward' compared to our own civilisation.[21] What is so often not realised is that they are <u>different</u> cultures from our own; we should be aware exactly what this difference implies, and that they may be highly advanced in their own terms, being well able to cope with the problems of their own society.

Many general histories of mathematics start with early number ideas fairly well-developed; with the Babylonian, or the Egyptians. Having little or no evidence to the contrary, they give the impression that mathematics started here. Some compare different number base systems used in early civilisations, and imply that primitive tribes could only count up to a certain number. (This is particularly true where 'finger tally' systems are being discussed.)[22] However, we find the critical reader's suspicions confirmed in a paper by Wolfers[23] where he discusses the counting systems of tribes in New Guinea. The first, and perhaps the most obvious fact that arises is that people can employ different counting systems for different kinds of

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21. Cajori (1896), (1-4) makes specific remarks about the abilities of the 'lower' and the 'higher' races.
22. These remarks are generally true of short histories of mathematics or histories written prior to the 1960's. A notable exception is Struik (1948).
23. Wolfers (1971).

calculations.[24] These different calculations often depend upon the different kinds of situations that arise. One example is given where a tribe "... employ a decimal system of numeration but with a separate set of terms to denote each number according to which of forty categories the item(s) being counted come(s) under. Other groups employ different bases according to what is being counted...."[25] The most commonly occurring bases are two, five and ten. Different counting systems may be used in transactions in everyday life, and in private or ceremonial situations. "The Mailu ... who normally count in tens, count certain foods, - taro, sweet potatoes, fish and coconuts counted for a feast - in groups of four."[26] Body-counting systems work to a number of different bases, and quite high numbers can be counted. Also, surprisingly (?) modulus systems have been discovered where the actual number names used are relatively few, but large numbers can be dealt with. Although the mathematical distinction between a base and a modulus is precise, it is not always clear how a particular system operates in practice. "The existence of modulus systems, or of counting systems with a physically present or visual, but no verbal base, may explain why some writers have assumed that particular groups of Papuans and New Guineans "cannot" count beyond a certain number." (For example,

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24. This has been known for some time. Wolfers examines the nature of these different counting systems, rather than showing them as examples of different bases in use.
25. op cit p.78
26. op.cit. p.79

the person counting may use a number of men, repeating the same number-words on a body tally system once for each.)[27]

Apart from the sophistication revealed by these studies, a final remark must be made about the need to count beyond a certain number. We must be aware, when criticising primitive number systems that also "There was not so much a limit to counting as a limit to the goods and the quantities that needed to be counted, or that particular groups of people wanted to measure."[28]

The development of written numbers - even the simplest tally system - represents a stage in the process of unwritten thought. It is not only possible, but necessary that before written records emerged, 'counting' was fairly well developed. We can speculate as to how people counted at this stage, with reference to body tally systems, but even this may have been at a fairly late stage in the story, and the original need to 'count' may not have been associated with activities like counting flocks. There is certainly evidence in the number names used[29] that the original use of these words was adjectival rather than nominal, conveying some aspect of the object being described.

The involvement of man (not only primitive man) in ritual is well known. Seidenberg[30] provides an interesting framework for the origin of counting in ritual, suggesting that the original need was not to count as we know it, but

27. op.cit. p.81

28. op.cit. p.82

29. Menninger (1969), (9-32)

30. Seidenberg (1962).

to call upon the participants in a ritual.

Seidenberg sees what we have discussed of primitive counting systems as the application of a device (which was already well-established) to another situation, namely, the 'shepherd and his flock'. The application of a device is seen to be an effect of the device and not a cause. "How can one ask 'How many ?' until one knows how to count ?"[31]

The occurrence of base two counting in widely separated peoples suggests a common origin in a particular type of ritual. (The evidence for the common origin need not concern us here.)[32] We also know that many 'primitive' peoples have a fear of being counted; the belief that by counting people, or knowing their name, one acquires power over them and can even kill them is not uncommon. The connection between number mysticism and ritual is also well established.[33]

These ideas are combined in a fascinating picture of the creation/fertility ritual where participants appear (or are called on by the 'priest') in pairs, male and female. "The original intention is to mimic a portion of the creation ritual. It is in this way that we envisage 'counting' to have become detached from the ritual and to have acquired its abstract or general character.

Secondly, we think that the higher counting may have started as a method of taking care of longer and longer

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31. ibid p.2
32. Siedenberg (1960). Though Wilder (in private correspondence) is sceptical of his evidence.
33. For example, the large literature on numerology, divination and so on.

processions (not with the idea of counting them, however). The base (which is not logically inherent in counting) corresponds to the number of persons in the basic ritual, and the higher counting derives from the continued repetition, with slight modifications, of this basic ritual."[34] However, we may criticise this theory, we can extract two ideas which may be relevant to our discussion. Namely, that (i) the 'need to count' as popularly imagined is not necessarily the origin of the act of counting, and (ii) ideas related to andassociated with 'counting' in the mind, may also be connected with rituals, ceremonies and games.[35] If we regard counting as the application of a device, perhaps we should look more carefully for the origins of this device in 'non-mathematical' situations.

With the gradual widening of the idea of what is 'mathematical' and what can be regarded as 'potentially mathematical', or looked upon in a 'mathematical way', we come to accept that many other things that primitive man did, and the way in which he regarded his world, could be seen as the sources of other (non-numerical) mathematical ideas. Early man was aware of very sophisticated relational aspects of the world which we have either forgotten, or which do not seem to be important in our civilization. Even today many peoples have highly developed classification systems, based on their relationship with the world they live in, empirical and highly relevant to their needs. The ability of so-called primitive people to provide investigators with lists of characteristics of animal and plant

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34. Siedenberg (1962) p.10

35. For example, children's 'counting out' rhymes.

life should not be surprising when their survival depends on being able to distinguish food and other useful things from their surroundings.[36]

This is only the tip of the iceberg. The ability to classify may be a pre-requisite for the formation of number concepts at a particular level, but is it not also at the foundation of a multiplicity of relational ideas which may result in what <u>we</u> call 'logic' or 'algebra' or 'geometry'?[37] The classification of objects and the perception of the

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36. Levi-Strauss (1966), Ch.II "The Logic of Totemic Classifications" gives many detailed examples of how precise native classifications of plants, animals, times and seasons can be.

37. For example, Gladwin (1970) is an investigation into the organisation of knowledge and modes of thought of a 'primitive' culture - the navigators and canoe-builders of Puluwat Atoll. The navigation of small canoes across the ocean without any of the normal aids requires a complex and well-organised body of knowledge. Gladwyn criticises current educational practice, claiming that the "Emphasis is (therefore) on measurement, with the qualities to be measured accepted as given. The possibility is thereby largely foreclosed of exploring other dimensions of thinking beyond those which are traditionally recognised within educational psychology." (p.216)

relations between them is a fundamental human activity. Also: "... the kind of logic in mythical thought is as rigorous as that of modern science, and (that) the difference lies, not in the quality of the intellectual process, but in the nature of the things to which it is applied ... The same logical processes operate in myth as in science, and (that) man has always been thinking equally well; the improvement lies not in an alleged progress of man's mind, but in the discovery of new areas to which it may apply its unchanged and unchanging powers."[38]

It may be stretching the point too far to claim these activities as mathematics, but perhaps we can help ourselves to a better appreciation of mathematics and how it arises in the world (our own inner world) by capitalising on our investigations of the thought and world of 'primitive' people. Not that I want to imply that children think like 'primitive' adults, but that we should become aware of the classificational and relational aspects of our own world (the world our children grow up in) so that we may help them by exploiting the aspects of the world that are relevant and meaningful to them. These aspects will, necessarily, be different at different stages, and will inevitably produce different effects; different mathematics, which we will have to recognise and respect for what it is. 'Number' is only one, and a relatively late aspect of the total situation.

We need therefore to review our philosophy of teaching mathematics at the primary level. These remarks have been taken from the point of view of mathematics rather than

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38. Levi-Strauss (1972) p.230 (underlining mine).

psychology or child development. Whatever we may learn about learning, can, in this context be separated out from what we can learn about mathematics, and the following remarks on the content of mathematics at primary level can be made:

i) We should be aware that the 'axiomatic structure development' dominates elementary mathematics. This has occurred through a particular view of mathematics insisting that there is only one true story. The result is pre-scriptive pedagogy - and a new orthodoxy.

ii) History of mathematics is not only based on documentary evidence of a mathematical nature, but also on 'non-mathematical' and even on 'non-documentary' evidence. Since our view of what mathematics is changes, so also does our view of what can be admitted as evidence.

iii) We should beware of the histories of mathematics that begin with 'counting'. Speculation about the early history of mathematics - or even the history of 'counting' - must pay attention to the vast quantity of 'pre-documentary' and 'non-mathematical' evidence that is available. Much of this suggests counting to be the application of a well-established device whose origins may not be the 'need to count'.

iv) Because of our traditional reverence for counting, and the idea that it is somehow the 'simplest' mathematics, we do not readily admit 'other kinds of mathematics' to be worthy of attention in elementary school. (Even if we know they exist.)

v) The sources of elementary mathematics (not only historically but in the contemporary sense also), do not necessarily lie in the area that we today recognise as mathematics; but also in human activities which can become literature, art, music, and other forms of communication when developed in different ways.

We need, therefore, to develop a number of alternatives - both in and to arithmetic - which are viable and valid mathematical activities, accessible to teachers and children alike. Historical perspective may enable us to produce those alternatives, and guard against any insistence that a particular alternative is necessarily correct.

c). Perception, Perspective, Projective Geometry

The development of techniques for representing three-dimensional space in the plane are recorded both in the history of art and the history of mathematics. It would seem, therefore, that this may be a fruitful area to explore, not only to widen one's appreciation of the origins of, and bases for certain mathematical theories, but also to emphasise the nature of the connections between mathematics, art, the psychology of perception, and the development of children's ideas of space.

Theories of perception, particularly visual perception, are discussed in Gregory[1] and Gibson.[2] A summary[3] puts forward concisely a new approach to the theory of perception. The former approaches, which on the whole attempt to describe the process of making sense of some abstract 'space' or 'form' are contrasted with an idea that emphasises

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1. Gregory (1966).
2. Gibson (1950)
3. Gibson (1965).

the invariance of perception with varying sensations. Sensory experience is a special self-conscious kind of awareness while perceptual experience is unself-conscious and direct. "The individual is bathed in a sea of energy at all times, and the stimulus energies that his receptors can pick up are a flowing array ... the flowing array has two components, one of change, one of non-change."[4] Gibson borrows two notions from mathematics, transformation and invariance under transformation. "The specific hypothesis is that the invariant component in a transformation carries information about an object and that the variant component carries information entirely, for example, about the relation of the perceiver to the object. When an observer attends to certain invariants he perceives objects; when he attends to certain variants he has sensations."[5]

The evidence supporting this hypothesis is extensively discussed in Gibson's (1950) book, where a section on the development of perception in children suggests that the child does not have to construct a constant world out of ever changing perspectives, but has to discover the properties of the world that are invariant under transformations.[6]

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4. ibid p.67
5. ibid p.68
6. An interesting comparison can be drawn between Gibson's theory of perception and the account given of the early experiences of the infant by Gattegno (1973), where transformation and invariance appear in a more obviously mathematical context.

It is interesting to note that the particular mathematical ideas that Gibson uses to develop his own ideas originated from one of the theories that he criticises, namely the various attempts to explain visual perception in terms of the efforts of the eye to discover the 'forms' or the Euclidean properties of the world.[7]

The theory of perception which gave rise to the technical developments of late Medieval and Renaissance periods, used Greek geometry and optical theory motivated by the religious idea that the eye is the window of the soul, to change the emphasis in painting from symbolic representation into a visually accurate record.[8]

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7. Gibson (1965) p.64 distinguishes three types of theory of perception; (i) those that appeal to innate ideas or the rational faculties of the mind for making sensory data intelligible; (ii) those appealing to past experience, memory or learning for supplementing and interpreting sensory data; (iii) the idea that sensory chaos is organised by a spontaneous process in the brain. The idea that Euclidean space is innate, and that the mind is informed not by sense impressions but by its own preconceptions, was a common assumption of Medieval optics.
8. The period of this change can be approximately given as mid-thirteenth to fifteenth century, i.e. from Duccio (1255-1318) to Della Francesca (1416-1492).

The stages in this development tell of a particularly interesting and well-motivated piece of applied mathematics.[9]

The analysis of children's art indicates that symbolism is foremost, and the perspective representation is not generally possible until the age of about fourteen.[10] However, a few attempts at giving an impression of depth or distance produce a 'plan' or 'terrace' where objects are arranged in rows, as it were, the foreground being in the first row, and the most distant objects being in the back row.[11] Kellog is quite adamant in insisting that the term perspective is inappropriate for describing children's art "because young children lack the ability to draw in true perspective."[12]

In the history of art there exists a parallel to this early stage of 'distance representation' to be seen in 'Terraced Perspective' where rows of dignitaries, saints, angels, etc., are placed one behind the other, but are

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9. The technical problem appears to have been solved by Brunelleschi about 1425. His pupil, Alberti in Della Pittura (1435) produced the first scientific writing on the new theory of perspective.
10. The principal works consulted here are Eng. (1954), Kellog (1970) and Jameson (1968).
11. Jameson (1968) p.46, figs.90, 91. The children were aged 7 and 6 respectively. Piaget interprets similar pictures quite differently (see below).
12. Kellog (1970) p.209.

still all the same size.[13]

It is, however, dangerous to make much of such parallels, because in an over-eager analysis of spatial representation, aspects of pictures like their conceptual organisation and symbolism and even in some cases the artist's materials, may make a more significant contribution than merely accurate representation of physical space.

The development of children's concepts of space has been analysed by Piaget,[14] where a progression from topological through projective to Euclidean concepts is described. The elementary topological perceptions are : "(1) proximity or near-byness, (2) separation, (3) order (or spatial succession), (4) enclosure or surrounding, (5) continuity."[15] "Projective space begins psychologically when the object or pattern ... begins to be thought of in relation to a 'point-of-view' ... (and) is concerned with the inter-coordination of objects separated in space ..."[16] While projective correspondences can be regarded as topological correspondences plus the conservation of straight lines, euclidean correspondences begin with the conservation of parallelism, and later include the conservation of angles

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13. Terraced perspective is shown well in Medieval paintings like 'Majesty' or 'The Annunciation' both by Simone Martini (1285-1344).
14. Piaget, J. The Child's Conception of Space. Routledge 1956.
15. Holloway (1967) p.3. The rest of the summary is also taken from this work.
16. ibid p.28.

and distances.[17] In a more general context, Piaget claims that this is the reverse of the historical process; that while topological aspects may be the first to be perceived by the child, it was the euclidean aspects of space that were first recognised in the history of mathematics.[18]

While it may be true that euclidean geometry was the first formalised system, it was certainly not the first kind of geometry discovered.[19] Visualisation we know was an important factor in pre-euclidean geometry,[20] and the so-called 'proofless' geometry of Greece, Egypt, Babylon, India and China requires some appreciation of the topological

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17. ibid p.55
18. Piaget (1950) "... historiquement, la géometrie euclidienne a precede de beaucoup la constitution de la géometrie projective, et celle-ci a precede de beaucoup la découverte de la topologie." p.237. The section (236-242) is a discussion of this idea.
19. Seidenberg (1962)
20. Szabo (1961)

and projective aspects.[21] There is obviously a difference between order of discovery and order of recording. Although pre-Greek geometers may have been aware of some topological and projective properties, they might have been considered trivial or unimportant at the time. Euclidean aspects of space seem easiest to record and manipulate, so it is little wonder that for a long time man has been persuaded of their necessary correspondence with reality.

The development of spatial relationships in 'pictorial space' in children is seen in three states.[22] The first contains only topological relationships; proportions,

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21. If we consider proximity, separation, order, enclosure and continuity as fundamental geometric intuitions, and the conservation of straight lines as one of the first experiences of objects in space, these are the topological and projective precursors of euclidean space. As soon as the construction of buildings became a necessary task, objects in space were seen to preserve affine and metric properties when manipulated. Many pre-euclidean proofs might easily be re-written in terms of the motion geometry used in schools today. A possible example is found in Seidenberg (1975). (289-291)
22. Holloway (1967) (9-11).

distances, perspectives, projections and sections are all absent. The second is a stage of intellectual realism[23] where children attempt to show 'everything that is there' - different points of view may be represented on the same picture, while attempts may be made to show both the inside and outside of closed objects. Projective and metric relationships are not coordinated here. Finally, attempts at visual realism appear, "... so late as to suggest that projective and euclidean notions are slow to emerge in the realm of representation in contrast to their development in perception."[24]

Little useful correspondence can be made between these developments and the history of art, for the historical products we have are the work of mature artists and depend on the cultural context for their significance. Even in the area of the pure technique of visual realism earlier theories have had to be revised in the light of new

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23. A similar idea appears in twentieth century art, but this is a deliberate technique, not a half-developed concept.

24. Holloway (1967) p.11. While these stages may be useful for the analysis of spatial concepts, the aspects of the world children are interested in recording, and their technique, or lack of it, need also to be considered.

evidence.[25]

It appears that the techniques of artists and craftsmen for the representation of distance and the correction of illusion appear in the fifth century BC., in the construction of buildings like the Parthenon[26] and the construction and use of scenery in the Greek theatre.[27] The techniques of foreshortening and 'aerial perspective' (the gradual diminution of intensity of colour with distance) were apparent in the wall-paintings of Pompeii.[28] The Renaissance artists had to rediscover these techniques, and the earlier attempts of the use of colour tones and the use of 'Vertical perspective' (parallel lines in nature meet on a verticalline in the centre of the picture) in Duccio and Giotto are quite striking in contrast with their Medieval precursors. Ghilberti's doors to the Baptistry in Florence at the end of the fourteenth century show how depth was also attempted by building the relief up in layers.

Brunelleschi is considered to be the founder of the system called 'focussed perspective' (all lines parallel in nature meet on a vanishing point on the horizon line), and Ucello is obviously fascinated by this technique.[29] Alberti's book gathers the techniques together and shows why it is necessary for a painter to know geometry.[30]

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25. The history of visual realism in art is not only about the theory of perspective in Medieval and Renaissance periods.
26. Fletcher (1961) p.95
27. Nicoll (1966) p.11.
28. First century A.D.
29. The famous 'Rout of San Romano' and his drawing 'Study of a Chalice' are excellent examples.
30. See note 9 above.

After writing on geometry, De Corporibus Regularibus (1487) Della Francesca produced a book on perspective De Prospettiva Pingendi (1470-1490), and his paintings are brilliant examples of the application of the new perspective techniques.[31] Other artists of the same period brought technique and experience together in a series of sensational applications of the new mathematical theory.[32]

Della Francesca's work contained practical procedures,[33] demonstrations and intuitive definitions which were to form the basis of Viator, De Artificiali Perspectiva, (1505) and the first work in northern Europe: Durer's Underweysung der Messung mid dem Zyrkel und Rychtscheyd (1525), to help artists with perspective. Durer's work contains a statement of the theory behind the known techniques, and it is fairly certain that he gained the Euclidean concept of the visual cone from a journey to Italy in about 1506.[34] A series of famous woodcuts demonstrate some of the practical techniques available.[35]

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31. For example 'Flagellation' and 'Architectural View of a City.'
32. For example, Boticelli, Leonardo, Micaelangelo, Raphael.
33. Here appears for the first time the remarkably simple 'distance point' or 'tiers points' construction for obtaining a true perspective based on the properties of similar triangles.
34. Panofsky (1956)
35. Kurth (1946).

By about 1635, we have the perspective techniques well enough known to be applied to stage scenery by Inigo Jones[36] but it seems that the particular problem here is one of moving actors against a perspective painted background - the reciprocity of size and distance.[37] Dubreuil's Perspective Practique (1649) seems to be the definitive work on the application of these techniques in the theatre of this period.

By this time, a series of theorems on perspective drawing had been developed which are little changed in the text books of today.[38]

Desargues' 'Brouillon Projet' of 1639 appears to be a development of an earlier pamphlet on perspective of 1636, and is usually considered to be the beginning of projective geometry as such.[39] It was, however, intended to be of use

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36. The designs for 'Florimene' in Nicoll (1966)p.107.
37. Gibson (1965) fig.3 p.66
38. For example see Kline (1972) (231-236)
39. G. Desargues 'Exemple de l'une des manieres universelles du S.G.D.L. touchant la practique de la perspective sans employer aucun tiers point, de distance n'y d'autre nature, qui soit hors du champ de l'ouvrage.' Paris, 1636. This practical pamphlet was followed by the 'Brouillon Projet d'une attiente aux euenemens des rencontres d'un cone avec un plan.' Paris 1639.

to artists and to this end employed a 'non-mathematical' language invented by Desargues which unfortunately failed to catch on, in spite of his energetic lectures and demonstrations and a popularisation in 1648 by his pupil Bosse.[40]

About the same time Pascal in two works was also developing the techniques of projection and section seen in Durer into a mathematical analysis of conic sections.[41]

These works were lost, and the sudden development of projective geometry in the nineteenth century began with the re-discovery of a work by la Hire, Sectiones Conicae (1685), which contains most of the familiar properties of conic sections synthetically proved and systematically established. Here also we find the first focus-distance definitions of conics, in contrast to the appolonian definitions as sections of cone.

The main responsibility for the revival of projective geometry lies with Poncelet in Traite des proprietes projectives des figures (1822) in which he produces the general formulation of the pole-polar transformation, and

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40. Bosse, A. 'Maniere universelle de M. Desargues pour pratiquer la perspective' Paris 1648.
41. The first, about 1639, was seen by Leibniz but lost some time after 1676. The second 'Essai Pour les Coniques' was lost and not rediscovered until 1779 see Oeuvres 1, 1908, (243-260). The contrast between the approaches of Pascal and Desargues is quite striking. Desargues appears mainly interested in practical applications, Pascal on the development of new geometrical methods.

in a later paper,[42] uses the 'method of reciprocal polars' as a transformation to establish new theorems.

The first significant difference between the mathematical theory of perspective and the projective geometry of Desargues is the introduction by Desargues of a convention of <u>points and lines at infinity</u>, (parallel lines meet at infinity,[43] and lines parallel in different directions meet on a line at infinity), which is an extension of the perspective idea that parallel lines meet at vanishing points on the horizon, enabling parallel planes to meet at a line at infinity. Thus Desargues was able to state a basic theorem of projective geometry which still bears his name.

The fundamental problem of projective geometry lies in the investigation of properties of figures which are constant under the projective transformation. None of the usual euclidean invariants (length, area, congruence, similarity) apply, and in this investigation Desargues appears to be the first to hit upon the important invariant of cross-ratio. Cross-ratio was known to Pappus and his work contains the theorem that a cross-ratio is the same for every transversal cutting line emanating from a given point, where all the transversals pass through the same point on one of the given lines.[44]

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42. Poncelet, J.V. 'Memoire sur la theorie generale des polaires reciproques'. Jour.fur.Math 4,1829, (1-71).
43. Parallel lines meeting at infinity and the principle of continuity appear in Kepler's 'Geometricae Pars Optica' of 1604. See Kline (1972) p.290
44. Book VII, prop.129. See Kline (1972)p.127.

It seems fairly obvious that Desargues' knowledge of Pappus, together with his idea of points and lines at infinity together produced the important realisation that cross-ratios were the property of figures that were preserved under the projective transformations.

In fact Book VII of Pappus contains a number of ideas known in the seventeenth century ('complete' quadrilaterals, harmonic sets of points) which were applied only to Euclidean space, but which were found to adapt usefully and powerfully to the new geometry. Most of this application however, had to wait until the nineteenth century.

Thus the second difference between the theory of perspective and the new geometry was the important idea of invariance under a transformation.

For a number of reasons the analytic geometry of Descartes greatly overshadowed Desargues' work,[45] but the idea of transformation seems to have come through.[46]

With the revival of synthetic geometry in the nineteenth century we find the following four ideas emerging as powerful principles whose application was responsible not only

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45. Descartes 'La Geometrie' (1637) was an appendix to the major philosophical work 'Discours de la Methode'. Not only was the general context more powerful, and the manner of its presentation more persuasive, Cartesian geometry provided a powerful tool for the development of infinitesimal methods into the calculus.
46. The 'transformation of axes' found in Fermat's coordinate geometry could equally well be a source of this idea. See Boyer (1968) (380-381).

for advances in geometry, but also in other branches of mathematics.

1. <u>Transformation</u>

The simple idea that the same figure looks different when viewed from different directions motivated not only the investigation of projective transformations, but also other kinds of transformation, even more general.

2. <u>Invariance</u>.

Having applied a transformation to a figure, what can one say about the new figure in relation to the old ? What remains constant ? Cross-ratio was the first invariant discovered, but as the nineteenth century progressed, the investigation of invariants of transformations became a particular branch of mathematics, apart from its geometric origins.[47]

3. <u>Continuity</u>

The principle of continuity was known and used much earlier[48] [illegible], but in its application by Poncelet[49] as an instrument for the

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47. Later in the nineteenth century this evolved into <u>invariant theory</u>.

48. Kepler, see note 43 above. Leibniz also believed that infinitesimal changes do not alter the geometric properties of figures, and Pascal's 1640 has this as a basic method.

49. Poncelet (1822).

discovery of new theorems motivated much discussion between the geometers and the analysts concerning the validity of the principle.[50] Poncelet introduced the idea of geometric continuity as a dynamic principle - if one can make a statement about a figure and then apply a transformation, since it is essentially the same figure, then a similar statement applies about the new situation. This led to the idea of points and lines becoming imaginary at infinity, and was used as a powerful heuristic principle in the establishment of new theorems.

4. Duality

Poncelet's 'method of reciprocal polars' led to the establishment of another principle which was used to establish a large number of new theorems. It had been observed that the replacement of the word 'point' by 'line' and 'line' by 'point' in projective geometry theorems produced other theorems which not only made sense, but could be proved to be true. Brianchon found the dual to Pascal's theorem in this way. Other ideas like point curve (the curve as locus of a moving point) and line curve (the curve as the result of intersecting tangents) arise from this reciprocal relation.

Gergonne appears to have been the first to state duality as a general principle applicable to all theorems

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50. The discussion was mainly about whether the 'principle of continuity' was a fact or just a heuristic principle, and lasted well into the nineteenth century.

involving non-metric properties, introducing the term duality to denote the relation between the original and the new theorem.[51]

Like continuity, the principle of duality continued to be used as a device to suggest new theorems, and standard works of the late nineteenth century continue to use the side by side layout of proofs originated by Gergonne.[52]

These four principles became the motivating factors behind much of nineteenth century mathematics, and continue in their contemporary versions, to be of importance today. To have some idea of the origins of these ideas and access to material on this particular phase of the development of mathematics could be of enormous pedagogical use today. Not only do we have a wide range of experience open to mathematical investigation from the point of view of the origins of these ideas in children, but also a large number of potential problem-situations for classroom exploitation at all levels.[53]

From the historical point of view, the development of

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51. Gergonne, J.D. Ann de Math.16, 1825-26 (209-231)

52. Cremona (1885).

53. For example, Elementary Science Study (McGraw-Hill, 1963). 'Light and Shadows' gives many examples of projective explorations for primary children, and the common techniques of projection and section and engineering drawing are available for older children.

projective geometry from euclidean space[54] is an example of a powerful generalisation. Applying the four principles above, mathematicians later demonstrated that a non-metric axiomatic projective geometry was possible, thus showing projective concepts to be logically more fundamental than euclidean.[55] This search for the 'essentials of space' led to the statement by Russell[56] that projective geometry was in a sense a property of the mind whereby it received and organised perceptions of space.

With this, the wheel turned full circle. Russell, as we now know, was overconfident in his claims, but at this point we see logically, mathematically and philosophically established the ideas that the renaissance artists set out to investigate. The teacher with access to this knowledge is in a powerful position, for not only is he able to judge

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54. There is no definite evidence so far that projective geometry grew directly out of the theory of perspective. Circumstantial evidence suggests that the key figures Kepler, Desargues and Pascal made strong conceptual links with perspective techniques. There is still much debate about this point.
55. This was due mainly to Von Standt's 'Geometry of Position' (1847) and Steiner's 'Systematic Development of the Dependence of Geometric Forms on One Another.'(1832)
56. Russell (1897) makes interesting reading. Kline's foreword to the Dover edition of 1956 is a useful precis.

the relative importance, both logically and psychologically, of certain mathematical concepts, but is also able to make critical assessments of theories of learning that use these concepts in their formulation.

d) Calculus: Metaphysics and Practice

1. Some Fundamental Ideas

Literature on the origins of the calculus is extensive. No single invention in mathematics has yet contributed so much to both theory and applications, so it is no wonder that so many mathematicians, educators, historians and philosophers have given their attention to the conceptual and technical problems surrounding its development. This discussion uses aspects of the calculus as an example of the way in which we might apply some of the models for teaching of Sect.3.

This is not the place to embark on yet another detailed analysis of the motivations and concepts of Newton and Leibniz, but from the literature available, we are now able to abstract a fair summary of the vital schemata in use in the seventeenth and eighteenth century.

It is possible to consider these schemata under two broad headings: technical and philosophical. More particularly, one might distinguish here two types of mathematical procedure: on the one hand we have that allowed for discovery; the procedures of the working mathematician, often appearing non-rigorous and even artful, and on the other that required to secure the foundations and to supply acceptable mathematical proof.

We know that one of the major aims of both the inventors of the calculus was to obtain from the 'inverse method of tangents' algorithms and tables of tangents and quadratures that would both survey and help to classify the large number of problems now within possible solution; part of their correspondence indicated their joint desire for the completion of such a program.[1]

The methods employed were different. Newton's 'conjunction of calculus with analytic geometry with expansions into infinite series',[2] forms the core of his method, and his particular use of infinitesimals, fluxions and prime and ultimate ratios are not successive attempts at more rigorous presentations, but can each be shown to have a particular and important role in his total scheme.[3]

Leibniz seems not to have attempted a variety of apparently different approaches, but to have used his 'differentials' as a development of the infinitesimal technique in a context where the parameters describing a curve could be selected from a large number of variables: axial co-ordinates, arc length, radius of curvature, area, etc., and where, unlike present practice, the normally accepted partitions of the variables were not always equal.[4] While Leibniz was concerned both with the development of the algorithms and with the problems of proof, it appears his immediate successors, having such a powerful technique at their command, paid more attention

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1. Scriba (1962).
2. Whiteside (1972)
3. Kitcher, (1973).
4. Bos, (1974). See esp. p.4-8.

to the development of the technique and its application to a range of hitherto difficult, insoluble and even undreamed of problems, employing a heuristic of differentials which is much more complex than previously imagined.[5]

For both Newton and Leibniz, the calculus was a technique for classification and analysis of geometrical problems, and it is important to note that curves were not considered as functions x y(x) where x is the independent variable.

The concept of function as a mapping, and the convention of a rectangular co-ordinate system, was generally absent until well into the eighteenth century. A curve was conceived as embodying a set of relationships between variable geometrical quantities, any one of which could be chosen as independent and which determines all the others by nature of their particular relationships described in a formula or equation. In the consideration of practical problems a commonly accepted underlying assumption was that curves were generated by motions of objects in time,[6] and the name 'fluxions' that Newton gave to his technique derives directly from such a background.

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5. op.cit. (53-64) Leibniz' own appreciation of the problems indicates a very clear distinction between the existence of infinitesimals, and their application to the solution of problems. He saw the calculus as a shorthand for the traditional proofs by exhaustion, and his proofs of the rules of the calculus were based on a 'law of continuity'.

6. The "Brachistochrone" problem is typical, posed by John Bernoulli in 1696.

Thus the kind of concern later shown for problems of continuity were mainly motivated by a redefinition of 'curve' in the dynamic sense, as 'function' in the analytic sense. For the exponents of the calculus in the seventeenth and early eighteenth century, there were no discontinous curves.[7] The development of the function concept has been discussed elsewhere;[8/9/10] suffice it to say that the change from proportional or dynamical definition, through equation to functional definition of curves is both a consequence of, and a motivation for, many practical and theoretical problems of the calculus.

It has been carefully argued[11] that Newton's calculus operated on two levels, a heuristic technique for the discovery of solutions to problems, and a rigorous procedure for the justification of the technique in terms of the 'method of the ancients'. The analytical method of the Ancients was a common heritage for both Newton and Leibniz, who searched for a way of presenting their discoveries so that they would be acceptable to the mathematical community.[12]

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7. The infinitely diminishing cycloid of a bouncing ball, for example was in this sense, no less continuous than the geometrical construction of a parabola. Of course, broken curves were known and discussed, but these 'monsters' were generally barred from the calculus at this time.

8. Boyer (1946)

9. Boyer (1956)

10. Grattan-Guinness, (1970)

11. Kitcher (1973) The following substantially represents Kitcher's views.

12. Famous precursors, Barrow & Huygens were strong in their belief that the developing art should be firmly founded on the geometry of Euclid and the algebra of Vieta.

The method of fluxions was a heuristic technique reclassifying the large number of problems to be tackled and reducing them to a set of fundamental problems which could be resolved in kinematical terms. Tangents and curvatures could be obtained by finding 'fluxions', and quadratures by finding the 'length of space described'.

Infinitesimals were used as a justification for the fluxional processes with the implicit admission that they were applied intuitively, and within the tradition of the 'demonstration', which was the use of an intermediate and commonly accepted device that explained algebraic practice and suggested further development.

It is clear from the repetitive nature of Newton's papers that he was involved in a search for a method of presentation of his new analysis so that it was both general and precise, and the height of his achievement became the method of prime and ultimate ratios conceived as a synthetic proof-method of the analytical theorems in the tradition of the ancients. He showed that all justifications using infinitesimals could be replaced by proofs using ultimate ratios. It has been claimed that in this achievement Newton almost produced a theory of limits.[13]

Thus: fluxions are the general framework for applicable technique and problem classification: infinitesimals provide intermediate justification and immediate reassurance, the demonstrations often suggesting further developments: ultimate ratios provide the rigorous and more lengthy proofs. The suggestion that this is the case, and that each of these aspects has an important place in Newton's theory[14]

13. Kitcher (1973) p.34 note 3.

14. Kitcher op.cit. p.34

corresponds with the common belief and experience of mathematicians in the way they work. Rigor is relative. The labels we use; justification, demonstration, proof and many others apply to the different levels of rigor we choose to work at. Any change in rigor must be motivated - an increase in response to a challenge, a decrease in response to tradition, security or expediency.

The inability of Berkeley to appreciate the range of subtlety is shown by his expectation that the mathematician should work at maximum rigor at all times.[15] This is not to deny that his criticism was useful, for eighteenth century English mathematics was swept by his challenge.[16] Berkeley showed himself familiar with the work of L'Hopital and other continental mathematicians, and included their methods in his presentation.

Differentials, as we have seen, are infinitely small variable geometric quantities, the relations between them defining a particular curve. Because of the absence of the concept of function in the Leibnizian claculus, no particular variable was considered as independent, upon which all the others depend,[17] and so in that calculus, the idea of derivative does not, and indeed could not, occur. Later the emergence of the function concept, the study of functions of more than one variable and the solution of

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15. Berkeley (1734.) See especially arts.1-4. A criticism of fluxions and infinitesimal techniques.
16. Cajori,F. (1919).
17. Current convention makes an equi-partition of a horizontal axis the basis of the independent variable.

differential equations produced so many technical problems with higher order differentials, that mathematicians were compelled to adopt a convention that consistently designated an independent variable, and the derivative was formulated.[18]

Further practical and conceptual problems brought about the introduction by Euler[19] of differential coefficients. This means that the differential of a particular variable is supposed constant, and for every variable y, we can write dy = pdx where p is a finite variable called the defferential coefficient.

The developing theory of limits, as expounded by Euler, D'Alembert and others was the rigorous proof-technique applicable to all cases where, for the sake of heuristic simplicity, differentials became infinitesimally small, and were required to disappear.[20] From hindsight, we know that the formulation of the limit concept in the eighteenth century allied to the clearer definition of function gave rise to the breakthrough in the mid-nineteenth century. The attempt of Lagrange to base all on power series and work at the extreme level of rigor failed,[21] and was

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18. Bos (1974)
19. Euler,L. Institutiones Calculi Differentialis, St. Petersburg 1755. Repr. in Opera Omnia ser.1. Vol.X. Leipzig-Berlin 1913. See especially ch.VI.
20. Early assumptions that (a) two quantities were equal if they differed by an infinitesimal amount and (b) infinitesimals obeyed laws of ordinary arithmetic were inconsistent, but judiciously and selectively used.
21. Lagrange,J.L. Theorie des fonctions analytiques.. Paris 1797.

balanced by that of Cauchy who still worked with infinitesimals while attempting to give rigorous proofs based on the developing limit concept.[22]

The inability of many later mathematicians to appreciate the problems and motivations of the original inventors, combined with the spectacular success of the heuristic techniques and the complexity of the technical problems so generated, showed the calculus of the early nineteenth century in a great deal of confusion. Too many things had changed for the old forms of argument to be accepted any longer, the relative rigor happily allowed in the seventeenth and early eighteenth century was now disturbing (for while the heuristic produced results, the justifications became increasingly tricky and less plausible), and the mathematical community was striving toward new formulations of fundamental concepts, and a hope for a new mode of rigor was emerging.

2. Some Nineteenth Century English Texts

The confusions of the nineteenth century become apparent when we look at some of the popular text books, for here we see the edge of research being directly transmitted to the student as probably never before or since.

The century opens with Robert Woodhouse's attempt to

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22. Cauchy, A.L. Cours D'Analyse, Paris 1821: Leçons sur le calcul différentiel, Paris 1829.

introduce the differential notation into England[23] adopting methods after the style of Lagrange, where the 'sound analytical principle of the calculus' were based on the multiplication of algebraic symbols. The resistance he encountered was strong, but he encouraged a group of young men to follow[24] Babbage; Herschel and Peacock produced an English translation of the well-known text book of Lacroix[25] which again was in the Lagrange tradition. This is a

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23. Woodhouse (1803). After a century of fluxional practice it had become clear that English mathematics was lagging due largely to the insistence on the use of this notation. This is an example of where notation inhibits development. Technical problems of writing and manipulating successive orders of fluxions became insurmountable. Woodhouse's attempt appears not to have been well received.

24. The true relations between Woodhouse and the Analytical Sooiety are not clear. There appears to be no record of Woodhouse ever being a member or attending meetings.

25. Lacroix, S.F. Traité elementaire du calcul différentiel et du calcul integral. Paris 1802. Second Edn. of 1806, tr. as An Elementary Treatise on the Differential and Integral Calculus. Cambridge 1816. First part: Differential Calculus. (Tr.Babbage.) Second part: Integral Calculus (Tr. Peacock.) The appendix of Lacroix on Differences and Series was replaced by an original treatise by Herschel on the same subject.

shortened version of an earlier work,[26] and the translators mention that "... he has substituted the method of limits of D'Alembert, in the place of the more correct and natural method of Lagrange ..."[27] Limit theory judged less rigorous here is used to demonstrate results which can be proved by the method of differences and series. We are told that "The Differential Calculus, and that of Differences, although forming two distinct branches of analytical investigation, have still a very near relation to each other, and when the former is considered in the light in which Leibniz presented it, or as depending on the theory of limits, it becomes a particular case of the latter."[28] Herschel then demonstrates how Taylor's theorem arises and claims that "... the analytical theory of the Differential Calculus no longer presents any difficulty, and accordingly we may perceive by this process in what manner this calculus results from that of differences."[29] The modes of operation are thus clearly separated: D'Alembert's limits for demonstration of principles and justification of working, and Lagrange's differences and series for the rigorous proofs.

In this treatment we can see that a new concept of function is being evolved, "In order to indicate that a quantity depends on one or several others, either by operations of any kind, <u>or by other relations</u>, <u>which it is impossible</u>

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26. Lacroix, S.F. <u>Traité du calcul différentiel et du calcul integral</u> 3 vols. Paris (1797-1800). Probably in collaboration with Lagrange, see note 21 above.
27. Lacroix (1816) Advertisement p.(iii).
28. op.cit. (539-540)
29. op.cit. p.541

to assign algebraically, but whose existence is determined by certain conditions, we call the first quantity a function of the others.", (underlining mine)[30] and that this idea is incorporated in the definition of the differential calculus as "... the finding of the limit of the ratios of the simultaneous increments of a function, and of the variable on which it depends ... in general the error arising from taking the differential (du) instead of the difference ($u' - u$) will be so much the less, the smaller we suppose the increment of the variable to be."[31]

The kinematical and infinitesimal ideas referred to above (p.206) are shown in a statement of the law of continuity, which itself defines the class of functions allowed in this analysis. "It was in the course of enquiries relative to curve lines that geometers first arrived at the differential calculus, which has since been exhibited under so many different points of view; but whatever may be the origin we assign to this calculus, it will always depend on an analytical fact antecedent to any hypothesis, ... and this fact is precisely that property which all functions possess, of admitting a limit in the ratio between their increments and that of the variable on which they depend. This limit, which is different for different functions, but constantly the same for the same function, and which is always independent of the absolute values of the increments themselves, characterises, in a peculiar manner, the course of the function in the different stages through which it may pass. In fact, the smaller the limits of the independent variable, the more nearly the successive values of the function approximate

30. op. cit. p.1

31. op. cit. p.5

to each other; the more does the function also approximate to coincidence with the law of continuity; and the more nearly does the ratio of its changes to that of the independent variable approximate to the limit assigned by the calculus. By the law of continuity is meant that which is observed in the description of lines by motion, and according to which the consecutive points of the same line succeed each other without any interval. The method of considering magnitude in analysis does not appear to admit of this law, since we always suppose an interval between two consecutive values of the same quantity; but the smaller this interval is, the more nearly we approach to the law of continuity, with which the limit accurately agrees; it is also in virtue of this law that the increments, although evanescent, still preserve the ratio to which they have gradually approached before they vanish."[32]

While the objects of analysis may be contin ous functions these functions may be implicit, defined by operations of any kind, or by "other relations" impossible to describe algebraically.[33] These ill-defined statements bring increased confusion later.

It is of particular interest that Lacroix and by implication the members of the Analytical Society, were happy to show two levels of operating in this expository work; the intuitive discovery method of limits, and the more rigorous proof method of differences and series.

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32. op.cit. (73-74).

33. I assume this means the usual operations of arithmetic $+, -, x, -, \sqrt{\ }. ()^n$ and algebra understood as generalised arithmetic.

Lardner's calculus text appeared next[34] and set out to be a simplified and shorter version of his predecessors. He gives a brief description of the principles of Newton's fluxions, D'Alembert's limits and Lagrange's series, viewing them as successive improvements in rigor, and adopts the differential notation saying that "differentials" ... are the same quantities as the "fluxions" in the Newtonian method differing only in notation and name." He omits discussion of "Leibniz's infinitesimal method .. because .. it is inferior in rigor to the others. Its validity consists in a kind of compensation of errors." The differential notation is used ".. in preference to that of the fluxional as well because it is generally received by the scientific world at present, as because of its superior simplicity and power. We shall, however, use the principles of all the three methods as they may seem best suited to the subject of investigation." In a footnote he adds, " "Wherever it can be used without too great complexity ... ... I have preferred the method of Lagrange, as being most rigorous, and free from metaphysical objections."[35] The subtle distinctions of the seventeenth and early eighteenth century faded, and while it may be believed by this text book writer that Lagrangian rigor is the ultimate, any method that seems 'best suited' will do to solve the problem. There is no clear distinction here (like there was in the earlier theories) of the separate roles of fluxions, limits and series; indeed, the only distinction apparent is that fluxions and limits are somehow less legitimate or respectable than series.

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34. Lardner (1825).

35. op. cit. (5-6).

Lardner defines function thus: "When two variable quantities enter the same investigation, they are frequently so related that the variation of either may be determined by that of the other. In other words, a relation may subsist between them, such that any particular value being assigned to either, the corresponding value of the other will be determined. In this case, either of the variables is said to be a function of the other ... The character F or f signifies a function, and F(x) or f(x) signifies a function of x, x being considered the variable. Thus, u = F(x) signifies that u is a function of x."[36] This given, the object of the differential calculus is to "determine the rate of the variation of a function relatively to that of its variable."[37] Although this was a much more abstract definition of function it was still strongly connected with geometric illustration.[38] Difficulties arise however, and amongst these we have a discussion and enumeration of singular points; points of inflexion, multiple points (two or more branches of a curve intersect), conjugate points (single isolated points), cusps and points of osculation, all where the differential coefficient at some stage assumes the form $\frac{0}{0}$.[39]

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36. op.cit. (2-3)

37. op.cit.p.3

38. Lardner makes no reference to Fourier whose Théorie analytique de la Chaleur appeared in 1822.

39. Lardner. op.cit. (137-144). Here he investigates cases where the differential coefficient becomes zero. He puts $0 = \frac{0}{0}$ p.138

The technicalities of treating curves assumed continuous by the method of series throw up a large number of problems and demonstrate the inadequacies of Lagrange's approach.

A further attempt at exposition of principles of the calculus in a simple and clear manner was made by De Morgan whose work[40] first appeared as a serialisation in 25 parts from 1836 to 1842 in the 'Library of Useful Knowledge'.

From the start, the rejection of series is clear, and the concept of limits is reinstated.

"The method of Lagrange, founded on a very defective demonstration of the possibility of expanding $\phi(x+h)$ in whole powers of h, had taken deep root in elementary works; it was the sacrifice of the clear and indubitable principle of limits to a phantom, the idea that an algebra without limits was purer than one in which that notion was introduced. But, independently of the idea of limits being absolutely necessary even to the proper conception of a convergent series, it must have been obvious enough to Lagrange himself, that all application of the science to concrete magnitude even in his own system, required the theory of limits."[41]

Basic principles are given in the second chapter entitled "On the General Theory of Functional Increments and Differentiation",[42] the first remarks being on function and continuity: "When any function of x is given, we can determine by common algebra the value which the function receives when x receives any given value, say a, and also

40. De Morgan (1842)

41. op. cit. preface (iv-v)

42. op. cit. (44-65).

the change of value which takes place when x <u>becomes</u> a + h, by which we merely mean when we pass from the consideration of the function a to that of the function a + h."[43] When x = a, $\phi(x)$ may have two kinds of values, (i) " a finite calculable value positive or negative" or (ii) .. "one of the varieties of form which arises out of our supposition being followed by an absence of all magnitude, or 0, in a place where the general form of the function would lead us to suppose there is some number or fraction to be operated on or with. Such forms are $\frac{o}{o}$, $\frac{1}{o}$, $a^{\frac{1}{o}}$, a^{o}, $(\frac{1}{o})^{o}$, $(\frac{1}{o})^{\frac{1}{o}}$, etc."[44]

The second possibility is overcome by the principle of limits and the following definition. "The function is said to have the value A when x has the value a, either when the common arithmetical sense of these phrases applies, or when by making x sufficiently near to a, we can make the function as near as we please to A. In the first case A is simply called a value, or ordinary value, of the function: in the second case A is called a <u>singular</u> value."[45]

In order to explicate this definition, the following postulates are given: "Postulate 1 - if ϕa be an ordinary value of ϕx, then h can always be taken so small that no singular value shall lie between ϕa and $\phi.(a + h)$, that is, no singular value shall correspond to any value of x between x = a and x = a + h."[46] He goes on: "The truth of this

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43. op. cit. p.44

44. loc.cit. These could arise for example from forms like $(1 - x)^{(1 - x)}$ at x = 1.

45. loc.cit.

46. op. cit. (44-45)

postulate is a matter of observation. We always find singular values separated by an infinite number of ordinary values... We have drawn the representation of a function below, so as to exhibit every variety of singular value, and more than the skill of the most practised algebraist would at present be able to find a function for. The stars mark the singular values, or rather the places at which there may possibly be a singular value; all other values are ordinary, however near the singular values they may approach in position. And we see that, however nearly a, the value of x may approach to b the value of x at one of the singular points, it must be possible to take a + h lying between a and b."[47]

There follows a diagram (below) showing a bizarre function with maxima, minima, points of inflexion, cusps, multiple points and asymptotes.

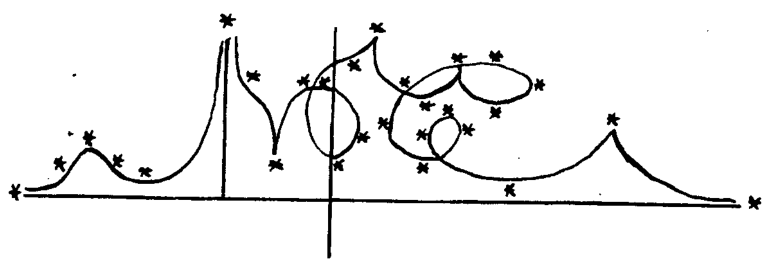

Fig.4.1 De Morgan (1842)p. 45

Thus, continuity is preserved in the neighbourhood of singular points.

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47. op. cit. p.45

"Postulate 2 - If $\emptyset a$ be any finite value of $\emptyset x$, it is always possible to take h so small that $\emptyset(a+h)$ shall be as near to $\emptyset a$ as we please, and that $\emptyset x$ shall remain finite from $x = a$ to $a + h$, and always lie between $\emptyset a$ and $\emptyset(a+h)$ in magnitude.

This again is part of our experience of algebraical functions. It is generally assumed under the Law of Continuity ... It is possible to imagine a function which does not observe this law, but we cannot, without further consideration of singular values, find the means of expressing it algebraically. For instance, in the following figure, the function represented by ABCDEF is discontinuous at B and D. But we have no means of expressing such a function in common algebra. We may call the law expressed in this postulate the law of continuity of value, to distinguish it from that of the next postulate; and we may say that functions which do not obey this law, if any, are discontinuous in value."[48]

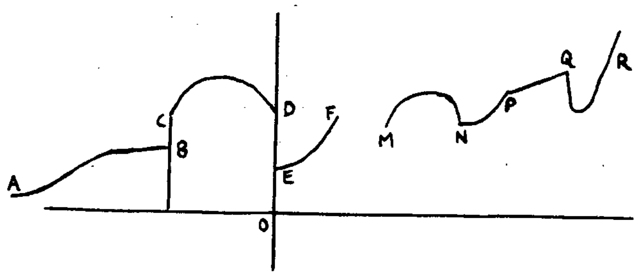

Fig. 4.2 De Morgan (1842) p.45

And finally we have:

"Postulate 3 - If any function follow one law for every value of x between $x = a$ and $x = a + h$, however small h may be, it follow the same law throughout: that is, the

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48. op. cit. (45-46).

curves of no two algebraical functions can entirely coincide with each other, for any arc, however small ... This we may call the law of continuity of form, or permanence of form.

Exceptions to this law may be represented but cannot yet be algebraically formed. As in MNPQR, we may conceive a function which is represented by an arc of a circle joined to one of a parabola, which is itself joined to part of a straight line, and so on. Such a function would be called discontinuous in form, and though not now exhibited algebraically, may actually occur in practice. Suppose, for instance, a spring of the form MNPQR fixed at the end M, and disturbed at the other end. The number of its vibrations per second might become a subject of enquiry."[49]

In these few pages we have De Morgan laying bare the state of the art as it was immediately prior to Weierstrass. Aware of the conceptual breakthrough of Fourier,[50] where a curve may be drawn and an algebraic expression be found to describe it, we first of all have an abstract definition of function as the mapping $x \to \phi x$ to include all the unusual situations conceivable. Then, in order to deal with the anomalous cases, it is necessary to make three postulates: Postulate 1 covers limiting values of ϕx at singular points and guarantees continuity of the function in their neighbourhood; Postulate 2 guarantees the continuity of ordinary algebraic functions, while Postulate 3 covers uniqueness and states that if the values of two functions are equal, the functions are identical.

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49. op. cit. p.46

50. A version of Fourier's theorem appears on (616-618).

The justification for these postulates is entirely intuitive, relying heavily on our "experience of algebraical functions", and it is upon these principles that all subsequent justifications and proofs lie. Later in the century these ideas were to be re-cast into a more 'respectable' axiomatic form.

For De Morgan, his main tool is the process of differentiation, or finding the differential coefficients of functions, while the theory of limits forms the basis of his proof-method. He states that "the limit of $\frac{\phi(x + \theta) - \phi x}{\theta}$ is called the differential coefficient of ϕx with respect to x" and that "every function either has a finite differential coefficient when x has the specific value a, or when it has a value, a + k where k may be as small as we please."[51] Thus every function is differentiable, but some are more differentiable than others, for there are the exceptions, the singular values that have to be dealt with separately.[52] The enormous labour and lengthy discussion taken over these cases is a consequence of the lack of precision of the postulates and the inability to shed geometrical preconceptions.

De Morgan goes to great pains to explain his procedures giving copious examples in some 850 pages of close print. I admire the perseverance of victorian students, for I marvel that anyone could digest such a book!

The last pre-Weierstrassian text of significance is that of Todhunter[53] whose book is considerably shorter than

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51. op. cit. p.48

52. op. cit. ch.X(172-183) and Ch.XIV(374-388)

53. Todhunter (1852).

De Morgan. While De Morgan exposed his innermost thoughts, his doubts and problems, Todhunter is quite uncompromising. It seems that by this stage the techniques of the calculus (or at least the strict proof-forms) have become so complex that they are no longer necessary or appropriate for the student. The definition of function, for example, is introduced by the notation $y = \phi(x)$ and further brief discussion distinguishes <u>explicit</u> and <u>implicit</u> functions in terms of algebraic formulae. Explicit functions are then classified as <u>algebraical</u> or <u>transcendental</u>, the former being obtained by the usual operations of arithmetic (including raising to powers and extraction of roots) while the latter are exponential, logarithmic or trigonometric functions.

The notion of <u>limit</u> is given more emphasis and some theorems and numerical examples are introduced. However, there is little motivation. Bald statements like:

"Limit of $(1 + \frac{1}{x})^x$. The following theorem, which we proceed to demonstrate, is very important."[54] are typical of the tone of the book. The emphasis is on demonstration by example, and continues in this way to submerge conceptual difficulties and foundational problems in a display of technical mastery. The fundamental ideas of function and limit are discussed in the short introductory chapter,[55] differentials appear much later as a shorthand for the usual notation of $\frac{dy}{dx}$ [56], and nowhere do we find the painstaking and revealing arguments of earlier writers. The

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54. op. cit. p.6

55. op. cit. (1-15)

56. op. cit. (340-344)

Doctrine of Limits is the true foundation of the calculus and the student is required to persevere to understand this truth.

Difficulties are briefly acknowledged at the end of the introductory chapter where Todhunter remarks that the apparent lack of practical applications in the earlier part of the book may lead the student to believe he has not correctly understood the elementary principles of the subject. "It may, therefore, be of some service to assure him that the difficulty of which he complains is probably owing much more to the nature of the subject than to his own want of comprehension .. he must be satisfied at first with reflecting upon the meaning of the definitions, and examining whether the deductions drawn by the writer from those definitions are correct ... we shall at first confine ourselves merely to the logical exercise of tracing the consequences of certain definitions."[57]

Talking of limits he admits that the student may have "a suspicion that the methods employed are only approximative, and therefore a doubt as to whether the results are absolutely true. .. In such a case all he can do is to fix his attention very carefully on some part of the subject as the theory of expansions for example, where specific important formulae are obtained. He must examine the demonstrations, and if he can find no flaw in them, he must allow that results <u>absolutely true and free from all approximation</u> can be legitimately derived by the doctrine of limits."[58]

57. op. cit. (11 - 12)

58. op. cit. p.12

Clearly, we have here the belief that ultimate rigor has been achieved in the theory of limits and that the student should learn to operate consistently and continually at this level.

3. The problem of relative rigor.

Towards the end of the nineteenth century calculus texts are beginning to contain elements of the recent work of Weierstrass.[59] This has obvious influence on a work such as Lamb's calculus,[60] published in the year of Weierstrass's death.

Here we find a definite attempt to simplify the presentation of the subject and to exercise the student in the kind of mathematics most useful for elementary applications in physics and engineering. Functions of a single variable only are the major concern, being defined when "one variable quantity is said to be a 'function' of another when, other things remain the same; if the value of the latter be fixed that of the former becomes determinate."[61] The first chapter is headed: 'Continuity', and firmly establishes the idea that the main study concerns continuous functions, the more awkward 'indeterminate forms' being omitted.[62]

59. Karl Weierstrass (1815-1897) is credited with the new formulation of the fundamental concepts of analysis. See Smithies, (1975).

60. Lamb (1898)

61. op. cit. p.12

62. op. cit. p.vi. "The omissions referred to include the general theory of indeterminate forms. This is somewhat tedious to establish rigorously; a good deal of it is very artificial; and, practically, the rules are not used by mathematicians who have recourse when the occasion arises, to more direct methods of evaluation."

The definition of continuous function is given thus: "Let x and y be corresponding values of the independent variable and of the function. Let δx be any admissible increment of x and δy the corresponding increment of y. Then if, σ being any positive quantity ϵ, different from zero, such that for all admissible values of δx which are less (in absolute value) than ϵ the value of δy will be less in absolute value than σ, the function is said to be 'continuous' for the particular value x of the independent variable.

Otherwise, if $\phi(x)$ be the function, the definition requires that it shall be possible to find a quantity ϵ such that

$$\left|\phi(x + h) - \phi(x)\right| < \sigma$$

for all admissible values of h such that $|h| < \epsilon$. The value of ϵ will be in general limited by that of σ, but it is implied that the condition can always be satisfied by <u>some</u> value of ϵ, however small σ may be."[63] Discontinuous functions exist, but are only admitted insofar as they can be applied to physical problems. The derived function, derivative or differential coefficient is defined as:

$$\phi'(x) = \lim_{h \to o} \frac{\phi(x + h) - \phi(x)}{h} \qquad 64$$

and <u>infinitesimals</u> are variables which tend to a limiting value of zero and are "equal when the limiting value of the ratio of one to the other is unity." [65]

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63. op. cit. p.16
64. op. cit. p.65
65. op. cit. p.61
67. op. cit. preface (vii-viii)

Differentials appear as approximations where

$$\frac{\delta y}{\delta x} = \phi'(x) + \sigma \quad (\delta y, \delta x \text{ increments})$$

so that as $\sigma \to 0$ it is more nearly true that

$$\frac{\delta y}{\delta x} = \phi'(x) \quad \text{or} \quad \delta y = \phi'(x)\, \delta x$$

which is often written $dy = \phi'(x)\ dx$

"The vanishing quantities dx, dy are called 'differentials'."[66]

The aim is to achieve a coherent presentation, "Considerable attention has been paid to the logic of the subject .. It is not claimed that the proofs of fundamental propositions which are here offered have the formal precision of statement which is <u>de rigeur</u> in the theory referred to; but it is hoped that in substance they will be found to be correct. Occasionally, where a rigorous proof of a theorem in its full generality would be long or intricate, it has been found possible, by introducing some additional condition into the statement, to simplify the argument, without really impairing the practical value of the theorem."[67]

This presentation set the final fashion that has predominated till the present, the original motivations largely concerning the problems of continuity generated by the redefinition of function, have been submerged.

To summarise, in the sixteenth and seventeenth century there was a well established tradition of <u>proof</u>, founded in Euclidean geometry and the algebra of Vieta; the working mathematician used <u>infinitesimals</u> (or indivisibles) as a shorthand method of justification, of checking results, or

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66. op. cit. p.132

67. op. cit. preface (vii-viii)

demonstrating procedures.[68] This also provided a heuristic for the classification and approach to problems.

When the 'Method of Inverse Tangents' was established, the heuristic changed, 'fluxions' and 'differentials' providing the re-classification and suggesting new approaches. At this time the three levels of working are distinct and generally accepted.

During the eighteenth century begins the separation of algebra and geometry, new contributions to the concept of function, and a host of technical problems, foundational, operational and in the choice and meaning of notation. On top of this we have demands to make the new calculus respectable, so that it becomes as rigorous as other parts of mathematics.[69]

The early nineteenth century shows a distinct lack of clarity as to which level to work at; this insecurity being a consequence of the fact that there was no commonly accepted proof-process to stabilise the foundational aspect.

Weierstrass's breakthrough occurs in the area of proof-process, the redefinitions of fundamental ideas suggesting reformulations of the technical processes; the demand for working at the level of maximum rigor seemed to be fulfilled. Today we know that this is not possible, we still have difficulties even in the 'elementary' calculus and the demand for maximum rigor seems unattainable.

Much of the time we are teaching, and even 'proving' theorems, we are working in Popper's first world (of

68. This was necessary on two counts: i) as a shortening of the proof itself, and ii) since the accepted fashion was to prove every result from the first principles.
69. The ultimate model being Euclidean geometry.

objects and problem-situations) and his second world (of belief-systems, creative processes and heuristic devices).[70]

So long as we are able to recognise the level (or the world) we are working in, we can come to terms with the objects and devices used. It is no longer necessary to demand a high level of rigor from a low level of working. In fact, heuristic devices cannot be rigorous in this sense, even though their use may be quite legitimate.[71]

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70. See above, Section 3.

71. We can make a case for the rigorisation of heuristic devices; the 'rehabilitation' of infinitesimals by Robinson (1966) Ch.X is an example. The appropriateness and relevance of this to Leibniz's theory is discussed by Bos (1974) where he clearly shows that these infinitesimals are not the same entities.

SECTION 5

Teaching History of Mathematics and teaching mathematics.

As indicated earlier[1] there have been a large number of papers, reports and other articles advocating the study of the history of mathematics and outlining its pedagogical implications. The purpose of this section is to collect and summarise the reasons given, and to discuss briefly the practical and pedagogical implications of the teaching or use of the history of mathematics at different levels.

a). Reports, Courses and Sources

Apart from the more general pedagogical writings[2] there have been a number of reports of official bodies on the teaching of mathematics, and on occasion, in this context, there has been mention of the history of mathematics. Of those examined,[3] very few make more than a passing mention that history can be 'useful' or 'interesting'. In my opinion, the most signficant of these reports are the I.A.A.M.[4] Teaching of Mathematics of 1957; the Ministry of Education Teaching of Mathematics in Secondary Schools of 1958; the Mathematical Association Second Report on the Teaching of Mechanics in Schools of 1965, and the A.T.C.D.E. Teaching Mathematics of 1967. All of these reports not only claim a definite place for the history of mathematics in

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1. See above Section 1.p. 82
2. See above Section 1.p. 20ff.
3. I refer to only those reports of organisations in Great Britain, like the numerous reports of the Mathematical Association.
4. Incorporated Association of Assistant Masters.

schools and teachers courses, but make suggestions and provide illustrations of the teaching and use of history, and also provide some guide to books for teachers and pupils.

In the I.A.A.M. report the general view is that mathematics is a fundamental part of human culture and a knowledge of the part mathematics has played in the progress of mankind is an essential part of education. The history of mathematics "should be brought in appropriately at various stages of the teaching of mathematics as a part of the subject as a whole." Mention is made of the possible advantage of considering the historical development of mathematical ideas "From the point of view of teaching..." though, apart from the reason already mentioned, it is not clear what else one hopes to gain, particularly since the "historical approach to a topic is not always the best one."[5]

Teachers need not be experts in the history of mathematics, but "Much of the history of mathematics can be taken casually as a natural part of the instruction of mathematics." A sample list of topics suggests the areas of mathematical history with which a teacher ought to be familiar.

There follows a separate section on the Philosophy of Mathematics, which, while it does not necessarily advocate such an approach in the classroom, points to the underlying questions on the nature of mathematics, mathematical proof, significant events, development of techniques, and examination

5. This, I assume is a reference to the biogenetic law.

of foundations that certainly have relevance for the teacher.[6] The fact that children can challenge apparently firmly established beliefs with simple questions should not really be surprising, and teachers should answer such questions as frankly as possible, pointing out the interrelation of various fields of mathematics, and showing that mathematics is still a living and growing subject.

The Ministry of Education report has by far the longest chapter devoted to a consideration of the history of mathematics. The reasons given for attention to be paid to the historical development of mathematics fall into two general areas:

(i) Reasons directly related to pupils learning: the cultural influence of mathematics and the social significance of mathematicians, from astrologer to statistician; the fact that some mathematics has been discarded, and the reasons why this is so; the process of mathematical modelling and the checking of results from models by observation and experiment; the fact that principles and methods have been used to obtain results often long before any analytic justification was available; the consideration of mistakes, paradoxes and controversies, and the insights given to pupils into individual mathematical creativity.

(ii) Methodological reasons: A teacher's knowledge of the history of mathematics gives continuity to a course, helping him to prevent teaching techniques in isolation or unrelated to further or later developments; knowledge of patterns of discovery improve a teacher's heuristic method;

6. Many of these ideas cannot really be separated from their historical context.

"knowing the history makes it easier for the teacher, 'so to teach that he creates the impression - the illusion, perhaps - that the pupil is discovering for himself what he is trying to teach him.'.."[7] The greatest advantage of a knowledge of history of mathematics is the realisation that mathematics is a branch of human culture important in its own right, with links with social history, art, literature, philosophy, and so on. With this knowledge the teacher can go a long way towards dispelling the fear of mathematics felt by so many pupils.

These two sets of reasons are, of course, not entirely mutually exclusive, but give a good idea of the obvious first benefits, and the more subtle advantages of the use of history.

The Second Report on the Teaching of Mechanics claims a definite place for the history of mechanics. The introduction says: "A knowledge of the history of the development of mechanics is invaluable to the teacher ... He should realise that interest lies not only in possible early anticipation of modern concepts, but also in trying to understand why men much cleverer than himself often took so long, and found it so difficult, to discover what seems obvious to him... The wise teacher will stimulate interest from time to time by reference to earlier developments and their attendant difficulties, and he will encourage an attitude of humility in looking back on the pioneers from an age which enjoys the fruition of their ideas. Perhaps the best help that can be derived from a knowledge of the

7. Ministry of Education (1958) p.144. The quotation is from C.A.Laisant, La Mathematique, Paris 1898.

history of the subject is the use of it to bring out some of the first principles in their earliest and simplest forms, so that pupils are encouraged to see the implications of these principles and the importance of precision in the enunciation of them."[8]

A later chapter by (J.R. Ravetz) gives an historical sketch of some problems and methods in mechanics from Aristotle to Newton. This also contains a bibliography whose contents are sufficient to provide stimulation over a wide range of subjects, certainly thoroughly covering the traditional school syllabus on the subject. In sketching the possibilities of study after Newton, Ravetz mentions several areas: celestial and theoretical mechanics, rational mechanics, applied and industrial mechanics, and text books, which served as a place for discussion of the foundations of mechanics. "From a study of these different traditions (and their interactions) we may come to illuminate pedagogical and conceptual problems in the elaborated structure of classical mechanics."

It is clear from the inclusion of this chapter that the authors think that it is the teacher's duty to increase his awareness of his subject, and that a good historical background helps him to appreciate the difficulties, development, and pedagogical implications of the study of mechanics.

This view is shared, in the wider context, by the authors of a short section in the A.T.C.D.E. report, Teaching Mathematics. While commending the lectures that might be given to enrich a main course, " .. this kind of approach can scarcely be considered an adequate basis for the

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8. Mathematical Association (1965)p.10

proper study of the history of mathematics. Indeed, any course unsupported by study and investigation on the part of the student can contribute little to the student's understanding of the development of mathematical concepts, techniques and proof structure, which is what the history of mathematics is about."[9]

It is clear from these reports that there are many advocates for the study of the history of mathematics. We may be able to judge the success of these exhortations by looking briefly at some examinations that have been set at different levels. In 1952, the new O-level G.C.E. mathematics examination of London University had a 2½ hour paper on history of mathematics. Candidates were asked to write one essay and answer four other questions. It is easy to criticise such an examination, and one has to be aware of the context and content of secondary school mathematics about this time. Even so, it is difficult to see how pupils might answer such questions as: "Describe how deductive geometry arose, and mention the parts played by some of the men who developed it in its early stages."[10] in the given time, without memorising teacher's dictated notes. Other questions like, "Give an account of <u>either</u> the Greek <u>or</u> the Egyptian number systems and the method of dealing with fractions in the system you have chosen."[11] may look simpler, but could be just as difficult to answer under examination conditions. Later, a short History of

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9. ATCDE (1967) p.24

10. London O-level G.C.E. History of Mathematics. 1952 Question 2.

11. London O-level G.C.E. History of Mathematics. 1955 Question 2

Mathematics was published[12] expressly to help pupils pass this examination. The book was divided into short chapters (some of only a couple of pages) with 'typical' examination questions interspersed. This examination appears to have been discontinued sometime after 1958[13]. Since the more serious pupils would have been doing 'real' mathematics, the less able probably found the history quite difficult, and the number of candidates did not justify its continuance.[14]

Advanced level examinations in mathematics are more demanding of both pupils and teachers time, and the inclusion of a history question in the Cambridge pure mathematics paper seems an anomaly.[15] The effort required to give a reasonable answer to one question, resulted in that too, being discontinued.

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12. Freeburg (1958) which takes mathematics up to the seventeenth century.
13. The University of London Schools Examinations Department was only able to find copies of these papers from Autumn 1952 to Autumn 1956.
14. Also, from about 1958 onwards many teachers were devoting their energies to the development of 'new maths' courses.
15. It has been particularly difficult to trace any history of mathematics examinations at Advanced level. This information was obtained from G. Kelloway at the Department of Education at Cambridge.

History of mathematics as a separate subject in the school curriculum seems to be a non-starter. The reasons for this are clear: not only is a fair mathematical background required for even the most elementary historical study, but time is at a premium, and more obviously utilitarian mathematics gains precedence. Other factors include the system which requires a timed examination rather than a long-term essay or project,[16] the general lack of historical knowledge of teachers, and resources at the right level for both teachers and pupils.

History of mathematics can be successful in schools as a background to mathematical courses,[17] and in other more subtle ways, but in the form described above is generally inappropriate. It may, however, be possible to include history in some new examination syllabuses for non-specialist sixth formers.[18]

For a long time educators of teachers have recommended a knowledge of history of mathematics as essential, and examinations at the certificate, B.Ed., and postgraduate levels have contained opportunities to answer questions or write essays both on history of mathematics and the uses of history in mathematical education. Unfortunately, apart from the institutions where there is a member of staff who is particularly interested in history of mathematics, these studies tend to be rather limited, and the

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16. This latter is now becoming possible with selected topics in the framework of examinations like the C.S.E. mode 3.
17. See below pp240 below and (251-252).
18. For example at the proposed N.level.

students efforts are sometimes quite trivial.[19] "The proper study of the history of mathematics must always involve constructive and creative endeavour on the part of individual students and student groups. Studies in this field should generate imaginative insight and will require patience and persistence of a high order in asking relevant questions, formulating problems, hunting for sources, analysing, classifying and interpreting material."[20] Such a level of activity will not be possible without staff who are aware of the enormous contribution historical studies can make to the professionalism of the teacher, and good library facilities; for "if the history of mathematics is to be studied seriously in a college then there must be at least as many history books on the shelves as there are textbooks on analysis or modern algebra or any other field of study."[21]

Fortunately, the resource situation is improving. At college level the more recent histories of mathematics provide good coverage of most fields.[22] While some suggest topics for self-study or use in the classroom,[23] others integrate history with the study of a particular area.[24]

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19. I quote my own and colleagues experience in examining many such students.
20. ATCDE (1967) p.24
21. ATCDE (1967) p.25
22. For example Boyer (1968) Kline (1972)
23. For example Eves (1969) Hallerberg (1969)
24. For example Eves and Newsom (1965) Greenberger (1973)

More specialist work has appeared recently,[25] and bibliographies help the investigator to locate books and materials.[26] Source books and reprints of original works are more easily available[27] and more journals contain articles with historical background.[28] Finally, the recent inception of the Open University course in the history of mathematics has provided colleges with a range of films, tapes, books and course units that have yet to be fully exploited.[29] While the resources are there, the major problem is one of cost. Books are expensive and even with present library allocations the proportion of money spent on books on the history of mathematics is quite small, unless for example, there happens to be a separate allocation to a department of history of science.

If colleges in general find it difficult to provide adequate resources, schools are in a far worse position. Not only are books costly, but the majority are inappropriate for school use,[30] and justifying the outlay on

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25. Novy (1973) and Menninger (1969).
26. May (1973) and Rogers (1975) in particular are appropriate at this level.
27. Struik (1969) and Whiteside's editions of Newton for example.
28. Apart from those devoted to history of science and mathematics more recent teachers journals like Int.J.Math. Educ. Sci. Technol and Mathematics in Schools are quite keen to publish historical material.
29. The sponsorship of cheap paperback editions of Wilder (1968) Kline (1954) Boyer (1968), Dedron and Itard (1974) and Popp (1975) is most welcome.
30. Histories like Boyer (1968) are useful but the majority of works already mentioned are suitable only for the best sixth form pupils and teachers.

such material is difficult in the face of other priorities. There is a real lack of reliable and appropriate material for use at school level.[31] The history of mathematics in schools can best be served, in my opinion, by producing materials like short topic books containing relevant historical background;[32] tape/slide programmes; collections of facsimilie documents;[33] posters and portraits;[34] and a few cheap paperback books on selected subjects like the history of modern algebra, for example, that can be read by fifth and sixth formers. Writing for teachers is also possible,[35] and while some recent books give much historical background a good history of mathematical education in

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31. While we may cite Dubbey (1970) and the tape/slide materials shown in Rogers (1975) (supplement 1) as good examples, there are a number of trivial and inaccurate books around. See my review of Morgan (1972) in Mathematics Teaching, 63, June 1973 p.77.
32. For example Tahta (1969)
33. 'Jackdaws' have produced a collection on 'Newton and Gravitation' (Ronan)
34. Useful for initiating questions and discussion, they are notoriously ephemeral and difficult to locate.
35. This continues to flourish, as already mentioned in 28 above.

Great Britain has yet to be written. These are the ways that background material can be provided to show teachers and pupils that mathematics has a history, and that its past is highly relevant to its present and future.

b) History of Mathematics for Teachers

While it is relatively easy to write a syllabus for a course containing what teachers ought to know, the reality is that both time and the student's energies are limited. The purpose of the following outline therefore is to provide relevant background for teachers on the historical development of some of the major areas of mathematics which appear in current certificate and B.Ed. syllabuses.[37] It is selective and not intended to be too detailed. Deeper individual study of particular areas of the history of mathematics would be possible in coursework assigned at different stages in the programme. It is hoped that the outline below will :

1) Portray the evolution of mathematical ideas as an ongoing human endeavour and a major part of our culture;
2) Contribute to the teacher's philosophical background by providing different views on the nature of mathematics and mathematical activity;
3) Provide pedagogical insights, motivations and examples for use in the classroom,
4) Give some idea of the causes and problems surrounding changes in mathematical education.

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37. I take as model the syllabuses for these examinations at the University of London. The mathematical content is fairly typical of similar courses throughout the country.

Such a course can be thought of in two sections, intended to be independent of each other if required.

(i) Elementary: intended for non-specialist primary and middle school teachers;

(ii) Advanced: intended for specialist teachers in middle and secondary schools.

One would normally expect those taking the advanced course to have followed the elemtary course, but within the courses themselves, there is little necessary order. In fact, the order of the suggested topics is probably best dictated by a correspondence with other mathematics covered, either in the professional, or specialist context.

Elementary Course.

It is assumed here that the mathematical background of students entering the elementary course is minimal. It is also expected that in general their attitudes towards mathematics may be poor, through lack of personal success of understanding.

The following topics are suggested so that they complement a student's professional studies and contrast with their probable recent school experience.

Arithmetic; Notations and Computation

Tally systems; heiroglyphic, symbolic, hindu-arabic notation; the origins of counting and number-names; decimal and other counting systems; finger reckoning, the use of the abacus; elementary operations and different methods in arithmetic; operations leading to unit fractions.

Use can be made here of some of the ideas in Section 4B.

Geometry: Perspective and the Geometry of Position

From perception to perspective, renaissance painting and the portrayal of reality; Euler and the bridges problem, connectivity and nearness.

Some of the material that can be used here is described in Section 4c. Euler's original short paper is available in translation.[38]

Measurement: The development of common weights and measures.

Motivations and problems in practical measurement; measures of value, time, length, area, volume, weight and mass; measures of likelihood and the origins of probability.

Quite a lot of material is available for example in Berriman (1953) and Zaslavsky (1973). Simple ideas of probability can be found in David (1962), and Rabinovitm (1973) describes a probabilistic method of making unequal shares 'fair'.

Assessment of such a course could be by project or essay. Collections of ideas and materials could be made for use in school.

Advanced Course

This course would normally require A-level mathematics or an equivalent standard. It is envisaged that this course would run in conjunction with a course of advanced mathematics, either certificate or B.Ed. Students would normally have followed the Elementary course at some earlier stage. Thus the Elementary and Advanced courses might run in consecutive years.

The following topics are suggested:

(i) General: Survey of the phases of mathematical history.

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38. Newman J.R. (Ed) (1960) (573-580)

The writing of history; sources; information storage and retrieval.

(ii) Algebra:

a) The solution of equations and the development of algebraic notation XV, XVI, XVII, centuries.

b) Complex numbers XVIII, XIX, centuries.

c) The rise of axiomatics XIX, century

This section shows the concurrent evolution of techniques and notation, and the application of standard methods (in particular, the extraction of roots) in new situations which gave rise to new types of numbers. The development of complex numbers and other generalised numbers gives a background to linear algebra, and the period of the algebraists Peacock to Peano shows the development of the axiomatic background.

The Open University films: "The Great Art" and "Quaternions - A Herald of Modern Algebra" can be used here.

(iii) Analysis:

a) Problems of tangents, quadrature and rectification.

b) Infinitesimal techniques in XVII; Newton's approximate solution of equations; Newton's versions of the calculus; Berkeley's objections.

c) Limit theory in XVIII, XIX; the beginnings of point set topology.

Note: This section is intended as a background to numerical methods, calculus and modern analysis.

(iv) Geometry

a) Invention of analytic geometry in XVII, Fermat and Descartes.

b) Projective Geometry XVIII, XIX, centuries.

c) Non-Euclidean Geometry, XIX, century.

This section gives an account of the development of a number of important concepts, principles and methods in contemporary mathematics; transformation, invariance, the principle of continuity and the idea of a mathematical model. It is also a good vehicle for the study of the development of methods of proof and standards of rigor. The section (b) follows on from the elementary course geometry. Part (c) connects with the earlier advanced section on algebra (ii)(c) above, and demonstrates the breakthrough to postulational systems.

(v) Mathematical Education

a) Historical survey of the reasons for teaching mathematics. Mathematics syllabuses and examinations.

b) Movements in mathematics teaching; Klein, Hilbert and the Gottingen school; teachers organisations,

c) Evolution of mathematical ideas; survey of general trends in geometry and algebra; later developments, e.g. statistics, finite mathematics and operational research.

This section represents aspects of mathematics and mathematics teaching developing as a result of a number of internal and external influences: different philosophies of mathematics, socio-economic demands external to mathematics, social events within mathematics (e.g. appointments, meetings, friendships, animosities etc.), technological developments (e.g. fast calculators, library systems, etc.), political decisions (e.g. education acts, financing of research, etc.). Much material for this section can be found in Griffiths and Howson (1975) and Frendenthal (1974).

With regard to this list of topics, it will be observed that a number of areas which might be considered to be important both historically and mathematically have been omitted. For example, no mention has been made of Greek mathematics, observational astronomy, the problems of force and motion, navigation and map-making, Babylonian and Chinese mathematics, and the rise of Newtonian mechanics. This is not to deny their importance in the development of elementary mathematics, but these areas are sufficiently documented in a more or less readily accessible form for them to provide a wide selection of problems and situations for research on the part of the student. A number of assignments given to students following the advanced course could be taken from these areas.

c) How We can Learn from the History of Mathematics

Any study of the history of mathematics offers opportunities for the contemplation of the evolution of mathematical structures. (In pure mathematics the structures are presented as complete. They may be constructed during a lecture, or in a course of lessons, but there is in general no opportunity to discuss the evolution of structures. This sense of constant growth is important in the teaching of mathematics.) The evolution of mathematical structures involves the gradual widening of concepts and changes in attitudes concerning the purpose and nature of mathematics. In investigating how a particular problem was solved, or a technique invented, we are involved in studying mathematical invention. The study of pure mathematics provides opportunity for the participation in and discussion of mathematical invention, but in the wider historical context, we have the possibility of considering the solution of problems which are so deeply embedded in our mathematical culture as to be taken for granted, their solutions obvious even to the beginner in mathematics. By studying the activity involved in the solution of many of the problems of elementary mathematics, we can generate attitudes toward the nature of mathematics and the regard for mathematical ideas. In a study of the history of mathematics we also become involved in problems in the philosophy of mathematics, not necessarily as an end in itself, nor at a very technical level, but as an essential aspect of the understanding and explanation of mathematical ideas. This dual involvement in the philosophy of mathematics and the study of mathematical activity leads to the area of mathematical epistemology, or how we come to know mathematical ideas, and the realisation of the power and potential of the mind. This leads us on to the communication of mathematical ideas and their acceptance or rejection by society. This concerns not only society at large, but more immediately the mathematical community of which the individual is a part. The status of axioms and definitions, the methods of proof, the techniques of construction of theories, the pertinent problems, become significant when we see them in the context where they originally appeared.

By relating the parts of mathematics to the whole, by suggesting retrospective-deductive explanations of the evolution of mathematical ideas, we become increasingly aware of the nature and quality of mathematical thought, and the kinds of conditions that are likely to encourage it. Because mathematics is continually growing we need to revise and replace our explanations regularly. New explanations may become apparent, not necessarily through the discovery of new historical evidence, but perhaps through some change in the philosophy of the investigator. We do not learn by vainly searching for laws of mathematical discovery,but by realising our explanations depend on the necessary fluidity of mathematical definitions.

d) What we Learn from the History of Mathematics

Many of the kinds of things we learn from the history of mathematics might be learnt from other sources within and without mathematics, and from the ways in which we are encouraged to learn mathematics in the first place. In particular, the area concerning the activity of mathematical invention can obviously be studied without reference to the history of mathematics. I would claim, however, that many of the following that might be encountered elsewhere have their most striking manifestation in the context of the history of mathematics. It would be futile to attempt a comprehensive list, but the following general areas are considered to be the most important:

i) The evolution of mathematical structures and the place of particular problems in their development.

ii) The significance of some fundamental problems and the kind of mathematical philosophy involved in their attempted solutions.

iii) The gradual changes in mathematical concepts and the breakthroughs connected with these changes.

iv) The motivations surrounding particular problems.

v) The generation of abstract mathematics from physical problems.

vi) The nature of mathematics as a mental activity and the necessary impermanence and uncertainty of the theoretical structures.

vii) Fashions in mathematics and the type of problems involved: the kind of mathematical activity produced as a result.

viii) The communication of mathematical ideas both within and without mathematics.

ix) The evolution of mathematics as a body of knowledge and the levels at which mathematics exists in the culture.

x) The relation of past mathematics to present mathematics and mathematical activity.

xi) The evolution of heuristic and mathematical education.

xii) The History of Mathematics as providing the data for curriculum modification.

The study of the history and philosophy of mathematics and mathematical activity itself brings us to the field of epistemology, the basis of pedagogy.

e) <u>Implications for Teaching</u>

It has never been denied that the history of mathematics forms a serious and fascinating study at research level, and that both postgraduate and undergraduate courses can demand a great deal, for a knowledge of both mathematics and of history is essential for a balanced view.[39]

The major emphasis of the argument presented here has been in the context of the training of teachers, and properly, this should be extended to the teachers of mathematics in the polytechnics and universities. For many of these, their researches lead them to examine previous work in their own field, and an essential preliminary to any research is a survey of similar problems and the attempts at their solution. Wilder (1972) claims that a knowledge of history can suggest new problem-areas or show that a field is declining, and if this is the case we have a serious argument for the inclusion of history of mathematics in the training of researchers as well as teachers. Thus there is a fair case that some aquaintance with the history of mathematics is a necessary part of the training of every mathematician, for many of the reasons summarised below are to be found in such discussions.[40]

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39. Some requirements are quite demanding. For example, for postgraduate work Truesdell asks for ".. a fluent reading knowledge of Latin, French and German; competence in mathematics equivalent to a good bachelor's degree in it as taught by mathematicians; some study of rational mechanics as it is understood today; and a good general view of the political, social, literary and artistic history of the period in which the candidate would concentrate." See Historia Mathematica 2(2) 1975 (192-193).

40. See Grattan-Guiness (1973).

While it was stated at the outset that many of the aims we may have in our teaching of mathematics have no necessary reliance on a teacher's knowledge of history, it does seem that much the easiest and most fruitful way of presenting a large part of mathematics is either against a background of history, or in the context of a historically based philosophy of mathematics. Teaching mathematics with an historical perspective is not merely attempting to imitate in teaching one's interpretation of a sequence of historical discovery, neither is it assuming that a growing child is passing through the same mathematical experiences as our ancestors; it is the constant awareness of the continuing and dynamic dialectic by which mathematics evolves.

The implications for teaching are profound. The reasons for the study and the use of history of mathematics fall into three main areas; philosophy, pedagogy and methodology.

I. Philosophy: concerning the fundamental nature of mathematics and mathematics in history.

i) Understanding of the nature of mathematics and of history aids the development of critical philosophy from ideology.

ii) Examination of different areas of mathematics in different periods emphasises the varying status of a theory and its stages of development.

iii) Critical examination of proofs and proof-structures assists our appreciation of the evolution of mathematical concepts and develops the dialectic necessary for the practice of mathematics.

II <u>Pedagogy</u>: Concerning the basis of the communication of mathematics.

i) Probing the logical and methodological difficulties at different stages in a theory enables us to make discoveries about the creative processes of mathematicians. Such discoveries may be useful when we examine the creative processes of children.

ii) Historical maturity enables us to criticise theories of teaching, many of which are founded on historical interpretations.

iii) Historical research enables us to suggest alternative forms of mathematics for inclusion in curricula.

III <u>Methodology</u>: Concerning the more practical aspects of the teaching-learning situation.

i) Critical examination of course structure from an historical viewpoint enables priorities to be decided and interrelationships made between topics.

ii) Historical perspective ensures that attention is paid to motivations and original problem-situations.

iii) History provides a fruitful source of connections with many other subject areas, and examples of the range and applications of mathematical models.

iv) Knowledge of the evolution of mathematical concepts and proofs furnishes data for the construction of heuristic.

v) Demonstration of developing mathematics increases its accessibility, promotes,interest and encourages a positive attitude.

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