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Equivariant Compactifications of Topological and Bitopological Transformation Groups

James Turton

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University of Cape Town

Abstract

A satisfactory theory of equivariant compactification for topological transformation groups has existed since the early 1980's, and it is with the presentation of the major results of this theory that the first part of this dissertation is concerned. This done, the second part begins with the introduction of the concept of bitopology, goes on to provide an outline of the theory of bitopological compactification as established by Salbany, and concludes with the development of a theory of equivariant compactification for what we have coined 'bitopological transformation groups', largely analogous to the theory for topological transformation groups.

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Chapter 1

Topological Transformation Groups

1.1 Introduction

The subject of topological transformation groups arose as an offshoot of the study of differential equations. Around the turn of the 20th century, mathematicians such as Henri Poincaré, Aleksandr Lyapunov and Garret Birkhoff developed techniques to study the behaviour of systems of differential equations without ever having to actually integrate them. These systems typically arose from classical dynamics, hence the current name of the theory that they then began: dynamical systems.

Dynamical systems grew to comprise both the continuous and the discrete. In the study of continuous dynamical systems one considers *flows* on a manifold called the *phase space* that are parameterised by a real time variable. Discrete dynamical systems, on the other hand, are usually specified via a phase space and an *iteration function* that maps phase space homeomorphically onto itself. The system then evolves as the iteration function is repeatedly applied.

It was noticed that in both of these cases, the ‘time’ variable ranges over a topological group (\mathbb{R} and \mathbb{Z} respectively, both with their additive group structure and usual topologies) that continuously parameterises transformations of the phase space. Substituting an arbitrary topological group G for the classical time and a topological space for the classical phase space, we obtain the structure known as a topological transformation group (or a G -space if only the phase space is under consideration). Since these topological transformation groups are essentially topological objects, it unsurprising that researchers began to ask, and answer, topological questions about them.

In particular, a theory of equivariant compactification for these structures, known as G -compactification, was developed in the late 1970's and early 1980's (see [9], [1]).

Just before these developments took place, the notion of compactification was extended in another direction by S. Salbany. Although bitopological spaces had been introduced by J. C. Kelly early on [19], a fruitful definition of compactness for such spaces came only with Salbany's thesis of 1970 [29]. Employing this definition, Salbany was able to develop a theory of bitopological compactification with many pleasing analogies to the topological theory.

As things stood then, we had two very satisfactory theories of compactification that were totally unrelated to each other. Having presented the relevant results from each of them in Chapters 2 and 3, we set about uniting the two in Chapter 4. We first define a bitopological transformation group by replacing the topological components of a topological transformation group with Salbany's bitopological equivalents, then seek a bitopological counterpart to each result from the theory of G -compactification presented in Chapter 2.

We present the fundamental result that a necessary and sufficient condition for a G -space to have a G -compactification is that it be initial with respect to its algebra of continuous G -uniform functions (see Theorem 2.2.13, [9]), and then give an analogous characterisation for the existence of G -bicompatifications of bitopological G -spaces in terms of initiality with respect to the bitopological equivalents of the continuous G -uniform functions (see Theorem 4.2.10).

The first example of Tychonoff G -space that is not G -Tychonoff was given in 1986 by M. Megrelishvili [24]. We present his construction in Example 2.2.15 and mention in Section 3.4 that it also provides an example of a pairwise Tychonoff bitopological G -space that is not pairwise G -Tychonoff.

J. de Vries was able to show that it is sufficient that the topological group G be locally compact for every G -space to have a G -compactification (see [8] or Corollary 2.3.6). We obtain a bitopological version of this result: if the parabitopological group G is locally bicompat and pairwise T_0 then every bitopological G -space is G -Tychonoff (see Corollary 4.3.10). It is important to note, however, that we were only able to prove this by first showing that such a parabitopological group is already a topological group (see Corollary 4.3.9).

It was established by Ju. M. Smirnov that there is an order isomorphism between the G -compactifications of the G -space X and the uniformly closed, point-separating, invariant subalgebras of the algebra of all continuous G -uniform functions (see [1] or Theorem 2.4.4). We introduce a partial order on the G -bicompatifications of a bitopological G -space and

show that, under this ordering, there is an order isomorphism between the G -bicompactifications and the uniformly closed, point-separating, invariant subsemi-algebras of the semi-algebra of all continuous G -uniform functions (see Theorem 4.4.4).

Proposition 2.4.5 proves that if a G -space has a G -compactification then a sufficient condition for it to have a least G -compactification is that it be locally compact. With Proposition 4.4.5 we reduce the problem of the existence of a G -bicompactifiable, locally bicompact bitopological G -space with no least G -bicompactification to the problem of the existence of a pairwise Tychonoff, locally bicompact bitopological space with no least bicompactification, but know of no solution to the latter problem.

It is possible to characterise the existence of G -compactifications by the admissibility of uniformities coarser than a special uniformity that is naturally induced by the action of a topological transformation group (see Theorem 2.5.1). The bitopological transformation groups that we have defined are naturally endowed with a special quasi-uniformity in a similar way, and Theorem 4.5.1 characterises the existence of G -bicompactifications in terms of the admissibility of quasi-uniformities coarser than it.

V. A. Chatyrko and K. L. Kozlov (see [5] or Section 2.4) showed that, in certain situations, the maximal G -compactification of a G -space is equivalent to some of the well-known classical T_2 compactifications of the underlying topological space. In Section 4.4, we exhibit some special cases where the maximal G -bicompactification of a bitopological G -space coincides with well-known bitopological compactifications of the underlying bitopological space.

It is interesting to determine what changes and what remains the same when we pass from a topologically symmetric setting to a topologically asymmetric setting. This dissertation fulfills a small part of this obviously very broad mandate. The following questions are intended as examples of what motivated our study of the subject of this work.

EXAMPLE 1.1.1 Is every classical continuous dynamical system (\mathbb{R} -space) a subsystem of some classical continuous dynamical system with a compact phase space? What about the classical discrete dynamical systems (\mathbb{Z} -spaces)? Does the same hold in the bitopological setting?

EXAMPLE 1.1.2 Do greatest and least G -compactifications always exist? What may be said of the order structure of the G -compactifications in between? When is it possible to obtain well-known T_2 compactifications of a G -space as maximal G -compactifications of the same space? What answers do these questions have in the bitopological setting?

1.2 Preliminaries

The fundamentals of topology are presumed: the reader in need of illumination is referred to Kelley [18].

DEFINITION 1.2.1 We shall take a *left topological transformation group* to be a triple, (G, X, Θ) , where G is a topological group, X is a T_2 topological space and $\Theta : G \times X \rightarrow X$ is a map satisfying the following conditions:

1. $\forall x \in X : \Theta(e, x) = x$.
2. $\forall g, h \in G \forall x \in X : \Theta(g, \Theta(h, x)) = \Theta(gh, x)$.
3. Θ is continuous.

We call G the *phase group*, X the *phase space* and Θ the *action* of G on X . In fact, these names are used even in the case when X is only a set and Θ only satisfies the first two conditions of Definition 1.2.1. For fixed phase group G , the pair (X, Θ) , or just X if the action is understood, is called a G -space. Note that any two G -spaces must have the same phase group G , but may have different actions.

For each $g \in G$, we have a continuous *transition* $\theta_g : X \rightarrow X$ defined by $\theta_g(x) = \Theta(g, x)$ and for each $x \in X$ a continuous *motion* $\theta^x : G \rightarrow X$ defined by $\theta^x(g) = \Theta(g, x)$.

NOTATION 1.2.2 Throughout this text, actions will be denoted by capital Greek letters and their corresponding transitions and motions by their lower case counterparts, as above.

Definition 1.2.1 referred to a *left* topological transformation group. This because of the ordering chosen for the composition rule (2). Had we (just as arbitrarily) chosen an action $\Psi : G \times X \rightarrow X$ that instead satisfied

$$\forall g, h \in G \forall x \in X : \Psi(h, \Psi(g, x)) = \Psi(x, gh),$$

then we would have a *right* topological transformation group. Given any right topological transformation group with action Ψ^r we can define a left topological transformation group with action Ψ^l , that has exactly the same group of transitions by setting $\Psi^l(g, x) = \Psi^r(x, g^{-1})$, and vice versa. The theory for the one variety carries over to the other via the obvious modification.

It is convenient to abbreviate the notation in certain cases and if the action Θ is clear from the context then we may write $g(x)$ or gx for $\Theta(g, x)$. In this notation, for example, conditions (1) and (2) in Definition 1.2.1 become

$$ex = x \text{ and } g(hx) = (gh)x$$

or, for a right G -space,

$$xe = x \text{ and } (xg)h = x(gh).$$

We extend this notation to subsets in the usual manner: if $H \subseteq G$ and $A \subseteq X$ then

$$H(A) = \{hx \in X : h \in H, x \in A\}.$$

A set $A \subseteq X$ is said to be *invariant* under G if $G(A) = A$.

Consider the transitions induced by the action Θ . We have $\theta_e = \text{id}_X$ and $\theta_g \circ \theta_h = \theta_{gh}$ by (1) and (2) from Definition 1.2.1. Since θ_g is continuous for any $g \in G$ and

$$\theta_g \circ \theta_{g^{-1}} = \theta_e = \text{id}_X = \theta_{g^{-1}} \circ \theta_g$$

we have that each θ_g is a homeomorphism of X . Denote by $\text{Homeo}(X)$ the group (under composition) of homeomorphisms of X . Then $g \mapsto \theta_g$ defines a group homomorphism

$$\sigma : G \rightarrow \text{Homeo}(X).$$

The kernel of this homomorphism is called the “kernel of the action Θ ”. The kernel consists of those elements of G that induce the identity transformation on X :

$$\ker \Theta = \{g \in G : gx = x \text{ for all } x \in X\}.$$

If the kernel of Θ is trivial (σ is injective) then we say that the action is *effective*. Every action naturally induces an effective action as follows.

PROPOSITION 1.2.3 *Let Θ be an action of G on X and let $N = \ker \Theta$. Then the action $\Theta/\ker \Theta : G/N \times X \rightarrow X$ defined by $(gN)x = gx$ is effective on X .*

Proof. Clearly the kernel of $\Theta/\ker \Theta$ is trivial. The projection map $\pi : G \rightarrow G/N$ is open, hence for U an open set in X we have that

$$(\Theta/\ker \Theta)^{-1}(U) = (\pi \times \text{id}_X)(\Theta^{-1}(U))$$

is also open and $\Theta/\ker \Theta$ is continuous.

DEFINITION 1.2.4 Let X and Y be G -spaces. A mapping $\varphi : X \rightarrow Y$ is called *equivariant* if it commutes with the group actions, that is,

$$\varphi(gx) = g\varphi(x) \text{ for all } g \in G \text{ and } x \in X.$$

It is clear that invariant sets are preserved by such maps. An equivariant map which is also a homeomorphism is called an *equivalence* or an *isomorphism* of G -spaces and in this case the inverse φ^{-1} of φ is also equivariant, for if $y = \varphi(x)$ then

$$\varphi^{-1}(gy) = \varphi^{-1}(g\varphi(x)) = \varphi^{-1}(\varphi(gx)) = gx = g\varphi^{-1}(y).$$

The concept of G -space isomorphism determines an equivalence relation for which we shall write $X \simeq Y$. If all that is required to make two G -spaces equivalent is an automorphism of G i.e. if there is an automorphism α (continuous) of G and a homeomorphism $\varphi : X \rightarrow Y$ with

$$\varphi(gx) = \alpha(g)\varphi(x)$$

then we say that they are *weakly equivalent*.

EXAMPLE 1.2.5 Let G be the cyclic group of order 5, whose elements are the fifth roots of unity. Let X be the unit circle in the complex plane. Then the action given by $(g, z) \mapsto gz$ is inequivalent to the action given by $(g, z) \mapsto g^2z$. One way of seeing this is by considering the images of the arcs $[0, g^n]$: it is not hard to show that a continuous map that commutes with the group actions cannot be bijective. However, these actions are weakly equivalent since $g \mapsto g^2$ is an automorphism of G .

DEFINITION 1.2.6 Let X be a G -space. Then for each $x \in X$, the set $Gx = \{gx : g \in G\} \subseteq X$ is the *orbit* of the point x under the action of G .

Clearly, a subset of the phase space is invariant if and only if it is the union of orbits. If $A \subseteq X$ is invariant then so is $X \setminus A$, since $g(X \setminus A) = X \setminus gA$ (θ_g is bijective). It follows from the fact that each $\theta_g : X \rightarrow X$ is a homeomorphism that

$$g(\text{cl} A) = \theta_g(\text{cl} A) = \text{cl} \theta_g(A) = \text{cl} gA = \text{cl} A.$$

From this we see that the invariance of A implies the invariance of $\text{cl} A$ and, combining the facts above, also the invariance of $\text{int} A$. Note that if $gx = hy$ for some $g, h \in G$ and $x, y \in X$ then for any $g' \in G$,

$$g'x = g'g^{-1}gx = g'g^{-1}hy \in Gy$$

so that $Gx \subseteq Gy$. Conversely, $Gy \subseteq Gx$. It follows that the orbits Gx and Gy are either disjoint or equal, and thus that they define a partition of the phase space.

1.3 Constructions

1. Let G be a T_2 topological group with group operation $\Omega : G \times G \rightarrow G$. Then G acts on itself (on the left or the right) via Ω , and is a G -space with transitions given by the (left or right) translates of G .
2. Let X be a G -space and let Z be a non-empty invariant set in X . Then

$$\Theta|_{G \times Z} : G \times Z \rightarrow Z$$

is a continuous action of G on Z . The G -space Z is then called a G -subspace of X . Note that $\Theta|_{G \times Z}$ is the unique action on Z making the inclusion mapping $Z \hookrightarrow X$ equivariant.

3. Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of G -spaces, each with associated action Θ_λ . Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ with the product topology and define $\Theta : G \times X \rightarrow X$ by

$$\Theta(g, x) = (\Theta_\lambda(g, x_\lambda))_{\lambda \in \Lambda}$$

for $g \in G$ and $x = (x_\lambda)_{\lambda \in \Lambda} \in X$. In other notation, $(gx)_\lambda = gx_\lambda$. Θ is a continuous action of G on X , called the *coordinate-wise* action. The coordinate-wise action is the unique action making all of the projections $\pi_\lambda : X \rightarrow X_\lambda$ equivariant.

4. Let \mathbb{R}^G be the space of all real-valued functions on the topological group G . The group G acts on \mathbb{R}^G , not necessarily continuously, by means of the action $\Theta : G \times \mathbb{R}^G \rightarrow \mathbb{R}^G$ defined by

$$\Theta(g, f)(t) = f(tg).$$

We check that Θ satisfies condition (2) of Definition 1.2.1:

$$\begin{aligned} ((gh)f)(t) &= f(t(gh)) \\ &= f((tg)h) \\ &= (hf)(tg) \\ &= (g(hf))(t) \end{aligned}$$

5. Let \mathbb{R}^X be the space of all real-valued functions on the G -space X . The group G acts on \mathbb{R}^X , not necessarily continuously, by means of the action $\Theta : G \times \mathbb{R}^X \rightarrow \mathbb{R}^X$ defined by

$$\Theta(g, f)(x) = f(g^{-1}x).$$

We check that Θ satisfies condition (2) of Definition 1.2.1:

$$\begin{aligned} ((gh)f)(x) &= f((gh)^{-1}x) \\ &= f(h^{-1}g^{-1}x) \\ &= (hf)(g^{-1}(x)) \\ &= (g(hf))(x) \end{aligned}$$

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Chapter 2

G -compactification

2.1 Preliminaries

In this chapter we shall review the theory of compact T_2 extensions for G -spaces. As such, we shall only be interested in Tychonoff G -spaces and so we introduce the concept of uniform structure at the outset.

DEFINITION 2.1.1 Let A and B be relations on a set X . We define the *composition* of A with B to be

$$A \circ B = \{(x, y) \in X \times X : (x, z) \in A \text{ and } (z, y) \in B \text{ for some } z \in X\}.$$

For the *inverse* of the relation A we write $A^{-1} = \{(y, x) : (x, y) \in A\}$. If $x \in X$ then we shall take $A[x]$ to mean $\{y \in X : (x, y) \in A\}$.

Note that the above definition for the composition of relations is the reverse of the usual definition one uses when dealing with functions. This does not usually lead to any confusion since it is mostly sufficient for the study of uniformities to consider only compositions of a relation with itself (see also [14]).

DEFINITION 2.1.2 A *uniform space* is a pair (X, \mathcal{U}) , where X is a set and the *uniformity* \mathcal{U} is a filter on $X \times X$. Each element of \mathcal{U} , being a subset of $X \times X$, is a relation on X . We may thus speak of the composition and inversion of elements of \mathcal{U} . The uniformity must satisfy the following conditions:

1. $\forall A \in \mathcal{U} : \Delta = \{(x, x) : x \in X\} \subseteq A$.
2. $\forall A \in \mathcal{U}, \exists B \in \mathcal{U} : B^2 = B \circ B \subseteq A$.
3. $A \in \mathcal{U} \Rightarrow A^{-1} \in \mathcal{U}$.

Every uniformity gives rise to a completely regular topology. Since we have already assumed that G -spaces are T_2 , we have that every uniformisable G -space is a Tychonoff space. Indeed, we might well have added to the above definition the requirement that any uniformity \mathcal{U} must be separated, that is $\bigcap \mathcal{U} = \Delta$, which is equivalent to requiring that the induced topology of \mathcal{U} be T_2 . We didn't, but we may rest safe in the knowledge that this condition will be fulfilled by our assumption that all G -spaces are Tychonoff.

It is well-known that the Tychonoff topological spaces are exactly those that have T_2 compactifications. When we compactify a G -space, however, we would like the compactification to preserve the group action in the sense of the following definition.

DEFINITION 2.1.3 Let X be a G -space. A G -compactification of X is a compact G -space Y such that X is isomorphic to a dense subspace of Y .

Then, by analogy to the standard topological case, we make the following definition.

DEFINITION 2.1.4 A G -space X is said to be G -Tychonoff if and only if it has a G -compactification.

We now cannot but follow in the footsteps of J. de Vries [10] and ask whether it is the case that every Tychonoff G -space is G -Tychonoff. A first attempt at answering this question might be to try the well-behaved Stone-Ćech compactification of X . Then for every $g \in G$, the transition $\theta_g : X \rightarrow X$ has a continuous extension $\tilde{\theta}_g : \beta X \rightarrow \beta X$. Since the equalities

$$\tilde{\theta}_e = \text{id}_{\beta X} \text{ and } \tilde{\theta}_g \circ \tilde{\theta}_h = \tilde{\theta}_{gh}$$

hold on a dense subset of βX , they hold on all of βX . Thus $\tilde{\Theta} : G \times \beta X \rightarrow \beta X$ defined by $\tilde{\Theta}(g, x) = \tilde{\theta}_g x$ is an action of G on βX extending the action of G on X and each transition $\tilde{\theta}_g$ is a homeomorphism of βX . However, $\tilde{\Theta}$ is not in general a continuous action of G on βX .

EXAMPLE 2.1.5 (de Vries [7]) Let $+$: $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be the natural addition mapping on \mathbb{R} . Then \mathbb{R} is an \mathbb{R} -space via the action $+$. Now consider the sets

$$A = \mathbb{N} \text{ and } B = \{n + 1/n : n \in \mathbb{N} \setminus \{0\}\}.$$

They are both closed and disjoint in \mathbb{R} and thus have disjoint closures in $\beta\mathbb{R}$, by the normality of \mathbb{R} . Let y be an accumulation point of A in $\beta\mathbb{R}$. If $+$ were continuous on $\mathbb{R} \times \beta\mathbb{R}$ at the point $(0, y)$ then $y = 0 \tilde{+} y$ would be an accumulation point of the sequence $(\frac{1}{n} \tilde{+} n)_{n \in \mathbb{N}}$ i.e. $y \in \text{cl} A \cap \text{cl} B$ - a contradiction.

2.2 Existence of G -compactifications

In the case that G is the trivial group, $\{e\}$, the question of the existence of G -compactifications for a G -space X reduces to the standard topological problem of the existence of T_2 compactifications of X . It is not clear, however, that more complicated groups will always allow G -compactifications of their G -spaces. We thus address ourselves to the problem of characterising the G -Tychonoff G -spaces.

Our approach will be analogous to the technique for solving the classical topological version of this problem: we shall attempt to identify a family of ‘elementary’ compact G -spaces and find a collection of equivariant maps to these spaces that separates the points and closed subsets of X . These elementary compact G -spaces take the place of the closed unit intervals in the classical construction in that we then use the evaluation map to equivariantly embed X into their product.

The spaces we seek vary from group to group and it is this realisation that leads us to consider the space $C(G)$ of all continuous real-valued functions on the acting group G . This space is a subspace of the space \mathbb{R}^G with action Ψ described in Construction (4).

NOTATION 2.2.1 Let (X, \mathcal{U}) be a uniform space and let \mathcal{Q}^s be the usual uniformity on \mathbb{R} . The set of uniformly continuous functions from (X, \mathcal{U}) to $(\mathbb{R}, \mathcal{Q}^s)$ will be denoted by $\mathcal{UC}(X, \mathcal{U})$ or just $\mathcal{UC}(X)$ if no ambiguity arises. A superscript ‘ \star ’ will always denote the set of all bounded members of a set of real-valued functions, whilst a subscript ‘ p ’ on a function space or operator will indicate that we are considering the space in question equipped with the topology of pointwise convergence, e.g. $C_p^\star(X)$, $\text{cl}_p A$. Finally, we shall denote the left and right uniformities on a topological group by \mathcal{L} and \mathcal{R} respectively.

In general, the action Ψ is not jointly continuous on $C_p(G)$ or even $\mathcal{UC}_p(G, \mathcal{R})$ though it is separately continuous on both of these spaces. We thus restrict our attention to the equicontinuous subsets of $C_p(G)$.

DEFINITION 2.2.2 Let X be a set. We say that $A \subseteq \mathbb{R}^X$ is *pointwise bounded* if and only if $A[x]$ is a bounded subset of \mathbb{R} for each $x \in X$.

LEMMA 2.2.3 (de Vries [8]) *Let Z be an invariant, equicontinuous subset of $C_p(G)$. Then $\Psi : G \times Z \rightarrow Z$ is jointly continuous. Moreover, $Z \subseteq \mathcal{UC}(G, \mathcal{R})$.*

Proof. Since we are working in the topology of pointwise convergence, it is sufficient to show that for each projection $\pi_{s_0} : \mathbb{R}^G \rightarrow \mathbb{R}$, where $s_0 \in G$, the

mapping $\pi_{s_0} \circ \Psi : G \times Z \rightarrow \mathbb{R}$ is continuous at the (arbitrarily chosen) point $(g_0, f_0) \in G \times Z$. To this end, let $(g, f) \in G \times Z$ and consider the inequality

$$|(gf)(s_0) - (g_0f_0)(s_0)| \leq |f(s_0g) - f(s_0g_0)| + |f(s_0g_0) - f_0(s_0g_0)|.$$

Let $\varepsilon > 0$. The condition on f that the second term on the right-hand side of the inequality be smaller than $\frac{\varepsilon}{2}$ defines a neighbourhood U of f_0 in Z . In view of the equicontinuity of Z at the point s_0g there is a neighbourhood V of g_0 in G such that the first term on the right-hand side of the inequality is less than $\frac{\varepsilon}{2}$. Hence the left-hand side is smaller than ε for all $(g, f) \in V \times U$ and the continuity of Ψ is proved.

To see that $Z \subseteq \mathcal{UC}(G, \mathcal{R})$, note that $f \in Z \Rightarrow Gf \subseteq Z$ so that the set Gf is equicontinuous. In particular, if we consider the point $e \in G$, we have that for every $\varepsilon > 0$ there exists $W \in \mathcal{N}_\varepsilon$ such that $|f(sg) - f(g)| < \varepsilon$ for any $s \in W$ and $g \in G$. This means precisely that $f \in \mathcal{UC}(G, \mathcal{R})$.

COROLLARY 2.2.4 (de Vries [8]) *Let Y be a pointwise bounded, invariant, equicontinuous subset of $C^*(G)$. Then $\text{cl}_p Y$ is a compact, invariant, equicontinuous subset of $\mathcal{UC}_p^*(G, \mathcal{R})$ and is thus a compact G -space under the action Ψ .*

Proof. We start with the observation that the equicontinuity of Y implies that of its pointwise closure in \mathbb{R}^G . Now, if $s \in G$, then the continuity of the projection $\pi_s : C_p(G) \rightarrow \mathbb{R}$ implies that

$$(\text{cl}_p Y)(s) = \pi_s(\text{cl}_p Y) \subseteq \text{cl}_{\pi_s}(Y) = \text{cl} Y(s),$$

which is a bounded set in \mathbb{R} . That $\text{cl}_p Y$ is compact follows from the fact that it is a closed subset of the compact product $\prod_{s \in G} \text{cl} Y(s)$. Finally, as we mentioned in Section 1.2, the invariance of Y implies that of $\text{cl}_p Y$. Combine these observations with Lemma 2.2.3 and the proof is complete.

DEFINITION 2.2.5 Let X be a G -space, \mathcal{Q}^s be the usual uniformity on \mathbb{R} and $f : X \rightarrow \mathbb{R}$ be continuous. We say that f is G -uniform if for any $A \in \mathcal{Q}^s$ there exists a $V \in \mathcal{N}_\varepsilon$ such that for all $x \in X$, $f(Vx) \subseteq A[f(x)]$.

NOTATION 2.2.6 Let X be a G -space and let C be any set of real-valued functions on X . Then C_G will denote the set of all G -uniform members of C .

LEMMA 2.2.7 (de Vries [8]) *Let X be a G -space with action Θ and let $f \in C(X)$. The following conditions are equivalent:*

1. f is G -uniform.

2. The set $\{f \circ \theta^x : x \in X\}$ is equicontinuous at $e \in G$.

3. The set $\{f \circ \theta^x : x \in X\}$ is right uniformly equicontinuous on G .

Moreover, if $f \in C^*(X)$ then $\{f \circ \theta^x : x \in X\}$ is pointwise bounded on G .

Proof. To see that (2) \Rightarrow (3) simply note that $f \circ \theta^x(Vg) = f \circ \theta^{gx}(V)$. The rest of the lemma is obvious.

LEMMA 2.2.8 (Smirnov [1]) *Let X be a compact G -space and let $f : X \rightarrow \mathbb{R}$ be continuous. Then f is G -uniform.*

Proof. Let Ω be the usual uniformity on \mathbb{R} , let $A \in \Omega$ and let $B \in \Omega$ be a symmetric open entourage such that $B^2 \subseteq A$. The open cover $\{f^{-1}(B[x]) : x \in \mathbb{R}\}$ has a finite subcover $\{f^{-1}(B[x_i]) : 1 \leq i \leq n\}$. Since the action of G on X is continuous we can find, for each $y \in X$, a neighbourhood $U_y \in \mathcal{N}_y$ and $V_y \in \mathcal{N}_e$ such that

$$V_y(U_y) \subseteq f^{-1}(B[x_i]) \text{ for some } 1 \leq i \leq n.$$

Let $\{U_{y_j} : 1 \leq j \leq k\}$ be a finite subcover of $\{U_y : y \in X\}$ and set $V = \bigcap_{j=1}^k V_{y_j}$. Then, for any $z \in X$, $f(z) \in f(Vz) \subseteq B[x_i]$ for some $x_i \in \mathbb{R}$ and thus

$$f(Vz) \subseteq B^2[f(z)] \subseteq A[f(z)].$$

COROLLARY 2.2.9 *If X is a compact G -space then $C(X) = C_G^*(X)$.*

LEMMA 2.2.10 (de Vries [8]) *Let X and Y be G -spaces with Y compact, and let $\varphi : X \rightarrow Y$ be a continuous equivariant map. Then $f \in C(Y) \Rightarrow f \circ \varphi \in C_G^*(X)$.*

Proof. Denote the action on X by Θ and the action on Y by Ψ . Then for every $x \in X$ we have

$$(f \circ \varphi) \circ \theta^x = f \circ \psi^{\varphi(x)}$$

and thus we obtain

$$\{(f \circ \varphi) \circ \theta^x : x \in X\} \subseteq \{f \circ \psi^y : y \in Y\}.$$

As the latter set is equicontinuous, so is the former. The G -uniformity of $f \circ \varphi$ follows from Lemma 2.2.8.

Let X be a G -space with action Θ and let $f \in C_G^*(X)$. Then it is obvious from Corollary 2.2.4 and the characterisation of G -uniform functions given by Lemma 2.2.7 that $\tilde{X}_f = \text{cl}_p \{f \circ \theta^x : x \in X\}$ is a closed, pointwise

bounded, invariant, equicontinuous subset of $C_p(G)$ and is thus a compact G -space.

PROPOSITION 2.2.11 (de Vries [8]) *Let X be a G -space with action Θ and let $f \in C_G^*(X)$. Then the mapping $\varphi_f : X \rightarrow \tilde{X}_f$ defined by $\varphi_f(x) = f \circ \theta^x$ is continuous and equivariant.*

Proof. That φ_f is continuous follows immediately from the observation that, for every $g \in G$, the mapping $x \mapsto (\varphi_f \circ \theta_g)(x) = f(gx)$ is continuous. Equivariance follows from

$$(\varphi_f(gx))(t) = f \circ \theta^{gx}(t) = f(tgx) = (gf \circ \theta^x)(t) = (g\varphi_f(x))(t).$$

PROPOSITION 2.2.12 (de Vries [8]) *Let X be a G -space. Then $C_G^*(X)$ separates the points and closed subsets of X if and only if the continuous equivariant maps from X to compact G -spaces separate the points and closed subsets of X .*

Proof. Let A be a closed non-empty subset of X , $x_0 \in X \setminus A$ and $f \in C_G^*(X)$ be such that $f(x_0) \notin \text{cl } f(A)$. This can be written as

$$(f \circ \theta^{x_0})(e) \notin \text{cl} \{(f \circ \theta^x)(e) : x \in A\}.$$

The continuity of the projection $\pi_e : \tilde{X}_f \rightarrow \mathbb{R}$ implies that $f \circ \theta^{x_0} \notin \text{cl} \{(f \circ \theta^x) : x \in A\}$. Thus, using the notation of Proposition 2.2.11, we have $\varphi_f(x_0) \notin \text{cl } \varphi_f(A)$, where φ_f is a continuous equivariant function to a compact G -space.

To prove the converse, consider a continuous equivariant map $\varphi : X \rightarrow Z$ where Z is a compact G -space and assume $\varphi(x_0) \notin \text{cl } \varphi(A)$. There exists $h \in C(Z)$ such that $h(x_0) = 0$ and $h(A) = \{1\}$. Now let $f = h \circ \varphi$. It follows from Lemma 2.2.10 that $f \in C_G^*(X)$ and it is clear that $f(x_0) \notin \text{cl } f(A)$.

THEOREM 2.2.13 (de Vries [8], [9]) *A G -space X is G -Tychonoff if and only if $C_G^*(X)$ separates the points and closed subsets of X .*

Proof. Clear from Proposition 2.2.12.

NOTATION 2.2.14 It follows from Proposition 2.2.12 that, for each $C \subseteq C_G^*$ that separates the points and closed subsets of X , we have a G -compactification of X with an equivariant embedding

$$\beta_C : X \hookrightarrow \prod_{f \in C} \tilde{X}_f$$

where β_C is the evaluation map with respect to $\{\varphi_f : f \in C\}$. This G -compactification will be written $\beta_C X$.

EXAMPLE 2.2.15 (Megrelishvili [24]) *A metrisable G -space X such that the set $C_G^*(X)$ does not separate the points and closed subsets of X .*

Let $\mathbb{I} = [0, 1]$ be the unit interval of real numbers with its usual topology; let $\text{Homeo}(\mathbb{I})$ be the group of homeomorphisms of \mathbb{I} with the topology of uniform convergence; let $S = \{1/n : n \in \mathbb{N}\}$ and let

$$G_1 = \{g \in \text{Homeo}(\mathbb{I}) : \forall s \in S, gs = s\}.$$

Then G_1 is a closed subgroup of $\text{Homeo}(\mathbb{I})$. The natural action $\Theta_1 : G_1 \times \mathbb{I} \rightarrow \mathbb{I}$ is continuous and has countably many orbits. All points of S and the point 0 are fixed. All the remaining orbits are the intervals of the form $(1/(n+1), 1/n)$ where $n \in \mathbb{N}$. The point 0 has a neighbourhood base consisting of the invariant intervals of the form $[0, 1/n], n \in \mathbb{N}$.

1. If $\{U_k : k \in \mathbb{N}\}$ is a set of open neighbourhoods of 0 in the G_1 -space \mathbb{I} such that $\forall k \in \mathbb{N} : G_1(U_k) \subseteq U_{k+1}$ then $U_{k_0+1} = \mathbb{I}$ if $k_0 \in \mathbb{N}$ is such that $[0, 1/k_0] \subseteq U_1$.

Let $\{(G_n, \mathbb{I}_n, \Theta_n) : n \in \mathbb{N}\}$ be a countable set of copies of $(G_1, \mathbb{I}, \Theta_1)$. We now define the product of topological groups $G = \prod \{G_n : n \in \mathbb{N}\}$ with natural projections denoted by $\pi_n : G \rightarrow G_n$ and the topological sum (disjoint union) $X = \bigcup \{\mathbb{I}_n : n \in \mathbb{N}\}$ with natural embeddings $i_n : \mathbb{I}_n \hookrightarrow X$. The action $\Theta : G \times X \rightarrow X$ is defined in the following natural way:

$$\Theta(g, x) = i_n(\Theta_n(\pi_n(g), x_0))$$

where $g \in G$, $x \in i_n(\mathbb{I}_n)$ and $i_n(x_0) = x$.

We now form a set Y by collapsing the set $\{i_n(0) : n \in \mathbb{N}\} \subseteq X$ to a point, while leaving the rest of X unchanged. The point $\{i_n(0) : n \in \mathbb{N}\} \in Y$ will be denoted by w . Let $p : X \rightarrow Y$ be the canonical projection. We define a topology on Y such that a neighbourhood base of $w \in Y$ is given by $\{A_k : k \in \mathbb{N}\}$, where

$$A_k = \bigcup \{p(i_n([0, 1/k])) : n \in \mathbb{N}\}.$$

At all other points we take the usual neighbourhoods. It is easy to verify that Y is homeomorphic to $J(\aleph_0)$ - the so-called metrisable hedgehog with countably many thorns.

We define an action $\bar{\Theta}$ of G on Y . Every point of the set $p^{-1}(w)$ is fixed. Therefore, there exists a unique action $\bar{\Theta}$ on Y under which p is equivariant. Formally,

$$\bar{\Theta}(g, x) = p(\Theta(g, p^{-1}(x))).$$

Each set A_k is invariant under $\bar{\Theta}$ so that $\bar{\Theta}$ is continuous at points of the form (g, w) , $g \in G$. The continuity of $\bar{\Theta}$ on the rest of Y is obvious and we thus have that Y is a Tychonoff G -space. It turns out that the closed subset $F = \{p(i_n(1)) : n \in \mathbb{N}\}$ of all “thorn tips” cannot be separated from the point w by a continuous G -uniform function.

We proceed by assuming the contrary: suppose $f \in C_G(X)$ is such that $f(F) = \{0\}$ and $f(w) = 1$. There exist a sequence $(U_n)_{n \in \mathbb{N}}$ of neighbourhoods of w and a sequence $(V_n)_{n \in \mathbb{N}}$ of neighbourhoods of the identity in G such that

$$\forall n \in \mathbb{N}, U_n \cap F = \emptyset \text{ and } V_n(U_n) \subseteq U_{n+1}.$$

By definition of the neighbourhood base at w , we have $A_{k_0} \subseteq U_1$ for any $k_0 \in \mathbb{N}$. Since G has the product topology, there exists $n_0 \in \mathbb{N}$ such that if $1 \leq k \leq k_0 + 1$ and $n_0 \leq n$ then $\pi_n(V_k) = G_n$. Using property (1) of the action Θ_1 , and the fact that the restriction of $\bar{\Theta}$ to each “thorn” $p(\mathbb{I}_m)$ is equivalent to Θ_1 , we see that

$$U_{k_0+1} \supseteq \bigcup \{p(i_m(\mathbb{I}_m)) : n_0 \leq m\}$$

and, in particular, $U_{k_0+1} \cap F \neq \emptyset$ - a contradiction.

COROLLARY 2.2.16 *There is a Tychonoff G -space that is not G -Tychonoff.*

2.3 A Sufficient Condition

DEFINITION 2.3.1 (de Vries [8]) Let X be a G -space with action Θ . A function $f \in C^*(X)$ is called *locally equicontinuous* if there exists a $V \in \mathcal{N}_e$ such that the family $\{f \circ \theta_g : g \in V\}$ is equicontinuous on X . The set of all positive locally equicontinuous functions on X will be denoted by $\mathcal{LE}^+(X)$.

Let $f \in \mathcal{LE}^+(X)$ with $V \in \mathcal{N}_e$ such that $\{f \circ \theta_g : g \in V\}$ is equicontinuous on X and define $\|f\| = \sup_{x \in X} \{f(x)\}$. The left uniformly continuous real-valued functions on G separate its points and closed subsets so there is a $\phi \in \mathcal{UC}(G, \mathbb{R})$ such that

$$\phi(G) \subseteq [0, \|f\| + 2], \quad \phi(e) = 0, \quad \phi(G \setminus V) = \{\|f\| + 2\}.$$

We now use ϕ to define a new function $\tilde{f}_\phi : X \rightarrow \mathbb{R}$ by

$$\tilde{f}_\phi(x) = \inf_{t \in G} \{\phi(t) + f(tx)\}.$$

Our aim is to show that $\tilde{f}_\phi \in C_G^*(X)$ and that, if we apply this construction to a subset of $\mathcal{LE}^+(X)$ that separates the points and closed subsets of X , then

we obtain a subset of $C_G^*(X)$ that separates the points and closed subsets of X .

LEMMA 2.3.2 (de Vries [8]) *Let $f \in \mathcal{LE}^+(X)$, $V \in \mathcal{N}_e$ and $\phi \in \mathcal{UC}(G, \mathcal{L})$ be as described in the construction above. Then $\tilde{f}_\phi \in C_G^*(X)$.*

Proof. Firstly note that for every $x \in X$ we have

$$0 \leq \tilde{f}_\phi(x) \leq \phi(e) + f(x) = f(x) \leq \|f\| \quad (2.3.1)$$

so that \tilde{f}_ϕ is bounded. In order to prove that \tilde{f}_ϕ is continuous we introduce the set

$$A_\phi = \{t \in G : \phi(t) < \|f\| + 1\}.$$

It is clear that $A_\phi \subseteq V$. Moreover, for all $t \in G \setminus A_\phi$ we have by inequality (2.3.1) and our definitions

$$\phi(t) + f(tx) \geq \|f\| + 1 + f(tx) \geq \|f\| + 1 \geq \tilde{f}_\phi(x) + 1$$

for all $x \in X$. This implies that, for any $x \in X$

$$\tilde{f}_\phi(x) = \inf_{t \in A_\phi} \{\phi(t) + f(tx)\}. \quad (2.3.2)$$

Now let $\varepsilon > 0$ and $x \in X$. By the local equicontinuity of f , there is a $W \in \mathcal{N}_x$ such that for all $y \in W$ and all $t \in V$, $|f(ty) - f(tx)| < \varepsilon$. It follows from equation (2.3.2) that there exists $r \in A_\phi \subseteq V$ such that

$$\phi(r) + f(rx) < \tilde{f}_\phi(x) + \varepsilon.$$

Fix any $y \in W$. Because $r \in V$ we have, by the choice of W , $f(ry) < f(rx) + \varepsilon$, hence

$$\tilde{f}_\phi(y) \leq \phi(r) + f(ry) < \phi(r) + f(rx) + \varepsilon \leq \tilde{f}_\phi(x) + 2\varepsilon.$$

Similarly, $\tilde{f}_\phi(x) < \tilde{f}_\phi(y) + 2\varepsilon$. This completes the proof that \tilde{f}_ϕ is continuous.

Finally, we show that \tilde{f}_ϕ is G -uniform. To this end, consider a point $(t, x) \in G \times X$. Then

$$\begin{aligned} \tilde{f}_\phi(tx) &= \inf_{s \in G} \{\phi(s) + f(stx)\} \\ &= \inf_{s \in G} \{\phi(st^{-1}) - \phi(s) + \phi(s) + f(stx)\} \\ &\geq \inf_{s \in G} \{\phi(st^{-1}) - \phi(s)\} + \tilde{f}_\phi(x). \end{aligned}$$

Now because $\phi \in \mathcal{UC}(G, \mathcal{L})$, there is, for any $\varepsilon > 0$, a $V \in \mathcal{N}_e$ such that $|\phi(st^{-1}) - \phi(s)| < \varepsilon$ for all $t \in V^{-1}$. It follows that $\tilde{f}_\phi(tx) > \tilde{f}_\phi(x) - \varepsilon$ for

all $x \in X$ and $t \in V^{-1}$. Now replace x by tx in this inequality, where $t \in V$; because then $t^{-1} \in V^{-1}$ we get

$$\tilde{f}_\phi(x) = \tilde{f}_\phi(t^{-1}tx) > \tilde{f}_\phi(tx) - \varepsilon.$$

Combining these results, we see that $|\tilde{f}_\phi(tx) - \tilde{f}_\phi(x)| < \varepsilon$ for all $x \in X$ and $t \in V \cap V^{-1}$. We have shown that $\tilde{f}_\phi \in C_G^*(X)$.

LEMMA 2.3.3 (de Vries [8]) *Let $g \in \mathcal{L}\mathcal{E}^+(X)$, let $F \subseteq X$ be closed and let $x_0 \in X \setminus F$ such that $g(x_0) \notin \text{cl}g(F)$. Then there exist a uniformly continuous $\psi : \mathbb{R} \rightarrow [0, 1]$ and a $\phi \in \mathcal{UC}(G, \mathcal{L})$ such that, letting $f = \psi \circ g$, we have $\tilde{f}_\phi(x_0) \notin \text{cl}\tilde{f}_\phi(F)$.*

Proof. Giving \mathbb{R} its usual uniformity, there exists a uniformly continuous $\psi : \mathbb{R} \rightarrow [0, 1]$ such that $\psi(g(x_0)) = 1$ and $\psi(\text{cl}g(F)) = \{0\}$. It is then easily seen that $f = \psi \circ g \in \mathcal{L}\mathcal{E}^+(X)$ with $f(x_0) = 1$ and $f(F) = \{0\}$. As $f(x_0) = 1$, there exists a $V \in \mathcal{N}_e$ such that $f(tx_0) > 1/2$ for all $t \in V$. Since $f \in \mathcal{L}\mathcal{E}^+(X)$, we may assume without loss of generality that $\{f \circ \theta_t : t \in V\}$ is equicontinuous on X .

As in the construction at the beginning of this section, we may now select $\phi \in \mathcal{UC}(G, \mathcal{L})$ and, because of the preceding lemma, consider $\tilde{f}_\phi \in C_G^*(X)$. Continuing, let A_ϕ be defined as in the proof of the preceding lemma. Then by equation (2.3.2) we infer that

$$\tilde{f}_\phi(x_0) = \inf_{t \in A_\phi} \{\phi(t) + f(tx_0)\}, \quad (2.3.3)$$

and because $A_\phi \subseteq V$, it follows that the right-hand side of (2.3.3) is at least $1/2$. In addition, by equation (2.3.1), we have for every $x \in F$ that $0 \leq \tilde{f}_\phi(x) \leq f(x) = 0$. Thus $\tilde{f}_\phi(F) = \{0\}$.

COROLLARY 2.3.4 (de Vries [8]) *If $\mathcal{L}\mathcal{E}^+(X)$ separates the points and closed subsets of X then so does $C_G^*(X)$.*

COROLLARY 2.3.5 (de Vries [8]) *If X is a Tychonoff G -space that admits a uniformity \mathcal{U} such that some $V \in \mathcal{N}_e$ acts equicontinuously on X with respect to \mathcal{U} then X is G -Tychonoff.*

Proof. In this case, if $f : X \rightarrow \mathbb{R}$ is uniformly continuous with respect to \mathcal{U} , bounded and non-negative then $f \in \mathcal{L}\mathcal{E}^+(X)$. Consequently, $\mathcal{L}\mathcal{E}^+(X)$ separates the points and closed subsets of X and the result follows.

COROLLARY 2.3.6 (de Vries [8], [9]) *If G is a locally compact topological group then every Tychonoff G -space is G -Tychonoff.*

Proof. The joint continuity of the action implies that every compact $V \in \mathcal{N}_e$ is equicontinuous on X [18].

This last corollary answers the topological questions posed by Example 1.1.1 in the affirmative since the usual topologies on \mathbb{R} and \mathbb{Z} , which are the ones used for classical dynamical systems, are both locally compact.

2.4 Ordering G -compactifications

As with the standard compactifications, we can put a partial order on the G -compactifications of the G -space X . We say that $\beta_1 X \leq \beta_2 X$ if there exists a continuous equivariant surjection $\varphi : \beta_2 X \rightarrow \beta_1 X$ such that $\varphi \circ \beta_2 = \beta_1$. If the mapping φ is a homeomorphism (or if $\beta_1 X \leq \beta_2 X \leq \beta_1 X$) then $\beta_1 X \simeq \beta_2 X$.

In fact, this ordering turns out to be equivalent to the usual ordering on T_2 compactifications when it is restricted to the G -compactifications. For if β_1 and β_2 are dense equivariant embeddings of the G -space X into the compact G -spaces $\beta_1 X$ and $\beta_2 X$ respectively, and $\varphi : \beta_2 X \rightarrow \beta_1 X$ is a continuous surjection that leaves X fixed then, for any $g \in G$, the mappings $\varphi \circ \theta_g$ and $\theta_g \circ \varphi$ agree on the dense subspace $\beta_2(X)$ and thus on the whole of $\beta_2 X$. Thus, φ is already equivariant. Conversely, it is clear that any equivariant surjection is already a surjection.

LEMMA 2.4.1 *Let X be a G -space. Then $C_G^*(X)$ is a uniformly closed sub-algebra of $C^*(X)$.*

Proof. To check that $C_G^*(X)$ is an algebra is straightforward. Now suppose that \mathcal{F} is a filter in $C_G^*(X)$ such that $\mathcal{F} \rightarrow f \in C^*(X)$ uniformly. Let \mathcal{U} be the usual uniformity on \mathbb{R} , let $B \in \mathcal{U}$ and let $A \in \mathcal{U}$ be a symmetric entourage such that $A^3 \subseteq B$. There is an $F \in \mathcal{F}$ such that if $g \in F$ then $(f(x), g(x)) \in A$ for all $x \in X$. Since g is G -uniform, there is a $V \in \mathcal{N}_e$ such that $g(Vx) \subseteq A[g(x)]$ for all $x \in X$. So

$$f(Vx) \subseteq A^{-1}[g(Vx)] \subseteq A^{-1} \circ A[g(x)] \subseteq A^{-1} \circ A^2[f(x)] = A^3[f(x)] \subseteq B[f(x)].$$

Thus $f \in C_G^*(X)$ and $C_G^*(X)$ is uniformly closed.

LEMMA 2.4.2 *Let X be a G -space. Then $C(X)$ is invariant under the action described in Construction 5.*

Proof. Let $f \in C(X)$, $g \in G$ and $x \in X$ with $x_\alpha \rightarrow x$. Then $(gf)(x_\alpha) = f(g^{-1}x_\alpha) \rightarrow f(g^{-1}x) = (gf)(x)$. We have shown that $gf \in C(X)$.

From now on we shall speak of the invariance of subsets of $C(X)$ for a G -space X , and take it to be understood that this invariance is meant in terms of the action described in Construction 5.

PROPOSITION 2.4.3 *Let X be a G -space and let $C \subseteq C_G^*(X)$ be invariant and separate the points and closed subsets of X . Then the G -compactification $\beta_C X$ is equivalent (as a compactification) to the classical T_2 compactification obtained from C .*

Proof. The classical T_2 compactification obtained from C is typically constructed by using the evaluation map to embed X as follows:

$$e : X \hookrightarrow \prod_{f \in C} \text{cl}_p f(X).$$

If we now define $\varphi : \beta_C(X) \rightarrow \text{cl}_p e(X)$ by $\varphi(\beta_C(x)) = e(x)$ and let $\pi_f : \mathbb{R}^C \rightarrow \mathbb{R}$, $\delta_f : \prod_{f \in C} \tilde{X}_f \rightarrow \tilde{X}_f$ and $\mu_g : \mathbb{R}^G \rightarrow \mathbb{R}$ be the natural projections on the given products then

$$\begin{aligned} \pi_f \circ \varphi(\beta_C(x)) &= \pi_f(e(x)) \\ &= f(x) \\ &= \mu_e(f \circ \theta^x) \\ &= \mu_e \circ \delta_f(\beta_C(x)) \end{aligned}$$

so that the equality $\pi_f \circ \varphi = \mu_e \circ \delta_f$ holds on a dense subspace of $\beta_C X$, and thus determines the continuous extension of φ to the whole of $\beta_C X$.

Because $\beta_C X$ is compact and $\prod_{f \in C} \text{cl}_p f(X)$ is T_2 , $\varphi : \beta_C X \rightarrow \varphi(\beta_C X) = \text{cl}_p e(X)$ is already a homeomorphism if it is bijective. Let $\tilde{x} \in \beta_C X$ and $\tilde{y} \in \beta_C X$ and suppose $\varphi(\tilde{x}) = \varphi(\tilde{y})$. We then have, for all $f \in C$, $\pi_f(\varphi(\tilde{x})) = \pi_f(\varphi(\tilde{y}))$ or $\mu_e \circ \delta_f(\tilde{x}) = \mu_e \circ \delta_f(\tilde{y})$. Now because for any $z \in X$,

$$\begin{aligned} \mu_g \circ \delta_f(\beta_C(z)) &= f(gz) \\ &= (g^{-1}f)(z) \\ &= \mu_e \circ \delta_{g^{-1}f}(\beta_C(z)), \end{aligned}$$

the equality $\mu_g \circ \delta_f = \mu_e \circ \delta_{g^{-1}f}$ holds on $\beta_C(X)$ and thus on $\beta_C X$. Invoking the invariance of C we find, for any $g \in G$

$$\begin{aligned} \mu_g \circ \delta_f(\tilde{x}) &= \mu_e \circ \delta_{g^{-1}f}(\tilde{x}) \\ &= \mu_e \circ \delta_{g^{-1}f}(\tilde{y}) \\ &= \mu_g \circ \delta_f(\tilde{y}) \end{aligned}$$

so $\tilde{x} = \tilde{y}$ and φ is bijective.

THEOREM 2.4.4 *The mapping $\beta_C X \leftrightarrow C$ is an order isomorphism between the G -compactifications of a G -space X and the uniformly closed invariant subalgebras of $C_G^*(X)$ that separate the points and closed subsets of X .*

Proof. It is well-known that the mapping $\beta_C X \leftrightarrow C$ is an order-isomorphism between the T_2 compactifications of a topological space X and the uniformly closed subalgebras of $C^*(X)$ that separate the points and closed subsets of X .

If X is a G -space then the algebra of continuous functions extendable to any G -compactification of X is a uniformly closed, invariant subalgebra of $C_G^*(X)$ that separates the points and closed subsets of X by Lemmas 2.4.1 and 2.4.2. Conversely, it follows from Proposition 2.4.3 that each uniformly closed, invariant subalgebra of $C_G^*(X)$ that separates the points and closed subsets of X gives rise to a G -compactification over which exactly that algebra is extendable.

With this and the comments made at the beginning of this section, it follows that the restriction of the classical order isomorphism to the G -compactifications of a G -space X is as advertised.

PROPOSITION 2.4.5 *If X is a locally compact, G -Tychonoff G -space then X has a least G -compactification.*

Proof. The one-point compactification of X is the least T_2 compactification of X . We then know from the classical theory of compactification that there is a least uniformly closed subalgebra C of $C^*(X)$ that separates the points and closed subsets of X . Since X is G -Tychonoff, we have by Theorem 2.2.13 and Lemma 2.4.1 that C is a subalgebra of $C_G^*(X)$ and we thus conclude by Theorem 2.4.4 that $\beta_C X$ is the least G -compactification of X .

2.5 G -compactifications as Completions

There is a uniformity on any G -space X that is induced by the phase group G in a natural way. For each $V \in \mathcal{N}_e$ define

$$A_V = \{(x, y) : y \in Vx\}.$$

Then the set $\{A_V : V \in \mathcal{N}_e\}$ is a base for a uniformity on X that we shall call \mathcal{U}_G^s . Note that the uniformity \mathcal{U}_G^s characterises the G -uniform functions on X in the sense that the G -uniform functions are exactly those functions

that are uniformly continuous with respect to \mathcal{U}_G^s . This allows us to give the following characterisation of the property ‘ G -Tychonoff’.

THEOREM 2.5.1 (de Vries [9]) *The G -space X is G -Tychonoff if and only if there is a uniformity \mathcal{U} on X , compatible with its topology, such that $\mathcal{U} \subseteq \mathcal{U}_G^s$.*

Proof. If $\mathcal{U} \subseteq \mathcal{U}_G^s$ then every bounded real-valued function on X that is uniformly continuous with respect to \mathcal{U} is uniformly continuous with respect to \mathcal{U}_G^s and hence G -uniform. Since $\mathcal{UC}^*(X, \mathcal{U})$ separates the points and closed subsets of X , so does the superset $C_G^*(X)$ and X is G -Tychonoff.

Conversely, if X is G -Tychonoff then $C_G^*(X)$ separates the points and closed subsets of X and thus the coarsest uniformity on X making every $f \in C_G^*(X)$ uniformly continuous is compatible with the topology of X . It is clear that this uniformity, having a base generated by G -uniform functions, is coarser than \mathcal{U}_G^s .

If the G -space X admits a uniformity \mathcal{U} such that $\mathcal{U} \subseteq \mathcal{U}_G^s$ it is said that the action Θ is *bounded* by \mathcal{U} . It is noteworthy that it is not always possible to bound the action by a uniformity compatible with the topology of X (not every Tychonoff G -space is G -Tychonoff), in spite of the fact that the topology induced by \mathcal{U}_G^s is always finer than the topology on X .

NOTATION 2.5.2 If X is a G -space then we shall write \mathcal{U}^* for the finest totally bounded uniformity admitted by X .

PROPOSITION 2.5.3 (Chatyrko & Kozlov [5]) *Let X be a G -space such that $\mathcal{U}^* \subseteq \mathcal{U}_G^s$. Then $\beta_{C_G^*(X)}X = \beta X$.*

Proof. In this case we have $C^*(X) = C_G^*(X)$. As every continuous bounded function can be extended to the completion of X with respect to the uniformity \mathcal{U}^* , so can every continuous, bounded G -uniform function. Since the completion of X with respect to the uniformity \mathcal{U}^* is βX , it follows from Theorem 2.4.4 that $\beta_{C_G^*(X)}X = \beta X$.

PROPOSITION 2.5.4 (Chatyrko & Kozlov [5]) *Let X be a G -space such that the uniformity \mathcal{U}_G^s is compatible with its topology. Then $\beta_{C_G^*(X)}X$ is the Samuel compactification of X with respect to U_G .*

Proof. Each function $f \in \mathcal{UC}^*(X, \mathcal{U}_G^s)$ can be extended to the Samuel compactification of X with respect to \mathcal{U}_G^s . We have already noted that $\mathcal{UC}^*(X, \mathcal{U}_G^s)$ is exactly the set of bounded G -uniform functions on X and since \mathcal{U}_G^s is compatible with the topology on X we must have $C_G^*(X) = \mathcal{UC}^*(X, \mathcal{U}_G^s)$. The result now follows from Theorem 2.4.4.

Chapter 3

Bitopological Spaces

3.1 Preliminaries

DEFINITION 3.1.1 Let X be a set and let \mathcal{T}_1 and \mathcal{T}_2 be topologies on X . Then the triple $(X, \mathcal{T}_1, \mathcal{T}_2)$ is a *bitopological space*. If $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(Y, \mathcal{S}_1, \mathcal{S}_2)$ are bitopological spaces then we say that $f : X \rightarrow Y$ is bicontinuous if and only if $f : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{S}_1)$ and $f : (X, \mathcal{T}_2) \rightarrow (Y, \mathcal{S}_2)$ are both continuous in the topological sense.

We define products of bitopological spaces so as to make them initial with respect to the natural projections:

$$\prod_{\lambda \in \Lambda} (X_\lambda, \mathcal{T}_\lambda, \mathcal{S}_\lambda) = \left(\prod_{\lambda \in \Lambda} X_\lambda, \prod_{\lambda \in \Lambda} \mathcal{T}_\lambda, \prod_{\lambda \in \Lambda} \mathcal{S}_\lambda \right).$$

EXAMPLE 3.1.2 Let $\mathcal{T}_u = \{(x, \infty) : x \in \mathbb{R}\} \cup \mathbb{R} \cup \emptyset$ (the so-called *upper* topology on \mathbb{R}) and let $\mathcal{T}_l = \{(-\infty, x) : x \in \mathbb{R}\} \cup \mathbb{R} \cup \emptyset$ (the so-called *lower* topology on \mathbb{R}). Then $(\mathbb{R}, \mathcal{T}_u, \mathcal{T}_l)$ is a bitopological space such that $\mathcal{T}_u \vee \mathcal{T}_l$ is the usual topology on \mathbb{R} .

NOTATION 3.1.3 If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is a bitopological space and $x \in X$ then we speak of the \mathcal{T}_1 -neighbourhoods of x and the \mathcal{T}_2 -neighbourhoods of x . Naturally, a \mathcal{T}_1 -neighbourhood of x is just a set $A \subseteq X$ such that $x \in A$ and there is a $B \in \mathcal{T}_1$ such that $B \subseteq A$. The definition of a \mathcal{T}_2 -neighbourhood is similar. In the same way, we have \mathcal{T}_1 -open sets, \mathcal{T}_2 -closed sets, *et cetera*.

Our notation for the canonical function spaces is to be understood in a bitopological sense from now on. In particular, when $(X, \mathcal{T}_1, \mathcal{T}_2)$ is a bitopological space, $BC(X)$ is the set of bicontinuous maps from $(X, \mathcal{T}_1, \mathcal{T}_2)$ to $(\mathbb{R}, \mathcal{T}_u, \mathcal{T}_l)$. Unless specified otherwise, function spaces will be equipped with the bitopology of pointwise convergence.

DEFINITION 3.1.4 (Salbany [29]) Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a bitopological space with $A \subseteq X$. We write $\text{cl}^s A$ for the closure of A in $\mathcal{T}_1 \vee \mathcal{T}_2$. The set A is *pairwise closed* if and only if $A = \text{cl}^s A$.

DEFINITION 3.1.5 (Salbany [29]) The bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *pairwise T_2* if and only if, for any $x, y \in X$ such that $x \neq y$, there exists a \mathcal{T}_1 -neighbourhood of x and a disjoint \mathcal{T}_2 -neighbourhood of y or there exists a \mathcal{T}_2 -neighbourhood of x and a disjoint \mathcal{T}_1 -neighbourhood of y .

The previously accepted definition of pairwise T_2 , as used by J.C. Kelly in the study of bitopology in the 60's [19], differed from Salbany's above in that the 'or' was an 'and'. This definition was found to be too strong - under it, a pairwise T_2 bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ with both \mathcal{T}_1 and \mathcal{T}_2 compact satisfied $\mathcal{T}_1 = \mathcal{T}_2$.

DEFINITION 3.1.6 (Kelly [19]) The bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *pairwise regular* if and only if each point has a \mathcal{T}_1 -neighbourhood base of \mathcal{T}_2 -closed sets and a \mathcal{T}_2 -neighbourhood base of \mathcal{T}_1 -closed sets.

DEFINITION 3.1.7 (Fletcher [13], Lane [22]) A bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *pairwise completely regular* if and only if, for each $x \in X$ and disjoint \mathcal{T}_1 -closed set F , there is a bicontinuous function

$$f : (X, \mathcal{T}_1, \mathcal{T}_2) \rightarrow (\mathbb{I}, \mathcal{T}_w, \mathcal{T}_l)$$

such that $f(x) = 1$ and $f(F) = \{0\}$; and, for each \mathcal{T}_2 -closed set E not containing x , there is a bicontinuous function

$$g : (X, \mathcal{T}_1, \mathcal{T}_2) \rightarrow (\mathbb{I}, \mathcal{T}_w, \mathcal{T}_l)$$

such that $f(E) = \{1\}$ and $f(x) = 0$.

Pairwise complete regularity is initiality with respect to bicontinuous maps into $(\mathbb{I}, \mathcal{T}_w, \mathcal{T}_l)$ [29]. When we say that a set of functions separates the points and closed subsets of a bitopological space, we mean this in the sense of the above definition.

DEFINITION 3.1.8 The bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *pairwise Tychonoff* if and only if $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise completely regular and pairwise T_2 .

DEFINITION 3.1.9 (Salbany [29]) The bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *bicompact* if and only if the topological space $(X, \mathcal{T}_1 \vee \mathcal{T}_2)$ is compact.

A number of other definitions for the bicomcompactness of a bitopological space had been proposed before Salbany gave the condition above. For example: Fletcher, Hoyle and Patty suggested in [12] that a bitopological space should be considered bicomcompact if any cover consisting of open sets from both topologies, and containing at least one non-empty open set from each, had a finite subcover. The objections to this definition are that it is not productive and that it makes the bitopological space $(\mathbb{R}, \mathcal{T}_u, \mathcal{T}_l)$ bicomcompact.

A better behaved notion of bicomcompactness was given by T. Bîrsan in [2]. It, however, fails to ensure that every bicontinuous map on a bicomcompact quasi-uniform space is quasi-uniformly continuous. Salbany's definition above suffers from none of these shortcomings and leads to a satisfactory theory of bicomcompactification.

DEFINITION 3.1.10 (Salbany [29]) A *paircover* of the bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an indexed family $\{(U_\lambda, V_\lambda) : \lambda \in \Lambda\}$ such that, for each $\lambda \in \Lambda$, $U_\lambda \in \mathcal{T}_1$ and $V_\lambda \in \mathcal{T}_2$ and, for each $x \in X$, there is a $\lambda \in \Lambda$ such that $x \in U_\lambda \cap V_\lambda$.

PROPOSITION 3.1.11 (Salbany [29]) *The bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is bicomcompact if and only if every paircover has a finite subcover.*

Proof. Let $\mathcal{T}_1 \vee \mathcal{T}_2$ be compact and let $\{(U_\lambda, V_\lambda) : \lambda \in \Lambda\}$ be a paircover of X . Then $\{U_\lambda \cap V_\lambda : \lambda \in \Lambda\}$ is a $\mathcal{T}_1 \vee \mathcal{T}_2$ open cover and so there is a finite set $\{\lambda_i : 1 \leq i \leq n\}$ such that $\{U_{\lambda_i} \cap V_{\lambda_i} : 1 \leq i \leq n\}$ covers X . It follows that $\{(U_{\lambda_i}, V_{\lambda_i}) : 1 \leq i \leq n\}$ is a finite subcover of $\{(U_\lambda, V_\lambda) : \lambda \in \Lambda\}$.

Conversely, let \mathcal{C} be a $(\mathcal{T}_1 \vee \mathcal{T}_2)$ -open cover of X . For each $x \in X$ there is a \mathcal{T}_1 -open set U_x and a \mathcal{T}_2 -open set V_x such that $x \in U_x \cap V_x \in \mathcal{C}$. Let $\{(U_{x_i}, V_{x_i}) : 1 \leq i \leq n\}$ be a finite subcover of the paircover $\{(U_x, V_x) : x \in X\}$. Then $\{U_{x_i} \cap V_{x_i} : 1 \leq i \leq n\}$ is a finite subcover of \mathcal{C} .

PROPOSITION 3.1.12 (Salbany [29]) *If a bitopological space is bicomcompact, pairwise regular and pairwise T_2 then it is pairwise Tychonoff.*

DEFINITION 3.1.13 (Salbany [29]) The bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *locally bicomcompact* if and only if every point has a bicomcompact \mathcal{T}_1 -neighbourhood and a bicomcompact \mathcal{T}_2 -neighbourhood.

3.2 Quasi-Uniformities

DEFINITION 3.2.1 A *quasi-uniform space* is a pair (X, \mathcal{U}) , where X is a set and the *quasi-uniformity* \mathcal{U} is a filter on $X \times X$ satisfying all of the conditions of Definition 2.1.2 except possibly the symmetry requirement (3).

If \mathcal{U} is a quasi-uniformity then so is its *conjugate* $\mathcal{U}^{-1} = \{A^{-1} : A \in \mathcal{U}\}$. A quasi-uniformity is a uniformity if and only if $\mathcal{U} = \mathcal{U}^{-1}$. Each quasi-uniformity \mathcal{U} on a set X naturally defines a bitopological space, namely $(X, \mathcal{T}(\mathcal{U}), \mathcal{T}(\mathcal{U}^{-1}))$. Conversely, we say that the bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ *admits* the quasi-uniformity \mathcal{U} if and only if $(X, \mathcal{T}_1, \mathcal{T}_2) = (X, \mathcal{T}(\mathcal{U}), \mathcal{T}(\mathcal{U}^{-1}))$. If a bitopological space admits some quasi-uniformity then it is said to be *quasi-uniformisable*.

THEOREM 3.2.2 (Lane [22]) *A bitopological space is quasi-uniformisable if and only if it is pairwise completely regular.*

NOTATION 3.2.3 For any quasi-uniformity \mathcal{U} let $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$. Let \mathcal{Q} denote the *upper quasi-uniformity* on \mathbb{R} , that is \mathcal{Q} is the quasi-uniformity generated by the base of all sets of the form

$$A_\varepsilon = \{(x, y) : x - y < \varepsilon\}$$

where $\varepsilon > 0$. Then we have $(\mathbb{R}, \mathcal{T}(\mathcal{Q}), \mathcal{T}(\mathcal{Q}^{-1})) = (\mathbb{R}, \mathcal{T}_u, \mathcal{T}_l)$.

We extend our earlier convention of writing $\mathcal{UC}(X, \mathcal{U})$ for the set of uniformly bicontinuous real-valued functions on the uniform space (X, \mathcal{U}) to quasi-uniform spaces in an obvious way: if \mathcal{U} is a quasi-uniformity then $\mathcal{QUC}(X, \mathcal{U})$ or just $\mathcal{QUC}(X)$ denotes the set of quasi-uniformly continuous functions from (X, \mathcal{U}) into $(\mathbb{R}, \mathcal{Q})$.

DEFINITION 3.2.4 Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a bitopological space, let (Y, \mathcal{U}) be a quasi-uniform space. A set of functions $C \subseteq Y^X$ is *equibicontinuous* at the point $x \in X$ if and only if for any $A \in \mathcal{U}$ there is a $U \in \mathcal{N}_x^{\mathcal{T}_1}$ and a $V \in \mathcal{N}_x^{\mathcal{T}_2}$ such that for all $f \in C$, $f(U) \subseteq A[f(x)]$ and $f(V) \subseteq A^{-1}[f(x)]$.

We shall make use of the fact that if (X, \mathcal{U}) is a quasi-uniform space then $\bigcap \mathcal{U}$ is a partial order, the *specialisation order*, on X . It is clear that $\bigcap \mathcal{U}$ is reflexive, and its transitivity is obvious when one realises that $x \leq y \Leftrightarrow x \in \text{cl}\{y\}$. The specialisation order is dependent only on the topology of \mathcal{U} .

3.3 Bicompactification and Bicompletion

DEFINITION 3.3.1 (Salbany [29]) A bicompactification of a bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is defined as a bicompact, pairwise Tychonoff bitopological space $(\tilde{X}, \tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2)$ such that $(X, \mathcal{T}_1, \mathcal{T}_2)$ is bihomeomorphic to a $(\tilde{\mathcal{T}}_1 \vee \tilde{\mathcal{T}}_2)$ -dense subspace of \tilde{X} .

We construct bicomcompactifications from point-separating sets of functions in the same manner as one would in topology. A detailed description may be found in [29]; we merely note that properties bicomcompact, pairwise T_2 and pairwise regular are productive and hereditary to pairwise closed subspaces, and that $(\mathbb{I}, \mathcal{T}_u|_{\mathbb{I}}, \mathcal{T}_l|_{\mathbb{I}})$ possesses all of them, as required by the usual construction.

We note in passing that the collections of bicontinuous functions that are naturally associated with bicomcompactifications are not in general algebras, as they are in the topological case. Rather, they are *semi-algebras*, closed under addition, multiplication and scaling by positive reals only.

LEMMA 3.3.2 *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a bitopological space and let C be a \mathcal{Q}^s -uniformly closed subsemi-algebra of $BC^*(X)$ that separates the points and closed subsets of X . Then there is a bicomcompactification $\beta_C X$ such that any $f \in BC^*(X)$ has a bicontinuous extension to $\beta_C X$ if and only if $f \in C$.*

Proof. Following Salbany [29], we have a bitopological embedding given by the evaluation map

$$e : X \hookrightarrow \prod_{f \in C} \text{cl}^s f(X).$$

As usual, we define the bicomcompact, pairwise Tychonoff space $\beta_C X = \text{cl}^s e(X)$ and we have the restriction of the projection $\pi_f : \beta_C X \rightarrow \mathbb{R}$ extending $f \in C$ over $\beta_C X$.

If we name the two topologies on $\beta_C X$ $\tilde{\mathcal{T}}_1$ and $\tilde{\mathcal{T}}_2$, then we also have the classical T_2 compactification $e : (X, \mathcal{T}_1 \vee \mathcal{T}_2) \hookrightarrow (\beta_C X, \tilde{\mathcal{T}}_1 \vee \tilde{\mathcal{T}}_2)$. It is well-known that $C(\beta_C X, \tilde{\mathcal{T}}_1 \vee \tilde{\mathcal{T}}_2)|_X = C$. Now, $BC(\beta_C X, \tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2) \subseteq C(\beta_C X, \tilde{\mathcal{T}}_1 \vee \tilde{\mathcal{T}}_2)$ so if $f \in BC(X, \mathcal{T}_1, \mathcal{T}_2)$ has a bicontinuous extension to $(\beta_C X, \tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2)$ then $f \in C$. We conclude that $f \in BC^*(X)$ extends over $\beta_C X$ if and only if $f \in C$.

PROPOSITION 3.3.3 *The mapping $\beta_C X \leftrightarrow C$ is an order isomorphism between the bicomcompactifications of a bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ and the \mathcal{Q}^s -uniformly closed subsemi-algebras of $C^*(X)$ that separate the points and closed subsets of X .*

Proof. Let $\beta_C X \leq \beta_{C'} X$. Then there is a bicontinuous surjection $\varphi : \beta_{C'} X \rightarrow \beta_C X$ that leaves X fixed. If $f \in C$ then, by the preceding lemma, there is a bicontinuous $\tilde{f} : \beta_C X \rightarrow \mathbb{R}$ that extends f over $\beta_C X$. Since $\tilde{f} \circ \varphi$ extends f over $\beta_{C'} X$, we have that $f \in C'$ and we may conclude that $C \subseteq C'$. Conversely, if $C \subseteq C'$ then the projection φ of the product $\prod_{f \in C'} \text{cl}^s f(X)$ onto the subproduct $\prod_{f \in C} \text{cl}^s f(X)$ is a bicontinuous surjection such that $\varphi|_{\beta_{C'} X}$ takes $\beta_{C'} X$ onto $\beta_C X$ and leaves X fixed. Thus $\beta_C X \leq \beta_{C'} X$.

The bijectivity of the mapping $\beta_C X \leftrightarrow C$ now follows automatically, but we still need to check that it has the correct range when restricted to the bicompletions. To do this it suffices to note that, for a given bicompletion, the semi-algebra of functions extendable to it is \mathcal{Q}^s -uniformly closed and separates the points and closed subsets of X . These facts follow from the observations that bicomplete, pairwise regular, pairwise T_2 spaces are pairwise Tychonoff and that the \mathcal{Q}^s -uniform limit of bicontinuous functions is bicontinuous.

It follows that every pairwise Tychonoff bitopological space X has a maximal bicompletion. As might be expected, this bicompletion plays the same role in bitopology as the Stone-Ćech compactification does in topology (see [15]).

DEFINITION 3.3.4 (Salbany [29]) The quasi-uniform space (X, \mathcal{U}) is *bicomplete* if and only if the uniform space (X, \mathcal{U}^s) is complete.

DEFINITION 3.3.5 (Salbany [29]) A bicompletion of the quasi-uniform space (X, \mathcal{U}) is a bicomplete quasi-uniform space $(\tilde{X}, \tilde{\mathcal{U}})$ such that (X, \mathcal{U}) is quasi-uniformly isomorphic to a $\tilde{\mathcal{U}}^s$ -dense subspace of $(\tilde{X}, \tilde{\mathcal{U}})$.

THEOREM 3.3.6 (Salbany [29]) *Every quasi-uniform space has a bicompletion. If a quasi-uniform space is pairwise T_2 then it has a unique pairwise T_2 bicompletion.*

Just as the T_2 compactifications of a Tychonoff topological space are the completions with respect to the totally bounded uniformities admitted by that space, so the bicompletions of a pairwise Tychonoff bitopological space are the bicompletions with respect to the totally bounded quasi-uniformities that it admits.

3.4 Bitopological Transformation Groups

NOTATION 3.4.1 Let G be a group and let \mathcal{T} be a topology on G . Then we have an inverse topology \mathcal{T}^{-1} on G , defined by $\mathcal{T}^{-1} = \{A^{-1} : A \in \mathcal{T}\}$.

DEFINITION 3.4.2 A *parabitopological group* $(G, \mathcal{T}, \mathcal{T}^{-1})$ is a bitopological space with carrier set G , where G is a group such that group multiplication $m : (G \times G, \mathcal{T} \times \mathcal{T}, \mathcal{T}^{-1} \times \mathcal{T}^{-1}) \rightarrow (G, \mathcal{T}, \mathcal{T}^{-1})$ is bicontinuous.

In fact, once the group multiplication is continuous with respect to one of the topologies on G , it is already bicontinuous. This follows from the fact that

group inversion is a homeomorphism between (G, \mathcal{T}) and (G, \mathcal{T}^{-1}) . From this it follows that parabitopological groups are just the well-known paratopological groups considered as bitopological spaces. This perspective was first considered by Raghavan and Reilly [28].

Let $(G, \mathcal{T}, \mathcal{T}^{-1})$ be a parabitopological group, let $V \in \mathcal{N}_e^{\mathcal{T}}$ and define

$$L_V = \{(g, h) : g^{-1}h \in V\} \text{ and } R_V = \{(g, h) : gh^{-1} \in V\}.$$

It is readily verified that the sets $\{L_V : V \in \mathcal{N}_e^{\mathcal{T}}\}$ and $\{R_V : V \in \mathcal{N}_e^{\mathcal{T}}\}$ are bases for quasi-uniformities, \mathcal{L} and \mathcal{R} respectively, such that $\mathcal{T}(\mathcal{L}) = \mathcal{T}(\mathcal{R}) = \mathcal{T}$. However, we also have the two conjugate quasi-uniformities \mathcal{L}^{-1} and \mathcal{R}^{-1} - they are easily seen to be the quasi-uniformities generated by the bases $\{L_V : V \in \mathcal{N}_e^{\mathcal{T}^{-1}}\}$ and $\{R_V : V \in \mathcal{N}_e^{\mathcal{T}^{-1}}\}$ so that $\mathcal{T}(\mathcal{L}^{-1}) = \mathcal{T}(\mathcal{R}^{-1}) = \mathcal{T}^{-1}$. For more information concerning these quasi-uniformities, see [23].

DEFINITION 3.4.3 Let $(G, \mathcal{T}, \mathcal{T}^{-1})$ be a parabitopological group. Further, let $(X, \mathcal{S}_1, \mathcal{S}_2)$ be a pairwise T_2 bitopological space and let $\Theta : G \times X \rightarrow X$ be a bicontinuous action of G on X . Then we shall say that the triple (G, X, Θ) is a *bitopological transformation group* and $(X, \mathcal{S}_1, \mathcal{S}_2)$ is a *bitopological G -space*.

Associated with any bitopological transformation group (G, X, Θ) is a topological transformation group $(G, X, \Theta)^s$, where G becomes a topological group under the topology $\mathcal{T} \vee \mathcal{T}^{-1}$ and $(X, \mathcal{S}_1, \mathcal{S}_2)$ becomes a T_2 space with the topology $\mathcal{S}_1 \vee \mathcal{S}_2$. Similarly, if $\Psi : G \times X \rightarrow X$ is a bicontinuous action of the topological group (G, \mathcal{T}) on the G -space (Y, \mathcal{S}) , then Ψ is also a bicontinuous action of the parabitopological group $(G, \mathcal{T}, \mathcal{T})$ on the bitopological G -space $(X, \mathcal{S}, \mathcal{S})$.

On the bitopological space G -space $(X, \mathcal{T}, \mathcal{T})$, a bitopological property generally holds if and only if its topological counterpart holds on (X, \mathcal{T}) . In particular, the sets $BC_G^*(X, \mathcal{T}, \mathcal{T})$ and $C_G^*(X, \mathcal{T})$ coincide and separate the points and closed subsets of $(G, \mathcal{T}, \mathcal{T})$ in the bitopological sense if and only if they separate the points and closed subsets of (G, \mathcal{T}) in the topological sense. Applying this realisation to the bitopological G -space associated with Megrelishvili's counterexample (2.2.15), we have immediately the following result.

PROPOSITION 3.4.4 *There exists a pairwise Tychonoff bitopological G -space that is not pairwise G -Tychonoff.*

DEFINITION 3.4.5 We shall say that two bitopological G -spaces are isomorphic if and only if there is an equivariant bihomeomorphism between them.

Chapter 4

G -bicompactification

4.1 Preliminaries

In this chapter we unite the theories of G -compactification and bicompactification to obtain a theory of equivariant bicompactification for bitopological G -spaces.

DEFINITION 4.1.1 Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a bitopological G -space. We define a G -bicompactification of $(X, \mathcal{T}_1, \mathcal{T}_2)$ to be a bicompact, pairwise Tychonoff bitopological G -space $(\tilde{X}, \tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2)$ such that $(X, \mathcal{T}_1, \mathcal{T}_2)$ is isomorphic to a $(\tilde{\mathcal{T}}_1 \vee \tilde{\mathcal{T}}_2)$ -dense subspace of $(\tilde{X}, \tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2)$.

Proceeding as we did in the topological case, we shall call a pairwise Tychonoff bitopological G -space *pairwise G -Tychonoff* if and only if it has a G -bicompactification and set about characterising these spaces.

4.2 Existence of G -bicompactifications

This section follows essentially the same programme as its topological counterpart, Section 2.2, and thus suggests that the bitopological definitions so far adopted were prudently chosen.

LEMMA 4.2.1 *Let Z be an invariant equibicontinuous subset of $BC_p(G)$. Then $\Psi : G \times Z \rightarrow Z$ is bicontinuous. Moreover, $Z \subseteq \mathcal{QUC}(G, \mathcal{R})$.*

Proof. Let us write $(G, \mathcal{T}, \mathcal{T}^{-1})$ for the acting group and $(Z, \mathcal{T}_1, \mathcal{T}_2)$ for the space Z with the bitopology of pointwise convergence. It is sufficient to show that for each projection $\pi_{s_0} : \mathbb{R}^G \rightarrow \mathbb{R}$, where $s_0 \in G$, the mapping $\pi_{s_0} \circ \Psi :$

$G \times Z \rightarrow \mathbb{R}$ is bicontinuous at the (arbitrarily chosen) point $(g_0, f_0) \in G \times Z$. To this end, let $(g, f) \in G \times Z$ and consider the equality

$$(g_0 f_0)(s_0) - (gf)(s_0) = [f(s_0 g_0) - f(s_0 g)] + [f_0(s_0 g_0) - f(s_0 g_0)].$$

Let $\varepsilon > 0$. The condition on f that the second bracketed term on the right-hand side of the equality be smaller than $\frac{\varepsilon}{2}$ defines a neighbourhood $U \in \mathcal{N}_{f_0}^{\mathcal{J}_1}$.

In view of the equibicontinuity of Z at the point $s_0 g$, there is a $V \in \mathcal{N}_{g_0}^{\mathcal{J}}$ such that the first bracketed term on the right-hand side of the equality is less than $\frac{\varepsilon}{2}$. Hence the left-hand side is smaller than ε for all $(g, f) \in V \times U$ and the upper half of the bicontinuity of Ψ is proved. The lower half is obtained by negating both sides of the above equality and carrying out the same argument.

To see that $Z \subseteq \mathcal{QUC}(G, \mathcal{R})$, note that $f \in Z \Rightarrow Gf \subseteq Z$ so that the set Gf is equibicontinuous. In particular, if we consider the point $e \in G$, we have that for every $\varepsilon > 0$ there exists $W \in \mathcal{N}_e^{\mathcal{J}}$ such that $f(g) - f(sg) < \varepsilon$ for any $s \in W$ and $g \in G$. This means precisely that $f \in \mathcal{QUC}(G, \mathcal{R})$.

COROLLARY 4.2.2 *Let Y be a pointwise bounded, invariant, equibicontinuous subset of $BC^*(G)$. Then $\text{cl}_p^s Y$ is a bicomact, invariant, equibicontinuous subset of $\mathcal{QUC}_p^*(G, \mathcal{R})$ and is thus a bicomact G -space under the action Ψ .*

Proof. We start with the observation that the equibicontinuity of Y implies that of $\text{cl}_p^s Y \subseteq \mathbb{R}^G$. Now, if $t \in G$, then the bicontinuity of the projection $\pi_t : BC_p(G) \rightarrow \mathbb{R}$ implies that

$$(\text{cl}_p^s Y)(t) = \pi_t(\text{cl}_p^s Y) \subseteq \text{cl}^s \pi_t(Y) = \text{cl}^s Y(t),$$

which is a bounded set in \mathbb{R} . That $\text{cl}_p^s Y$ is bicomact follows from that fact that it is a pairwise closed subset of the bicomact product $\prod_{t \in G} \text{cl}^s Y(t)$. Finally, it follows immediately from the remarks made in Section 1.2 that the invariance of Y implies that of $\text{cl}_p^s(Y)$. Combine these observations with Lemma 4.2.1 and the proof is complete.

DEFINITION 4.2.3 Let X be a bitopological G -space. We say that $f : X \rightarrow \mathbb{R}$ is G -uniform if and only if, for any $A \in \mathcal{Q}$, there exists $V \in \mathcal{N}_e^{\mathcal{J}}$ such that

$$\forall x \in X, f(Vx) \subseteq A[f(x)].$$

LEMMA 4.2.4 *Let X be a bitopological G -space with action Θ and let $f \in BC(X)$. The following conditions are equivalent:*

1. f is G -uniform.

2. The set $\{f \circ \theta^x : x \in X\}$ is equibicontinuous at $e \in G$.

3. The set $\{f \circ \theta^x : x \in X\}$ is right uniformly equibicontinuous on G .

Moreover, if $f \in BC^*(X)$ then $\{f \circ \theta^x : x \in X\}$ is pointwise bounded on G .

Proof. Identical to the proof of Lemma 2.2.7.

LEMMA 4.2.5 Let the paratopological group $(G, \mathcal{T}, \mathcal{T}^{-1})$ act on the bicom-
pact bitopological space $(X, \mathcal{S}_1, \mathcal{S}_2)$ and let $f : X \rightarrow \mathbb{R}$ be bicontinuous. Then
 f is G -uniform.

Proof. Let $A \in \mathcal{Q}$ and let $B \in \mathcal{Q}$ be an entourage that is open in $\mathcal{T}_l \times \mathcal{T}_u$ and
satisfies $B^2 \subseteq A$. Now,

$$\begin{aligned} B^2 &= \bigcup_{x \in \mathbb{R}} (B^{-1}[x] \times B[x]) \\ (f \times f)^{-1}(B^2) &= (f \times f)^{-1} \bigcup_{x \in \mathbb{R}} (B^{-1}[x] \times B[x]) \\ &= \bigcup_{x \in \mathbb{R}} (f \times f)^{-1}(B^{-1}[x] \times B[x]) \\ &= \bigcup_{x \in \mathbb{R}} f^{-1}(B^{-1}[x]) \times f^{-1}(B[x]). \end{aligned}$$

By Proposition 3.1.11, the paircover $\{(f^{-1}(B[x]), f^{-1}(B^{-1}[x])) : x \in \mathbb{R}\}$
has a finite subcover $\{(f^{-1}(B[x_i]), f^{-1}(B^{-1}[x_i])) : 1 \leq i \leq n\}$. By the
bicontinuity of the action we have, for each $x \in X$, a $V_x \in \mathcal{N}_e^{\mathcal{T}}$, a $U_x \in \mathcal{N}_x^{\mathcal{S}_1}$
and a $U'_x \in \mathcal{N}_x^{\mathcal{S}_2}$ such that

$$V_x^{-1}(U'_x) \times V_x(U_x) \subseteq f^{-1}(B^{-1}[x_i]) \times f^{-1}(B[x_i]) \text{ for some } 1 \leq i \leq n.$$

Let $\{(U_{x_j}, U'_{x_j}) : 1 \leq j \leq k\}$ be a finite subcover of the paircover $\{(U_x, U'_x) : x \in X\}$ and set $V = \bigcap_{j=1}^k V_{x_j}$. Then, for each $x \in X$, we have

$$V^{-1}x \times Vx \subseteq f^{-1}(B^{-1}[x_i]) \times f^{-1}(B[x_i]) \text{ for some } 1 \leq i \leq n$$

and thus $V^{-1}x \times Vx \subseteq (f \times f)^{-1}(B^2)$. Therefore, $f(Vx) \subseteq B^2[f(x)] \subseteq A[f(x)]$ for all $x \in X$.

COROLLARY 4.2.6 If X is a bicom-
pact bitopological G -space then $BC(X) = BC_G^*(X)$.

LEMMA 4.2.7 Let X and Y be bitopological G -spaces with Y bicom-
pact, and let $\varphi : X \rightarrow Y$ be a bicontinuous equivariant map. Then $f \in BC(Y) \Rightarrow f \circ \varphi \in BC_G^*(X)$.

Proof. Identical to the proof of Lemma 2.2.10.

Let X be a bitopological G -space with action Θ and let $f \in BC_G^*(X)$. Then it is immediate from Corollary 4.2.2 and the characterisation of G -uniform functions given by Lemma 4.2.4 that $\tilde{X}_f = \text{cl}_p^s \{f \circ \theta^x : x \in X\}$ is a pairwise closed, pointwise bounded, invariant, equibicontinuous subset of $BC_p(G)$ and is thus a bicomact, pairwise regular G -space.

PROPOSITION 4.2.8 *Let X be a bitopological G -space with action Θ and let $f \in BC_G^*(X)$. Then the mapping $\varphi_f : X \rightarrow \tilde{X}_f$ defined by $\varphi_f(x) = f \circ \theta^x$ is bicontinuous and equivariant.*

Proof. Identical to the proof of Proposition 2.2.11.

PROPOSITION 4.2.9 *Let X be a bitopological G -space. Then $BC_G^*(X)$ separates the points and closed subsets of X if and only if the bicontinuous equivariant maps from X to bicomact, pairwise regular G -spaces separate the points and closed subsets of X .*

Proof. Let A be a non-empty \mathcal{T}_1 -closed subset of X , $x_0 \notin A$ and $f \in BC_G^*(X)$ be such that $f(x_0) \notin \text{cl}_{\mathcal{T}_u} f(A)$. This can be written as

$$(f \circ \theta^{x_0})(e) \notin \text{cl}_{\mathcal{T}_u} \{(f \circ \theta^x)(e) : x \in A\}.$$

The bicontinuity of the projection

$$\pi_e : (\tilde{X}_f, \mathcal{S}_1, \mathcal{S}_2) \rightarrow (\mathbb{R}, \mathcal{T}_u, \mathcal{T}_l)$$

implies that $f \circ \theta^{x_0} \notin \text{cl}_{\mathcal{S}_1} \{(f \circ \theta^x) : x \in A\}$. Thus, using the notation of Proposition 4.2.8, we have $\varphi_f(x_0) \notin \text{cl}_{\mathcal{S}_1} \varphi_f(A)$, where φ_f is a bicontinuous equivariant function to a bicomact, pairwise regular G -space.

To prove the converse, consider a bicontinuous equivariant map

$$\varphi : (X, \mathcal{T}_1, \mathcal{T}_2) \rightarrow (Z, \mathcal{S}_1, \mathcal{S}_2)$$

where $(Z, \mathcal{S}_1, \mathcal{S}_2)$ is a bicomact, pairwise regular, bitopological G -space and assume $\varphi(x_0) \notin \text{cl}_{\mathcal{S}_1} \varphi(A)$. There exists $g \in BC(Z)$ such that $g(x_0) = 1$ and $g(A) = \{0\}$. Now let $f = g \circ \varphi$. It follows from Lemma 4.2.7 that $f \in BC_G^*(X)$ and it is clear that $f(x_0) \notin \text{cl}_{\mathcal{T}_u} f(A)$. The same proof applies when A is a non-empty \mathcal{T}_2 -closed subset of X .

THEOREM 4.2.10 *A bitopological G -space X is G -Tychonoff if and only if $BC_G^*(X)$ separates the points and closed subsets of X .*

Proof. Clear from Proposition 4.2.9.

4.3 A Sufficient Condition

The arguments of J. de Vries that allowed us to prove that if the topological group G is locally compact then every Tychonoff G -space is G -Tychonoff are heavily reliant on topological symmetry. In this section, we will attempt to work toward a similar result, but the reader will see that we cannot proceed without strong symmetry assumptions. Indeed, the preliminary results are only valid for the actions of symmetric paratopological groups - that is, paratopological groups of the form $(G, \mathcal{T}, \mathcal{T})$.

DEFINITION 4.3.1 Let the symmetric paratopological group $(G, \mathcal{T}, \mathcal{T})$ act on the bitopological G -space $(X, \mathcal{S}_1, \mathcal{S}_2)$ by means of the action Θ . A function $f \in BC^*(X)$ is called *locally equibicontinuous* if there exists a $V \in \mathcal{N}_e^{\mathcal{T}} = \mathcal{N}_e^{\mathcal{T}^{-1}} = \mathcal{N}_e$ such that the family $\{f \circ \theta_g : g \in V\}$ is equibicontinuous on X . The set of all positive locally equibicontinuous functions on X will be denoted by $\mathcal{LE}^+(X)$.

Let $f \in \mathcal{LE}^+(X)$ with $V \in \mathcal{N}_e$ such that $\{f \circ \theta_g : g \in V\}$ is equibicontinuous on X and define $\|f\| = \sup_{x \in X} \{|f(x)|\}$. The left (quasi-)uniformly bicontinuous real-valued functions on G separate its points and closed subsets, so there is a $\phi \in \mathcal{UC}(G, \mathcal{L})$ such that

$$\phi(G) \subseteq [0, \|f\| + 2], \quad \phi(e) = 0, \quad \phi(G \setminus V) = \{\|f\| + 2\}.$$

We now use ϕ to define a new function $\tilde{f}_\phi : X \rightarrow \mathbb{R}$ by

$$\tilde{f}_\phi(x) = \inf_{t \in G} \{\phi(t) + f(tx)\}.$$

Our aim is to show that $\tilde{f}_\phi \in BC_G^*(X)$ and that, if we apply this construction to a subset of $\mathcal{LE}^+(X)$ that separates the points and closed subsets of X , then we obtain a subset of $BC_G^*(X)$ that separates the points and closed subsets of X .

LEMMA 4.3.2 Let $f \in \mathcal{LE}^+(X)$, $V \in \mathcal{N}_e$ and $\phi \in \mathcal{UC}(G, \mathcal{L})$ be as described in the construction above. Then $\tilde{f}_\phi \in BC_G^*(X)$.

Proof. Firstly note that for every $x \in X$ we have

$$0 \leq \tilde{f}_\phi(x) \leq \phi(e) + f(x) = f(x) \leq \|f\| \tag{4.3.1}$$

so that \tilde{f}_ϕ is bounded. In order to prove that \tilde{f}_ϕ is bicontinuous we introduce the set

$$A_\phi = \{t \in G : \phi(t) < \|f\| + 1\}.$$

It is clear that $A_\phi \subseteq V$. Moreover, for all $t \in G \setminus A_\phi$ we have by inequality (4.3.1) and our definitions

$$\phi(t) + f(tx) \geq \|f\| + 1 + f(tx) \geq \|f\| + 1 \geq \tilde{f}_\phi(x) + 1$$

for all $x \in X$. This implies that, for any $x \in X$

$$\tilde{f}_\phi(x) = \inf_{t \in A_\phi} \{\phi(t) + f(tx)\}. \quad (4.3.2)$$

Now let $\varepsilon > 0$ and $x \in X$. By the local equibicontinuity of f , there is a $W \in \mathcal{N}_x^{\mathcal{S}_1}$ such that for all $y \in W$ and all $t \in V$, $f(tx) - f(ty) < \varepsilon$. Fix any $y \in W$. It follows from equation (4.3.2) that there exists $r \in A_\phi \subseteq V$ such that

$$\phi(r) + f(ry) < \tilde{f}_\phi(y) + \varepsilon.$$

Because $r \in V$ we have, by the choice of W , $f(rx) < f(ry) + \varepsilon$, hence

$$\tilde{f}_\phi(x) \leq \phi(r) + f(rx) < \phi(r) + f(ry) + \varepsilon \leq \tilde{f}_\phi(y) + 2\varepsilon.$$

The same argument but with $W \in \mathcal{N}_x^{\mathcal{S}_2}$ yields $\tilde{f}_\phi(y) < \tilde{f}_\phi(x) + 2\varepsilon$ and completes the proof that \tilde{f}_ϕ is bicontinuous.

Finally, we show that \tilde{f}_ϕ is G -uniform. To this end, consider a point $(t, x) \in G \times X$. Then

$$\begin{aligned} \tilde{f}_\phi(tx) &= \inf_{s \in G} \{\phi(s) + f(stx)\} \\ &= \inf_{s \in G} \{\phi(st^{-1}) - \phi(s) + \phi(s) + f(stx)\} \\ &\geq \inf_{s \in G} \{\phi(st^{-1}) - \phi(s)\} + \tilde{f}_\phi(x). \end{aligned}$$

Now because $\phi \in \mathcal{UC}(G, \mathcal{L})$, there is, for any $\varepsilon > 0$, a $V \in \mathcal{N}_e$ such that $|\phi(st^{-1}) - \phi(s)| < \varepsilon$ for all $t \in V^{-1}$. It follows that $\tilde{f}_\phi(tx) > \tilde{f}_\phi(x) - \varepsilon$ for all $x \in X$ and $t \in V^{-1} \in \mathcal{N}_e$. We have shown that $\tilde{f}_\phi \in BC_G^*(X)$.

LEMMA 4.3.3 *Let $(G, \mathcal{T}, \mathcal{T})$ act on $(X, \mathcal{S}_1, \mathcal{S}_2)$. If $g \in \mathcal{LE}^+(X)$ separates the point x_0 from an \mathcal{S}_1 -closed (\mathcal{S}_2 -closed) set $F \subseteq X$ then there is an $f \in BC_G^*(X)$ separating x_0 from F .*

Proof. Let F be an \mathcal{S}_1 -closed subset of X with $x_0 \notin F$. There exists a quasi-uniformly continuous $\psi : (\mathbb{R}, \mathcal{Q}) \rightarrow (\mathbb{I}, \mathcal{Q}|_{\mathbb{I}})$ such that $\psi(g(x_0)) = 1$ and $\psi(\text{cl}_{\mathcal{Q}}g(F)) = \{0\}$. It is then easily seen that $f = \psi \circ g \in \mathcal{LE}^+(X)$ with $f(x_0) = 1$ and $f(F) = \{0\}$. As $f(x_0) = 1$, there exists a $V \in \mathcal{N}_e^{\mathcal{T}}$ such that $f(tx_0) > 1/2$ for all $t \in V$, and since $f \in \mathcal{LE}^+(X)$, we may assume without loss of generality that $\{f \circ \theta_t : t \in V\}$ is equibicontinuous on X .

As in the construction at the beginning of this section, we may now select $\phi \in \mathcal{UC}(G, \mathcal{L})$ and, because of the preceding lemma, consider $\tilde{f}_\phi \in BC_G^*(X)$. Continuing, let A_ϕ be defined as in the proof of the preceding lemma. Then by equation (4.3.2) we infer that

$$\tilde{f}_\phi(x_0) = \inf_{t \in A_\phi} \{\phi(t) + f(tx_0)\}, \quad (4.3.3)$$

and because $A_\phi \subseteq V$, it follows that the right-hand side of (4.3.3) is at least $1/2$. In addition, by equation (4.3.1), we have for every $x \in F$ that $0 \leq \tilde{f}_\phi(x) \leq f(x) = 0$. Thus $\tilde{f}_\phi(F) = \{0\}$ and $\tilde{f}_\phi \notin \text{cl}_\Omega \tilde{f}_\phi(F)$.

Now suppose we carried out the same argument on the bitopological G -space $(X, \mathcal{S}_2, \mathcal{S}_1)$. For F an \mathcal{S}_2 -closed set with $x_0 \notin F$ we would obtain an $\tilde{f}_\phi \in BC_G^*(X, \mathcal{S}_2, \mathcal{S}_1)$ such that $\tilde{f}_\phi(x_0) \geq 1/2$ and $\tilde{f}_\phi(F) = \{0\}$. Now consider the function $(\tilde{f}_\phi(x_0) - \tilde{f}_\phi) \in BC^*(X, \mathcal{S}_1, \mathcal{S}_2)$. It is also G -uniform because the topology on the acting group is symmetric, and clearly $(\tilde{f}_\phi(x_0) - \tilde{f}_\phi)(x_0) \notin \text{cl}_{\Omega^{-1}}(\tilde{f}_\phi(x_0) - \tilde{f}_\phi)(F)$.

COROLLARY 4.3.4 *If $(G, \mathcal{T}, \mathcal{T})$ acts on $(X, \mathcal{S}_1, \mathcal{S}_2)$ and $\mathcal{LE}^+(X)$ separates the points and closed subsets of X then so does $BC_G^*(X)$.*

COROLLARY 4.3.5 *If $(G, \mathcal{T}, \mathcal{T})$ acts on a pairwise Tychonoff bitopological G -space X that admits a quasi-uniformity \mathcal{U} such that some $V \in \mathcal{N}_e$ acts equibicontinuously on X with respect to \mathcal{U} then X is G -Tychonoff.*

Proof. In this case, if $f : X \rightarrow \mathbb{R}$ is quasi-uniformly continuous with respect to \mathcal{U} , bounded and non-negative then $f \in \mathcal{LE}^+(X)$. Consequently, $\mathcal{LE}^+(X)$ separates the points and closed subsets of X and the result follows.

There can be no question that the results obtained in this section thus far are not what one initially would have liked to prove. After all, we are trying to develop an asymmetric theory of G -compactification, so our assumptions of symmetry are most undesirable! Nevertheless, we shall be able to obtain an analogue of de Vries's sufficient condition in Corollary 2.3.6.

LEMMA 4.3.6 *If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is a locally bicomact, pairwise regular bitopological space such that (X, \mathcal{T}_1) is a T_1 topological space then (X, \mathcal{T}_1) is a T_2 topological space.*

Proof. Suppose (X, \mathcal{T}_1) is not T_2 . Then there exist $x, y \in X$ such that $x \neq y$ and, for any $U \in \mathcal{N}_x^{\mathcal{T}_1}$ and $V \in \mathcal{N}_y^{\mathcal{T}_1}$, $U \cap V \neq \emptyset$. Let \mathcal{B}_x and \mathcal{B}_y be bicomact, pairwise closed bases for $\mathcal{N}_x^{\mathcal{T}_1}$ and $\mathcal{N}_y^{\mathcal{T}_1}$ respectively. By virtue of the local bicomactness of X , we may assume that $\bigcup \mathcal{B}_x$ and $\bigcup \mathcal{B}_y$ are

bicompact. Then the family $\mathcal{B}_x \cup \mathcal{B}_y$ of pairwise closed sets has the finite intersection property and each of its members is contained in the bicompact union $\bigcup \mathcal{B}_x \cup \bigcup \mathcal{B}_y$. Thus there exists $z \in \bigcap \mathcal{B}_x \cap \bigcap \mathcal{B}_y = \bigcap \mathcal{N}_x^{\mathcal{T}_1} \cap \bigcap \mathcal{N}_y^{\mathcal{T}_1}$, and X is not T_1 .

DEFINITION 4.3.7 A topological space (X, \mathcal{T}) is called *homogeneous* if and only if for any $x \in X$ and $y \in X$, there is a homeomorphism $\varphi : X \rightarrow X$ such that $\varphi(x) = y$.

LEMMA 4.3.8 *Let the locally bicompact quasi-uniform space (X, \mathcal{U}) be such that $(X, \mathcal{T}(\mathcal{U}))$ is a homogeneous T_0 topological space. Then $(X, \mathcal{T}(\mathcal{U}))$ is a T_2 topological space.*

Proof. Let \leq be the partial order induced by \mathcal{U} and let $x \in C \subseteq X$, where C is a chain in (X, \leq) . Consider the family $\{\uparrow y : y \geq x\}$. Each of its members is pairwise closed and is contained in the bicompact set $\uparrow x$. Furthermore, the family in question clearly has the finite intersection property. Thus there exists $z \in \bigcap \{\uparrow y : y \geq x\}$, and z is clearly an upper bound for C . We deduce by Zorn's Lemma that (X, \leq) must contain a maximal element. But then, by homogeneity, every element of X must be maximal. This combined with the assumption that $(X, \mathcal{T}(\mathcal{U}))$ is T_0 implies that $(X, \mathcal{T}(\mathcal{U}))$ is T_1 . The preceding lemma now tells us that $(X, \mathcal{T}(\mathcal{U}))$ is, in fact, already T_2 .

COROLLARY 4.3.9 *Let the paratopological group $(G, \mathcal{T}, \mathcal{T}^{-1})$ be locally bicompact and pairwise T_0 . Then $\mathcal{T} = \mathcal{T}^{-1}$.*

Proof. A pairwise regular, pairwise T_0 space is pairwise T_2 [29], so that both \mathcal{T} and \mathcal{T}^{-1} are T_0 . By the preceding lemmas, (G, \mathcal{T}) is a locally compact, T_2 space. The result now follows from a theorem of R. Ellis [11].

PROPOSITION 4.3.10 *Let the paratopological group $(G, \mathcal{T}, \mathcal{T}^{-1})$ be locally bicompact and pairwise T_0 . Then every pairwise Tychonoff bitopological G -space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise G -Tychonoff.*

Proof. We shall have the stated result if we can show that there is a $V \in \mathcal{N}_e^{\mathcal{T}} = \mathcal{N}_e$ that acts equibicontinuously on X with respect to a quasi-uniformity \mathcal{U} that is compatible with the bitopology on X . Let Θ denote the action of G on X , let $V \in \mathcal{N}_e^{\mathcal{T}}$ be bicompact, let $x \in X$ and let $A \in \mathcal{U}$ and $B \in \mathcal{U}$ with $B^2 \subseteq A$, where \mathcal{U} is a quasi-uniformity admitted by $(X, \mathcal{T}_1, \mathcal{T}_2)$.

By the joint bicontinuity of Θ , we have, for each $g \in V$, a $U_g \in \mathcal{N}_x^{\mathcal{T}_1}$ and a $W_g \in \mathcal{N}_g$ such that $W_g(U_g) \subseteq B[gx]$. The cover $\{W_g : g \in V\}$ has a finite subcover $\{W_{g_i} : 1 \leq i \leq n\}$. By Lebesgue's Covering Lemma there exists

a $W \in \mathcal{N}_e$ such that for any $g \in V$, $gW \subseteq W_{g_i}$ for some $1 \leq i \leq n$. Let $U = \bigcap_{i=1}^n U_{g_i}$.

Now consider the motion $\theta^x : (V, \mathcal{L}|_V) \rightarrow (X, \mathcal{U})$. Because it is bicontinuous and V is bicomact, Theorem 4.7 of [29] implies that it is quasi-uniformly continuous. It follows that there is a $W' \in \mathcal{N}_e$ such that $h \in gW' \Rightarrow hx \in B[gx]$ for all $g \in V$. By Corollary 4.3.9, there is a symmetric $V' \in \mathcal{N}_e$ such that $V' \subseteq W' \cap W$. Let $\{h_j V' : 1 \leq j \leq m\}$ cover the totally bounded V . Then $h \in V$ implies $h \in h_j V'$ for some j . Since $h_j V' \subseteq h_j W \subseteq W_{g_i}$ for some $1 \leq i \leq n$, it follows that $hU \subseteq W_{g_i} U \subseteq B[g_i x]$ and, since $g_i \in hV' \subseteq hW'$, $g_i x \in B[hx]$ so that $hU \subseteq B^2[hx] \subseteq A[hx]$.

The other half of the equibicontinuity of the action of V on X can be shown in exactly the same way. The proposition now follows from Corollaries 4.3.5 and 4.3.9.

Unfortunately, Corollary 4.3.10 is insufficient to answer the bitopological question posed in Example 1.1.1 because both $(\mathbb{R}, \mathcal{T}_u, \mathcal{T}_l)$ and $(\mathbb{Z}, \mathcal{T}_u|_{\mathbb{Z}}, \mathcal{T}_l|_{\mathbb{Z}})$ fail to be locally bicomact. We have been unable to answer the interesting question of whether every pairwise Tychonoff bitopological \mathbb{R} -space (\mathbb{Z} -space) is pairwise G -Tychonoff; it may be that a counterexample exists (see the conclusion). The following minor result is of little practical value.

PROPOSITION 4.3.11 *Let the parabitopological group $(G, \mathcal{T}, \mathcal{T}^{-1})$ have the property that $\bigcap \mathcal{N}_e^{\mathcal{T}} \cup (V \cap V^{-1}) \in \mathcal{N}_e^{\mathcal{T}}$ for any $V \in \mathcal{N}_e^{\mathcal{T}}$. Then it is sufficient for a bitopological G -space X to be pairwise G -Tychonoff that X admits a quasi-uniformity \mathcal{U} such that $\mathcal{U}^s \subseteq \mathcal{U}_G^s$.*

Proof. Let \mathcal{U} be a quasi-uniformity on X such that $\mathcal{U}^s \subseteq \mathcal{U}_G^s$ and let $A \in \mathcal{U}$. There is a $V \in \mathcal{N}_e^{\mathcal{T}}$ such that $(V \cap V^{-1})x \subseteq (A \cap A^{-1})[x] \subseteq A[x]$ for all $x \in X$. But $\bigcap \mathcal{N}_e^{\mathcal{T}} x \subseteq A[x]$ for all $x \in X$ by the joint bicontinuity of the action. We conclude that $\bigcap \mathcal{N}_e^{\mathcal{T}} \cup (V \cap V^{-1})[x] \subseteq A[x]$ for all $x \in X$.

COROLLARY 4.3.12 *It is sufficient for a bitopological \mathbb{R} -space (\mathbb{Z} -space) to be pairwise G -Tychonoff that it admits a quasi-uniformity \mathcal{U} such that $\mathcal{U}^s \subseteq \mathcal{U}_G^s$.*

4.4 Ordering G -bicomactifications

As with the standard bicomactifications, we can put a partial order on the G -bicomactifications of the bitopological G -space X . We say that $\beta_1 X \leq \beta_2 X$ if there exists a bicontinuous equivariant surjection $\varphi : \beta_2 X \rightarrow \beta_1 X$ such that $\varphi \circ \beta_2 = \beta_1$. If the mapping φ is a bihomeomorphism (or if $\beta_1 X \leq \beta_2 X \leq \beta_1 X$) then $\beta_1 X \simeq \beta_2 X$.

In fact, this ordering turns out to be equivalent to the usual ordering on bicomcompactifications when it is restricted to the G -bicomcompactifications. For if β_1 and β_2 are $(\mathcal{T}_1 \vee \mathcal{T}_2)$ - and $(\mathcal{S}_1 \vee \mathcal{S}_2)$ -dense equivariant bitopological embeddings of the bitopological G -space X into the bicomact G -spaces $(\beta_1 X, \mathcal{T}_1, \mathcal{T}_2)$ and $(\beta_2 X, \mathcal{S}_1, \mathcal{S}_2)$ respectively, and $\varphi : \beta_2 X \rightarrow \beta_1 X$ is a bicontinuous surjection that leaves X fixed then, for any $g \in G$, the mappings $\varphi \circ \theta_g$ and $\theta_g \circ \varphi$ agree on the $(\mathcal{S}_1 \vee \mathcal{S}_2)$ -dense subspace $\beta_2(X)$ and thus on the whole of $\beta_2 X$. Thus, φ is already equivariant. Conversely, it is clear that any equivariant surjection is already a surjection.

LEMMA 4.4.1 *Let X be a bitopological G -space. Then the set $BC_G^*(X)$ is a Ω^s -uniformly closed subsemi-algebra of $BC^*(X)$.*

Proof. To check that $BC_G^*(X)$ is a semi-algebra is straightforward. Now suppose that \mathcal{F} is a filter in $BC_G^*(X)$ such that $\mathcal{F} \rightarrow f \in BC^*(X)$ uniformly with respect to Ω^s . Let $B \in \Omega$ and let $A \in \Omega$ be an entourage such that $A^3 \subseteq B$. There is an $F \in \mathcal{F}$ such that if $g \in F$ then $(f(x), g(x)) \in A \cap A^{-1}$ for all $x \in X$. Since g is G -uniform, there is a $V \in \mathcal{N}_e^{\mathcal{T}}$ such that $g(Vx) \subseteq A[g(x)]$ for all $x \in X$. So

$$f(Vx) \subseteq A \cap A^{-1}[g(Vx)] \subseteq A^2[g(x)] \subseteq A^3[f(x)] \subseteq B[f(x)].$$

Thus $f \in BC_G^*(X)$ and $BC_G^*(X)$ is Ω^s -uniformly closed.

LEMMA 4.4.2 *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a bitopological G -space. Then $BC(X)$ is invariant under the action described in Construction 5.*

Proof. Let $f \in BC(X)$, $g \in G$, and $x \in X$ with $x_\alpha \rightarrow x$ with respect to, say, \mathcal{T}_1 . Then $(gf)(x_\alpha) = f(g^{-1}x_\alpha) \rightarrow f(g^{-1}x) = (gf)(x)$. The same proof applies when one considers convergence with respect to \mathcal{T}_2 and this proves that $gf \in BC(X)$.

From now on we shall speak of the invariance of subsets of $BC(X)$ for a bitopological G -space X , and take it to be understood that this invariance is meant in terms of the action described in Construction 5.

PROPOSITION 4.4.3 *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a bitopological G -space and let $C \subseteq BC_G^*(X)$ separate the points and closed subsets of X . Then it follows that the G -bicomcompactification $(\beta_C X, \tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2)$ is equivalent (as a bicomcompactification) to the classical bicomcompactification of $(X, \mathcal{T}_1, \mathcal{T}_2)$ obtained from C .*

Proof. The classical bicomactification of $(X, \mathcal{T}_1, \mathcal{T}_2)$ with respect to C is usually constructed by embedding X as follows

$$e : X \hookrightarrow \prod_{f \in C} \text{cl}_p^s f(X).$$

As in Proposition 2.4.3, we define $\varphi : \beta_C(X) \rightarrow \text{cl}_p^s e(X)$ by $\varphi(\beta_C(x)) = e(x)$ and let $\pi_f : \mathbb{R}^C \rightarrow \mathbb{R}$, $\delta_f : \prod_{f \in C} \tilde{X}_f \rightarrow \tilde{X}_f$ and $\mu_g : \mathbb{R}^G \rightarrow \mathbb{R}$ be the natural projections on the given products. The arguments of Proposition 2.4.3 applied to the G -space $(X, \mathcal{T}_1 \vee \mathcal{T}_2)$ are sufficient to prove that φ is a bicontinuous bijection, but are not enough to show that φ^{-1} is bicontinuous.

Recall that we established the equality $\mu_e \circ \delta_f = \pi_f \circ \varphi$ on a $\mathcal{T}_1 \vee \mathcal{T}_2$ -dense subspace of $\beta_C X$, which we may write $\mu_e \circ \delta_f \circ \varphi^{-1} = \pi_f$ now that we have the bijectivity of φ . A glance at our earlier calculations will reveal that the more general equality $\mu_g \circ \delta_f \circ \varphi^{-1} = \pi_{g^{-1}f}$, which is only well-defined because of the invariance of C , holds for all $g \in G$ and $e(x)$ where $x \in X$. The equality thus holds on the whole of $\beta_C X$ and therefore the bicontinuity of the each of the projections $\pi_{g^{-1}f}$ together implies the bicontinuity of φ^{-1} .

THEOREM 4.4.4 *The mapping $\beta_C X \leftrightarrow C$ is an order isomorphism between the G -bicomactifications of a bitopological G -space X and the \mathcal{Q}^s -uniformly closed invariant subsemi-algebras of $BC_G^*(X)$ that separate the points and closed subsets of X .*

Proof. We established with Proposition 3.3.3 that $\beta_C X \leftrightarrow C$ is an order-isomorphism between the bicomactifications of a bitopological space X and the \mathcal{Q}^s -uniformly closed subsemi-algebras of $BC^*(X)$ that separate the points and closed subsets of X .

If X is a bitopological G -space then the semi-algebra of bicontinuous functions extendable to any G -bicomactification of X is a \mathcal{Q}^s -uniformly closed, invariant subsemi-algebra of $BC_G^*(X)$ that separates the points and closed subsets of X by Lemmas 4.4.1 and 4.4.2. Conversely, it follows from Proposition 4.4.3 that each \mathcal{Q}^s -uniformly closed, invariant subsemi-algebra of $BC_G^*(X)$ that separates the points and closed subsets of X gives rise to a G -bicomactification over which exactly that semi-algebra is extendable.

With this and the comments made at the beginning of this section, it follows that the restriction of the mapping $\beta_C X \leftrightarrow C$ to the set of all G -bicomactifications of a G -space X is as advertised.

PROPOSITION 4.4.5 *There exists a pairwise G -Tychonoff, locally bicomact bitopological G -space with no least G -bicomactification if and only if there exists a pairwise Tychonoff, locally bicomact bitopological space with no least bicomactification.*

Proof. Let X be a pairwise Tychonoff, locally bicomact bitopological space with no least bicomactification. Then the trivial group $\{e\}$ acts on X and, because $BC^*(X) = BC_G^*(X)$, the set of G -bicomactifications of X is order isomorphic to the set of bicomactifications of X . Thus X has no least G -bicomactification.

On the other hand, suppose that X is a pairwise G -Tychonoff, locally bicomact bitopological G -space with no least G -bicomactification. If X had a least bicomactification then, by Proposition 3.3.3, there would be a least \mathcal{Q}^s -uniformly closed subalgebra of $BC^*(X)$ that separated the points and closed subsets of X . This algebra would be contained in every \mathcal{Q}^s -uniformly closed subalgebra of $BC_G^*(X)$ that separated the points and closed subsets of X , and would thus imply the existence of a least G -bicomactification of X by Theorem 4.4.4 - a contradiction.

We have been unable to find an example of a pairwise Tychonoff, locally bicomact bitopological space with no least bicomactification. Indeed, it seems that the existence of a one-point bicomactification for a pairwise Tychonoff, locally bicomact bitopological space may not always be guaranteed. As the preceding result shows, this problem is equivalent to its counterpart in the theory of bitopological G -spaces.

4.5 G -bicomactifications as Bicompletions

There is a quasi-uniformity on any bitopological G -space X that is induced by the phase group G in a natural way. For each $V \in \mathcal{N}_e$, define

$$A_V = \{(x, y) : y \in Vx\}.$$

Then the set $\{A_V : V \in \mathcal{N}_e\}$ is a base for a quasi-uniformity on X that we shall call \mathcal{U}_G . Note that the quasi-uniformity \mathcal{U}_G characterises the G -uniform functions on X in the sense that the G -uniform functions are exactly those functions that are quasi-uniformly continuous with respect to \mathcal{U}_G . This allows us to give the following characterisation of the property ‘pairwise G -Tychonoff’.

THEOREM 4.5.1 *The bitopological G -space X is pairwise G -Tychonoff if and only if it admits a quasi-uniformity \mathcal{U} such that $\mathcal{U} \subseteq \mathcal{U}_G$.*

Proof. If $\mathcal{U} \subseteq \mathcal{U}_G$ then every real-valued function on X that is quasi-uniformly continuous with respect to \mathcal{U} is quasi-uniformly continuous with respect to \mathcal{U}_G and hence G -uniform. Since $\mathcal{QUC}^*(X, \mathcal{U})$ separates the points

and closed subsets of X , so does the superset $BC_G^*(X)$ and X is pairwise G -Tychonoff.

Conversely, if X is pairwise G -Tychonoff then $BC_G^*(X)$ separates the points and closed subsets of X and thus the coarsest quasi-uniformity on X making every $f \in BC_G^*(X)$ quasi-uniformly continuous is compatible with the bitopology of X . It is clear that this quasi-uniformity, having a base generated by G -uniform functions, is coarser than \mathcal{U}_G .

NOTATION 4.5.2 If X is a bitopological G -space then we shall write \mathcal{U}^* for the finest totally bounded quasi-uniformity admitted by X .

PROPOSITION 4.5.3 Let X be a bitopological G -space such that $\mathcal{U}^* \subseteq \mathcal{U}_G$. Then $\beta_{BC_G^*(X)}X = \beta X$.

Proof. In this case we have $BC^*(X) = BC_G^*(X)$. As every bicontinuous bounded function can be extended to the bicompletion of X with respect to the quasi-uniformity \mathcal{U}^* , so can every bicontinuous, bounded G -uniform function. Since the bicompletion of X with respect to the quasi-uniformity \mathcal{U}^* is βX , it follows from Theorem 4.4.4 that $\beta_{BC_G^*(X)}X = \beta X$.

PROPOSITION 4.5.4 Let X be a bitopological G -space that admits the quasi-uniformity \mathcal{U}_G . Then $\beta_{BC_G^*(X)}X$ is the Samuel bicompletion of X with respect to \mathcal{U}_G .

Proof. Each function $f \in \mathcal{QUC}^*(X, \mathcal{U}_G)$ can be extended to the Samuel bicompletion of X with respect to \mathcal{U}_G . We have already noted that $\mathcal{QUC}^*(X, \mathcal{U}_G)$ is exactly the set of bounded G -uniform functions on X and since \mathcal{U}_G is compatible with the bitopology on X we must have $BC_G^*(X) = \mathcal{QUC}^*(X, \mathcal{U}_G)$. The result now follows from Theorem 4.4.4.

Chapter 5

Conclusion

By this dissertation we conclude that, just as the theory of bicomactification for bitopological spaces is largely analogous to the classical theory of T_2 compactification, the theory of G -bicomactification for bitopological G -spaces is largely analogous to the classical theory of G -compactification, although we did see some important deviations. That said, we shall mention a few other ways in which the theory of topological transformation groups may be extended to the asymmetric setting.

It has been found that the theory of quasi-uniform spaces when considered with just their induced topology, as opposed to the bitopology given by pairing the induced topology with the topology induced by the conjugate quasi-uniformity, is rather less tractable than its bitopological counterpart (see Künzi [20] for an overview). We would, in this case, be studying the actions of paratopological groups, as opposed to the parabitopological groups of this work. Of course, a theory of G -compactification would not be of interest if one retained the requirement that G -spaces must be T_2 , since every compact T_2 space is Tychonoff and we are back in the symmetric world. Research in this direction would be faced with many more difficulties than are encountered in bitopology.

Researchers in asymmetric topology have always been aware of the strong links between their field and the theory of partial orders. Indeed, L. Nachbin, in his seminal work *Topology and Order* [26], introduced quasi-uniformities precisely in order to study this relationship. Partial orders are also of interest in the theory of topological transformation groups - a partial order on the phase group captures our intuitive idea that time ‘flows in a certain direction’ (consider the usual order on \mathbb{R}), and allows us to speak of the ‘asymptotic behaviour’ of points in the phase space.

Given that each paratopological group has a naturally induced partial order, further research could investigate the actions of ‘partially ordered

paratopological transformation groups'. It may also be of interest to study the actions of the ordered topological group obtained from the symmetrisation of a paratopological group equipped with its induced partial order.

Bicompleteness is not the only notion of completeness for quasi-uniform spaces. Doitchinov has introduced the property *D-completeness* for the class of *quiet* quasi-uniform spaces. Subsequently, Künzi, Romaguera and Sipacheva [21] proved that a regular paratopological group is quiet when it is given its two-sided quasi-uniformity. It may be of interest to examine the extension of the actions of a regular paratopological group over its Doitchinov completion with respect to its two-sided quasi-uniformity.

We were unable to answer the intriguing question of whether or not every bitopological \mathbb{R} -space (\mathbb{Z} -space) is G -Tychonoff. One possible way of solving an even more general problem might be to obtain a bitopological version of the work recently done by M. Megrelishvili and T. Scarr [25], where a method for constructing G -spaces that are not G -Tychonoff is given.

University of Cape Town

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