An Application of Short Rate Modelling Involving Roll-Over Risk to Caplet Pricing

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy in the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.

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Abstract

The concept of roll-over risk encapsulates the risk that a bank sitting on an interbank panel may be unable to borrow at the interbank overnight reference rate at some point in the future. Roll-over risk is comprised of two separate risks: the risk that the bank may deteriorate in credit quality relative to the 'average' bank sitting on the interbank panel and the risk that the bank may experience worse liquidity than the 'average' panel bank. Roll-over risk has been offered as a possible explanation of basis spreads which have proliferated since the Global Financial Crisis. This dissertation makes use of the established methods in order incorporate roll-over risk in the pricing of a caplet based on an underlying reference rate. The caplet pricing function is compared to traditional discretised Monte Carlo techniques. The performance of the function proves more accurate and computationally efficient.

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Chapter 1

Introduction

A short rate refers to a rate of interest that applies over a brief period of time. In the limit this is embodied by the instantaneous rate – the varying rate at which some amount accrues interest at every instant. Naturally this is a strictly theoretical mathematical construct. For most practical purposes, the closest discrete incarnation of the short rate is the overnight rate offered between banks. The overnight rate fluctuates daily.

In the wake of the Global Financial Crisis (GFC), the traditional modelling of short rates needed to be revised. The presence of basis spreads – included in which is roll-over risk – means that textbook arbitrage conditions are no longer holding, indicating a flaw in the assumptions underlying these arbitrage opportunities. A basis spread is applied to one tenor leg of a same-currency swap. No-arbitrage arguments suggest that this spread should be zero but this assumption does not manifest in reality. A suggested explanation for this from Alfeus *et al.* (2020) is that this spread is due to *roll-over risk:* the risk that an entity may be unable to roll-over its short term borrowing at the reference overnight rate at some point in the future. Roll-over risk comprises of liquidity risk and downgrade risk. This recognition is important in furthering our understanding of what drives credit spreads and how to model them.

This dissertation will incorporate roll-over risk into derivative pricing by using the established affine transfrom method of Duffie *et al.* (2000). Specifically, a caplet will be priced based on an underlying interbank reference rate that explicitly accounts for roll-over risk as proposed by Alfeus *et al.* (2020). This framework will allow for the explicit modelling of each of the components of roll-over risk and hence allow for the risk to be taken into account when pricing the caplet.

The dissertation will proceed with an overview of the traditional literature on the broad topics of short rates in Chapter 2. This chapter will culminate in the introduction and explanation of roll-over risk. Chapter 3 will lay out the affine transform method that will be used in order to include roll-over risk in the pricing of a caplet. Finally, the results of the comparison between the affine transform method and traditional Monte Carlo methods will be presented and interrogated in Chapter 4.

Chapter 2

Literature Review

2.1 Establishing the Environment

Initial intuition behind the notion of a short rate can be gained by looking at forward rates. The forward rate can be defined as a rate accessible now, at time t, which allows us to lock in a specific interest rate at some point in the future, time T, and applies for the period of length ΔT. This rate is expressed as follows:

$$F(t; T, T + \Delta T).$$

The instantaneous forward rate is the forward rate that applies over an infinitesimally small period, i.e. $\Delta T \to 0$, expressed as

$$f(t, T) := \lim_{\Delta T \to 0} F(t; T, T + \Delta T).$$

If we represent the instantaneous short rate as $r(t)$, then clearly

$$r(t) = \lim_{T \to t} f(t, T).$$

Furthermore, we can view the short rate as the rate at which cash deposited in the bank will gain interest at every instant. Specifically, one unit invested at time t that accrues interest at an instantaneous rate $r(t)$ will be worth

$$\exp\left(\int_t^T r(s)\, ds\right) \tag{2.1}$$

at time T.

2.2 Traditional Short Rate Modelling

The aim of this section is to introduce the concept of short rate modelling so as to segue into modelling of roll-over risk in particular. The traditional modelling of short rates posit that the short rate follows some diffusion process. Vašíček (1977) establishes the first significant attempt at outlining a model for the short rate. The model outlines the process $r(t)$ as

$$dr(t) = \lambda(\mu - r(t))dt + \sigma dW(t),$$

where $W(t)$ is a Brownian Motion under the risk-neutral measure $\mathbb{Q}$. This model exhibits mean-reversion to the level μ. The rate at which the mean-reversion occurs is represented by the λ term.

Cox *et al.* (1985) put forward a similar model that displays mean reversion but does not allow for negative rates. Both models have a constant level μ to which they revert.

The assumption of a constant level of mean-reversion was relaxed with the model of Hull and White (1990). This model adds a dimension of time to the drift term, the level of mean-reversion and the volatility. Implicit in this is that the short-term interest rate dynamics can be calibrated using the term structure of interest rates and the term structure of forward rates (Hull and White, 1990).

The parameters of the process can be determined from the existing term structure through calibration. Herein lies the difference between the Hull and White (1990) model and its predecessors: the time-homogeneity of the Vasicek and CIR models makes calibration for different term structures difficult, if not impossible.

2.3 Short Rate Applications

Using the short rate framework outlined in Alfeus *et al.* (2020), we define as $r_c(t)$ as the continuously compounded short rate abstraction representing the interbank overnight (O/N) rate accessible to a bank on the interbank panel, i.e.,

$$r_c(t) = r(t) + \Lambda(t), \tag{2.2}$$

where $r(t)$ is the secured (risk-free) rate and $\Lambda(t)$ is a credit spread faced by an average member of the interbank panel. In other words, the addition of the $\Lambda(t)$ term introduces credit risk to the secured O/N rate. Importantly, $\Lambda(t)$ applies to the *average* member of the interbank panel and this panel can be updated with banks being added or dropped depending on a change in credit-worthiness. This fact will become pertinent when looking at the concept of roll-over risk.

Related to this is the overnight indexed swap (OIS). The OIS is an interest rate swap that allows the holder to receive a fixed rate of interest over the period of the swap in exchange for the reference floating rate. The floating rate referenced in the OIS is generally $r_c(t)$, the unsecured O/N rate. The OIS will be a tool used in taking advantage of the basis spread described later.

2.4 Roll-Over Risk

Roll-over risk describes the risk that an entity may not be able to borrow at some reference rate at some point in the future. Alfeus *et al.* (2020) posit that this risk comprises of liquidity risk and downgrade risk. Liquidity risk refers to the risk that an entity may not be able to find a counterparty willing to lend at the reference rate due to a lack of liquidity in the market (and explicitly for non-credit reasons). Downgrade risk refers to the risk that the bank in question may experience a downgrade of credit-worthiness relative to the rest of the interbank panel (for instance being dropped from the interbank panel) and hence be unable to access funds at the reference rate.

The existence of a spread applied to a floating-for-floating swap where each leg is in the same currency but different tenors - the 'frequency basis' – violates texbook no-arbitrage conditions. Chang and Schlögl (2015) note that generally the spread is applied to the shorter tenor of the basis swap. Table 2.1 is outlined by Alfeus *et al.* (2020) and is a demonstration of how to take advantage of the basis spread. Note that the term rate $L(t,T)$ refers to a simple rate which accrues interest between time t and time T. Generally, the term rate referred to in this dissertation will be the London Interbank Offered Rate (LIBOR). The floating payments are cancelled out and the arbitrageur is left to pocket the spread. This 'arbitrage' strategy is particular to continuous borrowing but can be easily generalised to any case where the reference entity is rolling-over borrowing at the shorter rate and lending at the longer rate. However, it is evident that this is not an arbitrage opportunity by definition, as it is not riskless. This is due to the fact that the strategy in Table 2.1 assumes the reference bank will be able to roll-over borrowing at $r_c(t)$ at all times

Action	Cashflows
Time t	
1. Borrow O/N	1
2. Enter Long OIS	0
3. Lend at term rate	-1
Net outcome	0
Time T	
1. Accumulation of rolled-over O/N borrowing	$-e^{\int_t^T r_c(s)ds}$
2. OIS payoff	$e^{\int_t^T r_c(s)ds} - (1 + (T-t)\text{OIS}(t,T))$
3. Loan repaid	$1 + (T-t)L(t,T)$
Net outcome	$(T-t)(L(t,T) - \text{OIS}(t,T))$

Tab. 2.1: Taking Advantage of the LIBOR-OIS Spread

throughout the duration of the strategy. However, due to the presence of roll-over risk, this assumption will not hold in general. As such, Alfeus *et al.* (2020) assert that the LIBOR-OIS spread is a compensation for roll-over risk. Bianchetti (2012) ratifies this fact, stating that the traditional no-arbitrage conditions do not hold and a basis is required as the explanation thereof. Grasselli and Miglietta (2016) show that the presence of the basis spread has led to the existence of multiple term structures, modernising the process of discounting cashflows and pricing contingent claims. This work has been further built-on by Crépey *et al.* (2012), who formulated a multi-curve model for interbank risk.

Initial efforts have been made to model the basis spread directly as a stochastic process. Chang and Schlögl (2015) explicitly model liquidity risk as the driver of the basis spread using an intensity model to describe the arrival time of liquidity shocks. Alfeus *et al.* (2020) build on this by making recourse to the underlying rollover risk.

The notion of roll-over risk is then introduced and denoted by $\pi(t)$, representing a spread over $r_c(t)$ which an arbitrary but fixed entity must pay when borrowing overnight. $\pi(t)$ itself is comprised of $\phi(t) + \lambda(t)$, where $\phi(t)$ is the idiosyncratic liquidity risk and $\lambda(t)$ is the idiosyncratic credit spread over $r_c(t)$. That is, a specific entity will pay

$$\begin{aligned} & r_c(t) + \pi(t) \\ &= r_c(t) + \lambda(t) + \phi(t) \end{aligned}$$

when borrowing overnight. [1]

It is worth emphasising that $\lambda(t)$ does not capture the credit risk of the bank in its entirety since $r_c(t)$ comprises of a systematic credit risk component $\Lambda(t)$. Rather, $\lambda(t)$ is the credit spread above and beyond what would be charged to an average bank on the interbank panel, indicating an increased relative credit risk of the bank in question. Similarly, $\phi(t)$ also represents an idiosyncratic funding risk since any systematic funding risk will be captured by $r_c(t)$.

In order to distinguish between these two factors, Alfeus *et al.* (2020) use credit default swaps which allow for the explicit modelling of the default intensity of the underlying entity. Note that Filipović and Trolle (2013) find that it is not reliably possible to identify $\Lambda(t)$ from market data and hence fix the figure at 5 bps (basis points), a choice followed by Alfeus *et al.* (2020). However, Backwell *et al.* (2019) see

[1] Alfeus *et al.* (2020) multiply all credit-risk factors by q, the loss given default. In the same paper, q is fixed to be 0.6. Without loss of generality, this dissertation assumes $q = 1$ in all cases for ease of reading and calculation. The 'true' value of q - the concept of loss given default - is outside the scope of the research. This assumption can be relaxed without further complicating calculations. Filipović and Trolle (2013) assume zero recovery – equivalent to $q = 1$ – for tractability.

this assumption as being too low, thereby ascribing too much of the credit spread to downgrade risk.

Alfeus *et al.* (2020) put forward the following derivation of certain relationships by slightly adjusting the 'arbitrage' argument outlined in Table 2.1. Once again the strategy will involve the following steps:

1. Borrowing O/N, rolling-over continuously.
2. Enter into a long OIS.
3. Lend at the term rate applying over the duration of the rolled-over borrowing.

As per Table 2.1, this costs 0 to enter into and has a certain payoff. However, the derivation of the proceeding expressions differs from Table 2.1 in that they will recognise roll-over risk as the driver of, and hence compensation for, the spread being exploited.

An entity borrowing at the short rate and constantly rolling principal and interest over from time t to maturity T will accrue to the amount

$$-e^{\int_t^T r_c(s)ds}e^{\int_t^T \pi(s)ds} \tag{2.3}$$

at time T. Note the addition of roll-over risk, encapsulated in the $\pi(s)$ term.

Since this borrowing is not collateralized it is discounted using $r_c(t)+\lambda(t)$ which leads to the expected present value of (2.3):

$$-\mathbb{E}\left[e^{\int_t^T \phi(s)ds}\middle|\mathcal{F}_t\right] \tag{2.4}$$

where $\mathbb{E}\left[\cdot\middle|\mathcal{F}_t\right]$ is the conditional expectation under a risk-neutral measure, $\mathbb{Q}$, that we assume to exist (under which discounted price processes are martingales).

We now analyse the second leg of the trading strategy, the OIS, from which we receive the floating O/N rate $r_c(t)$ and pay the fixed OIS rate over the duration of the strategy. The expected present value of the OIS cashflows are discounted using $r_c(t)$, since the transaction is collateralized.

$$\begin{aligned}&\mathbb{E}\left[e^{-\int_t^T r_c(s)ds}\left(e^{\int_t^T r_c(s)ds}-(1+(T-t)\text{OIS}(t,T))\right)\middle|\mathcal{F}_t\right]\\ =&\mathbb{E}\left[1-e^{-\int_t^T \phi(s)ds}-e^{-\int_t^T r_c(s)ds}\,(T-t)\text{OIS}(t,T)\middle|\mathcal{F}_t\right].\end{aligned} \tag{2.5}$$

Finally, the third leg of the trade. Lending at the present LIBOR term rate over the duration of the trade is credit-risky and hence must be discounted using $r_c(t)+\lambda(t)$. The expected present value is thus

$$\mathbb{E}\left[e^{-\int_t^T (r_c(s)+\lambda(s))ds}(1+(T-t)L(t,T))\middle|\mathcal{F}_t\right]. \tag{2.6}$$

Now that the expected present value of each of the legs of the strategy has been ascertained, recall that the strategy is the same as that from Table 2.1 and hence costs zero to set up. Also note that, unlike the situation in Table 2.1, the final payoff is not risk-free, as roll-over risk is considered compensation for the final payoff. We may thus combine (2.4), (2.5) and (2.6), arriving at the relationship:

$$\begin{aligned} 0 &= -\mathbb{E}\left[e^{\int_t^T \phi(s)ds} \big| \mathcal{F}_t\right] + \mathbb{E}\left[1 - e^{-\int_t^T \phi(s)ds} - e^{-\int_t^T r_c(s)ds}\,\text{OIS}(t,T)(T-t) \big| \mathcal{F}_t\right] \\ &\quad + \mathbb{E}\left[e^{-\int_t^T (r_c(s)+\lambda(s))ds}(1+(T-t)L(t,T)) \big| \mathcal{F}_t\right] \\ \Rightarrow \mathbb{E}\left[e^{\int_t^T \phi(s)ds} \big| \mathcal{F}_t\right] &= \mathbb{E}\big[e^{-\int_t^T (r_c(s)+\lambda(s))ds}(1+(T-t)L(t,T)) \\ &\quad - e^{-\int_t^T r_c(s)ds}(1+\text{OIS}(t,T)(T-t)) \big| \mathcal{F}_t\big]. \end{aligned} \tag{2.7}$$

Then, defining the discount factor implied by the overnight rate as

$$D^{\text{OIS}}(t,T) = \mathbb{E}_t\left[e^{-\int_t^T r_c(s)ds}\right]$$

and taking into account the fact that the value of the OIS at inception must be zero, leads to the following relationship:

$$\begin{aligned} \text{OIS}(t,T) &= \frac{1 - D^{\text{OIS}}(t,T)}{(T-t)D^{\text{OIS}}(t,T)} \\ \iff D^{\text{OIS}}(t,T) &= \frac{1}{1+(T-t)D^{\text{OIS}}(t,T)}. \end{aligned} \tag{2.8}$$

Since this dissertation is concerned with pricing a caplet, the proceeding equation is of utmost importance. Substituting (2.8) into (2.7), we have

$$L(t,T) = \frac{1}{(T-t)}\left(\frac{\mathbb{E}\left[e^{\int_t^T \phi(s)ds} \big| \mathcal{F}_t\right]}{\mathbb{E}\left[e^{-\int_t^T (r_c(s)+\lambda(s))ds} \big| \mathcal{F}_t\right]} - 1\right). \tag{2.9}$$

Clearly (2.9) allows for the term rate to be represented as a function of the constituent processes making up roll-over risk. This expression will be crucial in the pricing of the caplet.

Chapter 3

Pricing Implementation

3.1 Methodology

The rest of the dissertation will proceed as follows:

1. Price a caplet within the context of roll-over risk by using the methods of Duffie *et al.* (2000). The caplet will be written on an interbank reference term rate.

2. Incorporate roll-over risk informed by the work of Alfeus *et al.* (2020) outlined in the previous section in order to specify the model. Roll-over risk will be explicitly captured in the modelling of the underlying rate, as well as in the discounting of the option itself.

3. Obtain the parameters for the model from Alfeus *et al.* (2020) and Filipović and Trolle (2013) in order to reflect realistic dynamics.

4. Use traditional Monte Carlo techniques in order to price the same caplets and ascertain the efficacy of the caplet pricing function.

3.2 Option Pricing Framework

The general option pricing formula utilised in this dissertation is that put forward by Duffie *et al.* (2000). The framework allows for the pricing of instruments which depend on a multi-dimensional affine stochastic process of the form

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t) + dZ(t),$$

where $W(t)$ is a multi-dimensional $\mathbb{Q}$-standard Brownian motion, $\mu(X(t))$ and $\sigma(X(t))$ are affine functions of the reference stochastic process $X(t)$ and $Z(t)$ is a pure jump process based on some probability distribution and whose jumps arrive with intensity $\gamma(X_t)$. $\mathbb{Q}$ is a risk-neutral measure we assume to exist.

The function ψ, defined as follows:

$$\psi(u, X(t), t, T) = \mathbb{E}\left(\exp\left(\int_t^T -R(X(s))ds\right) e^{u\cdot X_T}\middle|\mathcal{F}_t\right), \tag{3.1}$$

is a key expression for the framework. Note that $R(X(t))$ is strictly an affine function of $X(t)$.

An expectation based on the terminal values of the stochastic processes is provided in (3.1). An advantage of the fact that we are dealing with affine processes is that we obtain

$$\psi(u, X(t), t, T) = e^{\alpha(t)+\beta(t)\cdot X(t)}. \tag{3.2}$$

That is, we can write the expectation as a scalar function of values known at time t. A point worth mentioning is that the caplet pricing function offers flexibility in that once $\alpha(t)$ and $\beta(t)$ have been established for the given dynamics of the multi-dimensional stochastic process $X(t)$, then various initial values of these processes may be inputted without having to run the function again.

This result is then used in the general option-pricing formula from Duffie *et al.* (2000):

$$\begin{aligned}
C(d, c, T) =& \mathbb{E}\left[\exp\left(-\int_0^T R(X(s))ds\right)(e^{d\cdot X(T)} - c)^+\right] \\
=& \mathbb{E}\left[\exp\left(-\int_0^T R(X(s))ds\right)(e^{d\cdot X(T)} - c)\mathbb{1}_{d\cdot X(T)\geq \ln(c)}\right] \\
=& \frac{\psi(d, X(0), 0, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\text{Im}[\psi(d - ivd, X(0), 0, T)e^{iv\ln(c)}]}{v}dv \\
& - c\left(\frac{\psi(0, X(0), 0, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\text{Im}[\psi(0 - ivd, X(0), 0, T)e^{iv\ln(c)}]}{v}dv\right),
\end{aligned} \tag{3.3}$$

where d is a vector of length n and c is a constant. Hence defining

$$\begin{aligned}
G_{a,b}(y; X_0, T) =& \frac{\psi(a, X(0), 0, T)}{2} \\
& - \frac{1}{\pi}\int_0^\infty \frac{\text{Im}\left[\psi(a + ivb, X(0), 0, T)e^{-ivy}\right]}{v}dv,
\end{aligned} \tag{3.4}$$

where a and b are vectors of length n, allows for the final general expression for the option price:

$$C(d, c, T) = G_{d,-d}(-\ln(c); X(0), T) - cG_{0,-d}(-\ln(c); X(0), T)$$

However, we are assessing a caplet with maturity T. This caplet will be written on the term rate $L(T, S)$, applying from time T to time S. Obtaining the expression for $L(T,S)$ by using (2.9), the payoff of the caplet will be

$$(S-T)\left(L(T,S)-K\right)^+ = (S-T)\left(\frac{1}{(S-T)}\left(\frac{\mathbb{E}\left[e^{\int_T^S \phi(s)ds}\middle|\mathcal{F}_T\right]}{\mathbb{E}\left[e^{-\int_T^S (r_c(s)+\lambda(s))ds}\middle|\mathcal{F}_T\right]}-1\right)-K\right)^+$$
$$= \left(\frac{\mathbb{E}\left[e^{\int_T^S \phi(s)ds}\middle|\mathcal{F}_T\right]}{\mathbb{E}\left[e^{-\int_T^S (r_c(s)+\lambda(s))ds}\middle|\mathcal{F}_T\right]}-(1+K(S-T))\right)^+.$$

From (3.2), the expectations in (2.9) can be written in the form $e^{\alpha(T)+\beta(T)\cdot X(T)}$. Let

$$\mathbb{E}\left[e^{\int_T^S \phi(s)ds}\middle|\mathcal{F}_T\right] = e^{\alpha_1(T)+\beta_1(T)\cdot X_T}$$
$$\mathbb{E}\left[e^{-\int_T^S (r_c(s)+\lambda(s))ds}\middle|\mathcal{F}_T\right] = e^{\alpha_2(T)+\beta_2(T)\cdot X_T}.$$

Hence, translating this into the correct format for input into (3.3) gives

$$(e^{d\cdot X(T)}-c)^+ = \left(\frac{\mathbb{E}\left[e^{\int_T^S \phi(s)ds}\middle|\mathcal{F}_T\right]}{\mathbb{E}\left[e^{-\int_T^S (r_c(s)+\lambda(s))ds}\middle|\mathcal{F}_T\right]}-(1+K(S-T))\right)^+$$
$$= e^{\alpha_1(T)-\alpha_2(T)}\left(e^{(\beta_1(T)-\beta_2(T))\cdot X(T)}-(1+K(S-T))e^{-(\alpha_1(T)-\alpha_2(T))}\right)^+.$$

According to Filipović and Trolle (2013), r_c is the appropriate rate at which to discount fully-collateralised derivative transaction and hence

$$R(X(t)) = r_c(t)$$

when substituting values into the ψ function in (3.5).

For conciseness, allow

$$e^{\alpha_1(T)-\alpha_2(T)+(\beta_1(T)-\beta_2(T))\cdot X(T)} = e^{\bar{\alpha}(T)+\bar{\beta}(T)\cdot X(T)}$$

and thus the present value of the caplet payoff is

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\int_0^T R(X(s))ds\right)(e^{d\cdot X(s)}-c)^+\right] \\
=\ & e^{\bar{\alpha}(T)}\mathbb{E}\left[\exp\left(-\int_0^T r_c(s)ds\right)(e^{\bar{\beta}(T)\cdot X(T)}-(1+K(S-T))e^{-\bar{\alpha}(T)})\mathbb{1}_{\bar{\beta}(T)\cdot X(T)\geq \ln(1+K(S-T))-\alpha(\bar{T})}\right] \\
=\ & e^{\bar{\alpha}(T)}\Big(G_{\bar{\beta}(T),-\bar{\beta}(T)}(-\ln(1+K(S-T))+\bar{\alpha}(T);X(0),T) \\
&-(1+K(S-T))e^{-\bar{\alpha}(T)}G_{0,-\bar{\beta}(T)}(-\ln(1+K(S-T))+\bar{\alpha}(T);X(0),T)\Big) \\
=\ & e^{\bar{\alpha}(T)}G_{\bar{\beta}(T),-\bar{\beta}(T)}(-\ln(1+K(S-T))+\bar{\alpha}(T);X(0),T) \\
&-(1+K(S-T))G_{0,-\bar{\beta}(T)}(-\ln(1+K(S-T))+\bar{\alpha}(T);X(0),T) \\
=\ & e^{\bar{\alpha}(T)}\times \\
&\left(\frac{\psi(\bar{\beta}(T),X(0),0,T)}{2}-\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}[\psi(\bar{\beta}(T)-iv\bar{\beta}(T),X(0),0,T)e^{iv(\ln(1+K(S-T))-\bar{\alpha}(T))}]}{v}dv\right) \\
&-(1+K(S-T))\times \\
&\left(\frac{\psi(0,X(0),0,T)}{2}-\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}[\psi(-iv\bar{\beta}(T),X(0),0,T)e^{iv(\ln(1+K(S-T))-\bar{\alpha}(T))}]}{v}dv\right).
\end{aligned} \tag{3.5}$$

Note that generally a caplet will be paid at time S, the maturity of the duration of the underlying interest rate. This dissertation has assumed payment at time T, since the pay-off is known at time T. However, one can readily convert this general caplet pricing function to be discounted from time S by making the adjustments outlined in Appendix A.

3.3 Specification of Model Dynamics

Alfeus *et al.* (2020) modelled the stochastic processes of the following form:

$$r_c(t) = a_0(t) + \sum_{i=1}^d a_i y_i(t)$$

$$\lambda(t) = b_0(t) + \sum_{i=1}^d b_i y_i(t)$$

$$\phi(t) = c_0(t) + \sum_{i=1}^d c_i y_i(t)$$

where the y_i follow CIR dynamics, i.e.

$$dy_i(t) = \kappa_i(\theta_i - y_i(t))dt + \sigma_i\sqrt{y_i(t)}dW_i(t),$$

each $W_i(t)(i = 1, ..., d)$ is an independent Wiener process and a_i, b_i and c_i are constants.

This model put forward reliable results and offered the advantage of being able to price instruments with tenors not necessarily available from market data which may offer an improvement over previous approaches such as Jakarasi *et al.* (2015).

This dissertation will make use of the calibrated parameters from Alfeus *et al.* (2020) for the $r_c(t)$ and $\phi(t)$ terms.

Filipović and Trolle (2013) include a jump term in their specification of the credit quality process $\lambda(t)$. In other words, they assume that a jump process governs the relative credit deterioration of an average bank that was a member of the panel at time 0 compared to the average credit quality of the periodically-refreshed panel. A jump signifies the dropping of the bank from the panel. In between jump times the credit quality of the bank will tend towards the average of the panel.

Clearly the use of a jump diffusion adds an added layer of realism to the model. While gradual credit deterioration (and the premium charged therefrom) is possible, it is most likely to come in the form of a sudden jump when a bank is dropped from the interbank panel. The dynamics of $\lambda(t)$ are as follows:

$$d\lambda(t) = -\kappa_\lambda \lambda(t)dt + dJ_\lambda(t),$$

where $J_\lambda(t)$ is a pure jump process with exponentially distributed jumps. The jump intensity is $\nu(t)$ which has dynamics

$$d\nu(t) = \kappa_\nu(\mu(t) - \nu(t))dt + \sigma_\nu\sqrt{\nu(t)}dW(t),$$

which is consistent with the methods of Filipović and Trolle (2013). Note that in their paper, $\Lambda(t)$ was used to denote the the total default risk for a particular entity which in this dissertation is denoted $\Lambda(t) + \lambda(t)$.

At initiation,

$$\lambda(0) = 0$$

since at the time of calibration the market aggregated average was considered by Alfeus *et al.* (2020) and by their definition of $\lambda(t)$ it must be zero. In other words, $\lambda(0) = 0$ implies the reference entity is able to borrow instantaneously at the market benchmark rate. A jump represents the credit downgrading of the formerly average entity. Clearly between jumps $\lambda(t)$ term tends back towards zero, indicating that the bank may be added back to the panel.

Once again the applicability of the affine transform methods of Duffie *et al.* (2000) become apparent as they are able to capture the effects of a jump term in the model and hence into the pricing of the caplet.

3.4 Monte Carlo Comparison

Since no closed-form solution for this option-pricing framework exists, the caplet function will be compared to a Monte Carlo method. In order to determine the present value, the following expectation will be evaluated:

$$\begin{aligned} &\mathbb{E}\left[e^{-\int_0^T r_c(s)ds}(S-T)(L(T,S)-K)^+\right] \\ =&\mathbb{E}\left[e^{-\int_0^T r_c(s)ds}(S-T)\left(\frac{1}{(S-T)}\left(\frac{\mathbb{E}\left[e^{\int_T^S \phi(s)ds}\right]}{\mathbb{E}\left[e^{-\int_T^S (r_c(s)+\lambda(s)q)ds}\right]}-1\right)-K\right)^+\right] \end{aligned} \tag{3.6}$$

which involves a nested expectation that will prove computationally difficult. This immediately offers an advantage of the methods of Duffie *et al.* (2000) as the inner expectation may be evaluated analytically due to its affine term structure. The methods are also easily applicable to the evaluation of the uncertain discounted pay-off. As such, the ψ function outlined in (3.1) will be used in order to determine the inner expectations. Figure 3.1 displays the efficacy of the ψ function when compared to Monte Carlo techniques, justifying its use replacing the nested expectation.

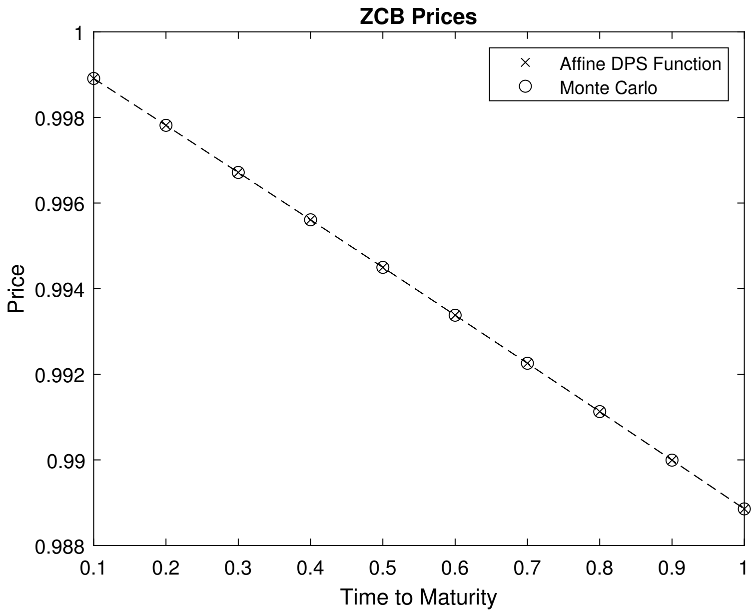

Fig. 3.1: Comparison of ZCB Prices of the Affine DPS Function and Monte Carlo Methods

The processes of the various stochastic rates will be evolved using discretization. The integrals themselves are evaluated using trapezoidal quadrature, i.e.,

$$\int_T^S f(t)\,dt = \sum_{j=1}^{i} (f(t_{j-1}) + f(t_j)) \frac{(t_j - t_{j-1})}{2}.$$

The integral in (3.5) is similarly computed with the added complication of truncating the integral due to the fact that its upper bound is infinite.

3.5 Setting Parameter Values

There are four processes which need to be incorporated into the pricing of the caplet: $r_c(t), \lambda(t), \phi(t), \nu(t)$, which represent the instantaneous unsecured rate, the credit spread, the liquidity spread and the intensity of the jump process of the credit spread, respectively. In order to incorporate a aspect of real-world application, these parameters will use figures from Alfeus *et al.* (2020) in the case of $r_c(t)$ and $\phi(t)$ and from Filipović and Trolle (2013) for the $\lambda(t)$ and $\nu(t)$ terms as this paper allows for a jump process. The jump process itself follows the intensity $\nu(t)$ with identically and independently distributed exponential jump sizes with mean $\frac{1}{\zeta_\lambda}$.

Chapter 4

Pricing Results

4.1 Computational Considerations

MATLABS's *integral* function is employed to numerically calculate the integral in (3.5). The effect of the size of the truncation will naturally affect the accuracy of the caplet pricing function. The effect of the accuracy of the function with respect to truncation size is shown in Figure 4.1. Clearly the marginal benefits of an increased

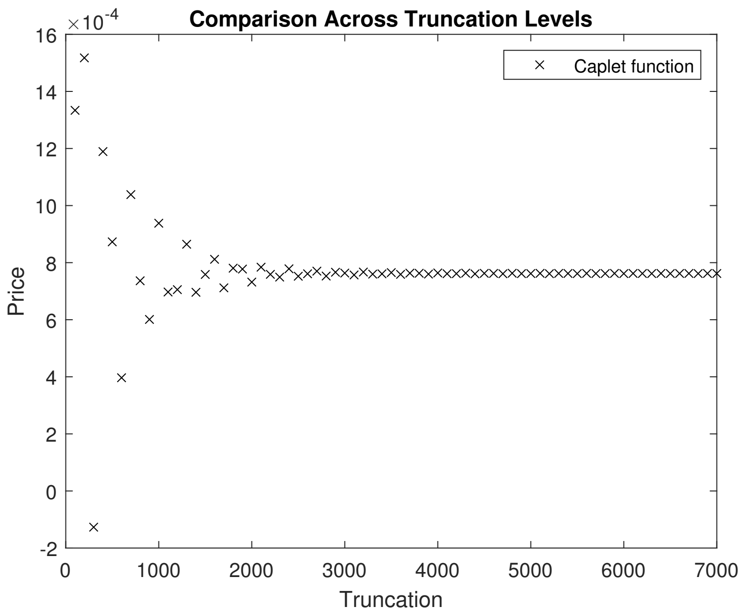

Fig. 4.1: Behaviour of Caplet Function Integrals

truncation size begin to drop off after 4000. Naturally too, there is a trade-off between a larger truncation and smaller differential against the speed of the function.

Computational issues proved an inordinate challenge for the Monte Carlo calculation. Since (3.6) involves a nested integral, the number of sample paths chosen for either integral is multiplicative. As such the decision was made to use the affine function from Duffie *et al.* (2000) to calculate the nested expectation, justified by Figure 3.1. Even so, the expectation does not converge quickly and the computation is time-consuming as demonstrated in Figure 4.2.

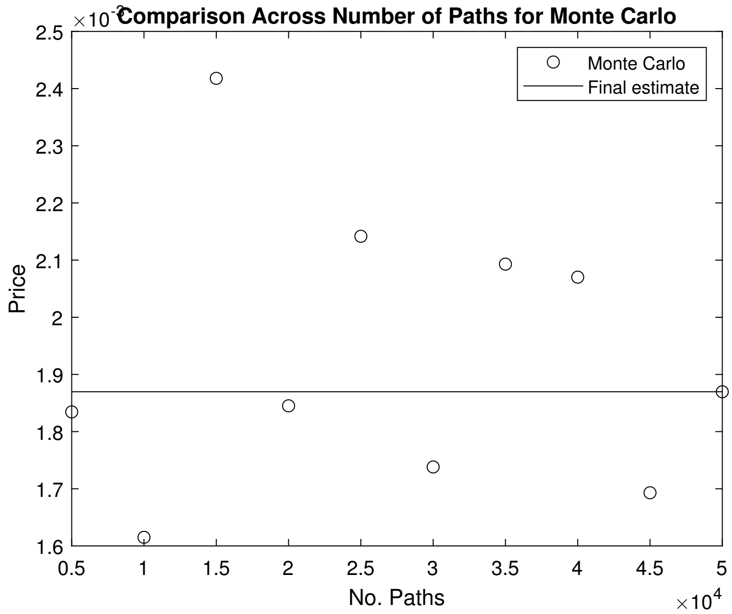

Fig. 4.2: Comparison of Monte Carlo Prices for Different Numbers of Sample Paths

4.2 Comparison

The results of the comparison between the two methods can be seen in Figure 4.2. The caplet pricing function is able to recover the same prices as the Monte Carlo method. 40000 Monte Carlo sample paths were used but there was still variability of the Monte Carlo pricing, demonstrated in Figure 4.2. As such, the prices recovered are not identical. However, there is clearly no bias one way or another and with greater computational efficiency the Monte Carlo prices would converge to the caplet pricing function. The caplet function proves clearly superior; able to deliver consistent results at times roughly 1200% faster than the Monte Carlo techniques.

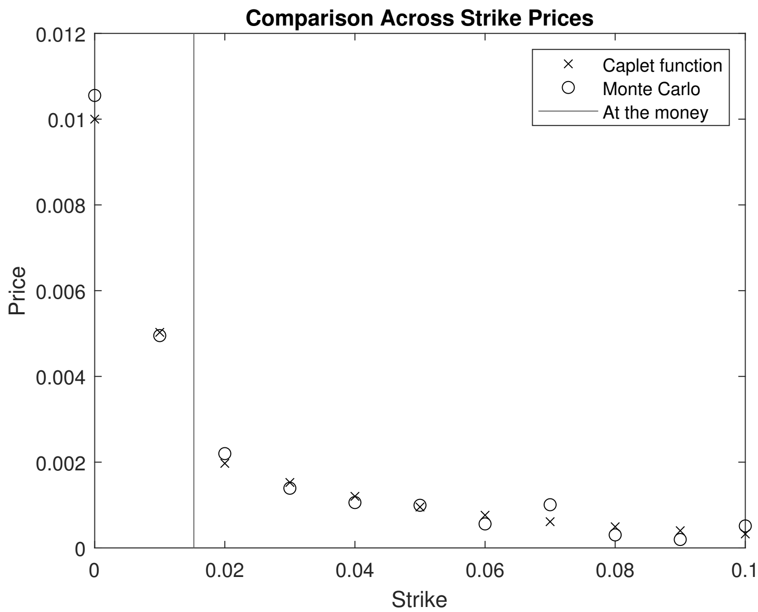

Fig. 4.3: Comparison of Prices at Various Strike Levels

Chapter 5

Conclusions

The goal at the beginning of this dissertation was to include roll-over risk in the pricing of a caplet by utilising the methods of Duffie *et al.* (2000) and avoiding the use of Monte Carlo techniques. After exploring this and comparing the function to Monte Carlo techniques, it is clear that the method is able to recover accurate results more reliably than Monte Carlo methods with greater computational efficiency.

The concept of roll-over risk was first introduced within the context of short rates in general. Mathematical derivations were found by incorporating roll-over risk and these were implemented in order to price the caplets and compare these prices to Monte Carlo techniques.

There is room for an extension of the research if one were to be able to obtain market data and compare the price of the caplets to that produced by the function. Furthermore, if one could harness better processing speed then the Monte Carlo techniques may be run with a greater number of sample paths, leading to a more accurate result and comparison.

The framework explored in this dissertation does put forward a method of incorporating roll-over risk into the pricing of a caplet. The function is able to produce prices that take into account an entity's risk of being dropped from the interbank panel as well as facing liquidity risk. The speed and accuracy of the function displays a marked improvement over Monte Carlo techniques and hence offers an efficient and flexible alternative to these techniques.

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Appendix A

Conversion to In-Arrears Payment

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(-\int_0^S r_c(s)\,ds\right)(S-T)(L(T,S)-K)^+\right]\\
=&\mathbb{E}\left[\exp\left(-\int_T^S r_c(s)\,ds\right)\exp\left(-\int_0^T r_c(s)\,ds\right)(S-T)(L(T,S)-K)^+\right]\\
=&\mathbb{E}\left[\mathbb{E}\left[\exp\left(-\int_T^S r_c(s)\,ds\right)\exp\left(-\int_0^T r_c(s)\,ds\right)(S-T)(L(T,S)-K)^+\middle|\mathcal{F}_T\right]\right]\\
=&\mathbb{E}\left[\mathbb{E}\left[\exp\left(-\int_T^S r_c(s)\,ds\right)\middle|\mathcal{F}_T\right]\exp\left(-\int_0^T r_c(s)\,ds\right)(S-T)(L(T,S)-K)^+\right]
\end{aligned}
$$

Then, denoting $\mathbb{E}\left[\exp\left(-\int_T^S r_c(s)\,ds\right)\middle|\mathcal{F}_T\right] = e^{\dot{\alpha}(T)+\dot{\beta}(T)\cdot X(T)}$ and $R(X_t) = r_c(t)$ and substituting in Expression 2.9 for $L(T,S)$, we have

$$
\begin{aligned}
=&\mathbb{E}\Bigg[e^{\dot{\alpha}(T)+\dot{\beta}(T)\cdot X(T)}\exp\left(-\int_0^T r_c(s)\,ds\right)(S-T)\\
&\left(\frac{1}{(S-T)}\left(\frac{\mathbb{E}\left[e^{\int_T^S \phi(s)ds}\middle|\mathcal{F}_T\right]}{\mathbb{E}\left[e^{-\int_T^S (r_c(s)+\lambda(s))ds}\middle|\mathcal{F}_T\right]}-1\right)-K\right)^+\Bigg]\\
=&\mathbb{E}\left[\exp\left(-\int_0^T r_c(s)\,ds\right)\left(e^{\dot{\alpha}(T)+\dot{\beta}(T)\cdot X(T)}e^{\bar{\alpha}(T)+\bar{\beta}(T)\cdot X(T)}-e^{\dot{\alpha}(T)+\dot{\beta}(T)\cdot X(T)}(1+K(S-T))\right)^+\right]\\
=&e^{\dot{\alpha}(T)+\bar{\alpha}(T)}\mathbb{E}\Bigg[\exp\left(-\int_0^T R(X(s))ds\right)(e^{(\dot{\beta}(T)+\bar{\beta}(T))\cdot X(T)}-(1+K(S-T))e^{-\bar{\alpha}(T)+\dot{\beta}(T)\cdot X(T)})\\
&\mathbb{1}_{(\dot{\beta}(T)+\bar{\beta}(T))\cdot X(T)\geq \ln(1+K(S-T))-\alpha(\bar{T})+\dot{\beta}(T)\cdot X(T)}\Bigg]\\
&-e^{\dot{\alpha}(T)+\bar{\alpha}(T)}\mathbb{E}\Bigg[\exp\left(-\int_0^T R(X(s))ds\right)(e^{(\dot{\beta}(T)+\bar{\beta}(T))\cdot X(T)}-(1+K(S-T))e^{-\bar{\alpha}(T)+\dot{\beta}(T)\cdot X(T)})\\
&\mathbb{1}_{\ln(1+K(S-T))-\alpha(\bar{T})+\dot{\beta}(T)\cdot X(T)\geq(\dot{\beta}(T)+\bar{\beta}(T))\cdot X(T)}\Bigg]
\end{aligned}
$$

which may be expressed and hence calculated as a modification of the G function introduced in Expression 3.3.