Interest-Rate Option Pricing Accounting For Jumps At Deterministic Times

Timothy Allman

A dissertation submitted to the Faculty of Commerce, University of Cape Town, in partial fulfilment of the requirements for the degree of Master of Philosophy.

March 15 2021

MPhil in Mathematical Finance,
University of Cape Town.

Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy in the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.

March 15 2021

Abstract

The short rate is central in the context of interest-rate markets as well as broader finance. As such, accurate modelling of this rate is of particular importance in the pricing of interest-rate options, especially during times of high volatility where increased demand is seen for simpler and lower risk investments. Recent interest has moved away from models of a pure continuous nature towards models that can account for discontinuities in the short rate. These are more representative of real world movements where the short rate is seen to jump due to current and scheduled market information. This dissertation examines this phenomenon in the context of a Vasicek short rate model and accounts for random-sized jumps at deterministic times following ideas similar to those introduced by Kim and Wright (2014). Finite difference methods are used successfully to find PDE solutions via backwards diffusion of the option value equation to its initial state. This procedure is implemented computationally and compared to Monte Carlo benchmark methods in order to assess its accuracy. In both non-jump and jump settings the method constructed was able to accurately price the call option specified and proved to be a viable means for pricing interest-rate options when stochastically-sized discontinuities are present at known times between inception and expiry. Furthermore the method showed that the stochastic discontinues in the short rate most notably affect the option price in the region around and just out of the money.

Keywords: Vasicek, short rate, stochastic discontinuities, finite difference method.

Acknowledgements

My sincerest thanks to my supervisor, Dr Alex Backwell, for his constant patience and support. His input was an indispensable source of guidance throughout this process and I am truly appreciative of his time in this regard.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Central to many topics in financial engineering is the idea of interest rates, their movements and the interplay of various economic factors that determine them. Of particular importance is the concept of the short rate or more specifically the instantaneous rate at which the bank account attracts interest for a given short-term deposit. This short rate is pivotal in the pricing of interest-rate options and fixed income securities which are themselves central to the idea of discounting involved in the pricing of other important instruments. In order to understand these various forms of pricing one must understand the actions of the interest rate and its causes and effects. Hence an examination of the methods of modelling the short rate are of high priority in any pricing setting.

In some of the classical paradigms of interest rate modelling, this exercise is well defined and one can often reach analytical solutions to pricing problems with little effort. However these closed form solutions by their very nature simplify the real world conditions and thus, while being very tractable, are far less useful in accurately portraying the real world dynamics of more complicated phenomena observed in reality. As such there exists a trade off between a model's tractability, and its ability to capture the more realistic dynamics of complex systems.

A significant challenge in the bond pricing and interest-rate option pricing environment is seen in the ”jumping” of interest rates, usually due to regulatory changes made by national reserve banks and government agencies. These jumps occur around policy announcement dates as stated in Piazzesi (2010) and therefore their timing is known prior to them happening. Their magnitudes, however, are random as it is unknown in advance how these changes will affect prevailing interest rates and whether these effects will be positive or negative.

This randomness poses non-trivial consequences in the context of mathematically modeling the aforementioned jumps and having said model be both plausible and tractable. Due to these characteristics, they present themselves in the literature as stochastic discontinuities at deterministic times and can be thought of as such for the purposes of the contingent claim pricing problem.

Therefore a means to account for this phenomenon, when modelling in a practical and accurate fashion, is important to further understand empirical short rate dynamics, bonds and the associated pricing thereof. This dissertation builds on the framework laid out in Kim and Wright (2014) and looks into the pricing of bonds in this setting and the extension of classical models of the short rate to account for these deterministically timed jumps, which once verified and implemented can then be used to price other more advanced instruments such as bond options in this less elementary setting.

Fig. 1.1: An image of the behaviour of the Daily Sterling overnight indexed average (SONIA) rate, a proxy for the short rate that is used in practice, for the period from January 2015 to February 2021, demonstrating the level shifts over time which are characteristic of the jumping notion that has been described.

https://www.bankofengland.co.uk/boeapps/database/fromshowcolumns.asp?Travel=NIxSUx&FromSeries=1&ToSeries=50&DAT=RNG&FD=1&FM=Jan&FY=2015&TD=31&TM=Dec&TY=2021&FNY=&CSVF=TT&html.x=89&html.y=39&C=5JK&Filter=N

Following this introduction, Chapter 2 defines the short rate and goes through a review of the various longstanding models for classical simplified interest rate dynamics and is followed by more recent research and literature of how deterministic jumps could be accounted for in the modelling of the short rate. Chapter 3 details the investigation and setup for the problem, its development and its practical implementation. The results of these findings are discussed in Chapter 4 with particular focus being that of the models accuracy in comparison to known methods and performance and calibration in different contexts. The final conclusions are drawn in Chapter 5 whereafter additional further suggestions are made for extended research on the topic and improvements to the study.

Chapter 2

Literature Review

2.1 The Short Rate, The Bank Account And The Price Of A Bond

2.1.1 The Short Rate

In order to understand interest-rate options one must have a sound grasp of the idea of the interest rates and the various constructs surrounding them. The forward rate is one such idea and is an interest rate which applies over a future time period defined from time T to time $T + \delta$ but is entered into at a prior time t which can be denoted as

$$F(t, T, T + \delta).$$

More theoretically one can let the limit of the length of the period tend to zero so as to move towards knowing the interest rate at any instant in time in the future, giving

$$f(t, T) = \lim_{\delta \to 0} F(t, T, T + \delta)$$

as denoted in Filipović (2009) as the expression for the instantaneous forward rate. To find an expression for the short rate, the time at which the infinitely short period starts must be brought back to the current time t and thus the short rate r_t can be written as

$$r_t = \lim_{T \to t} f(t, T).$$

More intuitively the short rate r_t is that instantaneous spot rate that prevails over the next infinitesimally small time and is loosely proxied by rates of the length of an overnight rate or shorter.

It is important to note that in the context of options where the underlying is a non-interest-rate product, the interest rate and its discount factor are usually assumed to be deterministic with respect to time, by justifying that the main variability of these products are the underlying themselves and the variability introduced by the interest rate is of a much lower order of magnitude to their pricing.

But it would be foolish to assume this in the context of products whose sole variability comes from the interest rate and its movements, as is the case for bonds and options written on them. Hence a means for modelling the short rate used in this situation is needed so as to quantify this randomness as will be presented later.

2.1.2 The Bank Account

Firmly tied to the short rate is the idea of the value of a bank account which can be assumed to evolved under the following dynamics:

$$B(t) = r_t B(t)dt \;\text{ with }\; B(0) = 1,$$

and thus is consequently described as in Brigo and Mercurio (2007) by

$$B(t) = e^{\int_0^t r_s ds},$$

where $B(t)$ is the value of the bank account or money market account after accumulating interest r_t at every instance from time zero to time t by starting out with an initial value of one.

Furthermore taking a ratio of the bank account value at time t and the bank account at time T, allows for the construction of a stochastic discount factor that applies between time t and T and is stated as

$$D(t,T) = \frac{B(t)}{B(T)} = e^{-\int_t^T r_s ds}$$

as defined in Brigo and Mercurio (2007) which denotes the amount of currency at time t that is equivalent to one unit at time T. Therfore knowledge of the short rate allows the discounting of any future payment for pricing purposes making it pivotal in continuous time modelling, due to its use in the numeraire bank account asset described initially.

2.1.3 The Price Of A Zero Coupon Bond

After defining these quantities they can now be utilised in practical applications. Brigo and Mercurio (2007) state the price of zero coupon bond as

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} | \mathcal{F}_t\right].$$

This is because a zero coupon bond has a maturity value at time T of one, which is then discounted to determine an initial price. As is shown, the bond price P can thus be thought of as an expectation under the risk-neutral measure, $\mathbb{Q}$, of the previously mentioned discount factor conditioned on all the information known at time t. And thus a zero coupon bond is simply that amount $P(t,T)$ that when paid at time t, will yield one unit of currency at the terminal time T.

Moreover this fact shows how integral it is to know the short rate dynamics under $\mathbb{Q}$, in both the wider context of financial mechanics and when concerned with the instruments that are constrained by them, and thus, demonstrates what an essential element the short rate is, in the pricing of bonds and valuing of bond options.

2.2 Affine Term Structure

The classical non-jump short rate models examined in this dissertation are all of a similar structure which allows easy simplification of the bond price, as was first introduced by Duffie and Kan (n.d.) and is described in Piazzesi (2010), to an expression with a relatively simple and tractable dependence on the current short rate. This is achieved through conditioning on the current short rate at time t, r_t, and leads to convenient analytical solutions when pricing zero coupon bonds. All of which are a consequence of the following definition found in Filipović (2009) that states that a short rate model is said to possess affine term structure (ATS) if bond prices are given by

$$P(t,T) = F(t, r_t, T)) = e^{-A(t,T)-B(t,T)r_t}, \tag{2.1}$$

where $A(t,T)$, $B(t,T)$ are sufficiently smooth and regular deterministic functions. A further proposition is that given short rate dynamics:

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t,$$

models that are of affine term structure are only such, if and only if the drift and volatility squared (given by the dynamics above) are also affine functions of the short rate r_t, i.e., if and only if we have:

$$\mu(t, r_t) = a(t)r_t + b(t),$$
$$\sigma^2(t, r_t) = c(t)r_t + d(t).$$

It follows that the coefficient functions for Equation 2.1 satisfy the following set of coupled differential equations (see, e.g., Björk (2009)):

$$\begin{aligned} A(t,T) &= b(t)B(t,T) - \frac{1}{2}d(t)B^2(t,T), \\ A(T,T) &= 0, \\ B(t,T) &= -a(t)B(t,T) + \frac{1}{2}c(t)B^2(t,T) - 1, \\ B(T,T) &= 0. \end{aligned}$$

2.3 Classical Affine Models Of The Short Rate

Vasicek (1977) proposes a general form of the term structure of interest rates, which states that under the risk-neutral measure $\mathbb{Q}$ the SDE for the short rate is given by :

$$dr_t = \alpha(\beta - r_t)dt + \sigma dW_t,$$

where β is a constant to which the path of r_t will eventually revert, α (usually assumed positive) is the rate of such reversion and σ is the volatility at the specific time instance and quantifies the magnitude of randomness associated with said time point. Furthermore W_t is a Wiener process under the risk-neutral measure and models the stochastic nature of future interest rates.

In particular this model is founded on the assumption that the short rate follows an Ornstein–Uhlenbeck stochastic process with constant parameters. Moreover the short rate can be shown to be Gaussian in its distribution and thus lends the model to being very tractable in most settings

Later Cox *et al.* (1985) proposed the so called CIR model which unlike the Vasicek did not allow the realisation of negative rates. This was specified via the inclusion of a square root term in the volatility of the short rate dynamics

$$dr_t = (\alpha - \beta r_t)dt + \sigma\sqrt{r_t}dW_t.$$

Noteworthy to mention is the fact that these constant parameters can be made time varying to allow for more calibration freedom to an initial term structure if needed. Furthermore in all the cases above, the aforementioned short rate is assumed to be continuous, but this is arguably not the truest reflection of reality. Thus new approaches for modeling discontinuous processes could extend and further insights into these definitions and be applicable in more general contexts as will be shown later.

2.4 Jump Diffusion Processes In Option Pricing

Jump diffusion processes are widely used in an array of disciplines including computer vision and the physical sciences to model processes involving both diffusion and discontinuity. An elementary discussion of jump diffusions with application to finance is presented now, in a similar vain to the presentation given by Piazzesi (2010), where a more thorough development on the topic can be found. A jump diffusion x_t solves

$$dx_t = \mu(x_t)dt + \sigma(x_t)dW_t + dJ_t, \tag{2.2}$$

where J_t is a pure jump process, W_t is an n-dimensional standard Brownian motion under $\mathbb{Q}$, μ is the drift of x, and $\sigma : D \to \mathbb{R}^{n\times n}$ is its volatility, while x_t is the time immediately preceding a jump time or more formally the left limit $x_t = \lim_{s\uparrow t} x_s$. The jump term J can come about in two different ways. The first is that jump times could be non-deterministic and be modelled by some Poisson process N^P where the parameter λ would represent its stochastic intensity. Heuristically, the conditional probability that a jump will occur in some interval t to $t + dt$ is given by $\lambda_t dt$ and so for very small intervals this means that the Poisson process takes on values as specified by

$$dN_t^P = \begin{cases} 0 & \text{with probability } 1 - \lambda dt, \\ 1 & \text{with probability } \lambda dt. \end{cases}$$

Intuitively this is merely saying that a jump will occur with probability λdt and that with the rest of the probability, $1 - \lambda dt$, the jump will not occur and thus not change anything about the diffusion. Models such as these have been employed in option pricing, as seen in Merton (1976) which demonstrates an approach to option pricing for stocks where the underlying stock S_t follows a jump diffusion as described above, and thus essentially augments the Black Scholes model to account for jumps. The solution for S_t is then found to be very similar to that of the non-jump diffusion model bar some of the algebraic adjustments based off of the jump parameters. And so using the usual mechanics they arrive at the price of a European call option as

$$C(S_0, T) = \sum_{n=0}^{\infty} C^* \left(S_0^n, T | N_T = n \right) P[N_T = n],$$

where

$$P[N_T = n] = \frac{(\lambda T)^n}{n!} e^{\lambda T}$$

is just the probability of the jumps from the Poisson density, given how it has been defined. This can be read as, the call price C, of an option with underlying stock

S_t, given that n many jumps are experienced by time T, is equal to the sum of the the regular Black Scholes call prices C^* times the probability associated with each option or, more holistically, the expectation over all possible call options.

The second way that the jump times may be modelled is through a completely deterministic counting process N^D. This would mean that the jumps would occur at known points in time. This is the topic under consideration in this dissertation and is very representative of short rate movements, where policy changes happen on set known dates and thus should be modeled accordingly. The process is similar to that which has be described above for the Poisson jumps case but simplifies down nicely to an equivalence under risk-neutral expectations as will be described later.

Thus the calculation of options prices in such jump diffusion models is essentially the same as before, i.e., the jumps are ignored if the time one is valuing the option is not a jump time, but then at the jump times, we explicitly take into consideration the jump term by summing over each individual possible diffusion that could occur after multiplying them by their probability of occurring.

2.5 Modelling The Short Rate And Accounting For Jumps At Deterministic Times

Interest as shown in Piazzesi (2001), Kou (2002), Johannes (2004) and more recently in Fontana *et al.* (2020), Jiao *et al.* (2017), Jiao and Li (2018) and Kim and Wright (2014) has moved to attempt to model the short rate and account for stochastic discontinuities both at known points in time and at random times. This is in part due to the need for accurate models when pricing fixed-income instruments and their derivatives at points in time where these shifts occur. Jumps in the short rate are caused by many factors including macro economic news, central bank meetings and surprising global events as noted in Piazzesi (2010), as well as other aspects like changes in a nation's banking regulation and the announcement of said changes as described in Fontana *et al.* (2020). Events like these cause jumps up or down by random amounts hence the framing of them as stochastic discontinuities – stochastic being related to the quality of having a random magnitude or size, either positive or negative, and discontinuous being that they jump abruptly, at known points in time. Bond pricing is more challenging in this setting as the continuous nature of the simple models explained prior do not carry over and as such more sophisticated reasoning and methods for modeling and pricing are described now, to account for this.

Jiao *et al.* (2017) extend the CIR model by adding jump terms driven by an α-stable Lévy process and arrive at explicit expressions for the bond price given its inclusion in the family of affine processes. Emphasis is placed on the self-exciting property of branching processes, which arise as the limit of Hawkes processes, and grants one the ability to describe observations in the sovereign bond market, specifically the persistency of low interest rates in the presence of large jumps.

A hybrid model for sovereign risks is proposed in Jiao and Li (2018), which examines the European debt crisis of 2009 and notes the impact of critical dates on a nation's ability to repay public debt, and thus lends insight into the country's probability of default. It combines well established structural and reduced-form methods and allows the cause of these negative jumps, seen as a fall in the sovereign bond yield on or slightly after these predetermined dates, to be characterised. This is done in a Markovian context, by separately categorising the sovereign solvency metric and idiosyncratic credit risk, whereafter closed-from formulas for the probability of default can be reached.

Fontana *et al.* (2020) describe the existence of two types of discontinuities; first that of a step-like jump from one level to another new level in alignment with monetary policy decisions and second a class of spike-like jumps unrelated to monetary policy which happen during maintenance periods (usually at month end) which are related to the fact that some jurisdictions require banks to hold a set value of deposits on their the central bank accounts at these fixed times. These cause sudden rises and falls or spikes in the short rate due to the increased liquidity need in the inter-bank market. Their model extends the framework introduced in Heath *et al.* (1992) and models jumps in the the whole forward curve. A multi-curve set up is also used which is a common approach in the literature when dealing with this type of problem.

Kim and Wright (2014) differs in that they have outlined a model which specifies an n-dimensional latent state vector x_t that follows a jump diffusion under the physical measure $\mathbb{P}$ given by:

$$dx_t = K(\theta - x_t)dt + \Sigma dW_t + \xi_t dN_t,$$

where W_t is an n-dimensional vector of independent standard Brownian motions, N_t is a counting process with jumps at deterministic times $t = T_i$ with $i = 1, 2, 3, ...$ ($dN_t = 1$ for $t = T_i, 0$ at other times), and ξ_{T_i} is an n-dimensional vector of random jump sizes and is assumed to be normally distributed thusly:

$$\xi_{T_i} \sim \mathcal{N}(\mu(x_{T_i-}), \Omega),$$

where $\Omega = \mathrm{TT}'$ and μ is an affine function of the state vector right before a jump given by

$$\mu(x_{t-}) = \gamma + \Gamma x_{t-}.$$

This makes the jump model state dependent, which, although useful in a general framework like Kim and Wright (2014) is unnecessary for the investigation of the basic effects of jumps on option pricing and its implementation and thus will be simplified by making the jumps size vector state independent later in this dissertation. From this the short rate is then given by

$$r_t = \rho_0 + \rho' x_t.$$

One can have set $p_0 = 0$ and $p = 1$ for an $n = 1$ model and then have short rate r_t be identical to the state vector x_t and thus further simplification of the multidimensional model to a one dimensional case can be achieved to assess the problem in its most basic form. The investigation outlined in this dissertation will limit itself to this one dimensional case.

From these facts Kim and Wright (2014) further prove that the state vector x_t follows the jump diffusion under the risk-neutral measure $\mathbb{Q}$ given by:

$$dx_t = K_Q(\theta_Q - x_t)dt + \Sigma dW_t^Q + \xi_t^Q dN_t,$$

by assuming that the pricing kernel is

$$\frac{dM_t}{M_t} = -r_t dt - \lambda_t' dW_t + J(\xi_t,\ x_{t-})dN_t,$$

where

$$J(\xi_t, x_{t-}) = \exp\left(-\psi_{t-}\mathrm{T}^{-1}(\xi_t - \mu(x_{t-})) - \frac{1}{2}\psi_{t-}'\psi_{t-} \right) - 1,$$

and where the jump size vector $\xi_{T_i}^Q$ has the distribution $\mathcal{N}(\mu_Q(x_{T_{i-}}),\ \Omega)$. and $\mu_Q(x_{t-}) = \gamma_Q + \Gamma_Q x_{t-}, K_Q = K + \Sigma\Lambda, \theta_Q = K_Q^{-1}(K\theta - \Sigma\lambda), \gamma_Q = \gamma - \mathrm{T}\psi$ and $\Gamma_Q = \Gamma - \mathrm{T}\Psi$.

Kim and Wright (2014) have further given expressions for closed form bond pricing derived from the expectation form given earlier. Assuming p jumps at times $T_1, T_2, ..., T_p$, between times t and maturity T then the time t price of a T-bond is given by

$$P(t,T) = \exp(a(t,T) + b(t,T)' x_t),$$

where $a(t,T)$ and $b(t,T)$ are deterministic.

Chapter 3

Methodology

This chapter highlights the objectives of the dissertation and the methods which will be used in attaining them. The essential topic under examination in this study is the replication of the results found in Kim and Wright (2014) and to further use the model defined in this work and apply it to the problem of pricing bond options. Supplementary comparison will be made to classical affine term structure pricing models and their performance in the same task, which will hopefully quantify the effectiveness of the Kim and Wright (2014) model and its usefulness for these results. In particular the main aims of the dissertation are to:

- Utilise the findings of Kim and Wright (2014) in the pricing of zero coupon bonds for a simple one dimensional case.
- Extend Kim and Wright (2014) by successfully implementing a bond option pricing scheme via finite difference methods.
- Compare error and deviations found in estimates from the analytical solutions if possible.
- Remark on the effectiveness when compared to classical methods for the same task.

3.1 Finite Difference Method

Finite difference methods are numerical techniques that are employed to solve differential and partial differential equations. A discretization of the problem space and the application of Taylor's theorem allows the approximation of partial derivatives as finite differences and translates PDE and ODE problems, which may be non-linear, into a set of linear difference equations which are possible to solve through computational matrix algebra techniques

3.1.1 Partial Derivative Approximations Via Taylor's Theorem

If one assumes a function $f : [x_1, x_2] \to \mathbb{R}$ and lets it be $k+1$ times differentiable on $[x_1, x_2]$, Then Taylor's theorem states that at a point $c \in [x_1, x_2]$ and for each x in this interval, there is

$$P_k(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(k)}(c)}{k!}(x-c)^k,$$

termed the Taylor polynomial of degree k about c. With that in mind the function f can be approximated by this polynomial by defining and adding some remainder term and thus written as

$$f(x) = P_k(x) + R_{k+1}(x),$$

where the remainder term $R_{k+1}(x)$ is well known to be of the form

$$R_{k+1}(x) = \frac{f^{k+1}(q)}{(n+1)!}(x-c)^{(k+1)},$$

for some q in between c and x. This is used in the case of the our option pricing problem where we have a function V as a function of a spacial variable x and time variable t. Applying Taylor's theorem and assuming at a point (x,t) and that $\delta_t > 0$, but is still small enough so that $(x, t \pm \delta_t)$ lies in the domain. The first degree Taylor's Polynomial of V about some point $c = (x,t)$ and $q = (x, t_q^+)$ is obtained as

$$\begin{aligned} V(x, t+\delta_t) &= V(x,t) + \frac{\partial V}{\partial t}(x,t)(t+\delta_t - t) + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(x, t_q^+)(t+\delta_t-t)^2 \\ &= V(x,t) + \delta_t \frac{\partial V}{\partial t}(x,t) + \delta_t^2 \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(x, t_q^+). \end{aligned} \tag{3.1}$$

Doing the same about a point $c = (x,t)$ and using $q = (x, t_q^-)$ we get

$$\begin{aligned} V(x, t-\delta_t) &= V(x,t) + \frac{\partial V}{\partial t}(x,t)(t-\delta_t - t) + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(x, t_q^-)(t-\delta_t-t)^2 \\ &= V(x,t) - \delta_t \frac{\partial V}{\partial t}(x,t) + \delta_t^2 \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(x, t_q^-), \end{aligned} \tag{3.2}$$

where t_q^+ and t_q^- are some quantities such that $t - \delta_t < t_q^- < t < t_q^+ < t + \delta_t$. Rearranging Equation 3.1 gives the forward difference approximation for $\frac{\partial V}{\partial t}$

$$\begin{aligned} \frac{\partial V}{\partial t}(x,t) &= \frac{V(x, t+\delta_t) - V(x,t)}{\delta_t} + \frac{1}{2}\delta_t \frac{\partial^2 V}{\partial t^2}(x, t_q^+) \\ &= \frac{V(x, t+\delta_t) - V(x,t)}{\delta_t} + O(\delta_t). \end{aligned}$$

Similarly Equation 3.2 gives the backward difference approximation for $\dfrac{\partial V}{\partial t}$ as

$$\frac{\partial V}{\partial t}(x,t) = \frac{V(x,t) - V(x,t-\delta_t)}{\delta_t} + O(\delta_t), \tag{3.3}$$

where $O(\delta_t)$ signify the remainder or error term being of the order of magnitude of the δ_t value. Taking Taylor polynomials of degree two yields

$$V(x,t-\delta_t) = V(x,t) - \delta_t \frac{\partial V}{\partial t}(x,t) + \frac{1}{2}\delta_t^2 \frac{\partial^2 V}{\partial t^2}(x,t) - \frac{1}{6}\delta_t^3 \frac{\partial^3 V}{\partial t^3}(x,\overline{t_q}^-), \tag{3.4}$$

and

$$V(x,t+\delta_t) = V(x,t) + \delta_t \frac{\partial V}{\partial t}(x,t) + \frac{1}{2}\delta_t^2 \frac{\partial^2 V}{\partial t^2}(x,t) + \frac{1}{6}\delta_t^3 \frac{\partial^3 V}{\partial t^3}(x,\overline{t_q}^+), \tag{3.5}$$

for some $\overline{t_q}^-$ and $\overline{t_q}^+$ satisfying $t - \delta_t < \overline{t_q}^- < t < \overline{t_q}^+ < t + \delta_t$. To get an approximation of $\dfrac{\partial V}{\partial t}$ using a regular central difference one takes Equation 3.4 and subracts it from Equation 3.5 to get

$$\begin{aligned} \frac{\partial V}{\partial t}(x,t) &= \frac{V(x,t+\delta_t) - V(x,t-\delta_t)}{2\delta_t} - \frac{1}{6}\delta_t^2 \left(\frac{\partial^3 V}{\partial t^3}(x,\overline{t_q}^+) + \frac{\partial^3 V}{\partial t^3}(x,\overline{t_q}^-) \right) \\ &= \frac{V(x,t+\delta_t) - V(x,t-\delta_t)}{2\delta_t} + O(\delta_t^2). \end{aligned} \tag{3.6}$$

Taking a Taylor polynomial of degree three in a similar fashion to before and focusing sing on the spacial variable x we obtain

$$\begin{aligned} V(x+\delta_x,t) = V(x,t) &+ \delta_x \frac{\partial V}{\partial x}(x,t) + \frac{1}{2}\delta_x^2 \frac{\partial^2 V}{\partial x^2}(x,t) \\ &+ \frac{1}{6}\delta_x^3 \frac{\partial^3 V}{\partial x^3}(x,t) + \frac{1}{24}\delta_x^4 \frac{\partial^4 V}{\partial x^4}(x_q^+,t), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} V(x-\delta_x,t) = V(x,t) &- \delta_x \frac{\partial V}{\partial x}(x,t) + \frac{1}{2}\delta_x^2 \frac{\partial^2 V}{\partial x^2}(x,t) \\ &- \frac{1}{6}\delta_x^3 \frac{\partial^3 V}{\partial x^3}(x,t) + \frac{1}{24}\delta_x^4 \frac{\partial^4 V}{\partial x^4}(x_q^-,t). \end{aligned} \tag{3.8}$$

Adding Equation 3.8 and 3.7 together, the approximation for the symmetric central difference is obtained as

$$\begin{aligned} \frac{\partial^2 V}{\partial x^2}(x,t) &= \frac{V(x-\delta_x,t) - 2V(x,t) + V(x+\delta_x,t)}{\delta_x^2} \\ &\quad - \frac{1}{12}\delta_x^2 \left(\frac{\partial^4 V}{\partial x^4}(x_q^-,t) + \frac{\partial^4 V}{\partial x^4}(x_q^+,t)\right) \\ &= \frac{V(x-\delta_x,t) - 2V(x,t) + V(x+\delta_x,t)}{\delta_x^2} + O(\delta_x^2). \end{aligned} \tag{3.9}$$

where again x_q^+ and x_q^- are some quantities such that $x-\delta_x < x_q^- < x < x_q^+ < x+\delta_x$. And $O(\delta_x^2)$ designates that the remainder or error term is of the order of magnitude of the δ_x^2 value.

3.1.2 Solving The PDE Numerically

In order to apply these approximations we consider the solution at a set of discrete points given by a grid spanning a set of values for the variables t and x. Concretely it can be assumed the spacial variables spans over N many evenly spaced values that range from a minimum value x_{min} up to a maximum value of x_{max} where δ_x will denote the distance between successive points. The other dimension will contain the time variable t which ranges over M many values starting at inception $t = 0$ and ending at the option expiry time T.

Given that we are dealing with a Markovian process we can let $V(t, r_t)$ denote the time t value of an option, and thus the PDE for the value of an interest-rate contingent claim for the Vasicek model seen in Chapter 2 is given as

$$0 = -r_t V(t, r_t) + \frac{\partial}{\partial t} V(t, r_t) + \frac{\partial}{\partial r_t} V(t, r_t)(\alpha(\beta - r_t)) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial r_t^2} V(t, r_t),$$

which is found via applying the Feynman-Kac Theorem as in Björk (2009) and ensures equivalence with the discounted risk-neutral expectation of the payoff. This PDE is of the parabolic form as described in Vetzal (1998) and thus the techniques laid out above have been employed since Brennan and Schwartz (1978), and are well suited to the option pricing problem, as well as being simple in their implementation and efficient in their computation.

For ease of notating the next sections we will write the approximations solved in Equations 3.3, 3.6 and 3.9 as

$$\frac{\partial}{\partial t} V(t, r_t) = \frac{V(t_{m+1}, r_n) - V(t_m, r_n)}{\delta_t} = \frac{V_{m+1}^n - V_m^n}{\delta_t}, \tag{3.10}$$

$$\frac{\partial}{\partial r_t} V(t, r_t) = \frac{V(t_m, r_{n+1}) - V(t_m, r_{n-1})}{2\delta_r} = \frac{V_m^{n+1} - V_m^{n-1}}{2\delta_r}, \tag{3.11}$$

$$\frac{\partial^2}{\partial r_t^2} V(t, r_t) = \frac{V(t_m, r_{n+1}) - 2V(t_m, r_n) + V(t_m, r_{n-1})}{\delta_r^2} = \frac{V_m^{n+1} - 2V_m^n - V_m^{n-1}}{\delta_r^2}, \tag{3.12}$$

where V_m^n is the value of the option at the node located at the mth time point and nth interest rate point. The choice of a symmetric central difference to approximate the double spacial derivative at a time t_{m+1} as in Equation 3.12, regular central difference for the first partial derivative with respect to the short rate at time step $m+1$ as in Equation 3.11, and backwards approximation for the time derivative at any short rate point r_n as in Equation 3.10, given our problem, is known as an explicit finite difference method. An implicit method could also be used which would have the added benefit of being unconditionally stable no matter our value of δ_t and δ_{r_t} as stated in Vetzal (1998). Expanding and simplifying we obtain

$$0 = -r_t V_{m+1}^n + \frac{V_{m+1}^n - V_m^n}{\delta_t} + \frac{V_{m+1}^{n+1} - V_{m+1}^{n-1}}{2\delta_r}\alpha(\beta - r_t) + \frac{V_{m+1}^{n+1} - 2V_{m+1}^n + V_{m+1}^{n-1}}{\delta_r^2}\frac{\sigma^2}{2}$$

$$0 = -r_t V_{m+1}^n + \frac{V_{m+1}^n}{\delta_t} - \frac{V_m^n}{\delta_t} + \frac{V_{m+1}^{n+1}}{2\delta_r}\alpha(\beta - r_t) - \frac{V_{m+1}^{n-1}}{2\delta_r}\alpha(\beta - r_t) + \frac{V_{m+1}^{n+1}}{\delta_r^2}\frac{\sigma^2}{2} - \frac{2V_{m+1}^n}{\delta_r^2}\frac{\sigma^2}{2} + \frac{V_{m+1}^{n-1}}{\delta_r^2}\frac{\sigma^2}{2}$$

$$+\frac{V_m^n}{\delta_t} = -r_t V_{m+1}^n + \frac{V_{m+1}^n}{\delta_t} - \frac{2V_{m+1}^n}{\delta_r^2}\frac{\sigma^2}{2} + \frac{V_{m+1}^{n+1}}{2\delta_r}\alpha(\beta - r_t) + \frac{V_{m+1}^{n+1}}{\delta_r^2}\frac{\sigma^2}{2} - \frac{V_{m+1}^{n-1}}{2\delta_r}\alpha(\beta - r_t) + \frac{V_{m+1}^{n-1}}{\delta_r^2}\frac{\sigma^2}{2}$$

$$\begin{aligned} V_m^n = &+ V_{m+1}^n\left(-r_t + \frac{1}{\delta_t} - \frac{2}{\delta_r^2}\frac{\sigma^2}{2}\right)\delta_t \\ &+ V_{m+1}^{n+1}\left(+\frac{1}{2\delta_r}\alpha(\beta - r_t) + \frac{1}{\delta_r^2}\frac{\sigma^2}{2}\right)\delta_t \\ &+ V_{m+1}^{n-1}\left(-\frac{1}{2\delta_r}\alpha(\beta - r_t) + \frac{1}{\delta_r^2}\frac{\sigma^2}{2}\right)\delta_t. \end{aligned}$$

Thus the problem has been reduced down to a set of difference equations where the value of the option at a time step m can be calculated by multiplying the option price one time step ahead (i.e., at time step $m+1$) with a suitable matrix F and adding the boundary vector for the time instance one time step ahead B_{m+1}, and thus can be represented as

$$V_m = F \times V_{m+1} + B_{m+1},$$

where $1 \leq m \leq M$ and so the equation represents all points in discretised time. In particular the $N \times M$ matrix F is tridiagonal due to the structure of the problem and is defined as

$$F = \begin{bmatrix} d_1 & u_1 & 0 & \dots & \dots & \dots & 0 \\ l_2 & d_2 & u_2 & 0 & & & \vdots \\ 0 & l_3 & d_3 & u_3 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & l_{N-2} & d_{N-2} & u_{N-2} & 0 \\ \vdots & & & 0 & l_{N-1} & d_{N-1} & u_{N-1} \\ 0 & \dots & \dots & \dots & 0 & l_N & d_N \end{bmatrix},$$

where the middle diagonal is given by

$$\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{N-1} \\ d_N \end{bmatrix} = \begin{bmatrix} \delta_t(-r_{\text{min}} + \frac{1}{\delta_t} - 2\frac{\sigma\sigma}{2\delta_r\delta_r}) \\ \delta_t(-r_{\text{min}} + \delta_r + \frac{1}{\delta_t} - 2\frac{\sigma\sigma}{2\delta_r\delta_r}) \\ \vdots \\ \delta_t(-r_{\text{max}} - \delta_r + \frac{1}{\delta_t} - 2\frac{\sigma\sigma}{2\delta_r\delta_r}) \\ \delta_t(-r_{\text{max}} + \frac{1}{\delta_t} - 2\frac{\sigma\sigma}{2\delta_r\delta_r}) \end{bmatrix},$$

the upper diagonal by

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} \delta_t(\alpha(\beta - r_{\text{min}})\frac{1}{2\delta_r} + \frac{\sigma\sigma}{2\delta_r\delta_r}) \\ \delta_t(\alpha(\beta - r_{\text{min}} + \delta_r)\frac{1}{2\delta_r} + \frac{\sigma\sigma}{2\delta_r\delta_r}) \\ \vdots \\ \delta_t(\alpha(\beta - r_{\text{max}} - \delta_r)\frac{1}{2\delta_r} + \frac{\sigma\sigma}{2\delta_r\delta_r}) \\ \delta_t(\alpha(\beta - r_{\text{max}})\frac{1}{2\delta_r} + \frac{\sigma\sigma}{2\delta_r\delta_r}) \end{bmatrix},$$

and the lower diagonal by

$$\begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_{N-1} \\ l_N \end{bmatrix} = \begin{bmatrix} \delta_t(-\alpha(\beta - r_{\text{min}})\frac{1}{2\delta_r} + \frac{\sigma\sigma}{2\delta_r\delta_r}) \\ \delta_t(-\alpha(\beta - r_{\text{min}} + \delta_r)\frac{1}{2\delta_r} + \frac{\sigma\sigma}{2\delta_r\delta_r}) \\ \vdots \\ \delta_t(-\alpha(\beta - r_{\text{max}} - \delta_r)\frac{1}{2\delta_r} + \frac{\sigma\sigma}{2\delta_r\delta_r}) \\ \delta_t(-\alpha(\beta - r_{\text{max}})\frac{1}{2\delta_r} + \frac{\sigma\sigma}{2\delta_r\delta_r}) \end{bmatrix}.$$

The boundary matrix is also dimensioned with N rows and M columns and contains each boundary column vector B_m to be used at each time instance m of the iterative algorithm and is constructed as follows:

$$\begin{bmatrix} \overset{B_1}{\begin{bmatrix} b_1^{\text{top}} \\ 0 \\ \vdots \\ 0 \\ b_1^{\text{bottom}} \end{bmatrix}} & \overset{B_2}{\begin{bmatrix} b_2^{\text{top}} \\ 0 \\ \vdots \\ 0 \\ b_2^{\text{bottom}} \end{bmatrix}} & \overset{\dots}{\dots} & \overset{B_{M-1}}{\begin{bmatrix} b_{M-1}^{\text{top}} \\ 0 \\ \vdots \\ 0 \\ b_{M-1}^{\text{bottom}} \end{bmatrix}} & \overset{B_M}{\begin{bmatrix} b_M^{\text{top}} \\ 0 \\ \vdots \\ 0 \\ b_M^{\text{bottom}} \end{bmatrix}} \end{bmatrix}.$$

Furthermore the top entry of each boundary column (i.e., the top row of the boundary matrix) is the value of the function at the x variable's lower boundary wherby in the option pricing scenario this relates to the value of the option just below the minimum value of the short rate r_{min}. It exists just outside of our solution space grid and hence each of the entries are the lower boundary condition of the option (explained in Section 3.1.5) evaluated at $r_{\text{min}} - \delta_r$ for each times step $t = 0, \delta_t, 2\delta_t, 3\delta_t, \ldots, (M-1)\delta_t = T$ multiplied by the first entry of the lower diagonal l_1 which gets truncated when constructing the matrix F. The top row of boundary matrix is therefore given as

$$\begin{bmatrix} b_1^{\text{top}} \\ b_2^{\text{top}} \\ \vdots \\ b_{M-1}^{\text{top}} \\ b_M^{\text{top}} \end{bmatrix} = \begin{bmatrix} l_1 \times V(r_{\text{min}} - \delta_r, t = 0) \\ l_1 \times V(r_{\text{min}} - \delta_r, t = \delta_t) \\ \vdots \\ l_1 \times V(r_{\text{min}} - \delta_r, t = (M-1)\delta_t) \\ l_1 \times V(r_{\text{min}} - \delta_r, t = T) \end{bmatrix}.$$

Analogously the bottom entry of each boundary column (i.e., the bottom row of the boundary matrix) is the value of the function at the x variable's upper boundary, whereby in the option pricing scenario this relates to the value of the option just above the maximum value of the short rate r_{max}. It exists just outside of our solution space grid and hence each of the entries are the upper boundary condition of the option (explained in Section 3.1.5) evaluated at $r_{\text{max}} + \delta_r$ for each times step $t = 0, \delta_t, 2\delta_t, 3\delta_t, \ldots, (M-1)\delta_t = T$ multiplied by the last entry of the upper diagonal u_N which gets truncated when constructing the matrix F. The bottom row of the boundary matrix is therfore given as

$$\begin{bmatrix} b_1^{\text{bottom}} \\ b_2^{\text{bottom}} \\ \vdots \\ b_{M-1}^{\text{bottom}} \\ b_M^{\text{bottom}} \end{bmatrix} = \begin{bmatrix} u_N \times V(r_{\text{max}} + \delta_r, t = 0) \\ u_N \times V(r_{\text{max}} + \delta_r, t = \delta_t) \\ \vdots \\ u_N \times V(r_{\text{max}} + \delta_r, t = (M-1)\delta_t) \\ u_N \times V(r_{\text{max}} + \delta_r, t = T) \end{bmatrix}.$$

Lastly V_m and V_{m+1} are the mth and $m+1$th columns of a larger $N \times M$ solution matrix containing the value of the option at all N discretized r_t values and along all M discrete time points as shown here:

$$\begin{array}{c} \\ V^1 \\ V^2 \\ \vdots \\ V^{N-1} \\ V^N \end{array} \left[\begin{array}{cccccccc} V_1 & V_2 & \ldots & V_m & V_{m+1} & \ldots & V_{M-1} & V_M \\ \begin{bmatrix} V_1^1 \\ V_1^2 \\ \vdots \\ V_1^{N-1} \\ V_1^N \end{bmatrix} & \begin{bmatrix} V_2^1 \\ V_2^2 \\ \vdots \\ V_2^{N-1} \\ V_2^N \end{bmatrix} & \ldots & \begin{bmatrix} V_m^1 \\ V_m^2 \\ \vdots \\ V_m^{N-1} \\ V_m^N \end{bmatrix} & \begin{bmatrix} V_{m+1}^1 \\ V_{m+1}^2 \\ \vdots \\ V_{m+1}^{N-1} \\ V_{m+1}^N \end{bmatrix} & \ldots & \begin{bmatrix} V_{M-1}^1 \\ V_{M-1}^2 \\ \vdots \\ V_{M-1}^{N-1} \\ V_{M-1}^N \end{bmatrix} & \begin{bmatrix} V_M^1 \\ V_M^2 \\ \vdots \\ V_M^{N-1} \\ V_M^N \end{bmatrix} \end{array} \right],$$

where $V_m = V_M = V_T$ is the terminal condition of the option and thus the solution throughout time can be found iteratively, given the specification of a suitable terminal and upper and lower boundary conditions. In the option pricing case the terminal condition is the known terminal payoff of the option, and thus prior time values can be solved by iterating through the difference equations from the terminal time to the initial time where $m = M, M-1, M-2, \ldots, 3, 2, 1$ at which point we would have diffused back to our desired answer, the initial price of the option and solved the problem. A bond call has been used in this dissertation as it is a fundamental interest-rate option. It is thus illustrative of the techniques employed in the pricing of a general interest-rate option problem when accounting for the effects of randomly-sized discontinuities at pre-determined times. Furthermore the methods employed in his dissertation are effective at solving any European style contingent claim problem, via the specification of a suitable terminal condition, according to the payoff of the option. And thus the method can be applied to other interest-rate options common in practice such as cap/floors and caplets/floorlets.

3.1.3 Augmenting The Iterative Algorithm To Account For Jumps At Deterministic Times

For the purposes of this dissertation it will be assumed that a stochastic discontinuity or jump in interest rate will only occur at some time point $t = T_i$ located in between the time of the option's inception $t = 0$ and the time of option expiry $t = T$. It will further be assumed that there are no jumps between time point T and the maturity of the bond S as we are soley focused in this dissertation on the pricing of an option with a known jump before its terminal date T (this could be generalised by specifying a more complicated terminal condition that accounts for the jumps remaining in the period from time T up to time S). One could envision this as the idea of having an option and knowing some macroeconomic announcement will occur before it expires and which will cause a jump in interest rates that then would need to be priced in effectively. Starting with the value of an option with underlying bond P, at time t, the result is found mathematically as follows:

$$\begin{aligned}
V(t, r_t) &= \mathbb{E}_t\left[e^{-\int_t^T r_u du} P\right] \\
&= \mathbb{E}_t\left[e^{-\int_t^g r_u du} e^{-\int_g^T r_u du} P\right] \\
&= \mathbb{E}_t\left[\mathbb{E}_g[e^{-\int_t^g r_u du} e^{-\int_g^T r_u du} P]\right] \\
&= \mathbb{E}_t\left[e^{-\int_t^g r_u du} \mathbb{E}_g[e^{-\int_g^T r_u du} P]\right] \\
&= \mathbb{E}_t\left[e^{-\int_t^g r_u du} V(g, r_g)\right],
\end{aligned}$$

where g is some time point between t and T. Setting $t = t^-$, denoting the time just before a jump, and $g = t^+$, denoting the time immediately after a jump, the following can be seen:

$$\begin{aligned}
V(t^-, r_{t^-}) &= \mathbb{E}_{t^-}\left[e^{-\int_{t^-}^{t^+} r_u du} V(t^+, r_{t^+})\right] \\
&= \mathbb{E}_{t^-}\left[V(t^+, r_{t^+})\right] \\
&= \mathbb{E}_{t^-}\left[V(t^+, r_{t^-} + J_t)\right] \\
0 &= \mathbb{E}_{t^-}\left[V(t^+, r_{t^-} + J_t) - V(t^-, r_{t^-})\right],
\end{aligned}$$

which is merely saying that, in taking probabilistic expectations this movement pre-jump and post-jump should be such that the risk-neutral effect is zero. This is because the expected return of the option under $\mathbb{Q}$ must be given by the short rate and this is violated if the jump causes a change in the option value that is not infinitesimally small. As such, the following condition must hold:

$$\mathbb{E}_{\mathbb{Q}}\left[V(t^+, r + J) - V(t^-, r)\right] = 0. \tag{3.13}$$

This means that in the finite difference method the price of the option directly before the jump can be taken as the probability weighted sum of the individual diffusion processes, over all possible diffusions which the current short rate could jump to. This is an expectation calculation and can be done as a matrix multiplication of the option value at the time immediately precceding the jump (worked out in the previous iteration) with a suitable $N \times N$ probability matrix P as is shown here:

$$V_{T_i^+} = P \times V_{T_i^-},$$

where T_i is a jump time and thus $V_{T_i^+}$ is defined as the option price immediately succeeding the instantaneous jump and $V_{T_i^-}$ is the value of the option at the time immediately preceding the jump. Each row in the probability matrix is a distribution relating to the jumps and is composed as follows:

$$P = \begin{bmatrix} \text{distribution}_1 \\ \text{distribution}_2 \\ \vdots \\ \text{distribution}_{N-1} \\ \text{distribution}_N \end{bmatrix} = \begin{bmatrix} p_1^1 & p_1^2 & \dots & p_1^{N-1} & p_1^N \\ p_2^1 & p_2^2 & \dots & p_2^{N-1} & p_2^N \\ \vdots & \vdots & & \vdots & \vdots \\ p_{N-1}^1 & p_{N-1}^2 & \dots & p_{N-1}^{N-1} & p_{N-1}^N \\ p_N^1 & p_N^2 & \dots & p_N^{N-1} & p_N^N \end{bmatrix},$$

where p_i^j represents the probability of a jumping from the ith to the jth short rate value in the ith distribution and the sum of each row equals one, i.e., $\sum_{j=1}^N p_i^j = 1$.

Consequently one must now consider the jump-size probability distribution, that is, the distribution that will be used in these rows and how they will be implemented in the simulation. This dissertation has followed Kim and Wright (2014) and taken the license of assuming normally distributed jumps distributed with a mean of γ and variance of Ω. Hence in the probability matrix used in this procedure, a normal curve is centred at each short rate value and thus the rate can jump up or down to neighbouring r_t values in a Gaussian fashion, as stimulated by the parameters specified.

A very minor problem arises in that, given a sufficiently large standard deviation, r_t values close to r_{min} will have a non-zero probability of jumping to a lower value and thus will effectively jump out of the finite difference mesh. Similarly values close to r_{max} have a non-zero probability of jumping to a higher value and hence out of the specified problem space grid. These non-zero probabilities lie outside the probability matrix and hence a small approximation error occurs when dealing with these extremes cases.

One way to address this specific issue is to interpolate the standard deviation of the jumps at the edges from zero towards the specified standard deviation $\sqrt{\Omega}$ in order to ensure that the probability of jumping out of the solution space is zero. Concretely we will take the first and last h many rows to interpolate over. The first and last rows will contain a very narrow Dirac-delta-like normal curve with zero standard deviation and is centred at the current r_t value. Moving down from rows one to h and up from rows N to $N - h + 1$ the standard deviation will gradually widen until the value is $\sqrt{\Omega}$ which will occur in the hth and hth last row. At that point the short rate is then sufficiently centred in our finite difference grid that the variance Ω does not lead to jumps outside of the grid and hence we are then no longer at the extremes and everything is as it was before. To be clear, this error at the extreme boundary is negligible and the interpolation is not consequential to our results, but the solution is provided as an aside in case valuing at extremes is desired.

This is the approach adopted in this dissertation as an addendum, where interpolation has been applied to the first and last $h = 15$ rows via linear interpolation as described by

$$f(x) = y_1 + (x - x_1)\left(\frac{y_2 - y_1}{x_2 - x_1}\right),$$

where (x_1, y_1) and (x_2, y_2) are the points between which we wish to interpolate which in the case described above are $(x_1 = 1, y_1 = 0)$ and $(x_2 = h, y_2 = \sqrt{\Omega})$ which allows for the creation of an interpolated jump standard deviation vector of $[0, \ldots, \sqrt{\Omega}]$ which can be applied to the first and last fifteen rows in order to further minimize error at extreme edges.

3.1.4 Terminal Condition

The terminal condition is the starting point for the finite difference algorithm and exploits the fact that some state of the system is known. For the interest-rate option pricing problem this known quantity is located at the terminal payoff, as at any point in time the option strike K is known and thus the payoff profile for various bond prices derived by different short rates is known at time T. For the call option examined here this terminal condition is given as

$$\max\Big(P(0, S - T, r_t) - K, 0\Big),$$

where as before T is the option expiry time and S is the maturity of the bond

3.1.5 Boundary Conditions

After starting at the terminal condition the finite difference algorithm moves backward in time and iterates over each value of the option until it reaches the initial price. To ensure the edges of the solution space are correct a lower and upper boundary condition is fixed according to the behaviour of the option at the two spacial variable extremes

For the call option priced in this dissertation, the upper boundary is the condition where the short rate r_t is at a maximum and thus the bond price a minimum hence leading to a zero condition at this boundary due to the call option being deeply out of the money as given by

$$V(r_t = r_{\text{max}+\delta+t}, t) = 0.$$

Conversely the lower boundary is the condition where the interest rate r_t is at a minimum and hence leads to a situation where the bond is at a maximum and thus the option is deeply in the money. The probability of going out of the money then becomes negligible as given by

$$V(r_t = r_{\min - \delta_r}, t) = P(t, S, r_t = r_{\min} - \delta_r) - KP(t, T, r_t = r_{\min} - \delta_r),$$

where T, S, and K are as before.

3.2 Bond Pricing Formula

The results above have assumed a known bond pricing function P, which is now specified. In this dissertation the problem examined revolves around two models, firstly the classical Vasicek model where jumps at known times are explicitly not included and secondly an extended Vasicek model which include the effects of discontinues. Fortunately however, for both the cases being examined this function P is the usual affine one and given as follows:

$$P(t, T, rt) = e^{-A(t,T)r_t + B(t,T)}.$$

However the functions for A and B are not entirely the same. For the classical Vasicek case these functions are well known and given by

$$A^*(t_1, t_2) = \frac{1}{\alpha}\left(1 - e^{-\alpha(t_2 - t_1)}\right),$$

$$B^*(t_1, t_2) = \left(\frac{\sigma^2}{2\alpha^2} - \beta\right)\Big((t_2 - t_1) - A^*(t_1, t_2)\Big) \\ \frac{-\sigma^2}{4\alpha}\left(A^*(t_1, t_2)\right)^2.$$

The environment where jumps at known times are to be considered is more complex than this. As described already the Kim and Wright (2014) framework can be reduced down to a one dimensional model that allow the pricing of bonds to account for discontinuities in the short rate at known times. This dissertation, does just that, and simplifies the framework to the the one dimensional case and thus the functions A and B are given by

$$A(t_1, t_2) = A^* t_1, t_2) + \sum_{t < T_i < T}\Bigg[- \alpha^{-1}\left(I - ^{-\alpha(t_2 - T_i)}\right)\gamma \\ + \frac{1}{2}\alpha^{-1}\left(I - e^{-\alpha(t_2 - T_i)}\right)\Omega\left(I - ^{-\alpha'(t_2 - T_i)}\right)\alpha^{-1\prime}\Bigg], \tag{3.14}$$

$$B(t_1, t_2) = B^*(t_1, t_2). \tag{3.15}$$

This result is taken directly from Kim and Wright (2014) and will hence be implemented to not only accurately price interest-rate options and their boundary and terminal conditions, but also examine the baseline effects that random discontinuities have on this type of problem. A noteworthy comment is that when there are no jumps between $t = 0$ and $t = T$ the second part of the extended model's A function essentially disappears as there are no jumps times T_i to sum over and hence the extended model collapses down into the classical one.

3.3 Benchmark Methods Used In Comparison

3.3.1 Closed Form Solution

For the classical Vasicek model the closed form price for an interest-rate option is well defined. For a call option, the initial price is given by the equation:

$$P(0, S, r_0)\Phi(d_1) - KP(0, T, r_0)\Phi(d_2),$$

where

$$\hat{\sigma} = \sigma\sqrt{(1 - e^{-2\alpha(T-0)})\frac{1}{2\alpha}}A^*(T, S),$$

$$d_1 = \left(\log\left(\frac{P(0, S, r_0)}{P(0, T, r_0)K}\right) + \frac{\hat{\sigma}^2}{2}\right)\frac{1}{\hat{\sigma}},$$

$$d_2 = d_1 - \hat{\sigma},$$

where K denotes the strike price and P is the price of a bond. Φ is the cumulative distribution function for the normal distribution, T is the expiry of the option and S the maturity of the bond underlying the option. Of course this does not apply in the case where jumps are to be included.

3.3.2 Monte Carlo Method

To benchmark the accuracy of the finite difference model in the extended Vasicek model, Monte Carlo simulation will be employed as there is no analytical solution for pricing a bond option when stochastic discontinuities are present between the option inception and option expiry. 500 000 sample paths of the short rate will be simulated and updated at each of the $M - 1$ time steps after the initial time. Additionally the standard deviation for the the Monte Carlo estimate will be calculated and used to infer a confidence interval useful in the assessment of a method's accuracy.

Chapter 4

Results And Discussion

After understanding the necessary theory and developing the specific details for a practical implementation, results could be generated for the pricing of a bond call option where the following parameters were used for the aforementioned models:

Vasicek Parameters
$\alpha = 0.1$ $\beta = 0.1$ $\sigma = 0.01$
Bond Call Option Parameters
$K = 0.9$ $T = 1$ $S = 2$
Jump Parameters
$\gamma = 0$ $\Omega = 0.0001$ $T_i = [0.4, 0.6]$
Finite Difference Parameters
$r_{\text{min}} = -0.2$ $r_{\text{max}} = 0.2$ $M = 300$ $N = 100$
Monte Carlo Parameters
Number of sample paths = 500 000

Tab. 4.1: Table listing all parameters used in the generation of the results of this dissertation.

The following results were obtained via the Matlab code found in Appendix A which was written in correspondence with the application details elucidated in

Chapter 3 using the parameters in Table 4.1.

4.1 Comparison Of Results When No Jumps Are Assumed

The initial implementation focused on attaining results for a classical Vasicek model where the short rate was assumed to be smooth and have no discontinuities. This, although unrealistic, was crucial in ensuring that the code written for both the Monte Carlo and finite difference methods were working correctly and would be able to be used in the more complex setting where the closed form solution would fall away.

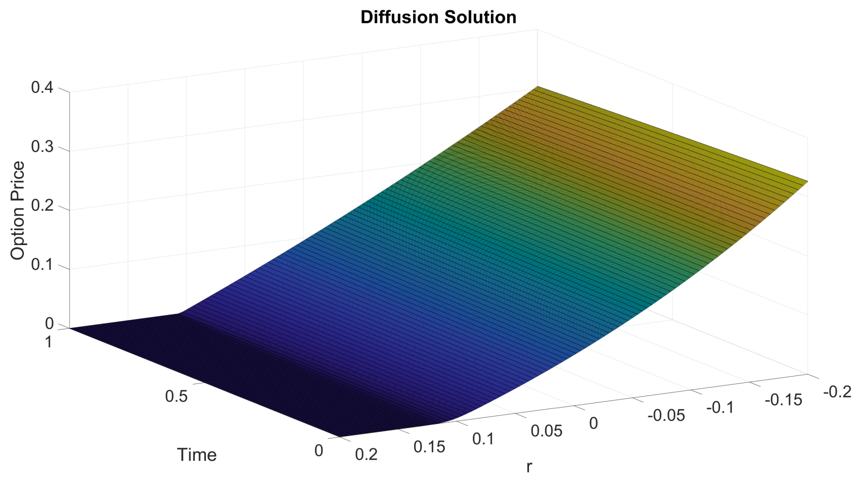

Fig. 4.1: Surface plot of diffusion solution obtained via finite difference methods in the context of a Vasicek short rate model with no jumps.

The bond pricing formula was correctly implemented and the finite difference method was successfully executed in the context of a bond call option. This was plotted exactly as expected and showed the price smoothly diffusing back in time over the range of r_t values and is seen in the surface plot shown in Figure 4.1.

Figure 4.2 shows the initial prices estimates of the finite difference and Monte Carlo methods as well as the value of the option given by the analytical solution and indicate all methods being in agreement.

Following this, the price estimates were compared to the price given by the closed form solution of a bond call. The Monte Carlo method price estimate and finite difference method price estimate were each then subtracted from the analytical solution and this error plotted in Figure 4.3 to assess the precision of the results. As can be seen these results confirm the agreement and accuracy stated earlier, in that the error of both methods in the non-jump setting is in the order of 10^{-5} indicating both methods being of sound accuracy for their desired purpose. Whats more, the finite difference results are well within the error bounds of the Monte Carlo and thus, are well within the precision required for the more complicated environment shown next.

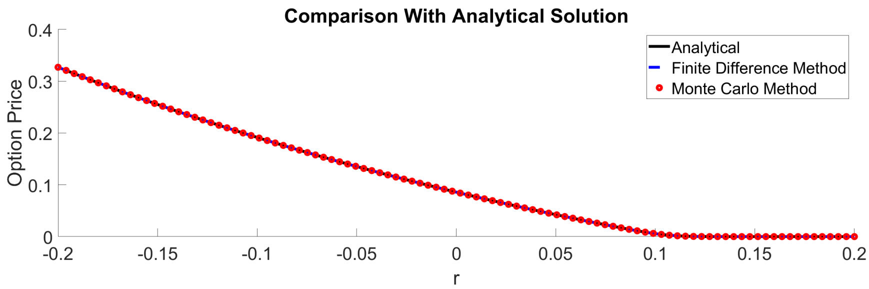

Fig. 4.2: Graph of option price estimates via Monte Carlo and finite difference methods in comparison to the closed form solution in the context of a Vasicek model with no jumps.

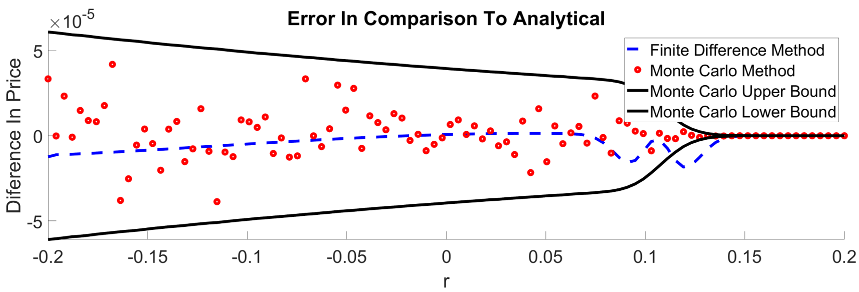

Fig. 4.3: Graph of error in Monte Carlo and finite difference methods in comparison to the closed form solution in the context of a Vasicek model with no jumps.

4.2 Comparison Of Results When Jumps At Deterministic Times Are Included

After the algorithm was seen to be working in the classical model case the extended model was used and the jumps at known times were added. These effects were assessed and any discrepancies from the previous plotted results are put forth now.

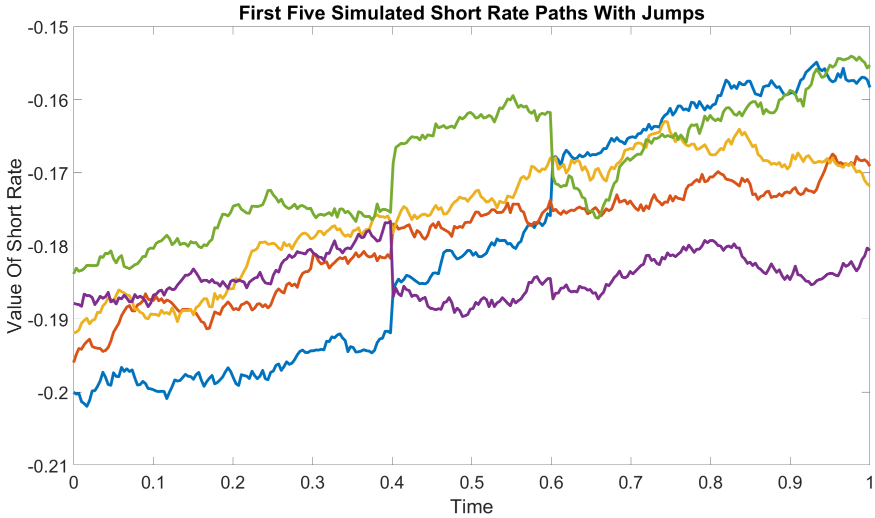

Fig. 4.4: Graph of some of the simulated short rate paths used in the Monte Carlo method each with a random sized jump at the specified known jump times.

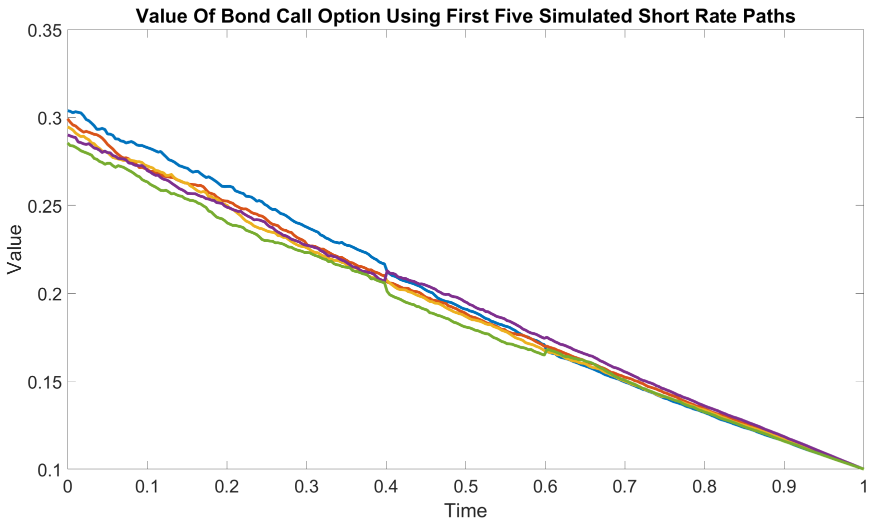

Fig. 4.5: Graph of the value of the bond option through time using the same set of simulated short rate paths.

A plot of the first five simulated short rates used in the Monte Carlo simulation was created and is shown in Figure 4.4. This was used to ensure that the short rate Monte Carlo jump implementation was working correctly as it was needed to benchmark the finite difference method in this setting. As one can see jumps are noted at the time points specified by the jump times vector T_i, as can be seen at times $t = T_1 = 0.4$ and $t = T_2 = 0.6$. Additionally, Figure 4.5 shows the value of the bond through time, using the same set of sample paths graphed in Figure 4.4, and is a useful visualisation of Equation 3.13 previously stated in Chapter 3.

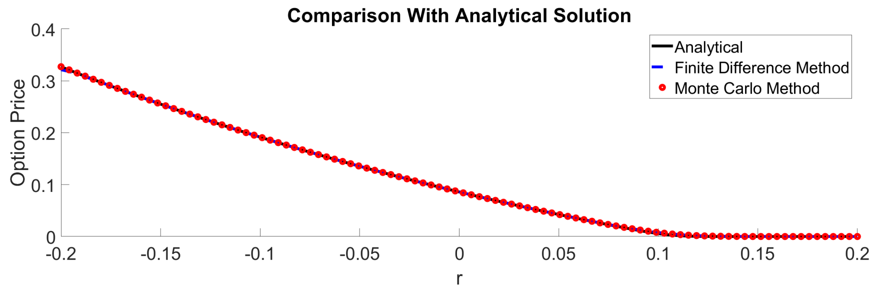

Fig. 4.6: Graph of option price estimates via Monte Carlo and finite difference methods when compared to the closed form solution in the context of the extended Vasicek model, now with the assumption of jumps at deterministic times.

To get a general idea of the the initial price estimates for the call option when deterministic jumps were included, a comparison plot for both Monte Carlo and finite difference methods was generated and is displayed in Figure 4.6. The classical analytical solution was still plotted on the graph to note any changes moving from non-jumps to jumps and seemed to indicate that jumps had minimal effect as all three methods appeared to be in close proximity to one another.

To asses this deeper, the extended model price estimates from the finite difference method and Monte Carlo were plotted and compared to the old closed form solution of the classical model by more technical means, by plotting the deviation from the analytical solution in both methods. This allowed insight to be gleaned into how the inclusion of jumps effect the solution, both methods deviated in the same manner for the majority of the graph and thus a clear difference was seen in the classical and extended cases as shown in Figure 4.7.

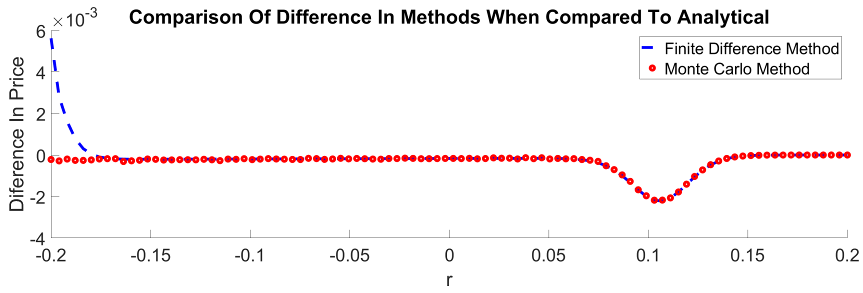

Fig. 4.7: Graph of the deviation in the Monte Carlo and finite difference methods when compared to the closed form solution in the context of the extended Vasicek model, now with the assumption of jumps at deterministic times.

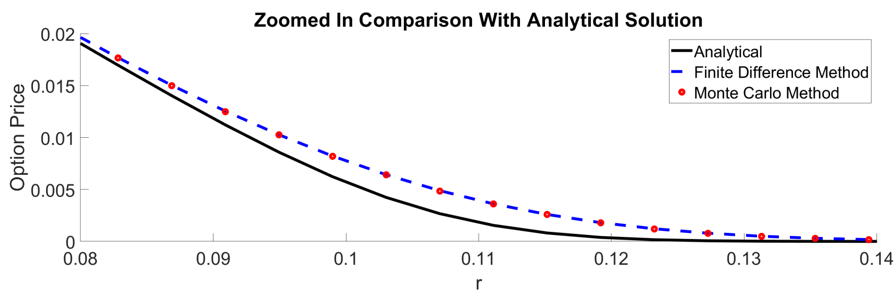

Fig. 4.8: Zoomed-in graph of the effects of deterministic jumps on the price of the option at the area around the at the money price.

The most prominent feature of this investigate was the performance of the estimates around the moneyness boundary of the option. After noting this a zoomed-in version of Figure 4.6 was created to better examine the area around the in the money/out the money region and is shown in Figure 4.8. With jumps there is much more potential for a move into the money which is why the price around this region is seen to change when moving to the model that includes them.

To benchmark the correctness of the finite difference method in the presence of a model that included jumps the previously mentioned Monte Carlo price estimates were subtracted from the finite difference method and the result plotted. This served to analyse the error of the method describe by this dissertation in Section 3.1 and is illustrated in Figure 4.9

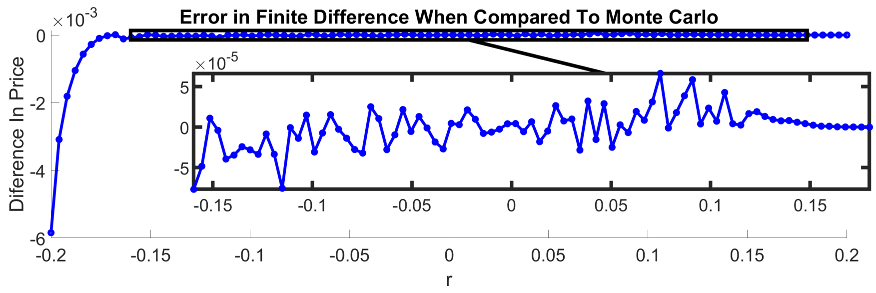

Fig. 4.9: Graph of error in finite difference method in comparison to the Monte Carlo method used as benchmark in the context of the extended Vasicek model, now with the assumption of jumps at deterministic times.

As shown, the finite difference method agreed very closely with the Monte Carlo estimates. This fact is demonstrated in that, again, the magnitude of the error in the plot as shown in the cutaway in Figure 4.9 is in the order of 10^-5. This shows that the implementation has delivered the same level of accuracy as it did in the case were no-jumps were assumed and is thus very effective at pricing in this problem.

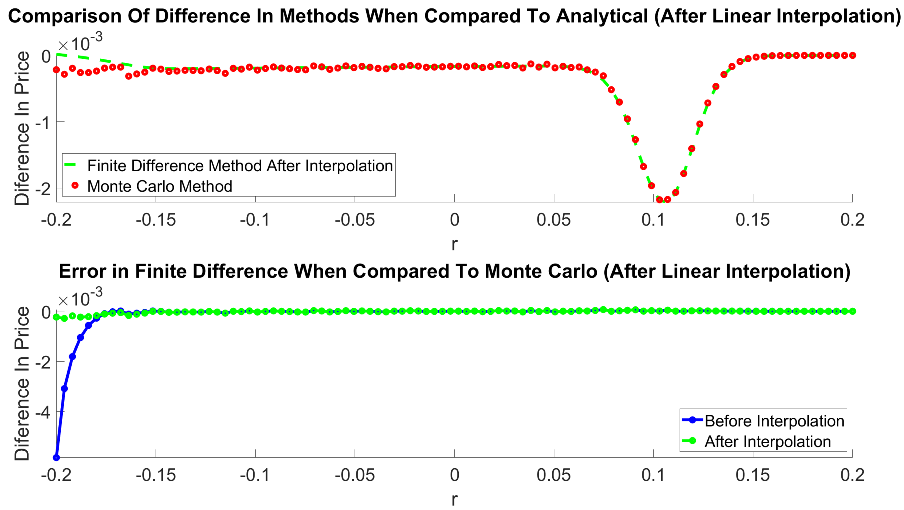

Fig. 4.10: Graph showing the effects of interpolating the probability matrix boundary edges using a linear interpolation of the jumps standard deviation to decrease the approximation error present in the previous results.

Finally the small deviation on the left of Figure 4.9 was looked at and linear interpolation methods were employed as described in Section 3.1.3. This reduced the error as shown in the comparative plots shown in Figure 4.10 but overall was unnecessary as the main method without interpolation worked well on the vast majority of short rate values as is shown in Figure 4.9

Chapter 5

Conclusions And Further Research

5.1 Conclusions

As shown in Chapter 4 the finite difference method implemented was able to effectively solve the PDE specified, given the terminal and boundary condition for the bond call. This means It correctly and smoothly diffused back in time, as was evident in the surface plot in Figure 4.1.

Consequently it was able to accurately price the call option under a classical Vasicek short rate model with no discontinuities. This is evident by the minimal error when comparing it to both Monte Carlo and the closed form solution specified in Section 3.3.1

In addition, it agreed closely to the Monte Carlo results for the case when jumps in the short rate where included in the model displayed in Figure 4.6. And thus can be said to have correctly priced call options in the extended Vasicek setting where the interest rate was allowed to jump.

Figure 4.8 shows an area of interest around the moneyness crossover of the option. From the above graphs in Section 4.2 It can be concluded that the discontinuities at known times cause a change in the option price at the point where the option is very minimally in the money or very minimally out the money. This is because the inclusion of a jump in the model means that a rate which was previous out of the money in the classical case can now jump to a rate that is now in the money and vice versa. This fact has to be accounted for when pricing the option and hence is the cause for the deviation seen in Figure 4.7 around the value $r_t = 0.1$. Furthermore the bigger the standard deviation the more prominent this effect as large standard deviation means that the spread of jumps is large and thus the magnitude of the jumps is big enough to cause rates even more out the money to be able to jump back into the money and vice versa.

A notable point to conclude further was that the deviation in the finite difference method and the Monte Carlo benchmark seen on the left hand side of Figure

4.9 was due to the probability matrix used in the expectation calculation in the jump inclusion stage of the finite difference method. More specifically it occurs due to the approximation made at the probability matrix boundaries and occurs when the probability distribution is truncated so that only some of the distribution is multiplied by each short rate value to obtain the expectation. This was successfully minimized by interpolation methods as shown in Figure 4.10 which show a reduction in the error at this boundary after interpolation and indicate an improvement of at least a factor of ten. Notable, though is that this only affects extreme values at the edge of simulation and is not, in actual fact, needed for the results obtained earlier because by the time we are at the extremes the option is so far in or out the money that the error improvement would be of limited utility.

In conclusion the study was successful in utilising the findings of Kim and Wright (2014) as planned, and a practical implementation was created to accurately price interest-rate options and account for random-sized jumps at deterministic times.

5.2 Further Research

The framework outlined in this dissertation serves as a basis for more complex research and can be extended through multiple means. Verification of the results for more complex options and payoffs can be done to assess error under different circumstances, while also benchmarking efficiency and performance when the option is more computationally intensive. Error at the edge values seen in Figure 4.9 were minimised as described in Section 5.1, but should pricing near extremes be necessary for a particular application, a more specific and completely different problem specification and solution should be devised, that takes into account the very specific nature of the extremes for such purposes. Different derivative approximations with better error could also be employed in the finite difference scheme and should be examined to determine if higher accuracy can be achieved and what tradeoffs exist in regards to this.

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Appendix A

First Appendix

A.1 Matlab Code

The following Matlab code was created to compute the necessary calculations and generate the included plots.

```
%DISSERTATION MATLAB SCRIPT
%TIMOTHY ALLMAN
%-----------------------------------------------------------
%PRICING BOND CALL OPTION USING FINITE DIFFERENCE METHOD
   WITH JUMPS IN THE
%SHORT RATE AT DETERMINISTIC TIMES
%-----------------------------------------------------------
%CLEAR AND CLOSE ALL PREVIOUS OUTPUTS
clc
clear
close all
format long
disp('START OF SCRIPT')

%GLOBAL SETTINGS FOR OUTPUT PLOTS
set(0,'defaultAxesFontSize',25);
set(0, 'DefaultLineLineWidth', 4);
optionname='CALL_'
saveimages=1; %1=SAVE IMAGES 0=DONT SAVE IMAGES

%OPTION PARAMETERS CONSTANTS
K=0.9;   %VALUE CHOSEN TO GIVE HOCKEY STICK BEND IN MIDDLE
T=1;     %OPTION MATURITY
S=2;     %BOND MATURITY

%FINITE DIFFERENCE PARAMETERS
rmin=-0.2;
rmax=0.2;
M=300;
N=100;
```

```
%MESH TIME AND SPACE INCREMENT PARAMETERS
delta_r=(rmax-rmin)/(N-1);
delta_t=T/(M-1);
rvector=rmin:delta_r:rmax;
timevector=0:delta_t:T;

%VASICEK PARAMETERS (dr=alpha(b-r)dt + sig dW)
sig=0.01;  %VOLATILITY
beta=0.1;  %MEAN (ROUGHLY=r0)
alpha=0.1;  %RATE OF MEAN REVERSION

%DETERMINISTIC JUMP DATES AND ADJUSTMENT VARIABLES
%JUMP SIZE PARAMETERS
jumpstddev=0.01;
jumpmean=0;
%JUMP TIME PARAMETERS
jumptimesvector=[0.4,0.6];
jumptimesvectordiscretized=ceil(jumptimesvector./delta_t);

%FOR OUTPUT PLOTS TO CONATIN LABEL IN FILENAME
if jumptimesvector
    fileprefix='JUMP_'
else
    fileprefix='NO_JUMP_'
end

%JUMP DISTRIBUTION PROBABILITY MATRIX POPULATION
rng(1)
probabilitymatrix=zeros(N);

%POPULATE NON INTERPOLATED PROBABILITY MATRIX
for rowindex=1:N
    %JUMP SIZE DISTRIBUTION
    probabilitymatrix(rowindex,:)=normpdf(rvector,jumpmean+
       rvector(rowindex),jumpstddev); %NORMALLY DISTRIBUTED
    probabilitymatrix(rowindex,:)=probabilitymatrix(rowindex
       ,:)./sum(probabilitymatrix(rowindex,:),2); %NORMALISE
end

%INTERPOLATION PARAMETERS
practicallyzero=0.00001;
numbertointerpolate=15;
interpolatedprobabilitymatrix=probabilitymatrix;
interpolationname=' Linear'
x1 = 1;
```

```
y1 = practicallyzero;
x2 = numbertointerpolate;
y2 = jumpstddev;

%LINEAR INTERPOLATION FOR PROBABILITY MATRIX BOUNDARY CASES
dy = y2 - y1;
dx = x2 - x1;
linearinterpolationfunction = @(x) y1+(x-x1).*dy/dx;

%GET INTERPOLATION
interpolatedjumpstddev=linearinterpolationfunction([x1:x2]);

%POPULATE INTERPLOATED PROBABILITY MARTRIX
for rowindex=1:N
    %INTERPOLATE TOP PROBABILITIES
    if rowindex < numbertointerpolate+1
        interpolatedprobabilitymatrix(rowindex,:)=normpdf(rvector,jumpmean+rvector(rowindex),interpolatedjumpstddev(rowindex));
        interpolatedprobabilitymatrix(rowindex,:)=interpolatedprobabilitymatrix(rowindex,:)./sum(interpolatedprobabilitymatrix(rowindex,:),2); %NORMALISE
    end
    %INTERPOLATE BOTTOM PROBABILITES
    if  N-numbertointerpolate<rowindex
        interpolatedprobabilitymatrix(rowindex,:)=normpdf(rvector,jumpmean+rvector(rowindex),interpolatedjumpstddev(N+1-rowindex));
        interpolatedprobabilitymatrix(rowindex,:)=interpolatedprobabilitymatrix(rowindex,:)./sum(interpolatedprobabilitymatrix(rowindex,:),2); %NORMALISE
    end
end


%ANONYMOUS FUNCTIONS
%FUNCTIONS FOR A B FOR CLASSICAL VASIEK MODEL i.e. NO JUMPS
vasicekA=@(t1, t2) 1/alpha*(1 - exp(-alpha*(t2-t1)));
vasicekB=@(t1, t2) ...
    (sig^2/2/alpha^2-beta).*((t2 - t1)...
    - vasicekA(t1, t2))...
    -sig^2/4/alpha*(vasicekA(t1, t2)).^2;

%ADDITIONAL VASICEK EXTENSION PARAMETERS
```

```
p0=0;
p=1;
Omega=jumpstddev.^2; %VARIANCE OF JUMPS
thetaQ=beta;
Ti=jumptimesvectordiscretized*delta_t; %JUMP TIMES IN MODEL
gammaQ=jumpmean;

%FUNCTIONS FOR A B FOR EXTENDING VASIEK MODEL i.e. INCLUDING
    JUMPS AT KNOWN TIMES
extendedvasicekA=@(t1,t2)vasicekA(t1,t2)...
    +sum(...
    -(p').*alpha^(-1).*(eye(1)-exp(-alpha.*(t2-Ti))).*gammaQ
       ...
    +0.5.*(p').*alpha^(-1).*(eye(1)-exp(-alpha.*(t2-Ti)))...
    .*Omega...
    .*(eye(1)-exp(-alpha'.*(t2-Ti))).*(alpha^(-1))'.*p...
    );
extendedvasicekB=@(t1,t2)vasicekB(t1,t2);

%P=@(t,T,rt) exp(-vasicekA(t,T).*rt+vasicekB(t,T));
P=@(t,T,rt) exp(-extendedvasicekA(t,T).*rt+extendedvasicekB(
   t,T));

%INITIAL CONDITION FUNCTION(S)
CONDITIONinit=@(r)max(P(0,S-T,r)-K,0); %TERMINAL CONDTION OF
    OPTION

%BOUNDARY CONDITION FUNCTION(S)
CONDITIONplusinf=@(r,t) zeros(size(t));     %OPTION VALUE AT
    r minus infinity
CONDITIONneginf=@(r,t) P(t,S,r)-K.*P(t,T,r);     %OPTION
   VALUE AT r plus infinity

%SET UP TRIDIAGONAL DIFFERENTIATION MATRICES
diagonalmiddle=delta_t.*(-rvector+1/delta_t-2*sig*sig/2/
   delta_r/delta_r);
diagonalplus=delta_t.*(alpha.*(beta-rvector)/2/delta_r+sig*
   sig/2/delta_r/delta_r);
diagonalminus=delta_t.*(-alpha.*(beta-rvector)/2/delta_r+sig
   *sig/2/delta_r/delta_r);
Fmatrix=diag(diagonalmiddle)+diag(diagonalminus(2:end),-1)+
   diag(diagonalplus(1:end-1),1);

%GET BOUNDARY VALUES
boundarytop=CONDITIONneginf(rmin-delta_r,timevector).*
   diagonalminus(1);
```

```
boundarybottom=CONDITIONplusinf(rmax+delta_r,timevector).*diagonalplus(end);

%INITIALISE SOULTION MATRIX FOR PRICING SURFACE
optionpricesurface=zeros(N,M);

%ADD IN TERMINAL CONITION ON RIGHT MOST COLUMN
optionpricesurface(:,end)=CONDITIONinit(rvector);

%INITIALISE BOUNDARY COLUMN MATRIX TO FIT DIMESIONS OF G AND SURFACE MATRIX
boundarycolumnmatrix=zeros(N,M);
%ADD TOP BOUNDARY CONDITION
boundarycolumnmatrix(1,:)=boundarytop;
%ADD BOTTOM BOUNDARY CONDITION
boundarycolumnmatrix(end,:)=boundarybottom;

%REPLICATE OPTION PRICING SURFACE FOR INTERPOLATED CASE
interpolatedoptionpricesurface=optionpricesurface;

for idx=M:-1:2

    %IF AT JUMP GET EXPECTATION OPTION PRICE VIA PROBABILITY MATRIX AND PLACE AT TIME OF JUMP
    if sum(idx==jumptimesvectordiscretized)
        %NON INTERPOLATED CASE
        ExpectedCallPriceAfterJump=probabilitymatrix*optionpricesurface(:,idx);
        optionpricesurface(:,idx)=ExpectedCallPriceAfterJump;

        %INTERPOLATED CASE
        interpolatedExpectedCallPriceAfterJump=interpolatedprobabilitymatrix*interpolatedoptionpricesurface(:,idx);
        interpolatedoptionpricesurface(:,idx)=interpolatedExpectedCallPriceAfterJump;
    end

    %GET TOP AND BOTTOM VALUES i.e BOUNDARY COLUMN
    boundarycolumn=boundarycolumnmatrix(:,idx);

    %NON INTERPOLATED CASE
    %GET ALREADY CALCULATED VALUE OF SURFACE TO USE IN PRESENT CALCULATION
    optionalreadycalculated=optionpricesurface(:,idx);
```

```
    %CALULATE AND INSERT COLUMN TO THE LEFT OF THE CURENT ONE
    optionpricesurface(:,idx-1)=Fmatrix*optionalreadycalculated+boundarycolumn;

    %INTERPOLATED CASE
    %GET ALREADY CALCULATED VALUE OF SURFACE TO USE IN PRESENT CALCULATION
    interpolatedoptionalreadycalculated=interpolatedoptionpricesurface(:,idx);
    %CALULATE AND INSERT COLUMN TO THE LEFT OF THE CURENT ONE
    interpolatedoptionpricesurface(:,idx-1)=Fmatrix*interpolatedoptionalreadycalculated+boundarycolumn;
end

%PLOT DIFFUSION SOLUTION/OPTION PRICING SURFACE
figure('Position',[0,0,1920,1080])
surf(timevector,rvector,optionpricesurface)
view(-120,28)
ylabel('r')
xlabel('Time')
zlabel('Option Price')
plottitle='Diffusion Solution';
title(plottitle);
filename=strcat('IMG_',fileprefix,optionname,plottitle,'.png');
if saveimages
    saveas(gcf,filename);
end

%ANALYTICAL SOLUTION
%BENCHMARK TO CHECK IF FINITIE DIFFERENCE METHOD WITHOUT JUMPS IS CORRECT
%CLOSED FORM PRICE OF BOND MODEL UNDER VASICEK MODEL
optionfinitedifference=optionpricesurface(:,1)';
interpolatedoptionfinitedifference=interpolatedoptionpricesurface(:,1)';
r0=rvector;

sighat=sig*sqrt((1-exp(-2*alpha*T-0))/2/alpha)*vasicekA(T,S);
d1=(log(P(0,S,r0)./P(0,T,r0)./K)+ sighat^2./2)./sighat;
d2=d1-sighat;
callanalytical=P(0,S,r0).*normcdf(d1)-K.*P(0,T,r0).*normcdf(d2);
```

```
optionanalytical=callanalytical;
differenceinanalyticalandfinitedifference=optionanalytical-
    optionfinitedifference;

%MONTE CARLO SOLUTION
%MONTE CARLO PARAMETERS
numberofpaths=500000;
numberofsteps=M-1;
r_t=repmat(r0,numberofpaths,1);
integral_rt=zeros(numberofpaths,1);

%SET UP LOADING SCREEN FOR LARGE PATH VALUES
loadingbar=waitbar(0,'Monte Carlo Running (incremeting r_t)
    Please Wait...');

%SET UP MATRIX TO STORE AN INSTANCE OF THE MONTE CARLO
    SIMULATION
shortratehistory=repmat(r0,numberofsteps+1,1);

%MONTE CARLO LOOP
for step=1:numberofsteps
    %GET RANDOM NUMBER FOR INCREMENT
    z1=randn(size(r_t));

    %STORE OLD SHORT RATE VALUES TO BE UPDATED
    oldr_t=r_t;
    incrementalr_t=alpha.*(beta-oldr_t).*delta_t+sqrt(delta_t
        ).*sig.*z1;

    %UPDATE THE OLD SHORT RATE VALUES TO NEW VALUES
    r_t=oldr_t+incrementalr_t;

    %IF AT JUMP TIME ADD JUMP TO SHORT RATE
    if sum(step==jumptimesvectordiscretized)
        disp(step)
        jump=jumpmean+jumpstddev.*randn(size(r_t));
        r_t=r_t+jump;
    end

    %ADD TO HISTORY MATRIX FOR VERIFICATION/VISUALISATION
    shortratehistory(step+1,:)=r_t(1,:);

    %INTEGRAL USING TRAPEZOIDAL RULE FOR DISCOUNTING
    integral_rt=integral_rt+(r_t+oldr_t)*delta_t/2;
    waitbar(step /numberofsteps)
end
```

```
close(loadingbar)

%GET ALL DISCOUNT FACTORS
DiscountFactors=exp(-integral_rt);

%GET n-MANY MONTE CARLO INSTANCES OF BOND PRICE
MonteCarloBondpriceTS=P(T,S,r_t);  %exp(-A(T,S).*rT+B(T,S));

%AVERGAE THE n-MANY PAYOFFS AND DISCOUNT BY YIELD TO GET
   MONTE CARLO ESTIMATE
optionmontecarlo=mean(DiscountFactors.*max(
   MonteCarloBondpriceTS-K,0));
%GET THE STANDARD DEVIATION TO PLOT BOUNDS
optionmontecarlostddev=std(DiscountFactors.*max(
   MonteCarloBondpriceTS-K,0))./sqrt(numberofpaths);


%PLOTS
figure('Position',[0,0,1920,1080/2])%PRICE COMPARISON PLOT
hold on
plot(rvector,optionanalytical,'k-','DisplayName','Analytical
   ')
plot(rvector,optionfinitedifference,'b--','DisplayName','
   Finite Difference Method')
plot(rvector,optionmontecarlo,'or','DisplayName','Monte
   Carlo Method')
ylabel('Option Price')
xlabel({'r'})
legend
plottitle='Comparison With Analytical Solution';
title(plottitle);
filename=strcat('IMG_',fileprefix,optionname,plottitle,'.png
   ');
if saveimages
    saveas(gcf,filename);
end

if jumptimesvector
    figure('Position',[0,0,1920,1080/2])%PRICE COMPARISON
       PLOT
    hold on
    plot(rvector,optionanalytical,'k-','DisplayName','
       Analytical')
    plot(rvector,optionfinitedifference,'b--','DisplayName',
       'Finite Difference Method')
    plot(rvector,optionmontecarlo,'or','DisplayName','Monte
```

```
        Carlo Method')
    ylabel('Option Price')
    xlabel({'r'})
    xlim([0.08 0.14])
    legend
    plottitle='Zoomed In Comparison With Analytical Solution';
    title(plottitle);
    filename=strcat('IMG_',fileprefix,optionname,plottitle,'.png');
    if saveimages
        saveas(gcf,filename);
    end
end

figure('Position',[0,0,1920,1080/2])%ANALYTICAL VS OTHER METHODS PLOT
hold on
plot(rvector,optionanalytical-optionfinitedifference,'b--','DisplayName','Finite Difference Method')
plot(rvector,optionanalytical-optionmontecarlo,'or','DisplayName','Monte Carlo Method')

ylabel('Diference In Price')
xlabel('r')
legend
if jumptimesvector
    plottitle='Comparison Of Difference In Methods When Compared To Analytical';
else
    plottitle='Error In Comparison To Analytical';
    plot(rvector,3.*optionmontecarlostddev,'-k','DisplayName','Monte Carlo Upper Bound')
    plot(rvector,-3.*optionmontecarlostddev,'-k','DisplayName','Monte Carlo Lower Bound')
end
title(plottitle);
filename=strcat('IMG_',fileprefix,optionname,plottitle,'.png');
if saveimages
    saveas(gcf,filename);
end
hold off

figure('Position',[0,0,1920,1080/2])%FINITE DIFFERENCE - MONTE CARLO PLOT
```

```
hold on
plot(rvector,optionfinitedifference-optionmontecarlo,'b-o','
    DisplayName','Finite Diff option minus Monte Carlo option
    ')
ylabel('Diference In Price')
xlabel('r')
%legend
plottitle='Error in Finite Difference When Compared To Monte
     Carlo';
title(plottitle);
%DRAW ZOOMED IN BOX FOR JUMP SEETING TO SHOW ERROR IS THE
    SAME
if jumptimesvector
    %BOX AROUND AREA OF INTEREST IN UNZOOMED PLOT
    offset=15*delta_r;
    boxxlower=-0.16;
    boxxupper=0.18;
    boxwidth=boxxupper-boxxlower;
    boxindex = (boxxlower<rvector)&(rvector<boxxupper);
    padding=min(diff(optionfinitedifference(boxindex)-
        optionmontecarlo(boxindex)));
    boxmax=max(optionfinitedifference(boxindex)-
        optionmontecarlo(boxindex))+abs(padding);
    boxmin=min(optionfinitedifference(boxindex)-
        optionmontecarlo(boxindex))-abs(padding);
    boxheight=boxmax-boxmin;
    ymax=max(optionfinitedifference-optionmontecarlo);
    ymin=min(optionfinitedifference-optionmontecarlo);
    rectangle('Position',[boxxlower boxmin boxwidth
        boxheight],'Linewidth',5);
    plot([boxxlower+boxwidth/2 max(rvector)],[boxmin (ymax+
        ymin)/2],'k','Linewidth',5)

    %ZOOMED IN PLOT
    graphright=0.905;
    graphleft=0.131;
    graphwidth=graphright-graphleft;
    zoomleft=(boxxlower-rmin)/(rmax-rmin);
    zoomright=(boxxupper-rmin)/(rmax-rmin);
    zoombottom=0.35;
    zoomwidth=(zoomright-zoomleft)*graphwidth;
    zoomheight=0.4;
    axes('Position',[graphleft+offset+zoomleft*graphwidth
        zoombottom zoomwidth zoomheight],'Linewidth',5);
    box on %PUT BOX AROUND AXES
    hold on
```

```
    plot(rvector(boxindex),optionfinitedifference(boxindex)-
       optionmontecarlo(boxindex),'b-o') %PLOT ON ZOOMED
       AXES
    axis tight
end

filename=strcat('IMG_',fileprefix,optionname,plottitle,'.png
   ');
if saveimages
    saveas(gcf,filename);
end
hold off

%SHORTRATE PATH PLOT
figure('Position',[0,0,1920,1080])
plot(timevector,shortratehistory(:,1:5)')
ylabel('Value Of Short Rate')
xlabel('Time')
%legend
plottitle='First Five Simulated Short Rate Paths With Jumps'
   ;
title(plottitle);
filename=strcat('IMG_',fileprefix,optionname,plottitle,'.png
   ');
if saveimages
    saveas(gcf,filename);
end

%VALUE OF BOND OPTION PLOT (USING SHORTRATE PATH PLOT VALUES
   )
valueofbond=P(timevector,T,shortratehistory(:,1:5)')
valueofbondoption=max(valueofbond-K,0)
figure('Position',[0,0,1920,1080])
plot(timevector,valueofbondoption)
ylabel('Value')
xlabel('Time')
%legend
plottitle='Value Of Bond Call Option Using First Five
   Simulated Short Rate Paths';
title(plottitle);
filename=strcat('IMG_',fileprefix,optionname,plottitle,'.png
   ');
if saveimages
    saveas(gcf,filename);
end

```

```
%INTERPOLATED PLOTS
figure('Position',[0,0,1920,1080])%ANALYTICAL VS OTHER
    METHODS PLOT
subplot(2,1,1)
hold on
plot(rvector,optionanalytical-
    interpolatedoptionfinitedifference,'g--','DisplayName','
    Finite Difference Method After Interpolation')
plot(rvector,optionanalytical-optionmontecarlo,'or','
    DisplayName','Monte Carlo Method')
plottitle=strcat('Comparison Of Difference In Methods When
    Compared To Analytical (After',interpolationname,'
    Interpolation)');
title({plottitle,' '});
legend('Location','southwest')
ylabel('Diference In Price')
xlabel('r')

subplot(2,1,2)%ERROR PLOT
hold on
plot(rvector,optionfinitedifference-optionmontecarlo,'b-o','
    DisplayName','Before Interpolation')
plot(rvector,interpolatedoptionfinitedifference-
    optionmontecarlo,'g--o','DisplayName','After
    Interpolation')
plottitle=strcat('Error in Finite Difference When Compared
    To Monte Carlo (After',interpolationname,' Interpolation)
    ');
title({plottitle,''});
legend('Location','southeast')
ylabel('Diference In Price')
xlabel('r')

filename=strcat('IMG_',fileprefix,optionname,
    interpolationname,plottitle,'.png');
if saveimages
     saveas(gcf,filename);
end
```