UNIVERSITY OF CAPE TOWN

Department of Mathematics

(STRONGLY) ZERO-DIMENSIONAL ORDERED SPACES

by

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INTRODUCTION AND SUMMARY

Introduction

The relationship between transitive uniform spaces and zero-dimensional topological spaces was first established by Banaschewski [1957], and was later investigated by Levine [1969]. The theory of transitive quasi-uniform spaces is treated in [Fletcher and Lindgren 1972], [Brümmer 1984] and [Künzi 1990, 1992a, 1992b,1993]; a convenient presentation for our purpose is to be found in [Fletcher and Lindgren 1982]. After Reilly [1972] introduced the notion of zero-dimensionality in bitopological spaces, Bîrsan [1974] and Halpin [1974] studied the relationship between transitive quasi-uniform spaces and zero-dimensional bitopological spaces.

In this thesis we define a notion of zero-dimensionality in ordered topological spaces and examine the relationship between transitive quasi-uniform spaces and zero-dimensional ordered topological spaces. To a large extent , our presentation is influenced by the situation in bitopological spaces (cf. [Halpin 1974] and [Bîrsan 1974]), and uses the commutative diagrams which occur in [Schauerte 1988] and [Brümmer 1977, 1982]. We also study strongly zero-dimensional ordered topological spaces and their relation with functorial quasi-uniformities. In this respect, our results are influenced by those of [Fora 1984], [Banaschewski and Brümmer 1990] and [Künzi 1990] for strongly zero-dimensional bitopological spaces.

Summary

Chapter 1. This chapter introduces the basic concepts needed in the body of the thesis.

Chapter 2. The first section of this chapter contains results about the relations between quasi-uniform spaces, completely regular bispaces and completely regular ordered spaces, mostly following the development by Schauerte [1988]. Building on the work of McCallion [1972], Lane [1965] and Salbany [1984], we describe the action of the functor "L : **CTopOrd** $\longrightarrow$ **Cr2Top**" on objects.

Chapter 3. In this chapter we define, categorically, the notion of a zero-dimensional ordered space and establish its internal characterizations. We further characterize zero-dimensional ordered spaces as those ordered spaces which admit transitive quasi-uniformities. We establish the restriction of the functor diagrams which occur in [Brümmer 1982] and [Schauerte 1988] from (quasi-) uniform spaces and complete regularity to transitive (quasi-) uniform spaces and zero-dimensionality. We use the functor "M : **Cr2Top** $\longrightarrow$ **CTopOrd**" which is found in [Salbany 1984] and [Schauerte 1988] to study the notion of (total) order-disconnectedness and its relation to zero-dimensionality in the realm of compactness.

Chapter 4. In this chapter we give several characterizations of a strongly zero-dimensional ordered space and establish that the coarsest functorial quasi-uniformity on **CTopOrd** is transitive precisely on the strongly zero-dimensional ordered spaces. These results are in analogy with those proved in [Fora 1984] and [Banaschewski and Brümmer 1990] for bitopological spaces.

At the end of each chapter we give notes of remarks and acknowledgements.

CHAPTER 1

Notations, terminology and preliminaries

1.1 Notations and terminology

The notations and terminology which we use come from [Brümmer 1977, 1979, 1982], [McCallion 1972] and [Schauerte 1988]. We use the subscripts "t" and "z" for the restrictions to transitivity and zero-dimensionality. We change the notation for the bitopological space $(\mathbf{R}, l, u)$ from ([Brümmer 1979], [Salbany 1984],[Schauerte 1988] and others) to $(\mathbf{R}, i, d)$ for reasons given in Remark 1.2.5 of the preliminaries. We apply the words *increasing (decreasing)* to a set and *order preserving (reversing)* to a function. Following [Banaschewski, Brümmer and Hardie 1983] we apply terms such as *(completely) regular, zero-dimensional, compact* to bitopological spaces without the traditional qualification "pairwise", because this word is redundant in the given context. We follow [Schauerte 1988], and deviate from the important sources [Nachbin 1965] and [Fletcher and Lindgren 1982], when defining *ordered spaces* and *complete regularity* in such spaces. In fact, Schauerte [1988, 1993+] points out how much terminological disagreement there is in the literature on ordered spaces. For emphasis we retain the term "*order-disconnected*" from [Priestley 1970] and [Burgess and McCartan 1970].

1.2 Preliminaries

Ordered spaces. An *ordered topological space (or ordered space)* is a triple $(X, \tau, \leq)$, where X is a set, τ a topology on X and $\leq$ is a preorder (i.e. a reflexive and transitive relation) with a closed graph in the product topology on $X \times X$. If $\leq$ is antisymmetric then we call $(X, \tau, \leq)$ a *partially ordered space* . Nachbin [1965] showed that each partially ordered space has Hausdorff topology; the converse is not true. Given ordered spaces $(X, \tau, \leq)$ and $(X', \tau', \leq')$, we say that a function $f : X \longrightarrow X'$ is *order preserving (reversing)* if for all $x, y \in X$, $x \leq y \Longrightarrow f(x) \leq f(y)$ $(f(y) \leq f(x))$ and is continuous if $f : (X, \tau) \longrightarrow (X', \tau')$ is *continuous.* A subset B of X is said to be *increasing (decreasing)* if for all $x, y \in X$, $x \leq y$ and $x \in B \Longrightarrow y \in B$ (for all $x, y \in X$, $x \leq y$ and $y \in B \Longrightarrow x \in B$). We say that $(X, \tau, \leq)$ is *compact* if τ is compact. We use $i(x)$ and $d(x)$ to denote the sets $\{y \in X \mid x \leq y\}$ and $\{y \in X \mid y \leq x\}$. For every ordered space $(X, \tau, \leq)$ there are two natural topologies $\mathcal{U}_\tau$ and $\mathcal{L}_\tau$ given by $\mathcal{U}_\tau = \{U \in \tau \mid U \text{ is increasing }\}$ and $\mathcal{L}_\tau = \{L \in \tau \mid L \text{ is decreasing }\}$. It is easy to check that $\mathcal{U}_\tau$ and $\mathcal{L}_\tau$ are indeed topologies on X. We call the topologies $\mathcal{U}_\tau$ and $\mathcal{L}_\tau$ on X the *upper topology* and the *lower topology* respectively. If τ is a topology on X then the order $\leq_\tau$ defined by $x \leq_\tau y \Longleftrightarrow x \in cl_\tau\{y\}$ is called the *specialization order* with respect to τ. If $A \subseteq X$ and τ is a topology on X then τ_A denotes the subspace topology. An ordered space $(X, \tau, \leq)$ is said to be *convex* if its topology τ has as subbase the family $\mathcal{U}_\tau \cup \mathcal{L}_\tau$. If τ and η are topologies on X we shall write $\tau \leq \eta$ to mean that τ is coarser than η. We sometimes use X for $(X, \tau, \leq)$ if it is clear that we refer to an ordered space.

We will denote the real line by $\mathbf{R}$ and we will denote the usual order on $\mathbf{R}$ by $\leq^\mu$ and the usual topology on $\mathbf{R}$ by μ. For a real-valued order preserving function f on an ordered space X, we write $0 \leq f \leq 1$ to mean $f[X] \subseteq [0, 1]$.

1.2.1 Example

We will henceforth denote the ordered space ($\mathbf{R}$, usual topology, usual order) by $\mathbf{R}_0$. For a subset $\mathbf{A}$ of $\mathbf{R}$, $\mathbf{A}_0$ denotes an ordered subspace of $\mathbf{R}_0$, e.g. $\mathbf{I}_0 =$ ($[0,1]$, usual topology, usual order), $\mathbf{D}_0 =$ ($\{0,1\}$, discrete topology, usual order).

1.2.2 Proposition [Nachbin 1965]

Let $(X,\tau,\leq)$ be an ordered space. Then

(a) For all $x,y \in X$ such that $x \not\geq y$, there exists an increasing neighbourhood U of y and a decreasing neighbourhood L of x which are disjoint.

(b) The sets $i(x)$ and $d(x)$ are closed, for all $x \in X$.

1.2.3 Corollary

Let $(X,\tau,\leq)$ be an ordered space, $x,y \in X$ with $y \not\geq x$. Then there is a set $U \in \mathcal{U}_\tau$ such that $x \in U$ and $y \notin U$.

Proof. Let $x,y \in X$, $y \not\geq x$. By the proposition 1.2.2 $d(y)$ is closed. Since $y \not\geq x$, $x \in X - d(y)$, which is τ-open and $\leq$-increasing. Furthemore $y \notin X - d(y)$. Therefore putting $U = X - d(y)$ we have the result. □

1.2.4 Proposition

Let $(X,\tau,\leq)$ be an ordered space. Then $\leq = \leq_{\mathcal{U}_\tau}$.

Proof. Let $x,y \in X$ with $x \not\leq y$. By corollary 1.2.3 there is an increasing open neighbourhood U of x such that $y \notin U$. Thus $x \notin cl_{\mathcal{U}_\tau}\{y\}$, i.e. $x \not\leq_{\mathcal{U}_\tau} y$.

Suppose that $x \not\leq_{\mathcal{U}_\tau} y$. Then there exists $U \in \mathcal{U}_\tau$ such that $x \in U$ and $y \notin U$. Then $x \not\leq y$ since U is increasing with respect to $\leq$. Therefore for all $x,y \in X$ we have $x \leq y \iff x \leq_{\mathcal{U}_\tau} y$. □

Bitopological spaces. A *bitopological space(or bispace)* is a triple (X, τ_1, τ_2), where X is a set, τ_1 and τ_2 are topologies on X. A *bicontinuous function* $f : (X, \tau_1, \tau_2) \longrightarrow (X', \tau_1', \tau_2')$ is a function between the underlying sets such that $f : (X, \tau_1) \longrightarrow (X', \tau_1')$ and $f : (X, \tau_2) \longrightarrow (X', \tau_2')$ are both continuous. We use X to denote a bitopological space when no confusion can arise.

Let i be the topology on $\mathbf{R}$ having as a base $\{(a, \infty) \mid a \in \mathbf{R}\}$ and d be the topology having as a base $\{(-\infty, a) \mid a \in \mathbf{R}\}$. Then we denote the bitopological space $(\mathbf{R}, i, d)$ by $\mathbf{R}_b$. For a subset $\mathbf{A}$ of $\mathbf{R}$, $\mathbf{A}_b$ denotes the subbispace of $\mathbf{R}_b$.

1.2.5 **Remark**

The topologies i and d above were called the "lower topology" and the "upper topology" respectively (see [Brümmer 1979], [Salbany 1984] and [Schauerte 1988]),and were denoted by l and u respectively. However, this can cause confusion since one would expect the upper topology to have a basis of upper (i.e. increasing) sets and a lower topology to have a basis of lower (i.e. decreasing) sets. Therefore we have used the notation and terminology that best agrees with our intuition.

Categories and Functors. For the category $\mathbf{X}$ we denote the class of objects of $\mathbf{X}$ by $Ob\mathbf{X}$, the class of morphisms of $\mathbf{X}$ by $Mor\mathbf{X}$ and the set of morphisms from A to B by $\mathbf{X}(A, B)$. The reader is referred to the list of named categories and functors at the end of this thesis. These concrete categories have (unnamed) forgetful functors to **Set**. A functor $W : \mathbf{X} \longrightarrow \mathbf{Y}$ is called a *concrete* functor if it commutes with the forgetful functors. A source $(f_\alpha : X \longrightarrow X_\alpha)_{\alpha \in J}$ is *initial* if and only if for any object Y in $\mathbf{X}$ and any set map $h : Y \longrightarrow X$ such that $f_\alpha \circ h$ is a morphism in $\mathbf{X}$ for every $\alpha \in J$, h is a morphism in $\mathbf{X}$.

NOTES.

Salbany [1984 p.487] has the same result as proposition 1.2.4, but uses the assumption of monotonic normality, which we drop.

CHAPTER 2

Completely regular ordered spaces

In this chapter we look at the relations between quasi-uniform spaces, completely regular bitopological spaces and completely regular ordered spaces, focusing our attention on some of the results which we will need in the subsequent chapters. The full embedding L : **CTopOrd** $\longrightarrow$ **Cr2Top** is first defined by the spanning construction (i.e. categorically) and afterward characterised in terms of objects (i.e. internally).

2.1 Relations between Quu, Cr2Top and CTopOrd

Basic concepts and results about quasi-uniform spaces are taken from [Fletcher and Lindgren 1982].

2.1.1 Definition

A *quasi-uniformity* on a set X is a filter $\mathcal{U}$ on $X \times X$ with the following properties:
(a) Each member of $\mathcal{U}$ contains $\triangle$ ($\triangle = \{(x,x) \mid x \in X\}$).
(b) If $U \in \mathcal{U}$, then $V \circ V \subseteq U$ for some $V \in \mathcal{U}$ ($V \circ U = \{(x,y) \in X \times X \mid$ for some $z, (x,z) \in U$ and $(z,y) \in V\}$).

The pair $(X,\mathcal{U})$ is called a *quasi-uniform space.* If $\mathcal{U}$ is a quasi-uniformity on X, then $\mathcal{U}^{-1} = \{U^{-1} \mid U \in \mathcal{U}\}$ is also a quasi-uniformity on X. A quasi-uniformity $\mathcal{U}$ is a *uniformity* provided $\mathcal{U} = \mathcal{U}^{-1}$. A subfamily $\mathcal{B}$ of a quasi-uniformity $\mathcal{U}$ is a *base* for

$\mathcal{U}$ if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$. A subset $U \subseteq X \times X$ is said to be *transitive* if $U \circ U \subseteq U$. A function $f : X \longrightarrow Y$ between two quasi-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is *uniformly continuous* if for each $V \in \mathcal{V}$, $(f \times f)^{-1}(V) \in \mathcal{U}$. With these morphisms, the quasi-uniform spaces form a category which we shall denote by **Quu**; its full subcategory of uniform spaces is denoted by **Unif.** For each $U \in \mathcal{U}$ and for each $x \in X$, $U(x) = \{y \in X \mid (x,y) \in U\}$, and for each subset A of X, $U[A] = \bigcup\{U(x) \mid x \in A\}$. The family $\{U(x) \mid U \in \mathcal{U}\}$ is a neighborhood system at x for the topology denoted by $\tau(\mathcal{U})$. We thus have a concrete functor $\overline{T} : \mathbf{Quu} \longrightarrow \mathbf{Cr2Top}$ defined by $\overline{T}(X,\mathcal{U}) = (X, \tau(\mathcal{U}), \tau(\mathcal{U}^{-1}))$ for each $\mathcal{U}$ in **Quu** corresponding to the well known concrete functor $T : \mathbf{Unif} \longrightarrow \mathbf{Creg}$ given by $T(X,\mathcal{U}) = (X, \tau(\mathcal{U}))$ for each $\mathcal{U}$ in **Unif.** We say that a quasi-uniform space $(X,\mathcal{U})$ is *bicomplete* if $(X, \mathcal{U} \vee \mathcal{U}^{-1})$ is a complete uniform space. A quasi-uniform space $(X,\mathcal{U})$ is *totally bounded* if $(X, \mathcal{U} \vee \mathcal{U}^{-1})$ is a totally bounded uniform space. The *increasing quasi-uniformity* on the real line $\mathbf{R}$ is the quasi-uniformity whose basis consists of the sets of the form $\{(x,y) \in \mathbf{R} \times \mathbf{R} \mid x < y + \epsilon\}$, $\epsilon > 0$. The resulting quasi-uniform space will be denoted by $\mathbf{R}_q$ and a subset $\mathbf{A}$ of $\mathbf{R}$ will determine a quasi-uniform subspace $\mathbf{A}_q$ of $\mathbf{R}_q$.

The following functors occur in [Brümmer 1977]:

$\overline{\gamma} : \mathbf{Quu} \longrightarrow \mathbf{Quu}$:- sends a quasi-uniform space to its bicompletion.

$\overline{p} : \mathbf{Quu} \longrightarrow \mathbf{Quu}$:- totally bounded bireflector.

$\gamma : \mathbf{Unif} \longrightarrow \mathbf{Unif}$:- sends a uniform space to its completion.

2.1.2 Definition

A quasi-uniformity $\mathcal{U}$ on a set X is said to be *transitive* if it has a (sub-)base of transitive elements of $\mathcal{U}$.

2.1.3 Definition

A bitopological space (X, τ_1, τ_2) is called *completely regular* iff the source **BiTop**$((X, \tau_1, \tau_2), \mathbf{I}_b)$ is initial. These bispaces together with the bicontinuous maps form the category **Cr2Top**.

2.1.4 Remark

The above definition is equivalent to the following condition (due to Lane [1967]): For each $x \in X$ and τ_1-closed set E with $x \notin E$, there exists a bicontinuous $f : (X, \tau_1, \tau_2) \longrightarrow \mathbf{I}_b$ such that $f(x) = 1$ and $f[E] \subseteq \{0\}$; and for each $x \in X$ and τ_2-closed set F with $x \notin F$, there exists a bicontinuous $g : (X, \tau_1, \tau_2) \longrightarrow \mathbf{I}_b$ such that $g(x) = 0$ and $g[F] \subseteq \{1\}$. The equivalence was observed by Salbany [1970].

2.1.5 Definition

An ordered space $(X, \tau, \leq)$ is a *completely regular ordered space* iff the source **TopOrd**$(X, \mathbf{I}_0)$ is initial.

2.1.6 Remark

The above definition is equivalent (cf. [Schauerte 1988]) to the following two conditions which were introduced by Nachbin [1965]:

(a) If $a \in X$ and U is a neighborhood of a there exist continuous, real-valued functions f and g on X, f order preserving and g order reversing, such that (i) $0 \leq f \leq 1$, $0 \leq g \leq 1$, (ii) $f(a) = 1$, $g(a) = 1$ and (iii) $f(x) \wedge g(x) = 0$ if $x \in X - U$.

(b) If $x, y \in X$ and $x \leq y$ is false, there exists a continuous order preserving real-valued function f on X such that $f(x) > f(y)$.

2.1.7 **Definition** [Schauerte 1988]

(a) **CTopOrd** denotes the full subcategory of completely regular ordered spaces in **TopOrd**.

(b) $V : \mathbf{Quu} \longrightarrow \mathbf{CTopOrd}$ denotes the concrete functor given by $V(X,\mathcal{U}) = (X, \tau(\mathcal{U} \vee \mathcal{U}^{-1}), \bigcap\mathcal{U})$ for all $(X,\mathcal{U}) \in \mathbf{Quu}$.

An ordered space $(X, \tau, \leq)$ is said to *admit* (or be *determined* by) a quasi-uniformity $\mathcal{U}$ if $V(X,\mathcal{U}) = (X, \tau, \leq)$. Nachbin [1965] proved that an ordered space admits some quasi-uniformity iff it is completely regular.

2.1.8 **Proposition** [Nachbin 1948c, 1965], [Fletcher and Lindgren 1982]

A compact partially ordered space admits a unique quasi-uniformity. □

Salbany [1984] studied functors between **Cr2Top** and **CTopOrd**. The following functor from [Schauerte 1988] is a modification of one of the functors originally studied by Salbany [1984].

2.1.9 **Definition**

For each $X \in \mathbf{Cr2Top}$, let MX be the **CTopOrd**-space which is the domain of the initial lift of the source $\mathbf{Cr2Top}(X, \mathbf{I}_b)$ to the codomain $\mathbf{I}_0$ in **CTopOrd**. Let M act on morphisms by preserving set maps. Then $M : \mathbf{Cr2Top} \longrightarrow \mathbf{CTopOrd}$ is a functor.

The following lemma, which was first proved by [Salbany 1970, 1984], is of prime importance in determining how the functor M transforms different concepts (e.g. "total order-disconnectedness") between **Cr2Top** and **CTopOrd**. Schauerte [1988] supplied a direct proof:

2.1.10 Lemma [Schauerte 1988]

Let $(X, \tau_1, \tau_2) \in$ ***Cr2Top***. *Then* $x \in cl_{\tau_1}\{y\} \Longleftrightarrow y \in cl_{\tau_2}\{x\}$.

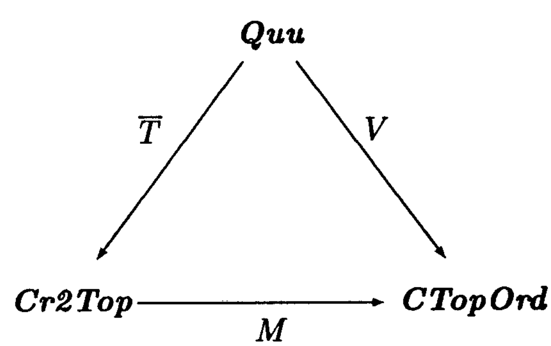
Proof. "$\Rightarrow$" Suppose $x \in cl_{\tau_1}\{y\}$ but $y \notin cl_{\tau_2}\{x\}$. Since (X, τ_1, τ_2) is completely regular there exists a bicontinuous $f : (X, \tau_1, \tau_2) \longrightarrow I_b$ such that $y \in f^{-1}(\{0\})$ and $cl_{\tau_2}\{x\} \subseteq f^{-1}(\{1\})$. Now $f[cl_{\tau_1}\{y\}] \subseteq cl_i\{f(y)\} = cl_i\{0\} = \{0\}$. But $x \in cl_{\tau_1}\{y\}$ implies $f(x) = 0$, which contradicts $f[cl_{\tau_2}\{x\}] = 1$.
"$\Leftarrow$" Similar. $\square$

2.1.11 Proposition [Schauerte 1988]

For $(X, \tau_1, \tau_2) \in$ ***Cr2Top***, $M((X, \tau_1, \tau_2)) = (X, \tau_1 \vee \tau_2, \leq_{\tau_1})$. $\square$

2.1.12 Proposition [Schauerte 1988]

The following diagram commutes:



Proof. Let $(X, \mathcal{U}) \in$ **Quu**. Then $M\overline{T}(X, \mathcal{U}) = M(X, \tau(\mathcal{U}), \tau(\mathcal{U}^{-1})) = (X, \tau(\mathcal{U} \vee \mathcal{U}^{-1}), \leq_{\tau(\mathcal{U})})$. On the other hand $V(X, \mathcal{U}) = (X, \tau(\mathcal{U} \vee \mathcal{U}^{-1}), \bigcap \mathcal{U})$, so we only have to show that $\leq_{\tau(\mathcal{U})} = \bigcap \mathcal{U}$. Suppose that $x \not\leq_{\tau(\mathcal{U})} y$. Then there exists a $\tau(\mathcal{U})$-open set containing x but not y. The collection $\{U(x) \mid U \in \mathcal{U}\}$ forms a neighborhood base at x in $\tau(\mathcal{U})$, so there exists $U_o \in \mathcal{U}$ with $x \in U_o(x)$ but $y \notin U_o(x)$. Thus $(x, y) \notin U_o$

and so $(x,y) \notin \bigcap\mathcal{U}$. Therefore $\bigcap\mathcal{U} \subseteq\ \leq_{\tau(\mathcal{U})}$. If $(x,y) \notin \bigcap\mathcal{U}$ then there exists $U_o \in \mathcal{U}$ such that $(x,y) \notin U_o$, i.e. $y \notin U_o(x)$. $U_o(x)$ is then a $\tau(\mathcal{U})$-neighborhood of x not containing y, so $x \not\leq_{\tau(\mathcal{U})} y$. Hence $\leq_{\tau(\mathcal{U})} = \bigcap\mathcal{U}$. □

2.1.13 **Proposition** [Schauerte 1988]

*(a) The functor M : **Cr2Top** $\longrightarrow$ **CTopOrd** preserves initial sources.*

*(b) The functor V : **Quu** $\longrightarrow$ **CTopOrd** preserves initial sources.*

Proof. (a) Let the source $(f_\alpha : (X,\tau,\kappa) \longrightarrow (X_\alpha,\tau_\alpha,\kappa_\alpha))_{\alpha\in J}$ be initial in **Cr2Top**. Then $(f_\alpha : (X,\tau \vee \kappa) \longrightarrow (X_\alpha, \tau_\alpha \vee \kappa_\alpha))_{\alpha\in J}$ is initial. It remains to show that the order is initial, i.e. $x \leq_\tau y \iff \forall\alpha \in J\ f_\alpha(x) \leq_{\tau_\alpha} f_\alpha(y)$. Clearly $x \leq_\tau y \Rightarrow f_\alpha(x) \leq_{\tau_\alpha} f_\alpha(y)$ since by proposition 2.1.11 each f_α is order preserving. Conversely suppose that $x \not\leq_\tau y$. Then by definition $x \notin cl_\tau\{y\}$. Then there exists $U \in \tau$ such that $x \in U$ but $y \notin U$. Since $(f_\alpha)_J$ is initial for the first topologies, there exist $\alpha_1, ..., \alpha_n \in J$, $U_j \in \tau_{\alpha_j}$ $(j = 1, ..., n)$ such that $x \in \bigcap_{j=1}^n f_j^{-1}[U_j] \subseteq U$. Now since $y \notin U$, there exists $j \in \{1,...,n\}$ such that $y \notin f_{\alpha_j}{}^{-1}[U_j]$ i.e. $f_{\alpha_j}(y) \notin V_j$; whereas clearly $f_{\alpha_j}(x) \in V_j$; so $f_{\alpha_j}(x) \not\leq_{\tau_\alpha} f_{\alpha_j}(y)$; as required.

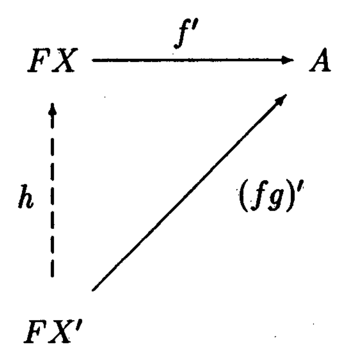
(b) This follows by Proposition 2.1.12 since both M and $\overline{T}$ preserve initial sources. □

Hušek [1967] and Brümmer [1971] introduced the spanning construction for the concrete functors. We give a brief outline of this construction:

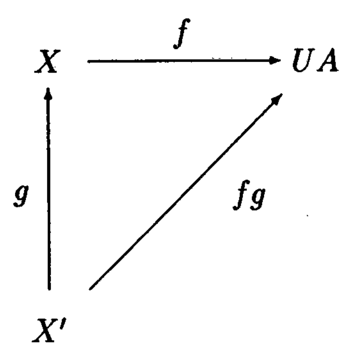
Construction of a section of a given functor. Let **X** and **Y** be categories, U : **Y** $\longrightarrow$ **X** and E : **X** $\longrightarrow$ **Set** be faithful amnestic functors so that $W = EU$ is topological and U preserves initial sources. Then we can define a functor F : **X** $\longrightarrow$ **Y** as follows:

Let $\mathcal{A}$ be a class of objects of $\mathbf{Y}$. For any $X \in Ob\mathbf{X}$ there exists a unique W-initial source $(f' : B \longrightarrow A)$ satisfying $Wf' = Ef$ and $WB = EX$, where f ranges through $\mathbf{X}(X, UA)$ and A through $\mathcal{A}$. Let $FX = B$.

For any fixed morphism $g : X' \longrightarrow X$ in $\mathbf{X}$, consider the following diagrams:



we get the diagram



for all $f \in \mathbf{X}(X, UA)$, $A \in \mathcal{A}$.

The lower diagram commutes for all the given f. The source $(f' : FX \longrightarrow A)$ is the one given by the definition of FX. The source $((fg)' : FX' \longrightarrow A)$ consists of the maps $fg : X' \longrightarrow UA$ regarded as morphisms in $\mathbf{Y}$. They are morphisms in $\mathbf{Y}$ by definition of FX'. Now the W-initiality of $(f' : FX \longrightarrow A)$ gives us a unique morphism $h : FX' \longrightarrow FX$ satisfying $Wh = Eg$. We let $Fg = h$.

It is easily seen that F is a functor; we call it the functor *spanned* by $\mathcal{A}$ and write $F = <\mathcal{A}>_U$.

Schauerte [1988] used this construction for the following functors and categories:

(a) $\mathbf{Y} = \mathbf{Unif}$, $\mathbf{X} = \mathbf{Creg}$, $U = T$.

(b) $\mathbf{Y} = \mathbf{Quu}$, $\mathbf{X} = \mathbf{Cr2Top}$, $U = \overline{T}$.

(c) $\mathbf{Y} = \mathbf{Quu}$, $\mathbf{X} = \mathbf{CTopOrd}$, $U = V$.

(d) $\mathbf{Y} = \mathbf{Cr2Top}$, $\mathbf{X} = \mathbf{CTopOrd}$, $U = M$.

2.1.14 **Definition** [Brümmer 1976]

Let $\mathbf{X}$ and $\mathbf{Y}$ be categories, $U : \mathbf{Y} \longrightarrow \mathbf{X}$ and $E : \mathbf{X} \longrightarrow \mathbf{Set}$ be amnestic faithful functors so that $W = EU$ is topological and U preserves initial sources.

An object X_0 is called a *U-pivot* iff $\{X_0\}$ is E-initially dense in $\mathbf{X}$ and there is exactly one object Y_0 in $\mathbf{Y}$ satisfying $U(Y_0) = X_0$.

2.1.15 **Proposition** [Brümmer 1976]

Let $\mathbf{X}$, $\mathbf{Y}$, U and E be as in definition 2.1.14 . Let $\mathcal{A}$ be a class of objects of $\mathbf{Y}$ and $F = < \mathcal{A} >_U$. Then the following are equivalent:

(a) F is a U-section i.e. $UF = 1$.

(b) $U\mathcal{A}$ is E-initially dense in $\mathbf{X}$.

In case a U-pivot X_o exists, the above are equivalent to:

(c) $UFX_o = X_o$. □

It is well known that the functor $\overline{T} : \mathbf{Quu} \longrightarrow \mathbf{Cr2Top}$ preserves initial sources. So in the light of proposition 2.1.13, (Schauerte[1988]) gave the following definitions:

2.1.16 **Definition**

$\overline{C^*} = < \{\mathbf{I}_q\} >_{\overline{T}}$, $\overline{C} = < \{\mathbf{R}_q\} >_{\overline{T}}$, $\overline{\phi} = < Ob\mathbf{Quu} >_{\overline{T}}$

$C^{*\dagger} = < \{\mathbf{I}_q\} >_V$, $C^{\dagger} = < \{\mathbf{R}_q\} >_V$, $\phi^{\dagger} = < Ob\mathbf{Quu} >_V$, $L = < \{\mathbf{I}_b\} >_M$

2.1.17 **Proposition** [Schauerte 1988]

*(a) The functor L : **CTopOrd** $\longrightarrow$ **Cr2Top** is the unique right inverse of M, and is left adjoint to M.*

*(b) Let $\mathcal{A} \subseteq Ob$**Quu**. Then $< \mathcal{A} >_V$ is a V-section iff $< \mathcal{A} >_{\overline{T}}$ is a $\overline{T}$-section and in this case $< \mathcal{A} >_{\overline{T}} \circ L = < \mathcal{A} >_V$.*

*(c) The functor L is a full embedding of **CTopOrd** into **Cr2Top** as bicoreflective subcategory with coreflection functor LM.* □

2.2 About the embedding of CTopOrd into Cr2Top.

McCallion's definition of "ordered space" ([McCallion 1972]) did not require the order to be closed. We will apply his results to ordered spaces in the sense of 1.2, i.e. with closed order.

2.2.1 **Definition** [McCallion 1972]

Let $(X, \tau, \leq)$ be an ordered space. A subset Z of X is called a *decreasing (increasing) zero set* in $(X, \tau, \leq)$ when there is an order preserving continuous function from X to $\mathbf{R}$ such that $Z = \{x \in X \mid f(x) \leq 0\}$ $(Z = \{x \in X \mid f(x) \geq 0\})$.

In the above definition for an increasing zero set we can replace order preserving by order reversing and have $Z = \{x \in X \mid f(x) \leq 0\}$.

The set of all decreasing (increasing) zero sets will be denoted by $\mathcal{A}_0$ ($\mathcal{B}_0$).

2.2.2 **Definition** [McCallion 1972]

If $(X, \tau, \leq)$ is an ordered space and τ_1, τ_2 are topologies on X then (τ_1, τ_2) is called an *order defining pair* whenever $\tau_1, \tau_2 \subseteq \tau$ and the following statements are equivalent:

(i) $x \in cl_{\tau_1}\{y\}$;

(ii) $x \leq y$;

(iii) $y \in cl_{\tau_2}\{x\}$.

From lemma 2.1.10 and the definition 2.2.2, we have the following proposition:

2.2.3 Proposition

*Let $(X, \tau_1, \tau_2) \in$ **Cr2Top**. Then for the ordered space $(X, \tau_1 \vee \tau_2, \leq_{\tau_1})$, (τ_1, τ_2) is an order defining pair.* □

2.2.4 Lemma

If $f : (X, \tau, \leq) \longrightarrow (X', \tau', \leq')$ is continuous and order preserving then $f^{-1}[\mathcal{A}_0'] \subseteq \mathcal{A}_0$ and $f^{-1}[\mathcal{B}_0'] \subseteq \mathcal{B}_0$ where $\mathcal{A}_0'$ and $\mathcal{B}_0'$ consist of the decreasing (increasing) zero sets in $(X', \tau', \leq')$ respectively.

Proof. Let $A' \in \mathcal{A}_0'$. Then $A' = \{y \in X' \mid g(y) \leq 0\}$ for some continuous order preserving function $g : (X', \tau', \leq') \longrightarrow R_0$. Then

$$\begin{aligned} f^{-1}[A'] &= \{x \in X \mid f(x) \in A'\} \\ &= \{x \in X \mid g(f(x)) \leq 0\} \\ &= \{x \in X \mid gf(x) \leq 0\}. \end{aligned}$$

Therefore $f^{-1}[A'] \in \mathcal{A}_0$ since $gf : (X, \tau, \leq) \longrightarrow R_0$ is continuous and order preserving. Similarly for $B' \in \mathcal{B}_0'$ we have $f^{-1}[B'] \in \mathcal{B}_0$. □

2.2.5 Definition

(a) A set $Z \subseteq X$ in a bitopological space (X, τ_1, τ_2) is called an *upper zero set* of X if there exists a bicontinuous function $f : (X, \tau_1, \tau_2) \longrightarrow (\mathbf{R}, i, d)$ such that $Z = f^{-1}[[0, \infty)]$. The complement of an upper zero set is called a *lower cozero set*.

(b) A set $Z \subseteq X$ in a bitopological space (X, τ_1, τ_2) is called a *lower zero set* of X if there exists a bicontinuous function $f : (X, \tau_1, \tau_2) \longrightarrow (\mathbf{R}, i, d)$ such that $Z = f^{-1}[(-\infty, 0]]$. The complement of a lower zero set is called an *upper cozero set.*

2.2.6 **Remark**

In consequence of our choice of having $\mathbf{R}_b = (\mathbf{R}, i, d)$, our "upper zero sets" are what Lane [1967] in the case of a bispace (X, τ_1, τ_2) called "τ_2-zero sets" and what Fora [1984] called "p-lower zero sets"; Fora was using $(\mathbf{R}, d, i)$ and Lane in effect was using $(\mathbf{R}, i, d)$. The dual remark applies to our "lower zero sets". When citing results from authors who used $(\mathbf{R}, d, i)$ we shall automatically translate their terminology into ours.

2.2.7 **Proposition** [Lane 1967]

A bitopological space (X, τ_1, τ_2) is completely regular if and only if the lower zero sets form a base for the τ_1-closed sets and the upper zero sets form a base for the τ_2-closed sets. □

We now aim to prove an internal description of the full embedding $L : \mathbf{CTopOrd} \longrightarrow \mathbf{Cr2Top}$, which was to be expected from a proposition in [Salbany 1984, p.486]. We first have to prove some intermediate results. Let $(X, \tau, \leq)$ be a completely regular ordered space. Consider the topology $\tau_{\mathcal{A}_0}$ having $\mathcal{A}_0$ as a base for the closed sets and the topology $\tau_{\mathcal{B}_0}$ on X having $\mathcal{B}_0$ as a base for the closed sets.

2.2.8 **Lemma** [McCallion 1972]

If $(X, \tau, \leq)$ is an ordered space and $A \in \mathcal{A}_0$ (respectively $B \in \mathcal{B}_0$) then there is a continuous order preserving (reversing) function g (respectively h) from X into $\boldsymbol{R}$

such that $g \geq 0$ *and* $A = \{x \in X \mid g(x) = 0\}$ *(respectively* $h \geq 0$ *and* $B = \{x \in X \mid h(x) = 0\}$*).*

Proof. We have $A = \{x \in X \mid f(x) \leq 0\}$ and $B = \{x \in X \mid k(x) \leq 0\}$ where f is a continuous order preserving function from X into $\mathbf{R}$ and k is a continuous order reversing function from X into $\mathbf{R}$. By putting $g(x) = f(x) \vee 0$ and $h(x) = k(x) \vee 0$ for all $x \in X$, we have $A = \{x \in X \mid g(x) = 0\}$ and $B = \{x \in X \mid h(x) = 0\}$ with $g \geq 0$ and $h \geq 0$. Clearly g and h are continuous order preserving and continuous order reversing respectively. □

2.2.9 **Proposition** [McCallion 1972]

An ordered space $(X, \tau, \leq)$ *is completely regular if and only if* $(\tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$ *is an order defining pair and* $\tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0} = \tau$.

Proof. Suppose that $(X, \tau, \leq)$ is a completely regular ordered space. Let $x, y \in X$ with $x \leq y$. Clearly $x \in cl_{\tau_{\mathcal{A}_0}}\{y\}$ and $y \in cl_{\tau_{\mathcal{B}_0}}\{x\}$ since each member of $\mathcal{A}_0$ is decreasing and each member of $\mathcal{B}_0$ is increasing. If $x \not\leq y$ there is an order preserving continuous function f from X into $\mathbf{R}$ such that $f(y) < f(x)$. We can assume without loss of generality that $f(y) = 0$ and $f(x) = 1$. Now let $A = \{u \in X \mid f(u) \leq 0\}$. Then $A \in \mathcal{A}_0, y \in A$ and $x \notin A$ and thus $x \notin cl_{\tau_{\mathcal{A}_0}}\{y\}$. By a similar argument $x \leq y$ if and only if $y \in cl_{\tau_{\mathcal{B}_0}}\{x\}$.

If G is τ-open, $x \in G$, then there are two continuous functions $h, g : X \longrightarrow \mathbf{R}$, with h order preserving and g order reversing such that $0 \leq h \leq 1, 0 \leq g \leq 1$, $h(x) = 1 = g(x)$ and $h(y) \wedge g(y) = 0$ if $y \in X - G$. Put $A_1 = \{u \in X \mid h(u) \leq 0\}; B_1 = \{u \in X \mid g(u) \leq 0\}$, then $x \in (X - A_1) \cap (X - B_1) \subseteq G$ and thus $\tau \leq \tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0}$. Since $\tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0} \leq \tau$ we have $\tau = \tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0}$.

On the other hand let $(\tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$ be an order defining pair with $\tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0} = \tau$. If $x, y \in X, x \not\leq y, x \notin cl_{\tau_{\mathcal{A}_0}}\{y\}$ then there exists an $A \in \mathcal{A}_0$ such that $y \in A, x \notin A$.

Since $A \in \mathcal{A}_0$ there is an order preserving function, p, from X into $\mathbf{R}$ such that $p(x) > p(y)$. If $x \in X$ and U is a τ-neighborhood of x, then since $\tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0} = \tau$ there are sets A, B in $\mathcal{A}_0, \mathcal{B}_0$ respectively such that $x \in (X - A) \cap (X - B) \subseteq U$. Now by lemma 2.2.8 there is a continuous order preserving function f from X into R and a continuous order reversing function g from X into R such that $f, g \geq 0$ and $A = \{u \in X \mid f(u) = 0\}, B = \{u \in X \mid g(u) = 0\}$. Now put $h = (f/f(x)) \wedge 1$, $k = (g/g(x)) \wedge 1$, then $h(k)$ is a continuous order preserving(reversing) function from X into R such that $0 \leq h \leq 1, 0 \leq k \leq 1, h(x) = 1 = k(x)$ and $u \in X - U$ implies $h(u) \wedge k(u) = 0$. □

2.2.10 **Theorem**

For $(X, \tau, \leq) \in$ ***CTopOrd***, $L(X, \tau, \leq) = (X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$.

Proof. Let $(X, \tau, \leq) \in$ **CTopOrd**. By definition of L the source $(f : L(X, \tau, \leq) \longrightarrow \mathbf{I}_b \mid f \in \mathbf{CTopOrd}((X, \tau, \leq), \mathbf{I}_o))$ is initial. We now show that $(X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0}) \in$ **Cr2Top**. By proposition 2.2.7 it suffices to show that the decreasing zero sets and the increasing zero sets of $(X, \tau, \leq)$ coincide with the lower and the upper zero sets of $(X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$ respectively. Let A be a decreasing zero set, i.e. $A = f^{-1}[(-\infty, 0]]$ where $f : (X, \tau, \leq) \longrightarrow \mathbf{R}_o$ is continuous and order preserving. We show that $f : (X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0}) \longrightarrow \mathbf{R}_b$ is bicontinuous. Consider a set $(-\infty, a], a \in \mathbf{R}$, i.e. a basic closed set with respect to i. Then $f^{-1}[(-\infty, a]] = (f - a)^{-1}[(-\infty, 0]]$ is a decreasing zero set and thus $\tau_{\mathcal{A}_0}$-closed. Similarly for a basic closed set $[a, \infty)$, with respect to d, $f^{-1}[[a, \infty)]$ is $\tau_{\mathcal{B}_0}$-closed. Hence $f : (X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0}) \longrightarrow \mathbf{R}_b$ is bicontinuous and so A is a lower zero set. Conversely suppose that A is a lower zero set of $(X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$, i.e. $A = f^{-1}[(-\infty, 0]]$ where $f : (X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0}) \longrightarrow \mathbf{R}_b$ is bicontinuous. By applying the functor M, we see from proposition 2.1.11 that

$f : (X, \tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0}, \leq_{\tau_{\mathcal{A}_0}}) \longrightarrow \mathbf{R}_o$ is continuous and order preserving. Moreover by proposition 2.2.9 $(X, \tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0}, \leq_{\tau_{\mathcal{A}_0}}) = (X, \tau, \leq)$. Thus A is a decreasing zero set of $(X, \tau, \leq)$. Similarly the increasing zero sets of $(X, \tau, \leq)$ coincide with the upper zero sets of $(X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$. Therefore $(X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0}) \in$ **Cr2Top** and thus the source **Cr2Top**$((X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0}), \mathbf{I}_b)$ is initial. The above argument shows that **Cr2Top**$((X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0}), \mathbf{I}_b) =$ **CTopOrd**$((X, \tau, \leq), \mathbf{I}_o)$. By initiality of the source $(f : L(X, \tau, \leq) \longrightarrow \mathbf{I}_b \mid f \in$ **CTopOrd**$((X, \tau, \leq), \mathbf{I}_0))$, we have $L(X, \tau, \leq) = (X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$. □

NOTES

(1) The functors M and L come from [Salbany 1984] and were later studied by Schauerte [1988]. The only slight change in Schauerte [1988] was the use of $(\mathbf{R}, i, d)$ instead of $(\mathbf{R}, d, i)$ from Salbany [1984]. This change does not make a significant difference.

(2) The term "order defining pair" was introduced by McCallion [1972]. As a result of lemma 2.1.10 we formulated proposition 2.2.3.

(3) The notions of increasing and decreasing zero sets come from [McCallion 1972], and upper zero sets and lower zero sets were introduced by Lane [1965]. These sets enabled us to describe the functor L in terms of objects, i.e. internally, in theorem 2.2.10. This result does not occur in [Salbany 1984] nor in [Schauerte 1988].

(4) Charlton [1973, Proposition 1.2] proved lemma 2.1.10 for "pairwise" R_0 spaces. He defined a functor from **BiTop** to **TopOrd** that acts on objects in the same way as the functor M [Definition 5.5].

CHAPTER 3

Zero-dimensional ordered spaces

The relationship between zero-dimensionality in **Top** and transitivity in **Unif** was established by Banaschewski [1957]. Reilly [1972] introduced the notion of zero-dimensionality in **BiTop**; Bîrsan [1974] and Halpin [1974] have studied the relationship between the latter notion and transitivity in **Quu**. We will now investigate the relationship between zero-dimensionality in **TopOrd** and transitivity in **Quu**. In this chapter we will also study the notion of total disconnectedness which is closely related to zero-dimensionality.

3.1 Zero-dimensional bitopological spaces and zero-dimensional ordered spaces

3.1.1 **Definition** [Reilly 1972]

For a bitopological space (X, τ_1, τ_2), τ_i *is zero-dimensional with respect to* τ_k if τ_i has a base of τ_k-closed sets, i.e. for each point $x \in X$ and each τ_i-open set A containing x there is a τ_k-closed τ_i-open set G such that $x \in G \subseteq A$, where $i, k = 1, 2$ and $i \neq k$.

3.1.2 **Definition** [Reilly 1972]

Let $(X, \tau_1, \tau_2) \in$ **BiTop**. (X, τ_1, τ_2) is a *zero-dimensional bitopological space* iff τ_1 is zero-dimensional with respect to τ_2 and τ_2 is zero-dimensional with respect to τ_1.

The following two propositions were proved independently by Halpin[1974] and Bîrsan[1974].

3.1.3 **Proposition** [Halpin 1974], [Bîrsan 1974]
A bitopological space (X, τ_1, τ_2) is zero-dimensional iff the source ***BiTop***$((X, \tau_1, \tau_2), \boldsymbol{D}_b)$ *is initial.*

Proof. Suppose (X, τ_1, τ_2) is a zero-dimensional bitopological space. Let A be a basic τ_1-open τ_2-closed set. Now define $f : X \longrightarrow \mathbf{D}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Then $f^{-1}(\{1\}) = A \in \tau_1$ and $f^{-1}(\{0\}) = X - A \in \tau_2$. Hence f is bicontinuous. Thus A is the preimage of $\{1\}$ under a bicontinuous map. Similarly we can show that if B is a basic τ_2-open τ_1-closed set then B is the preimage of $\{0\}$ under a bicontinuous map. Therefore the source $\mathbf{BiTop}((X, \tau_1, \tau_2), \mathbf{D}_b)$ is initial.
Conversely suppose that the source $\mathbf{BiTop}((X, \tau_1, \tau_2), \mathbf{D}_b)$ is initial. Then the subbasic τ_2 open sets are of the form $f^{-1}(\{0\})$ where f is a bicontinuous map. Now $X - f^{-1}(\{0\}) = f^{-1}(\{1\})$ which is τ_1-open. Thus $f^{-1}(\{0\})$ is τ_1-closed and τ_2 has a subbase of τ_2-open τ_1-closed sets. Similarly τ_1 has a subbase of τ_1-open τ_2-closed sets. Hence (X, τ_1, τ_2) is a zero-dimensional bitopological space. □

3.1.4 **Proposition** [Halpin 1974], [Bîrsan 1974].
A bitopological space is zero-dimensional if and only if it admits a transitive quasi-uniformity.

Proof. Suppose (X, τ_1, τ_2) admits a transitive quasi-uniformity $\mathcal{U}$. Let $\mathcal{V}$ be a transitive base. For each $V \in \mathcal{V}$ and for each $x \in X$, $V(x)$ is $\tau(\mathcal{U})$-open $\tau(\mathcal{U}^{-1})$-closed, and hence the family $\{V(x) \mid V \in \mathcal{V}, x \in X\}$ is a base for $\tau(\mathcal{U})$ consisting of $\tau(\mathcal{U}^{-1})$-closed sets. Similarly the family $\{V^{-1}(x) \mid V \in \mathcal{V}, x \in X\}$ is a base for $\tau(\mathcal{U}^{-1})$ consisting of $\tau(\mathcal{U})$-closed sets.

Conversely, suppose that (X, τ_1, τ_2) is zero-dimensional. It is easy to show that τ_1 has a base $\mathcal{B}$ of τ_2-closed sets such that the family $\mathcal{B}' = \{B' \mid X - B' \in \mathcal{B}\}$ is a base for τ_2 consisting of τ_1-closed sets. Let $\mathcal{U}$ be a quasi-uniformity with the subbase $\mathcal{S} = \{S_B \mid B \in \mathcal{B}\}$, where $S_B = B \times B \cup (X - B \times X)$. It is easy to show that $\mathcal{S}$ is a transitive subbase, and thus $\mathcal{U}$ is transitive. Now consider the bitopological space $(X, \tau(\mathcal{U}), \tau(\mathcal{U}^{-1})$. Clearly, $\tau(\mathcal{U})$ has a base $\mathcal{B}$ and $\tau(\mathcal{U}^{-1})$ has a base $\mathcal{B}'$, and thus $\tau(\mathcal{U}) = \tau_1$ and $\tau(\mathcal{U}^{-1}) = \tau_2$. □

Proposition 3.1.3 motivates the following definition.

3.1.5 Definition

An ordered space $(X, \tau, \leq)$ is a *zero-dimensional ordered space* iff the source $\mathbf{TopOrd}((X, \tau, \leq), \mathbf{D}_0)$ is initial.

3.1.6 Proposition

*The functor M : **Cr2Top** $\longrightarrow$ **CTopOrd** sends zero-dimensional bispaces to zero-dimensional ordered spaces.*

Proof. Let X be a zero-dimensional bitopological space. Then the source $\mathbf{Cr2Top}(X, \mathbf{D}_b)$ is initial. Applying the functor M : **Cr2Top** $\longrightarrow$ **CTopOrd** we have that the source $\mathbf{CTopOrd}(MX, \mathbf{D}_0)$ is initial, since the functor M preserves initial sources by proposition 2.1.13. Thus MX is zero-dimensional. □

3.1.7 Theorem

An ordered space $(X, \tau, \leq)$ is zero-dimensional if and only if τ has a subbase consisting of clopen increasing and clopen decreasing sets and $\leq = \leq_{\mathcal{U}'}$ where $\mathcal{U}'$ is the topology generated by the clopen increasing sets, i.e. $\leq$ is the specialization order with respect to $\mathcal{U}'$.

Proof. "$\Rightarrow$": Suppose that $(X, \tau, \leq)$ is a zero-dimensional ordered space. Then the source $\mathcal{A} = \mathbf{CTopOrd}((X, \tau, \leq), D_0)$ is initial. Thus τ has a subbase $\mathcal{S} = \{f^{-1}[V] \mid V \text{ open in } D_0, f \in \mathcal{A}\}$. It is clear that $\mathcal{S}$ consists of clopen increasing and clopen decreasing sets. Let $\mathcal{U}'$ be the topology which has the subbase (in fact a base since the finite intersection of clopen increasing sets is again clopen increasing) $\{f^{-1}(\{1\}) \mid f \in \mathcal{A}\}$. Clearly every clopen increasing set belongs to this family. We now show that $\leq \; = \; \leq_{\mathcal{U}'}$, where the order $\leq$ is given by the initiality of the source, i.e. $\forall x, y \in X, x \leq y \Leftrightarrow \forall f \in \mathcal{A}, f(x) \leq^{\mu} f(y)$. Let $x, y \in X$. Now if $x \not\leq y$ then there exists f in $\mathcal{A}$ such that $f(x) \not\leq^{\mu} f(y)$, i.e. $f(x) = 1$ and $f(y) = 0$. Since $x \in f^{-1}(\{1\})$ and $y \in f^{-1}(\{0\})$, $x \notin cl_{\mathcal{U}'}\{y\}$. Therefore $x \not\leq_{\mathcal{U}'} y$. Conversely suppose that $x \not\leq_{\mathcal{U}'} y$. Then $x \notin cl_{\mathcal{U}'}\{y\}$. Thus there exists a $U \in \mathcal{U}'$ such that $x \in U$ and $y \notin U$. Therefore $x \not\leq y$ since U is $\leq$-increasing. Hence $\leq \; = \; \leq_{\mathcal{U}'}$.

"$\Leftarrow$": Suppose τ has a subbase consisting of clopen increasing and clopen decreasing sets and $\leq = \leq_{\mathcal{U}'}$ where $\mathcal{U}'$ is the topology generated by all clopen increasing sets. Let $\mathcal{L}'$ be the topology generated by all clopen decreasing sets. Then $\tau = \mathcal{U}' \vee \mathcal{L}'$. Thus $(X, \tau, \leq) = (X, \mathcal{U}' \vee \mathcal{L}', \leq_{\mathcal{U}'})$, since $\leq \; = \; \leq_{\mathcal{U}'}$. By proposition 3.1.6 it suffices to show that $(X, \mathcal{U}', \mathcal{L}')$ is a zero-dimensional bitopological space. As remarked in the first part of the proof, we have the bases $\mathcal{B}_1$ and $\mathcal{B}_2$ for $\mathcal{U}'$ and $\mathcal{L}'$ respectively, where $\mathcal{B}_1$ consists of all clopen increasing sets and $\mathcal{B}_2$ consists of all clopen decreasing sets. Let $B \in \mathcal{B}_1$. Then $X - B$ is τ-clopen and decreasing, and hence $X - B \in \mathcal{B}_2$, so that B is $\mathcal{L}'$-closed. Therefore $\mathcal{U}'$ has a base consisting of $\mathcal{L}'$-closed sets. Similarly

Proof. (a) $\Rightarrow$ (b): Follows from corollary 3.1.9.

(b) $\Rightarrow$ (c): If (b) holds, there is by proposition 3.1.4 a transitive quasi-uniformity $\mathcal{U}$ with $\overline{T}(X,\mathcal{U}) = (X,\tau_1,\tau_2)$. So $(X,\tau,\leq) = M\overline{T}(X,\mathcal{U}) = V(X,\mathcal{U})$, by proposition 2.1.12. Therefore (c) holds.

(c) $\Rightarrow$ (a): Suppose $(X,\tau,\leq)$ admits a transitive quasi-uniformity $\mathcal{U}$, i.e. $V(X,\mathcal{U}) = (X,\tau,\leq)$. Thus $(X,\tau,\leq) = M(\overline{T}(X,\mathcal{U}))$. We know that $\overline{T}(X,\mathcal{U})$ is a zero-dimensional bitopological space (proposition 3.1.4). By proposition 3.1.6 $(X,\tau,\leq)$ is a zero-dimensional ordered space.

Let **ZTopOrd** be the full subcategory of **TopOrd** which consists of zero-dimensional ordered spaces and order preserving maps. Every zero-dimensional ordered space is a completely regular ordered spaces, since if $(X,\tau,\leq) \in$ **ZTopOrd** then the source

$$((X,\tau,\leq) \xrightarrow{f} \mathbf{D_0} \xrightarrow{i} \mathbf{I_0} \mid f \in \mathbf{TopOrd}((X,\tau,\leq),D_0))$$

where i is the inclusion map, is initial (since composition of initial sources is initial). So **ZTopOrd** is a full subcategory of **CTopOrd**. □

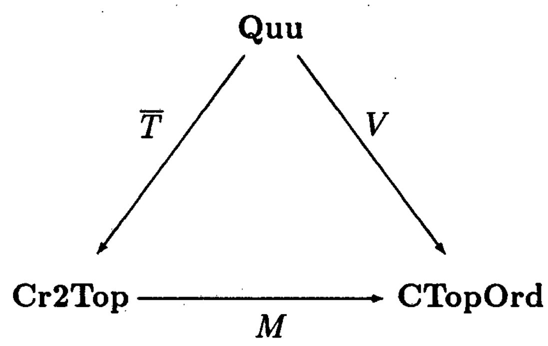
3.1.11 Proposition

ZTopOrd is a topological category.

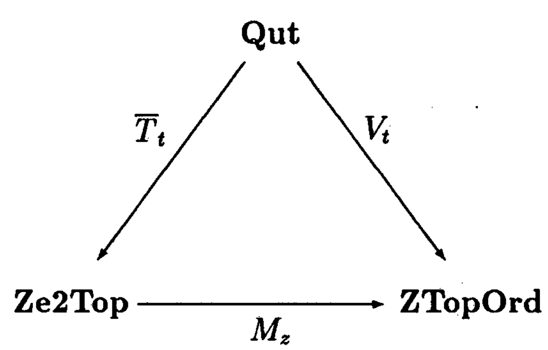
Proof. By definition 3.1.5 and the above argument, **ZTopOrd** is the initial hull of $\{D_0\}$ in **CTopOrd**. Since over **Set**, initial hulls are bireflective hulls, we have that **ZTopOrd** is a bireflective subcategory of **CTopOrd**. Since **CTopOrd** is topological (see Schauerte [1988] p.18), **ZTopOrd** is topological (because any bireflective subcategory of a topological category over **Set** is topological). □

3.2 Restricting quasi-uniform(uniform) spaces and complete regularity to transitive quasi-uniform(transitive uniform) spaces and zero-dimensionality

Using propositions 3.1.10 and 3.1.4 we can restrict the diagram



to



where **Qut** is the full subcategory of transitive quasi-uniform spaces, and **Ze2Top** and **ZTopOrd** are the obvious full subcategories of zero-dimensional objects. There is a similar restriction T_t : **Unift** $\longrightarrow$ **ZTop** of T : **Unif** $\longrightarrow$ **Creg.**

3.2.1 Proposition

The functors $\overline{T}_t$: ***Qut*** $\longrightarrow$ ***Ze2Top***, M_z : ***Ze2Top*** $\longrightarrow$ ***ZTopOrd***, V_t : ***Quu*** $\longrightarrow$ ***ZTopOrd*** *and* T_t : ***Unift*** $\longrightarrow$ ***ZTop*** *preserve initial sources.* □

In the light of proposition 3.2.1 we can apply proposition 2.1.15 to construct sections of the functors T_t, $\overline{T}_t$, V_t, M_z.

3.2.2 Examples

(a) A compact T_2-space admits a unique uniformity. Thus $\mathbf{D}_d$, the discrete two point topological space admits a unique uniform space $\mathbf{D}_s$, which is transtive. Further, $\mathbf{D}_d$ is initially dense in **ZTop**. Therefore $\mathbf{D}_d$ is a T_t-pivot.

(b) By proposition 3.1.10 $\mathbf{D}_0$ admits a unique transitive quasi-uniformity. Since $\mathbf{D}_0$ is initially dense in **ZTopOrd**, $\mathbf{D}_0$ is a V_t-pivot.

(c) A compact regular bitopological space admits a unique quasi-uniformity [Salbany 1970]. Since $\mathbf{D}_b$ is a zero-dimensional bitopological space it admits a unique transitive quasi-uniformity (proposition 3.1.4). Since $\mathbf{D}_b$ is initially dense in **Ze2Top**, it is a $\overline{T}_t$-pivot.

(d) Since $M_z\overline{T}_t = V_t$ and $\mathbf{D}_0$ admits a unique transitive quasi-uniformity, it also admits a unique zero-dimensional bitopological space (namely $\mathbf{D}_b$). Thus $\mathbf{D}_0$ is an M_z- pivot since it is initially dense in **ZTopOrd**.

We define sections for T_t, $\overline{T}_t$, V_t and M_z as follows:

3.2.3 Definition

(a) $C_t = <\{\mathbf{D}_s\}>_{T_t}$, $\overline{C_t^*} = <\{\mathbf{D}_q\}>_{\overline{T}_t}$, $C_t^{*\dagger} = <\{\mathbf{D}_q\}>_{V_t}$, $L_z = <\{\mathbf{D}_b\}>_{M_z}$.

Each function in $\mathbf{Ze2Top}(X, \mathbf{D}_b)$ is a function in $\mathbf{ZTopOrd}(M_z, \mathbf{D}_o)$ since $M_z g = g \in M_z(\mathbf{Ze2Top}(X, \mathbf{D}_b)) \subseteq \mathbf{ZTopOrd}(M_z X, M_z \mathbf{D}_b) = \mathbf{ZTopOrd}(M_z X, \mathbf{D}_o)$. Therefore $L_z M_z X \geq X$. Hence $L_z M_z \geq \mathbf{1}$.

We give an example to show that $L_z M_z \neq \mathbf{1}$. Let $\mathbf{R}$ be the real line equipped with the usual topology τ and let τ^* be the topology having as a base the closed sets in $(\mathbf{R}, \tau)$. Then $(\mathbf{R}, \tau, \tau^*)$ is a zero-dimensional bitopological space. Since τ is a T_1-topology, the topology τ^* and the order $\leq_\tau$ are discrete.

Then $L_z M_z(\mathbf{R}, \tau, \tau^*) = L_z(\mathbf{R}, \tau \vee \tau^*, \leq_\tau) = L_z(\mathbf{R}, \tau^*, \leq_\tau)$

$= L_z(\mathbf{R}$, discrete topology,discrete order$) = (\mathbf{R}$, discrete topology,discrete topology$)$

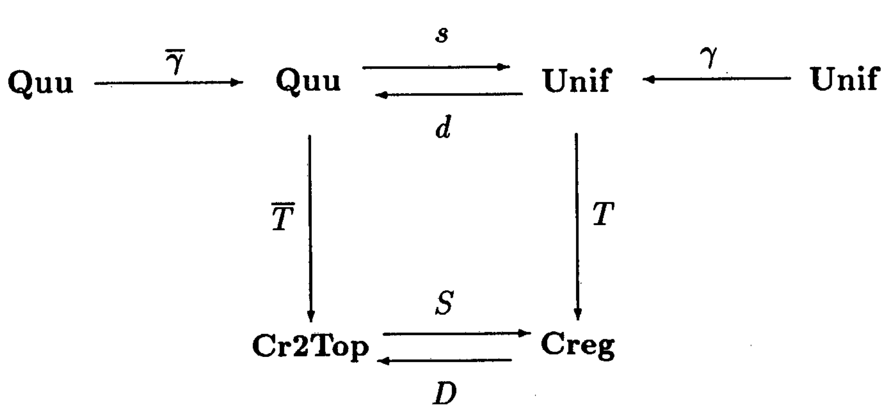
$\neq (\mathbf{R}, \tau, \tau^*)$. □

3.2.7 **Proposition**

L_z is the unique right inverse of M_z.

Proof. By proposition L_z is the coarsest M_z-section. We show that it is also the finest. Let F be any M_z-section. Then $L_z = L_z(M_z F) = (L_z M_z)F \geq F$ since $L_z M_z \geq \mathbf{1}$. Therefore $L_z = F$. □

We want to restrict the diagram



(*)

from [Brümmer 1977] (where d is the natural inclusion functor, s the concrete symmetrization functor given by $s(X,\mathcal{U}) = (X,\mathcal{U} \vee \mathcal{U}^{-1})$, D is the concrete functor given by $D(X,\tau) = (X,\tau,\tau)$, S is the concrete functor given by $S(X,\tau_1,\tau_2) = (X,\tau_1 \vee \tau_2)$ and $\overline{\gamma}$ and γ are the reflectors onto the (bi)-complete spaces) to the diagram where **Quu** is replaced by **Qut**, **Unif** is replaced by **Unift**, **Cr2Top** is replaced by **Ze2Top** and **Creg** replaced by **ZTop**. □

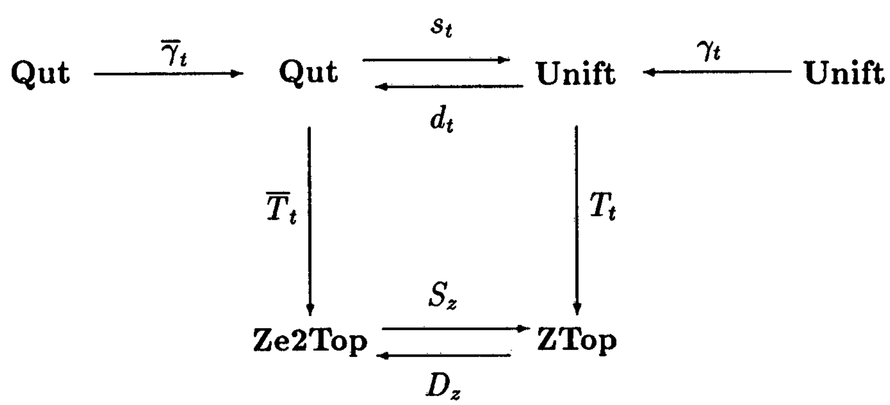
3.2.8 **Proposition** [Brümmer 1977]

$sd = \mathit{1}$; $\overline{T}d = DT$; $Ts = S\overline{T}$; $d\gamma = \overline{\gamma}d$; $\gamma s = s\overline{\gamma}$. □

3.2.9 **Corollary**

$s_t d_t = \mathit{1}$; $\overline{T}_t d_t = D_z T_t$; $T_t s_t = S_z \overline{T}_t$; $d_t \gamma_t = \overline{\gamma}_t d_t$; $\gamma_t s_t = s_t \overline{\gamma}_t$. □

Therefore the diagram (*) can be restricted to the diagram



3.2.10 **Definition** [Schauerte 1988]

(a) Let B : **CTopOrd** $\longrightarrow$ **Creg** be the concrete forgetful functor given by $B(X,\tau,\leq) = (X,\tau)$.

(b) Let J : **Creg** $\longrightarrow$ **CTopOrd** be the concrete functor given by $J(X,\tau) = (X,\tau,cl\Delta)$, where the relation $cl\Delta$ is the closure of the diagonal in $(X,\tau)\times(X,\tau)$.

Schauerte[1988] checked that B and J are are indeed functors.

3.2.11 **Proposition** [Schauerte 1988]

(a) J is the finest right inverse of B.

(b) $JT = Vd$.

(c) $Ts = BV$. □

If we let the functors B_z : **ZTopOrd** $\longrightarrow$ **ZTop** and J_z : **ZTop** $\longrightarrow$ **ZTopOrd** denote the restrictions of the functors B and J, then we get the following corollary:

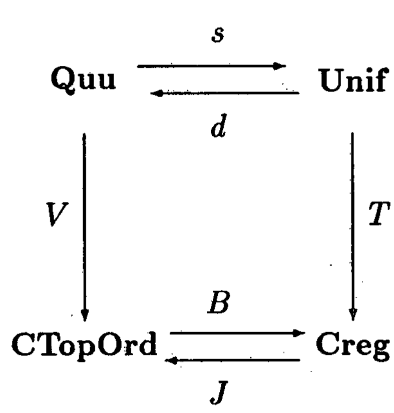
3.2.12 **Corollary**

(a) J_z is the finest right inverse of B_z.

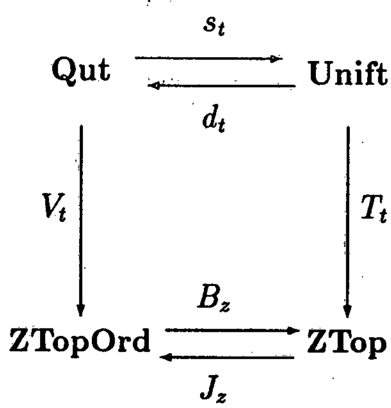
(b) $J_zT_t = V_td_t$.

(c) $T_ts_t = B_zV_t$. □

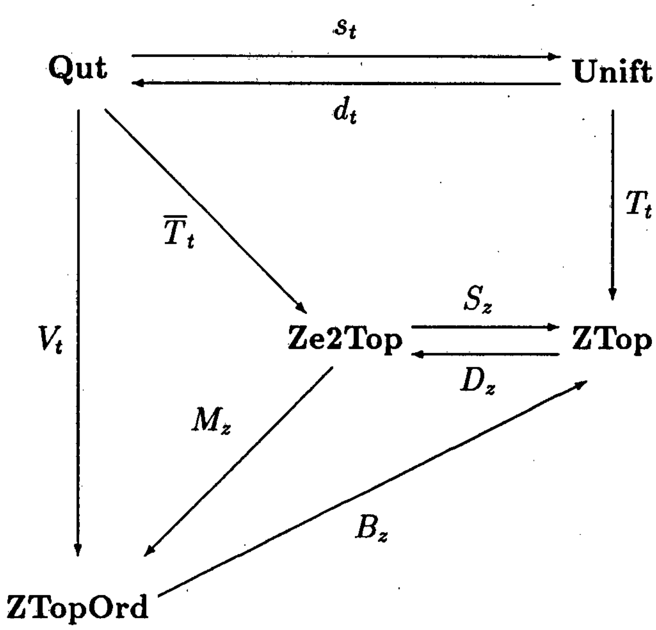
Now the diagram



from [Schauerte 1988] becomes



Combining the previous diagrams we get the following diagram:



which summarizes the relations between zero-dimensionality in **Top**, **TopOrd** and **BiTop** and transitivity in **Quu** and **Unif**.

3.3 Disconnected bitopological spaces and order-disconnected ordered spaces

3.3.1 **Definition** [Swart 1971]

Let (X, τ_1, τ_2) be a bitopological space.

(a) A pair of non-empty disjoint sets A and B such that A is τ_1-open and B is τ_2-open and $X = A \cup B$ is called a *disconnection* of X. We write A/B if the sets A and B satisfy this condition.

(b) (X, τ_1, τ_2) is *disconnected* if there is a disconnection of X.

(c) (X, τ_1, τ_2) is *totally disconnected* if for every two distinct points x and y of X there is a disconnection A/B with $x \in A$ and $y \in B$.

(d) (X, τ_1, τ_2) is *connected* if and only if it is not disconnected.

3.3.2 **Definition** [Burgess and McCartan 1970]

Let $(X, \tau, \leq)$ be a topological ordered space.

(a) $(X, \tau, \leq)$ is said to be *order-separated* if and only if there exists disjoint non-empty τ-closed subsets A and B of X with A increasing and B decreasing in X, such that $X = A \cup B$.

(b) $(X, \tau, \leq)$ is *order-connected* if it is not order-separated.

(c) We shall use the term "*order-disconnected* " to coincide with "order-separated".

3.3.3 **Proposition**

A topological ordered space $(X, \tau, \leq)$ is order-connected if and only if every continuous order preserving function $f : (X, \tau, \leq) \longrightarrow \mathbf{D}_0$ is constant.

Proof. Suppose that $(X, \tau, \leq)$ is not order-connected. Then there are disjoint non-empty τ-closed subsets A and B of X, with A increasing and B decreasing such that $X = A \cup B$. Since $A = X - B$ and $B = X - A$ we have that A and B are clopen. Then the function $f : X \longrightarrow \{0, 1\}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in B \\ 1 & \text{if } x \in A \end{cases}$$

is a morphism in $\mathbf{TopOrd}((X, \tau, \leq), \mathbf{D}_0)$. Clearly f is not constant.

Conversely suppose that there is a continuous order preserving function $f : (X, \tau, \leq) \longrightarrow D_0$ which is not constant. Then $f^{-1}(\{0\})$ is closed and decreasing, and $f^{-1}(\{1\})$ is closed and increasing, they are disjoint, and $X = f^{-1}(\{0\}) \cup f^{-1}(\{1\})$. Hence $(X, \tau, \leq)$ is not order-connected. □

3.3.4 **Proposition**

*The functor M : **Cr2Top** $\longrightarrow$ **CTopOrd** sends disconnected bitopological spaces to order-disconnected ordered spaces.*

Proof. Let $(X, \tau_1, \tau_2) \in$ **Cr2Top** be a disconnected bitopological space. Then we have disjoint non-empty sets A and B such that A is τ_1-open and B is τ_2-open and $X = A \cup B$. We want to show that $(X, \tau_1 \vee \tau_2, \leq_{\tau_1})$ is order-disconnected. Since A and B are $\tau_1 \vee \tau_2$-closed we only have to show that under the order $\leq_{\tau_1}$, A is increasing and B is decreasing. Suppose that $x \leq_{\tau_1} y$ where $y \in B$ and $x \in X$. Then $x \in cl_{\tau_1}\{y\} \subseteq B$ since B is τ_1-closed. Therefore B is decreasing. Similarly A is increasing. Hence $(X, \tau_1 \vee \tau_2, \leq_{\tau_1})$ is order-disconnected. □

3.3.5 **Definition** [Priestley 1970]

A topological ordered space $(X, \tau, \leq)$ is said to be *totally order-disconnected* if, given $x, y \in X$ with $x \not\leq y$, there exist disjoint τ-clopen sets U, L with U increasing and L decreasing such that $x \in U$, $y \in L$.

3.3.6 **Proposition**

*The functor M : **Cr2Top** $\longrightarrow$ **CTopOrd** sends totally disconnected bitopological spaces to totally order-disconnected ordered spaces.*

Proof. Suppose that a completely regular bitopological space (X, τ_1, τ_2) is totally disconnected. Let $x, y \in X$ and $x \not\leq_{\tau_1} y$. By assumption there is a disconnection

$X = A/B$ with $x \in A$ and $y \in B$. By the argument used in the previous proof, A is increasing and B is decreasing with respect to $\leq_{\tau_1}$. Since A and B are τ_2-closed and τ_1-closed respectively they are $\tau_1 \vee \tau_2$-closed. Therefore A and B are $\tau_1 \vee \tau_2$-clopen. Hence $(X, \tau_1 \vee \tau_2, \leq_{\tau_1})$ is totally order-disconnected. □

3.3.7 Example

Consider the ordered space $\mathbf{R}_0$. Its ordered subspace of rational numbers is totally order-disconnected. Similarly its ordered subspace of irrational numbers is totally order-disconnected.

3.3.8 Proposition

If $(X, \tau, \leq)$ is a partially ordered space and is totally order-disconnected, then (X, τ) is totally disconnected.

Proof. Let $(X, \tau, \leq)$ be totally order-disconnected. To prove that (X, τ) is totally disconnected it is enough to show that the maximal connected sets are the singletons. Let C be a subset of X with at least two points. Let $x, y \in C$, $x \neq y$. Since we have a partial order, either $x \not\geq y$ or $y \not\geq x$. Without loss of generality we can assume that $x \not\geq y$. Then there exist τ-clopen sets U and L, with U increasing and L decreasing such that $x \in U$, $y \in L$. Since $C \cap U$ is clopen in C with $C \cap U \neq \emptyset$ we have that C is not connected. Therefore (X, τ) is totally disconnected. □

3.3.9 Proposition [Priestley 1970]

If $(X, \tau, \leq)$ is a compact partially ordered space then $(X, \tau, \leq)$ is totally order-disconnected if and only if the upper topology has τ-clopen base. □

3.3.10 Proposition

For a compact partially ordered space $(X, \tau, \leq)$ the following are equivalent:

(a) $(X, \tau, \leq)$ is totally order-disconnected.

(b) $(X, \tau, \leq)$ is zero-dimensional.

(c) $(X, \tau, \leq)$ admits a transitive quasi-uniformity.

(d) $(X, \tau, \leq)$ admits a zero-dimensional bitopolgy.

(e) The upper topology of $(X, \tau, \leq)$ has a τ-clopen base.

Proof. $(b) \Leftrightarrow (c) \Leftrightarrow (d)$: Theorem 3.1.10.

$(a) \Leftrightarrow (c)$: This is proved in [Künzi and Brümmer 1987, proposition 1].

$(a) \Leftrightarrow (e)$: Proposition 3.3.9 above, due to [Priestly 1970]. □

NOTES

(1) Halpin [1974] and Bîrsan [1974] proved proposition 3.1.3 and proposition 3.1.4 independently. Bîrsan makes use of "principal topologies" in his proof.

(2) Our definition 3.1.5 is motivated by proposition 3.1.3 and the fact that the functor M sends $\{\mathbf{D}_b\}$ to $\{\mathbf{D}_0\}$. The results concerning the relationship between **Qut, Ze2Top** and **ZTopOrd** (i.e. propositions 3.1.6 and 3.1.11, theorems 3.1.7 and 3.1.10, and corollaries 3.1.8 and 3.1.9) are to our knowledge, not in the literature. The present author formulated and proved theorem 3.1.7 with the aim of giving an internal characterization of zero-dimensionality for ordered spaces.

(3) The results in section 3.2 concerning the restrictions to transitivity and zero-dimensionality are ours.

(4) In section 3.3 we look at the behaviour of the functor M with respect to disconnectedness and we further investigate the notions of total order-disconnectedness and zero-dimensionality in **TopOrd** in the realm of compactness, hence proposition 3.3.10. Except for those that are credited to others, the results are our own.

(5) Proposition 3.3.3 is essentially stated, but not proved, in [Burgess and McCartan 1970].

CHAPTER 4

Strongly zero-dimensional ordered spaces

In this chapter we study strong zero-dimensionality in bitopological spaces and ordered spaces. We define the strongly zero-dimensional ordered spaces and give different characterizations of these. We further prove a characterization in terms of the coarsest functorial quasi-uniformity on these ordered spaces, analogous to a result proved by Brümmer and Banaschewski [1990] for bitopological spaces.

4.1 Strongly zero-dimensional bitopological spaces

Strongly zero-dimensional bitopological spaces have been studied by Fora [1984], Brümmer and Banaschewski [1990] and more recently by Künzi [1992a]. For the purposes of comparison later on with ordered spaces, we give a definition of a strongly zero-dimensional bitopological space which looks slightly different from the definition by Fora [1984]. It is easy to show that the two definitions are equivalent, given that our terminology differs from Fora's, as already explained in remark 2.2.6.

4.1.1 Definition

A bitopological space (X, τ_1, τ_2) is called *strongly zero-dimensional* if for any lower zero set Z_1 and any upper zero set Z_2 such that $Z_1 \cap Z_2 = \emptyset$, there exists a τ_1-closed τ_2-open set V such that $Z_1 \subseteq V$, $Z_2 \subseteq X - V$.

4.1.2 Remark

We state without proof the following theorem from [Fora 1984], translated into our terminology, which gives the characterization of strongly zero-dimensional bitopological spaces. We shall later give a proof for the characterization of strongly zero-dimensional ordered spaces.

4.1.3 Theorem [Fora 1984]

For a bitopological space (X, τ_1, τ_2), the following are equivalent:

(a) X is strongly zero-dimensional.

(b) For any lower zero set Z_1 and any upper zero set Z_2 such that $Z_1 \cap Z_2 = \emptyset$, there exists a τ_1-closed τ_2-open and there exists a τ_2-closed τ_1-open set U such that $Z_1 \subseteq V, Z_2 \subseteq U$ and $U \cap V = \emptyset$.

(c) For any lower zero set Z_1 and any upper zero set Z_2 such that $Z_1 \cap Z_2 = \emptyset$ there exists a set V which is a countable intersection of τ_1-closed τ_2-open sets and there exists a set U which is a countable intersection of τ_2-closed τ_1-open sets such that $Z_1 \subseteq V, Z_2 \subseteq U, Z_2 \cap V = \emptyset$ and $Z_1 \cap V = \emptyset$.

(d) For any lower zero set Z_1 and any upper zero set Z_2 such that $Z_1 \cap Z_2 = \emptyset$, there exists a set V which is a countable union of τ_1-open τ_2-closed sets and there exists a set U which is a countable union of τ_2-open τ_1-closed sets such that $Z_1 \subseteq U, Z_2 \subseteq V, Z_2 \cap U = \emptyset$ and $Z_1 \cap V = \emptyset$.

(e) Every lower zero set set of X is a countable intersection of τ_1-closed τ_2-open sets of X and every upper zero set of X is a countable intersection of τ_2-closed τ_1-open sets of X.

(f) Every upper cozero set of X is a countable union of τ_2-closed τ_1-open sets of X and every lower cozero set of X is a countable union of τ_1-closed τ_2-open sets of X. □

For a given completely regular bispace X we denote by $J = C_b(X, \mathbf{I}_b)$ the set of all bicontinuous functions from X into $\mathbf{I}_b$. Let $e : X \longrightarrow \mathbf{I}_b^J$ be the evaluation map. Then e is an initial map. Now put $P = \mathbf{I}_b^J$ and define $\overline{\beta}X = cl_{SP}e[X]$, where $SP = \mathbf{I}^J$ is the topological space carrying the join of the two topologies in P. Then $\overline{\beta}X$ equipped with the induced bitopology from P is a compact completely regular T_0 bitopological space. The bispace $\overline{\beta}X$ together with the initial map $X \longrightarrow \overline{\beta}X$ is called the *bi-Stone-Čech compactification* of X and we shall call the corresponding functor $\overline{\beta}$: **Cr2Top** $\longrightarrow$ **Cr2Top** the *bi-Stone-Čech compactification functor* [Salbany 1970].

4.1.4 Remark

It is well known that in **Tych**, the notion of strong zero-dimensionality of a topological space (X, τ) is equivalent to requiring that the Stone-Čech compactification of (X, τ) be zero-dimensional (cf. [Walker 1974], [Engelking 1977]). Brümmer and Banaschewski [1990] defined a bitopological space (X, τ_1, τ_2) to be strongly zero-dimensional iff the Stone-Čech bicompactification of (X, τ_1, τ_2) is zero-dimensional. Using their definition they proved that the coarsest $\overline{T}$-section is transitive precisely on the strongly zero-dimensional bitopological space.

4.1.5 Proposition [Salbany 1970],[Brümmer 1978]

(a) $\overline{\gamma}\overline{C}^* = \overline{C}^*\overline{\beta}$.

(b) $\overline{C}^* = \overline{p}\overline{C}$. □

4.1.6 Proposition [Brümmer and Banaschewski 1990]

For $(X, \tau_1, \tau_2) \in$ ***Cr2Top****,* $\overline{\beta}X$ *is zero-dimensional if and only if* $\overline{C}^*X$ *is transitive.*

Proof. By proposition 4.1.5 we know that $\overline{\gamma}\overline{\mathcal{C}}^* = \overline{\mathcal{C}}^*\overline{\beta}$. If $\overline{\beta}X$ is zero-dimensional then it admits a transitive quasi-uniformity (proposition 3.1.4). Since is $\overline{\beta}X$ compact regular it admits just one quasi-uniformity. Thus $\overline{\gamma}\overline{\mathcal{C}}^*X = \overline{\mathcal{C}}^*\overline{\beta}X$ is transitive. The reflection map from $\overline{\mathcal{C}}^*X$ into $\overline{\gamma}\overline{\mathcal{C}}^*X$ is initial, whence $\overline{\mathcal{C}}^*X$ is transitive. Conversely, since $\overline{\gamma}$ preserves transitivity, if $\overline{\mathcal{C}}^*X$ is transitive so is $\overline{\mathcal{C}}^*\overline{\beta}X$. Thus $\overline{\beta}X$ admits a transitive quasi-uniformity, and is hence zero-dimensional. □

4.1.7 **Proposition** [Fora 1984, Theorem 3.7]

(a) A set $Z \subseteq X$ is an upper zero set in a bitopological space (X, τ_1, τ_2) iff there exists a bicontinuous function $f : (X, \tau_1, \tau_2) \longrightarrow \mathbf{I}_b$ such that $Z = f^{-1}(\{1\}$.

(b) A set $Z \subseteq X$ is a lower zero set in a bitopological space (X, τ_1, τ_2) iff there exists a bicontinuous function $f : (X, \tau_1, \tau_2) \longrightarrow \mathbf{I}_b$ such that $Z = f^{-1}(\{0\}$. □

4.1.8 **Proposition**

For a bitopological space (X, τ_1, τ_2), the following are equivalent:

(a) (X, τ_1, τ_2) is strongly zero-dimensional.

(b) For each bicontinuous map $f : (X, \tau_1, \tau_2) \longrightarrow (I, i_I, d_I)$ there exists a set $U \in \tau_1$ such that $X - U \in \tau_2$, $f^{-1}(\{0\}) \subseteq X - U$ and $f^{-1}(\{1\}) \subseteq U$.

(c) $\overline{\beta}X$ is zero-dimensional.

Proof. $(a) \Longrightarrow (b)$: Suppose that (X, τ_1, τ_2) is strongly zero-dimensional. Let $f : (X, \tau_1, \tau_2) \longrightarrow \mathbf{I}_b$ be a bicontinuous function. By proposition 4.1.7 $f^{-1}(\{0\})$ is a lower zero set and $f^{-1}(\{1\})$ is an upper zero set. Since $f^{-1}(\{0\}) \cap f^{-1}(\{1\}) = \emptyset$, by (a) there exists a τ_1-closed τ_2-open set V such that $f^{-1}(\{0\}) \subseteq V$ and $f^{-1}(\{1\}) \subseteq X - V$. If we put $U = X - V$ then we have $U \in \tau_1$ with $X - U \in \tau_2$, $f^{-1}(\{0\}) \subseteq X - U$ and $f^{-1}(\{1\}) \subseteq U$.

$(b) \Longrightarrow (a)$: Suppose that (b) holds. Let Z_1 be a lower zero set and let Z_2 be an upper zero set, with $Z_1 \cap Z_2 = \emptyset$. Then there exist bicontinuous functions $h, g : X \longrightarrow \mathbf{I}_b$ such that $Z_1 = h^{-1}(\{0\})$ and $Z_2 = g^{-1}(\{1\})$ (proposition 4.1.7). Since $h^{-1}(\{0\}) \cap g^{-1}(\{1\}) = \emptyset$, the function $f : X \longrightarrow \mathbf{I}_b$ defined by

$$f(x) = \frac{h(x)}{1 - g(x) + h(x)}$$

is bicontinuous, and $Z_1 = h^{-1}(\{0\}) = f^{-1}(\{0\})$ and $Z_2 = g^{-1}(\{1\}) = f^{-1}(\{1\})$. By (b) there is a τ_1-open τ_2-closed set U such that $Z_1 = f^{-1}(\{0\}) \subseteq X - U$ and $Z_2 = f^{-1}(\{1\}) \subseteq U$. Let $V = X - U$. Then V is τ_1-closed τ_2-open, $Z_1 \subseteq V$ and $Z_2 \subseteq U = X - V$. Hence (X, τ_1, τ_2) is strongly zero-dimensional.

$(b) \Longleftrightarrow (c)$: This is proved by Künzi [1992a, lemma 1]. □

4.1.9 Remark

(a) The bicontinuity of the function

$$f(x) = \frac{h(x)}{1 - g(x) + h(x)}$$

in the above proof was stated by Künzi [1992a] without proof, referring to an equivalent result in the proof of [Lane 1967, proposition 2.8].

(b) Proposition 4.1.8 shows that the definition of strong zero-dimensionality of a bitopological space due to [Fora 1984] is equivalent to the definition due to [Brümmer and Banaschewski 1990].

(c) Brümmer and Banaschewski [1990] noted that a given bitopological space X, is strongly zero-dimensional whenever $\overline{C}X$ is transitive (recall that $\overline{C}X$ carries the quasi-uniformity which is initial for the bicontinuousmaps into $\mathbf{R}_q$). They then asked whether the converse is also true. Künzi [1992a] gave an affirmative answer to their question:

4.1.10 **Proposition** [Künzi 1992a]

*Let $(X, \tau_1, \tau_2) \in$ **Cr2Top**. If $\overline{\beta}X$ is a zero-dimensional bitopological space then $\overline{C}X$ is transitive.* □

4.2 Nachbin-Stone-Čech ordered compactification using increasing and decreasing zero sets

Throughout this section we assume that all ordered spaces are completely regular partially ordered spaces . We shall abbreviate partially ordered spaces by "pospaces". Let X be a pospace and let Y be a compact pospace. If there is a **TopOrd**-embedding $h : X \longrightarrow Y$ such that $h[X]$ is dense in Y with respect to the topology, we call the pair (h, Y) an *ordered compactification* of X. We denote by $K = C_o(X, \mathbf{I}_0)$ the set of all continuous order preserving maps from X into $\mathbf{I}_0$. Now put $P = \mathbf{I}_0^K$ and let $e : X \longrightarrow \mathbf{I}_0^K$ be the evaluation map. Then e is a **TopOrd**-embedding. Now if we define $\beta^{\uparrow}X = cl_P e[X]$ then $\beta^{\uparrow}X$ is a compact pospace [Nachbin 1948b, 1965]. A detailed proof of the following result is given in [Fletcher and Lindgren 1982]:

4.2.1 **Theorem** [Nachbin 1948a, 1965]

Let $(X, \tau, \leq)$ be a completely regular pospace. Then the following are equivalent:

(a) For $f \in C_o(X, I_0)$, there exists a unique $\overline{f} \in C_o(\beta^{\uparrow}X, I_0)$ such that $\overline{f} \circ e = f$.

(b) For $f \in C_o(X, Y)$ with Y a compact pospace, there exists a unique $\overline{f} \in C_o(\beta^{\uparrow}X, Y)$ such that $\overline{f} \circ e = f$.

(c) $\beta^{\uparrow}X$ is the largest ordered compactification. □

From (b) above, it follows that a completely regular pospace X is compact iff $X = \beta^{\uparrow}X$. We shall call $\beta^{\uparrow}X$ the *Nachbin Stone-Čech ordered compactification* and $\beta^{\uparrow}$ the

Nachbin Stone-Čech ordered compactification functor on the category of completely regular pospaces.

We give a construction of the Nachbin-Stone-Čech ordered compactification for a completely regular ordered space due to McCallion[1972] using decreasing and increasing zero sets. We recall from chapter 1 that $\mathcal{A}_0$ and $\mathcal{B}_0$ represent the collections of decreasing zero sets and increasing zero sets respectively.

4.2.2 **Definition** [McCallion 1972]

Let $(X, \tau, \leq)$ be a partially ordered space, $\mathcal{A}$ a family of decreasing τ-closed sets of X and $\mathcal{B}$ a family of increasing τ-closed sets of X. Then $\mathcal{A} \cup \mathcal{B}$ is called a *normally ordered subbase* for $(X, \tau, \leq)$ if the following conditions are satisfied:

(i) $\mathcal{A}$ ($\mathcal{B}$) is a base for the closed sets of the topology $\tau_{\mathcal{A}}$ ($\tau_{\mathcal{B}}$) on X such that $\tau_{\mathcal{A}} \vee \tau_{\mathcal{B}} = \tau$ and $(\tau_{\mathcal{A}}, \tau_{\mathcal{B}})$ is an order defining pair.

(ii) Given any $\tau_{\mathcal{A}}$ ($\tau_{\mathcal{B}}$)-closed set $F \subseteq X$ and $x \in X - F$ there is a set B (A) in $\mathcal{B}$ ($\mathcal{A}$) such that $x \in B$ (A) and $B \cap F = \emptyset$ ($A \cap F = \emptyset$).

(iii) Given A and B in $\mathcal{A}$ and $\mathcal{B}$ with $A \cap B = \emptyset$, there are sets $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$ such that $A \subseteq A', B \subseteq B', A \cap B' = \emptyset = A' \cap B$ and $A' \cup B' = X$.

4.2.3 **Lemma** [McCallion 1972, Lemma 1.2]

If $(X, \tau, \leq)$ is an ordered space, $A \in \mathcal{A}_0$, $B \in \mathcal{B}_0$ with $A \cap B = \emptyset$, then there exists an order preserving continuous function $f : X \longrightarrow \boldsymbol{R}_0$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$. □

4.2.4 **Corollary** [McCallion 1972]

Let (X, τ) be a partially ordered space and let $A \in \mathcal{A}_0$, $B \subseteq \mathcal{B}_0$ be such that $A \cap B = \emptyset$. Then there is an $A' \in \mathcal{A}_0$ and a $B' \in \mathcal{B}_0$ such that $A \subseteq A'$, $B \subseteq B'$, $A' \cup B' = X$ and $A \cap B' = \emptyset = A' \cap B$. □

4.2.5 **Proposition** [McCallion 1972]

For any completely regular partially ordered space $(X, \tau, \leq)$, $\mathcal{A}_0 \cup \mathcal{B}_0$ *is a normally ordered subbase.* □

4.2.6 **Definition** [McCallion 1972]

Let $\mathcal{D}$ and $\mathcal{I}$ be non-empty subsets of $\mathcal{A}_0$ and $\mathcal{B}_0$ respectively. An ordered pair $(\mathcal{D}, \mathcal{I})$ is an $[\mathcal{A}_0, \mathcal{B}_0]$ *family* whenever $D \cap I \neq \emptyset$ for all $D \in \mathcal{D}$ and for all $I \in \mathcal{I}$.

4.2.7 **Remark**

The ordered pairs in the above definition are partially ordered by a relation written $(\mathcal{D}, \mathcal{I}) \subseteq (\mathcal{D}', \mathcal{I}')$ iff $\mathcal{D} \subseteq \mathcal{D}'$ and $\mathcal{I} \subseteq \mathcal{I}'$. An $[\mathcal{A}_0, \mathcal{B}_0]$ family is *maximal* if there is no other $[\mathcal{A}_0, \mathcal{B}_0]$ family properly "containing" it. It then follows from Zorn's lemma that every $[\mathcal{A}_0, \mathcal{B}_0]$ family is "contained" in a maximal $[\mathcal{A}_0, \mathcal{B}_0]$ family. If $x \in X$ then the set of members of $\mathcal{A}_0$ and $\mathcal{B}_0$ containing x are denoted by $\mathcal{A}_0^x$ and $\mathcal{B}_0^x$ respectively.

4.2.8 **Proposition** [McCallion 1972]

If $(\mathcal{D}, \mathcal{I})$ *is a maximal* $[\mathcal{A}_0, \mathcal{B}_0]$ *family and* $A \in \mathcal{A}_0$ $(B \in \mathcal{B}_0)$ *such that* $A \cap I \neq \emptyset$ *for all* $I \in \mathcal{I}$ $(B \cap D \neq \emptyset$ *for all* $D \in \mathcal{D})$ *then* $A \in \mathcal{D}$ $(B \in \mathcal{I})$.

Proof. Let $A \in \mathcal{A}_0$ be such that $A \cap I \neq \emptyset$ for all $I \in \mathcal{I}$ and put $\mathcal{D}' = \mathcal{D} \cup \{A\}$. Then $(\mathcal{D}', \mathcal{I})$ is an $[\mathcal{A}_0, \mathcal{B}_0]$ family and thus by the maximality of $(\mathcal{D}, \mathcal{I})$, $A \in \mathcal{D}$. A similar argument suffices to show that $B \in \mathcal{I}$. □

4.2.9 **Proposition** [McCallion 1972]

If X *is a completely regular pospace, then* $(\mathcal{A}_0^x, \mathcal{B}_0^x)$ *is a maximal* $[\mathcal{A}_0, \mathcal{B}_0]$ *family for each* $x \in X$.

Proof. It is obvious that $(\mathcal{A}_0^x, \mathcal{B}_0^x)$ is an $[\mathcal{A}_0, \mathcal{B}_0]$ family and thus we need only show the maximality. If $(\mathcal{A}_0^x, \mathcal{B}_0^x)$ is not a maximal $[\mathcal{A}_0, \mathcal{B}_0]$ family then there is an $[\mathcal{A}_0, \mathcal{B}_0]$ family $(\mathcal{D}, \mathcal{I})$ such that $(\mathcal{A}_0^x, \mathcal{B}_0^x) \subset (\mathcal{D}, \mathcal{I})$. If $\mathcal{A}_0^x \neq \mathcal{D}$ then there exists a $D \in \mathcal{D}$ such that $D \notin \mathcal{A}_0^x$, and thus $x \notin D$. By proposition 4.2.5 and part (ii) of definition 4.2.2 there is a $B \in \mathcal{B}_0$ such that $x \in B$ and $B \cap D = \emptyset$. Then $B \in \mathcal{B}_0^x$ and thus $B \in \mathcal{I}, D \in \mathcal{D}, B \cap D = \emptyset$, which contradicts our assumption that $(\mathcal{D}, \mathcal{I})$ is an $[\mathcal{A}_0, \mathcal{B}_0]$ family. A similar argument suffices if $\mathcal{B}_0^x \neq \mathcal{I}$. □

4.2.10 **Proposition** [McCallion 1972]

Let $(X, \tau, \leq)$ be a pospace. Then,

(a) If $x \in X$, then $\bigcap \mathcal{A}_0^x = d(x)$ and $\bigcap \mathcal{B}_0^x = i(x)$.

(b) If $x \in X$, then $\bigcap(\mathcal{A}_0^x \cup \mathcal{B}_0^x) = \{x\}$.

Now let $C_{\mathcal{A}_0}^{\mathcal{B}_0} X$ be the set of all maximal $[\mathcal{A}_0, \mathcal{B}_0]$ families on $(X, \tau, \leq)$. It follows from propositions 4.2.9 and 4.2.10 that there is a one-to-one correspondence between the maximal $[\mathcal{A}_0, \mathcal{B}_0]$ families of the form $(\mathcal{A}_0^x, \mathcal{B}_0^x)$, $x \in X$ and the points of X.

We now denote members of $C_{\mathcal{A}_0}^{\mathcal{B}_0} X$ by $p = (\mathcal{A}_p, \mathcal{B}_p)$. This notation enables us to identify any point $x \in X$ with the pair $(\mathcal{A}_0^x, \mathcal{B}_0^x)$. For each $A \in \mathcal{A}_0$ and for each $B \in \mathcal{B}_0$, let $\phi(A)$ and $\psi(B)$ be defined by $\phi(A) = \{p \in C_{\mathcal{A}_0}^{\mathcal{B}_0} X \mid A \in \mathcal{A}_p\}$ and $\psi(B) = \{p \in C_{\mathcal{A}_0}^{\mathcal{B}_0} X \mid B \in \mathcal{B}_p\}$.

4.2.11 **Lemma** [McCallion 1972]

If $p, q \in C_{\mathcal{A}_0}^{\mathcal{B}_0} X$ then $\mathcal{A}_q \subseteq \mathcal{A}_p$ if and only if $\mathcal{B}_p \subseteq \mathcal{B}_q$. □

We now define the relation R on $C_{\mathcal{A}_0}^{\mathcal{B}_0} X$ by:

pRq if and only if $\mathcal{A}_q \subseteq \mathcal{A}_p$.

4.2.12 **Proposition** [McCallion 1972]

The relation R is a partial order on $C^{\mathcal{B}_0}_{\mathcal{A}_0}X$ and further for $x, y \in X$, $x \leq y$ if and only if xRy. □

We denote the partial order R on $C^{\mathcal{B}_0}_{\mathcal{A}_0}X$ by $\leq$ from now on.

4.2.13 **Lemma**

(a) If $A \in \mathcal{A}_0$, $B \in \mathcal{B}_0$ and $A \cup B = X$, then $\phi(A) \cup \psi(B) = C^{\mathcal{B}_0}_{\mathcal{A}_0}X$.

(b) The family $\{\phi(A) \mid A \in \mathcal{A}_0\} \cup \{\psi(B) \mid B \in \mathcal{B}_0\}$ is a subbase for the closed sets of a topology η on $C^{\mathcal{B}_0}_{\mathcal{A}_0}X$.

Proof. See [McCallion 1972]. □

4.2.14 **Theorem** [McCallion 1972]

$(C^{\mathcal{B}_0}_{\mathcal{A}_0}X, \eta, \leq)$ is a compact pospace.

Proof. We will show that every subbase for a closed filter has non-empty intersection. Suppose that $\{\phi(A_k) : A_k \in \mathcal{A}_0,\ k \in K\} \cup \{\psi(B_l : B_l \in \mathcal{B}_0,\ l \in L\}$ has the finite intersection property. Now put $\mathcal{D} = \{A_k : k \in K\}, \mathcal{I} = \{B_l : l \in L\}$. Then $(\mathcal{D}, \mathcal{I})$ is an $[\mathcal{A}_0, \mathcal{B}_0]$ family and thus there is a maximal $[\mathcal{A}_0, \mathcal{B}_0]$ family $(\mathcal{A}_p, \mathcal{B}_p) = p \in C^{\mathcal{B}_0}_{\mathcal{A}_0}X$ such that $(\mathcal{D}, \mathcal{I}) \subseteq (\mathcal{A}_p, \mathcal{B}_p)$. Thus $p \in \phi(A_k) \cap \psi(B_l)$ for each $k \in K, l \in L$, and hence $(C^{\mathcal{B}_0}_{\mathcal{A}_0}X, \eta, \leq)$ is compact.

To prove that the graph $G(\leq)$ of the order is closed, consider $p \not\leq q$ in $C^{\mathcal{B}_0}_{\mathcal{A}_0}$. Then $\mathcal{A}_q \ /\!\subseteq \mathcal{A}_p$ and hence there is an $A_q \in \mathcal{A}_q$ such that $A_q \notin \mathcal{A}_p$. Now by proposition 4.2.8 there exists a $B_p \in \mathcal{B}_p$ such that $A_q \cap B_p = \emptyset$. Then by part (iii) of definition 4.2.2 there are set A, B in $\mathcal{A}_0$, $\mathcal{B}_0$ respectively such that $A_q \subseteq B$, $B_p \subseteq B$, $A \cup B = X$ and $A_q \cap B = \emptyset = A \cap B_p$. Now put $U = C^{\mathcal{B}_0}_{\mathcal{A}_0} - \phi(A)$, $V = C^{\mathcal{B}_0}_{\mathcal{A}_0} - \psi(B)$. Then U, V are η-open neighbourhoods of of p, q respectively. Thus $(p, q) \in U \times V$

and $(U \times V) \cap G(\leq) = \emptyset$, since U is increasing and V is decreasing (being the complements of decreasing and increasing sets respectively). Hence $G(\leq)$ is closed

□

4.2.15 **Proposition** [McCallion 1972]

η is the interval topology on $(C^{\mathcal{B}_0}_{\mathcal{A}_0}X, \leq)$. In particular, for each $A \in \mathcal{A}_0$ and $B \in \mathcal{B}_0$ there exists $p, q \in C^{\mathcal{B}_0}_{\mathcal{A}_0}X$ such that $\phi(A) = d(p)$ and $\psi(B) = i(q)$. □

4.2.16 **Lemma** [McCallion 1972]

Let $f : X \longrightarrow I_0$ be order preserving and continuous. Define functions $F, G : C^{\mathcal{B}_0}_{\mathcal{A}_0}X \longrightarrow I_0$ by:

$$F(p) = \inf_{A \in \mathcal{A}_p} \{\sup f[A]\} \text{ and}$$

$$G(p) = \sup_{B \in \mathcal{B}_p} \{\inf f[B]\}.$$

Then (a) F is order preserving and $F(x) = f(x)$, for all $x \in X$, and

(b) $F = G$.

Proof. (a) Let $p, q \in C^{\mathcal{B}_0}_{\mathcal{A}_0}X$ with $p \leq q$. Then $\mathcal{A}_q \subseteq \mathcal{A}_p$. Let $S = \{\sup f(A) \mid A \in \mathcal{A}_q\}$ and let $T = \{\sup f(A) : A \in \mathcal{A}_p\}$. Then $S \subseteq T$ and hence $F(p) = \inf T \leq \inf S = F(q)$. Let $x \in X$, then $A = \{y \in X \mid f(y) \leq f(x)\}$ is a member of $\mathcal{A}^x_0$ and thus $F(x) \leq \sup f[A] \leq f(x)$. Conversely $F(x) = \inf_{A \in \mathcal{A}^x_0}\{\sup f[A]\} \geq f(x)$.
(b) Let $p \in C^{\mathcal{B}_0}_{\mathcal{A}_0}X$. Then for all $A \in \mathcal{A}_p, B \in \mathcal{B}_p$ we have

$$\sup f[A] \geq \sup f[A \cap B] \geq \inf f[A \cap B] \geq \inf f[B].$$

Thus $\inf_{A \in \mathcal{A}_p}\{\sup f[A]\} \geq \inf f[B]$ and $F \geq G$.
Conversely, given $\epsilon > 0$, put $a_\epsilon = F(p) - \epsilon, A_\epsilon = \{x \in X : f(x) \leq a_\epsilon\}$ and $B_\epsilon = \{x \in X : f(x) \geq a_\epsilon\}$. Now $A_\epsilon \cup B_\epsilon = X$ and by lemma 4.2.13 $\phi(A_\epsilon) \cup \psi(B_\epsilon) =$

$C_{\mathcal{A}_0}^{\mathcal{B}_0}X$. Hence either $p \in \phi(A_\epsilon)$ or $p \in \psi(B_\epsilon)$. If $p \in \phi(A_\epsilon)$, then $A_\epsilon \in \mathcal{A}_p$ and thus $F(p) \leq \sup f(A_\epsilon) \leq a_\epsilon < F(p)$. Hence for each $\epsilon > 0$, $B_\epsilon \in \mathcal{B}_p$ and thus $G(p) \geq \inf f(B_\epsilon) \geq a_\epsilon = F(p) - \epsilon$. It follows that $G \geq F$. □

4.2.17 **Theorem** [McCallion 1972]

Let X be a completely regular pospace and let $f : X \longrightarrow \mathbf{I}_0$ be an order preserving continuous function. Then there is an order preserving continuous function $F : C_{\mathcal{A}_0}^{\mathcal{B}_0}X \longrightarrow \mathbf{I}_0$ such that $f(x) = F(x)$ for each $x \in X$.

Proof. Let F be as in the above lemma. It remains to show that F is continuous. Let $p \in C_{\mathcal{A}_0}^{\mathcal{B}_0}X$ and let K be a neighbourhood of $F(p)$ where K is of the form $[0, a], a \in \mathbf{I}$. Put $K' = [b, 1]$ where $b \in R$ and $F(p) \leq b \leq a$. Then $K \cup K' = \mathbf{I}$ and $f^{-1}[K] \cup f^{-1}[K'] = X$. Now since $f^{-1}[K] \in \mathcal{A}_0, f^{-1}[K'] \in \mathcal{B}_0$ we have by lemma 4.2.13 that:

$$\phi(f^{-1}[K]) \cup \psi(f^{-1}[K']) = C_{\mathcal{A}_0}^{\mathcal{B}_0}X.$$

Now $p \notin \psi(f^{-1}[K'])$ since otherwise $f^{-1}[K'] \in \mathcal{B}_p$ and thus $G(p) \geq \inf f(f^{-1}[K']) \geq \inf K' > F(p)$. If $U = C_{\mathcal{A}_0}^{\mathcal{B}_0}X - \psi(f^{-1}[K'])$ then U is clearly an η-neighbourhood of p. If $q \in U$ then by , $q \in \phi(f^{-1}[K])$ and hence $f^{-1}[K] \in \mathcal{A}_q$. Thus $F(q) \leq \sup(f^{-1}(K)) \leq \sup K$ and hence $F(q) \in K$. It follows that $p \in U \subseteq F^{-1}[K]$.
If K is a neighbourhood of $F(p)$ of the form $[a, 1], a \in \mathbf{I}$, then by an argument similar to the above, there is a η-neighbourhood V of p such that $p \in V \subseteq F^{-1}[K]$ and thus F is continuous. □

4.2.18 **Corollary** [McCallion 1972]

If $(X, \tau, \leq)$ is a completely regular pospace then the closure of X in $(C_{\mathcal{A}_0}^{\mathcal{B}_0}X, \eta, \leq)$ is equivalent to the Nachbin-Stone-Čech ordered compactification of $(X, \tau, \leq)$.

Proof. This follows from the extension property given in the above theorem. □

4.3 Nachbin-Stone-Čech ordered compactification using maximal o-completely regular filters

In order to prove the main theorem in this section we give an outline of another construction of the Nachbin-Stone-Čech ordered compactification due to Choe and Hong [1976]. Again we consider only completely regular pospaces.

A filter $\mathcal{F}$ on a completely regular pospace $(X, \tau, \leq)$ is said to be *o-completely regular* if $\mathcal{F}$ has an open filterbase $\mathcal{B}$ such that for each $U \in \mathcal{B}$ there exists a $V \in \mathcal{B}$ with $V \subseteq U$, and there exist $f_1, ..., f_n \in C_o(X, [-1,1]_o)$ such that $f_i[V] = \{0\}$ for $i \in \{1, ..., n\}$ and $X - U \subseteq \bigcup_{i=1}^{n} f_i^{-1}[\{-1,1\}]$.

The following results were shown in [Choe and Hong 1976]:

(a) Every neighbourhood filter of a completely regular pospace is a maximal o-completely regular filter.

(b) A filter $\mathcal{M}$ on a completely regular pospace contains a maximal o-completely regular filter iff $f(\mathcal{M})$ is convergent for each continuous order preserving function $f : X \longrightarrow [-1,1]$.

(c) For a completely regular pospace X, we have that X is compact iff every maximal o-completely regular filter on X is convergent.

Given a maximal o-completely regular filter $\mathcal{M}$ on a pospace $(X, \tau, \leq)$ and $f \in C_o(X, [-1,1]_o)$ we will denote the limit of the filter $f(\mathcal{M})$ by $\lim f(\mathcal{M})$. Let $\beta_0(X)$ be the set of all maximal o-completely regular filters on X, endowed with the topology τ^* generated by $\{U^* : U \in \tau\}$, where $U^* = \{\mathcal{M} \in \beta_0(X) : U \in \mathcal{M}\}$, and with an order $\leq$ defined as follows: $\mathcal{M} \leq \mathcal{N}$ in $\beta_0(X)$ if and only if $\lim f(\mathcal{M}) \leq \lim f(\mathcal{N})$ for all $f \in C_o(X, [-1,1]_o)$.

Let $e : X \longrightarrow \beta_0 X$ be the map defined by $e(x) = \mathcal{N}_x$, where $\mathcal{N}_x$ is the neighborhood filter at x. Then e is a **TopOrd**-embedding, dense with respect to the topology, and $\beta_0 X$ is a compact pospace. Also for every continuous order preserving function

$f : X \longrightarrow [-1,1]$ there is a unique extension $\overline{f} : \beta_0(X) \longrightarrow [-1,1]$ of f defined by $\overline{f}(\mathcal{M}) = \lim f(\mathcal{M})$. Thus the pair $(e, \beta_0 X)$ is equivalent to the Nachbin-Stone-Čech ordered compactification of X.

4.3.1 Lemma

If U is open in $(X, \tau, \leq)$ then $e^{-1}[U^*] = U$.

Proof.

$$\begin{aligned} x \in e^{-1}[U^*] &\iff e(x) \in U^* \\ &\iff \mathcal{N}_x \in U^* \\ &\iff U \in \mathcal{N}_x \\ &\iff x \in U. \end{aligned}$$

□

4.3.2 Theorem

Let $(X, \tau, \leq)$ be a completely regular pospace. If $U \subseteq X$ is clopen and increasing in $(X, \tau, \leq)$ then $cl_{\beta_0 X} e[U]$ is clopen and increasing in $\beta_0 X$.

Proof. Let $U \subseteq X$ be clopen and increasing in $(X, \tau, \leq)$. Define $\chi_U : X \longrightarrow [-1,1]$ by:

$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

χ_U is obviously continuous. We show that it is order preserving: Let $x, y \in X$ be such that $x \leq y$ and $x \in U$. If $\chi_U(x) \not\leq \chi_U(y)$ then $\chi_U(x) = 1$ and $\chi_U(y) = 0$. This implies that $x \in U$ and $y \notin U$ contradicting the fact that U is increasing. If $x \leq y$ and $x \notin U$ then $\chi_U(x) = 0$ and thus $\chi_U(x) \leq \chi_U(y)$. Hence χ_U is order preserving and there is an extension, $\overline{\chi}_U$, of χ_U from $\beta_0 X$ into $[-1,1]$ given by $\overline{\chi}_U(\mathcal{M}) =$

$\lim \chi_U(\mathcal{M})$ for $\mathcal{M} \in \beta_0 X$. We now show that $cl_{\beta_0 X}[U] = \{\mathcal{M} \in \beta_0 X \mid \overline{\chi}_U(\mathcal{M}) = 1\}$. We know that $(\overline{\chi}_U \circ e)[U] = \chi_U[U] = \{1\}$, therefore $e[U] \subseteq \overline{\chi}_U{}^{-1}(\{1\})$. This implies that $cl_{\beta_0 X}\{e[U]\} \subseteq \overline{\chi}_U{}^{-1}\{1\}$. We show that $\overline{\chi}_U{}^{-1}\{1\} \subseteq cl_{\beta_0 X}\{e[U]\}$. Now,

$$\begin{aligned}
\mathcal{M} \in \overline{\chi}_U{}^{-1}\{1\} &\implies \overline{\chi}_U(\mathcal{M}) = 1 \\
&\implies \lim \chi_U(\mathcal{M}) = 1 \\
&\implies (\exists\, M \in \mathcal{M})(M \subseteq U) \text{ since } \chi_U \text{ is 2-valued} \\
&\implies U \in \mathcal{M} \\
&\implies \mathcal{M} \in U^*.
\end{aligned}$$

Suppose $\mathcal{M} \notin cl_{\beta_0 X}\{e[U]\}$. Then there exists an open set $V \subseteq X$ such that $\mathcal{M} \in V^*$ and $V^* \cap e[U] = \emptyset$. Now $e^{-1}[V^*] \cap U = \emptyset$, and so $V \cap U = \emptyset$ by lemma 4.3.1. Since $V \in \mathcal{M}$ and $\mathcal{M}$ is a filter we have a contradiction. Therefore $\mathcal{M} \in cl_{\beta_0 X}\{e[U]\}$ and we conclude that $cl_{\beta_0 X}\{e[U]\} = \{\mathcal{M} \in \beta_0 X \mid \overline{\chi}_U(\mathcal{M}) = 1\}$. Thus $\overline{\chi}_U$ is the characteristic function of $e[U]$ because it can only take values in $\{0,1\}$, since it is a dense extension of χ_U which takes values in $\{0,1\}$. Since $\overline{\chi}_U$ is continuous, $cl_{\beta_0 X}\{e[U]\}$ is clopen. We now show that $cl_{\beta_0 X}\{e[U]\}$ is increasing. Let $\mathcal{M} \leq \mathcal{N}$ with $\mathcal{M} \in cl_{\beta_0 X}\{e[U]\}$. Then $1 = \overline{\chi}_U(\mathcal{M}) \leq \overline{\chi}_U(\mathcal{N}) \leq 1$ since $\overline{\chi}_U$ is order preserving and $\{0,1\}$-valued. Thus $\overline{\chi}_U(\mathcal{N}) = 1$ and hence $\mathcal{N} \in cl_{\beta_0 X}\{e[U]\}$. So $cl_{\beta_0 X}\{e[U]\}$ is clopen and increasing. □

4.4 Characterization of strongly zero-dimensional partially ordered spaces.

In this section we will use the characterization of the Nachbin-Stone-Čech ordered compactification of a completely regular pospace $(X, \tau, \leq)$ which is due to McCallion [1972]. We recall that this ordered compactification is denoted by $\beta^{\uparrow} X$. We shall take advantage of the fact that any two Nachbin-Stone-Čech ordered compactifications of a completely regular pospace are isomorphic under a **TopOrd**-isomorphism which leaves the embedded space fixed. We shall revert to the notations of section 4.2, but denoting by $e : X \longrightarrow \beta^{\dagger}X$ the version of the ordered compactification constructed there.

4.4.1 Definition

We define a completely regular pospace $(X, \tau, \leq)$ to be *strongly zero-dimensional* if $\beta^{\dagger}X$ is a zero-dimensional ordered space.

4.4.2 Lemma

(a) If $A \in \mathcal{A}_0$ *and* $B \in \mathcal{B}_0$ *then* $e[A] \subseteq \phi(A)$ *and* $e[B] \subseteq \psi(B)$.

(b) If $A_1 \subseteq A_2$ *are in* $\mathcal{A}_0$ *and* $B_1 \subseteq B_2$ *are in* $\mathcal{B}_0$*, then* $\phi(A_1) \subseteq \phi(A_2)$ *and* $\psi(B_1) \subseteq \psi(B_2)$.

(c) If $A \in \mathcal{A}_0$ *and* $B \in \mathcal{B}_0$ *then* $A \cap B = \emptyset$ *iff* $\phi(A) \cap \psi(B) = \emptyset$.

Proof. (a) Let $p \in e[A]$. Then $p = e(x)$ for some $x \in A$, i.e. $p = (\mathcal{A}_0^x, \mathcal{B}_0^x)$, so $A \in \mathcal{A}_0^x$, and hence $p \in \phi(A)$. Therefore $e[A] \subseteq \phi(A)$. Similarly $e(B) \subseteq \psi(B)$.

(b) Let $p \in \psi(B_1)$. Then $B_1 \in \mathcal{B}_p$ and $B_2 \in \mathcal{B}_p$ since $(\mathcal{A}_p, \mathcal{B}_p)$ is a maximal $[\mathcal{A}_0, \mathcal{B}_0]$ family and $(\mathcal{A}_p, \mathcal{B}_p \cup \{B_2\})$ is an $[\mathcal{A}_0, \mathcal{B}_0]$ family. Therefore $p \in \psi(B_2)$ and hence $\psi(B_1) \subseteq \psi(B_2)$. Similarly $\phi(A_1) \subseteq \phi(A_2)$.

(c) Let $A \cap B = \emptyset$. Suppose that there exists a $p \in \phi(A) \cap \psi(B)$. Then $p \in \phi(A)$ and $p \in \psi(B)$, i.e. $A \in \mathcal{A}_p$ and $B \in \mathcal{B}_p$, and $A \cap B \neq \emptyset$. This contradicts $A \cap B = \emptyset$, and therefore $A \cap B = \emptyset$ implies that $\phi(A) \cap \psi(B) = \emptyset$.

The reverse implication follows from (a) above. □

4.4.3 Theorem

Let $(X, \tau, \leq)$ be a completely regular pospace. Then $(X, \tau, \leq)$ is a strongly zero-dimensional ordered space if and only if for any $A \in \mathcal{A}_0$ and for any $B \in \mathcal{B}_0$ such that $A \cap B = \emptyset$ there exists a clopen increasing set $U \subseteq X$ such that $B \subseteq U$ and $U \cap A = \emptyset$.

Proof. Suppose that $A \in \mathcal{A}_0$, $B \in \mathcal{B}_0$ are such that $A \cap B = \emptyset$ and also that $\beta^{\uparrow}X$ is a zero-dimensional ordered space. By the above lemma $\phi(A) \cap \psi(B) = \emptyset$. By proposition 4.2.15 we have that $\phi(A) = d(q)$ for some $q \in \phi(A)$ and $\psi(B) = i(p)$ for some $p \in \psi(B)$. Then $p \notin \phi(A)$ and thus $p \not\leq q$. Since $\beta^{\uparrow}X$ is totally order-disconnected (proposition 3.3.10) there exist disjoint clopen sets U and L, with U increasing and L decreasing such that $p \in U$ and $q \in L$. Now, by lemma 4.4.2, $e[B] \subseteq \psi(B) \subseteq U$ and $e[A] \subseteq \phi(A) \subseteq L$.

Therefore we have a clopen increasing set, U, such that $e[B] \subseteq U$ and $U \cap e[A] = \emptyset$. Then $e^{-1}[U]$ is a clopen increasing set containing B and disjoint from A.

Conversely, suppose that for any $A \in \mathcal{A}_0$ and for any $B \in \mathcal{B}_0$ such that $A \cap B = \emptyset$ there exists a clopen increasing set $U \subseteq X$ such that $B \subseteq U$ and $U \cap A = \emptyset$. Suppose that $p \not\leq q$ where $p, q \in \beta^{\uparrow}X$. Then $\mathcal{A}_q \not\subseteq \mathcal{A}_p$. We can now find an $A' \in \mathcal{A}_q$ such that $A' \notin \mathcal{A}_p$, and hence by proposition 4.2.8 there exists a $B' \in \mathcal{B}_p$ such that $A' \cap B' = \emptyset$. By our assumption there is a clopen increasing set $U \subseteq X$ such that $B' \subseteq U$ and $U \cap A' = \emptyset$. By theorem 4.3.2, $cl_{\beta^{\uparrow}X} e[U]$ is clopen and increasing. We now show that $p \in cl_{\beta^{\uparrow}X} e[U]$ and $q \notin cl_{\beta^{\uparrow}X} e[U]$. We have seen in the proof of theorem 4.3.2 that $cl_{\beta^{\uparrow}X} e[U] = \{r \in \beta^{\uparrow}X \mid G_{\chi_U}(r) = 1\}$, where G_{χ_U} is the extension of the characteristic function χ_U, as defined in lemma 4.2.16. Suppose that $p \notin cl_{\beta^{\uparrow}X} e[U]$. Now $G_{\chi_U}(p) = 0$ and so $\sup_{B \in \mathcal{B}_p} \{\inf \chi_U[B]\} = 0$. Then $\inf \chi_U[B] = 0$ for all $B \in \mathcal{B}_p$ and hence $B \cap (X - U) \neq \emptyset$ for all $B \in \mathcal{B}_p$. In particular $B' \cap (X - U) \neq \emptyset$. But this contradicts the fact that $B' \subseteq U$. Therefore $p \in cl_{\beta^{\uparrow}X}e[U]$.

$$\begin{aligned} \text{Now } q \in cl_{\beta^{\uparrow}X}e[U] &\implies G_{\chi_U}(q) = 1 \\ &\implies \sup_{B \in \mathcal{B}_q} \{\inf \chi_U[B]\} = 1 \\ &\implies \inf \chi_U[B^*] = 1 \text{ for some } B^* \in \mathcal{B}_q \\ &\implies B^* \subseteq U. \end{aligned}$$

However $B^* \cap A' \neq \emptyset$ since $A' \in \mathcal{A}_q$ (recall definition 4.2.6 of an $[\mathcal{A}_0, caB]$-family). But this contradicts the fact that $U \cap A' = \emptyset$. Therefore $q \notin cl_{\beta^{\uparrow}X}e[U]$. Hence $\beta^{\uparrow}X$ is totally order-disconnected, i.e. $\beta^{\uparrow}X$ is a zero-dimensional ordered space. We can therefore conclude that X is a strongly zero-dimensional ordered space.

□

4.4.4 Remark

We give further characterizations of strongly zero-dimensional ordered spaces similar to the ones given by [Fora 1984] for strongly zero-dimensional bitopological spaces. For this we need some preliminaries and lemmas about the increasing zero sets and the decreasing zero sets. We shall call the complements of the increasing zero sets and decreasing zero sets in a completely regular ordered space, the *decreasing cozero sets* and the *increasing cozero sets* respectively. Let $(X, \tau, \leq)$ be a completely regular ordered space and let $a \in \mathbf{R}$. Set $A := \{x \in X \mid f(x) \leq a\}$ and set $B := \{x \in X \mid f(x) \geq a\}$, where $f : (X, \tau, \leq) \longrightarrow \mathbf{R}_0$ is a continuous order preserving function. Then $A \in \mathcal{A}_0$ and $B \in \mathcal{B}_0$, and thus A is a decreasing zero set and B is an increasing zero set. This follows by observing that the function g from X into R defined by $g(x) = f(x) - a$ is continuous and order preserving.

4.4.5 Lemma

Let $(X, \tau, \leq)$ be a completely regular ordered space. Then, *(a) A subset A of X is a decreasing zero set iff there is a continuous order preserving function* $f : (X, \tau, \leq) \longrightarrow I_0$ *such that* $A = f^{-1}(\{0\})$.

(b) A subset B of X is an increasing zero set iff there is a continuous order preserving function $f : (X, \tau, \leq) \longrightarrow I_0$ *such that* $B = f^{-1}(\{1\})$.

Proof. (a) Let $A \subseteq X$ be a decreasing zero set. Then $A = \{x \in X \mid h(x) \leq 0\}$ for some continuous order preserving function $h : (X, \tau, \leq) \longrightarrow \mathbf{R}_0$. Let the $g : \mathbf{R}_0 \longrightarrow \mathbf{I}_0$ be given by:

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

It is easy to see that $g : \mathbf{R}_0 \longrightarrow \mathbf{I}_0$ is continuous and order preserving. The function $f : (X, \tau, \leq) \longrightarrow \mathbf{I}_0$ defined by $f = g \circ h$ is continuous and order preserving, and $A = f^{-1}\{0\}$ as required. Conversely, let $f : (X, \tau, \leq) \longrightarrow \mathbf{I}_0$ be a continuous order preserving function such that $A = f^{-1}\{0\}$. Let $g : (X, \tau, \leq) \longrightarrow \mathbf{R}_0$ be given by $g = i \circ f$, where $i : \mathbf{I}_0 \hookrightarrow \mathbf{R}_0$ is the inclusion map. Then the function g is continuous and order preserving, and furthermore $A = \{x \in X : g(x) \leq 0\}$.

(b) This can be proved in a similar way with the function g above given by:

$$g(x) = \begin{cases} 0 & \text{if } x < -1 \\ x+1 & \text{if } -1 \leq x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

□

4.4.6 Remark

The functions used in the above proof were also used in proposition 4.1.7 for bispaces.

4.4.7 Theorem

For a completely regular pospace $(X, \tau, \leq)$ the following are equivalent:

(a) X is strongly zero-dimensional.

(b) For any $A \in \mathcal{A}_0, B \in \mathcal{B}_0$ such that $A \cap B = \emptyset$ there exists a clopen increasing set $U \subseteq X$ such that $B \subseteq U$ and $A \subseteq X - U$.

(c) For any $A \in \mathcal{A}_0, B \in \mathcal{B}_0$ such that $A \cap B = \emptyset$ there exists a clopen decreasing set $V \subseteq X$ and there exists a clopen increasing set $U \subseteq X$ such that $A \subseteq V$, $B \subseteq U$ and $U \cap V = \emptyset$.

(d) For any $A \in \mathcal{A}_0, B \in \mathcal{B}_0$ such that $A \cap B = \emptyset$, there exists a set V which is a countable intersection of clopen decreasing sets and there exists a set U which is a countable intersection of clopen increasing sets such that $A \subseteq V$, $B \subseteq U$, $V \cap B = \emptyset$ and $U \cap A = \emptyset$.

(e) For any $A \in \mathcal{A}_0, B \in \mathcal{B}_0$ such that $A \cap B = \emptyset$ there exists a set V which is a countable union of clopen increasing sets and there exists a set U which is a countable union of clopen decreasing sets such that $A \subseteq U$, $B \subseteq V$, $U \cap B = \emptyset$ and $V \cap A = \emptyset$.

(f) Every $A \in \mathcal{A}_0$ is a countable intersection of clopen decreasing sets and every $B \in \mathcal{B}_0$ is a countable intersection of clopen increasing sets.

(g) Every increasing cozero set of X is a countable union of clopen increasing sets of X and every decreasing cozero set of X is a countable union of clopen decreasing sets of X.

Proof.

(a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c) follows from theorem 4.4.3.

(c) $\Longrightarrow$ (d): Let $A \in \mathcal{A}_0, B \in \mathcal{B}_0$ and $A \cap B = \emptyset$. By lemma 4.2.3 there is a continuous order preserving function $f : (X, \tau, \leq) \longrightarrow \mathbf{I}_0$ such that $A \subseteq f^{-1}\{0\}$ and $B \subseteq f^{-1}\{1\}$. We know that

$$\{0\} = \bigcap_{n=1}^{\infty} [0, \frac{1}{2n}].$$

Then

$$A \subseteq f^{-1}(\{0\}) = \bigcap_{n=1}^{\infty} f^{-1}[[0, \frac{1}{2n}]].$$

We also have that each $f^{-1}[[0, \frac{1}{2n}]] = A_n$ is a decreasing zero set disjoint from B. Thus $A_n \cap B = \emptyset$ for all n. By our assumption, for each n there is a clopen decreasing set V_n and a clopen increasing set U_n such that $A_n \subseteq V_n$, $B \subseteq U_n$ and $V_n \cap U_n = \emptyset$. Then $A \subseteq f^{-1}(\{0\}) = \bigcap_{n=1}^{\infty} A_n \subseteq \bigcap_{n=1}^{\infty} V_n = V$, $B \subseteq \bigcap_{n=1}^{\infty} U_n = U$, $V \cap B = \emptyset$ and $U \cap A = \emptyset$.

(d) $\Longrightarrow$ (e): This follows by taking complements.

(e) $\Longrightarrow$ (f) Suppose (e) holds and let $B \in \mathcal{B}_0$. By lemma 4.4.5 there exists a continuous order preserving function $f : (X, \tau, \leq) \longrightarrow \mathbf{I}_0$ such that $B \subseteq f^{-1}(\{1\})$. Fix $n \in N$. Now $f^{-1}[[0, \frac{n-1}{n}]]$ is a decreasing zero set and by (e) there exist clopen decreasing sets $V_{kn}, k \in N$ such that $f^{-1}[0, \frac{n-1}{n}] \subseteq \bigcup_{k=1}^{\infty} V_{kn}$ and $\bigcup_{k=1}^{\infty}[V_{kn} \cap B] = \emptyset$. We show that $B = \bigcap_{n=1}^{\infty}(X - V_{kn})$. Clearly $B \subseteq \bigcap_{k=1}^{\infty}(X - V_{kn})$. Now,

$$\begin{aligned} x \in \bigcap_{k=1}^{\infty}(X - V_{kn}) &\Longrightarrow x \in (X - V_{kn}) \text{ for all } k \\ &\Longrightarrow x \notin V_{kn} \text{ for all } k \\ &\Longrightarrow x \notin f^{-1}[[0, 1 - 1/n]] \\ &\Longrightarrow f(x) > 1 - 1/n. \end{aligned}$$

This holds for every $n \in \mathbf{N}$, and so $f(x) \geq 1$, i.e. $f(x) = 1$, i.e. $x \in B$. Hence $B = \bigcap_{k=1}^{\infty}(X - V_{kn})$. Since each V_{kn} is clopen and decreasing, each $X - V_{kn}$ is clopen and increasing. If $A \in \mathcal{A}_0$, then by lemma 4.4.5 there exists a continuous order preserving $f : (X, \tau, \leq) \longrightarrow \mathbf{I}_0$ such that $A = f^{-1}(\{0\})$. Since $f^{-1}[\frac{1}{n}, 1]$ is an increasing zero set, by (e) there exist clopen increasing sets $U_{kn}, k \in N$ such that $f^{-1}[\frac{1}{n}, 1] \subseteq \bigcup_{k=1}^{\infty} U_{kn}$ and $\bigcup_{k=1}^{\infty} V_{kn} \cap A = \emptyset$. As in the previous case $A = \bigcap_{k=1}^{\infty}(X - U_{kn})$ and each $X - U_{kn}$ is clopen and decreasing.

(f) $\Longrightarrow$ (g): Obvious.

(g) $\Longrightarrow$ (c): Suppose that (g) holds. Let $A \in \mathcal{A}_0, B \in \mathcal{B}_0$ and $A \cap B = \emptyset$. By (g) there exist clopen increasing sets $U_n, n \in N$ and there exist clopen decreasing sets $V_n, n \in N$ such that $X - A = \bigcup_{n=1}^{\infty} U_n$ and $X - B = \bigcup_{n=1}^{\infty} V_n$. Since $A \cap B = \emptyset$, we have $X = X - (A \cap B) = (X - A) \cup (X - B) = \bigcup_{n=1}^{\infty}(U_n \cup V_n)$. Now define $U_n' = U_n - (V_1 \cup V_2 \cup ... \cup V_{n-1}), n \neq 1$, and define $V_n' = V_n - (U_1 \cup U_2 \cup ... \cup U_n), n \neq 1$. Furthermore let $U_1' = U_1$ and let $V_1' = V_1$. Then U_n' and V_n' are clearly clopen for all n. We show that U_n' is increasing and V_n' is decreasing. The case $n = 1$ is obvious since U_1 and V_1 are increasing and decreasing respectively. Let $x \leq y$ and $x \in U_n'$. Then $x \in U_n$ and $x \notin (V_1 \cup V_2 \cup ... \cup V_{n-1}), n \neq 1$. If $y \notin U_n'$ then we have the following three cases:

Case 1: $y \in U_n$ and $y \in V_1 \cup V_2 \cup ... \cup V_{n-1}$. But this will imply that $x \in V_1 \cup V_2 \cup ... \cup V_{n-1}$ since $V_1 \cup V_2 \cup ... \cup V_{n-1}$ is decreasing, which contradicts our assumption.
Case 2: $y \notin U_n$ and $y \in V_1 \cup V_2 \cup ... \cup V_{n-1}$. This case leads to the same contradiction.
Case 3: $y \notin U_n$ and $y \notin V_1 \cup V_2 \cup ... \cup V_{n-1}$. This cannot happen since U_n is increasing.
In all cases we have a contradiction and thus $y \in U_n'$. Hence U_n' is increasing. Similarly V_n' is decreasing. It is clear that $U_n \cap V_n = \emptyset$ for all $n \in N$. Letting $U = \bigcup_{n=1}^{\infty} U_n'$ and $V = \bigcup_{n=1}^{\infty} V_n$, we have $U \cap V = \emptyset$. We now show that $A \subseteq V$ and $B \subseteq U$. Let $x \in A$. Then $x \notin \bigcup_{n=1}^{\infty} U_n$, so that $x \in \bigcup_{n=1}^{\infty} V_n$ since $X = \bigcup_{n=1}^{\infty}(U_n \cup V_n)$. Now we have a smallest $n_0 \in \mathbf{N}$ for which $x \in V_{n_0}$. Therefore $x \in V_{n_0}'$, and thus $x \in \bigcup_{n=1}^{\infty} U_n = V$. Thus $A \subseteq V$, and similarly $B \subseteq U$. Furthermore we have $X = U \cup V$. Therefore we have found a set U which is clopen and increasing, and a set V which is clopen and decreasing satisfying (c). $\square$

4.5 Strongly zero-dimensional ordered spaces and V-sections.

In this section we give the analogue of proposition 4.1.6, which is due to [Brümmer and Banaschewski 1990], for partially ordered spaces.

4.5.1 **Proposition** [Salbany 1984]

$M \circ \overline{\beta} \circ L = \beta^\dagger$.

Proof. Given $(X, \tau, \leq) \in$ **CTopOrd**, let $(X, \tau_1, \tau_2) = L(X, \tau, \leq)$. Let $\overline{\beta}(X, \tau_1, \tau_2) = (\overline{X}, \overline{\tau}_1, \overline{\tau}_2)$. Then $(\overline{X}, \overline{\tau}_1, \overline{\tau}_2)$ is the bi-Stone-Čech compactification of (X, τ_1, τ_2), so that X is $\overline{\tau}_1 \vee \overline{\tau}_2$ dense in $(\overline{X}, \overline{\tau}_1, \overline{\tau}_2)$ and $\overline{\tau}_1$ restricted to X is τ_1, and $\overline{\tau}_2$ restricted to X is τ_2. This implies that the order $\leq_{\overline{\tau}_1}$ restricted to X is $\leq_{\tau_1}$, so that $\leq_{\overline{\tau}_1}$ restricted to X is $\leq$. Now $M(\overline{X}, \overline{\tau}_1, \overline{\tau}_2) = (\overline{X}, \overline{\tau}_1 \vee \overline{\tau}_2, \leq_{\overline{\tau}_1})$. Put $\overline{\tau} = \overline{\tau}_1 \vee \overline{\tau}_2$. Thus $(\overline{X}, \overline{\tau}, \leq_{\overline{\tau}_1})$ is an order compactification of $(X, \tau, \leq)$. We show that any $f : (X, \tau, \leq) \longrightarrow \mathbf{I}_0$ can be extended to $\overline{f} : (\overline{X}, \overline{\tau}, \leq_{\overline{\tau}_1}) \longrightarrow \mathbf{I}_0$, which would show that $\beta^\dagger(X, \tau, \leq) = (\overline{X}, \overline{\tau}, \leq_{\overline{\tau}_1})$. Now $f : (X, \tau, \leq) \longrightarrow \mathbf{I}_0$ gives a map $Lf : (X, \tau_1, \tau_2) \longrightarrow (I, i_I, d_I)$, which can be extended to a map $\overline{f} : (\overline{X}, \overline{\tau}_1, \overline{\tau}_2) \longrightarrow (I, i_I, d_I)$, hence $M\overline{f} : (\overline{X}, \overline{\tau}_1 \vee \overline{\tau}_2, \leq_{\overline{\tau}_1}) \longrightarrow \mathbf{I}_0$ with $M\overline{f}(x) = f(x)$ for each $x \in X$. □

4.5.2 **Corollary** [Salbany 1984]

$L \circ \beta^\dagger = \overline{\beta} \circ L$. □

4.5.3 **Corollary**

$\overline{\gamma} C^{*\dagger} = C^{*\dagger} \beta^\dagger$.

Proof.

$$
\begin{aligned}
\overline{\gamma} C^{*\dagger} &= \overline{\gamma}\overline{C^*} L \text{ by proposition 2.1.17 } (b) \\
&= \overline{C^*}\overline{\beta} L \text{ by proposition 4.1.5 } (a) \\
&= \overline{C^*} L \beta^\dagger \text{ by corollary 4.5.2} \\
&= C^{*\dagger} \beta^\dagger \text{ by proposition 2.1.17 } (b).
\end{aligned}
$$

□

4.5.4 **Theorem**

Let $(X, \tau, \leq)$ be a completely regular pospace. Then $C^{*\dagger}X$ is transitive iff $(X, \tau, \leq)$ is a strongly zero-dimensional ordered space.

Proof. $\Longleftarrow$: Let X be a strongly zero-dimensional ordered space. Then, by definition $\beta^{\dagger}X$ is is a zero-dimensional ordered space and thus admits a transitive quasi-uniformity. Since $\beta^{\dagger}X$ is a compact ordered space, $\beta^{\dagger}X$ admits a unique quasi-uniformity. By corollary 4.5.3 $\overline{\gamma}C^{*\dagger}X = C^{*\dagger}\beta^{\dagger}X$. Therefore $\overline{\gamma}C^{*\dagger}X$ is transitive since $C^{*\dagger}\beta^{\dagger}X$ is transitive. Since the reflection map from $C^{*\dagger}X$ into $\overline{\gamma}C^{*\dagger}X$ is initial, $C^{*\dagger}X$ is transitive.

$\Longrightarrow$: Suppose that $C^{*\dagger}X$ is transitive. Then $\overline{C}^{*}LX$ is transitive by proposition 2.1.17(b). Then LX is a strongly zero-dimensional bispace by proposition 4.1.6 and thus $\overline{\beta}LX$ is a zero-dimensional bispace. Hence $L\beta^{\dagger}X$ is a zero-dimensional bispace by corollary 4.5.2. By applying the functor M : **Cr2Top** $\longrightarrow$ **CTopOrd** we have that $\beta^{\dagger}X$ is a zero-dimensional ordered space. Therefore $(X, \tau, \leq)$ is a strongly zero-dimensional ordered space.

□

4.5.5 **Proposition**

Let $(X, \tau, \leq)$ be an ordered space. If $C^{\dagger}(X, \tau, \leq)$ is transitive then $(X, \tau, \leq)$ is a strongly zero-dimensional ordered space.

Proof. The proof is similar to the proof of the forward implication in theorem 4.5.4. □

NOTES

(1) The work of this chapter was largely motivated by the situation in **Top** and **BiTop** (see [Fletcher and Lindren 1982], [Walker 1974], [Engelking 1977]) for **Top**, and see [Fora 1984], [Brümmer and Banaschewski 1990] and [Künzi 1992a] for **BiTop**) where in both cases we see a clear exposition of the internal characterisations of the notion of strong zero-dimensionality.

(2) We used McCallion's construction of the Nachbin-Stone-Čech ordered compactification [McCallion 1972], because it is precisely through the decreasing and increasing zero sets that we wanted to characterize strongly zero-dimensional ordered spaces. On the other hand we used the construction from [Choe and Hong 1976] in proving our theorem 4.3.2 since it was easier to handle the open sets in this construction. That result enabled us to prove our theorem 4.4.3. We have not seen either of the results in the literature.

(3) The different characterizations in theorem 4.4.7 also seem to be new and are in analogy with proposition 4.1.3 .

(4) In section 4.5 we used the commutativity of the diagram in proposition 2.1.12 to obtain the analogue of proposition 4.1.6 for ordered spaces. We believe there is an analogue of proposition 4.1.10 in ordered spaces though we could not prove it.

(5) In chapter 4 we have for convenience often restricted results to partially ordered spaces, which could also be proved for ordered spaces (i.e., as always, spaces with a closed graph).

List of Categories, Functors and Symbols

Category	Objects	Morphisms
Set	sets	functions
Top	topological spaces	continuous functions
Creg	completely regular topological spaces	continuous functions
ZTop	zero-dimensional topological spaces	continuous functions
TopOrd	ordered spaces	continuous order-preserving functions
CTopOrd	completely regular ordered spaces	continuous order-preserving functions
ZTopOrd	zero-dimensional ordered spaces	continuous order-preserving functions
BiTop	bitopological spaces	bicontinuous functions
Cr2Top	completely regular bispaces	bicontinuous functions
Ze2Top	zero-dimensional bispaces	bicontinuous functions
Unif	uniform spaces	uniformly continuous functions
Unift	transitive uniform spaces	uniformly continuous functions
Quu	quasi-uniform spaces	uniformly continuous functions
Qut	transitive quasi-uniform spaces	uniformly continuous functions

Table of functors:	Definitions	pages
$T : \mathbf{Unif} \longrightarrow \mathbf{Creg}$		7
$T_t : \mathbf{Unift} \longrightarrow \mathbf{ZTop}$		27
$\overline{T} : \mathbf{Quu} \longrightarrow \mathbf{Cr2Top}$		7
$\overline{T}_t : \mathbf{Qut} \longrightarrow \mathbf{Ze2Top}$		27
$V : \mathbf{Quu} \longrightarrow \mathbf{CTopOrd}$	2.1.7	
$V_t : \mathbf{Qut} \longrightarrow \mathbf{ZTopOrd}$		27
$s : \mathbf{Quu} \longrightarrow \mathbf{Unif}$		31
$s_t : \mathbf{Qut} \longrightarrow \mathbf{Unift}$		31
$d : \mathbf{Unif} \longrightarrow \mathbf{Quu}$		31
$M : \mathbf{Cr2Top} \longrightarrow \mathbf{CTopOrd}$	2.1.9	
$M_z : \mathbf{Ze2Top} \longrightarrow \mathbf{ZTopOrd}$		27
$\overline{C}^*, \overline{C}, \overline{\phi} : \mathbf{Cr2Top} \longrightarrow \mathbf{Quu}$	2.1.16	
$\overline{C}_t^*, \overline{\phi}_t : \mathbf{Ze2Top} \longrightarrow \mathbf{Qut}$	3.2.3	
$C^{*\dagger}, C^{\dagger}, \phi^{\dagger} : \mathbf{CTopOrd} \longrightarrow \mathbf{Quu}$	2.1.16	
$C_t^{*\dagger}, \phi_t^{\dagger} : \mathbf{ZTopOrd} \longrightarrow \mathbf{Qut}$	3.2.3	
$C_t, \phi_t : \mathbf{ZTop} \longrightarrow \mathbf{Unift}$	3.2.3	
$L : \mathbf{CTopOrd} \longrightarrow \mathbf{Cr2Top}$	2.1.16	
$L_z : \mathbf{ZTopOrd} \longrightarrow \mathbf{Ze2Top}$	3.2.3	
$\overline{\gamma} : \mathbf{Quu} \longrightarrow \mathbf{Quu}$	2.1.1	
$\overline{p} : \mathbf{Quu} \longrightarrow \mathbf{Quu}$	2.1.1	
$\gamma : \mathbf{Unif} \longrightarrow \mathbf{Unif}$	2.1.1	

Other special symbols	**Definitions**	**pages**
$\mathbf{R}_0$, $\mathbf{I}_0$, $\mathbf{D}_0$	1.2	
$\mathbf{R}_b$, $\mathbf{I}_b$, $\mathbf{D}_b$		4
$\mathbf{R}_q$, $\mathbf{I}_q$, $\mathbf{D}_q$		7
i, d	1.2	
$i(x)$, $d(x)$	1.2	
$\mathcal{U}_\tau$, $\mathcal{L}_\tau$	1.2	
$\leq_\tau$, $\leq^\mu$	1.2	

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