ALGORITHMIC RANDOMNESS ON COMPUTABLE METRIC SPACES AND HYPERSPACES.

THOMAS BIRCH

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FACS Lab
Mathematics Department
University of Cape Town

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Supervisor: Vasco Brattka.

Contents

1. Introduction

The field of algorithmic randomness first began as an attempt to study what it means for a real number to be random. The question of which properties a random real should have has been debated since the turn of the previous century. One major problem has always been that the word "random" has no strict definition in any language, indeed in different languages the concept could mean anything from unlikely to unpredictable, from inconsequential to representative. Even within the English language the meaning of the word random depends almost entirely upon context. In the 1960's and 1970's, with the fields of probability theory, measure theory, and computability theory allowing for more formal definitions, the three forerunners in terms of our intuitive understanding of what properties a random sequence should have were that a random sequence should either conform to known probability distributions ([1]), or be incompressible ([2]), or be unpredictable ([3], [4]).

It was von Mises [5] who initially proposed that a random real when viewed as a sequence of 0's and 1's in the Cantor space should satisfy a set of statistical "randomness" tests such as the law of large numbers, the frequency stability property and the law of averages. It was not until later when Martin-Löf [1] discovered that the statistical tests suggested by von Mises could be viewed as an effective measure zero set on the Cantor space, the theory of which we shall introduce in section 5.1, that a formal mathematical definition of a random real was defined.

The notion of incompressibility was first defined by Kolmogorov for finite strings [6]. In his paper Kolmogorov [7] proposed that the complexity of a string could be defined by the smallest definition of the string that exists on a given Turing machine. A random finite sequence was then defined as a sequence for which the complexity of a string was larger than the length of the string itself. Or in other words that a random string is one that cannot be compressed by a computer program. This notion was later generalized to infinite sequences by Levin [2] as prefix free complexity and prefix free randomness.

The last definition of randomness that we shall consider is that of unpredictability. The idea is that for no random sequence is there a betting strategy such that an infinite amount of money could be won by betting on the outcome of the next digit in the sequence. This concept was formalized by Schnorr [3] as an effective martingale. An effective martingale is a computable map f (see Chapter 2.1 for definitions of computability and symbols) from $\{0,1\}$ to $\mathbb{R}^+$ where for all $w \in \{0,1\}$ we have that $f(w) = \frac{f(w^\frown 0)+f(w^\frown 1)}{2}$ and for the empty string ε we have that $f(\varepsilon) = 1$. An infinite sequence $\sigma \in \{0,1\}^\omega$ is then random if for all effective martingales (betting strategies) f we have that $\limsup_{n\in\mathbb{N}} f(\sigma\lceil_n) < \infty$.

Since the 1970's many other stronger and weaker forms of randomness have been studied (for an excellent study on the relationship between the differing forms of randomness see [8]). In 1976 Chaitin [9] proved that the definitions of randomness with regard to effective martingales, prefix free complexity and Martin-Löf tests were equivalent characterisations of the same intuitive concept of randomness. And with that the focus shifted from what is a random real to what are the properties of random reals.

A more recent approach to the study of algorithmic randomness has been to adapt Martin-Löf's intuitive notion of randomness to other spaces such as the space continuous functions ([10]), random walks ([11],[12]), computable topological measure spaces ([13]), and as we discuss in section 5.1 in computable metric spaces.

In this text we shall be focusing on generalizing Martin-Löf randomness to computable metric spaces with arbitrary measure (for examples of this type of generalization see Gács [14], Rojas and Hoyrup [15]. The aim of this generalization is to define algorithmic randomness on the hyperspace of non-empty compact subsets of a computable metric space, the study of which was first proposed by Barmpalias et al. [16] at the University of Florida in their work on the random closed subsets of the Cantor space. Much work has been done in the study of random sets with authors such as Diamondstone and Kjos-Hanssen [17] continuing the Florida approach, whilst others such as Axon [18] and Cenzer and Broadhead [19] have been studying the use of capacities to define hyperspace measures for use in randomness tests.

Lastly in section 6.4 we shall be looking at the work done by Hertling and Weihrauch [13] on universal randomness tests in effective topological measure spaces and relate their results to randomness on computable metric measure spaces and in particular to the randomness of compact sets in the hyperspace of non-empty compact subsets of computable metric spaces.

2. **Computable Analysis**

2.1. **Computablity & Representations.**

In Section 2.1 we shall begin by stating a few basic definitions from classical computability theory, then moving on to Type-2 Theory of Effectivity (TTE) where we shall lay the groundwork for computability in hyperspaces and algorithmic randomness in arbitrary metric spaces.

It is expected that the reader is familiar with the basic computability notions of a partial function $f :\subseteq \mathbb{N} \to \mathbb{N}$ being computable and a subset $A \subseteq \mathbb{N}$ being computably enumerable (c.e.). We shall be generalizing these concepts to computable partial maps and computably enumerable subsets on and of arbitrary sets (see [20], [21] for two excellent introductory texts on these subjects). It should be noted that in this thesis the notation $\mathbb{N}$ refers to the set of natural numbers including 0, i.e. $\mathbb{N} = \{0, 1, 2, ...\}$. We define Cantor's pairing function as a bijective function $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, where $\pi(a, b) := \frac{1}{2}(a + b)(a + b + 1) + b$. The standard notation for $\pi(a, b)$ is $\langle a, b\rangle$. The inverse functions of the Cantor pairing function are denoted as

$$\pi_1 : \mathbb{N} \to \mathbb{N},\ \langle a, b\rangle \mapsto a,$$
$$\pi_2 : \mathbb{N} \to \mathbb{N},\ \langle a, b\rangle \mapsto b.$$

The Cantor pairing function can, by induction, be applied to higher dimensions and is written as $\langle x_1, x_2, ..., x_n\rangle := \langle x_1, \langle x_2, \langle ...., \langle x_{n-1}, x_n\rangle\rangle\rangle\rangle$. Two sequences $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ can be combined into a single sequence $\langle (x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}\rangle$ by

$$\begin{array}{rcl} \langle (x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}\rangle \quad (2n) & := & x_n \\ \langle (x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}\rangle \quad (2n+1) & := & y_n \end{array}$$

and infinitely many sequences p_i in $\mathbb{N}$ can be combined into a single sequence $\langle p_0, p_1, ...\rangle$ by

$$\langle p_0, p_1, ...\rangle(\langle n, k\rangle) := p_k(n)$$

A numbering of a set X is a partial surjective map from the natural numbers to the set X. A function $f :\subseteq X \to Y$ is computable with respect to the numberings ν_X and ν_Y of X and Y respectively if there exists a computable function $g :\subseteq \mathbb{N} \to \mathbb{N}$ such that $\nu_Y(g(n)) = f(\nu_X(n))$ for all $n \in \operatorname{dom}(f \circ \nu_X)$. A similar definition uses the set of all finite strings from a finite alphabet instead of the natural numbers and the concept is called a notation. A notation of a set X is a (partial) surjective mapping $\nu :\subseteq \Sigma^* \to X$ using a finite alphabet Σ. A function $f :\subseteq X \to Y$ where the sets X and Y have notations ν_X and ν_Y is (ν_X, ν_Y)-computable if there exists a computable function $g :\subseteq \Sigma^* \to \Sigma^*$ such that the following diagram commutes:

$$\begin{array}{ccc} \Sigma^* & \xrightarrow{g} & \Sigma^* \\ {\scriptstyle \nu_X}\downarrow & & \downarrow{\scriptstyle \nu_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

That is that $\nu_Y(g(\sigma)) = f(\nu_X(\sigma))$ for all $\sigma \in \mathrm{dom}(f \circ \nu_X)$.

A finite string w is a map $w : \{0, 1, ..., n\} \to \Sigma$ and the length of the string is $|w| = n + 1$. We define an infinite string σ as a map from the natural numbers to the alphabet Σ; note that the length of the string is denoted as $|\sigma|$ and is equal to infinity. The collection of all finite strings containing letters from the alphabet Σ is denoted Σ^* and the collection of all infinite strings is denoted by Σ^ω. If w is a finite string of length n then for $m \leq n$ the partial string $w_{\lceil m}$ is defined as $w_{\lceil m} = w(0)...w(m-1)$, the partial string $w_{\lceil 0} = \varepsilon$ where ε denotes the empty string. A finite string $w \in \Sigma^*$ and an infinite string $\sigma \in \Sigma^\omega$ can be joined to make a new infinite string $w^\frown\sigma$ where

$$w^\frown\sigma(n) := \begin{cases} w(n) & \text{if } n < |w| \\ \sigma(n - |w|) & \text{otherwise} \end{cases}$$

For the purpose of this thesis we shall restrict ourselves to using numberings. A few examples of canonical numberings that we shall be using are:

2.1.1. **Examples.**

$$\nu_{\mathbb{N}} : \mathbb{N} \to \mathbb{N},\ \langle n \rangle \mapsto n$$

$$\nu_{\mathbb{Z}} : \mathbb{N} \to \mathbb{Z},\ \langle n, k \rangle \mapsto n - k$$

$$\nu_{\mathbb{Q}} : \mathbb{N} \to \mathbb{Q},\ \langle n, k, l \rangle \mapsto \frac{n-k}{l+1}$$

$$\nu_{\mathbb{Q}^+} : \mathbb{N} \to \mathbb{Q}^+ \text{ (The set of non-negative rationals)},\ \langle n, k \rangle \mapsto \frac{n}{k+1}$$

The obvious drawback to using notations and numberings is that of cardinality, as $\mathbb{N}$ and Σ^* are both at most countable we can only express countable sets. The solution to this is the concept of representations where we use a sequence of natural numbers to represent points in a space.

2.1.2. **Definition.**
A **representation** of a set X is a (partial) surjective map $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$.

A few examples of representations are the standard representation of real numbers $\rho_{\mathbb{R}} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ where $(\langle x_i, r_i \rangle)_{i\in\mathbb{N}}$ is a description of x if and only if $(\langle x_i, r_i \rangle)_{i\in\mathbb{N}}$ is a list of all the pairs that describe rational open balls containing x, the $\rho_< :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ representation which describes real numbers by a list of all rationals less than the described point and the $\rho_> :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ representation which similarly describes real numbers by the rationals greater than the described point. That is where:

$$\rho_{\mathbb{R}}((\langle x_i, r_i \rangle)_{i\in\mathbb{N}}) := x \iff \{\langle a, r \rangle \in \mathbb{N} : |x - \nu_{\mathbb{Q}}(a)| < \nu_{\mathbb{Q}^+}(r)\} = \{\langle x_i, r_i \rangle : i \in \mathbb{N}\}$$
$$\rho_<((x_i)_{i\in\mathbb{N}}) := x \iff \{a \in \mathbb{N} : \nu_{\mathbb{Q}}(a) < x\} = \{x_i : i \in \mathbb{N}\}$$
$$\rho_>((x_i)_{i\in\mathbb{N}}) := x \iff \{a \in \mathbb{N} : x < \nu_{\mathbb{Q}}(a)\} = \{x_i : i \in \mathbb{N}\}$$

The Baire space is the space of all infinite sequences of natural numbers with the product topology and can be represented by using the identity function. A function f from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$ is said to be computable if and only if there exists a total computable monotone function $g : \mathbb{N}^* \to \mathbb{N}^*$ such that for all $p \in \mathrm{dom}(f)$ we have that $f(p) := \sup_{n\in\mathbb{N}} g(p_{|n})$, where $p_{|n}$ is the initial n terms of the sequence p. Here $\sup_{i\in\mathbb{N}} g(p_{|i}) = q$ is defined if and only if $(\forall i)\ g(p_{|i})$ is a prefix for q and $(\forall j)(\exists i)$ such that the length of $g(p_{|i})$ is larger than j. In other words $\sup_{n\in\mathbb{N}} g(p_{|n})$ is only defined when $\sup_{n\in\mathbb{N}} g(p_{|n}) \in \mathbb{N}^{\mathbb{N}}$.

A function $f :\subseteq \mathbb{N}^* \to \mathbb{N}^*$ is computable if and only if it is (ν^*, ν^*)-computable where $\nu^* : \mathbb{N} \to \mathbb{N}^*$ is a numbering of the set of finite sequences defined by:

$$\nu^*(\langle 0, n \rangle) := \varepsilon$$
$$\nu^*(\langle i, \langle x_1, ..., x_i \rangle \rangle) := (x_1, ..., x_i)$$

where ε is a symbol denoting the empty sequence.

2.1.3. **Definition.**
If we let δ_X and δ_Y be representations of the sets X and Y then we have the following definitions:

(1) $x \in X$ is δ_X-computable if and only if there exists a computable $p \in \mathbb{N}^{\mathbb{N}}$ such that $x = \delta_X(p)$.

(2) $f :\subseteq X \to Y$ is (δ_X, δ_Y)-computable if and only if there exists a computable function $g :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that
$$f \circ \delta_X(p) = \delta_Y \circ g(p), \text{ for all } p \in \text{dom}(f \circ \delta_X).$$

(3) $f :\subseteq X \to Y$ is (δ_X, δ_Y)-continuous if and only if there exists a continuous function $g :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that
$$f \circ \delta_X(p) = \delta_Y \circ g(p), \text{ for all } p \in \text{dom}(f \circ \delta_X)$$

In the sections that follow we shall be using a number of common representations. We can now define some representations that allow us to construct new representations from old. Three popular constructions on representations that we shall be using in this text are the product representation, the sequence representation and the function representation.

Product Representation: Let δ_X and δ_Y be representations for the sets X and Y. Then the product representation is $[\delta_X, \delta_Y] :\subseteq \mathbb{N}^{\mathbb{N}} \to X \times Y$ where
$$[\delta_X, \delta_Y](\langle p, q \rangle) := (\delta_X(p), \delta_Y(q))$$
for all $p, q \in \mathbb{N}^{\mathbb{N}}$.

Sequence Representation: Let δ_X be a representation of the set X. Then the sequence representation is $\delta_X^{\mathbb{N}} :\subseteq \mathbb{N}^{\mathbb{N}} \to X^{\mathbb{N}}$ where
$$\delta_X^{\mathbb{N}}(\langle p_0, p_1, ... \rangle) := (\delta_X(p_i))_{i \in \mathbb{N}}$$
for all sequences $(p_i)_{i \in \mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$.

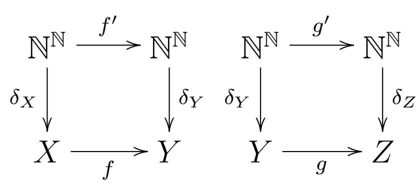
Function Representation: Let δ_X and δ_Y be representations for the sets X and Y. The function representation $[\delta_X \to \delta_Y]$ is a representation of all the total (δ_X, δ_Y)-continuous functions from X to Y and is defined using a standard representation of all the total continuous functions from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}^\mathbb{N}$ whose domains are effective G_δ sets (Prop. 6.15 [21]).

These constructions on representations allow us to confer computability notions to functions, sequences and products. For example we say that a sequence $(x_i)_{i\in\mathbb{N}}$ in X is δ_X-computable if and only if it admits a computable name with respect to $\delta_X^\mathbb{N}$.

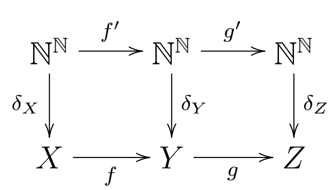
2.1.4. **Theorem.** *(Theorem 3.1.6* [20]*)*
Let δ_X, δ_Y and δ_Z be numberings or representations of the sets X, Y and Z with $f :\subseteq X \to Y$ being (δ_X, δ_Y)-computable and $g :\subseteq Y \to Z$ being (δ_Y, δ_Z)-computable then the function $g \circ f :\subseteq X \to Z$ is (δ_X, δ_Z)-computable.

Thus the computable functions in the Type-2 Theory of Effectivity are closed under composition. This can best be shown by the use of the following diagrams:

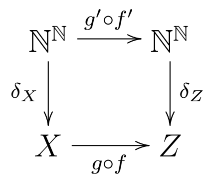
Since $f :\subseteq X \to Y$ is (δ_X, δ_Y)-computable and $g :\subseteq Y \to Z$ is (δ_Y, δ_Z)-computable there exist suitable computable functions $f', g' :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that the following two diagrams commute.



And as $f(\delta_X(p)) = \delta_Y(f'(p))$ for all $p \in \mathrm{dom}(f \circ \delta_X)$ and $g(\delta_Y(q)) = \delta_Z(g'(q))$ for all $q \in \mathrm{dom}(g \circ \delta_Y)$ we get that $g(f(\delta_X(p))) = g(\delta_Y(f'(p))) = \delta_Z(g'(f'(p)))$ for all $p \in \mathrm{dom}(g \circ f \circ \delta_X)$ and hence the following diagram commutes.



This is the same as saying that the following diagram commutes.

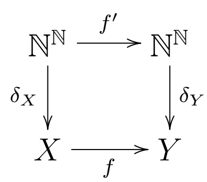


Thus $g' \circ f'$ is a partial computable function such that $g \circ f \circ \delta_X(p) = \delta_Z \circ g' \circ f'(p)$ for all $p \in \text{dom}(g \circ f \circ \delta_X)$ and hence $g \circ f$ is (δ_X, δ_Z)-computable.

Next, we show a very useful result that combines the sequence representation with computable functions. It states that computable functions preserve the computability of sequences. This is an extension of the rather obvious result that a computable function preserves computable points.

2.1.5. **Theorem.** *(Corollary 4.23* [21]*)*
Let δ_X and δ_Y be representations of the sets X and Y. If $f :\subseteq X \to Y$ is (δ_X, δ_Y)-computable and the sequence $(x_n)_{n\in\mathbb{N}}$ is δ_X-computable then the sequence $(f(x_n))_{n\in\mathbb{N}}$ is δ_Y-computable.

Proof. If $(x_n)_{n\in\mathbb{N}}$ is δ_X-computable then there exists a computable $p = \langle p_0, p_1, ...\rangle \in \mathbb{N}^{\mathbb{N}}$ such that $(x_n)_{n\in\mathbb{N}} = \delta_X^{\mathbb{N}}(p) = \delta_X^{\mathbb{N}}(\langle p_0, p_1, ...\rangle) = (\delta_X(p_n))_{n\in\mathbb{N}}$. Since there exists a partial computable function $f' :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that the following diagram commutes:



we have that $(f(x_n))_{n\in\mathbb{N}} = (f(\delta_X(p_n)))_{n\in\mathbb{N}} = (\delta_Y(f'(p_n)))_{n\in\mathbb{N}} =$
$\delta_Y^{\mathbb{N}}(\langle f'(p_0), f'(p_1), ...\rangle)$. Therefore the sequence $(f(x_n))_{n\in\mathbb{N}}$ is δ_Y-computable as needed. □

In next definition we define reducibility and equivalence of representations of the same set.

2.1.6. **Definition.**
Let $\delta, \delta' :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ be representations of a set X. We say that δ is reducible to

δ' (written $\delta \leqslant \delta'$) if there exists a computable function $f :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\delta(p) = \delta'(f(p))$ for all $p \in \mathrm{dom}(\delta)$.

We say that δ is equivalent to δ' (written $\delta \equiv \delta'$) if $\delta \leqslant \delta'$ and $\delta' \leqslant \delta$.

2.1.7. **Definition.**
On a set X with representation (numbering) δ_X a real valued function $f :\subseteq X \to \mathbb{R}$ is upper semi-continuous if and only if it is $(\delta_X, \rho_>)$-continuous and similarly it is lower semi-continuous if and only if it is $(\delta_X, \rho_<)$-continuous.

The last definition in this section is that of upper and lower semi-computable functions. We limit this definition to functions that map to the reals or rationals as we only need this concept in the context of metrics which shall be defined in the next section.

2.1.8. **Definition.**
On a set X with representation (numbering) δ_X a real valued function $f :\subseteq X \to \mathbb{R}$ is upper semi-computable if and only if it is $(\delta_X, \rho_>)$-computable and similarly it is lower semi-computable if and only if it is $(\delta_X, \rho_<)$-computable.

It should be noted that if a function is both upper and lower semi-computable then it is computable in terms of the standard representations of the reals and hence in terms of the Cauchy representation (Definition 2.2.18). It is also easy to see that if a function is upper (lower) semi-computable then it is upper (lower) semi-continuous.

2.2. **Computable Metric Spaces.**

In this section we shall give a definition of computable metric spaces and relate this definition to the concepts of Type-2 Theory of Effectivity as described in the previous section. We shall begin by giving the standard definitions of metric and topological spaces and then define the notation that shall be used throughout this thesis for open and closed balls, complements of sets, metric topologies and closures of sets.

2.2.1. **Definition.**
A metric space is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}^+$ is a function such that for all $x, y, z \in X$ the following hold:

- $d(x, y) \geq 0$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) + d(y, z) \geq d(x, z)$

If d satisfies the above criterion then d is called a metric on X.

Throughout the thesis we shall be using arbitrary metric spaces, but in a few instances we shall restrict ourselves to specific ones. Here are a few of the metric spaces that we shall be using as examples in the text.

2.2.2. **Examples.**

(1) In $\mathbb{R}$ we define the usual metric $d_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ as the absolute distance between two points, i.e. let $x, y \in \mathbb{R}$ then $d_{\mathbb{R}}(x, y) := |x - y|$.

(2) In $\mathbb{R}^n$ we define the usual metric $d_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$ as the square root of the sum of the squares of the distance between the components, i.e. let $(x_1, ..., x_n), (y_1, ..., y_n) \in \mathbb{R}^n$ then

$$d_{\mathbb{R}^n}((x_1, ..., x_n), (y_1, ..., y_n)) := \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}.$$

(3) In the product of metric spaces (X, d_X) and (Y, d_Y) we define the maximum metric $d_{X \times Y} : (X \times Y)^2 \to \mathbb{R}^+$ as

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) := \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

(4) In the space $\{0,1\}^\omega$ we define the "Cantor" metric $d_C : \{0,1\}^\omega \times \{0,1\}^\omega \to \mathbb{R}^+$ as $d_C(p,q) := 2^{-i}$ where i is the smallest number such that $p(i) \neq q(i)$ if $p \neq q$ and 0 otherwise.

(5) In a set X the discrete metric $d : X \times X \to \mathbb{R}^+$ can be defined as

$$d(x,y) := \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{otherwise.} \end{cases}$$

2.2.3. **Definition.** (Def. 13 [22])
Two sets A and B are said to be positively separated in a metric space (X, d) if

$$\inf_{a\in A,\ b\in B} d(a,b) > 0$$

2.2.4. **Definition.**
A topology on a set X is a collection τ of subsets of X where the following conditions hold:

- An arbitrary union of a family $(U_i)_{i\in I}$ of elements of τ itself belongs to τ,
$$\text{i.e. } \bigcup_{i\in I} U_i \in \tau.$$
- The intersection of every finite family $(U_i)_{i<k}$ of elements of τ belongs to τ,
$$\text{i.e. } \bigcap_{i<k} U_i \in \tau.$$
- $\emptyset \in \tau$ and $X \in \tau$.

We call any element $U \in \tau$ an open set, and (X, τ) a topological space. A set F whose complement $F' := X - F = X \backslash F$ is an open set is called closed.

Both topological and metric spaces have been intensively studied and their properties are well documented in the literature and many textbooks (see [23], [24] and [25] as examples). In what follows a certain amount of basic knowledge about topological and metric spaces is assumed though nothing beyond what can be simply derived from the definitions.

2.2.5. **Lemma.**
In a topological space (X, τ) *with a subset* $Y \subseteq X$ *the collection* $\tau_Y = \{G \cap Y : G \in \tau\}$ *is a topology on the set* Y*. This is known as the subspace topology.*

Proof.
We now show that the subspace topology fulfills all the requirements of being a topology.
Let $(V_i)_{i \in I}$ be a family of elements from τ_Y, and let $(U_i)_{i \in I}$ be a family of elements from τ such that $U_i \cap Y = V_i$. Then $\bigcup_{i \in I} U_i \in \tau$ and $\bigcup_{i \in I}(U_i \cap Y) = (\bigcup_{i \in I} U_i) \cap Y \in \tau_Y$ as needed.
Let $(V_i)_{i<k}$ be a family of elements from τ_Y, and let $(U_i)_{i<k}$ be a family of elements from τ such that $U_i \cap Y = V_i$. Then $\bigcap_{i<k} U_i \in \tau$ and $\bigcap_{i<k}(U_i \cap Y) = (\bigcap_{i<k} U_i) \cap Y \in \tau_Y$ as needed.
Finally $\emptyset \in \tau$ therefore $\emptyset = \emptyset \cap Y \in \tau_Y$ and $X \in \tau$ therefore $Y = X \cap Y \in \tau_Y$. □

2.2.6. **Definition.**
In a topological space (X, τ) a collection of sets $\mathcal{B} \subseteq \tau$ is a base for the topology τ (or alternatively a base for the topological space X) if

$$\tau = \left\{ \bigcup_{U \in \mathcal{C}} U : \mathcal{C} \subseteq \mathcal{B} \right\}.$$

A collection of sets $\mathcal{B}$ in X is a base for some topology if and only if $X = \bigcup_{B \in \mathcal{B}} B$ and for all non-disjoint sets $U, V \in \mathcal{B}$ with $x \in U \cap V$ there exists a set $B \in \mathcal{B}$ such that $x \in B \subseteq U \cap V$ (Theorem 5.3 [24]).

As we have mentioned earlier countability plays an important role in computable analysis. Thus the existence a countable base for the topology allows us to easily apply computability notions to the topological space.

2.2.7. **Definition.**
A topological space (X, τ) is said to be second countable if there exists a countable base for the topology.

This implies that we would ideally like to find a topology for metric spaces which is second countable. It is to this end that we define the open and closed balls in a metric space.

2.2.8. **Definition.**
In a metric space (X, d) we denote the open ball with centre $x \in X$ and radius $r \in \mathbb{R}$ as a set $B(x, r) \subseteq X$ where

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

And similarly we denote the closed ball with centre $x \in X$ and radius $r \in \mathbb{R}$ as $B[x, r] \subseteq X$ where

$$B[x, r] = \{y \in X : d(x, y) \leqslant r\}.$$

Using open balls we can show that any metric space is also a topological space. One such way to do this is to use the family of open balls as a base for the topology. Alternatively we can use the following definition.

2.2.9. **Definition.**
Let (X, d) be a metric space, then we can define a canonical topology on X called the metric topology τ_d. The open sets in the metric topology are those sets $U \subseteq X$ for which any $x \in U$ is such that there exists a radius r such that $B(x, r) \subseteq U$.

It should be obvious that the two definitions for the canonical topologies are equivalent as any set with the above property is an arbitrary union of open balls and any open set generated from the base of open balls has the property (due to the triangle inequality of metrics) that any point in the open set is the centre of an open ball contained in the set.

2.2.10. **Definition.**
If X is a topological space then the closure of a set $G \subseteq X$ is the set

$$\overline{G} = \bigcap\{A \subseteq X : \ A \text{ is closed and } G \subseteq A\}$$

Remark 1. It should be noted that the closure of an open ball is not necessarily the closed ball with same centre and radius. As an example take the discrete metric on the reals (Examples 2.2.2(5)) then the closure of the open ball $B(0, 1)$ is the singleton $\{0\}$ and the closed ball with same centre and radius is the entire set, i.e. $B[0, 1] = \mathbb{R}$.

A compact set in the theory of metric spaces is a closed and totally bounded complete set. An analogous characterization of compact sets in the context of topological spaces is as follows: a subset of a topological space is compact if and only if for every open cover of the subset there exists a finite subcover which also covers the set ([24]). It should be noted that in metric spaces a compact set has the property of always being contained in a closed ball, this is a consequence of the set being totally bounded.

2.2.11. **Proposition.**
Let (X, d) be a metric space and $K \subseteq X$ a compact subset. Then for all $\varepsilon > 0$ there exists a finite subset $F \subseteq K$ such that

$$K \subseteq \bigcup_{x \in F} B(x, \varepsilon).$$

Proof.
Let $K \subseteq X$ be compact and fix $\varepsilon > 0$.
Define $\mathcal{G} = \{B(x, \varepsilon) : x \in K\}$. Then $\mathcal{G}$ is an open cover of K and since K is compact there exists a finite subcover $\mathcal{F}$ of $\mathcal{G}$ that covers K. Then $F = \{x \in K : B(x, \varepsilon) \in \mathcal{F}\}$ is a finite subset of K such that

$$K \subseteq \bigcup_{x \in F} B(x, \varepsilon).$$

□

Type-2 Theory of Effectivity builds a framework for talking about the computability of finite and infinite objects. This is then generalized to allow these objects to act as names for the points in arbitrary spaces, the limitation being that these spaces must have at most continuum cardinality. We put a further restriction on the spaces that we shall using by requiring that the spaces be separable.

2.2.12. **Definition.**
Let (X, τ) be a topological space then X is said to be separable if and only if there is a countable subset $Q \subseteq X$ such that $\overline{Q} = X$. A metric space (X, d) is separable if in the topological space (X, τ_d) generated by the metric d, there exists a set Q such that $\overline{Q} = X$, we denote this separable metric space as (X, Q, d).

We can now weaken our initial requirements for a standard base of the metric topology generated from a metric space so that instead of using the family of all open balls we can limit ourselves to using just those open balls with rational radius and centre from a separable dense subset of the space. This has the added advantage of being a countable collection from which any open set in the metric topology can be written as the union of countably many elements from this collection.

2.2.13. **Theorem.**
Let (X,Q,d) be a separable metric space with metric topology τ_d. Let $\mathcal{B}$ be the set of all open balls $B(x,r)$ where $x \in Q$ and $r \in \mathbb{Q}^+$. Then $\mathcal{B}$ is a base for the metric topology τ_d.

Proof. Let $U \in \tau_d$. Then for all $x \in U$ there exists a radius $r_x > 0$ such that $B(x, r_x) \subseteq U$. Let $q_x \in \mathbb{Q}^+$ and $x_q \in Q$ be such that $q_x < \frac{r_x}{2}$ and $d(x_q, x) < q_x$. The existence of x_q is guaranteed by Q being dense in X. By the triangle inequality of metrics we guarantee that $x \in B(x_q, q_x) \subseteq B(x, r_x) \subseteq U$.

Then $\mathcal{C}_U := \{B(x_q, q_x) : x \in U\}$ is such that $\bigcup \mathcal{C}_U = U$.

We therefore have that for any $U \in \tau_d$ the collection of elements $\mathcal{C}_U$ of $\mathcal{B}$ is such that $\bigcup \mathcal{C}_U = U$. Therefore every open set can be generated by an arbitrary union of elements of $\mathcal{B}$. Hence $\mathcal{B}$ is a base for the metric topology. □

2.2.14. **Corollary.**
If (X, Q, d) is a separable metric space then a set $U \subseteq X$ is an open set in the topology induced by the metric d if and only if there exists sequences $(q_i)_{i\in\mathbb{N}}$ in Q and $(r_i)_{i\in\mathbb{N}}$ in $\mathbb{Q}^+$ such that

$$U = \bigcup_{i\in\mathbb{N}} B(q_i, r_i)$$

So in a separable metric space any point can be "approximated" as the limit of a Cauchy sequence from the dense subset, but in order for us to be able to represent the space in the context of Type-2 Theory of Effectivity we also need that this dense subset be enumerable. In a metric space another natural property that we would like is for the metric itself to be in some way computable. This leads us to the next definition in which we restrict ourselves to the metric spaces which have an enumerable dense subset and a metric that is computable on this subset. It is interesting to note that nearly all the spaces in which real world applications of computability are being sought and "meaningful" questions are being asked are of this type.

2.2.15. **Definition.** (Computable Metric Space)

(X, Q, d, α) is a computable metric space (CMS) if the following three conditions are met:

(1) (X, Q, d) is a separable metric space.
(2) $\alpha : \mathbb{N} \to Q$ is a numbering of Q.
(3) $d \circ (\alpha \times \alpha) : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is $(\nu_{\mathbb{N}}, \nu_{\mathbb{N}}, \rho_{\mathbb{R}})$-computable.

There is another useful characterization of computable metric spaces requiring that the metric be both upper and lower semi-computable.

2.2.16. **Lemma.** *(Lemma 4.4.8* [26], *Def. 2.1* [27]*)*
Let (X, Q, d) be a separable metric space with a total numbering $\nu_Q : \mathbb{N} \to Q$ then (X, Q, d, ν_Q) is a computable metric space if and only if the sets

$$D_< := \{(i, j, k) \in \mathbb{N}^3 : d(\nu_Q(i), \nu_Q(j)) < \nu_{\mathbb{Q}^+}(k)\}$$

$$D_> := \{(i, j, k) \in \mathbb{N}^3 : \nu_{\mathbb{Q}^+}(k) < d(\nu_Q(i), \nu_Q(j))\}$$

are c.e.

There are many common computable metric spaces, what follows is a few examples that we shall be using in the text.

2.2.17. **Examples.**

(1) The space $(\mathbb{Q}, \mathbb{Q}, d_{\mathbb{Q}}, \nu_{\mathbb{Q}})$ of rational numbers where $d_{\mathbb{Q}}$ is the Euclidean metric on $\mathbb{R}$ restricted to the rationals and $\nu_{\mathbb{Q}}$ is as defined in Examples 2.1.1.

(2) The Euclidean space $(\mathbb{R}, \mathbb{Q}, d_{\mathbb{R}}, \nu_{\mathbb{Q}})$ where $d_{\mathbb{R}}$ is the usual Euclidean metric as defined in Examples 2.2.2(1).

(3) The product space of two computable metric spaces (X, Q_X, d_X, α_X) and (Y, Q_Y, d_Y, α_Y) is a computable metric space $(X \times Y, Q_X \times Q_Y, d_{X\times Y}, \alpha_{X\times Y})$

where $\alpha_{X\times Y}(\langle a,b\rangle) := (\alpha_X(a), \alpha_Y(b))$ and $d_{X\times Y}$ is the maximum metric (Examples 2.2.2(3)).

(4) The Cantor space $(\{0,1\}^\omega, \{0,1\}^*0^\omega, d_C, \nu)$ where d_C is the "Cantor" metric (Examples 2.2.2(4)) and $\nu : \mathbb{N} \to \{0,1\}^*0^\omega$, $n \mapsto 2_n 0^\omega$ (where 2_n is the inverted binary expansion of n, i.e. $2_n := (x_1, ..., x_i)$, where $\sum_{j\leq i} 2^j \cdot x_j = n$ and $2_0 := \varepsilon$).
The open balls in the Cantor space are $N_\sigma := \{p \in \{0,1\}^\omega : (\exists w \in \{0,1\}^\omega)\ \sigma^\frown w = p\}$, where $\sigma \in \{0,1\}^*$.

The Cantor space can be generalized by replacing the set $\{0,1\}$ by an arbitrary finite alphabet Σ and 2_n with a numbering of all the finite sequences of elements from Σ. The Cauchy representation and the standard representation of the Cantor space and the Euclidian space are used extensively in the literature on both Martin-Löf randomness and computable analysis. We shall now define the same notions here for generalized computable metric spaces.

2.2.18. **Definition.**
(Cauchy and Standard Representations of Computable Metric Spaces)
The Cauchy representation of a computable metric space (X, Q, d, α_Q) is $\rho_X :\subseteq \mathbb{N}^\mathbb{N} \to X$ defined by

$$\rho_X((x_i)_{i\in\mathbb{N}}) = x \iff (\forall i)\ d(x, \alpha_Q(x_i)) \leq 2^{-i}$$

The standard representation of points in a computable metric space (X, Q, d, α) is $\delta_\alpha :\subseteq \mathbb{N}^\mathbb{N} \to X$ where

$$\delta_\alpha((\langle x_i, r_i\rangle)_{i\in\mathbb{N}}) = x \iff \{\langle y, r\rangle \in \mathbb{N} : x \in B(\alpha(y), \nu_{\mathbb{Q}^+}(r))\} = \{\langle x_i, r_i\rangle : i \in \mathbb{N}\}.$$

Thus the standard and Cauchy representations are representations for a computable metric space using the numbering of the dense subset and the metric. It is for this reason that we limited ourselves to those metric spaces with an effectively dense subset and a metric computable with respect to the numbering of the dense subset.

Remark 2. If we have two CMS's (X, Q, d_X, ν_Q) and (Y, Q_Y, d_Y, ν_Y) with Cauchy representations ρ_X and ρ_Y, then the (ρ_X, ρ_Y)-continuous functions are exactly the ordinary continuous functions (Prop. 7.4 [21]).

2.2.19. **Definition.**
Let (X, τ) be a topological space and let $\tau_>$ and $\tau_<$ be the topologies on the reals generated from the bases $\{(-\infty, r) : r \in \mathbb{R}\}$ and $\{(r, \infty) : r \in \mathbb{R}\}$ respectively. A real valued function $f : X \to \mathbb{R}$ is called an upper semi-continuous function if it is $(\tau, \tau_>)$-continuous and lower semi-continuous if it is $(\tau, \tau_<)$-continuous.

We can now show that upper semi-continuity on topological spaces and upper semi-continuity on represented spaces are equivalent characterizations on computable metric spaces.

2.2.20. **Lemma.**
Let (X, Q, d_X, ν_Q) be a computable metric space and $f : X \to \mathbb{R}$ a real valued function. Then the following hold:

(1) *f is $(\nu, \rho_<)$-continuous if and only if it is $(\tau_d, \tau_<)$-continuous.*

(2) *f is $(\nu, \rho_>)$-continuous if and only if it is $(\tau_d, \tau_>)$-continuous.*

Proof.
The proof that f is $(\nu, \rho_<)$-continuous if and only if it is $(\tau_d, \tau_<)$-continuous follows from Definition 4.1.3 [20] and Theorem 3.2.7 [20], which implies that $\rho_<$ is an admissible representation with respect to $\tau_<$, and Theorem 3.2.11 [20] which finishes the proof. Similarly we get that f is $(\nu, \rho_>)$-continuous if and only if it is $(\tau, \tau_>)$-continuous. $\square$

An equivalent characterization of upper and lower semi-continuity on a metric space X is that a real valued function $f : X \to \mathbb{R}$ is upper semi-continuous if for all $x \in X$ and convergent sequence $x_n \to x$ we have that $\lim_{n\to\infty} \sup_{m>n} f(x_m) = f(x)$, or equivalently if $\lim_{n\to\infty} \sup_{m>n} f(x_m) \leqslant f(x)$ for all $x \in X$. The function f is lower semi-continuous if $-f$ is upper semi-continuous (see [28], [24]).

The following definition is a natural consequence of Corollary 2.2.14. It states that a c.e. open set is an open set which can be described using a c.e. subset of $\mathbb{N}$.

2.2.21. **Definition.**
In a CMS (X, Q, d_X, ν_Q) a set $U \subseteq X$ is (called) a c.e. open set if there exists a c.e. subset $A \subseteq \mathbb{N}^2$ where

$$U = \bigcup_{(n,k)\in A} B(\nu_Q(n), \nu_{\mathbb{Q}^+}(k)).$$

In a topological space a G_δ set is the intersection of a sequence of open sets and a F_σ set is the union of a sequence of closed sets.

2.2.22. **Definition.**
A topological space (X, τ) is a Hausdorff space if for any two distinct points $x, y \in X$ there exists two open sets $G_1, G_2 \in \tau$ such that $G_1 \cap G_2 = \emptyset$ and $x \in G_1$ and $y \in G_2$.

We now go on to prove that computable metric spaces are second countable Hausdorff spaces. The first thing we do is prove that computable metric spaces are Hausdorff.

2.2.23. **Lemma.**
If (X, Q, d_X) is a separable metric space then X is a Hausdorff space.

Proof.
Let (X, Q, d_X) be a separable metric space. If $x, y \in X$ are such that $x \neq y$ then $d_X(x, y) > 0$. Let $c = d_X(x, y)$. Then the open balls $B(x, \frac{c}{2})$ and $B(y, \frac{c}{2})$ are open subsets of X. Assume $z \in B(x, \frac{c}{2}) \cap B(y, \frac{c}{2})$. Then $c = d_X(x, y) \leq d_X(x, z) + d_X(z, y) < \frac{c}{2} + \frac{c}{2} = c$ which is clearly impossible. Therefore $B(x, \frac{c}{2}) \cap B(x, \frac{c}{2}) = \emptyset$ as needed.
Therefore X is a Hausdorff space. □

As we promised earlier we can now show that computable metric spaces have the desirable property of being second countable.

2.2.24. **Lemma.**
If (X, Q, d_X) is a separable metric space then X is second countable.

Proof.
In any separable metric space (X, Q, d_X) the collection $\{B(q, r) : q \in Q,\ r \in \mathbb{Q}^+\}$ is a countable base for the metric topology generated from the metric d_X. □

So computable metric spaces are second countable Hausdorff spaces, but we would still like a further property and that is for computable metric spaces to be locally compact as well.

2.2.25. **Definition.**
Let (X, τ) be a Hausdorff topological space. The space X is said to be locally compact if for each element $x \in X$ there exists an open set, contained in a compact set, that contains the point x.

Unfortunately not all computable metric spaces are locally compact. A very easy example of this is the CMS $(\mathbb{Q}, \mathbb{Q}, d_{\mathbb{Q}}, \mu_{\mathbb{Q}})$ as defined in example 1 in 2.2.17. In this space all of the compact subsets have empty interior and hence no open set can be contained in them. Thus $(\mathbb{Q}, \mathbb{Q}, d_{\mathbb{Q}}, \mu_{\mathbb{Q}})$ is not locally compact.

Note that in the text that follows we shall be using computable metric spaces, compact computable metric spaces, and locally compact computable metric spaces depending on the situation, though the level of compactness being used shall always be clearly mentioned. The following lemma is the well known fact that compact topological spaces are locally compact.

2.2.26. **Lemma.**
If a Hausdorff topological space (X, τ) is compact then it is locally compact.

Hence a compact CMS is locally compact. Thus compact computable metric spaces and locally compact computable metric spaces are locally-compact second-countable Hausdorff spaces or as it is sometimes referred to in the literature a LCHS space.

3. **Hyperspaces**

In chapter 3 we shall be exploring the concepts of the hyperspaces of compact sets of a metric space and open sets of a metric space. We shall include the standard representations of these hyperspaces and show a few equivalent definitions of c.e. subsets, computable subsets and computable points in the hyperspaces.

3.1. **Open Subsets.**

We designate the set of all open subsets of a topological space (X, τ) as $\mathcal{O}(X)$.

3.1.1. **Definition.**
With a computable metric space (X, Q, d_X, ν_Q) the representation of the set of all open subsets $\mathcal{O}(X)$ is designated $\vartheta :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathcal{O}(X)$ where for any sequence $\langle q_i, r_i \rangle_{i\in\mathbb{N}}$ of natural numbers, i.e. an element of $\mathbb{N}^{\mathbb{N}}$, we have that

$$\vartheta(\langle q_i, r_i \rangle_{i\in\mathbb{N}}) := \bigcup_{i\in\mathbb{N}} B(\nu_Q(q_i), \nu_{\mathbb{Q}^+}(r_i))$$

One important aspect of this definition, that shall be used later in this section, is that the empty set can be represented by any open ball with a zero radius, i.e. $B(\nu_Q(n), \nu_{\mathbb{Q}^+}(0)) = \emptyset$ for any $n \in \mathrm{dom}(\nu_Q)$. This is in contrast to the representation of compact sets, which we define in the next section, where we specifically leave out the empty set. With this representation of open sets we can now see that in a metric space the c.e. open sets are just the ϑ-computable sets and that the computable sequences of c.e. open sets are just the $\vartheta^{\mathbb{N}}$-computable sequences of sets.

Another important characterisation of the open sets is used to define the pre-image representation which essentially represents open sets as the zero set of a total real valued continuous function from the set, and is equivalent to our standard representation of the open subsets ϑ (Theorem 3.10 [27]).

3.1.2. **Definition.** (Pre-image Representation)
Let δ_X be the Cauchy representation of a CMS (X, Q, d_X, ν_Q) and p is a sequence of natural numbers. Then the pre-image representation is $\vartheta' :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathcal{O}(X)$ where

$$\vartheta'(p) = U \iff ([\delta_X \to \rho_{\mathbb{R}}](p))^{-1}\{0\} = X\backslash U.$$

The intersection property was defined by Hertling and Weihrauch [13] and is used to define computable functions in computable topological measure spaces. We shall later be showing that any computable metric space with the Hausdorff measure and topology induced by the metric is a computable topological measure space as defined by Hertling and Weihrauch and it always satisfies the 'intersection property.'

3.1.3. **Definition.**
In a topological space (X, τ), we say that a sequence $(U_i)_{i\in\mathbb{N}}$ of open sets in X satisfies the intersection property if and only if there exists a c.e. set $A \subseteq \mathbb{N}$ such that

$$U_i \cap U_j = \bigcup\{U_k : \langle i, j, k\rangle \in A\} \text{ for all } i, j \in \mathbb{N}$$

We say that a topological space (X, τ) with a base $\mathcal{B}$ and total numbering $\nu_{\mathcal{B}} : \mathbb{N} \to \mathcal{B}$ satisfies the intersection property if the numbering of the base for the topology satisfies the intersection property.

3.1.4. **Theorem.**
The set functions $\bigcup : \mathcal{O}(X) \times \mathcal{O}(X) \to \mathcal{O}(X)$ *and* $\bigcap : \mathcal{O}(X) \times \mathcal{O}(X) \to \mathcal{O}(X)$ *are* $([\vartheta, \vartheta], \vartheta)$*-computable.*

Proof.

- $\bigcup$**:** Let p and q be two sequences of natural numbers that are the description of the two open sets U and V in X. Then we can obtain the union of two open sets in a natural way by $\bigcup(\vartheta(p), \vartheta(q)) := \vartheta(\langle p, q\rangle)$ and since combining two sequences of natural numbers into one sequence is computable we have that $\bigcup$ is $([\vartheta, \vartheta], \vartheta)$-computable.

- $\bigcap$**:** Let p and q be two sequences of natural numbers that are the pre-image description of the two open sets U and V in X. Then the two following equivalences hold

$$\vartheta'(p) = U \iff ([\delta_X \to \rho_{\mathbb{R}}](p))^{-1}\{0\} = U'$$
$$\vartheta'(q) = V \iff ([\delta_X \to \rho_{\mathbb{R}}](q))^{-1}\{0\} = V'$$

We can now obtain the set function $\bigcap :\subseteq \mathcal{O}(X) \times \mathcal{O}(X) \to \mathcal{O}(X)$ as

$$\bigcap(\vartheta(p), \vartheta(q)) := (([\delta_X \to \rho_{\mathbb{R}}](p) \cdot [\delta_Y \to \rho_{\mathbb{R}}](q))^{-1}\{0\})'.$$

Which is $([\vartheta', \vartheta'], \vartheta')$-computable, because the product is a computable function on the reals (Theorem 4.14 [21]) and any computable function can be lifted to the space of total real-valued continuous functions using evaluation and type conversion (Prop. 6.15 [21]), and hence the intersection is $([\vartheta, \vartheta], \vartheta)$-computable, since $\vartheta' \equiv \vartheta$.

□

3.1.5. **Corollary.**
If (X, Q, d_X, ν_Q) is a computable metric space then with the standard base for the metric topology and associated numbering it satisfies the intersection property.

Proof. The base $\mathcal{B}$ for the metric topology τ_{d_X} as defined in Theorem 2.2.13 has a total numbering $B_{\langle a,r\rangle} : \mathbb{N} \to \mathcal{B}$, $\langle a, r\rangle \mapsto B(\nu_Q(a), \nu_{\mathbb{Q}^+}(r))$. The sequence $(B_i)_{i\in\mathbb{N}}$ is a $\vartheta^{\mathbb{N}}$-computable sequence. Therefore using Theorem 3.1.4 we have that there exists a partial computable function $g :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for all $\langle p, q\rangle \in \mathrm{dom}(\bigcap \circ [\vartheta, \vartheta])$, $\vartheta(g(\langle p, q\rangle)) = \bigcap(\vartheta(p), \vartheta(q))$. We define a new computable function $g' : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ as $g'(\langle i, j\rangle) := g(\langle i^{\mathbb{N}}, j^{\mathbb{N}}\rangle)$. We can then define the set $A := \{\langle i, j, k\rangle \in \mathbb{N} : (\exists l \in \mathbb{N})\ k = g'(\langle i, j\rangle)(l)\}$ which is c.e. (by parallelization) and has the property that for all $i, j \in \mathbb{N}$,

$$B_i \cap B_j = \bigcup \{B_k : \langle i, j, k\rangle \in A\}.$$

Which proves that $(B_i)_{i\in\mathbb{N}}$ satisfies the intersection property and therefore (X, Q, d_X, ν_Q) satisfies the intersection property.

□

3.2. **Compact Subsets.**

In this section we define the hyperspace of non-empty compact subsets. We shall show that this hyperspace, when generated from a computable metric space, is itself a computable metric space. In order to do this though we need a canonical metric on the hyperspace generated from the metric on the original set and a dense subset generated from the original dense subset. For this we use the Hausdorff metric and the set of all finite subsets of the dense subset.

We then go on to show that the topology generated by the Hausdorff metric on the hyperspace of all non-empty compact subsets of a computable metric space is equivalent to the Fell topology on the hyperspace.

3.2.1. **Definition.**

We denote the set of all non-empty compact subsets of a topological space (X, τ) as $\mathcal{K}(X)$. And the set of all compact subsets of a topological space (X, τ) as $\mathcal{K}^0(X)$.

It should be noted that $\mathcal{K}^0(X) = \mathcal{K}(X) \cup \{\emptyset\}$. In what follows we shall often refer to the hyperspace of non-empty compact subsets of a metric space (X, d). In this case, unless explicitly stated otherwise, we shall implicitly be using the topological space (X, τ_d) where τ_d is the metric topology generated from the metric d.

There is a standard method for generating a metric for compact sets in a metric space from the original metric. We shall use this method to define a metric for the hyperspace of compact subsets in a metric space.

3.2.2. **Definition.**

In a metric space (X, d) the Hausdorff metric $d_H : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathbb{R}$ generated by a metric d is defined as

$$d_H(A, B) := \max\{d(A, B), d(B, A)\}$$

$$\text{where } d(A, B) := \sup_{a \in A} d(a, B)$$

$$\text{and } d(a, B) := \inf_{b \in B} d(a, b)$$

Thus the metric we use in the hyperspace can be thought of as the largest shortest distance from one compact set to another. So the smaller the Hausdorff distance between two sets the closer they are to being the same set.

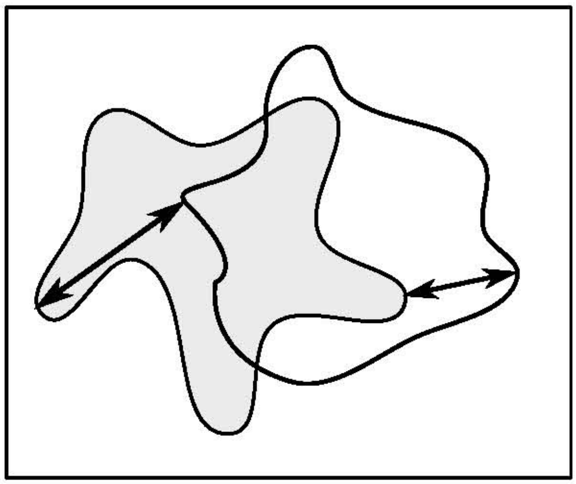


FIGURE 1. The two lines are the largest shortest distance from either set to the other. The Hausdorff distance is then the largest between the two.

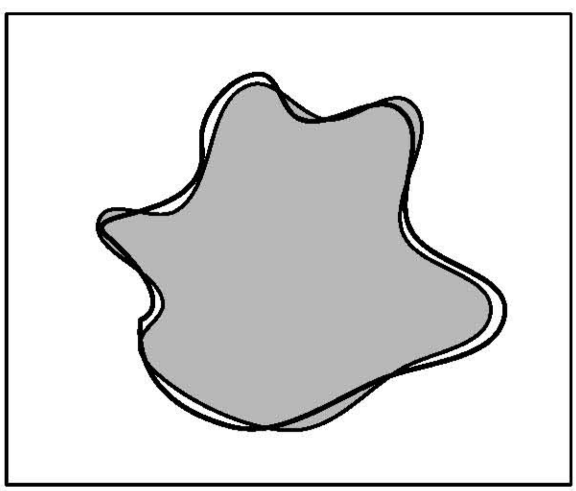


FIGURE 2. An example of two sets in $\mathbb{R}^2$ which have a small Hausdorff distance.

The singletons in the hyperspace are particularly easy to work with as they have the property that the distance between them and any set is the supremum between that point and all the points in the other set.

3.2.3. **Lemma.**
If (X, d) is a metric space, and d_H is the Hausdorff metric induced by d on the Hyperspace of compact sets $\mathcal{K}(X)$ then for any $x \in X$, $B \in \mathcal{K}(X)$ we have that

$$d_H(\{x\}, B) = \sup\{d(x, b) : b \in B\}$$

Proof. Let $B \in \mathcal{K}(X)$

$$\begin{aligned} d(\{x\}, B) &= \sup\{\inf\{d(a, b) : b \in B\} : a \in \{x\}\} \\ &= \inf\{d(x, b) : b \in B\} \\ d(B, \{x\}) &= \sup\{\inf\{d(b, a) : a \in \{x\}\} : b \in B\} \\ &= \sup\{d(b, x) : b \in B\} \\ &= \sup\{d(x, b) : b \in B\} \end{aligned}$$

$$\implies d_H(\{x\}, B) = \max\{\inf\{d(x, b) : b \in B\}, \sup\{d(x, b) : b \in B\}\}$$
$$= \sup\{d(x, b) : b \in B\}$$ □

In particular we have that $d_H(\{x\}, \{y\}) = d(x, y)$.

In other papers on the algorithmic randomness of sets previous authors have limited themselves to the closed subsets of the Cantor space or the real interval from 0 to 1 (see [13], [16], [10], [14] for examples). As both the Cantor space and the real interval are compact, these closed subsets are themselves compact. In generalizing the study to all metric spaces we shall be limiting ourselves to the compact subsets, the reasons being made clear in later sections. In order to do this we need a canonical way in which to represent the compact subsets as a computable metric space. For this we shall first need an enumerable dense subset in the hyperspace.

3.2.4. **Theorem.** *(Section 4.4.7* [26]*)*
In a separable metric space (X, Q, d) the set of non-empty finite subsets of Q (called ζ or $F(Q)$) is dense in $\mathcal{K}(X)$. Therefore $(\mathcal{K}(X), \zeta, d_H)$ is a separable metric space.

Note that the reason why we do not allow the empty set is simply so that the Hausdorff metric is well defined. We shall now define a standard method for enumerating the dense subset obtained in Theorem 3.2.4.

3.2.5. **Definition.**
In the separable metric space $(\mathcal{K}(X), \zeta, d_H)$ generated by the computable metric space (X, Q, d, q) we define the standard numbering of the non-empty finite subsets of Q as $\nu_\zeta : \mathbb{N} \to \zeta$ where $\nu_\zeta\langle k, \langle x_0, ..., x_k\rangle\rangle = \{q(x_0), ..., q(x_k)\}$.

So using our standard method of generating a representation from a numbering we get that the standard representation of the non-empty compact subsets of X is $\rho_\zeta :\subseteq \mathbb{N}^\mathbb{N} \to \mathcal{K}(X)$ where $\rho_\zeta(\langle x_i, r_i\rangle_{i\in\mathbb{N}}) = K \iff \{\langle x, r\rangle \in \mathbb{N} : K \in B(\nu_\zeta(x), \nu_{\mathbb{Q}^+}(r))\} = \{\langle x_i, r_i\rangle : i \in \mathbb{N}\}$.

We are now ready to show that the hyperspace of all non-empty compact sets is a natural choice for the extension of a computable metric space to a hyperspace, i.e. that the hyperspace consisting of non-empty compact subsets of a computable metric space is itself a computable metric space.

3.2.6. **Theorem.**
If (X, Q, d_X, ν_Q) is a computable metric space then so is the hyperspace $(\mathcal{K}(X), \zeta, d_H, \nu_\zeta)$.

Proof. We already know that ζ is dense in $\mathcal{K}(X)$ when using the metric topology induced by d_H giving us that $(\mathcal{K}(X), \zeta, d_H)$ is a separable metric space and $\nu_\zeta : \mathbb{N} \to \zeta$ is a total numbering of ζ. So all we need to do is show that $d_H \circ (\nu_\zeta, \nu_\zeta)$ is $(\nu_\mathbb{N}, \nu_\mathbb{N}, \rho_\mathbb{R})$-computable.

We have that $\nu_\zeta(n)$ is a finite set for all $n \in \mathbb{N}$ and so

$$d_H(\nu_\zeta(i), \nu_\zeta(j)) = \max\left\{\max_{n\le k}\min_{m\le l} d_X(\nu_Q(x_n), \nu_Q(y_m)), \max_{m\le l}\min_{n\le k} d_X(\nu_Q(x_n), \nu_Q(y_m))\right\}$$

where $i = \langle k, \langle x_0, ..., x_k\rangle\rangle$ and $j = \langle l, \langle y_0, ..., y_l\rangle\rangle$.

The functions $\max : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\min : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are both $(\rho_\mathbb{R}, \rho_\mathbb{R}, \rho_\mathbb{R})$-computable (Theorem 4.14 and Lemma 4.12 [21]) and $d_X \circ (\nu_Q, \nu_Q)$ is $(\nu_\mathbb{N}, \nu_\mathbb{N}, \rho_\mathbb{R})$-computable. Thus $d_H \circ (\nu_\zeta, \nu_\zeta)$ is $(\nu_\mathbb{N}, \nu_\mathbb{N}, \rho_\mathbb{R})$-computable since it can be written as a finite composition of the functions max, min and d_X (Theorem 2.1.4).

□

The next theorem is a useful result which shows that the Hausdorff metric preserves the completeness of the underlying metric space.

3.2.7. **Theorem.** *(Section 4.4.7 [26])*
The space $(\mathcal{K}(X), d_H)$ is complete if the space (X, d) is complete.

In 1975 Matheron ([29] Chapter 2) proves that in a locally compact metric space if there exists a base such that the closures of the basic open sets are compact then the hyperspace of all closed subsets of the metric space is compact. We can now use a similar approach to show that if the underlying metric space is locally compact and the closure of the open balls are compact subsets then the hyperspace of non-empty compact subsets is locally compact.

3.2.8. **Lemma.**
In a locally compact separable metric space (X, Q, d) if the base $\{B(x, q) : x \in Q$ and $q \in \mathbb{Q}^+\}$ is such that the closure of the basic open sets $\overline{B(x, q)}$ are compact for all $x \in Q$ and $q \in \mathbb{Q}^+$ then the hyperspace $\mathcal{K}(X)$ is locally compact.

Proof.
Let (X, Q, d) be a locally compact separable metric space such that the closures $\overline{B(x, q)}$ of open balls are compact for all $x \in Q$ and $q \in \mathbb{Q}^+$. Since (X, Q, d) is locally compact, it is complete and hence $(\mathcal{K}(X), d_H)$ is complete too.

Let $K \in \mathcal{K}(X)$, $r \in \mathbb{Q}$. We claim that $B[K, r]$ is compact. Since $(\mathcal{K}(X), d_H)$ is complete, it suffices to show that $B[K, r]$ is totally bounded.

Fix $\varepsilon > 0$.

There exists $x \in Q$ and $p \in \mathbb{Q}$ such that $B(x, p) \supseteq \bigcup_{k \in K} B(k, r + \varepsilon)$ because K is compact. Therefore $\overline{B(x, p)}$ is a compact subset of X and the open balls $B(q, \varepsilon)$ where $q \in Q \cap \overline{B(x, p)}$ form an open covering of the compact set $\overline{B(x, p)}$. Let the points $q_1, ..., q_n \in Q \cap \overline{B(x, p)}$ be a finite collection of points such that the open balls $B(q_i, \varepsilon)$ form a finite subcovering of $\overline{B(x, p)}$. We now define the collection of sets $F_0, ..., F_m$ as the collection of all non-empty finite combinations of the points $q_1, ..., q_n$.

Let $S \in B[K, r]$, then $d_H(S, K) \leqslant r$. By the compactness of S and K we have that for all $s \in S$ there exists $k \in K$ such that $d(s, k) \leqslant r$ and $k \in B(x, p)$. Therefore there exists $i \leqslant n$ such that $s \in B(q_i, \varepsilon)$. So for every $s \in S$ there exists a q_i such that $d(s, q_i) < \varepsilon$.

We now construct a non-empty finite set $F = \{q_i : \ d(q_i, S) < \varepsilon\}$.

Since $(F_i)_{i \leqslant m}$ contains every non-empty finite combination of the points $q_1, ..., q_n$ we have that there exists $i \leqslant m$ such that $F = F_i$. Therefore $d_H(S, F_i) < \varepsilon$ and hence $S \in \bigcup_{i \leqslant m} B(F_i, \varepsilon)$.

Therefore for all $\varepsilon > 0$ the closed set $B[K, r]$ can be covered by finitely many open balls with radius ε. Therefore $B[K, r]$ is totally bounded and hence compact.

Thus every point $K \in \mathcal{K}(X)$ has a compact neighbourhood, therefore $\mathcal{K}(X)$ is locally compact. □

We have already defined what a c.e. open set is. What follows is a definition for a concept that allows us to speak of a compact set being co-c.e. The intuition behind this concept is that a compact set is co-c.e. if its complement is c.e. This is equivalent to there being an enumeration of all the open coverings of a set consisting of finitely many open balls with centre from the dense subset and rational radius.

3.2.9. **Definition.**
In a CMS (X, Q, d, q) a compact set K is co-c.e. compact if the set of finite coverings $\{\langle k, \langle\langle x_1, r_1\rangle, ..., \langle x_k, r_k\rangle\rangle\rangle : K \subseteq \bigcup_{i=1}^{k} B(q(x_i), \nu_{\mathbb{Q}^+}(r_i))\}$ is c.e.

3.2.10. **Lemma.** *In a hyperspace $(\mathcal{K}(X), d_H)$ of non-empty compact subsets of a metric space (X, d), if $K \in \mathcal{K}(X)$, then*

$$B(K, r) = \left\{ F \in \mathcal{K}(X) : \ K \subseteq \bigcup_{x \in F} B(x, r) \text{ and } (\forall x \in F)(B(x, r) \cap K \neq \emptyset) \right\}.$$

Proof.

Let $K \in \mathcal{K}(X)$ and $r > 0$.

Let $F \in \mathcal{K}(X)$ then

$$\begin{aligned} & \sup_{f\in F}\inf_{k\in K} d(f,k) < r \\ \iff & (\forall f \in F)(\exists k \in K)\ d(f,k) < r \\ \iff & (\forall f \in F)\ B(f,r) \cap K \neq \emptyset \end{aligned}$$

and

$$\begin{aligned} & \sup_{k\in K}\inf_{f\in F} d(f,k) < r \\ \iff & (\forall k \in K)(\exists f \in F)\ d(f,k) < r \\ \iff & K \subseteq \bigcup_{f\in F} B(f,r). \end{aligned}$$

Therefore $F \in B(K,r)$ if and only if $K \subseteq \bigcup_{f\in F} B(f,r)$ and $(\forall f \in F)\ B(f,r) \cap K \neq \emptyset$.

□

We now look at upper semi-continuity on compact hyperspaces of non-empty compact subsets of a metric space. We should note that the hyperspace $\mathcal{K}(X)$ generated from a metric space (X,d) is a partially ordered set under set inclusion.

3.2.11. **Lemma.**
Let $(\mathcal{K}(X), d_H)$ be the metric space generated from the compact metric space (X,d) and $(K_n)_{n\in\mathbb{N}}$ be a decreasing sequence in the hyperspace. Then the following are equivalent

(1) $K_n \to K$.

(2) $K = \bigcap_{i=0}^{\infty} \overline{\bigcup_{n>i} K_n}$.

Proof.
(1) $\implies$ (2)

Let $K_n \to K$ then $(\forall \varepsilon > 0)$ $(\exists N_\varepsilon \in \mathbb{N})$ such that $(\forall n > N_\varepsilon)$ $d_H(K_n, K) < \varepsilon$. Let $x \in K$ then for each $n \in \mathbb{N}$ we pick an $m > N_{2^{-n}}$ and $x_n \in K_m$ such that

$d(x, x_n) < 2^{-n}$. Therefore $x_n \to x$ and $x \in \overline{\bigcup_{m>i} K_m}$ for all $i \in \mathbb{N}$. Therefore $x \in \bigcap_{i=0}^{\infty} \overline{\bigcup_{n>i} K_n}$.

We now let $x \in \bigcap_{i=0}^{\infty} \overline{\bigcup_{n>i} K_n}$ then for all $i \in \mathbb{N}$ we have that $x \in \overline{\bigcup_{n>i} K_n}$. So for all $i \in \mathbb{N}$ we choose an $x_i \in \bigcup_{n>N_{2^{-i}}} K_n$ such that $d(x_i, x) < 2^{-i}$. Note that this point exists since for all $n > N_{2^{-i}}$ we have that $d_H(K_n, K) < 2^{-i}$. Then $x_i \to x$ and $x \in K$ since K is compact.

Therefore $K = \bigcap_{i=0}^{\infty} \overline{\bigcup_{n>i} K_n}$.

(2) $\Longrightarrow$ (1)

Let $K = \bigcap_{i=0}^{\infty} \overline{\bigcup_{n>i} K_n}$. Assume that there exists an $\varepsilon > 0$ such that $\forall N \in \mathbb{N}$ we have that $\exists n > N$ such that $d_H(K_n, K) \geqslant \varepsilon$. But $(\overline{\bigcup_{n>i} K_n})_{i\in\mathbb{N}}$ is a decreasing sequence, so $K = \bigcap_{i=0}^{\infty} \overline{\bigcup_{n>i} K_n} = \lim_{i\to\infty} \overline{\bigcup_{n>i} K_n}$. And since $(K_n)_{n\in\mathbb{N}}$ is decreasing sequence we have that $K = \overline{\lim_{i\to\infty} K_n}$. But this obviously contradicts our assumption, therefore $K_n \to K$ as needed.

$\square$

3.2.12. **Lemma.**
Let $(\mathcal{K}(X), d_H)$ be the metric space generated from the compact metric space (X, d) and $f : \mathcal{K}(X) \to \mathbb{R}$ a real valued function. If for all $A \subseteq B$ we have that $f(A) \leqslant f(B)$ then the following are equivalent

(1) *For all $K \in \mathcal{K}(X)$ and decreasing sequence $(K_i)_{i\in\mathbb{N}}$ converging to K we have that $\lim_{n\to\infty} f(K_n) = f(K)$.*

(2) *f is upper semi-continuous.*

Proof.
(2) $\Longrightarrow$ (1)

This follows immediately from the characterization of upper semi-continuity defined after Lemma 2.2.20.

(1) $\Longrightarrow$ (2)

Let $f : \mathcal{K}(X) \to \mathbb{R}$ be such that for all $A \subseteq B$ we have that $f(A) \leqslant f(B)$ and for all $K \in \mathcal{K}(X)$ and decreasing sequence $(K_i)_{i\in\mathbb{N}}$ converging to K we have that $\lim_{n\to\infty} \sup_{m>n} f(K_m) \leqslant f(K)$.

Let $K \in \mathcal{K}(X)$ and $(K_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}(X)$ such that $K_n \to K$. We now define the sequence $F_i = \overline{\bigcup_{n>i} K_n}$. Note that since X is compact we have that the closures of the sets $\bigcup_{n>i} K_n$ are also compact and hence in $\mathcal{K}(X)$. Then $K = \bigcap_{i\in\mathbb{N}} F_i$ by Lemma 3.2.11 and for all $i \in \mathbb{N}$ we have that $K_i \subseteq F_i$ and hence that $f(K_i) \leqslant f(F_i)$. Since $(K_i)_{i\in\mathbb{N}}$ is a decreasing sequence and hence that $(F_i)_{i\in\mathbb{N}}$ is a decreasing sequence we have that $K = \bigcap_{i\in\mathbb{N}} F_n = \bigcap_{i\in\mathbb{N}} \overline{\bigcup_{n>i} F_n}$, therefore by Lemma 3.2.11 we have that $(F_n)_{n\in\mathbb{N}}$ converges to K. Therefore $\lim_{n\to\infty} f(F_n) \leqslant f(K)$ and hence

$$\lim_{n\to\infty} f(K_n) \leqslant \lim_{n\to\infty} f(F_n) \leqslant f(K).$$

Therefore f is upper semi-continuous.

□

As we shall see in the next section the previous lemma shows the connection between the Hausdorff metric topology and the Fell topology on the hyperspace of non-empty compact subsets.

3.3. **The Fell Topology.**

In this section we define the upper, lower Fell topologies and the Fell topology on the hyperspace of non-empty compact subsets and prove some useful results with the Fell topology.

We begin by defining the hit and miss set operations on the hyperspace of compact subsets and the hyperspace of non-empty compact subsets.

3.3.1. **Definition.**
Let (X, τ) be a topological space and $\mathcal{K}(X)$ the collection of non-empty compact subsets of X then we can define three subsets of the hyperspace as follows:

$$\mathcal{K}^A := \{K \in \mathcal{K}(X) : K \cap A = \emptyset\}$$
$$\mathcal{K}_A := \{K \in \mathcal{K}(X) : K \cap A \neq \emptyset\}$$
$$\mathcal{K}_B^A := \mathcal{K}^A \cap \mathcal{K}_B$$

The set $\mathcal{K}_A$ is the collection of all non-empty compact sets which hit a given set A, the set $\mathcal{K}^A$ is the collection of all non-empty compact sets which miss the set A and the set $\mathcal{K}_B^A$ is the collection of all non-empty compact sets which both hit the set B and miss the set A.

We define the hit and miss operations on the hyperspace of compact sets $\mathcal{K}^0(X)$ similarly, but allowing the collections to include the empty set.

$$\mathcal{K}^A := \{K \in \mathcal{K}^0(X) : K \cap A = \emptyset\}$$
$$\mathcal{K}_A := \{K \in \mathcal{K}^0(X) : K \cap A \neq \emptyset\}$$
$$\mathcal{K}_B^A := \mathcal{K}^A \cap \mathcal{K}_B$$

The natural question, now that we have defined these two set operations, is how do the hit and miss operations behave with respect to our previously defined set operations such as union and intersection. In the next proposition we show that the union of sets that *hit* and the intersection of sets that *miss* are in a sense symmetric, whilst the union of sets that *miss* and the intersection of sets that *hit* are not so well behaved.

3.3.2. **Proposition.** *(Proposition 2.4.3 [18])*
If $(A_i)_{i\in\mathbb{N}}$ is a sequence of subsets of a topological space (X,τ) then the following hold:

(1) $\bigcap_{i\in\mathbb{N}} \mathcal{K}^{A_i} = \mathcal{K}^{\bigcup_{i\in\mathbb{N}} A_i}$

(2) $\bigcup_{i\in\mathbb{N}} \mathcal{K}_{A_i} = \mathcal{K}_{\bigcup_{i\in\mathbb{N}} A_i}$

(3) $\bigcup_{i\in\mathbb{N}} \mathcal{K}^{A_i} \subseteq \mathcal{K}^{\bigcap_{i\in\mathbb{N}} A_i}$

(4) $\bigcap_{i\in\mathbb{N}} \mathcal{K}_{A_i} \supseteq \mathcal{K}_{\bigcap_{i\in\mathbb{N}} A_i}$

Proof.

(1) If $A \in \bigcap_{i\in\mathbb{N}} \mathcal{K}^{A_i}$ then $A \cap A_i = \emptyset$ for all $i \in \mathbb{N}$, hence $A \in \mathcal{K}^{\bigcup_{i\in\mathbb{N}} A_i}$. Similarly if $A \in \mathcal{K}^{\bigcup_{i\in\mathbb{N}} A_i}$ then $A \cap A_i = \emptyset$ for all $i \in \mathbb{N}$ and hence $A \in \mathcal{K}^{A_i}$ for all $i \in \mathbb{N}$, therefore $A \in \bigcap_{i\in\mathbb{N}} \mathcal{K}^{A_i}$. Therefore $\bigcap_{i\in\mathbb{N}} \mathcal{K}^{A_i} = \mathcal{K}^{\bigcup_{i\in\mathbb{N}} A_i}$.

(2) If $A \in \bigcup_{i\in\mathbb{N}} \mathcal{K}_{A_i}$ then there exists an $j \in \mathbb{N}$ such that $A \cap A_j \neq \emptyset$, therefore $A \cap \bigcup_{i\in\mathbb{N}} A_i \neq \emptyset$ and hence $A \in \mathcal{K}_{\bigcup_{i\in\mathbb{N}} A_i}$. Similarly if $A \in \mathcal{K}_{\bigcup_{i\in\mathbb{N}} A_i}$ then $A \cap \bigcup_{i\in\mathbb{N}} A_i \neq \emptyset$, therefore there exists a $j \in \mathbb{N}$ such that $A \cap A_j \neq \emptyset$ and therefore $A \in \mathcal{K}_{A_j}$. Therefore $A \in \bigcup_{i\in\mathbb{N}} \mathcal{K}_{A_i}$. Therefore $\bigcup_{i\in\mathbb{N}} \mathcal{K}_{A_i} = \mathcal{K}_{\bigcup_{i\in\mathbb{N}} A_i}$.

(3) If $A \in \bigcup_{i\in\mathbb{N}} \mathcal{K}^{A_i}$ then there exists a $j \in \mathbb{N}$ such that $A \in \mathcal{K}^{A_j}$. Therefore $A \cap A_j = \emptyset$. Therefore $A \cap \bigcap_{i\in\mathbb{N}} A_i = \emptyset$. Therefore $A \in \mathcal{K}^{\bigcap_{i\in\mathbb{N}} A_i}$. Therefore $\bigcup_{i\in\mathbb{N}} \mathcal{K}^{A_i} \subseteq \mathcal{K}^{\bigcap_{i\in\mathbb{N}} A_i}$.

(4) If $A \in \mathcal{K}_{\bigcap_{i\in\mathbb{N}} A_i}$ then $A \cap \bigcap_{i\in\mathbb{N}} A_i \neq \emptyset$. Let $x \in A \cap \bigcap_{i\in\mathbb{N}} A_i$. Then x is a witness that A hits each set A_i and hence for all $i \in \mathbb{N}$ the set $A \in \mathcal{K}_{A_i}$. Therefore $A \in \bigcap_{i\in\mathbb{N}} \mathcal{K}_{A_i}$. Therefore $\mathcal{K}_{\bigcap_{i\in\mathbb{N}} A_i} \subseteq \bigcap_{i\in\mathbb{N}} \mathcal{K}_{A_i}$.

□

Remark 3. Note that the previous proof remains unchanged whether we are using the hit and miss sets with respect to the compact subsets or the non-empty compact subsets. Unless otherwise stated the following theorems apply equally to both instances.

We now consider the power set of X as a partially ordered set with regard to set inclusion and show that the hit and miss operations are in a certain sense order preserving.

3.3.3. **Proposition.**
If $A_1, A_2 \subseteq X$ then the following hold:

(1) $\mathcal{K}_{A_1} \subseteq \mathcal{K}_{A_2}$ *if and only if* $A_1 \subseteq A_2$
.
(2) $\mathcal{K}^{A_1} \subseteq \mathcal{K}^{A_2}$ *if and only if* $A_1 \supseteq A_2$.

Proof.

(1) ('$\Rightarrow$')
Let $\mathcal{K}_{A_1} \subseteq \mathcal{K}_{A_2}$. If $x \in A_1$ then $\{x\} \in \mathcal{K}_{A_1}$ and hence $\{x\} \in \mathcal{K}_{A_2}$. Therefore $\{x\} \cap A_2 \neq \emptyset$ and hence $x \in A_2$. Therefore $A_1 \subseteq A_2$.
('$\Leftarrow$')
Let $A_1 \subseteq A_2$. If $A \in \mathcal{K}_{A_1}$ then $A \cap A_1 \neq \emptyset$. But since $A_1 \subseteq A_2$ we have that $A \cap A_2 \neq \emptyset$. Therefore $A \in \mathcal{K}_{A_2}$. Therefore $\mathcal{K}_{A_1} \subseteq \mathcal{K}_{A_2}$.

(2) ('$\Rightarrow$')
Let $\mathcal{K}^{A_1} \subseteq \mathcal{K}^{A_2}$. If $x \notin A_1$ then $\{x\} \cap A_1 = \emptyset$ and hence $\{x\} \in \mathcal{K}_{A_1}$. Therefore $\{x\} \in \mathcal{K}_{A_2}$ and hence $\{x\} \cap A_2 = \emptyset$. Therefore $x \notin A_2$. Therefore $A_2 \subseteq A_1$.
('$\Leftarrow$')
Let $A_1 \supseteq A_2$. If $A \in \mathcal{K}^{A_1}$ then $A \cap A_1 = \emptyset$. But since $A_2 \subseteq A_1$ we have that $A \cap A_2 = \emptyset$. Therefore $A \in \mathcal{K}^{A_2}$. Therefore $\mathcal{K}^{A_1} \subseteq \mathcal{K}^{A_2}$.

□

We shall use these order preserving properties of the hit and miss set operations to prove the connection between capacity functionals and monotonicity (4.4.3). Similarly we shall use the following lemmas to prove the connection between capacity functionals and completely alternating functionals (4.4.6).

The next lemma shows us that the set induced by the hit operation on the union of two sets can be described as the union of two disjoint sets on the hyperspace of non-empty compact subsets and the hyperspace of compact subsets in a natural way.

3.3.4. **Lemma.**
If $A, B \subseteq X$ *then* $\mathcal{K}_A \cup \mathcal{K}_B^A = \mathcal{K}_{A\cup B}$ *and* $\mathcal{K}_A \cap \mathcal{K}_B^A = \emptyset$.

Proof.
Let $K \in \mathcal{K}_{A\cup B}$.
Then $K \cap (A \cup B) \neq \emptyset$ so $K \cap A \neq \emptyset$ or $K \cap B \neq \emptyset$. If $K \cap A \neq \emptyset$ then $K \in \mathcal{K}_A$. If $K \cap A = \emptyset$ then $K \cap B \neq \emptyset$ and $K \in \mathcal{K}_B^A$. Therefore $K \in \mathcal{K}_A \cup \mathcal{K}_B^A$.

Let $K \in \mathcal{K}_A \cup \mathcal{K}_B^A$.
If $K \in \mathcal{K}_A$ then $K \cap A \neq \emptyset$ and hence $K \cap (A \cup B) \neq \emptyset$. If $K \in \mathcal{K}_B^A$ then $K \cap A = \emptyset$ and $K \cap B \neq \emptyset$ hence $K \cap (A \cup B) \neq \emptyset$. Therefore $K \in \mathcal{K}_{A\cup B}$.

Therefore $\mathcal{K}_A \cup \mathcal{K}_B^A = \mathcal{K}_{A\cup B}$.

Let $K \in \mathcal{K}_A$ then $K \cap A \neq \emptyset$ and hence $K \notin \mathcal{K}_B^A$.

Let $K \in \mathcal{K}_B^A$ then $K \cap A = \emptyset$ and hence $K \notin \mathcal{K}_A$.

Therefore $\mathcal{K}_A \cap \mathcal{K}_B^A = \emptyset$. □

The next lemma is slightly more esoteric than the last, showing us that the miss operation on a set can be written as the union of two disjoint sets by the use of an arbitrary subset of the underlying space.

3.3.5. **Lemma.**
If $A, B \subseteq X$ *then* $\mathcal{K}_B^A \cup \mathcal{K}^{A\cup B} = \mathcal{K}^A$ *and* $\mathcal{K}_B^A \cap \mathcal{K}^{A\cup B} = \emptyset$.

Proof.
Let $K \in \mathcal{K}^A$.
Then $K \cap A = \emptyset$ and either $K \cap B = \emptyset$ or $K \cap B \neq \emptyset$. If $K \cap B = \emptyset$ then $K \in \mathcal{K}^{A\cup B}$. If $K \cap B \neq \emptyset$ then $K \in \mathcal{K}_B^A$. Therefore $K \in \mathcal{K}_B^A \cup \mathcal{K}^{A\cup B}$. Therefore $\mathcal{K}_B^A \cup \mathcal{K}^{A\cup B} \subseteq \mathcal{K}^A$.

Let $K \in \mathcal{K}_B^A \cup \mathcal{K}^{A\cup B}$.
Then $K \cap A = \emptyset$. Therefore $K \in \mathcal{K}^A$. Therefore $\mathcal{K}^A \subseteq \mathcal{K}_B^A \cup \mathcal{K}^{A\cup B}$.

Therefore $\mathcal{K}_B^A \cup \mathcal{K}^{A\cup B} = \mathcal{K}^A$.

Let $K \in \mathcal{K}_B^A$ then $K \cap B \neq \emptyset$, therefore $K \notin \mathcal{K}^{A \cup B}$.

Let $K \in \mathcal{K}^{A \cup B}$ then $K \cap B = \emptyset$, therefore $K \notin \mathcal{K}_B^A$. □

We shall now use the hit and miss set operations to define the basic open sets of the upper and lower Fell topologies.

3.3.6. **Definition.**
Let (X, τ) be a topological space. The upper Fell topology on the hyperspace $\mathcal{K}^0(X)$ ($\mathcal{K}(X)$) is generated from the base $\{\mathcal{K}^K : K \in \mathcal{K}^0(X)\}$, the collection of all subsets of compact sets (non-empty compact sets) which hit a compact subset of X. The lower Fell topology is generated from the sub-base $\{\mathcal{K}_G : G \in \mathcal{O}(X)\}$, the collection of all subsets of compact sets (non-compact subsets) which miss an open set.

Remark 4. It should be noted that $\{\mathcal{K}^K : K \in \mathcal{K}^0(X)\}$ is a base. This is due to the union of two compact sets being compact and Proposition 3.3.2(1), which together show that the intersection of two elements from $\{\mathcal{K}^K : K \in \mathcal{K}^0(X)\}$ is itself an element of $\{\mathcal{K}^K : K \in \mathcal{K}^0(X)\}$.

And now by combining the basic open sets of the upper and lower Fell topologies we can define the Fell topology on the hyperspace of non-empty compact subsets and the hyperspace of compact subsets.

3.3.7. **Definition.**
The Fell topology on the hyperspace of compact subsets (non-empty compact subsets) is the smallest topology which contains both the upper and lower Fell topologies, and can be generated from the sub-base of $\{\mathcal{K}^K : K \in \mathcal{K}^0(X)\} \bigcup \{\mathcal{K}_G : G \in \mathcal{O}(X)\}$, the union of the base for the upper Fell topology and the sub-base for the lower Fell topology.

Thus the Fell topology is generated from the sub-base of the base for the upper Fell topology joined with the sub-base for the lower Fell topology. There is an alternative equivalent characterisation of the Fell topology which we describe below that is generated from a base as opposed to the sub-base used in the above characterisation.

3.3.8. **Proposition.**
In the hyperspaces $\mathcal{K}(X)$ or $\mathcal{K}^0(X)$ generated from a topological space (X,τ) the collection $\{\mathcal{K}^K \cap \bigcap_{i\leqslant n} \mathcal{K}_{G_i} : K \in \mathcal{K}^0(X)$ and $(\forall i \leqslant n)\ G_i \in \mathcal{O}(X)\}$ forms a base for the Fell topology.

Proof.
We observe that $\{X\} \subseteq \mathcal{O}(X)$ and $\mathcal{K}_X = \mathcal{K}(X)$. Hence for all $K \in \mathcal{K}^0(X)$ we have that $\mathcal{K}^K = \mathcal{K}^K \cap \mathcal{K}_X \in \{\mathcal{K}^K \cap \bigcap_{i\leqslant n} \mathcal{K}_{G_i} : K \in \mathcal{K}^0(X)$ and $(\forall i \leqslant n)\ G_i \in \mathcal{O}(X)\}$.

Similarly we observe that $\{\emptyset\} \subseteq \mathcal{K}^0(X)$ and $\mathcal{K}^\emptyset = \mathcal{K}(X)$. Hence for all $G \in \mathcal{O}(X)$ we have that $\mathcal{K}_G = \mathcal{K}^\emptyset \cap \mathcal{K}_G \in \{\mathcal{K}^K \cap \bigcap_{i\leqslant n} \mathcal{K}_{G_i} : K \in \mathcal{K}^0(X)$ and $(\forall i \leqslant n)\ G_i \in \mathcal{O}(X)\}$.

Thus the sub-base of the Fell topology is contained in our collection. That is that $\{\mathcal{K}^K : K \in \mathcal{K}^0(X)\} \bigcup \{\mathcal{K}_G : G \in \mathcal{O}(X)\} \subseteq \{\mathcal{K}^K \cap \bigcap_{i\leqslant n} \mathcal{K}_{G_i} : K \in \mathcal{K}^0(X)$ and $(\forall i \leqslant n)\ G_i \in \mathcal{O}(X)\}$.

We finish the proof by noticing that every finite intersection of elements from the subbase is in the collection $\mathcal{K}^K \cap \bigcap_{i\leqslant n} \mathcal{K}_{G_i}$. Hence our collection forms a base for the Fell topology generated by the sub-base.

□

For simplicity's sake we denote sets of the form $\mathcal{K}^K \cap \bigcap_{i\leqslant n} \mathcal{K}_{G_i}$ where $K \in \mathcal{K}^0(X)$ and $(\forall i \leqslant n)\ G_i \in \mathcal{O}(X)$, as $\mathcal{K}^K_{G_1,G_2,\ldots,G_n}$.

We note that the Fell topology on the hyperspace of non-empty compact subsets is merely the subspace topology of the Fell topology on the hyperspace of compact subsets when it is restricted to the hyperspace of non-empty compact subsets.

The following proposition shows the connection between the Fell topology and the Hausdorff metric topology on the hyperspace of non-empty compact subsets. It should be noted that there is the requirement that the underlying topological metric space be compact.

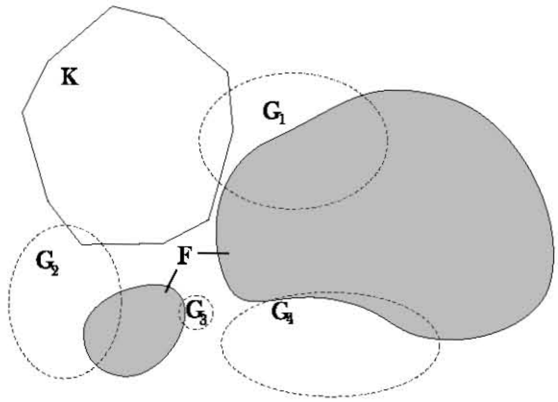


FIGURE 3. The set F is an element of the basic open set $\mathcal{K}^K_{G_1,G_2,G_3,G_4}$. Note that the basic open set $\mathcal{K}^K_{G_1,G_2,G_3,G_4}$ consists of all the non-empty compact subsets that miss K and hit each of the open sets G_1, G_2, G_3 and G_4.

3.3.9. **Proposition.** *(Theorem 3.93* [30]*)*
In the hyperspace $\mathcal{K}(X)$ generated from a compact metric space (X, d) the Fell topology $\mathcal{F}$ coincides with the metric topology τ_{d_H} generated by the Hausdorff metric d_H.

We can begin relaxing the condition that the underlying space be compact when dealing with computable metric spaces that have the property that the closure of the basic open balls are compact. We use this slightly less restrictive condition in the following lemma.

3.3.10. **Lemma.** *(Lemma 2.4.7* [18]*)*
If a locally compact separable metric space (X, Q, d_X) with the base $\{B(x, q) : x \in Q \text{ and } q \in \mathbb{Q}^+\}$ is such that the closure of the basic open sets $\overline{B(x, q)}$ are compact for all $x \in Q$ and $q \in \mathbb{Q}^+$ then the hyperspace $\mathcal{K}(X)$ is locally compact, Hausdorff and second countable with the Fell topology being generated from the sub-base $\{\mathcal{K}^{B(x,q)} : x \in Q \text{ and } q \in \mathbb{Q}^+\} \bigcup \{\mathcal{K}_{B(x,q)} : x \in Q \text{ and } q \in \mathbb{Q}^+\}$.

Proof. We begin by noting that the hyperspace is locally compact (Lemma 3.2.8) and that since the hyperspace is itself a separable metric space (Theorem 3.2.4) we have that it is both Hausdorff (Lemma 2.2.23) and second countable (Lemma 2.2.24).

We now show that the Fell topology on the hyperspace $(\mathcal{K}(X), \zeta, d_H, \nu_\zeta)$ can be generated from the sub-base $\{\mathcal{K}^{\overline{B(x,q)}} : x \in Q \text{ and } q \in \mathbb{Q}^+\} \bigcup \{\mathcal{K}_{B(x,q)} : x \in Q \text{ and } q \in \mathbb{Q}^+\}$. The standard sub-base for the Fell topology as defined in Definition 3.3.7 is $\{\mathcal{K}^K : K \in \mathcal{K}^0(X)\} \bigcup \{\mathcal{K}_G : G \in \mathcal{O}(X)\}$.

Let $K \in \mathcal{K}^0(X)$ and $G \in \mathcal{O}(X)$. Since we can write G as the union of basic open sets $B(x,q)$ where $(x,q) \in \mathcal{J} \subseteq Q \times \mathbb{Q}^+$, and using Lemma 3.3.2 we get that

$$\mathcal{K}_G = \mathcal{K}_{\bigcup_{(x,q)\in\mathcal{J}} B(x,q)} = \bigcup_{(x,q)\in\mathcal{J}} \mathcal{K}_{B(x,q)}.$$

Thus any of the sub-basic elements from the set $\{\mathcal{K}_G : G \in \mathcal{O}(X)\}$ which makes up part of our standard sub-base for the Fell topology can be generated from the potential sub-base $\{\mathcal{K}^{\overline{B(x,q)}} : x \in Q \text{ and } q \in \mathbb{Q}^+\} \bigcup \{\mathcal{K}_{B(x,q)} : x \in Q \text{ and } q \in \mathbb{Q}^+\}$.

Let $K \in \mathcal{K}(X)$ and $F \in \mathcal{K}^K$. Then $F \cap K = \emptyset$.

Since X is a metric space and every metric space is T_4 we have that there exists $U, V \in \mathcal{O}(X)$ such that $K \subseteq U$, $F \subseteq V$, and $U \cap V = \emptyset$ and that in particular $\overline{U} \cap F = \emptyset$.

We can write U as the union of basic open sets $B(x_i, q_i)$ where for all $i \in \mathbb{N}$ we have that $x_i \in Q$ and $q_i \in \mathbb{Q}^+$. By the compactness of K there exists an $n \in \mathbb{N}$ such that

$$K \subseteq \bigcup_{i \leqslant n} B(x_i, q_i) \subseteq \bigcup_{i \leqslant n} \overline{B(x_i, q_i)} \subseteq \overline{U}.$$

Therefore

$$F \in \bigcap_{i \leqslant n} \mathcal{K}^{\overline{B(x_i,q_i)}} = \mathcal{K}^{\bigcup_{i \leqslant n} \overline{B(x_i,q_i)}} \subseteq \mathcal{K}^K.$$

Thus any of the sub-basic elements from the set $\{\mathcal{K}^K : K \in \mathcal{K}^0(X)\}$ which makes up part of our standard sub-base for the Fell topology can be generated from the sub-base $\{\mathcal{K}^{\overline{B(x,q)}} : x \in Q \text{ and } q \in \mathbb{Q}^+\} \bigcup \{\mathcal{K}_{B(x,q)} : x \in Q \text{ and } q \in \mathbb{Q}^+\}$.

Therefore $\{\mathcal{K}^{\overline{B(x,q)}} : x \in Q \text{ and } q \in \mathbb{Q}^+\} \bigcup \{\mathcal{K}_{B(x,q)} : x \in Q \text{ and } q \in \mathbb{Q}^+\}$ is a sub-base for the Fell topology. $\square$

4. **Measures, Hausdorff Measures and Capacities**

4.1. **A Few Basic Measure Theoretic Results.**

In this chapter we shall introduce a method of generating a measure for a metric space from its metric. We shall first lay groundwork for general Measure Theory with a few basic definitions and measure theoretic results. After this we shall introduce a canonical construction of a measure from a metric, discuss a few of its more useful properties and finally show a sufficient condition for a map between two spaces to preserve continuity in measure.

A σ-algebra is a collection of subsets of a set which is closed under complement and countable union and contains the empty set. With any set, the collection containing just the entire set and the empty set is a σ-algebra, likewise the power set of any set, though these are trivial ones. A typical example of a σ-algebra is the collection of all countable and co-countable subsets of an uncountable space, though collections like the finite subsets and the open subsets are usually not σ-algebras.

4.1.1. **Definition.** (σ-Algebra)
A collection $\mathcal{F}$ of subsets of a set X is called a σ-algebra if the following hold:

(1) $X \in \mathcal{F}$
(2) $A \in \mathcal{F} \implies A' \in \mathcal{F}$
(3) $(A_i)_{i\in\mathbb{N}}$ in $\mathcal{F} \implies \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}$

A measure, which wil be properly defined next, can be thought of as a way to measure the volume of a set. A very common use of measures, and indeed the standard measures in $\mathbb{R}^n$ give rise to this concept, is to measure the length, area or volumes of a set depending on context. Measure theory has very close ties with calculus. Another common use of measures is in probability theory and statistical analysis. In this context the idea of a measure as an indicator of volume is particularly apt.

4.1.2. **Definition.** (Measure & Measure Space)

Measure: A measure on a set X with σ-algebra $\mathcal{G}$ is a set function $\mu : \mathcal{G} \to \mathbb{R} \cup \{\infty\}$ which has the following properties:

(1) $\mu(\emptyset) = 0$

(2) $(\forall (A_i)_{i\in\mathbb{N}} \in \mathcal{G}^{\mathbb{N}})\ (A_i \cap A_j = \emptyset,\ i \neq j \implies \mu(\bigcup_i A_i) = \sum_i \mu(A_i))$ (σ-additivity)

Measure Space: A measure space is a tuple $(X, \mathcal{G}, \mu)$ where μ is a measure on the set X.

In a measure space $(X, \mathcal{G}, \mu)$ the sets in the σ-algebra $\mathcal{G}$ are called μ-measurable sets.

4.1.3. **Lemma.** *(Proposition 214A* [31]*)*
Let $(X, \mathcal{G}, \mu)$ be a measure space and $Y \subseteq X$. Then $\mathcal{G}_Y = \{G \cap Y : G \in \mathcal{G}\}$ is a σ-algebra on Y, the restriction μ_Y of the measure μ to the subset Y is a measure on Y and $(Y, \mathcal{G}_Y, \mu_Y)$ is a measure space.

In a topological space (X, τ), the Borel σ-algebra $\mathcal{B}(X)$ is the collection of all countable unions and complements of the closed and open sets. Or in other words, it is the smallest family of sets which contains all the open sets and is closed under countable unions and complementation and it is a σ-algebra [32].

4.1.4. **Definition.**
In the hyperspace of non-empty compact subsets $(\mathcal{K}(X), \zeta, d_H, \nu_\zeta)$ generated from a CMS (X, Q, d_X, ν_Q), the Borel σ-algebra generated from the metric topology induced by the Hausdorff metric is denoted as $\mathcal{B}(\mathcal{K})$.

If we order the σ-algebra of a measure space by set inclusion then the measure on that space is necessarily order preserving when seen as a map onto the ordered set of the non-negative reals.

4.1.5. **Proposition.**
In a measure space $(X, \mathcal{G}, \mu)$ if $A,\ B \in \mathcal{G}$ are such that $A \subseteq B$ then $\mu(A) \leqslant \mu(B)$.

Proof. Let $A \subseteq B \in \mathcal{G}$ then $B \backslash A \in \mathcal{G}$ and $A \cap (B \backslash A) = \emptyset$. Therefore $\mu(B) = \mu(A) + \mu(B \backslash A)$ and since $\mu(B \backslash A) \geqslant 0$ we have that $\mu(B) \geqslant \mu(A)$. ☐

In what follows we shall be giving the definitions for what it means for a measure μ in a topological Hausdorff space (X, τ) with Borel σ-algebra $\mathcal{B}(X)$ to be σ-finite, a Borel measure, locally finite, inner regular, outer regular, regular and a Radon measure, respectively.

σ-Finite: μ is σ-finite if there exists a sequence of subsets $(A_i)_i$ in $\mathcal{G}$ such that $\bigcup_{i\in\mathbb{N}} A_i = X$ and $\mu(A_i) < \infty$ for all $i \in \mathbb{N}$.

Borel Measure: μ is a Borel measure if $\mu(K) < \infty$ for all compact $K \subseteq X$.

Locally Finite: μ is locally finite if for every $x \in X$ there exists an open set $G \in \tau$ such that $x \in G$ and $\mu(G) < \infty$.

Inner Regular: μ is inner regular if for every $B \in \mathcal{B}(X)$ we have that

$$\mu(B) = \sup\{\mu(K) : K \text{ is compact in } X \text{ and } K \subseteq B\}$$

Outer Regular: μ is outer regular if for every $B \in \mathcal{B}(X)$ we have that

$$\mu(B) = \inf\{\mu(U) : U \subseteq X \text{ is open and } B \subseteq U\}$$

Regular: μ is regular if it is both inner and outer regular.

Radon Measure: μ is a Radon measure if it is both inner regular and locally finite.

Remark 5. Note that the condition that a space X be a Hausdorff space ensures that all compact sets are closed and thus in the Borel algebra.

The two most important definitions in the collection above, in the context of this work, is that of a Borel measure and that of a σ-finite measure.

4.1.6. **Lemma.** *(Lemma 25.4* [32]*)*
In a topological Hausdorff space (X, τ) with Borel σ-algebra $\mathcal{B}(X)$, if every point in X has a countable neighbourhood basis then every inner regular Borel measure on X is locally finite and hence a Radon measure.

4.1.7. **Theorem.**
In a metric space (X, d) with Borel σ-algebra $\mathcal{B}(X)$ generated from τ_d every inner regular Borel measure on X is locally finite and hence a Radon measure.

Proof. This follows directly from Lemma 4.1.6 and the fact that for every point x in the computable metric space X the collection of open balls centred on x $\{B(x, q) : q \in \mathbb{Q}\}$ forms a countable neighbourhood basis for x. □

A Polish space is a space (X, τ) whose topology has a countable base $\{B_i : i \in \mathbb{N}\}$ which can be generated by a complete metric d_τ, i.e. there exists a countable subset A of X such that $\{B_i : i \in \mathbb{N}\} = \{B(x, q) : x \in A,\ q \in \mathbb{Q}^+\}$. It follows that in any CMS the topology generated by the metric has a countable base. Thus any complete CMS is a Polish space.

4.1.8. **Lemma.** *(Theorem 26.3* [32]*)*
In a Polish space X every locally finite Borel measure is a σ-finite Radon measure.

4.1.9. **Lemma.** *(Corollary 26.4* [32]*)*
In a Polish space X every Radon measure is outer regular.

We shall later show that every Hausdorff measure on a complete CMS is locally finite and hence that it is a regular Radon measure. We shall later show that our concept of randomness relies on the computability of the space as well as the measure that we define on it. Specifically we shall be looking at collections of sets whose members tends towards a set of measure 0. In order to compare different notions of randomness we therefore need to be able to see how two measures compare when we have incredibly small sets in the sense of their respective measures. It is to that end that we introduce the concept of measures being absolutely continuous in each other.

4.1.10. **Definition.** Absolutely Continuous in Measure
Let $(X, \mathcal{G}, \mu_1)$ and $(X, \mathcal{G}, \mu_2)$ be measure spaces. Then μ_1 is absolutely continuous with respect to μ_2 (written $\mu_1 \ll \mu_2$) iff

$$(\forall A \in \mathcal{G})\ \ (\mu_2(A) = 0 \Longrightarrow \mu_1(A) = 0)$$

There is an equivalent definition of absolute continuity for measures that are σ-finite.

4.1.11. **Proposition.** *(Theorem 17.8* [32]*)*
Let $(X, \mathcal{G}, \mu_1)$ and $(X, \mathcal{G}, \mu_2)$ be measure spaces where μ_1 is a finite measure on the space X, then $\mu_1 \ll \mu_2$ iff

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall A \in \mathcal{G})\ (\mu_2(A) < \delta \Longrightarrow \mu_1(A) < \varepsilon)$$

4.1.12. **Definition.**
If μ_1 and μ_2 are two finite measures on a topological space X equipped with its Borel σ-algebra, then we say that μ_1 is computably absolutely continuous with respect to μ_2, written $\mu_1 \ll_c \mu_2$, iff there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$(\forall n \in \mathbb{N})(\forall A \in \mathcal{B}(X))\ (\mu_2(A) < 2^{-f(n)} \Longrightarrow \mu_1(A) < 2^{-n})$$

Henceforth $(\forall A \in \mathcal{B}(X))$ will be omitted when using any of the above formulations if there is no chance of confusion.

4.1.13. **Definition.**
Let X and Y be topological spaces with μ a Borel measure on the space X and $f : X \to Y$ a continuous function, then μf^{-1} is defined as $\mu f^{-1} : \mathcal{B}(Y) \to \mathbb{R}$ where $\mu f^{-1}(A) := \mu(f)(A) := \mu(f^{-1}(A))$.

We now show that the function defined above is itself a measure on the topological space (Y, τ_Y), with its associated Borel σ-algebra.

4.1.14. **Theorem.**
If X and Y be topological spaces with μ a Borel measure on the space X and τ_X being the family of open sets which generate the Borel σ-algebra $\mathcal{B}(X)$ and (Y, τ_Y) is a topological space with $f : X \to Y$ a continuous function from X to Y then μf^{-1} is a measure on $\mathcal{B}(Y)$.

Proof. Note that by definition $f^{-1}(\emptyset) = \{x \in X : f(x) \in \emptyset\} = \emptyset$.
Therefore $\mu f^{-1}(\emptyset) = \mu(\emptyset) = 0$

Let $(A_i)_{i\in\mathbb{N}}$ be a family of disjoint sets from the Borel σ-algebra $\mathcal{B}(Y)$ generated by the open sets in Y. Since f is continuous we have that for any open (closed) set $G \subseteq Y$ the set $f^{-1}(G)$ is open (closed) set in X and in particular $f^{-1}(Y)$ is open in X. Therefore $f^{-1}(\mathcal{B}(Y)) \subseteq \mathcal{B}(X)$. So for all $i \in \mathbb{N}$ we have that $f^{-1}(A_i)$ is an element of $\mathcal{B}(X)$. Let $i, j \in \mathbb{N}$ where $i \neq j$ then if $A_i \cap A_j = \emptyset$ we have that

$$\begin{aligned} f^{-1}(A_i) \cap f^{-1}(A_j) &= \{x \in X : f(x) \in A_i\} \cap \{x \in X : f(x) \in A_j\} \\ &= \{x \in X : f(x) \in A_i \cap A_j\} \\ &= \{x \in X : f(x) \in \emptyset\} \\ &= \emptyset. \end{aligned}$$

Therefore $\mu(\bigcup_i f^{-1}(A_i)) = \sum_i \mu(A_i)$.

Therefore μf^{-1} is a measure on Y. □

4.2. **The Hausdorff Measure and Its Properties.**

We shall now introduce the aforementioned canonical construction of a measure from a metric. After this a few of its more useful properties will be discussed including a sufficient condition for a map between two spaces to preserve continuity in measure.

4.2.1. **Definition.** Hausdorff Measure (Section 264K [31])
Let (X, d) be a metric space. The Hausdorff measure μ_d induced by a metric d is defined as follows:

Let $A \in \mathcal{B}(X)$.

$$\mu_d(A) := \sup_{\delta>0} \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(U_i) : A \subseteq \bigcup_{i=1}^{\infty} U_i \ \& \ (\forall i)\ U_i \text{ open and } \operatorname{diam}(U_i) < \delta \right\}$$

Where the diameter of a set A is:

$$\operatorname{diam}(A) := \sup_{x,y \in A} d(x, y)$$

and

$$\operatorname{diam}(\emptyset) := 0.$$

We now have that any computable metric space (X, Q, d, ν) or metric space (X, d) is also a measure space (X, μ_d). The next theorem states the existence of a few measures that are equivalent to the Hausdorff measure in any metric space.

4.2.2. **Theorem.** *(Theorem 28 [22])*
Let (X,d) be a metric space. The measures $\overline{\mu_d}$, μ_d^X and $\mu_d^=$ defined for any $A \in \mathcal{B}(X)$ as:

$$\overline{\mu_d}(A) := \sup_{\delta>0} \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(F_i) : A \subseteq \bigcup_{i=1}^{\infty} F_i \ \&\ (\forall i)\ F_i \textit{ closed and } \operatorname{diam}(F_i) < \delta \right\}$$

$$\mu_d^X(A) := \sup_{\delta>0} \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(E_i) : A \subseteq \bigcup_{i=1}^{\infty} E_i \ \&\ (\forall i)\ \operatorname{diam}(E_i) < \delta \right\}$$

$$\mu_d^=(A) := \sup_{\delta>0} \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(E_i) : A = \bigcup_{i=1}^{\infty} E_i \ \&\ (\forall i)\ \operatorname{diam}(E_i) < \delta \right\}$$

are such that for any $A \in \mathcal{B}(X)$ we have that $\mu_d(A) = \overline{\mu_d}(A) = \mu_d^X(A) = \mu_d^=(A)$.

A function from one measure space to another is called a measurable map or measurable function if the pre-image of any measurable set is measurable. Thus from Theorem 4.1.14 we know that any continuous function between metric spaces equipped with their respective Hausdorff measures is a measurable function.

Since the inverse map of a continuous function preserves the Hausdorff measurability of sets, the next question is whether the continuous functions allows us to compare sets of very small measure. As the following lemma proves this is indeed the case.

4.2.3. **Theorem.**
Let (X, d_X) and (Y, d_Y) be metric spaces with $f : X \to Y$ a continuous function such that for some $k > 0$ and $t > 0$ we have that $d_X(x, y) \leqslant k \cdot d_Y(f(x), f(y))^t$ for all $x, y \in X$. Then the Hausdorff measures μ_X and μ_Y induced by d_X and d_Y respectively are such that $\mu_X^{\frac{1}{t}} f^{-1} \ll_c \mu_Y$.

Proof. Define (X, d_X), (Y, d_Y), $f : X \to Y$ and k as above.

Let $U \subseteq X$, then

$$\begin{aligned} \operatorname{diam}(U)^{\frac{1}{t}} &= (\sup\{d_X(x,y) \mid x, y \in U\})^{\frac{1}{t}} \\ &\leqslant \sup\left\{d_X(x,y)^{\frac{1}{t}} \mid f(x), f(y) \in f(U)\right\} \\ &\leqslant \sup\left\{k^{\frac{1}{t}} \cdot d_Y(f(x), f(y)) \mid f(x), f(y) \in f(U)\right\} \\ &= k^{\frac{1}{t}} \cdot \operatorname{diam}(f(U)) \end{aligned}$$

If $V \subseteq Y$ then $f(f^{-1}(V)) \subseteq V$ and thus $\operatorname{diam}(f(f^{-1}(V))) \leqslant \operatorname{diam}(V)$. Putting the two previous results together we get that if $V \subseteq Y$ then

$$\operatorname{diam}(f^{-1}(V))^{\frac{1}{t}} \leqslant k^{\frac{1}{t}} \cdot \operatorname{diam}(f(f^{-1}(V))) \leqslant k^{\frac{1}{t}} \cdot \operatorname{diam}(V_i).$$

Fix $n \in \mathbb{N}$.
Let $A \subseteq Y$ be such that $\mu_Y(A) < \frac{2^{-n+1}}{k^{\frac{1}{t}}}$, then

$$\sup_{\alpha>0}\inf\left\{\sum \operatorname{diam}(V_i) : A \subseteq \bigcup_i V_i \ \&\ (\forall i)\ V_i \text{ open and } \operatorname{diam}(V_i) < \alpha\right\} < \frac{2^{-n+1}}{k^{\frac{1}{t}}}$$

$$\Rightarrow \quad (\forall \alpha > 0)\ \inf\left\{\sum \operatorname{diam}(V_i) : A \subseteq \bigcup_i V_i \ \&\ (\forall i)\ V_i \text{ open and } \operatorname{diam}(V_i) < \alpha\right\} < \frac{2^{-n+1}}{k^{\frac{1}{t}}}$$

$$\Rightarrow \quad (\forall \alpha > 0)\ \inf\left\{\sum \operatorname{diam}(f^{-1}(V_i))^{\frac{1}{t}} : A \subseteq \bigcup_i V_i \ \&\ (\forall i)\ V_i \text{ open and } \operatorname{diam}(V_i) < \alpha\right\} \leqslant$$

$$\inf\left\{k^{\frac{1}{t}} \cdot \sum \operatorname{diam}(V_i) : A \subseteq \bigcup_i V_i \ \&\ (\forall i)\ V_i \text{ open and } \operatorname{diam}(V_i) < \alpha\right\} < 2^{-n+1}$$

$$\Rightarrow \quad (\forall \alpha > 0)\ \inf\left\{\sum \operatorname{diam}(f^{-1}(V_i))^{\frac{1}{t}} : f^{-1}(A) \subseteq \bigcup_i f^{-1}(V_i) \ \&\ (\forall i)\ V_i \text{ open and } \operatorname{diam}(f^{-1}(V_i))^{\frac{1}{t}} < \alpha\right\} < 2^{-n+1}$$

$$\Rightarrow \quad (\forall \alpha > 0)\ \inf\left\{\sum \operatorname{diam}(U_i)^{\frac{1}{t}} : f^{-1}(A) \subseteq \bigcup_i U_i \ \&\ (\forall i)\ U_i \text{ open and } \operatorname{diam}(U_i)^{\frac{1}{t}} < \alpha\right\} < 2^{-n+1}$$

$$\Rightarrow \quad \sup_{\alpha>0}\inf\left\{\sum \operatorname{diam}(U_i)^{\frac{1}{t}} : f^{-1}(A) \subseteq \bigcup_i U_i \ \&\ (\forall i)\ U_i \text{ open and } \operatorname{diam}(U_i)^{\frac{1}{t}} < \alpha\right\} \leqslant 2^{-n+1}$$

$$\Rightarrow \quad \mu_X^{\frac{1}{t}}(f^{-1}(A)) \leqslant 2^{-n+1}$$

$$\Rightarrow \quad \mu_X^{\frac{1}{t}}(f^{-1}(A)) < 2^{-n}$$

Therefore $\mu_X^{\frac{1}{t}} f^{-1} \ll_c \mu_Y$. □

4.2.4. **Corollary.**
Let (X, d_X) and (Y, d_Y) be metric spaces with $f : X \to Y$ a continuous function such that for some $k > 0$ we have that $d_X(x, y) \leqslant k \cdot d_Y(f(x), f(y))$ for all $x, y \in X$. Then the Hausdorff measures μ_X and μ_Y induced by d_X and d_Y respectively are such that $\mu_X f^{-1} \ll_c \mu_Y$

In the sections following we shall denote the Hausdorff measure of a computable metric hyperspace $(\mathcal{K}(X), \zeta, d_H, \nu_\zeta)$ generated by a CMS (X, Q, d, q) as $\mathbf{h}_d$. Remember that, when there is no chance for confusion, the Borel algebra $\mathcal{B}(\mathcal{K}(X))$ on the hyperspace is defined as $\mathcal{B}(\mathcal{K})$.

4.3. **Capacities.**

In the previous section we introduced a method of constructing a measure on the hyperspace of non-empty compact subsets using the Hausdorff measure of the Hausdorff metric and whilst this is a natural construction in other areas of mathematics, we shall show in section 6.1 that in hyperspaces the Hausdorff measure leads to trivially random and non-random compact sets. In the following sections we shall look at another method of constructing a measure on the hyperspace of non-empty compact subsets, in this approach we shall use the notion of a capacity functional and a Choquet capacity to generate a natural measure on the hyperspace from the Hausdorff measure of the underlying space.

We begin by defining a "random closed set" or RACS a concept that is not equivalent to the concept of a random closed set that we shall define in following sections. In order to avoid confusion we shall refer to the probability and measure theoretic concept as a RACS without any further reference to the acronym's origin.

4.3.1. **Definition.** (Definition 1.1' [33])
Let $(X, \mathcal{G}, \tau, \mu)$ be a compact topological measure space with $\mu(X) = 1$. Let $\mathcal{K}^0(X)$ be the hyperspace of compact subsets equipped with the Fell topology and associated Borel σ-algebra $\mathcal{B}(\mathcal{K}^0)$. We define a RACS as a measurable map $\Psi : X \to \mathcal{K}^0(X)$.

We now begin defining the image measure on the hyperspace of compact subsets that we generate from a RACS and the Hausdorff measure of the underlying metric space.

4.3.2. **Definition.** (Appendix E [33])
Let $(X, \mathcal{G}, \tau, \mu)$ be a compact topological measure space with $\mu(X) = 1$. Let $\mathcal{K}^0(X)$ be the hyperspace of compact subsets equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{K}^0)$. Let $\Psi : X \to \mathcal{K}^0(X)$ be a RACS from X to $\mathcal{K}^0(X)$. We then define the image measure $\mu_\Psi : \mathcal{B}(\mathcal{K}^0) \to [0, 1]$ as the measure where $\mu_\Psi(\mathcal{A}) = \mu(\Psi^{-1}(\mathcal{A}))$.

It is a trivial exercise to adjust this measure so that the measure of $\mathcal{K}^0(X)$ is equal to 1 and hence have that μ_Ψ is a probability measure. Since Ψ is a measurable map the set $\Psi^{-1}(\mathcal{K}^0(X))$ is in the σ-algebra $\mathcal{B}(X)$. Hence $\mu(\Psi^{-1}(\mathcal{K}^0(X))) \leqslant 1$. As long as we don't have the trivial case where for all sets $\mathcal{A} \in \mathcal{B}(\mathcal{K}^0)$ the image measure is 0 then the measure of $\Psi^{-1}(\mathcal{K}^0(X))$ is equal to some constant $c \in (0, 1]$. If we then let $\mu_\Psi(\mathcal{A}) = \frac{1}{c}\mu(\Psi^{-1}(\mathcal{A}))$. The measure $\mu_\Psi(\mathcal{K}(X)) = \frac{1}{c}\mu(\Psi^{-1}(\mathcal{K}^0(X))) = \frac{1}{c}c = 1$. Though in most cases this adjustment will not be needed as we commonly use measurable maps Ψ with the property that $\Psi^{-1}(\mathcal{K}(X)) = X$ and hence the measure of the set $\mathcal{K}^0(X)$ is $\mu_\Psi(\mathcal{K}^0(X)) = \mu_d(\Psi^{-1}(\mathcal{K}^0(X))) = \mu(X) = 1$.

A functional is a real valued function, that is a map from an arbitrary set to the reals. A capacity functional is a functional on the metric space with the additional property that for any compact subset of the metric space it agrees with the image measure (generated from a RACS and the Hausdorff measure on the metric space) on the set in the hyperspace of compact subsets generated by the hit operation on the compact subset.

4.3.3. **Definition.**
Let (X, d_X) be a compact metric space with the Hausdorff measure μ_d satisfying $\mu_d(X) = 1$, the hyperspace $\mathcal{K}^0(X)$ be equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{K}^0)$ and a RACS $\Psi : X \to \mathcal{K}^0(X)$. We can then define the capacity functional of Ψ, $T_\Psi : \mathcal{K}^0(X) \to [0, 1]$, where $T_\Psi(K) = \mu_\Psi(\mathcal{K}_K)$ for all $K \in \mathcal{K}^0(X)$.

4.4. **Choquet Capacities.**

In this section we define a Choquet capacity, introduce the regularized Choquet capacity theorem for computable metric spaces and define a computable Choquet capacity. We shall later use these notions to define a measure on the hyperspace of non-empty compact subsets that we shall use to define a random compact set.

We begin by defining two operations on compact subsets which have been referred to as the *successive differences* operations by Molchanov [33].

4.4.1. **Definition.**

The operation Δ is defined on a collection of compact subsets $K, K_1, K_2, ..., K_n$ with respect to particular functional $T : \mathcal{K}^0(X) \to [0,1]$. We define the operation Δ as $\Delta_{K_1}T(K) = T(K) - T(K \cup K_1)$ and inductively on n when $n \geqslant 2$ by

$$\Delta_{K_n} \cdots \Delta_{K_2}\Delta_{K_1}T(K) = \Delta_{K_{n-1}} \cdots \Delta_{K_2}\Delta_{K_1}T(K) - \Delta_{K_{n-1}} \cdots \Delta_{K_2}\Delta_{K_1}T(K \cup K_n).$$

The operation ∇ is defined on a collection of compact subsets $K, K_1, K_2, ..., K_n$ with respect to particular functional $T : \mathcal{K}^0(X) \to [0,1]$. We define the operation ∇ as $\nabla_{K_1}T(K) = T(K) - T(K \cap K_1)$ and inductively on n when $n \geqslant 2$ by

$$\nabla_{K_n} \cdots \nabla_{K_2}\nabla_{K_1}T(K) = \nabla_{K_{n-1}} \cdots \nabla_{K_2}\nabla_{K_1}T(K) - \nabla_{K_{n-1}} \cdots \nabla_{K_2}\nabla_{K_1}T(K \cap K_n).$$

We now introduce a fundamental concept in the study of RACS and capacities. In this text we refer to real valued functions as functionals. A functional is said to be completely alternating if the successive difference is always negative. We shall later be using this classification of functionals to completely characterise the 'nice' functionals that induce Borel probability measures on a hyperspace of non-empty compact subsets.

4.4.2. **Definition.**

A functional $T : \mathcal{K}^0(X) \to [0,1]$ is completely alternating if for all collections of compact subsets the successive difference is less than or equal to 0. That is that $\Delta_{K_n} \cdots \Delta_{K_2}\Delta_{K_1}T(K) \leqslant 0$ for all $K, K_1, ..., K_n \in \mathcal{K}^0(X)$.

A functional $T : \mathcal{K}^0(X) \to [0,1]$ is completely $\cap$-alternating if for all collections of compact subsets the intersection successive difference is less than or equal to 0. That is that $\nabla_{K_n} \cdots \nabla_{K_2}\nabla_{K_1}T(K) \leqslant 0$ for all $K, K_1, ..., K_n \in \mathcal{K}^0(X)$.

A functional is called monotone when $K_1 \subseteq K_2$ implies that $T(K_1) \leqslant T(K_2)$ for all compact subsets K_1 and K_2. We can now show that any capacity functional is necessarily monotone.

4.4.3. **Proposition.**
Let (X, d_X) be a compact metric space with a Hausdorff measure μ_d satisfying $\mu_d(X) = 1$, and let $\mathcal{K}^0(X)$ be the hyperspace of compact subsets equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{K}^0)$. Let $\Psi : X \to \mathcal{K}^0(X)$ be a RACS. If T_Ψ is a capacity functional then it is monotone.

Proof. Let K_1 and K_2 be subsets such that $K_1 \subseteq K_2$ then by Proposition 3.3.3 we have that $\mathcal{K}_{K_1} \subseteq \mathcal{K}_{K_2}$. Then $\mu_\Psi(\mathcal{K}_{K_1}) \leqslant \mu_\Psi(\mathcal{K}_{K_2})$ and hence we have that $T_\Psi(K_1) \leqslant T_\Psi(K_2)$ as needed.

□

We can also introduce the dual notions for completely alternating functionals and completely $\cap$-alternating functionals. These dual notions are referred to as being completely $\cup$-monotone and completely monotone respectively.

4.4.4. **Definition.**
A functional $T : \mathcal{K}^0(X) \to [0, 1]$ is completely $\cup$-monotone if for all collections of compact subsets the successive difference is greater than or equal to 0. That is that $\Delta_{K_n} \cdots \Delta_{K_2}\Delta_{K_1}T(K) \geqslant 0$ for all $K, K_1, ..., K_n \in \mathcal{K}^0(X)$.

A functional $T : \mathcal{K}^0(X) \to [0, 1]$ is completely monotone if for all collections of compact subsets the intersection successive difference is greater than or equal to 0. That is that $\nabla_{K_n} \cdots \nabla_{K_2}\nabla_{K_1}T(K) \geqslant 0$ for all $K, K_1, ..., K_n \in \mathcal{K}^0(X)$.

The following proposition allows for an alternate characterisation of completely alternating and completely $\cap$-alternating functionals. The proof uses Molchanov's Theorem G.8 [33] which states that if f is a non-negative function on an abelian semigroup S then f is completely alternating w.r.t. S if and only if e^{-tf} is completely monotone w.r.t. S for all $t > 0$.

4.4.5. **Proposition.**
A functional $T : \mathcal{K}^0(X) \to [0, 1]$ is completely alternating if and only if $-e^{tT}$ is

completely $\cup$-monotone for all $t > 0$. And similarly a functional $T : \mathcal{K}^0(X) \to [0,1]$ is completely $\cap$-alternating if and only if $-e^{tT}$ is completely monotone for all $t > 0$.

Proof.
This is a simple application of Molchanov's Theorem G.8 [33]. □

We now show that a capacity functional defined on a compact metric space is necessarily completely alternating.

4.4.6. **Lemma.**
Let $(X, \mathcal{G}, \tau, \mu)$ be a compact topological measure space with $\mu(X) = 1$. Let $\mathcal{K}^0(X)$ be the hyperspace of compact subsets equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{K}^0)$. Let $\Psi : X \to \mathcal{K}^0(X)$ be a RACS from X to $\mathcal{K}^0(X)$. If T_Ψ is a capacity functional then it is completely alternating.

Proof.
Let T_Ψ be a capacity functional. Then T_Ψ is monotone (Proposition 4.4.3). Let $K, K_1 \in \mathcal{K}^0(X)$ then $\Delta_{K_1} T_\Psi(K) = T_\Psi(K) - T_\Psi(K \cup K_1)$ and since $K \subseteq K \cup K_1$ and T_Ψ is monotone we have that $T_\Psi(K) \leqslant T_\Psi(K \cup K_1)$ and hence $\Delta_{K_1} T_\Psi(K) \leqslant 0$.

We now look at the measure which induces the capacity functional T_Ψ,

$$T_\Psi(K \cup K_1) = \mu_\Psi(\mathcal{K}_{K \cup K_1}),$$

$$T_\Psi(K) = \mu_\Psi(\mathcal{K}_K),$$

and $\mu_\Psi(\mathcal{K}_{K\cup K_1}) = \mu_\Psi(\mathcal{K}_K) + \mu_\Psi(\mathcal{K}^K_{K_1})$ (Lemma 3.3.4). Therefore $T_\Psi(K \cup K_1) - T_\Psi(K) = \mu_\Psi(\mathcal{K}^K_{K_1})$.

We now begin an inductive argument where we assume for $n \geqslant 2$ and an arbitrary collection of compact sets $K, K_1, K_2, ..., K_n \in \mathcal{K}^0(X)$ that the operation $\Delta_{K_n} \cdots \Delta_{K_2} \Delta_{K_1} T_\Psi(K) \leqslant 0$ and $\Delta_{K_n} \cdots \Delta_{K_2} \Delta_{K_1} T_\Psi(K) = -\mu_\Psi(\mathcal{K}^K_{K_1,...,K_n})$.

Let $K, K_1, K_2, ..., K_n, K_{n+1} \in \mathcal{K}^0(X)$ be a collection of compact sets, then

$$\begin{aligned} \Delta_{K_{n+1}} \cdots \Delta_{K_1} T_\Psi(K) &= \Delta_{K_n} \cdots \Delta_{K_1} T_\Psi(K) - \Delta_{K_n} \cdots \Delta_{K_1} T_\Psi(K \cup K_{n+1}) \\ &= -\mu_\Psi(\mathcal{K}^K_{K_1,...,K_n}) + \mu_\Psi(\mathcal{K}^{K\cup K_{n+1}}_{K_1,...,K_n}). \end{aligned}$$

We know from Lemma 3.3.5 that $\mu_\Psi(\mathcal{K}^K_{K_1,...,K_n,K_{n+1}}) + \mu_\Psi(\mathcal{K}^{K\cup K_{n+1}}_{K_1,...,K_n}) = \mu_\Psi(\mathcal{K}^K_{K_1,...,K_n})$. Therefore $\Delta_{K_{n+1}} \cdots \Delta_{K_1} T_\Psi(K) = -\mu_\Psi(\mathcal{K}^K_{K_1,...,K_n,K_{n+1}})$ and since μ_Ψ is non-negative we have that $\Delta_{K_{n+1}} \cdots \ \Delta_{K_1} T_\Psi(K) \leqslant 0$.

Therefore the capacity functional T_Ψ is completely alternating.

□

Which brings us to the Choquet capacity theorem on hyperspaces. This is similar to the usual Choquet capacity theorem [33] with the only difference being that we have restricted ourselves to the hyperspace of compact subsets of compact metric spaces. The reason for this restriction appears later in Section 6.3 where we look at the randomness of non-empty compact subsets of computable metric spaces.

4.4.7. **Theorem.** *(Choquet Capacity Theorem on CMS hyperspaces.)*
In a compact metric space (X, d) with the Hausdorff measure μ_d satisfying $\mu_d(X) = 1$ and hyperspace $\mathcal{K}^0(X)$ equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{K}^0)$. Let $T : \mathcal{K}^0(X) \to [0, 1]$ be a functional on the hyperspace of compact subsets. Then T is the capacity functional of a unique Borel measure $\mu_T : \mathcal{B}(\mathcal{K}^0(X)) \to [0, 1]$ with $\mu_T(\mathcal{K}^0(X)) = 1$ and $\mu_T(\mathcal{K}_K) = T(K)$ for all $K \in \mathcal{K}^0(X)$ if and only if the following three conditions are met:

(1) *$T(\emptyset) = 0$.*

(2) *T is upper semi-continuous.*

(3) *T is completely alternating.*

Proof.
('⇒')
Let T be the capacity functional of a unique Borel measure $\mu_T : \mathcal{K}^0(X) \to [0, 1]$ with $\mu_T(\mathcal{K}^0(X)) = 1$ and $\mu_T(\mathcal{K}_K) = T(K)$ for all $K \in \mathcal{K}^0(X)$.

(1) $T(\emptyset) = \mu_T(\mathcal{K}_\emptyset) = \mu_T(\emptyset) = 0$

(2) Let $(K_i)_{i\in\mathbb{N}}$ be a decreasing sequence of compact subsets such that $\bigcap_{i\in\mathbb{N}} K_i = K$ where K is a compact set. By Proposition 3.2.11 we have that $K_n \to K$ and by Proposition 3.3.3 we have that $\mathcal{K}_K \subseteq \mathcal{K}_{K_i} \subseteq \mathcal{K}_{K_j}$ for all $i > j$. Then by Proposition 3.3.2 we have that $\bigcap_{i\in\mathbb{N}} \mathcal{K}_{K_i} \supseteq \mathcal{K}_{\cap_{i\in\mathbb{N}} K_i} = \mathcal{K}_K$.

Assume that $(\bigcap_{i\in\mathbb{N}} \mathcal{K}_{K_i}) \nsubseteq \mathcal{K}_K$. Then there exists a $K' \in (\bigcap_{i\in\mathbb{N}} \mathcal{K}_{K_i})$ such that $K' \cap K = \emptyset$. For each $i \in \mathbb{N}$ let $x_i \in K'$ be a witness that $K' \cap K_i \neq \emptyset$. Since K' is a compact subset there exists a subsequence $(y_i)_{i\in\mathbb{N}}$ of $(x_i)_{i\in\mathbb{N}}$ such that the limit of the subsequence exists in K'. Let $\lim_{i\to\infty} y_i = x$. Fix $k \in \mathbb{N}$. For all $j > k$ the witnesses $x_j \in K_k$ and since K_k is a compact subset we have that $\lim_{i\to\infty, i>k} y_i = x \in K_k$. Therefore $x \in K_i$ for all $i \in \mathbb{N}$. Therefore $x \in \bigcap_{i\in\mathbb{N}} K_i = K$. Which contradicts our original assumption. Therefore $(\bigcap_{i\in\mathbb{N}} \mathcal{K}_{K_i}) \subseteq \mathcal{K}_K$.

Therefore $(\bigcap_{i\in\mathbb{N}} \mathcal{K}_{K_i}) = \mathcal{K}_K$ and hence $\lim_{i\in\mathbb{N}} T(K_i) = \lim_{i\in\mathbb{N}} \mu_T(\mathcal{K}_{K_i})$ and by the convergence of measures $\lim_{i\in\mathbb{N}} \mu_T(\mathcal{K}_{K_i}) = \mu_T(\bigcap_{i\in\mathbb{N}} \mathcal{K}_{K_i})$. Therefore

$$\lim_{i\in\mathbb{N}} T(K_i) = \mu_T(\mathcal{K}_K) = T(K).$$

Therefore by the compactness of X and Lemma 3.2.12 we have that T is upper semi-continuous.

(3) Since T is a capacity functional we have that T is completely alternating by Lemma 4.4.6.

('$\Leftarrow$')
Let $T : \mathcal{K}^0(X) \to [0,1]$ be a functional on the hyperspace of compact subsets such that

(1) $T(\emptyset) = 0$.

(2) T is upper semi-continuous.

(3) T is completely alternating.

We now define an extension $\bar{T} : \mathcal{B}(X) \to [0,1]$ of T onto the Borel algebra of (X, τ_d) by defining $\bar{T}(G) = \sup\{T(K) : K \in \mathcal{K}^0(X) \text{ and } K \subseteq G\}$ for all $G \in \mathcal{B}(X)$.

We can now define a measure $\mu : \mathcal{B}(\mathcal{K}^0) \to \mathbb{R}^+$ on the Borel algebra generated from the Fell topology on the hyperspace of compact subsets. Let $K \in \mathcal{K}^0(X)$ and $G_1, G_2, ..., G_n \in \mathcal{O}(X)$, then $\mathcal{K}^K_{G_1,G_2,...,G_n}$ is a basic open set of the Fell topology, and indeed all the basic open sets of the Fell topology can be described by some finite collection of this form. The measure μ is then defined on the basic open sets as $\mu(\mathcal{K}^K_{G_1,G_2,...,G_n}) = -\Delta_{G_n} \cdots \Delta_{G_2}\Delta_{G_1}\bar{T}(K)$.

Note that the successive difference operation is equivalent to reducing the measure $\mu(\mathcal{K}^K_{G_1,G_2,...,G_n})$ to its basic components of the form $\mu(\mathcal{K}_{G_i})$ and $\mu(\mathcal{K}^K)$ using the identity $\mu(A \cap B) = \mu(A) + \mu(B) - \mu(A \cup B)$, the distributive laws and the properties of the hit and miss sets outlined in Proposition 3.3.2. These basic components are then defined as $\mu(\mathcal{K}_{G_i}) = \bar{T}(G_i)$ and $\mu(\mathcal{K}^K) = \bar{T}(K)$ thus giving us that $\mu(\mathcal{K}^K_{G_1,G_2,\ldots,G_n}) = -\Delta_{G_n} \cdots \Delta_{G_2}\Delta_{G_1}\bar{T}(K)$.

We now extend the definition of μ to all Borel sets. To do this we define the collection of all basic open subsets of $\mathcal{K}^0(X)$ as $\mathcal{G}(\mathcal{K}^0(X)) = \{\mathcal{K}^K_{G_1,G_2,\ldots,G_n} : K \in \mathcal{K}^0(X)$ and $G_1, G_2, ..., G_n \in \mathcal{O}(X)\}$. For each $\mathcal{U} \in \mathcal{B}(\mathcal{K})$ let $\mu(\mathcal{U}) = \sup\{\mu(\mathcal{V}) : \mathcal{V} \subseteq \mathcal{U}$ and $\mathcal{V} \in \mathcal{G}(\mathcal{K}(X))\}$.

To show that μ is a measure we first note that $\mu(\emptyset) = \mu(\mathcal{K}_\emptyset) = 0$. For the proof that μ is σ-additive see Molchanov (Lemma 1.21 [33]).

Let $\mu_T(\mathcal{U}) = \frac{\mu(\mathcal{U})}{\mu(\mathcal{K}(X))}$ for all $\mathcal{U} \in \mathcal{B}(\mathcal{K}^0)$. Then $\mu_T(\mathcal{K}^0(X)) = \frac{\mu(\mathcal{K}^0(X))}{\mu(\mathcal{K}^0(X))} = 1$. For all $\mathcal{U} \in \mathcal{B}(\mathcal{K}^0)$ we have that $\mu(\mathcal{U}) \geqslant 0$ since T is completely alternating and hence $\mu_T(\mathcal{U}) \geqslant 0$. And for all $K \in \mathcal{K}^0(X)$ we have that $\mu_T(\mathcal{K}_K) = \frac{\mu(\mathcal{K}_K)}{\mu(\mathcal{K}(X))} = \frac{-\Delta_K \bar{T}(\emptyset)}{\mu(\mathcal{K}_X)} = \frac{-\bar{T}(\emptyset)+\bar{T}(K\cup\emptyset)}{\bar{T}(X)} = \frac{\bar{T}(K)}{\bar{T}(X)} = \frac{T(K)}{T(X)}$. Thus if $T(X) = 1$ then $\mu_T(\mathcal{K}_K) = T(K)$.

Therefore $\mu_T : \mathcal{B}(\mathcal{K}^0) \to [0, 1]$ is a Borel measure with $\mu_T(\mathcal{K}^0(X)) = 1$ and $\mu_T(\mathcal{K}_K) = T(K)$ for all $K \in \mathcal{K}^0(X)$. If we now define a RACS Ψ as the identity map then T is the capacity functional of the measure μ_T. □

As one can see the Choquet capacity theorem allows us to generate a unique measure on the hyperspace of compact subsets from a capacity functional. The nice thing with this measure as opposed to the Hausdorff measure generated from the Hausdorff metric is that the capacity functional which is defined on the compact subsets of the underlying metric space bounds the measure of those sets on the hyperspace which have been generated from the hit operation on a compact subset of the underlying metric space. In particular this means that if the underlying metric space is compact then the functional of the entire set is bound by 1. This is because the hit operation on the whole underlying compact set gives us the set containing the entire hyperspace, therefore we have that the measure of the entire hyperspace is also bound by 1. Therefore the measure induced by the capacity functional is an outer regular σ-finite Radon measure (Lemmas 4.1.8 and 4.1.9).

We now classify the functionals which satisfy the properties of the Choquet capacity theorem.

4.4.8. **Definition.**
In a compact CMS (X, Q, d_X, ν_Q) with the Hausdorff measure μ_d satisfying $\mu_d(X) = 1$ and hyperspace $\mathcal{K}^0(X)$ equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{K}^0)$. If a functional $T : \mathcal{K}^0(X) \to [0,1]$ with $\mu_T(\mathcal{K}^0(X)) = 1$ satisfies the three conditions in the Choquet Capacity Theorem (Theorem 4.4.7) then we call T a Choquet capacity and μ_T a Choquet measure.

If a functional has the added property of being upper semi-computable then we call the resulting measure a computable Choquet measure.

The next result from Choquet gives us a way to construct Choquet capacities from upper semi-continuous functions.

4.4.9. **Proposition.** *(Proposition 1.1.11. & Lemma 4.1.7.* [33]*)*
Let (X, τ) be a locally-compact second-countable Hausdorff space and $f : X \to [0,1]$ an upper semi-continuous function, then the function $T_f : \mathcal{K}^0(X) \to \mathbb{R}$ defined as

$$T_f(K) = \begin{cases} \sup_{x \in K} f(x) & \text{if } K \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

is a Choquet capacity.

Another way in which we can construct a Choquet capacity on the hyperspace of closed subsets is by using a Borel measure defined on the underlying space. A typical approach is to use the generalized Poisson process developed by Matheron [29] and later used by Axon [18] in his approach to algorithmic randomness on closed sets.

4.4.10. **Proposition.** *(Proposition 2.4.23* [18]*)*
Let μ be a Borel measure on (X, τ) such that the measure is finite for all compact subsets of X. Then $T : \mathcal{K}^0(X) \to \mathbb{R}$ defined as $T(K) = 1 - e^{-\mu(K)}$ for all $K \in \mathcal{K}^0(X)$ is a Choquet capacity.

4.4.11. **Proposition.**
In a compact CMS (X, Q, d_X, ν_Q) with the Hausdorff measure μ_d satisfying $\mu_d(X) = 1$ and hyperspace $\mathcal{K}^0(X)$ equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{K}^0)$. If a functional $T : \mathcal{K}^0(X) \to [0,1]$ is a capacity functional of the Borel measure $\mu_T : \mathcal{B}(\mathcal{K}^0) \to [0,1]$

then the restriction $\mu_T^* : \mathcal{B}(\mathcal{K}) \to [0,1]$ of μ_T to the hyperspace of non-empty compact subsets is a measure on $\mathcal{K}(X)$ and $(\mathcal{K}(X), \zeta, d_H, \nu_\zeta)$ is a computable metric space with Borel σ-algebra $\mathcal{B}(\mathcal{K})$, such that $\mu_T^*(\mathcal{K}_K) = T(K)$ for all $K \in \mathcal{K}(X)$.

Proof.
This result follows directly from the properties of the subspace measure and subspace topology. □

5. Martin-Löf Randomness

5.1. Martin-Löf Randomness for Computable Metric Spaces.

Martin-Löf randomness is a randomness concept first defined using Lebesgue measures by Martin-Löf in 1966 and then extended to probability measures by Levin in 1973. Since then much work has been done in the field of algorithmic randomness. And whilst there are other randomness concepts such as Schnorr randomness, Weak-2 randomness and others that have equal validity in this setting, we shall be focusing on Martin-Löf randomness, even though a number of the results in this and following sections can be used with other randomness concepts with a few minor changes.

We shall begin by giving the standard definition for Martin-Löf randomness in the generalized Cantor space Σ^ω.

5.1.1. **Definition.**
A subset $A \subseteq \Sigma^\omega$ of the Cantor space Σ^ω is a nullset if there exists a $W \subseteq \mathbb{N} \times \Sigma^*$ where for all n, $W_n = \{\sigma : (n, \sigma) \in W\}$ is such that $A \subseteq \bigcup_{\sigma \in W_n} N_\sigma$ and $\sum_{\sigma \in W_n} |\Sigma|^{-|\sigma|} \leqslant 2^{-n}$. For easier notation in the following we shall refer to the set W as the nullset.

5.1.2. **Definition.**
A Martin-Löf test is a c.e. set $W \subseteq \mathbb{N} \times \Sigma^*$ such that W satisfies the condition of being a nullset. A Martin-Löf test W is said to satisfy a point $x \in \Sigma^\omega$ if $x \in \bigcup_{\sigma \in W_n} N_\sigma$ for all $n \in \mathbb{N}$.

Thus a point in a generalized Cantor space $x \in \Sigma^\omega$ is satisfied by a Martin-Löf test $W \subseteq \mathbb{N} \times \Sigma^*$ if $x \in \bigcap_{n \in \mathbb{N}} \bigcup_{\sigma \in W_n} N_\sigma$.

5.1.3. **Definition.** (Martin-Löf Randomness)
A point $x \in \Sigma^\omega$ is Martin-Löf random if there is no Martin-Löf test W which satisfies the point $x \in \Sigma^\omega$.

We shall now define Martin-Löf randomness in computable metric spaces, which we shall later show to be a proper generalization of Martin-Löf randomness in the Cantor space.

5.1.4. **Definition.** (Martin-Löf Test in Computable Metric Space)
A CMS-randomness test on a computable metric space (X, Q, d_X, ν_Q) is a computable sequence $(U_i)_i$ of c.e. open sets U_i such that $\mu_d(U_i) \leqslant 2^{-i}$ for all $i \in \mathbb{N}$. A CMS-randomness test $(U_i)_i$ is said to satisfy a point $x \in X$ if $x \in \bigcap_{i \in \mathbb{N}} U_i$.

A point in a CMS satisfies a CMS-randomness test if there exists a computable sequence of c.e. open sets each of which contains the point and each of whose Hausdorff measure is bound by 2^{-n}.

5.1.5. **Definition.** (Martin-Löf Randomness in Computable Metric Spaces)
For a computable metric space (X, Q, d_X, ν_Q) an element $x \in X$ is CMS-random if and only if $x \notin \bigcap_{i \in \mathbb{N}} U_i$ for any CMS-randomness test $(U_i)_i$ on X. Or in other words, a point $x \in X$ is CMS-random if and only if no CMS-randomness test satisfies it.

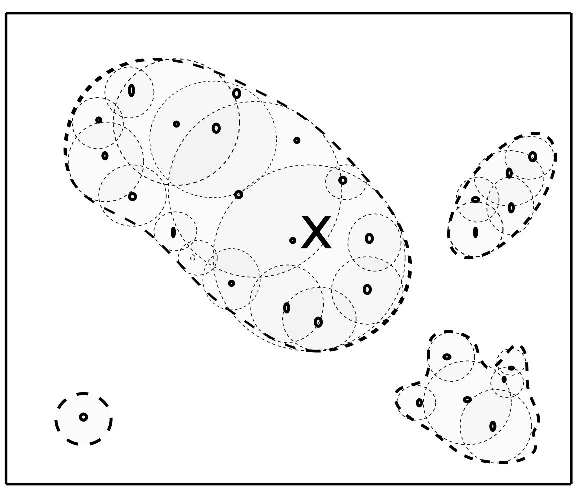


FIGURE 4. The point x is contained in a c.e. open set which is the union of open balls with centre from the enumerable dense subset and rational radius.

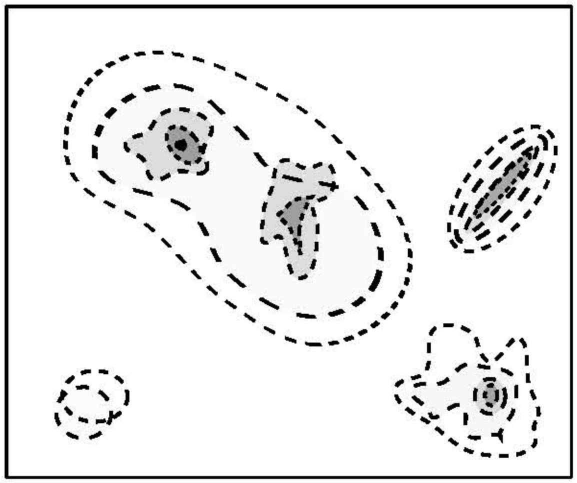


FIGURE 5. An example of a CMS randomness test's first few c.e. open sets, with the darker shading indicating a later open set in the sequence. Note that a CMS-randomness test need not single out a point. One randomness test could contain many non-random points.

5.1.6. **Examples.** Examples of non-random points in common metric spaces:

(1) In the CMS $(\mathbb{R}, \mathbb{Q}, d_{\mathbb{R}}, \nu_{\mathbb{Q}})$ the rational numbers $q \in \mathbb{Q}$ are non-random.
For each $q \in \mathbb{Q}$ set $U_i^q := B(q, 2^{-i-1})$. Then the sequence $(U_i^q)_{i\in\mathbb{N}}$ is a randomness test for q.

(2) In any CMS (X, Q, d_X, ν_Q) the computable points are non-random.
Let $x \in X$ be a computable point then let $(q_i)_{i\in\mathbb{N}}$ be a sequence in Q rapidly converging to x. Then set $U_{i+1} := B(q_i, 2^{-i})$. We then have that $(U_i)_{i\in\mathbb{N}}$ is a randomness test for x.

(3) Thus in $(\mathbb{R}, \mathbb{Q}, d_{\mathbb{R}}, \nu_{\mathbb{Q}})$ the points π, e and $\sqrt{2}$ are all non-random.
They are non-random as there are algorithms which can approximate them with arbitrary precision using rational numbers.

5.2. **CMS-Randomness in Cantor Space.**

We shall now show the relationship between our generalization of Martin-Löf randomness and the usual definition of Martin-Löf randomness in the Cantor space.

5.2.1. **Theorem.**
A point $x \in \Sigma^\omega$ is Martin-Löf non-random if and only if it is CMS-non-random in the space $(\Sigma^\omega, \Sigma^* 0^\omega, d_\Sigma, \nu_\Sigma)$.

Proof.
In the CMS $(\Sigma^\omega, \Sigma^* 0^\omega, d_\Sigma, \nu_\Sigma)$ let $x \in \Sigma^\omega$.

('$\Rightarrow$')

Let x be ML non-random.

Then there exists a c.e. set $W \subseteq \mathbb{N} \times \Sigma^*$ such that $x \in \bigcap_{i \in \mathbb{N}} \bigcup_{\Sigma \in W_i} \mathcal{N}_\sigma$.

We now define a sequence of sets $(U_i)_i \in \mathbb{N}$ as follows:

$$U_n = \bigcup_{\sigma \in W_n} \mathcal{N}_\sigma = \bigcup_{\sigma \in W_n} B(\sigma 000..., 2^{|\sigma|-1})$$

Then for all $n \in \mathbb{N}$ the set U_n is a c.e. open set and the sequence $(U_i)_{i \in \mathbb{N}}$ is a CMS randomness test satisfying the point x since

(1) $(U_i)_{i \in \mathbb{N}}$ is a computable sequence of c.e. open sets

(2) $x \in \bigcap_{i \in \mathbb{N}} U_i$

(3) $\mu_{d_\Sigma}(U_i) \leqslant 2^{-i}$ for all $i \in \mathbb{N}$.

Therefore the point x is CMS non-random.

('$\Leftarrow$')

Let x be a CMS non-random point and $(U_i)_{i \in \mathbb{N}}$ a CMS randomness test satisfying x.

Since for each $i \in \mathbb{N}$ the open set U_i is c.e. we have that there exists a c.e. sequence $((w_0^i, q_0^i), (w_1^i, q_1^i), ...)$ in $\Sigma^* \times \mathbb{Q}^+$ such that $U_i = \bigcup_{n \in \mathbb{N}} B(w_n^i, q_n^i)$ and that for each pair (w_n^i, q_n^i) we can computably find a $\sigma_n \in \Sigma^*$ such that $\mathcal{N}_{\sigma_n} = B(w_n^i, q_n^i)$.

Note that for any $w \in \Sigma^*$ and $q \in \mathbb{Q}^+$ we can find an $n \in \mathbb{N}$ such that $2^{-n-1} \leqslant q \leqslant 2^{-n}$ and that if we take the first n digits of $w^\frown 0^\omega$ then the resulting word $w' = w^\frown 0^\omega\lceil_n$ is such that $\mathcal{N}_{w'} = B(w, q)$.

We now define the set $W = \{(i, \sigma) \in \mathbb{N} \times \Sigma^* : \ (\exists j \in \mathbb{N}^+)\ \mathcal{N}_\sigma = B(w_j^i, q_j^i)\}$.

In order to insure that W is a nullset we need to limit the number of repeated partial strings in the final set, since the requirement that $\sum_{\sigma \in W_n} |\Sigma|^{-|\sigma|} \leqslant 2^{-n}$ has not been guaranteed.

To this end we create a new set W' by listing the elements of the c.e. set W where for each element (i, σ) in W we do the following:

(1) If there exists an element (j, α) already in W' such that there exists $k \leqslant |\sigma|$ such that $i = j$ and $\sigma\lceil_k = \alpha$ then leave (i, σ) out of W'.

(2) If there exists an element (j, α) already in W' such that there exists $k \leqslant |\alpha|$ such that $i = j$ and $\alpha\lceil_k = \sigma$ then for all $w \in \Sigma^{|\alpha|-|\sigma|}$, excepting the w such that $\sigma^\frown w = \alpha$, add $(i, \sigma^\frown w)$ to W_i.

(3) If neither (1) or (2) was true then put (i, σ) in W'.

We now have that W' is a c.e. set that satisfies the condition of being a nullset and that it satisfies the point x. Therefore x is ML non-random. □

Thus the ML-random points in the Cantor space with the usual measure are exactly those points that are CMS-random with the Cantor space metric as defined in Example 2.2.17 (4). This supports the claim that our generalization is a valid approach to studying randomness in computable metric spaces.

6. **Randomness of Non-Empty Compact Subsets**

6.1. **The Hausdorff Measure of a Hausdorff Metric.**

When looking at the randomness of non-empty compact subsets of a computable metric space it seems natural to use the Hausdorff measure of the Hausdorff metric in line with our definition of CMS-randomness in Definition 5.1.5. Unfortunately this leads to a trivial notion of randomness, where the non-empty compact subsets are either all random or are all non-random, in many hyperspaces.

We begin by clearly defining the CMS-randomness on a hyperspace with the Hausdorff metric. This is merely an application of Definition 5.1.5 on the CMS $(\mathcal{K}(X), \zeta, d_H, \nu_\zeta)$.

6.1.1. **Definition.**
Let (X, Q, d_X, ν_Q) be a CMS which generates $(\mathcal{K}(X), \zeta, d_H, \nu_\zeta)$ the hyperspace of non-empty compact subsets and $\mathbf{h}_d$ the Hausdorff measure of the Hausdorff metric on the hyperspace. By our earlier definition of a CMS-randomness test we can define the CMS-randomness test on the hyperspace ($\mathbf{h}_d$-randomness) as a computable sequence $(\mathcal{U}_i)_{i \in \mathbb{N}}$ of c.e. open sets $\mathcal{U}_i$ such that $\mathbf{h}_d(\mathcal{U}_i) \leqslant 2^{-i}$ for all $i \in \mathbb{N}$.

We now define two types of spaces, connected and discrete. We shall later use these definitions to show how $\mathbf{h}_d$-randomness behaves on different types of spaces.

6.1.2. **Definition.** (Connected Space)
We call a topological space connected if the space cannot be written as the union of two disjoint non-empty open sets. A topological space is totally disconnected if every non-empty connected subset of the space is a singleton.

6.1.3. **Definition.** (Discrete Space)
Let (X, d) be a metric space. A point in the space is isolated if there is some $r > 0$ for which the open ball centered on the point with radius r is a singleton. We call the space perfect if no point in the space is isolated. A space in which every point is isolated is called discrete.

It is easy to see that if a metric space is connected and has more than one point then it is perfect. The converse is not necessarily true. As an example we take the space of rational numbers with its usual metric. The rationals are perfect since no point in the space has an open neighbourhood containing only itself, but it is also totally disconnected because if we take any two points $x, y \in \mathbb{Q}$ then there exists an irrational number r in between the two numbers and the open sets $(-\infty, r) \cap \mathbb{Q}$ and $(r, \infty) \cap \mathbb{Q}$ cover the entire space and separate the points.

6.1.4. **Definition.** (Continuous Space)
Let (X, d) be a metric space. We call the space continuous if the space cannot be written as the union of two disjoint non-empty open sets $U, V \in \mathcal{O}(X)$ where

$$\inf_{x \in U, y \in V} d(x, y) > 0.$$

We refer to this requirement on the disjoint open subsets as being that the open subsets are not positively separated. A space that is connected is continuous, but a continuous space need not be connected, as in the previous example we can use the set of rationals with the usual metric as a counter example.

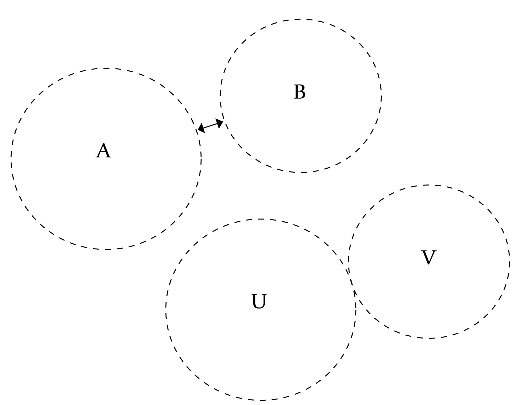


FIGURE 6. The open sets A and B are an example of positively separated sets, hence $A \cup B$ is neither connected nor continuous. The open sets U and V are an example of disjoint open sets whose distance between them is 0. Thus $U \cup V$ is continuous but not connected.

An important example of a connected hyperspace is the space of non-empty compact subsets of the reals $\mathcal{K}(\mathbb{R})$ equipped with the metric topology induced by the Hausdorff metric d_H.

6.1.5. **Definition.** (Proper Space)
Let (X, d) be a metric space. The space is called proper if for every $x \in X$ and $r \in \mathbb{R}^+$ the closed ball $B[x, r]$ is compact.

We can now begin looking at the properties that the metric and eventually that the Hausdorff measure has in these different spaces.

6.1.6. **Lemma.**
If (X, d) is a connected metric space and $a, b \in X$ with $d(a, b) = k$ then for all $\varepsilon \leqslant k$ there exists a $y \in X$ such that $d(a, y) = \varepsilon$.

Proof.
Let $a, b \in X$ be such that $d(a, b) = k$.
Obviously if $\varepsilon = k$ then $b \in X$ is such that $d(a, b) = \varepsilon$.

Assume that there exists an $\varepsilon < k$ such that $d(a, y) \neq \varepsilon$ for all $y \in X$.
Then $B(a, \varepsilon) = B[a, \varepsilon]$ so $B(a, \varepsilon)$ and $X \backslash B[a, \varepsilon]$ are disjoint open sets such that $B(a, \varepsilon) \cup (X \backslash B[a, \varepsilon]) = X$. Which contradicts the connectedness of X.

Therefore our assumption is false and for all $\varepsilon \leqslant k$ there exists a $y \in X$ such that $d(a, y) = \varepsilon$.

□

We have thus shown that in a connected space if there exists two points a given distance apart in the space then for any distance less than that we can find points between the two. Of particular interest to us is the case of an open ball centered on a point x with a radius r that is less than half the diameter of the space. In this case we see, using the triangle inequality, that there must exist points in the space whose distance from x is arbitrarily close to r.

6.1.7. **Lemma.**
Let (X, d) be a connected metric space with $\operatorname{diam}(X) > 0$. If $x \in X$ and $0 < 2r < \operatorname{diam}(X)$ then $\operatorname{diam}(B(x, r)) \geqslant r$.

Proof.
Let $x \in X$ and $2r < \text{diam}(X)$.

Therefore there exists $a, b \in X$ such that $d(a,b) = 2r$ by Lemma 6.1.6.

Using the triangle inequality we get that

$$d(a,x) + d(x,b) \geqslant 2r$$

Therefore either $d(a,x) \geqslant r$ or $d(x,b) \geqslant r$.

Therefore by Lemma 6.1.6 we know that for all $\varepsilon < r$ there exists a $y \in X$ such that $d(x,y) = \varepsilon$ and hence that $\sup\{d(x,y) : \ y \in X \text{ and } d(x,y) < r\} = r$. Therefore $\text{diam}(B(x,r)) \geqslant r$.

□

We can now see, using the two previous lemmas, that if an open ball $B(x,r)$ has a radius less than half the diameter of the connected space then for any distance $0 \leqslant \varepsilon < r$ there exists a point $y \in B(x,r)$ for which $d(x,y) = \varepsilon$.

6.1.8. **Lemma.**
Let (X,d) be a connected metric space, $x \in X$ and $0 < 2r < \text{diam}(X)$. If V is the connected component of $B(x,r)$ containing x then for all $\varepsilon < r$ there exists a $y \in V$ such that $d(x,y) = \varepsilon$.

Proof.
Let V be the connected component of $B(x,r)$ containing x. If $B(x,r) \neq V$ then $B(x,r)$ is not connected so there exists a non-empty open subset U of X such that $V \cap U = \emptyset$, $B(x,r) \cap U \neq \emptyset$, and $(B(x,r) \cap V) \cup (B(x,r) \cap U) = B(x,r)$. If $B(x,r) = V$ then let $U = \emptyset$.

Assume that there exists $\varepsilon < r$ such that for all $y \in V$ we have that $d(x,y) \neq \varepsilon$. Then $B(x,\varepsilon) \subseteq (B(x,\varepsilon) \cap V) \cup U$ with $B(x,\varepsilon) \cap V$ being non-empty since $x \in V$ and $(X \backslash B[x,\varepsilon]) \cup U$ is non-empty since $\varepsilon < r < \text{diam}(X)$.

Note that $B(x,\varepsilon) \cap V = B[x,\varepsilon] \cap V$ so all the boundary elements of $B(x,\varepsilon)$ are in U. Therefore $B(x,\varepsilon) \cap V$ and $(X \backslash B[x,\varepsilon]) \cup U$ are non-empty open sets such that $(B(x,\varepsilon) \cap V) \cup ((X \backslash B[x,\varepsilon]) \cup U) = X$ and $(B(x,\varepsilon) \cap V) \cap ((X \backslash B[x,\varepsilon]) \cup U) = \emptyset$ which contradicts the connectedness of X. Therefore for all $\varepsilon < r$ there exists a $y \in V$ such that $d(x,y) = \varepsilon$. □

We now prove that the sums of the diameters of the open sets that cover an open ball is greater than or equal to the radius of the open ball.

6.1.9. **Lemma.**
Let (X, d) be a connected proper metric space with $\text{diam}(X) > 0$*. Let $x \in X$ and $0 < 2r <$* $\text{diam}(X)$*. If $(U_i)_{i\in\mathbb{N}}$ is a collection of open sets which cover $B(x, r)$ then*

$$\sum_{i\in\mathbb{N}} \text{diam}(U_i) \geqslant r.$$

Proof.
Let $(U'_i)_{i\in\mathbb{N}}$ be a collection of open sets which cover $B(x, r)$. Fix $r > \varepsilon > 0$. Then let $(U_i)_{i\leqslant k}$ be the finite subcover which covers the compact set $B[x, r-\varepsilon]$ and V be the connected component of $B(x, r-\varepsilon)$ which contains x.

Then $\bigcup_{i\leqslant k}(V \cap U_i) = V$ and $\sum_{i\leqslant k} \text{diam}(V \cap U_i) \leqslant \sum_{i\in\mathbb{N}} \text{diam}(U'_i)$.

Fix $\delta > 0$. By Lemma 6.1.8 and the connectedness of V we have that there exists a $y_\varepsilon \in V$ such that $d(x, y_\varepsilon) = r - \varepsilon - \delta$.

Let $j_0 \in \mathbb{N}$ be such that $x \in U_{j_0}$. We now define $V_0 = V \cap U_{j_0}$.

We now inductively define the indexes j_i and the sets V_i as follows:

(1) Let $j_{i+1} \in \mathbb{N}$ be such that for all $k \leqslant i$ we have that $j_{i+1} \neq j_k$ and $(V \cap U_{j_{i+1}}) \cap V_i \neq \emptyset$.

(2) Let $V_{i+1} = V \cap U_{j_{i+1}}$.

(3) If no such j_{i+1} is possible we choose a different j_i.

We end the induction when $y_\varepsilon \in V_n$ for some $n \in \mathbb{N}$. Note that this construction can be easily defined as a simple tree search algorithm and furthermore that the connectedness of V and that there are only finitely many open sets U_i to check guarantees a finite path satisfying $x \in V_0$ and $y_\varepsilon \in V_n$.

We now choose points in the resulting overlaps. For each $i < n$ we choose a point $a_i \in V_i \cap V_{i+1}$. Then for each $i < n$ we have that $\text{diam}(V_{i+1}) \geqslant d(a_i, a_{i+1})$ and that

$\text{diam}(V_0) \geqslant d(x, a_0)$ and $\text{diam}(V_n) \geqslant d(a_{k-1}, y_\varepsilon)$.

Therefore

$$\begin{aligned}\sum_{i\in\mathbb{N}} \text{diam}(U_i) &\geqslant \sum_{i\leqslant n} \text{diam}(V_i) \\ &= \text{diam}(V_0) + \text{diam}(V_1) + ... + \text{diam}(V_{n-1}) + \text{diam}(V_n) \\ &\geqslant d(x, a_0) + d(a_0, a_1) + ... + d(a_{n-2}, a_{n-1}) + d(a_{n-1}, y_\varepsilon) \\ &\geqslant d(x, y_\varepsilon) \text{ (by the triangle inequality)} \\ &= r - \varepsilon - \delta.\end{aligned}$$

And since δ and ε can be arbitrarily small we have that

$$\sum_{i\in\mathbb{N}} \text{diam}(U_i) \geqslant r.$$

□

The consequence of the above lemma is that the Hausdorff measure of any open ball in a connected proper hyperspace is greater than or equal to the radius of that ball, if the radius is less than half the diameter of the hyperspace. We shall shortly use this fact to prove that Hausdorff measure of any open ball in the hyperspace is in fact infinite.

But first we must show if an underlying metric space is proper then the hyperspace of non-empty compact subsets is proper as well. For this proof we shall be using an equivalent definition of compact sets in which a closed set in a metric space is compact if and only if it is complete and totally bounded (that is that for any $\varepsilon > 0$ the closed set can be covered by finitely many open balls of a radius ε).

6.1.10. **Lemma.**
Let (X, d) be a proper metric space then $(\mathcal{K}(X), d_H)$ is a proper metric space.

Proof.
Let (X, d) be a proper metric space. Let $K \in \mathcal{K}(X)$ and $r \in \mathbb{R}^+$. Let $\varepsilon > 0$.

By the compactness of K we have that there exists $x \in X$ and $\delta \in \mathbb{R}^+$ such that $B[x, \delta] \supseteq \{y \in X : (\exists k \in K)\ d(y, k) \leqslant r\}$. Since $B[x, \delta]$ is compact there exists a finite collection of points $(x_i)_{i<k}$ in X such that $(B(x_i, \varepsilon))_{i<k}$ covers the set $B[x, \delta]$.

We now define a finite collection $(F_i)_{i<m}$ of compact sets by taking all non-empty combinations of the points from $(x_i)_{i<k}$. Thus for any $I \subseteq \{0, 1, ..., k-1\}$ we have

that there exists a $j < m$ such that $F_j = \{x_i : \ i \in I\}$.

Let $F \in B[K, r]$. Then $F \subseteq \{y \in X : \ (\exists k \in K) \ d(y, k) \leqslant r\} \subseteq B[x, \delta]$ and hence there exists $j < m$ such that the following hold

$$(\forall a \in F)(\exists b \in F_j) \ d(a, b) < \varepsilon.$$
$$(\forall a \in F_j)(\exists b \in F) \ d(a, b) < \varepsilon.$$

Therefore $d_H(F_j, F) < \varepsilon$. Therefore $F \in B(F_j, \varepsilon)$.

Therefore $(B(F_i, \varepsilon))_{i<m}$ is a finite open cover of the set $B[K, r]$.

Since ε was arbitrary the closed ball $B[K, r]$ is compact.

Therefore $(\mathcal{K}(X), d_H)$ is a proper metric space.

□

A variation of the following result was first noticed in what was a little known paper written by Christoph Bandt and Gebreselassie Baraki in 1986 [34].

6.1.11. **Lemma.**
Let (X, d) be a connected proper metric space with $\operatorname{diam}(X) > 0$ and $\mathcal{K}(X)$ connected. If $0 < 2r < \operatorname{diam}(X)$ and $K \in \mathcal{K}(X)$ then $\boldsymbol{h}_d(B(K, r)) = \infty$.

Proof.
Let $2r < \operatorname{diam}(X)$ and $K \in \mathcal{K}(X)$.

We begin by constructing 31 compact sets which are elements of $B(K, r)$. To begin we choose one point $x \in K$ and let $F = K \cap (X \backslash B(x, \frac{11r}{12}))$. It is easy to see that F is a compact set.

By Lemma 6.1.7 we know that $\operatorname{diam}(B(x, \frac{7r}{8})) \geqslant \frac{7}{8}r$. We now choose six points $a_0, ..., a_5$ where $a_0 = x$ and for each $i = 1, 2, 3, 4, 5$ the point $a_i \in B(x, \frac{7r}{8}) \backslash (\bigcup_{j<i} B(a_j, \frac{r}{12}))$. Note that we can guarantee that these five points exist by the triangle inequality of metrics and Lemma 6.1.6. That is to say that if only four points existed that satisfied the criterion above then $B(x, \frac{7r}{8}) \backslash (\bigcup_{j<5} B(a_j, \frac{r}{12})) = \emptyset$. Therefore $B(x, \frac{7r}{8}) \subseteq \bigcup_{j<5} B(a_j, \frac{r}{12})$ and $\operatorname{diam}(B(a_0, \frac{r}{12})) + \operatorname{diam}(B(a_1, \frac{r}{12})) + \operatorname{diam}(B(a_2, \frac{r}{12})) +$

$\text{diam}(B(a_3, \frac{r}{12})) + \text{diam}(B(a_4, \frac{r}{12})) \leqslant 5 \times \frac{2r}{12} < \frac{7r}{8} \leqslant \text{diam}(B(x, \frac{7r}{8}))$ which contradicts Lemma 6.1.9.

We next define the 31 compact sets $K_0, ..., K_{30}$ of our construction by joining all possible combinations of the five points $a_1, ..., a_5$ with the set $F \cup \{a_0\}$. For example:

$$\begin{aligned}
K_0 &= F \cup \{a_0, a_1\} \\
K_1 &= F \cup \{a_0, a_1, a_2\} \\
K_2 &= F \cup \{a_0, a_2, a_3\} \\
K_5 &= F \cup \{a_0, a_1, a_2, a_4\} \\
&. \\
&. \\
&. \\
K_{29} &= F \cup \{a_0, a_1, a_2, a_3, a_4\} \\
K_{30} &= F \cup \{a_0, a_1, a_2, a_3, a_4, a_5\}
\end{aligned}$$

Note that for all $i \in \{0, 1, ..., 30\}$ that $d_H(K_i, K) < \frac{11r}{12}$ since for all $y \in K$ either $y \in F$ or $d(y, a_0) < \frac{11r}{12}$ and for all $y \in K_i$ either $y \in F \cup \{a_0\} \subseteq K$ or $d(y, x) < \frac{7r}{8} < \frac{11r}{12}$.

We now prove that for any $i, j \in \{0, 1, ..., 30\}$ **where** $i \neq j$ **that** $B(K_i, \frac{r}{24}) \cap B(K_j, \frac{r}{24}) = \emptyset$**.**

Fix $i, j \in \{0, 1, ..., 30\}$ such that $i \neq j$.

Assume that $B(K_i, \frac{r}{24}) \cap B(K_j, \frac{r}{24}) \neq \emptyset$ and let $L \in B(K_i, \frac{r}{24}) \cap B(K_j, \frac{r}{24})$. Then $d_H(K_i, L) < \frac{r}{24}$ and $d_H(K_j, L) < \frac{r}{24}$. But this means that the following hold:

$$(\forall a \in L)(\exists b \in K_i) \;\; d(a, b) < \frac{r}{24} \qquad (1)$$
$$(\forall a \in K_i)(\exists b \in L) \;\; d(a, b) < \frac{r}{24} \qquad (2)$$
$$(\forall a \in L)(\exists b \in K_j) \;\; d(a, b) < \frac{r}{24} \qquad (3)$$
$$(\forall a \in K_j)(\exists b \in L) \;\; d(a, b) < \frac{r}{24} \qquad (4)$$

But then we have that if $a_k \in K_i$ for some $k \in \{0,1,2,3,4,5\}$ then by (2) there exists $b \in L$ such that $d(a_k, b) < \frac{r}{24}$. But then by (3) there exists a $c \in K_j$ such that $d(b,c) < \frac{r}{24}$. And by the triangle inequality we have that $d(a_k, c) < \frac{r}{12}$. But the only element of X within $\frac{r}{12}$ of a_k in any of the K_n's is a_k. Therefore $a_k \in K_j$. Similarly if any $a_k \in K_j$ then $a_k \in K_i$. But this means that $K_i = K_j$ which is impossible since $i \neq j$. Therefore our assumption is false and $B(K_i, \frac{r}{24}) \cap B(K_j, \frac{r}{24}) = \emptyset$.

We next show that $B(K_i, \frac{r}{24}) \subset B(K, r)$ for all $i \in \{0, 1, ..., 30\}$.

Fix $i \in \{0, 1, ..., 30\}$.

Let $L \in B(K_i, \frac{r}{24})$ then $d_H(L, K_i) < \frac{r}{24}$ and $d_H(K_i, K) < \frac{11r}{12}$ hence, by the triangle inequality,

$$r > d_H(L, K_i) + d_H(K_i, K) \geqslant d_H(L, K)$$

Therefore $L \in B(K, r)$. Therefore $B(K_i, \frac{r}{24}) \subset B(K, r)$.

We now show that for any $K \in \mathcal{K}(X)$ and $2r < \operatorname{diam}(X)$ the Hausdorff measure of the ball is greater than $\frac{31}{24}r$.

We now have 31 pairwise disjoint open balls all of which are contained in $B(K, r)$. Thus

$$\begin{aligned}
\mathbf{h}_d(B(K,r)) &\geqslant \mathbf{h}_d\Big(\bigcup_{i<31} B(K_i, \frac{r}{24})\Big) \\
&= \sum_{i<31} \mathbf{h}_d(B(K_i, \frac{r}{24})) && (*) \\
&\geqslant \sum_{i<31} \frac{r}{24} && \text{(Lemma 6.1.9 and Lemma 6.1.10)} \\
&= \frac{31}{24} r.
\end{aligned}$$

Lastly we show that $\mathbf{h}_d(B(K, r)) = \infty$.

We now use the above method recursively on the open balls in step (*) in order to get the following inequality:

$$\mathbf{h}_d(B(K,r)) \geqslant \frac{31^n}{24^n} r.$$

Where n is the number of times the sets in step (*) were split into 31 smaller sets whose radii are $\frac{1}{24}$ the radius of the previously created set.

Thus as we take larger and larger values of n we get that

$$\mathbf{h}_d(B(K,r)) \geqslant \lim_{n\to\infty} \frac{31^n}{24^n} r = \infty.$$

□

6.1.12. **Theorem.** *If (X, Q, d_X, ν_Q) is a connected proper CMS with the hyperspace of non-empty compact subsets $(\mathcal{K}(X), \zeta, d_H, \nu_\zeta)$ being connected then every compact set $K \in \mathcal{K}(X)$ is CMS-random.*

Proof.
Assume $K \in \mathcal{K}(X)$ be CMS-nonrandom. Let $(U_i)_{i\in\mathbb{N}}$ be a CMS-randomness test for K.

Fix $i \in \mathbb{N}$.

There exists a sequence $((F_j, q_j))_{j\in\mathbb{N}}$ in $\zeta \times \mathbb{Q}^+$ such that $U_i = \bigcup_{j\in\mathbb{N}} B(F_j, q_j)$. Note that $\bigcap_{i\in\mathbb{N}} U_i \neq \emptyset$. So there is at least one $j \in \mathbb{N}$ such that $q_j > 0$, else $U_i = \emptyset$. Fix $j \in \mathbb{N}$ such that $q_j > 0$.

Then by Lemma 6.1.11 we know that $\mathbf{h}_d(B(F_j, q_j)) = \infty$ and since $B(F_j, q_j) \subseteq U_i$ we have that $\mathbf{h}_d(U_i) = \infty$.

And since this is true for all $i \in \mathbb{N}$ we have that $\mathbf{h}_d(U_i) > 2^{-i}$ for all $i \in \mathbb{N}$, which is a contradiction. Therefore $\bigcap_{i\in\mathbb{N}} U_i = \emptyset$ and hence $K \notin \bigcap_{i\in\mathbb{N}} U_i$. Therefore K is CMS random. □

We have now proven that in connected computable metric spaces we get a trivial concept of randomness. Therefore in the hyperspace $(\mathcal{K}([0,1]), F(\mathbb{Q} \cap [0,1]), d_H, \nu_{F(\mathbb{Q}\cap[0,1])})$ of the unit interval in $\mathbb{R}$ with the usual Euclidean metric every compact subset is random, including the whole space and the singletons. Obviously this is not a very useful notion of randomness in the hyperspaces.

Having shown the problem with connected spaces we now look at another class of spaces, namely the discrete spaces. We begin by showing that if the underlying space is discrete then the hyperspace of non-empty compact subsets is discrete as well.

6.1.13. **Lemma.**
If a separable metric space (X, d) is discrete then the hyperspace of non-empty compact subsets $(\mathcal{K}(X), d_H)$ is a discrete metric space as well.

Proof.
Let (X, d) be a discrete space, then for all $x \in X$ there exists a $\delta_x > 0$ such that $B(x, \delta_x) = \{x\}$.

Let $K \in \mathcal{K}(X)$. Since X is countable and K is compact we have that K is finite. If we then set $\delta = \min\{\delta_x : x \in X\}$ then $B(K, \delta) = \{K\}$. Therefore $\mathcal{K}(X)$ is discrete.

□

We can now prove that the Hausdorff measure of any subset of a discrete space is 0.

6.1.14. **Lemma.**
If (X, d) is a discrete separable metric space then for all $A \subseteq X$ the Hausdorff measure $\mu_d(A) = 0$.

Proof.
Let (X, d) be a discrete space, then for all $x \in X$ there exists a $\delta_x > 0$ such that $B(x, \delta_x) = \{x\}$.

In order to prove that for all $A \subseteq X$ the Hausdorff measure is 0, we need only look at the Hausdorff measure of the entire space. Since X is countable, the collection $\mathcal{U} = \{B(x, \delta_x) : x \in X\}$ is a countable collection of open sets that cover X.

For all $x \in X$ and $\delta \leqslant \delta_x$ we have that $B(x, \delta)$ is a singleton, therefore for all $\varepsilon > 0$

$$\inf\left\{\sum_{i\in\mathbb{N}} \operatorname{diam}(U_i) : \bigcup_{i\in\mathbb{N}} U_i \supseteq X \text{ and } (\forall i \in \mathbb{N})\ U_i \text{ is open and } \operatorname{diam}(U_i) < \epsilon\right\}$$
$$\leqslant \sum_{x\in X} \operatorname{diam}(\{x\}) = 0.$$

Therefore $\mu_d(X) = 0$.

□

6.1.15. **Theorem.** *In a CMS* (X, Q, d_X, ν_Q) *with hyperspace of non-empty compact subsets* $(\mathcal{K}(X), \zeta, d_H, \nu_\zeta)$ *let* $\mathcal{U} \in \mathcal{O}(\mathcal{K}(X))$ *be such that for all* $K \in \mathcal{U}$ *and for all* $x \in K$ *the point* x *is isolated in* X. *Then* $\boldsymbol{h}_d(\mathcal{U}) = 0$.

Proof.
Let (X, Q, d_X, ν_Q) be a CMS and $(\mathcal{K}(X), \zeta, d_H, \nu_\zeta)$ the hyperspace of non-empty compact subsets. Let $\mathcal{U} \in \mathcal{O}(\mathcal{K}(X))$ be such that for all $K \in \mathcal{U}$ and for all $x \in K$ the point x is isolated in X. Then the subspace $A = \{x \in X : (\exists K \in \mathcal{U}) x \in K\} \subseteq X$ is discrete and hence $\mathcal{K}(A)$ is a discrete subspace of $\mathcal{K}(X)$ (Lemma 6.1.13).

We also have that $\mathcal{U} \subseteq \mathcal{K}(A)$. Therefore by Lemma 6.1.14 we have that $\mathbf{h}_{d|\mathcal{K}(A)}(\mathcal{U}) \leqslant \mathbf{h}_{d|\mathcal{K}(A)}(\mathcal{K}(A)) = 0$ where $\mathbf{h}_{d|\mathcal{K}(A)}$ is the subspace measure. But since $\mathcal{U}$ is an open set in $\mathcal{K}(X)$ we have $\mathbf{h}_d(\mathcal{U}) = \mathbf{h}_{d|\mathcal{K}(A)}(\mathcal{U}) = 0$.

□

Thus if every compact subset, that is an element of an open subset in the hyperspace, is discrete then the measure of the open set is 0 in the hyperspace of non-empty compact subsets with the Hausdorff measure of the Hausdorff metric.

We have thus shown that in the hyperspace $\mathcal{K}(\mathbb{R})$ with the Euclidean metric on the underlying space that all the non-empty compact subsets are random. And that this is true even in the compact subspace $[0, 1]$. We can now see that every point in the hyperspace generated from $\mathbb{N}$ with its usual metric is non-random. That is that every non-empty finite subset of $\mathbb{N}$ is non-random in the hyperspace.

6.2. **Computable Metric Hyperspaces with Arbitrary Measure.**

We have now shown that our definition of CMS-randomness (Definition 5.1.5) leads to trivial randomness notions in hyperspaces of non-empty compact subsets generated from computable metric spaces. We thus generalize our definition of randomness to allow other measures besides the Hausdorff measure. To show this difference we define a computable metric measure space as a computable metric space with arbitrary measure.

6.2.1. **Definition.** (Computable Metric Measure Space)
(X, Q, d, α, ν) is a computable metric measure space (CMMS) if the following four conditions are met:

(1) (X, Q, d) is a separable metric space.
(2) $\alpha : \mathbb{N} \to Q$ is a numbering of Q.
(3) $d \circ (\alpha \times \alpha) : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is $(\nu_{\mathbb{N}}, \nu_{\mathbb{N}}, \rho_{\mathbb{R}})$-computable.
(4) ν is a measure defined on the Borel σ-algebra $\mathcal{B}(X)$.

6.2.2. **Definition.** (Martin-Löf Test in Computable Metric Measure Space)
A CMMS-randomness test on a computable metric measure space $(X, Q, d_X, \mu_X, \nu_Q)$ is a computable sequence $(U_i)_i$ of c.e. open sets U_i such that $\mu_X(U_i) \leqslant 2^{-i}$ for all $i \in \mathbb{N}$. A CMS-randomness test $(U_i)_i$ is said to satisfy a point $x \in X$ if $x \in \bigcap_{i\in\mathbb{N}} U_i$.

A point in a CMMS satisfies a CMMS-randomness test if there exists a computable sequence of c.e. open sets each of which contains the point and each of whose measure is bound by 2^{-n}.

6.2.3. **Definition.** (Martin-Löf Randomness in Computable Metric Measure Spaces)
For a computable metric measure space $(X, Q, d_X, \mu_X, \nu_Q)$ an element $x \in X$ is CMMS-random if and only if $x \notin \bigcap_{i\in\mathbb{N}} U_i$ for any CMMS-randomness test $(U_i)_i$ on X. Or in other words, a point $x \in X$ is CMMS-random if and only if no CMMS-randomness test satisfies it.

Alternatively we could label a CMMS-random test as a μ-CMS-random test and a $CMMS$-random point as a μ-CMS-random point. This allows us to highlight exactly which measure we are using when confusion could arise. Thus any CMS-random test is a μ_d-CMS-random test and a random compact set defined using the Hausdorff measure on the hyperspace of non-empty compact sets is a $\mathbf{h}_d$-CMS-random point in $\mathcal{K}(X)$. In what follows we shall implicitly use the notation that a μ-CMS-random compact set is a μ-CMS-random point in the hyperspace $\mathcal{K}(X)$ where $\mu : \mathcal{B}(\mathcal{K}(X)) \to \mathbb{R}$ is a measure on the hyperspace.

Having defined randomness on computable metric spaces with arbitrary measures we will now begin by looking at randomness in a few commonly used metric spaces and measures.

6.3. **Examples of μ-CMS-randomn compact sets.**

We have already seen that in $\mathbf{h}_d$-CMS-randomness of compact sets every set $K \in \mathcal{K}(X)$ is a $\mathbf{h}_d$-CMS-random compact set if the underlying space is connected and has a positive diameter. And that every set $K \in \mathcal{K}(X)$ is a $\mathbf{h}_d$-CMS-nonrandom compact set if the underlying space is discrete. Here we give an example of an easy measure that gives us both random and nonrandom compact sets.

The point inclusion measure.

Let $(\mathbb{R}, \mathbb{Q}, d_{\mathbb{R}}, \nu_{\mathbb{Q}})$ be the usual Euclidean computable metric space and for some fixed $x \in X$ we define a point inclusion measure on the hyperspace of non-empty compact subsets as

$$\mu_x(\mathcal{A}) = \begin{cases} 1 & \text{if } x \in K \text{ for some } K \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}$$

Then a set $K \in \mathcal{K}(\mathbb{R})$ is a μ_x-CMS-random compact set if and only if $x \in K$.

We can prove this by noting that if $x \in K$ then any open set $\mathcal{U}$ that contains K has measure 1 and hence no μ_x-CMS-randomness test satisfies K. For the other direction we note that the union of open balls with radius 2^{-n} and centres that are finite subsets of $\mathbb{Q}$ whose members are each a distance of at least 2^{-n} away from x is a c.e. open set in $\mathcal{K}(\mathbb{R})$ whose measure is 0. So if $x \notin K$ then it follows that eventually for a large enough $n \in \mathbb{N}$ the set K will be contained in the open set and hence we can generate a μ_x-CMS-randomness test that satisfies K.

The Hausdorff Measure.

We can use the Hausdorff measure of the Hausdorff metric to produce an example of a space that has both random and non-random compact subsets. An easy example of this would be the hyperspace generated from $([0,1] \cup \mathbb{N}, (\mathbb{Q} \cap [0,1]) \cup \mathbb{N}, d_{\mathbb{R}}, \nu_{\mathbb{Q}})$ where every compact set in the underlying space containing points from $[0,1]$ is $\mathbf{h}_d$-CMS-random and those that do not are $\mathbf{h}_d$-CMS-nonrandom.

The Florida approach.

In 2006 Barmpalias, Brodhead, Cenzer, Dayshti and Weber [16] devised what shall be referred to as the Florida approach. In this approach they use a bijective mapping of all closed sets in the Cantor space to the space of $3^{\mathbb{N}}$. They then define a random closed set as being random if and only if it's code in $3^{\mathbb{N}}$ is random.

A few new symbols and concepts must be introduced to define the Florida approach. For any tree T, $[T]$ is the set of infinite strings for which any finite prefix is an element of T. For a finite binary string $\sigma \in 2^*$ we define the set $I(\sigma) = \{x \in 2^{\mathbb{N}} : \sigma \subset x\}$. For a closed set $Q \subseteq 2^{\mathbb{N}}$ the tree T_Q is the binary tree defined as $\{\sigma : P \cap I(\sigma) \neq \emptyset\}$. We can then define the tree $T = T_Q$ where $Q = [T]$.

The mapping $E : \mathcal{K}(2^{\mathbb{N}}) \to 3^{\mathbb{N}},\ E(Q) = x_Q$ is defined as follows. Let $\lambda = \sigma_0, \sigma_1, ...$ enumerate the elements of T in lexicographical order. The sequence x_Q can now be defined recursively by $x_Q(n) = 0$ if $\sigma_n^\frown 0 \in T_Q$ and $\sigma_n^\frown 1 \notin T_Q$, $x_Q(n) = 1$ if $\sigma_n^\frown 0 \notin T_Q$ and $\sigma_n^\frown 1 \in T_Q$ and $x_Q(n) = 2$ if $\sigma_n^\frown 0 \in T_Q$ and $\sigma_n^\frown 1 \in T_Q$.

6.3.1. **Definition.** (Closed Random Subsets of the Cantor Space)
A compact set $Q \subseteq 2^{\mathbb{N}}$ is random (in the Florida approach) if and only if its code $x_Q \in 3^{\mathbb{N}}$ is random in the usual Martin-Löf sense.

We call a compact set $Q \subseteq 2^{\mathbb{N}}$ Florida-random if it is random in the Florida approach and Florida-nonrandom if it is not.

We now define two computable functions that will allow us to transfer randomness tests between $3^{\mathbb{N}}$ and $\mathcal{K}(2^{\mathbb{N}})$.

6.3.2. **Proposition.**
There exists two functions $\Omega : \mathcal{O}(3^{\mathbb{N}}) \to \mathcal{O}(\mathcal{K}(2^{\mathbb{N}}))$ *and* $\Theta : \mathcal{O}(\mathcal{K}(2^{\mathbb{N}})) \to \mathcal{O}(3^{\mathbb{N}})$ *such that*

(1) Ω *is* $(\vartheta_{3^{\mathbb{N}}}, \vartheta_{\mathcal{K}(2^{\mathbb{N}})})$*-computable.*

(2) Θ *is* $(\vartheta_{\mathcal{K}(2^{\mathbb{N}})}, \vartheta_{3^{\mathbb{N}}})$*-computable.*

(3) $(\forall\, \mathcal{U} \in \mathcal{O}(\mathcal{K}(2^{\mathbb{N}})))\ \Theta(\mathcal{U}) = \{x_K :\ K \in \mathcal{U}\}$.

(4) $(\forall\ U \in \mathcal{O}(3^{\mathbb{N}}))\ \Theta(\Omega(U)) = U$.

Proof.
(1) Let $U \in \mathcal{O}(3^{\mathbb{N}})$. Then we can rewrite U as the union of basic open balls,

$$U = \bigcup_{i \in \mathbb{N}} N_{\sigma_i}$$

where $\sigma_i \in 3^*$ for all $i \in \mathbb{N}$.

For each i we create a finite tree T_i of depth $|\sigma_i|$. Beginning with the full binary tree of length $|\sigma_i|$ we start at the top element $w = \varepsilon$ and $j = 0$. We then proceed through the tree lexicographically as follows:

(1) Let w be the current node.

(2) If $\sigma_{i\lceil j} = 0$ then remove the all nodes with the prefix $w^\frown 1$.

(3) If $\sigma_{i\lceil j} = 1$ then remove the all nodes with the prefix $w^\frown 0$.

(4) If $\sigma_{i\lceil j} = 2$ then do nothing.

(5) Increment j and proceed to the next available node.

Let $F_i = \{w^\frown 0^\omega :\ w \in 2^* \text{ is a node on the tree } T_i \text{ of length } |\sigma_i|\}$.

We then define our first function $\Omega(U) = \bigcup_{i \in \mathbb{N}} B(F_i, 2^{-|\sigma_i|})$.

It is easy to see that if U is c.e. then we can use the above algorithm to transform the $\vartheta_{3^{\mathbb{N}}}$-name of U to a $\vartheta_{\mathcal{K}(2^{\mathbb{N}})}$-name for $\Omega(U)$. Therefore Ω is $(\vartheta_{3^{\mathbb{N}}}, \vartheta_{\mathcal{K}(2^{\mathbb{N}})})$-computable.

(2) Let $\mathcal{U} \in \mathcal{O}(\mathcal{K}(2^{\mathbb{N}}))$. Then we can rewrite $\mathcal{U}$ as the union of basic open balls,

$$\mathcal{U} = \bigcup_{i \in \mathbb{N}} B(F_i, r_i)$$

where $F_i \in F(2^{\mathbb{N}})$ and $r_i \in \mathbb{Q}^+$ for all $i \in \mathbb{N}$.

For each $i \in \mathbb{N}$ we define the following,

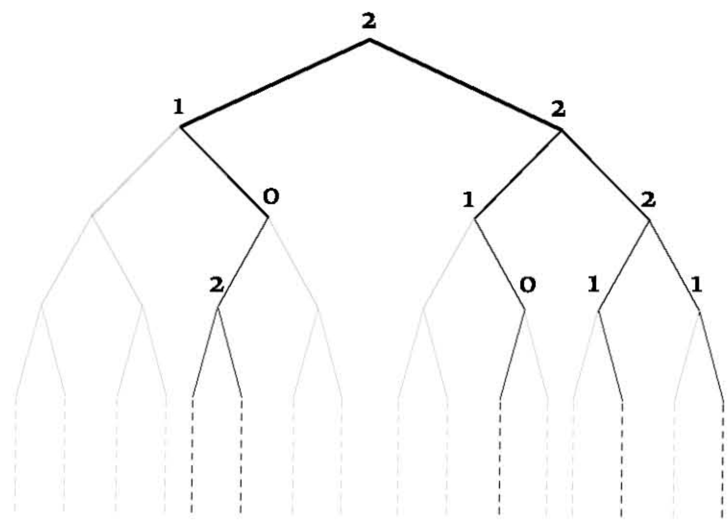


Figure 7. The tree above is an example of a tree T generated from the open ball $N_{2120122011}$ using construction in the definition of Ω.

(1) n_i where $n_i = \min\{n \in \mathbb{N} : \ n \geqslant \log_2 r\}$

(2) V_i where $V_i = \{v \in 2^{n_i} : \ (\exists \sigma \in F_i) \ v = \sigma_{\lceil m}\}$.

(3) W_i where $W_i = \{w \in 2^{n_i+1} : \ (\exists v \in V_i) \ w = v^\frown 0 \text{ or } w = v^\frown 1\}$.

Then V_i is finite and has at most 2^{n_i} many elements and W_i has at most 2^{n_i+1} many elements. We now define A_{ij} as the collection of all possible sets which satisfy the following,

(1) $(\forall w \in A_{ij}) \ w \in W_i$.

(2) $(\forall v \in V_i)(\exists w \in A_{ij}) \ v^\frown 0 = w$ or $v^\frown 1 = w$.

Then since there are only finitely many possible combinations of the elements from W_i we have that there are only finitely many A_{ij}'s.

We then define our second function as follows,

$$\Theta(\mathcal{U}) = \bigcup_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} N_{x_{A_{ij}}}.$$

Note that $B(F_i, r_i) = \bigcup_{j<\infty}\{F \in \mathcal{K}(2^{\mathbb{N}}) : (\forall x \in F)(\exists w \in A_{ij})\ x\lceil_{n_i+1} = w$ and $(\forall w \in V_i)(\exists x \in F)\ x\lceil_{n_i} = w\}$. Or in other words the open ball $B(F_i, r_i)$ is equal to the union of all sets indexed over j for which every point in every compact set in the set has a prefix in A_{ij} and every point in V_i has a point in the compact set for which it is a prefix.

Since each A_{ij} is a c.e. set and $K \mapsto x_K$ is a computable function it is easy to see that we can computably transform a $\vartheta_{\mathcal{K}(2^{\mathbb{N}})}$-name for $\mathcal{U}$ to a $\vartheta_{3^{\mathbb{N}}}$-name for $\Theta(\mathcal{U})$. Therefore Θ is $(\vartheta_{\mathcal{K}(2^{\mathbb{N}})}, \vartheta_{3^{\mathbb{N}}})$-computable.

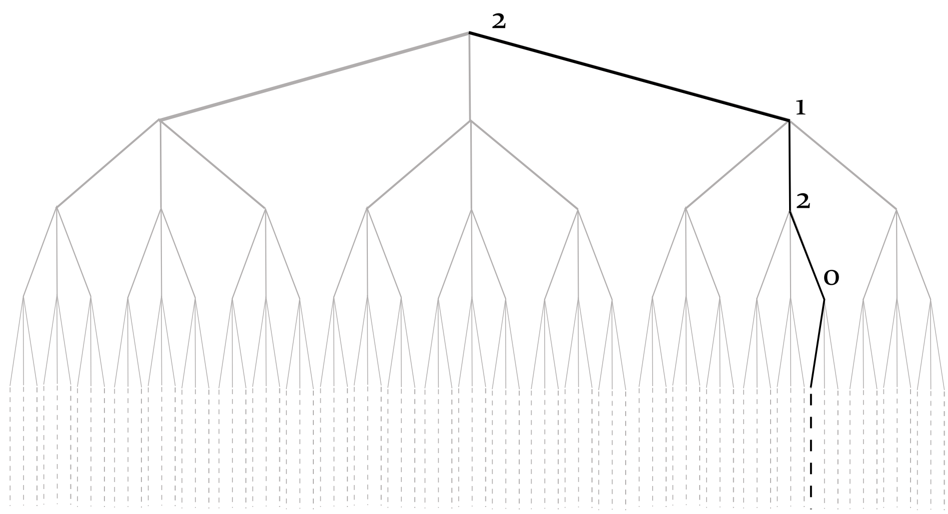


FIGURE 8. The open ball above is an example of an open ball N_{x_A} whose finite path $x_A = 2120$ was generated from the open ball with centre $F = \{010, 10, 11\}$ and radius $r = \frac{1}{8}$ by using the construction from the definition of Θ.

(3) Let $\mathcal{U} \in \mathcal{O}(\mathcal{K}(2^{\mathbb{N}}))$ and use the construction in (2) to define $\Theta(\mathcal{U})$. Let $K \in \mathcal{K}(2^{\mathbb{N}})$.

(" $\supseteq$ ")

Let $x_K \in \{x_F : F \in \mathcal{U}\}$ then $K \in \mathcal{U}$ and there exists an $i \in \mathbb{N}$ such that $K \in B(F_i, r_i)$. Therefore we have that for all $w \in V_i$ there exists an $x \in K$ such that $x_{\lceil n_i} = w$. Therefore T_K and $T_{A_{ij}}$ agree on the first n levels for all j. Therefore

$$d_{3^{\mathbb{N}}}(x_K, x_{A_{ij}}) < 2^{-n_i} \leqslant 2^{-|x_{A_{ij}}|}.$$

Therefore $x_K \in N_{x_{A_{ij}}}$ for all j.

Therefore $x_K \in \Theta(\mathcal{U})$ and hence $\Theta(\mathcal{U}) \supseteq \{x_F : F \in \mathcal{U}\}$.

(" $\subseteq$ ")

Let $x_K \in \Theta(\mathcal{U})$ then there exists $i \in \mathbb{N}$ and a j such that $x_K \in N_{x_{A_{ij}}}$. Therefore $d_{3^{\mathbb{N}}}(x_K, x_{A_{ij}}) < 2^{-|x_{A_{ij}}|}$. Therefore T_K and T_{A_ij} agree on all nodes up until the n^{th} level. Therefore for all $v \in V_i$ there exists a $x \in K$ such that $x_{\lceil n_i} = v$ and for all $x \in K$ there exists a $v \in V_i$ such that $x_{\lceil n_i} = v$.

Therefore $K \in B(F_i, r_i) \subseteq \mathcal{U}$.

Therefore $x_K \in \{x_F : F \in \mathcal{U}\}$ and hence $\Theta(\mathcal{U}) \subseteq \{x_F : F \in \mathcal{U}\}$.

(4) Let $U \in \mathcal{O}(3^{\mathbb{N}})$. Then we can rewrite U as the union of basic open balls,

$$U = \bigcup_{i \in \mathbb{N}} N_{\sigma_i}$$

where $\sigma_i \in 3^*$ for all $i \in \mathbb{N}$.

By (3) above we know that $\Theta(\Omega(U)) = \{x_K : K \in \Omega(U)\}$. We now need to show that $\{x_K : K \in \Omega(U)\} = U$.

(" $\supseteq$ ")

Let $x \in U$. Then there exists an $i \in \mathbb{N}$ such that $x \in N_{\sigma_i}$. Therefore for all $j \leqslant |\sigma_i|$ we have that $x_{\lceil j} = \sigma_{i\lceil j}$. Thus there exists a $K \in B(F_i, 2^{-|\sigma_i|}) \subseteq \Omega(U)$ such that for all $j \in \mathbb{N}$ $x_{\lceil j} = x_{K\lceil j}$.

Therefore $\{x_K : \ K \in \Omega(U)\} \supseteq U$.

(" $\subseteq$ ")

Let $x \in \{x_K : \ K \in \Omega(U)\}$. Then there exists $K \in \Omega(U)$ such that $x = x_K$. And there exists an $i \in \mathbb{N}$ such that $K \in B(F_i, 2^{-|\sigma_i|}) \subseteq \Omega(U)$. But then in particular $x = x_K \in \mathcal{N}_{\sigma_i} \subseteq U$.

Therefore $\{x_K : \ K \in \Omega(U)\} \subseteq U$.

□

We next define measure on the space $\mathcal{K}(2^{\mathbb{N}})$.

6.3.3. **Definition.**
We define the image measure $\mu_{\mathcal{K}(2^{\mathbb{N}})} : \mathcal{K}(2^{\mathbb{N}}) \to \mathbb{R}^+$ as follows,

$$(\forall\ \mathcal{A} \in \mathcal{B}(\mathcal{K}(2^{\mathbb{N}})))\ \mu_{\mathcal{K}(2^{\mathbb{N}})}(\mathcal{A}) = \mu_{3^{\mathbb{N}}}(E(\mathcal{A})).$$

We can now show that our two functions defined in Proposition 6.3.2 do indeed transform randomness tests between $3^{\mathbb{N}}$ and $\mathcal{K}(2^{\mathbb{N}})$. We use this to show that a compact subset of $2^{\mathbb{N}}$ is Florida-random if and only if it is $\mu_{\mathcal{K}(2^{\mathbb{N}})}$-CMS-random.

6.3.4. **Theorem.**
A compact set $K \in \mathcal{K}(2^{\mathbb{N}})$ is Florida-random if and only if it is $\mu_{\mathcal{K}(2^{\mathbb{N}})}$-CMS-random in the CMMS $(\mathcal{K}(2^{\mathbb{N}}), F(2^{\mathbb{N}}), d_{2^{\mathbb{N}}}, \mu_{\mathcal{K}(2^{\mathbb{N}})}, \nu_{F(2^{\mathbb{N}})})$.

Proof.

("$\Leftarrow$")

Assume that $K \in \mathcal{K}(2^{\mathbb{N}})$ is Florida-nonrandom. Then x_K is non-random in $3^{\mathbb{N}}$. Therefore there exists a computable sequence of c.e. open sets $(U_i)_{i\in\mathbb{N}}$ such that $\mu_{3^{\mathbb{N}}}(U_i) \leqslant 2^{-i}$ and $x_K \in U_i$ for all $i \in \mathbb{N}$.

We now show that $(\Omega(U_i))_{i\in\mathbb{N}}$ is a randomness test for K in $\mathcal{K}(2^{\mathbb{N}})$.

(1) $\mu_{\mathcal{K}(2^{\mathbb{N}})}(\Omega(U_i)) = \mu_{3^{\mathbb{N}}}(\{x_A : A \in \Omega(U_i)\}) = \mu_{3^{\mathbb{N}}}(\Theta(\Omega(U_i))) = \mu_{3^{\mathbb{N}}}(U_i) \leqslant 2^{-i}$

(2) $x_K \in U_i$ so $x_K \in \Theta(\Omega(U_i)) = \{x_A : A \in \Omega(U_i)\}$. Therefore $K \in \Omega(U_i)$.

(3) Since $(U_i)_{i\in\mathbb{N}}$ is $\vartheta^{\mathbb{N}}_{3^{\mathbb{N}}}$-computable and Ω is $(\vartheta_{3^{\mathbb{N}}}, \vartheta_{\mathcal{K}(2^{\mathbb{N}})})$-computable we have by Theorem 2.1.5 that $(\Omega(U_i))_{i\in\mathbb{N}}$ is $\vartheta^{\mathbb{N}}_{\mathcal{K}(2^{\mathbb{N}})}$-computable.

Therefore $(\Omega(U_i))_{i\in\mathbb{N}}$ is a randomness test for K in $\mathcal{K}(2^{\mathbb{N}})$ and hence K is $\mu_{\mathcal{K}(2^{\mathbb{N}})}$-CMS-nonrandom.

("$\Rightarrow$")

Assume that $K \in \mathcal{K}(2^{\mathbb{N}})$ is $\mu_{\mathcal{K}(2^{\mathbb{N}})}$-CMS-nonrandom. Therefore there exists a computable sequence of c.e. open sets $(\mathcal{U}_i)_{i\in\mathbb{N}}$ such that $\mu_{\mathcal{K}(2^{\mathbb{N}})}(\mathcal{U}_i) \leqslant 2^{-i}$ and $K \in \mathcal{U}_i$ for all $i \in \mathbb{N}$.

We now show that $(\Theta(\mathcal{U}_i))_{i\in\mathbb{N}}$ is a randomness test for x_K in $3^{\mathbb{N}}$.

(1) $\mu_{3^{\mathbb{N}}}(\Theta(\mathcal{U}_i)) = \mu_{3^{\mathbb{N}}}(\{x_A : A \in \mathcal{U}_i\}) = \mu_{\mathcal{K}(2^{\mathbb{N}})}(\mathcal{U}_i) \leqslant 2^{-i}$

(2) $\Theta(\mathcal{U}_i) = \{x_A : A \in \mathcal{U}_i\}$ and since $K \in \mathcal{U}_i$ we have that $x_K \in \Theta(\mathcal{U}_i)$.

(3) Since $(\mathcal{U}_i)_{i\in\mathbb{N}}$ is $\vartheta^{\mathbb{N}}_{\mathcal{K}(2^{\mathbb{N}})}$-computable and Θ is $(\vartheta_{\mathcal{K}(2^{\mathbb{N}})}, \vartheta_{3^{\mathbb{N}}})$-computable we have by Theorem 2.1.5 that $(\Theta(\mathcal{U}_i))_{i\in\mathbb{N}}$ is $\vartheta^{\mathbb{N}}_{3^{\mathbb{N}}}$-computable.

Therefore $(\Theta(\mathcal{U}_i))_{i\in\mathbb{N}}$ is a randomness test for x_K in $3^{\mathbb{N}}$ and hence x_K is Florida-nonrandom. □

Thus Florida randomness is an example of CMMS-randomness on the hyperspace of non-empty compact subsets of the Cantor space.

Capacities

We can use Choquet capacities to generate upper semi-continuous measures on the hyperspaces of non-empty compact subsets.

(1) In a locally-compact CMS (X, Q, d_X, ν_Q). Let $f : X \to [0, 1]$ be an upper semi-continuous function then by Proposition 4.4.9 the functional T_f is

a Choquet capacity on $\mathcal{K}^0(X)$. The hyperspace $(\mathcal{K}(X), \zeta, d_H, \nu_\zeta)$ equipped with the measure μ_{T_f} is a CMMS. Let $(F_i)_{i\in\mathbb{N}}$ be a computable sequence of elements from ζ for which $\sup_{x\in F_i} f(x) < 2^{-i}$ for all $i \in \mathbb{N}$. Then we have that the sequence of open sets $(\mathcal{K}_{F_i})_{i\in\mathbb{N}}$ is a randomness test in the hyperspace.

(2) In a compact CMS (X, Q, d_X, ν_Q) with the Hausdorff measure μ_d satisfying $\mu_d(X) = 1$ and hyperspace $\mathcal{K}^0(X)$ equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{K}^0)$. If a functional $T : \mathcal{K}^0(X) \to [0, 1]$ is a capacity functional of the Borel measure $\mu_T : \mathcal{B}(\mathcal{K}^0) \to [0, 1]$ then the restriction $\mu_T^* : \mathcal{B}(\mathcal{K}) \to [0, 1]$ of μ_T to the hyperspace of non-empty compact subsets is a measure on $\mathcal{K}(X)$ and $(\mathcal{K}(X), \zeta, d_H, \nu_\zeta, \mu_T^*)$ is a computable metric measure space (Proposition 4.4.11).

Thus Choquet capacities allow us to generate measures on computable metric hyperspaces with their respective randomness tests being in part defined by the upper semi-continuous function in the underlying space.

6.4. **Universal Randomness Tests.**

The following definition and theorem are slight modifications of Hertling and Weihrauch's Definition 3.9 and Theorem 3.10 in their 2003 paper on the randomness of effective topological spaces with measure. We shall be using their Theorem 3.10, which states that any effective topological measure space with an upper semi-computable measure has a universal randomness test, and our definition of the numbering $B_{\langle n,r\rangle} : \mathbb{N} \to \mathcal{B}$ of a base for the metric topology from the proof that every computable metric space satisfies the intersection property (Corollary 3.1.5) to prove that every computable metric space has a universal CMS randomness test.

6.4.1. **Definition.** (Universal Randomness Test)
In a CMS (X, Q, d_X, ν_Q) a randomness test $(U_i)_{i\in\mathbb{N}}$ is called universal if and only if for any randomness test $(V_i)_{i\in\mathbb{N}}$ on (X, Q, d_X, ν_Q) there exists a number c such that $V_{i+c} \subseteq U_i$ for all $i \in \mathbb{N}$.

We can now use Hertling and Weihrauch's Theorem 3.10 to prove that any computable metric space equipped with an upper semi-computable measure has a universal randomness test.

6.4.2. **Theorem.**
Any CMS (X, Q, d_X, ν_Q) *equipped with an upper semi-computable measure has a universal randomness test.*

Proof.
The proof of this theorem lies in the fact that any CMS equipped with an upper semi-computable measure and metric topology is an effective topological measure space as defined by Hertling and Weihrauch [13]. And since the measure is $(\vartheta, \rho_>)$-computable on the set of all open sets and (X, Q, d_X, ν_Q) satisfies the intersection property (Corollary 3.1.5) we have by Hertling and Weihrauch's Theorem 3.10 that there exists a universal randomness test. □

Thus any Hausdorff measure which is $(\vartheta, \rho_>)$-computable has a universal randomness test. It should come as no surprise that any completely isolating computable metric space or any connected proper comptuable metric space has an upper semi-computable Hausdorff measure on the hyperspace. Hence these spaces have universal randomness tests, though they are trivial. In the case where a hyperspace has a connected proper computable metric underlying space the universal randomness test consists of only the empty set, and in the case of a totally isolating underlying space the universal randomness test consists of only the entire hyperspace.

These results suggest that a potentially fruitful avenue of research could be where we consider under what conditions does an upper semi-continuous functional on the underlying space generate a measure from the Choquet capacity that is an upper semi-computable measure on the hyperspace. Or in other words, under what conditions does an upper semi-continuous functional on the underlying space generate a Choquet capacity and measure on the hyperspace that guarantees that that hyperspace has a universal randomness test.

Another area for investigation suggested through this research is to assess the conditions required for a computable metric space with an arbitrary measure to have a randomness concept in which the union of two non-random sets is non-random.

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