Hedging Performance of Interest-Rate Models

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MPhil in Mathematical Finance,
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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy in the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.

May 31, 2016

Abstract

This dissertation is a hedging back-study which assesses the effectiveness of interest-rate modelling and the hedging of interest-rate derivatives. Caps that trade in the Johannesburg swap market are hedged using two short-rate models, namely the Hull and White (1990) one-factor model and the subsequent Hull and White (1994) two-factor extension. This is achieved by using the equivalent Gaussian additive-factor models (G1++ and G2++) outlined by Brigo and Mercurio (2007). The hedges are constructed using different combinations of theoretical zero-coupon bonds. A flexible factor hedging method is proposed by the author and the bucket hedging technique detailed by Driessen, Klaasen and Melenberg (2003) is tested. The results obtained support the claims made by Gupta and Subrahmanyam (2005), Fan, Gupta and Ritchken (2007) and others in the literature that multi-factor models outperform one-factor models in hedging interest-rate derivatives. It is also shown that the choice of hedge instruments can significantly influence hedge performance. Notably, a larger set of hedge instruments and the use of hedge instruments with the same maturity as the derivative improve hedging accuracy. However, no evidence to support the finding of Driessen *et al.* (2003) that a larger set of hedge instruments can remove the need for a multi-factor model is found.

Acknowledgements

I thank Alex Backwell for his guidance as a supervisor. The clarity and insight he provided was invaluable. I must also acknowledge Professor David Taylor and the entire University of Cape Town Mathematical Finance section for providing an extraordinary environment in which to learn.

The debt of gratitude I owe to my parents is immeasurable. Their unconditional support has been instrumental throughout my education.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

The term structure of interest rates is driven by multiple factors. As such, it is widely held that interest rate derivative securities should be priced and hedged using multi-factor models. Despite this belief, the development of many term structure models has primarily been motivated by their analytical tractability. While these models have provided significant theoretical insight, their empirical hedging performance has remained largely untested. This has led to much controversy regarding the best models for pricing and hedging caps and other interest-rate securities.

This dissertation empirically analyses the hedging performance of both a one-factor and a two-factor interest-rate model. In particular, the short-rate models due to Hull and White (1994) are tested. A number of caps across a range of both strike rates and maturities (taken from the Johannesburg swap market) are hedged using different hedging techniques. Specifically, the factor and bucket hedging techniques described in the study of Driessen, Klaasen and Melenberg (2003) are used as the results of this study have been contended by other authors. An additional technique, which the author has named *flexible* factor hedging, is also tested for comparison with these more widely used methodologies. This technique was considered by Fan, Gupta and Ritchken (2003) in their work on unspanned stochastic volatility, but has not appeared in the more prominent works on hedging performance.

Notable studies that fail to reproduce the results of Driessen *et al.* (2003) are due to Gupta and Subrahmanyam (2005) and Fan, Gupta and Ritchken (2007). These authors find that multi-factor term structure models outperform one-factor models regardless of the hedging technique used. This directly opposes the suggestion made by Driessen *et al.* (2003) that the use of a larger set of hedging instruments leads to one-factor models performing as well as multi-factor models.

The controversy regarding the best possible model for hedging is the basis for the primary research problem addressed by this dissertation. This problem is best expressed in the form of two open-ended research questions:

1. Is a one-factor interest-rate model sufficient for hedging or is a multi-factor model necessary?

2. Is the hedging technique employed more important than the interest-rate model that is chosen?

The results obtained imply definite answers to these questions. The two-factor model significantly outperforms the one-factor model regardless of the hedging technique used, strongly indicating the importance of multi-factor specifications for hedging. Further, the bucket hedging and flexible factor hedging techniques consistently outperform the vanilla factor hedging method. This evidence suggests that the use of a well-designed hedging methodology can markedly improve hedging performance. However, the magnitude of improvement is not great enough to allow the one-factor model to perform as well as the two-factor model. Thus, the results agree with the findings of authors opposing the suggestion of Driessen *et al.* (2003) that the use of particular hedging techniques removes the need for multi-factor interest-rate models.

In addition, the author's suggested flexible factor hedging concept is shown to be marginally superior to the more convoluted bucket hedging approach of Driessen *et al.* (2003). The flexible hedging technique is computationally easier to implement and is able to achieve stronger hedging performance using a smaller set of hedging instruments.

The dissertation is organised as follows. Chapter 2 reviews the relevant empirical literature in the area with a particular focus on three significant studies. In Chapter 3, an overview of the two term structure models and their implementation is presented. Chapter 4 describes the data used in this study, along with a rigorous explanation of the chosen model calibration and hedging methodologies. The hedging results are presented and summarised in Chapter 5, and Chapter 6 contains concluding remarks.

Chapter 2

Empirical Hedging Studies

The empirical pricing performance of term structure models has received considerable attention in the literature. However, similar research relating to the hedging performance of the models has lagged behind. This is largely due to the fact that most of the relevant derivatives are traded in over-the-counter markets where data is often not systematically recorded. Only in the past two decades have empirical studies of hedging performance become feasible with some authors undertaking hedging back-studies on exchange-traded data. Noteworthy studies in this area (henceforth referred to as the *primary studies*) are due to Driessen *et al.* (2003), Gupta and Subrahmanyam (2005) and Fan *et al.* (2007) and will inform critical methodological choices throughout this dissertation.

Driessen *et al.* (2003) test one-factor and multi-factor Heath, Jarrow and Morton (1992) models using at-the-money (ATM) cap and swaption volatilities. They employ two specific hedging methodologies. The first is factor hedging which uses the minimum number of hedge instruments. For instance, a two factor model requires at least two hedge instruments to form a theoretically perfect hedge. The second is bucket hedging which involves using a larger number of hedge instruments with specific maturities. Interestingly, they find that the choice and number of hedge instruments affects hedging performance more than the particular interest rate model (and, as a result, number of factors) chosen. This result is not replicated by the other primary studies. Some authors suggest that the result may be due to the study of Driessen *et al.* (2003) being limited to ATM options. The effect of the strike rate is extremely important as many model imperfections are more prominent when analysing options away-from-the-money. This controversial result aside, Driessen *et al.* (2003) do state that multi-factor models outperform single-factor models when the number of hedge instruments is small.

Gupta and Subrahmanyam (2005) conduct a study similar to that of Driessen *et al.* (2003) using multiple short rate and forward rate models, along with one LIBOR market model. Critically, they incorporate daily data on US cap and floor prices across both maturities and strike rates. Their results indicate that, while a one-factor framework provides accurate pricing results, the addition of a second stochastic factor leads to a much larger reduction in hedging errors. The modelling of additional factors allows for the inclusion of possible twists in the yield curve while calculating sensitivities. Thus, they strongly contend the claim of Driessen *et al.* (2003) that an adequately calibrated and specified one-factor model is as effective at hedging interest-rate derivatives as multi-factor models when a large number of hedge instruments is used.

Fan *et al.* (2007) attempt to answer the controversy regarding model selection for hedging by expanding upon the work of Gupta and Subrahmanyam (2005). Despite their focus on swaptions, they also include data on US cap prices and test the hedging performance of 18 different models. These include one, two, three and four factor Heath, Jarrow and Morton (1992) models. Their results indicate that, even if the same number of hedging instruments is used in both one-factor and multi-factor specifications, there are significant hedging benefits when using multi-factor models. They emphasise the fact that away-from-the-money options provide significant information about skew effects which leads to models that incorporate level dependence in their volatility structure performing consistently better across choices of hedge instruments.

It is important to highlight two fundamental principles that the primary studies – and, consequently, this dissertation – observe. Firstly, time-inhomogeneous models must be used for hedging. The time-dependency of the model is used to calibrate to the initial yield curve. Secondly, calibration of at least some of the model parameters is done on each hedge date. This allows the model to reproduce the most recent market information in the form of the prevailing cap prices. The model itself makes no allowance for such parameter changes. However, we violate this implicit assumption in order to attain a close fit and to replicate accepted market practice. Abiding by these fundamentals allows one to make meaningful comparisons to the earlier empirical hedging studies.

Additionally, the use of cap *and* swaption data in the primary studies is noteworthy. Brigo and Mercurio (2007) illustrate that swaptions, unlike caps, contain information regarding the correlation between forward rates. This implies that multi-factor models should provide superior performance when hedging swaptions (in most cases) as the additional stochastic factors can incorporate this correlation. However, Choy, Dun and Schlögl (2004) contest this notion for European swaptions by stating that, while correlation affects the swaption price in theory, the effect of this in practice is negligible. In addition, authors such as Longstaff, Santa-Clara and Schwartz (2001) find the impact of correlation on Bermudan and other swaptions to be similarly minimal.

Theoretically, one-factor models may be able to hedge caps adequately as there is no forward rate correlation to be considered for these derivatives. Despite this, Driessen *et al.* (2003) suggest that one-factor models can adequately hedge *both* swaptions and caps if the bucket hedging approach is used. Thus, their controversial result cannot be attributed to the use of caps alone. Importantly, both Gupta and Subrahmanyam (2005) and Fan *et al.* (2007) show that caps are not hedged well by one-factor models and that the use of multi-factor specifications significantly improves hedging performance.

While the primary studies are the most relevant to this dissertation - in terms of data used, methodology and model types - a number of authors have followed on from their work. For example, the hedging effectiveness of LIBOR market models (see Brace, Gatarek and Musiela (1997)) has been tested by An and Suo (2008) and Pelsser and Pietersz (2010), while the ability of SABR models to hedge interest-rate caps is investigated by Wu (2012). These studies, while related to the work that follows, extend beyond the scope of this dissertation.

Chapter 3

Short-Rate Term Structure Models

Given the large number of interest-rate models that have been developed – ranging from short-rate specifications such as the famous work of Cox, Ingersoll and Ross (1985) to the more recent LIBOR market model work of Brace *et al.* (1997), Jamshidian (1997) and Miltersen, Sandmann and Sondermann (1997) – it is necessary to restrict this dissertation to a specific subset of models. Consequently, the one-factor and two-factor short-rate models due to Hull and White have been chosen. These models lend themselves to easy numerical implementation, but also suffer from some drawbacks – most notably that there is a theoretical possibility of the short rate going below zero (although this is no longer necessarily considered a drawback). This, coupled with the fact that both models have a Gaussian distribution for the short rate, allows for a fair comparison to be made between the two. Critically, this selection of models enables us to focus on one of our areas of research: the effect on hedging performance of having more than one stochastic factor in the term structure model. The effect of employing different hedging techniques can also be easily investigated and the use of only two models keeps the scale of the dissertation manageable.

What follows is a brief outline of the two Hull and White models, with a focus on their implementation and use in this study. We follow Brigo and Mercurio (2007) in viewing these models in terms of their underlying Gaussian additive factors. This allows for convenient implementation using a number of analytical formulae. The derivation of many of the closed-form solutions that follow is dealt with concisely in this chapter, but the interested reader can find some elaboration in the appendix.

3.1 Hull and White one-factor (G1++)

Hull and White (1990) aimed to satisfy the need for an exact fit to an observed yield curve by introducing a time-varying parameter in the classic model of Vasicek

(1977). Some basic manipulation (see Chapter 3 of Brigo and Mercurio (2007)) of the risk-neutral dynamics assumed in their paper allows for the instantaneous short-rate process to be written as

$$r_t = x_t + \alpha(t),$$

where $\alpha(\cdot)$ is a deterministic function of time and $\{x_t\}$ is a stochastic process (specifically an Ornstein and Uhlenbeck (1930) process) that reverts around a mean of zero:

$$dx_t = -ax_t dt + \sigma dW_t, \quad x_0 = 0.$$

Here, $a, \sigma \in \mathbb{R}$ and $\{W_t\}$ is a Brownian Motion under the risk-neutral measure $\mathbb{Q}$. One can easily interpret a and σ as the speed of mean reversion parameter and volatility parameter respectively. The function $\alpha(\cdot)$ is used to calibrate the model to a given initial yield curve.

A simple closed-form solution for the price (at time t) of a zero-coupon bond (ZCB) paying one unit at time T is available. This is derived using the usual expectation

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r_s ds}|\mathcal{F}_t],$$

which Brigo and Mercurio (2007) show to be given by

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}, \quad B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right].$$

The function $A(t,T)$ is irrelevant for our work as bond prices will be determined from the yield curves in our dataset and our primary concern is the sensitivity with respect to the stochastic factor x_t. This can easily be seen as

$$\frac{\partial P(t,T)}{\partial x_t} = -B(t,T)A(t,T)e^{-B(t,T)r_t} = -B(t,T)P(t,T).$$

Using these bond prices, along with a standard change-of-numeraire technique, we can then find the price of a put option on a ZCB using the pricing formula (with expectation taken under the forward measure $\mathbb{Q}_T$):

$$ZBP(t,T,S,X) = P(t,T)\mathbb{E}_{\mathbb{Q}_T}\left[(X - P(T,S))^+ |\mathcal{F}_t\right].$$

Here t represents the valuation time, T the option maturity, S the underlying bond maturity ($S > T$) and X the strike price. Again, Brigo and Mercurio (2007) provide a closed-form expression for the expectation showing that

$$ZBP(t,T,S,X) = XP(t,T)\Phi(-h+\sigma_p) - P(t,S)\Phi(-h),$$

where

$$\sigma_p = \sigma\sqrt{\frac{1-e^{-2a(T-t)}}{2a}}B(T,S),$$
$$h = \frac{1}{\sigma_p}\log\left(\frac{P(t,S)}{P(t,T)X}\right) + \frac{\sigma_p}{2}.$$

Here the function $B(\cdot,\cdot)$ is taken from the expression for bond prices and $\Phi(\cdot)$ denotes the standard normal cumulative distribution function.

This dissertation is concerned with the hedging of caps as they are one of the most liquidly traded interest-rate derivatives. Each cap consists of a number of caplets which are essentially call options on a simple interest rate. Analogously, a floor is an interest-rate security consisting of a number of floorlets, with each floorlet being a put option on a simple interest rate.

Caps can be shown to be equivalent to a portfolio of European ZCB put options - see Chapter 2 of Brigo and Mercurio (2007). We now have a mechanism for calculating cap prices. To this end, we denote by $\mathcal{T} = \{t_0, t_1, \ldots, t_n\}$ the set of differences between valuation date t and the date on which the i-th caplet's cash flow is realised. Moreover, we denote by τ_i the year fraction between cash flow times from t_{i-1} to t_i. Following Brigo and Mercurio (2007), we can obtain the price at time $t \leq t_0$ of a cap struck at X, nominal value N, with

$$Cap(t,\mathcal{T},N,X) = N\sum_{i=1}^{n}(1+X\tau_i)ZBP\left(t,t_{i-1},t_i,\frac{1}{1+X\tau_i}\right). \tag{3.1}$$

Finally, to calculate cap sensitivities we deviate from the use of closed-form solutions and adopt a finite-difference – specifically a central-difference – approach. Representing cap prices as a function of the Gaussian factor x_t implies a central-difference technique of the form:

$$\frac{\partial Cap}{\partial x_t} \approx \frac{Cap(x_t+\Delta x) - Cap(x_t-\Delta x)}{2\Delta x}.$$

This is accommodated by adjusting the bond price input into the cap pricing function. We can calculate the following, where a superscript $+$ denotes the Gaussian factor bumped up by Δx:

$$\begin{aligned} P(t,T)^+ &= A(t,T)e^{-B(t,T)(x_t+\Delta x+\alpha(t))} \\ &= A(t,T)e^{-B(t,T)r_t}e^{-B(t,T)\Delta x} \\ &= P(t,T)e^{-B(t,T)\Delta x}. \end{aligned}$$

A similar argument shows that the bond price with Gaussian factor bumped down (denoted with a superscript $-$) is

$$P(t,T)^- = P(t,T)e^{-B(t,T)(-\Delta x)}.$$

Selecting an appropriately small value for Δx, the bumped bond prices can be used to calculate corresponding cap prices for input into the above central-differencing scheme. This allows for the necessary cap sensitivities to be determined.

3.2 Hull and White two-factor (G2++)

Similarly to the one-factor case, the additive two-factor Gaussian model outlined by Brigo and Mercurio (2007) is equivalent to the two-factor model of Hull and White (1994). Formulating the model as the sum of two Gaussian factors provides some insight and intuition regarding the model's interpretation as each factor has its own speed of mean reversion and volatility parameter. In addition, there is a substantial improvement in analytical tractability. This makes pricing and, more importantly, hedging considerably easier. This section will be largely similar to the previous section in its sequential development of useful techniques. We write the instantaneous short-rate process as

$$r_t = x_t + y_t + \varphi(t),$$

where $\varphi(\cdot)$ is a deterministic function of time and both $\{x_t\}$ and $\{y_t\}$ are stochastic processes. As before, these are Ornstein-Uhlenbeck processes reverting around mean zero. Therefore, they satisfy

$$\begin{aligned} dx_t &= -ax_t dt + \sigma dW_t^1, \quad x_0 = 0, \\ dy_t &= -by_t dt + \eta dW_t^2, \quad y_0 = 0. \end{aligned}$$

Here, $\{W_t^1\}$ and $\{W_t^2\}$ are Brownian Motions under $\mathbb{Q}$ with instantaneous correlation parameter ρ:

$$dW_t^1 dW_t^2 = \rho dt.$$

Further, a, b, σ and η are real constants and the function $\varphi(\cdot)$ is again used to calibrate the model to a term structure of interest rates. The closed form solution for ZCB prices in the two-factor case is slightly more complicated:

$$\begin{aligned} P(t,T) = \exp\Bigg\{ &- \int_t^T \varphi(u)du - \frac{1-e^{-a(T-t)}}{a} x_t \\ &- \frac{1-e^{-b(T-t)}}{b} y_t + \frac{1}{2} V(t,T) \Bigg\}. \end{aligned}$$

Analogous to the one-factor case, the function $V(\cdot,\cdot)$ and the integral of $\varphi(\cdot)$ do not need to be considered as we are only concerned with the bond sensitivities with respect to the two Gaussian factors x_t and y_t. These are easily determined using market bond prices taken from a yield curve:

$$\frac{\partial P(t,T)}{\partial x_t} = -\frac{1-e^{-a(T-t)}}{a}P(t,T),$$
$$\frac{\partial P(t,T)}{\partial y_t} = -\frac{1-e^{-b(T-t)}}{b}P(t,T).$$

The formula for the price of a put option on a ZCB (with inputs as specified in the previous section) is

$$\begin{aligned} ZBP(t,T,S,X) = &- P(t,S)\Phi\left(\frac{\log\frac{XP(t,T)}{P(t,S)}}{\Sigma(t,T,S)} - \frac{1}{2}\Sigma(t,T,S)\right) \\ &+ P(t,T)X\Phi\left(\frac{\log\frac{XP(t,T)}{P(t,S)}}{\Sigma(t,T,S)} + \frac{1}{2}\Sigma(t,T,S)\right), \end{aligned}$$

where $\Sigma(t,T,S)^2$ takes the following form:

$$\begin{aligned} \Sigma(t,T,S)^2 = &+ \frac{\sigma^2}{2a^3}\left[1-e^{-a(S-T)}\right]^2\left[1-e^{-2a(T-t)}\right] \\ &+ \frac{\eta^2}{2b^3}\left[1-e^{-b(S-T)}\right]^2\left[1-e^{-2b(T-t)}\right] \\ &+ 2\rho\frac{\sigma\eta}{ab(a+b)}\left[1-e^{-a(S-T)}\right]\left[1-e^{-b(S-T)}\right]\left[1-e^{-(a+b)(T-t)}\right]. \end{aligned}$$

The technique used to convert put option prices into cap prices is identical to the previous the section. Thus, we construct cap prices by making use of Equation (3.1).

As before, we employ a central-difference technique to calculate cap sensitivities with respect to the *two* Gaussian factors x_t and y_t:

$$\frac{\partial Cap}{\partial x_t} \approx \frac{Cap(x_t+\Delta x) - Cap(x_t-\Delta x)}{2\Delta x},$$
$$\frac{\partial Cap}{\partial y_t} \approx \frac{Cap(y_t+\Delta y) - Cap(y_t-\Delta y)}{2\Delta y}.$$

We now include x and y in the superscript notation for bumped bond prices to clarify which factor is being bumped:

$$
\begin{aligned}
P(t,T)^{x+} &= \exp\left\{ -\int_t^T \varphi(u)du - \frac{1-e^{-a(T-t)}}{a}(x_t+\Delta x) \right.\\
&\qquad \left. - \frac{1-e^{-b(T-t)}}{b} y_t + \frac{1}{2}V(t,T) \right\}\\
&= \exp\left\{ -\int_t^T \varphi(u)du - \frac{1-e^{-a(T-t)}}{a}x_t \right.\\
&\qquad \left. - \frac{1-e^{-b(T-t)}}{b} y_t + \frac{1}{2}V(t,T) \right\} \exp\left\{ -\frac{1-e^{-a(T-t)}}{a}\Delta x \right\}\\
&= P(t,T)\exp\left\{ -\frac{1-e^{-a(T-t)}}{a}\Delta x \right\}.
\end{aligned}
$$

Similarly, it is easily seen that

$$
\begin{aligned}
P(t,T)^{x-} &= P(t,T)\exp\left\{ -\frac{1-e^{-a(T-t)}}{a}(-\Delta x) \right\},\\
P(t,T)^{y+} &= P(t,T)\exp\left\{ -\frac{1-e^{-b(T-t)}}{b}\Delta y \right\},\\
P(t,T)^{y-} &= P(t,T)\exp\left\{ -\frac{1-e^{-b(T-t)}}{b}(-\Delta y) \right\}.
\end{aligned}
$$

Choosing an appropriately small value for both Δx and Δy, bumped cap prices can be determined and used in the above central-difference schemes to calculate the relevant sensitivities.

It must be stressed that the Hull and White (1994) two-factor model is an uncomplicated multi-factor term structure model in that it does not contain any jump processes or stochastic volatility component. This allows us to isolate the effect of the additional stochastic factor and illustrate whether the multi-factor aspect of the model is significant. Intuitively, each factor has a particular mean reversion rate which dictates how it affects the yield curve. It is this added ability to influence the yield curve that is the fundamental difference between the one- and two-factor models.

Chapter 4

Testing Methodology

This chapter makes use of a number of sections to detail the investigative approach adopted in the dissertation. A discussion of the data used is included here as it directly informs the methodological choices.

4.1 Data

The data consist of a closely spaced implied volatility surface from the Johannesburg swap market and corresponding yield curve for a period of nearly three years (2012-2015). This continuously compounded yield curve is bootstrapped daily from liquid instruments and extends out to a maturity of 30 years. We follow the primary studies in that the yields are used to determine ZCBs which are then used as our hedging instruments.

The set of cap volatilities chosen for testing range in maturity from 1-year out to 10-year (in increments of one year) and three specific strike rates are chosen. The selection of strike rates is made so as to ensure that the models are tested using both at-the-money and away-from-the-money options:

- 4.5% strike rate, in-the-money (ITM),
- 6% strike rate, *approximately* at-the-money (ATM),
- 7.5% strike rate, out-the-money (OTM).

Thus, this dissertation does not suffer from the problem of only testing ATM options which Gupta and Subrahmanyam (2005) suggest may be the reason for the controversial results in the study of Driessen *et al.* (2003). The volatilities, converted into cap prices using the standard Black (1976) formula, are not highly liquid. At a daily frequency there is some staleness and, as a result, we use the volatilities on a weekly basis to construct hedges. Figure 4.1 shows yields for a few horizons and the corresponding bond prices, while Figure 4.2 shows cap volatilities and corresponding cap prices. Both figures are plotted across the almost 700 days used in the study.

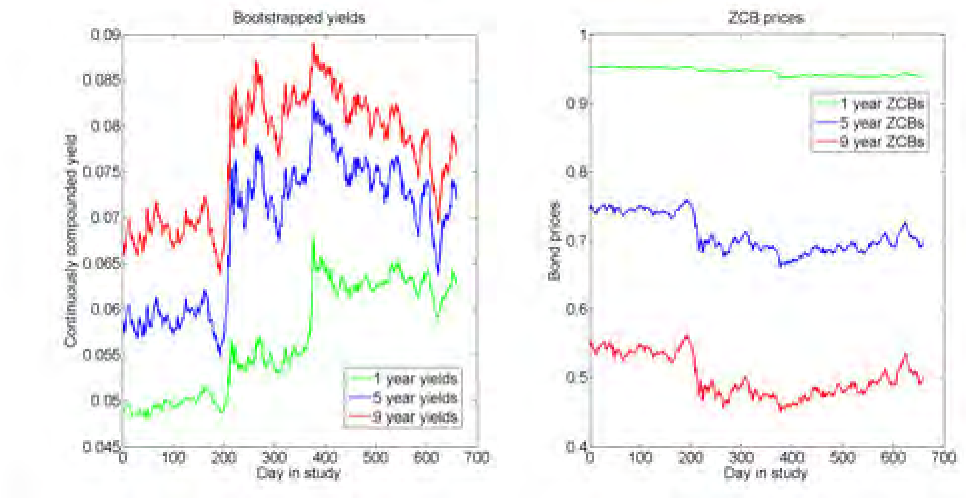

Fig. 4.1: Data: Yield curves and corresponding ZCB prices.

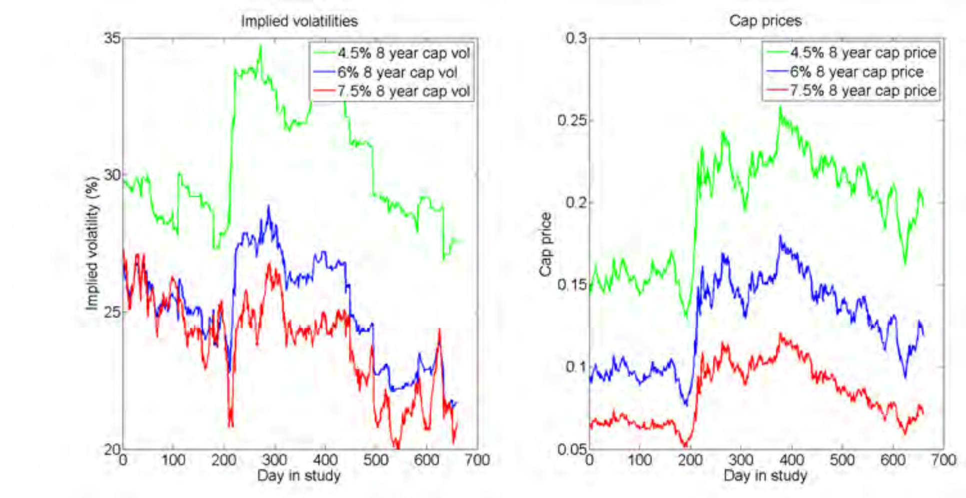

Fig. 4.2: Data: Implied volatilities and corresponding cap prices.

4.2 Calibration

The models are calibrated on each hedge date to a yield curve using the deterministic functions $\alpha(\cdot)$ and $\varphi(\cdot)$ and to prevailing cap prices using the remaining model parameters. The deterministic functions do not, in fact, have to be calculated explicitly as only the ZCB prices depend on them and these bond prices can be taken directly from the observed yield curve.

Calibration of the model parameters involves minimising the sum of squared pricing errors by varying all of the parameters:

$$\sum_{i=1}^{n}(\text{Market price}_i - \text{Model price}_i)^2.$$

While some authors suggest minimising the *relative* pricing errors, we follow the above approach as outlined by Acar and Natcheva-Acar (2009).

In particular, we calibrate to *difference cap* prices as done in a number of hedging studies. These difference caps are obtained by simply taking the difference between the consecutive caps in our study. Thus, the 1-year cap is itself the first difference cap and consists of four caplets with payment dates in three, six, nine and twelve months' time. The second difference cap is obtained by subtracting the 1-year cap from the 2-year cap. As a result, it consists of four caplets with payment dates in 15, 18, 21 and 24 months' time. The third difference cap is obtained by subtracting the 2-year cap from the 3-year cap, the fourth by subtracting the 3-year cap from the 4-year cap and so on up to the tenth difference cap. Each difference cap is therefore an identical portfolio of four options, except that the time to maturity for each option is incremented by one year.

The G1++ model has only two parameters (a and σ) and cannot fit the ten difference cap prices sufficiently well. Therefore, the ten difference caps (at a particular strike) are partitioned into the first three, the next three and the last four. This results in different sets of parameters for each of the three partitions and a particularly good fit. The G2++ model has five parameters (a, σ, b, η and ρ) and, consequently, it can match the ten difference caps well. Brigo and Mercurio (2007) mention that the correlation parameter here often approaches its lower bound of -1 when calibrating – as evinced in our study – but this is not found to be problematic. The calibration for both models is done across all three strike rates, meaning that three different sets of parameters are obtained in each case and are used to separately hedge the caps at each strike.

An example of this calibration procedure is shown in Figure 4.3. An excellent fit to the term structure of difference cap prices is presented for an ATM cap 350 days into our study using the G2++ model. The only noticeable deviation of model price from market price is in the eighth difference cap where the market price hump is difficult to match. Nevertheless the calibration is clearly successful. The calibrated parameters in this example are $\sigma = 0.03992$, $a = 1.31943$, $\eta = 0.02513$, $b = 0.04221$ and $\rho = -0.87632$.

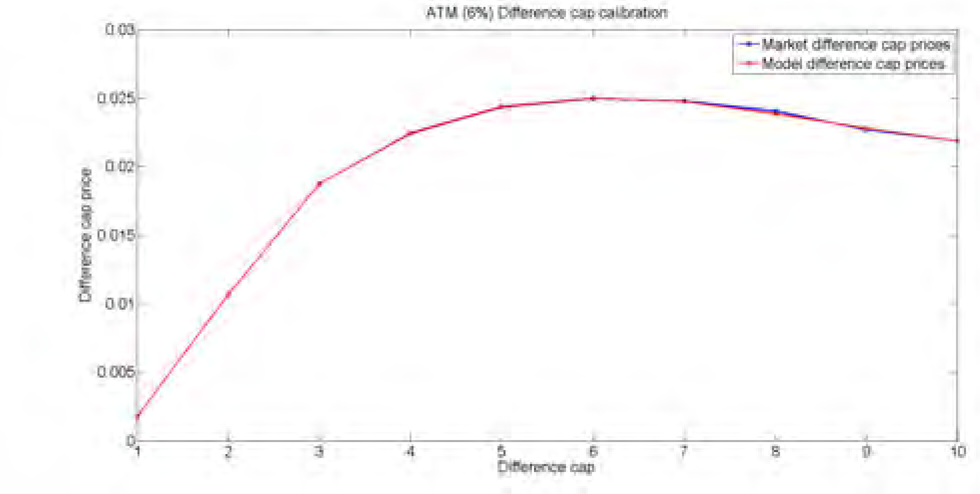

Fig. 4.3: Calibration: An example of an excellent fit for the G2++ model.

4.3 Hedging techniques

Sensitivities for each cap with respect to the Gaussian factors present in the model are calculated using the calibrated parameters. That is, denoting the day in study by t and for all cap maturities $i \in \{1, 2, \dots, 10\}$, we calculate

$$\frac{\partial c_{t,i}}{\partial x_t},$$

in the one-factor case and

$$\frac{\partial c_{t,i}}{\partial x_t}, \frac{\partial c_{t,i}}{\partial y_t},$$

in the two-factor case. This is done using the central-difference approach discussed in the previous chapter. Different combinations of ZCBs are then used to delta-hedge these exposures. Similarly to the sensitivities above, denoting a general ZCB maturity date by T_j, we calculate

$$\frac{\partial P(t, T_j)}{\partial x_t},$$

in the one-factor case and

$$\frac{\partial P(t, T_j)}{\partial x_t}, \frac{\partial P(t, T_j)}{\partial y_t},$$

in the two-factor case using the simple closed-form solutions outlined previously. We require the total sum of the sensitivities in our hedge portfolios to be zero. The hedged portfolio will therefore, according to the model, have no exposure to the sources of randomness and will grow continuously at the short rate. However, there is no unique choice for T_j despite authors such as Dudenhausen, Schlögl and

Schlögl (1999) proposing the use of ”natural” hedge instruments which mature on the payment dates of the instrument's cash flows. In fact, many different combinations of ZCBs can be used to achieve overall neutrality to the stochastic processes. The choice of ZCBs used for hedging (and, hence, maturities T_j) is determined by the hedging methodology currently being employed.

To test the controversial results of Driessen *et al.* (2003), the hedging methodologies outlined in their study are used in this dissertation. They suggest a *factor hedging* approach where the minimum number of ZCBs is used and a *bucket hedging* approach where a surplus of ZCBs is used. By employing the same techniques, we can find evidence which either supports or contradicts their findings. Furthermore, an additional hedging technique – *flexible factor hedging* – is proposed. This technique is a slight modification of the minimal hedge of Driessen *et al.* (2003) which takes into account the maturity of the cap being hedged.

4.3.1 Factor hedging

In factor hedging, the number of chosen hedge instruments is equivalent to the number of factors in the underlying term structure model. For the one-factor model, we have only one cap sensitivity and hedge using one ZCB. The 5-year bond is chosen and hedge portfolio holdings ($\lambda^5_{t,i}$) are easily constructed via:

$$\lambda^5_{t,i} = -\left(\frac{\partial P(t,5)}{\partial x_t}\right)^{-1} \frac{\partial c_{t,i}}{\partial x_t}.$$

This clearly results in an overall portfolio sensitivity of zero. In the two-factor model, two ZCBs are needed to simultaneously hedge the two cap sensitivities. The 1-year and 10-year bonds are chosen as we would like the two ZCBs to be exposed to different ends of the yield curve. Similarly to the above calculation, hedge portfolio holdings are constructed as follows:

$$\begin{bmatrix} \lambda^1_{t,i} \\ \lambda^{10}_{t,i} \end{bmatrix} = -\begin{bmatrix} \frac{\partial P(t,1)}{\partial x_t} & \frac{\partial P(t,10)}{\partial x_t} \\ \frac{\partial P(t,1)}{\partial y_t} & \frac{\partial P(t,10)}{\partial y_t} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial c_{t,i}}{\partial x_t} \\ \frac{\partial c_{t,i}}{\partial y_t} \end{bmatrix}.$$

In both cases, the difference between the price of the relevant cap and the cost of the ZCBs used to construct the hedge is then invested in the cash account. Moving one week forward then allows for the calculation of hedge residuals (deviations from zero, similar to a profit and loss calculation).

4.3.2 Flexible factor hedging

Flexible factor hedging is a simple extension of factor hedging in which the ZCBs used to construct the hedge portfolios have the same (or similar) maturity to that of the underlying cap. This is easily achieved in the one-factor case as we only require one ZCB to hedge and simply choose the ZCB with the same maturity as the cap:

$$\lambda_{t,i}^{i} = -\left(\frac{\partial P(t,i)}{\partial x_t}\right)^{-1} \frac{\partial c_{t,i}}{\partial x_t}.$$

In the two-factor case, we require two ZCBs. Thus, in addition to the relevant ZCB, we choose the ZCB with maturity one year after the maturity of the cap. This restricts exposure to a small section of the yield curve, but also provides adequate spacing between the two ZCBs. The hedge portfolio holdings are easily seen as:

$$\begin{bmatrix} \lambda_{t,i}^{i} \\ \lambda_{t,i}^{i+1} \end{bmatrix} = -\begin{bmatrix} \frac{\partial P(t,i)}{\partial x_t} & \frac{\partial P(t,i+1)}{\partial x_t} \\ \frac{\partial P(t,i)}{\partial y_t} & \frac{\partial P(t,i+1)}{\partial y_t} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial c_{t,i}}{\partial x_t} \\ \frac{\partial c_{t,i}}{\partial y_t} \end{bmatrix}.$$

The subsequent calculation of hedge residuals is identical to the vanilla factor hedging case.

This flexible technique is also a reflection of real-world concerns. According to the model, the choice of hedge instrument is irrelevant. However, in practice one must actually decide on particular hedging instruments. By choosing the hedging instruments to have specific maturities which depend on the derivative being hedged, we are employing a methodology that is closer to real-world practices.

4.3.3 Bucket hedging

In bucket hedging, each hedge is constructed using bonds with maturities corresponding to the cash flow dates of the underlying derivative. For example, a 1-year cap consisting of quarterly caplets would be hedged with zero coupon bonds of maturity three months, six months, nine months and one year for a total of four hedge instruments. The system used to calculate hedge residuals is now under-specified (unlike in the factor hedging case where a unique solution is evident). Thus, we simply group the ZCBs used for hedging into one group in the one-factor case and two groups in the two-factor case. We can then calculate one holding value ($\lambda_{t,i}^1$) applied to all ZCBs for G1++ hedges, and two holding values ($\lambda_{t,i}^1, \lambda_{t,i}^2$) applied to the two groupings of ZCBs for G2++ hedges. An example of this concept, when hedging a one-year cap using the G1++ model, is shown below:

$$\lambda_{t,1}^{1} = -\left(\frac{\partial P(t,0.25)}{\partial x_t} + \frac{\partial P(t,0.5)}{\partial x_t} + \frac{\partial P(t,0.75)}{\partial x_t} + \frac{\partial P(t,1)}{\partial x_t}\right)^{-1} \frac{\partial c_{t,1}}{\partial x_t}.$$

Similarly, when hedging a one-year cap using the G2++ model:

$$\begin{bmatrix} \lambda_{t,1}^{1} \\ \lambda_{t,1}^{2} \end{bmatrix} = -\begin{bmatrix} \frac{\partial P(t,0.25)}{\partial x_t} + \frac{\partial P(t,0.75)}{\partial x_t} & \frac{\partial P(t,0.5)}{\partial x_t} + \frac{\partial P(t,1)}{\partial x_t} \\ \frac{\partial P(t,0.25)}{\partial y_t} + \frac{\partial P(t,0.75)}{\partial y_t} & \frac{\partial P(t,0.5)}{\partial y_t} + \frac{\partial P(t,1)}{\partial y_t} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial c_{t,1}}{\partial x_t} \\ \frac{\partial c_{t,1}}{\partial y_t} \end{bmatrix}.$$

Note that the relevant ZCBs are grouped in such a way that the total holding in bonds maturing in the first half of the year is equivalent to the total holding in bonds maturing in the last half of the year. Theoretically, any grouping is admissible, but the overlapping grouping shown above yielded the best hedging results.

As before, the difference between the cap price and hedge cost is deposited in the cash account and, after moving one week forward, hedging residuals are determined.

4.4 Performance measure

In order to assess the performance of a given hedge, a *hedging* R^2 metric is used. This metric determines how much variation a given hedging technique is able to remove. The calculation of the R^2 value involves constructing an unhedged control portfolio to compare with a given hedging technique's portfolio. For an unhedged portfolio, the present value of the cap that is being hedged is simply deposited in the cash account and, after moving one week forward, a set of unhedged residuals is calculated. These residuals, along with the hedged residuals from the given delta-hedged portfolio, are then used to construct an R^2 value. The value is calculated by summing the squares of both sets of residuals:

$$R^2 = 1 - \frac{SS_{hedged}}{SS_{unhedged}}.$$

Clearly, values closer to one are an indicator of good hedge performance. The hedged residuals are then significantly smaller than the unhedged residuals, im-

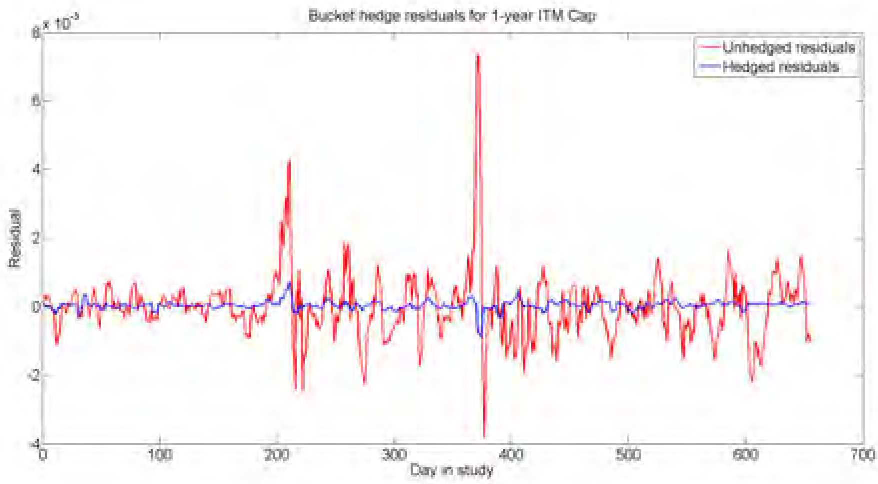

Fig. 4.4: Residual plot for a hedge constructed using the G2++ model.

plying that the implemented hedge has greatly reduced variation. In the extreme case where $SS_{hedged} = 0$ we have that $R^2 = 1$ and 100% of variation has been removed by the implemented hedge. Consequently, values closer to zero indicate poor hedge performance and negative R^2 values actually imply that the hedge has *increased* portfolio variation.

An example of the residuals used to construct this performance measure is shown in Figure 4.4. This 1-year ITM bucket hedge using the G2++ model was clearly very effective as the hedged residuals are significantly smaller than the unhedged control residuals. The R^2 value for this example was 0.98046, indicating that the G2++ model has hedged the cap excellently.

Chapter 5

Results

Given the model and methodological choices outlined above, this section will report the obtained hedging results. The hedging R^2 values are grouped in a manner that allows for direct answering of the two main research questions. In particular, Table 5.1 displays the results of the factor hedging technique across the three chosen strike rates and all ten difference cap maturities for both the G1++ and the G2++ models. Tables 5.2 and 5.3 are then formatted in a similar manner, but indicate the results of the flexible factor hedging and bucket hedging techniques respectively. More detailed analysis, including a number of summary statistics, follows after these tables have been presented.

	G1++			**G2++**		
Maturity	R^2_{ITM}	R^2_{ATM}	R^2_{OTM}	R^2_{ITM}	R^2_{ATM}	R^2_{OTM}
1	0.591	0.436	0.410	0.981	0.737	0.465
2	0.618	0.601	0.594	0.709	0.601	0.320
3	0.817	0.819	0.776	0.831	0.780	0.603
4	0.848	0.840	0.724	0.865	0.830	0.683
5	0.808	0.778	0.698	0.853	0.826	0.777
6	0.587	0.566	0.509	0.794	0.749	0.697
7	0.343	0.323	0.329	0.567	0.504	0.494
8	0.221	0.260	0.232	0.421	0.383	0.348
9	0.038	0.266	0.248	0.213	0.420	0.345
10	0.086	0.180	0.234	0.138	0.276	0.277

Tab. 5.1: Model results using the **factor hedging** technique.

Maturity	**G1++** R^2_{ITM}	R^2_{ATM}	R^2_{OTM}	**G2++** R^2_{ITM}	R^2_{ATM}	R^2_{OTM}
1	0.985	0.850	0.673	0.982	0.854	0.672
2	0.947	0.880	0.741	0.987	0.921	0.758
3	0.874	0.836	0.710	0.989	0.960	0.835
4	0.821	0.815	0.700	0.962	0.954	0.812
5	0.808	0.778	0.698	0.967	0.932	0.837
6	0.689	0.651	0.578	0.944	0.883	0.855
7	0.475	0.440	0.449	0.921	0.858	0.890
8	0.359	0.361	0.349	0.934	0.773	0.869
9	0.150	0.392	0.359	0.582	0.894	0.852
10	0.148	0.273	0.327	0.418	0.573	0.762

Tab. 5.2: Model results using the **flexible factor hedging** technique.

Maturity	**G1++** R^2_{ITM}	R^2_{ATM}	R^2_{OTM}	**G2++** R^2_{ITM}	R^2_{ATM}	R^2_{OTM}
1	0.918	0.768	0.594	0.980	0.892	0.686
2	0.917	0.848	0.717	0.983	0.918	0.749
3	0.796	0.757	0.650	0.986	0.958	0.825
4	0.793	0.787	0.672	0.962	0.950	0.798
5	0.786	0.755	0.676	0.967	0.931	0.830
6	0.656	0.623	0.554	0.941	0.876	0.846
7	0.446	0.412	0.420	0.916	0.856	0.885
8	0.333	0.340	0.324	0.928	0.769	0.866
9	0.135	0.373	0.341	0.577	0.891	0.844
10	0.142	0.264	0.316	0.417	0.566	0.758

Tab. 5.3: Model results using the **bucket hedging** technique.

Two peculiarities in the results must be highlighted. Firstly, the R^2 values for OTM derivatives are lower than their ATM and ITM counterparts. This is explained by Carr, Gabaix and Wu (2011), who suggest that jump processes are necessary to accurately capture the pricing behaviour of far OTM options. The significant jump in price that occurs when such options move into the money cannot be incorporated by the short-rate models we have used and this unpredictability is reflected in the slightly poorer hedging results.

Secondly, there is a significant drop off in R^2 values for ITM difference caps with maturities nine and ten years (for all three hedging techniques). This peculiarity is attributable to a problematic calibration. The price hump is far more pronounced for these derivatives and, as shown in Figure 5.1, the models struggle to reproduce this. The hedging performance of the models is found to be highly dependent on the quality of calibration. Accuracy to within a very fine margin (usually 1%) is required to ensure meaningful results. Given that this problematic calibration is isolated to a very small section of the results and is consistent across models and hedging techniques, it does not meaningfully impact upon the analysis that follows.

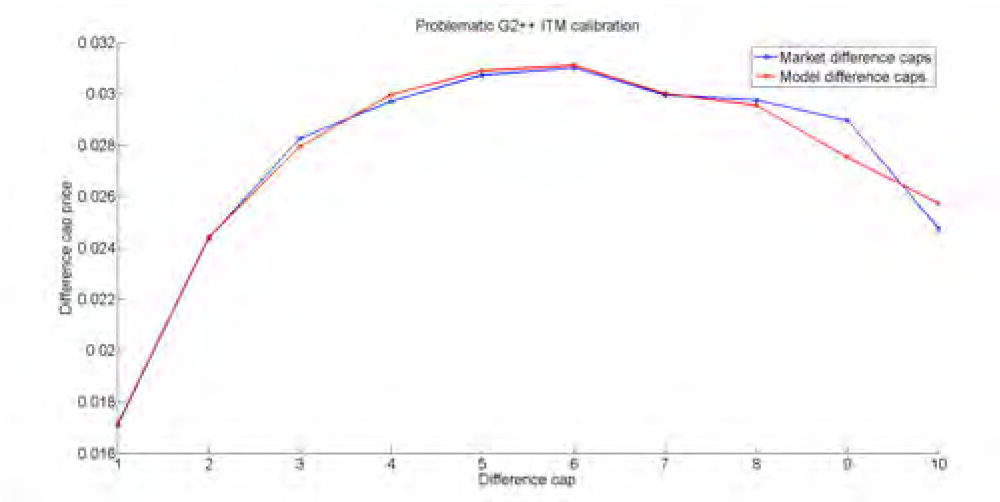

Fig. 5.1: Problematic ITM calibration for the G2++ model.

5.1 Model effect

This section contrasts results obtained using the G1++ model against those obtained using the G2++ model. Clearly there are significant performance benefits when using G2++ irrespective of the hedging technique that is employed. This is best reflected in the mean R^2 values for the respective models shown in Table 5.4.

	G1++			G2++		
Means	R^2_{ITM}	R^2_{ATM}	R^2_{OTM}	R^2_{ITM}	R^2_{ATM}	R^2_{OTM}
	0.571	0.576	0.520	0.791	0.777	0.708
Total mean		R^2			R^2	
		0.556			0.759	

Tab. 5.4: Model results summary

Across all three strikes the two-factor model produces markedly superior results to the one-factor model. The inclusion of the additional stochastic factor allows our hedges to reduce roughly 20% more variation on average. The difference in performance is most notable when observing the results for bucket and flexible factor hedging. While the R^2 values for the G1++ model decrease steadily with increasing maturity, the values for the G2++ model remain impressively consistent. This, coupled with the excellent hedges produced by the G2++ model for shorter maturities, provides strong evidence to support the belief that multi-factor term structure models are necessary for hedging.

5.2 Hedging technique effect

This section compares the results obtained when using the factor, flexible factor and bucket hedging techniques. We begin by considering the means for the two techniques used in the work of Driessen *et al.* (2003) shown in Table 5.5 (G1++ case) and Table 5.6 (G2++ case).

	Factor			Bucket		
Means	R^2_{ITM}	R^2_{ATM}	R^2_{OTM}	R^2_{ITM}	R^2_{ATM}	R^2_{OTM}
	0.496	0.507	0.476	0.592	0.593	0.526
Total mean		R^2			R^2	
		0.493			0.570	

Tab. 5.5: G1++ hedging technique results summary – factor versus bucket hedging

There is a distinct advantage when using the bucket hedging technique for both the one-factor and the two-factor model. The method is particularly effective when using the G2++ model, as shown by the increase of roughly 0.25 in the mean R^2

	Factor			Bucket		
Means	R^2_{ITM}	R^2_{ATM}	R^2_{OTM}	R^2_{ITM}	R^2_{ATM}	R^2_{OTM}
	0.637	0.611	0.501	0.866	0.861	0.808
Total mean		R^2			R^2	
		0.583			0.845	

Tab. 5.6: G2++ hedging technique results summary – factor versus bucket hedging

values. However, while the bucket hedging technique does improve the hedging performance of the G1++ model, there is no evidence to support the claim made by Driessen *et al.* (2003) that a one-factor model can perform as well as multi-factor specifications when adopting this approach. In fact, single-factor bucket hedges do not even perform as well as the two-factor factor hedges (on average). As a result, this dissertation supports authors such as Gupta and Subrahmanyam (2005) and Fan *et al.* (2007) in suggesting that multi-factor interest-rate models always outperform one-factor models, irrespective of the hedging technique that is employed.

We now turn our attention to assessing the performance of the proposed flexible factor hedging approach. The relevant summary statistics for this technique are presented in Table 5.7.

	G1++			G2++		
Means	R^2_{ITM}	R^2_{ATM}	R^2_{OTM}	R^2_{ITM}	R^2_{ATM}	R^2_{OTM}
	0.626	0.627	0.558	0.869	0.860	0.814
Total mean		R^2			R^2	
		0.604			0.848	

Tab. 5.7: Flexible factor hedging technique results summary

Comparing these total means to the total means for bucket hedging presented on the previous page, we see that, for both models, this technique is able to remove more variation. In the G1++ case the mean R^2 increases by 0.03 and in the G2++ case the mean R^2 increases by 0.003. While these are not vast improvements, they are clear indicators that the flexible factor approach is at least as effective as the bucket hedging approach. Given that the author's proposed technique is easier to implement and less computationally intensive, a strong argument can be made for this approach to be used in lieu of the needlessly complicated bucket hedging method.

In addition, the results using this methodology further confirm the need for multi-factor term structure models when hedging. While the use of flexible factor hedging does significantly improve the performance of the one-factor model, the two-factor model still clearly provides superior results. This is evident across all three chosen strike rates. Thus, we can state with a high degree of confidence that multi-factor models allow for superior hedging performance.

Finally, as a clear and concise way of summarising the dissertation's findings, we present the average results across all three strike rates for each difference cap maturity. Figure 5.2 displays the results obtained for the two models and Figure 5.3 displays the results obtained when using the different hedging techniques.

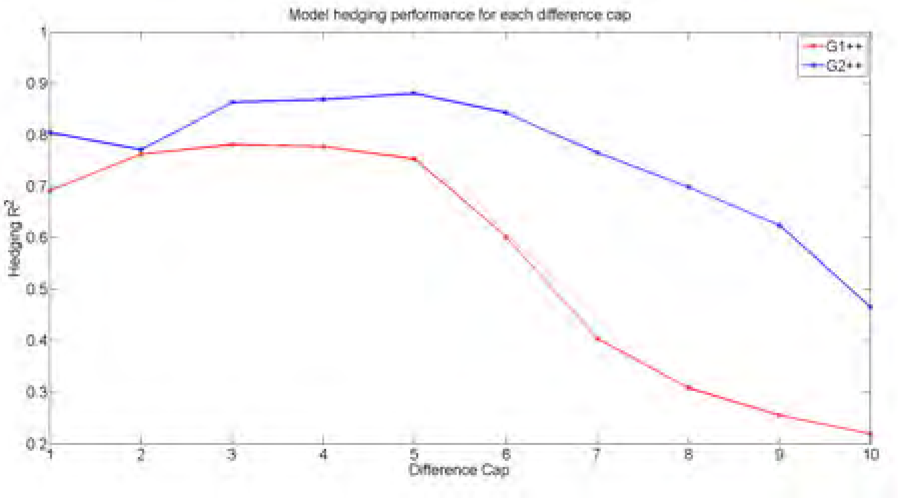

Fig. 5.2: Mean results for each difference cap by model.

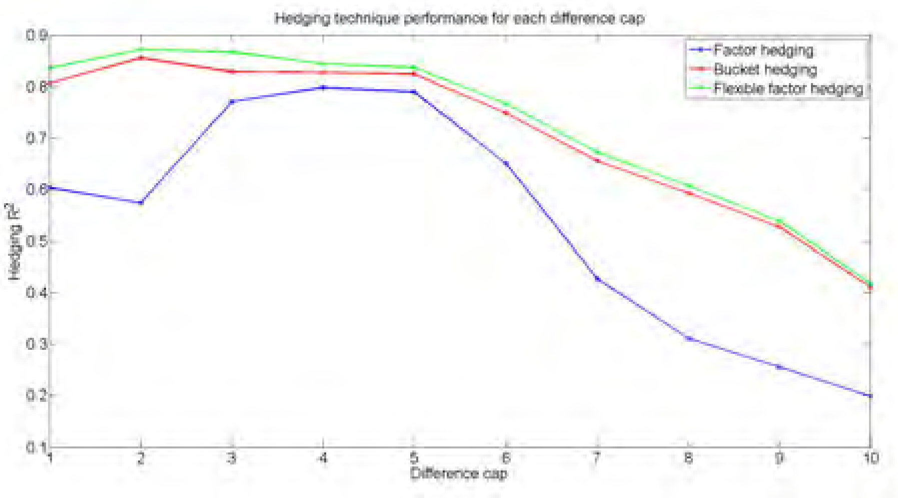

Fig. 5.3: Mean results for each difference cap by hedging technique.

As expected, we see that the G2++ model dominates the G1++ model. The significant drop off in G1++ performance for the caps of later maturity is evident, while the one-factor model is more competitive when hedging caps of shorter maturity. The G2++ model's ability to reliably remove over 50% of the variation in the hedge portfolios is strongly indicative of its suitability for hedging. This is remarkable given the model's relative simplicity and lack of more advanced features, such as the inclusion of stochastic volatility.

Interestingly, Figure 5.3 shows factor hedging providing strong results for the difference caps of maturity three, four and five years. This is attributable to the choice of the 5-year bond when implementing the G1++ factor hedging scheme. The ZCB is exposed to a similar point on the yield curve as the cap being hedged and this leads to better hedging performance. The author's flexible factor hedging approach is shown to dominate the methods suggested by Driessen *et al.* (2003). Both flexible factor hedging and bucket hedging consistently remove over 50% of the variation in the hedge portfolios, implying that they are highly effective hedging methodologies

Chapter 6

Discussion

This dissertation has attempted to empirically test the hedging performance of interest-rate models. A brief discussion of the controversy in the literature regarding the number of stochastic factors required for a model to adequately hedge is followed by a thorough introduction to the one- and two-factor models used for our investigation. The dissertation's key contribution is its detailed explanation of the implementation and assessment of different hedging techniques, including a flexible extension made to popular methods used in the literature.

The results of this dissertation provide evidence to support the (largely untested) belief that multi-factor term structure models are necessary when hedging interest-rate derivative securities. The assertion made by Driessen *et al.* (2003) regarding the ability of one-factor models to perform as well as multi-factor models when a large set of hedge instruments is used is found to be questionable. Rather, the results align with authors such as Gupta and Subrahmanyam (2005) and Fan *et al.* (2007) who find multi-factor models to reliably outperform one-factor models. In particular, the G2++ model is shown to dominate the G1++ model in terms of hedging performance.

Notably, the proposed *flexible* factor hedging technique is shown to outperform hedging techniques used by other authors. While the approach's performance is only marginally superior to the bucket hedging technique of Driessen *et al.* (2003), the method is significantly easier to implement. The results also illustrate that a well-designed hedging technique can markedly improve hedging performance.

Natural extensions to the research would include testing more recently developed models, such as the LIBOR market models due to Jamshidian (1997) and Brace *et al.* (1997). Expanding upon the concept of flexible factor hedging by extending other hedging techniques in a similar fashion may also yield interesting results.

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Appendix

This brief appendix illustrates how realisations of the short-rate can be generated using the one-factor Hull and White (1990) model. The G1++ formulation is not used here so as to allow for an easier interpretation of the result. A similar result for the two-factor Hull and White (1994) follows thereafter.

The instantaneous short-rate process in the one-factor case evolves under risk-neutral measure $\mathbb{Q}$ according to

$$dr_t = [\theta(t) - ar_t]dt + \sigma dW_t,$$

where a and σ are positive constants and $\theta(\cdot)$ is chosen to fit the current term structure of interest rates. The market instantaneous forward rate for maturity T (at time 0) $f^M(0,T)$ is found in the usual manner,

$$f^M(0,T) = -\frac{\partial \log P^M(0,T)}{\partial T},$$

where $P^M(0,T)$ denotes the corresponding ZCB price. It can be shown that we must have:

$$\theta(t) = \frac{\partial f^M(0,t)}{\partial T} + af^M(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

Note that $\frac{\partial f^M}{\partial T}$ denotes the partial derivative of f^M with respect to its second argument. Integrating the short-rate dynamics yields:

$$\begin{aligned} r_t &= r_s e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta(u)du + \sigma \int_s^t e^{-a(t-u)}dW_u \\ &= r_s e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)}dW_u, \end{aligned}$$

where

$$\alpha(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2.$$

Therefore, r_t conditional on $\mathcal{F}_s$ is normally distributed with mean and variance given by:

$$\begin{aligned} \mathbb{E}[r_t|\mathcal{F}_s] &= r_s e^{-(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)}, \\ \mathbb{V}ar[r_t|\mathcal{F}_s] &= \frac{\sigma^2}{2a}[1 - e^{-2a(t-s)}]. \end{aligned}$$

This short-rate distribution can now be used to determine a number of model-specific solutions.

For the two-factor case, we make use of the G2++ formulation described in Chapter 3 to avoid unnecessary complexity. The short-rate dynamics can be found in this Chapter. Simple integration of the two Ornstein-Uhlenbeck processes yields:

$$r_t = x_s e^{-a(t-s)} + y_s e^{-b(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW_u^1 + \eta \int_s^t e^{-b(t-u)} dW_u^2 + \varphi(t)$$

Consequently, r_t conditional on $\mathcal{F}_s$ is normally distributed with mean and variance given by:

$$\begin{aligned}\mathbb{E}[r_t|\mathcal{F}_s] &= x_s e^{-a(t-s)} + y_s e^{-b(t-s)} + \varphi(t),\\ \mathbb{V}ar[r_t|\mathcal{F}_s] &= \frac{\sigma^2}{2a}[1 - e^{-2a(t-s)}] + \frac{\eta^2}{2b}[1 - e^{-2b(t-s)}] + 2\rho\frac{\sigma\eta}{a+b}[1 - e^{-(a+b)(t-s)}].\end{aligned}$$

As before, this distribution can be used to derive a number of analytical solutions.