Long term orbital inclination and eccentricity oscillations of the planets in our solar system

Abstract

The orbits of the planets in our solar system are not in the same plane, therefore natural torques stemming from Newton's gravitational forces exist to pull them all back to the same plane. This causes the inclinations of the planet orbits to oscillate with potentially long periods and very small damping, because the friction in space is very small. Orbital inclination changes are known for some planets in terms of current rates of change, but the oscillation periods are not well published. They can however be predicted with proper dynamic simulations of the solar system.

A three-dimensional dynamic simulation was developed for our solar system capable of handling 12 objects, where all objects affect all other objects. Each object was considered to be a point mass which proved to be an adequate approximation for this study. Initial orbital radii, eccentricities and speeds were set according to known values. The validity of the simulation was demonstrated in terms of short term characteristics such as sidereal periods of planets as well as long term characteristics such as the orbital inclination and eccentricity oscillation periods of Jupiter and Saturn. A significantly more accurate result, than given on approximate analytical grounds in a well-known solar system dynamics textbook, was found for the latter period.

Empirical formulas were developed from the simulation results for both these periods for three-object solar type systems. They are very accurate for the Sun, Jupiter and Saturn as well as for some other comparable systems.

Introduction

Some publications address the aspects of long term effects in our solar system by concentrating more on approximate analytical solutions. The advantage of simulations instead is that they can work well for multi-object systems as our solar system, without having to go for the type of approximations required by analytical solutions. Very accurate so-called ephemerides simulations[1] do exist for restricted time spans, but there is little indication that they can handle large time spans as are required for the investigation of long term effects of 100 000 years and more. Some extremely sophisticated solar system simulations exist[2] for even millions of year simulations, but they are not available to researchers in general at any affordable price if available at all. The type of investigations that was done in this paper required a simulation that was accurate enough, but does not have to compete with the best or the most expensive in the world. It is totally affordable and can run on basically all modern PCs. The findings described are novel or to better accuracy than what are found in previous publications as far as literature studies revealed.

Simulations also use approximations in order not to end up with large amounts of unnecessary code with associated very long execution times. It is very important to validate simulations before attempting to claim findings resulting from them. The validity of the simulation used in this investigation was demonstrated by:

- Good correlation between the resultant simulated sidereal periods of the planets and their known values – therefore short term validation;
- Relatively very small three-dimensional movement of the centre of mass of the complete solar system (that should ideally be zero), even after long simulation periods – therefore long-term verification; and
- Good correlation with some of the few published long term effects that fall in the time span of interest for this investigation.

Block diagrams of the simulation

Following is a block diagram of the simulation of one of the 12 simulated celestial objects, showing the X axis, and the effects of all other 11 objects on this one. Adding Y and Z axes represent the 3-dimensional scenario. Twelve similar block diagrams represent the full system. The X, Y and Z axes system is an inertial one – it is non-rotating and not moving, because the centre of it is the centre of mass of all the objects.

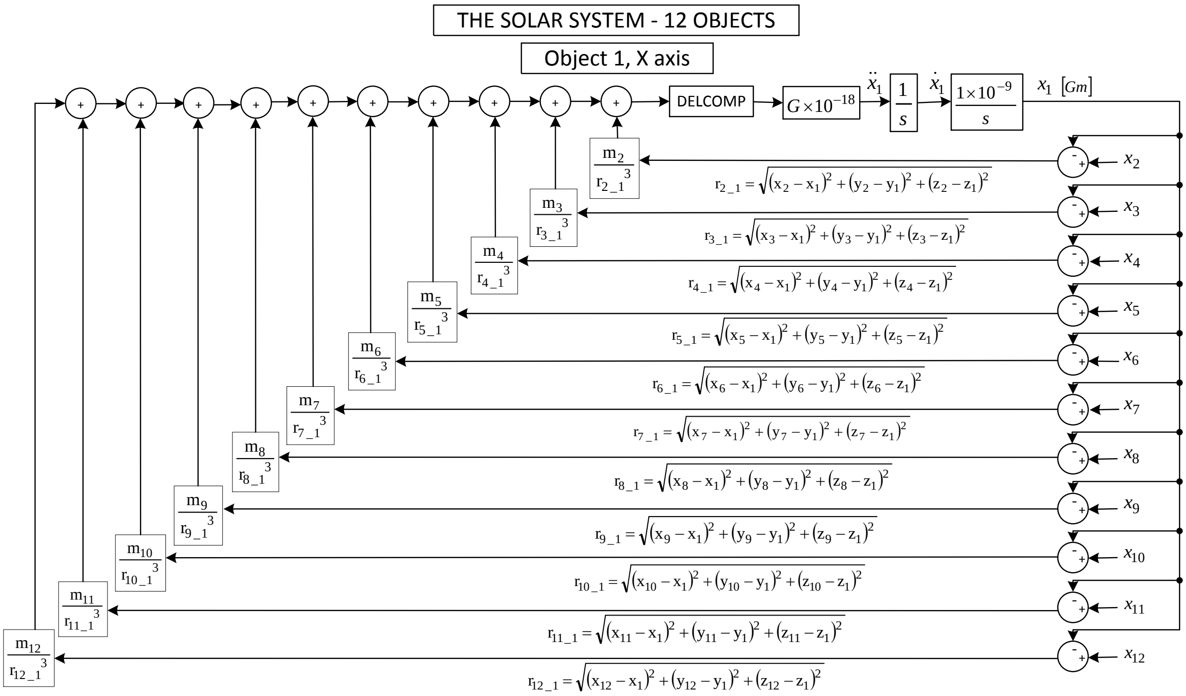

Figure 1: Block diagram of x-axis of 1 of 12 objects

It is possible to generate a solar system simulation from this information and published data of the Sun and all the planets with commercially available software, but the more difficult tasks are:

- To get all the initial conditions correct;
- To do all the required transformations between different axes systems;
- To develop code to calculate special results for example the sidereal periods and orbital inclinations; and
- To display the results meaningful on a PC screen.

Comparison of sidereal periods and centre of mass movement

The sidereal period of a planet is the time it takes to orbit the Sun. The simulation results found in this investigation compared as follows with the known sidereal periods.

Table 1: Comparison of sidereal periods

		Known	Simulation
Earth	year	1.000	1.0008 – 1.0009
Moon	day	27.32	27.2 – 27.4
Mercury	year	0.241	0.2415
Venus	year	0.615	0.616

Mars	year	1.881	1.885
Jupiter	year	11.86	11.87 – 11.88
Saturn	year	29.46	29.35 – 29.41
Uranus	year	84.02	84.41 – 84.44
Neptune	year	164.8	163.4
Pluto	year	247.7	251 – 252

The comparisons are good enough that the simulation is deemed to be sufficiently accurate for this study regarding this validity test.

The movement of the centre of mass in the simulation over about 126 800 years was found to be less than 100 m. This is very small relative to any other movement, for example the smallest other movement is that of the Sun itself, which is about 1 Gm maximum from its starting point due to it being influenced by the planets orbiting it. 100 m relative to 1 Gm is negligible. This verification test result regarding the accuracy of the simulation was thus also found to be sufficient.

The final validations regarding longer term effects are described in the text to follow.

Orbital inclination changes

The initial orbits of the inner planets up to Mars shown as a complete ellipse can be seen in the following XYZ figure.

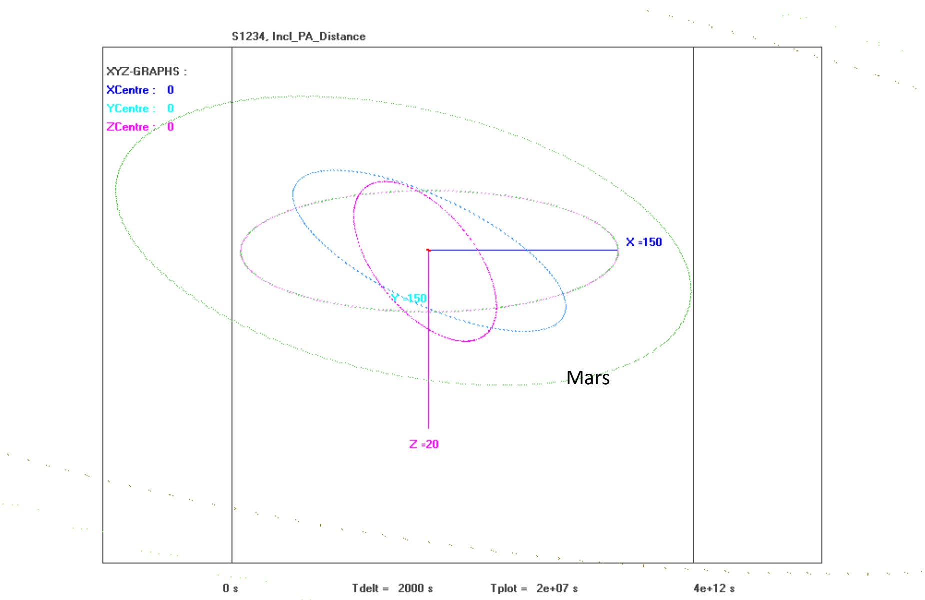

Figure 2: Initial orbits of inner planets

The Z axis has a smaller scale than the X and Y axes; therefore the orbital inclinations are exaggerated in this representation.

The future progress of the orbits looks like the following for the next 10 800 years; clearly showing the expected orbital changes over time.

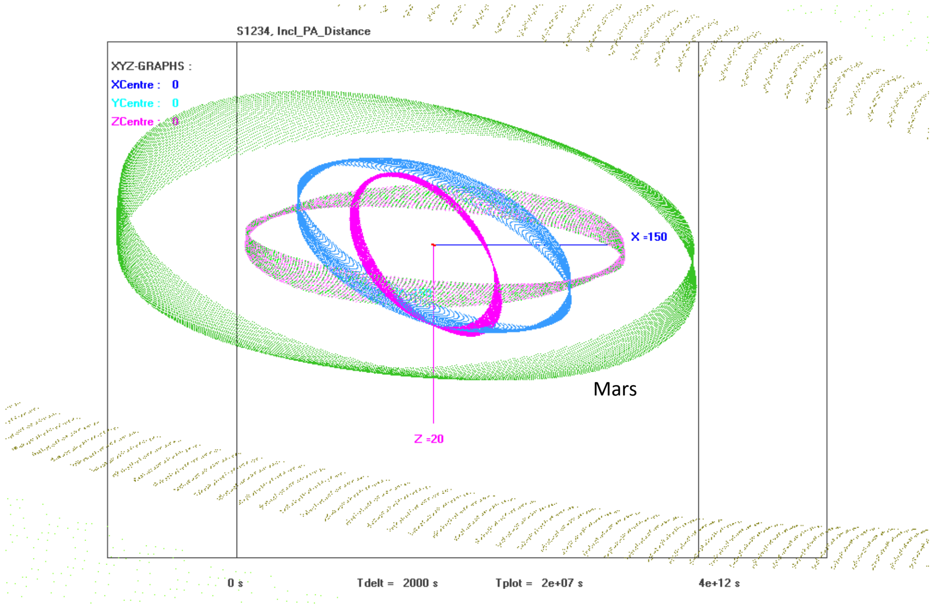

Figure 3: Future progress of orbits of inner planets

The initial orbits of all the planets up to Neptune can be seen in the following XYZ figure.

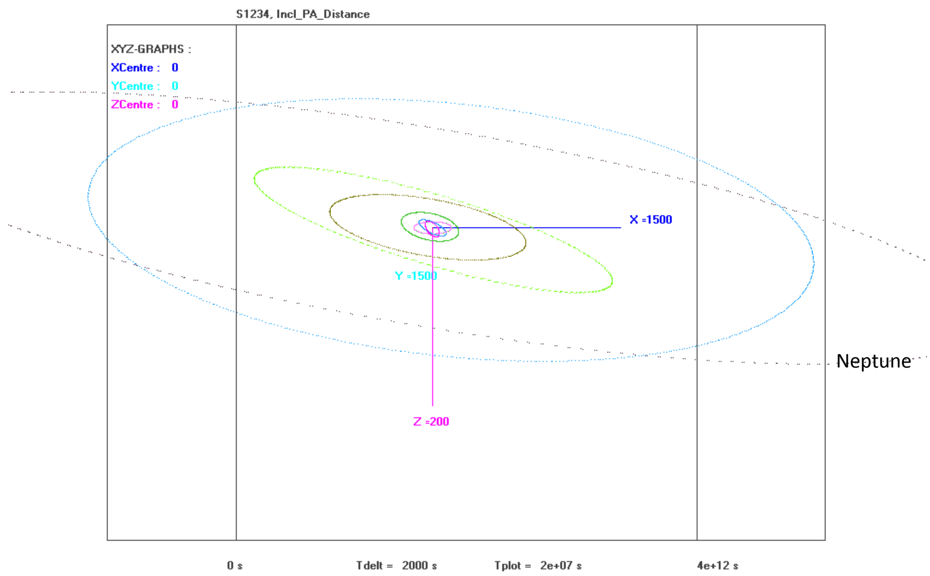

Figure 4: Initial orbits of all the planets

Again the Z axis has a smaller scale than the X and Y axes; therefore the orbital inclinations are exaggerated as before.

The future progress of the orbits looks like the following for the next 12 700 years; clearly showing the orbital changes over time of even the outer planets.

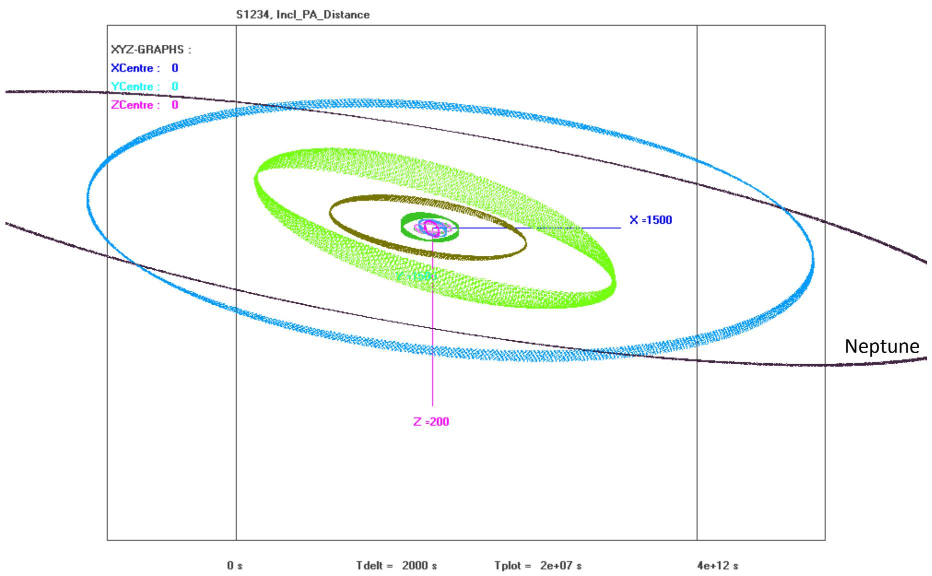

Figure 5: Future progress of orbits of all the planets

The following simulation results show the inclinations of all the planets over time. Periodicity is clear for most of them, but will probably show for all of them if plotted over longer time spans. The vertical scales are in degrees, and were chosen differently for all the planets to get conveniently sized signals.

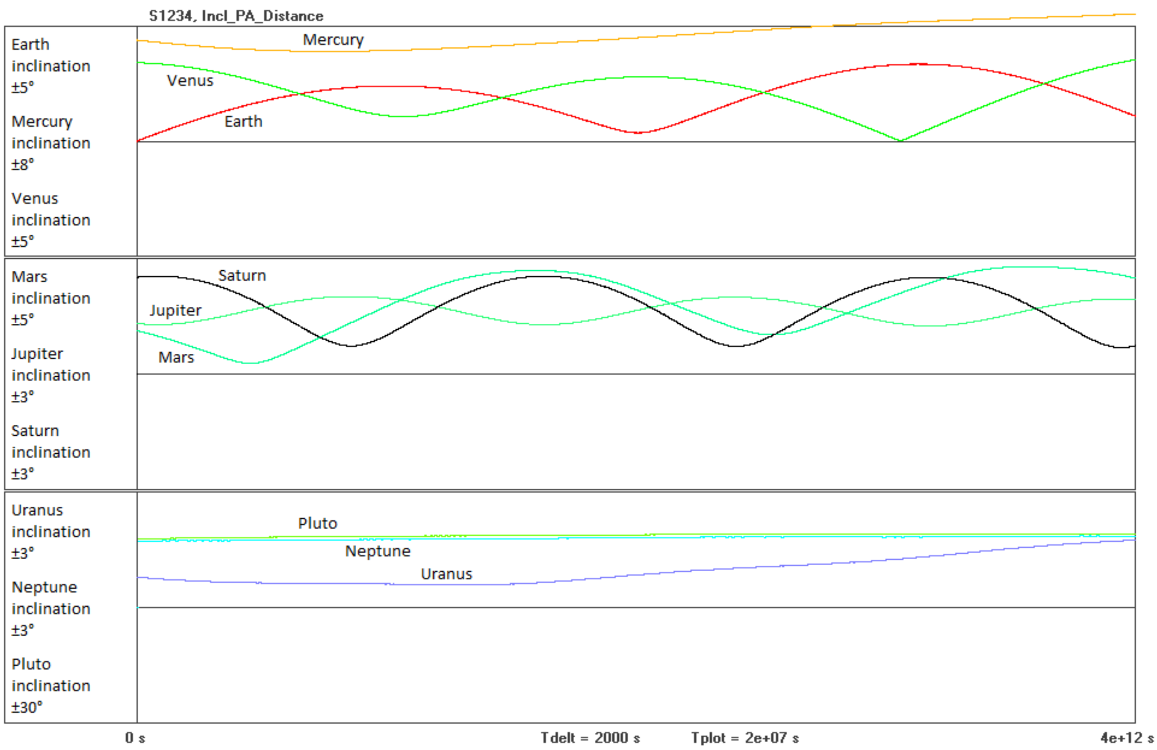

Figure 6: Orbital inclinations over time

Venus and the Earth affect one another strongly, both having periods of about 65 000 years (63 000 to 67 000), but this period is not very constant as was found with longer simulations not shown here. See Muller[3] for some confirmation of this where a period of 70 000 years is mentioned relative to the current ecliptic, and Horner[4] where a period of 67 000 years can be derived from the graphs in his conference paper. Jupiter and Saturn run very synchronized but out of phase with periods of about 48 900 years. See Murray and Dermott[5], page 307, Figure 7.11, where a period of 50 500 years was found. Mars also shows strong periodicity (63 000 to 67 000 years), but its period is also not very constant as would be visible by simulating over longer time spans.

These changes in inclinations are caused by the planets having different inclinations in the beginning. This can be proved by simulating what will happen when all planets start with 0° inclinations. The inclinations will stay at 0° although the perihelions of the orbits will rotate, as was seen with simulations not shown here. The rotation of the perihelions starting with the current orbital inclinations will be seen in simulation results to follow.

Orbital inclination change rates

The current inclination change rates of the planets can better be seen by zooming in on the initial part of the previous graphs, after subtracting the current initial inclinations; therefore all graphs start at zero inclination. Figure 7 contains the results.

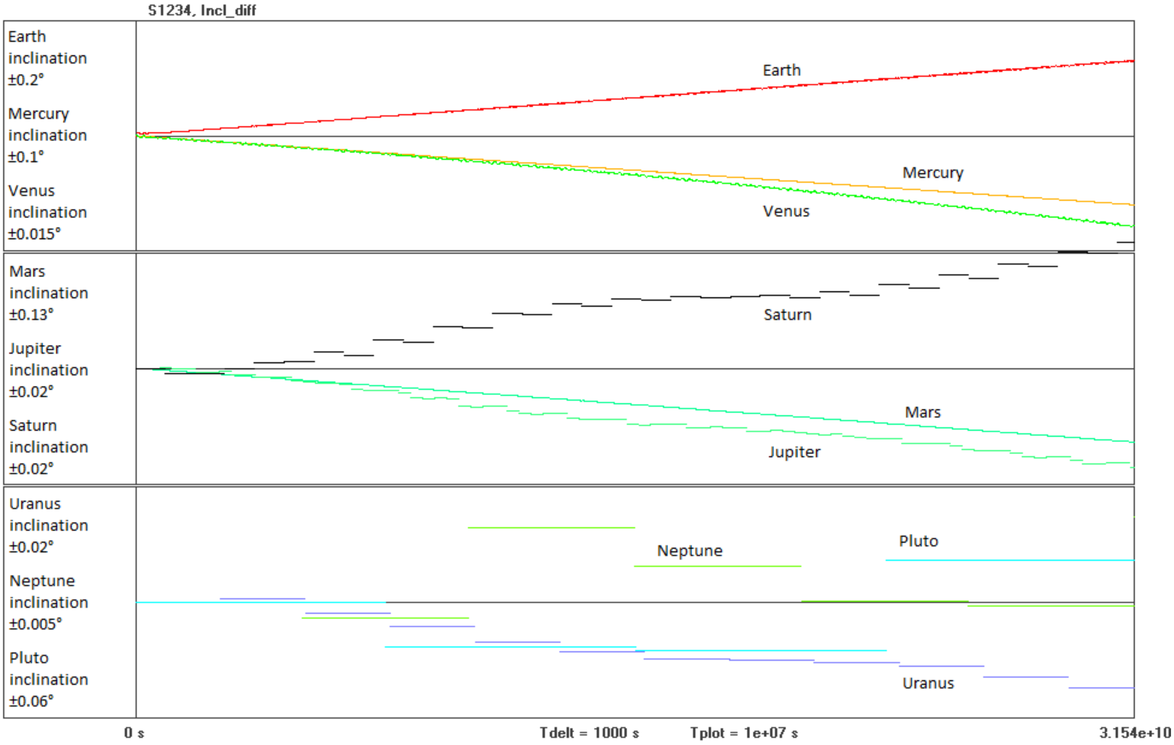

Figure 7: Short term inclinations to determine inclination rates

It is hard to determine current inclination rates for Uranus, Neptune and Pluto because their sidereal periods are so big and the method that was used in simulation to determine the inclinations calculated them only when their orbits crossed the current ecliptic. Therefore only an indication based on average changes can be given for their inclination rates over a chosen relatively long time period. It was decided to determine the average rate for each object over the next 1000 years (3.154E10 s). The only published inclination rates that could be found, are those of Bannister.[6] It is not known over what periods Bannister's results were determined; therefore the comparison in the next table is simply a rough indication of the two sets of results. The non-linear curve of Saturn's inclination in Figure 7 above is a clear demonstration that the change rates are not constant even over relatively small time spans.

Table 2: Comparison of inclination rates

Planet	Bannister[6] [in arcsecs/century]	Simulation [in arcsecs/century]
Mercury	-23.51	-21.3
Venus	-2.86	-4.2
Earth	-46.94	+47.0
Mars	-25.47	-29.6

Jupiter	-4.15	-6.0
Saturn	+6.11	+7.8
Uranus	-2.09	-5.3
Neptune	-3.64	-0.1
Pluto	+11.07	+7.5

Because the Earth's inclination is per definition currently 0°, it is a matter of choice what the sign of its rate must be. While the one side of the orbit will go up, the other side will go down. But actually, since inclination is defined to be positive, the positive sign is more correct, so that in 1000 years, the inclination of the Earth should be about +0.13° relative to the current ecliptic, not -0.13°. The conclusion is therefore that the Bannister[6] internet publication can be trusted regarding the Earth's inclination rate value, but not necessarily regarding its sign.

Investigating a three-object system regarding inclination oscillations

A three-object system, consisting of the Sun, Jupiter and Saturn was investigated next. The decision to narrow down to this investigation at this point followed from the suspicion that Jupiter and Saturn reacted as a pair with very small influence from the other planets. (This will be shown just below to be quite the case.) The intention was to discover empirically some correlations between the period of orbital inclination oscillation of such a pair and some orbital and planet parameters. In every case when some parameter was changed, it was reset back to its nominal value for the following case.

The gyroscopic nature of this system could be seen clearly. The orbits of Jupiter and Saturn rotated about an axis closely perpendicular to the axis of the natural torque between them. While the one orbit rotated clockwise, the other rotated anti-clockwise.

The period of the orbital inclination oscillation of these two planets was found to be about 49 350 years, which compares well with the 48 900 years mentioned before for the same two in the complete solar system simulation. The oscillation periods of both planets are the same because their oscillations are exactly out of phase and stay like that. The first finding is that this three-object system is a good approximation of the complete system as far as inclination oscillation for Jupiter and Saturn is concerned.

The next investigation was to see how sensitive the inclination period is for initial inclination angles. The initial inclination angle of Saturn was changed by 1° from 2.494° to 3.494°. An inclination period of 49 480 years was found, quite the same as the 49 350 before. The second finding is that the inclination period is sufficiently independent of initial inclinations at least for up to 40% variation in initial inclination.

Next the initial ascending node angle of Saturn was changed by 30° from 113.64° to 143.64°. The ascending node is where the orbit is crossing the ecliptic from below to above. An inclination period of 49 480 years was found, much the same as the 49 350 before. The third finding is that the inclination period is sufficiently independent of initial ascending node angles at least for up to 30° variation.

Again the Sun, Jupiter and Saturn only were simulated, but with Jupiter and Saturn having double their known masses. The simulation time graphs showed an inclination period of about 24 660 years. With only the mass of Jupiter double its known value, the inclination period was about 28 810 years. With only the mass of Saturn double its known value, the inclination period was about 38 330 years. With the masses of Jupiter and Saturn restored, but the mass of the Sun 1.3 times what it is known to be, the inclination period was found to be about 56 350 years. In all these cases of mass changes, the initial velocities of the planets were changed accordingly so that the shapes of the planetary orbits were retained. Based on these results, the fourth finding is that the changed inclination period is approximately as follows related to the nominal period and the masses, with the primed variables the changed values.

$$P'_{JS\_incl} = P_{JS\_incl} \cdot \sqrt{\frac{M'_{Sun}}{M_{Sun}} \cdot \frac{M_{Jup}+M_{Sat}}{M'_{Jup}+M'_{Sat}}}$$

with M_{Sun} the mass of the Sun, M_{Jup} the mass of Jupiter and M_{Sat} the mass of Saturn.

The following table summarizes the simulation results in comparison with the formula.

Table 3: Comparison of simulation and formula inclination periods regarding mass

Modified masses	Simulation period [in years]	Formula period [in years]	Difference
$M'_{Sun}=M_{Sun}$, $M'_{Jup}=M_{Jup}$, $M'_{Sat}=M_{Sat}$	49 350	49 350	0%
$M'_{Sun}=1.3\text{x}M_{Sun}$, $M'_{Jup}=M_{Jup}$, $M'_{Sat}=M_{Sat}$	56 350	56 270	-0.1%
$M'_{Sun}=M_{Sun}$, $M'_{Jup}=2\text{x}M_{Jup}$, $M'_{Sat}=2\text{x}M_{Sat}$	24 660	24 680	+0.1%
$M'_{Sun}=M_{Sun}$, $M'_{Jup}=2\text{x}M_{Jup}$, $M'_{Sat}=M_{Sat}$	28 810	29 410	+2.1%
$M'_{Sun}=M_{Sun}$, $M'_{Jup}=M_{Jup}$, $M'_{Sat}=2\text{x}M_{Sat}$	38 330	37 380	-2.5%

These results correlate well enough, within 3%, so that the proposed formula is deemed sufficiently accurate.

Next again only the Sun, Jupiter and Saturn were simulated, but with the orbits of Jupiter and Saturn having 1.1 times their average distances to the Sun. The initial speeds of the planets were adjusted as a function of the average distance so as to keep the orbit shapes the same. This means that the sidereal period of the relevant planet also changed as a result. The simulation time graphs showed an inclination period of about 56 810 years. With only the average distance of Jupiter 1.1 times its known value (meaning it was closer to Saturn), the inclination period was about 34 920 years. With only the average distance of Saturn 1.1 times its known value, the inclination period was about 77 730 years. With both the average distances of Jupiter and Saturn 200 Gm more than their known values, the inclination period was found to be about 42 220 years. With both the average distances of Jupiter and Saturn 100 Gm less than their known values, the inclination period was found to be about 54 980 years. Based on these results, the fifth finding is that the changed inclination period is approximately as follows related to the nominal period, the masses and the average distances, with the primed variables the changed values. The power factor was found by changing it to get a good fit after seeing that a factor of 1 gave far too small an effect for the difference in orbit radii.

$$P'_{JS\_incl} = P_{JS\_incl} \cdot \sqrt{\frac{M'_{Sun}}{M_{Sun}}} \cdot \frac{M_{Jup}+M_{Sat}}{M'_{Jup}+M'_{Sat}} \cdot \left[\frac{R'_{aveSat}-R'_{aveJup}}{R_{aveSat}-R_{aveJup}}\right]^{2.402} \cdot \frac{R_{JSCOMave}}{R'_{JSCOMave}}$$

with R_{aveSat} the average distance of Saturn from the Sun, R_{aveJup} the average distance of Jupiter from the Sun and $R_{JSCOMave}$ the mass weighted average distance of Jupiter and Saturn from the Sun: $R_{JSCOMave} = \frac{R_{aveJup} \cdot M_{Jup} + R_{aveSat} \cdot M_{Sat}}{M_{Jup}+M_{Sat}}$

The following table summarizes the simulation results in comparison with the formula.

Table 4: Comparison of simulation and formula inclination periods regarding average distance from the Sun

Modified average distances from the Sun	Simulation period [in years]	Formula period [in years]	Difference
$R'_{aveSat}=R_{aveSat}$, $R'_{aveJup}=R_{aveJup}$	49 350	49 350	0%
$R'_{aveSat}=1.1\text{x}R_{aveSat}$, $R'_{aveJup}=1.1\text{x}R_{aveJup}$	56 810	56 410	-0.7%
$R'_{aveSat}=R_{aveSat}$, $R'_{aveJup}=1.1\text{x}R_{aveJup}$	34 920	34 100	-2.3%
$R'_{aveSat}=1.1\text{x}R_{aveSat}$, $R'_{aveJup}=R_{aveJup}$	77 730	76 840	-1.1%

R'_{aveSat}=R_{aveSat}+200, R'_{aveJup}=R_{aveJup}+200	42 220	40 560	-3.9%
R'_{aveSat}=R_{aveSat} -100, R'_{aveJup}=R_{aveJup} -100	54 980	55 310	+0.6%

These results correlate well enough, within 4%, so that also this proposed formula is deemed sufficiently accurate.

Approximate absolute empirical formula for inclination period

Based on the last proposed relative formula (new period relative to old period), the following absolute formula is proposed for a three-object system. It is written here in terms of general planets x and y, but x and y can of course refer to Jupiter and Saturn or to any two other planets that are relatively close to one another. The constants in the formula were adjusted so that the formula also complies with the simulation results of the three-object systems consisting of the Sun, Venus and the Earth, and consisting of the Sun, Uranus and Neptune. The constant K_{p_incl} was determined to be K_{p_incl} = 7.742E12 to give the period P in years with masses M in kg and distances R in Gm.

$$P_{xy\_incl} = K_{p\_incl} \cdot \frac{\sqrt{M_{Sun}}}{M_x + M_y} \cdot \frac{|R_{xave} - R_{yave}|^{2.216}}{R_{xyCOMave}{}^{0.736}}$$

with M_{Sun} the mass of the Sun, M_x the mass of planet x, M_y the mass of planet y, R_{xave} the average distance of planet x from the Sun, R_{yave} the average distance of planet y from the Sun and $R_{xyCOMave}$ the mass weighted average distance of planets x and y from the Sun:

$$R_{xyCOMave} = \frac{R_{xave} \cdot M_x + R_{yave} \cdot M_y}{M_x + M_y}$$

The accuracy of the formula is demonstrated in the following table.

Table 5: Comparison of simulation and absolute formula periods for inclination

	Simulation period [in years]	Formula period [in years]	Difference
Sun, Earth and Venus only	106 700	106 600	-0.1%
Sun, Jupiter and Saturn only	49 350	49 350	0%
Sun, Jupiter and Saturn in the complete solar system	48 900	49 350	-0.9%
Sun, Uranus and Neptune only	1 759 000	1 759 000	0%

These results correlate very well (less than 1% differences) but further work will have to be done to determine the robustness of the formula against parameter changes. It may result in a more complicated formula.

It must however be stressed that it cannot be applied for Venus and the Earth or for Uranus and Neptune in the complete solar system. It is only valid for a three-object system or a system that has two planets (e.g. Jupiter and Saturn) that are orders more massive than the other planets. Their orbit sizes must also not be too different or too similar. The suggestion is that one should be careful if the calculated period is less than 20 000 year or more than 2 000 000 year. Some future work should properly determine the validity ranges of the formula.

Orbital eccentricity changes

The complete solar system simulation was run again, but only the Earth, Jupiter and Saturn's graphs are plotted in Figure 8. Graphs 1 to 3 contain the inclinations, 4 to 6 the perihelion angles and graphs 7 to 9 contain the variations in the distance from the Sun. The latter is the distance from the Sun for each planet minus the average distance from the Sun. It is therefore a good indication of the eccentricity of the planet's orbit.

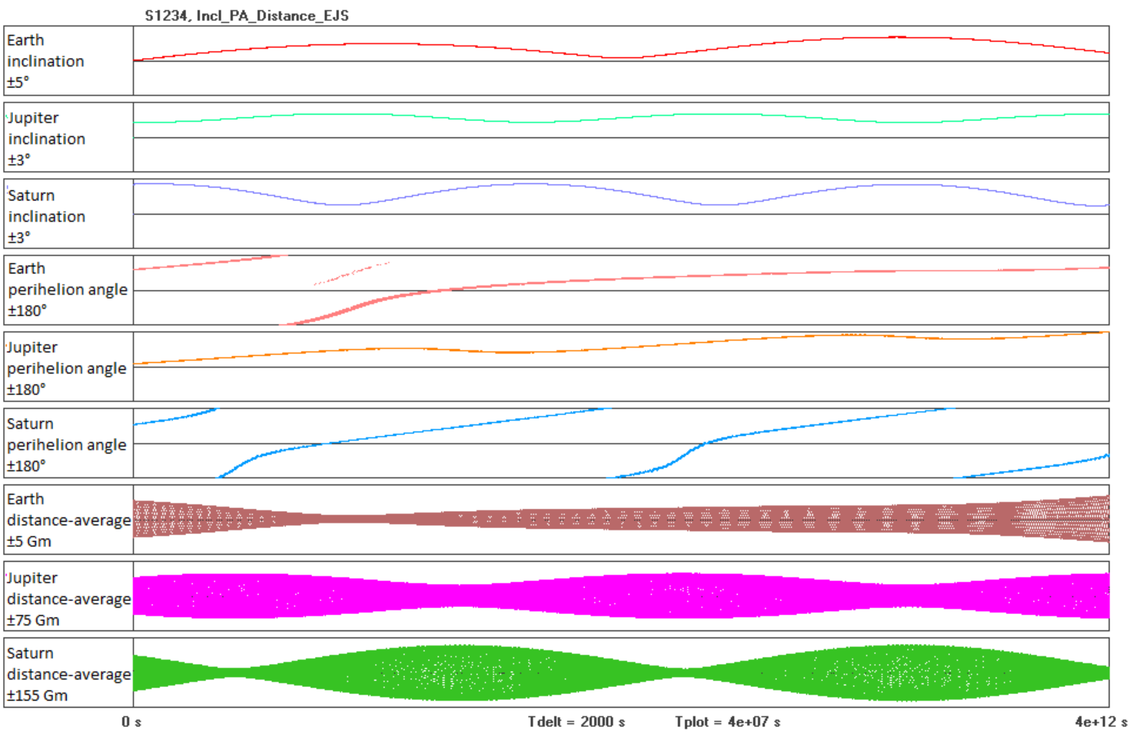

Figure 8: Inclination, perihelion and eccentricity changes in the solar system

From these graphs can be determined that the eccentricity period of the Jupiter and Saturn pair is about 58 650 years. In Murray and Dermott[5], page 307, Figure 7.11, a period of 55 400 years was found. The correlation is reasonable given the approximated nature of the formulas in the referenced textbook.

The Earth's eccentricity is showing in Figure 8 to change significantly in the 126 800 years, but on this scale no oscillation period can be defined. The reason is the influence of the other planets. Proper periodic oscillations for them will appear when only a three-object system of the Sun, Earth and Venus is simulated.

Investigating a three-object system regarding eccentricity oscillations

By following the same reasoning as with the inclination oscillations, three-object systems were investigated regarding eccentricities.

The three-object system of the Sun, Jupiter and Saturn was simulated to compare the eccentricity results with those in the complete solar system.

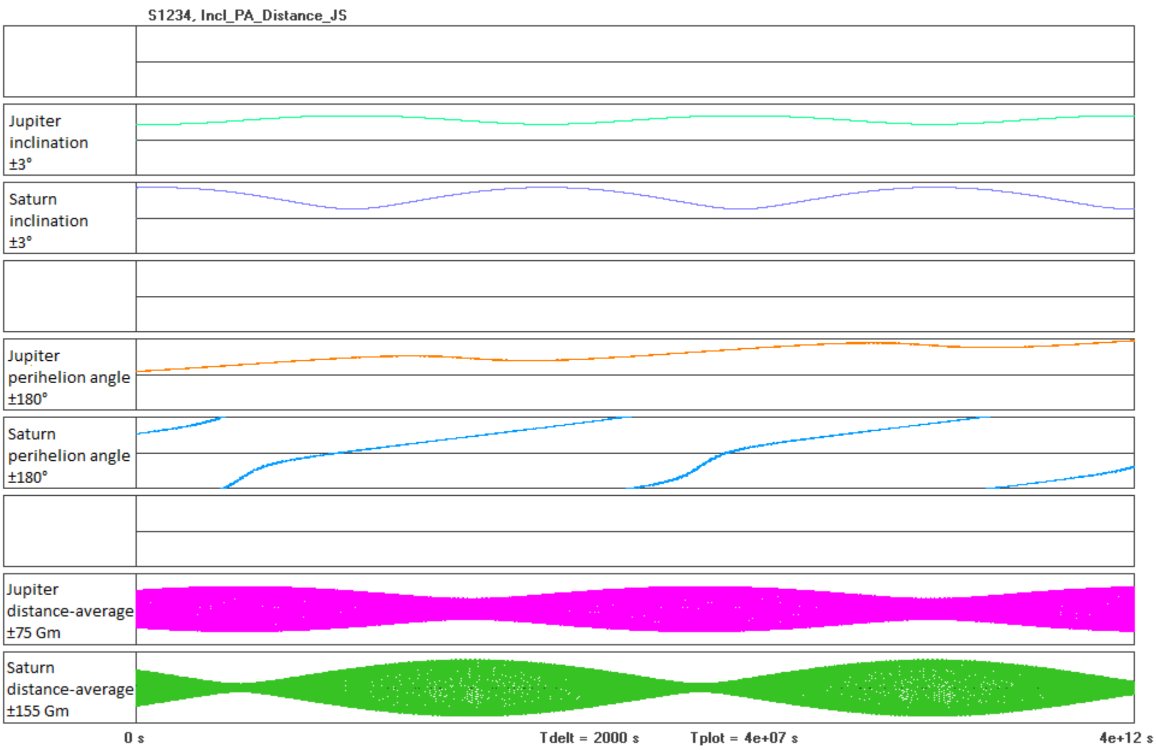

Figure 9: Inclination, perihelion and eccentricity changes in the Sun-Jupiter-Saturn system

The eccentricity period of the Jupiter and Saturn pair is about 58 500 years. The difference between the last two graphs (indicating the eccentricities of Jupiter and Saturn) in Figure 9 and Figure 8 is so small that it can't be seen on the scales shown here. Contrary to this in Murray and Dermott[5], page 282, Figure 7.1, a period of 70 100 years was found, but on page 283 the authors point out that this number is only an approximation. They later suggest that the perturbations from Uranus and Neptune and the almost 2:5 ratio (commensurability) between the sidereal periods of Jupiter and Saturn, which is not represented in their approximate analytical solutions, will make a considerable difference. They do this to explain the huge difference in eccentricity periods between their Figures 7.1 and 7.11 where the Jupiter and Saturn eccentricities are given in the three-object and complete solar systems respectively.

It is unknown what the commensurability correction will do, because it is not discussed in the textbook, but from the simulation results above, it is clear that adding Uranus and Neptune to the system won't make much difference at all to the eccentricity oscillation period of Jupiter and Saturn. The simulation can be trusted because it gave a result closer to the answer for the complete solar system which Murray and Dermott[5] showed on page 307, Figure 7.11. A period of about 55 400 years can be derived from the eccentricity graph in that figure. It originated from the 1950 publication of Brouwer and Van Woerkom[7], who derived more complicated and accurate, but still approximated analytical formulas.

In Figure 9 above, a perihelion angular acceleration can be seen every time the eccentricity is a minimum for both orbits. For Jupiter the acceleration is negative and for Saturn it is positive. The period of this perihelion acceleration is exactly the period of the eccentricity oscillation. It coincides with the eccentricity of the corresponding planet's orbit at its minimum. This was found for all the simulation runs in this investigation. It feels intuitively correct because with the orbit closest to a circle, its moment of inertia should be the least.

The next investigation was to see how sensitive the eccentricity period is for initial inclination angles. The initial inclination angle of Saturn was changed by 1° from 2.494° to 3.494°. An eccentricity period of 58 630 years was found, quite the same as the 58 500 before. In the extreme case of setting the initial inclination angles of both Jupiter and Saturn to zero, the eccentricity oscillation period was found to be 58 490 years, therefore the eccentricity period is definitely insensitive to the initial inclination angles.

Next the initial ascending node angle of Saturn was changed by 30° from 113.64° to 143.64°. An eccentricity period of 58 640 years was found, quite the same as the 58 500 before.

Again the Sun, Jupiter and Saturn only were simulated, but with Jupiter and Saturn having double their known masses. The simulation time graphs showed an eccentricity period of about 27 030 years. With only the mass of Jupiter double its known value, the eccentricity period was about 29 020 years. With only the mass of Saturn double its known value, the eccentricity period was about 48 410 years. With the masses of Jupiter and Saturn restored, but the mass of the Sun 1.3 times what it is known to be, the eccentricity period was found to be about 68 420 years. In all these cases of mass changes, the initial velocities of the planets were changed accordingly so that the shapes of the planetary orbits were retained. Based on these results, the finding is that the changed eccentricity period is approximately as follows related to the nominal period and the masses, with the primed variables the changed values.

$$P'_{JS\_ecc} = P_{JS\_ecc} \cdot \sqrt{\frac{M'_{Sun}}{M_{Sun}}} \cdot \frac{M_{Jup}+M_{Sat}}{M'_{Jup}+M'_{Sat}}$$

with M_{Sun} the mass of the Sun, M_{Jup} the mass of Jupiter and M_{Sat} the mass of Saturn.

The following table summarizes the simulation results in comparison with the formula.

Table 6: Comparison of simulation and formula eccentricity periods regarding mass

Modified masses	Simulation period [in years]	Formula period [in years]	Difference
$M'_{Sun}=M_{Sun}$, $M'_{Jup}=M_{Jup}$, $M'_{Sat}=M_{Sat}$	58 500	58 500	0%
$M'_{Sun}=1.3\text{x}M_{Sun}$, $M'_{Jup}=M_{Jup}$, $M'_{Sat}=M_{Sat}$	68 420	66 700	-2.5%
$M'_{Sun}=M_{Sun}$, $M'_{Jup}=2\text{x}M_{Jup}$, $M'_{Sat}=2\text{x}M_{Sat}$	27 030	29 250	+8.2%
$M'_{Sun}=M_{Sun}$, $M'_{Jup}=2\text{x}M_{Jup}$, $M'_{Sat}=M_{Sat}$	29 020	33 050	+13.9%
$M'_{Sun}=M_{Sun}$, $M'_{Jup}=M_{Jup}$, $M'_{Sat}=2\text{x}M_{Sat}$	48 410	47 560	-1.8%

These results don’t correlate very well (less than 14% difference). But they show that the basic form of the formula is sufficient.

Next again only the Sun, Jupiter and Saturn were simulated, but with the orbits of Jupiter and Saturn having 1.1 times their average distances to the Sun. The initial speeds of the planets were adjusted as a function of the average distance so as to keep the orbit shapes the same. This means that the sidereal period of the relevant planet also changed as a result. The simulation time graphs showed an eccentricity period of about 67 500 years. With only the average distance of Jupiter 1.1 times its known value (meaning it was closer to

Saturn), the eccentricity period was about 40 700 years. With only the average distance of Saturn 1.1 times its known value, the inclination period was about 104 200 years. With both the average distances of Jupiter and Saturn 200 Gm more than their known values, the inclination period was found to be about 48 490 years. With both the average distances of Jupiter and Saturn 100 Gm less than their known values, the inclination period was found to be about 75 750 years. Based on these results, the finding is that the changed inclination period is approximately as follows related to the nominal period, the masses and the average distances, with the primed variables the changed values. The power factor was simply taken as a first attempt to be the same as for the inclination period formula. It is true that a slightly better fit with the simulation results can be found by changing it a bit, but since the power factor is going to be modified later on with the same procedure as was done for the inclination formula, it was not beneficial to optimize it now.

$$P'_{JS\_ecc} = P_{JS\_ecc} \cdot \sqrt{\frac{M'_{Sun}}{M_{Sun}}} \cdot \frac{M_{Jup}+M_{Sat}}{M'_{Jup}+M'_{Sat}} \cdot \left[\frac{R'_{aveSat}-R'_{aveJup}}{R_{aveSat}-R_{aveJup}}\right]^{2.402} \cdot \frac{R_{JSCOMave}}{R'_{JSCOMave}}$$

with R_{aveSat} the average distance of Saturn from the Sun, R_{aveJup} the average distance of Jupiter from the Sun and $R_{JSCOMave}$ the mass weighted average distance of Jupiter and Saturn from the Sun: $R_{JSCOMave} = \frac{R_{aveJup} \cdot M_{Jup} + R_{aveSat} \cdot M_{Sat}}{M_{Jup}+M_{Sat}}$

The following table summarizes the simulation results in comparison with the formula.

Table 7: Comparison of simulation and formula eccentricity periods regarding average distance from the Sun

Modified average distances from the Sun	Simulation period [in years]	Formula period [in years]	Difference
$R'_{aveSat}=R_{aveSat}$, $R'_{aveJup}=R_{aveJup}$	58 500	58 500	0%
$R'_{aveSat}=1.1 \text{x} R_{aveSat}$, $R'_{aveJup}=1.1 \text{x} R_{aveJup}$	67 500	66 860	-0.9%
$R'_{aveSat}=R_{aveSat}$, $R'_{aveJup}=1.1 \text{x} R_{aveJup}$	40 700	40 420	-0.7%
$R'_{aveSat}=1.1 \text{x} R_{aveSat}$, $R'_{aveJup}=R_{aveJup}$	104 200	91 090	-12.6%
$R'_{aveSat}=R_{aveSat}+200$, $R'_{aveJup}=R_{aveJup}+200$	48 490	48 080	-0.8%
$R'_{aveSat}=R_{aveSat}-100$, $R'_{aveJup}=R_{aveJup}-100$	75 750	65 570	-13.4%

These results don't correlate very well although reasonably well (less than 14% difference); so the proposed formula was deemed sufficiently accurate to proceed with the investigation.

Next the three-object system of the Sun, Earth and Venus was simulated.

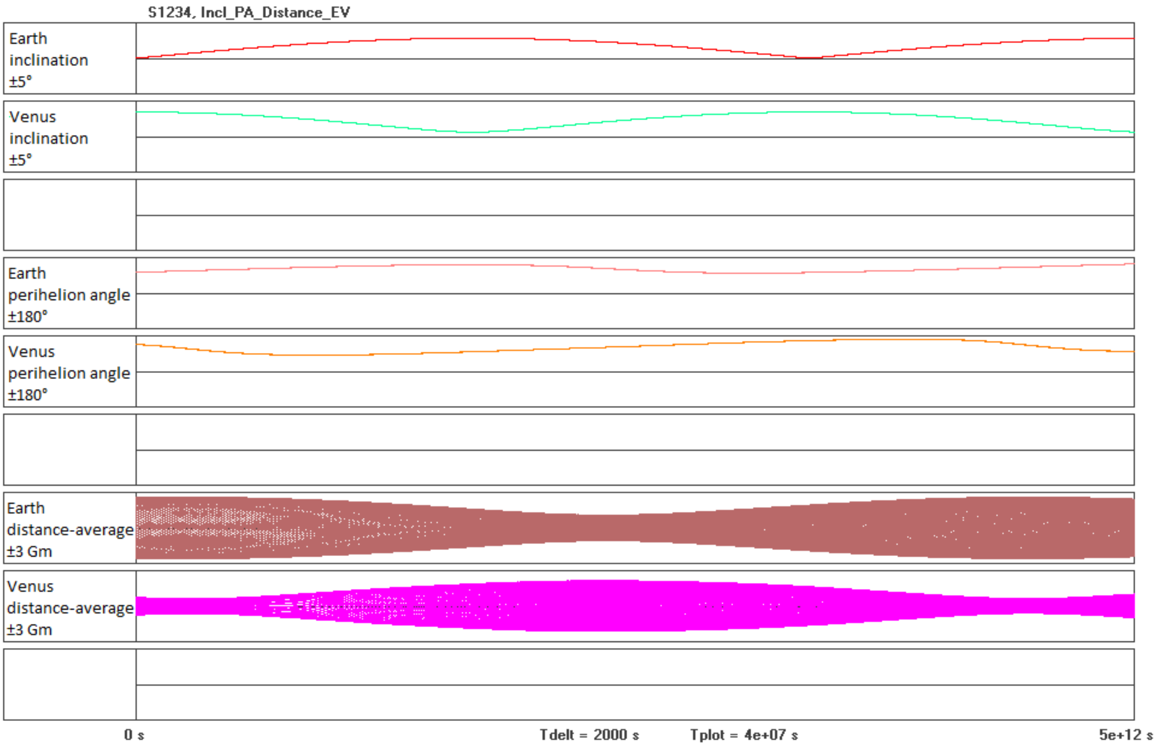

Figure 10: Inclination, perihelion and eccentricity changes in the Sun-Earth-Venus system

The eccentricity period of the Earth and Venus pair is about 133 000 years. The difference between the 3rd last graph (indicating the eccentricity of the Earth) in Figure 10 and Figure 8 is huge because the masses of the Earth and Venus are dominated by the larger planets in the complete solar system represented in Figure 8.

Approximate absolute empirical formula for eccentricity period

Based on the last proposed relative formula (new period relative to old period), the following absolute formula is proposed for a three-object system. It is written here in terms of general planets x and y, but x and y can of course refer to Jupiter and Saturn or to any two other planets that are relatively close to one another. The constants in the formula were adjusted so that the formula also complies with the simulation results of the three-object systems consisting of the Sun, Venus and the Earth, and consisting of the Sun, Uranus and Neptune. The constant K_{p_ecc} was determined to be K_{p_ecc} = 1.970E13 to give the period P in years with masses M in kg and distances R in Gm.

$$P_{xy\_ecc} = K_{p\_ecc} \cdot \frac{\sqrt{M_{Sun}}}{M_x + M_y} \cdot \frac{|R_{xave} - R_{yave}|^{2.403}}{R_{xyCOMave}{}^{1.025}}$$

with $R_{xyCOMave}$ the mass weighted average distance of planets x and y from the Sun:

$$R_{xyCOMave} = \frac{R_{xave} \cdot M_x + R_{yave} \cdot M_y}{M_x + M_y}$$

The accuracy of the formula is demonstrated in the following table.

Table 8: Comparison of simulation and absolute formula periods for eccentricity

	Simulation period [in years]	Formula period [in years]	Difference
Sun, Earth and Venus only	133 000	133 000	0%
Sun, Jupiter and Saturn only	58 500	58 500	0%
Sun, Jupiter and Saturn in the complete solar system	58 300	58 500	+0.3%
Sun, Uranus and Neptune only	1 645 000	1 654 000	+0.5%

These results correlate very well (less than 1% differences) but further work will have to be done to determine the robustness of the formula against parameter changes. It may result in a more complicated formula.

It must however be stressed that this formula too, similar to the inclination period formula, cannot be applied for Venus and the Earth or Uranus and Neptune in the complete solar system. It is only valid for a three-object system or a system that has two planets (e.g. Jupiter and Saturn) that are orders more massive than the other planets. Their orbit sizes must also not be too different. The suggestion is again that one should be careful if the calculated period is less than 20 000 year or more than 2 000 000 year. These types of validity ranges must be properly determined in future investigations.

Conclusions

This paper demonstrates how modelling and simulation of dynamics can be used to find empirical formulas for certain behaviour. The simulations however must be accurate representations of reality. Formulas are very useful in seeing trends and parameter dependence and can give results must faster than simulations which often have very long execution times. The accessibility of formulas is also much better in general than simulations, not only because of costs, but also because of ease of use.

The particular formulas derived in this paper are for the periods of oscillation of planetary inclinations and eccentricities, applicable to three-object systems. As far as the author could find, these formulas are novel. They are very accurate (better than 1%) over quite a large range of three-object systems, covering from as light weight and close to the Sun as Venus and the Earth, through as massive as Jupiter and Saturn, to as distant from the Sun as Uranus and Neptune. However, more work and potential improvements will have to be done regarding the limits of the validity of the formulas. Inclination and eccentricity periods differ from one another, but eccentricity oscillations have the same periods as the repetitions of prominent perihelion angular accelerations. The latter two periods are the same because these angular accelerations occur every time the corresponding orbit’s eccentricity is a minimum.

The derived formulas are also applicable to Jupiter and Saturn in our complete solar system because these two planets are dominant regarding mass relative to the other planets. It would be senseless to attempt to derive similar formulas for the other planets in our system, because neither their inclination nor their eccentricity periods are constant due to the influences of the other planets. Simulations show what will happen with all of their future orbital inclinations and eccentricities on the assumption of no extra-ordinary events.

Good correlation between the simulation results of the inclination oscillation period for Jupiter and Saturn and published results in papers and a well-known and highly regarded textbook was obtained. The corresponding oscillation periods of some of the other planets in our solar system, although not constant, were also determined by the simulation. In certain scientific fields the periods of the Earth’s orbit are of particular interest, but it is important to know that this period is not constant. The average orbital inclination rates over the next 1000 years of all the planets were also determined because there is published though very scarce information on this; only one internet publication could be found. Good correlation with the simulation results was also demonstrated for these. Correlation could be found with the published orbital eccentricity periods in the above-mentioned textbook, although the simulation result was clearly more accurate than one of the main approximate analytical methods described in the book.

Based on the predicted future solar system behaviour in this paper some conclusions may be drawn regarding the history of our solar system, on the assumption that no extra-ordinary events with solar system impact happened in the past. The same holds true regarding the predicted future.

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