Alternative Distributions in the Black-Litterman Model of Asset Allocation *

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Abstract

The Black Litterman Model is a quantitative method of asset allocation which in its formulation assumes that asset returns are normally distributed. In reality, research has shown that asset returns especially from the equity markets are skewed and have high kurtosis i.e fat tails. In this thesis we replace the normal distribution assumption in the calculation of the prior equilibrium returns used in the model with a more general distribution which captures the skewness and fat tails exhibited by stock data. We consider the α-stable distributions as an alternative distribution to the normal distribution. Consequently we also consider alternative measures of risk, the Value at Risk and the Conditional Value at Risk other than the variance used in the normal case. We applied the model to a South African setting and found that in the Black Litterman Model the α-stable distributions together with the Value at Risk and Conditional Value at Risk as risk measures gave the best allocation with higher values for the Sharpe Ratios for the resultant portfolios. The portfolio that resulted from using the normal distribution and the variance of stock returns as a risk measure, had the lowest Sharpe Ratio.
Keywords(Black Litterman, Bayesian Estimation, α-stable distributions, VaR, CVaR)

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1 Introduction

Modern portfolio theory is mainly based on the work of Harry Markowitz [Markowitz, 1952]. According to Markowitz investors only care about the first two moments of an asset returns distribution, the mean and the variance . In portfolio selection, their goal is to minimize the portfolio risk, as measured by the standard deviation of the return distribution, for a certain required return or equivalently to maximize the portfolio expected return as measured by the mean of the return distribution, for a certain acceptable level of risk. For that reason this method of choosing investments is known as the *mean variance approach* or *mean variance optimization* method.

The mean variance approach revolutionized portfolio selection and asset allocation and it gave investors a standard way of ranking investments. But implicit in its formulation is the assumption that asset returns are normally distributed. According to [Tobin, 1956] and [Hanoch and Levy, 1969], the mean-variance criterion is valid if and only if the distribution of returns is of a two parameter family. [Hanoch and Levy, 1969] went on to prove that mean-variance optimization will only be optimal if the individual asset returns distributions considered are all Gaussian. Although the normal distribution has been used since the time of Markowitz to model asset returns, there is overwhelming evidence from historical data which show that stock returns are not normally distributed. The normal distribution is perfectly symmetric about its mean but historical stock returns are skewed. Individual stocks frequently have returns which are skewed to the right and stock indexes have returns skewed to the left. The normal distribution has tails that gradually flattens out as you move away from the mean but asset returns from the stock markets on the other hand have excessive kurtosis and exhibit fat tails. This means that in reality there is more probability in the tails stock returns distributions than would be justified by a normal distribution.

Besides the distributional inadequacy of the normal distribution, there are other problems cited by academics and practitioners that arose when they tried to use mean variance optimization in practice. It is a well known fact that adding assets that are less correlated with those already in a portfolio reduces its variance i.e the risk. This is the rationale behind diversification. Mean-variance optimization sometimes result in portfolios with extremely large long and short positions in only a few assets which contradicts the logic of diversification. These types of portfolios are referred to as *concentrated portfolios* and can lead to large losses if one do not take heed of them. As noted in, [Black and Litterman, 1992] if one tries to put constraints against these short positions, the mean variance model sometimes results in corner solutions with zero weights in many assets and unreasonably large weights in small capitalized stocks. These problems come from the fact that the inputs to the optimization, the vector of expected returns and the covariance matrix, are usually forecasts estimated from historical data and according to [Black and Litterman, 1992], historical estimates are bad predictors for future behavior.

The other problem with the mean variance criterion is that the model requires inputs for estimated expected returns, variance and covariance of each and every asset included in the optimization. Estimating these inputs can be a very cumbersome exercise to do in practice because of the very large number of parameters that will be needed to be estimated, and also having too many estimated inputs will translate into higher estimation error. As noted in [Michaud, 1989] small changes in the values of the inputs can significantly cause a very large change in the weights of the resultant portfolio. Mean-Variance optimization on the other hand assumes that inputs have no estimation error which as according to [Michaud, 1989] is not generally true . [Best and Grauer, 1991] showed that a small increment in the mean can change the weights of a portfolio into a completely new and with completely different composition to the one before. They deduced that there is a direct relationship between the change in the weights of a portfolio and the change in the expected return. [Michaud, 1989] investigated the impact of the change in the covariance matrix on mean-variance optimized portfolios. He found that sometimes estimation of covariance matrices from historical data can produce covariance matrices that are singular or invertible. These are just a few of the reasons, why the popularity of the mean variance method has greatly diminished.

The Black Litterman Model is on the other hand a quantitative asset allocation model that avoids some of the problems of mean variance optimization. The model was formulated by members of the Quantitative Research Strategies at Goldman Sachs, Fischer Black and Robert Litterman and published in their three papers [Black and Litterman, 1991b], [Black and Litterman, 1991a], and [Black and Litterman, 1992]. The starting point in the model, is that the market is in equilibrium and the expected equilibrium market returns are those given by an equilibrium asset pricing model. In the original formulation of the Black-Litterman Model the equilibrium asset pricing model they assumed is the Capital Asset Pricing Model(CAPM). The CAPM is described in [Lintner, 1965], [Mossin, 1966] , [Treynor, 1961] and [Sharpe, 1964]. The CAPM assumes that asset returns are normally distributed. The investor according to [Black and Litterman, 1992] then express views on each and every asset he/she intends to invest in so as to tilt these equilibrium returns towards his/her preferences, depending on the information that he/she is able to get hold of. An investor with no views, i.e who is neutral to the market, argues [Black and Litterman, 1992] will choose to follow a passive strategy in which they just hold the assets according to their weight in the market portfolio. In that way they are guaranteed to get the return on the market. We explain the Black Litterman model in Section 3.

The Black Litterman assumes that the asset returns are normally distributed. But as we mentioned earlier asset returns especially stock returns are skewed and have fat tails. Therefore there is a natural need to extend the Black Litterman Model to include non-normal distributional properties in asset returns. There are generally two modifications followed in the literature on how to do this extension. The first approach concentrates on the views and is called the Copula Opinion Pooling Method, due to [Meucci, 2005]. The idea in Meucci's approach is to allow the views to follow any distribution other than the normal distribution assumed in the original model. This distribution can be any arbitrary one even a skewed or fat tailed one. The views are also assumed to be dependent and their dependence structure is modeled by a weight function called a copula function. Once the views have been estimated they can then be combined with the equilibrium returns as in the original Black Litterman Model. In this framework there is no need to specify distributional assumptions of the equilibrium market returns. They will inherit the distribution of the view returns in a natural way. The expected returns that results from this combination will have the general distributional properties specified. More on this method can be found in [Meucci, 2005], [Meucci, 2008] and other papers by Meucci.

The second approach concentrates on the equilibrium market returns and tries to assume general distributional properties for these equilibrium market returns other than the normal distribution assumed in the original model. The approach was first proposed in [Giacometti et al., 2007]. In this approach the equilibrium returns are not assumed to be those given by an equilibrium asset pricing or from a normal distribution as in the original model. This is the approach we are going to investigate in this thesis with special emphasis on the South African Market. In particular we are going to assume that the asset returns are from the α-stable distributions. This is because the α-stable distributions are flexible in modeling empirical stock returns data and includes parameters of skewness and kurtosis. Definitions and properties are of the α-stable distributions are going to be given in a later section and fitted to South African market stock returns data.

We are going to start with an overview of the mean variance approach and then we go into the original Black Litterman Model as formulated by Black and Litterman [Black and Litterman, 1992] with normally distributed stock returns and together with the assumption that the CAPM is the equilibrium model that priced the expected returns observed. Basic assumptions and definitions are going to be given underlying the model followed by general results and proofs. We then consider the α-stable distributions as an alternative distribution to the normal distribution in the section after that. We describe the α-stable distributions, their basic properties and definitions and how the parameters of the distributions can be estimated for empirical data. The variance is the measure of risk used when asset returns are assumed to be normally distributed. We consider other risk measures, the Value at Risk(VaR) and the Conditional Value at Risk (CVaR). Basics definitions of VaR and CVaR are going to be given with a detailed explanation on how they can be used in the α-stable distribution case to get the equilibrium market returns. Formulas for the equilibrium expected returns are then going to be derived for the three cases, the normal distribution with covariance matrix, and variance as a risk measure, α-stable distributions with dispersion matrix and VaR as a risk measure and then finally α-stable distributions with dispersion matrix and CVaR as a risk measure. These equilibrium returns will be used as inputs in the Black Litterman Model implementation in the last section. We find that the portfolio that result from using the α-stable distributions with the alternative VaR and the CVaR as risk measures had a higher Sharpe Ratio than the one for the normal distribution with the variance as a risk measure.

2 Mean Variance Optimization

The mean variance approach is a criteria used by investors to choose investments. There are two ways they can do that according to [Markowitz, 1952],

(i) Maximize expected return for some level of risk

(ii) Minimize the portfolio risk for a certain required expected return

Let the number of assets to be invested in be N and let $\mathbf{R}$ the $N \times 1$ vector of expected returns and $N \times N$ covariance matrix $\mathbf{\Sigma}$. Let the weights of all the assets used to form a portfolio be collected in a vector $\mathbf{w}$, then the portfolio return and variance can be written as $\mathbf{w}'\mathbb{E}(\mathbf{R})$ and $\mathbf{w}'\mathbf{\Sigma}\mathbf{w}$ respectively. The portfolio allocation problem, using mean variance, can be formulated as solution to either of the following two optimization problems.

(i) Expected Return

$$\max_{\mathbf{w}\in\mathbb{R}^N} \mathbf{w}'\mathbb{E}(\mathbf{R}) \quad s.t \qquad \mathbf{w}'\mathbf{\Sigma}\mathbf{w} = \sigma^2. \tag{1}$$

(ii) Risk

$$\min_{\mathbf{w}\in\mathbb{R}^N} \frac{1}{2}\mathbf{w}'\mathbf{\Sigma}\mathbf{w} \qquad s.t \qquad \mathbf{w}'\mathbb{E}(\mathbf{R}) = \mu. \tag{2}$$

Let λ be the risk aversion coefficient of the investor. Assuming the investors' preferences can be described by a quadratic utility function. The above two problems can be combined into a problem of maximizing the investor's expected utility of wealth $\mathbb{E}(U(\mathbf{w}'\mathbb{E}(\mathbf{R})))$, i.e

$$\max_{\mathbf{w}\in\mathbb{R}^N} \mathbb{E}(U(\mathbf{w}'\mathbb{E}(\mathbf{R}))) = \max_{\mathbf{w}\in\mathbb{R}^N} \left(\mathbf{w}'\mathbb{E}(\mathbf{R}) - \frac{1}{2}\lambda\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\right) \tag{3}$$

The resultant portfolio weights from this optimization are then

$$\mathbf{w}^* = (\lambda\boldsymbol{\Sigma})^{-1}\mathbb{E}(\mathbf{R}). \tag{4}$$

Remark It is worth noting that in order for the above optimization to be feasible, it necessary that the covariance matrix $\boldsymbol{\Sigma}$ be non singular, i.e $det(\boldsymbol{\Sigma}) \neq 0$ and positive definite.

3 The Black Litterman Model

The Black Litterman model was formulated by Fischer Black and Robert Litterman at Goldman Sachs in the early 1990s. In the original papers by Black and Litterman the model is not explained much mathematically and it took people a long time to see how useful the model can be in asset allocation. The first good exposition was given by [Satchell and Scowcroft, 2000] and since then there has been considerable interest in the model in the academia and industry with good papers by [Meucci, 2005], [Idzorek, 2004], [Giacometti et al., 2007] and [Walters, 2008] and others coming out in the last decade. The model consists of two main stages

(1) Calculating the vector of expected returns given that the market is in equilibrium.

(2) Incorporating the investors' views to twist the equilibrium market returns towards that of the investor.

3.1 Equilibrium Returns

Consider a market of N assets with a vector of portfolio weights $\mathbf{w} \in \mathbb{R}$. The returns of the assets are then a random variable with expected return $\mathbb{E}(\mathbf{R})$. Assuming that the variance of the returns, $Var(\mathbf{R}) = \mathbf{\Sigma}$ exists and is well defined and following [Satchell and Scowcroft, 2000], the return distribution is then assumed to be

$$\mathbf{R} \sim N(\mathbb{E}(\mathbf{R}), \mathbf{\Sigma}). \tag{5}$$

Black and Litterman assumes that market is in or always moving towards equilibrium and so $\mathbb{E}(\mathbf{R})$ itself is also a random variable with the mean $\mathbf{\Pi}$ the equilibrium risk premiums. The variance of $\mathbb{E}(\mathbf{R})$ is assumed to be proportional to the variance of the returns $Var(\mathbf{R}) = \mathbf{\Sigma}$ through a constant of proportionality τ, i.e

$$Var(\mathbb{E}(\mathbf{R})) \propto \mathbf{\Sigma} \tag{6}$$

$$\Rightarrow Var(\mathbb{E}(\mathbf{R})) = \tau\mathbf{\Sigma}. \tag{7}$$

And therefore

$$\mathbb{E}(\mathbf{R}) \sim N(\mathbf{\Pi}, \tau\mathbf{\Sigma}). \tag{8}$$

Assuming returns are normally distributed and that the covariance matrix is well defined, guarantees that the equilibrium expected returns are those given by the CAPM.

$$\mathbf{\Pi} = \mathbb{E}(\mathbf{R}) - R_f\mathbf{1} = \beta\left(\mathbb{E}(R_M) - R_f\right)$$

Where $\beta = \dfrac{cov(\mathbf{R}, \mathbf{R}'\mathbf{w})}{\sigma_M^2}$.

3.2 The Views

There are two types of views that can be specified by the investor, an absolute and a relative view. An absolute view states the specific value of the expected return for the asset the view is on. For example asset A is going to achieve an expected return of 10%. A relative view states how a certain asset is going to perform relative to another asset, for example asset A is going to outperform asset B by 5%. Views can be expressed on each of the assets as long as they are specified relative to the vector of expected returns $\mathbb{E}(\mathbf{R})$, or relative to each other, and the investor must be able to express a level of certainty in the views. These two cases are the ones commonly used ways of formulating views in practice. In the Copula Opinion Pooling Framework of **Meucci**, there is the possibility of formulating views as uniform values on ranges for example, the return of asset A is going to be between 0.25 and 0.5.

Lets consider the case where K views are formulated, K is assumed to be less than N. Let $\mathbf{P}$ be the $K \times N$ matrix with entries which shows which assets the views are on. A relative view is given a value of 1 on the column corresponding to the asset where the view is on. Relative views are entered as a positive numbers on the column of the assets that are expected outperform and corresponding negative values are entered on the assets expected to be outperformed. Each row of $\mathbf{P}$ must sum to 1 for absolute views and zero for relative views. Let $\mathbf{Q}$ be the K dimensional vector containing the view specific values and let the uncertainty of the views be represented by a normally distributed error term ϵ with mean the zero vector and variance covariance matrix $\mathbf{\Omega}$.
The views can then be written as

$$\mathbf{P}\mathbb{E}(\mathbf{R}) = \mathbf{Q} + \epsilon, \quad \epsilon \sim N(\mathbf{0}, \mathbf{\Omega}) \tag{9}$$

where $\mathbb{E}(\mathbf{R})$ is the vector of assets expected returns.
The quantities in $\mathbf{P}\mathbb{E}(\mathbf{R})$ are called the **view returns**. Now to get the distribution of the view returns we take the expected value and variance of the above equation.

$$\begin{aligned} \mathbb{E}\mathbf{P}'\mathbb{E}(\mathbf{R})) &= \mathbb{E}(\mathbf{Q} + \epsilon) = \mathbf{Q} & (10) \\ Var((\mathbf{P}'\mathbb{E}(\mathbf{R}))) &= Var(\epsilon) = \mathbf{\Omega} & (11) \end{aligned}$$

Since the error term is assumed to be normally distributed, the view returns are then also normally distributed with mean $\mathbf{Q}$ and variance $\mathbf{\Omega}$.

$$\mathbf{P}'\mathbb{E}(\mathbf{R})|\mathbb{E}(\mathbf{R}) \sim N(\mathbf{Q}, \mathbf{\Omega}). \tag{12}$$

Besides having the number of views to be less than the number of assets available, i.e $K < N$, another important assumption made in the Black-Litterman model is that the views should be mutually uncorrelated. Having mutually uncorrelated views ensures that the views covariance matrix $\mathbf{\Omega}$ is diagonal in nature and the diagonal elements are then considered to be the uncertainty in the corresponding views.

3.3 The Black Litterman Assumptions

The details of above two sections can be summarized into two important assumptions underlying the Black Litterman Model. These are that

A1 The equilibrium returns are normally distributed with mean the equilibrium premiums $\mathbf{\Pi}$ and variance $\tau\mathbf{\Sigma}$

$$\mathbb{E}(\mathbf{R}) \sim N(\mathbf{\Pi}, \tau\mathbf{\Sigma}). \tag{13}$$

A2 The view returns are normally distributed with mean $\mathbf{Q}$ and variance $\mathbf{\Omega}$

$$\mathbf{P}'\mathbb{E}(\mathbf{R})|\mathbb{E}(\mathbf{R}) \sim N(\mathbf{Q}, \mathbf{\Omega}). \tag{14}$$

4 Combining Views with Equilibrium Returns

There are two approaches used to combine the views with market equilibrium returns in the model, the Bayesian Approach explained in detail in [Satchell and Scowcroft, 2000] and the mixed estimation approach suggested by [Black and Litterman, 1991a]) and proposed by [Henry and Goldberg Arthur, 1961]. We follow the Bayesian Approach and for a reader interested in the mixed estimation method detailed proofs can be found in [Walters, 2008].

4.1 The Bayesian Approach

The Bayesian approach is based on Bayes' Theorem which states that

Theorem 4.2 Bayes' Theorem *Let A and B be any two events in a sample space, $\mathbf{\Omega}$, then the* **posterior probability** *of A given B , $\mathbb{P}(\mathbf{A}|\mathbf{B})$ is given by*

$$\mathbb{P}(\mathbf{A}|\mathbf{B}) = \frac{\mathbb{P}(\mathbf{B}|\mathbf{A})\mathbb{P}(\mathbf{A})}{\mathbb{P}(\mathbf{B})} \tag{15}$$

where $\mathbb{P}(\mathbf{A})$ is called the **prior probability** *of A. $\mathbb{P}(\mathbf{B}|\mathbf{A})$ is often referred to as the conditional probability of B given A or the* **likelihood** *of A and $\mathbb{P}(\mathbf{B})$ is just a normalizing constant called the marginal probability of B. Therefore Bayes' Theorem can be summarized as*

$$\mathbb{P}(\mathbf{A}|\mathbf{B}) \propto \ell(\mathbf{A}|\mathbf{B})\mathbb{P}(\mathbf{A}). \tag{16}$$

The posterior distribution is proportional to the likelihood times the prior distribution.

4.3 Certain Views

If the investor is certain about his/her views, for example they might have entered a derivative contract like a forward contract, from which the payment is certain, it implies that the views have a zero standard deviation and the elements of the view covariance matrix $\mathbf{\Omega}$ are all zeros. In this case the views can be expressed in a deterministic form or as known constants. The optimization problem to be solved, according to [Black and Litterman, 1992] is then

$$\min_{\mathbb{E}(\mathbf{R})} \left(\mathbb{E}(\mathbf{R}) - \mathbf{\Pi}\right)' \tau \mathbf{\Sigma} \left(\mathbb{E}(\mathbf{R}) - \mathbf{\Pi}\right) \qquad s.t. \quad \mathbf{P}'\mathbb{E}(\mathbf{R}) = \mathbf{Q}. \tag{17}$$

Theorem 4.4 *The solution $\mathbb{E}(\mathbf{R})$ to the above optimization problem is*

$$\mathbb{E}(\mathbf{R}) = \mathbf{\Pi} + \mathbf{\Sigma}^{-1}\mathbf{P}'(\mathbf{P}\mathbf{\Sigma})^{-1}\mathbf{P}')^{-1}(\mathbf{Q} - \mathbf{P}\mathbf{\Pi}). \tag{18}$$

Proof Let ζ be a $K \times 1$ vector of constants. Using the method of Lagrange Multipliers, with multiplier ζ, the Lagrangian to the above optimization problem can be formed as

$$L(\zeta, \mathbb{E}(\mathbf{R})) = (\mathbb{E}(\mathbf{R}) - \mathbf{\Pi})'\tau\mathbf{\Sigma}(\mathbb{E}(\mathbf{R}) - \mathbf{\Pi}) - \zeta(\mathbf{P}\mathbb{E}(\mathbf{R}) - \mathbf{Q}). \tag{19}$$

Which can be expanded using laws of matrix multiplication to

$$\begin{aligned} L(\zeta, \mathbb{E}(\mathbf{R})) &= (\mathbb{E}(\mathbf{R}))'\tau\mathbf{\Sigma}\mathbb{E}(\mathbf{R}) - (\mathbb{E}(\mathbf{R}'\tau\mathbf{\Sigma}\mathbf{\Pi} \\ &\quad -\mathbf{\Pi}'\tau\mathbf{\Sigma}\mathbb{E}(\mathbf{R}) + \mathbf{\Pi}'\tau\mathbf{\Sigma}\mathbf{\Pi} - \zeta'\mathbf{P}\mathbb{E}(\mathbf{R}) - \zeta'\mathbf{Q}. \end{aligned}$$

Applying the first order conditions (FOC's), we have

$$\frac{\partial L(\zeta, \mathbb{E}(\mathbf{R}))}{\partial \mathbb{E}(\mathbf{R})} = \tau\mathbf{\Sigma}\mathbb{E}(\mathbf{R}) + \tau\mathbf{\Sigma}\mathbb{E}(\mathbf{R}) - \tau\mathbf{\Sigma\Pi} - \tau\mathbf{\Sigma\Pi} - \mathbf{P}'\zeta = 0 \tag{20}$$

$$\frac{\partial L(\zeta, \mathbb{E}(\mathbf{R}))}{\partial \zeta} = \mathbf{P}\mathbb{E}(\mathbf{R}) - \mathbf{Q} = 0. \tag{21}$$

Which implies that

$$2\tau\mathbf{\Sigma}\mathbb{E}(\mathbf{R}) - 2\tau\mathbf{\Sigma\Pi} - \mathbf{P}'\zeta = 0 \tag{22}$$

$$\Rightarrow 2\tau\mathbf{\Sigma}(\mathbb{E}(\mathbf{R}) - \mathbf{\Pi}) = \mathbf{P}'\zeta \tag{23}$$

$$\Rightarrow (\mathbb{E}(\mathbf{R}) - \mathbf{\Pi}) = \frac{1}{2\tau}\mathbf{\Sigma}^{-1}\mathbf{P}'\zeta \tag{24}$$

$$\tag{25}$$

$$\Rightarrow \mathbb{E}(\mathbf{R}) = \mathbf{\Pi} + \frac{1}{2\tau}\mathbf{\Sigma}^{-1}\mathbf{P}'\zeta \tag{26}$$

and

$$\mathbf{P}\mathbb{E}(\mathbf{R}) = \mathbf{Q}. \tag{27}$$

Substituting the expression for $\mathbb{E}(\mathbf{R})$ from equation 26 in equation 27 above we have

$$\mathbf{P}'\left(\mathbf{\Pi} + \frac{1}{2\tau}\mathbf{\Sigma}^{-1}\zeta\mathbf{P}'\right) = \mathbf{Q} \tag{28}$$

$$\Rightarrow 2\tau\mathbf{P\Pi} + \mathbf{P\Sigma}^{-1}\mathbf{P}\zeta = 2\tau\mathbf{Q} \tag{29}$$

$$\Rightarrow \mathbf{P\Sigma}^{-1}\mathbf{P}\zeta = 2\tau(\mathbf{Q} - \mathbf{P\Pi}) \tag{30}$$

$$\Rightarrow \zeta = (\mathbf{P\Sigma}^{-1}\mathbf{P}')^{-1}2\tau(\mathbf{Q} - \mathbf{P\Pi}). \tag{31}$$

Therefore substituting this value of ζ in our previous expression of $\mathbb{E}(\mathbf{R})$ in equation 26 we get

$$\begin{aligned}\mathbb{E}(\mathbf{R}) &= \mathbf{\Pi} + \frac{1}{2\tau}\mathbf{\Sigma}^{-1}\mathbf{P}'(\mathbf{P\Sigma}^{-1}\mathbf{P}')^{-1}2\tau(\mathbf{Q} - \mathbf{P\Pi}) \\ &= \mathbf{\Pi} + \mathbf{\Sigma}^{-1}\mathbf{P}'(\mathbf{P\Sigma}^{-1}\mathbf{P}')^{-1}(\mathbf{Q} - \mathbf{P\Pi}).\end{aligned}$$

∎

Collorary 4.4.1 *If the investor has no views then*

$$\mathbb{E}(\mathbf{R}) = \mathbf{\Pi} \tag{32}$$

Proof If there are no views then the view matrix $\mathbf{P}$ is a zero $\mathbf{0}$ and

$$\begin{aligned}\mathbb{E}(\mathbf{R}) &= \mathbf{\Pi} + \frac{1}{2\tau}\mathbf{\Sigma}^{-1}\mathbf{P}'(\mathbf{P\Sigma}^{-1}\mathbf{P}')^{-1}2\tau(\mathbf{Q} - \mathbf{P\Pi}) \\ &= \mathbf{\Pi} + \frac{1}{2\tau}\mathbf{\Sigma}^{-1}\mathbf{0}'(\mathbf{0\Sigma}^{-1}\mathbf{0}')^{-1}2\tau(\mathbf{Q} - \mathbf{0}\ \mathbf{\Pi}) \\ &= \mathbf{\Pi}.\end{aligned}$$

∎

4.5 Uncertain Views

The case of uncertain views is slightly different in that we have to include an error term ϵ in the views to model this uncertainty. The optimization problem is to be solved is then

$$\begin{aligned} \min_{\mathbb{E}(\mathbf{R})} \quad & (\mathbb{E}(\mathbf{R}) - \mathbf{\Pi})\tau\mathbf{\Sigma}(\mathbb{E}(\mathbf{R}) - \mathbf{\Pi}) \\ s.t. \quad & \mathbf{P}\mathbb{E}(\mathbf{R}) = \mathbf{Q} + \epsilon. \end{aligned}$$

Theorem 4.6 *The solution* $\mathbb{E}(\mathbf{R})$ *to the above optimization problem, under the assumption of normality has the following property*

$$\mathbb{E}(\mathbf{R})|(\mathbf{P}\mathbb{E}(\mathbf{R}) \sim N(\mu_{\mathbf{BL}}, \mathbf{\Sigma_{BL}}) \tag{33}$$

where

$$\mu_{\mathbf{BL}} = [(\tau\mathbf{\Sigma})^{-1} + \mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}]^{-1}[(\tau\mathbf{\Omega})^{-1}\mathbf{\Pi} + \mathbf{P}'\mathbf{\Omega}\mathbf{Q}] \tag{34}$$

$$\mathbf{\Sigma_{BL}} = [(\tau\mathbf{\Sigma})^{-1} + \mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}]^{-1}. \tag{35}$$

This theorem can be proved in two ways. The first method is by using a Generalized Least Squares (GLS) technique called the mixed estimation method. It was proposed and derived by Henry and Goldberg [Henry and Goldberg Arthur, 1961], thoroughly explained in Theil (1971). This is also the method suggested by [Black and Litterman, 1992].
The second method is by using Bayesian Estimation, and this is the method we are going to utilize in the following proof. We are going to follow closely [Satchell and Scowcroft, 2000]. Although the paper is heavily criticized in [Walters, 2008], it is rich in mathematical intuition and content. We are also going to use ideas from [Salomons, 2007] and [Christodoulakis, 2002].

Proof From the assumptions of normality $A1$ and $A2$ made above we have

$$\mathbb{E}(\mathbf{R}) \sim N(\mathbf{\Pi}, \tau\mathbf{\Sigma}) \tag{36}$$

And

$$\big(\mathbf{P}'\mathbb{E}(\mathbf{R})|\mathbb{E}(\mathbf{R}) = \mathbf{\Pi}\big) \sim N(\mathbf{Q}, \mathbf{\Omega}) \tag{37}$$

Thus the multivariate **probability density functions** (pdf) of these two random variables can be written as

$$pdf(\mathbb{E}(\mathbf{R})) = \frac{1}{\sqrt{2\mathbf{\Pi}^N|\tau\mathbf{\Sigma}|}} Exp\left(-\frac{1}{2}(\mathbb{E}(\mathbf{R}) - \mathbf{\Pi})'(\tau\mathbf{\Sigma})^{-1}(\mathbb{E}(\mathbf{R}) - \mathbf{\Pi})\right) \tag{38}$$

$$pdf(\mathbf{P}\mathbb{E}(\mathbf{R})|\mathbf{\Pi}) = \frac{1}{\sqrt{2\mathbf{\Pi}^K|\mathbf{\Omega}|}} Exp\left(-\frac{1}{2}(\mathbf{P}\mathbb{E}(\mathbf{R}) - \mathbf{Q})'\mathbf{\Omega}^{-1}(\mathbf{P}\mathbb{E}(\mathbf{R}) - \mathbf{Q})\right). \tag{39}$$

In the Black-Litterman model it is assumed that investors formulate their views using knowledge of equilibrium expected returns. The equilibrium returns are considered the prior expected returns and they will then be updated using the views to get the posterior expected returns. Thus by Bayes' Theorem

$$\begin{aligned}
\mathbb{P}(\mathbb{E}(\mathbf{R})|\mathbf{P}\mathbb{E}(\mathbf{R})) &= \frac{\mathbb{P}(\mathbf{P}\mathbb{E}(\mathbf{R})|\mathbf{\Pi})\mathbb{P}(\mathbb{E}(\mathbf{R}))}{\mathbb{P}(\mathbf{P}\mathbb{E}(\mathbf{R}))} \\
&= \frac{1}{\mathbb{P}(\mathbf{P}\mathbb{E}(\mathbf{R}))}\left[\frac{1}{\sqrt{2\mathbf{\Pi}^K|\mathbf{\Omega}|}}exp\left(-\frac{1}{2}(\mathbf{P}\mathbb{E}(\mathbf{R})-\mathbf{Q})'\mathbf{\Omega}^{-1}(\mathbf{P}\mathbb{E}(\mathbf{R})-\mathbf{Q})\right)\right] \\
&\quad \times\left[\frac{1}{\sqrt{2\mathbf{\Pi}^N|\tau\mathbf{\Sigma}|}}exp\left(-\frac{1}{2}\left(\mathbb{E}(\mathbf{R})-\mathbf{\Pi}\right)'\left(\tau\mathbf{\Sigma}\right)^{-1}\left(\mathbb{E}(\mathbf{R})-\mathbf{\Pi}\right)\right)\right] \\
&\propto \mathbb{P}(\mathbf{P}\mathbb{E}(\mathbf{R})|\mathbf{\Pi})\mathbb{P}(\mathbb{E}(\mathbf{R}))
\end{aligned}$$

and so

$$\begin{aligned}
\mathbb{P}(\mathbb{E}(\mathbf{R})|\mathbf{P}\mathbb{E}(\mathbf{R})) &\propto Exp\Big[-\frac{1}{2}\left(\mathbf{P}\mathbb{E}(\mathbf{R})-\mathbf{Q}\right)'\mathbf{\Omega}^{-1}\left(\mathbf{P}\mathbb{E}(\mathbf{R})-\mathbf{Q}\right) \\
&\quad -\frac{1}{2}\left(\mathbb{E}(\mathbf{R}-\mathbf{\Pi})\right)'\left(\tau\mathbf{\Sigma}\right)^{-1}\left(\mathbb{E}(\mathbf{R}-\mathbf{\Pi})\right)\Big] \\
&\propto Exp\left[-\frac{1}{2}\varphi\right]
\end{aligned}$$

where we have written the exponent as φ excluding the $\frac{1}{2}$ coefficient.

$$\varphi = \left(\mathbf{P}\mathbb{E}(\mathbf{R})-\mathbf{Q}\right)'\mathbf{\Omega}^{-1}\left(\mathbf{P}\mathbb{E}(\mathbf{R})-\mathbf{Q}\right)-\left(\mathbf{\Pi}-\mathbb{E}(\mathbf{R})\right)'\left(\tau\mathbf{\Sigma}\right)^{-1}\left(\mathbf{\Pi}-\mathbb{E}(\mathbf{R})\right). \tag{40}$$

Expanding the brackets in φ using matrix algebra we have

$$\begin{aligned}
\varphi &= (\mathbf{P}\mathbb{E}(\mathbf{R}))'\mathbf{\Omega}^{-1}\mathbf{P}\mathbb{E}(\mathbf{R})-(\mathbf{P}\mathbb{E}(\mathbf{R}))'\mathbf{\Omega}^{-1}\mathbf{Q} \\
&\quad -\mathbf{Q}'\mathbf{\Omega}^{-1}(\mathbf{P}\mathbb{E}(\mathbf{R}))+\mathbf{Q}'\mathbf{\Omega}^{-1}\mathbf{Q}-\mathbf{\Pi}'(\tau\mathbf{\Sigma})^{-1}\mathbf{\Pi}-\mathbf{\Pi}'(\tau\mathbf{\Sigma})^{-1}(\mathbb{E}(\mathbf{R})) \\
&\quad -(\mathbb{E}(\mathbf{R}))'(\tau\mathbf{\Sigma})^{-1}\mathbf{\Pi}+(\mathbb{E}(\mathbf{R}))'(\tau\mathbf{\Sigma})^{-1}(\mathbb{E}(\mathbf{R}))).
\end{aligned}$$

The matrices $\mathbf{\Omega}$ and $\mathbf{\Sigma}$ are symmetric, so collecting like terms thus rearranging we have

$$\begin{aligned}
\varphi &= (\mathbf{P}\mathbb{E}(\mathbf{R}))'\mathbf{\Omega}^{-1}\mathbf{P}\mathbb{E}(\mathbf{R})-2\mathbf{Q}'\mathbf{\Omega}^{-1}(\mathbf{P}\mathbb{E}(\mathbf{R}))+\mathbf{Q}'\mathbf{\Omega}^{-1}\mathbf{Q} \\
&\quad -2\mathbf{\Pi}'(\tau\mathbf{\Sigma})^{-1}(\mathbb{E}(\mathbf{R}))+(\mathbb{E}(\mathbf{R}))'(\tau\mathbf{\Sigma})^{-1}(\mathbb{E}(\mathbf{R}))+\mathbf{\Pi}'(\tau\mathbf{\Sigma})^{-1}\mathbf{\Pi}.
\end{aligned}$$

This can be further simplified by factorization to

$$\begin{aligned}
\varphi &= \mathbb{E}(\mathbf{R})'[\mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}+(\tau\mathbf{\Sigma})^{-1}]\mathbb{E}(\mathbf{R}) && (41) \\
&\quad -2[\mathbf{Q}'\mathbf{\Omega}^{-1}\mathbf{P}+\mathbf{\Pi}'(\tau\mathbf{\Sigma})^{-1}]\mathbb{E}(\mathbf{R})+\mathbf{Q}'\mathbf{\Omega}^{-1}\mathbf{Q}+\mathbf{\Pi}'(\tau\mathbf{\Sigma})^{-1}\mathbf{\Pi}. && (42)
\end{aligned}$$

Now Let

$$\begin{aligned}
\mathbf{A} &= \mathbf{Q}'\mathbf{\Omega}^{-1}\mathbf{Q}+\mathbf{\Pi}'(\tau\mathbf{\Sigma})^{-1}\mathbf{\Pi} && (43) \\
\mathbf{C} &= \mathbf{Q}'\mathbf{\Omega}^{-1}\mathbf{P}+\mathbf{\Pi}'(\tau\mathbf{\Sigma})^{-1} && (44) \\
\mathbf{H} &= \mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}+(\tau\mathbf{\Sigma})^{-1}. && (45)
\end{aligned}$$

$\mathbf{H}$ is symmetric, that is $\mathbf{H}=\mathbf{H}'$ because $\mathbf{\Omega}$ and $\mathbf{\Sigma}$ are.
The exponent φ can be written as

$$
\begin{aligned}
\varphi &= \mathbb{E}(\mathbf{R})'\mathbf{H}\mathbb{E}(\mathbf{R}) - 2\mathbf{C}'\mathbb{E}(\mathbf{R}) + \mathbf{A} \\
&= \mathbb{E}(\mathbf{R})'\mathbf{H}'\mathbb{E}(\mathbf{R}) - 2\mathbf{C}'\mathbb{E}(\mathbf{R}) + \mathbf{A} \\
&= (\mathbb{E}(\mathbf{R})\mathbf{H})'\mathbf{H}^{-1}\mathbf{H}\mathbb{E}(\mathbf{R}) - 2\mathbf{C}'\mathbf{H}^{-1}\mathbf{H}\mathbb{E}(\mathbf{R}) + \mathbf{A} \\
&= (\mathbb{E}(\mathbf{R})\mathbf{H})'\mathbf{H}^{-1}(\mathbf{H}\mathbb{E}(\mathbf{R})) - 2\mathbf{C}'\mathbf{H}^{-1}(\mathbf{H}\mathbb{E}(\mathbf{R})) + \mathbf{C}'\mathbf{H}^{-1}\mathbf{C} - \mathbf{C}'\mathbf{H}^{-1}\mathbf{C} + \mathbf{A} \\
&= (\mathbb{E}(\mathbf{R}) - \mathbf{H}^{-1}\mathbf{C})'\mathbf{H}(\mathbb{E}(\mathbf{R}) - \mathbf{H}^{-1}\mathbf{C}) + A - \mathbf{C}'\mathbf{H}^{-1}\mathbf{C},
\end{aligned}
$$

where we have added and subtracted $\mathbf{C}'\mathbf{H}^{-1}\mathbf{C}$ in the third line and multiplied the first two terms in the first equation by $I = \mathbf{H}^{-1}\mathbf{H}$.

Thus the posterior density satisfies

$$
\begin{aligned}
\mathbb{P}(\mathbb{E}(\mathbf{R})|\mathbf{P}\mathbb{E}(\mathbf{R})) &= const \times (Exp(\mathbf{A} - \mathbf{C}'\mathbf{H}^{-1}\mathbf{C})) \times Exp\left[-\frac{1}{2}(\mathbb{E}(\mathbf{R}) - \mathbf{H}^{-1}\mathbf{C})'\mathbf{H}(\mathbb{E}(\mathbf{R}) - \mathbf{H}^{-1}\mathbf{C})\right] \\
&= const \times Exp[-\frac{1}{2}(\mathbb{E}(\mathbf{R}) - \mathbf{H}^{-1}\mathbf{C})'\mathbf{H}(\mathbb{E}(\mathbf{R}) - \mathbf{H}^{-1}\mathbf{C})] \\
&\propto Exp\left[-\frac{1}{2}(\mathbb{E}(\mathbf{R}) - \mathbf{H}^{-1}\mathbf{C})'\mathbf{H}(\mathbb{E}(\mathbf{R}) - \mathbf{H}^{-1}\mathbf{C})\right],
\end{aligned}
$$

where *const* is an expression of constants.
Hence $\mathbb{E}(\mathbf{R})|\mathbf{P}\mathbb{E}(\mathbf{R})$ is multivariate normally distributed with mean $\mathbf{H}^{-1}\mathbf{C}$ and covariance matrix $\mathbf{H}^{-1}$. That is

$$
\mathbb{E}(\mathbf{R})|(\mathbf{P}\mathbb{E}(\mathbf{R}) \sim N(\mu_{\mathbf{BL}}, \mathbf{\Sigma}_{\mathbf{BL}}) \tag{46}
$$

where

$$
\mu_{\mathbf{BL}} = \mathbf{H}^{-1}\mathbf{C} = [(\tau\mathbf{\Sigma})^{-1} + \mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}]^{-1}[(\tau\mathbf{\Omega})^{-1}\mathbf{\Pi} + \mathbf{P}'\mathbf{\Omega}\mathbf{Q}], \quad \textit{and} \tag{47}
$$

$$
\mathbf{\Sigma}_{\mathbf{BL}} = \mathbf{H}^{-1} = [(\tau\mathbf{\Sigma})^{-1} + \mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}]^{-1}. \tag{48}
$$

Which completes the proof. ∎

A detailed proof with explanations can be found in [Walters, 2008] and corresponding computer code in MATLAB, R, C++ and a JAVA applet can be found on the website maintained by Jay Walters, [Walters, 2008] , www.blacklitterman.org.

The posterior weights of each of the assets can then be obtained by optimization replacing the prior vector of expected returns with the new posterior returns and the prior variance covariance matrix with the corresponding posterior variance covariance matrix.

5 Stable Distributions

As mentioned in a previous section the normal distribution and other common probability distributions used in practice fail to account or explain the heavy tails and massive skewness found in empirical data. The α-stable distributions are a more general class of distributions which allows for skewness and heavy tails in financial modeling. Some of the common distributions that are widely used in practice, like the Gaussian, Cauchy and Levy are part of this larger family of distributions. The α-stable distributions were first used by Paul Levy to investigate the the **Generalized Central Limit Theorem** for independent and identically distributed random variables.

Their use in finance and economics was advocated in the 1960's by [Mandelbrot, 1963], [Fama, 1965] and recently [Embrechts et al., 1999]. They have also found application in engineering and telecommunications as in [Zolotarev, 1986], [Willinger et al., 1998], and [Samorodnitsky and Taqqu, 1994]. In most cases in practice [Nolan, 1998], α-stable distributions have been found to fit suspected heavy tailed and skewed data very well.

5.1 Univariate Stable Distributions

The α-stable distributions are described by four parameters, α called the *index of stability* or the characteristic exponent, β the *skewness parameter*, γ the *scale parameter* and δ the *location parameter*. There are many different ways of defining α-stable distributions called the S parameterizations, but in this thesis we are going to concentrate on the $S0$ parameterization used in [Nolan, 1999b]. We are going to follow the notation used in [Rimmer and Nolan, 2005] and [Kring et al., 2008], with some modifications

Definition 5.2 *Let $X_1, X_2, X_3, \ldots, X_n$ and X be sequence of independent and identically distributed random variables. If there exists real numbers $c_i > 0$ and $d_i \in \mathbb{R}$ such that, for all $i = 1, 2, 3, \ldots, n$,*

$$X_1 + X_2 + X_3 + \ldots + X_i \stackrel{d}{=} c_i X + d_i \tag{49}$$

then the random variable X is said to be α-stable distributed or in some literature just stable distributed. X is strictly stable if for all $i = 1, 2, 3, \ldots, n$, $d_i = 0$ and symmetric stable if $X \stackrel{d}{=} -X$. It is written $X \sim S(\alpha, \beta, \gamma, \delta)$ to justify the dependence on the four parameters.

The definition can be interpreted as to say $X_1 + X_2 + X_3 + \ldots + X_n$ has the same type of distribution as X.

Often this definition is not that useful in estimation and simulations, and a more widely used definition is through their characteristic function.

Definition 5.3 *A random variable is said to be α-stable, $X \sim S(\alpha, \beta, \gamma, \delta)$, if and only if X has the characteristic function*

$$\varphi_X(\mathbf{t}) = \mathbb{E}(exp(iXt)) = \begin{cases} exp(-\gamma^\alpha |t|^\alpha [1 - i\beta(\tan\frac{\pi\alpha}{2})(sign(t))] + i\delta t), & if \quad \alpha \neq 1 \\ & ,\alpha \in [0,2] \\ exp(-\gamma|t|[1 + i\beta\frac{\pi\alpha}{2}(sign(t))\ln|t|] + i\delta t), & if \quad \alpha = 1. \end{cases} \tag{50}$$

where $\alpha \in (0, 2], \beta \in [-1, 1], \gamma \in (0, \infty)$ and $\delta \in \mathbb{R}$.
If $\beta = 0$ then the distribution is called symmetric stable, if $\beta > 0$ it is called right-skewed and if $\beta < 0$ it is called left-skewed.

These two definitions are equivalent and the proof for their equivalence can be found in [Samorodnitsky and Taqqu, 1994].

The probability densities of α-stable random variables exists and they are found by inverting the above characteristic function.

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \varphi_X(t) dt \tag{51}$$

Evaluating the integral on the right hand side in the above equation is not a straight forward exercise to do in practice. In most cases analytical closed formulas for probability densities and distributions functions are unknown. Numerical methods like the Fast Fourier Transform(FFT) method,[Nolan, 1999b], [Nolan, 1999a] and numerical integration [Rachev and Mittnik, 2000] are used to approximate the density functions of the α-stable distributions. Cases where the closed formulas are known, include the Gaussian(Normal) Distribution, $\alpha = 2$, the Cauchy Distribution, $\alpha = 1$ and the Levy distribution, $\alpha = \frac{1}{2}$. Having no closed formula for the PDFs and CDFs as noted in [Nolan, 1998], should not discourage the use of the α-stable distributions in practice. The same is true for the normal distribution which has no known closed formula for standard normal distributions function(CDF) but there are tables and computer algorithms to evaluate it for various range of the parameters.

Moments of any distribution are very important quantities, they summarize its properties and behavior. For most univariate probability distributions the first two moments are enough to describe them. With the stable distribution not all moments exists.This is because due to heavy tails, the integral representation $\mathbb{E}(X^p) = \int_{-\infty}^{\infty} x^p f(x)dx$ for the expectation may not be bounded and might diverge. Thus as noted in [Nolan, 1999b] it is quite useful sometimes to use what are called **fractional absolute moments** , $\mathbb{E}(|X|^p) = \int_{-\infty}^{\infty} |x|^p f(x)dx$ where $p \in \mathbb{R}$. Closed formulas for the values of p where the fractional absolute moment exists, can be found in [Zolotarev, 1986].

5.4 Multivariate Stable Distributions

The multivariate α-stable distributions can be defined by extending the univariate definitions in a natural way. As is in the univariate case the multivariate Gaussian distribution is a special case of the multivariate stable distribution and any linear combination of multivariate stable random variables is multivariate stable distributed. The pdfs and cdfs are also not known in multivariate cases and one has to work with characteristic functions as in the univariate case.

The representation of the characteristic function in the multivariate case includes a finite measure on the unit sphere called the **spectral measure** which describes the dependence structure of the different α-stable random vectors. The spectral measure is very difficult to estimate even in low dimensions and attempts to estimate it have been made in [Nolan et al., 2001] and [Nolan, 1998]. This makes multivariate stable distributions difficult to use in financial modeling, see [Kring et al., 2008].

Thus in order to do serious practical work, one has to confine themselves to certain subclasses of the multivariate α-stable distributions where the spectral measure has an amicable and easier representation which can be estimated without problems. One such subclass, is the **sub-Gaussian** α-stable family of distributions obtained by multiplying a Gaussian vector with a α-stable random vector $W^{1/2}$, where $W \sim S(\alpha, \beta, \gamma, \delta)$ and $\beta > 0$.

The two definitions in the univariate case can also be restated in the multivariate case in the following way.

Definition 5.5 *A random vector $\mathbf{X}' = (X_1, X_2, X_3, \ldots, X_n)$ is said to be an α-stable random vector in $\mathbb{R}^n$ if for any real numbers $a \in \mathbb{R}^+, b \in \mathbb{R}^+$ there exists a vector of constants $\mathbf{d} \in \mathbb{R}^n$ and a real number $c \in \mathbb{R}^+$ such that*

$$a\mathbf{X}^{(1)} + b\mathbf{X}^{(2)} \stackrel{d}{=} c\mathbf{X} + \mathbf{d} \tag{52}$$

where $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are independent copies of $\mathbf{X}$. Similarly as in the univariate case an α-stable random vector $\mathbf{X}$ is called symmetric stable if

$$\mathbb{P}(\mathbf{X} \in \mathbf{A}) = \mathbb{P}(-\mathbf{X} \in \mathbf{A}) \tag{53}$$

for all Borel sets $\mathbf{A} \in \mathbb{R}^n$

Theorem 5.6 *The random vector $\mathbf{X}' = (X_1, X_2, X_3, \ldots, X_n)$ is an α-stable random vector in $\mathbb{R}^n$ if there exist a unique finite measure Γ, called the spectral measure, on the unit sphere S^N and a unique vector $\delta \in \mathbb{R}^n$ such that*

$$\begin{aligned}\varphi_{\mathbf{X}}(\mathbf{t}) &= \mathbb{E}(exp(i\mathbf{t}'\mathbf{X})) \\ &= \begin{cases} Exp\{-\int_{S^N} |t,s|^\alpha [1 - i(\tan\frac{\pi\alpha}{2})(sign(t,s))]\Gamma(ds) + i(t,\delta)\}, & if \quad \alpha \neq 1 \\ & , \alpha \in [0,2] \\ Exp\{-\int_{S^N} |t,s|(1 + i(\frac{2}{\pi})(sign(t,s)) \ln|(t,s)|)\Gamma(ds) + i(t,\delta)\}, & if \quad \alpha = 1. \end{cases}\end{aligned}$$

5.7 Sub-Gaussian α-Stable Distributions

The sub-Gaussian α-stable Distribution class is one of the subclasses of the α-stable distributions where the spectral measure becomes easier to estimate and makes it easier to fit to data. The following definitions are due to [Kring et al., 2008],

Definition 5.8 *Let $\mathbf{Z}$ be a Gaussian random vector with mean zero and variance covariance matrix $\mathbf{\Sigma}$ and let $W \in S(\frac{\alpha}{2}, (\cos\frac{\pi\alpha}{4})^{2/\pi}, 1, 0)$ be a totally skewed random variable independent of Z. Then the random vector*

$$\mathbf{X} = \delta + \mathbf{Z}\sqrt{W} \tag{54}$$

is said to be a sub-Gaussian α-stable random vector. The distribution of $\mathbf{X}$ is called the α-stable sub-Gaussian distribution. The matrix $\mathbf{\Sigma}$ is called the **dispersion matrix**.

The Gaussian random vector passes on its dependence structure to the α-stable sub-Gaussian random vector there by making the spectral measure easier to estimate.

Theorem 5.9 *An α-stable sub-Gaussian random vector with location parameter $\delta \in \mathbb{R}^n$ has the characteristic function*

$$\mathbb{E}(e^{i\mathbf{t}'\mathbf{X}}) = e^{i\mathbf{t}'\delta - (\frac{1}{2}\mathbf{t}'\mathbf{\Sigma}\mathbf{t})^{\frac{\alpha}{2}}} \tag{55}$$

where $\Sigma_{ij} = \mathbb{E}Z_iZ_j, i,j = 1,2,3,\ldots,n$ are the covariances of the underlying Gaussian random vectors $(Z_1, Z_2, \ldots, Z_n$.

Theorem 5.10 *Let* $\mathbf{X} \in \mathbb{R}^n$ *be an* α*-stable sub-Gaussian random vector with location parameter* $\delta \in \mathbb{R}^n$ *and dispersion matrix* $\mathbf{\Sigma}$*. Then for all* $a \in \mathbb{R}^n$*, we have* $a'\mathbf{X} \sim S(\alpha, \beta(a), \gamma(a), \delta(a)$ *where*

(i)

$$\gamma(a) = \left(\frac{1}{2}\mathbf{a}'\mathbf{\Sigma}\mathbf{a}\right)^{\frac{1}{2}}$$

(ii)

$$\beta(a) = 0$$

(iii)

$$\delta(a) = \mathbf{a}'\delta$$

Proof The proof which is given in [Kring et al., 2008] and we explain it here.
The distribution function of $a'\mathbf{X}$ is determined by its characteristic function

$$\begin{aligned}
\varphi_{\mathbf{X}}(\mathbf{a}'\mathbf{t}) &= \mathbb{E}(e^{i(\mathbf{ta})'\mathbf{X}}) \\
&= e^{i(\mathbf{ta})'\delta} e^{-(|\frac{1}{2}(\mathbf{ta})'\mathbf{\Sigma}\mathbf{ta}|)^{\frac{\alpha}{2}}} \\
&= e^{i(\mathbf{ta})'\delta} e^{-(|\frac{1}{2}\mathbf{t^2}\mathbf{a}'\mathbf{\Sigma}\mathbf{ta}|)^{\frac{\alpha}{2}}} \\
&= e^{(-|\mathbf{t}|^{\alpha}|(\frac{1}{2}(\mathbf{a})'\mathbf{\Sigma}\mathbf{a})^{\frac{1}{2}}|^{\alpha} + i(\mathbf{ta})'\delta)}
\end{aligned}$$

Let $\gamma(a) = \left(\frac{1}{2}a'\mathbf{\Sigma}a\right)^{\frac{1}{2}}$,$\beta(a) = 0$,$\delta(a) = a'\delta$, then $\forall \in \mathbb{R}$ we must have

$$\begin{aligned}
\varphi_{\mathbf{a}'\mathbf{X}}(\mathbf{t}) &= e^{(-|\mathbf{t}|^{\alpha}(\gamma(a)^{\alpha}[1-0] + i\mathbf{t}\delta(\mathbf{a}))} \\
&= Exp(-\gamma(a)^{\alpha}|t|^{\alpha}\left[1 - i\beta(a)(\tan\frac{\pi\alpha}{2})(sign(t))\right] + i\delta(a)t)
\end{aligned}$$

Which is the characteristic function of an α-stable random vector with parameters $\alpha, \gamma(a), \beta(a), \gamma(a)$ and δ. ∎

5.11 Elliptical Distributions and Spherical Random Variables

There is a large volume of literature on *elliptical* distributions since they are naturally an extension of the normal distribution, which again is a special case of the class.

Definition 5.12 *Let* $\mathbf{X} = (X_1, X_2, X_3, \ldots, X_n)$ *be a random vector then if for every orthogonal matrix* $U \in \mathbb{R}^{n\times n}$ *we have*

$$UX \stackrel{d}{=} X \tag{56}$$

then $\mathbf{X}$ *is said to have a spherical distribution.*

Definition 5.13 *A random vector* $\mathbf{X} = (X_1, X_2, X_3, \ldots, X_n)$ *is said to be elliptically distributed if there exists a matrix of constants* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *and vector of constants* $\delta \in \mathbb{R}^n$ *such that*

$$\mathbf{X} = \mathbf{A}Y + \delta \tag{57}$$

where Y *is a spherical random variable.*

As noted in [Kring et al., 2008] elliptical random variables can be constructed by multivariate affine transformations of spherical random variables. The following theorem gives a characterization of spherical distributions and so of elliptical distributions also.

Theorem 5.14 *A random variable* $\mathbf{X}$ *is spherical if and only if there exists a scalar valued function* φ *such that for all* $\mathbf{t} \in \mathbb{R}^n$,

$$\mathbb{E}(e^{i\mathbf{t}'\mathbf{X}}) = \varphi(\mathbf{t}'\mathbf{t}) = \varphi(t_1^2 + t_2^2 + \ldots + t_n^2) \tag{58}$$

and if and only if for all $\mathbf{a} \in \mathbb{R}^n$,

$$\mathbf{a}'\mathbf{X} = ||a||X \tag{59}$$

where $||a||$ *is the norm which in the Euclidean Space is the Euclidean norm.*

The proofs for these results and others are found in [Rachev, 2003].
The function φ is called the **characteristic** generator of the spherical random variable and the notation used is $X \in S_n(\varphi)$ since the characteristic function determines the distribution of a random variable.

Collorary 5.14.1 *Let* Y *be a spherical random variable with the characteristic generator* φ *and let* $\mathbf{A}$ *be a matrix of constants. Consider an elliptically distributed random variable* $\mathbf{X} = \delta + \mathbf{A}Y$ *then the characteristic function of* $\mathbf{X}$ *is given by*

$$\phi_X(\mathbf{t}) = \mathbb{E}(e^{i\mathbf{t}'\mathbf{X}}) = e^{i\mathbf{t}'\delta}\varphi(\mathbf{t}'\mathbf{\Sigma}\mathbf{t}) \tag{60}$$

where $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}'$,

Proof The characteristic function of $\mathbf{X}$ is defined as

$$\begin{aligned}
\phi_X(\mathbf{t}) &= \mathbb{E}(e^{i\mathbf{t}'\mathbf{X}}) \\
&= \mathbb{E}(e^{i\mathbf{t}'(\delta+\mathbf{AY})}) \\
&= e^{i\mathbf{t}'\delta}\mathbb{E}(e^{i(\mathbf{A}'\mathbf{t})'Y}) \\
&= e^{i\mathbf{t}'\delta}\varphi(((\mathbf{A}'\mathbf{t}))'((\mathbf{A}'\mathbf{t}))) \\
&= e^{i\mathbf{t}'\delta}\varphi(\mathbf{t}'\mathbf{A}\mathbf{A}'\mathbf{t}) \\
&= e^{i\mathbf{t}'\delta}\varphi(\mathbf{t}'\mathbf{\Sigma}\mathbf{t}),
\end{aligned}$$

which is the characteristic function of $\mathbf{X}$ given in the theorem.

The characteristic function determines the distribution of a random variable, so an elliptical distribution is written as $X \sim E_d(\delta, \Sigma, \varphi)$. The parameter δ remains the location parameter as in the general stable distribution and Σ is also called the dispersion matrix. The first and second moments do not necessarily exists, but if they do the location parameter is the mean and the dispersion matrix is then the covariance matrix.

In order that the elliptical distribution representation be unique it is necessary that to place restrictions on the dispersion matrix $\mathbf{\Sigma}$. From [Kring et al., 2008] it is stated that the necessary condition which must hold is that $det(\mathbf{\Sigma}) = 1$.

5.15 Some Properties of Elliptical Distributions

The elliptical distributions have some of the nice properties of the normal distribution.

Theorem 5.16 *Let* $\mathbf{X}$ *be an elliptical distributed random vector* $\mathbf{X} \sim E_d(\delta, \Sigma, \varphi)$ *then any affine linear combination formed from the* $\mathbf{X}$ *remains elliptical and has the same characteristic generator* φ*, that is if* $\mathbf{B} \in \mathbb{R}^{k \times n}$ *and* $\mathbf{b} \in \mathbb{R}^k$ *then*

$$\mathbf{BX} + b \sim E_k(\mathbf{B}\delta + \mathbf{b}, \mathbf{B\Sigma B}', \varphi) \tag{61}$$

Proof The result when $\mathbf{B}$ is a vector is given in [Fang et al., 1990] and a matrix extension is given in [Landsman and Valdez, 2003]. The characteristic function of $\mathbf{BX} + b$ is given by

$$\begin{aligned}
\phi_{\mathbf{BX}+\mathbf{b}}(\mathbf{t}) &= \mathbb{E}(e^{i\mathbf{t}'(\mathbf{BX}+\mathbf{b})}) \\
&= \mathbb{E}(e^{i\mathbf{t}'(\mathbf{B}(\delta+\mathbf{AY})+\mathbf{b})}) \\
&= e^{i\mathbf{t}'(\mathbf{B}\delta+\mathbf{b})}\mathbb{E}(e^{i(\mathbf{BA}'\mathbf{t})'Y}) \\
&= e^{i\mathbf{t}'(\mathbf{B}\delta+\mathbf{b})}\varphi(((\mathbf{BAt}))'((\mathbf{B}'\mathbf{A}'\mathbf{t}))) \\
&= e^{i\mathbf{t}'(\mathbf{B}\delta+\mathbf{b})}\varphi(\mathbf{t}'\mathbf{BAA}'\mathbf{B}'\mathbf{t}) \\
&= e^{i\mathbf{t}'(\mathbf{B}\delta+\mathbf{b})}\varphi(\mathbf{t}'\mathbf{B\Sigma B}'\mathbf{t})
\end{aligned}$$

and thus as in the corollary we have

$$\mathbf{BX} + \mathbf{b} \sim E_k(\mathbf{B}\delta + \mathbf{b}, \mathbf{B\Sigma B}', \varphi) \tag{62}$$

6 Alternative Measures of Risk

In portfolio optimization the measure of risk used in the normal distribution case is the variance or in the multivariate case the covariance matrix. When we extend to non-normal distributions or sub-Gaussian α-stable distributions we have to consider other risk measures. Other measures of risk than the variance have been proposed in the literature, [Rockafellar and Uryasev, 2000], [Rachev et al., 2008],and others. A measure of risk is useful if it satisfies some certain minimum properties called the **axioms of coherence** proposed in [Artzner et al., 1999]. The risk measure must satisfy the properties of translation invariance, sub-additivity, positive homogeneity and monotonicity.

Axioms of coherence can be summarized as to say

1. **Translation Invariance**, Adding a constant to a loss from an investment does not change the risk
2. **Sub-additivity**, The risk of a portfolio is at most the combined risks of the individual constituent investments.
3. **Positive Homogeneity**, If you multiply an investment by a factor, the risk increases by this factor.
4. **Monotonicity**, Larger losses translate into higher risk

The two most commonly used measures are the Value at Risk(VaR) and the Conditional Value at Risk(CVaR). These two measures satisfy the above axioms of coherence except for VaR which does not satisfy the axiom of sub-additivity. Thus VaR is not a coherent measure of risk. Let us define the VaR and CVaR of a portfolio.

Definition 6.1 Value at Risk *Let R be the return of a portfolio that can be realized in a particular investment horizon. The Value At Risk(VaR) of the portfolio is defined implicitly as that quantity that satisfies the following*

$$\mathbb{P}(R < -VaR_\epsilon(R)) = 1 - \epsilon \tag{63}$$

for some level of significance ϵ.

Definition 6.2 The Conditional Value at Risk *The conditional value at risk(CVaR) of a portfolio is defined as*

$$CVaR_\epsilon(R) = \mathbb{E}(R|R < VaR_\epsilon(R)) \tag{64}$$

for some level of significance ϵ.

These definitions of VaR and CVaR using the portfolio loss distribution, can be found in [Rachev et al., 2008]. Although returns of most financial assets are not normally distributed with fat tails and excess kurtosis they can be described by sub-Gaussian α-stable distributions. In the case of assets returns with sub-Gaussian α-stable distributions it is stated in [Rockafellar and Uryasev, 2000], [Giacometti et al., 2007] and explained further in [Rachev et al., 2008] that CVaR has the following representation.

Theorem 6.3 *Let $\mathbf{X}$ be a sub-Gaussian α-stable random vector with location vector μ. Let Y be an α-stable spherical random variable with skewness parameter β, i.e $Y \sim S(\alpha, \beta, 1, 0)$ such that*

$$CVaR_\epsilon(Y) = \frac{1}{p}\bar{G}(\frac{1}{2}y_q^2) \tag{65}$$

$$\bar{G} = \begin{cases} G(\infty) - G(x),, & if \quad G(\infty) < \infty \\ 0, & if \quad G(\infty) = \infty, \end{cases} \tag{66}$$

$G(x) = \int_0^x g_1(2u)du, y_q$ is the q-quantile satisfying $F_y(y_q)) = \int_{-\infty}^{y_q} g_1(u^2)du = q$ and $q = 1 - p$.

Let $\mathbf{w} \in \mathbb{R}^n$, and $\mathbf{X'w}$ be a random variable with dispersion matrix $\mathbf{V}$. Then we must have

$$CVaR_\epsilon(\mathbf{X'w}) = \sqrt{\mathbf{w'Vw}}CVaR_\epsilon(Y) - \mathbf{w'}\mu. \tag{67}$$

The theorem is given and proved in [Stoyanov et al., 2006].

Tabulated values of CVaR for various levels of significance and various parameters α and β can also be found in [Stoyanov et al., 2006]. We explain how to compute the VaR of a spherical α-stable random variable in the empirical analysis section.

A useful property stated in [Giacometti et al., 2007] about the α-stable distributions is that

$$CVaR_\epsilon(\gamma\mathbf{X} + \mu) = \gamma CVaR_\epsilon(\mathbf{X}) - \mu \tag{68}$$

where

$$\gamma\mathbf{X} + \mu \sim S(\alpha, \beta, \gamma, \mu). \tag{69}$$

This property also hold for VaR.

Thus the optimization problems given earlier for the variance can be reformulated now using the measures of risk here defined, the VaR and the CVaR as

$$\max_{\mathbf{w}} \mathbb{E}(U(\mathbf{w})) = \max_{\mathbf{w}} \left(\mathbf{w}'\mathbf{\Pi} - \frac{\lambda}{2}\rho(\mathbf{R}'\mathbf{w}) \right). \tag{70}$$

where $\rho(\mathbf{R}'\mathbf{w})$ is the appropriate measure of risk under consideration VaR or CVaR and λ is the coefficient of risk aversion see [Giacometti et al., 2007].

6.4 Mean VaR optimization

For the case of VaR and making use of the properties stated above, it follows that the optimization problem is

$$\begin{aligned} \max_{\mathbf{w}} \mathbb{E}(U(\mathbf{w})) &= \max_{\mathbf{w}} \left(\mathbf{w}'\mathbf{\Pi} - \frac{\lambda}{2} VaR_{\epsilon}(\mathbf{R}'\mathbf{w}) \right) \\ &= \max_{\mathbf{w}} \left[\mathbf{w}'\mathbf{\Pi} - \frac{\lambda}{2} \left(\sqrt{\mathbf{w}'V\mathbf{w}} VaR_{\epsilon}(Y) - \mathbf{w}'\mu \right) \right]. \end{aligned}$$

The Lagrangian to the problem is

$$L(\mathbf{w}, \lambda) = \mathbf{w}'\mathbf{\Pi} - \frac{\lambda}{2} \left(\sqrt{\mathbf{w}'V\mathbf{w}} VaR_{\epsilon}(Y) - \mathbf{w}'\mu \right). \tag{71}$$

Applying the first order conditions using matrix differentiation rules we have

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{w}} &= \mathbf{\Pi} - \frac{\lambda}{2} \left(\frac{1}{2} (\mathbf{w}V + \mathbf{w}V)(\mathbf{w}'V\mathbf{w})^{-\frac{1}{2}} (VaR_{\epsilon}(Y) - \mu \right) = 0 \\ \Rightarrow \quad & \mathbf{\Pi} - \frac{\lambda}{2} \left(\frac{1}{2} \frac{2\mathbf{w}V}{\sqrt{\mathbf{w}'V\mathbf{w}}} VaR_{\epsilon}(Y) - \mu \right) = 0. \end{aligned}$$

Thus the equilibrium returns are then given by

$$\mathbf{\Pi} = \frac{\lambda}{2} \left(\frac{\mathbf{w}V}{\sqrt{\mathbf{w}'V\mathbf{w}}} VaR_{\epsilon}(Y) - \mu \right) \tag{72}$$

6.5 Mean CVaR optimization

And similarly for the case of CVaR, we have, and again making use of the properties stated earlier

$$\begin{aligned} \max_{\mathbf{w}} \mathbb{E}(U(\mathbf{w})) &= \max_{\mathbf{w}} \left(\mathbf{w}'\mathbf{\Pi} - \frac{\lambda}{2} CVaR_{\epsilon}(\mathbf{R}'\mathbf{w}) \right) \\ &= \max_{\mathbf{w}} \left[\mathbf{w}'\mathbf{\Pi} - \frac{\lambda}{2} \left(\sqrt{\mathbf{w}'V\mathbf{w}} CVaR_{\epsilon}(Y) - \mathbf{w}'\mu \right) \right]. \end{aligned}$$

The Lagrangian is

$$L(\mathbf{w}, \lambda) = \mathbf{w}'\mathbf{\Pi} - \frac{\lambda}{2} \left(\sqrt{\mathbf{w}'V\mathbf{w}} CVaR_{\epsilon}(Y) - \mathbf{w}'\mu \right) \tag{73}$$

Applying the first order conditions we have

$$\begin{aligned}\frac{\partial L}{\partial \mathbf{w}} &= \mathbf{\Pi} - \frac{\lambda}{2}\left(\frac{1}{2}(\mathbf{w}V + \mathbf{w}V)(\mathbf{w}'V\mathbf{w})^{-\frac{1}{2}}(CVaR_{\epsilon}(Y) - \mu\right) = 0 \\ \Rightarrow & \quad \mathbf{\Pi} - \frac{\lambda}{2}\left(\frac{1}{2}\frac{2\mathbf{w}V}{\sqrt{\mathbf{w}'V\mathbf{w})}}CVaR_{\epsilon}(Y) - \mu\right) = 0.\end{aligned}$$

Thus

$$\mathbf{\Pi} = \frac{\lambda}{2}\left(\frac{\mathbf{w}V}{\sqrt{\mathbf{w}'V\mathbf{w}}}CVaR_{\epsilon}(Y) - \mu\right). \tag{74}$$

The equilibrium returns given by the above expressions can then be used in the Black Litterman model as prior expected returns. These together with the views will results in returns that reflects the investors' views for the three different cases the normal distribution with variance as a risk measure, α-stable distribution with VaR as a risk measure and α-stable distributions with CVaR as a risk measure.

7 Application to The South African Market

The data used in the empirical analysis are the monthly closing stock prices of the top ten constituent stocks of the RAFI Top 40 Index on the Johannesburg Stock Exchange. The stocks were ranked by their weight in the Index according to a Johannesburg fact sheet released on 30 August 2010 on jse.co.za. The rationale behind taking this index is that it has been shown historically that this index tracks the Johannesburg All Share Index very well in terms of risk and return and it would make a good representation of the South African Market. Only the top ten stocks were used in our portfolio allocation problem. This is reasonable from a portfolio management point of view since the average number of stocks which are held by most South African portfolio managers is 15. The extension to include all stocks in the index can be carried out in a natural way.

Since the RAFI index is constructed relative to the Johannesburg Stock Exchange All Share Index (ALSI) we use the log returns of the ALSI as returns of our market portfolio. We also use the South African Government Bond(R157) rates as to represent the risk free interest on the South African market . This is in line with what is done in practice. The period considered for parameter estimation is from 01/01/2000 to 31/12/2010. The month of January 2011 is used as a hypothetical investment horizon to test the performance of the portfolio formed from the ten stocks. The data was downloaded from DataStream and the logarithmic returns for each stock were calculated in Excel by taking the logarithm of the current price divided by the previous price. Let P_{t-1} be the previous closing price of the stock and P_t the current price, the log return R_t is calculated as

$$R_t = \log\left(\frac{P_t}{P_{t-1}}\right). \tag{75}$$

All the returns for all the stocks were collected into a matrix of returns for analysis.

7.1 Exploratory Data Analysis

All the exploratory data analysis was done in R. The returns data was subjected to normality tests using the Bera -Jarques Test. We chose the Bera -Jarques Test because its test statistic is constructed using the parameters of skewness and kurtosis which is appropriate for this investigation. The calculations were done using the bjtest function in R-package fBasics. The summary statistics, mean, variance, skewness, kurtosis and variance were also calculated in R and the results placed in Table16.

We found out from the normality tests that we rejected the assumption of normality in 21 of the 40 stocks in the Top 40 Index at 5% significance level. This is slightly above half and conforms with our suspicion that stock returns are not generally normally distributed. In almost all of the stocks the skewness parameter was not zero which also evidence of non-normality since the skewness of a normal distribution is zero.

After we were convinced that the returns were not normally distributed we went on to try and fit the α-stable distributions for each of the stock returns series. Generally the two methods that can be used to fit the α-stable distributions to empirical data are the maximum likelihood method, [Rachev and Mittnik, 2000] and the quantile method of [McCulloch, 1986]. The methods involves fitting discrete points of the probability density function using numerical methods and the characteristic function, and then interpolating between points. In the maximum likelihood method these discrete points are used to construct the likelihood function which can then be optimized using a relevant optimization algorithm. More on how to do this can be found in [Nolan, 1998] and [Nolan, 2006] and [Rachev and Mittnik, 2000]. In R, all these two methods can be implemented using the package fBasics. We used the maximum likelihood method in our estimation. All the top 40 stocks were fitted to the α-stable distributions and the parameters collected and put in Table 17.

How flexible the α-stable distributions are in modeling the stock returns can be seen from the two examples BHP Billiton Figure 1 and Richemont Figure2. In Figure 1, we can see that the BHP stock returns are skewed to the left. Fitting a symmetric normal distribution to such data would have been a huge misrepresentation of this skewness. The α-stable distribution fitted captures this skewness such that all the returns are represented in the fit and the left fat tail of the returns is also captured.

The same happens with Richemont which has a right fat tail and right skewed returns. The α-stable distribution properly models these features so that the returns are fully fitted by the distribution. We did such plots for all of the ten stocks used in our analysis and found that the α-stable distributions were able to capture all the basic properties of all the stock returns.

In the Black Litterman implementation we require the single multivariate index of stability α for the estimation of the dispersion matrix , the VaR and CVaR of the α-stable distributions. According to [Kring et al., 2008] this parameter α can be estimated as the average of the constituent univariate random variables. That is

$$\tilde{\alpha} = \frac{1}{N} \sum_{k=1}^{N} \alpha_k \tag{76}$$

For all of the top 40 stocks for the period under consideration we found that the average α was 1.9035. This is the α that we use throughout in this analysis. A different α to use would have been the average α of the top ten stocks.

7.2 Estimation of inputs to Black Litterman

Our portfolio consisted of the RAFI Index top ten stocks mentioned earlier and the vector of expected returns of these stocks was used in the estimation of the inputs to the Black Litterman Model. The average return for each stock returns was calculated in R and placed into a vector,$E(\mathbf{R})$ and this was used as a sample estimate for the mean vector for both the normal distribution and the α-stable distribution. The sample mean as mentioned in [Kring et al., 2008] has nice large sample properties as a good estimator for the population mean.

7.2.1 The Equilibrium Returns

As noted earlier, there are two stages followed in the implementation of the Black Litterman model. The estimation of equilibrium market returns and then estimation of the market implied view returns. We start with the equilibrium returns. The equilibrium returns were derived earlier as the results from the following three formulas for the three different risk measures.

1. **Variance as a risk measure**

$$\mathbf{\Pi} = \lambda \mathbf{V} \mathbf{w} \tag{77}$$

2. **CVaR as a risk measure**

$$\mathbf{\Pi} = \frac{\lambda}{2}\left(\frac{\mathbf{Vw}}{\sqrt{\mathbf{wVw}}} CVaR_{\delta} - E(\mathbf{R})\right) \tag{78}$$

3. **VaR as a risk measure**

$$\mathbf{\Pi} = \frac{\lambda}{2}\left(\frac{\mathbf{Vw}}{\sqrt{\mathbf{wVw}}} VaR_{\delta} - E(\mathbf{R})\right) \tag{79}$$

where $CVaR_{\delta}$ and VaR_{δ} are tabulated standard values for the α-stable distributions. In order for us to compute the equilibrium returns from the above formulas we need the estimates of $\mathbf{V}$ the variance covariance matrix for the normal distribution case, and the dispersion matrix for the α-stable distributions case.

7.2.2 Variance Covariance Matrix

Let $\tilde{R_{ik}} = (R_{ik} - E(R_i))$ be the k-th centered return for the i-th asset return series, the commonly used sample estimator for the variance covariance matrix $\tilde{V}$ has elements v_{ij} given by

$$v_{ij} = \frac{1}{N}\sum_{k=1}^{N}(R_{ik} - E(R_i))(R_{kj} - E(R_j)), \qquad for \;\; i \neq j \tag{80}$$

and

$$v_{jj} = \frac{1}{N}\sum_{k=1}^{N}(R_{jk} - E(R_j))^2, \qquad for \;\; i = j \tag{81}$$

where .

For the actual computation of this estimate we used the in built function `cov` in R and the resulting variance covariance matrix is given in Table 12.

7.2.3 Dispersion Matrix

As mentioned on the exposition on α-stable distributions, stock returns $R = [R_1, R_2, R_3, \ldots, R_N]$ can be modeled using a sub Gaussian α-stable distribution with an index of stability α such that $1 < \alpha < 2$.

The characteristic function is given by

$$\phi_{\mathbf{X}}(\mathbf{t}) = e^{i\mathbf{t}'\delta - (\frac{1}{2}\mathbf{t}'\mathbf{V}\mathbf{t})^{\frac{\alpha}{2}}} \tag{82}$$

where $\mathbf{V} = [v_{ij}]$ is the *dispersion matrix* for the returns and δ is the location vector.

There are problems encountered when one tries in practice, to estimate the dispersion matrix $\mathbf{V}$ for the α- stable distributions as noted before, see [Ortobelli et al., 2004], [Lamantia et al., 2005] and [Giacometti et al., 2007]. The problems are related to the fact that the spectral measure for multivariate distributions is not easy to estimate especially for the the α-stable distributions with infinite variance.

For the sub-Gaussian α-stable distributions we use a moment estimator for the dispersion matrix proposed in the three papers, [Ortobelli et al., 2004], [Lamantia et al., 2005] and [Giacometti et al., 2007].

Let $\tilde{R_{jk}} = R_{jk} - E(R_j)$ be the k-th centered observation of the j-th asset, then the elements $[v_{ij}]$ of the dispersion matrix $\mathbf{V}$ are given by

$$v_{ij} = (v_{jj})^{2-p} A(p) \frac{1}{N} \sum_{k=1}^{N} \tilde{R_{ik}} |\tilde{R_{jk}}|^{p-1} sgn(\tilde{R_{jk}}), \qquad 1 < p < \alpha \tag{83}$$

For $i \neq j$, where $A(p) = \dfrac{\Gamma(1-\frac{p}{2})\sqrt(\pi)}{2^p \Gamma(1-\frac{p}{\alpha})\Gamma(\frac{p+1}{2})}$. And

$$v_{jj} = \left(A(p) \frac{1}{N} \sum_{k=1}^{N} |\tilde{R_{jk}}|^p \right)^{\frac{2}{p}}, \qquad 1 < p < 2 \tag{84}$$

when $i = j$.

We implemented the above estimate of the dispersion in R using a value of $p = 1.3065$. This is, according to our analysis, the value of p that gave the maximum value for the coefficient of dispersion for each of the 10 stocks we used. The task of finding p can be made more rigorous by using the optimization methods suggested in [Kring et al., 2008]. The resulting dispersion matrix is given in Table 13.

7.2.4 Which coefficient of risk aversion λ to use?

If the coefficient of risk aversion coefficient λ is zero the investor is risk neutral, if $\lambda > 0$ the investor is risk averse and if $\lambda < 0$ the investor is a risk seeker. According to Black and Litterman [Black and Litterman, 1992] the value of λ to be used in the model can be calculated as the market risk premium i.e the excess mean return divided by the variance of the market returns. [Giacometti et al., 2007] calculated the risk aversion parameter of the S&P 500 Index in the US and found the value of λ to be 36.29. This is comparatively a very large number compared to the 0.32 used in [Black and Litterman, 1992]. As noted in [Giacometti et al., 2007] in equations 77, 78 and 79, λ acts as a scaling factor and a large λ increases the equilibrium returns. If λ is allowed to increase to very large values, it will scale the equilibrium returns to unrealistic levels.

In our case we use the estimate of λ derived in [Salomons, 2007]. The λ is a result of solving the optimization problem for the expected return using the method of Lagrange multipliers.

$$\begin{aligned} &\max_{\mathbf{w} \in \mathbb{R}^N} \mathbf{w}'\mathbb{E}(\mathbf{R}) \\ &s.t. \quad \frac{1}{2}\mathbf{w}'\mathbf{\Sigma}\mathbf{w} = \sigma^2 \end{aligned}$$

The Lagrange multiplier λ, which is our risk aversion coefficient can be easily shown, from the above problem, to be

$$\lambda = \sqrt{\frac{1}{2\sigma^2}(\mathbb{E}(\mathbf{R}))'\mathbf{\Sigma}^{-1}\mathbb{E}(\mathbf{R})} \tag{85}$$

where the parameters carry the usual meaning introduced before.

This we found that gives a way of calculating the risk aversion coefficient of each individual investor by just specifying the level of risk σ they will be willing to accept. In our application using the ten stocks, we chose our risk to be 11% and it gave us the value of λ of 6.369972, which gave reasonable estimates for our equilibrium returns. This is the λ that we use for all our estimations.

7.2.5 VaR Estimation for the α-Stable Distributions

The definition of value at risk (VaR) says, for an acceptable level of significance ϵ ,it is the quantity that satisfies

$$\mathbb{P}(R < -VaR_\epsilon(R)) = 1 - \epsilon \tag{86}$$

where R is the portfolio target return. We can see that estimating VaR is equivalent to estimating the lower order quantiles of the portfolio return distribution. Methods of calculating VaR used in practice concentrate on constructing the distribution of the portfolio returns. Common examples, are the Delta Method , historical simulation method and Monte Carlo Simulation methods see [Mittnik et al., 2002]. For a normal distribution of portfolio returns with say mean μ and variance σ^2, the VaR can be estimated as

$$VaR_\epsilon(R) = z_{1-\epsilon}\sigma + \mu \tag{87}$$

where $z_{1-\epsilon}$ is the $1-\epsilon$-th quantile from the standard normal distribution. We can make use of this property to estimate the VaR for a sub-Gaussian α-stable portfolio returns distribution with location parameter γ as

$$VaR_\epsilon(R)) = z_{1-\epsilon}\gamma \tag{88}$$

where $z_{1-\epsilon}$ is the $1-\epsilon$-th quantile from the standardized α-stable distribution, $S(\alpha, \beta, 1, 0)$. The standardized α-stable random variable is also called a spherical α-stable random variable. Further details of this can be found in [Mittnik et al., 2002].

In our case following [Mittnik et al., 2002] we found that for a sub-Gaussian α-stable random variable, $\beta = 0$ and the scale parameter γ in the VaR calculation is 1. This implies that

$$VaR_\epsilon(R)) = z_{1-\epsilon} \tag{89}$$

Thus the VaR we want to estimate for a spherical random variable is just minus a corresponding quantile from the standardized α-stable distribution, $S(\alpha, \beta, 1, 0)$. This is the value we will substitute in the calculation of equilibrium market returns with VaR as a risk measure. For more information on this VaR approximation one should see, [Mittnik et al., 2002]. We use VaR estimates at 1% and at 5%.

7.2.6 CVaR Estimation for the α-Stable Distributions

According to [Stoyanov et al., 2006] for a spherical α-stable random variable , $R \sim S(\alpha, \beta, 1, 0)$ the conditional value at risk (CVaR) at significance level ϵ can be computed as

$$CVaR_\epsilon(R)) = \frac{\alpha}{1-\alpha}\frac{|VaR_\epsilon(R)|}{\pi\epsilon}\int_{-\theta_0}^{\frac{\pi}{2}} g(\theta) exp(-|VaR_\epsilon|^{\frac{\alpha}{\alpha-1}} v(\theta))d\theta \tag{90}$$

where

$$g(\theta) = \frac{\sin(\alpha(\theta_0 + \theta) - 2\theta)}{\sin(\alpha(\theta_0 + \theta))} - \frac{\alpha\cos^2\theta}{sin^2\alpha(\theta_0 + \theta)}, \tag{91}$$

$$v(\theta) = (\cos\alpha\theta_0)^{\frac{1}{\alpha-1}} \left(\frac{\cos\theta)}{\sin(\alpha(\theta_0+\theta))}\right)^{\frac{\alpha}{1-\alpha}} \frac{\cos(\alpha\theta_0 + (\alpha-1)\theta)}{\cos\theta} \tag{92}$$

$$\theta_0 = \frac{1}{\alpha}\arctan(\beta\tan\frac{\pi\alpha}{2}) - Z_{1-\epsilon}, \qquad \beta = -sign(VaR_\epsilon(R))\beta. \tag{93}$$

These long and strenuous calculations require time and computing power to be performed with accuracy. But fortunately as is case in the standard normal distribution, one does not need to do all these calculations in practice. There are standardized tables for CVaR estimates in [Stoyanov et al., 2006] for various values of α and β, for a random variable distributed $S(\alpha,\beta,1,0)$. We used these tables to get the standardized α-stable CVaR estimates for the two cases we are working with 1% and at 5%. In our case for the sub-Gaussian class, $\beta = 0$, see [Stoyanov et al., 2006] and our universal calculated α is 1.906 $\approx$ 1.91. The tabulated estimates for this case are $CVaR_{0.05} = 3.3746$ and $CVaR_{0.01} = 5.5477$.

With all these estimates we were able to estimate the various equilibrium market returns $\mathbf{\Pi}$ for the fives cases, the normal distribution, the α-stable distribution with risk measure $CVaR_{0.05}$, the α-stable distribution with risk measure $CVaR_{0.01}$, the α-stable distribution with risk measure $VaR_{0.05}$ and the α-stable distribution with risk measure $VaR_{0.01}$. The results are given in Table 3 and Table 4. These will form the inputs to the Black Litterman Model together with the returns implied by the views.

7.2.7 The Views

As an illustration we formulated the views using the end of January 2011 realized monthly returns for each of the ten stocks. We pretended as if we are on 31 December 2010 and our investment horizon is the month of January 2011 and suppose that we were able to guess the end of January 2011 monthly returns exactly. Of course this is a hypothetical situation and too idealistic, real views should be formulated using information from the market press or other sources. The January 2011 monthly returns for the the ten stocks were as in Table 5.

Based on January 2011 returns we formulate the following four views, two relative and two absolute.

1. The average return of the two mining stocks, Anglo American and BHP Billiton of 0.21599, will be greater than the return of the oil and gas producer Sasol of 0.002365227, by 0.019233.

2. The sum of the returns of the two life assurance stocks Old Mutual and Sanlam of 0.067853296 will exceed the sum of returns of the two banks Standard Bank and FirstRand of -0.022928101 by 0.090781397.

3. The return of beverages stock SABMiller will be -0.017690479.

4. The return of telecommunications stock MTN Group will be -0.088784871.

These views can be interpreted and written in terms of the matrices P and Q as

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{94}$$

and

$$\mathbf{Q} = \begin{pmatrix} 0.019233 \\ -0.01769 \\ 0.090781 \\ -0.08878 \end{pmatrix}. \tag{95}$$

We suppose that there is enough information for us to say that we have 100% confidence in the first view, 90% confidence in the second, 100% confidence in the third and 90% confidence in the fourth.

7.3 Results and Interpretation

In the Black Litterman the views the equilibrium prior returns $\mathbf{\Pi}$ are combined with the view returns $P\mathbb{E}(\mathbf{R})$ to get the posterior estimated returns $\mu_{\mathbf{BL}}$ and dispersion matrix $\mathbf{\Sigma_{BL}}^{\mu}$ given by

$$\mu_{\mathbf{BL}} = [(\tau\mathbf{\Sigma})^{-1} + \mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}]^{-1}[(\tau\mathbf{\Omega})^{-1}\mathbf{\Pi} + \mathbf{P}'\mathbf{\Omega}\mathbf{Q}], \quad \textit{and} \tag{96}$$

$$\mathbf{\Sigma_{BL}}^{\mu} = [(\tau\mathbf{\Sigma})^{-1} + \mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}]^{-1}. \tag{97}$$

Where $\tau\mathbf{\Sigma}$ is the prior dispersion matrix and $\mathbf{\Omega}$ is the views covariance matrix. The actual posterior dispersion matrix of the posterior returns $\mu_{\mathbf{BL}}$ is given by $\mathbf{\Sigma_{BL}} = \mathbf{\Sigma} + \mathbf{\Sigma_{BL}}^{\mu}$ see [Meucci, 2008].

We did this implementation in R using the package BLCOP to get the posterior returns for each of the three cases the normal distribution with variance as a risk measure, α-stable distributions with CVaR and VaR as risk measures. We did this for the two levels of significance $\epsilon = 5\%$ and $\epsilon = 1\%$. The results are summarized in Table 6 and Table 7.

The posterior weights of the stocks after applying the Black Litterman Model, can be obtained by reverse optimization. The optimization inputs are the posterior vector of expected returns and the posterior covariance matrix Table14 for the normal distribution and dispersion matrix Table15 , for the α-stable distributions.

In R this optimization can be carried out using the package fPortfolio. We performed this optimization first, without the constraint against short positions and got the results in Table 8. We can see from Figure 3 that the equilibrium returns under the normal distribution interprets the views in such a way that a large short position has to be taken in Standard Bank and a corresponding long position in Old Mutual. This looks reasonable when one looks at the views because they favor these positions.

But however the situation changes dramatically when we consider the α-stable distributions, illustrated in Figure 4 for the case of CVaR at 5%. Although the short position in Standard Bank and the long position in Old Mutual are still favoured they have been greatly reduced and other views fully reflected. The most interesting feature is how the α-stable distributions manages to capture the views according to their magnitude. This is illustrated for example in the two mining stocks Anglo American and BHP Billiton in Figure 4 first two columns, which have a higher combined expected return and so are favored. The α-stable distributions are also able to fully capture the neutral view on Richemont with remarkable accuracy. The other stocks are given the appropriate weights according to the views.

The case for VaR at 5% is similar but not as good as the one for CVaR at 5%. This is because as mentioned earlier though VaR is a widely used measure of risk, it is not a coherent one. It is still able to capture the stock allocations in a better way than the variance in the normal distribution case. When we consider the measures at 1% Figure 6 and Figure 7, we see that the pattern is still the same but with small changes that reflects that the 1% loss target is a very difficult one to achieve in practice. The weights for all the cases are summarized in Table 8 and Table 9.

As mentioned in the introduction having arbitrarily large long and short positions in certain stocks is an undesirable feature of mean variance optimization. Some portfolio managers especially pension and public fund managers have restrictions against taking short positions. We went on to impose the constraint against short positions and we got remarkable results with the α-stable distributions. We found out that when we used the normal distribution the resulting weights were that we should only invest in the two mining stocks Anglo American and BHP Billiton with weights 0.5202 and 0.4798 respectively, see Table 10. But when we looked at the α-stable distributions we saw that the weights were evenly distributed among the stocks we had positive views on and a value of zero on the negative one.

7.4 Performance Measurement

There are various measures that can be used to measure the performance of a portfolio. We used the Sharpe Ratio as performance measure to compare the various portfolios that resulted from the reverse optimization with constraint on against short positions. We computed the portfolio expected returns and the corresponding standard deviation in R and placed them in Table 11. At 5% significance level the portfolio that results from using the VaR as a risk measure had the highest Sharpe Ratio of 1.026506024 followed by the portfolio formed using CVaR as a risk measure which had a Sharpe Ratio of 0.593457944. The portfolio that resulted from using the normal distribution and variance as a risk measure had the lowest Sharpe Ratio of 0.155787641. At 1% significance the pattern remained the same but the Sharpe Ratio was greatly reduced for the two alternative risk measures the VaR and CVaR to 0.650234742 and 0.404545455 respectively. This is because the 1% level of significance is a very low value of losses that can result from an investment compared to the 5%. But still even at 1% the two Sharpe Ratios for the VaR and CVaR are greater than the normal distribution and variance Sharpe Ratio value of 0.155787641.

8 Conclusion

In this investigation we started with the Black Litterman model which assumes that returns are normally distributed and we have extended the model to cases where the returns are assumed to follow α-stable distributions. Using the alternative risk measures the VaR and CVaR we have managed to deduce that using the α-stable distribution greatly improves the asset allocation weighting. Using the Sharpe Ratio as measure to compare the performance of the portfolios that resulted from using the Various risk measures we have deduced that the portfolio that resulted from using the α-stable distributions together with VaR as a risk measure had the highest Sharpe Ratio. The portfolio that resulted from using the α-stable distributions and the CVaR as a risk measure had the second highest Sharpe Ratio. The normal distribution and variance as a measure of risk gave a portfolio with the lowest Sharpe Ratio.

Company Name	Sector	RAFI40 Weight	ALSI weight
Anglo American	Mining	8.72%	14.51%
BHP Billiton	Mining	8.41%	13.97%
Standard Bank	Banks	6.3	3.55%
Sasol	Oil and Gas	5.13%	6.39%
Richemont	Personal Goods	3.36%	5.91%
Sanlam	Insurance	3.37%	1.01%
SAB Miller	Beverages	3.28%	4.4%
Old Mutual	Insurance	3.33%	2%
Firstrand	Banks	3.32%	1.8%
MTN Group	Telecoms	3.09%	6.01%

Table 1: The weights of the ten stocks in the RAFI40 Index and the ALSI Index

Company Name	α	β	γ	δ
Anglo American	1.89848097	-0.99990000	0.07230666	0.01862512
BHP Billiton	1.99990000	-0.02177520	0.06626487	0.01566929
Standard Bank	1.89783964	-0.99990000	0.04866543	0.02465096
Sasol	1.999900000	-0.212301357	0.048201946	0.008505126
Richemont	1.999900000	0.490044682	0.101138178	0.005300856
Sanlam	1.84909580	-0.99990000	0.04309404	0.01830945
SAB Miller	1.999900000	-0.333200402	0.051487455	0.007613129
Old Mutual	1.7244588621	-0.0827049903	0.0479235005	0.0005827264
Firstrand	1.999900000	-0.351564139	0.053692363	0.006807343
MTN Group	1.82799541	0.13859084	0.06714286	0.01135468

Table 2: Estimated stable distribution parameters for each return time series of the ten stocks used

Company Name	Normal	Stable CVAR 5%	Stable VAR 5%
ANGLO.AMERICAN	0.02545568	0.039481328	0.067531814
BHP.BILLITON	0.02201244	0.030557814	0.053913437
STANDARD.BANK	0.009922869	0.003991578	0.009111648
SASOL	0.01732698	0.01362927	0.025579663
RICHEMONT.SECS	0.01532191	0.00931566	0.018245719
SANLAM	0.007610591	0.00104625	0.003786739
SABMILLER	0.01032474	0.002267018	0.005886446
OLD.MUTUAL	0.014062	0.007107105	0.012244068
FIRSTRAND	0.009550017	0.006493942	0.012535644
MTN.GROUP	0.01401893	0.014610712	0.027693565

Table 3: Equilibrium Returns for the three cases at 5% level

Company Name	Normal	Stable CVAR 1%	Stable VAR 1%
ANGLO.AMERICAN	0.02545568	0.026938654	0.043186802
BHP.BILLITON	0.02201244	0.020114431	0.033643096
STANDARD.BANK	0.009922869	0.001702157	0.00466794
SASOL	0.01732698	0.008285694	0.015207919
RICHEMONT.SECS	0.01532191	0.005322616	0.010495322
SANLAM	0.007610591	-0.00017915	0.001408269
SABMILLER	0.01032474	0.000648604	0.002745145
OLD.MUTUAL	0.014062	0.00481013	0.007785698
FIRSTRAND	0.009550017	0.003792416	0.007292052
MTN.GROUP	0.01401893	0.008760762	0.016338959

Table 4: Equilibrium Returns for The three cases at 1% level

Company	Return
ANGLO AMERICAN	0.013321946
BHP BILLITON	0.029875123
SASOL	0.002365227
SABMILLER	-0.017690479
RICHEMONT SECS.	0
STANDARD BK.GP	-0.021902459
FIRSTRAND	-0.001025641
OLD MUTUAL	0.093979601
SANLAM	-0.026126305
MTN GROUP	-0.088784871

Table 5: January 2011 returns for each of the ten stocks

Company Name	Normal	Stable CVAR 5%	Stable VAR 5%
ANGLO.AMERICAN	0.018835268	0.038434686	0.065110774
BHP.BILLITON	0.01655653	0.029821848	0.052014207
STANDARD.BANK	-0.010037789	-0.001336164	0.003567702
SASOL	0.010762478	0.013321723	0.02689576
RICHEMONT.SECS	0.005423891	0.008392074	0.017173752
SANLAM	-0.002393355	0.003513995	0.006176194
SABMILLER	0.004799626	0.000421937	0.003710388
OLD.MUTUAL	0.00483659	0.010394095	0.015432741
FIRSTRAND	-0.010573974	0.000948547	0.006756284
MTN.GROUP	-0.023121964	-0.003803112	0.006920352

Table 6: Posterior Equilibrium Return Estimates for the three cases at 5%

Company Name	Normal	Stable CVAR 1%	Stable VAR 1%
ANGLO.AMERICAN	0.018835268	0.026506569	0.041958602
BHP.BILLITON	0.01655653	0.019898614	0.032753462
STANDARD.BANK	-0.010037789	-0.00352891	-0.000688363
SASOL	0.010762478	0.007252141	0.015114856
RICHEMONT.SECS	0.005423891	0.004465378	0.009552135
SANLAM	-0.002393355	0.002323602	0.003865672
SABMILLER	0.004799626	-0.001048482	0.000856342
OLD.MUTUAL	0.00483659	0.008141083	0.011059701
FIRSTRAND	-0.010573974	-0.001648362	0.00171575
MTN.GROUP	-0.023121964	-0.008598071	-0.002386541

Table 7: Posterior Equilibrium Return Estimates for the three cases at 1%

Company	Normal		Stable -CVAR 1%		Stable-VAR 1%	
Name	Prior	Posterior	Prior	Posterior	Prior	Posterior
ANGLO.AMERICAN	1.3262165	0.43799445	7.2269827	4.8151363	4.9793023	3.374746
BHP.BILLITON	0.9295536	0.78559303	6.1204253	4.1596444	4.0570352	2.8381043
STANDARD.BANK	1.3943435	-2.28738634	0.4133114	-0.9462681	-0.0301038	-1.2396
SASOL	0.2304106	-0.01379819	1.6831533	1.168562	0.9895346	0.5880351
RICHEMONT.SECS	0.2302718	0.40050593	0.9444727	0.9439703	0.4688186	0.6115364
SANLAM	0.2392425	1.107274	-2.1306595	0.4205342	-1.5760981	0.7577636
SABMILLER	-0.4059217	0.01669069	-2.8925328	-1.5994625	-2.0463013	-1.0276735
OLD.MUTUAL	-0.1986264	1.64015271	-0.5209516	1.5668734	0.0096172	1.8845372
FIRSTRAND	-0.8050603	0.05367516	-0.5423568	-0.8678648	-0.2032244	-0.6743979
MTN.GROUP	0.1799195	-1.64591354	1.0917766	-1.4740891	0.7227487	-1.6101561

Table 8: Weights from optimization using posterior returns with short positions allowed at 5%

Company	Normal		Stable -CVAR 1%		Stable-VAR 1%	
Name	Prior	Posterior	Prior	Posterior	Prior	Posterior
ANGLO.AMERICAN	1.3262165	0.43799445	12.253704	8.0364308	7.8910134	5.2406698
BHP.BILLITON	0.9295536	0.78559303	10.734999	7.1151419	6.7300112	4.5500661
STANDARD.BANK	1.3943435	-2.28738634	1.404967	-0.2902596	0.5443092	-0.8596093
SASOL	0.2304106	-0.01379819	3.234365	2.4668547	1.8880687	1.3400667
RICHEMONT.SECS	0.2302718	0.40050593	2.008227	1.6874269	1.0849949	1.042181
SANLAM	0.2392425	1.107274	-3.370883	-0.3336468	-2.2944932	0.3209068
SABMILLER	-0.4059217	0.01669069	-4.785048	-2.8782137	-3.1425344	-1.7683857
OLD.MUTUAL	-0.1986264	1.64015271	-1.707518	0.8564486	-0.6776972	1.4730261
FIRSTRAND	-0.8050603	0.05367516	-1.300794	-1.3005351	-0.6425464	-0.9250206
MTN.GROUP	0.1799195	-1.64591354	1.917072	-1.1697885	1.2007983	-1.433891

Table 9: Weights from optimization using posterior returns with short positions allowed at 1%

Company	**Normal**		**Stable -CVAR 5%**		**Stable-VAR 5%**	
Name	Prior	Posterior	Prior	Posterior	Prior	Posterior
ANGLO.AMERICAN	0.2434	0.4798	0.3106	0.3424	0.3435	0.3650
BHP.BILLITON	0.2344	0.5202	0.2711	0.3000	0.2861	0.3067
STANDARD.BANK	0.0596	0.0000	0.024	0.0000	0.0004	0.0000
SASOL	0.1074	0.0000	0.0983	0.0938	0.0898	0.0566
RICHEMONT.SECS	0.0992	0.0000	0.0790	0.0784	0.0637	0.0472
SANLAM	0.0172	0.0000	0.0000	0.0472	0.0000	0.0460
SABMILLER	0.0339	0.0000	0.0000	0.0000	0.0000	0.0000
OLD.MUTUAL	0.0302	0.0000	0.0513	0.1382	0.0623	0.1785
FIRSTRAND	0.0739	0.0000	0.0687	0.0000	0.0627	0.0000
MTN.GROUP	0.1009	0.0000	0.0971	0.0000	0.0915	0.0000

Table 10: Posterior weights for the three different risk measures with constraint against short positions

Portfolio	**Expected Return**	**Standard Deviation**	**Sharpe Ratio**
Normal	0.0179	0.1149	0.155787641
Stable CVAR 5%	0.0254	0.0428	0.593457944
Stable VAR 5%	0.0426	0.0415	1.026506024
Stable CVAR 1%	0.0178	0.044	0.404545455
Stable VAR 1%	0.0277	0.0426	0.650234742

Table 11: Portfolio risk and return for all the five cases with the corresponding Sharpe Ratios

Company	Anglo	BHP	STD.BK	SASOL	RICHEMNT	SANLAM	SABMIL	OLD.M	FirstRand	MTN
ANGLO	0.0118613	0.0079338	0.0018839	0.0062925	0.0050578	0.0009006	0.0038554	0.0043349	0.0017803	0.0026006
BHP	0.0079338	0.0091955	0.0018242	0.0055812	0.0038520	0.0019603	0.0030196	0.0034509	0.0016919	0.0025627
STD.BANK	0.0018839	0.0018242	0.0057075	0.0014333	0.0023824	0.0034869	0.0017795	0.0035178	0.0050746	0.0039456
SASOL	0.0062925	0.0055812	0.0014333	0.0082378	0.0036947	0.0005638	0.0023379	0.0026801	0.0012148	0.0012906
RICHEMNT	0.0050578	0.0038521	0.0023824	0.0036947	0.0071387	0.0017865	0.0024783	0.0031463	0.0019225	0.0030265
SANLAM	0.0009006	0.0019603	0.0034869	0.0005638	0.0017864	0.0049645	0.0018181	0.0033155	0.0036741	0.0036102
SABMIL	0.0038554	0.0030196	0.0017795	0.0023379	0.0024783	0.0018181	0.0043823	0.0022802	0.0019012	0.000829
OLD.M	0.0043349	0.0034509	0.0035178	0.0026801	0.0031463	0.0033155	0.0022802	0.0066496	0.0034264	0.0044062
Firstrand	0.0017803	0.0016919	0.0050746	0.0012148	0.0019225	0.0036741	0.0019013	0.0034265	0.0060379	0.0038379
MTN	0.0026006	0.0025627	0.0039456	0.0012906	0.0030265	0.0036102	0.0008291	0.0044062	0.0038379	0.0126918

Table 12: Prior Variance Covariance matrix assuming assuming return follow normal distribution

Company	ANGLO	BHP	STD.BK	SASOL	RICHEMNT	SANLAM	SABMIL	OLD.M	Firstrand	MTN
ANGLO	0.0048597	0.00052258	0.00008987	0.00039344	0.000263	0.0000493	0.00016658	0.00020459	0.0001138	0.000215
BHP	0.00052258	0.00409451	0.0000853	0.00034566	0.00019902	0.00009019	0.00013783	0.00017698	0.0000932	0.0001657
STD.BK	0.00008987	0.0000853	0.00235643	0.0000801	0.00012286	0.0001780	0.00009256	0.00016929	0.00027056	0.0002512
SASOL	0.00039344	0.0003456	0.0000801	0.00351185	0.00019097	0.0000467	0.00011251	0.00014806	0.00007651	0.0001394
RICHEMNT	0.00026303	0.00019902	0.0001228	0.00019097	0.0027374	0.00009937	0.00010967	0.0001724	0.0001208	0.0002307
SANLAM	0.0000493	0.00009019	0.000178	0.0000467	0.00009937	0.00203055	0.00008889	0.00016952	0.0001872	0.0002378
SABMIL	0.00016658	0.0001378	0.0000925	0.0001125	0.0001096	0.0000888	0.0017490	0.0001156	0.0001064	0.0001052
OLD.MUT	0.0002045	0.0001769	0.0001693	0.0001481	0.0001724	0.0001695	0.0001156	0.0025597	0.000176	0.0002875
Firstrand	0.0001138	0.0000932	0.0002705	0.0000765	0.0001208	0.0001872	0.0001064	0.0001761	0.0024742	0.0002546
MTN	0.000215	0.0001657	0.0002512	0.0001395	0.0002307	0.0002378	0.0001053	0.0002875	0.0002546	0.004801

Table 13: Prior Dispersion Matrix assuming α-stable distributions

Company	ANGLO	BHP	STD.BK	SASOL	RICHEMNT	SANLAM	SABMIL	OLD.M	FIRSTRND	MTN
ANGLO	0.0171946	0.0114005	0.0025706	0.0093823	0.0072624	0.0010423	0.0053801	0.0061002	0.0024055	0.0033372
BHP	0.0114005	0.0133697	0.0025191	0.0083231	0.0054985	0.00266671	0.0042051	0.0048166	0.002314	0.003304
STD.BK	0.0025706	0.0025191	0.00821	0.0020263	0.0033344	0.0049886	0.002486	0.0050104	0.0072587	0.0051833
SASOL	0.0093823	0.0083230	0.0020263	0.0121138	0.0054063	0.0007322	0.00334	0.0038808	0.0017007	0.0017163
RICHENT	0.0072624	0.0054985	0.0033344	0.0054063	0.01046	0.0024347	0.0034736	0.0044116	0.0026432	0.0039569
SANLAM	0.0010423	0.002666	0.0049886	0.0007322	0.0024347	0.0071897	0.0025289	0.0046470	0.00527074	0.0047316
SABMIL	0.0053802	0.0042051	0.002486	0.0033441	0.0034736	0.0025289	0.0061896	0.0031682	0.0026586	0.0010354
OLD.M	0.0061002	0.0048166	0.0050104	0.0038808	0.0044116	0.0046471	0.0031682	0.0095471	0.0048766	0.0057701
FIRSTRND	0.0024055	0.002314	0.0072587	0.0017007	0.0026432	0.005270	0.0026586	0.0048766	0.0086998	0.0050387
MTN	0.0033372	0.003304	0.0051833	0.0017163	0.0039569	0.0047311	0.0010354	0.0057701	0.0050387	0.016713

Table 14: Posterior Variance Covariance Matrix for the normal distribution

Company	ANGLO	BHP	STD.BK	SASOL	RICHEMNT	SANLAM	SABMIL	OLD.M	Firstrand	MTN
ANGLO	0.0071837	0.0006935	0.0001337	0.0007320	0.0003914	0.0000712	0.000241	0.0003028	0.0001686	0.0003012
BHP	0.0006935	0.0060646	0.0001276	0.0006397	0.0002959	0.0001326	0.0002000	0.0002615	0.0001388	0.000232
STD.BK	0.0001337	0.0001276	0.0035067	0.00012	0.0001833	0.0002871	0.000135	0.0002801	0.0003766	0.0003545
SASOL	0.000732	0.0006397	0.000120	0.0050717	0.000288	0.0000702	0.0001666	0.0002231	0.000116	0.0001991
RICHENT	0.0003914	0.0002959	0.0001833	0.000288	0.0041048	0.0001475	0.0001601	0.000256	0.0001802	0.0003254
SANLAM	0.0000712	0.0001326	0.0002871	0.0000702	0.0001475	0.0030274	0.00012960	0.0002304	0.0003018	0.0003352
SABMIL	0.0002417	0.0002000	0.000135	0.0001666	0.000160	0.0001296	0.0025595	0.0001686	0.0001553	0.0001453
OLD.MUT	0.0003028	0.0002615	0.0002801	0.0002231	0.000256	0.0002304	0.0001686	0.0038087	0.0002916	0.0004053
FIRSTRAND	0.0001686	0.0001388	0.0003766	0.000116	0.0001802	0.0003018	0.0001553	0.0002916	0.0036809	0.0003593
MTN	0.0003012	0.000232	0.0003545	0.0001991	0.0003254	0.0003352	0.0001453	0.00040531	0.0003593	0.0067749

Table 15: Posterior Dispersion Matrix for the α-stable distributions

Company Name	*Mean*	*Volatility*	*Skewness*	*Kurtosis*	B-J Test	P-Value
Absa	0.011946	0.076732	-0.2707321	0.9298967	6.9936	0.03029
African Rainbow	0.0130681	0.11145898	-0.6215514	1.651985	24.8509	4.015e-06
African Bank	0.0078854	0.1084696	-0.3533219	1.834881	22.6921	1.182e-05
Anglo American	0.00868173	0.1088315	-0.4035145	0.8502538	8.1549	0.01695
Anglo Platinum	0.0100989	0.1318525	-0.8257028	2.877692	63.3295	1.776e-14
AngloGold ASh	0.0061086	0.1020454	-0.1329275	-0.4218122	1.1912	0.5512
Arcelor Mittal	0.02745442	0.15883344	0.8737074	6.422952	252.5537	¡2.2e-16
Aspen Pharmacare	0.0242948	0.08891484	0.5287666	1.99169	29.6073	3.723e-07
BHP Billiton	0.0156674	0.0937144	-0.2179152	-0.3412982	1.5529	0.46
Bidvest Group	0.0073661	0.06551627	-0.2609509	0.2392568	1.9781	0.3719
Capital Shopcts	0.00316295	0.0762588	-0.674388	2.928128	52.1055	4.847e-12
Exxaro Resources	0.0264051	0.1238566	0.06382155	0.5774475	1.1825	0.5536
Firstrand	0.0068051	0.07593425	-0.07582282	0.0086844	0.145	0.93
Gold Fields	0.0104907	0.1184802	0.0787438	-0.6408605	2.1326	0.3443
Growth Point	0.01099033	0.0598596	0.3068457	0.7541375	5.7035	0.05774
Harmony Gold	0.0053066	0.1430342	0.1455868	-0.0833780	0.4816	0.786
Impala Platinum	0.01459924	0.12747141	-0.5599749	1.579260	21.8655	1.786e-05
Investec	0.00024759	0.09355025	0.0117734	0.9160716	5.2084	0.07396
Investec(JSE)	0.0070112	0.0941414	0.3412424	2.040386	21.1305	2.580e-05
Kumba Iron Ore	0.02642163	0.1223439	-0.5754108	0.1635975	3.0011	0.223
Lonmin	0.009037425	0.135717880	-1.012532	3.444624	91.4667	¡2.2e-16
Massmart	0.01928369	0.08588142	-0.5734136	1.813854	25.466	2.952e-06
Mondi	-0.006733025	0.116735372	-0.3587934	0.3670506	1.4409	0.4865
Mondi(JSE)	-0.00700732	0.1431377	0.06905475	-0.2941402	0.0773	0.962
MTN Group	0.0125394	0.1030734	-0.02834209	0.7467352	3.5516	0.1694
Naspers	0.01420464	0.13300232	-1.49293	6.80842	314.1974	¡2.2e-16
Nedbank	5.362917e-05	7.676898e-02	-0.05942383	-0.3626493	0.6445	0.7245
Old Mutual	-0.0013681	0.079770940	-0.1506015	1.391275	12.1272	0.002326
Pick N Pay	0.01159377	0.06604991	0.3089795	2.312144	33.4207	5.531e-08
Reinet INV	-0.00335647	0.07831944	-1.758718	5.44932	54.1976	1.703e-12
Remgro	0.01649436	0.05303197	0.0308497	0.7625104	3.4793	0.1756
Richemont	0.0130133	0.0850244	-0.6532102	1.041172	16.1802	0.0003066
RMB	0.007610	0.07281651	-0.3027566	-0.0254776	2.0501	0.3588
SAB Miller	0.0098837	0.0662865	-0.5740415	0.9316686	12.7375	0.001714
Sanlam	0.0085036	0.06817020	0.03210603	0.1679944	0.2799	0.8694
Sasol	0.0139802	0.0884375	0.0873474	-0.00582725	0.1794	0.9142
Shoprite	0.0181598	0.0728000	-0.4237392	0.5015104	5.6963	0.05795
Standard Bank	0.0110124	0.0730076	0.0912614	0.329995	0.9802	0.6126
Steinhoff Intl	0.0098969	0.0937356	-0.2635373	1.864045	22.0769	1.607e-05
Tiger Brands	0.0108174	0.06118180	-0.08671135	-0.6258787	2.063	0.3565
Truworths	0.01797535	0.0814429	-0.3227945	1.661389	18.728	8.576e-05
Vodacom GP	0.0136642	0.04943579	0.3896394	-1.242734	1.3362	0.5127

Table 16: Summary Statistics and Bera-Jarques Test Statistics of each return time series

Company Name	α	β	γ	δ
Absa	1.94882806	-0.99990000	0.05197417	0.01589427
African Rainbow	1.86701362	-0.99990000	0.07119862	0.02667923
African Bank	1.75078866	-0.26984558	0.06593592	0.01375309
Anglo American	1.89848097	-0.99990000	0.07230666	0.01862512
Anglo Platinum	1.88160751	-0.99990000	0.08340676	0.02482863
AngloGold Ashanti	1.99990000	-0.01356947	0.07215584	0.00610969
Arcelor Mittal	1.74888514	0.05981904	0.08735619	0.02273351
Aspen Pharmacare	1.99990000	0.12362114	0.06794466	0.03003280
BHP Billiton	1.99990000	-0.02177520	0.06626487	0.01566929
Bidvest Group	1.99990000	-0.01027481	0.04734888	0.01438418
Capital Shopcts	1.741926811	-0.175325978	0.044492248	0.007827659
Exxaro Resources	1.99990000	0.03159855	0.08716974	0.02640431
Firstrand	1.999900000	-0.351564139	0.053692363	0.006807343
Gold Fields	1.999900000	0.005802811	0.083777860	0.010490495
Growth Point	1.905641909	0.999900000	0.040278566	0.006019475
Harmony Gold	1.999900000	0.490044682	0.101138178	0.005300856
Impala Platinum	1.89606982	-0.99990000	0.08281054	0.02717343
Investec	1.844380315	-0.152026269	0.060691198	0.001479879
Investec JSE	11.99990000	0.009644896	0.062413452	0.026198835
Kumba Iron Ore	1.79093945	-0.99990000	0.07847848	0.04674535
Lonmin	1.67369271	-0.22913221	0.07368496	0.02173051
Massmart	1.894429811	0.999900000	0.048775946	-0.005503878
Mondi	1.887096949	-0.999900000	0.078085928	0.004560854
Mondi JSE	1.999900000	-0.261201663	0.101212318	-0.007004725
MTN Group	1.82799541	0.13859084	0.06714286	0.01135468
Naspers	1.69520213	-0.38963615	0.07012960	0.02849169
Nedbank	1.999900	-4.583708e-03	5.428319e-02	5.405239e-05
Old Mutual	1.7244588621	-0.0827049903	0.0479235005	0.0005827264
Pick N Pay	1.894429811	0.999900000	0.048775946	-0.005503878
Reinet INV	0.9370000	-0.4290000	0.025851433	-0.008507887
Remgro	1.99990000	-0.90020756	0.03749798	0.01649871
Richemont	1.999900000	0.490044682	0.101138178	0.005300856
RMB	1.76207915	-0.80732434	0.05244835	0.02659399
SAB Miller	1.999900000	-0.333200402	0.051487455	0.007613129
Sanlam	1.84909580	-0.99990000	0.04309404	0.01830945
Sasol	1.999900000	-0.212301357	0.048201946	0.008505126
Shoprite	1.999900000	0.006772673	0.062532550	0.013978649
Standard Bank	1.89783964	-0.99990000	0.04866543	0.02465096
Steinhoff Intl	1.99990000	0.23498580	0.07132603	0.01669365
Tiger Brands	1.99990000	0.205619451	0.041737224	0.007571972
Truworths	1.85492095	-0.14906498	0.05227091	0.02025437
Vodacom GP	1.99990000	-0.25401814	0.03893575	0.01570981

Table 17: Estimated stable distribution parameters for each return time series of all the Top 40 stocks on Johannesburg Stock Exchangen (JSE)

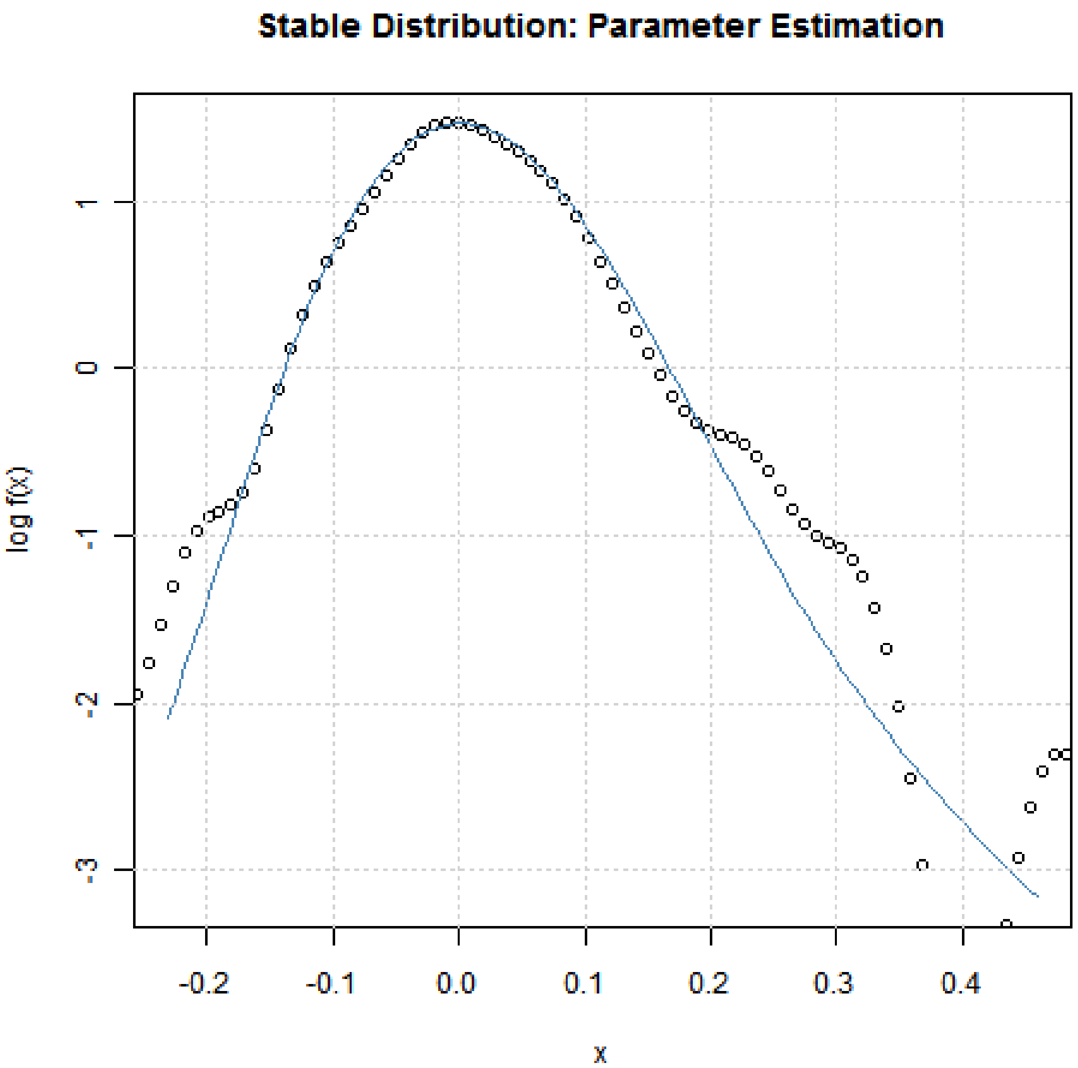

Figure 1: BHP Billiton returns fitted with an α-Stable Distribution

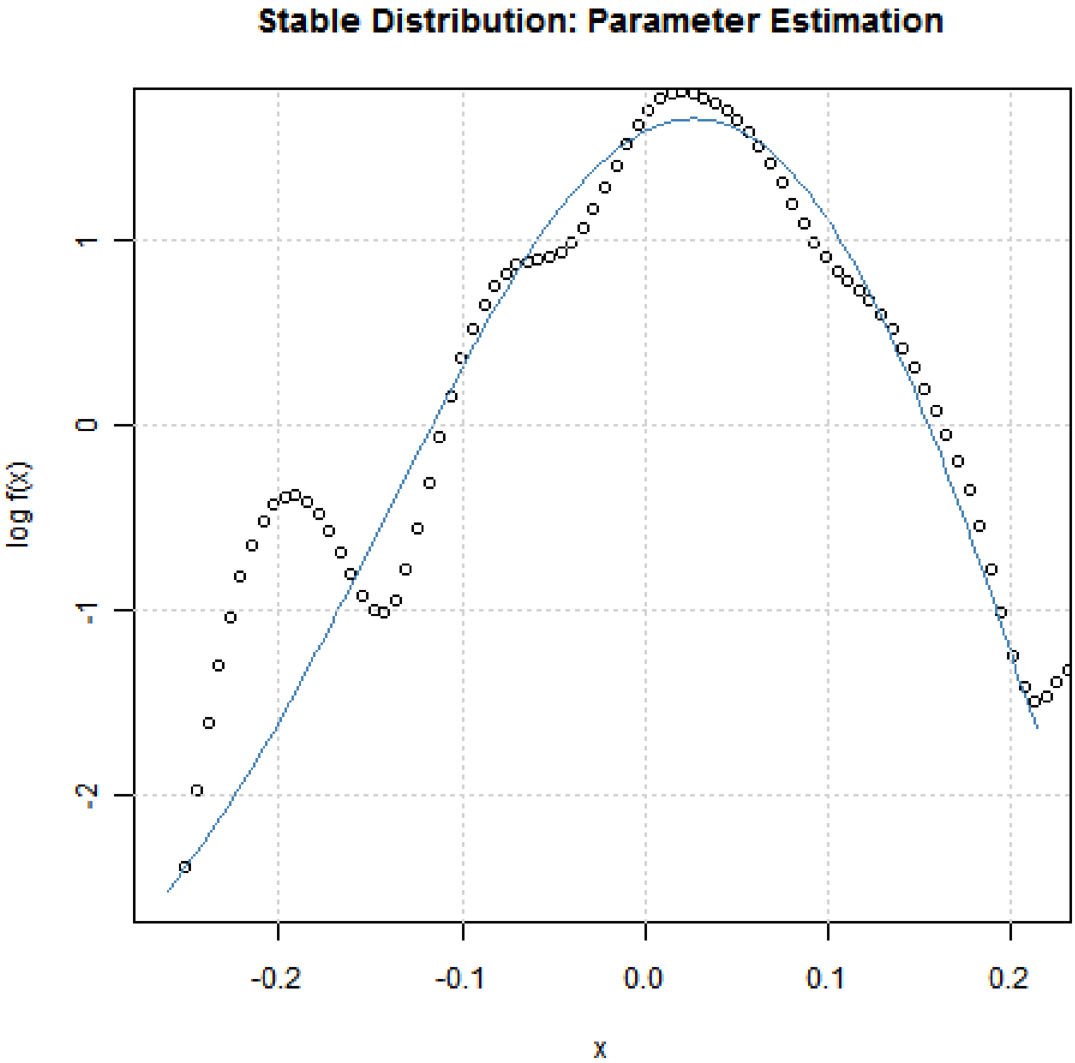

Figure 2: Richemont returns fitted with an α-Stable Distribution

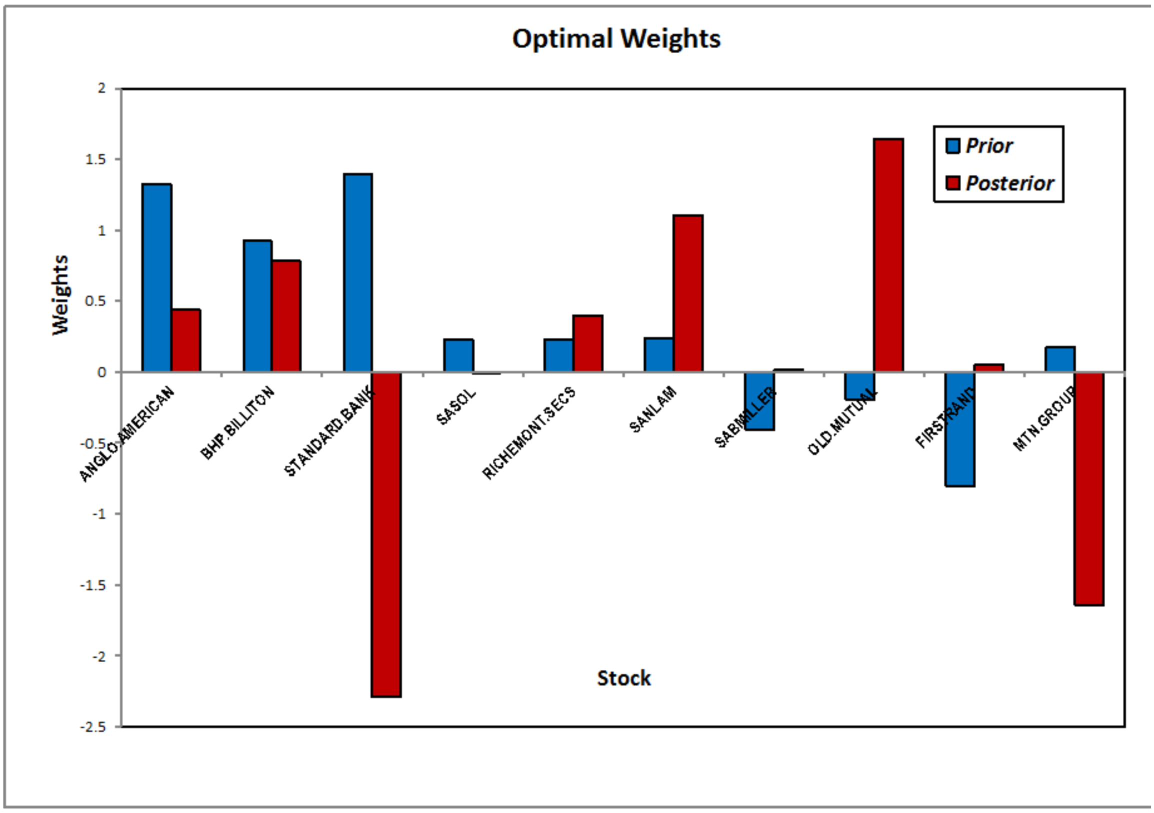

Figure 3: Posterior weight allocation that result from using the normal distribution with variance as a risk measure and short positions allowed.

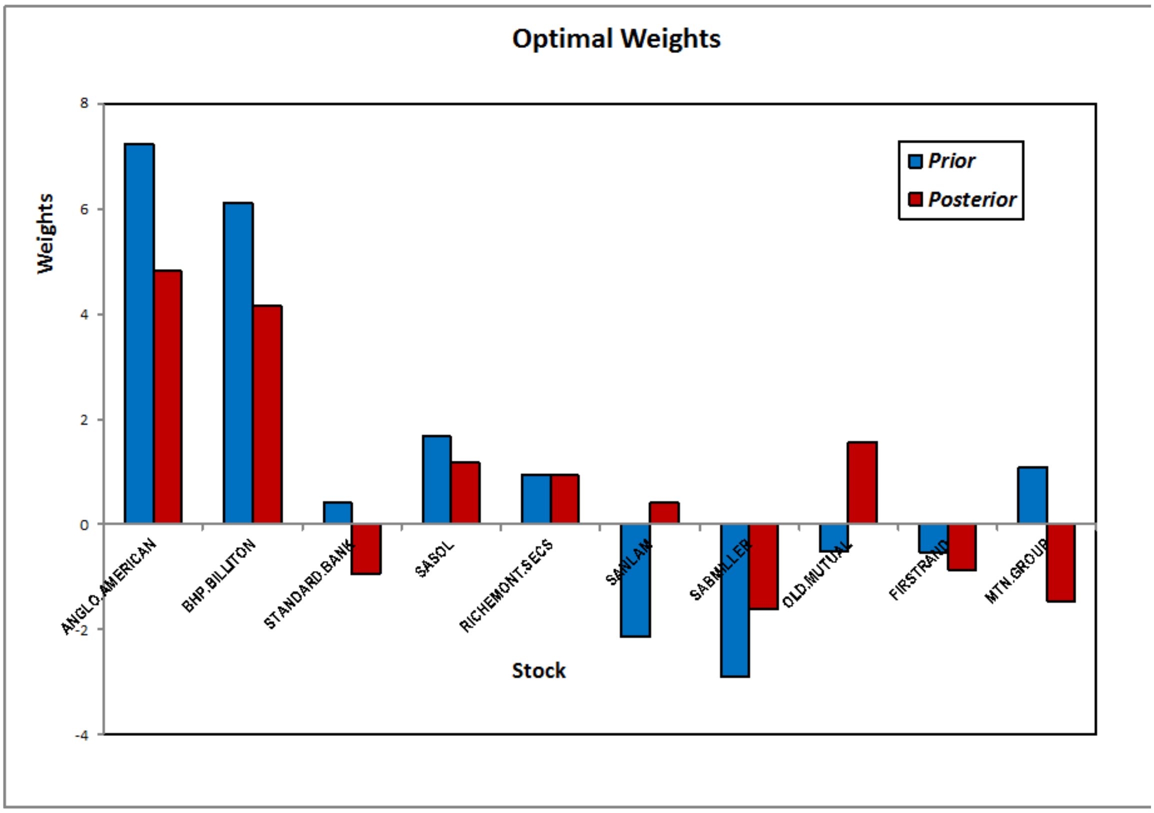

Figure 4: Posterior weight allocations that result from using the α-stable distributions, with CVAR at 5% as a risk measure and short positions allowed.

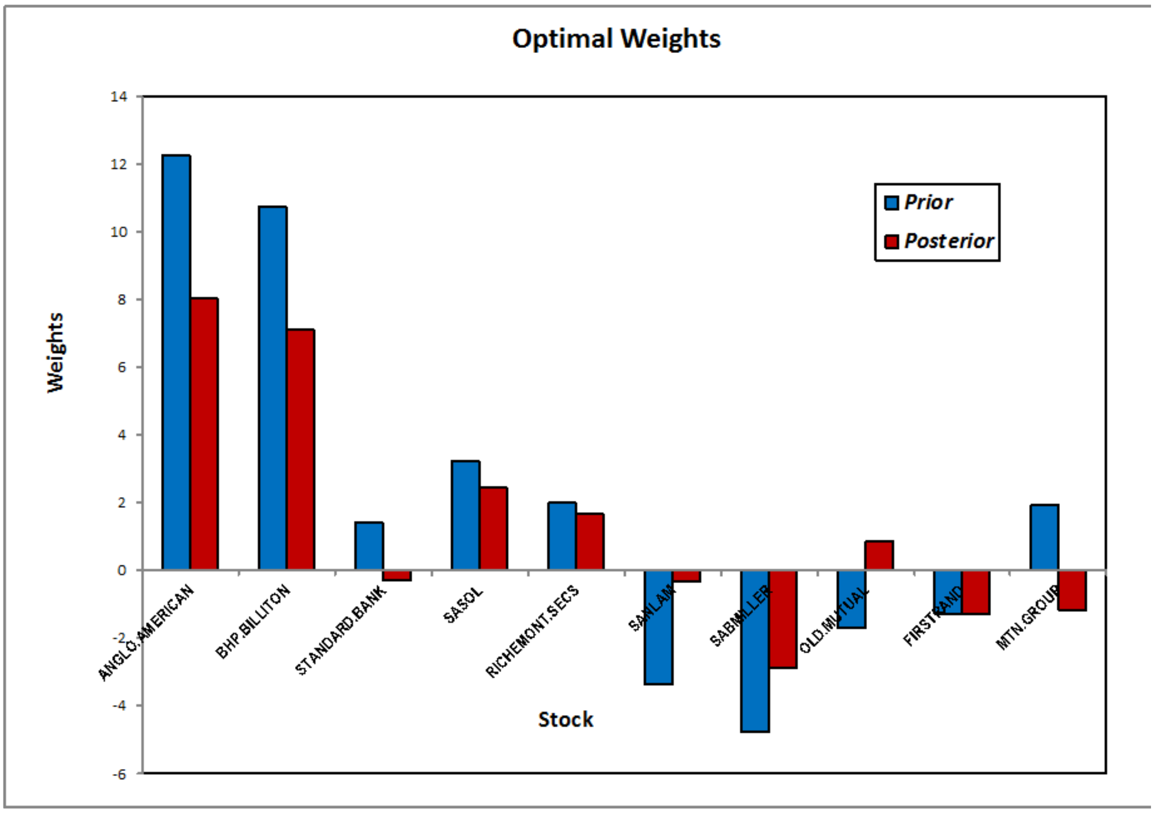

Figure 5: Posterior weight allocations that result from using the α-Stable Distributions, with VAR at 5% as a risk measure and short positions allowed.

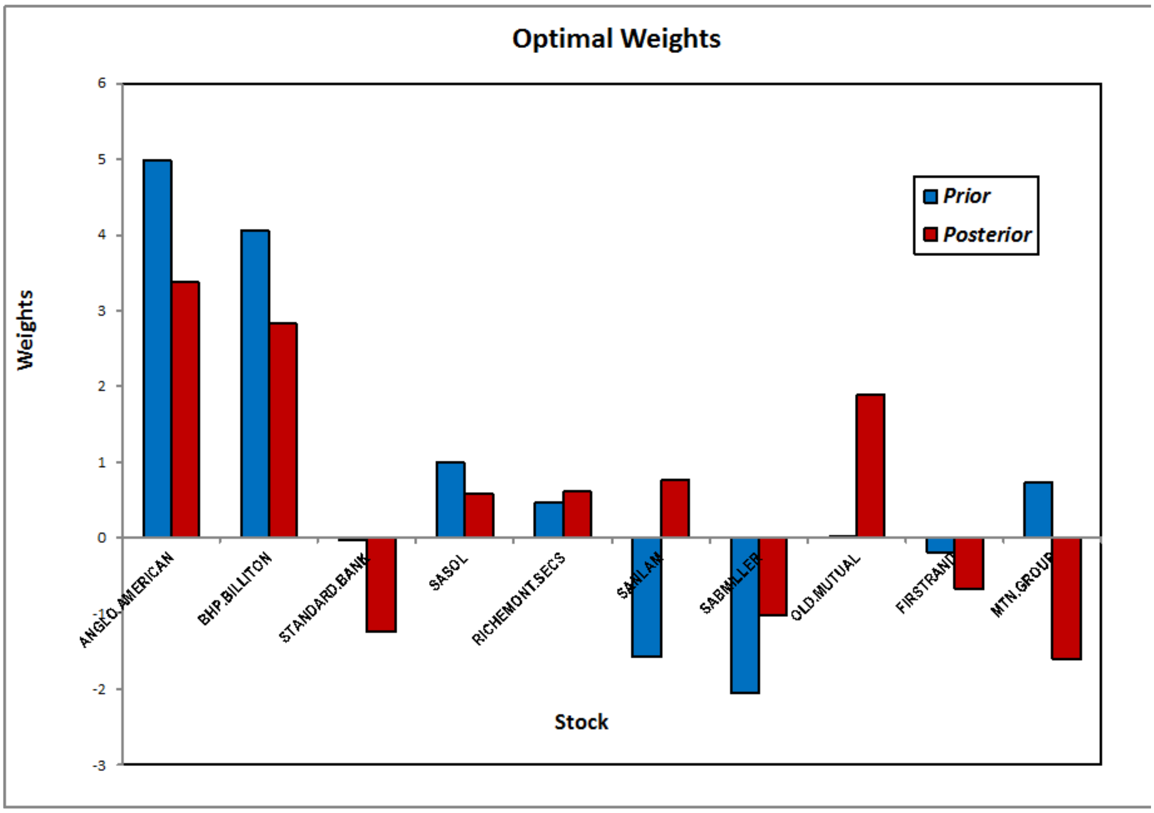

Figure 6: Posterior weight allocation that result from using the α-Stable Distributions, with CVAR at 1% as a risk measure and short positions allowed.

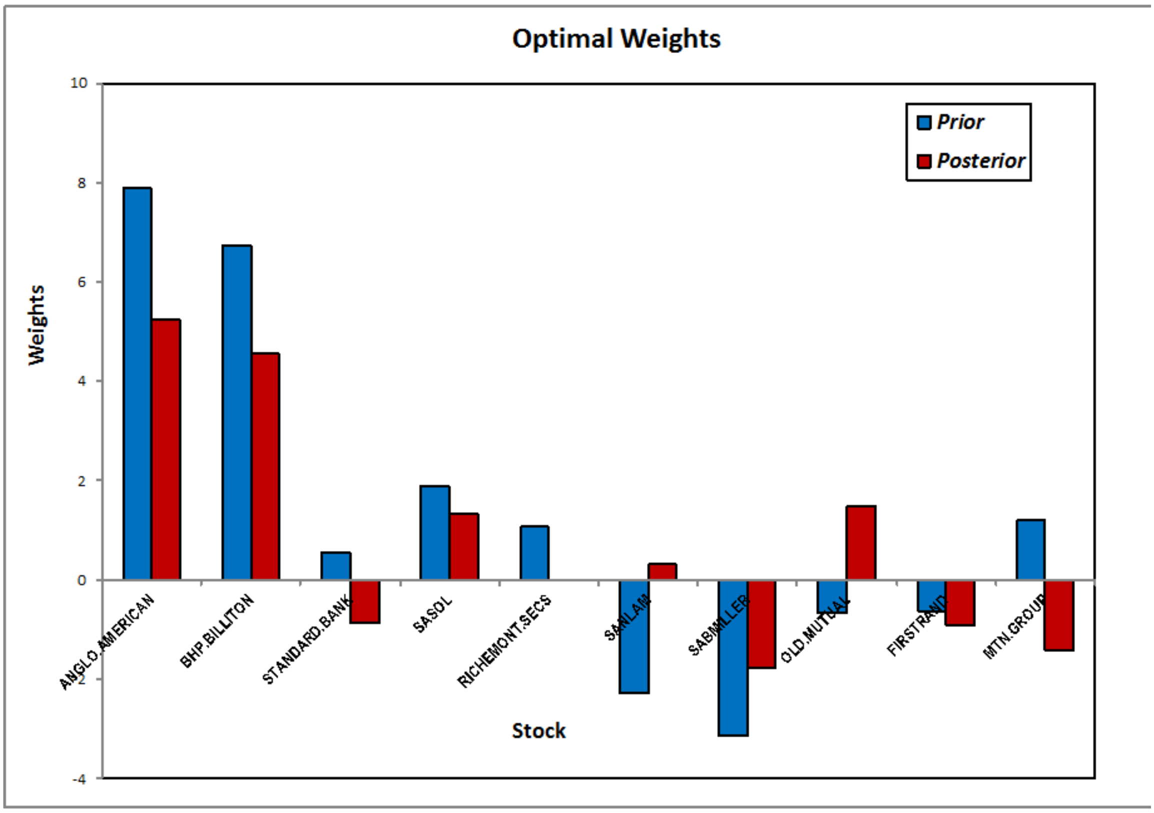

Figure 7: Posterior weight allocation that result from using the α-Stable Distributions, with VAR at 1% as a risk measure and short positions allowed.

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