Analysis of Convertible Bonds

by

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Analysis of Convertible Bonds

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The convertible bond is a complex instrument with many embedded features. These features can result in the convertible bond behaving like both debt and equity, while affording the issuer as well as the holder certain guarantees against total loss. The review of valuation techniques specifically for convertible bonds started almost three decades ago with the paper by Brennan and Schwartz [8], while more recent models take almost all possible situations into account.

Other than trying to price convertible bonds more accurately, the literature has also tried to focus on the reasons behind convertible bonds issue. The rationale behind convertible bond's issue, as well as a market study is conducted.

The older and more current pricing techniques are examined, and the latest techniques are implemented and compared. We see how the effect of an issuer's default, and the consequences thereof, can be made to influence the possible value of a convertible bond. We also investigate the effect of foreign exchange risk on cross currency convertible bonds and apply previous techniques to the newer methods of valuation.

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Chapter 1

Introduction

Convertible bonds are complex instruments that exhibit features of both debt and equity. Convertible bonds receive coupons and a principle payment similar to that of straight debt. However, should the holder decide to convert their convertible bonds into a predetermined amount of shares, the instrument ceases to exist. Most convertible bonds have the added feature that the issuer can call the convertible bond back, and a put option allowing the holder to put the bond back to the issuer. These added features complicate the analysis and pricing of convertible bonds.

Since we consider a convertible bond as part debt and part equity, its value is subject to a fluctuating stock price and interest rate. While no one knows what the instrument will mature as, many different investors get involved either for the debt or equity benefits that the convertible bond offers. A share price increase would mean the convertible bond gets converted and ends up as shares, otherwise the holder would keep receiving coupons and the convertible bond would mature with a principle payment. In case the company seems likely to default on the coupon payments, the holder sometimes has the right to put the instrument back to the issuer.

We may want to consider a convertible bond as a combination of a bond, and a warrant on the company's stock with strike equal to conversion value. While this is a straightforward description it is not an accurate approach to pricing this instrument. Since the conversion, call and put options on the convertible bond allow the instrument to terminate early, the cash flows and warrant like options will expire immediately; this is not taken into account when using a bond plus a warrant to value the convertible bond, since it is the best of a bond and a warrant.

The first Black-Scholes type pricing of convertible bonds started with the papers of Brennan and Schwartz [7] and Ingersoll [22, 23]. Ingersoll uses arbitrage arguments to derive optimal call and conversion strategies, then derive analytical solutions for the value of a convertible bond in some special cases. Brennan and Schwartz [7] derive a partial differential equation for the price of a convertible bond with call provisions, coupons as well as dividends. Finite difference methods are then used to solve the partial differential equations subject to certain boundary and final conditions. Later they extended this by adding a stochastic interest rate [8], and conclude that the addition of this extra stochastic factor adds complexity to the model with little more accuracy. Nyborg [32] later extended this model by introducing coupons that are financed by selling risky assets, as well as dividends that are a constant fraction of the risky asset.

Brennan and Schwartz [9], Nyborg [31], Asquith [1] and Jensen and Meckling [25] examine reasons for the issue of convertible bonds, as well as their influence on the companies management. They state and justify reasons like sweetening debt, delaying equity and changing risk patterns for the issue of convertible debt. Companies that believe their stock price will rise issue convertible bonds as a method of selling common stock at a price above the current market. Another rationale that has been investigated is that lower coupons sweeten debt for the issuer, making it easier for higher risk companies to raise capital.

We may think of convertible bonds as derivatives of both the underlying interest rate and equity. A two factor Black-Scholes model is often used since it accounts for both stochastic interest rate and fluctuating equity prices. We account for default risk in the Black-Scholes model by using a credit spread in discounting the derivatives value in time. Only part of a convertible bonds value is exposed to default risk. This is due to future cash flows not being guaranteed, since there is a possibility of early termination due to the put or call provisions embedded in the convertible bond. This and other methods like that of Merton [28], who take default risk into account by also applying it to the whole instrument, is invalid for the pricing of a convertible bond since only the part related to future cash payments is exposed to default risk.

The next generation of models that attempts to include default risk started with Tsiveriotis and Fernandes [35] who decompose the convertible bond into two components. A cash component and an equity component. These yield two coupled partial differential equations that allowed for different discount factors to be applied to each. Where a riskless discount factor would be applied to the equity component, a risky discount factor would be applied to the cash component, since only payment is subject to default risk. While this model introduced default risk to some extent, it did not model what would actually happen in the event of default. Ayache, Forsyth and Vetzal [3] presented a model that not only handeled the event of default, but also how much of the bond is recovered if default occured. Yigitbasioglu [37] extended the Tsiveriotis and Fernande's [35] approach to include cross currency convertibles. This technique is easily applied on the Ayache, Forsyth and Vetzal [3] approach so that their model can also value cross currency convertibles.

This dissertation is arranged as follows: Chapter 2 describes the basics behind the convertible bond, and uses a binomial tree to show the effect that changing the stock price has on its value and the greeks, largely due to [11]. Chapter 2 also investigates the effect that a volatile equity and a fluctuating interest has on the instrument. Chapter 3 describes the size of the convertible bond market at present, and discusses the rationale behind the issue of convertible bonds. Chapter 4 investigates the optimal strategies for both issuer and holder, as well as one and two factor models for valuation. Chapter 4 also introduces the boundary conditions and final conditions necessary in valuation. Chapter 5 discusses the current models by Ayache, Forsyth and Vetzal [3], Tsiveriotis and Fernandes [35] and Yigitbasioglu [37]. The method of Ayache, Forsyth and Vetzal [3] is extended to price cross-currency convertibles in the same manner that Yigitbasioglu [37] extended Tsiveriotis and Fernandes [35]. These models include default and foreign exchange risk. Chapter 6 concludes by presenting finite difference methods to implement the models of Chapter 5.

Chapter 2

The Convertible Bond Basics

This chapter introduces the basic concepts of convertible bonds and how they perform when variables like maturity, share price, volatility and interest rate are changed. We investigate the effect of introducing the issuer's call back option, as well as the less common put option. We also give an overview of the convertible bond greeks, and investigate the effect of time decay on the greeks. The graphs presented were obtained by pricing with the binomial tree approach, which doesn't produce accurate results but is accurate enough for the purposes of this discussion.

2.1 The Basic Convertible Bond

To illustrate the market terminology and how the convertible bond behaves under different conditions we consider the following example.

Nominal value: \$1000, **Maturity:** 4 Years, **Market price:** \$1100
Coupon: 8%, **Coupon frequency:** Semi-Annual
Conversion ratio: 125, **Conversion price:** \$8
Call price: 1100, **Call period:** After 2 Years
Put price: 1050, **Put period:** 3 years from issue only

The conversion price is just the nominal value divided by the conversion ratio. The market price of \$1100 means that the price at issue is 110% of the nominal \$1000. Therefore this convertible bond will cost \$1100 and can be converted into 125 shares at a later date. Consider buying this convertible bond and immediately converting the instrument into the shares. If the current

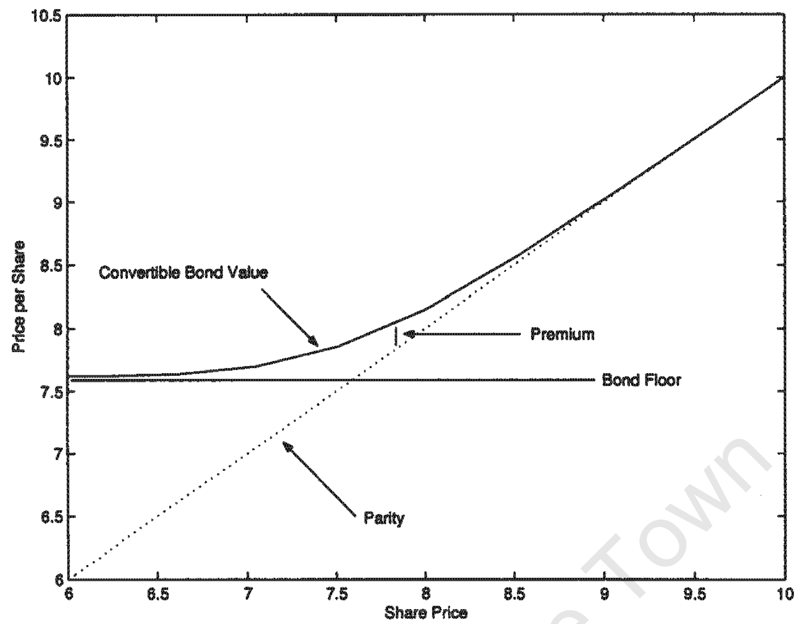

Figure 2.1: Convertible Bond value due to share price variation

share price is 7.8 we are left with shares to the value of $975. In this instance we seem to have lost $125. The $125 is known as the premium and the $975 as the parity, we can see the payment of the premium as an advantage since the share price may fall over the period of the convertible bond. The embedded option gives us a method of avoiding this risk, and at the same time allows us to profit from a share increase.

The bond floor shown in figure 2.1 is the value of a straight bond with none of the extra features of a convertible bond. This bond floor is a lower bound to the value of the convertible bond since the extra features give the holder more rights which raises the value. Note that this might not be true if the issuers right to call is the only option available on the convertible. We also need to know that the bond floor will move if the interest rate changes. All of the attributes of the bond floor are easily understood if you consider the valuation of a straight bond, which is by the present value of all the bonds cash flows.

Figure 2.1 shows us the relationship between the value of the convertible bond and the share price. At a high share price the convertible bond behaves almost exactly like the share while at a low share price the convertible bond's value will decrease at a lower rate. The fact that

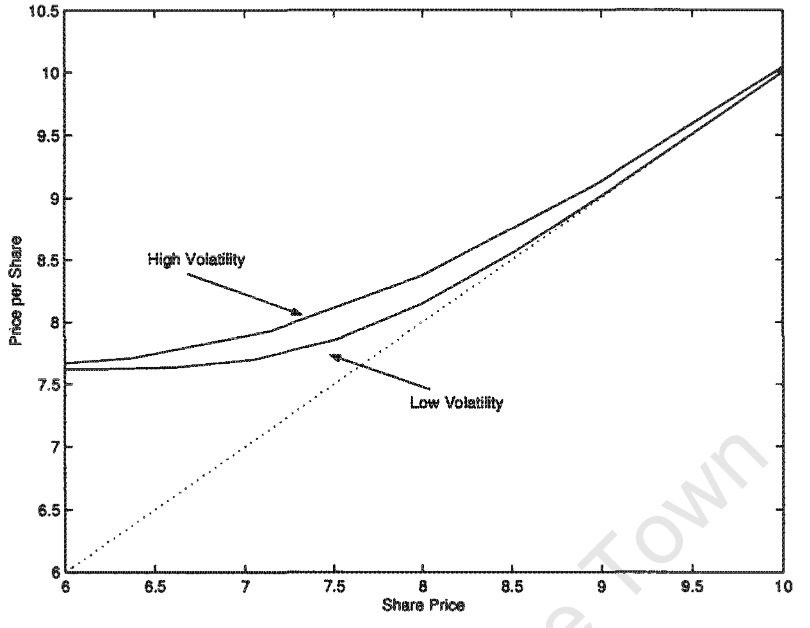

Figure 2.2: Convertible Bond value variation due to changing volatility in share

the convertible bond behaves like equity at a high share price means that an investor can take advantage of a rising market. While the fact that at a low share price the convertible bond levels out at what is known as the *bond floor* (the level at which convertible bonds tend to at low share prices), will give the investor some protection against a falling market.

"The convertible bondholder is perhaps best thought of as a kind of fair-weather stockholder and foul-weather bondholder." - Brennan and Schwartz [9]

In figure 2.1 we see that the value of the share is equivalent to the parity portion of the convertible bond: The vertical difference between the value of the share and the value of the convertible bond is the premium portion of the value of the convertible bond. Investors pay the premium for the advantage of being exposed to a share price increase but not a decrease; this premium is low at high share prices while more prominent near low share prices.

2.1.1 Maturity

Convertible bonds with a longer maturity behave more like equity, this is due to the fact that the nominal payout has less influence with longer maturities, due to present value. Therefore if the maturity of a convertible bond is increased then the value of the convertible bond is reduced

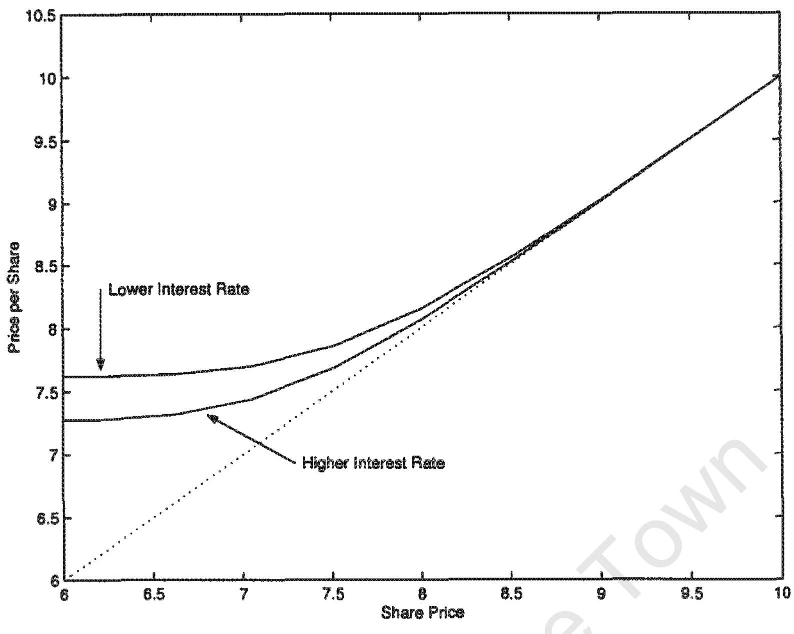

Figure 2.3: Convertible Bond value variation due to changing interest rate

as the bond floor is lowered. The bond floor is lowered since the influence of the present value of each cash flow is reduced. Investors will want to pay more for the shorter dated convertible bond since the parity is not far above the bond floor and so the instrument is not behaving strictly like equity. This is because the convertible is mostly a straight bond at lower maturity since the nominal payout is large in comparison to the value of the other features.

2.1.2 Coupon

The typical coupon frequency is semi-annual. A higher coupon raises the bond price and the bond floor, lowering the convertible bond's premium. On the other hand, a smaller coupon decreases its value and lowers the bond floor, thereby increasing the convertible bond's premium.

2.1.3 The Effect of Volatile Equity

The higher curved line in figure 2.2 represents the value of a convertible bond on a highly volatile share. The lower curved line is on a share with lower volatility. The price of the convertible bond increases with the volatility of the share as we expect, since a higher volatility yields a higher chance of ending with a price significantly greater than the conversion price.

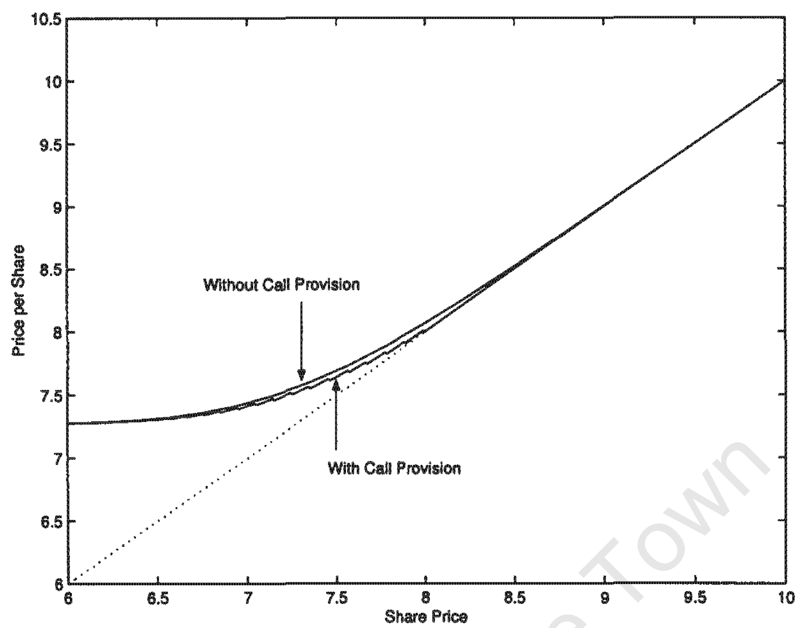

Figure 2.4: Convertible Bond value variation due to Call Option

2.1.4 The Effect of Fluctuating Interest Rate

A convertible bond is sensitive to interest rate changes just like a straight bond: the bond price falls when interest rates rise. We investigate the effect of interest rate changes in figure 2.3. The higher curve represents the price at lower interest rates, and at high share prices with the convertible bond trading like equity. It is almost completely independent of interest rate changes, while at low share prices the price is almost as dependent as that of a straight bond. It is important to note that since the bond price is dependent on the interest rate at low share value, the bond floor will move. This may remove the advantage of investing in convertible bonds for downside protection, since interest rate fluctuations may drop the bond floor to a lower level.

2.1.5 The Call Back Option

The bulk of convertible bonds issued have call features embedded allowing the issuer to repurchase the bonds at a particular price, called the call price, thereby creating a more flexible capital structure for the issuing company. Such a call option often comes with the protection of a hard non-call period, which disallows the issuer from calling the bond before a certain period

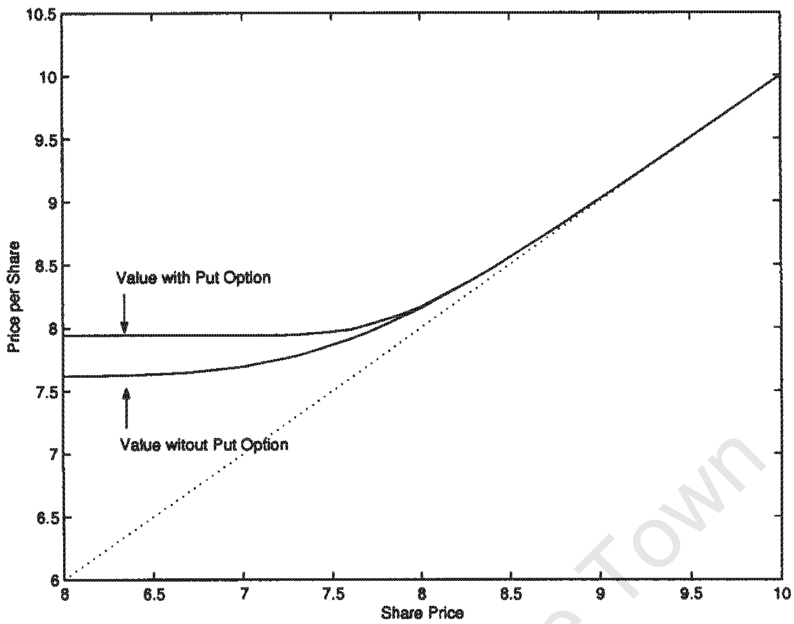

Figure 2.5: Convertible Bond value variation due to Put Option

expires. Some bonds are also issued with soft call provisions where the issuer may call the bond only if the equity trades higher than a certain price level called the trigger price, for longer than a pre-specified period. Since the call price is usually lower that the conversion value, the holder would convert and receive the higher amount. Convertible bonds are almost always issued with a soft call period, giving the holder time to convert to shares after the issuer has called, this lowers the value of the call slightly and increases the value of the bond. This call period also changes the rationale behind a company calling the bond, and is discussed in more detail in chapter 3.

The call schedule sets out the prices and dates of the call price as it varies with time. These are agreed upon before sale. The introduction of a call back option invariably lowers the price of the convertible bond since the option is the issuer's and not the investors. We can see in figure 2.4 that when the share price is low, the convertible bond value will be lower than the premium value and the call option will have no effect, while at higher share prices the bond would be trading at parity anyway. Therefore no effect will be visible.

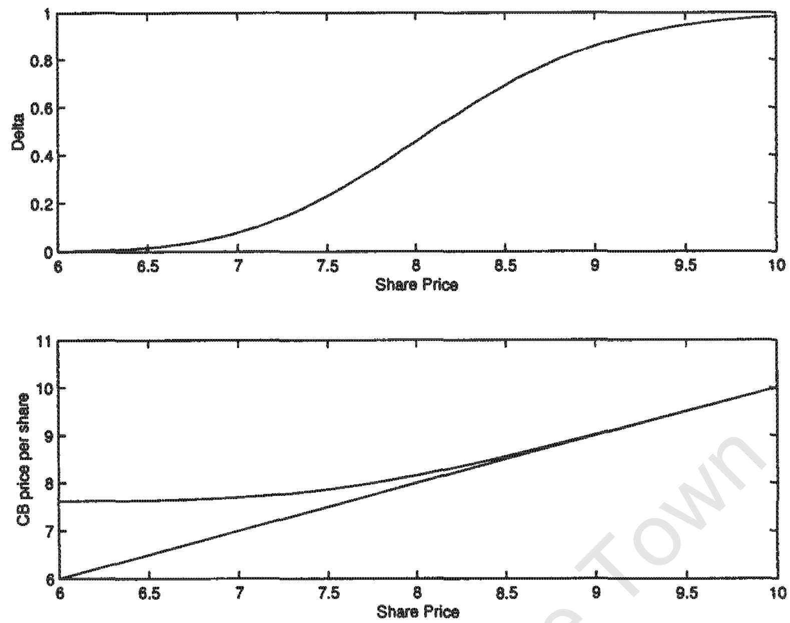

Figure 2.6: Delta and the Convertible Bond Value

2.1.6 The Holder's Put Option

Put provisions are less common in convertible bonds than call features; the put provision allows the holder to put the bond back to the company, at a particular price known as the put price, on predetermined dates described in the put schedule. Figure 2.5 shows that a convertible bond with a put option has a higher value than one without a put option. This is to be expected since the extra option to the buyer should increase the value of the instrument. The put option has most influence at lower share prices since receiving cash today is always better than receiving cash at some later date. Convertible bonds with high coupons have less valuable put options, since it would be more valuable to receive the coupons and the nominal instead of the shares. A low share value may yield less profit at expiry than receiving the nominal.

2.1.7 Cross Currency Convertible Bonds

These are convertible bonds which are convertible into shares in one currency but redeemable into cash in another currency. Since the exchange rate fluctuates, the conversion price also changes constantly, making pricing more complex.

2.1.8 Refix Clauses

Refix clauses were first issued by companies in Japan, and were designed to make the instrument more attractive to investors. Refix clauses alter the conversion ratio or conversion price of the convertible bond subject to the share price level on specific days of the life of the instrument. Usually if the average share price for a certain predetermined period trades below a predetermined price, then the conversion price is decreased, thus increasing the conversion ratio and giving the investor access to more shares. The reset feature therefore gives investors protection from a decline in the share price. Since this feature gives the holder more rights, it increases the value of the convertible bond and so increases the premium paid for the bond.

2.1.9 Dilution

The conversion rights offered by a convertible bond represent a call on the issuers shares. However any possible future dilution caused by the issue of new shares by the corporation is already reflected in the share price. Dilution will already in effect have lowered the share price and volatility by the time the bonds get converted. In a study into the rationale behind the issue of convertible bonds [32], we see that conversion into shares through convertible debt causes less dilution than the outright issue of new stock, see Section 3.2. We therefore ignore the effect caused by dilution in the valuation method.

2.1.10 Default and Foreign Exchange Risk

The payments made by the issuer of a convertible bond are not guaranteed, this creates default risk. While we can always convert to shares in this event, the shares would most probably also be worthless if the company has already defaulted on its debt. Default risk is an issue that complicates the valuation of convertible bonds and is discussed in detail in Chapter 5.

A cross currency convertible bond pay cash amounts from a different country, this introduces foreign exchange risk. This also complicates the valuation of convertible bonds since a fluctuating exchange rate would compromise the coupon payments as well as the nominal. The models presented in Chapter 5 explain how to deal with this kind of risk.

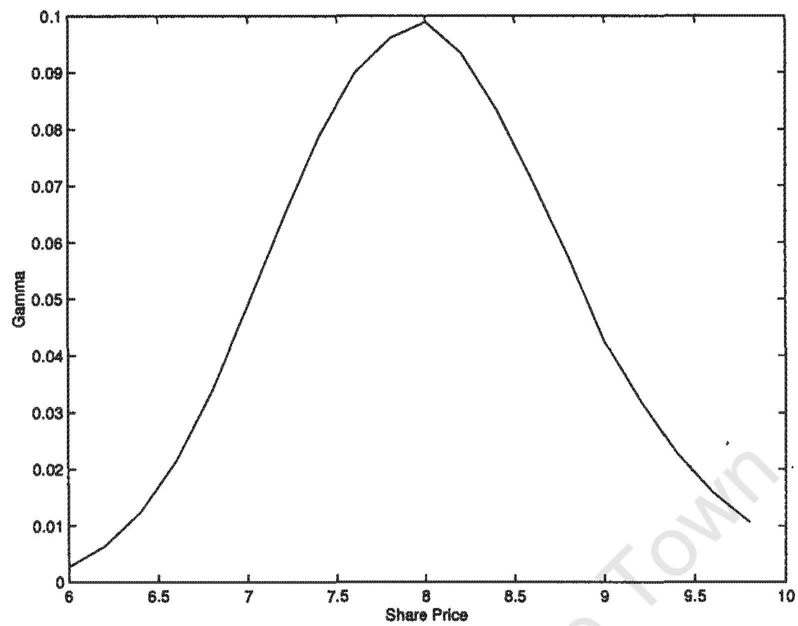

Figure 2.7: Gamma of the Convertible Bond

2.2 The Convertible Bond Greeks

This section assumes for illustration purposes that the underlying share pays no dividends, that the convertible bond pays no coupons, and that there are no call or put options embedded in the instrument. The price sensitivities were calculated by two applications of the binomial method, and varying the relative parameter between each calculation.

2.2.1 Delta

The delta, or sensitivity to a change in the underlying share price, tells us how many shares to sell short to hedge out a long position in a convertible bond. If we are long a convertible bond with a conversion ratio of 100, and the delta at the time is 0.2, we need to short 20 shares to be hedged against a shift in the underlying share price.

Since the value of delta is the first derivative of the value of the convertible bond, we see that the delta approaches zero as the value of the underlying share gets lower. This is because the value of the convertible bond will begin to tend to the bond floor, and will thus have a zero derivative, shown in figure 2.6. When the value of the underlying share moves higher the value

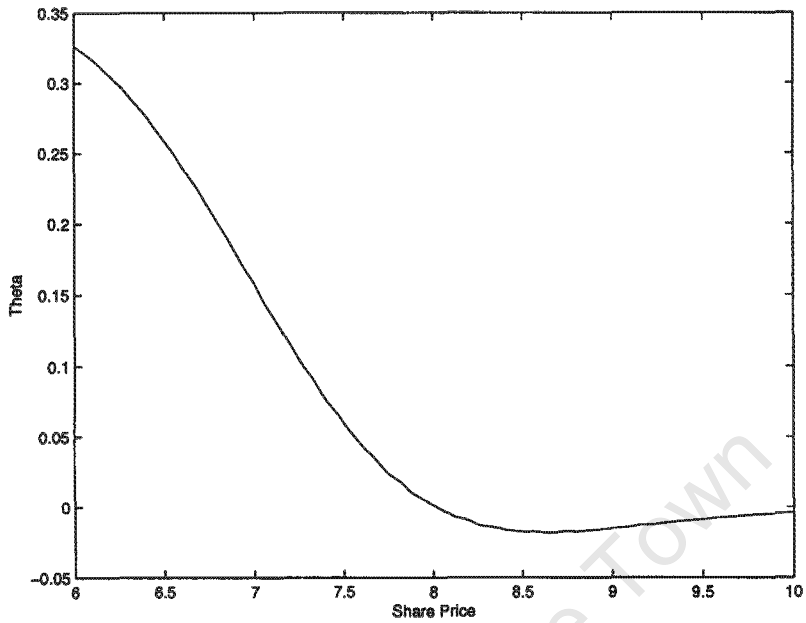

Figure 2.8: Theta of the Convertible Bond

of the convertible bond will tend towards it. Therefore the value of delta will approach one. This once again shows us that as the share price drops, the convertible bond becomes less dependent on the value of the underlying share. Hence delta becomes zero, and the delta will tend to one as the share price rises. That confirms that the convertible bond value behaves exactly like the share as share prices increase.

2.2.2 Gamma

The gamma, the rate of change of the delta, shown in figure 2.7, shows us that gamma is at its largest around the conversion price, while dropping to zero as the share price rises or falls. Looking at the delta graph we see why this happens since gamma is the first derivative of delta.

2.2.3 Theta

Next we investigate the time decay or theta of a convertible bond. Since most derivative instruments with an option component are influenced by time decay, we should see the same for convertible bonds. Figure 2.8 illustrates that the theta of a convertible bond is positive when share prices are low and negative when the underlying share price is high. We now investigate

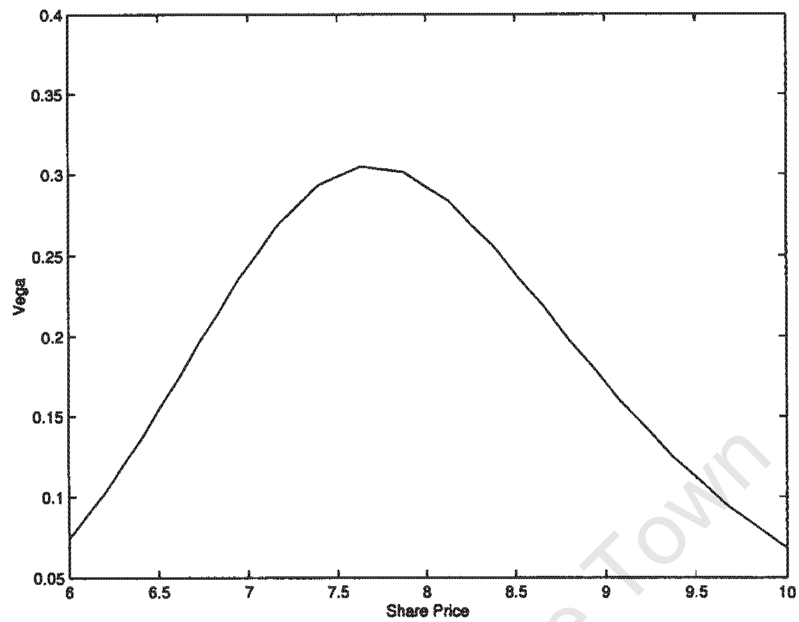

Figure 2.9: Vega of the Convertible Bond

this behavior starting at low share value; the call option component at low share value will have no worth and will therefore have no time decay. The convertible bond behaves purely like a zero coupon bond redeemable at the conversion price. As time passes, the discounting effect of the nominal payout at maturity will decrease, and so the zero coupon bond would increases in value so that theta must be positive.

At higher share prices the call option component begins to have an effect. Equity options always suffer time decay and this reduces the positive theta effects of the bond. The higher the share value we consider, the more the positive theta effect of the zero is canceled out by the negative theta effect of the call option. At highest share prices the effect of the option begins to diminish and the two effects cancel each other in the limit. At very high share value the convertible bond suffers no time decay and behaves like straight equity.

2.2.4 Vega

Since the price of call options on equity are dependent on volatility, it is natural to assume that convertible bonds are as well. We next investigate the vega, or sensitivity to fluctuating

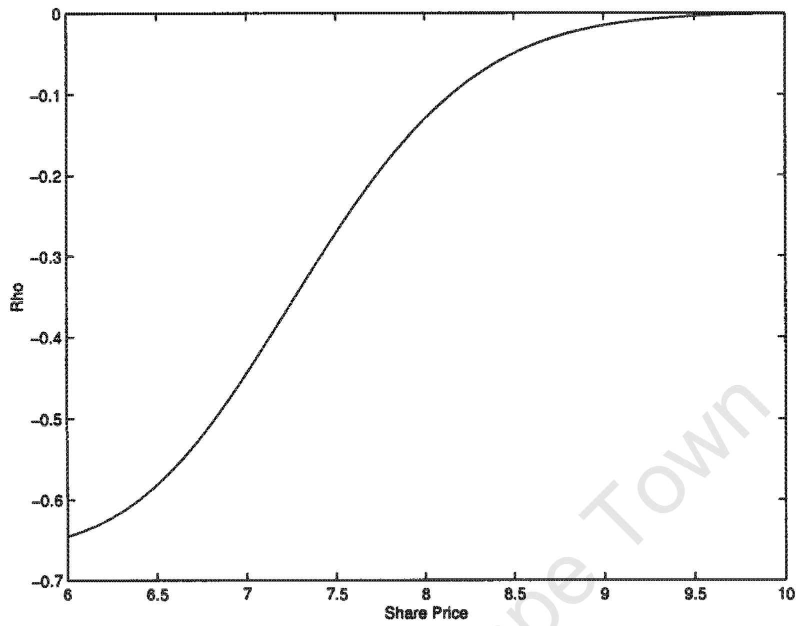

Figure 2.10: Rho of the Convertible Bond

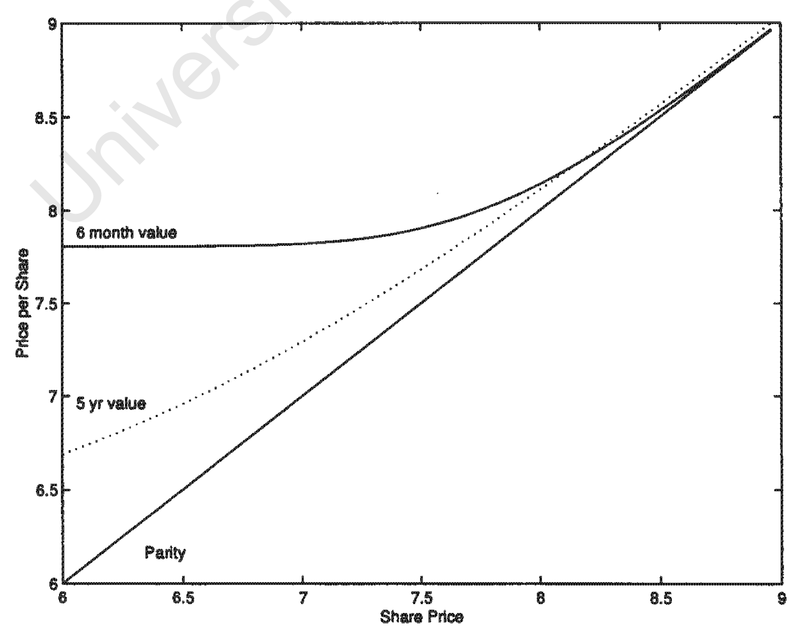

Figure 2.11: Price of the Convertible Bond as time passes

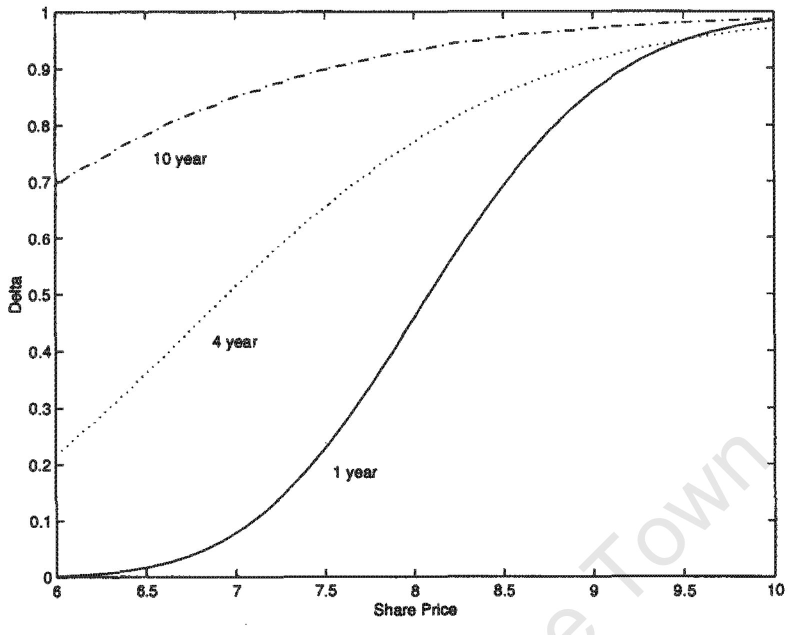

Figure 2.12: Delta at various times to expiry

volatility of a convertible bond. Figure 2.9 shows the vega of a convertible bond with varying share value. Here we can see that it is at a maximum around the conversion price and that it decreases in a non-linear manner at higher and lower share value. This reflects the property of the embedded call option which is most sensitive to volatility changes at or near the exercise or conversion price.

2.2.5 Rho

We consider a convertible bond a derivative of the share price more than the interest rate, as shown in Chapter 4. Therefore we treat rho as we would any other derivative on the share price, and not as that of a bond. We know that if interest rates increase then bond prices will fall. At very low share prices when the instrument is behaving like a straight bond, the convertible bonds sensitivity to an interest rate movement, called rho, is a maximum. We see this in figure 2.10. We also see that at very high share prices when the instrument is behaving like straight equity, that rho tends to zero.

A delta hedged portfolio is long convertible bonds and short shares, and will be immune to

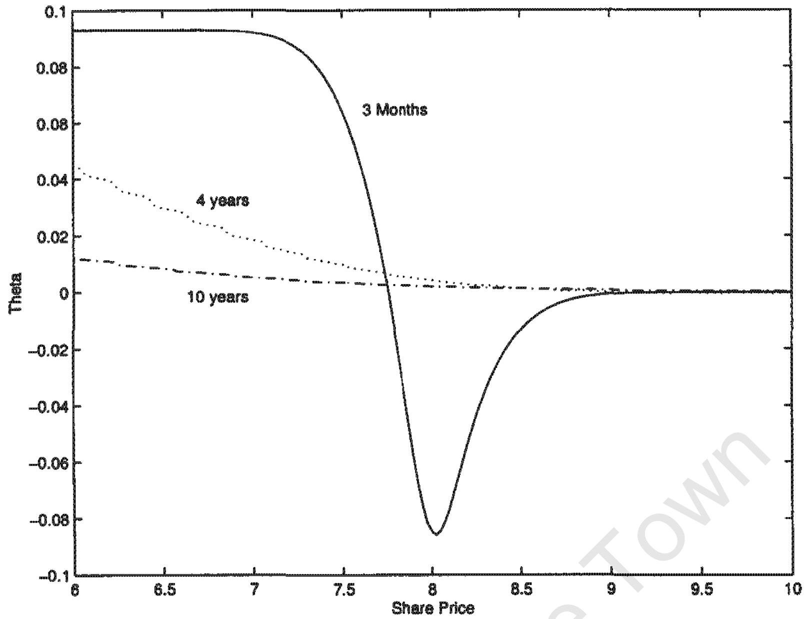

Figure 2.13: Theta at various times to expiry

changes in the share price but not to changes in the interest rate. To hedge out this interest rate risk we could use interest rate futures or interest rate put options. Interest rate futures decrease in price when interest rates increase, and increase in price when interest rates decrease, while interest rate put's pay the opposite way.

2.2.6 Decay of the Greeks

We investigate the effect of time on the convertible bond sensitivities. Figure 2.11 shows how a standard convertible bond price changes over time. We notice immediately that if the share price is low, the convertible bond increases in value with less time to maturity. This is more visible the lower the value of the underlying shares. The change of delta seen in figure 2.12 shows that at low share price with 10 years to expiry the convertible bond price curve has a moderate slope with delta positive. With passing time the instrument increases in value and at the same time the curve becomes more horizontal. At low share prices the delta must decrease as time passes, as we can see in figure 2.12. This is due to the fact that the principle is still far away with longer maturities, and we require more shares to hedge the convertible bond than we would nearer the maturity date.

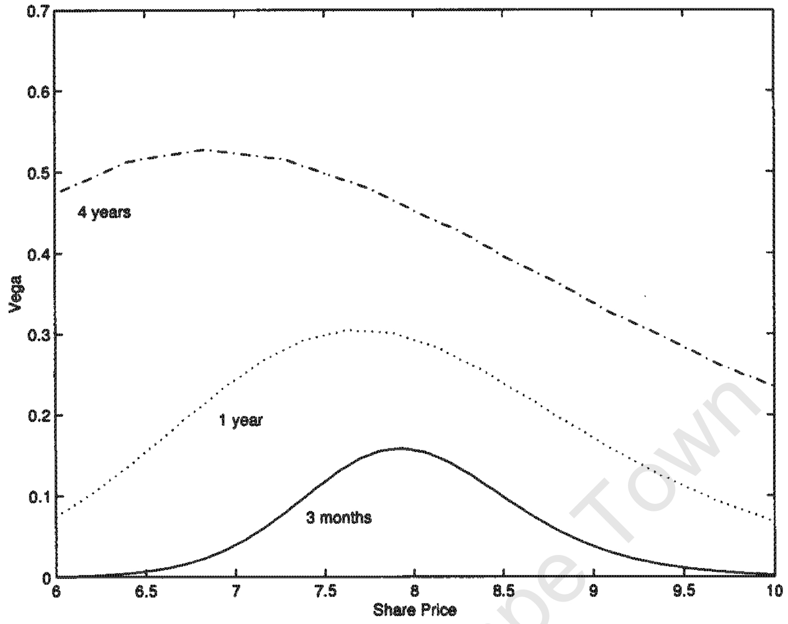

Figure 2.14: Vega at various times to expiry

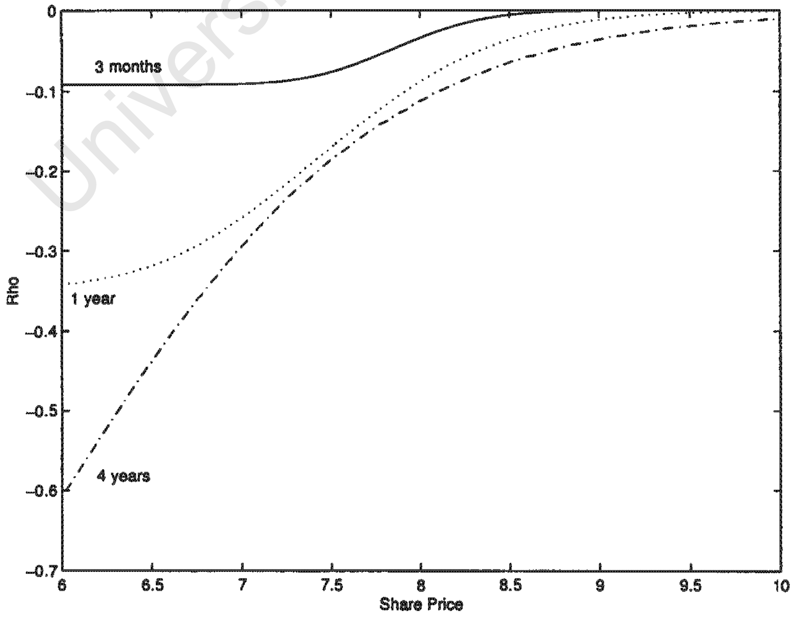

Figure 2.15: Rho at various times to expiry

Figures 2.14 show us that the vega implies that longer-dated convertible bonds are much more sensitive to volatility inputs than short-dated ones. So if we incorrectly specify volatility by a small percentage, it will have a much greater effect on longer-dated convertible bond than on a short-dated one. Figure 2.15 shows us how the rho changes over time and we see that longer-dated convertible bonds are more susceptible to interest rate risk. The effect is more pronounced the lower the share price.

The next chapter describes the state of the convertible bond market at present, as well as describing possible rationale behind the issue of convertibles instead of straight bonds.

Chapter 3

The Current Convertible Bond Market

3.1 The Market at Present

The global convertible bond market[1] size in July 2003 was $607 billion, made up mostly by the American market. The market size per country is given in table 3.1, and we see that the American market makes up more than fifty percent of the global market.

Country	Size ($Bil)
America	305
Europe	177
Japan	77
Asia (Excl. Japan)	31
Canada	20

Table 3.1: Estimated Global Market Size

The American market is currently at $305 billion, almost double what it was 5 years before when it was $161 billion, while in 1988 it was only $54 billion. The growth of the American market is shown in figure 3.1, while the number of convertible bonds issued per year in the American market is shown in figure 3.2.

[1]Data taken from www.convertbond.com [14].

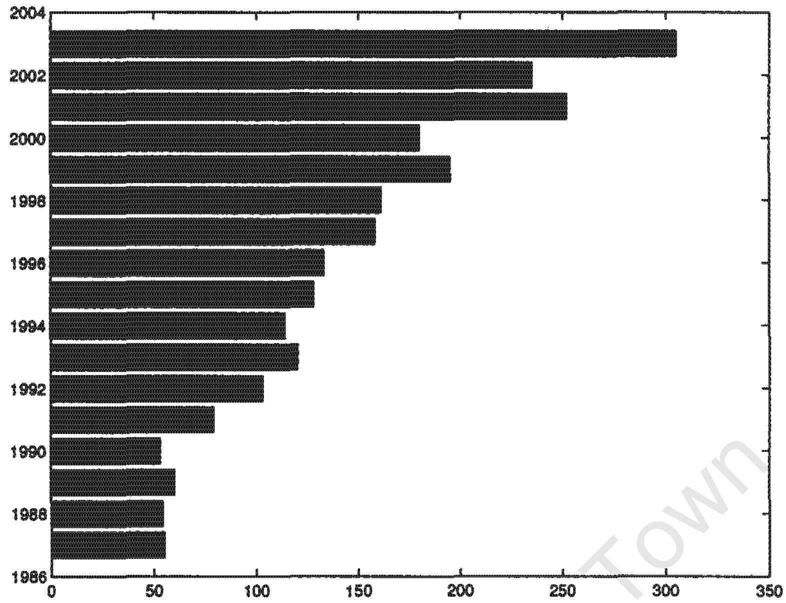

Figure 3.1: Estimated Market Capitalization of Convertible Bonds

Convertible bonds are not the only type of convertible instruments at issue; other types include zero coupon convertible bonds, mandatory convertibles bond and convertible preferred stocks among others. A convertible preferred has features similar to that of a convertible bond, while at the same time paying quarterly coupons and sometimes even a long or perpetual maturity. Mandatory convertibles or Participating Equity Preferred Shares (PEPS) are a type of convertible security which provides investors with a high current income and equity like participation. Table 3.1 shows the number of convertible instruments at issue.

Type	Number
Convertible Preferred Stocks	118
Convertible Bonds	607
Zero Coupon Convertible Bonds	98
Mandatory Convertibles	79

Table 3.2: Number of Convertible Securities at Issue

Figure 3.3 shows us the breakdown of the convertible bond market in America per sector. We can see that the information technology and health care sectors are primarily responsible for the issue of convertible bonds.

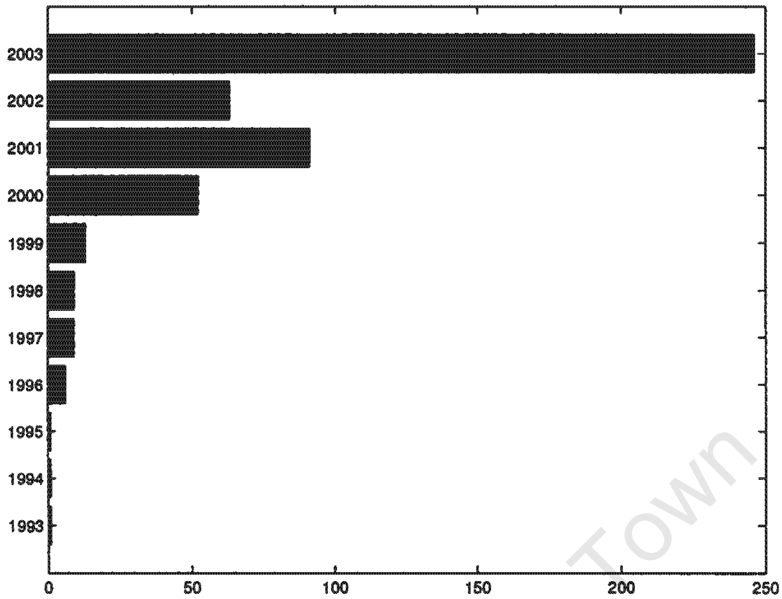

Figure 3.2: Number of Convertible Bonds Issued per Year

The performance of the convertible bond market is summarized in figure 3.4.

Issuing convertible debt is a way for small, fast growing and non-creditworthy companies to obtain capital. As we see in table 3.1, that 58% and 44% of American convertible bonds are not rated by Standard & Poor's and Moody's respectively and therefore have junk bond status. Noting that very few convertible bonds have an A credit rating, with the bulk of the rated convertible bonds as B rated[2].

Rating	Moody	S&P
A	13	27
B	203	239
C or Lower	38	71
Not Rated	353	270

Table 3.3: Rating Status of US Convertible Bonds

[2]A Rated: Moody A1 to Aaa, S&P A to AAA, B Rated: Moody B1 to Baa3, S&P B to BBB-, C Rated: Moody C to Caa3, S&P C to CCC- as well as D

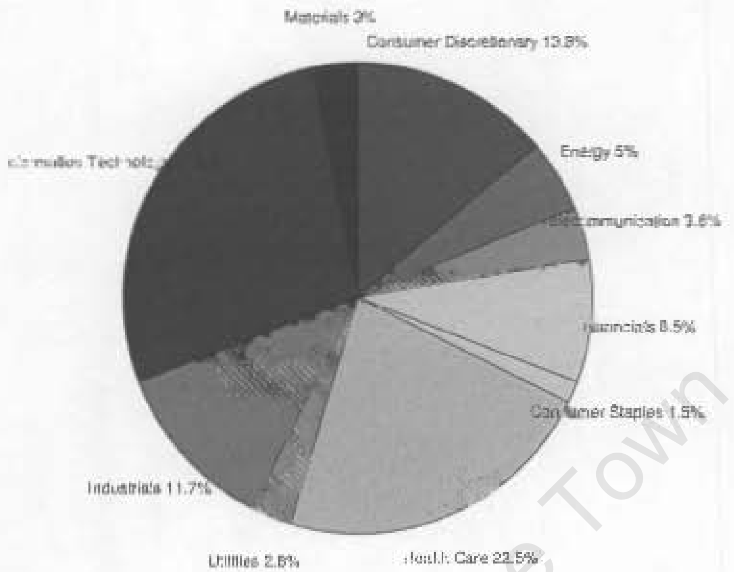

Figure 3.3: Convertible Bond Issue per Sector

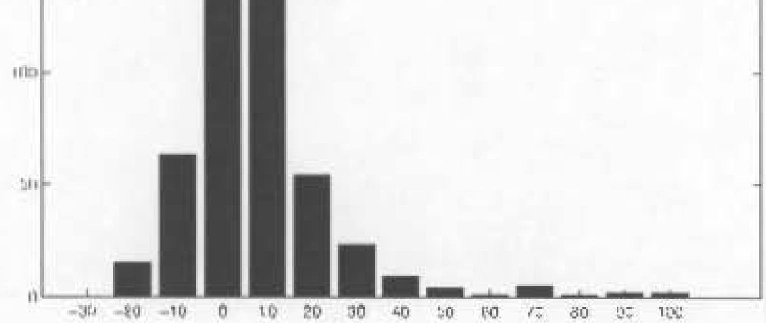

Figure 3.4: Performance of the American Convertible Bond Market

3.2 Why Convertible Bonds are Issued

Investing in equity has the advantage that it comes with a certain amount of insurance against complete loss. Straight debt has less security since a flailing company would default on its larger debt. This implies that convertible bonds would be issued by firms of medium quality, since its hybrid nature of equity and bonds would give the investor more options. The findings of three studies[3] into the rationale behind companies issuing convertible bonds was summarized by Nyborg [32] and is given in table 3.2. In the next two sections we examine the two main reasons

Reason	1955	1966	1977
Delayed Equity	82%	68%	40%
Sweeten Debt	9%	27%	37%
Other	9%	5%	23%

Table 3.4: Reasons for Companies Issuing Convertible Bonds

thought to motivate companies to issue convertible bonds more closely, as well as a third more recent justification.

3.2.1 Delayed Equity

We see that one of the most common but declining reasons for issuing a convertible bond is to add delayed equity to the firm's capital structure. This is due to the fact that companies believe that their stock price will rise, and convertibles present a method of selling common stock at a price above the existing market. If the stock price rises and the bonds are converted, or can be forced to convert, then new stock is issued with less dilution than stock issued in the usual manner since the terms of conversion were set initially at lesser terms. If the stock price doesn't rise the bonds will not be converted and the nominal and coupons will have to be paid at a lower fee, unless the issuer is still able to force conversion with the call feature.

Asquith [1] examines a set of convertibles issued over a three year period and finds that calling does occur as soon as the conversion value exceeds the call price by a certain premium. There is no cash flow advantage to not calling. The premium is obvious since there is always a risk

[3]See Nyborg [32] for references to these studies.

that the share price may fall below the call price in the call notice period. This premium is dependent on the volatility of the share price. Asquith [1] finds that twice the average monthly standard deviation of the share is usual for the premium. The cash flow advantage could be that the dividend payments on the converted shares could be more than the after-tax coupon payments on the unconverted bond. The cash flow advantage can be altered by reducing the future dividends at the same time as forcing conversion. Both of these actions have a negative impact on the company in the marketplace.

On the other hand, the benefit of convertible bonds as delayed equity is only apparent if the conversion is not forced by the call feature. Nyborg [31] finds that forcing conversion is an indication of the issuing companies need to prepare for a fall in their stock price. Falling stock would leave the issuer with unconverted convertibles, and future coupon payments. This means that not forcing conversion by calling implies confidence on the issuer's part that stock will rise and conversion will be voluntary. A part of Nyborg's [32] study finds that many firms will not force conversion. They will rather give the holder an incentive to convert by raising the future dividends, avoiding any negative market reaction while at the same time sending a positive signal to investors about the state of the company.

If firms do delay conversion it means that any pricing model that does not take this into account will undervalue the convertible bond. This could be rectified by assuming that conversion is forced as soon the conversion value rises above the premium discussed above.

3.2.2 Sweetening Debt

A growing reason for the issue of convertible bonds is that it is a less costly way of raising capital than straight debt, since the coupons that the issuer needs to pay are usually lower than that of normal bonds. Nyborg [32] notes that this is a misconception since Modigliani and Miller [29] show that in perfect markets convertible debt cannot be a cheaper source of financing than straight debt or equity. The lower coupons are merely brought on by the value of the right of conversion, but might be advantageous to defer costs.

3.2.3 Changes in Risk

Brennan and Schwartz [9] bring to light a convincing argument concerning the changes in risk and the rationale behind the issue of convertibles. They argue that higher risk and greater price volatility will decrease the value of any straight debt they might issue, while any warrant on the companies stock will value higher because of increased risk. Therefore the effect of higher risk on a convertible bonds debt component is offset by the right of conversion, and will be unaffected by the risk of the issuing firm. This dynamic allows companies that the market perceives as risky to raise capital on better terms than straight debt.

The following argument has been brought forward by Jensen and Meckling [25], it is due to the fact that the management of a company usually holds shares in the company. Since downside risk is held mostly by the holders of straight debt and upside returns are mostly profitable to the shareholders, it is in their interest to increase the risk in their future operations. The future risk pattern of a company is of less concern to a convertible bondholder though, since the right to conversion into the companies shares would be used if the shares increase in value.

Chapter 4

Pricing Models

4.1 Optimal Strategies

The valuation of a convertible bond is greatly simplified if we firstly determine the call and conversion strategies that the corporation and investor will follow. If we know that the investor will never convert the convertible bond prior to maturity, its value would clearly be affected. The ideas on investor and issuer's strategies in this section were first presented by Brennan and Schwartz [7] and extended by Ingersoll [22]. We begin by making the assumption that both the corporation and the investor expect each other to pursue the most optimal strategy and that they do so themselves as well. Then we can derive useful call and conversion strategies.

We make the assumption that the investor's optimal conversion strategy will be to maximize the value of the convertible bond at each instant in time. The issuer will try to minimize the value of the convertible bond at all times. The Modigliani-Miller theorem [29] states that the market value of any corporation's securities is independent of its capital structure. Also, no tactic followed by the corporation in relation to the composition of its capital structure can alter the value of its securities. This implies that the value of the convertible bonds are independant of the particular call and conversion strategies followed.

The corporation's equity is equal to the difference between the market value of the corporation and the value of the corporation's bonds and convertibles. Therefore by minimizing the value of the outstanding convertible bonds they will in turn be maximizing the value of the firms equity, given that the value of the bonds stay constant. The Modigliani-Miller theorem also states that a corporation cannot increase its market value by securing part of its capital through the sale of bonds. In this case the conversion of the bonds into shares cannot alter the total value of the corporation. Therefore the value of a share after the conversion of the bond is given by the value of the corporation before the conversion, divided by the number of shares outstanding after conversion has taken place. The following lemma, presented by Brennan and Schwartz, gives a strong condition on the value of the bond

Lemma

The only optimal time to convert an uncalled convertible bond would be at maturity, prior to either a dividend date or an adverse change in the conversion terms.

The proof by Brennan and Schwartz [7] relies on the fact that it is never optimal to convert the bond if its market value exceeds its conversion value. An adverse change is a change in the convertible bond's terms that reduces the value of the convertible bond to the holder. An uncalled bond will only sell at conversion value immediately prior to a dividend date or adverse change in conversion terms. If this were not true then the return on the bond up to the next dividend date or conversion term change would display stochastic dominance over the return on the underlying common stock. If the bond value is currently selling at conversion value, its rate of return up to the next dividend date or conversion change can never fall below the rate of return of the stock. This is stochastic dominance and is due to the fact that the bond value can never fall below the conversion value, since if it did it would simply be converted.

The Lemma makes computation easier, since conversion need only be considered at discrete dividend dates, at a call, or at an adverse change in the conversion terms. Since the investor can still choose between the call price and the conversion value if the corporation calls the bond for redemption, the value of the bond if called will be given by the maximum of the call price and the conversion value of the bond.

Lemma

The Corporation's optimal call strategy would be to call the bond as soon as its value equals the call price.

This is only true under theoretical conditions and not in practice as discussed below. The optimal call strategy is one that minimizes the value of the convertible bond. The minimum value of the convertible bond, if called, is the call price. This is because an issuer would never call the bond if it would give the holder more value, and if the bonds were left uncalled at a market value exceeding their value if called, their value would clearly not be minimized. Therefore we accomplish the optimal strategy by calling the convertible bonds when their value if not called equals their value if called; this is the strategy which minimizes the value of the bonds.

If the bond becomes callable and the conversion value exceeds the call price, then the bond will be called immediately since the uncalled bond would sell for more than the conversion value. Conversion value is then the value if called, hence minimizing the value of the bond. Note that we have assumed that the bonds may be called instantaneously. However the corporation is usually required to give notice of their intention to call. This introduces a new risk: the conversion value of the bonds will have fallen below the call price in the time between giving notice and the actual call. Also the bonds will not be redeemed but rather converted, therefore convertible bonds are sometimes called only after the stock price has risen above some premium. If we assume an efficient market and that the strategies mentioned above are used, then the bond will sell at a price equal to its value if called when the call is exercised. If this were not true we could have arbitrage profit between the pre-call bond price and the value if called.

4.2 Tree Pricing

A tree for the stock price can be constructed in the usual way, with interest rates deterministic and the life of the tree set equal to the life of the convertible bond. The value of the convertible bond at the final nodes is calculated based on any conversion options available to the holder. As we roll back through the tree we test whether it would be optimal for the issuer to call at this point, as well as testing whether conversion is allowed and would be optimal for the holder. If it is optimal for the issuer to call, conversion is also allowed, therefore we need to check if conversion were optimal. Other issues like a put feature, dividends and coupons would be handled in a similar manner. Note that dividends cannot be dealt with well in the tree approach since the tree would not recombine. Recombination is possible but not discussed here.

One of the complications in using this method is the discount rate used in the tree. If a convertible is certain to remain a bond it would be appropriate to use a risky discount rate reflecting the issuers credit rating. On the other hand, if the bond were sure to be converted we would use the risk-free interest rate instead. This causes complications since we cannot know when the bond will be converted, called or put back to the issuer.

Cheung and Nelken [10] propose a combination of two binary trees, an interest rate binomial tree as well as a stock price binomial tree. This would model the convertible bond price's dependency on both the stock price as well as the interest rates, but would still fail to account for the issuer's credit risk. Chueng and Nelken adopted the interest rate model developed by Kalotay, Williams and Fabozzi [26] as well as the basic stock price tree developed by Cox, Ross and Rubinstein [12] to combine together into a single *quadro-tree*. Each node of the tree has four descendants, two with stock prices stationary and interest rates changing, as well as two with stationary interest rates and moving stock prices. This would mean the first node would have both the interest rate and the stock price rising. The second would see the interest rate falling and the stock price rising. The third would again see the interest rate rising but the stock price falling. Finally the fourth node would have both the interest rate as well as the stock price falling. Cheung and Nelken [10] then use the same techniques as in the basic binomial tree to price the convertible bond.

The use of binomial trees for pricing is useful for the most basic of convertible bonds, but is too basic for the pricing of more complicated issues. Factors like credit and default risk, as well as interest rate risk are difficult if not impossible to include in binomial methods. For that reason we need to look at more theoretically complicated models, some of which will be presented next.

4.3 The one-factor Model

Assuming that interest rates are deterministic, and that the value of the convertible bond depends on the share price as well as the time to expiry, we can apply standard Black-Scholes analysis, where the hedging portfolio we use would be long one convertible bond and short $\frac{\partial C}{\partial S}$ of shares. Where C represents the convertible bond, and S the value of a share. Then we have eliminated the random element from the portfolio and in the usual manner we obtain the following partial differential inequality

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC \leq 0.$$

Brennan and Schwartz [7] implement a similar model by modeling the value of the firm V as a stochastic process, where V represents the aggregate market value of the firms outstanding securities. They assume that the convertible bond value C is a function of V and t and that V follows the stochastic process

$$\frac{\mathrm{d}V}{V} = \mu \mathrm{d}t + \sigma \mathrm{d}W,$$

with W a Brownian motion. Then by once again applying the Black-Scholes methodology they obtain a similar partial differential equation for the value of the convertible bond given by

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 C}{\partial V^2} + rV\frac{\partial C}{\partial V} - rC = 0.$$

The total value of the firm influences the convertible bond value through its effect on the probability of default and on the conversion value of the bonds, while it influences the value of straight debt only through the probability of default. The disadvantage in this model is that the firm value is not directly observable and that the capital structure of a firm is very difficult to model. The partial differential inequality needs to be solved numerically, subject to the boundary conditions discussed later in the chapter.

4.4 The two-factor Model

The two-factor model first presented by Brennan and Schwartz [8] incorporates a stochastic interest rate while modelling the value of the firm, as well as introducing call features and voluntary conversion prior to maturity. The model was extended by Nyborg [32] to incorporate a put feature, discussed later in the chapter with all the other boundary conditions. This model is more accurate since the interest rate affects the bond component of the convertible through the discount factor of future returns.

The convertible can be converted voluntarily at any time and called at the call price denoted $CP(t)$, as well as put back to the firm by investors at the put price denoted $PP(t)$. Note that we can set a put protection period by letting $PP(t) = 0$ for t smaller than the expiry of the protection period, or whenever no put is allowed we can set it to zero for just that period. The call back feature can be disabled for a similar reason by setting $CP(t) = \infty$.

Brennan and Schwartz [8] modelled the value of the firm V with the stochastic process

$$\mathrm{d}V = (\mu_V V - Q(V, r, t))\mathrm{d}t + \sigma_V \mathrm{d}W_V, \tag{4.1}$$

where μ_V and σ_V are the mean and volatility of the total rate of return on the firms value. $Q(V, r, t)$ is the total rate of cash distributions to the firms security holders, and r is the short rate with its own Wiener process $\mathrm{d}W_V$. The cash distribution function $Q(V, r, t)$ is defined as follows

$$Q(V, r, t) = c(r, t) + b(r, t) + d(V, t),$$

where $c(r, t)$ and $b(r, t)$ represents the coupon payments to the convertibles and straight bonds respectively, and $d(V, t)$ the dividends to the shares. The short rate is modeled with mean reversion as follows

$$\mathrm{d}r = \alpha(\mu_r - r)\mathrm{d}t + r\sigma_r \mathrm{d}W_r,$$

with W_r a standard Brownian motion, μ_r and σ_r the mean and volatility of the short rate, and α the model's adjustment coefficient. The calibration coefficient sets the rate of reversion. The Wiener process is again $\mathrm{d}W_r$.

We now begin to derive the partial differential equation for the value of the convertible bond by letting $D_i(r, t)$ with $i = 1, 2$ denote the market value of two default free discount bonds. We obtain the instantaneous change in the value of these bonds by Ito's Lemma, this is given by

$$\frac{\mathrm{d}D_i}{D_i} = \mu_{D_i}\mathrm{d}t + \sigma_{D_i}\mathrm{d}W_r.$$

We define μ and σ as the mean and volatility of the variable used in the subscript. Now consider a portfolio with no initial investment, formed by borrowing $x_1 + x_2$ at the current interest rate

r, and then investing the amounts x_i in bond i. The instantaneous return on this portfolio is given by

$$[x_1(\mu_{D_1} - r) + x_2(\mu_{D_2} - r)]\,\mathrm{d}t + [x_1\sigma_{D_1} + x_2\sigma_{D_2}]\,\mathrm{d}W_r,$$

so if we choose x_1 and x_2 so that

$$(x_1\sigma_{D_1} + x_2\sigma_{D_2}) = 0,$$

then we have eliminated the random element in the portfolio. So to avoid the possibility of arbitrage the return on this portfolio must be zero, which is only possible if

$$\frac{\mu_{D_1} - r}{\sigma_{D_1}} = \frac{\mu_{D_2} - r}{\sigma_{D_2}} = \lambda(r,t), \tag{4.2}$$

where λ is known as the market price of interest rate risk, which is the reward to variability ratio of a portfolio whose rate of return is perfectly correlated with changes in the interest rate, like any long term bond. This is one of the main drawbacks in using this type of model, since it means that we require knowledge of the market price of interest rate risk and this requires calibration.

Now given that the only two sources of uncertainty are the value of the firm and the interest rate, we see that the value of a convertible bond may be written as a function of these variables, as well as time. Then by using Ito's Lemma, the instantaneous rate of return on the convertible bond is given by

$$\mathrm{d}C(V,r,t) = \mu_c \mathrm{d}t + C_V V(t)\sigma_V \mathrm{d}W_V + C_r r\sigma_r \mathrm{d}W_r \tag{4.3}$$

where

$$\mu_c = \Big[C_t + C_r\alpha(\mu_r - r) + C_V(\mu_V - Q(V,t)) + \frac{1}{2}C_{rr}r^2\sigma_r^2 + C_{Vr}V(t)r\rho\sigma_V\sigma_r + \frac{1}{2}C_{VV}V^2(t)\sigma_V^2\Big] \quad , \tag{4.4}$$

with ρ the instantaneous correlation between $\mathrm{d}W_r$ and $\mathrm{d}W_v$.

Now we construct a portfolio with zero initial investment again by investing x_C in convertibles, x_G in default free bonds and x_V in shares by borrowing at the instantaneous interest rate. Now by taking into account the rate of coupon payment on the convertible $c(r,t)$, and by using equations (4.1), (4.3) and (4.4) we can obtain the instantaneous return on the portfolio, and eliminate all the random components as before to avoid the possibility of arbitrage to obtain the pricing equation

$$\frac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 C}{\partial V^2} + \rho r \sigma_V \sigma_r V \frac{\partial^2 C}{\partial V \partial r} + \frac{1}{2} r^2 \sigma_r^2 \frac{\partial^2 C}{\partial r^2} + [\alpha(\mu_r - r) - \lambda r \sigma_r] \frac{\partial C}{\partial r} + \frac{\partial C}{\partial V}[rV - Q] + c(r,t) - rC + \frac{\partial C}{\partial t} \leq 0.$$

The obstacle in this model is that solving the partial differential inequality, depends on estimating ρ and λ, as well as the drift and volatility terms of the stochastic processes. Note that since the short rate is not a tradable instrument, its growth rate does not drop out of the equation as the growth rate of the firm value does. The partial differential inequality we have derived is still dependent on the boundary conditions which we will discuss in section (4.5).

The modeling of the share price instead of the value of the firm follows exactly like that of the one-factor model. See Wilmott [36] for a complete derivation. Wilmott [36] also shows how to include an instantaneous risk of default into a convertible bond pricing model by using a Poisson process. We model the instantaneous risk of default with the hazard rate p, which defines the probability of default between t and $t + \mathrm{d}t$ given that default has not yet occurred. Introducing the extra deterministic factor leads us to

$$\frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 C}{\partial S^2} + \rho \sigma_S \sigma_r S \frac{\partial^2 C}{\partial S \partial r} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 C}{\partial r^2} + [\mu_r - \lambda \sigma_r] \frac{\partial C}{\partial r} + rS\frac{\partial C}{\partial S} - (r+p)C + \frac{\partial C}{\partial t} \leq 0,$$

with slightly different assumptions about the short rate and no inclusion of Brennan and Schwartz's Q term. The market sees a small share price as a sign of a company in trouble so convertible bond prices usually fall a great deal. The default model above yields a convertible bond price close to that of a non-convertible bond in case of a small stock price. Wilmott [36] recommends making the hazard rate dependent on the stock price, so that we can set $p(S) \to \infty$ as $S \to 0$.

The inclusion of a stochastic interest rate makes the model more complex and increases the computational cost of solving the partial differential inequality. Brennan and Schwartz [8] found through numerical implementation of both the one and two-factor models, that inclusion of a stochastic interest rate produces a very slight improvement in accuracy. This computation is certainly more possible now, but still too restrictive for very little reward. Brennan and Schwartz

[8] accomplished this result through calculating the value of a convertible bond using a constant interest rate, as well as a stochastic interest rate. Their findings were such that the errors from the use of a deterministic interest rate were likely to be slight, and that it may be preferable to use the simpler model. This means that we might be better served including the hazard rate as a second stochastic factor instead of the interest rate.

4.5 Boundary Conditions

The following boundary conditions are necessary to enforce the effect of the call, put and conversion options embedded in the convertible bond.

4.5.1 The Call Condition

We know from the second Lemma in section 4.1 that it is optimal to call the convertibles as soon its value is equal to the call price[1], hence we have

$$C(V,r,t) \leq CP(t).$$

4.5.2 The Put Condition

Since the investors optimal strategy will be to maximize the value of the convertible bond at each instant in time, the investor would want to put the convertible back to the firm as soon as optimal. Optimality would occur when the convertible bond's value if not put falls below the put price. This yields

$$C(V,r,t) \geq PP(t).$$

4.5.3 The Conversion Condition

The conversion ratio $R(t)$ of the convertible bond is the number of shares that each bondholder can convert the bond into at time t, this leads us to

$$C(V,r,t) \geq R(t)S(r,t).$$

[1]Noting that a premium is added here if there is a call notice period.

4.5.4 The Maturity Condition

In the case that we have used the firm value to model the price of the convertible bond, we have assumed that the straight bond is senior to the convertible bond. The convertible bond will only be paid off in full or even partially at maturity if the value of the firm is larger than the principal of the straight bond. Thus if P denotes the face value of the straight bond, the payoff at maturity of the convertible bond is given by

$$C(V,r,T) = \left\{ \begin{array}{lll|l} R(t)S(r,t) & \text{if } R(t)S(r,t) \geq F & & \text{Conversion } > \text{ Face} \\ F & \text{if } F + P < V < R(t)S(r,t) & & \text{Face } > \text{Conversion} \\ V - P & \text{if } P < V < F + P & & \text{Partial Payment} \\ 0 & \text{if } V < P & & \text{No Payment} \end{array} \right.$$

Note the different reasons for payment, where partial payment and no payment means the straight bonds were paid out first. The investor can still convert if the conversion value is greater than the face value of the convertible bond. In the case that we have used the equity value to model the price of the convertible bond we use only the first two conditions above.

4.5.5 The Bankruptcy Condition

Assuming that the convertible bond rules are such that the investors will receive a fraction k of the par value when the firm goes bankrupt. If the firm goes bankrupt the firms value falls to the sum of the par value of the straight debt, and k times the par value of the convertible, therefore

$$C(V,r,t) = kF \qquad \text{if} \qquad V = P + kF.$$

Wilmott [36] suggests setting fixed stock level S^*, and if the share price should reach this level the company would default. In that case we would use

$$C(S^*,t) = 0.$$

4.6 Conclusion

The models discussed in this section are unrealistic since they do not incorporate default risk. Although using a model that does not incorporate default risk and pricing with one that does, allows us to measure the discount in price due to default risk. Another problem we encounter is that we are required to measure the market price of interest rate risk using something like the capital asset pricing model. This is complicated and error prone. The next chapter will introduce methods that include default risk while not requiring the market price of interest rate risk.

Chapter 5

Models Including Default and Foreign Exchange Risk

5.1 Dealing with Default Risk

The more traditional techniques used in pricing contingent claims use a credit spread on top of the risk free interest rate. This would reduce the value of the instrument due to the fact that the issuer might default. The convertible bond on the other hand is comprised of both bond and equity properties. Therefore the coupon and nominal payments are subject to default risk, while the equity upside offered by the convertible bond is not. Simply using the credit spread of a non-convertible bond would therefore limit the equity upside of the convertible bond.

The issue of default is the reason that most methods of valuation fail to capture the necessary default risk inherent in the convertible bond. Creating the necessity to split the convertible bond into two components, and attempt to value each component separately. Tsiveriotis and Fernandes [35] were the first to attempt this, but failed to capture what actually occurs in the event of default. What Tsiveriotis and Fernandes [35] were able to do was split the convertible bond into two components, and then apply the risky interest rate to only part of the instrument. This part was considered the cash component, responsible for all payments and subject to default risk. The other component, the equity component, had no cash payments and was therefore not subject to default risk. The Tsiveriotis and Fernandes [35] method is discussed in the next section.

5.2 The Tsiveriotis and Fernandes Approach

This section describes an approach, introduced by Tsiveriotis and Fernandes [35], that uses the market observed credit spreads of straight bonds in the valuation of convertible bonds to try describe the default risk. The crux of this method is the fact that the value of the future cash payments that a rational investor will choose to receive can be viewed as a derivative on the underlying equity and interest rate. This results in using one additional Black-Scholes equation for each different credit class of payments involved in the convertible bond, as well as the original Black-Sholes equation for the entire convertible bond. Thereby we only need the market observed credit spreads corresponding to the various credit classes involved in the convertible's expected cash flows, as well as the volatilities of the underlying.

Tsiveriotis and Fernandes [35] define a related security made up of the cash-only part of the convertible bond. The holder of this new security is entitled to only the cash flows that an optimally behaving holder of the actual convertible bond would receive, so by this definition the value of this new security is determined by the behavior of the convertible bond price as well as the stock price and time. The price B of the new security should follow the same Black-Scholes equation as C since the new security is a derivative on the same underlying. The significant difference is that the new security involves only cash payments, and no other features. Tsiveriotis and Fernandes were then able to apply the risky discount rate to only this security by using the market obtainable credit spread. Having defined the new security, we can see that $E = C - B$ would represent the value of the equity part of the convertible bond, and should be discounted using the risk-free rate since no defaultable payments are made. This leads us to a new system of Black-Scholes equations and allows us to price the convertible bond through two separate partial differential equations for B and E. Note that the value of the entire convertible bond is given by $C = B + E$, the risky and risk free components.

To derive the pricing equations we assume that the underlying stock price follows a stochastic process given by

$$dS = (r-q)S dt + \sigma S dW, \tag{5.1}$$

and that the value of the equity and cash components of the convertible bond are functions of the stock price and time. Therefore by using Itô's lemma we obtain

$$dE = \left[\frac{\partial E}{\partial t} + (r-q)S\frac{\partial E}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 E}{\partial S^2}\right] dt + \sigma S \frac{\partial E}{\partial S} dW, \tag{5.2}$$

and

$$dB = \left[\frac{\partial B}{\partial t} + (r-q)S\frac{\partial B}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2}\right] dt + \sigma S \frac{\partial B}{\partial S} dW. \tag{5.3}$$

Next we construct a hedging portfolio that is long one convertible bond C and short $\frac{\partial C}{\partial S}$ shares,

$$\Pi = C - \frac{\partial C}{\partial S} S$$

so that we can split the portfolio into two separate components $\Pi = \Pi_1 + \Pi_2$ given by

$$\begin{aligned}\Pi_1 &= B - \frac{\partial B}{\partial S} S, \\ \Pi_2 &= E - \frac{\partial E}{\partial S} S.\end{aligned}$$

The change in value of the two portfolios over the period dt is given by

$$d\Pi_1 = dB - \frac{\partial B}{\partial S} dS, \tag{5.4}$$

$$d\Pi_2 = dE - \frac{\partial E}{\partial S} dS, \tag{5.5}$$

so that we can substitute (5.1), (5.2) and (5.3) into (5.4) and (5.5) to obtain

$$d\Pi_1 = \left[\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2}\right] dt,$$

and

$$d\Pi_2 = \left[\frac{\partial E}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 E}{\partial S^2}\right] dt.$$

The two parts of portfolio Π_1 should earn at separate rates, the cash component B should earn at the risky discount rate $(r+s)$ while the shares should earn at the risk-free rate. Dividends also need to be taken into account in the equity component. Portfolio Π_2 also earns $(r-q)$ on the $\frac{\partial E}{\partial S}$ shares, but the equity component should be discounted by the risk-free rate. Therefore

$$d\Pi_1 = \left[(r+s)B - (r-q)\frac{\partial B}{\partial S} S\right] dt, \tag{5.6}$$

and

$$d\Pi_2 = \left[rE - (r-q)\frac{\partial E}{\partial S}S \right] dt. \tag{5.7}$$

Now by setting equations (5.4) and (5.6), as well as (5.5) and (5.7) equal, we obtain the separate pricing equations for the cash and equity components of the convertible bond. These are

$$\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} + (r-q)S\frac{\partial B}{\partial S} - (r+s)B = 0, \tag{5.8}$$

and

$$\frac{\partial E}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 E}{\partial S^2} + (r-q)S\frac{\partial E}{\partial S} - rE = 0. \tag{5.9}$$

Now since the value of the convertible is given by $C = E + B$, we are left with the pricing equation of the convertible bond as

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-q)S\frac{\partial C}{\partial S} - rC - sB = 0. \tag{5.10}$$

The fact that B still appears in the pricing equation, means that equations (5.8) and (5.10) are coupled. Therefore to obtain the value of the convertible bond, we need to solve equations (5.8) and (5.10) simultaneously. Another approach would be to solve equations (5.9) and (5.8) and sum their solutions to obtain the value of the convertible bond. This approach would be no different and offer no benefit, it would only require that different final and boundary conditions be applied. Note that if some cash payments are of a different credit class then an extra instance of equation (5.8) needs to be added with the applicable credit spread.

We see in figure 5.1 that the fact that we discount the cash component with a risky discount factor lowers the price of the convertible bond at low share prices. Since the convertible bond behaves more like equity at high share prices we expect that the risky discount factor would not have a large influence.

To introduce the final and boundary conditions for the equations (5.8) and (5.10), we consider a convertible bond that pays a principal F, if not yet converted into R shares by maturity. Otherwise it is called at a price CP, or put by the holder for PP. The convertible bond pays a fixed coupon c at times t_i, and a call or put may only occur after the times t_c and t_p respectively. These allow us to introduce the necessary conditions, starting with the conditions due to maturity;

$$C(S,T) = \max(RS, F), \tag{5.11}$$

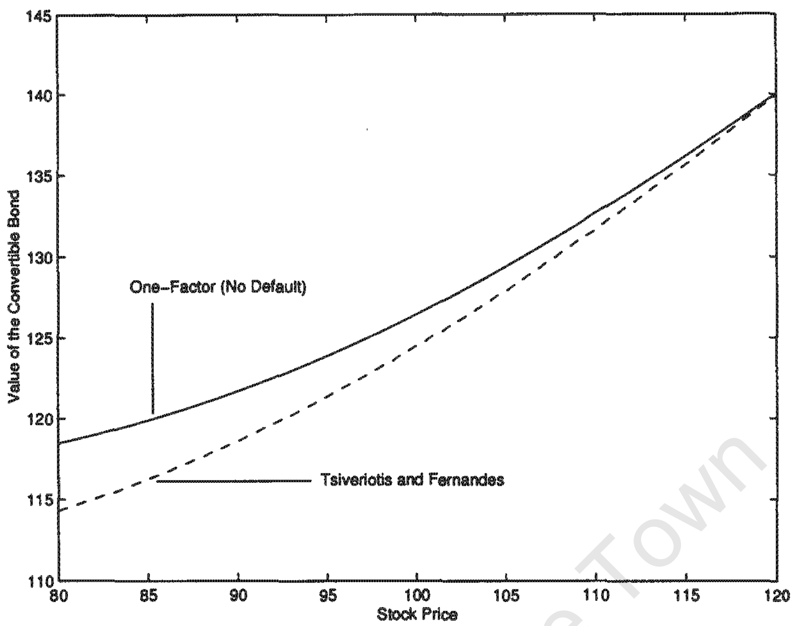

Figure 5.1: Tsiveriotis and Fernandes Vs. One-Factor

and

$$B(S,T) = \begin{cases} 0 & \text{if } RS \geq F \\ F & \text{otherwise} \end{cases} \quad . \tag{5.12}$$

Equations (5.11) and (5.12) show that if the conversion value is less than the principal payment, then the issuer will never convert. Although as soon as the shares the holder will receive at conversion have greater value than the final principal payment, the holder will convert and take the shares while the cash only instrument is immediately worthless.

We also have a lower bound on the value of the convertible

$$C \geq RS, \tag{5.13}$$

while the cash component B will become zero in the case of conversion. This argument is in line with the rules of conversion which state that the convertible bond's value cannot fall below the conversion value RS. Since if it did, the holder would immediately convert to the shares, and conversion immediately nulls the value of the synthetic instrument B.

The issuer's callability option in the model gives rise to the following constraints for $t \in [t_c, T]$,

$$C \leq \max(CP, RS). \tag{5.14}$$

We assume that the holder has the right to convert once the issuer has called the convertible bond. Equation (5.14) is in line with the fact that the issuer would never allow the conversion value to become greater than the call price, since it is in the issuers best interest to call the convertible bond and pay CP, thereby securing the more valuable shares. This means that the value of the convertible bond is bound from above by the maximum of the two since the holder still has the option to convert if the bond is called. This then implies that we also have to have the following condition if the bond has been called,

$$B = 0 \quad \text{if} \quad C \geq CP, \tag{5.15}$$

since if $C \geq CP$ then the issuer would call the bond, cash flows would stop and the cash only part of the convertible would lose its value.

The holder's right to put the convertible bond back to the issuer gives rise to the following constraints over $t \in [t_p, T]$

$$C \geq PP, \tag{5.16}$$

since the value of the convertible bond is bound below by the put price. This follows because if the value of the convertible bond dropped below the put price, the holder would immediately put the bond back to the issuer. Thus, once the convertible has been put back to the issuer we have

$$B = PP, \tag{5.17}$$

since the last cash payment would be the put price.

The credit spread s, due to default risk is not necessarily the same as the spread on straight bonds by the same issuer. The difference is due to the fact that straight bonds are always senior to convertible bonds, so their credit spreads can differ. Therefore the spread s should be higher than that of straight debt. Ayache, Forsyth and Vetzal [3] suggest setting $s = p(1 - \mathit{Recovery})$, where p is a hazard rate and *Recovery* is the portion of the bond that is recovered upon default. The Ayache, Forsyth and Vetzal [3] approach presents a complete description of how to handle default, and is discussed in section 5.4.

While the Tsiveriotis and Fernandes [35] method successfully splits the convertible bond into cash and equity parts, thus applying the credit spread to the risky cash component only, it fails to model the event of default and what actually happens when the issuer defaults. Ayache, Forsyth and Vetzal [2] note that in this approach the value of the convertible bond depends on the value of the credit spread, while the value of the credit spread depends on the value of the convertible bond. Another element missing here is what happens to the stock price when the issuer defaults on the convertible bond. Surely the value of the share price should fall, and upon default the convertible bond holder should expect some measure of remuneration. Later in the chapter the hedge model by Ayache, Forsyth and Vetzal [3] is presented. This model values the convertible while taking default and recovery into account as well as introducing a hazard rate to model default risk.

The next section presents an extension to the Tsiveriotis and Fernandes [35] model by including the possibility of cross-currency convertibles. Foreign exchange risk is present in international convertibles since these convertibles pay coupons and the nominal value in domestic currency, while converting into foreign equity. Yigitbasioglu [37] propose extending the Tsiveriotis and Fernandes [35] approach of dealing with credit risk by introducing a similar cash only synthetic asset.

5.3 Cross-Currency Extension to Tsiveriotis and Fernandes

The Tsiveriotis and Fernandes [35] method has been extended by Yigitbasioglu [37], who presented a two-factor coupled partial differential equation approach, with a change of numeraire to price international convertibles. International convertibles are discussed more in Section 2.1.7. Yigitbasioglu [37] uses a change of numeraire to deal with this foreign exchange risk. This relies on the stochastic behavior of the foreign equity in the domestic currency; hence there is no volatility smile available from the market. Yigitbasioglu [37] suggests backing out the price of options struck on this process, by using the implied volatility of foreign equity options in a foreign currency, as well as foreign vanilla exchange options. The prices obtained through numeric techniques are then used to back out the implied volatility of the foreign equity in the domestic currency.

We assume that the foreign stock price follows the stochastic process

$$\mathrm{d}S_f = (r_f - q)S_f\mathrm{d}t + \sigma_f S_f \mathrm{d}W, \tag{5.18}$$

and that the exchange rate follows

$$\mathrm{d}X = \alpha X\mathrm{d}t + \sigma_X X\mathrm{d}W. \tag{5.19}$$

Now we denote the domestic price of the foreign equity as $\tilde{S}_f$, and apply Itô's lemma to $\tilde{S}_f = S_f X$ to obtain

$$\mathrm{d}\tilde{S}_f = \frac{\partial \tilde{S}_f}{\partial X}\mathrm{d}X + \frac{\partial \tilde{S}_f}{\partial S_f}\mathrm{d}S_f + \frac{\partial^2 \tilde{S}_f}{\partial X \partial S_f}\mathrm{d}X\mathrm{d}S_f,$$

we then insert (5.18) and (5.19), as well as the correlations, to obtain

$$\begin{aligned}\mathrm{d}\tilde{S}_f &= (\alpha + r_f - q + \sigma_f\sigma_X^T)S_f X\mathrm{d}t + (\sigma_f + \sigma_X)S_f X\mathrm{d}W \\ &= (\alpha + r_f - q + \sigma_f\sigma_X^T)\tilde{S}_f\mathrm{d}t + (\sigma_f + \sigma_X)\tilde{S}_f\mathrm{d}W.\end{aligned}$$

Now for the domestic price of the foreign equity to have the correct risk neutral drift $r_d - q$, we need to introduce the correct measure transformation, i.e.

$$(\sigma_f + \sigma_X)\mathrm{d}\bar{W} = (\sigma_f + \sigma_X)\mathrm{d}W + (r_d - \alpha - r_f - \sigma_f\sigma_X^T)\mathrm{d}t,$$

with $\mathrm{d}\bar{W}$ the domestic brownian motion associated with the risk neutral measure. This yields the dynamics

$$\mathrm{d}\tilde{S}_f = (r_d - q)\tilde{S}_f\mathrm{d}t + (\sigma_f + \sigma_X)\tilde{S}_f\mathrm{d}W. \tag{5.20}$$

Now assuming that the value of the convertible bond is dependent on the value of the foreign equity in domestic currency, as well as the domestic interest rate and time, Yigitbasioglu [37] obtains the stochastic dynamics of the convertible bond $\mathrm{d}C$ by using Itô's Lemma. Yigitbasioglu [37] also assumes that

$$\mathrm{d}\tilde{S}_f\mathrm{d}r = \tilde{S}_f(\sigma_f + \sigma_X)w\sqrt{r}\hat{\rho}\mathrm{d}t,$$

where $\hat{\rho}$ is the correlation between the interest rate and the value of the foreign equity in domestic currency. Since Brennan and Schwartz [8] have shown that the inclusion of the interest rate has almost no effect, it has been current practice to ignore it. We thus restrict our attention to the case where interest rates are deterministic to avoid complication.

Now following the same derivation of section 5.2, we define the cash and equity parts of the convertible bond. Their processes are given by

$$\begin{aligned}
\mathrm{d}E &= \frac{\partial E}{\partial t}\mathrm{d}t + \frac{\partial E}{\partial \tilde{S}_f}\mathrm{d}\tilde{S}_f + \frac{1}{2}(\sigma_f + \sigma_X)^2 \tilde{S}_f^2 \frac{\partial^2 E}{\partial \tilde{S}_f^2}\mathrm{d}t \\
&= \left[\frac{\partial E}{\partial t} + (r_d - q)\tilde{S}_f \frac{\partial E}{\partial \tilde{S}_f} + \frac{1}{2}(\sigma_X^2 + 2\rho\sigma_X\sigma_f + \sigma_f^2)\tilde{S}_f^2 \frac{\partial^2 E}{\partial S^2}\right]\mathrm{d}t \qquad (5.21)\\
&\quad +(\sigma_f + \sigma_X)\tilde{S}_f \frac{\partial E}{\partial S}\mathrm{d}W,
\end{aligned}$$

and similarly

$$\mathrm{d}B = \left[\frac{\partial B}{\partial t} + (r_d - q)\tilde{S}_f \frac{\partial B}{\partial \tilde{S}_f} + \frac{1}{2}(\sigma_X^2 + 2\rho\sigma_X\sigma_f + \sigma_f^2)\tilde{S}_f^2 \frac{\partial^2 B}{\partial S^2}\right]\mathrm{d}t + (\sigma_f + \sigma_X)\tilde{S}_f \frac{\partial B}{\partial S}\mathrm{d}W. \quad (5.22)$$

We then use the same portfolio argument as in section 5.2, yielding a similar change in the portfolio's value over the time increment $\mathrm{d}t$, so

$$\begin{aligned}
\mathrm{d}\Pi_1 &= \mathrm{d}B - \frac{\partial B}{\partial \tilde{S}_f}\tilde{S}_f \\
&= \left[\frac{\partial B}{\partial t} + \frac{1}{2}(\sigma_X^2 + 2\rho\sigma_X\sigma_f + \sigma_f^2)\tilde{S}_f^2 \frac{\partial^2 B}{\partial \tilde{S}_f^2}\right]\mathrm{d}t
\end{aligned}$$

as well as

$$\begin{aligned}
\mathrm{d}\Pi_2 &= \mathrm{d}E - \frac{\partial E}{\partial \tilde{S}_f}\tilde{S}_f \\
&= \left[\frac{\partial E}{\partial t} + \frac{1}{2}(\sigma_X^2 + 2\rho\sigma_X\sigma_f + \sigma_f^2)\tilde{S}_f^2 \frac{\partial^2 E}{\partial \tilde{S}_f^2}\right]\mathrm{d}t.
\end{aligned}$$

The value of the two portfolios should grow by a risk free rate or arbitrage would be possible, therefore

$$\begin{aligned}
\mathrm{d}\Pi_1 &= \left[(r_f + s)B - (r_f - q)\frac{\partial B}{\partial \tilde{S}_f}\right]\mathrm{d}t \\
\mathrm{d}\Pi_2 &= \left[r_f E - (r_f - q)\frac{\partial E}{\partial \tilde{S}_f}\right]\mathrm{d}t.
\end{aligned}$$

We note that the cash part of the convertible bond is discounted by the risky amount, while the equity part is discounted by the risk-free amount. This is similar to the Tsiveriotis and Fernandes [35] approach.

Equating these two portfolios will yield the partial differential equation for the cash and equity components as

$$\frac{\partial B}{\partial t}+\frac{1}{2}(\sigma_X^2+2\rho\sigma_X\sigma_f+\sigma_f^2)\tilde{S}_f^2\frac{\partial^2 B}{\partial \tilde{S}_f^2}+(r_f-q)\tilde{S}_f\frac{\partial B}{\partial \tilde{S}^f}-(r_f+s)B=0,$$

and

$$\frac{\partial E}{\partial t}+\frac{1}{2}(\sigma_X^2+2\rho\sigma_X\sigma_f+\sigma_f^2)\tilde{S}_f^2\frac{\partial^2 E}{\partial \tilde{S}_f^2}+(r_f-q)\tilde{S}_f\frac{\partial E}{\partial \tilde{S}^f}-r_fE=0.$$

This leads us to the default aware international pricing equation given by Yigitbasioglu [37]

$$\frac{\partial C}{\partial t}+\frac{1}{2}(\sigma_X^2+2\rho\sigma_X\sigma_f+\sigma_f^2)\tilde{S}_f^2\frac{\partial^2 C}{\partial \tilde{S}_f^2}+(r_f-q)\tilde{S}_f\frac{\partial C}{\partial \tilde{S}^f}-r_fC-sB=0,$$

where the boundary conditions are the same as that of the Tsiveriotis and Fernandes [35] approach.

The next section describes the hedge model presented by Ayache, Forsyth and Vetzal [3], this model takes into account default risk as well as the possibility of recovery in the case of default. They also improve on the Tsiveriotis and Fernandes [35] approach by adding the fact that the stock value should drop as soon as the company defaults on the convertible bond.

5.4 The Hedge Model

The method presented by Ayache, Forsyth and Vetzal [3] relies on simple hedging arguments and a risk free rate as a known deterministic function. For clear presentation they assume that the real world and risk neutral default probabilities are equal, which is not the case, but can be corrected by adjusting their pricing equations as discussed in Hull [21]. The probability of default in the time period dt, between t and $t+\mathrm{d}t$, is given by a deterministic function $p(S,t)$, known as the hazard rate. We take S^- and S^+ as the stock price immediately before and immediately after default occurs, and η as the default rate where $\eta=1$ is considered to be total default, and $0\leq\eta<1$ is considered partial default. The default rate measures how much the stock rate jumps upon default of the issuing firm, this yields

$$S^+=(1-\eta)S^-.$$

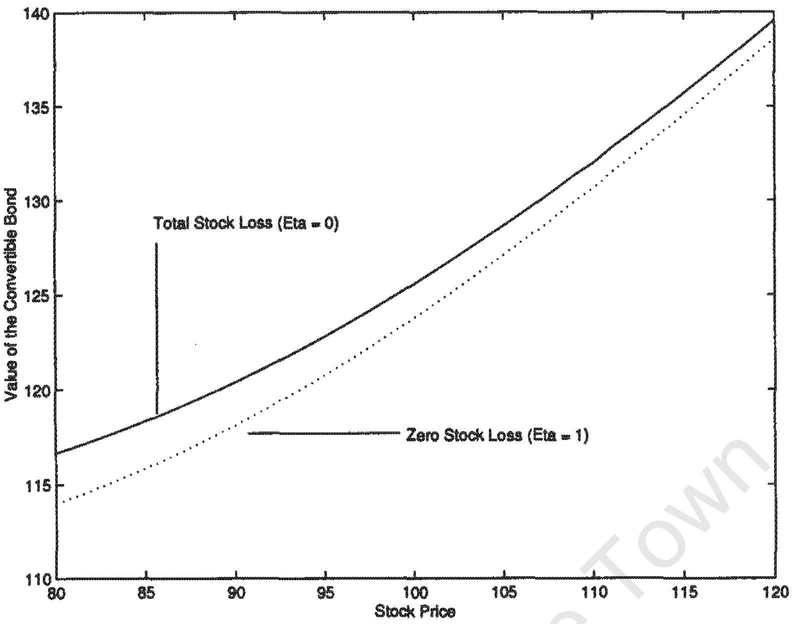

Figure 5.2: Effect of the Default Ratio

Figure 5.2 shows the effect of the introduction of the default ratio. We see that the convertible bonds will have a higher value if we expect the event of default not to have any effect on the value of the equity.

We also assume that the convertible bond holder will have the option of receiving either shares worth $\kappa(1-\eta)S$, or a recovery amount upon default. The recovery amount κX is assumed to be a fraction κ of the cash component of the convertible bond. This fraction is called the recovery factor. Figure 5.3 exhibits the effect of the recovery rate. It shows that if we expect to recover some portion of the convertible bond's value upon default, we will obviously value the convertible bond higher.

The Ayache, Forsyth and Vetzal [3] model can be formulated in the same manner as the Tsiveriotis and Fernandes [35] model, starting with the construction of a simple hedging portfolio for both the cash and equity components of the convertible bond. We begin with the assumption that the stock process is driven by equation (5.1), and the cash and equity processes by equations (5.2) and (5.3). Then by constructing a hedging portfolios as we did in Section 5.2, we assume that with probability $(1-p\mathrm{d}t)$ the change in value of the portfolios over a period $\mathrm{d}t$ is

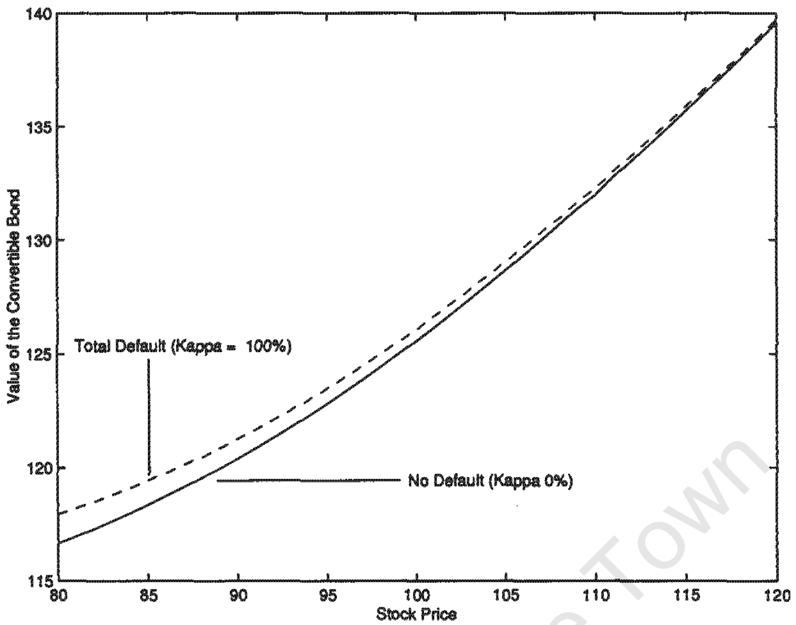

Figure 5.3: Effect of the Recovery Rate

given by equations (5.4) and (5.5). Then we assume that with probability pdt, the issuer will default, and the change in the value of the portfolio will be given by

$$d\Pi_1 = -B + \kappa B + \eta \frac{\partial B}{\partial S} S. \tag{5.23}$$

Since we lose all future cash payments except for the recovery fraction and we are short η less of the shares, the second portfolio's change is given by

$$d\Pi_2 = -E + \eta \frac{\partial E}{\partial S} S + \max[R\,S(1-\eta) - \kappa B, 0]. \tag{5.24}$$

This is due to the fact that we lose the equity component but regain the higher of the post-default conversion value or recovered bond component. This will yield the total change in the value of the portfolio as

$$\begin{aligned} d\Pi_1 &= (1-pdt)\left[dB - \frac{\partial B}{\partial S} dS\right] + pdt\left[-B + \kappa B + \eta \frac{\partial B}{\partial S} S\right] && (5.25) \\ d\Pi_2 &= (1-pdt)\left[dE - \frac{\partial E}{\partial S} dS\right] + pdt\left[-E + \eta \frac{\partial E}{\partial S} S + \max[R\,S(1-\eta) - \kappa B, 0]\right], && (5.26) \end{aligned}$$

and by substituting equations (5.1), (5.2) and (5.3) into these we obtain

$$\begin{aligned} d\Pi_1 &= \left[\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} - pB + p\kappa B + p\eta S \frac{\partial B}{\partial S}\right] dt && (5.27) \\ d\Pi_2 &= \left[\frac{\partial E}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 E}{\partial S^2} - pE + p\eta S \frac{\partial E}{\partial S} + p\max[R\,S(1-\eta) - \kappa B, 0]\right] dt. && (5.28) \end{aligned}$$

Now Ayache, Forsyth and Fetzal [3] describe the portfolios as earning the risk free rate on both the cash and equity component of the convertible bond, so they earn

$$d\Pi_1 = \left[rB - (r-q)\frac{\partial B}{\partial S}S\right] dt, \tag{5.29}$$

and

$$d\Pi_2 = \left[rE - (r-q)\frac{\partial E}{\partial S}S\right] dt. \tag{5.30}$$

This is where the two methods differ. Tsiveriotis and Fernandes [3] use the risky rate to discount the cash component of the convertible by using the credit spread, while Ayache, Forsyth and Fetzal [3] discount both the cash and equity components at the risk free rate, accounting for default risk by their introduction of the hazard rate.

Equating equations (5.27) and (5.29), as well as equations (5.28) and (5.30) we obtain the pricing equations of the cash and equity components as

$$\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} + (r - q + p\eta)S\frac{\partial B}{\partial S} - (r+p)B + p\kappa B = 0, \tag{5.31}$$

and

$$\frac{\partial E}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 E}{\partial S^2} + (r - q + p\eta)S\frac{\partial E}{\partial S} - (r+p)E + p\max[R\,S(1-\eta) - \kappa B, 0] = 0. \tag{5.32}$$

This leaves us with the default aware pricing equation that allows for share value loss and recovery of a fraction of the cash component

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - q + p\eta)S\frac{\partial C}{\partial S} - (r+p)C + p\max[R\,S(1-\eta), \kappa B] = 0. \tag{5.33}$$

We should note that the recovery upon default is not necessarily a portion of the value of the cash component, it could just as easily be a fraction of the face value of the convertible. In the case where the recovery on default is not a fraction of the cash component, the separate valuation equations of the cash and equity parts decouple, and there would be no need to formulate the model using two separate portfolios. In that case we can use a single hedging portfolio, since default would no longer have a separate impact on the bond component of the convertible. In this case we would be able to derive equation (5.33) with κX instead of κB, where X is the face value of the convertible or even any predetermined value that is not linked into the cash or equity components separately.

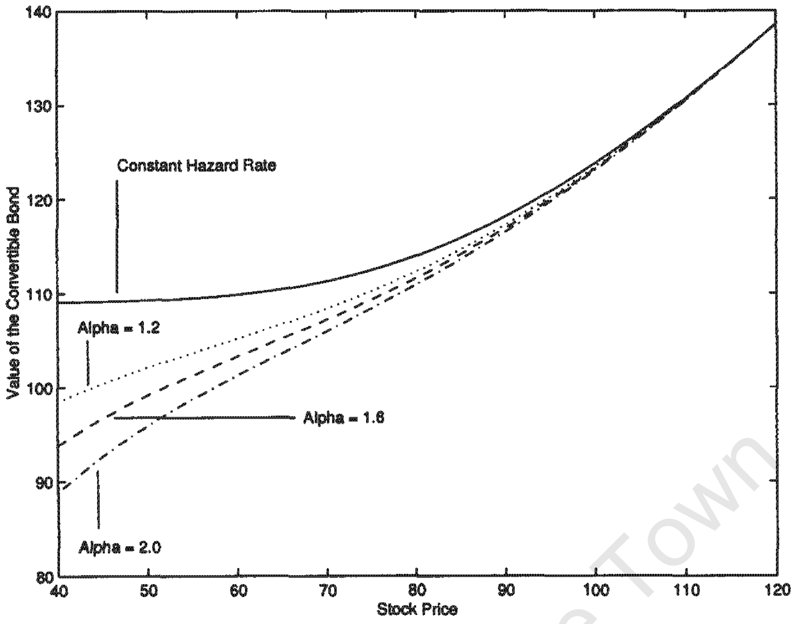

Figure 5.4: Comparison of the Effect of the Calibration Constant

Ayache, Forsyth and Vetzal [3] recommend using a hazard rate that increases as the stock price decreases, since a lower share price implies the company is more likely to default. An economical model for the hazard rate observed by Muromachi [30] is given by

$$p(S) = p_0 \left(\frac{S}{S_0}\right)^{\alpha},$$

where p_0 is the initial estimate of the hazard rate at the current stock price and α is a calibration constant. Muromachi [30] found that typical values for α are between 1.2 and 2. Figure 5.4 shows the effect of different calibration constants on the value of the convertible bond.

Figure 5.5 demonstrates the value of convertible bonds under the hedge model and compares it to the one-factor no-default model and the Tsiveriotis and Fernandes [35] model. We see that at high share prices the models converge since all the bonds will be converted, while the hedge model's price stays around the Tsiveriotis and Fernandes [35] model's price, depending on the default ratio.

The next section presents the same cross-currency extension that Yigitbasioglu [37] applied to the Tsiveriotis and Fernandes approach [35], applied to the hedge model.

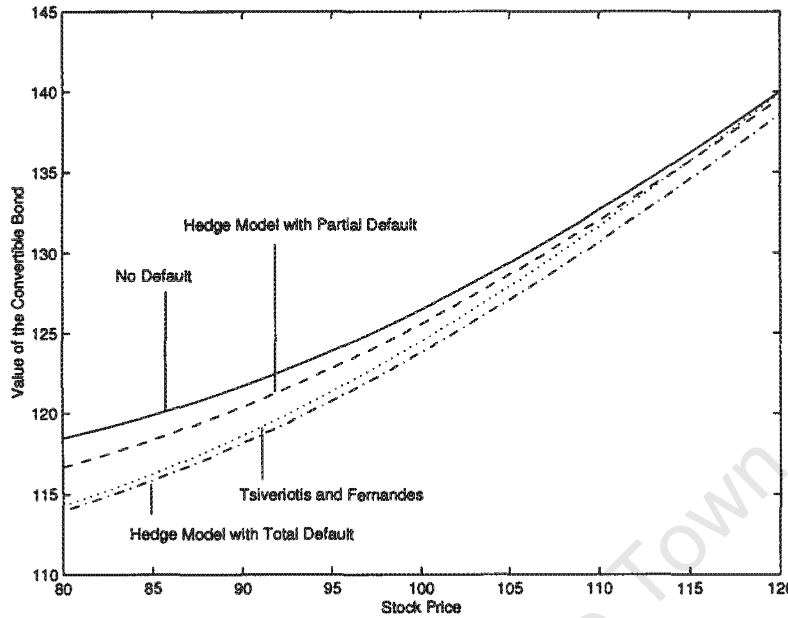

Figure 5.5: Comparison Between the Different Models

5.5 Cross-Currency Extension to the Hedge Model

We can easily extend the hedge model [3] to include cross-currency convertible bonds in the same way that Yigitbasioglu [37] extended the Tsiveriotis and Fernandes approach [35].

We begin by assuming that the foreign stock price and exchange rate follows the stochastic processes given by equations (5.18) and (5.19). Then by inserting equations (5.20), (5.21) and (5.22) into the change in value of the portfolios of the hedge model, given by equations (5.25) and (5.26). We are thus able to equate them to the arbitrage portfolio, earning risk free interest as before, to obtain the value of the cash and equity components of the convertible bond, given by

$$\frac{\partial B}{\partial t} + \frac{1}{2}(\sigma_X^2 + 2\rho\sigma_X\sigma_f + \sigma_f^2)\tilde{S}_f^2\frac{\partial^2 B}{\partial \tilde{S}_f^2} + (r_f - q + p\eta)\tilde{S}_f\frac{\partial B}{\partial \tilde{S}^f} - (r_f + p)B + pRB = 0, \quad (5.34)$$

and

$$\begin{aligned}\frac{\partial E}{\partial t} + \frac{1}{2}(\sigma_X^2 + 2\rho\sigma_X\sigma_f + \sigma_f^2)\tilde{S}_f^2\frac{\partial^2 E}{\partial \tilde{S}_f^2} + (r_f - q \quad &+ \quad p\eta)\tilde{S}_f\frac{\partial E}{\partial \tilde{S}^f} - (r_f + p)E \\ &+ \quad p\max[R\,\tilde{S}_f(1-\eta) - \kappa B, 0] = 0.\end{aligned}$$

This leads us to the adjusted default aware international pricing equation given by Ayache, Forsyth and Fetzal [3]

$$\frac{\partial C}{\partial t} + \frac{1}{2}(\sigma_X^2 + 2\rho\sigma_X\sigma_f + \sigma_f^2)\tilde{S}_f^2\frac{\partial^2 C}{\partial \tilde{S}_f^2} + (r_f - q \quad + \quad p\eta)\tilde{S}_f\frac{\partial C}{\partial \tilde{S}^f} - (r_f + p)C$$
$$+ \quad p\max[R\,\tilde{S}_f(1-\eta), \kappa B] = 0, \quad (5.35)$$

where the boundary conditions are the same as that of the Hedge model [3] approach.

5.6 Conclusion

This chapter has presented the Tsiveriotis and Fernandes [35] approach and highlighted its drawbacks and presented an extension to include cross-currency convertibles by Yigitbasioglu. We have introduced the hedge model by Ayache, Forsyth and Vetzal [3] and showed the effects of the newly introduced parameters, as well as extending the hedge model to incorporate cross-currency convertibles in the same manner as Yigitbasioglu [37]. The next chapter will discuss how these models are implemented.

Chapter 6

Implementation

6.1 The Solution Method

The models presented in the previous chapter require us to solve two coupled partial differential equations, one for the cash component of the convertible bond and another for the equity component. We could solve an equation for the value of the convertible bond, and another for the equity component as described in chapter 4, this would only require different boundary and final conditions and makes no difference to the solution. In this section we focus on solving the cash and equity components seperately, for example equations (5.31) and (5.32). We begin by creating a finite difference grid for each of the equations as in figure 6.1, and will continue by using the method of finite differences.

The finite difference mesh is two-dimensional in S and t, and we use the index i to describe the dimension in S, and k to describe the *inverted* time dimension. We let k terminate at $K = \frac{T-t}{\Delta t}$ and i terminate at approximately four times the current share price. We need to invert time because the maturity condition can then supply us with the initial conditions on the mesh, the boundary conditions discussed in section 6.3 will supply us with the values of the nodes at the first and last point.

The grid points will not always fall exactly on dividend and coupon dates, so we need to add mesh lines like at point 2 in figure 6.1. Noting that we need to make an adjustment for Δt over

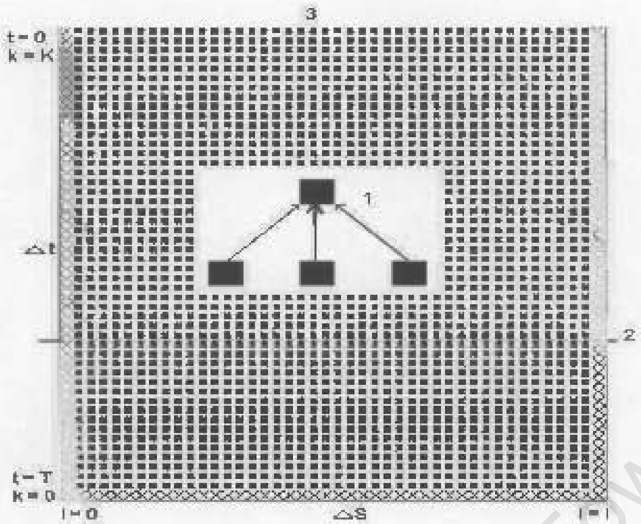

Figure 6.1: The Finite Difference Grid Used in Implementation

those points since we are not progressing in steps of Δt around those points anymore.

After calculating the initial nodes and the boundary nodes, we use the Crank-Nicolson finite difference method to compute the interior points of the mesh for each of the equations. The nature of the equations that we are solving is such that each of the interior nodes not yet calculated depends on three previous nodes that have already been calculated, as in point 1 in figure 6.1. The Crank-Nicolson method is presented in the next section.

Once we have calculated all the nodes for a specific k, we apply the minimum and maximum constraints presented in section 6.4. These minimum and maximum constraints exist as a result of the call, put and conversion rights embedded in the convertible bond.

Once we have calculated all the values of all the nodes for both the equations, we create a new grid by adding the two calculated grids together node by node. We are able to do this because we know that the convertible value is equal to $C = B + E$. We then read the solution straight off the new mesh, somewhere along point 3 in figure 6.1.

To implement the Tsiveriotis and Fernandes method [35], we would need to solve equations (5.8) and (5.9), while for the one-factor model we need only solve one equation. To solve the hedge model we need to solve equations (5.31) and (5.32), and then add the solutions together for the final value of the convertible bond.

6.2 The Crank-Nicolson Finite Difference Method

All of the methods discussed in Chapter 5 call on us to solve equations of the form

$$\mathcal{M}(X) = \frac{\partial X}{\partial t} + a(S,t)\frac{\partial^2 X}{\partial S^2} + b(S,t)\frac{\partial X}{\partial S} + c(S,t)X, \tag{6.1}$$

to compute the other points on the grid. The Crank-Nicolson scheme of equation (6.1) is given in Wilmott [36] and [33] as

$$\begin{aligned} \Delta t\mathcal{M}(X) &- A_i^{k+1}X_{i-1}^{k+1} - (B_i^{k+1}-1)X_i^{k+1} - C_i^{k+1}X_{i+1}^{k+1} \\ &= A_i^k X_{i-1}^k + (B_i^k + 1)X_i^k + C_i^k X_{i+1}^k, \end{aligned} \tag{6.2}$$

with

$$\begin{aligned} A_i^k &= \frac{1}{2}v_1 a_i^k - \frac{1}{4}v_2 b_i^k \\ B_i^k &= -v_1 a_i^k + \frac{1}{2}\Delta t v_2 c_i^k \\ C_i^k &= \frac{1}{2}v_1 a_i^k + \frac{1}{4}v_2 b_i^k \\ v_1 &= \frac{\Delta t}{(\Delta S)^2} \\ v_2 &= \frac{\Delta t}{\Delta S}, \end{aligned}$$

and with a_i^k, b_i^k and c_i^k given by a, b and c in equation (6.1) respectively. For example equation (5.35) would yield

$$\begin{aligned} a_i^k &= \frac{1}{2}(\sigma_X^2 + 2\rho\sigma_X\sigma_f + \sigma_f^2)(i\Delta S)^2 \\ b_i^k &= (r_f - q + p\eta)i\Delta S \\ c_i^k &= -(r_f + p). \end{aligned}$$

Note that if we are at a step where we have inserted a coupon or dividend date, we need to use another Δt at this point.

The term $\mathcal{M}(X)$ on the left hand side of equation (6.1) would also depend on the equation we want to solve, for example in equation (5.34) we would need to discretisize

$$\mathcal{M}(B) = p\kappa B,$$

which would yield

$$\begin{aligned} \mathcal{M}^d(B) &= \frac{1}{2}p\kappa\left(B_i^k + B_i^{k+1}\right) \\ &= \frac{1}{2}pRB_i^k + \frac{1}{2}p\kappa B_i^{k+1} \\ &= \mathcal{M}_{k,i}^d(B) + \mathcal{M}_{k+1,i}^d(B). \end{aligned}$$

The Crank-Nicolson method can be written in matrix form, this will ease computation as it yields a tri-diagonal matrix on either side of the equation which can be solved with a tri-diagonal solver. We write the Crank-Nicolson equation (6.2) in matrix form as it is given in Wilmott [36], that is

$$M_L^{k+1}v^{k+1} = M_R^k v^k, \tag{6.3}$$

where

$$M_L^{k+1} = \begin{pmatrix} \mathcal{A}_L^1 & \mathcal{B}_L^1 & \mathcal{C}_L^1 & 0 & \cdot & \cdot & \cdot \\ 0 & \mathcal{A}_L^2 & \mathcal{B}_L^2 & \mathcal{C}_L^2 & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & 0 & \mathcal{A}_L^{I-2} & \mathcal{B}_L^{I-2} & \mathcal{C}_L^{I-2} & 0 \\ \cdot & \cdot & \cdot & 0 & \mathcal{A}_L^{I-1} & \mathcal{B}_L^{I-1} & \mathcal{C}_L^{I-1} \end{pmatrix}, v^k = \begin{pmatrix} X_0^{k+1} \\ X_1^{k+1} \\ \cdot \\ \cdot \\ \cdot \\ X_{I-1}^{k+1} \\ X_I^{k+1} \end{pmatrix}$$

and

$$M_R^k = \begin{pmatrix} \mathcal{A}_R^1 & \mathcal{B}_R^1 & \mathcal{C}_R^1 & 0 & \cdot & \cdot & \cdot \\ 0 & \mathcal{A}_R^2 & \mathcal{B}_R^2 & \mathcal{C}_R^2 & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & 0 & \mathcal{A}_R^{I-2} & \mathcal{B}_R^{I-2} & \mathcal{C}_R^{I-2} & 0 \\ \cdot & \cdot & \cdot & 0 & \mathcal{A}_R^{I-1} & \mathcal{B}_R^{I-1} & \mathcal{C}_R^{I-1} \end{pmatrix},$$

with

$$\mathcal{A}_L^j = -(A_j^{k+1} + \mathcal{M}_{k+1,j}^d(X))$$

$$
\begin{aligned}
\mathcal{B}_L^j &= -(B_j^{k+1} + 1 + \mathcal{M}_{k+1,j}^d(X)) \\
\mathcal{C}_L^j &= -(C_j^{k+1} + \mathcal{M}_{k+1,j}^d(X)) \\
\mathcal{A}_R^j &= A_j^k + \mathcal{M}_{k,j}^d(X) \\
\mathcal{B}_R^j &= B_j^k + 1 + \mathcal{M}_{k,j}^d(X) \\
\mathcal{C}_R^j &= C_j^k + \mathcal{M}_{k,j}^d(X).
\end{aligned}
$$

We will know the boundary conditions at every $i = 0$ and $k = 0$, as well as at every $i = I$. Following from the discussion in section 4.5, if we simplify the left hand side of equation (6.3) to ensure that we build the mesh with the right conditions, then we have

$$M_L^{k+1}v^{k+1} + b^{k+1} = M_R^k v^k + b^k,$$

where

$$M_L^{k+1}v^{k+1} + b^{k+1} = \begin{pmatrix} \mathcal{B}_L^1 & \mathcal{C}_L^1 & 0 & \cdot & \cdot \\ \mathcal{A}_L^2 & \mathcal{B}_L^2 & \mathcal{C}_L^2 & 0 & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & \mathcal{A}_L^{I-2} & \mathcal{B}_L^{I-2} & \mathcal{C}_L^{I-2} \\ \cdot & \cdot & 0 & \mathcal{A}_L^{I-1} & \mathcal{B}_L^{I-1} \end{pmatrix} \begin{pmatrix} X_1^{k+1} \\ \cdot \\ \cdot \\ \cdot \\ X_{I-1}^{k+1} \end{pmatrix} + \begin{pmatrix} \mathcal{A}_R^1 X_0^{k+1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \mathcal{C}_L^{I-1} X_I^{k+1} \end{pmatrix}.$$

Similar changes should be applied to $M_R^k v^k + b^k$. Then M_L^{k+1} and M_R^k are square of size $I-2$, and we can solve it directly. We rewrite the equation as

$$
\begin{aligned}
M_L^{k+1}v^{k+1} &= M_R^k v^k + b^k - b^{k+1} \\
&= q_k,
\end{aligned}
\tag{6.4}
$$

and we can calculate M^{k+1} and q_k already. So all we need to do is invert M^{k+1} and calculate the new values of V^k. We then continue this process until $k = K$ has been calculated and look to the correct node to find the solution.

Note that this is only true in the case of no American type conditions. In the next section we see that we can apply the American conversion condition in an ad hoc manner, but this reduces the accuracy. We actually need to apply the American condition using a method like Succesive Over

Relaxation (SOR) to solve a linear complimentarity problem. This would allow us to change the value of each node with the exact optimal exercise condition. This was not within the scope of this dissertation.

We can use LU decomposition to hasten the inversion of the matrix, the process decomposes the matrix M^{k+1} into two matrices L and U, and a complete description of the process can be found in Wilmott [36]. Here L is lower triangular matrix and U is upper triangular. This is especially simple because M^{k+1} is a tri-diagonal matrix, which means it can be decomposed very quickly. The process is as follows

$$\begin{aligned} Mv &= q \\ LUv &= q \\ Uv &= L^{-1}q \\ v &= U^{-1}(L^{-1}q). \end{aligned}$$

This is purely to save on computation and is not necessary to obtain a result. We can also solve the system directly since M is tri-diagonal.

We use this method to compute the grid values at $k = 1$, where the final conditions have already supplied us with the values at $k = 0$. Each calculation depends on 3 points from the previous step, like in point 1 in the figure. We continue in this manner until we have calculated the values of each grid point.

6.3 The Boundary and Jump Conditions

We need to have boundary conditions to begin the implementation, we know from the maturity condition of a convertible bond that the value at maturity should be the higher of the conversion value and the bond value, so we have

$$B_i^0 = \begin{cases} F + \frac{c}{n} & \text{if there is a coupon at maturity} \\ F & \text{else} \end{cases},$$

where c is the size of that coupon. The equity part requires

$$E_i^0 = \max[Ri\Delta S - B_i^0, 0].$$

This would yield the final result, which matches the maturity condition:

$$C_i^0 = \begin{cases} \max[Ri\Delta S, F + \frac{c}{n}] & \text{if there is a coupon at maturity} \\ \max[Ri\Delta S, F] & \text{else} \end{cases}.$$

Since we are working in inverse time, the maturity condition is our initial condition when using the Crank-Nicolson method. This would mean that we require the boundary conditions along the sides next, i.e. when $i = 0$ and $i = I$ while k is non-zero. We know from the bankruptcy condition that if the stock price has dropped to zero then the convertible bond value should also be zero unless a put is active, therefore we have

$$B_0^k = \begin{cases} \max[0, PP + AccInt(PP, k)] & \text{If the put is active at } k \\ 0 & \text{else} \end{cases},$$

and

$$E_0^k = 0.$$

Ayache, Forsyth and Vetzal [3] define the accrued interest on the payment as

$$AccInt(K_p, t) = K_p \frac{t - t_p}{t_n - t_p},$$

where t_n and t_p are the next and previous payments of the same nature as the payment in question, and K_p is the payment at t_n.

As the value of the stock increases the convertible bond's value will behave more like stock, so the value of the convertible bond is linear in S. This gives us

$$B_I^k = 2B_{I-1}^k - B_{I-2}^k$$
$$E_I^k = 2E_{I-1}^k - E_{I-2}^k.$$

Which gives us all the boundary conditions we need to start computing the mesh points. The next section describes the jump conditions we need to enforce while calculating the inner mesh points.

The only jump conditions we need to adhere to are those of the coupons and dividends. Since the dividend is discrete and it is added to each node at the mesh point k corresponding to the coupon dates, we have this simple addition

$$B_i^k = B_i^k + c.$$

We need to ensure that we have a grid line at the time of coupon and dividend payments, so we have to add the number of coupons and dividends that occur in the life of the convertible bond to the maximum grid constant K, but make sure not to add the coupons and dividends that already fall exactly on the grid. Note that the time steps before and after any added gridlines will have a different Δt, so we would need to alter the Crank-Nicolson equations on those points.

6.4 The Minimum and Maximum Constraints

We need to check that each mesh point falls within the minimum and maximum bounds; since we assume that the issuer will never allow the convertible bond's value to exceed the call price we have

$$B_i^{k+1} = \min[CP, B_i^{k+1}],$$

and

$$E_i^{k+1} = \min[E_i^{k+1}, \max[RS, CP] - B_i^{k+1}],$$

if the call is active. As mentioned earlier this is accomplished by setting $CP = \infty$ while it is not active. This simply means that if the call is active, then the convertible bond's value cannot fall lower than the conversion value, or if the call price is higher than the conversion value then it cannot fall lower than the call price.

If the put is active we have the following two constraints on the bond and equity components

$$B_i^{k+1} = \max[B_i^{k+1}, PP - E_i^{k+1}]$$
$$E_i^{k+1} = \max[RS - B_i^{k+1}, E_i^{k+1}],$$

since this would yield

$$C_i^{k+1} = \max[RS, PP],$$

which means if the put is active the value of the convertible bond cannot fall below either the conversion value or the put price.

6.5 Numerical Example

This section uses the algorithms described in this chapter to evaluate the numerical example detailed in table 6.1.

Nominal	100
Maturity	5 Years
Coupon	4% Semi-Annual
Continuous Dividend	3%
Current Share Price	20
Share Volatility	20%
Interest Rate	5%
Call Price	110
Call Period	Active after 2 years
Put Price	105
Put Period	Active at 3 years
Conversion Ratio	1
η	0%
p	0.02
α	1
κ	0

Table 6.1: Numerical Example

We next investigate the difference between the hedge model with total and partial default, as well as the Tsiveriotis and Fernandes and zero-default methods. We also compare these models with differing mesh sizes to compare rates of convergence.

We note that these results are consistent with what we expected by looking at figure 5.5. The zero-default model over values the convertible bond since it doesn't take default into account. The hedge model values the convertible bond higher or lower than the Tsiveriotis and Fernandes approach, depending on the stock loss rate η. We also note that convergence is quick with relatively small mesh sizes. Mesh properties like $\Delta S = 0.1$ and $\Delta t = 0.01$ will produce a mesh with around 3 million nodes, easily handled computationally. Remember that the computation

ΔS	Δt	Hedge Model $\eta = 1$	Hedge Model $\eta = 0$	T & F	Zero Default
1	0.1	120.9418633	122.9135767	121.5190327	123.8581244
0.5	0.1	120.9426562	122.9144726	121.5200104	123.8586652
0.5	0.01	120.7860531	122.7905524	121.3555066	124.2115067
0.1	0.01	120.7862049	122.7907367	121.3556393	124.2116349

Table 6.2: Numerical Results

will require a matrix inversion, which could be very problematic with large mesh sizes. This highlights the importance of the LU decomposition discussed earlier.

η	Hedge Model $S = 20$	Hedge Model $S = 120$
0%	109.6056156	134.8855875
100%	108.7456959	133.2880993

Table 6.3: Comparison of Stock Loss Rate

Table 6.3 compares the stock loss rate at different share levels. We note that if we assume that the share price stays unchanged during default, the convertible is valued higher than it would be otherwise. These results confirm the discussion in chapter 5 and with figure 5.2.

Recovery Percentage	Hedge Model $S = 20$	Hedge Model $S = 120$
0%	109.6056156	134.8855875
50%	111.7701339	134.8851494
100%	114.8378997	135.2129409

Table 6.4: Comparison of Recovery Percentages

Table 6.4 compares different recovery percentages at varying share levels. We notice that if we assume higher recovery upon default, the convertible is valued higher. We also note that the recovery percentage has a more prominent effect at lower share prices. This is consistent with the discussion in Chapter 5 and the results of figure 5.3.

Table 6.5 compares the hedge model with varying calibration constants at differing share prices.

Shareprice	$\alpha = 1.2$	$\alpha = 1.6$	$\alpha = 2.0$
20	74.68273013	74.06569639	73.36085383
40	83.13712991	82.5113108	81.82112261
60	91.93405399	91.28900411	90.60070194
80	102.0827334	101.4744572	100.842064
100	114.9445964	114.4737966	113.9974485
120	130.6834086	130.3877925	130.0851711

Table 6.5: Comparison of Calibration Constant

At high share prices we notice that there is little difference between the resulting prices, while at lower share price there is a very prominent effect in the convertible bond's value. This is consistent with the theoretical discussion in chapter 5, as well as with figure 5.4.

The next section shows the code used to implement the hedge model, it is very similar to the approach used by Tsiveriotis and Fernandes approach.

6.6 Program Code: Binomial method used in Chapter 2

The following code is the implementation of the Binomial tree used to produce the graphs in Chapter 2.

```
Private Sub CommandButton1_Click()
    Unload Me
End Sub

Private Sub CommandButton2_Click()

    Worksheets("BondCalculator").Activate

    '''''''''''' Clear the cells on worksheet
    ClearContents = Range("A50").Value + 1
    For j = 1 To ClearContents
        For i = 1 To j
```

```
        Cells(2 * i + 5, j + 1).Clear
        Cells(2 * i + 4, j + 1).Clear
    Next
Next
'''''''''''''''''''''''''''''''''''''''''


'''''''''' Input Constants from worksheet
Nominal = NominalTextBox.Value
ConversionRatio = ConversionRatioTextBox.Value
MaturityOfCB = MaturityTextBox
SharePrice = SharePriceTextBox.Value
VolatilityOfStock = VolatilityTextBox.Value
InterestRate = InterestRateTextBox.Value
NumberOfPeriods = NumberOfPeriodsTextBox.Value
coupon = CouponTextBox.Value
Numofcoupons = CouponNumberTextBox.Value
Dividend = DividendTextBox.Value
NumberOfDividend = DividendPerYearTextBox.Value
Hardcall = CInt(HardCallTextBox.Value)
Premium = HardPremiumTextBox.Value
PutDate = PutOptionTextBox.Value

CouponFlag = CouponToggle.Value
DividendFlag = DividendToggle.Value
HardCallFlag = CallToggle.Value
PutFlag = PutToggle.Value
''''''''''''''''''''''''''''''''''''


''''''''' Create Trees
Dim CBTree() As Double
ReDim CBTree(NumberOfPeriods, NumberOfPeriods)
Dim StockTree() As Double
```

```
ReDim StockTree(NumberOfPeriods, NumberOfPeriods)
'''''''''

'''''' Precompute constants and error checks
dt = MaturityOfCB / NumberOfPeriods
ConversionPrice = Nominal / ConversionRatio
CouponAmount = ((Nominal * coupon) / Numofcoupons) / ConversionRatio
NextCouponDate = MaturityOfCB
NextDividendDate = MaturityOfCB

'''' Compute necessary constants for Binomial method
disc = (1 + InterestRate) ^ dt - 1
u = Exp(VolatilityOfStock * Sqr(dt))
d = 1 / u
p = (1 + disc - d) / (u - d)
q = 1 - p

'''' This block creates the asset tree, taking care of any dividends paid
'''' Dividends are paid at period N and then worked backwards
'''' If there is no dividend it will have no effect
For i = NumberOfPeriods To 0 Step -1
    If (DividendFlag = True) And ((dt * i) <= NextDividendDate) And (i > 0) Then
        StockTree(i, i) = SharePrice * (d ^ i) * (1 - Dividend)
        NextDividendDate = NextDividendDate - 1 / NumberOfDividend
    Else
        StockTree(i, i) = SharePrice * (d ^ i)
    End If
    For j = i - 1 To 0 Step -1
        StockTree(i, j) = StockTree(i, j + 1) * u / d
    Next
Next
'''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''
```

```
'''' Apply European Option at last timestep
For j = 0 To NumberOfPeriods
    If CouponFlag = True Then
        CBTree(NumberOfPeriods, j) = ...
        WorksheetFunction.Max(StockTree(NumberOfPeriods, j) ...
        , ConversionPrice + CouponAmount)
    Else
        CBTree(NumberOfPeriods, j) = ...
        WorksheetFunction.Max(StockTree(NumberOfPeriods, j) ...
        , ConversionPrice)
    End If
Next
If CouponFlag = True Then
    NextCouponDate = NextCouponDate - 1 / Numofcoupons
End If

'''' Step back through the tree
For i = (NumberOfPeriods - 1) To 0 Step -1
    For j = 0 To i

        '''' Pay Coupon Here
        If (CouponFlag = False) Then
                CBTree(i, j) = (p * CBTree(i + 1, j) + q * ...
                CBTree(i + 1, j + 1)) / (1 + disc)
                If (dt * i > Hardcall) And (HardCallFlag = True) ...
                And (StockTree(i, j) >= (ConversionPrice * ...
                (1 + Premium))) Then
                    CBTree(i, j) = StockTree(i, j)
                End If
                If (dt * i = PutDate) And (PutFlag = True) ...
                And (CBTree(i, j) < ConversionPrice) Then
```

```
            CBTree(i, j) = ConversionPrice
        End If
Else
    If ((dt * i) <= NextCouponDate) Then
        CBTree(i, j) = (p * CBTree(i + 1, j) + q * ...
        CBTree(i + 1, j + 1)) / (1 + disc) + CouponAmount
        If (dt * i > Hardcall) And (HardCallFlag = True) ...
        And (StockTree(i, j) >= (ConversionPrice * ...
        (1 + Premium))) Then
            CBTree(i, j) = StockTree(i, j)
        End If
        If (dt * i = PutDate) And (PutFlag = True) And ...
        (CBTree(i, j) < ConversionPrice) Then
            CBTree(i, j) = ConversionPrice
        End If
        If j = 0 Then
            NextCouponDate = NextCouponDate - 1 / Numofcoupons
        End If
    Else
        CBTree(i, j) = (p * CBTree(i + 1, j) + q * ...
        CBTree(i + 1, j + 1)) / (1 + disc)
        If (dt * i > Hardcall) And (HardCallFlag = True) ...
        And (StockTree(i, j) >= (ConversionPrice * ...
        (1 + Premium))) Then
            CBTree(i, j) = StockTree(i, j)
        End If
        If (dt * i = PutDate) And (PutFlag = True) And ...
        (CBTree(i, j) < ConversionPrice) Then
            CBTree(i, j) = ConversionPrice
        End If
    End If
End If
```

```
    Next
Next


If OutputCheckBox.Value = True Then
    '''''' Output Tree ''''''
    For i = 0 To NumberOfPeriods
        For j = 0 To i
            Cells(6 + 2 * j, 2 + i).Value = StockTree(i, j)
            Cells(6 + 2 * j, 2 + i).Select
            Selection.Interior.ColorIndex = 35

            '''''''''''''' Border ''''''''''''''''
            Cells(6 + 2 * j, 2 + i).Select
            Selection.Borders(xlDiagonalDown).LineStyle = xlNone
            Selection.Borders(xlDiagonalUp).LineStyle = xlNone
            With Selection.Borders(xlEdgeLeft)
                .LineStyle = xlContinuous
                .Weight = xlThick
                .ColorIndex = xlAutomatic
            End With
            With Selection.Borders(xlEdgeTop)
                .LineStyle = xlContinuous
                .Weight = xlThick
                .ColorIndex = xlAutomatic
            End With
            With Selection.Borders(xlEdgeBottom)
                .LineStyle = xlContinuous
                .Weight = xlThin
                .ColorIndex = xlAutomatic
            End With
            With Selection.Borders(xlEdgeRight)
```

```
    .LineStyle = xlContinuous
    .Weight = xlThick
    .ColorIndex = xlAutomatic
End With
'''''''''''''' Border End ''''''''''''''''

Cells(7 + 2 * j, 2 + i).Value = CBTree(i, j)
Cells(7 + 2 * j, 2 + i).Select
Selection.Interior.ColorIndex = 36
'''''''''''''' Border ''''''''''''''''
Cells(7 + 2 * j, 2 + i).Select
Selection.Borders(xlDiagonalDown).LineStyle = xlNone
Selection.Borders(xlDiagonalUp).LineStyle = xlNone
With Selection.Borders(xlEdgeLeft)
    .LineStyle = xlContinuous
    .Weight = xlThick
    .ColorIndex = xlAutomatic
End With
With Selection.Borders(xlEdgeTop)
    .LineStyle = xlContinuous
    .Weight = xlThin
    .ColorIndex = xlAutomatic
End With
With Selection.Borders(xlEdgeBottom)
    .LineStyle = xlContinuous
    .Weight = xlThick
    .ColorIndex = xlAutomatic
End With
With Selection.Borders(xlEdgeRight)
    .LineStyle = xlContinuous
    .Weight = xlThick
    .ColorIndex = xlAutomatic
```

```
                End With
                ''''''''''''''' Border End '''''''''''''''''
            Next
        Next
        Range("O3").Select
        Range("A50") = NumberOfPeriods
    End If
    ''''''  Output Tree ''''''

    PriceTextBox.Caption = Round(100 * CBTree(0, 0) / SharePrice, 3)
    ParityTextBox.Caption = Round(SharePrice * ConversionRatio, 3)
    PremiumTextBox.Caption = Round((CBTree(0, 0) / SharePrice) * ...
    Nominal - SharePrice * ConversionRatio, 3)



End Sub



Private Sub ConversionRatioTextBox_Change()
    If (NominalTextBox.Value <> "") And ...
    (ConversionRatioTextBox.Value <> "") Then
        ConversionPriceTextBox.Caption = ...
        Round(NominalTextBox.Value / ConversionRatioTextBox.Value, 3)
    End If
End Sub

Private Sub NominalTextBox_Change()
    If (NominalTextBox.Value <> "") And ...
    (ConversionRatioTextBox.Value <> "") Then
        ConversionPriceTextBox.Caption = ...
```

```
        Round(NominalTextBox.Value / ConversionRatioTextBox.Value, 3)
    End If
End Sub
```

6.7 Program Code: Finite diference approach

The following code is implemented in Visual Basic for Excel, this code was completed in a rush, and in no way values any convertible bond. The code can only handle a 5 year CB with semi annual coupons. This was done so that a grading deadline could be made. Please do not in any way take this code as the correct method of implementation.

```
ReDim FirstZ(Maxi) '=== FirstZ and FirstE are the final mesh line that we set
ReDim FirstE(Maxi) '=== up using the final conditions, for both the equity and
                   '=== cash components.
ii = 0
Do
    FirstZ(ii) = Zwithkzero()  '=== Set node to final cash condition
    FirstE(ii) = Ewithkzero()  '=== Set node to final equity condition
    ii = ii + 1
Loop While ii <= Maxi

ActualTime = T + Deltat '=== We need to keep track of time since were
                        '=== stepping backwards.
'=== Now we step through from k = 0 to the current time, or k = K
For kk = 0 To MaxK
    '=== Time Keeper
    ActualTime = ActualTime - Deltat
    '=== Check Put and Call Times and Set price if Active
    If (ActualTime = PutTime) And PutExists Then
        PutPrice = PutPriceHolder
```

```
Else
    PutPrice = 0
End If
If (ActualTime >= CallTime) And CallExists Then
    CallPrice = CallPriceHolder
Else
    CallPrice = 1000000000
End If

'=== Insert new row of Z's and E's
If kk > 0 Then '=== Should only enter if not the first mesh line, since the
               '=== first mesh line has already been created
    For ii = 1 To Maxi - 1
        '=== This step is ignored the first time, therefore the first time
        '=== this code is run, NewZ and NewE will exist. They are the new
        '=== values for our mesh line.
        FirstZ(ii) = NewZ(ii)
        FirstE(ii) = NewE(ii)
    Next ii
    '=== Now we need to apply our boundary conditions to the nodes where
    '=== i = 0 and i = I
    FirstZ(0) = Zwithizero(kk)
    FirstZ(Maxi) = 2 * FirstZ(Maxi - 1) - FirstZ(Maxi - 2)
    FirstE(0) = Ewithizero()
    FirstE(Maxi) = 2 * FirstE(Maxi - 1) - FirstE(Maxi - 2)
End If
'=== Now we apply the coupon, if one exists at this node.
If AnnualCoupon Then
    If Round(kk * Deltat - Int(kk * Deltat), 5) = 0 Then
        For ii = 0 To Maxi
            FirstZ(ii) = FirstZ(ii) + Coupon
        Next ii
```

```
    End If
ElseIf SemiAnnualCoupon Then
    If (Round(kk * Deltat - Int(kk * Deltat), 5) = 0)
                Or (Round(kk * Deltat - Int(kk * Deltat), 5) = 0.5) Then
        For ii = 0 To Maxi
            FirstZ(ii) = FirstZ(ii) + Coupon
        Next ii
    End If
End If
' === Now we construc the Right hand side vector v_{k}
ReDim RightMatrix(1 To Maxi - 1)
ReDim RightMatrixE(1 To Maxi - 1)
For ii = 1 To Maxi - 1
    RightMatrix(ii) = FirstZ(ii)
    RightMatrixE(ii) = FirstE(ii)
Next ii
'=== Now we multiply the big M matrix on the right with v_{k}
For ii = 1 To Maxi - 1
    RightMatrix(ii) = BigA(ii) * FirstZ(ii - 1) + (BigB(ii)
      + 0.5 * Deltat * P(ii) * RecoveryPercentage + 1)
      * FirstZ(ii) + BigC(ii) * FirstZ(ii + 1)
    RightMatrixE(ii) = BigA(ii) * FirstE(ii - 1) + (BigB(ii) + 1)
      * FirstE(ii) + BigC(ii) * FirstE(ii + 1) + Deltat
      * P(ii) * WorksheetFunction.Max(ConversionRatio * (ii)
      * DeltaS * (1 - ShareLossRatio) - RecoveryPercentage
      * FirstZ(ii), 0)
Next ii
'=== Now we add the b^{k} term and subtract the b^{k+1} term to obtain q_{k}
RightMatrix(1) = RightMatrix(1) + BigA(1) * Zwithizero(kk + 1)
RightMatrixE(1) = RightMatrixE(1) + BigA(1) * Ewithizero()
'=== Build the Big Matrix on right hand side
'=== (As Diagonals so that we can do LU factorization Easily)
```

```
If kk = 0 Then ' ONLY USE THIS LOOP IF MATRIX IS TIME INVARIANT
    ReDim Diag(1 To Maxi - 1)
    ReDim SuperDiag(1 To Maxi - 1)
    ReDim SubDiag(1 To Maxi - 1)
    ReDim DiagE(1 To Maxi - 1)
    ReDim SuperDiagE(1 To Maxi - 1)
    ReDim SubDiagE(1 To Maxi - 1)
    For ii = 1 To Maxi - 1
        Diag(ii) = -(BigB(ii) + 0.5 * Deltat * P(ii) * RecoveryPercentage - 1)
        DiagE(ii) = -(BigB(ii) - 1)
    Next ii
    For ii = 1 To Maxi - 1
        SubDiag(ii) = -BigA(ii + 1)
        SubDiagE(ii) = -BigA(ii + 1)
    Next ii
    For ii = 1 To Maxi - 1
        SuperDiag(ii) = -BigC(ii)
        SuperDiagE(ii) = -BigC(ii)
    Next ii
    '=== Do LU Factorisation
    '=== This code thanks to Paul Wilmott on Derivatives
    ReDim d(1 To Maxi - 1)
    ReDim u(1 To Maxi - 1)
    ReDim l(1 To Maxi - 1)
    ReDim dE(1 To Maxi - 1)
    ReDim uE(1 To Maxi - 1)
    ReDim lE(1 To Maxi - 1)
    d(1) = Diag(1)
    dE(1) = DiagE(1)
    For ii = 2 To Maxi - 1
        u(ii - 1) = SuperDiag(ii - 1)
        l(ii) = SubDiag(ii - 1) / d(ii - 1)
```

```
        d(ii) = Diag(ii) - l(ii) * SuperDiag(ii - 1)
        uE(ii - 1) = SuperDiagE(ii - 1)
        lE(ii) = SubDiagE(ii - 1) / dE(ii - 1)
        dE(ii) = DiagE(ii) - lE(ii) * SuperDiagE(ii - 1)
    Next ii
End If
'=== Multiply L and U and obtain new RHS
'=== This code thanks to Wilmott book
ReDim NewZ(0 To Maxi)
ReDim w(1 To Maxi - 1)
ReDim NewE(0 To Maxi)
ReDim wE(1 To Maxi - 1)
w(1) = RightMatrix(1)
wE(1) = RightMatrixE(1)
For ii = 2 To Maxi - 1
   w(ii) = RightMatrix(ii) - l(ii) * w(ii - 1)
    wE(ii) = RightMatrixE(ii) - lE(ii) * wE(ii - 1)
Next ii
NewZ(Maxi - 1) = w(Maxi - 1) / d(Maxi - 1)
NewE(Maxi - 1) = wE(Maxi - 1) / dE(Maxi - 1)
For ii = Maxi - 2 To 1 Step -1
    NewZ(ii) = (w(ii) - u(ii) * NewZ(ii + 1)) / d(ii)
    NewE(ii) = (wE(ii) - uE(ii) * NewE(ii + 1)) / dE(ii)
Next ii
'=== Now add the boundary values at i=0 and i=I
NewZ(0) = Zwithizero(kk)
NewZ(Maxi) = 2 * NewZ(Maxi - 1) - NewZ(Maxi - 2)
NewE(0) = Ewithizero()
NewE(Maxi) = 2 * NewE(Maxi - 1) - NewE(Maxi - 2)
'=== Check Minimum and Maximum Value Constraints
For ii = 0 To Maxi
    '== Minimum
```

```
        NewZ(ii) = WorksheetFunction.Min(CallPrice + CallAccInt(), NewZ(ii))
        If (ConversionRatio * (DeltaS * ii) < PutPrice) Then
            NewZ(ii) =
              WorksheetFunction.Max(NewZ(ii), PutPrice + PutAccInt() - NewE(ii))
        Else
            NewE(ii) =
              WorksheetFunction.Max(ConversionRatio
                 * (DeltaS * ii) - NewZ(ii), NewE(ii))
        End If
        '== Maximum
        NewE(ii)
          = WorksheetFunction.Min(NewE(ii), WorksheetFunction.Max(ConversionRatio
            * (DeltaS * ii), CallPrice + CallAccInt()) - NewZ(ii))
    Next ii
Next kk

Dim i As Double, Solution As Double
i = StockPrice / DeltaS
Solution = NewZ(i) + NewE(i)
```

The following are the supporting functions:

```
Function BigA(i)
    BigA = 0.5 * (Deltat / (DeltaS ^ 2)) * LittleA(i)
           - 0.25 * (Deltat / DeltaS) * LittleB(i)
End Function

Function BigB(i)
    BigB = -(Deltat / (DeltaS ^ 2)) * LittleA(i)
           + 0.5 * Deltat * LittleC(i)
End Function
```

```
Function BigC(i)
    BigC = 0.5 * (Deltat / (DeltaS ^ 2)) * LittleA(i)
           + 0.25 * (Deltat / DeltaS) * LittleB(i)
End Function

Function P(i)
P = HazardRate * (StockPrice / (i * DeltaS)) ^ (Alpha)
End Function

Function LittleA(i)
    LittleA = 0.5 * (Sigma ^ 2) * (i * DeltaS) ^ 2
End Function

Function LittleB(i)
    LittleB = (r - q) * i * DeltaS + P(i) * ShareLossRatio * i * DeltaS
End Function

Function LittleC(i)
    LittleC = -(r + P(i))
End Function

Function Zwithkzero()
    Zwithkzero = Nominal + Coupon ' Coupon gets dealt with later
End Function

Function Ewithkzero()
    Ewithkzero = WorksheetFunction.Max(ConversionRatio * Maxi * DeltaS
                 - Zwithkzero(), 0)
End Function

Function Zwithizero(k)
    If k = 0 Then
```

```
        Zwithizero = Zwithkzero()
    Else
        Zwithizero = WorksheetFunction.Max(0, PutPrice + PutAccInt())
    End If
End Function

Function Ewithizero()
    Ewithizero = 0
End Function

Function CallAccInt()
    Dim PreviousCoupon As Double, NextCoupon As Double
    If CallPrice < 1000000000 Then
        If OnCouponLine Then
            CallAccInt = 0
        ElseIf AfterCouponLine Then
            NextCoupon = Int(ActualTime)
            PreviousCoupon = Int(ActualTime) + 0.5
            CallAccInt =
             Coupon * (ActualTime - PreviousCoupon) / (NextCoupon - PreviousCoupon)
        Else
            NextCoupon = Int(ActualTime) + 0.5
            PreviousCoupon = Int(ActualTime) + 1
            CallAccInt =
             Coupon * (ActualTime - PreviousCoupon) / (NextCoupon - PreviousCoupon)
        End If
    Else
        CallAccInt = 0
    End If
End Function

Function PutAccInt()
```

```
    Dim PreviousCoupon As Double, NextCoupon As Double
    If PutPrice > 0 Then
        If OnCouponLine Then
            PutAccInt = 0
        ElseIf AfterCouponLine Then
            NextCoupon = Int(ActualTime)
            PreviousCoupon = Int(ActualTime) + 0.5
            PutAccInt =
              Coupon * (ActualTime - PreviousCoupon) / (NextCoupon - PreviousCoupon)
        Else
            NextCoupon = Int(ActualTime) + 0.5
            PreviousCoupon = Int(ActualTime) + 1
            PutAccInt =
              Coupon * (ActualTime - PreviousCoupon) / (NextCoupon - PreviousCoupon)
        End If
    Else
        PutAccInt = 0
    End If
End Function
```

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