33

An Elementary Approach To Some Aspects Of Heegaard Floer Homology

By

Huygens Christian Ravelomanana

THESIS PRESENTED FOR THE DEGREE OF MASTER OF SCIENCE

IN THE DEPARTMENT OF MATHEMATICS

UNIVERSITY OF CAPE TOWN

SUPERVISOR: Dr. DAVID T. GAY (PP Prof. Christopher Gilmour)

February 2010

Abstract

Heegaard Floer homology is a new invariant for 3-manifolds and links introduced by Ozsvath and Szabo. It counts pseudo-holomorphic Whitney disks in the symmetric product of the Heegaard surface of the underlying manifold. This thesis is about showing the existence of a natural complex structure on the symmetric product $\mathrm{Sym}^g(\Sigma_g)$ of a surface Σ_g and mainly studying the moduli space of the set of holomorphic representatives of Whitney disks for special cases of domains from the Heegaard diagram which are bigons and squares. These moduli spaces are relevant for the computation of the boundary map in Heegaard Floer Homology. All of this is done using basic tools from differential topology and complex analysis. We will end this thesis by briefly describing some of the analysis required to ground this work in the more general theory of J-holomorphic curves and partial differential operators.

Contents

Introduction

Heegaard Floer homology is a recent invariant of closed and oriented 3-manifolds and links. It was first introduced in [OS04c] and [OS04b]. In [OS06], Ozsvath and Szabo give a first introduction to Heegaard Floer homology and [OS05] is a continuation of it. This homology is based on a variation of Floer homology for Lagrangian intersections and makes use of Gromov's theory of pseudo-holomorphic disks. To each closed and oriented 3-manifold with some extra data it assigns an abelian group which is up to isomorphism independent of these extra data. Two homeomorphic 3-manifolds will give isomorphic groups. The aim of this thesis is not to give an account of all these developments but to focus on Heegaard Floer homology and a particular aspect. As the title suggests it gives an elementary approach to some aspects of the theory. By elementary approach one means an approach which comes from simple intuitions. We do not discuss the analysis involved in the Heegaard Floer theory, the independence with respect to analytic data and will not give a combinatorial description. With such an approach we still can get some information, especially for the computation of the boundary map.

Two important things in the Heegaard Floer homology are the symmetric product of the Heegaard surface and the J-holomorphic Whitney disks on the symmetric product. The main results of this thesis are, first the proof that the symmetric product $\mathrm{Sym}^g(\Sigma_g)$ of a Riemann surface has a natural complex structure, which is the direct image of the product structure on the g-fold Cartesian product of the surface. This is done in Chapter 5. The proof we give involves some concepts of complex analysis such as analytic sets and analytic continuation. It is elementary in the sense that we give an explicit construction of the complex charts.

The second one involves the counting of holomorphic disks in Heegaard Floer homology. Using the complex structure from Chapter 5 one proves that for the *domain* corresponding to *convex bigons* or *convex squares*, there is exactly a 1-parameter family of holomorphic Whitney disks.

This is done by applying elementary tools from complex analysis and differential topology. We make a discussion on the fact that if there is at least one non-convex vertex then the dimension increases at least by one and it becomes irrelevant for computing the boundary map. Chapter 7 contains the principal part of this.

We end with a discussion on the larger context of counting J-holomorphic curves in almost-complex manifolds. We give a brief description of the problem of moduli space of J-holomorphic curves.

For the prerequisites, we give in Chapter 1 some definitions and result on manifolds; in Chapter 1 the section on connections is for the last chapter on moduli space of pseudo-holomorphic curves. Chapter 2 is about Morse theory focusing on Morse homology and Chapter 3 about complex and almost-complex structures; references for this chapter are [Wel80], [MS04], [MS98]. Some basic background in general topology is needed for the discussion on symmetric product. In the last chapter, we use the notion of the first Chern class which is beyond the scope of this thesis, for this, the reader should look at extra references such as [Wel80], [Mor01].

Chapter 1

Manifolds

This is a review of some notions related to manifolds. References for manifolds, differentiable structure, transversality, vector bundles, metrics and connections can be found in the literature [Mil65] [GP74] [Rha60].

All the topological spaces in this chapter will be assumed to be Hausdorff and para-compact *i.e* every open cover has a locally finite open refinement. We denote

$$\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_n \geq 0\}.$$

1.1 Basic Structures

Definition 1.1. *A topological manifold of dimension n is a topological space M in which each point has a neighbourhood homeomorphic to an open set in $\mathbb{R}^n_+$.*

The boundary of M denoted ∂M is the set of points which correspond to points in $\{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_n = 0\}$.

M is called a topological manifold with boundary if $\partial M \neq \emptyset$ and without boundary if $\partial M = \emptyset$.

We also use the terminology n-dimensional manifold or n-manifold for a topological manifold of dimension n.

Definition 1.2. *A manifold M is called closed if it is compact without boundary.*

Definition 1.3. *Let $p \in \mathbb{N}$. A C^p (resp. smooth) atlas on a topological manifold M of dimension n is a family $(U_k, \varphi_k)_k$ such that:*

1. *$(U_k)_k$ is an open cover of M,*
2. *for each k, φ_k is an homeomorphism of U_k onto an open set of $\mathbb{R}^n_+$,*
3. *for each pair l, k of indices, $\varphi_k \circ \varphi_l^{-1} : \varphi_k (U_k \cap U_l) \longrightarrow \varphi_l (U_k \cap U_l)$ is a C^p (resp. smooth) map.*

The pair (U_k, φ_k) is called a local chart and the n-tuple $(x_1, \cdots, x_n)$ of components of φ_k is called a local coordinate system.

The map $\varphi_k \circ \varphi_l^{-1}$ is called a transition map.

We define an equivalence relation on the set of atlases of class C^p (resp. smooth atlases) on the same manifold by:

$$(U_k, \varphi_k)_k \sim (V_l, \phi_l)_l \qquad \text{iff} \qquad \text{for every } l, k, \ \varphi_k \circ \phi_l^{-1} \text{ is a diffeomorphism.}$$

Definition 1.4. *An n-dimensional differentiable manifold of class C^p (resp. a smooth manifold) is an n-dimensional topological manifold equipped with an equivalence class of C^p (resp. smooth) atlases.*

We are specially interested in smooth manifolds. All the definitions which come below can be applied to C^p manifolds.

Definition 1.5. *A smooth manifold M is called orientable if it possesses an atlas $(U_k, \varphi_k)_k$ such that each transition map $\varphi_k \circ \varphi_l^{-1}$ has a strictly positive Jacobian.*

Definition 1.6. *Let N and M be two smooth manifolds. A continuous map $f : M \longrightarrow N$ is a smooth map if at each point $x \in M$, there exists a local chart (U, φ) at x and (V, ψ) at $f(x)$ such that the map $\psi \circ f \circ \varphi^{-1}$ is smooth.*

A diffeomorphism between M and N is an homeomorphism $f : M \longrightarrow N$ such that f is smooth and f^{-1} is smooth.

These definitions do not depend on the choice of charts.

Definition 1.7. *Let M be an m-dimensional smooth manifold. A smooth submanifold of M of dimension $n \leq m$ is a subset $N \subset M$ such that at each point $x \in N$ there exists a local chart (U, φ) with the property $\varphi(U \cap N) = \varphi(U) \cap \left(\mathbb{R}^n_+ \times \underline{0}\right)$.*

We can think of a smooth submanifold of M as a subset of M which has a structure of a smooth manifold with respect to the induced topology.

Let M be a smooth m-manifold and $x \in M$. Let $\mathcal{P}_x$ be the set of differentiable paths $\gamma : (-\epsilon_\gamma, \epsilon_\gamma) \longrightarrow M$ with $\gamma(0) = x$. We define an equivalence relation $\mathcal{R}$ on $\mathcal{P}_x$ by

$$\gamma_0 \mathcal{R} \gamma_1 \text{ iff there exist a local chart } (U, \varphi) \text{ at } x \text{ such that } (\varphi \circ \gamma_0)'(0) = (\varphi \circ \gamma_1)'(0).$$

Definition 1.8. *The tangent space of M at x denoted T_xM is the set of equivalence classes $\mathcal{P}_x/\mathcal{R}$.*

The tangent space T_xM has a natural structure of an m-dimensional real vector space.

Definition 1.9. *For each $x \in M$, the dual $(T_xM)^*$ of T_xM denoted T^*_xM is called the cotangent space of M at x.*

For a positive integer p, we denote $\bigwedge^p T^*_xM$ the space of p-antilinear forms on T_xM, $End(T_xM)$ the space of endomorphisms of T_xM and $S^2T^*_xM$ the space of symmetric bilinear forms on T_xM

Definition 1.10. *Let M and N be two smooth manifolds of dimension m and n with atlases $(U_k, \varphi_k)_k$ and (V_l, ψ_l) respectively. The product structure on $M \times N$ is the natural structure of smooth $n + m$ dimensional manifold given by the atlas $(U_k \times V_l, \varphi_k \times \psi_l)_{(k,l)}$ where*

$$\varphi_k \times \psi_l : (x, y) \in U_k \times V_l \mapsto (\varphi_k(x), \psi_k(y)).$$

By induction we can define the product structure on a finite product $M_1 \times \cdots \times M_r$ of smooth manifold.

Let $M \times N$ be equipped with the product structure, the tangent space at $(x, y) \in M \times N$ is

$$T_{(x,y)}(M \times N) = T_xM \oplus T_yN.$$

Definition 1.11. *Let $f : M \longrightarrow N$ be a C^1-map between smooth manifolds, the differential $\mathrm{d}f_x$ of f at $x \in M$ is the linear map*

$$\begin{aligned} \mathrm{d}f_x : T_xM &\longrightarrow T_{f(x)}N \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned}$$

The rank of f at x is the rank of $\mathrm{d}f_x$.

1.2 Transversality

Definition 1.12. *Let M be an m-dimensional smooth manifold and N_1, N_2 two submanifolds of M. N_1 is transverse to N_2 and we denote $N_1 \pitchfork N_2$ if*

$$T_xN_1 + T_xN_2 = T_xM \qquad \text{for each } x \in N_1 \cap N_2.$$

Definition 1.13. *Let M and N be two smooth manifolds. Let $f : M \longrightarrow N$ be a differentiable map and L a smooth submanifold of N. The map f is transverse to L and we denote $f \pitchfork L$ if for every $x \in f^{-1}(L)$ we have:*

$$T_{f(x)}L + \mathrm{d}f_x(T_xM) = T_{f(x)}N.$$

If $dim(M) + dim(L) < dim(N)$ (resp. $dim(N_2) + dim(N_1) < dim(M)$) then $f \pitchfork L$ (resp. $N_1 \pitchfork N_2$) means that $f(M) \cap L = \emptyset$ (resp. $N_1 \cap N_2 = \emptyset$).

Properties

- If $dim(N_2) + dim(N_1) = dim(M)$ and $N_1 \pitchfork N_2$ then $N_1 \cap N_2$ is a submanifold of dimension

$$dim(N_1 \cap N_2) = dim(N_1) + dim(N_2) - dim(M).$$

- If $f \pitchfork L$ and $f(M) \cap L \neq \emptyset$ then $f^{-1}(L)$ is a submanifold of M of dimension

$$dim\left(f^{-1}(L)\right) = dim(M) - dim(N) + dim(L).$$

- f is transverse to L if and only if for every $x \in f^{-1}(L)$ the induced map

$$T_xN \xrightarrow{\mathrm{d}f_x} T_{f(x)}M \longrightarrow T_{f(x)}M/T_{f(x)}L$$

is surjective.

Remark 1.14. *The notion of transversality depends on the ambient space. For instance two transverse surfaces in $\mathbb{R}^3$ have an intersection of dimension 1 but in $\mathbb{R}^4$ they have dimension 0 intersection.*

1.3 Vector Bundles

In what follows, $\mathbb{K}$ designs either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.15. *A surjective map $\pi : E \longrightarrow M$ between smooth manifolds is called a (smooth) vector bundle of fibre type $\mathbb{K}^p$ if there exist an open cover $(U_j)_j$ of M and smooth diffeomorphisms $\phi_j : \pi^{-1}(U_j) \longrightarrow U_j \times \mathbb{K}^p$ such that:*

1. *For each $x \in M$, $E_x = \pi^{-1}(x)$ is a $\mathbb{K}$-vector space isomorphic to $\mathbb{K}^p$,*
2. *for each $x \in U_j$, $\phi_j(\pi^{-1}(x)) = x \times \mathbb{K}^p$, and the restriction $pr_2 \circ \phi_j|_{E_x} : E_x \longrightarrow \mathbb{K}^p$ is a $\mathbb{K}$-linear isomorphism.*
3. *for each pair j, k, $\phi_j \circ \phi_k^{-1}$ can be defined by*

$$\begin{aligned}(U_j \cap U_k) \times \mathbb{K}^p &\longrightarrow (U_j \cap U_k) \times \mathbb{K}^p \\ (x, a) &\mapsto (x, g_{jk}(x)(a))\end{aligned}$$

where $g_{jk} : U_j \cap U_k \longrightarrow \mathrm{GL}(p, \mathbb{K})$ is a (smooth) continuous map.

For $x \in M$, $E_x = \pi^{-1}(x)$ is called the fibre over x.

Definition 1.16. *A section of a vector bundle $\pi : E \longrightarrow M$, is a continuous (smooth) map $s : M \longrightarrow E$ which satisfies $\pi \circ s = Id_M$ i.e such that $s(x) \in E_x$ for each $x \in M$.*

Let M be an n-dimensional manifold and let

$$TM := \bigcup_{x \in M} \{x\} \times T_x M.$$

Let $\pi : TM \longrightarrow M$ be the projection on M, *i.e* $\pi(y) = x$ if $y \in T_x M$. The space TM has a natural structure of $2n$-dimensional manifold such that $\pi : TM \longrightarrow M$ is a real vector bundle over M of fibre type $\mathbb{R}^n$.

Definition 1.17. *The manifold TM is called the tangent bundle of M.*

Definition 1.18. *By analogy with the tangent bundle we can also define the cotangent bundle T^*M of M where*

$$T^*M := \bigcup_{x \in M} \{x\} \times T_x^* M$$

Similarly we have [1]

$$\bigwedge^p T^*M := \bigcup_{x\in M} \{x\} \times \bigwedge^p T_x^*M,$$

$$S^2T^*M := \bigcup_{x\in M} \{x\} \times S^2T_x^*M$$

and $$End(TM) := \bigcup_{x\in M} \{x\} \times End(T_xM),$$

which all have a natural structure of smooth manifolds defining real vector bundles over M via the canonical projection.

Definition 1.19. *A Riemannian metric on a smooth manifold M is a section g of the bundle $S^2T^*M \longrightarrow M$ such that for each $x \in M$, $g_x : T_xM \times T_xM \longrightarrow \mathbb{R}$ is a positive definite symmetric bilinear form.*

A manifold with a Riemannian metric g is called a Riemannian manifold.

Sometimes we denote a Riemannian manifold M with Riemannian metric g by (M, g).

Theorem 1.20. *Every smooth manifold can be equipped with a Riemannian metric.*

A proof of this theorem can be found in [Rha60]. The proof uses the fact that the manifold is para-compact which we have chosen to be part of the definition.

Definition 1.21. *A vector field on a manifold M is a section of the tangent bundle. A local vector field is a section of the tangent bundle which is defined only on an open subset of M.*

A vector field assigns to each point x a vector in the tangent space T_xM. We denote $\mathfrak{X}(M)$ the set of all vector fields on M. The space $\mathfrak{X}(M)$ is a real vector space with point-wise addition and multiplication.

Definition 1.22. *Let M be a smooth manifold and $p \in M$. A local flow of a vector field X at p is a map*

$$\psi : I \times U \longrightarrow M$$
$$(t, x) \longmapsto \psi_t(x) := \psi(t, x)$$

such that for every $x \in U$

$$\frac{\mathrm{d}\,\psi_t(x)}{\mathrm{d}t}_{|t=0} = X(x)$$

where U is an open neighbourhood of p and $I \subset \mathbb{R}$ is an open interval containing 0 with $\psi(0, x) = x$.

[1] Here $S^2T_x^*M$ denotes the space of symmetric bilinear forms on T_xM.

Let $f : M \longrightarrow \mathbb{R}$ be a smooth map on a Riemannian manifold (M, g). The differential $\mathrm{d}f_x : T_xM \longrightarrow \mathbb{R}$ of f at each point is a linear form so $\mathrm{d}f \in T^*M$. On the other hand for each point x, g_x is a non degenerate bilinear form on T_xM therefore it induces an isomorphism

$$\begin{aligned}\tilde{g} : TM &\longrightarrow T^*M \\ \nu &\longmapsto g(\nu, .)\end{aligned}$$

Thus we can make the following definition.

Definition 1.23. *The gradient vector field of f with respect to g is the unique smooth vector field*

$$\nabla_g f := \tilde{g}^{-1}(\mathrm{d}f)$$

Definition 1.24. *Let (M, g) be a Riemannian manifold and $f : M \longrightarrow \mathbb{R}$ be a smooth map. A gradient flow of f with respect to g is a flow of the gradient vector field $\nabla_g f$.*

1.4 Connections

Definition 1.25. *Let M be an n-dimensional smooth manifold. A bilinear map*

$$\begin{aligned}\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ (X, Y) &\longmapsto \nabla_X Y := \nabla(X, Y)\end{aligned}$$

is called a connection on M if

$$\begin{aligned}\nabla_{fX} Y &= f \nabla_X Y \\ \nabla_X (fY) &= f \nabla_X Y + X(f) Y\end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M, \mathbb{R})$.

Proposition 1.26. *Every smooth manifold possesses a connection.*

A proof of this can be found in [Rha60]. Again the proof uses the fact that our manifolds are paracompact.

Definition 1.27. *Let ∇ be a connection on a manifold M and $\gamma : [0, 1] \longrightarrow M$ a parametrized curve on M. Given a vector $v \in T_{\gamma(0)}M$ the parallel transport along γ is the unique vector field X along γ solution of the differential equation*

$$\begin{cases} \nabla_{\gamma'(t)} X(t) = 0 \\ X(0) = v \end{cases}$$

A connection gives a way to lift a path to the tangent bundle TM. In particular it produces an isomorphism

$$\widehat{\gamma} : T_x M \longrightarrow T_y M$$

where $x = \gamma(0)$, $y = \gamma(1)$. A vector $v \in T_x M$ is mapped to $X(y)$ where X is the parallel transport along γ with initial condition $X(0) = v$.

This isomorphism is called *parallel transport* from x to y along γ.

Definition 1.28. *Let ∇ be a connection on a smooth manifold M and γ a smooth path in M. A smooth vector field X along γ is parallel along γ if it satisfies:*

$$\nabla_{\gamma'(t)} X = 0.$$

A vector field $X \in \mathfrak{X}(M)$ acts on a smooth map $f \in C^\infty(M, \mathbb{R})$ by

$$(X.f)(x) := \mathrm{d}f(X(x)), \qquad x \in M.$$

We then define the Lie bracket $[X, Y]$ of two vector fields X and Y to be the unique vector field with the property

$$[X, Y].f = X.(Y.f) - Y.(X.f)$$

Definition 1.29. *A connection ∇ on a smooth manifold M is symmetric if for every vector fields X and Y*

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Definition 1.30. *Let (M, g) be a Riemannian manifold. A connection ∇ on M is compatible with the Riemannian metric g if $g(X, X')$ is constant for every smooth path γ in M and every vector fields X, X' parallel along γ.*

A connection is compatible with the Riemannian metric g if the scalar product is preserved under the map of parallel transport $\widehat{\gamma} : T_{\gamma(0)} M \longrightarrow T_{\gamma(1)} M$, *i.e*

$$g(\widehat{\gamma}(v), \widehat{\gamma}(w)) = g(v, w)$$

Theorem 1.31 (Levi-Civita). *Let g be a Riemann metric on a manifold M. There exists a unique connection ∇ on M such that*

1. *∇ is symmetric,*

2. ∇ *is compatible with the Riemannian metric* g.

We call such connection a Levi-Civita connection on M.

This last theorem is a fundamental theorem in Riemannian geometry.

Chapter 2

Morse Theory

Morse theory is an important tool for the study of the topology of manifolds. One uses it in Chapter 6. In this review we follow mainly the theories in [Mat02] and [Sch], and [Ban04].

2.1 Morse Functions

Let M be a smooth m-manifold without boundary and $f : M \longrightarrow \mathbb{R}$ a smooth function.

Definition 2.1. *A point $p \in M$ is a critical point of f if there exists a local coordinates system $(x_1, \cdots, x_m)$ around p such that*

$$\frac{\partial f}{\partial x_1}(p) = \cdots = \frac{\partial f}{\partial x_m}(p) = 0.$$

A critical value of f is the image of a critical point.

This definition is independent of the choice of coordinates.

Definition 2.2. *Let p be a critical point of $f : M \longrightarrow \mathbb{R}$, the Hessian of f at p with respect to the local coordinates $(x_1, \cdots, x_m)$ is the matrix:*

$$H_f(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{i,j}$$

Remark 2.3. *Let $(x_1, \cdots, x_m)$ and $(y_1, \cdots, y_m)$ be two distinct sets of local coordinates around a point p. Let $H_f(p)$ and $H'_f(p)$ be the Hessians corresponding to the two coordinates. $H_f(p)$ and*

$H'_f(p)$ are related by

$$H_f(p) = J(p)^t H'_f(p) J(p)$$

where $J(p)$ is the Jacobian of the local coordinates transform from $(y_1, \cdots, y_m)$ to $(x_1, \cdots, x_m)$.

Definition 2.4. *A critical point p of a smooth function $f : M \longrightarrow \mathbb{R}$ is called non degenerate if $H_f(p)$ is non singular. If not it is called degenerate.*

Definition 2.5. *A smooth function $f : M \longrightarrow \mathbb{R}$ is a Morse function if every critical point is non degenerate.*

One can prove that Morse functions always exist, see [Mil69], [Sch] and [Ban04], and that they are sufficiently generic such that for each real valued function on M there always exists a Morse function close to it.

Theorem 2.6 (Morse lemma). *Let $f : M \longrightarrow \mathbb{R}$ be a Morse function. At every critical point p there exist local coordinates $(x_1, \cdots, x_m)$ with respect to which f has the expression*

$$f = f(p) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_m^2$$

Definition 2.7. *The number i is called the index of the non degenerate critical point p of f and is denoted by $ind(p)$.*

Around a non-degenerate critical point p the Hessians H and H' obtained from two different local coordinates $(x_1, \cdots, x_m)$ and $(y_1, \cdots, y_m)$ are related by

$$H = {}^tJ(p) H' J(p)$$

where $J(p)$ is the Jacobian matrix of the transition map from $(x_1, \cdots, x_m)$ to $(y_1, \cdots, y_m)$, is invertible since p is a nondegenerate critical point. Thus by Sylvester's law of inertia the index is independent of the choice of local coordinates.

Proposition 2.8. *The critical points of a Morse function are isolated.*

In particular if M is compact then the number of critical points is finite.

We can reconstruct a smooth orientable manifold via a Morse function. Our interest here is in Morse homology, for readers interested to learn more about other aspects of Morse theory we refer to [Mil69]. Some references for Morse homology are [Sch] and [Ban04].

2.2 Morse Homology

Let us consider a Riemannian metric g on M. Let $f : M \longrightarrow \mathbb{R}$ be a Morse function on M. Let $-\nabla f$ be the negative gradient of f with respect to g and ψ its negative gradient flow.

Definition 2.9. *Let $p \in M$ be a critical point of f. We define the stable and the unstable manifold of f at p to be respectively:*

$$S_p = \{x \in M | \lim_{t \to +\infty} \psi(t, x) = p\}$$

$$U_p = \{x \in M | \lim_{t \to -\infty} \psi(t, x) = p\}$$

One can prove, see [AR67] that S_p and U_p are respectively an $ind(p)$-dimensional and an $(m - ind(p))$-dimensional embedded open disk in M.

Definition 2.10. *The pair (f, g) is Morse-Smale if for every pair of critical points p and q, S_p is transverse to U_q.*

From now on we assume that the pair (f, g) is Morse-Smale.

Definition 2.11. *Let p and q be two critical points of f. A gradient flow line from p to q is a differentiable map $\phi : \mathbb{R} \longrightarrow M$ such that*

$$\phi'(t) = -\nabla f(\phi(t))$$

$$\lim_{t \to -\infty} \phi(t) = p \quad \text{and} \quad \lim_{t \to +\infty} \phi(t) = q$$

$\mathbb{R}$ acts on the set of gradient flow lines from p to q by translation we denote $\mathcal{M}(p, q)$ the quotient with respect to this $\mathbb{R}$ action. One can check, see for instance [AR67], that $\mathcal{M}(p, q) = (U_p \cap S_q) / \mathbb{R}$ and for $p \neq q$

$$\dim(U_p \cap S_q) = ind(p) - ind(q).$$

We are interested in the case $ind(p) - ind(q) = 1$. Here $\dim \mathcal{M}(p, q) = 0$, because of the free $\mathbb{R}$ action and the Morse-Smale condition. We can then count the unparametrized gradient flow lines modulo a choice of orientation and some compactness assumption. The issue of orientation is discussed in [Sch]. We have (see [Sch]) a canonical isomorphism

$$\begin{aligned} TU_p &\simeq T(U_p \cap S_q) \oplus TM/TS_q \\ &\simeq T_\phi \mathcal{M}(p, q) \oplus T_\phi \oplus T_q U_q. \end{aligned}$$

The orientation of $\mathcal{M}(p, q)$ is chosen such that this isomorphism is orientation preserving.

Let $Crit_k$ be the set of index k critical points of f and C_k the free Abelian group generated by $Crit_k$. The Morse chain complex is

$$\mathcal{C}_M = \bigoplus_{k \in \mathbf{Z}} C_k.$$

We define the boundary map $\partial : \mathcal{C}_M \longrightarrow \mathcal{C}_M$ by

$$\partial p = \sum_{q \in Crit_{k-1}} \sharp \mathcal{M}(p, q) \, \cdot q$$

for $p \in Crit_k$.

One can prove, see [Sch], that $\partial^2 = 0$, so we can take the homology of this complex.

Definition 2.12. *The Morse homology of M with respect to (f, g) is* $\ker \partial / \mathrm{im} \partial$.

The main interest in Morse homology is described by the following theorem.

Theorem 2.13. *The Morse homology is independent of the pair (f, g) and is isomorphic to the singular homology of M.*

For the details the readers can consult [Sch].

Chapter 3

Complex and Almost Complex Structures

Let M be a $2n$-dimensional smooth manifold. We can then transform any atlas on M to an atlas which identifies a neighbourhood of each point to an open subset of $\mathbb{C}^n$. Although in general, there is no chance that the transition maps are holomorphic, one can in some cases make the tangent bundle TM into a complex vector bundle over M.

3.1 Complex Structures on Real Vector Spaces

Definition 3.1. *Let E be a real vector space. A complex structure on E is an element $J \in End(E)$ such that $J^2 = -Id$.*

A complex structure J on E determines a structure of complex vector space. The complex scalar multiplication is given by

$$(a+ib)\ v = av + bJv, \qquad a+ib \in \mathbb{C}, \ \ v \in E.$$

Conversely if F is a complex vector space the multiplication by $i \in \mathbb{C}$ defines a complex structure on the underlying real vector space.

In finite dimension, complex structures exist only on even dimensional real vector spaces.

The natural complex structure on $\mathbb{R}^{2n}$ is given by:

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

Where I_n is the identity on $\mathbb{R}^n$.

3.2 Complex and Almost Complex Manifolds

Definition 3.2. *Let M be a $2n$-dimensional smooth manifold. A complex structure on M is an atlas $(U_k, \varphi_k)_k$ such that*

- *φ_k is an homeomorphism of U_k onto an open subset of $\mathbb{C}^n$,*
- *the transition maps $\varphi_k \circ \varphi_l^{-1} : \varphi_k(U_k \cap U_l) \longrightarrow \varphi_l(U_k \cap U_l)$ are holomorphic maps between open subsets of $\mathbb{C}^n$.*

  The n-tuple $(z_1, \cdots, z_n)$ of components of φ_k is called local complex coordinates.

  The integer n is the complex dimension of M.

Definition 3.3. *A smooth manifold with a complex structure is called a complex manifold.*

Proposition 3.4. *Complex manifolds are orientable.*

Proof. Since the transition maps are biholomorphic they have strictly positive Jacobians. □

In particular a compatible complex atlas induces a canonical orientation on the underlying real manifold of a complex manifold due to this positivity of the Jacobians of the transition maps.

Example 3.5. *1. Any open subset of $\mathbb{C}^n$ is a complex manifold.*

2. $\mathbb{CP}^n$ the space of complex lines in $\mathbb{C}^{n+1}$ is a complex manifold.

3. Oriented surfaces are complex manifolds.

Definition 3.6. *Let L be another complex manifold of complex dimension q. A continuous map $f : M \longrightarrow L$ is an holomorphic map if at each point $x \in M$, there exists a local chart (U, φ) at x and (V, ψ) at $f(x)$, which are all compatible with the holomorphic structures, such that the map $\psi \circ f \circ \varphi^{-1}$ is an holomorphic map between open subsets of $\mathbb{C}^n$ and $\mathbb{C}^q$.*

Definition 3.7. *A biholomorphism is an homeomorphism f such that $f : M \longrightarrow L$ and f^{-1} are holomorphic.*

We shall use the terms conformally equivalent and biholomorphic interchangeably.

A smooth map $f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is holomorphic if and only if its differential $\mathrm{d}f$ is $\mathbb{C}$ linear or equivalently if $\mathrm{d}f$ commutes with the natural complex structure J_0 on $\mathbb{R}^{2n}$

$$\mathrm{d}f \circ J_0 = J_0 \circ \mathrm{d}f.$$

Now let M be a complex manifold, $x \in M$ and (U_k, φ_k), (U_l, φ_k) two complex charts at x. From the previous discussion, the differential of the transition map $\varphi_k \circ \varphi_l^{-1}$ is $\mathbb{C}$-linear and commutes with J_0 since it is holomorphic.

The real linear map $J_x : T_xM \longrightarrow T_xM$ defined by

$$J_x := (\mathrm{d}\varphi_l)_{\varphi_l(x)} \circ J_0 \circ (\mathrm{d}\varphi_x)_k$$

satisfies $J_x^2 = -Id$. Therefore this is a complex structure on the real vector space T_xM. This linear map does not depend on the choice of charts since the transition maps commute with J_0.

We can then define a section of the vector bundle $End(TM)$ by

$$J : M \longrightarrow End(TM)$$
$$x \longmapsto J_x$$

In general the existence of such a section J does not guarantee that the manifold can be endowed with a complex structure.

Definition 3.8. *Let r be a positive integer. An almost complex structure of class C^r on M is a section J of class C^r of the vector bundle $End(TM)$ such that $J(x)^2 = -Id$ for every $x \in M$.*

(M, J) is called an almost complex manifold.

If there is no ambiguity about the almost complex structure J we simply say that M is an almost complex manifold. Similarly if there is no ambiguity on the class C^r of the almost complex structure or if it is not relevant in the discussion we simply say that we have an almost complex structure.

In local coordinates we can think of J as a map to the space of $2n \times 2n$ real matrices.

It is obvious that an almost complex manifold or a complex manifold must be of even dimension.

Remark 3.9. *A complex manifold has a natural structure of almost complex manifold but an almost complex manifold does not have in general a compatible complex structure. When it is the case we say that the almost complex structure is integrable. The class of surfaces form a particular case since every almost complex structure on an orientable surface is integrable.*

There are a lot of even dimensional manifolds which do not admit almost complex structures. For instance S^2 and S^6 are the only spheres which admit almost complex structures [May99] p. 212 of the pdf version.

Definition 3.10. *Let (M, J) and (M', J') be two almost complex manifolds. A differentiable map $f : M \longrightarrow M'$ is called (J, J')-holomorphic if its differential $\mathrm{d}f_x : T_xM \longrightarrow T_{f(x)}M$ at each point $x \in M$ satisfies the relation*

$$\mathrm{d}f_x \circ J_x = J'_{f(x)} \circ \mathrm{d}f_x.$$

When there is no confusion we say that f is pseudo-holomorphic.

When all the manifolds are smooth, the regularity of pseudo-holomorphic curves will come from the regularity of the almost complex structures.

Given two arbitrary almost complex manifolds (M, J) and (M', J') there are in general no pseudo-holomorphic maps between M and M'. However, when M is a Riemann surface Σ there are infinitely many pseudo-holomorphic maps $\Sigma \longrightarrow M'$ and these maps are called pseudo-holomorphic curves.

Proposition 3.11. *Let (M, J) be an almost complex manifold and M' a smooth manifold which can be endowed with an almost complex structure. Let $f : M \longrightarrow M'$ be a local diffeomorphism. There exists a unique almost complex structure J' on M' which makes f into a pseudo-holomorphic map. J' is determined by*

$$J'_x = \mathrm{d}f_x \circ J_x \circ \mathrm{d}f_x^{-1}.$$

Similarly if the almost complex structure J' of M' is fixed then there is a unique almost complex structure J on M such that f is pseudo-holomorphic. J is determined by

$$J_x = \mathrm{d}f_x^{-1} \circ J'_x \circ \mathrm{d}f_x.$$

Chapter 4

Symmetric Products of Spaces

4.1 Definitions and Examples

Definition 4.1. *Let X be a topological space and $n \in \mathbb{N}$. The n-fold symmetric product of X denoted* $\mathrm{Sym}^n(X)$ *is the quotient of X^n by the action of the n-th symmetric group $\mathfrak{S}_n$, equipped with the quotient topology.*

The permutation group $\mathfrak{S}_n$ acts on X^n by permuting the components. Two elements $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ of X^n are equivalent if there exists a permutation $\sigma \in \mathfrak{S}_n$ such that

$$(y_1, \cdots, y_n) = \left(x_{\sigma(1)}, \cdots, x_{\sigma(n)}\right).$$

This action is by homeomorphism since for a given $\sigma \in \mathfrak{S}_n$, the map $(x_1, \cdots, x_n) \mapsto \left(x_{\sigma(1)}, \cdots, x_{\sigma(n)}\right)$ is an homeomorphism. We denote the class of $(x_1, \cdots, x_n)$ under this action by $x_1 + \cdots + x_n$, so we can view an element of $\mathrm{Sym}^n(X)$ as a formal linear combination $\sum_i n_i x_i$ where $n_i \in \mathbb{N}$, $x_i \in X$, $\sum_i n_i = n$ and $x_i \neq x_j$ if $i \neq j$. We use the same convention for the 'class' of a product $A_1 \times \cdots \times A_n \subset X^n$. If we forget about the topology, the n-fold symmetric products is just the set of n-tuples of unordered points of X.

The diagonal Δ in X^n is the set of points where at least two components are equal,

$$\Delta = \{(x_1, \cdots x_n) \mid x_i = x_j \text{ for some } i \neq j\}$$

The diagonal in $\mathrm{Sym}^n(X)$ is the image of the one in X^n by the canonical projection,

$$X^n \longrightarrow X^n/\mathfrak{S}_n = \mathrm{Sym}^n(X).$$

When the space X is based, which means that we specify a special point $a \in X$ then we have the natural inclusions

$$\mathrm{Sym}^{n-1}X \longrightarrow \mathrm{Sym}^{n}(X)$$

$$x_1 + \cdots + x_n \longmapsto a + x_1 + \cdots + x_n$$

which is continuous. Analogous result can be obtained when we specify more than one point. If we specify k points $a_1, \cdots, a_k \in X$, $k \leq n$, then we have the map

$$\mathrm{Sym}^{n-k}X \longrightarrow \mathrm{Sym}^{n}(X)$$

$$x_1 + \cdots + x_{n-k} \longmapsto a_1 + \cdots + a_k + x_1 + \cdots + x_{n-k}$$

which is also continuous.

Example 4.2. *The n-fold symmetric product of the sphere S^2 is homeomorphic to the n-th complex projective space $\mathbb{CP}^n$.*

S^2 is the same as $\mathbb{CP}^1$ which can be identified with the set of non-zero degree one polynomials over $\mathbb{C}$ modulo scale by non-zero complex numbers. An element $[a : b] \in \mathbb{CP}^1$ is then identified with the polynomial $aX + b$ modulo scale. Using this, to an element $x_1 + \cdots + x_n \in \mathrm{Sym}^n\mathbb{CP}^1$ we associate the element $[w_0 : \cdots : w_n] \in \mathbb{CP}^n$ where $w_0, \cdots, w_n$ are the coefficients of the polynomial

$$(a_1X + b_1)(a_2X + b_2)\cdots(a_nX + b_n),$$

the binomial $a_kX + b_k$ representing the element $x_k = [a_k : b_k] \in \mathbb{CP}^1$ for $k = 1, \cdots, n$. This gives a bijection between $\mathrm{Sym}^n\mathbb{CP}^1$ and $\mathbb{CP}^n$ using the fact that $\mathbb{C}$ is an algebraically closed field. One can check that this is the identification needed.

It will be proved that symmetric product of Riemann surfaces are complex manifolds.

Similarly we have,

Example 4.3. *The n-fold symmetric product of $\mathbb{C}$ is homeomorphic to $\mathbb{C}^n$.*

We will prove later that $\mathrm{Sym}^n\mathbb{C}$ is a complex manifold biholomorphic to $\mathbb{C}^n$.

Example 4.4. *The second symmetric product of the unit circle S^1 is the Möbius band.*

Let $I = [0, 1]$, consider $S^1 \times S^1$ as the unit square $I \times I$ with opposite sides identified as in the following figure:

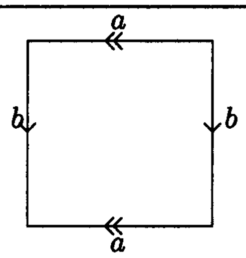


Figure 4.1: $S^1 \times S^1$

$(x_1, x_2) \sim (x_2, x_1)$ so $a \sim b$ and a fundamental domain will be the triangle in Figure 4.2.

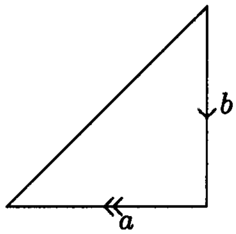


Figure 4.2: Fundamental domain for $\mathrm{Sym}^2 S^1$

We cut this triangle into two triangles and glue them according to the orientation of the arrows.

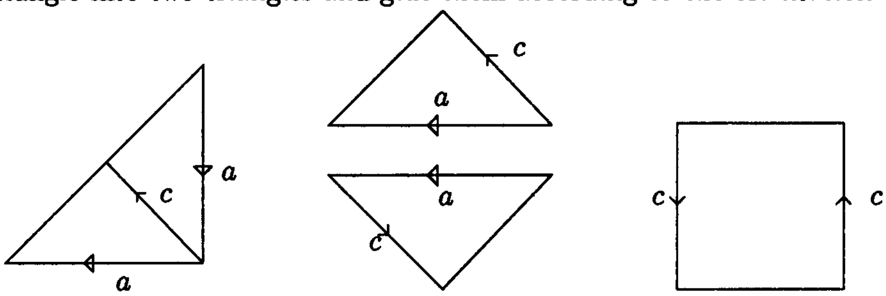


Figure 4.3:

The result is a Möbius band as shown in Figure 4.3.

Notice that in our example S^1 is a closed and oriented manifold but its second symmetric product is a non-oriented manifold with boundary.

4.2 Basic Properties

Proposition 4.5. *The canonical projection* $\pi : X^n \longrightarrow \mathrm{Sym}^n(X)$ *is an open and a closed map.*

Proof. Let U be an open set in X^n.

$$\pi^{-1}(\pi(U)) = \bigcup_{\sigma \in \mathfrak{S}_n} \sigma U$$

which is open since it is the union of open sets. Then $\pi(U)$ is open by the definition of quotient topology. Thus π is an open map.

Let F be a closed set in X^n.

$$\pi^{-1}(\pi(F)) = \bigcup_{\sigma \in \mathfrak{S}_n} \sigma F$$

which is a finite union of closed set since $\mathfrak{S}_n$ is finite, hence it is closed. Then $\pi(F)$ is closed by definition of quotient topology . Thus π is a closed map. □

Proposition 4.6. *If X is Hausdorff then* $\mathrm{Sym}^n(X)$ *is Hausdorff.*

Proof. Let $x, y \in \mathrm{Sym}^n(X)$ such that $x \neq y$, if $x = x_1 + \cdots + x_n$ and $y = y_1 + \cdots + y_n$, this means that

$$\{x_1, \cdots, x_n\} \neq \{y_1, \cdots, y_n\}$$

$$i.e \qquad \{x_1, \cdots, x_n\} \not\subseteq \{y_1, \cdots, y_n\} \text{ or } \{y_1, \cdots, y_n\} \not\subseteq \{x_1, \cdots, x_n\}.$$

For the first case, there exists $l \in \{1, \cdots, n\}$ such that

$$x_l \neq y_k, \text{ for all } k \in \{1, \cdots, n\}$$

We can assume $l = 1$ without loss of generality. Since X is Hausdorff, there exist for each k an open neighbourhood N_k of y_k and N'_k of x_1 in X such that $N_k \cap N'_k = \emptyset$. Let us take

$$O_1 = N'_1 \cap \cdots \cap N'_n$$

O_1 is an open neighbourhood of x_1 satisfying $O_1 \cap N_k = \emptyset$ for each k. Now, consider open neighbourhood $O_2, \cdots, O_n$ of $x_2, \cdots, x_n$ respectively, such that $O_j = O_1$ if $x_j = x_1$.

Let $N = N_1 + \cdots + N_n$, and $O = O_1 + O_2 + \cdots + O_n$, N is an open neighbourhood of y and O is an open neighbourhood of x.

$$N \cap O = \{a_1 + \cdots + a_n \mid \text{for each } k,\ a_k \in N_j \cap O_i \text{ for some } i, j\}$$

but $O_1 \cap N_j = \emptyset$ for all j and at least one $a_j \in O_1$. Thus $N \cap O = \emptyset$.
For the case

$$\{y_1, \cdots, y_n\} \not\subseteq \{x_1, \cdots, x_n\}.$$

we do the same reasoning.
Therefore we can find a neighbourhood O of x and N of y such that $N \cap O = \emptyset$. □

Suppose that X is second countable, then so is X^n. Let us consider a countable basis $\mathfrak{U}$ for X^n, then $\pi(\mathfrak{U})$ is a countable basis for $\mathrm{Sym}^n(X)$. For, let $\bar{x} \in \mathrm{Sym}^n(X)$, N an open neighbourhood of $\bar{x}$ and $x \in \pi^{-1}(\bar{x})$. $\pi^{-1}(N)$ is an open neighbourhood of x then there exist $U \in \mathfrak{U}$ such that $U \subset \pi^{-1}(N)$. Therefore $\pi(U) \subset N = \pi(\pi^{-1}(N))$. It is obvious that $\pi(\mathfrak{U})$ is also countable. Thus, we have proven the following:

Proposition 4.7. *If the topology of X is countable then the topology of $\mathrm{Sym}^n(X)$ is countable.*

Lemma 4.8. *If X is Hausdorff, then $\pi : X^n \setminus \Delta \longrightarrow \mathrm{Sym}^n(X)$ is a local homeomorphism.*

Proof. Let $(x_1, \cdots, x_n) \in X^n \setminus \Delta$. Since the x_i's are distinct and X is Hausdorff, we can find open neighbourhoods $N_1, \cdots, N_n$ of $x_1, \cdots, x_n$ such that

$$\heartsuit \qquad N_i \cap N_j = \emptyset \text{ if } i \neq j.$$

Let $N = N_1 \times \cdots \times N_n$. The map $\pi : N \longrightarrow \pi(N)$ is continuous, open and surjective. It suffices to prove that it is injective. Let $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n) \in N$ such that

$$\pi(a_1, \cdots, a_n) = \pi(b_1, \cdots, b_n)$$

then

$$\{a_1, \cdots, a_n\} = \{b_1, \cdots, b_n\}.$$

Let $i \in \{1, \cdots, n\}$ there exists j such that $a_i = b_j$. Therefore $a_i \in N_i \cap N_j$, but the condition $\heartsuit$ forces $i = j$, hence $a_i = b_i$. Thus

$$(a_1, \cdots, a_n) = (b_1, \cdots, b_n).$$

□

We need the following lemma to establish the next theorem.

Lemma 4.9. *Let $f : E \longrightarrow B$ be a local homeomorphism such that E is Hausdorff and the cardinality of the preimage of a point is a finite constant. Then f is a covering map.*

Proof. Let us suppose that all the preimages of a point have constant positive cardinality $m \in \mathbb{N}$. Let $y \in B$ and $f^{-1}(y) = \{x_1, \cdots, x_m\}$. Since f is a local homeomorphism, for each $i \in \{1, \cdots, m\}$ there exists an open neighbourhood U_i of x_i such that $f_{|U_i} : U_i \longrightarrow f(U_i)$ is an homeomorphism. Because E is Hausdorff we can choose the U_i to be pairwise disjoint.
Let

$$V = f(U_1) \cap \cdots \cap f(U_m),$$

V is open since f is open being a local homeomorphism. Let $x \in f^{-1}(V)$, then $f(x) \in f(U_1) \cap \cdots \cap f(U_m)$ and we can find for each i an $a_i \in U_i$ such that $f(x) = f(a_i)$. Since the U_i are disjoints, the a_i are all distinct. Then since the number of preimage is constant we have $f^{-1}(\{f(x)\}) = \{a_1, \cdots, a_m\}$. Therefore $x \in \left(f^{-1}(V) \cap U_1\right) \cup \cdots \cup \left(f^{-1}(V) \cap U_m\right)$. Thus $f^{-1}(V) \subset \left(f^{-1}(V) \cap U_1\right) \cup \cdots \cup \left(f^{-1}(V) \cap U_m\right)$. Since the converse inclusion is trivial we have the equality

$$f^{-1}(V) = \left(f^{-1}(V) \cap U_1\right) \cup \cdots \cup \left(f^{-1}(V) \cap U_m\right).$$

Since this is a disjoint union, the number m is independent of the point $y \in B$ and the restriction of f to $f^{-1}(V) \cap U_i$ is an homeomorphism, so f is a covering map. □

Theorem 4.10. *If X is Hausdorff, then $\pi : X^n \setminus \Delta \longrightarrow \mathrm{Sym}^n(X) \setminus \pi(\Delta)$ is a covering map.*

Proof. X^n is Hausdorff, the number of points in the pre-image of a point in $\mathrm{Sym}^n(X) \setminus \pi(\Delta)$ is constant equal to $n!$ and from Lemma 6.2.1 the map is a local homeomorphism. □

More precisely the canonical projection $\pi : X^n \longrightarrow \mathrm{Sym}^n(X)$ is a $n!$-sheeted branched covering with the diagonal in X^n as set of ramification points and the diagonal in $\mathrm{Sym}^n(X)$ as set of branched points.

If X is compact then so is $\mathrm{Sym}^n(X)$ since products of compact sets are compact sets and the image of a compact set by a continuous map to an Hausdorff space is a compact set. The same result holds for local compactness, local compactness of X implies local compactness of $\mathrm{Sym}^n(X)$. We have analogous situations for connectedness, local connectedness, path connectedness and local path connectedness.

The following property is also important,

Proposition 4.11. *If X is Hausdorff, then the canonical projection $\pi : X^n \longrightarrow \mathrm{Sym}^n(X)$ is a proper map.*

Proof. The canonical projection π is a closed map and for each $y \in \mathrm{Sym}^n(X)$, $\pi^{-1}(y)$ is finite then compact.

□

In general if X is a manifold $\mathrm{Sym}^n(X)$ is not necessarily a manifold, it may have singularities on the diagonal. Even if X is a closed orientable manifold and $\mathrm{Sym}^n(X)$ is a manifold, it will not follow that $\mathrm{Sym}^n(X)$ is closed and orientable. It will be compact but it may have boundary and may not be orientable as in the case of the second symmetric product of S^1. The case of surfaces will be the exception.

Chapter 5

Symmetric Products of Surfaces

It is a well known fact that any oriented smooth surface is a complex one dimensional manifold or Riemann surface. Let Σ be a Riemann surface, the n-fold symmetric product of Σ is the space of degree n effective divisors of Σ and is strongly related to its Jacobian. These notions from algebraic geometry may suggest that $\mathrm{Sym}^n(\Sigma)$ has nice properties. In fact, for a closed and oriented manifold M it was proved in [Wag80] that $\mathrm{Sym}^n M$ is a closed and oriented manifold if and only if $\dim M = 2$. In [Tre99] it is proved that the symmetric product of a surface is a complex manifold. Here we will prove an analogous result, namely that $\mathrm{Sym}^n(\Sigma)$ has a particular complex structure coming from the one on Σ^n, precisely:

Theorem 5.1. *Let Σ be a Riemann surface and n a positive integer. Then $\mathrm{Sym}^n(\Sigma)$ has a natural complex structure which is, outside the diagonal, the direct image of the product structure on Σ^n by the canonical projection.*

5.1 Symmetric Product of the Complex Line

Let n be a positive integer, $\mathrm{Sym}^n(\mathbb{C})$ is in one to one correspondence with $\mathbb{C}^n$ because $\mathbb{C}$ is an algebraically closed field. In fact, elements of $\mathrm{Sym}^n(\mathbb{C})$ are n-tuples of unordered points of $\mathbb{C}$ and they can be seen as the roots of a monic complex polynomial of degree n. Conversely, if we have a degree n monic complex polynomial then it has exactly n roots in $\mathbb{C}$ counting with multiplicity. These n roots are unordered elements of $\mathbb{C}$ so they constitute an element of $\mathrm{Sym}^n(\mathbb{C})$.

Let P be a monic complex polynomial of degree n with roots $z_1, \cdots, z_n$.

$$P(X) = X^n + w_1 X^{n-1} + \cdots + w_n$$

The relations between the coefficients and the roots are given by :

$$w_k = (-1)^k \sum_{1 \leq j_1 < \cdots < j_k \leq n} z_{j_1} \cdots z_{j_k},$$

The right hand side is known as the k-th elementary symmetric polynomial of n variables if we are not considering the minus sign.

It is clear from the expression that $w = (w_1, \cdots, w_n)$ is an holomorphic function of $(z_1, \cdots, z_n) \in \mathbb{C}^n$.

Since the set of all monic complex polynomials of degree n is a complex n dimensional vector space we can identify it with $\mathbb{C}^n$ as a complex manifold. We have then a direct surjective holomorphic map

$$\begin{aligned} f : \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ z &\longmapsto (w_1, \cdots, w_n) \end{aligned}$$

which associates ordered roots and coefficients of a polynomial. The map is not injective since another ordering of the roots gives the same polynomial but it is surjective. Obviously this map passes to the quotient by the action of $\mathfrak{S}_n$ since it is constant in each equivalence class. Let π be the canonical projection, we have :

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{C}^n \\ \mathrm{Sym}^n(\mathbb{C}) \arrow[ur, dotted, "\tilde{f}"'] & \end{tikzcd}$$

The diagram is a commutative diagram of continuous functions and $\tilde{f}$ is unique and bijective. $\tilde{f}$ is surjective since f is surjective. $\tilde{f}$ is injective since pre-images of a point in $\mathbb{C}^n$ by f are in the same equivalence class. f is a non constant holomorphic map but in general non constant holomorphic maps between high dimensional complex spaces are not necessarily open unless it is an holomorphic map to the complex plane. [1]

[1] If D is a domain of $\mathbb{C}^n$, a non constant holomorphic map $D \longrightarrow \mathbb{C}$ is an open map but a non constant holomorphic map $D \longrightarrow \mathbb{C}^m$ where $m > 1$ is not necessarily open, for instance the map $(z, w) \mapsto (z, zw)$ is not open.

So we cannot deduce immediately that $\tilde{f}$ is open. To prove that $\tilde{f}^{-1}$ is continuous we need to use the following properties.

Proposition 5.2. *Let X, Y, Z be three topological spaces. Let $h : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two maps such that h is surjective, continuous and closed. Then, g is continuous if and only if $g \circ h$ is continuous.*

Proof. If g is continuous then it is immediate that $g \circ h$ is continuous.

Conversely suppose that $g \circ h$ is continuous. Let C be a closed set in Z

$$\begin{aligned} g^{-1}(C) &= h\left(h^{-1}\left(g^{-1}(C)\right)\right) \\ &= h\left((g \circ h)^{-1}(C)\right) \end{aligned}$$

$g \circ h$ is continuous then $(g \circ h)^{-1}(C)$ is closed, h is closed then $h\left((g \circ h)^{-1}(C)\right)$ is closed. Thus $g^{-1}(C)$ is closed. □

Lemma 5.3. *The map $f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is a proper map.*

Proof. We use the fact that compact sets of $\mathbb{C}^n$ are the closed and bounded subsets. Let $K \subset \mathbb{C}^n$ be compact. Since K is closed and f continuous $f^{-1}(K)$ is closed, it then suffices to prove that it is bounded. Let $M \geq 0$ such that

$$\sup_{w \in K} \| w \|_\infty \leq M.$$

Let $z \in f^{-1}(K)$ and z_k one component of z, if $|z_k| \leq 1$ we can assume $|z_k| \leq M$, we can then consider the case $|z_k| > 1$. z_k is the root of a monic polynomial with ordered coefficients $(w_1, \cdots, w_n) \in K$, then

$$z_k^n + w_1 z_k^{n-1} + \cdots + w_n = 0$$

hence

$$\begin{aligned} |z_k|^n &\leq |w_1|\,|z_k|^{n-1} + \cdots + |w_n| \\ |z_k|^n &\leq M\left(\,|z_k|^{n-1} + \cdots + 1\right) \\ |z_k|^n &\leq M\frac{|z_k|^n - 1}{|z_k| - 1} \\ |z_k|^n &\leq M\frac{|z_k|^n}{|z_k| - 1} \end{aligned}$$

then $|z_k| - 1 \leq M$ *i.e* $|z_k| \leq M + 1$.

Therefore, since z and z_k are taken arbitrary

$$\| z \|_\infty \leq M+1 \quad \text{for all} \quad z \in f^{-1}(K).$$

□

Lemma 5.4. *The map $\tilde{f}^{-1}$ is continuous.*

Proof. $\tilde{f}^{-1} \circ f = \pi$ is continuous, f is closed since proper and is surjective, then from Proposition 5.2 $\tilde{f}^{-1}$ is continuous.

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{C}^n \arrow[dl, dotted, "\tilde{f}^{-1}"'] \\ \mathrm{Sym}^n(\mathbb{C}) & \end{tikzcd}$$

□

Thus we have proven that $\tilde{f}$ is an homeomorphism between $\mathrm{Sym}^n(\mathbb{C})$ and $\mathbb{C}^n$. The following result follows immediately:

Theorem 5.5. $\mathrm{Sym}^n(\mathbb{C})$ *is a complex manifold conformally equivalent to $\mathbb{C}^n$ with an atlas made by a unique chart $\left(\mathrm{Sym}^n(\mathbb{C}), \tilde{f}\right)$.*

Proof. $\tilde{f}$ is an homeomorphism and the overlap map is the identity map of $\mathbb{C}^n$ which is biholomorphic

$$\begin{tikzcd} & \mathrm{Sym}^n(C^n) \arrow[dr, "\tilde{f}"] & \\ \mathbb{C}^n \arrow[ur, "\tilde{f}^{-1}"] \arrow[rr, "Id"'] & & \mathbb{C}^n \end{tikzcd}$$

□

Corollary 5.6. *The symmetric products of an open subset of $\mathbb{C}$ is a complex manifold.*

5.2 Complex Charts on the Symmetric Product of a Surface

Given a Riemann surface Σ and a positive integer n. Σ^n is equipped with an atlas made by products of charts on Σ. Let $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ be such an atlas, it is straightforward that $(\pi(U_\alpha))_{\alpha\in A}$ is a covering of the n-fold symmetric product of Σ.

Now π will denote the canonical projection of the n-fold product of a surface ($\mathbb{C}$ and U_α's included) to its n-fold symmetric product. f will denote both the function in the unique chart on $\mathrm{Sym}^n\mathbb{C}$ and its restriction to an arbitrary open subset.

5.2.1 Construction of the Chart Maps

Lemma 5.7. *Let $\alpha \in A$, the map $f \circ \pi \circ \varphi_\alpha : U_\alpha \longrightarrow \mathbb{C}^n$ satisfies:*

$$\forall p \in U_\alpha, \forall \sigma \in \mathfrak{S}_n, \qquad f \circ \pi \circ \varphi_\alpha (p) = f \circ \pi \circ \varphi_\alpha (\sigma p)$$

Proof. Since f is injective it suffices to prove that $\pi \circ \varphi_\alpha (p) = \pi \circ \varphi_\alpha (\sigma p)$.
Let $p = (p_1, \cdots, p_n) \in U_\alpha$ and $\sigma \in \mathfrak{S}_n$. In this proof $\varphi_{\sigma(k)}$ (resp. φ_k) denotes the map corresponding to the local chart at the point $p_{\sigma(k)}$ (resp. p_k) of Σ.

$$\begin{aligned} \varphi_\alpha (p) &= (\varphi_1 (p_1), \cdots, \varphi_n (p_n)) \\ \varphi_\alpha (\sigma p) &= \left(\varphi_{\sigma(1)} \left(p_{\sigma(1)}\right), \cdots, \varphi_{\sigma(n)} \left(p_{\sigma(n)}\right)\right) \end{aligned}$$

then

$$\pi \circ \varphi_\alpha (\sigma p) = \varphi_{\sigma(1)} \left(p_{\sigma(1)}\right) + \cdots + \varphi_{\sigma(n)} \left(p_{\sigma(n)}\right)$$

and after reordering the terms

$$\begin{aligned} \pi \circ \varphi_\alpha (\sigma p) &= \varphi_1 (p_1) + \cdots + \varphi_n (p_n) \\ &= \pi \circ \varphi_\alpha (p). \end{aligned}$$

□

Theorem 5.8. *Let $\alpha \in A$, $N_\alpha = \pi (U_\alpha) \subset \mathbb{C}^n$. There exists a unique ψ_α such that the following diagram commutes*

$$\begin{tikzcd} U_\alpha \arrow[r, "\pi"] \arrow[d, "\varphi_\alpha"'] & \pi (U_\alpha) \arrow[dddl, dotted, "\psi_\alpha"] \\ N_\alpha \arrow[d, "\pi"'] & \\ \pi (N_\alpha) \arrow[d, "f"'] & \\ \mathbb{C}^n & \end{tikzcd}$$

Moreover ψ_α is an homeomorphism onto its image.

Proof. From the lemma 5.7 the map $f \circ \pi \circ \varphi_\alpha$ factorize through the quotient to give a unique continuous injection ψ_α. This map is open since $f \circ \pi \circ \varphi_\alpha$ is open, then it is an homeomorphism onto its image. □

Corollary 5.9. $\mathrm{Sym}^n\Sigma$ *is a topological manifold equipped with the complex atlas* $\left((\pi(U_\alpha))_{\alpha\in A}, \psi_\alpha\right)$.

5.2.2 Property of the Chart Maps

To prove that the transition maps of the complex atlas on $\mathrm{Sym}^n(\Sigma)$ are biholomorphic we need to understand the meaning of: $\pi(U_\alpha) \cap \pi(U_\beta)$, for $\alpha, \beta \in A$.

$$\pi^{-1}(\pi(U_\alpha) \cap \pi(U_\beta)) = \pi^{-1}(\pi(U_\alpha)) \cap \pi^{-1}(\pi(U_\beta))$$

but

$$\pi^{-1}(\pi(U_\alpha)) = \bigcup_{\sigma\in\mathfrak{S}_n} \sigma U_\alpha$$

$$\pi^{-1}(\pi(U_\beta)) = \bigcup_{\sigma'\in\mathfrak{S}_n} \sigma' U_\beta$$

then

$$\pi^{-1}(\pi(U_\alpha) \cap \pi(U_\beta)) = \bigcup_{(\sigma,\sigma')\in\mathfrak{S}_n\times\mathfrak{S}_n} \sigma U_\alpha \cap \sigma' U_\beta.$$

If $\pi(U_\alpha) \cap \pi(U_\beta) \neq \emptyset$, we can find a non-empty subset $\mathfrak{I} \subset \mathfrak{S}_n \times \mathfrak{S}_n$ such that

$$\pi^{-1}(\pi(U_\alpha) \cap \pi(U_\beta)) = \bigcup_{(\sigma,\sigma')\in\mathfrak{I}} \sigma U_\alpha \cap \sigma' U_\beta$$

$$\text{and} \qquad \forall (\sigma,\sigma') \in \mathfrak{I}, \quad \sigma U_\alpha \cap \sigma' U_\beta \neq \emptyset.$$

On the other hand, since π is surjective, for any $\alpha, \beta \in A$

$$\pi\left(\pi^{-1}(\pi(U_\alpha) \cap \pi(U_\beta))\right) = \pi(U_\alpha) \cap \pi(U_\beta)$$

thus

$$\begin{aligned} \pi(U_\alpha) \cap \pi(U_\beta) &= \pi\left(\bigcup_{(\sigma,\sigma')\in\mathfrak{I}} \sigma U_\alpha \cap \sigma' U_\beta\right) \\ &= \bigcup_{(\sigma,\sigma')\in\mathfrak{I}} \pi(\sigma U_\alpha \cap \sigma' U_\beta). \end{aligned}$$

Therefore, since $\pi(\sigma U_\alpha \cap \sigma' U_\beta)$ is open for $(\sigma, \sigma') \in \mathfrak{I}$, it suffices to prove that the transition maps are holomorphic on each of these intersections to prove that they are holomorphic.

The proof of Theorem 5.1 is then reduced to the following :

Lemma 5.10. *Let $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$. The transition map on $\pi(U_\alpha \cap U_\beta)$ is holomorphic. That is, we have the following diagram where $\psi_{\alpha\beta} = \psi_\alpha^{-1} \circ \psi_\beta$ is holomorphic.*

$$\begin{tikzcd}
 & \pi(U_\alpha \cap U_\beta) \arrow[dr, "\psi_\beta"] & \\
O_\alpha \arrow[ur, "\psi_\alpha^{-1}"] \arrow[rr, "\psi_{\alpha\beta}"'] & & O_\beta
\end{tikzcd}$$

$O_\alpha = \psi_\alpha(\pi(U_\alpha \cap U_\beta))$, $O_\beta = \psi_\beta(\pi(U_\alpha \cap U_\beta))$.

Discussion of the lemma. We have the following commutative diagram where $\varphi_{\alpha\beta} = \varphi_\alpha^{-1} \circ \varphi_\beta$ is biholomorphic:

$$\begin{tikzcd}
 & U_\alpha \cap U_\beta \arrow[dr, "\varphi_\beta"] & \\
N_\alpha \arrow[ur, "\varphi_\alpha^{-1}"] \arrow[rr, "\varphi_{\alpha\beta}"'] & & N_\beta
\end{tikzcd}$$

By construction of ψ_α and ψ_β the diagram below commutes

$$\begin{tikzcd}
 & & \pi(U_\alpha \cap U_\beta) \arrow[dddrr, "\psi_\beta"] & & \\
 & & U_\alpha \cap U_\beta \arrow[u, "\pi"] \arrow[dr, "\varphi_\beta"] & & \\
 & N_\alpha \arrow[ur, "\varphi_\alpha^{-1}"] \arrow[rr, "\varphi_{\alpha\beta}"'] \arrow[dl, "\bar{\pi}"] & & N_\beta \arrow[dr, "\bar{\pi}"] & \\
O_\alpha \arrow[uuurr, "\psi_\alpha^{-1}"] \arrow[rrrr, "\psi_{\alpha\beta}"'] & & & & O_\beta
\end{tikzcd}$$

$\bar{\pi} = f \circ \pi$ where f is the map which associate roots to polynomial.

The transition map $\psi_{\alpha\beta}$ then satisfies the following property by construction :

$$\psi_{\alpha\beta}\left(\bar{\pi}\left(y\right)\right) = \bar{\pi}\left(\varphi_{\alpha\beta}\left(y\right)\right), \qquad \text{for all } y \in N_\alpha. \tag{5.1}$$

By diagonal of a subset of $\mathbb{C}^n$ (resp. $\mathrm{Sym}^n(\mathbb{C})$) we mean the intersection of the corresponding subset with the diagonal in $\mathbb{C}^n$ (resp. $\mathrm{Sym}^n(\mathbb{C})$). Since we use product of charts on Σ^n, the diagonal 'in' N_α is the direct image of the diagonal 'in' U_α. Let $\tilde{\Delta} = \bar{\pi}(\Delta)$, where Δ is the diagonal in N_α.

We can now state the following lemma :

Lemma 5.11. *The transition map $\psi_{\alpha\beta}$ is holomorphic on $N_\alpha \setminus \tilde{\Delta}$.*

Proof. Since being holomorphic is a local property, we will prove the result for a point $z \in N_\alpha \setminus \tilde{\Delta}$. π is a covering map on $N_\alpha \setminus \tilde{\Delta}$, f is an homeomorphism and $\bar{\pi} = f \circ \pi$ is holomorphic then it is a local biholomorphism.

Let $z \in N_\alpha \setminus \tilde{\Delta}$, there exists a local invert g_z of $\bar{\pi}$ at z which is holomorphic. Therefore, from the equation 5.1 we have

$$\psi_{\alpha\beta}(w) = \pi \circ \varphi_{\alpha\beta} \circ g_z(w)$$

for all w in an open neighbourhood of z where g_z is defined. Since the composition $\pi \circ \varphi_{\alpha\beta} \circ g_z$ is holomorphic, $\psi_{\alpha\beta}$ is holomorphic on a neighbourhood of z.

□

The next step is to show that that $\psi_{\alpha\beta}$ is holomorphic on its entire domain O_α. This seems obvious because it is continuous and the diagonal in $\mathbb{C}^n$ is nowhere dense and nowhere separating, but the proof (at least our proof) requires some arguments from the theory of complex analysis in several variables. In the next section we will review these tools.

5.3 Digression in Complex Analysis

The aim of this section is to give a brief account of analytic continuation in higher dimension. It will be about the generalization of the Riemann theorem for removable singularity. We will not discuss other analytic continuations like Hartogs's continuation. We follow [Kra92] and [Kau83].

5.3.1 Zeros of Analytic Functions

For an open set $U \in \mathbb{C}^n$, $H(U)$ will denote the set of holomorphic function from U to $\mathbb{C}$.

Proposition 5.12 (Principle of Analytic continuation). *Let D be a domain*[1] *of $\mathbb{C}^n$ and $F \in H(D)$, if F vanishes on some open set $V \subset D$ then $F \equiv 0$.*

Proof. If F vanishes on V then all the complex partial derivatives of F vanishes on V since V is open.
For $k = (k_1, \cdots k_n) \in \mathbb{N}^n$, we denote $|k| = k_1 + \cdots + k_n$, and

$$D^k F = \frac{\partial^{|k|}}{\partial z^{k_1} \ldots \partial z^{k}} F$$

Let

$$E = \left\{ w \in D \mid D^k F(w) = 0, \quad \forall k \in \mathbb{N}^n \right\}$$

$V \subset E$ then $E \neq \emptyset$.

All the partial complex derivatives of F are holomorphic then continuous. The zero set of a continuous function is closed then E is closed since intersection of closed sets.

Let $z \in E$, since F is analytic at z:

$$F(\zeta) = \sum_{k_1, \cdots, k_n = 0}^{\infty} \frac{D^k F(z)}{k_1! \cdots k_n!} \, (\zeta_1 - z_1)^{k_1} \cdots (\zeta_n - z_n)^{k_n}$$

for all ζ in an open neighbourhood $N \subset D$ of z. But since $D^k F(z) = 0$, $F(\zeta) = 0$ for all $\zeta \in N$, and it follows that $N \subset E$. Therefore E is open. Thus $E = D$ since D is connected. □

Remark 5.13. *In the hypothesis, requiring that F is identically zero on an open subset is equivalent to saying that there exists $w_0 \in D$ such that F and all its partial complex derivatives vanish at w_0.*

In one dimension, the zeroes of a non identically null analytic function are isolated. When we go to higher dimensions this property collapses. For instance the zero set of the map $(z, w) \mapsto zw$ $(\mathbb{C} \times \{0\}) \cup (\{0\} \times \mathbb{C})$. Intuitively for the zeroes of an analytic function in one variable to be isolated means that they form a zero dimensional space, but $0 = 1 - 1$, so we could expect that in $\mathbb{C}^n$ the zeroes of an analytic function form possibly an $(n-1)$ dimensional space. In fact if we have an

[1] An open and connected subset

holomorphic map $F : \mathbb{C}^n \longrightarrow \mathbb{C}$ which vanishes at some point a with $\mathrm{d}F(a) \neq 0$, by analogy with the real case we can expect that the zeros of F form an $n-1$ dimensional complex manifold.

If $\mathrm{d}F(a) = 0$ and F is not identically zero, we can use the local description of the function at a like in one variable. In fact, let $a = (a', a_n)$, $a' \in \mathbb{C}^{n-1}$, by applying a complex affine change of coordinates if necessary, there is $p \in \mathbb{N}$ [1] such that $z_n \mapsto F(a', z_n)$ has a zero of order p at a_n.

Now by the Rouché theorem[2] in one variable and continuity of F, for each small enough $\epsilon > 0$ there exists $\delta > 0$ such that for each $z' \in B(a', \delta) \subset \mathbb{C}^{n-1}$, $z_n \mapsto F(z', z_n)$ has exactly p roots counting with multiplicity in the disc $D(a_n, \epsilon) \subset \mathbb{C}$. We have then proved the following

Proposition 5.14. *The zeroes of an analytic function in more than one variable are not isolated.*

From the above discussion, the zero set of a non identically null holomorphic function has the size of an $n-1$ dimensional ball. To deal with smaller sets, we need to consider the common zero set of a collection of holomorphic functions, finite collection in our case. This will lead us to the notion of analytic sets.

5.3.2 Analytic Sets and Riemann Removable Singularity

Definition 5.15. *Let D be a domain of $\mathbb{C}^n$. A subset A of D is called an analytic set if for every $z \in D$, there exists a neighbourhood U_z of z and a finite tuple of analytic functions $f_1, \cdots, f_p \in H(U_z)$ such that*

$$A \cap U_z = \{w \in U_z | f_1(w) = \cdots = f_p(w) = 0\}.$$

A proper analytic subset of D is an analytic subset which is different from D.

Let D be a domain in $\mathbb{C}^n$. Analytic sets in D are closed in D. $\emptyset$ is of course an analytic set but when we talk about analytic we always mean a non-empty analytic set unless stated otherwise. $\mathbb{C}^n$ is an analytic set, complex affine subspaces of $\mathbb{C}^n$ are analytic sets. Intersections of D with complex affine subspaces of $\mathbb{C}^n$ are analytic sets and D itself is an analytic set. Any intersection and locally

[1] this order p of the zero depends upon the choice of affine parametrization, two different parametrizations may give different orders.

[2] Here is a statement of the theorem: Let D be a domain in $\mathbb{C}$, $g, f \in H(D)$, $c \in V, r > 0$ such $\bar{D}(c, r) \subset D$ and $|f(z) - g(z)| < |f(z)|$ on $\partial\bar{D}(c, r)$, then f and g have the same number of zero in $D(c, r)$ counting with multiplicity. Two points $a', b' \in \mathbb{C}^{n-1}$ determine two holomorphic functions $F(a', z_n)$ and $F(b', z_n)$ of z_n and the continuity of F delivers the inequality in the hypothesis of Rouché theorem.

finite union of analytic sets are analytic sets, for instance the diagonal in $\mathbb{C}^n$ is an analytic set. Inverse images of analytic sets under holomorphic mapping are analytic set. Images of analytic sets under biholomorphic mapping are analytic set but moreover:

Proposition 5.16. *Images of analytic sets under proper holomorphic mapping are analytic sets.*

We are not going to give the proof of this complex analysis fact but the interested reader should consult [Chi89].

Now, instead of defining the dimension of an analytic set we will define formally its co-dimension.

Definition 5.17. *An analytic set A in a domain D of $\mathbb{C}^n$ has co-dimension q at $a \in A$, and we denote* $\operatorname{codim}_a A = q$ *if there exists a q-dimensional, but no $(q+1)$-dimensional, affine subspace L of $\mathbb{C}^n$ such that a is an isolated point of $L \cap A$.*

We then define the co-dimension of A to be:

$$\operatorname{codim} A = \min_{a \in A} \operatorname{codim}_a A.$$

Proposition 5.18. *For an analytic set A of a domain D,* $\operatorname{codim} A = 0$ *if and only if $A = D$.*

Proof. If $A = D$, it is trivial that $\operatorname{codim} A = 0$. Suppose $\operatorname{codim} A = 0$, let $a \in A$ such that $\operatorname{codim}_a A = 0$. Let U_a be a neighbourhood of a as in the definition of analytic set and $f_1, \cdots, f_p$ the defining functions of A. Let L be a complex line through a, by applying a complex affine transformation if necessary we can assume that $L = \mathbb{C} \times \underline{0}$. By hypothesis a is a non isolated zero of the f_i on $A \cap L$, then by the one variable analytic continuation $f_i \equiv 0$ on $L \cap U_a$. Since this is true for all complex line through a and U_a can be covered by complex lines, $f_i \equiv 0$ on U_a. Then $U_a \subset A$ and $int\,(A) \neq \emptyset$. Let us show that $int\,(A)$ is closed. Let $z \in clos\,(int\,(A)) \cap D$, let U_z be the domain of the defining functions of A at z. $U_z \cap int\,(A)$ is open and non empty and the defining functions at z vanish on this set, therefore by analytic continuation, they are identically zero on U_z. Thus $U_z \subset A$ and it follows that $z \in int\,(A)$. Therefore $int\,(A)$ is closed and since D is connected, $A = D$.

□

Corollary 5.19. *If A is a proper analytic subset of D then $int\,(A) = \emptyset$ and $D \setminus A$ is dense in D.*

Theorem 5.20 (Riemann Removable Singularity). *If A is a proper analytic set of a domain $D \subset \mathbb{C}^n$, and g is an holomorphic function on $D \setminus A$ which is locally bounded at A, then g is extensible to a unique holomorphic function on D.*

Proof. By corollary 5.19 if the extension exists then it is unique.

Let $a \in A$. Since $\text{codim} A = 1$, there exists a complex line L such that a is an isolated point of $A \cap L$, by applying a complex affine transformation if necessary we may assume that $a = 0$ and $L = \underline{0} \times \mathbb{C}$. We can find a polydisc[1] $P = P^{n-1}(r) \times P^1(r) \subset D$ with closure contained in D such that

$$A \cap (\underline{0} \times clos(P^1(r))) = \{0\}.$$

Let $T = \partial P^1(r)$, $A \cap (\underline{0} \times T) = \emptyset$. Since T is compact, there exist $\epsilon > 0$ and an annular neighbourhood N of T such that $P'(\epsilon) \times N \subset P \setminus A$.

Let $a' \in P'(\epsilon)$ be fixed. The defining functions of A are analytic functions of one variable on $P^1(r)$. If the intersection is not discrete, the zero set of the corresponding functions possesses an accumulation point, then by the one variable analytic continuation, they are identically zero on $a' \times (P^1(r))$ which is impossible since $a' \times N \subset P \setminus A$ and $N \cap (P^1(r))$ is non-empty as N is an annular neighbourhood of T. Therefore $(a' \times P^1(r)) \cap A$ is discrete. The map

$$h : (z', z_n) \longmapsto \frac{1}{2\pi i} \int_T \frac{g(z', \zeta)}{\zeta - z_n} d\zeta$$

defined on $P'(\epsilon) \times P^1 \setminus A$ is a continuous function. Since $P'(\epsilon) \times T \subset P \setminus A$ we can do complex differentiation on h with respect to the variable $z' \in P'(\epsilon)$. The same thing holds for z_n in P^1 when we fix z' except for some isolated points in A. Since g is locally bounded at A, by the one variable Riemann removable singularity h is an analytic continuation of g on all P^1 as a function of the one variable z_n. Then h is an analytic continuation of g on $P'(\epsilon) \times P^1$ by Cauchy integral formula.

We have proved that g is extensible to an analytic function in a neighbourhood of each $a \in A$, thus g is extensible to an analytic function on D. □

5.4 End of the Proof of Theorem 5.1

Lemma 5.21. *The image $\overline{\pi}(\Delta)$ in O_α (resp. O_β) of the diagonal Δ in N_α (resp. N_β) is a proper analytic subset of O_α (resp. O_β).*

[1] Cartesian product of disks in $\mathbb{C}$

Proof. The diagonal is an analytic set since union of complex hyperplane and $\bar{\pi}$ is a proper holomorphic map, then by proposition 5.16 the result holds. □

Lemma 5.22. *The transition map $\psi_{\alpha\beta}$ is holomorphic.*

Proof. We have proved that it is holomorphic except on the image of the diagonal. Since it is continuous it is locally bounded on this image which is a proper analytic set of O_α. Therefore we can apply the Riemann removable singularity principle.

□

This ends the proof of Theorem 5.1.

Chapter 6

Heegaard Diagrams and Floer Homology

Our interest in the symmetric product of surfaces and their natural complex structure is motivated by Heegaard Floer homology. This is a recent invariant of 3-manifolds and knots in 3-manifolds which counts holomorphic disks in the symmetric product of surface. The surface in question is the Heegaard surface which comes from the Heegaard splitting of the underlying manifold.

6.1 Heegaard Diagram

We assume that all manifolds are connected. Let us denote B^k the k-dimensional ball and S^k the k-dimensional sphere. A *n-dimensional k-handle* is the product:

$$H^{n,k} = B^k \times B^{n-k}$$

with boundary

$$\partial H^{n,k} = \left(S^{k-1} \times B^{n-k}\right) \bigcup \left(B^k \times S^{n-k-1}\right)$$

$S^{k-1} \times B^{n-k}$ is called the *attaching region.* Given a smooth n-manifold M with boundary, attaching a k-handle to M means:

- Choose an embedding $\varphi : \left(S^{k-1} \times B^{n-k}\right) \longrightarrow M$
- Glue $H^{n,k}$ to M along φ to get a new manifold M'

$$M' = M \amalg H^{n,k}/p \sim \varphi(p).$$

We can always get a smooth manifold by smoothing corners if needed. If we change the attaching map, M' is determined up to diffeomorphism by isotopy class of φ.

For the case of a 3-manifold ($n = 3$), a *genus g handlebody* is a 3-manifold homeomorphic to a 3-ball with g 1-handles attached. Its boundary is a genus g surface.

Definition 6.1. *A genus g Heegaard splitting of a 3-manifold Y is a decomposition of Y into union of two genus g handlebodies glued together along their boundaries via an orientation reversing diffeomorphism of these boundaries.*

The following theorem justifies the definition.

Theorem 6.2. *Every closed oriented 3-manifold Y admits a Heegaard splitting.*

It can be proven by using triangulation of the 3-manifold. We take a regular neighbourhood of the graph of the triangulation[1] as one of the handlebody. The other handlebody is the complement of this neighbourhood.

Another approach is the use of Morse functions which is more appropriate in our context. We chose a self indexing Morse function $f : Y \longrightarrow \mathbb{R}$ with one index 0, one index 3 and the same number of index 1 and index 2 critical points. The two handlebodies can be choosing to be $f^{-1}([0, 3/2])$ and $f^{-1}([3/2, 3])$. The genus of the Heegaard splitting will be the number of index 1 critical points.

The common boundary $f^{-1}(3/2)$ of the handlebodies is called the *Heegaard surface* of the splitting.

Example 6.3. *The 3-sphere is the unique 3-manifold which has a genus zero Heegaard splitting.*

Let p, q be two coprime integers. The group $\mathbb{Z}/p\mathbb{Z}$ acts on $S^3 = \{(z, w) \in \mathbb{C}^2 | |z^2| + |w^2| = 1\}$ by

$$e^{2i\pi/p} \, . \, (z, w) = \left(e^{2i\pi/p} z, e^{2iq\pi/p} w\right)$$

The resulting quotient of S^3 is a smooth 3-manifold called the lens space $L(p, q)$. Lens spaces and $S^2 \times S^1$ are the 3-manifolds which admit genus one Heegaard splitting.

[1] This graph is made by the vertices and the edges of the triangulation

It is obvious that, by gluing two genus g handlebodies H_0 and H_1 with boundaries Σ_0 and Σ_1 via an orientation reversing diffeomorphism $\psi : \Sigma_0 \longrightarrow \Sigma_1$ of the boundaries, we get a closed oriented 3-manifold.

Definition 6.4. *A set of attaching circles for a handlebody H_g is a collection of g simple closed curves $\gamma_1, \cdots, \gamma_g$ in $\Sigma_g = \partial H_g$ such that:*

- *They are disjoint from each other.*
- *$\Sigma_g - \gamma_1 - \cdots - \gamma_g$ is connected.*
- *They bound disjoints embedded disk in H_g.*

Let H_g be a genus g handlebody and $\Sigma_g = \partial H_g$. Let $c_1, \cdots, c_g \subset \Sigma_g$ be a set of attaching circles for H_g. The Heegaard splitting of a 3-manifold Y can also be described as an embedding $\psi : \Sigma_g \hookrightarrow Y$ such that $Y \setminus \Sigma_g$ has 2 components and the closure A and B of the components are both diffeomorphic to H_g.

Let us choose two diffeomorphisms $\varphi_A : A \longrightarrow H_g$, $\varphi_B : B \longrightarrow H_g$ and let

$$\alpha_i = \psi^{-1}\left(\varphi_A^{-1}(c_i)\right), \qquad \beta_i = \psi^{-1}\left(\varphi_B^{-1}(c_i)\right).$$

We can choose ψ so that $\varphi_A \circ \psi = id_{\Sigma_g}$. Let us denote $\underline{\alpha} = \{\alpha_1, \cdots, \alpha_g$ and $\underline{\beta} = \{\beta_1, \cdots, \beta_g\}$.

Proposition 6.5. *The 3-manifold Y is determined up to diffeomorphism by $\left(\Sigma_g, \underline{\alpha}, \underline{\beta}\right)$.*

Proof. Let $Y', \psi', \varphi_{A'}, \varphi_{B'}$, be another splitting with the same diagram $\left(\Sigma_g, \underline{\alpha}, \underline{\beta}\right)$ chosen such that $\varphi_{A'} \circ \psi' = id_{\Sigma_g}$. We want to define a diffeomorphism between M and M'. Let us consider the restriction of the map $\varphi_{A'}^{-1} \circ \varphi_A$ on the boundary $\partial A = \partial B$, by pre-composing with φ_B^{-1} and post-composing with $\varphi_{B'}$ we get a self-diffeomorphism of Σ_g,

$$\varphi_{B'} \circ \varphi_{A'}^{-1} \circ \varphi_A \circ \varphi_B^{-1} : \Sigma_g \longrightarrow \Sigma_g$$

The result will be a consequence of the following lemma.

Lemma 6.6. *Given a diffeomorphism $h : \Sigma_g \longrightarrow \Sigma_g$ such that $h(c_i) = c_i$ for all i, then h extends to a diffeomorphism $H_g \longrightarrow H_g$.*

Proof. Diffeomorphisms of S^1 are isotopic to either the identity map or a reflection, then by isotoping h near the c_i in a collar neighbourhood of ∂H_g we can assume that $h_{|c_i}$ is either the identity map or a reflection. Since the identity map and a reflection extend over the disk, $h_{|c_i}$ can be extended over a "disk" in H_g. By doing this in a narrow annuli around c_i one can extend $h_{|c_i}$ over a disk cross interval. Therefore h extends over H_g minus disjoint union of 3-ball. It then suffices to extend over the 3-balls. Since we already have diffeomorphism of the boundary for a ball, by the fact that diffeomorphism of S^2 is isotopic either to the identity or a reflection, one can extend over the 3-ball. □

By the lemma the map $\varphi_{B'} \circ \varphi_{A'}^{-1} \circ \varphi_A \circ \varphi_B^{-1}$ extend to a self-diffeomorphism F of H_g. For, we just need to prove that

$$\varphi_{B'} \circ \varphi_{A'}^{-1} \circ \varphi_A \circ \varphi_B^{-1}(c_i) = c_i.$$

In fact

$$\varphi_{B'} \circ \varphi_{A'}^{-1} \circ \varphi_A \circ \varphi_B^{-1} = \varphi_{B'} \circ \psi' \circ \psi'^{-1} \circ \varphi_{A'}^{-1} \circ \varphi_A \circ \psi \circ \psi^{-1} \circ \varphi_B^{-1}$$

Then

$$\begin{aligned}\varphi_{B'} \circ \varphi_{A'}^{-1} \circ \varphi_A \circ \varphi_B^{-1}(c_i) &= \varphi_{B'} \circ \psi' \circ \psi'^{-1} \circ \varphi_{A'}^{-1} \circ \varphi_A \circ \psi \circ \psi^{-1} \circ \varphi_B^{-1}(c_i) \\ &= \varphi_{B'} \circ \psi' \circ \psi'^{-1} \circ \varphi_{A'}^{-1} \circ \varphi_A \circ \psi(\beta_i)\end{aligned}$$

Using the fact that $\psi'^{-1} \circ \varphi_{A'}^{-1} = \varphi_A \circ \psi = id_\Sigma$ we obtain

$$\begin{aligned}\varphi_{B'} \circ \varphi_{A'}^{-1} \circ \varphi_A \circ \varphi_B^{-1}(c_i) &= \varphi_{B'} \circ \psi'(\beta_i) \\ &= \varphi_{B'} \circ \psi' \circ \psi'^{-1} \circ \varphi_{B'}^{-1}(c_i) \\ &= c_i.\end{aligned}$$

Now by pre-composing with φ_B and post-composing with $\varphi_{B'}^{-1}$ we get a diffeomorphism $G : B \longrightarrow B'$ as shown in the following diagram

$$\begin{tikzcd} H_g \arrow[r, "F"] & H_g \arrow[d, "\varphi_{B'}^{-1}"] \\ B \arrow[u, "\varphi_B"] \arrow[r, "G"'] & B' \end{tikzcd}$$

G coincides with $\varphi_{A'}^{-1} \circ \varphi_A$ on the boundary $\partial A = \partial B$. Thus we can choose the diffeomorphism between Y and Y' to be G on B and $\varphi_{A'}^{-1} \circ \varphi_A$ on A. □

Definition 6.7. *Let Y be a 3-manifold with a genus g Heegaard splitting $(\partial H_0, H_0, H_1)$. A compatible Heegaard diagram for this splitting is given by the data $(\Sigma_g, \underline{\alpha}, \underline{\beta})$ in Proposition 6.5.*

Here are some justifications for the terminology. We can picture a surface (closed orientable) on a plane by a disk with $2g-1$ holes (or small disks removed) and identify the exterior boundary with one component of the interior boundary, the remaining boundary components are identified two by two. So a Heegaard diagram becomes a disk with holes and curves drawn on it. A genus g surface can also be seen as S^3 with $2g$ disks removed such that the boundaries are identified two by two. This gives another way to picture a Heegaard diagram by just considering S^3 as the one point compactification of the plane.

One Heegaard diagram determines uniquely a 3-manifold up to diffeomorphism but one 3-manifold gives rise to different Heegaard diagrams due to the large choice of triangulations and Morse functions. However there is an equivalence relation on the set of diagram such that different classes determine distinct manifolds. There are three modifications on a Heegaard diagram that do not change the underlying manifold:

- *Isotopy.* This moves the attaching circles in a 1-parameter family such that they remain disjoints.
- *Handle-slide.* Choose two curves γ_1 and γ_2. Replace γ_1 by a simple closed curves $\tilde{\gamma}$ such that
  - $\tilde{\gamma}$ is disjoint from $\gamma_1, \gamma_2, \cdots, \gamma_g$
  - $\tilde{\gamma}$, γ_1 and γ_2 bound an embedded pair of pants (disk with two holes) in $\Sigma - \gamma_1 - \gamma_2 - \cdots - \gamma_g$
- *Stabilization.* This is a connected sum with a torus. This increases the genus of the Heegaard diagram by one and add a new meridional and longitudinal curves which are of trivial type.

Two Heegaard diagram are called equivalent if they are related by finite sequences of these modifications. For more details we refer to [Rol76].

We have the following facts which have been proved first by James Singer in [Sin33]. Here by two 3-manifolds being equivalent we mean diffeomorphic.

Theorem 6.8. *Two Heegaard diagrams arising from two equivalent 3-manifolds are equivalent.*

Theorem 6.9. *Two Heegaard diagrams which give rise to two equivalent 3-manifold are equivalent.*

Theorem 6.10. *If two 3-manifolds give rise to equivalent Heegaard diagrams then they are equivalent.*

Remark 6.11. *There is also the notion of k- pointed Heegaard diagram for a positive integer k. They are Heegaard diagrams with marked points $z_1, \cdots, z_k \in \Sigma_g - \underline{\alpha} - \underline{\beta}$ on the Heegaard surface. For this case, during handle-slide the base points must be outside the pair of pant region and during isotopy the base points must be disjoint from the curves. These new moves are called pointed handle slide and pointed isotopy. With stabilization these new moves also define an equivalence relation on the set of pointed Heegaard diagrams.*

Example 6.12. *Here we give some examples of Heegaard diagrams for some classical 3-manifolds.*

- *Heegaard diagram for $S^2 \times S^1$.*

  We cut the sphere S^2 into two disks along the equator. Then $S^2 \times S^1$ splits into two pieces, each of these pieces is a circle cross a disk which is homeomorphic to a solid torus. The α and β curves are then parallel copies of the equator. Then $S^2 \times S^1$ has a Heegaard diagram which corresponds to a torus with α and β curves which are parallel copy of each other. The plane diagram is shown in the following figure where we identify the two circles which delimit the shadowed annulus to represent the torus.

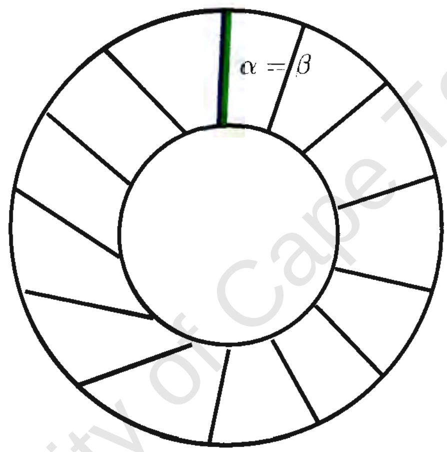


Figure 6.1: genus 1 Heegaard diagram for $S^2 \times S^1$.

- *Heegaard diagram for $L(7,2)$*

  This Heegaard diagram is obtained from the one in [BM80] p. 48.

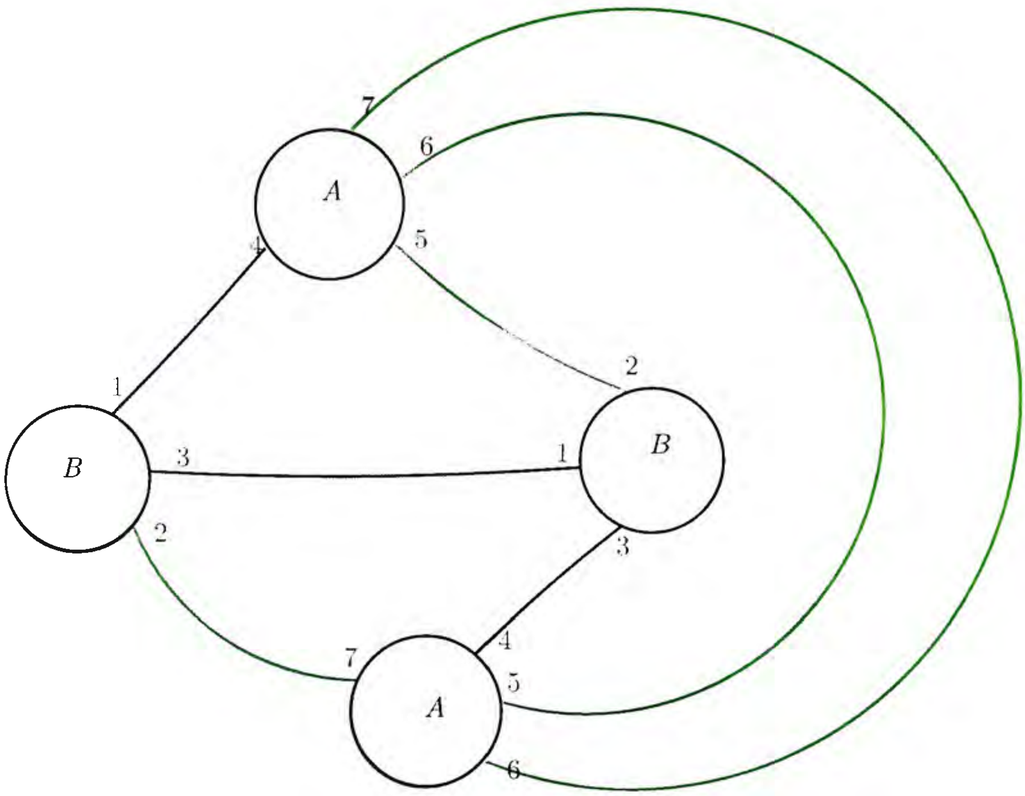


Figure 6.2: genus 2 Heegaard diagram for $L(7,2)$.

- *Heegaard diagram for the Poincaré sphere.*

  We can find the complete and original discussion, proof included, about the Poincaré sphere and its Heegaard diagram in [GL53]. This Heegaard diagram is also cited in [Rol76]. Here is a slightly modified version of the corresponding diagram. We are not precise about the orientation.

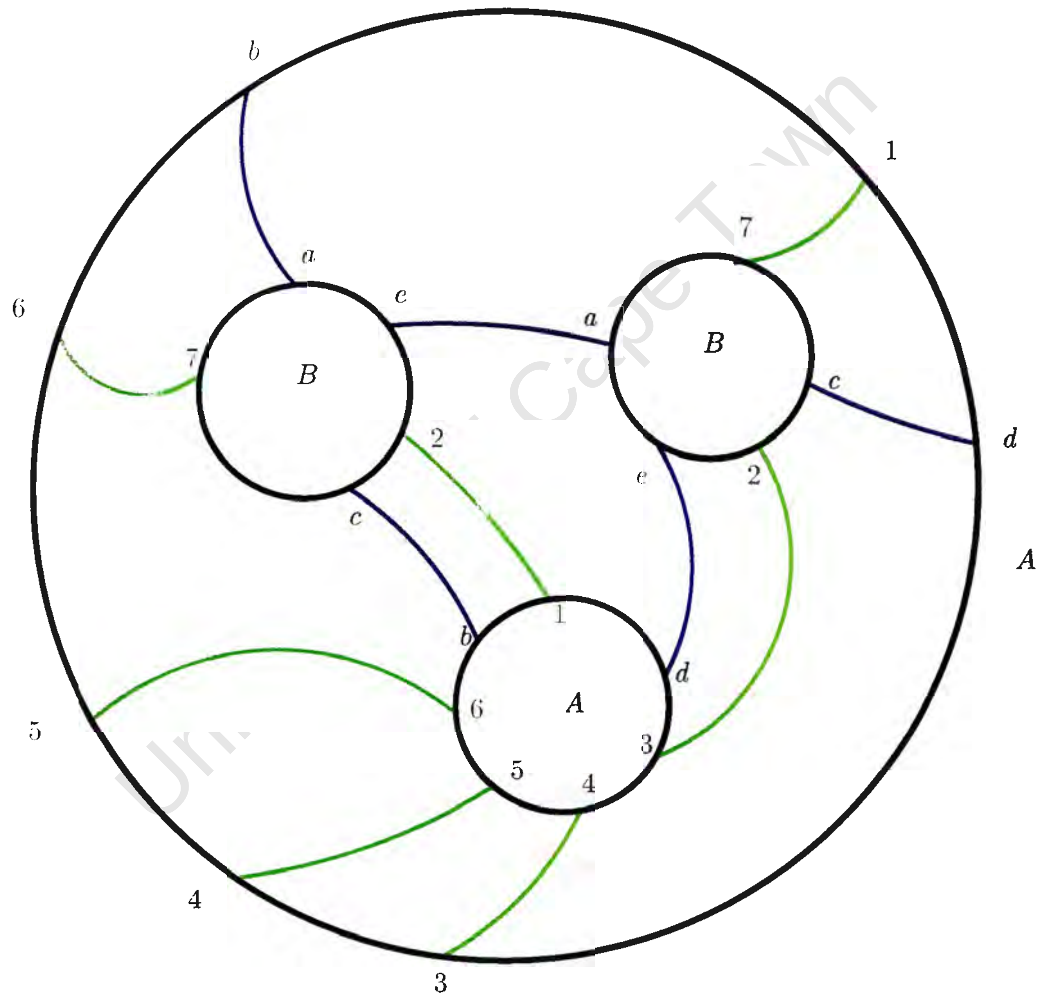


Figure 6.3: genus 2 Heegaard diagram for the Poincaré sphere.

- *Heegaard diagram for the three torus* $S^1 \times S^1 \times S^1$.

  We only draw the β *curves, the* α *curves are parallel to the boundary circles. The same colour indicate the same curve.*

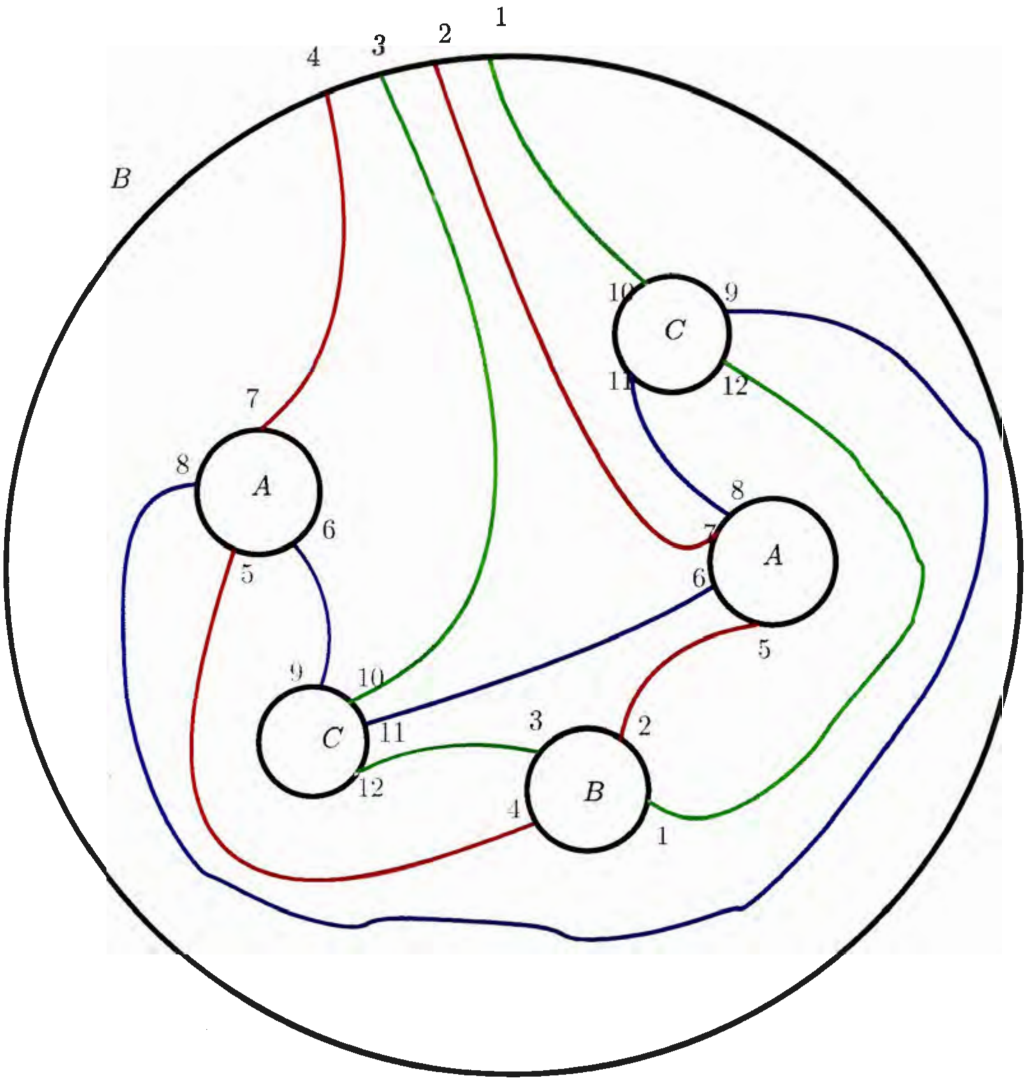
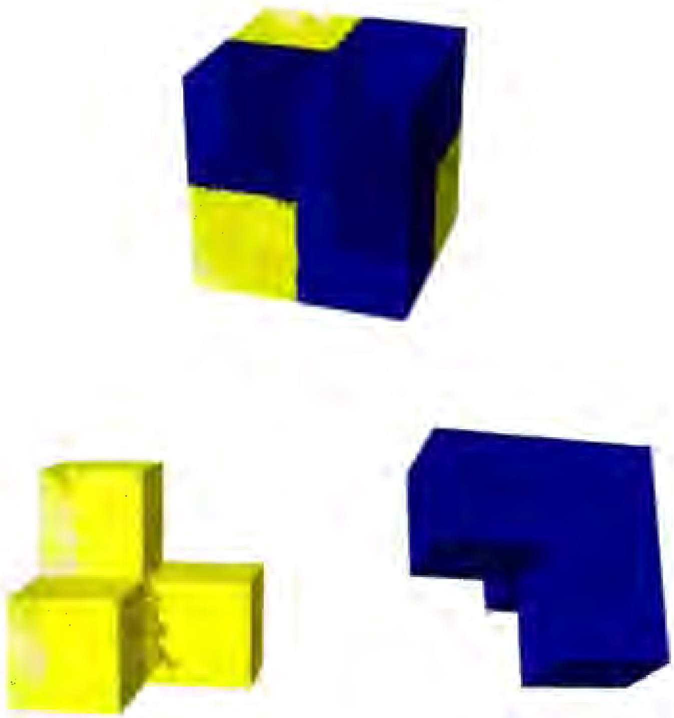


Figure 6.4: genus 3 Heegaard diagram for $S^1 \times S^1 \times S^1$.

We are now going to prove this last claim.

We consider the three torus as the quotient of a cube by identification of opposite faces.

We can split this cube in two parts as shown in the following figure.



Because we have identified opposite faces of the cube, each of the above part is a solid bretzel with three holes. The α and β curves are then the curves shown in the figure below

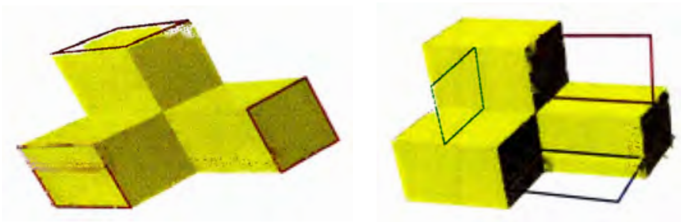
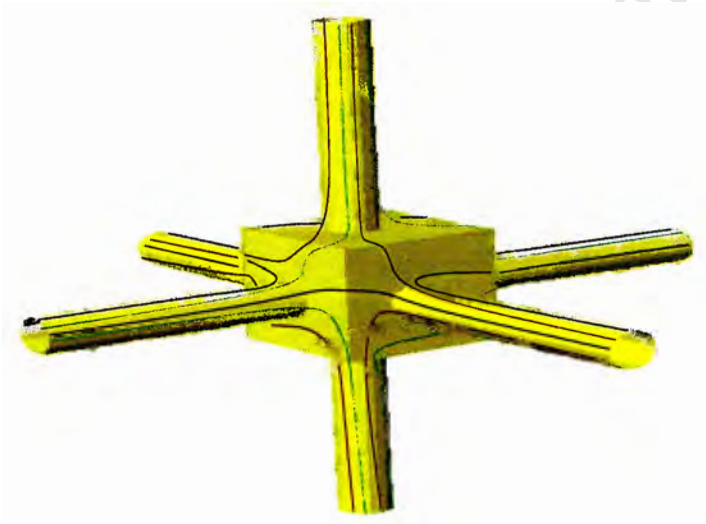


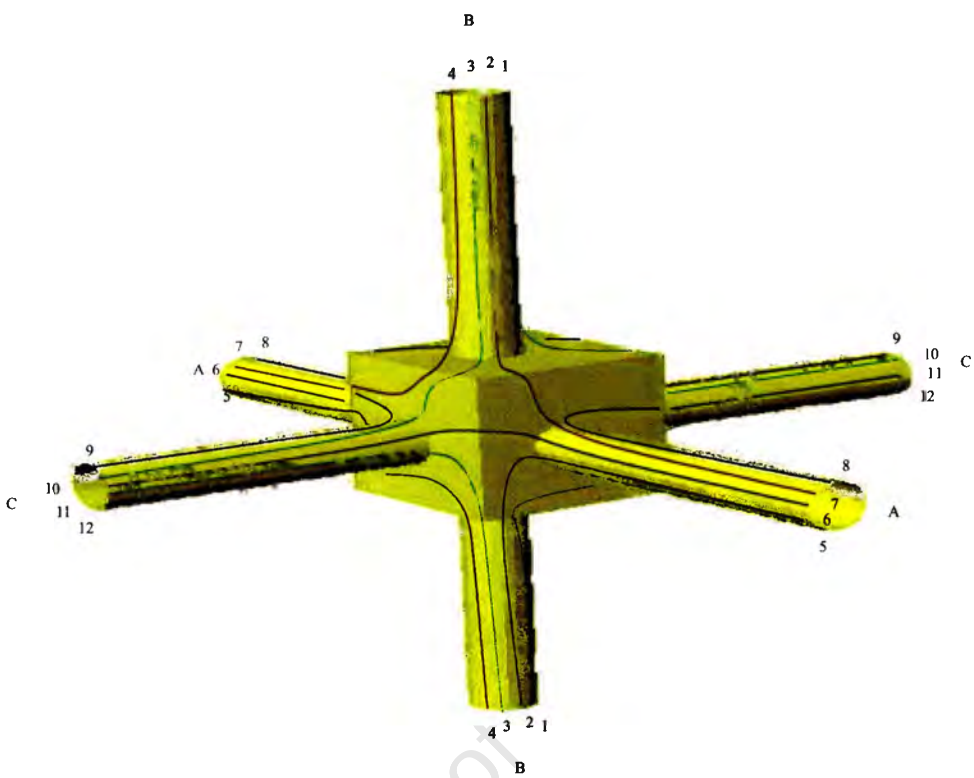
Figure 6.5: α curves on the left and β curves on the right.

From this one can obtain the followibg figure where we draw only the β curves.



We identify the ends of cylinder which correspond to opposite faces of the cube in the centre.

The next figure is obtained by labelling opposite faces of the cylinder with the same capital letter, and labelling the end points of the curves which are identified with the same number.



From this we deduce our Heegaard diagram for the three torus.

6.2 Domains and Heegaard Floer Homology

6.2.1 Domains

Let $D_1, \cdots, D_k$ be the closure of the connected components of $\Sigma - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$.

Definition 6.13. *A domain is a linear combination of the D_i's with integer coefficients.*

Definition 6.14. *A domain is said to be positive if all the coefficients are greater or equal to 0.*

The set of all domains is then the free Abelian group generated by the set all D_i's. The boundary of a domain is a linear combination of arcs contained in the α or β curves with integer coefficients.

Definition 6.15. *A periodic domain is a domain with boundary a linear combination of entire α curves and entire β curves and the component containing the base point has coefficient zeros.*

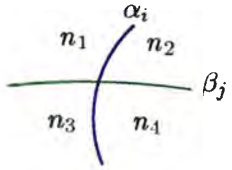
Theorem 6.16. *The set of periodic domains is a subgroup isomorphic to $H_2(Y, \mathbb{Z})$.*

Proof. Let $\mathfrak{P}$ be the set of periodic domains. It is obvious that $\mathfrak{P}$ forms a subgroup of the group of domains.

We want a map $f : \mathfrak{P} \longrightarrow H_2(Y, \mathbb{Z})$.

◇ Let us give first a description of $H_2(Y, \mathbb{Z})$ which relates to α and β curves.

The situation at one intersection point of α and β is described on the figure below, where the label on the left and right of the α are the coefficient of the corresponding domains.



For a periodic domain each sub-arc of the α's have the same coefficient so $n_1 - n_2 = n_3 - n_4$ and this is equivalent to $n_1 - n_3 = n_2 - n_4$ *i.e* each sub-arc of the β's have also the same coefficient. We use the definition of $H_2(Y, \mathbb{Z})$ from Morse complex considering the Morse function and Riemann metric associated to the Heegaard splitting. A reference for Morse theory and homology are [Mil69] and [Sch]. Let C_i be the free Abelian group generated by the index i critical points and $Crit_i$ the set of index i critical points. We have the Morse complex:

$$0 \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

where

$$\partial_k q = \sum_{p \in Crit_{k-1}} \sharp\mathcal{M}(q,p) \ . \ p$$

with $\mathcal{M}(q,p)$ being the space of gradient flow lines from q to p. For ∂_3, since the flow lines appear in cancelling pairs we have $\partial_3 q = 0$ then $\operatorname{Im} \partial_3 = 0$ and therefore $H_2(Y,\mathbb{Z}) = \ker \partial_2$. The index 1 critical points "correspond" to the α's and the index 2 to the β's, so an element C_2 (resp. C_1) can be expressed as linear combination of the β's (resp. α's). Let $q = \sum b_i \ \beta_i \in C_2$,

$$\begin{aligned}
\partial_2 q &= \sum b_i \ \partial_2 \beta_i \\
&= \sum_i b_i \sum_{p \in Crit_1} \sharp\mathcal{M}(\beta_i, p) \ . \ p \\
&= \sum_i b_i \sum_j \sharp\mathcal{M}(\beta_i, \alpha_j) \ . \ \alpha_j \\
&= \sum_j \left[\sum_i b_i \ \sharp\mathcal{M}(\beta_i, \alpha_j)\right] \ . \ \alpha_j,
\end{aligned}$$

when we do not specify the index set for i and j we are doing the summation on all the critical points of the corresponding index.

All the flow lines have to pass through Σ and the intersection of the descending manifold of the index two critical points and the ascending manifolds of the index one critical points (modulo $\mathbb{R}$ action) are precisely the intersection of the α and β curves, therefore :

$$\sharp\mathcal{M}(\beta_i, \alpha_j) = \beta_i \circ \alpha_j$$

where we use $\circ$ to denote the intersection operation. Thus

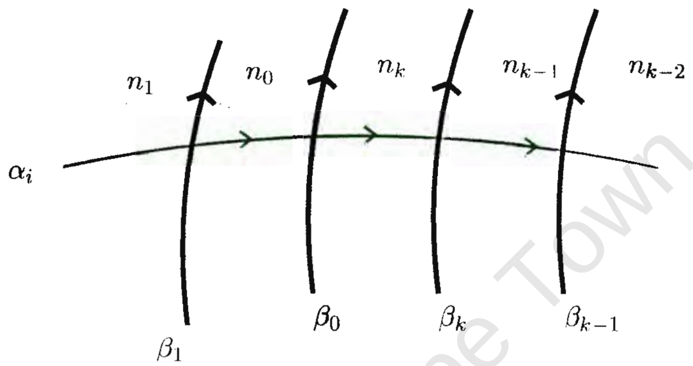
$$\ker \partial_2 = \langle \sum b_i \ \beta_i \mid \sum b_i \ \beta_i \circ \alpha \ = \ 0, \text{ for all } \alpha \text{ curves} \rangle$$

◇ Now we can construct the desired map f, taking into account the previous description. Let $P \in \mathfrak{P}$. $\partial P = \partial_\alpha P + \partial_\beta P$, where $\partial_\alpha P$ is the component from the α curves and $\partial_\beta P$ the component from the β curves. We define

$$f(P) = \partial_\beta P$$

We need to check first that this is indeed an element of $H_2(Y, \mathbb{Z})$.

Let $\partial_\beta P = b_0\beta_0 + \cdots + b_k\beta_k$ and let α be one "component" of $\partial_\alpha P$. First we fix an orientation for α, from the left to the right on the figure . We give an orientation to the β's such that (α, β) is positively oriented. The situation is shown on the figure



Now we assign a sign correction σ_j to each β_j as follows:

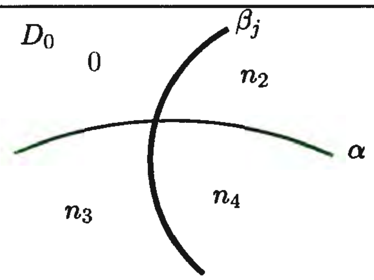
$$\sigma_j = \begin{cases} 1 & \text{if the orientation of } \beta_j \text{ coincide with the exact one} \\ -1 & \text{otherwise} \end{cases}$$

then $\beta_j \circ \alpha = -\sigma_j$, in another hand $b_j = \sigma_j (n_j - n_{j-1})$. Therefore

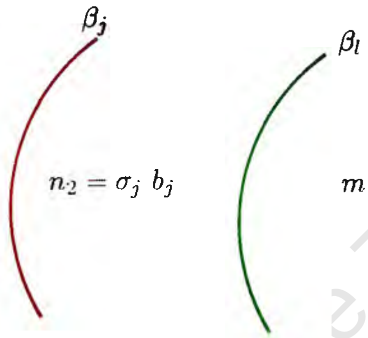
$$\begin{aligned} \sum_j b_j \ \beta_j \circ \alpha &= \sum_j \sigma_j \ (n_j - n_{j-1}) \ (-\sigma_j) \\ &= -\sum_j \sigma_j^2 \ (n_j - n_{j-1}) \\ &= -\sum_j (n_j - n_{j-1}) \\ &= 0 \end{aligned}$$

Thus $f(P) = \partial_\beta P \in H_2(Y, \mathbb{Z})$, and the map is well defined. It is easy to see that this is an homomorphism. To prove that it is a bijection we define its inverse as follows:

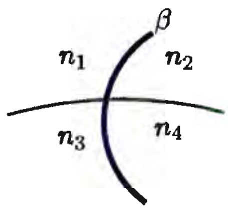
Let D_0 be the component containing the base point. Let $\sum_\nu b_\nu \beta_\nu \in H_2(Y, \mathbb{Z}) = \ker \partial_2$. Across D_0 we have the following situation:



Since $b_j = \sigma_j (n_j - n_{j-1})$, $b_j = \sigma_j n_2$ or equivalently $n_2 = \sigma_j b_j$. Considering the case



we have $b_l = \sigma_l (n_2 - m) = \sigma_l (\sigma_j b_j - m)$, then $m = -\sigma_l b_l + \sigma_j b_j$. Since the α curves do not disconnect the surface, we can go from one component[1] to another by just crossing β arcs. Therefore doing the above process repeatedly determine uniquely all the coefficient of the components. The last step is to prove that the domain obtained is a periodic domain. We have $n_0 = 0$. At an intersection point of α and β curve we have



Let b be the coefficient of the β curve, $b = (n_1 - n_2)\sigma = (n_3 - n_4)\sigma$. σ being the sign correction. The same holds for the α's.

$\square$

This theorem is an illustration of how the knowledge of the Heegaard diagram can give informa-

[1] closure of one component of $\Sigma - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$

tion about the underlying 3-manifold. In this simple case it is the second homology group. One can try to construct invariant of three manifolds via Heegaard Diagram[2]. The Heegaard Floer Homology is such an invariant. The definition of this invariant requires the use of the g-fold symmetric product of the Heegaard surface.

6.2.2 Holomorphic Disks in Symmetric Product

Let us consider the g-fold symmetric product of the Heegaard surface. It is as we have seen in the previous chapter a complex manifold or if we are less restrictive an almost complex manifold.

Definition 6.17. *A submanifold N of a complex manifold (M, J) is called totally real if for each $x \in N$*

$$T_xN \cap JT_xN = 0.$$

The g-tuple of α and β curves form inside the symmetric product two embedded totally real tori

$$\begin{aligned} \mathbb{T}_\alpha &= [\alpha_1 \times \cdots \times \alpha_g] \\ \mathbb{T}_\beta &= [\beta_1 \times \cdots \times \beta_g] \end{aligned}$$

where the brackets mean their representatives in $\mathrm{Sym}^g\Sigma$. Since α (resp. β) curves do not intersect among themselves $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are away from the diagonal.

Let $\mathbb{D}$ be the unit disk in the complex plane, closed or open according to the context. Let S_+, S_- be the arcs in the boundary of $\mathbb{D}$ corresponding to $\mathrm{Im}[z] \geq 0$ and $\mathrm{Im}[z] \leq 0$.

Definition 6.18. *Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. A Whitney disk connecting x to y is a continuous map:*

$$u : \mathbb{D} \longrightarrow \mathrm{Sym}^g\Sigma$$

which send $-i$ to x, i to y, S_+ inside $\mathbb{T}_\alpha$ and S_- inside $\mathbb{T}_\beta$ We denote $\pi_2(x, y)$ the set of homotopy classes of Whitney disks connecting x to y and $V_z = z \times \mathrm{Sym}^{g-1}(\Sigma)$ for $z \in \Sigma$. We define the multiplicity of $\phi \in \pi_2(x, y)$ at z to be the integer :

$$n_z(\phi) = \sharp u^{-1}(\mathbb{D} \cap V_z)$$

where u is a smooth representative chosen transverse to V_z.

[2]The equivalence class of a Heegaard Diagram is itself a three manifold invariant

If ϕ admits a (pseudo-)holomorphic representative then its multiplicity is positive since holomorphic maps preserve orientation, or more generally, J-holomorphic curves intersect J-holomorphic submanifolds of codimension 2 positively.

Definition 6.19. *The domain of a homotopy class $\phi \in \pi_2(x, y)$ is the formal linear combination*

$$\mathcal{D}(\phi) := \sum_{i=1}^{k} n_{z_i}(\phi)$$

where $z_i \in D_i$ are points in the interior of D_i.

For $\phi \in \pi_2(x, y)$ we define $\mathcal{M}(\phi)$ to be the set of (pseudo-) holomorphic representatives of ϕ.

The automorphism of the unit disk is $\mathrm{PSL}(2, \mathbb{R})$ so the subgroup preserving i and $-i$ is isomorphic to $\mathbb{R}$. Therefore $\mathbb{R}$ acts on $\mathcal{M}(\phi)$ by re-parameterization of the unit disk:

$$g \, . \, u = u \circ g$$

We denote $\widehat{\mathcal{M}}(\phi)$ the quotient of $\mathcal{M}(\phi)$ by this $\mathbb{R}$ action. Ozsváth and Szabó specify a set of almost complex structures that includes those induced by complex structures on Σ. They prove that, for a dense subset of these almost complex structures, $\mathcal{M}(\phi)$ is a smooth manifold whose dimension equals a certain index called the Maslov index of ϕ, which we will not define here. A result from Gromov says that in every homotopy class ϕ of Maslov index 1 the set $\widehat{\mathcal{M}}(\phi)$ is finite (compact 0-dimensional). We will omit the discussion about the genericity of complex structure and Gromov result.

6.2.3 Heegaard Floer Homology

Heegaard Floer homology is an invariant for closed oriented 3-manifolds which has been extended to links and to cobordism invariants for 4-manifolds. It was first developed from Floer homology which is an homology obtained by counting pseudo-holomorphic disks in a symplectic manifold with boundary condition on Lagrangian submanifolds. Heegaard Floer homology uses the Heegaard Diagram as the source of the data needed for Floer Homology, the symplectic manifold is the g-fold symmetric product of the Heegaard surface, the totally real tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ play the role of Lagrangian submanifolds and the holomorphic disks are Whitney disks connecting two elements $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. We construct a chain complex with boundary counting pseudo-holomorphic Whitney disks. The homology obtained is then a diffeomorphism invariant of 3-manifolds which is the Heegaard Floer Homology. The result is independent of the choice of analytic data: Riemann metric, Morse function, almost complex and symplectic structure etc., so we have a topological invariant. However we will not discuss this issue of invariance.

Now, we give a short description of Heegaard Floer theory with some "recent" results.

Let Y be a 3-manifold with pointed Heegaard diagram $D = (\Sigma_g, \alpha, \beta, z)$. We define a complex $\widehat{CF}(D)$ over $\mathbb{Z}/\mathbb{Z}_2$ (resp. $\mathbb{Z}$) freely generated by intersection points of the torus $\mathbb{T}_\alpha = [\alpha_1 \times \cdots \times \alpha_g]$ and $\mathbb{T}_\beta = [\beta_1 \times \cdots \times \beta_g]$ [1] which "play" the role of Lagrangian submanifold in the symmetric product $\mathrm{Sym}^g(\Sigma_g) = \Sigma_g^{\times g}/\mathfrak{S}_g$.

Now we can think of $\mathcal{M}(\phi)$ and $\widehat{\mathcal{M}}(\phi)$ as pseudo-holomorphic representatives with respect to a certain almost complex structure J on $\mathrm{Sym}^g(\Sigma_g)$.

We define a boundary operator on the complex by

$$\widehat{\partial}(x) = \sum \sharp\widehat{\mathcal{M}}(\phi) \, . \, y$$

where the summation runs over all $\phi \in \pi_2(x,y)$, $y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\mu(\phi) = 1$ and $n_z(\phi) = 0$.

Theorem 6.20 (Ozsváth-Szabó). *For appropriate choice of analytic data, $\widehat{\partial}^2 = 0$.*

For integer coefficients we need a choice of coherent orientation to allow us to count the points with sign.

We have then a chain complex and can take homology.

Theorem 6.21 (Ozsváth-Szabó). *$H_*\left(\widehat{CF}(D), \widehat{\partial}\right) = \widehat{HF}(Y)$ is a diffeomorphism invariant for Y.*

A refinement of this homology is the one obtained by taking $\mathbb{Z}/2\mathbb{Z}[U]$ (resp. $\mathbb{Z}[U]$) as ring of coefficients. The complex obtained is denoted CF^-. For $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $d \in \mathbb{Z}$ we define the space $\mathcal{M}(x,y,d)$ to be the set of holomorphic representative of Whitney disk ϕ connecting x to y with $n_z(\phi) = d$. The holomorphic assumption insures that $d \geq 0$. $\mathcal{M}(x,y,d)$ also admit an $\mathbb{R}$ action and the quotient space is denoted $\widehat{\mathcal{M}}(x,y,d)$.

[1] $\alpha = \{\alpha_1, \cdots, \alpha_g\}$ and $\beta = \{\beta_1, \cdots, \beta_g\}$

The boundary for this new complex is then defined by:

$$\partial^{-}(x) = \sum_{y \in \mathbf{T}_\alpha \cap \mathbf{T}_\beta} \sum_{d \in \mathbf{Z}} n_{xy}(d)\ U^d \cdot y$$

where

$$n_{xy}(d) = \begin{cases} 0 \text{ if } \dim \mathcal{M}(x, y, d) > 1 \\ \\ \sharp\widehat{\mathcal{M}}(x, y, d) \quad \text{otherwise} \end{cases}$$

This definition of ∂^{-} only make sense for diagrams for which there are only finitely many y with $n_{xy} \neq 0$ for a given x. However Sarkar and Wang prove that every closed and oriented 3-manifold admit such diagrams which are called admissible.

Theorem 6.22. *Every 3-manifold admit a Heegaard diagram for which $\partial^{-2} = 0$.*

The homology obtained $H_*(CF^-, \partial^-) = HF^-(Y)$ is also an invariant for 3-manifolds.

We can also define $HF^-(Y, K)$ for a knot $K \subset Y$. We use a two pointed Heegaard diagram instead of one.

Now we state the results which relate to combinatorial descriptions.

Theorem 6.23 (Sarkar-Wang). *Every 3-manifold has a Heegaard diagram for which $\widehat{HF}(Y)$ can be computed combinatorially.*

Theorem 6.24 (Ozsváth-Szabó). *For every 3-manifold Y there exists a Heegaard diagram for which $HF^-(Y)/U^2$ and $HF^-(Y)/U^3$ can be computed combinatorially.*

Theorem 6.25 (Manolescu-Ozsváth-Sarkar). *The homology $HF^-(S^3, K)$ can be computed combinatorially for knot K in S^3.*

Theorem 6.26 (Manolescu-Ozsváth-Sarkar-Thurston). *The homology $HF^-(S^3, K)$ can be defined purely combinatorially for knot K in S^3.*

In the perspective of trying to understand how we can go from an analytic setting to a purely combinatorial one, in the next chapter we will focus on how the choice of special types of domains like bigons and squares may give part of an elucidation of the fact that holomorphic rigidity can lead to combinatorial descriptions. For this purpose we will consider only holomorphic disks and we think of the symmetric product $\mathrm{Sym}^g(\Sigma_g)$ as equipped with the complex structure discussed in chapter 5.

Chapter 7

Holomorphic Domains

7.1 Preliminary

Let us consider a continuous map $u : \mathbb{D} \longrightarrow \mathrm{Sym}^g\Sigma$. We define $\widehat{D} \subset \mathbb{D} \times \Sigma$ by

$$\widehat{D} = \{(z,w) \in \mathbb{D} \times \Sigma \mid w \in \{w_1, \cdots, w_g\} \text{ with } u(z) = w_1 + \cdots + w_g\}$$

Then we have the natural maps $P : \widehat{D} \longrightarrow \mathbb{D}$ and $\hat{u} : \widehat{D} \longrightarrow \Sigma$ which are the restrictions of the first and the second projection respectively. It can be checked that P is a g-fold branched cover of $\mathbb{D}$ if the map u is away from the diagonal except for discrete number of points, which is the case for our Whitney disks. We have the fact that

$$u(z) = \hat{u}(z, w_1) + \cdots + \hat{u}(z, w_g)$$

where $\{(z, w_1), \cdots, (z, w_g)\} = P^{-1}(z)$, some w_i's may be the same according to multiplicity. Conversely a g-fold branched cover $P : \widehat{D} \longrightarrow \mathbb{D}$ and a map $\hat{u} : \widehat{D} \longrightarrow \Sigma$ determine uniquely a map u from the disk to the symmetric product.

We get the following property from the above construction and the definition of intersection number.

Proposition 7.1. *For u holomorphic, if $w \in \Sigma$ such that $u \pitchfork V_w$, then*

$$n_w(u) = \sharp\hat{u}^{-1}(w).$$

The holomorphic hypothesis is just to have the sign of the intersection points positive. For only smooth u we must pay attention to signs.

Proof.

$$n_w(u) = \sum_{q \in u^{-1}(V_w)} sign(q)$$

but since the signs are positive due to the holomorphic hypothesis, it is just the number of elements in the preimage. By the same reason $\sharp\hat{u}^{-1}(w)$ is just the number of preimages of w. Now, by definition of $\widehat{D}$ and $\hat{u}$, element of $\hat{u}^{-1}(w)$ are the ordered pair (z, w) for which $u(z) \in V_w$. This proves the desired equality. □

The study of maps $u : \mathbb{D} \longrightarrow \mathrm{Sym}^g\Sigma$ can then be reduced to the study of $\hat{u}$ and the knowledge of the g-fold branched cover $\widehat{D}$ of $\mathbb{D}$.

$\widehat{D}$ may be disconnected but it may happen that there is only one component for which the restriction of $\hat{u}$ is not a constant map; this is the most interesting case for our purpose. A particular care will be taken when this component is a topological disk.

A *non trivial component* is a component of $\widehat{D}$ for which the restriction of $\hat{u}$ is not a constant map.

Let us suppose that $\widehat{D}$ has only one non trivial component which can be chosen to be a disk and which we still denote by $\widehat{D}$. Let X be the universal covering of the Riemann surface Σ and π the natural covering map from X onto Σ. Let $z \in \widehat{D}$, $b = \hat{u}(z)$ and $w \in \pi^{-1}(b)$. Since $\widehat{D}$ is simply connected we have a unique lift $\tilde{u} : \widehat{D} \longrightarrow X$ of $\hat{u}$ such that $\tilde{u}(z) = w$. In particular if the map $\hat{u}$ is injective then $\tilde{u}$ is also injective.
The fact that the universal cover X is either $\mathbb{C}$, $\mathbb{D}$ or $\mathbb{CP}^1$ allows us to completely reduce the study of $\hat{u}$ to the study of a map from the unit disk $\mathbb{D}$ to the complex plane. Because we are interested in the case where u is holomorphic, our work will concentrate on holomorphic maps from the unit disk in the standard sense using usual tools of complex analysis and holomorphic branched covers of the unit disk. In this perspective an m-gon in Σ will be treated as an m-gon in $\mathbb{C}$.

After this preliminary discussion let us now start with the study of holomorphic representatives of the domains defined in the previous chapter. From our discussion this study is reduced to the study of domains in the plane or precisely of m-gons in the plane for nice domains.

7.2 Domain with Holomorphic Representative

Proposition 7.2. *For $w \in \Sigma$ not a branched point for $\hat{u}$ and such that none of $z \in \hat{u}^{-1}(w)$ are ramification points for P, u is transverse to V_w.*

Proof. We have to prove that for every $z' \in u^{-1}(V_w)$

$$Du_{z'}(T_{z'}\mathbb{D}) \oplus T_{u(z')}V_w = T_{u(z')}\mathrm{Sym}^g\Sigma.$$

By construction of P and $\hat{u}$, an element $z' \in u^{-1}(V_w)$ has to be of the form $P(z)$ where $z \in \hat{u}^{-1}(w)$. By assumption z' is not a branched point of P. Therefore we can find an open neighbourhood U of z', g pairwise disjoint open sets $U_1, \cdots, U_g$ in $\widehat{D}$ and maps $\hat{u}_k : U_k \longrightarrow \Sigma$, $k = 1, \cdots, g$, such that the restriction $P : U_k \longrightarrow U$ and the map u_k are diffeomorphisms for each k, and the restriction of $\hat{u}$ to $U_1 \amalg \cdots \amalg U_k$ is the map $\hat{u}_1 \amalg \cdots \amalg \hat{u}_g$. The fact that the U_k's are pairwise disjoint implies that the restriction of u to U is away from the diagonal. Therefore, since we can always choose U to be simply connected (a small disk for instance) and since the natural projection $\Sigma^g \longrightarrow \mathrm{Sym}^g\Sigma$ is a covering map outside the diagonal, we have a lift $\bar{u} : U \longrightarrow \Sigma^g$ up to a choice of ordering.

Now, let us pick a basis (a, b) of $T_{z'}U$, for each $k = 1, \cdots, g$ we lift this basis to a basis (a_k, b_k) of $T_{z_k}U_k$ via P and then to a basis (c_k, d_k) of $T_{\hat{u}_k(z_k)}\Sigma$ via $\hat{u}_k$, where $z_k \in P^{-1}(z')$. By construction of $\hat{u}$ and P, we get :

$$D\bar{u}_{z'}(a) = (c_1 \cdots c_g) = c$$

$$D\bar{u}_{z'}(b) = (d_1 \cdots d_g) = d$$

Since each pair c_k, d_k are linearly independent, c and d are linearly independent. If we chose the ordering such that w is the first component of $u(z')$, then

$$T_{\bar{u}(z')}w \times \Sigma^{(g-1)} = 0 \times T_{\hat{u}_2(z_2)}\Sigma \times \cdots \times T_{\hat{u}_g(z_g)}\Sigma$$

Let $p = (\hat{u}_2, \cdots, \hat{u}_g)$, we have

$$\langle c, d \rangle \oplus 0 \times T_p\Sigma^{(g-1)} = T_{\bar{u}(z')}\Sigma^g.$$

The natural projection $\Sigma^g \longrightarrow \mathrm{Sym}^g\Sigma$ is a local diffeomorphism outside the diagonal therefore the above equality implies

$$Du_{z'}(T_{z'}\mathbb{D}) \oplus T_{u(z')}V_w = T_{u(z')}\mathrm{Sym}^g\Sigma.$$

□

Theorem 7.3. *Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\phi \in \pi_2(x, y)$. Suppose ϕ has an holomorphic representative u for which the corresponding $\widehat{D}$ has only one non trivial component. If the coefficients involved in $\mathcal{D}(\phi)$ are only 0 or 1, then $\hat{u}$ is an embedding on the non trivial component.*

Proof. Let $w \in \Sigma$, such that $u \pitchfork V_w$, we have $n_w(\phi) = \sharp\hat{u}^{-1}(w) = 1$, where

$$\sharp\hat{u}^{-1}(w) = \sum_{q \in \hat{u}^{-1}(w)} sign(q) \ .$$

Since ϕ has an holomorphic representative $sign(q) \geq 0$. Therefore $\hat{u}^{-1}(w)$ has only one element and this element must belong to the non trivial component. Proposition 7.2 tells us that this holds for all except maybe for finitely many $w \in \Sigma$, the exception will be for the images of the trivial components. Therefore $\hat{u}$ is injective on the non trivial component, hence it is an embedding. □

This theorem is a consequence of proposition 7.2. For domains with coefficient only 0 or 1, this theorem with the discussion in the preliminary of this chapter allow us to consider only biholomorphic maps between domains in $\mathbb{C}$, *i.e* Riemann maps for suitably nice domains.

Proposition 7.4. *Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. If $\mathcal{D}$ is the domain of a homotopy class $\phi \in \pi_2(x, y)$ with holomorphic representative u, then $\mathcal{D}$ is the image under $\hat{u}$ of the non-trivial components of $\widehat{D}$.*

Proof. This is directly related to the fact that $n_w(\phi) = \sharp\hat{u}^{-1}(w)$. Since $n_w(\phi)$ is the multiplicity of the domain containing w and this is non zero if and only if $\sharp\hat{u}^{-1}(w)$ is non zero, the image of $\hat{u}$ must be $\mathcal{D}$. □

Thus, if the domain $\mathcal{D}(\phi)$ is a $2m$-gon (with multiplicity 0 or 1 in all the component), we have to look for an holomorphic map $\hat{u} : \widehat{D} \longrightarrow \Sigma$ with the given $2m$-gon as image. We are interested in the existence of such holomorphic maps and such that the corresponding moduli space $\mathcal{M}(\phi)$ is relevant for computing the boundary map in Heegaard Floer homology. We are interested in the cases when $m = 1$ and 2 *i.e* bigons and squares.

We are now thinking of $\hat{u}$ as a map to the complex plane instead of Σ. From earlier discussion we know that $\hat{u}$ has to be an embedding, so an holomorphic $\hat{u}$ needs to be a Riemann map. We are interested in the set $\mathcal{M}(\phi)$ for the two cases of bigons and squares. How do we describe all of its elements, is it a smooth manifold and of which dimension, and how about $\widehat{\mathcal{M}}(\phi)$? We will not do a completely exhaustive treatment but will concentrate on some cases which are the most relevant. The remaining sections will be devoted to these questions.

7.3 Assumption on Boundaries

Before doing the bigon case let us discuss an important point which is relevant for both the square case and the bigon case.

Definition 7.5. *A domain D of the complex plane is called finitely connected along its boundary if for every point $z \in \partial D$ and every $r > 0$, there exists $\epsilon \in (0, r)$ such that $D \cap D(z, \epsilon)$ intersects at most finitely many components of $D \cap D(z, r)$.*

We have the following important theorem which gives a necessary and sufficient condition for extending a conformal map continuously over the boundary. This theorem is stated and proved in [P.P91] page 441.

Theorem 7.6. *Let f be a conformal mapping of the open unit disk $\mathbb{D}$ onto a domain D in $\mathbb{C}$. Then f can be extended to a continuous mapping $\tilde{f}$ of $\bar{\mathbb{D}}$ onto $\bar{D}$ if and only if D is finitely connected along its boundary.*

By the above theorem, if a square or a bigon domain is not finitely connected along its boundary then no Riemann map which maps onto this domain can be extended to the boundary. Thus the square or the bigon made no contribution for the computation of the boundary map since they cannot represent holomorphic Whitney disks.

Due to this fact we will always assume that our domains are finitely connected along their boundary.

7.4 Bigon Case

Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$,

$$x = x_1 + x_2 + \cdots + x_g$$

$$y = y_1 + x_2 + \cdots + x_g$$

where $x_2, \cdots, x_g$ are $g-1$ distinct points in Σ.

Let $\mathcal{D}$ be a domain which is a bigon with x_1 and y_1 as vertices and the union of two arcs α and β as boundary[1]. We want to construct a holomorphic Whitney disk u with domain $\mathcal{D}$.

[1]We still use the notation α, β for the arcs on Σ even if before we were thinking of α and β as collection of curves $\{\alpha_1, \cdots, \alpha_g\}$ and $\{\beta_1, \cdots, \beta_g\}$

We choose the cover $\widehat{D}$ to be the disjoint union of g copies of $\mathbb{D}$. By the Riemann mapping theorem we can find a biholomorphic map $f : \mathbb{D} \longrightarrow \mathcal{D}$. By Theorem 7.6 this Riemann map extends continuously over the boundary since we assume that all the domains we are considering are finitely connected along their boundaries. By doing a reparametrization of the unit disk if necessary we can assume that $-i$ is mapped to x, i is mapped to y, S_+ is mapped inside α and S_- is mapped inside β.

Remark 7.7. *All other maps satisfying these properties can be obtained by pre-composition with automorphisms of $\mathbb{D}$ which fix $-i$, i and send the point we have fixed in S_+ to a point in α. There is exactly a 1-parameter family of such automorphisms.*

Here the g-fold branched cover $P : \widehat{D} \longrightarrow \mathbb{D}$ is just the g-disjoint union of the identity map of $\mathbb{D}$:

$$P = \mathrm{Id}_{\mathbb{D}} \amalg \cdots \amalg \mathrm{Id}_{\mathbb{D}}$$

We define the map $\hat{u}$ to be

$$\hat{u} \;=\; f \;\amalg\; \hat{u}_2 \;\amalg \cdots \amalg\; \hat{u}_g$$

where $\hat{u}_k$ is a constant map mapping $\mathbb{D}$ to x_k for $k = 2, \cdots, g$. Now define $u : \mathbb{D} \longrightarrow \mathrm{Sym}^g\Sigma$ by:

$$u(z) = f(z) + x_2 + \cdots + x_g$$

Since f is a biholomorphism it is immediate that u is holomorphic. We have then proved the following:

Theorem 7.8. *For a bigon domain $\mathcal{D}$, which is finitely connected along its boundary, there exists an holomorphic representative and all holomorphic representatives are obtained by the above construction.*

Proof. The existence follows from the above discussion. We will just clarify that if we have an holomorphic representative u, then u can be obtained by the above manner. Let u be an holomorphic representative of $\mathcal{D}$. The multiplicity of u at every point w inside $\mathcal{D}$ is equal to one since $\mathcal{D}$ is a bigon. Therefore for each $w \in \mathcal{D}$ the identity

$$n_w(u) = \sharp\hat{u}^{-1}(w)$$

implies

$$\sharp \hat{u}^{-1}(w) = 1.$$

Thus $\hat{u}$ must be an holomorphic embedding. Then the preimage $\hat{u}^{-1}(\mathcal{D})$ of $\mathcal{D}$ under $\hat{u}$ has to be conformally equivalent to the unit disk. We chose $\hat{u}^{-1}(\mathcal{D})$ to be the non trivial component and we identify it with $\mathbb{D}$. Then we can consider the restriction of $\hat{u}$ on $\hat{u}^{-1}(\mathcal{D})$ as a Riemann map $f : \mathbb{D} \longrightarrow \mathcal{D}$. It follows that $\hat{u}$ can be written as

$$\hat{u} = f \amalg \hat{u}_2 \amalg \cdots \amalg \hat{u}_g$$

and that u satisfies for every $z \in \mathbb{D}$

$$u(z) = f(z) + x_2 + \cdots + x_g.$$

□

Definition 7.9. *We call an m-gon convex if it has at all its vertices's an interior angle less than π.*

For a non convex domain we can choose a Riemann map f with image not necessarily the interior of the domain but its interior minus some portion of the α (resp. β) arcs which start at a vertex with interior angle greater than π and goes inside the domain. An example of this situation is pictured by the figure below:

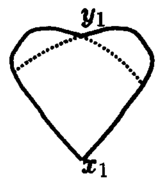


Figure 7.1: heart shaped domain

Therefore for each possible region (image of f) there is one unparameterized Riemann map.

7.4.1 Convex Bigons

Now suppose that we have a convex bigon as domain. The fact that the subset of $\mathrm{Aut}(\mathbb{D}) = \mathrm{PSL}(2,\mathbb{R})$ which has the properties mentioned in remark 7.7 forms a one dimensional smooth manifold implies that $\mathcal{M}(u)$ is a one dimensional smooth manifold as well. Thus we can state the following.

Theorem 7.10. *Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$,*

$$x = x_1 + x_2 + \cdots + x_g$$

$$y = y_1 + x_2 + \cdots + x_g$$

If $\mathcal{B}$ is a domain which is a convex bigon with vertices x_1 and y_1, then there is an holomorphic Whitney disk u representing $\mathcal{B}$ and $\mathcal{M}(u)$ is a one dimensional smooth manifold.

Proposition 7.11. *With the hypothesis of Theorem 7.10, $\widehat{\mathcal{M}}(u)$ has only one element.*

Proof. This is also a direct consequence of remark 7.7. □

7.4.2 Non Convex Bigons

In this subsection we want to justify that some types of non convex bigons are not relevant for calculating the Heegaard Floer boundary map even if they always have an holomorphic representative. We will argue that the set $\mathcal{M}(\phi)$ cannot be a one dimensional smooth manifold. Let us consider first the simple case of figure 7.1

Proposition 7.12. *Let ϕ be a representative of the domain in figure 7.1. $\widehat{\mathcal{M}}(\phi)$ is homeomorphic to an open interval, and therefore $\mathcal{M}(\phi)$ is not a one dimensional manifold.*

Proof. $\widehat{\mathcal{M}}(\phi)$ is the set of unparameterized curves. So its elements can be distinguished by their image meaning that for $[u], [u'] \in \widehat{\mathcal{M}}(\phi)$,

$$[u] \neq [u'] \text{ if and only if } u(\mathbb{D}) \neq u'(\mathbb{D})$$

Let $f : \mathbb{D} \longrightarrow \mathcal{D}$ (resp. $f : \mathbb{D} \longrightarrow \mathcal{D}'$) be the holomorphic map which appear in the definition of u (resp. u') via $\widehat{D}$ and $\hat{u}$ (resp. $\hat{u}'$). The equivalence above implies that

$$[u] \neq [u'] \text{ if and only if } f(\mathbb{D}) \neq f'(\mathbb{D})$$

by the construction of u (resp. u') discussed in the beginning of the section.

Now the choice of an image $f(\mathbb{D})$ is completely determined by the endpoint of the arcs (α or β) which goes from one vertex and end up inside the domain as shown in the figure:

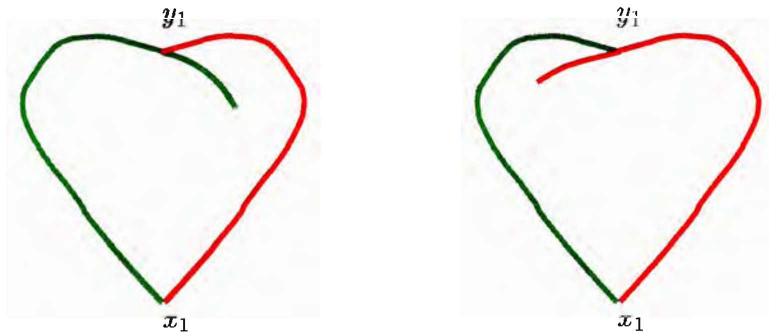
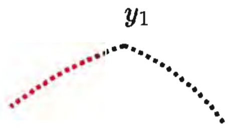


Figure 7.2: Two types of images for the map f

Note that we cannot have both arcs α and β going inside the domain since we require $f(S_+) \subset \alpha$ and $f(S_-) \subset \beta$. The set of all possible end points is the open arc inside the domain in figure 7.1



which is homeomorphic to an open interval. The Riemann mapping theorem tells us that we can always find such an holomorphic f since the regions we are considering here for each point of the "interval" are all simply connected. The Theorem 7.6 tells us that this map indeed extends to the boundary of the region and with the desired properties. Thus $\widehat{\mathcal{M}}(\phi)$ is homeomorphic to an open interval.

Since for each unparameterized map $[u]$ we have a one parameter family of parametrized ones by pre-composing with automorphisms of $\mathbb{D}$ preserving i and $-i$, we have implicitly shown the existence of a continuous map:

$$\begin{array}{c}\mathcal{M}(\phi)\\ \big\downarrow\\ \mathcal{I}\end{array}$$

where $\mathcal{I}$ is an open interval and such that each fibre is a one dimensional manifold. Therefore $\mathcal{M}(\phi)$ cannot be a one dimensional manifold.

□

There are two simple types of non-convex bigons for which $\mathcal{M}(\phi)$ cannot be a one dimensional manifold. The first one is shown in figure 7.1, the second one is shown in the following figure.

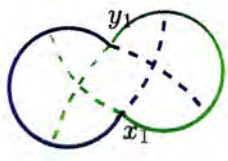


Figure 7.3: second type of non convex bigon

By analogous reasoning we can prove that for the second case $\mathcal{M}(\phi)$ cannot be a one dimensional manifold.

7.5 Square Case

As for the bigon case, we assume that our squares are finitely connected along their boundaries. We will distinguish between convex and non convex squares. The non convex squares are also not relevant for computing the boundary map in Heegaard Floer homology.

Let $\mathcal{S}$ be a domain which is a square with vertices v_1, v_2, v_3, v_4. Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that:

$$x = v_1 + v_2 + x_3 + \cdots + x_g$$

$$y = v_3 + v_4 + x_3 + \cdots + x_g$$

We are going to construct an holomorphic Whitney disk u that represents $\mathcal{S}$. We choose $\widehat{D}$ to be the disjoint union of $g-1$ copies of the unit disk $\mathbb{D}$:

$$\widehat{D} = \underbrace{\mathbb{D} \amalg \cdots \amalg \mathbb{D}}_{g-1}$$

We know that $\hat{u}$ must be an holomorphic embedding (*i.e* a Riemann map) restricted to the non trivial component of $\widehat{D}$ if there is only one. By the Riemann mapping theorem, we can find a biholomorphic map $f : \mathbb{D} \longrightarrow \mathcal{S}$, so the map $\hat{u}$ is defined to be :

$$f \amalg \hat{u}_3 \amalg \cdots \amalg \hat{u}_g$$

where u_k is a constant map mapping $\mathbb{D}$ to x_k for each $k = 3, \cdots, g$. To complete the data for the definition of u we need to specify the g-fold branched cover $P : \widehat{D} \longrightarrow \mathbb{D}$. We define

$$P = B \amalg \underbrace{\mathrm{Id}_{\mathbb{D}} \amalg \cdots \amalg \mathrm{Id}_{\mathbb{D}}}_{g-2}$$

where B is a 2-fold holomorphic branched cover of $\mathbb{D}$ by itself which maps the preimage under $\hat{u}$ of the vertices to $-i$ and i. In other words, if $z_1,\ z_2,\ z_3,\ z_4$ are the preimages of $v_1,\ v_2,\ _3,\ v_4$, then:

$$\begin{aligned} B(z_1) &= -i \\ B(z_2) &= -i \\ B(z_3) &= i \\ B(z_4) &= i \end{aligned} \tag{7.1}$$

Remark 7.13. *It is known from complex analysis that the 2-fold holomorphic branched cover space of the unit disk is the unit disk. That is why we have chosen $g-1$ disjoint union of $\mathbb{D}$ in our definition of $\widehat{D}$ for the square case.*

To continue further the study we need to determine all the 2-fold holomorphic branched covers of $\mathbb{D}$ by itself.

7.5.1 Holomorphic 2-fold Branched Cover of $\mathbb{D}$ by Itself

Any proper holomorphic map is a branched cover and a branched cover is proper if and only if it has finite degree.

Since we are looking for 2-fold holomorphic branched covers, it suffices to consider proper holomorphic maps from $\mathbb{D}$ to itself. By analogy with the fact that a map $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is proper if and only if $||g(x)|| \to \infty$ as $||x|| \to \infty$, we have the following characterization.

Proposition 7.14. *Let $g : \mathbb{D} \longrightarrow \mathbb{D}$ be a continuous function. g is proper if and only if $|g(z)| \to 1$ as $|z| \to 1$.*

Proof. $\diamond$ Let us suppose that g is a proper map. Let $\varepsilon < 1$, $g^{-1}\left(\bar{D}(0,\varepsilon)\right)$ is compact in $\mathbb{D}$ then there exists $\delta < 1$ such that $g^{-1}\left(\bar{D}(0,\varepsilon)\right) \subset \bar{D}(0,\delta)$. Therefore, $|z| > \delta$ implies $|g(z)| > \varepsilon$. Thus $|g(z)| \to 1$ as $|z| \to 1$.

- Conversely suppose that $|g(z)| \longrightarrow 1$ as $|z| \longrightarrow 1$. Let $K \subset \mathbb{D}$ be compact. There is $\varepsilon < 1$ such that $K \subset \bar{D}(0,\varepsilon)$. By the limit hypothesis we can find $\delta < 1$ such that $g^{-1}(K) \subset \bar{D}(0,\delta)$. Finally since g is continuous $g^{-1}(K)$ is closed in $\mathbb{D}$.

□

Lemma 7.15. *If $g : \mathbb{D} \longrightarrow \mathbb{D}$ is a proper holomorphic map, then the zeros of g are finite.*

Proof. Zeros of holomorphic maps are isolated and since g is compact they form a compact subset, therefore they are finite. □

Let $g : \mathbb{D} \longrightarrow \mathbb{D}$ be a proper holomorphic map. Let $a_1, \cdots, a_n$ be the zeros of g counted with multiplicity. Let us consider the map:

$$h(z) = \prod_{k=1}^{n} \frac{z - a_k}{1 - \bar{a}_k z}$$

It is immediate that $|h(z)| \longrightarrow 1$ as $z \longrightarrow 1$ since the modulus of each term tends to 1 as $z \longrightarrow 1$. The singularities of h/g and g/h are removable, so they can be extended to holomorphic maps on the unit disk. In addition $|h/g|$ (resp.$|g/h|$) tends to 1 as $|z| \longrightarrow 1$, so according to the maximum principle:

$$|\frac{g}{h}| \leq 1 \quad \text{and} \quad |\frac{h}{g}| \leq 1$$

hence

$$|\frac{g}{h}| = 1 \quad \text{and} \quad g = \zeta\, h \quad \text{for} \quad \zeta \in S^1$$

Thus we can state the following theorem.

Theorem 7.16. *A n-fold holomorphic branched cover of the unit disk $\mathbb{D}$ by itself is of the form:*

$$z \mapsto \zeta \prod_{k=1}^{n} \frac{z - a_k}{1 - \bar{a}_k z}$$

where $\zeta \in S^1$ *and* $a_k \in \mathbb{D}$ *for* $k = 1, \cdots, n$.

Such a map is known as a finite *Blaschke product* and the number n is called the *degree* of the product.

Proposition 7.17. *Let $m \in \mathrm{Aut}(\mathbb{D})$ and B a finite Blaschke product. Then $m \circ B$ and $B \circ m$ are Blaschke product of the same degree.*

Proof. $B \circ m$ and $m \circ B$ are still proper holomorphic maps from $\mathbb{D}$ to itself, so they are Blaschke products. Composing with automorphisms of $\mathbb{D}$ does not change the number of zeros so they have the same degree as B. □

A 2-fold holomorphic branched cover of $\mathbb{D}$ is then a Blaschke product of the form:

$$B(z) = \zeta \frac{z - a_1}{1 - \bar{a_1} z} \frac{z - a_2}{1 - \bar{a_2} z}$$

where $\zeta \in S^1$ and $a_1,\ a_2 \in \mathbb{D}$. The relations

$$B(z_1) = -i$$
$$B(z_2) = -i$$
$$B(z_3) = i$$
$$B(z_4) = i$$

give 4 complex equations of 5 real variables (imaginary and real part of a_1, a_2 and argument of ζ). We may expect 8 real equations of 5 variables but since the relations take place on the unit circle we have exactly 4 relations on 5 unknown. In particular a 5-th independent relation will determine B uniquely.

7.5.2 The Set of Holomorphic Representatives

Now an holomorphic representative u for the square domain S can be defined as in the preliminary since we have clearly defined $\widehat{D}$, $\hat{u}$ and P. More precisely we have the following:

$$u(z) = f(z_1) + f(z_2) + x_3 + \cdots + x_g$$

where $\{z_1, z_2\} = B^{-1}(z)$, with B being a degree 2 Blaschke product satisfying equation (7.1) and $f : \mathbb{D} \longrightarrow \Sigma^1$ a Riemann map.

We summarize this in this theorem which is the same as for the bigon case but the proof is not as trivial as for this case.

Theorem 7.18. *For a square domain S there exists holomorphic representatives and all holomorphic representatives are obtained from the above construction.*

[1]we are putting Σ but we are allowed to use usual Riemann map between regions of $\mathbb{C}$

Proof. It suffices to prove that the map u defined above is indeed holomorphic. Let $z \in \mathbb{D}$ not a branched point for B, and $\{z_1, z_2\} = B^{-1}(z)$. We can find open neighbourhood U_1 of z_1 and U_2 of z_2 such that:

$$U_1 \cap U_2 = \emptyset \quad \text{and} \quad B_{|U_1} \text{ (resp. } B_{U_2}) \text{ is a biholomorphism on its image}$$

on the other hand since $x_3, \cdots, x_g$ are all distinct, we can find pairwise disjoint open neighbourhood $U_3, \cdots, U_g$ of them which are disjoint from U_1 and U_2 as well. Therefore we can find an open neighbourhood U of z such that u factorizes locally as shown by the commutative diagram:

$$\begin{tikzcd} & U_1 \times \cdots \times U_g \arrow[d, "\pi"] \\ U \arrow[ur, "\tilde{u}"] \arrow[r, "u"'] & \mathrm{Sym}^g\Sigma \end{tikzcd}$$

where π is the canonical projection which is holomorphic from Chapter 5 and $\tilde{u}$ is the map defined by:

$$\tilde{u}(w) = \left(B_{|U_1}(w),\ B_{|U_2}(w),\ x_3,\ \cdots,\ x_g\right)$$

which is holomorphic. Therefore u is holomorphic at z.

We have proved that u is holomorphic on $\mathbb{D}$ minus the branch points. Since u is continuous and there is only one branched point, by analytic continuation u is holomorphic on the entire disk $\mathbb{D}$. $\square$

Let us assume that $\mathcal{S}$ is a convex square so that the image of f for a given u is the entire interior of $\mathcal{S}$.

Let u' be another holomorphic representative of $\mathcal{S}$. We have:

$$u'(z) = g(s_1) + g(s_2) + x_3 + \cdots + x_g$$

with $\{s_1, s_2\} = B'^{-1}(z)$, B' and g having the same properties as B and f.

We are going to prove that $u = u' \circ \rho$ for some $\rho \in \mathrm{Aut}(\mathbb{D})$ fixing i and $-i$. For, we express B in term of B'. Let a, b, c, d (resp. a', b', c', d') be the inverse images of the four vertices v_1, v_2, v_3, v_4 of $\mathcal{S}$ by f (resp. g) and let w be a point in the boundary of $\mathbb{D}$ different from a, b, c, d.

$g^{-1} \circ f$ is an automorphism of $\mathbb{D}$ mapping a, b, c, d to a', b', c', d'. $B' \circ g^{-1} \circ f$ is a Blaschke product from proposition 7.17 and it satisfies (7.1). There is one automorphism ρ^{-1} of $\mathbb{D}$ mapping

$B' \circ g^{-1} \circ f(w)$ to $B(w)$ and fixing i, $-i$. The composition $\rho^{-1} \circ B' \circ g^{-1} \circ f$ is again a Blaschke product from proposition 7.17, and in addition it maps a, b, c, d to $-i, i$ and w to $B(w)$ so we have 5 independent relations. Since a Blaschke product is uniquely determined by 5 parameters there is only one which satisfies this property. B satisfies the same relations so we must have:

$$B = \rho^{-1} \circ B' \circ g^{-1} \circ f \tag{7.2}$$

Let $z \in \mathbb{D}$,

$$u' \circ \rho(z) = g(t_1) + g(t_2) + x_3 + \cdots + x_g$$

where $\{t_1, t_2\} = B'^{-1}(\rho(z))$

On the other hand

$$u(z) = f(z_1) + f(z_2) + x_3 + \cdots + x_g$$

with $\{z_1, z_2\} = B^{-1}(z)$. Using equation (7.2) we get

$$\{z_1, z_2\} = f^{-1}\left(g\left(B'^{-1}(\rho(z))\right)\right)$$

therefore

$$\begin{aligned} f(z_1) + f(z_2) &:= f(\{z_1, z_2\}) \\ &= f\left(f^{-1}\left(g\left(B'^{-1}(\rho(z))\right)\right)\right) \\ &= g\left(B'^{-1}(\rho(z))\right), \qquad \text{using the fact that } f \text{ is a bijection} \end{aligned}$$

but

$$g\left(B'^{-1}(\rho(z))\right) = g(\{t_1, t_2\}) \ = \{g(t_1), g(t_2)\} =: g(t_1) + g(t_2)$$

finally

$$\begin{aligned} u(z) &= g(t_1) + g(t_2) + x_3 + \cdots + x_g \\ &= u' \circ \rho(z) \end{aligned}$$

Consequently u and u' are in the same equivalence class of the $\mathbb{R}$ action on $\mathcal{M}(\phi)$ for a Whitney disk ϕ with domain $\mathcal{S}$. Thus we have proved the following

Theorem 7.19. *Let $\mathcal{S}$ be a convex square. If ϕ is a Whitney disk with domain $\mathcal{S}$, then $\widehat{\mathcal{M}}(\phi)$ has only one element.*

Smoothness of the Set of Holomorphic Representatives

Theorem 7.18 and 7.8 are just saying that $\mathcal{M}(\phi)$ is non empty for these cases. But knowing that it is non empty is not enough, it needs to be a smooth manifold with the correct dimension (one).

Theorem 7.20. *If the domain $\mathcal{S}$ is a convex square with holomorphic representative u, then $\mathcal{M}(u)$ is a one dimensional smooth manifold.*

Proof. What really matters in the definition of an holomorphic representative u for a convex square is the 2-fold branched covering G which is a degree 2 Blaschke product. This is because change of the Riemann map is equivalent to a change of the Blaschke product, which can be deduced from the proof of theorem 7.19. Therefore fixing an arbitrary Riemann map $f : \mathbb{D} \longrightarrow \mathcal{S}$, $\mathcal{M}(u)$ is completely determined by the set of all possible degree 2 Blaschke products B satisfying relation 7.1. This means that there is a one to one correspondence between $\mathcal{M}(u)$ and this set.

It suffices then to prove that the set of degree 2 Blaschke product

$$B(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z} \frac{z-b}{1-\bar{b}z}$$

satisfying relation 7.1 is a one dimensional smooth manifold. Let $p_1, p_2, q_1, q_2 \in \partial\mathbb{D}$ be the points which correspond to the vertices of $\mathcal{S}$, the preimages of $i, i, -i, -i$ under B.

Consider the map

$$\begin{aligned} G : S^1 \times \mathbb{D} \times \mathbb{D} &\longrightarrow S^1 \times S^1 \times S^1 \times S^1 \\ (\theta, a, b) &\longmapsto (B(p_1), B(p_2), B(q_1), B(q_2)) \end{aligned}$$

G is smooth and $G^{-1}(i, i, -i, -i)$ is the space of maps in one to one correspondence with $\mathcal{M}(u)$. In order to prove that it is a one dimensional manifold, we need to show that $\mathrm{d}G$ is surjective or has rank 4 at a preimage of $(i, i, -i, -i)$. To simplify the proof we chose to work for a map $G : S^1 \times \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{C}^4$, so we can express the differential with respect to a and b in terms of ∂_a, $\bar{\partial}_a$ and ∂_b, $\bar{\partial}_b$.

For a map

$$g(\theta, a, b) = e^{i\theta} \frac{z-a}{1-\bar{a}z} \frac{z-b}{1-\bar{b}z}, \text{ where } z \text{ is fixed}$$

we have

$$\begin{aligned}
\partial_a g &= -e^{i\theta} \frac{1}{1-\bar{a}z} \frac{z-b}{1-\bar{b}z} \\
\partial_b g &= -e^{i\theta} \frac{1}{1-\bar{b}z} \frac{z-a}{1-\bar{a}z} \\
\bar{\partial}_a g &= e^{i\theta} (z-a) \frac{z}{(1-\bar{a}z)^2} \frac{z-b}{1-\bar{b}z} \\
\bar{\partial}_b g &= e^{i\theta} (z-b) \frac{z}{\left(1-\bar{b}z\right)^2} \frac{z-a}{1-\bar{a}z}
\end{aligned}$$

using the relation

$$\begin{aligned}
B(p_1) = B(p_2) &= i \\
B(q_1) = B(q_2) &= -i
\end{aligned}$$

we get

$$\begin{aligned}
\frac{p_k-b}{1-\bar{b}p_k} &= ie^{-i\theta} \frac{1-\bar{a}p_k}{p_k-a} \\
\frac{q_k-b}{1-\bar{b}q_k} &= -ie^{-i\theta} \frac{1-\bar{a}q_k}{q_k-a}
\end{aligned}$$

for $k=1,2$. Let $x_1=\theta$, $x_2=a$, $x_3=\bar{a}$, $x_4=b$, $x_5=\bar{b}$, $G_1=B(p_1)$, $G_2=B(p_2)$, $G_3=B(q_1)$, $G_4=B(q_2)$.

The matrix of partial derivative

$$\mathsf{D}G = \left(\frac{\partial G_l}{\partial x_k}\right)_{l,k}$$

can then be written at a preimage t of $(i,i,-i,-i)$ as:

$$\mathsf{D}G_t = \begin{pmatrix}
-1 & \frac{-i}{p_1-a} & \frac{ip_1}{1-\bar{a}p_1} & \frac{-i}{p_1-b} & \frac{ip_1}{1-\bar{b}p_1} \\
-1 & \frac{-i}{p_2-a} & \frac{ip_2}{1-\bar{a}p_2} & \frac{-i}{p_2-b} & \frac{ip_2}{1-\bar{b}p_2} \\
1 & \frac{i}{q_1-a} & \frac{-iq_1}{1-\bar{a}q_1} & \frac{i}{q_1-b} & \frac{-iq_1}{1-\bar{b}q_1} \\
1 & \frac{i}{q_2-a} & \frac{-iq_2}{1-\bar{a}q_2} & \frac{i}{q_2-b} & \frac{-iq_2}{1-\bar{b}q_2}
\end{pmatrix}$$

We aim to prove that $\mathsf{D}G_t$ has rank 4 which is equivalent to proving that its transpose has rank 4. This transpose is given by:

$$
\begin{pmatrix}
-1 & -1 & 1 & 1 \\
\frac{-i}{p_1-a} & \frac{-i}{p_2-a} & \frac{i}{q_1-a} & \frac{i}{q_1-a} \\
\frac{ip_1}{1-\bar{a}p_1} & \frac{ip_2}{1-\bar{a}p_2} & \frac{-iq_1}{1-\bar{a}q_1} & \frac{-iq_2}{1-\bar{a}q_2} \\
\frac{-i}{p_1-b} & \frac{-i}{p_2-b} & \frac{i}{q_1-b} & \frac{i}{q_1-b} \\
\frac{ip_1}{1-bp_1} & \frac{ip_2}{1-bp_2} & \frac{-iq_1}{1-bq_1} & \frac{-iq_2}{1-bq_2}
\end{pmatrix}
$$

Let us denote C_k the k-th column of a matrix. After replacing

$$C_2 \text{ by } C_2 + C_3$$

$$C_3 \text{ by } C_1 + C_3$$

$$C_4 \text{ by } C_1 + C_4$$

we get the matrix

$$
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
\frac{-i}{p_1-a} & \frac{-i}{p_2-a} + \frac{i}{q_1-a} & \frac{-i}{p_1-a} + \frac{i}{q_1-a} & \frac{-i}{p_1-a} + \frac{i}{q_1-a} \\
\frac{ip_1}{1-\bar{a}p_1} & \frac{ip_2}{1-\bar{a}p_2} + \frac{-iq_1}{1-\bar{a}q_1} & \frac{ip_1}{1-\bar{a}p_1} + \frac{-iq_1}{1-\bar{a}q_1} & \frac{ip_1}{1-\bar{a}p_1} + \frac{-iq_2}{1-\bar{a}q_2} \\
\frac{-i}{p_1-b} & \frac{-i}{p_2-b} + \frac{i}{q_1-b} & \frac{-i}{p_1-b} + \frac{i}{q_1-b} & \frac{-i}{p_1-b} + \frac{i}{q_1-b} \\
\frac{ip_1}{1-bp_1} & \frac{ip_2}{1-bp_2} + \frac{-iq_1}{1-bq_1} & \frac{ip_1}{1-bp_1} + \frac{-iq_1}{1-bq_1} & \frac{ip_1}{1-bp_1} + \frac{-iq_2}{1-bq_2}
\end{pmatrix}
$$

and after simplification, the matrix obtained by removing the first row and the first column is:

$$
i\begin{pmatrix}
\frac{p_2-q_1}{(p_2-a)\,(q_1-a)} & \frac{p_1-q_1}{(p_1-a)\,(q_1-a)} & \frac{p_1-q_2}{(p_1-a)\,(q_2-a)} \\
\frac{p_2-q_1}{(1-\bar{a}p_2)\,(1-\bar{a}q_1)} & \frac{p_1-q_1}{(1-\bar{a}p_1)\,(1-\bar{a}q_1)} & \frac{p_12-q_2}{(1-\bar{a}p_1)\,(1-\bar{a}q_2)} \\
\frac{p_2-q_1}{(p_2-b)\,(q_1-b)} & \frac{p_1-q_1}{(p_1-b)\,(q_1-b)} & \frac{p_1-q_2}{(p_1-b)\,(q_2-b)} \\
\frac{p_2-q_1}{(1-\bar{b}p_2)\,(1-\bar{b}q_1)} & \frac{p_1-q_1}{(1-\bar{b}p_1)\,(1-\bar{b}q_1)} & \frac{p_12-q_2}{(1-\bar{b}p_1)\,(1-\bar{b}q_2)}
\end{pmatrix}
$$

It is enough to show that the matrix obtained from the above by taking out the i factor and by removing the last row has rank 3, *i.e* the matrix

$$
\begin{pmatrix}
\frac{p_2-q_1}{(p_2-a)\,(q_1-a)} & \frac{p_1-q_1}{(p_1-a)\,(q_1-a)} & \frac{p_1-q_2}{(p_1-a)\,(q_2-a)} \\
\frac{p_2-q_1}{(1-\bar{a}p_2)\,(1-\bar{a}q_1)} & \frac{p_1-q_1}{(1-\bar{a}p_1)\,(1-\bar{a}q_1)} & \frac{p_1-q_2}{(1-\bar{a}p_1)\,(1-\bar{a}q_2)} \\
\frac{p_2-q_1}{(p_2-b)\,(q_1-b)} & \frac{p_1-q_1}{(p_1-b)\,(q_1-b)} & \frac{p_1-q_2}{(p_1-b)\,(q_2-b)}
\end{pmatrix}
$$

Taking out the column factors $(p_2 - q_1)$, $(p_1 - q_1)$ and $(p_1 - q_2)$ does not change the rank. We are left with the matrix:

$$
\begin{pmatrix}
\frac{1}{(p_2-a)\,(q_1-a)} & \frac{1}{(p_1-a)\,(q_1-a)} & \frac{1}{(p_1-a)\,(q_2-a)} \\
\frac{1}{(1-\bar{a}p_2)\,(1-\bar{a}q_1)} & \frac{1}{(1-\bar{a}p_1)\,(1-\bar{a}q_1)} & \frac{1}{(1-\bar{a}p_1)\,(1-\bar{a}q_2)} \\
\frac{1}{(p_2-b)\,(q_1-b)} & \frac{1}{(p_1-b)\,(q_1-b)} & \frac{1}{(p_1-b)\,(q_2-b)}
\end{pmatrix}
$$

Replacing

$$C_3 \text{ by } (q_1 - a)\,C_2 - (q_2 - a)\,C_3$$
$$C_2 \text{ by } (p_2 - a)\,C_1 - (p_1 - a)\,C_2$$

yields

$$\begin{pmatrix} \square & 0 & 0 \\ \square & \frac{(p_2-a)}{(1-\bar{a}q_1)\,(1-\bar{a}p_2)} - \frac{(p_1-a)}{(1-\bar{a}p_1)\,(1-\bar{a}q_1)} & \frac{(q_1-a)}{(p_1-a)\,(q_2-a)} - \frac{(q_2-a)}{(1-\bar{a}p_1)\,(1-\bar{a}q_2)} \\ \square & \frac{(p_2-a)}{(p_2-b)\,(q_1-b)} - \frac{(p_1-a)}{(p_1-b)\,(q_1-b)} & \frac{(q_1-a)}{(p_1-b)\,(q_1-b)} - \frac{(q_2-a)}{(p_1-b)\,(q_2-b)} \end{pmatrix}$$

where we do not change the first column. Considering the 2 by 2 matrix on the right below the zeros and putting in the same denominator gives:

$$\begin{pmatrix} \frac{(1-a\bar{a})\,(p_2-p_1)}{(1-\bar{a}p_2)\,(1-\bar{a}p_1)\,(1-\bar{a}q_1)} & \frac{(1-a\bar{a})\,(q_1-q_2)}{(1-\bar{a}p_1)\,(1-\bar{a}q_1)\,(1-\bar{a}q_2)} \\ \frac{(a-b)\,(p_2-p_1)}{(p_2-b)\,(p_1-b)\,(q_1-b)} & \frac{(a-b)\,(q_1-q_2)}{(p_1-b)\,(q_1-b)\,(q_2-b)} \end{pmatrix}$$

After taking out row and column common factors we obtain:

$$\begin{pmatrix} \frac{1}{1-\bar{a}p_2} & \frac{1}{1-\bar{a}q_2} \\ \frac{1}{p_2-b} & \frac{1}{q_2-b} \end{pmatrix}$$

Replacing C_2 by $1/\,(1-\bar{a}q_2)\,C_1 - 1/\,(1-\bar{a}p_2)\,C_2$ gives:

$$\begin{pmatrix} \square & 0 \\ \square & \frac{1}{(1-\bar{a}q_2)\,(p_2-b)} - \frac{1}{(1-\bar{a}p_2)\,(q_2-b)} \end{pmatrix}$$

Then after putting in the same denominator:

$$\begin{pmatrix} \square & 0 \\ \square & \frac{(q_2-p_2)\,(1-\bar{a}b)}{(1-\bar{a}q_2)\,(1-\bar{a}p_2)\,(p_2-b)\,(q_2-b)} \end{pmatrix}$$

Since $a, b \in \mathbb{D}$ and p_2 and q_2 are distinct,

$$(q_2 - p_2)\;(1 - \bar{a}b) \neq 0$$

Thus this matrix has rank 2 and it follows that DG has rank 4 by basic results from linear algebra.

Since $S^1 \subset \mathbb{C}$ the map $G : S^1 \times \mathbb{D} \times \mathbb{D} \longrightarrow S^1 \times S^1 \times S^1 \times S^1$ has rank 4 at the desired point.

Therefore $G^{-1}(i, i, -i, -i)$ is a smooth one dimensional real manifold.

$\square$

Non Convex Squares

The case of non convex squares has some similarities with the case of non convex bigons. By the same argument as for bigons, with at least one non convex vertex the space $\mathcal{M}(\phi)$ may fail to be a one dimensional manifold. But one needs to make a more careful treatment which we will not do here.

7.6 The Correct Mathematical Setting

We made the choice of topology for $\mathcal{M}(\phi)$ in a very simple way. The special treatment we have applied probably does not generalize to similar problems like counting (pseudo-)holomorphic curves with other conditions and in other kinds of (almost) complex manifolds.

To understand the full features of $\mathcal{M}(\phi)$ we need to set up mathematical structures in which it fits nicely and naturally. First of all, holomorphic disks involve choices of complex structures which we have chosen so far to be the one from Chapter 5 for the symmetric product. In practice complex structures are too rigid since they can lose their "holomorphy" with small deformations. So using almost complex structures is more appropriate. In fact as we have seen earlier Heegaard Floer homology counts pseudo-holomorphic disks for a fixed almost complex structure. Not all almost complex structures are relevant for Floer homology. However the set of almost complex structures used in the theory form a generic set (set of second category in Baire sense), so a small perturbation of a given almost complex structure will give a correct one.

Let j be the complex structure on $\mathbb{D}$ and J the almost complex structure on $\mathrm{Sym}^g(\Sigma_g)$. We are then interested in Whitney J-holomorphic disks, *i.e* Whitney disks u with the property

$$\bar{\partial}_J u = 0$$

$$\text{where} \quad \bar{\partial}_J u = \frac{1}{2}\left(\mathrm{d}u + J \circ \mathrm{d}u \circ j\right) \quad \text{is the complex antilinear part of } \mathrm{d}u.$$

The operator $\bar{\partial}_J$ is naturally a map defined on the space of smooth functions $\mathcal{C}^\infty(\mathbb{D}, \mathrm{Sym}^g(\Sigma_g))$ which can be endowed with the C^∞-topology.
In our case we are interested in the subset $\mathcal{B} \subset \mathcal{C}^\infty(\mathbb{D}, \mathrm{Sym}^g(\Sigma_g))$ of the maps that represent the relative homology class $[\phi(\mathbb{D})] \in H_2(\mathrm{Sym}^g(\Sigma_g), \mathbb{T}_\alpha \cup \mathbb{T}_\beta)$ for $\phi \in \pi_2(x, y)$ and $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The space $\mathcal{B}$ has the structure of an infinite dimensional manifold. Its tangent space at a map u is the space of smooth vector fields along u

$$T_u\mathcal{B} = \Omega^0(\mathbb{D}, u^*T\mathrm{Sym}^g(\Sigma_g))$$

We can define over $\mathcal{B}$ an infinite dimensional fibre bundle $\mathcal{E} \longrightarrow \mathcal{B}$ with fibre over $u \in \mathcal{B}$

$$\mathcal{E}_u := \Omega^{0,1}(\mathbb{D}, u^*T\mathrm{Sym}^g(\Sigma_g))$$

the space of smooth J-anti-linear one form on $\mathbb{D}$ with value in $u^*T\mathrm{Sym}^g(\Sigma_g)$.

We can then think of the operator $\bar{\partial}_J$ as defining a section $\mathcal{S}$ of this bundle:

$$\mathcal{S} : u \in \mathcal{B} \longmapsto \left(u, \bar{\partial}_J u\right) \in \mathcal{E}_u$$

$\mathcal{M}(\phi)$ is therefore the zeros of this section. After some modifications on the spaces and the problem, one proves that the operator $\bar{\partial}_J$ is a Fredholm operator or more precisely it possesses a linearization which is a Fredholm operator. Then using the implicit function theorem for infinite dimensional manifolds one can prove that $\mathcal{M}(\phi)$ is a smooth manifold with dimension the index of the linearization of $\bar{\partial}_J$. The computation of the index is done via a modified version of Riemann-Roch theorem for complex vector bundles over a Riemann surface. Next we quotient with the $\mathbb{R}$ action, if the dimension is one a result from Gromov (Gromov compactness theorem) says that the quotient is compact zero dimensional, we can then count the pseudo-holomorphic Whitney disks. This is the most natural "geometric" description of the moduli space $\mathcal{M}(\phi)$ which can be found in the literature [MS04].

For the analysis the unit disk $\mathbb{D}$ becomes sometimes difficult to work with, so if needed we replace it by a conformally equivalent domain which is the vertical strip $\Delta = [-1, 1] \times \mathbb{R}$, we then need to replace the conditions $u(-i) = x$ and $u(i) = y$ by the fact that u lift to the tangent space at x and y at the extremities of the strip (neighbourhood of the two infinities). With this we lose a bit of the geometry of $\mathbb{D}$, for instance we cannot visualize the boundary at infinity. However the group of automorphism preserving the two points at infinity becomes simpler: the set of vertical translations, and the analysis is much easier.

The analysis techniques required for the theory need some properties that the smooth setting in which we have discussed does not allow us to use. Then one needs to enlarge the space by including non smooth functions. So one uses Whitney disks belonging to the space $W^{k,p}(\Delta, \mathrm{Sym}^g(\Sigma_g))$ of distributions on Δ with values in $\mathrm{Sym}^g(\Sigma_g)$, and the subspace of $W^{k,p}(\Delta, \mathrm{Sym}^g(\Sigma_g))$ corresponding to infinitesimal deformations of such Whitney disks. The last space is the completion of the smooth bundle discussed earlier with respect to the Sobolev $W^{k,p}$-norm. It appears that the $W^{k,p}$-topology and the C^∞-topology are equivalent on the solution space. A nice consequence of this is that we are now working in Banach manifolds instead of Féchet manifolds. This extension into distributions is justified by the fact that if the complex structure J is smooth then the "weak" solutions of $\bar{\partial}_J u = 0$ are smooth *i.e* elements of $C^\infty(\Delta, \mathrm{Sym}^g(\Sigma_g))$. This result is a modified version of one in [MS04] and is linked to the boundary condition *i.e* the boundary of $\mathbb{D}$ or Δ is sent to the union of two totally real submanifolds $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ of half the real dimension of $\mathrm{Sym}^g(\Sigma_g)$. Let us state this result which appears in [MS04].

Let M be a $2n$-dimensional manifold with almost complex structure J and L an n-dimensional J-totally real submanifold of M *i.e* $TL \cap J(TL) = 0$. Let R be a Riemann surface possibly open with complex structure j.

Theorem 7.21. *Let $p > 2$ and suppose J is of class C^l for $l \geq 2$. If $u : R \longrightarrow M$ is a $W^{1,p}$ solution of $\bar{\partial}_J u = 0$ with boundary condition $u(\partial R) \subset L$, then u is of class $W^{l,p}$. In particular if $l = \infty$ then u is smooth.*

So we just have to choose a smooth generic almost complex structure on $\mathrm{Sym}^g(\Sigma_g)$. The extension of the theory into distributions does not change the moduli space $\mathcal{M}(\phi)$ of J-holomorphic representative of ϕ and its $\mathbb{R}$-quotient $\widehat{\mathcal{M}}(\phi)$.

The next chapter will be a brief account on pseudo-holomorphic curves.

Chapter 8

Pseudo-Holomorphic Curves

In this chapter we just want to concentrate on the spirit of the problem so we will restrict ourselves to smooth maps, avoiding Sobolev embedding and Sobolev completion. Our exposition will only consider pseudo-holomorphic curves for a fixed Riemann surface (Σ, j) with fixed complex structure. Let us start with brief comments on Fredholm theory.

8.1 Fredholm Index Theory

For this section we follow partially [MS04], [Sch56] and [Gro67]. [Sch56] is a nice and readable reference for an introduction to general Fredholm theory. For someone who would like to go deep in the theory, the best reference is probably [Gro67].

Definition 8.1. *A Banach manifold modelled on a Banach space X is a Hausdorff topological space $\mathcal{M}$ together with an open covering $\{U_\alpha\}$ and homeomorphisms: $\varphi_\alpha : U_\alpha \longrightarrow N_\alpha \subset X$ with the property that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a smooth map between open set of X.*

An ordered pair $(U_\alpha, \varphi_\alpha)$ is called a local chart and we can define atlases and maximal atlases as for usual manifolds.

Definition 8.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be two Banach manifolds. A continuous map $f : \mathcal{U} \subset \mathcal{M} \longrightarrow \mathcal{N}$ is a differentiable (resp. smooth) map if:*

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1} \quad : \quad \varphi_\alpha \left(f^{-1} \left(U_\alpha \right) \cap \mathcal{U} \right) \subset X \longrightarrow X$$

is a differentiable (resp. smooth) map for every chart $(U_\alpha, \varphi_\alpha)$ and (V_β, ψ_β) of $\mathcal{M}$ and $\mathcal{N}$.

The definition of submanifold of a Banach manifold is slightly different from usual

Definition 8.3. *A subspace $\mathcal{N}$ of a Banach manifold $\mathcal{M}$ is called a submanifold of $\mathcal{M}$ if we have a topological splitting $X = E \oplus F$ and for each $x \in \mathcal{N}$, there is a chart (U, φ) in $\mathcal{M}$ such that $x \in U$ and for some $p \in F$:*

$$\varphi(U \cap \mathcal{N}) = \varphi(U) \cap (E \times \{p\}).$$

We have similar things as all the different structures which come with usual manifolds.

Let X and Y be two Banach spaces.

Definition 8.4. *Let $A : X \longrightarrow Y$ be a linear operator. A is called a Fredholm operator if it is closed and*

1. *it has a closed range*
2. *its domain is dense in X*
3. *it has finite kernel and cokernel.*

Definition 8.5. *The index of a Fredholm operator A is :*

$$ind\,(A) = \dim\ker A - \dim\operatorname{coker} A$$

Theorem 8.6 (Inverse Function Theorem). *Let $f : U \subset X \longrightarrow Y$ be a differentiable map. If $Df\,(a)$ is an isomorphism for some $a \in U$, then there exists an open neighbourhood U_0 of a such that $f_{|U_0}$ is a diffeomorphism on its image.*

Definition 8.7. *A differentiable map $f : \mathcal{M} \longrightarrow \mathcal{N}$ between Banach manifolds is called Fredholm if $Df\,(x)$ is a Fredholm operator for each $x \in \mathcal{M}$.*

The index of f is $ind\,(Df\,(x))$ for $x \in \mathcal{M}$.

Definition 8.8. *Let $f : U \subset X \longrightarrow Y$ be a Fredholm map. $y \in Y$ is a regular value for f if $Df\,(x)$ is surjective and has a right inverse for all $x \in f^{-1}\,(y)$.*

Theorem 8.9 (Implicit Function Theorem). *If y is a regular value for a differentiable map $f : U \subset X \longrightarrow Y$, then $f^{-1}\,(y) =: \mathcal{M}$ is a Banach manifold and $T_x\mathcal{M} = \ker Df\,(x)$.*

In particular if f is a Fredholm operator then $\mathcal{M}$ has finite dimension and $\dim \mathcal{M} = ind\,(f)$.

Basic Properties of the Index

Let X, Y, Z be Banach spaces and $A : X \longrightarrow Y$ a Fredholm operator.

Theorem 8.10. *If $B : Y \longrightarrow Z$ is Fredholm operators, then $B \circ A$ is a Fredholm operator and*

$$ind\,(B \circ A) = ind\,(B) + ind\,(A)\,.$$

Theorem 8.11. *If $K : X \longrightarrow Y$ is a compact operator, then $A + K$ is a Fredholm operator and*

$$ind\,(A + K) = ind\,(A)$$

Theorem 8.12. *There is $\epsilon > 0$ such that for every bounded operator $T : X \longrightarrow Y$ with $||T|| < \epsilon$, $A + T$ is Fredholm and*

$$ind\,(A + T) = ind\,(A)$$

$$\dim \ker\,(A + T) \leq \dim \ker A$$

The idea is to prove that the operator $\bar{\partial}_J$ is Fredholm corresponding to a precise linearization for given boundary conditions. The index of this operator can then be used to define topological invariants, like the Heegaard Floer homology which corresponds to Whitney disks in the symmetric product of Heegaard surface, Gromov-Witten invariants for symplectic manifolds.

8.2 Cauchy-Riemann Equation

Let M be a $2n$-dimensional manifold with almost complex structure J. Let us consider a smooth map $u : \Sigma \longrightarrow M$. The differential of u splits into two parts, one is complex linear with respect to the complex structure J and the other is complex antilinear

$$\mathrm{d}u = \partial_J u + \bar{\partial}_J u$$

$$\text{where} \qquad \bar{\partial}_J u = \frac{1}{2}\left(\mathrm{d}u + J \circ \mathrm{d}u \circ j\right), \qquad \partial_J u = \frac{1}{2}\left(\mathrm{d}u - J \circ \mathrm{d}u \circ j\right)$$

u is called a *J-holomorphic curve* if it satisfies:

$$\bar{\partial}_J u = 0$$

This is known as the *Cauchy-Riemann equation.*

We are now going to give a local description of J-holomorphic curves. Let $z = s + it$ be a conformal coordinate on Σ and i the standard complex structure on $\mathbb{C}$. The differential of a smooth map $u : \Sigma \longrightarrow M$ in local coordinates is,

$$du = \frac{\partial u}{\partial s} \circ \mathrm{d}s \ + \ \frac{\partial u}{\partial t} \circ \mathrm{d}t$$

The matrix representation of the action of i is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Therefore

$$\begin{aligned} \mathrm{d}s \circ i &= -\mathrm{d}t \\ \mathrm{d}t \circ i &= \mathrm{d}s \end{aligned}$$

It follows that

$$J \circ \mathrm{d}u \circ i = -J \circ \frac{\partial u}{\partial s} \circ \mathrm{d}t + J \circ \frac{\partial u}{\partial t} \circ \mathrm{d}s$$

From this we get a local expression for $\bar{\partial}_J$,

$$\bar{\partial}_J u = \frac{1}{2}\left(\frac{\partial u}{\partial s}\mathrm{d}s \ + \ \frac{\partial u}{\partial t}\mathrm{d}t - J(u)\,\frac{\partial u}{\partial s}\mathrm{d}t + J(u)\,\frac{\partial u}{\partial t}\mathrm{d}s\right)$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial s} + J(u)\,\frac{\partial u}{\partial t}\right)\mathrm{d}s + \frac{1}{2}\left(\frac{\partial u}{\partial t} - J(u)\,\frac{\partial u}{\partial s}\right)\mathrm{d}t$$

Hence the equation $\bar{\partial}_J u = 0$ is equivalent to

$$\begin{cases} \frac{\partial u}{\partial s} + J(u)\,\frac{\partial u}{\partial t} = 0 \\ \\ \frac{\partial u}{\partial t} - J(u)\,\frac{\partial u}{\partial s} = 0 \end{cases}$$

The two equations are the same due to the fact that $J^2 = -Id$. Therefore we can state the following local characterization.

Proposition 8.13. *A smooth map $u : \Sigma \longrightarrow M$ is J-holomorphic if and only if in conformal coordinate $z = s + it$ it satisfies*

$$\frac{\partial u}{\partial s} + J(u)\,\frac{\partial u}{\partial t} = 0$$

As in standard terminology a *critical point* for a J-holomorphic curve u is a point $w \in \Sigma$ for which $du(w) = 0$.

Proposition 8.14. *The set of preimages of critical values of a J-holomorphic curve $u : \Sigma \longrightarrow M$ is discrete.*

There are two types of J-holomorphic curves:

◇ **Multiply-covered curves**, which are curves that can be written as composition of a non trivial branched covering $\pi : \Sigma \longrightarrow \Sigma'$ and another J-holomorphic curve $\tilde{u} : \Sigma' \longrightarrow M$

$$\Sigma \xrightarrow{\pi} \Sigma' \xrightarrow{\tilde{u}} M$$

◇ **"Somewhere injective curves"**, they are curves u for which there is point $w \in \Sigma$ such that:

$$du(w), \quad u^{-1}(u(z)) = \{w\}$$

such point is called an **injective point**. We call **simple** this type of curves.

Simple curves have the following important property.

Proposition 8.15. *The set of injective points of a simple curve is dense.*

A proof of this can be found in [MS04] and [MS98]. [MS04] particularly gives two different proofs.

8.3 The Moduli Space and the Linearization of $\bar{\partial}_J$

The term "moduli space" refers in general to a geometric description of some class of objects corresponding to solutions of certain mathematical problems. Here the problem can be thought of as the resolutions of the Cauchy-Riemann equation for maps from surfaces to an almost-complex manifold, possibly with some boundary conditions and some extra requirements.

Let M be a $2n$-dimensional manifold with almost complex structure J and $A \in H_2(M, \mathbb{Z})$. We are interested in the space $\mathcal{X}(A)$ of smooth maps $u : \Sigma \longrightarrow M$ which represents the homology class A. Equipped with the C^∞ topology $\mathcal{X}(A)$ is an infinite dimensional manifold with tangent space $T_u\mathcal{X}(A) = \mathcal{C}^\infty(\Sigma, u^*TM)$ at u. We can consider the delbar operator $\bar{\partial}_J$ as an element of the space $\Omega^{0,1}(\Sigma, u^*TM)$ of smooth M-valued J-antilinear 1-form on Σ. Since elements of $\mathcal{X}(A)$ are the zeros of $\bar{\partial}_J$ we can consider the infinite dimensional vector bundle

$$\mathcal{E} \longrightarrow \mathcal{X}(A)$$

The fibre at $u \in \mathcal{X}(A)$ is $\mathcal{E}_u = \Omega^{0,1}(\Sigma, u^*TM)$. The operator $\bar{\partial}_J$ is then a section of this bundle and the J-holomorphic curves representing A are the zero of this section. For $\bar{\partial}_J^{-1}(0)$ to be a smooth manifold we require that this section is transverse to the zero section *i.e* the composite map

$$\mathcal{C}^\infty(\Sigma, u^*TM) \xrightarrow{D\bar{\partial}_J} T_{(u,0)}\mathcal{E} = T_u\mathcal{X}(A) \oplus \mathcal{E}_u \xrightarrow{Pr} \mathcal{E}_u$$

is linear and surjective for every solution u of the Cauchy-Riemann equation representing the homology class A. We denote $D_{u,0}$ this composition map. $D\bar{\partial}_J : \mathcal{C}^\infty(\Sigma, u^*TM) \longrightarrow T_{(u,0)}\mathcal{E}$ is the differential of the section $\bar{\partial}_J : \mathcal{X}(A) \longrightarrow \mathcal{E}$. For now $D_{u,0}$ is just defined for J-holomorphic curve u. To extend $D_{u,0}$ to general smooth maps we need to make a choice of splitting of the tangent space $T_{(u,\bar{\partial}_J u)}\mathcal{E}$ into horizontal and vertical subspaces. For, we need to chose a connection on TM which should preserve J and such that the fibres of $\mathcal{E}$ are invariant under parallel transport. Now to do this extension let us consider the Levi-Civita connection ∇ on TM with respect to certain metric. From this we derive a complex linear connection $\widehat{\nabla}$ defined by

$$\widehat{\nabla}_Y X := \nabla_Y X - \frac{1}{2} J(\nabla_Y J) X$$

Let $\xi \in \mathcal{C}^\infty(\Sigma, u^*TM)$. Let us consider the complex bundle isomorphism

$$\phi_u(\xi) : u^*TM \longrightarrow \exp_u(\xi)^* TM$$

given by parallel transport along the geodesics $s \mapsto \exp_{u(z)}(s\xi(z))$ with respect to $\widehat{\nabla}$. The vertical part of the section $\mathcal{S} : u \mapsto (u, \bar{\partial}_J u)$ with respect to $\widehat{\nabla}$ for general smooth maps $u : \Sigma \longrightarrow M$ is given by

$$\begin{aligned} \bar{\partial}_u : \mathcal{C}^\infty(\Sigma, u^*TM) &\longrightarrow \Omega^{0,1}(\Sigma, u^*TM) \\ \xi &\mapsto \phi_u(\xi)^{-1} \bar{\partial}_J(\exp_u(\xi)) \end{aligned}$$

Definition 8.16. *The linearization of $\bar{\partial}_J$ at u is the linear map*

$$D_u := D\bar{\partial}_u(0) : \mathcal{C}^\infty(\Sigma, u^*TM) \longrightarrow \Omega^{0,1}(\Sigma, u^*TM)$$

The maps D_u and $\bar{\partial}_u$ have the same domain and range but D_u is linear and $\bar{\partial}_u$ is not. When u is J-holomorphic $D_u = D_{u,0}$. In particular when 0 is a regular value for $\bar{\partial}_J$ the tangent space at u to $\bar{\partial}_J^{-1}(0)$ is $\ker D_u$ by the implicit function theorem [Theorem 8.9].

Remark 8.17. *To make use of the formulation in terms of distributions we just replace all the space of maps by the corresponding Sobolev completion with respect to the Sobolev $W^{k,p}$-norm. In fact due to elliptic regularity* [Theorem 7.21] *the kernel and the cokernel of the operator D_u do not depend on the choice of space in which they are defined.*

In general the moduli space of J-holomorphic curves representing A fail to be a smooth manifold. This is due to the existence of multiply-covered curves. To avoid this problem there are two approaches. One is to perturb the Cauchy-Riemann equation by adding a perturbation term ν, *i.e* by considering the equation

$$\bar{\partial}_J u = \nu.$$

The other approach which is used in [MS04] and [MS98] is to consider only simple curves. This last approach is the most commonly used. We follow this last approach. We consider then the moduli space $\mathcal{M}(A,J)$ of simple J-holomorphic curves representing the homology class A. In this case there exists a subset $\mathcal{J}_{reg}$ of the set of almost complex structure on M which contains a countable intersection of open and dense subset and such that for each $J \in \mathcal{J}_{reg}$, $\mathcal{M}(A,J)$ is a smooth finite dimensional manifold. We then choose a generic almost complex structure J. The linearization D_u is then well defined and is a Fredholm operator. The Fredholm index of D_u is given by the Riemann-Roch theorem [MS04].

Let $c_1 \in H^2(M,\mathbb{Z})$ be the first Chern class of the tangent bundle TM. For a 2-dimensional submanifold S, $\langle c_1, [S]\rangle$ is the topological degree of the restriction of TM to S as complex vector bundle. For a closed Riemann surface of genus g the index of D_u is given by

$$ind(D_u) = n(2-2g) + 2\langle c_1, A\rangle$$

where $\langle c_1, A\rangle = \langle c_1, [u(\Sigma)]\rangle$. This index is called the *virtual dimension* of $\mathcal{M}(A,J)$.

For non compact surfaces one needs to introduce what we call the *boundary Maslov index* [MS04].

In order to define invariants from $\mathcal{M}(A,J)$ we need to have some compactness results. In general $\mathcal{M}(A,J)$ is not compact; in some cases however the quotient $\mathcal{M}(A,J)/Aut(\Sigma)$ is compact where $Aut\,(\Sigma)$ is the automorphism group of the surface Σ. The main obstruction of compactness is the phenomena of bubbling which is describes in [MS04] and [MS98].

Conclusion

In this thesis it has been shown that one can deduce some information about the moduli space of holomorphic representatives $\mathcal{M}(\phi)$ of an element $\phi \in \pi_2(x, y)$ by reducing the theory to the use of the Riemann mapping theorem. This is due to the fact that the disks which we are interested in possess special properties inherited from the Heegaard diagram. Their boundary must be made by α and β arcs so that they represent domains of the Heegaard Diagram. From this result, $\mathcal{M}(\phi)$ forms a 1-parameter family for convex bigons and squares and the $\mathbb{R}$-quotient $\widehat{\mathcal{M}}(\phi)$ contains only one element. The fact that we use a special complex structure on $\mathrm{Sym}^g(\Sigma_g)$ which is inherited directly from product structure is another important point. It allows us to get holomorphic maps in our constructions. This may suggest that the combinatorialization of the computation of the boundary map in Heegaard Floer theory is strongly related to the Riemann mapping theorem.

In the general theory, the virtual dimension of the moduli space is computed via the Riemann-Roch theorem or some of its variations. One can ask if in general there are correspondences between such kind of problems as counting J-holomorphic curves in some almost-complex or symplectic manifolds and the study of Riemann surfaces and complex vector bundles over Riemann surfaces which possess some algebraic and combinatorial aspects. If such correspondence exists, one can ask whether the stronger the relation, the more combinatorial is the theory.

Acknowledgements

I am grateful for the supervision of Dr. David Gay. Thanks also to Prof. András Stipsicz and to Vera Vertesi. Thanks to Dr. Gérard Razafimanantsoa and his mathematical heritage. I do not forget Jan for his help with LaTeX and Frances Aron.

Ho an'ireo izay akaiky ny foko.

For those which I do not forget and who do not forget me.

References

[AR67] R. H. Abraham and Robbin, *Transversal mapping and flow*, Benjamin: Reading, Ma, USA, 1967.

[Ban04] Augusting Banyaga, *Lectures on Morse homology*, Kluwer Academic Publisher, USA, 2004.

[BM80] Joan S. Birman and José Mariá Montesinos, *On minimal heegaard splittings*, Michigan Math. J. **27** (1980), 47–57.

[Car61] Henri Cartan, *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes*, Hermann, Paris, 1961.

[Chi89] E.M Chirka, *Complex analytic sets*, Kluwer Academic Publishers Group, Dordrecht, 1989, Mathematics and its applications (Soviet Series), vol. 46.

[GL53] René Garnier and Jean Leray, *Oeuvres de Henri Poincaré*, Published by the Académie des Sciences, La section de géométrie (1953), 183–498.

[GP74] V. Guillemin and A. Pollack, *Differential topology*, Prentice-Hall, United States, 1974.

[Gro67] Alexander Grothendieck, *La théorie de Fredholm*, Bulletin de la Société Mathématique de France **84** (1967), 319–384.

[Kau83] Ludger Kaup, *Holomorphic functions of several variables*, Walter de Gruyter, Berlin, 1983.

[Kra92] Steven G. Krantz, *Function theory of several complex variables*, AMS Chelsea Publishing, USA, 1992.

[Lip06] R. Lipshitz, *A cylindrical reformulation of Heegaard Floer homology*, Geom. Topol. **10** (2006), 955–1097.

[Liv93] C. Livingston, *Knot theory*, Carus Mathematical Monographs, Washington DC, 1993.

[Mat02] Yukio Matsumoto, *An introduction to Morse theory*, Translations of Mathematical Monographs, A.M.S **208** (2002).

[May99] J. P. May, *A concise course in algebraic topology*, The University of Chicago Press, USA, 1999, available at www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf.

[Mil65] J. Milnor, *Topology from the differentiable viewpoint*, University Press of Virginia, Charlottesville, 1965.

[Mil69] ———, *Morse theory*, Princeton University Press, United States, 1969.

[Mor01] Shigeyuki Morita, *Geometry of characteristic classes*, Translations of Mathematical Monographs, A.M.S **199** (2001).

[MOS09] Ciprian Manolescu, Peter Ozsváth, and Sucharit Sarkar, *A combinatorial description of knot Floer homology*, Annals of Mathematics **169** (2009), 633–660.

[MOST07] C. Manolescu, P. S. Ozsváth, Z. Szabó, and D. P. Thurston, *On combinatorial link Floer homology*, Geom. Topol. **11** (2007), 2339–2412.

[MS98] D. McDuff and D. Salamon, *Introduction to symplectic topology*, second edition, Oxford Science Publications, UK, 1998.

[MS04] ———, *J-holomorphic curves and symplectic topology*, AMS Colloquium Publications **52** (2004).

[OS04a] P. S. Ozsváth and Z. Szabó, *Heegaard diagrams and holomorphic discs*, GT/0403029, in Different faces of geometry Int. Math. Ser . (N.Y.) Kluwer/Plenum (2004), 301–348.

[OS04b] ———, *Holomorphic disks and three-manifold invariants: Properties and applications*, Ann. of Math. **159** (2004), 1159–1245.

[OS04c] ———, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. **159** (2004), 1027–1158.

[OS05] ______, *Lectures on Heegaard Floer homology*, Clay Mathematics Proceedings **5** (2005), 1027–1158.

[OS06] ______, *An introduction to Heegaard Floer homology*, Clay mathematics proceedings **5** (2006), In Floer Homology, Gauge Theory, and Low-dimensional Topology.

[P.P91] Bruce P.Palka, *An introduction to complex function theory*, Springer, New York, 1991.

[Rha60] G.De Rham, *Variétés différentiables*, Hermann, Paris, 1960.

[Rol76] D. Rolfsen, *Knots and links*, Publish or Perish, Berkeley, 1976.

[RS93] Joel Robbin and Dietmar Salamon, *The Maslov index for paths*, Topology **32 no. 4** (1993), 827–844.

[Sch] Matthias Schwarz, *Morse homology*, Progress in Mathematics, vol. 111, Birkhäuser Verlag, Basel and Boston, MA, 1993.

[Sch56] Martin Schechter, *Basic theory of Fredholm operators*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 21, no 2 (1956), 261–280.

[Sin33] J. Singer, *Three-dimensional manifolds and their Heegaard-diagrams*, Trans. Amer. Math. Soc. **35** (1933).

[Tre99] K. Trencevski, *Permutation products of 1-dimensional complex manifolds*, Contributions, Mac. Acad. Sci. Arts. XX (1999).

[Wag80] C. H. Wagner, *Symmetric, cyclic and permutation products of manifolds*, Dissert. Math. (Rozprawy Mat.) **182** (1980), 3–48.

[Wel80] Raymond O'Neil Wells, *Differential analysis on complex manifolds*, Springer, New York, 1980.