

The Vyncke et al. Solution for Pricing European-style Arithmetic Asian Options

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Abstract

This paper investigates the European-style arithmetic Asian option pricing solution of Vyncke, Dhaene, and Goovaerts (2004) who apply the concept of comonotonicity to obtain upper and lower bounds for the true option price. A moment-matching formula is used to find a weighted average solution of the two bounds, thereby obtaining a fast approximation to the true price. This method is implemented and tested against an accurate Monte Carlo benchmark and compared with some other well-known closed-form approximations. Although a summary of some of the theoretical aspects underpinning the solution is provided to build intuitive understanding, the focus of the paper lies instead in the empirical analysis. The Vyncke et al. solution is found to be very accurate across a range of input parameters and outperforms competing solutions in some important cases, most notably high volatility and long maturities.

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1 Introduction

Asian options (also referred to in the literature as “Average Rate” or “Average Price” options) are path-dependent derivative securities whose settlement price is calculated with reference to the average price of the underlying security over a given time interval.

Asian options come in a variety of styles. “Fixed Strike” options have a payoff at maturity of $\max[\eta(A_\tau - K), 0]$ where A_τ is the average price function of the underlying asset at time τ and K is the fixed strike. η is a binary indicator taking the value of 1 for a call and -1 for a put. We also find “Average Strike” options with a terminal payoff of $\max[\eta(S_\tau - A_\tau), 0]$ where S_τ is the terminal price of the underlying asset and A_τ is the averaged strike. This function can be calculated as either a geometric or arithmetic average, and can be weighted if desired. This average can be either discretely or continuously sampled. Only the former is a practicable reality although much of the literature deals exclusively with the continuous case. We also differentiate between European or American contracts. In this paper we deal exclusively with Fixed-Strike European Asian options with discrete sampling.

Asian options are considered to be “Exotic” in nature due to their path-dependent nature and it is common to find Asian characteristics embedded in more sophisticated financial instruments and derivatives.

The averaging feature of these contracts causes a smoothing effect which reduces vulnerability to the impact of large price shocks at or near maturity. Asian features are often found in financial structures with thinly-traded underlying securities in order to make them more robust against manipulation.

As pointed out in Nielsen and Sandmann (2003) and Lord (2006), Asian options can also be used in a unit-linked insurance context to guarantee a minimum rate of return of a long-term investment plan where periodic payments are invested in risky investment funds. These plans are regularly used in the retirement annuity space, and quantifying the risk of these embedded options is becoming increasingly important.

A further potential use for Asian derivatives is the hedging of corporate cash flows against substantial adverse price movements of some commodity (e.g. gold) over a certain time period. An Asian option is cheaper than the corresponding portfolio of vanilla options due to the fact that the volatility of

the average asset is lower than that of the underlying asset. Asians are also easier to hedge as they are single instruments with easily-determined hedging parameters. Figure 1 demonstrates that a European Asian put option has a lower value than the corresponding European vanilla put for differing values of K , the strike price, although this does not necessarily hold if the Asian option is priced while already inside the averaging period (Turnbull and Wakeman, 1991). Underlying assets for Asian options contracts are typically commodities such as oil and gold, foreign exchange rates and interest-rates (Zhang, 1997). They are almost always traded over-the-counter and are among the Exotic options most frequently traded with the outstanding volume estimated to be in the range of five to ten billion U.S. dollars (Milevsky and Posner, 1998).

Valuing Asian options has long been a problematic area of research because there exists no closed-form solution for the arithmetic option price in the Black-Scholes economy. The reason for this is that the arithmetic average is a sum of correlated lognormal random variables with no analytical probability density expression, thus precluding the pricing techniques stemming from Black and Scholes (1973). The geometric average has no such problem as it is a product of lognormal prices which is itself lognormal, and is shown

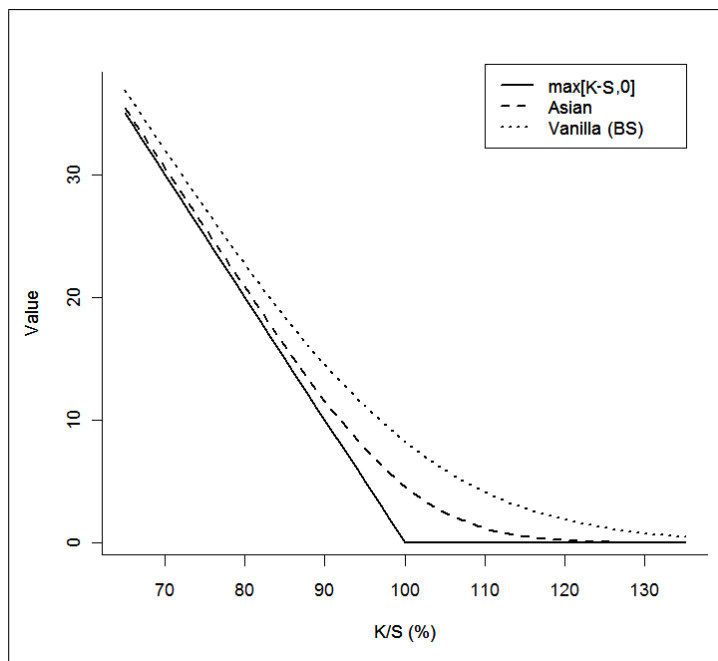


Figure 1: Relative values of Asian and Vanilla European Put Options

by Kemna and Vorst (1990) to have a closed-form pricing solution.

This paper considers the pricing method of Dhaene, Denuit, Goovaerts, Kaas, and Vyncke (2002a) and Vyncke, Dhaene, and Goovaerts (2004) who derive tight bounds for the true arithmetic option price using a form of Jamshidian's trick (1989) together with the statistical theory of comonotonic variables and the concept of convex ordering. We test their closed-form solutions against benchmark prices generated by Monte Carlo simulation with a geometric control variate. Other relevant arithmetic Asian option pricing methods (Turnbull and Wakeman, 1991; Curran, 1994) are also used for further reference.

We begin our investigation in the next section by surveying the literature on the subject of Asian option pricing. We follow this in Section 3 with a brief foray into the theoretical machinery required to competently handle this pricing method within the Black-Scholes universe. The resultant Vyncke et al. solution will then be presented. Section 4 goes on to describe and present the numerical analysis as we test and compare their pricing method, followed by a summary and conclusions in Section 5.

2 Review of Asian Option Pricing Literature

Confronted by the problem of finding a tractable and accurate solution to the Asian option valuation problem where the average has a non-normal probability law, has resulted in substantial amounts of research into Asian option pricing since 1990. The existing literature appears to fall into five primary areas of research.

Firstly, there are those that use a Monte-Carlo methodology. Inspired by Boyle (1977) to use simulation in an option-pricing context, Kemna and Vorst (1990) price fixed-strike arithmetic European Asian options by successfully using a control variate based on the corresponding geometric Asian options. See also Broadie and Glasserman (1996), Dufresne and Vazquez-Abad (1998), and Lapeyre and Temam (1999) for alternatives and extensions to this approach. These methods are flexible and accurate but suffer from significant calculation times relative to some other methods. They are often used in subsequent literature (as is done in this paper) to provide benchmark prices against which to compare prices obtained via other methods.

A second approach is to employ numerical methods to evaluate the probability law of the average price. Carverhill and Clewlow (1990) attempt to use a Fast Fourier Transform methodology to evaluate the convolution of the density. Geman and Yor (1993) express the option value as a triple integral and use a Laplace transform inversion algorithm to find a solution. However they experience numerical problems for pricing options with low volatilities and short maturities. These problems are independent of the inversion technique chosen and result from a slowly decaying oscillatory integrand. While both these methods are fairly accurate they are extremely slow and difficult to implement. Furthermore these approaches often rely on the assumption that stock prices are independent, and do not work sufficiently well for multi-asset derivatives such as basket, rainbow or portfolio options. Rogers and Shi (1995) manage to exploit a scaling property in order to express the option value as a parabolic PDE which they then solve using numerical methods with decent accuracy. Večer (2001) also investigates the PDE approach with some success.

The third approach involves using the geometric option price to approximate the arithmetic option price. Ruttiens (1990) and Vorst (1990) advocate approximating the price of an arithmetic Asian option by modifying the solution to the geometric Asian pricing problem. While being significantly

faster than previous attempts these methods can suffer from inaccuracy at high volatility and maturity values, resulting in overpriced put options and underpriced call options. These methods also suffer from the problem of only satisfying Asian put-call parity at expiry and hence admit arbitrage.

The fourth methodology is to approximate the density for the arithmetic average by replacing the unknown distribution with another. Turnbull and Wakeman (1991) were the first to employ this approach and use an Edgeworth series expansion to approximate the average density function by matching the first four moments. Levy (1992) adopts a very similar approach but instead uses a 2-moment Wilkinson approximation of the Lognormal distribution with almost identical results. An Inverse Gaussian approximation with the correct first two moments is used by Jacques (1996) with results comparable with Levy (1992) as long as the parameters are chosen in the same range. These approaches have the advantage of being extremely fast and quite easy to implement at the cost of some inaccuracy for large volatilities and maturities. Vyncke et al. (2004) point out that these approximations have two structural disadvantages. Firstly if the parameters are in a certain range the approximations can be smaller than the theoretical lower bound and secondly the approximations work only if the underlying asset process follows Geometric Brownian Motion. Nevertheless these methods are used extensively by practitioners worldwide (Haug, 1997).

The fifth and final approach, and the one which this paper will consider, involves the derivation of bounds on the true option price and the approximation of the option price using a function of one or both of these upper or lower bounds. Curran (1992, 1994) tries to solve the pricing problem by determining a tight lower bound by conditioning on the geometric mean price. Rogers and Shi (1995) also find a suitably accurate lower bound by conditioning on a zero-mean variable. Thompson (1999) provides an alternative derivation which leads to a simpler expression for the bounds as well as introducing a new upper bound. Nielsen and Sandmann (2003) continue in this vein, developing and comparing bounds by conditioning the maturity payment of the Asian option on the geometric average. Furthermore they express both upper and lower bounds as a portfolio of delayed payment European call options by exploiting a form of Jamshidian's trick (1989).

3 Theoretical Development

In this section we consider and present the necessary theoretical development and solution of Vyncke, Dhaene, and Goovaerts (2004).

3.1 Preliminary Theory

We consider a form of the Black-Scholes (1973) economy with deterministic time-dependent drift and volatility parameters. The underlying asset price (typically a dividend-paying stock) is modelled with a Geometric Brownian Motion stochastic process $\{S_t, t \geq 0\}$ with the following dynamics:

$$dS = S(\mu_t - q_t)dt + S\sigma_t dW_t^{\mathbb{P}}$$

where μ_t is the drift parameter (annual and continuously compounded), q_t represents the continuous annual dividend yield, σ_t represents the annual volatility of the underlying asset price and $W_t^{\mathbb{P}}$ is a standard Weiner process under the real-world probability measure \mathbb{P} . All parameters are deterministic and time dependant, implying term structures for the drift, dividend yield and volatility. This implies a minor reworking of the solution in the literature (which assumes constant parameters) to handle the more flexible parameter term structures.

Although the standard Black-Scholes assumptions of constant drift, dividend and volatility parameters are extended to cater for term-structures, we retain the rest of the usual assumptions such as negligible transaction costs and taxes.

Classical martingale pricing theory of Harrison and Kreps (1979) and Harrison and Pliska (1981) permits us to change the probability measure from the real-world \mathbb{P} -measure to the risk-neutral \mathbb{Q} -measure under which the discounted underlying price process is a martingale and therefore has no drift. This allows us to dispense with the μ_t parameter and replace it with the risk-free rate of interest r_t , observable in the government bond market. The underlying asset price now has dynamics:

$$dS = S(r_t - q_t)dt + S\sigma_t dW_t^{\mathbb{Q}}$$

The value of a standard European arithmetic Asian option maturing at time τ can then be expressed as

$$V = e^{-r_\tau \tau} \mathbb{E}^{\mathbb{Q}} [(\eta(A_\tau - K))^+] \quad (1)$$

where

- $(A_\tau - K)^+$ denotes the convex function $\max(A_\tau - K, 0)$
- $K (\geq 0)$ is the option strike price
- r_τ is the risk-free yield for maturity τ
- η is a binary indicator taking the value of 1 for a call and -1 for a put option
- $\mathbb{E}^{\mathbb{Q}}[\cdot]$ is the expectation operator under the risk-neutral probability measure \mathbb{Q} .

The Asian option is written on the average function A_τ of the underlying asset price. The continuous form is given by

$$A_\tau = \frac{1}{\tau} \int_{t=0}^{\tau} S_t dt$$

It is impossible to make sense of this purely theoretical expression in practical applications so we use a discretely sampled version of this function below.

Recall that $\ln S_t$ is normally distributed under the risk-neutral probability measure \mathbb{Q} implying that S_t has a lognormal probability density. This causes us to conclude that the averaging function random variable A_τ has no discernable closed-form density function because a sum (and correspondingly an average) of lognormal random variables is itself no longer lognormal. Evaluating the expectations term of Equation (1) in closed-form requires an analytical density function. Vyncke et al. (2004) overcome this problem by developing closed-form approximating bounds.

3.2 Vital Theoretical Concepts

The closed-form solution of Vyncke et al. (2004) is centred around the concepts of convex ordering and comonotonicity, covered in detail in Dhaene et al. (2002b). Even though a comprehensive theoretical development lies outside the scope of this paper, a brief summary of the salient points are

presented in order to provide the reader with some intuition as to the conceptual foundations of the solution.

We define the notion of convex order used extensively in the actuarial field of stop-loss reinsurance theory:

Definition 1 (Convex Order). *A random variable X is said to precede another random variable Y in the **convex order** if and only if:*

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[Y] && \text{and} \\ \mathbb{E}[(X - K)^+] &\leq \mathbb{E}[(Y - K)^+] \end{aligned}$$

for $K \in \mathbb{R}$. We denote this convex order by $X \leq_{cx} Y$

Extensive use is made of convex ordering in order to find bounds for the true option price.

The concept of comonotonicity is defined as

Definition 2 (Comonotonicity). *A random vector \mathbf{S} (S_1, \dots, S_m) is said to be **comonotonic** and the individual random variables S_i ($i = 1, \dots, m$) are **mutually comonotonic** if*

1. *The m -variate CDF is*

$$F_{\mathbf{S}}(\mathbf{s}) = \min\{F_1(s_1), F_2(s_2), \dots, F_m(s_m)\}, \quad \forall \mathbf{S} \in \mathbb{R}^n$$

2. *There exists a random variable Z and non-decreasing functions $g_1, \dots, g_m : \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$(S_1, \dots, S_m) \stackrel{d}{\sim} (g_1(Z), \dots, g_m(Z))$$

3. *For any uniform $[0,1]$ random variable U we have*

$$(S_1, \dots, S_m) \stackrel{d}{\sim} (F_1^{-1}(U), \dots, F_m^{-1}(U))$$

where equality in distribution is denoted by $\stackrel{d}{\sim}$ and the inverse of a random variable's density function $F_S(s) = \Pr[S \leq s]$ is defined as

$$F_S^{-1}(p) = \inf\{s \in \mathbb{R} | F_S(s) \geq p\}, \quad p \in [0, 1]$$

The concept of comonotonicity allows us to express a random variable with a higher or lower convex ordering to the option value as a sum of inverse marginal density functions with an easily obtainable probability law, thus enabling us to find closed form expressions for upper and lower bounds. For more detail concerning the concept of comonotonicity refer to Dhaene et al. (2002b) as it lies outside the scope of this paper.

The following theorem is crucial:

Theorem 1. *Consider a sum of m dependent random variables $X = \sum_{i=1}^m S_i$ and define*

$$\begin{aligned} X^c &= F_{S_1}^{-1}(U) + F_{S_2}^{-1}(U) + \cdots + F_{S_m}^{-1}(U) && \text{(Comonotonicity)} \\ X^l &= \mathbb{E}[S_1|\Lambda] + \mathbb{E}[S_2|\Lambda] + \cdots + \mathbb{E}[S_m|\Lambda] && \text{(Conditioning)} \end{aligned}$$

where $U \sim \text{Uniform}(0,1)$ and where Λ is an arbitrary random variable independent of U .

Then

$$\mathbb{E}[(X^l - K)^+] \leq_{cx} \mathbb{E}[(X - K)^+] \leq_{cx} \mathbb{E}[(X^c - K)^+]$$

for all $K \in \mathbb{R}$. Also,

$$\mathbb{E}[X^l] = \mathbb{E}[X] = \mathbb{E}[X^c]$$

Proof. See Dhaene et al. (2002b). □

Theorem 1 enables a convex ordering for the random variables X^c and X^l (which both enjoy closed-form probability density functions) which Vyncke et al. (2004) exploit in order to find an upper bound (using X^c) and a lower bound (using X^l) similar to that found by Rogers and Shi (1995). The intuitive appeal behind this method is easily discerned by noting the similarity of the convex ordering condition to that of an expected option payoff.

3.3 The Closed-Form Solution of Vyncke et al.

Set-up and Definitions

We consider a discrete set of n time points along the time interval $[0, \tau]$ such that asset prices are observed at time points $\{0 = t_0 < t_1 < \cdots < t_{n-1} = \tau\}$.

The option expires at time $\tau = t_{n-1}$ and we simplify notation by defining

$S_{t_i} \triangleq S_i$ as the underlying asset price at time t_i . Note that S_0 then denotes the initial price of the underlying asset at $t = 0$. Similarly we let $r_{t_i} \triangleq r_i$, $q_{t_i} \triangleq q_i$ and $\sigma_{t_i} \triangleq \sigma_i$, representing the discrete term structures of the riskless rate, dividend yield and volatility parameters respectively.

We choose to calculate the average function as the discretely-sampled arithmetic weighted average of the underlying asset prices at time $\tau = t_{n-1}$ over the last m time points $\{t_{n-m} < t_{n-m+1} < \dots < t_{n-1}\}$ such that

$$A_\tau = \sum_{i=n-m}^{n-1} w_i S_i$$

where typically $w_i = \frac{1}{m}$ for all i such that

$$A_\tau = \frac{1}{m} \sum_{i=n-m}^{n-1} S_i$$

Note that A_τ is itself a random variable with an unknown density function although it is possible to calculate its statistical moments.

The discrete time points are often conveniently chosen as days such that $n = 120$ and $m = 30$ denote an Asian option with a time to maturity of 120 days and an averaging period of 30 days. If the average is taken over the entire remaining life of the option then $m = n$. Section 3.4.2 presents the straightforward solution to the case if the option is priced inside the averaging period (i.e. $m > n$).

For the sake of notational neatness we define

$$m_i^- \triangleq r_i - q_i - \frac{1}{2}\sigma_i^2$$

Finally we let $N(\cdot)$ denote the standard normal CDF.

The Upper Bound

We define a comonotonic expression for the average function at time τ

$$A_\tau^c = \sum_{i=n-m}^{n-1} w_i F_{S_i}^{-1}(U)$$

We then apply Theorem 1 to Equation (1) in order to find the upper bound:

$$\begin{aligned} V &= e^{-r_{n-1}\tau} \mathbb{E}^{\mathbb{Q}}[(\eta(A_\tau - K)^+)] \\ &\leq e^{-r_{n-1}\tau} \mathbb{E}^{\mathbb{Q}}[(\eta(A_\tau^c - K)^+)] =: V^u \end{aligned} \quad (\text{Theorem 1})$$

The theory on comonotonic random variables in Kaas et al. (2000) makes it possible to determine the exact probability law for A_τ^c . We then exploit the independence of A_τ^c to find the following closed-form solution for the upper bound as in Vyncke et al. (2004):

$$V^u = e^{-r_{n-1}\tau} \eta \left[S_0 \sum_{i=n-m}^{n-1} w_i e^{(r_i - q_i)t_i} N(\eta(\sigma_i \sqrt{t_i} - y^*)) - KN(-\eta y^*) \right] \quad (2)$$

$$\text{where} \quad K = S_0 \sum_{i=n-m}^{n-1} w_i \exp(m_i^- t_i + \sigma_i \sqrt{t_i} y^*)$$

We find y^* by applying a numerical procedure (such as Newton's method) to the above expression.

We can interpret V^u as an arithmetic average of vanilla European options with adjusted parameters obtained via applying a technique similar to Jamshidian (1989).

This upper bound can be made tighter by adding a further conditioning argument to the mix (Dhaene et al., 2002a,b) but Vyncke et al. (2004) show that it makes little impact on the final answer at the cost of extra calculations and so we choose to ignore it here.

The Lower Bound

The lower bound calculation requires us to condition on the zero-mean random variable Λ defined by:

$$\Lambda = \sum_{i=n-m}^{n-1} e^{m_i^- t_i} W_i^{\mathbb{Q}}$$

The variance of Λ is then given by

$$\sigma_{\Lambda}^2 = \sum_{i=n-m}^{n-1} \sum_{j=n-m}^{n-1} e^{m_i^- t_i + m_j^- t_j} \min(t_i, t_j)$$

We shall require the correlation vector between the Wiener Process $W_i^{\mathbb{Q}}$ and Λ which is calculated as:

$$\rho_i = \frac{\sum_{j=n-m}^{n-1} e^{m_j^- t_j} \min(t_i, t_j)}{\sigma_{\Lambda} \sqrt{t_i}}$$

In order to find a closed-form expression for the lower bound of the option price we define a new average function random variable A_{τ}^l :

$$A_{\tau}^l = \sum_{i=n-m}^{n-1} \mathbb{E}^{\mathbb{Q}}[A_i | \Lambda]$$

The conditioned random variable $A_i | \Lambda$ follows a lognormal distribution such that A_{τ}^l is a comonotonic sum of lognormal variables with an easily-obtained density function as a sum of inverse marginal distributions. Invoking Theorem 1 we conclude that $A_{\tau}^l \leq_{cx} A_{\tau}$.

So we then have

$$\begin{aligned} V &= e^{-r_{n-1}\tau} \mathbb{E}^{\mathbb{Q}}[(\eta(A_{\tau} - K)^+)] \\ &\geq e^{-r_{n-1}\tau} \mathbb{E}^{\mathbb{Q}}[(\eta(A_{\tau}^l - K)^+)] =: V^l \quad (\text{From Theorem 1}) \end{aligned}$$

and knowing A_{τ}^l 's comonotonic probability distribution enables the following expression for the lower bound to the price of a European-style arithmetic Asian option maturing at time τ (Dhaene et al., 2002a; Vyncke et al., 2004):

$$V^l = e^{-r_{n-1}\tau} \eta \left[S_0 \sum_{i=n-m}^{n-1} w_i e^{(r_i - q_i)t_i} N(\eta(\sigma_i \rho_i \sqrt{t_i} - y^*)) - KN(-\eta y^*) \right] \quad (3)$$

$$\text{where} \quad K = S_0 \sum_{i=n-m}^{n-1} w_i \exp \left[(r_i - q_i - \frac{1}{2} \sigma_i^2 \rho_i^2) t_i + \sigma_i \rho_i \sqrt{t_i} y^* \right]$$

Again we need to use some numerical technique to determine y^* .

Moment Matched Approximation

Vyncke et al. (2004) found the upper bound to be inaccurate while the lower bound was extremely accurate, particularly for options with low volatility and term parameters.

The best results were obtained by combining the two estimates into a weighted average of the two prices using moment matching.

Theorem 1 suggests that the first moments of the random average functions A_τ , A_τ^c and A_τ^l match already so all that remains is to match the second moments. The variance expressions relevant to the actual average function and the lower and upper approximations are given by:

$$\begin{aligned}\text{Var}[A] &= \sum_{i=n-m}^{n-1} \sum_{j=n-m}^{n-1} e^{(r_i-q_i)t_i+(r_j-q_j)t_j} (e^{\sigma_i^2 \min(t_i,t_j)} - 1) \\ \text{Var}[A^l] &= \sum_{i=n-m}^{n-1} \sum_{j=n-m}^{n-1} e^{(r_i-q_i)t_i+(r_j-q_j)t_j} (e^{\sigma_i \sigma_j \rho_i \rho_j \sqrt{t_i t_j}} - 1) \\ \text{Var}[A^c] &= \sum_{i=n-m}^{n-1} \sum_{j=n-m}^{n-1} e^{(r_i-q_i)t_i+(r_j-q_j)t_j} (e^{\sigma_i \sigma_j \sqrt{t_i t_j}} - 1)\end{aligned}$$

and the weight β is given by

$$\beta = \frac{\text{Var}(A^c) - \text{Var}(A)}{\text{Var}(A^c) - \text{Var}(A^l)}$$

The approximate option price V is then calculated as

$$V = \beta \cdot V^l + (1 - \beta) \cdot V^u \quad (4)$$

where β is the weight given to the lower-bound of the price (V^l) and $(1 - \beta)$ is the weight attached to the upper-bound of the price (V^u).

3.4 Some Theoretical Side-Shows

3.4.1 Asian Put-Call Parity

A fundamental requirement of a good option-pricing model is that it satisfies the iconic put-call parity relationship throughout its life. We present the case for Asian options here which routinely appears in the literature (Levy, 1992; Vyncke et al., 2004).

If we define forward prices as

$$\begin{aligned} f_i &= \mathbb{E}^{\mathbb{Q}}[S_i] \\ &= S_0 e^{(r_i - q_i)t_i} \end{aligned}$$

then the put-call parity relationship for Asian Options with expiry at time τ is given by

$$V_{Put} = V_{Call} + e^{-(r_\tau - q_\tau)\tau} \left(K - \sum_{i=n-m}^{n-1} f_i \right) \quad (5)$$

3.4.2 Pricing an Asian Option Inside the Averaging Period (i.e. $n < m$)

The solution presented and the numerical analysis that follows in Section 4 deal only with the case of pricing the option at or before the start of the averaging period (i.e. $n \geq m$). In order to price Asian options in the case $n < m$ we need to develop the following theory (West, 2009).

Assume we have already observed p asset prices and define

$$\bar{S} = \frac{1}{p} \sum_{i=1}^p S_i$$

Define A_f as the arithmetic average of observations remaining in the option life so

$$A_f = \frac{1}{m-p} \sum_{i=p+1}^m S_i$$

Knowing that the full average is

$$A = \frac{p\bar{S} + (m - p)A_f}{m}$$

allows us to write

$$\begin{aligned} A - K &= \frac{p\bar{S} + (m - n)A_f}{m} - K \\ &= \frac{m - p}{m}(A_f - K^*) \end{aligned}$$

where
$$K^* = \frac{m}{m - p}K - \frac{p}{m - p}\bar{S}$$

Thus we conclude that an option with maturity n already inside an averaging period of m points is merely equivalent to issuing $\frac{m-p}{m}$ new Asian options with modified strike K^* .

Note that if $K^* < 0$ then the option will be exercised with certainty as there is a 100% probability of a positive payoff. The Asian option can in this case be priced as an Asian-style forward contract (West, 2009).

3.4.3 Hedging and Replicating Portfolios

It is important to take note of the fact that these pricing formulae lose much practical appeal if the options cannot be hedged. In order to accurately and effectively hedge an Asian option one would require a replicating portfolio and therefore accurate representations of the Greeks, which measure sensitivities of the option price to various market variables. Vyncke et al. (2004, Section 5) provide the detail we choose to omit for lack of space.

4 Numerical Application, Testing and Results

In this section we test and analyse the Vyncke et al. (2004) solution using numerical illustrations within the Black-Scholes economy.

4.1 Methodology

We make use of the solution presented in Section 3 and consider the prices of European Asian call options (implying $\eta = 1$). Illustrating the results for put options would merely involve setting η equal to -1 or alternatively making use of the put-call parity relationship for Asian options presented in Equation (5). Furthermore, the study only considers the case of pricing the option at or before the start of the averaging period (i.e. $n \geq m$). It is a simple matter to extend the theory to cater for the case of $n < m$ (see Section 3.4.2) but we exclude it in order to maintain some semblance of parsimony.

We test the accuracy of the option values given by the bounds and approximations against Monte Carlo (MC) estimates (each based on 10000 paths) in which a control variate based on the geometric average is used to reduce the variance (Kemna and Vorst, 1990). The Mersenne Twister pseudo-random number generator is used to ensure true randomness and remove as much bias as possible from the MC estimates. Each MC estimate has a corresponding standard error (s.e.) and Vyncke et al. (2004) remind us that the asymptotic 95% confidence interval is given by 1.96 standard errors on either side of the estimate while the range between the theoretical lower and upper bounds contain the exact price with certainty.

In order to gauge the accuracy of the Vyncke et al. solution relative to other arithmetic Asian option pricing methodologies we also include the distribution-approximating results of Turnbull and Wakeman (TW) as well as Curran's method of conditioning on the geometric average. The theoretical development of these methods lie outside the scope of this paper but the interested reader is invited to consult their research directly (Turnbull and Wakeman, 1991; Curran, 1994). Convenient closed-form solutions to these methods are given in Haug (1997).

In order to facilitate comparisons with the literature the numerical calculations in this section have been made using flat term structures for r , q and σ . We choose a time unit of 1 day (implying daily averaging) with input parameters n denoting the number of days until option maturity and m de-

noting the number of days in the averaging period. The initial underlying asset price is chosen to be $S_0 = 100$ and the annual risk-free rate of interest (continuously compounded) is $r = \ln(1.09)$ which corresponds to an effective annual rate of 9%.

4.2 Accuracy Analysis

We present the moment-based approximation presented in Equation (4) and denote it by MM. The upper and lower bounds to the true option price given in Equations (2) and (3) are included in the analysis and are denoted as UB and LB respectively.

We test across three primary cases, corresponding to option prices with short ($n = 30$ days), medium ($n = 120$ days) and long ($n = 360$ days) times to expiry. The short term case is presented in Table 1 and the medium and long term cases are presented in Tables 2 and 3. Within each case we consider short, medium and long averaging periods by varying m . It seems prudent that we evaluate the Vyncke et al. solution in light of the high volatility levels experienced during the recent financial crisis so we consider three values of volatility (0.2, 0.4 and 0.8). Finally, we test across three values of the strike price K (90, 100 and 110).

In each case an indication is given as to whether or not the MC estimate has violated the lower bound (LB) of the true option price. This would occur due to the random nature of each MC estimate (as measured by its standard error). Factors that increase the standard error (such as higher values of σ) therefore induce a higher chance of violation, especially if the LB is extremely close to the true price. The upper bound (UB) is never violated.

The results in Table 1 demonstrate that the extremely short maturity parameter of $n = 30$ days results in MC standard errors that are extremely small, indicating that the MC estimate of the true option price is a very good one. It also exhibits extremely accurate prices for the Vyncke et al. solution (relative to the MC estimate) across all values of m , σ and K . The LB is found to be extremely close to MM and the MC estimate in all cases, while the UB is always significantly greater than the true price, especially for higher values of σ . The CU and TW approximations are reasonably accurate but lose accuracy for higher values of σ , although the CU approximation always marginally outperforms TW. We see that the Monte Carlo estimate only violates LB three times, each in a high-volatility scenario.

Table 1: A Comparison of Arithmetic Asian Option Pricing Methods. The Vyncke et al. Moment-Matched approximation (MM) with lower (LB) and upper bounds (UB) are compared to Turnbull and Wakeman (TW) and Curran (CU) approximations. MC estimates violating the LB are marked with *. ($S_0 = 100$, $r = \ln(1.09)$, $q = 0$)

Short-Term Case ($n = 30$ days)								
m	σ	K	MM	LB	UB	TW	CU	MC (s.e. $\times 10^4$)
10	0.2	90	10.554	10.554	10.560	10.555	10.554	10.554 (0.84)
		100	2.345	2.345	2.410	2.351	2.349	2.345 (0.81)
		110	0.087	0.087	0.103	0.089	0.088	0.087 (0.30)
	0.4	90	11.205	11.205	11.275	11.212	11.210	11.206 (3.38)
		100	4.368	4.368	4.499	4.381	4.376	4.368 (3.25)
		110	1.126	1.126	1.221	1.135	1.132	1.126 (2.41)
	0.8	90	14.080	14.080	14.295	14.103	14.094	14.081 (14.60)
		100	8.414	8.414	8.676	8.441	8.431	8.415 (13.80)
		110	4.648	4.647	4.897	4.671	4.663	4.647 (12.55)
20	0.2	90	10.418	10.418	10.425	10.419	10.419	10.418 (1.62)
		100	1.974	1.974	2.114	1.983	1.977	1.975 (1.65)
		110	0.031	0.031	0.051	0.032	0.031	0.031 (0.55)
	0.4	90	10.806	10.806	10.932	10.816	10.809	10.807 (6.70)
		100	3.691	3.691	3.970	3.708	3.696	3.691 (6.44)
		110	0.698	0.698	0.875	0.705	0.701	0.698 (5.14)
	0.8	90	13.008	13.007	13.448	13.044	13.016	13.010 (28.05)
		100	7.126	7.126	7.683	7.162	7.136	7.130 (27.19)
		110	3.468	3.467	3.981	3.492	3.477	3.466 (25.77)
30	0.2	90	10.294	10.294	10.296	10.294	10.294	10.294 (2.45)
		100	1.537	1.537	1.742	1.548	1.537	1.537 (2.49)
		110	0.004	0.004	0.014	0.004	0.004	0.004 (0.62)
	0.4	90	10.434	10.434	10.562	10.446	10.434	10.435 (9.44)
		100	2.880	2.880	3.290	2.903	2.880	2.879 (9.68)
		110	0.299	0.299	0.494	0.299	0.299	0.299 (8.42)
	0.8	90	11.787	11.786	12.383	11.861	11.786	11.783 (38.32)
		100	5.567	5.566	6.385	5.622	5.566	5.570 (43.91)
		110	2.145	2.144	2.840	2.153	2.144	2.146 (41.25)

Table 2: A Comparison of Arithmetic Asian Option Pricing Methods. The Vyncke et al. Moment-Matched approximation (MM) with lower (LB) and upper bounds (UB) are compared to Turnbull and Wakeman (TW) and Curran (CU) approximations. MC estimates violating the LB are marked with *. ($S_0 = 100$, $r = \ln(1.09)$, $q = 0$)

Medium-Term Case ($n = 120$ days)								
m	σ	K	MM	LB	UB	TW	CU	MC (s.e. $\times 10^4$)
30	0.2	90	12.677	12.677	12.720	12.679	12.678	12.677 (2.53)
		100	5.461	5.461	5.556	5.464	5.463	5.461 (2.50)
		110	1.625	1.625	1.707	1.628	1.627	1.625 (2.13)
	0.4	90	15.359	15.359	15.512	15.365	15.363	15.361 (10.84)
		100	9.511	9.511	9.704	9.519	9.516	9.513 (10.88)
		110	5.480	5.479	5.672	5.486	5.484	5.478 (9.65) *
	0.8	90	22.373	22.372	22.718	22.390	22.381	22.374 (50.05)
		100	17.619	17.619	18.000	17.637	17.628	17.620 (48.29)
		110	13.778	13.777	14.172	13.794	13.787	13.774 (48.29) *
60	0.2	90	12.179	12.179	12.260	12.181	12.179	12.179 (5.30)
		100	4.826	4.826	5.025	4.831	4.828	4.826 (5.20)
		110	1.192	1.192	1.351	1.194	1.193	1.192 (4.23)
	0.4	90	14.423	14.423	14.734	14.435	14.426	14.421 (21.25) *
		100	8.452	8.452	8.855	8.463	8.455	8.450 (21.05) *
		110	4.500	4.500	4.894	4.506	4.503	4.501 (20.79)
	0.8	90	20.594	20.591	21.312	20.634	20.598	20.593 (100.72)
		100	15.716	15.713	16.512	15.751	15.720	15.715 (100.09)
		110	11.858	11.855	12.677	11.885	11.863	11.851 (97.15) *
120	0.2	90	11.207	11.207	11.302	11.214	11.207	11.207 (10.29)
		100	3.368	3.367	3.761	3.378	3.367	3.367 (11.05)
		110	0.386	0.386	0.607	0.379	0.386	0.387 (9.77)
	0.4	90	12.339	12.338	12.883	12.393	12.338	12.337 (42.32) *
		100	5.940	5.939	6.737	5.971	5.939	5.945 (45.85)
		110	2.325	2.324	3.031	2.310	2.324	2.325 (44.18)
	0.8	90	16.310	16.300	17.707	16.519	16.299	16.296 (95.62) *
		100	11.102	11.090	12.678	11.238	11.090	11.099 (93.52)
		110	7.331	7.320	8.897	7.375	7.321	7.338 (98.27)

Table 3: A Comparison of Arithmetic Asian Option Pricing Methods. The Vyncke et al. Moment-Matched approximation (MM) with lower (LB) and upper bounds (UB) are compared to Turnbull and Wakeman (TW) and Curran (CU) approximations. MC estimates violating the LB are marked with *. ($S_0 = 100$, $r = \ln(1.09)$, $q = 0$)

Long-Term Case ($n = 360$ days)								
m	σ	K	MM	LB	UB	TW	CU	MC (s.e. $\times 10^4$)
30	0.2	90	18.540	18.540	18.572	18.542	18.541	18.541 (2.72)
		100	11.916	11.916	11.965	11.918	11.918	11.916 (2.74)
		110	7.052	7.052	7.107	7.054	7.053	7.052 (2.54)
	0.4	90	24.038	24.038	24.126	24.041	24.041	24.039 (12.27)
		100	18.893	18.893	18.995	18.896	18.896	18.895 (12.54)
		110	14.692	14.692	14.801	14.696	14.695	14.693 (12.21)
	0.8	90	36.671	36.670	36.853	36.679	36.676	36.676 (72.79)
		100	32.855	32.855	33.052	32.864	32.861	32.859 (69.81)
		110	29.496	29.496	29.703	29.505	29.502	29.498 (77.80)
60	0.2	90	18.110	18.110	18.174	18.111	18.111	18.111 (5.63)
		100	11.472	11.472	11.570	11.473	11.473	11.471 (5.30) *
		110	6.646	6.646	6.758	6.648	6.648	6.647 (5.28)
	0.4	90	23.411	23.410	23.589	23.415	23.413	23.412 (25.75)
		100	18.241	18.241	18.448	18.246	18.243	18.241 (24.24)
		110	14.044	14.044	14.266	14.049	14.047	14.044 (24.18)
	0.8	90	35.660	35.658	36.029	35.675	35.664	35.669 (115.17)
		100	31.802	31.800	32.201	31.818	31.806	31.801 (114.57)
		110	28.416	28.414	28.836	28.431	28.420	28.412 (115.82) *
120	0.2	90	17.244	17.244	17.372	17.246	17.244	17.244 (11.25)
		100	10.569	10.569	10.773	10.572	10.570	10.568 (10.84) *
		110	5.830	5.830	6.059	5.832	5.831	5.829 (10.79) *
	0.4	90	22.128	22.127	22.494	22.139	22.130	22.128 (50.00)
		100	16.904	16.903	17.331	16.914	16.906	16.903 (51.38)
		110	12.719	12.718	13.174	12.727	12.721	12.726 (48.37)
	0.8	90	33.566	33.559	34.326	33.614	33.564	33.548 (164.92) *
		100	29.618	29.611	30.440	29.665	29.616	29.595 (173.21) *
		110	26.178	26.170	27.043	26.222	26.175	26.172 (189.29)

In the medium-term case presented in Table 2 we note that the case of $m = 30$ and $\sigma = 0.2$ correspond directly to the results presented in Vyncke et al. (Table 1, pg. 132), and that we obtain their exact values for LB, UB and MM. As σ increases to 0.4 and the more extreme value of 0.8 we see that the MC value tends to violate the theoretical LB value to a greater extent (0 times for $\sigma = 0.2$ vs. 7 times for $\sigma = 0.4$ and $\sigma = 0.8$) as the greater σ values increase the standard error of the MC estimate. MM and LB demonstrate excellent accuracy across the board and show particularly good results relative to TW and CU for high values of σ and out-of-the-money options.

Table 3 exhibits the case for long-dated Asian options with n of 360 days. Excellent congruency is obtained for the MM and LB prices across all parameters, with particular outperformance of the TW and CU approximations at higher levels of volatility. The UB performs even worse than before which is a phenomenon attributable to decreasing comonotonicity in the dependency structure of the average price function at longer maturities (Dhaene et al., 2002a). It should be noted that the long-term MC estimates for $\sigma = 0.8$ experience significant standard errors and thus perhaps do not provide an extremely reliable estimate of the true price, although they still shed some light on high-volatility pricing for closed-form solutions.

In order to perform a better overall assessment of the different bounds and approximating methods we assume that the true option price is given by the Monte Carlo estimate and calculate the absolute difference across all the varying parameters of n , m , σ and K . Table 4 summarizes the results for the short ($n = 30$), medium ($n = 120$) and long term ($n = 360$) cases as well as the overall (Total) results across all three cases. The results consistently suggest that the moment-matched approximation (MM) and the lower bound (LB) outperform the other methods by a significant margin. Curran's method (CU) is third with an absolute error of roughly 5 times that of MM. Turnbull and Wakeman's (TW) four-moment-matched approximation performs worse, yielding an error of almost 40 times that of MM. We note that the upper bound (UB) is consistently the least accurate of the 5 methods by a very large margin and would suggest that it only be used in the construction of the moment-based approximation (MM).

We see that the absolute error for each of the pricing methods increase with the term (n). It is evident that MM outperforms the other methods significantly for short-dated Asian options ($n = 30$ days) while the lower bound becomes extremely accurate for options with longer terms. Very notably we find that this study suggests that the MM solution performs very well in envi-

Table 4: Absolute Error vs. MC Benchmark.
 $(S_0 = 0, r = \ln(1.09), q = 0)$

Term (n)	Method	Abs Err	Rank
30	MM	0.0050467	1
	LB	0.0116537	2
	CU	0.1019072	3
	TW	0.4257118	4
	UB	6.4197985	5
120	MM	0.0238278	1
	LB	0.0269187	2
	CU	0.0563417	3
	TW	0.7075339	4
	UB	13.0738943	5
360	LB	0.0151550	1
	MM	0.0212436	2
	CU	0.0744456	3
	TW	0.3045927	4
	UB	7.3713569	5
Total	MM	0.0501181	1
	LB	0.0537274	2
	CU	0.2326944	3
	TW	1.4378384	4
	UB	26.8650498	5

ronments of high volatility and for options with longer terms, a phenomenon which is not highlighted by Vyncke et al. in the original 2004 paper.

4.3 Parameter Sensitivity Analysis

We now turn to further analysis in order to investigate the sensitivity of the MM, TW and CU Asian pricing methods to changes in the input parameters. We set base parameters of $S_0 = 100$, $K = 100$, $r = \ln(1.09)$, $\sigma = 0.3$, $n = 120$ and $m = 30$ which are constant unless subject to analysis. In particular we investigate the accuracy of each method (relative to the MC estimate) across the option strike (K), risk-free rate of interest (r), volatility (σ), number of days to maturity (n) and number of days in the averaging period (m).

4.3.1 Sensitivity of Accuracy to Strike and Interest Rate

Figure 2 shows the accuracy of the three pricing methods to changes in strike and the risk-free interest rate with interesting results. The graph on the left suggests that the TW and CU methods overestimate the true price for the Asian option for a range of K between 80 and 150 corresponding to percentage terms (relative to S_0) of 80% - 150%. The moment-matched solution of Vyncke et al. denoted by MM does not exhibit such behaviour and enjoys fairly even accuracy across the range of K , implying that whether an option is deep in-the-money or not has no discernable impact on the Vyncke et al. solution. Note that the methods are tested against the corresponding MC estimates and therefore a significant amount of the jagged behaviour has to do with the standard error of the simulations and not necessarily due to irregular results of the closed-form pricing algorithms.

The graph on the right of Figure 2 demonstrates that the accuracy of none of the three pricing methods is significantly affected if the risk-free rate parameter (r) is changed, with the error between MM, CU and TW remaining relatively constant apart from the random effect attributable to the Monte Carlo standard errors. We note that once again the Vyncke et al. solution outperforms the others across the entire r spectrum.

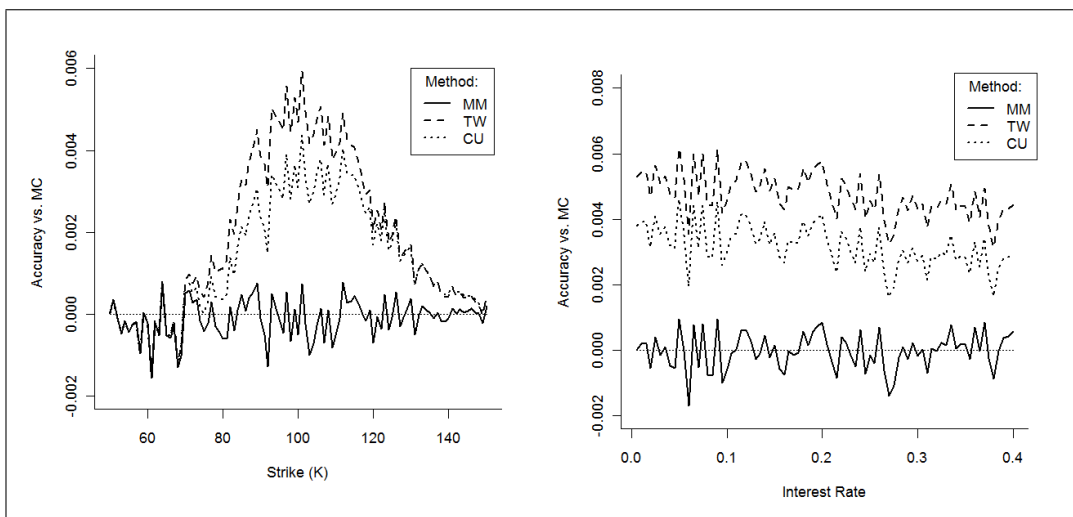


Figure 2: Outperformance of Vyncke et al. solution (MM) of the CU and TW solutions across Strike (K) and Interest Rate (r) parameters. ($S_0 = 0$, $\sigma = 0.3$, $q = 0$, $n = 120$, $m = 30$).

4.3.2 Sensitivity of Accuracy to Volatility

The case for changing volatility is even more startling. Figure 3 demonstrates that as volatility increases along the range 0 – 90% the TW prices begin to diverge from the true value. This is confirmed in Table 2 as high values of σ result in rather severe mis-pricings for the TW solution. A similar phenomenon is experienced by the CU method, but to a lesser extent. This is to be expected as it seems plausible that any approximations made in obtaining the closed form solutions (such as the distributional assumption in Turnbull and Wakeman) would be increasingly violated as the volatility increases. Therefore we are surprised to see that the Vyncke et al. solution (MM) does not exhibit such diverging behaviour - a very pleasing characteristic. The random variation around 0 increases for higher volatilities and is caused by increasing MC standard errors.

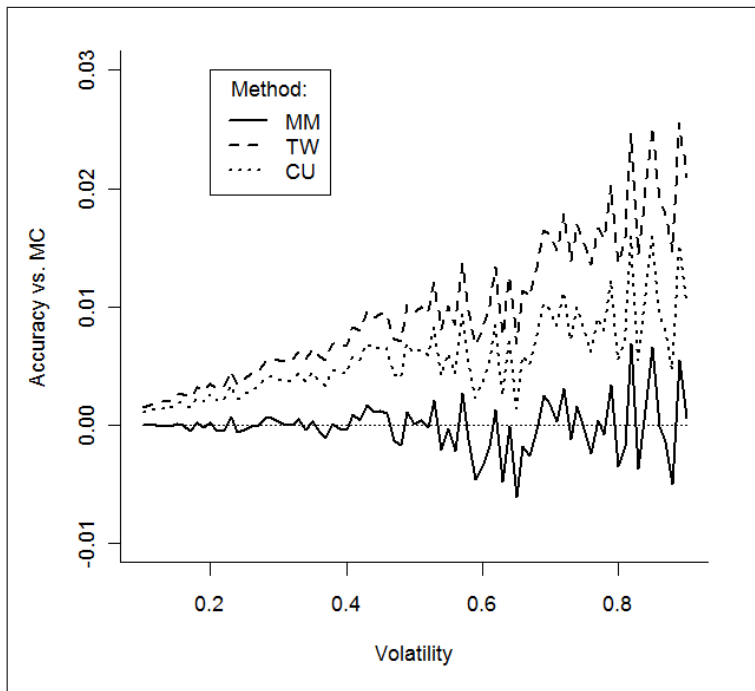


Figure 3: Outperformance of Vyncke et al. solution (MM) of the CU and TW solutions for increasing volatility.

($S_0 = 0$, $K = 100$, $r = \ln(1.09)$, $q = 0$, $n = 120$, $m = 30$).

4.3.3 Sensitivity of Accuracy to Term and Averaging Period

Figure 4 examines the sensitivity of the accuracy of the closed-form solutions if we vary the time to expiry (n) and the length of the averaging period (m). The investigation depicted in the graph on the left suggests that the TW method performs poorly for short-dated Asian options but converges to the CU method as n increases. Once again we note the evenness of the accuracy of the Vyncke et al. solution across all values of n .

The right hand side of Figure 4 provides an interesting analysis of the accuracy across a range of m (the averaging period). Due to the base value of the parameter n being 120 we can consider the range of m from 0 to 120 in percentage terms to be 0-100%. We find the TW method diverging significantly from the true price as the averaging period increases, probably due to the increasing violation of the assumption of lognormality for the arithmetic average stock price as the number of prices in the average increase. The CU method is fairly stable and converges to the solution of Vyncke et al. as the averaging period exceeds 70%. The MM solution is again stable and accurate across the whole spectrum of m .

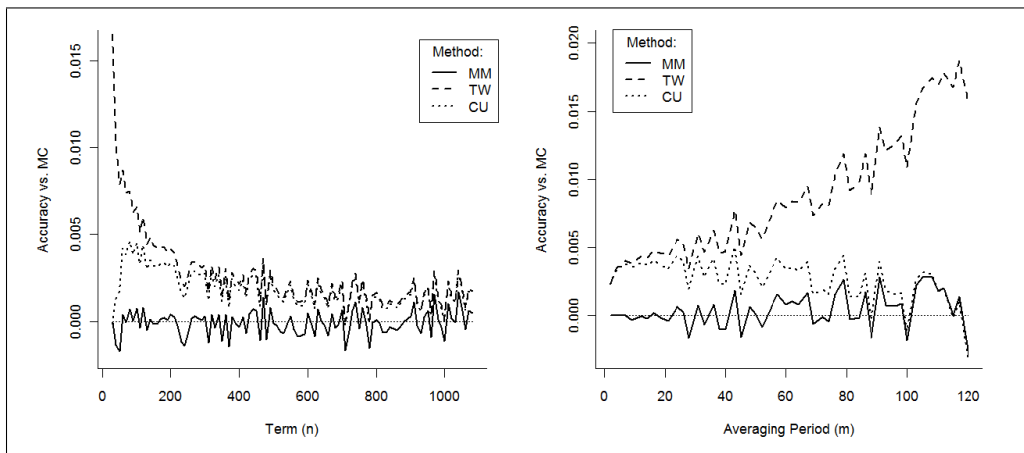


Figure 4: Outperformance of Vyncke et al. solution (MM) of the CU and TW solutions for increasing term (n) and averaging period (m). ($S_0 = 0$, $K = 100$, $\sigma = 0.3$, $r = \ln(1.09)$, $q = 0$).

4.4 Bound Divergence

The theory presented thus far is supported by the numerical results in demonstrating that the true price of a European Asian call option is bounded below

by the theoretical lower bound (LB) and above by the upper bound (UB). Figure 5 shows that these bounds converge when the option is very far in- or out-of-the-money but tends to diverge for strike values slightly greater than S_0 . The degree of divergence is positively related to the volatility parameter which would make it likely that the MM approximation should become inaccurate with increasing volatility as the two building-blocks (LB and UB) diverge. This is not supported by the results in Tables 1-3 and Figure 3 and so we conclude that the moment-matching process adapts to the high volatility to an extent sufficient to ensure consistent accuracy.

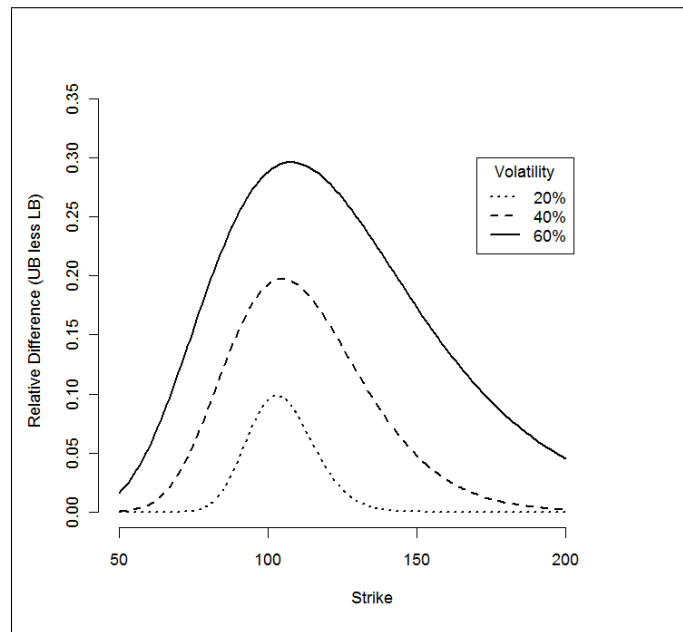


Figure 5: Absolute differences between the upper (UB) and lower bound (LB) of an Asian Option for low, medium and high volatility environments. ($n = 120$ days, $m = 30$ days, $r = \ln(1.09)$, $q = 0$)

5 Summary and Conclusion

European-style arithmetic Asian options are introduced and we highlight the difficulty in pricing them arising from not knowing the distribution of a sum of lognormal variables. Various different techniques and methods for overcoming this difficulty are reviewed in the literature before presenting the solution of Vyncke, Dhaene, and Goovaerts (2004).

Their implemented solution appears to be extremely accurate when compared to Monte Carlo estimates with low standard error, outperforming a selection of other arithmetic Asian option pricing techniques. In particular the Vyncke et al. solution demonstrates a constant accuracy across varying strike levels, suggesting consistent accuracy in option pricing, regardless of whether the option is in- or out-the-money. The constant accuracy across volatility commends the solution as a good one in times of higher than normal market volatility, as was experienced in the wake of the sub-prime turmoil. The solution also performed excellently across varying interest rates, terms and averaging periods, outperforming similar closed-form approximations in each case, at times quite substantially.

While falling outside the scope of this paper it would be logical to consider possible extensions in further research. In particular it would be interesting to investigate the impact of arbitrage-free stochastic interest rate, dividend yield and volatility parameters to the Vyncke et al. solution relative to estimates from Monte Carlo simulation exercises. Also, analysing the accuracy of the solution to the effect of volatility shocks would be especially interesting in the light of the recent financial crisis. It would also be prudent to have a closer look at the impact that pricing inside the averaging period (as discussed in Section 3.4.2) would have on accuracy for this solution.

The Vyncke et al. solution performs with excellent accuracy and is easy to implement. The fact that it is a closed-form solution means that it is extremely fast and would be recommended for practical application where large books of derivatives need to be valued regularly.

A Appendix

A.1 Code

Many lines of code were written in order to successfully implement and test the numerical aspects of this paper and while it would be an inappropriate use of the allocated word-count to include it all here we provide a small sample of some of the most vital code.

The following VBA code snippet was used to find the approximating bounds for the Vyncke et al. solution:

```
Function BoundCalc(S As Double, K As Double, RateA() As Double, _
qA() As Double, VolA() As Double, TimeA() As Double, _
WeightA() As Double, Eta As Integer, n As Integer, m As Integer)

Dim i As Integer, y As Double, UB As Double, Sum As Double, mA() As Double

ReDim mA(n - 1)

    For i = 0 To (n - 1)
        mA(i) = (RateA(i) - qA(i) - 0.5 * (VolA(i) ^ 2))
    Next i

    'Apply Newton method to find y*:
y = -0.1
    For i = 1 To 7
        y = y - f(S, K, m, n, mA, VolA, TimeA, WeightA, y) / _
            fDash(S, m, n, mA, VolA, TimeA, WeightA, y)
    Next i

    Sum = 0
    For i = (n - m) To (n - 1)
        Sum = Sum + WeightA(i) * Exp((RateA(i) - qA(i)) * TimeA(i)) * _
            CDF(Eta * (VolA(i) * Sqr(TimeA(i)) - y))
    Next i

    BoundCalc = Exp(-RateA(n - 1) * TimeA(n - 1)) * Eta * _
(S * Sum - K * CDF(-Eta * y))

End Function
```

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