APPROACHES TO REFLECTIVE HULLS OF SUBCATEGORIES

by

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"So it was: Eilenberg, coming from topology, met Mac Lane, coming from algebra; together they chanced upon a problem involving both these subjects. From this collision, of ideas and of individuals, both homological algebra and category theory developed."

Saunders Mac Lane (1989)

To Vlasta and Michèle.

In memory of my late father, who completed his Ph.D. 20 years ago.

Contents

Introduction and Summary

Reflective subcategories originated as a formal mathematical concept in the 1960's. Perhaps the first abstract definition of reflectivity can be attributed to P. Freyd who, in [Freyd 1960] and [Freyd 1964], gave a definition in terms of *reflection arrows.* Already in [Isbell 1964] the general definition of a reflective subcategory was applied to some concrete situations, and used to formulate one of the first problems concerning reflectivity, namely, whether the intersection of (full, isomorphism-closed) reflective subcategories of the category of uniform spaces is again reflective. This problem, together with analogous questions posed in other contexts (e.g., by H. Herrlich for the category of topological spaces), led to the formulation of the *reflective hull problem* for subcategories in general, namely, whether a given subcategory is contained in a *smallest* reflective supercategory.

Much of the research concerned with reflectivity and the reflective hull problem considers sufficient conditions for a category such that the reflectivity of (certain) subcategories can be described and the existence of reflective hulls can be guaranteed. These conditions are usually given in terms of (co)completeness and (co)wellpoweredness (see, for example, [Tholen 1987], [Kelly 1987]). A primary objective of this thesis is to provide sufficient and *necessary* conditions, formulated in subcategory-related terms, for the reflectivity of a given subcategory, and for the characterisation of the existence of reflective hulls. Our approach to finding appropriate descriptions of reflective hulls is essentially a constructive one, in the sense that we attempt to "generate" the reflective hull of a given subcategory (and hence give a concrete description of the hull) by means of certain closure processes applied to the given subcategory. We should also emphasise that our philosophy is not a conservative one in that, apart from applications of our constructions to particular situations, we make as few global assumptions as possible in our considerations.

Intuitively, there are several ways in which reflectivity can be viewed as a mathematical concept; the results in this thesis emphasise these points of view. First, reflectivity may be viewed as a completeness property, i.e., as a kind of limit procedure; we study the correspondence between reflective hulls and closures of subcategories under certain types of limits. Reflectivity may also be considered as a cocompleteness property; appropriately we also consider the closure of a given subcategory under certain kinds of colimits and its relation to a possible reflective hull. Both of these constructions are generalisations of natural descriptions of reflection arrows in the special case of partially-ordered classes. Finally, reflectivity can be considered as a (subcategory-related) factorisation property; in this context we consider closures of a subcategory in terms of factorisations relative to the given subcategory, and, related to this, closures under special kinds of colimits relative to the given subcategory. In this thesis we also obtain results concerning the relation between reflectivity and weaker concepts; in particular results concerning intersections of reflective subcategories, and reflective hulls of almost reflective subcategories, are given, and applied to concrete situations, for example, the following problem posed in [Rosický and Tholen 1988] : Is the category of complete Boolean algebras an intersection of reflective subcategories of the category of frames ?

We give a survey of the subsequent Chapters :

In **Chapter 0** we fix notation, recall a few definitions, and make some basic observations which may be of use in subsequent Chapters of the thesis.

In **Chapter 1** we follow the point of view that reflectivity is a kind of completeness or cocompleteness property, and attempt to approach possible reflective hulls via closures of subcategories formed by so-called *canonical* (resp. *reflecting*) *limits* and *approaching colimits.* Using the formal criterion for the existence of a left adjoint functor as a starting point, we describe the reflectivity of a given subcategory in terms of canonical limits, define the *canonical* (resp. *reflecting*) *limit closure* of a given subcategory, and characterise the reflectivity of these closures in terms of canonical (resp. reflecting) limits. We show that the respective closures need not, in general, coincide with an existing reflective hull, and make some comparisons between the respective closures, exhibiting an example which shows that the canonical limit closure may properly contain the reflecting limit closure. Finally, we deduce a universal property for the reflecting limit closure, namely that of a Kan extension. Reflectivity is characterised by the existence of approaching colimits; we then study the closure of a given subcategory under approaching colimits, and show that this closure is a better "approximation" of the given subcategory to a possible reflective hull in that it always contains the orthogonal closure of the given subcategory. We show that the existence of reflecting limits implies the existence of approaching colimits but not vice versa, and define an iterated version of closure under approaching colimits in order to gain a better approximation of the given subcategory to a possible reflective hull. Canonical limit closures and approaching colimit closures are compared to other well-known concepts related to reflectivity, for example, limit closures and orthogonal closures. Observations concerning the relation between canonical limit closures and reflective hulls of small subcategories are deduced.

In **Chapter 2**, we study the correspondence between intersections of reflective subcategories and orthogonal subcategories, and also give an explicit description of reflective hulls of almost reflective subcategories in the presence of a factorisation structure for morphisms. The class of complete Boolean algebras, considered as a subcategory of the class of frames, serves as a motivating example for the material in this Chapter. It is shown that the complete Boolean algebras are an example of an orthogonal subcategory which is not an intersection of reflective subcategories of the category of frames. We then show that this result can be generalised to a categorical statement concerning the kinds of subcategories in question. In an attempt to understand almost reflective subcategories in the category of frames which contain the class of complete Boolean algebras, we deduce that for any almost reflective subcategory of frames which contains the class of complete Boolean algebras, an almost reflection of the 3-element frame is an extremal monomorphism.

In **Chapter 3**, we work with the fundamental notion of a factorisation structure *relative* to a given subcategory, that is, a factorisation structure for sources with codomains in the subcategory. After deducing some basic facts about relative factorisations, we study the correspondence between relative factorisations and reflective subcategories. We characterise the reflectivity of a subcategory by the existence of relative factorisations, and further give descriptions of reflective supercategories in terms of relative factorisations. A key observation is that a factorisation structure $(E, \mathbf{M})$ relative to a fixed subcategory induces an E-reflective hull of the subcategory. The reflective supercategories of a fixed subcategory are shown to be in correspondence to a distinguished collection of factorisation structures relative to the given subcategory, namely, the so-called *orthogonal* factorisation structures. From this correspondence, we deduce that a subcategory has a reflective hull if and only if there exists a finest orthogonal factorisation structure relative to the given subcategory. In addition, we show that the existence of a finest (not necessarily orthogonal) relative factorisation structure implies the existence of a reflective hull. As consequences, we characterise the reflectivity of the orthogonal closure of a given subcategory in terms of relative factorisations, and obtain necessary and sufficient conditions, in terms of relative factorisations, for the intersection of a collection of reflective subcategories to be reflective.

In **Chapter 4**, we introduce the new notion of *multiple pushout* (resp. *cointersection*) relative to a subcategory. This definition enables us to characterise relative factorisation structures by the existence of relative multiple pushouts. The approaching colimits of Chapter 1 are shown to be instances of relative cointersections, which can intuitively be viewed as "best approximations" of an object to a given subcategory. We show that a composition-closed class of E morphisms induces a relative factorisation structure if and only if relative E-multiple pushouts exist. For a relative factorisation structure $(E, \mathbf{M})$, we describe the objects of the E-reflective hull as relative E-cointersections. Finally, we show that a subcategory has a reflective hull if and only if there exists a largest composition-closed class E of morphisms orthogonal to the given subcategory, such that relative E-multiple pushouts exist; in that case the objects of the reflective hull are precisely the relative cointersections of sources of morphisms orthogonal to the subcategory.

Chapter 0

Preliminaries

This Chapter is an overview of assumptions, terminology and results which will be used in subsequent Chapters of the thesis. Our primary reference for basic categorical terminology is [Adámek, Herrlich and Strecker 1990], in particular, we use the well-established terminology of sources, sinks and factorisation structures. The reader may also consult [Mac Lane 1971] and [Herrlich and Strecker 1979] for general categorical definitions and concepts. Concerning foundational matters, our approach will be the one taken in [Adámek, Herrlich and Strecker 1990], p. 5 - 9, that is, we assume a hierarchy of sets, classes and conglomerates.

Notation and Basic Definitions.

We shall reserve capital script letters (usually $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$) for denoting categories. In the sequel, unless otherwise stated, we will be considering subcategories of a category $\mathcal{B}$ which are assumed to be *full* and *isomorphism-closed* (all examples of subcategories mentioned in this thesis satisfy these conditions). In a deviation from standard notation, we shall, given an object B of a category $\mathcal{B}$ and a subcategory $\mathcal{A} \subset \mathcal{B}$, denote the source of all $\mathcal{B}$-morphisms with domain B and codomain in $\mathcal{A}$ by $All(B, \mathcal{A})$; the source $All(B, \mathcal{A})$ shall conveniently be called the *all-source from B to $\mathcal{A}$*. Recall that a subcategory $\mathcal{A}$ of $\mathcal{B}$ is said to be *reflective* in $\mathcal{B}$ if, for each $\mathcal{B}$-object B there exists an *$\mathcal{A}$-reflection arrow* for B, i.e., a member $r : B \to A$ of $All(B, \mathcal{A})$ through which $All(B, \mathcal{A})$ uniquely factorises. If $\mathcal{A}$ is a reflective subcategory of $\mathcal{B}$, the associated reflection functor will frequently be denoted by r, the $\mathcal{A}$-reflection arrow for a $\mathcal{B}$-object B denoted by $r_B : B \to rB$.

Limits and colimits will normally be identified via their associated limit sources (resp. colimit sinks). If $(f_i : B \to B_i)_I$ and $(g_j : C \to C_j)_J$ are sources in a category $\mathcal{B}$, and $h : B \to C$ is a $\mathcal{B}$-morphism, then h is said to be a *source map* if for each $j \in J$ there exists $i \in I$ such that $f_i = g_j \cdot h$. Finally, throughout the text, the symbol "$\subset$" is taken to include the possibility of equality.

Concepts Related To Reflectivity.

Let $\mathcal{A}$ be a subcategory of $\mathcal{B}$. Then $\mathcal{A}$ is said to be *closed under the formation of limits in* $\mathcal{B}$ (or, *limit-closed in* $\mathcal{B}$) if, for every diagram $D : I \to \mathcal{B}$ which has a limit in $\mathcal{B}$, with the property that $D(i) \in \mathcal{A}$ for each $i \in I$, it follows that the limit object corresponding to the limit of D belongs to $\mathcal{A}$.

A $\mathcal{B}$-object A is said to be *orthogonal* with respect to a $\mathcal{B}$-morphism $p : B \to C$ if for each $\mathcal{B}$-morphism $f : B \to A$ there exists a unique $\mathcal{B}$-morphism $\bar{f} : C \to A$ satisfying $f = \bar{f} \cdot p$. Given a class $\mathcal{H}$ of $\mathcal{B}$-morphisms, $\mathcal{A}$ is said to be *orthogonal* with respect to $\mathcal{H}$ (written $\mathcal{A} = \mathcal{H}_\perp$) if $\mathcal{A}$ consists of precisely those $\mathcal{B}$-objects which are orthogonal with respect to every element of $\mathcal{H}$. The class of all $\mathcal{B}$-morphisms orthogonal with respect to $\mathcal{A}$ is denoted by $\mathcal{A}^\perp$, and morphisms in $\mathcal{A}^\perp$ will be called *$\mathcal{A}$-orthogonal.* Note that the $\mathcal{A}$-orthogonal morphisms $p : B \to C$ are exactly those epimorphisms relative to $\mathcal{A}$ that are first factors of the all-source $All(B, \mathcal{A})$ (where an epimorphism *relative* to a given subcategory $\mathcal{A}$ is a $\mathcal{B}$-morphism $p : B \to C$ such that for any pair $g, h : C \rightrightarrows A$ of $\mathcal{B}$-morphisms from B to an $\mathcal{A}$-object A, $g \cdot p = h \cdot p$ implies $g = h$). Note also that the class $\mathcal{A}^\perp$ is closed under composition. The orthogonality relation defines a Galois correspondence between subcategories of $\mathcal{B}$ and morphism classes in $\mathcal{B}$; orthogonal subcategories are closed under this correspondence, in particular, we have $\mathcal{A} = (\mathcal{A}^\perp)_\perp$ for an orthogonal subcategory of $\mathcal{B}$. For more details, see [Freyd and Kelly 1972], [Tholen 1983], [Tholen 1986] and [Tholen 1987].

An appropriate weakening of the concept of orthogonality yields the notion of *injectivity class* : A $\mathcal{B}$-object A is said to be *injective* with respect to a $\mathcal{B}$-morphism $p : B \to C$ if for each $\mathcal{B}$-morphism $f : B \to A$ there exists a (not necessarily unique) $\mathcal{B}$-morphism $\bar{f} : C \to A$ satisfying $f = \bar{f} \cdot p$. Given a class $\mathcal{H}$ of $\mathcal{B}$-morphisms, $\mathcal{A}$ is said to be *injective with respect to* $\mathcal{H}$ (written : $\mathcal{A} = Inj(\mathcal{H})$) if $\mathcal{A}$ consists of precisely those $\mathcal{B}$-objects which are injective with respect to every element of $\mathcal{H}$.

A *prereflection* (T, η) on a category $\mathcal{B}$ consists of an endofunctor $T : \mathcal{B} \to \mathcal{B}$ and a natural transformation $\eta : Id_\mathcal{B} \to T$, with the property that for every $f : B \to C$ and

$g : TB \to TC$ in $\mathcal{B}$, $g \cdot \eta_B = \eta_C \cdot f$ implies $g = Tf$. A subcategory $\mathcal{A}$ of $\mathcal{B}$ is said to be *prereflective* in $\mathcal{B}$ if $\mathcal{A} = \{B \in Ob(\mathcal{B}) \mid \eta_B \text{ is a } \mathcal{B}\text{-isomorphism}\}$ (see [Tholen 1983], [Tholen 1986]).

Following [Herrlich 1993a], a subcategory $\mathcal{A}$ of $\mathcal{B}$ is called *weakly reflective* in $\mathcal{B}$ if for every $\mathcal{B}$-object there exists a $\mathcal{B}$-morphism $r_B : B \to A_B$ from B to an $\mathcal{A}$-object A_B with the property that for every $\mathcal{B}$-morphism $f : B \to A$ from B to an $\mathcal{A}$-object A there exists a (not necessarily unique) $\mathcal{A}$-morphism $\bar{f} : A_B \to A$ such that $f = \bar{f} \cdot r_B$. If $\mathcal{A}$ is in addition closed under retracts in $\mathcal{B}$, then $\mathcal{A}$ is called *almost reflective* in $\mathcal{B}$. An almost reflective subcategory $\mathcal{A}$ of $\mathcal{B}$ for which every almost $\mathcal{A}$-reflection is a monomorphism in $\mathcal{B}$ will be called *almost monoreflective* in $\mathcal{B}$.

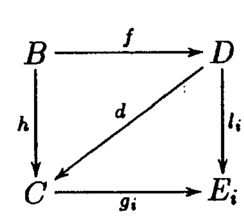
The following sequences of implications for subcategories are well-established (see, e.g., [Rosický and Tholen 1988], [Herrlich 1993a], [Adámek and Rosický 1993]) :

(1) reflective $\Rightarrow$ prereflective $\Rightarrow$ orthogonal $\Rightarrow$ limit-closed;
(2) reflective $\Rightarrow$ intersection of reflective subcategories $\Rightarrow$ orthogonal;
(3) almost reflective $\Rightarrow$ intersection of almost reflective subcategories $\Rightarrow$ injectivity class $\Rightarrow$ closed under products and retracts.

For counter-examples showing that the implications in (1) to (3) need not in general be reversible, see also the cited references.

Orthogonality.

In a simultaneous generalisation of the diagonal condition with respect to morphism factorisation structures and the orthogonality relation defined on classes of morphisms in [Freyd and Kelly 1972], we say that a $\mathcal{B}$-morphism $f : B \to D$ is *orthogonal* to a $\mathcal{B}$-source $(g_i : C \to E_i)_I$ (written $f \downarrow (g_i)$) if given any $\mathcal{B}$-morphism $h : B \to C$ and any $\mathcal{B}$-source $(l_i : D \to E_i)_I$ such that $g_i \cdot h = l_i \cdot f$ for each $i \in I$, there exists a unique diagonal $d : D \to C$ such that



$h = d \cdot f$ and $l_i = g_i \cdot d$ for each $i \in I$. Given a class E of morphisms, the collection of all $\mathcal{B}$-sources which are orthogonal to every member of E shall be denoted by $E^{\downarrow}$. Dually, for a conglomerate M of $\mathcal{B}$-sources, we shall denote by $M^{\uparrow}$ the class of all $\mathcal{B}$-morphisms which are orthogonal to every member of M.

Chapter 1

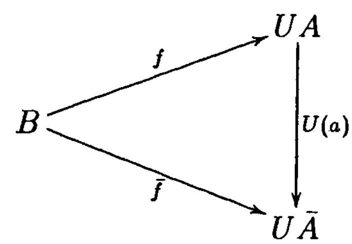
Subcategories Generated Via Canonical Limits and Approaching Colimits

1 Canonical Limit Closures.

We begin our study of approaches to reflectivity (via concepts weaker than reflectivity) by considering closures of subcategories with respect to so-called *canonical limits.* The study of such closures will be seen, inter alia, to incorporate the study of codense subcategories. In the hierarchy of concepts related to reflectivity that shall be studied, the concept of canonical limit closure is the furthest away from reflectivity in the sense that the subcategories generated via closure under canonical (resp. reflecting) limits (to be discussed in this Section) are contained in the subcategories generated by many other closure processes (e.g. limit closures, orthogonal closures) that occur.

Before commencing with the definitions, we attempt to clarify to some extent the origins of closure via canonical limits, at the same time justifying why the concept should be viewed as an "approach" to reflectivity. For the origins of the concept, one need look no further than [Mac Lane 1971], Chapter X, p. 230, where a "formal criterion for the existence of an adjoint" is given, which dates back to [Bénabou 1965].

Given a functor $U : \mathcal{A} \to \mathcal{B}$, let B be a $\mathcal{B}$-object. Denote by $(B \downarrow U)$ the comma category with objects pairs (f, A), where $f : B \to UA$ is a $\mathcal{B}$-morphism, and morphisms $a : (f, A) \to (\bar{f}, \bar{A})$ those $\mathcal{A}$-morphisms $a : A \to \bar{A}$ such that the diagram



commutes, i.e., $\bar{f} = U(a) \cdot f$. Define a diagram $D_B^U : (B \downarrow U) \to \mathcal{A}$ by $D_B^U(f, A) = A$, and $D_B^U(a : (f, A) \to (\bar{f}, \bar{A})) = a$. The diagram $D_B^U : (B \downarrow U) \to \mathcal{A}$ shall be called a *U-canonical diagram*, and the limit of $D_B^U : (B \downarrow U) \to \mathcal{A}$, if it exists, shall conveniently be called a *U-canonical limit.* We state Bénabou's formal criterion for the existence of an adjoint in a slightly modified form :

1.1 Theorem. *For each $\mathcal{B}$-object B, the following conditions are equivalent :*

(1) there exists a U-universal morphism over B;

(2) (a) the limit of UD_B^U exists in $\mathcal{B}$, and

(b) U creates the limit of UD_B^U.

PROOF. (1) $\Rightarrow$ (2) : suppose that there exists a U-universal morphism for B. Then by Theorem 2, p. 230 of [Mac Lane 1971] the limit of D_B^U exists in $\mathcal{A}$. From the proof of [Adámek, Herrlich and Strecker 1990] 18.9 the limit of UD_B^U exists in $\mathcal{B}$, and U clearly creates this limit. Hence conditions (a) and (b) of (2) are satisfied.
(2) $\Rightarrow$ (1) : from conditions (a) and (b) it follows that the limit of D_B^U exists in $\mathcal{A}$. Hence the argument used in Theorem 2, p. 230 of [Mac Lane 1971] can be applied to show that there exists a U-universal morphism over B. □

We shall apply Theorem 1.1 to the situation where $\mathcal{A}$ is a (full and isomorphism-closed) subcategory of $\mathcal{B}$. Given $B \in \mathcal{B}$, let $(B \downarrow \mathcal{A})$ denote the comma category of $\mathcal{A}$-objects under B. In a slight modification of the concept of a U-canonical diagram defined above, we define an *$\mathcal{A}$-canonical diagram* (or simply, *canonical diagram*) to be a diagram $D_B^{\mathcal{A}} : (B \downarrow \mathcal{A}) \to \mathcal{B}$, which sends an object (f, A) of $(B \downarrow \mathcal{A})$ to A, and a morphism $a : (f, A) \to (\bar{f}, \bar{A})$ of $(B \downarrow \mathcal{A})$ to a. We shall denote this diagram by D_B when the context is clear. Limits of such diagrams will be referred to as *($\mathcal{A}$)-canonical limits.* As a consequence of 1.1 we obtain :

1.2 Proposition. *The following conditions are equivalent :*

(1) B has an $\mathcal{A}$-reflection;

(2) the limit of D_B exists in $\mathcal{B}$, and the limit object belongs to $\mathcal{A}$. □

Hence we obtain :

1.3 Corollary. *$\mathcal{A}$ is reflective in $\mathcal{B}$ iff $\mathcal{B}$ has, and $\mathcal{A}$ is closed in $\mathcal{B}$ under the formation of, $\mathcal{A}$-canonical limits.* □

1.4 Remark. If $\mathcal{A}$ and $\mathcal{B}$ in 1.3 above are assumed to be partially-ordered classes, then 1.3 reduces to the following : $\mathcal{A}$ is reflective in $\mathcal{B}$ iff for each $B \in \mathcal{B}$, $\bigwedge\{A \in \mathcal{A} \mid B \leq A\}$ exists and belongs to $\mathcal{A}$, where $\bigwedge$ denotes infimum in $\mathcal{B}$. This special case will be discussed again in Section 2 of this Chapter.

1.5 Notation. Henceforth, for the sake of convenience, we shall write any source $(l_{(f,A)})_{(f,A)\in(B\downarrow\mathcal{A})}$ which is indexed by the comma category $(B \downarrow \mathcal{A})$ as $(l_f)_{(B\downarrow\mathcal{A})}$ (note that since $\mathcal{A}$ is embedded in $\mathcal{B}$, in this case we may abbreviate (f, A) to f without disrupting the indexing via $(B \downarrow \mathcal{A})$).

We now come to the definition of codensity :

1.6 Definition. [Mac Lane 1971],[Adámek, Herrlich and Strecker 1990] A subcategory $\mathcal{A}$ of $\mathcal{B}$ is said to *codense* in $\mathcal{B}$ if for each $\mathcal{B}$-object B, the (self-indexed) source $All(B, \mathcal{A})$ of all $\mathcal{B}$-morphisms with domain B which have codomains in $\mathcal{A}$ is the limit source for D_B.

The above condition for codensity may be reformulated in an alternative manner : suppose that for a given $\mathcal{B}$-object B, the limit of D_B exists; denote the limit source by $(l_f : L \to A_f)_{(B\downarrow\mathcal{A})}$. Since $All(B, \mathcal{A})$ determines a natural source for D_B (note that the members of $All(B, \mathcal{A})$ are in bijective correspondence with the objects of $(B \downarrow \mathcal{A})$), there exists by the limit property of $(l_f)_{(B\downarrow\mathcal{A})}$ a unique $r_B : B \to L$ such that for each $(f, A) \in (B \downarrow \mathcal{A})$, $f = l_f \cdot r_B$. We call r_B the *canonical morphism* from B to the limit object L. Then,

1.7 Proposition. *Let $B \in \mathcal{B}$, and suppose that the limit $(L, (l_f)_{(B\downarrow\mathcal{A})})$ of D_B exists. Then the following conditions are equivalent :*

(1) $All(L,\mathcal{A})$ *is the limit source for the* $\mathcal{A}$*-canonical diagram* D_L*, i.e.,* $(L,(l_f)_{(B\downarrow\mathcal{A})}) = All(L,\mathcal{A})$*;*

(2) the canonical morphism $r_B : B \to L$ *is an epimorphism relative to* $\mathcal{A}$*, i.e., for any* $\mathcal{B}$*-morphisms* $p,q : L \rightrightarrows A$ *from* L *to an* $\mathcal{A}$*-object* A*,* $p \cdot r_B = q \cdot r_B$ *implies* $p = q$*.*

PROOF. (1) $\Rightarrow$ (2) : let $p,q : L \rightrightarrows A$ be $\mathcal{B}$-morphisms from L to an $\mathcal{A}$-object A, such that $p \cdot r_B = q \cdot r_B$. Since $(L,l_f)_{(B\downarrow\mathcal{A})} = All(L,\mathcal{A})$, $p = l_f$ and $q = l_g$ for some f and g respectively in $(B \downarrow \mathcal{A})$. Thus, $f = l_f \cdot r_B = p \cdot r_B = q \cdot r_B = l_g \cdot r_B = g$; hence $p = l_f = l_g = q$.
(2) $\Rightarrow$ (1) : let $p : L \to A$ be a member of $All(L,\mathcal{A})$. Then $(p \cdot r_B, A) \in (B \downarrow \mathcal{A})$ and $p \cdot r_B = l_{p\cdot r_B} \cdot r_B$, hence, since r_B is an epimorphism relative to $\mathcal{A}$, and $A \in \mathcal{A}$, $p = l_{p\cdot r_B}$; thus we have shown that p belongs to $(l_f)_{(B\downarrow\mathcal{A})}$, so $(L,(l_f)_{(B\downarrow\mathcal{A})}) = All(L,\mathcal{A})$. □

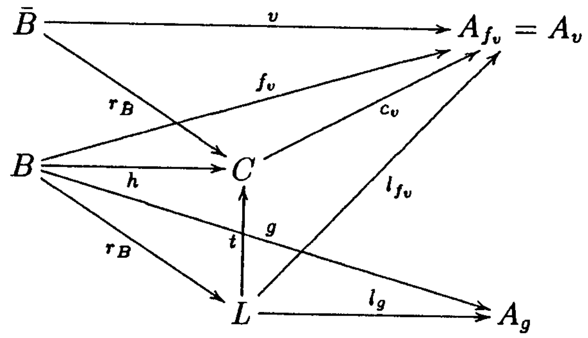
Henceforth, we shall say that $\mathcal{A}$-canonical limits satisfying the equivalent conditions in 1.7 above have the $\mathcal{A}$-*reflecting property*, alternatively, that these limits are $\mathcal{A}$-*reflecting*; a canonical morphism from B that satisfies (2) of 1.7 will be called an $\mathcal{A}$-*reflecting morphism* for B. Let $\mathbf{CL}(\mathcal{A})$ (resp. $\mathbf{RL}(\mathcal{A})$) denote the full subcategory of $\mathcal{B}$ consisting of all limit objects of $\mathcal{A}$-canonical (resp. $\mathcal{A}$-reflecting) diagrams. Note then that $\mathcal{A} \subset \mathbf{RL}(\mathcal{A}) \subset \mathbf{CL}(\mathcal{A})$. Further, since by 1.7 above every $\mathbf{RL}(\mathcal{A})$-object L is by definition the limit object of the limit source for D_L, $\mathcal{A}$ is codense in $\mathbf{RL}(\mathcal{A})$. The category $\mathbf{RL}(\mathcal{A})$ is also the largest subcategory of $\mathcal{B}$ in which $\mathcal{A}$ is codense.

1.8 Proposition. *Equivalent are :*

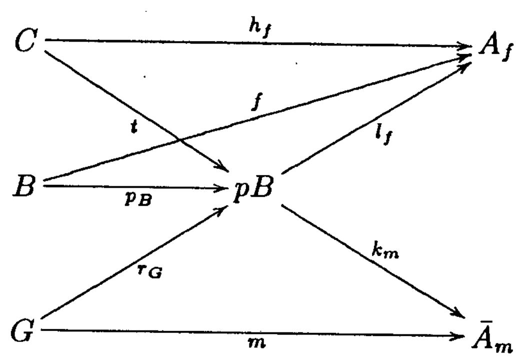
(1) $\mathcal{B}$ *has* $\mathcal{A}$*-reflecting limits of all* $\mathcal{A}$*-canonical diagrams;*

(2) $\mathbf{RL}(\mathcal{A})$ *is reflective in* $\mathcal{B}$*.*

PROOF. (1) $\Rightarrow$ (2) : Let $B \in \mathcal{B}$; let $(l_f : L \to A_f)_{(B\downarrow\mathcal{A})}$ be the limit source for D_B. We shall show that the canonical morphism $r_B : B \to L$ (see the discussion prior to 1.7) is the $\mathbf{RL}(\mathcal{A})$-reflection morphism for B.



Let $h : B \to C$ be a $\mathcal{B}$-morphism with $C \in \mathbf{RL}(\mathcal{A})$. Then C is the limit object for a limit source $(c_v : C \to A_v)_{(\bar{B}\downarrow\mathcal{A})}$ of $D_{\bar{B}}$ for some $\bar{B} \in \mathcal{B}$. For each $(v, A_v) \in (\bar{B} \downarrow \mathcal{A})$, $(c_v \cdot h, A_v) \in (B \downarrow \mathcal{A})$; denote this composition by f_v. Then to each $v \in (\bar{B} \downarrow \mathcal{A})$ there corresponds an $l_{f_v} : L \to A_{f_v} = A_v$ such that $l_{f_v} \cdot r_B = f_v$ $(= c_v \cdot h)$ (as shown in the above diagram). The source $(l_{f_v})_{(\bar{B}\downarrow\mathcal{A})}$ is natural for $D_{\bar{B}}$, so since $(c_v)_{(\bar{B}\downarrow\mathcal{A})}$ is the limit of $D_{\bar{B}}$, there exists a unique $t : L \to C$ such that $l_{f_v} = c_v \cdot t$ for each $v \in (\bar{B} \downarrow \mathcal{A})$. For each $v \in (\bar{B} \downarrow \mathcal{A})$ we have $c_v \cdot h = l_{f_v} \cdot r_B = c_v \cdot t \cdot r_B$, hence since $(c_v)_{(\bar{B}\downarrow\mathcal{A})}$ is in particular a mono-source, $h = t \cdot r_B$. Now consider a $\mathcal{B}$-morphism $s : L \to C$ such that $s \cdot r_B = h$. Then, for all $v \in (\bar{B} \downarrow \mathcal{A})$, $c_v \cdot s \cdot r_B = c_v \cdot h = c_v \cdot t \cdot r_B$, hence $c_v \cdot s = c_v \cdot t$ (since r_B is an epimorphism relative to $\mathcal{A}$), and so $s = t$ since $(c_v)_{(\bar{B}\downarrow\mathcal{A})}$ is a mono-source.
(2) $\Rightarrow$ (1) :

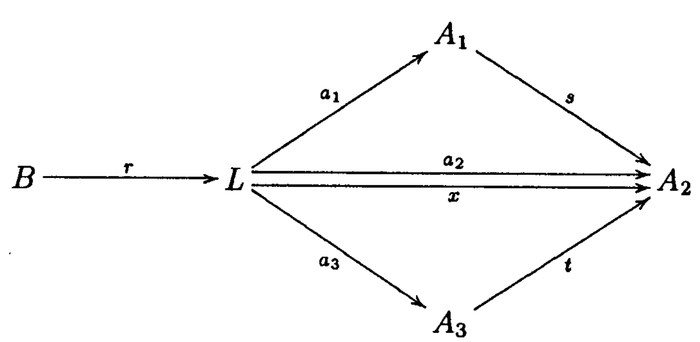


Let $p_B : B \to pB$ be the $\mathbf{RL}(\mathcal{A})$-reflection for B. For each $(f, A_f) \in (B \downarrow \mathcal{A})$ there exists a (unique) $l_f : pB \to A_f$ such that $f = l_f \cdot p_B$. Since p_B is an epimorphism relative to $\mathcal{A}$, $(pB, (l_f)_{(B\downarrow\mathcal{A})}) = All(pB, \mathcal{A})$: for any member $s : pB \to A$ of $All(pB, \mathcal{A})$, we have $s \cdot p_B = l_{s \cdot p_B} \cdot p_B$, hence $s = l_{s \cdot p_B}$. Now, pB is an $\mathbf{RL}(\mathcal{A})$-object, hence the limit object for the $\mathcal{A}$-reflecting limit $(pB, (k_m)_{(G\downarrow\mathcal{A})})$ of D_G for some $\mathcal{B}$-object G, so by 1.7 $(pB, (k_m)_{(G\downarrow\mathcal{A})}) = All(pB, \mathcal{A}) = (pB, (l_f)_{(B\downarrow\mathcal{A})})$. We wish to show that $(pB, (l_f)_{(B\downarrow\mathcal{A})})$ is the limit of D_B. Since p_B is an epimorphism relative to $\mathcal{A}$, $(pB, (l_f)_{(B\downarrow\mathcal{A})})$ is natural for

D_B. Let $(C, (h_f)_{(B\downarrow\mathcal{A})})$ be natural for D_B. Then $(C, (h_{k_m \cdot p_B})_{(G\downarrow\mathcal{A})})$ is natural for D_G : let $(m, \bar{A}_m)$, $(n, \bar{A}_n)$ be $(G \downarrow \mathcal{A})$-objects, and suppose that $a : \bar{A}_m \to \bar{A}_n$ is an $\mathcal{A}$-morphism such that $n = a \cdot m$. Now, $(k_m \cdot p_B, \bar{A}_m)$ and $(k_n \cdot p_B, \bar{A}_n)$ are $(B \downarrow \mathcal{A})$-objects, and moreover, since $k_n = a \cdot k_m$ (naturality of $(pB, (k_m)_{(G\downarrow\mathcal{A})})$ for D_G), $k_n \cdot p_B = a \cdot k_m \cdot p_B$ (i.e., $a : k_m \cdot p_B \to k_n \cdot p_B$ is a $(B \downarrow \mathcal{A})$-morphism), hence $h_{k_n \cdot p_B} = a \cdot h_{k_m \cdot p_B}$ since $(C, (h_f)_{(B\downarrow\mathcal{A})})$ is natural for D_B. So by the limit property of $(pB, (k_m)_{(G\downarrow\mathcal{A})})$ there exists a unique $t : C \to pB$ such that $k_m \cdot t = h_{k_m \cdot p_B}$ for each m in $(G \downarrow \mathcal{A})$. Since $(pB, (k_m)_{(G\downarrow\mathcal{A})}) = (pB, (l_f)_{(B\downarrow\mathcal{A})})$, it follows that $t : C \to pB$ is the unique morphism such that $h_f = t \cdot l_f$ for each $f \in (B \downarrow \mathcal{A})$. □

We have seen that every $\mathcal{A}$-reflecting limit of a canonical diagram is a fortiori the limit of that diagram. The next example shows that an $\mathcal{A}$-canonical limit need not be an $\mathcal{A}$-reflecting limit :

1.9 Example. Consider the following situation,



where $\mathcal{B}$ is given by the entire diagram and the object class of $\mathcal{A}$ is defined to be $\{A_1, A_2, A_3\}$. In the above diagram, the morphism $x : L \to A_2$ has the property that $x \neq a_2$, but $x \cdot r = a_2 \cdot r$; further, we have $s \cdot a_1 = a_2 = t \cdot a_3$. Then $Ob(B \downarrow \mathcal{A}) = \{(a_1 \cdot r, A_1), (a_2 \cdot r, A_2), (a_3 \cdot r, A_3)\}$, and (omitting identities) $Mor(B \downarrow \mathcal{A}) = \{s, t\}$. It can be checked that $(L, \{a_1, a_2, a_3\})$ is the limit of D_B, but since r is not an epimorphism relative to $\mathcal{A}$, $(L, \{a_1, a_2, a_3\})$ is not the $\mathcal{A}$-reflecting limit of D_B (this limit does not exist here).

In order for the constructions $\mathbf{CL}(\mathcal{A})$ and $\mathbf{RL}(\mathcal{A})$ to be considered as "approximations" to reflective subcategories of $\mathcal{B}$ containing $\mathcal{A}$, $\mathbf{CL}(\mathcal{A})$ and $\mathbf{RL}(\mathcal{A})$ should at least be contained in every reflective subcategory of $\mathcal{B}$ which contains $\mathcal{A}$ (and hence also in the reflective hull of $\mathcal{A}$ in $\mathcal{B}$, if it exists). To this effect, we have :

1.10 Proposition. $\mathbf{CL}(\mathcal{A})$*, hence* $\mathbf{RL}(\mathcal{A})$*, is contained in every reflective subcategory of* $\mathcal{B}$ *which contains* $\mathcal{A}$*.*

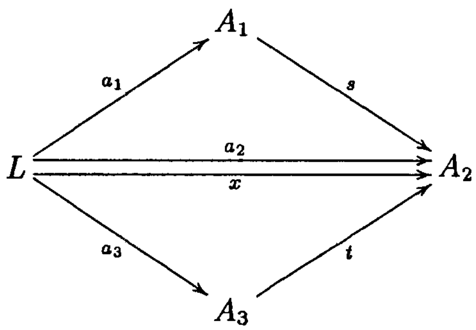
PROOF. This follows since reflective subcategories in $\mathcal{B}$ are closed under the formation of all limits. □

From 1.8 and 1.10 we obtain :

1.11 Proposition. *If* $\mathbf{RL}(\mathcal{A})$ *is reflective in* $\mathcal{B}$*, then* $\mathbf{RL}(\mathcal{A}) = \mathbf{CL}(\mathcal{A})$*, and* $\mathbf{RL}(\mathcal{A})$ *is the reflective hull of* $\mathcal{A}$ *in* $\mathcal{B}$*. In particular, if* $\mathcal{B}$ *has* $\mathcal{A}$*-reflecting limits, then* $\mathcal{A}$ *has a reflective hull in* $\mathcal{B}$*, namely,* $\mathbf{RL}(\mathcal{A})$*.* □

In contrast to the situation in 1.11 above, the reflectivity of $\mathbf{CL}(\mathcal{A})$ need not imply the reflectivity of $\mathbf{RL}(\mathcal{A})$ (and hence need not imply that $\mathbf{RL}(\mathcal{A})$ and $\mathbf{CL}(\mathcal{A})$ coincide). Consider the following modification of 1.9 :

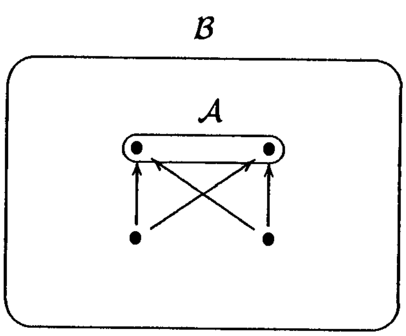
1.12 Example. Take $\mathcal{A}$ to be as in 1.9, and $\mathcal{B}$ to be the full subcategory $\{L, A_1, A_2, A_3\}$ of the whole category shown in 1.9 :



From the data in the diagram, $\mathbf{CL}(\mathcal{A}) = \mathcal{B}$, i.e., $\mathbf{CL}(\mathcal{A})$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$; however, $\mathbf{RL}(\mathcal{A}) = \mathcal{A}$.

Note that in the situation that the reflective hull of $\mathcal{A}$ in $\mathcal{B}$ exists, the respective closures $\mathbf{CL}(\mathcal{A})$ and $\mathbf{RL}(\mathcal{A})$ need not, in general, coincide with the reflective hull. Let $\mathcal{A}$ and $\mathcal{B}$ be as in Example 1.9. It follows from the calculations in 1.9 that $\mathbf{RL}(\mathcal{A}) = \mathcal{A}$ and $\mathbf{CL}(\mathcal{A}) = \mathcal{A} \cup \{L\}$, i.e., closing up under $\mathcal{A}$-reflecting limits has no effect in this case. The reflective hull of $\mathcal{A}$ in $\mathcal{B}$ in both examples 1.9 and 1.12 is, however, the entire category $\mathcal{B}$. These examples also exhibit a situation where $\mathbf{RL}(\mathcal{A})$ is properly contained in $\mathbf{CL}(\mathcal{A})$. Another instance showing $\mathbf{CL}(\mathcal{A})$ and $\mathbf{RL}(\mathcal{A})$ to be properly contained in the reflective hull of $\mathcal{A}$ in $\mathcal{B}$ is the following :

1.13 Example.



Let $\mathcal{A}$ and $\mathcal{B}$ be as in the above diagram. Then $\mathbf{CL}(\mathcal{A}) = \mathbf{RL}(\mathcal{A}) = \mathcal{A}$, whereas $\mathcal{B}$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$.

In Proposition 1.8 it was shown that $\mathbf{CL}(\mathcal{A})$ is reflective in $\mathcal{B}$ if and only if $\mathcal{B}$ has all $\mathcal{A}$-reflecting limits. In this case, $\mathbf{CL}(\mathcal{A})$ coincides with $\mathbf{RL}(\mathcal{A})$. So the condition "$\mathcal{B}$ has all $\mathcal{A}$-canonical limits" is alone not sufficient for the reflectivity of $\mathbf{CL}(\mathcal{A})$. However, we do have the following, as the proof of 1.8 immediately shows :

1.14 Proposition. *If $\mathcal{B}$ has all $\mathcal{A}$-canonical limits, then* $\mathbf{CL}(\mathcal{A})$ *is weakly reflective in $\mathcal{B}$.* □

Next, we show that the reflectivity of $\mathbf{RL}(\mathcal{A})$ (hence $\mathbf{CL}(\mathcal{A})$) may depend on set-theoretical hypotheses : in Example 12E of [Adámek, Herrlich and Strecker 1990], the following statements are presented as equivalent :

(1) the category with object class $\{\mathbf{N}\}$, (where $\mathbf{N}$ denotes the set of natural numbers) and morphisms all functions from $\mathbf{N}$ to $\mathbf{N}$ is a codense subcategory of **Set**;

(2) no cardinal is measurable, that is, every ultrafilter which is closed under the formation of countable meets is trivial, i.e., closed under the formation of all meets.

We note that (1) above is equivalent to the assertion that $\mathbf{RL}(\{\mathbf{N}\})$ (hence $\mathbf{CL}(\{\mathbf{N}\})$ is reflective (hence the reflective hull of $\{\mathbf{N}\}$) in **Set** :

Sufficiency of (1) : if $\{\mathbf{N}\}$ is codense in **Set**, then by the remarks following 1.7 **Set** = $\mathbf{RL}(\{\mathbf{N}\})$, hence by 1.11 $\mathbf{RL}(\{\mathbf{N}\})$ is the reflective hull of $\{\mathbf{N}\}$ in **Set**.

Necessity of (1) : suppose that $\mathbf{RL}(\{\mathrm{N}\})$ is reflective in **Set**. We show that $\mathbf{RL}(\{\mathrm{N}\})$ = **Set**. The forward inclusion is clear. For the reverse inclusion, first note that by 1.8 **Set** has all $\{\mathrm{N}\}$-reflecting limits. So, given a set S, the $\{\mathrm{N}\}$-reflecting limit of the $\{\mathrm{N}\}$-canonical diagram D_S exists in **Set**. Let $r_S : S \to L_S$ denote the $\{\mathrm{N}\}$-reflecting morphism from S into the associated $\{\mathrm{N}\}$-reflecting limit object. We shall show that r_S is a bijection : this is certainly the case if S is empty since the empty set is the limit object for the $\{\mathrm{N}\}$-reflecting limit of its own canonical diagram. If S is not empty, then the source $All(S, \{\mathrm{N}\})$ of all maps from S to $\{\mathrm{N}\}$ is a mono-source : for distinct elements x, y in S, the map k sending x to 0 and all remaining elements s in S to 1 separates x and y. Since $All(S, \{\mathrm{N}\})$ factorises through r_S, it follows that r_S is injective, hence a section, i.e., there exists $t : L_S \to S$ such that $t \cdot r_S = id_S$. Now, if $(L_S, (l_f)_{(S\downarrow\{\mathrm{N}\})})$ denotes the $\{\mathrm{N}\}$-reflecting limit source for D_S, then for each f in $(S \downarrow \{\mathrm{N}\})$, $l_f \cdot r_S \cdot t \cdot r_S = l_f \cdot r_S$, hence $l_f \cdot r_S \cdot t = l_f$ since r_S is an epimorphism relative to $\{\mathrm{N}\}$. But then $r_S \cdot t = id_{L_S}$ since $(l_f)_{(S\downarrow\{\mathrm{N}\})}$ is in particular a mono-source. Hence every set is, up to isomorphism, the limit object for the $\{\mathrm{N}\}$-reflecting limit of its own $\{\mathrm{N}\}$-canonical diagram. So **Set** = $\mathbf{RL}(\{\mathrm{N}\})$, and consequently from the definition of $\mathbf{RL}(\{\mathrm{N}\})$ it follows that $\{\mathrm{N}\}$ is codense in **Set**.

1.15 Remark. Hence if (1) or (2) above does not hold, $\mathbf{RL}(\{\mathrm{N}\})$ is properly contained in **Set**. Note that **Set** is always the reflective hull of $\{\mathrm{N}\}$ in **Set**, since **Set** is the limit closure of $\{\mathrm{N}\}$ in **Set** (**Set** is complete and wellpowered, and every set is an extremal subobject of a power of N).

Finally, we consider a universal property of $\mathbf{RL}(\mathcal{A})$, in terms of Kan extension (note that in the following it is not necessary to make any assumptions concerning the existence of $\mathcal{A}$-canonical limits in $\mathcal{B}$) :

1.16 Proposition. *The inclusion* $\mathbf{RL}(\mathcal{A}) \hookrightarrow \mathcal{B}$ *is the right Kan extension of* $\mathcal{A} \hookrightarrow \mathcal{B}$ *along* $\mathcal{A} \hookrightarrow \mathbf{RL}(\mathcal{A})$.

PROOF. Denote by E the inclusion from $\mathbf{RL}(\mathcal{A})$ into $\mathcal{B}$. Note first that the inclusion $\mathcal{A} \hookrightarrow \mathcal{B} = E \mid \mathcal{A}$ (where $E \mid \mathcal{A}$ is the restriction of E to $\mathcal{A}$) coincides with $\mathcal{A} \hookrightarrow \mathbf{RL}(\mathcal{A})$ followed by E. We wish to show that E, together with the identity transformation from $\mathcal{A} \hookrightarrow \mathbf{RL}(\mathcal{A}) \hookrightarrow \mathcal{B}$ to $E \mid \mathcal{A}$, is the right Kan extension of $\mathcal{A} \hookrightarrow \mathcal{B}$ along $\mathcal{A} \hookrightarrow \mathbf{RL}(\mathcal{A})$. Consider a functor $S : \mathbf{RL}(\mathcal{A}) \to \mathcal{B}$ and a natural transformation $\epsilon : S \mid \mathcal{A} \to E \mid \mathcal{A}$. Our goal is the definition of a suitable transformation $\sigma : S \to E$. Let $L \in \mathbf{RL}(\mathcal{A})$, i.e.,

L is the limit object for the $\mathcal{A}$-reflecting limit $(L,(l_f)_{(B\downarrow\mathcal{A})})$ of D_B for some $B \in \mathcal{B}$. Note that the source $(SL,(\epsilon_A \cdot S(l_f))_{(B\downarrow\mathcal{A})})$ is natural for D_B : given a $(B \downarrow \mathcal{A})$-morphism $a : (f, A) \to (\bar{f}, \bar{A})$, we have the following string of equalities :

$$\begin{aligned} a \cdot \epsilon_A \cdot S(l_f) &= \epsilon_{\bar{A}} \cdot S(a) \cdot S(l_f) \quad \text{(naturality of } \epsilon) \\ &= \epsilon_{\bar{A}} \cdot S(a \cdot l_f) \\ &= \epsilon_{\bar{A}} \cdot S(l_{\bar{f}}) \quad \text{(naturality of } (l_f)_{(B\downarrow\mathcal{A})}) \text{ for } D_B) \end{aligned}$$

Hence there exists a unique $\sigma_L : SL \to L$ such that $\epsilon_A \cdot S(l_f) = l_f \cdot \sigma_L$ for each $(f, A) \in (B \downarrow \mathcal{A})$. Next, we must show that the morphisms $(\sigma_L)_{L\in\mathbf{RL}(\mathcal{A})}$ define a natural transformation : let $h : L \to \bar{L}$ be a morphism in $\mathbf{RL}(\mathcal{A})$, where $(L,(l_f)_{(B\downarrow\mathcal{A})})$ and $(\bar{L},(m_g)_{(C\downarrow\mathcal{A})})$ are $\mathcal{A}$-reflecting limits for $\mathcal{B}$-objects B and C respectively. Let $(g, A) \in (C \downarrow \mathcal{A})$. Since $(l_f)_{(B\downarrow\mathcal{A})} = All(L,\mathcal{A})$ by 1.7, $m_g \cdot h = l$ for some $l \in (l_f)_{(B\downarrow\mathcal{A})}$. Then,

$$\begin{aligned} m_g \cdot \sigma_{\bar{L}} \cdot S(h) &= \epsilon_A \cdot S(m_g) \cdot S(h) \\ &= \epsilon_A \cdot S(m_g \cdot h) \\ &= \epsilon_A \cdot S(l) \\ &= l \cdot \sigma_L \\ &= m_g \cdot h \cdot \sigma_L \end{aligned}$$

So, $m_g \cdot \sigma_{\bar{L}} \cdot S(h) = m_g \cdot h \cdot \sigma_L$ for each $g \in (C \downarrow \mathcal{A})$, hence since $(m_g)_{(C\downarrow\mathcal{A})}$ is in particular a mono-source, $\sigma_{\bar{L}} \cdot S(h) = h \cdot \sigma_L$. Clearly, $\epsilon = \sigma \mid \mathcal{A}$, since each $\mathcal{A}$-object is the $\mathcal{A}$-reflecting limit object for its $\mathcal{A}$-canonical diagram. To see the uniqueness of σ with respect to this property, let $\gamma : S \to E$ be such that $\gamma \mid \mathcal{A} = \epsilon$. For an $\mathbf{RL}(\mathcal{A})$-object L, where $(L,(l_f)_{(B\downarrow\mathcal{A})})$ is the limit of D_B for some $B \in \mathcal{B}$, we have $l_f \cdot \gamma_L = \gamma_A \cdot S(l_f) = \epsilon_A \cdot S(l_f) = l_f \cdot \sigma_L$ for each $(f, A) \in (B \downarrow \mathcal{A})$. Hence, since $(l_f)_{(B\downarrow\mathcal{A})}$ is in particular a mono-source, $\gamma_L = \sigma_L$. □

1.17 Remark. Proposition 1.16 above bears some resemblance to Corollary 4, page 235 of [Mac Lane 1971] (where a right Kan extension of an arbitrary functor along a full embedding is constructed, under the assumption that certain canonical limits exist); compare also Proposition 1, page 242 of [Mac Lane 1971].

2 An Approach via Colimits.

In this Section we study closures of subcategories $\mathcal{A}$ of $\mathcal{B}$ via colimits of special diagrams, the schemes of these diagrams depending on the class $\mathcal{A}^{\perp}$ of morphisms orthogonal with respect to $\mathcal{A}$ (see Chapter 0). Given a subcategory $\mathcal{A}$ of $\mathcal{B}$, let $B \in \mathcal{B}$, and denote by $\mathcal{A}^{\perp}_{B}$ the full subcategory of the comma category $(B \downarrow \mathcal{B})$, with objects pairs (p, B_p), where $p : B \to B_p$ belongs to $\mathcal{A}^{\perp}$, and morphisms $f : (p, B_p) \to (q, B_q)$ those $\mathcal{B}$-morphisms $f : B_p \to B_q$ satisfying $q = f \cdot p$. Analogous to the procedure for canonical diagrams, we define as a diagram for the scheme $\mathcal{A}^{\perp}_{B}$ the associated projection functor $D^{\perp}_{B} : \mathcal{A}^{\perp}_{B} \to \mathcal{B}$ by $(p, B_p) \mapsto B_p$ $((p, B_p) \in Ob(\mathcal{A}^{\perp}_{B}))$, $f : (p, B_p) \to (q, B_q) \mapsto f$ $(f : (p, B_p) \to (q, B_q) \in Mor(\mathcal{A}^{\perp}_{B}))$. Diagrams of this type will be referred to as *$\mathcal{A}$-approaching diagrams*, and colimits of these diagrams will be called *$\mathcal{A}$-approaching colimits.* The canonical morphism with respect to any such colimit will be called an *$\mathcal{A}$-approaching morphism* (alternatively, a *codiagonal morphism*).

2.1 Lemma. *Every $\mathcal{A}$-approaching morphism belongs to $\mathcal{A}^{\perp}$.*

PROOF. Let $B \in \mathcal{B}$ and suppose that $D^{\perp}_{B}$ has a colimit $(c_p : B_p \to K)_{\mathcal{A}^{\perp}_{B}}$. Note that since $(id_B, B) \in \mathcal{A}^{\perp}_{B}$, $c_B = c_{id_B}$ is the $\mathcal{A}$-approaching morphism associated with $(c_p)_{\mathcal{A}^{\perp}_{B}}$, with $c_B = c_p \cdot p$ for each $p \in \mathcal{A}^{\perp}_{B}$. We must show that $c_B \in \mathcal{A}^{\perp}$. Let $f : B \to A$ be a $\mathcal{B}$-morphism with codomain in $\mathcal{A}$. Then for each $p \in \mathcal{A}^{\perp}_{B}$, there exists a unique $m_p : B_p \to A$ with $f = m_p \cdot p$. The sink $(m_p)_{\mathcal{A}^{\perp}_{B}}$ is natural for $D^{\perp}_{B}$, since morphisms in $\mathcal{A}^{\perp}$ are in particular epimorphisms relative to $\mathcal{A}$. Now $(c_p)_{\mathcal{A}^{\perp}_{B}}$ is the colimit of $D^{\perp}_{B}$, so there exists a unique $s : K \to A$ such that $m_p = s \cdot c_p$ for each $p \in \mathcal{A}^{\perp}_{B}$. In particular, we have $f = m_{id_B} = s \cdot c_{id_B} = s \cdot c_B$. Let $t : K \to A$ be such that $t \cdot c_B = s \cdot c_B$. Then $t \cdot c_p \cdot p = s \cdot c_p \cdot p$ for each $p \in \mathcal{A}^{\perp}_{B}$, so $t \cdot c_p = s \cdot c_p$ for each $p \in \mathcal{A}^{\perp}_{B}$, and since $(c_p)_{\mathcal{A}^{\perp}_{B}}$ is in particular an epi-sink, $t = s$. □

2.2 Proposition. *The following conditions are equivalent, for a $\mathcal{B}$-object B :*

(1) B has an $\mathcal{A}$-reflection;

(2) the colimit of $D^{\perp}_{B}$ exists and has colimit object belonging to $\mathcal{A}$.

PROOF. (1) $\Rightarrow$ (2) : let $r : B \to A$ be the $\mathcal{A}$-reflection for B. For each $p : B \to B_p$ in $\mathcal{A}_B^\perp$, there exists a unique $c_p : B_p \to A$ such that $r = c_p \cdot p$. We will show that $((c_p)_{\mathcal{A}_B^\perp}, A)$ is the colimit of $D_B^\perp$.

Naturality of $(c_p)_{\mathcal{A}_B^\perp}$: let $f : (p, B_p) \to (q, B_q)$ be a morphism in $\mathcal{A}_B^\perp$. From $c_q \cdot q = r = c_p \cdot p$ and $q = f \cdot p$ it follows that $c_p \cdot p = c_q \cdot f \cdot p$, hence since $p \in \mathcal{A}^\perp$, $c_p = c_q \cdot f$.

Universality of $(c_p)_{\mathcal{A}_B^\perp}$: let $(m_p : B_p \to C)_{\mathcal{A}_B^\perp}$ be a natural sink for $D_B^\perp$. For each $p \in \mathcal{A}_B^\perp$, $c_p : (p, B_p) \to (r, A)$ is a $\mathcal{A}_B^\perp$-morphism, so by the naturality of $(m_p)_{\mathcal{A}_B^\perp}$, $m_p = m_r \cdot c_p$. Let $h : A \to C$ be such that $m_p = h \cdot c_p$ for each $p \in \mathcal{A}_B^\perp$. Now , $c_r : (r, L) \to (r, L)$ is a $\mathcal{A}_B^\perp$-morphism, i.e., $c_r \cdot r = r$, hence since $r \in \mathcal{A}^\perp$, $c_r = id_A$. So we have $m_r = h \cdot c_r = h$.
(2) $\Rightarrow$ (1) : by 2.1 above, the $\mathcal{A}$-approaching morphism $c_B : B \to K$ for B belongs to $\mathcal{A}^\perp$, and by (2) $K \in \mathcal{A}$, hence c_B is the $\mathcal{A}$-reflection morphism for B. □

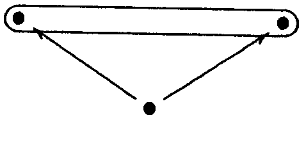
2.3 Corollary. *The following conditions are equivalent :*

(1) $\mathcal{A}$ is reflective in $\mathcal{B}$;

(2) for each $\mathcal{B}$-object B, the colimit of $D_B^\perp$ exists and has colimit object in $\mathcal{A}$. □

The requirement that for each $\mathcal{B}$-object B the colimit of $D_B^\perp$ must have colimit object in $\mathcal{A}$ is essential :

2.4 Example. Let $\mathcal{B}$ be given by all the data in the diagram below, and let $\mathcal{A}$ be as shown.

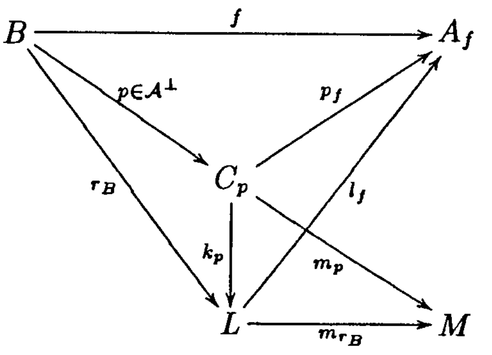


Then $\mathcal{B}$ has colimits of all $\mathcal{A}$-approaching diagrams, but $\mathcal{A}$ is not reflective in $\mathcal{B}$.

2.5 Remark. Consider again the special case where $\mathcal{B}$ and $\mathcal{A}$ are partially-ordered classes. In 1.4 it was noted that an $\mathcal{A}$-reflection object A_B for a $\mathcal{B}$-object B can be characterised as the infimum (with respect to $\mathcal{B}$) of all those $\mathcal{A}$-objects A which are greater than or equal to B. By 2.2, we can characterise $\mathcal{A}$-reflections in the present context as follows : a $\mathcal{B}$ object B has an $\mathcal{A}$-reflection if and only if $\bigvee\{C \in \mathcal{B} \mid B \leq C$ and for all $A \in \mathcal{A}$ $(B \leq A \Rightarrow C \leq A)\}$ exists and belongs to $\mathcal{A}$ (where $\bigvee$ is taken with respect to $\mathcal{B}$). So in the present restricted situation there is a kind of duality between attaining reflections as canonical limit objects, and as approaching colimit objects. In fact, we have $\bigwedge\{A \in \mathcal{A} \mid B \leq A\} = \bigvee\{C \in \mathcal{B} \mid B \leq C$ and for all $A \in \mathcal{A}$ $(B \leq A \Rightarrow C \leq A)\}$ (if both the respective elements exist), even when the respective operations produce an element outside $\mathcal{A}$. It shall subsequently be shown that in the general situation of categories $\mathcal{A}$ and $\mathcal{B}$, the respective limit and colimit objects need not coincide.

2.6 Proposition. *Let $B \in \mathcal{B}$. If the $\mathcal{A}$-reflecting limit of D_B exists, then the colimit of $D_B^{\perp}$ exists, and the codomain of the colimit sink coincides, up to isomorphism, with the domain of the $\mathcal{A}$-reflecting limit source, i.e., the $\mathcal{A}$-reflecting morphism for B coincides (up to isomorphism) with the $\mathcal{A}$-approaching morphism for B.*

PROOF. Let $(L, (l_f)_{(B\downarrow\mathcal{A})})$ be the limit source for D_B.



Let $p : B \to C_p$ be in $\mathcal{A}^{\perp}$. For each $(f, A) \in (B \downarrow \mathcal{A})$, denote by p_f the unique morphism such that $f = p_f \cdot p$. Since p is an epimorphism relative to $\mathcal{A}$, $(p_f)_{(B\downarrow\mathcal{A})}$ is natural for D_B. Since $(l_f)_{(B\downarrow\mathcal{A})})$ is the limit of D_B, there exists a unique $k_p : C_p \to L$ such that $l_f \cdot k_p = p_f$ for each $f \in (B \downarrow \mathcal{A})$. We now show that $((k_p)_{p\in\mathcal{A}_B^{\perp}}, L)$ is the colimit of $D_B^{\perp}$.

Naturality : let $s : (p, C_p) \to (q, C_q)$ be a morphism in $\mathcal{A}_B^{\perp}$. Then, for each $f \in (B \downarrow \mathcal{A})$, $p_f \cdot p = f = q_f \cdot q = q_f \cdot s \cdot p$, so since $p \in \mathcal{A}^{\perp}$ we have $p_f = q_f \cdot s$ (for each $f \in (B \downarrow \mathcal{A})$). Now, applying equalities derived above, $l_f \cdot k_p = p_f = q_f \cdot s = l_f \cdot k_q \cdot s$ for each $f \in (B \downarrow \mathcal{A})$, so since $(l_f)_{(B\downarrow\mathcal{A})}$ is a monosource, $k_p = k_q \cdot s$.

Universality : let $((m_p)_{\mathcal{A}_B^{\perp}}, M)$ be a natural sink for $D_B^{\perp}$. For each $p \in \mathcal{A}_B^{\perp}$, $k_p : (p, C_p) \to (r_B, L)$ is a morphism in $\mathcal{A}_B^{\perp}$ (recall that r_B is the canonical morphism from B to the limit object L), since $r_B \in \mathcal{A}^{\perp}$. Hence for each $p \in \mathcal{A}_B^{\perp}$, $m_p = m_{r_B} \cdot k_p$. Given $t : L \to M$

with $t \cdot k_p = m_p$ for each $p \in \mathcal{A}_B^\perp$, we have in particular that $t \cdot k_{r_B} = m_{r_B}$, hence $t = m_{r_B}$ since $k_{r_B} = id_L$. This holds since $l_f \cdot k_{r_B} \cdot r_B = l_f \cdot r_B$ implies that $l_f \cdot k_{r_B} = l_f$ for all $f \in (B \downarrow \mathcal{A})$, hence $k_{r_B} = id_L$. □

The converse to 2.6 above, namely, that every $\mathcal{A}$-approaching colimit is an $\mathcal{A}$-reflecting limit, need not hold : consider again the categories $\mathcal{A}$ and $\mathcal{B}$ of Example 1.9. The only $\mathcal{A}$-orthogonal morphism with domain B is id_B; the colimit object for the colimit of $D_B^\perp$ is then B. However, the $\mathcal{A}$-reflecting limit of D_B does not exist. Moreover, as discussed in 1.9, the canonical limit of D_B is $(L, \{a_1, a_2, a_3\})$.

Let $\mathbf{K}(\mathcal{A})$ denote the closure of $\mathcal{A}$ in $\mathcal{B}$ under all $\mathcal{A}$-approaching colimits, i.e., the subcategory of $\mathcal{B}$ obtained by adding to $\mathcal{A}$ all $\mathcal{B}$-objects B for which id_B is the $\mathcal{A}$-approaching morphism for $D_B^\perp$, equivalently, the subcategory of $\mathcal{B}$ obtained by adding to $\mathcal{A}$ all objects which are the colimit objects of $\mathcal{A}$-approaching colimits.

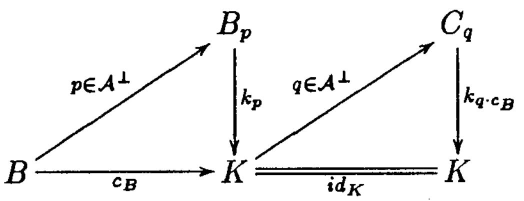
The following is an immediate consequence of 2.6 :

2.7 Corollary. *If $\mathcal{B}$ has $\mathcal{A}$-reflecting limits of all canonical diagrams, then* $\mathbf{CL}(\mathcal{A}) = \mathbf{RL}(\mathcal{A}) = \mathbf{K}(\mathcal{A})$. □

In contrast to the situation in the previous Section, where the limit $(L, l_f)_{(B\downarrow\mathcal{A})}$ of a canonical diagram D_B for some $B \in \mathcal{B}$ was not necessarily the limit of D_L (thus necessitating the distinction between $\mathcal{A}$-canonical limits and $\mathcal{A}$-reflecting limits), $\mathcal{A}$-approaching colimits are better behaved :

2.8 Proposition. *A $\mathcal{B}$-object K is the colimit object for the colimit of $D_B^\perp$ for some $B \in \mathcal{B}$ iff K is the colimit object for the colimit of $D_K^\perp$.*

PROOF. "if" : clear.
"only if" :

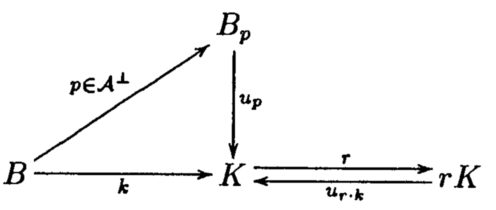


Suppose that $((k_p)_{\mathcal{A}_B^\perp}, K)$ is the colimit of $D_B^\perp$ for some $B \in \mathcal{B}$. Let $q \in \mathcal{A}_K^\perp$; let c_B denote the codiagonal corresponding to the colimit of $D_B^\perp$. Then $c_B \in \mathcal{A}^\perp$ by 2.1. Since

$c_B \in \mathcal{A}^\perp$ and $q \in \mathcal{A}^\perp$, it follows that $q \cdot c_B \in \mathcal{A}^\perp$ (see Chapter 0), hence there exists $k_{q \cdot c_B} : C \to K$ with $c_B = k_{q\cdot c_B} \cdot q \cdot c_B$, i.e., $k_{q\cdot c_B} : (c_B, K) \to (c_B, K)$ is a morphism in $\mathcal{A}^\perp_B$, hence since $(k_p)_{\mathcal{A}^\perp}$ is a natural sink for $D^\perp_B$, $k_{c_B} \cdot (k_{q\cdot c_B} \cdot q) = k_{c_B}$. But $k_{c_B} = id_K$ (for all $p \in \mathcal{A}^\perp_B$, $p : (p, B_p) \to (c_B, K)$ is a morphism in $\mathcal{A}^\perp_B$, i.e., $k_{c_B} \cdot k_p = k_p$, so since $(k_p)_{\mathcal{A}^\perp_B}$ is an epi-sink, $k_{c_B} = id_K$), so $k_{q\cdot c_B} \cdot q = id_K$. Thus $(k_{q\cdot c_B})_{\mathcal{A}^\perp_K}$, including the morphism $k_{id_K \cdot c_B} = id_K$, is natural for $D^\perp_K$. If $((g_q)_{\mathcal{A}^\perp_K}, E)$ is another natural sink for $D^\perp_K$, then g_{id_K} is the unique morphism h such that $g_q = h \cdot k_{q\cdot c_B}$ for each $q \in \mathcal{A}^\perp_K$, hence K is the colimit object for the colimit $(k_{q\cdot c_B})_{\mathcal{A}^\perp_K}$ of $D^\perp_K$. □

2.9 Proposition. *For each reflective $\mathcal{C} \subset \mathcal{B}$ with $\mathcal{A} \subset \mathcal{C}$, $\mathbf{K}(\mathcal{A}) \subset \mathcal{C}$.*

PROOF. Suppose that $\mathcal{C}$ is a reflective subcategory of $\mathcal{B}$ containing $\mathcal{A}$. Let $K \in \mathbf{K}(\mathcal{A})$. Then K is the colimit object for the colimit $((u_p)_{\mathcal{A}^\perp_B}, K)$ of $D^\perp_B$ for some $B \in \mathcal{B}$. Let $k : B \to K$ denote the associated codiagonal. Then $k \in \mathcal{A}^\perp_B$ by 2.1. Let $r : K \to rK$ denote the $\mathcal{C}$-reflection of K.



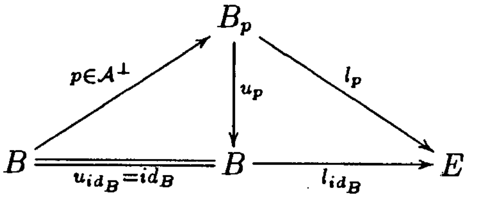
Since $\mathcal{A}^\perp$ is closed with respect to composition, and $r \in \mathcal{C}^\perp \subset \mathcal{A}^\perp$, $r \cdot k \in \mathcal{A}^\perp$. As in the proof of 2.8, we have $u_k = id_K$.

Now, note that since $r \cdot u_p : (p, B_p) \to (r\cdot k, rK)$ (for each $p \in \mathcal{A}^\perp_B$) and $u_{r\cdot k} : (r\cdot k, rK) \to (k, K)$ are $\mathcal{A}^\perp_B$-morphisms, $u_{r\cdot k} \cdot r \cdot u_p : (p, B_p) \to (k, K)$ is an $\mathcal{A}^\perp_B$-morphism for each $p \in \mathcal{A}^\perp_B$, so by the naturality of $(u_p)_{\mathcal{A}^\perp_B}$, $u_p = u_k \cdot u_{r\cdot k} \cdot r \cdot u_p = u_{r\cdot k} \cdot r \cdot u_p$ for each $p \in \mathcal{A}^\perp_B$. So, since $(u_p)_{\mathcal{A}^\perp_B}$ is an epi-sink, $u_{r\cdot k} \cdot r = id_K$, i.e., r is a section, hence an isomorphism. So, $K \in \mathcal{C}$. □

The following observation will be of subsequent use :

2.10 Proposition. $(\mathcal{A}^\perp)_\perp \subset \mathbf{K}(\mathcal{A})$.

PROOF. Let $B \in (\mathcal{A}^\perp)_\perp$. Then for each $p : B \to B_p$ in $\mathcal{A}^\perp$, there exists a unique $u_p : B_p \to B$ such that $u_p \cdot p = id_B$. In fact, the sink $(u_p)_{\mathcal{A}^\perp_B}$ is natural for $D^\perp_B : \mathcal{A}^\perp_B \to \mathcal{B}$, since if $m : (p, B_p) \to (q, B_q)$ is an $\mathcal{A}_B^{\perp}$-morphism (i.e., $m \cdot p = q$), then $u_q \cdot m \cdot p = u_q \cdot q = id_B = u_p \cdot p$, hence $u_q \cdot m = u_p$ (since $B \in (\mathcal{A}^{\perp})_{\perp}$ and p is an epimorphism relative to $\mathcal{A}$). We show that $(u_p)_{\mathcal{A}_B^{\perp}}$ is the colimit of $D_B^{\perp}$, with associated codiagonal morphism id_B. So, let $((l_p)_{\mathcal{A}_B^{\perp}}, E)$ be another natural sink for $D_B^{\perp}$. Note that for each $p \in \mathcal{A}_B^{\perp}$, $u_p : (p, B_p) \to (id_B, B)$ is a $\mathcal{A}_B^{\perp}$-morphism, hence for each $p \in \mathcal{A}_B^{\perp}$, $l_{id_B} \cdot u_p = l_p$.

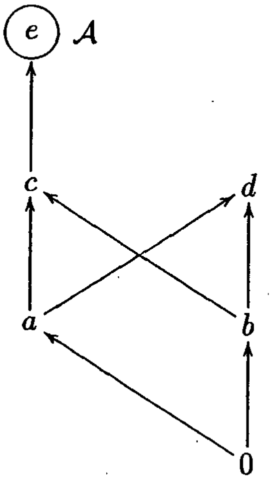


To show that l_{id_B} is the unique morphism with this property, let $t : B \to E$ be such that $l_p = s \cdot u_p$ for each $p \in \mathcal{A}_B^{\perp}$. Then in particular, $l_{id_B} = s \cdot u_{id_B} = s$. □

2.11 Remark. It may be asked whether there are categories $\mathcal{B}$ in which for every subcategory $\mathcal{A}$ of $\mathcal{B}$, the closure $\mathbf{K}(\mathcal{A})$ exhausts the reflective hull of $\mathcal{A}$ in $\mathcal{B}$, i.e., the reflective hull may be "reached" via the $\mathcal{A}$-approaching colimits. In [Adámek, Rosický and Trnková 1988] it has been shown that if $\mathcal{B}$ is locally presentable, and the Weak Vopênka Principle (a large cardinal principle) is assumed, then every limit-closed subcategory of $\mathcal{B}$ (hence every orthogonal subcategory of $\mathcal{B}$) is reflective in $\mathcal{B}$. So, if $\mathcal{A}$ is a subcategory of a locally presentable category $\mathcal{B}$, and the Weak Vopênka Principle is assumed, then since $(\mathcal{A}^{\perp})_{\perp}$ is reflective in $\mathcal{B}$, by 2.9 and 2.10 it follows that $(\mathcal{A}^{\perp})_{\perp} = \mathbf{K}(\mathcal{A})$, hence $\mathbf{K}(\mathcal{A})$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$, i.e., the reflections for the reflective hull of $\mathcal{A}$ in $\mathcal{B}$ are exactly the $\mathcal{A}$-approaching morphisms.

Next, we make some further observations regarding the closure $\mathbf{K}(\mathcal{A})$. By 2.10 it follows that, for any subcategory $\mathcal{A}$, the orthogonal closure $(\mathcal{A}^{\perp})_{\perp}$ of $\mathcal{A}$ in $\mathcal{B}$ is contained in $\mathbf{K}(\mathcal{A})$. This inclusion is in general not reversible :

2.12 Example. In the following diagram



$Ob(\mathcal{B}) = \{0, a, b, c, d, e\}$; $\mathcal{A}$ and the non-identity morphisms in $\mathcal{B}$ are as shown, and $\mathcal{A}^{\perp}$ consists of all arrows in $\mathcal{B}$ which do not have codomain d. A straightforward calculation gives :

(1) $(\mathcal{A}^{\perp})_{\perp} = \{e\}$, and

(2) $\mathbf{K}(\mathcal{A}) = \{e, d\}$.

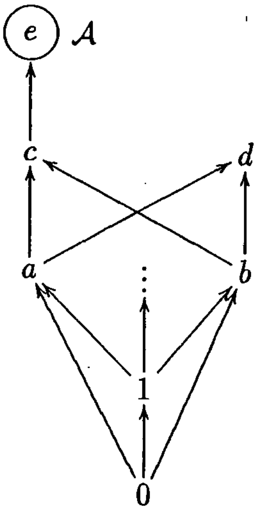
2.13 Remark. Example 2.12 above also provides an example of a non-reflective orthogonal subcategory which has a reflective hull.

As illustrated by 2.12, $\mathbf{K}(\mathcal{A})$ intuitively reaches "closer" to a possible reflective hull of $\mathcal{A}$ in $\mathcal{B}$ than $(\mathcal{A}^{\perp})_{\perp}$. On the other hand, notice also in 2.12 above that $\mathbf{K}(\mathcal{A})$ is itself not reflective in $\mathcal{B}$. However, a repeated application of the operator $\mathbf{K}$ does lead to the reflective hull of $\mathcal{A}$ in $\mathcal{B}$ in 2.12 above : let $\mathbf{K}^2(\mathcal{A}) = \mathbf{K}(\mathbf{K}(\mathcal{A}))$ denote the subcategory of $\mathcal{B}$ obtained by adding all existing colimits of $\mathbf{K}(\mathcal{A})$-approaching diagrams; analogously, define $\mathbf{K}^3(\mathcal{A}) = \mathbf{K}(\mathbf{K}^2(\mathcal{A}))$, $\mathbf{K}^4(\mathcal{A}) = \mathbf{K}(\mathbf{K}^3(\mathcal{A}))$. Then it can be verified that :

$$\begin{aligned}\mathbf{K}^2(\mathcal{A}) &= \{a, b, d, e\},\\ \mathbf{K}^3(\mathcal{A}) &= \{0, a, b, d, e\},\\ \mathbf{K}^4(\mathcal{A}) &= \mathbf{K}^3(\mathcal{A}).\end{aligned}$$

In fact, $\mathbf{K}^3(\mathcal{A})$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$. Now consider the following modification to 2.12 :

2.14 Example.



In the above situation, with $\mathcal{B}$ extended to include (as shown above) all the ordinal numbers, $\mathcal{A}$ does not have a reflective hull in $\mathcal{B}$ ($\mathcal{B}$ is reflective in $\mathcal{B}$, but then so are $\mathcal{B} - \{0\}$, $\mathcal{B} - \{0,1\}$, ...). We also see that, as in 2.12, $\mathbf{K}^2(\mathcal{A}) = \{a,b,d,e\}$, but that $\mathbf{K}^3(\mathcal{A}) = \{a,b,d,e\} = \mathbf{K}^2(\mathcal{A})$, so in this example the operator $\mathbf{K}$ does eventually become idempotent. Note also that in this instance $\mathcal{B}$ is an example of a category which does not have colimits of all chains (the collection $\mathcal{A}_0^{\perp}$ of all morphisms orthogonal to $\mathcal{A}$ with domain 0 contains a nonterminating chain, hence to construct a reflection morphism for 0 via iteration of the operator $\mathbf{K}$ is not possible).

The above examples suggest that, more generally, it may be possible to approach the reflective hull of $\mathcal{A}$ in $\mathcal{B}$ (in case it exists) via an iterated application of the operator $\mathbf{K}$. Define the following "iterated" versions of the operator $\mathbf{K}$:

$$\begin{aligned}
\mathbf{K}^0(\mathcal{A}) &= \mathcal{A}, \\
\mathbf{K}^1(\mathcal{A}) &= \mathbf{K}(\mathcal{A}), \\
\mathbf{K}^{\alpha+1}(\mathcal{A}) &= \mathbf{K}(\mathbf{K}^{\alpha}(\mathcal{A})), \alpha \text{ a finite ordinal}, \\
\mathbf{K}^{\lambda}(\mathcal{A}) &= \bigcup_{\alpha<\lambda} \mathbf{K}^{\alpha}(\mathcal{A}), \lambda \text{ a limit ordinal.}
\end{aligned}$$

2.15 Notation. When there is a possibility of confusion, we denote, for a $\mathcal{B}$-object B and a subcategory $\mathcal{A}$ of $\mathcal{B}$, the associated $\mathcal{A}$-approaching diagram by $D_B^{\mathcal{A}^\perp}$.

First, we have an appropriate generalisation of 2.9 :

2.16 Proposition. *For every ordinal number α, $\mathbf{K}^\alpha(\mathcal{A})$ is contained in every reflective subcategory $\mathcal{C}$ of $\mathcal{B}$ which contains $\mathcal{A}$.*

PROOF. We proceed via transfinite induction. The case $\alpha = 0$ is trivial, and the proof for the step $\alpha \to \alpha + 1$ follows immediately from 2.9. Now let λ be a limit ordinal, with $\mathbf{K}^\beta(\mathcal{A}) \subset \mathcal{C}$ for every ordinal $\beta < \lambda$. Let $K \in \mathbf{K}^\lambda(\mathcal{A})$, and denote by $r : K \to rK$ the $\mathcal{C}$-reflection for K. So, for some $B \in \mathcal{B}$ and $\beta < \lambda$, K is the colimit object for the colimit $((u_p)_{\mathbf{K}^\beta(\mathcal{A})^\perp_B}, K)$ of $D_B^{\mathbf{K}^\beta(\mathcal{A})^\perp}$ with associated codiagonal $k : B \to K$. Now, $k \in \mathbf{K}^\beta(\mathcal{A})^\perp$ by 2.1, $r \in \mathcal{C}^\perp$, and $\mathcal{C}^\perp \subset \mathbf{K}^\beta(\mathcal{A})^\perp$, hence $r \cdot k \in \mathbf{K}^\beta(\mathcal{A})^\perp$, and an argument analogous to that used in the proof of 2.9 shows that r is an isomorphism; so $K \in \mathcal{C}$. □

2.17 Corollary. *For every ordinal number α, the following conditions are equivalent :*

(1) $\mathbf{K}^\alpha(\mathcal{A})$ *is the reflective hull of* $\mathcal{A}$ *in* $\mathcal{B}$;

(2) (a) For each $B \in \mathcal{B}$, the colimit of $D_B^{\mathbf{K}^\alpha(\mathcal{A})^\perp}$ exists, and

(b) $\mathbf{K}(\mathbf{K}^\alpha(\mathcal{A})) = \mathbf{K}^\alpha(\mathcal{A})$.

PROOF. Immediate from 2.3 and 2.16. □

3 Some Comparisons and Modifications.

We make some comparisons between the closure processes described in the previous two Sections, and also investigate how these relate to well-known closures such as limit closures and orthogonal closures.

The processes of closing up under canonical and $\mathcal{A}$-reflecting limits respectively are not well-behaved (in general) in comparison to closing up under other types of limits. Clearly $\mathbf{RL}(\mathcal{A})$ and $\mathbf{CL}(\mathcal{A})$ are contained in $\mathbf{L}(\mathcal{A})$, the smallest limit-closed subcategory of $\mathcal{B}$ containing $\mathcal{A}$. Since the reflectivity of $\mathbf{L}(\mathcal{A})$ itself may depend on set-theoretical hypotheses (see, e.g., [Adámek, Rosický and Trnková 1988]), this fact already places $\mathbf{RL}(\mathcal{A})$ and $\mathbf{CL}(\mathcal{A})$ far away from attaining (or approaching) a possible reflective hull. To show that the inclusion of $\mathbf{CL}(\mathcal{A})$ in $\mathbf{L}(\mathcal{A})$ is proper, we show via the following example that $\mathbf{CL}(\mathcal{A})$ need not be closed with respect to retracts :

3.1 Example. In the category **Bool** of Boolean algebras and Boolean homomorphisms, let $\{\mathbf{2}\}$ denote the one-object subcategory consisting of the 2-element Boolean algebra $\mathbf{2}$. Note then that the only Boolean homomorphism from $\mathbf{2}$ to itself is the identity. Hence $\{\mathbf{2}\}$-canonical diagrams in **Bool** are discrete, and so $\{\mathbf{2}\}$-canonical limits in **Bool** are products. Further, note that for any set I, the I-fold power $\mathbf{2}^I$ of $\mathbf{2}$ is the limit of its own $\{\mathbf{2}\}$-canonical diagram, hence belongs to $\mathbf{CL}(\{\mathbf{2}\})$. So $\mathbf{CL}(\{\mathbf{2}\})$ consists of precisely those Boolean algebras which are powers of $\mathbf{2}$. Now let B be any complete *non-atomic* Boolean algebra. Then there exists an embedding m of Boolean algebras from B into a Boolean algebra of form $\mathcal{P}(X)$, for some set X (see, e.g. [Davey and Priestley 1990] 10.4); note that $\mathcal{P}(X)$ is isomorphic to a power of $\mathbf{2}$. Now, the complete Boolean algebras are precisely those Boolean algebras which are injective with respect to embeddings in **Bool** ([Adámek, Herrlich and Strecker 1990] 9.3(2)(a)), so B is injective with respect to m, hence a retract of the given power of $\mathbf{2}$, but lies outside $\mathbf{CL}(\{\mathbf{2}\})$.

The relationships between the various concepts that have been considered (for an arbitrary subcategory $\mathcal{A}$ of $\mathcal{B}$) in Sections 1 and 2 may be summarised by the following sequence of containments :

$$\mathbf{RL}(\mathcal{A}) \subset \mathbf{CL}(\mathcal{A}) \subset \mathbf{L}(\mathcal{A}) \subset (\mathcal{A}^{\perp})_{\perp} \subset \mathbf{K}(\mathcal{A})$$

where $\mathbf{L}(\mathcal{A})$ denotes the closure of $\mathcal{A}$ in $\mathcal{B}$ under all existing limits. To see that the first, second and fourth inclusions are proper, see 1.9, 3.1 and 2.12 respectively. That $\mathbf{L}(\mathcal{A}) \subset (\mathcal{A}^{\perp})_{\perp}$ is well-known; some examples of non-orthogonal limit-closed subcategories are presented in [Rosický and Tholen 1988].

We shall introduce a variant of the closedness properties which we have been considering, which are better behaved (e.g. closedness under retracts) than the concepts mentioned above. These modified closedness properties for subcategories are comparable to the property of being closed under limits of all diagrams (limit-closed subcategories have appeared extensively in the literature) : consider, for $B \in \mathcal{B}$ and a full subcategory $\mathcal{I} \subset (B \downarrow \mathcal{A})$, the diagram $D_B^{\mathcal{I}} : \mathcal{I} \to \mathcal{B}$, defined in the same way as the canonical diagrams $D_B^{\mathcal{A}}$ defined in Section 1; we shall reserve the term *subcanonical diagram* for any diagram of this form. It is well-known that if $\mathcal{B}$ is complete, then every small limit-closed subcategory $\mathcal{A}$ of $\mathcal{B}$ is reflective in $\mathcal{B}$ (this observation is an immediate consequence of the appropriate Adjoint Functor Theorem - see, for example, [Adámek, Herrlich and Strecker 1990] 18.12). Our immediate objective is to state an analogue of this fact (using a previous result in this Chapter) for those subcategories $\mathcal{A} \subset \mathcal{B}$ which are closed with respect to limits of subcanonical diagrams.

3.2 Lemma. *Let $\mathcal{A} \subset \mathcal{B}$ be closed under the formation of limits of subcanonical diagrams. Then $\mathcal{A}$ is closed in $\mathcal{B}$ under the formation of equalizers.*

PROOF. Suppose that $g, h : C \rightrightarrows D$ is a pair of $\mathcal{A}$-morphisms; let (E, e) denote the equalizer of g and h. Let $\mathcal{I}$ be the full subcategory of $(E \downarrow \mathcal{A})$ with object class $\{(e, C), (g \cdot e = h \cdot e, D)\}$. We show that the source $(E, (e : E \to C, g \cdot e : E \to D))$ is the limit of $D_E^{\mathcal{I}}$. So suppose that $(B, (k_e : B \to C, k_{g \cdot e} : B \to D))$ is natural for $D_E^{\mathcal{I}}$. Since $g : (e, C) \to (g \cdot e, D)$ and $h : (e, C) \to (g \cdot e, D)$ are morphisms in $\mathcal{I}$, we have $k_{g \cdot e} = g \cdot k_e = h \cdot k_e$ by naturality, i.e., k_e equalizes g and h, hence there exists by the equalizer property of (E, e) a unique $t : B \to E$ with the property that $k_e = e \cdot t$. Further, we have $k_{g \cdot e} = g \cdot k_e = g \cdot e \cdot t$, and given any $d : B \to E$ such that $k_e = e \cdot d$ and $k_{g \cdot e} = g \cdot e \cdot d$, it follows from the equalizer property of (E, e) that $d = t$. Hence $(E, (e : E \to C, g \cdot e : E \to D))$ is the limit of $D_E^{\mathcal{I}}$, and consequently E must belong to $\mathcal{A}$. □

Our analogue to the well-known result for small limit-closed subcategories follows :

3.3 Proposition. *Let $\mathcal{B}$ be complete. Then every small subcategory $\mathcal{A}$ of $\mathcal{B}$ which is closed under the formation of limits of subcanonical diagrams is reflective in $\mathcal{B}$.*

PROOF. Let $\mathcal{A} \subset \mathcal{B}$ be small and closed with respect to limits of subcanonical diagrams. By 3.2 $\mathcal{A}$ is closed under equalizers, hence in particular $\mathcal{A}$ is closed under retracts. Since $\mathcal{A}$ is small, it follows that for each $\mathcal{B}$-object B the limit of $D_B^{\mathcal{A}}$ exists, and the associated limit object belongs to $\mathcal{A}$. Since $\mathcal{A} = \mathbf{CL}(\mathcal{A})$, it follows by 1.14 that $\mathcal{A}$ is weakly reflective in $\mathcal{B}$. Hence $\mathcal{A}$ is almost reflective in $\mathcal{B}$. In [Adámek and Rosický 1993] it is remarked that an almost reflective subcategory which is in addition closed with respect to equalizers is a reflective subcategory; so $\mathcal{A}$ is reflective in $\mathcal{B}$. □

Another pleasant property of subcategories which are closed under subcanonical limits is the following :

3.4 Proposition. *The intersection of subcategories closed with respect to limits of subcanonical diagrams is itself closed with respect to limits of subcanonical diagrams.*

PROOF. Let $\mathcal{A} = \cap_I \mathcal{A}_i$, where for each $i \in I$, $\mathcal{A}_i$ is closed under limits of subcanonical diagrams. Let $(L, (l_j)_{\mathcal{J}})$ be the limit of a subcanonical diagram $D_B^{\mathcal{J}} : \mathcal{J} \to \mathcal{A}$, say, where $B \in \mathcal{B}$. Since $\mathcal{A} \subset \mathcal{A}_i$ for all $i \in I$, $\mathcal{J} \subset (B \downarrow \mathcal{A}_i)$ for each $i \in I$, hence the limit object L belongs to $\mathcal{A}_i$ for each $i \in I$. So, $L \in \cap_I \mathcal{A}_i = \mathcal{A}$. □

Next we consider the role of the construction $\mathbf{CL}(\mathcal{A})$ in the context of reflective hulls of small subcategories. Recall (see, e.g., [Tholen 1987], [Kelly 1987]) that if $\mathcal{B}$ is complete and wellpowered (resp. strongly complete), then for every small subcategory $\mathcal{A}$ of $\mathcal{B}$, the limit closure $\mathbf{L}(\mathcal{A})$ of $\mathcal{A}$ in $\mathcal{B}$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$. In this case, every object of the reflective hull appears as an extremal subobject of a product of $\mathcal{A}$-objects. If $\mathbf{P}(\mathcal{A})$ denotes the subcategory of $\mathcal{B}$ obtained by adding all products of $\mathcal{A}$-objects to $\mathcal{A}$, then $\bar{\mathcal{S}}(\mathbf{P}(\mathcal{A}))$, the subcategory consisting of all extremal subobjects of $\mathbf{P}(\mathcal{A})$-objects, is the epireflective hull of $\mathcal{A}$ in $\mathcal{B}$ (see [Tholen 1987]). Note then that $\mathbf{CL}(\mathcal{A}) \subset \bar{\mathcal{S}}(\mathbf{P}(\mathcal{A}))$, hence $\bar{\mathcal{S}}(\mathbf{CL}(\mathcal{A})) \subset \bar{\mathcal{S}}(\mathbf{P}(\mathcal{A}))$. On the other hand, $\bar{\mathcal{S}}(\mathbf{P}(\mathcal{A})) \subset \bar{\mathcal{S}}(\mathbf{CL}(\mathcal{A}))$ since $\bar{\mathcal{S}}(\mathbf{CL}(\mathcal{A}))$ is an epireflective subcategory of $\mathcal{B}$ containing $\mathcal{A}$. In the situation under consideration, the (reflection) objects of the reflective hull $\mathbf{L}(\mathcal{A})$ can be related to $\mathcal{A}$-canonical limit objects in the following manner :

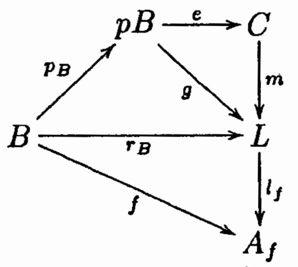
3.5 Proposition. *Let $\mathcal{A}$ be a small subcategory of a complete and wellpowered category $\mathcal{B}$, and let $\mathbf{L}(\mathcal{A})$ denote the limit closure (hence the reflective hull) of $\mathcal{A}$ in $\mathcal{B}$. Then the following conditions hold :*

(1) For each $\mathcal{B}$-object B, the $\mathbf{L}(\mathcal{A})$-reflection object pB for B is an extremal subobject of the $\mathcal{A}$-canonical limit object for D_B.

(2) Every $\mathbf{L}(\mathcal{A})$-object R occurs as an extremal subobject of the $\mathcal{A}$-canonical limit object for D_R.

PROOF. Note that (2) follows directly from (1). To see (1), let $B \in \mathcal{B}$; denote by $(L, (l_f)_{(B \downarrow \mathcal{A})})$ the $\mathcal{A}$-canonical limit of D_B, and let $p_B : B \to pB$ (resp. $r_B : B \to L$) denote the $\mathbf{L}(\mathcal{A})$-reflection (resp. $\mathcal{A}$-canonical) morphism for B. Since L belongs to

$\mathbf{L}(\mathcal{A})$, there exists a unique $\mathcal{B}$-morphism $g : pB \to L$ such that $r_B = g \cdot pB$. Since $\mathcal{B}$ is complete and wellpowered, it follows (see [Adámek, Herrlich and Strecker 1990] 14.21) that (Epi, Extremal Mono) is a morphism factorisation structure on $\mathcal{B}$. Let $m \cdot e : pB \to C \to L$ denote the (Epi, Extremal Mono)-factorisation of g.



Note that $e \cdot p_B$ an epimorphism relative to $\mathcal{A}$, and further belongs to $\mathcal{A}^{\perp}$ since for each $f \in (B \downarrow \mathcal{A})$, $l_f \cdot m$ is (the unique morphism) $\bar{f}$ such that $f = \bar{f} \cdot e \cdot p_B$. Now, since $\mathbf{L}(\mathcal{A})$ is reflective in $\mathcal{B}$, we have that $\mathbf{L}(\mathcal{A})$ coincides with $(\mathcal{A}^{\perp})_{\perp}$, hence $pB \in (\mathcal{A}^{\perp})_{\perp}$, and there exists a unique $\mathcal{B}$-morphism $s : C \to pB$ such that $p_B = s \cdot e \cdot p_B$, so since p_B is an epimorphism relative to $\mathbf{L}(\mathcal{A})$, it follows that $s \cdot e = id_{pB}$, i.e., e is a section, hence an isomorphism. Consequently g is an extremal monomorphism. □

3.6 Remark. (Freely Added Canonical Limits)
Instead of closing up under the formation of $\mathcal{A}$-canonical limit objects, one can freely add to $\mathcal{A}$ all existing $\mathcal{A}$-canonical limit sources in $\mathcal{B}$. More precisely, one may consider the category $(\mathcal{A}^*, U^*)$, the *free canonical limit closure of $\mathcal{A}$ over $\mathcal{B}$*, with objects the $\mathcal{A}$-canonical limit sources and morphisms the usual source maps, together with the obvious projection functor $U^* : \mathcal{A}^* \to \mathcal{B}$. There is a natural full embedding $E : \mathcal{A} \hookrightarrow \mathcal{A}^*$, defined by $E(A) = (A, (a)_{(A \downarrow \mathcal{A})})$ and $E(f) = f$. Without exhibiting proofs, we mention the following observations with reference to some results in Section 1 of this Chapter :
(1) $(\mathcal{A}^*, U^*)$ has free objects if and only if $\mathcal{B}$ has $\mathcal{A}$-canonical limits (compare 1.8 and 1.15);
(2) $U^* : \mathcal{A}^* \to \mathcal{B}$ is a full embedding if and only if each $\mathcal{A}$-canonical morphism is an epimorphism relative to $\mathcal{A}$. In this case $\mathcal{A}^*$ coincides, up to isomorphism, with $\mathbf{RL}(\mathcal{A})$.

Chapter 2

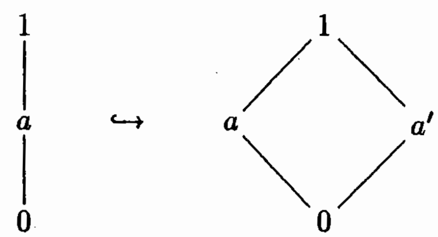
Intersections of (Almost) Reflective Subcategories

In this Chapter we exhibit some negative results concerning the relationship between orthogonal (resp. prereflective) subcategories and intersections of reflective subcategories. In "nice" categories satisfying appropriate (co)completeness and smallness conditions, the above concepts are equivalent (it is always true that a subcategory which is an intersection of reflective subcategories is an orthogonal subcategory) : for example, in **Top** it is well-known that every subcategory which is orthogonal with respect to a set of morphisms is reflective, consequently every orthogonal subcategory of **Top** is an intersection of reflective subcategories of **Top** (see [Freyd and Kelly 1972], [Herrlich 1993a]).

It shall be shown in Section 1 that in a non-cowellpowered category there may exist subcategories which are even orthogonal with respect to a single morphism, but which are not intersections of reflective subcategories. Some of the observations in Section 1 below have already appeared in [Vajner 1993]. In Section 2 we generalise the considerations of Section 1 to give sufficient conditions, in categorical terms, for an orthogonal (resp. prereflective) subcategory *not* to be an intersection of reflective subcategories. Section 3 considers the as yet unsolved question : is the class of complete Boolean Algebras an intersection of *almost* reflective subcategories of the category of frames ? Hitherto a major difficulty with this question has been lack of information about almost reflective subcategories of frames containing the class of complete Boolean Algebras. Some observations are deduced in this regard from new general results (also of independent interest) concerning reflective hulls of almost reflective subcategories.

1 Complete Boolean Algebras and Frames.

This Section deals with the following interesting example of an orthogonal (resp. prereflective) subcategory which is not an intersection of reflective subcategories : let **CBool** denote the category of complete Boolean algebras (considered as a full subcategory of the category **Frm** of frames and frame homomorphisms). It is well-known (see, for example, [Johnstone 1982] p. 57) that **CBool** is not a reflective subcategory of **Frm**. In [Tholen 1986] and [Rosický and Tholen 1988] it has been noted that **CBool** is a prereflective (see Chapter 0) subcategory of **Frm**, a suitable prereflection for a frame L given by the embedding of L into its frame of congruences, equivalently, the embedding of L into its frame of nuclei ([Banaschewski, Frith and Gilmour 1987], [Tholen 1986]). In addition, it is a straightforward exercise to verify that the complete Boolean algebras are precisely those frames which are orthogonal with respect to the morphism



(see also [Rosický and Tholen 1988], [Adámek and Rosický 1993]), i.e., a frame B is Boolean if and only if every frame homomorphism from the 3-element chain to B factorises uniquely through the above embedding.

In [Rosický and Tholen 1988] the following question was left open : is **CBool** an *intersection* of reflective subcategories of **Frm** ? We give a negative answer to this question.

We recall from [Banaschewski, Frith and Gilmour 1987] some basic facts concerning congruence frames. Given a frame L, let $c_L : L \hookrightarrow \mathcal{C}L$ denote the embedding of L into the congruence frame $\mathcal{C}L$ of L, defined by the assignment $a \mapsto \nabla_a = \{(x,y) \mid x \vee a = y \vee a\}$ $(a \in L)$. For each $a \in L$, the congruence $\Delta_a = \{(x,y) \mid x \wedge a = y \wedge a\}$ is the complement of ∇_a in $\mathcal{C}L$, i.e., $\nabla_a \cap \Delta_a = \Delta = \{(x,x) \mid x \in L\}$, the bottom element of $\mathcal{C}L$, and $\nabla_a \vee \Delta_a = L \times L$, the top element of $\mathcal{C}L$. In other words, every element in the image of L under c_L has a complement in $\mathcal{C}L$.

1.1 Lemma. *For each frame L, the following hold :*
(1) $c_L : L \hookrightarrow \mathcal{C}L$ *is universal among the frame homomorphisms with domain L which have in their codomain a complement for each element in the image of L.*
(2) $c_L : L \hookrightarrow \mathcal{C}L$ *is an epimorphism of frames.*

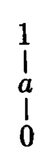
PROOF. (1) Given a frame homomorphism $f : L \to M$ such that each member of the image of L under f is complemented in M, define, for $\rho \in \mathcal{C}L$, $\bar{f}(\rho) = \bigvee\{f(b) \wedge f(a)' \mid a\rho b \text{ and } a \leq b\}$ (where $f(a)'$ denotes the complement of $f(a)$ in M). In [Frith 1987] it is shown that $\bar{f} : \mathcal{C}L \to M$ is the unique frame homomorphism satisfying $\bar{f} \cdot c_L = f$.
(2) (See also [Banaschewski, Frith and Gilmour 1987].) Let $g, h : \mathcal{C}L \rightrightarrows M$ be homomorphisms with $g \cdot c_L = h \cdot c_L$. Now there exists a frame embedding $e : M \hookrightarrow B$ of M into a complete Boolean algebra B (see [Johnstone 1982] p. 53). So we have $e \cdot g \cdot c_L = e \cdot h \cdot c_L$, hence from (1) it follows that $e \cdot g = e \cdot h$; consequently, since e is monic, $g = h$. □

Concerning reflective subcategories of **Frm** which contain the class of complete Boolean algebras, we have :

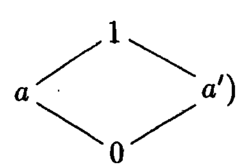
1.2 Lemma. *Let $\mathcal{A}$ be reflective in* **Frm**, *with* **CBool** $\subset \mathcal{A}$. *Then every $\mathcal{A}$-reflection is an epimorphism in* **Frm**.

PROOF. Apply the argument used in 1.1, part (2), but use the reflectivity of $\mathcal{A}$, and the inclusion **CBool** $\subset \mathcal{A}$, instead of 1.1, part (1). □

1.3 Notation. Denote the 3-element chain



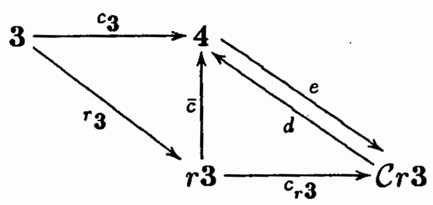
(resp. the complete Boolean algebra



by **3** (resp. **4**).

1.4 Proposition. *Let $\mathcal{A}$ be reflective in* **Frm**, *with* **CBool** $\subset \mathcal{A}$. *Then* $\mathbf{3} \in \mathcal{A}$.

PROOF. Let $r_3 : \mathbf{3} \to r\mathbf{3}$ denote the $\mathcal{A}$-reflection for $\mathbf{3}$. Recall (see, e.g., [Frith 1987]), that $C\mathbf{3}$ is, up to isomorphism, the complete Boolean algebra $\mathbf{4}$. Since **CBool** $\subset \mathcal{A}$, $\mathbf{4} \in \mathcal{A}$, hence there exists a unique morphism $\bar{c} : r\mathbf{3} \to \mathbf{4}$ such that $c_3 = \bar{c} \cdot r_3$. Note that since c_3 is an embedding, r_3 is an embedding. Now consider the embedding $c_{r3} : r\mathbf{3} \hookrightarrow Cr\mathbf{3}$ of $r\mathbf{3}$ into its congruence frame :



Each element in the image of $\mathbf{3}$ under $c_{r3} \cdot r_3$ has a complement in $Cr\mathbf{3}$, so by the universal property of c_3 (1.1 (1)), there exists a unique frame homomorphism $e : \mathbf{4} \to Cr\mathbf{3}$ such that $e \cdot c_3 = c_{r3} \cdot r_3$. Since r_3 is an epimorphism (1.2) and c_{r3} is an epimorphism (1.1 (2)), $c_{r3} \cdot r_3$ $(= e \cdot c_3)$ is an epimorphism, hence e is an epimorphism.

Now, $\mathbf{4}$ is complemented, so by the universality of c_{r3} (1.1 (1)), there exists a unique frame homomorphism $d : Cr\mathbf{3} \to \mathbf{4}$ such that $\bar{c} = d \cdot c_{r3}$. So, $d \cdot e \cdot c_3 = d \cdot c_{r3} \cdot r_3 = \bar{c} \cdot r_3 = c_3$. Hence, since c_3 is an epimorphism (1.1 (2)), $d \cdot e = id_4$, i.e., e is a section, hence an isomorphism. Consequently, $Cr\mathbf{3} \simeq \mathbf{4}$, i.e., $\bar{c} : r\mathbf{3} \to \mathbf{4}$ is an embedding, and $\mathbf{4}$ is, up to isomorphism, the congruence frame of $r\mathbf{3}$.

There are thus only two possibilities for $r\mathbf{3}$: (1) $r\mathbf{3} \simeq \mathbf{4}$, i.e., c_3 is, up to isomorphism, the $\mathcal{A}$-reflection arrow for $\mathbf{3}$; or (2) $r\mathbf{3} \simeq \mathbf{3}$.

Now if (1) was true, then every $\mathcal{A}$-object would be injective with respect to the morphism $c_3 : \mathbf{3} \hookrightarrow \mathbf{4}$, hence, since **CBool** $= \{c_3\}_\perp$), we would have $\mathcal{A} =$ **CBool** - contradicting the non-reflectivity of **CBool** in **Frm**.

So the only possibility is (2), namely, that r_3 is an isomorphism, i.e., $\mathbf{3} \in \mathcal{A}$. □

1.5 Corollary. **CBool** *is not an intersection of reflective subcategories of* **Frm**.

PROOF. By 1.4, $\mathbf{3}$ belongs to every reflective subcategory $\mathcal{A}$ of **Frm** which contains **CBool**, hence to the intersection of any collection of such subcategories. But $\mathbf{3}$ does not belong to **CBool**, hence **CBool** cannot be the intersection of a collection of reflective subcategories of **Frm**. □

In [Rosický and Tholen 1988], a category is said to be *weakly cowellpowered* if it is cowellpowered with respect to strong epimorphisms (for the definition of strong epimorphism, see, e.g., [Adámek, Herrlich and Strecker 1990]). In **Frm**, the strong epimorphisms are precisely the surjective frame homomorphisms, since **Frm** is a (Regular Epi, Mono-sources)-category (recall that **Frm** is monadic over **Set**). So we obtain :

1.6 Corollary. *The category* **Frm** *is an example of a cocomplete, weakly cowellpowered category in which there exists a prereflective subcategory which is not an intersection of reflective subcategories.* □

1.7 Remark. Corollary 1.5 also shows that in **Frm**, subcategories which are orthogonal with respect to a set of morphisms (equivalently, prereflective subcategories by [Rosický and Tholen 1988] 3.4 and 4.3) may be far from reflective. This is in contrast to the situation in the category of topological spaces, where every subcategory orthogonal with respect to a set of morphisms is in fact reflective (see [Freyd and Kelly 1972]).

We do not know whether there exists a non-trivial reflective subcategory of **Frm** which contains **CBool**, i.e., whether **Frm** is in fact the reflective hull of **CBool** in **Frm**. We can, however, make the following observation :

1.8 Proposition. **Frm** *is the strongly epireflective hull of* **CBool** *in* **Frm**.

PROOF. From [Johnstone 1982] p. 53, every frame can be embedded into a complete Boolean algebra. Since the strongly epireflective hull of **CBool** in **Frm** is closed under embeddings, the assertion follows by [Adámek, Herrlich and Strecker 1990] 16.8 . □

2 Some Generalisations.

In this Section we generalise 1.5 in Section 1 to a categorical setting, presenting conditions which ensure that an orthogonal (resp. prereflective) subcategory is not an intersection of reflective subcategories. These conditions can thus be used as a partial "test" to determine whether a given subcategory is not an intersection of reflective subcategories. In the sequel, we assume $\mathcal{A}$ to be a *non-reflective* subcategory of $\mathcal{B}$.

2.1 Notation. Denote by $\mathcal{S}(\mathcal{A})$ the class of all those $\mathcal{B}$-objects B for which there exists a monomorphism $m : B \to A$ into an $\mathcal{A}$-object A.

2.2 Lemma. *Let $\mathcal{C}$ be a reflective subcategory of $\mathcal{B}$, with $\mathcal{A} \subset \mathcal{C}$. Then the $\mathcal{C}$-reflection morphism for every $\mathcal{B}$-object is an epimorphism relative to $\mathcal{S}(\mathcal{A})$.*

PROOF. Let $B \in \mathcal{B}$, and let $r_B : B \to rB$ denote the $\mathcal{C}$-reflection for B. Suppose that $g, h : rB \rightrightarrows D$ are $\mathcal{B}$-morphisms such that $g \cdot r_B = h \cdot r_B$, and $D \in \mathcal{S}(\mathcal{A})$. So, there exists a monomorphism $m : D \to A$ in $\mathcal{B}$ with $A \in \mathcal{A}$. Hence, $m \cdot g \cdot r_B = m \cdot h \cdot r_B$ implies $m \cdot g = m \cdot h$ since $m \cdot g$ and $m \cdot h$ are $\mathcal{C}$-morphisms; consequently, $g = h$. □

Next, some additional terminology is required :

2.3 Definition. (1) Let E be a class of $\mathcal{B}$-morphisms. Then E is said to have the *locally orthogonal extension property* if for each $\mathcal{B}$-object B there exists a morphism $e_B : B \to E_B$ in E, called the *locally orthogonal E-extension of* B, with the following universal property : given a $\mathcal{B}$-morphism $f : C \to B$ and a morphism $e : C \to D$ which belongs to E, there exists a unique $t : D \to E_B$ such that $t \cdot e = e_B \cdot f$, i.e., such that the diagram

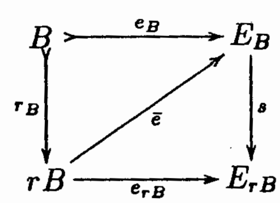
$$\begin{array}{ccc} C & \xrightarrow{\;e\;} & D \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle t} \\ B & \xrightarrow[\;e_B\;]{} & E_B \end{array}$$

commutes.
(2) Let $m : B \to C$ be a non-trivial monomorphism in $\mathcal{B}$, i.e., (B, m) is a proper subobject of C. Then (B, m) (or, simply m when the context is clear) is called a *maximal* subobject of C provided that there is no proper subobject of C strictly larger than m, that is, if there is no non-trivial factorisation of m through a proper subobject of C.

2.4 Remark. The locally orthogonal extension property defined in 2.3 (1) above has appeared in the literature, albeit under a different name : in [Bousfield 1977], [Tholen 1983] and [Tholen 1986] a class of morphisms is called a *localisation class* if it has the locally orthogonal extension property, and locally orthogonal extensions are called *localisations*; the term localisation is usually reserved for a reflection for which the reflection functor preserves finite limits, hence the alternative terminology used here. It should also be remarked (see, e.g., [Tholen 1986]) that there is a 1-1 correspondence between prereflections and localisation classes.

2.5 Proposition. *Suppose that $\mathcal{A} = E_\perp$, where E has the locally orthogonal extension property with associated locally orthogonal E-extensions $e_B : B \to E_B$ such that*
(1) $\{E_B \mid B \in \mathcal{B}\} \subset \mathcal{S}(\mathcal{A})$, and
(2) for each $\mathcal{B}$-object B, $E_B \in \mathcal{A}$ implies e_B is monic.
If there exists $B \in \mathcal{B}$ with $E_B \in \mathcal{A}$ and e_B a maximal subobject of E_B, then B belongs to every reflective subcategory $\mathcal{C}$ of $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{C}$ and $\mathcal{C} \not\subset \{e_B\}_\perp$.

PROOF. Let $\mathcal{C}$ be reflective in $\mathcal{B}$, with $\mathcal{A} \subset \mathcal{C}$, and $\mathcal{C} \not\subset \{e_B\}_\perp$. Denote the $\mathcal{C}$-reflection for B by $r_B : B \to rB$. Since $E_B \in \mathcal{A}$, and $\mathcal{A} \subset \mathcal{C}$, there exists by the universality of r_B a unique morphism $\bar{e} : rB \to E_B$ such that $e_B = \bar{e} \cdot r_B$. By the locally orthogonal extension property of e_{rB}, there exists a unique $s : E_B \to E_{rB}$ such that $s \cdot e_B = e_{rB} \cdot r_B$. Also note that since e_{rB} is orthogonal with respect to $\mathcal{A}$, and $E_B \in \mathcal{A}$, there exists a unique $t : E_{rB} \to E_B$ such that $\bar{e} = t \cdot e_{rB}$.



Now, we have $e_{rB} \cdot r_B = s \cdot e_B = s \cdot \bar{e} \cdot r_B$, hence, since by 2.2 r_B is an epimorphism with respect to $\mathcal{S}(\mathcal{A})$, and $E_{rB} \in \mathcal{S}(\mathcal{A})$ by assumption, $e_{rB} = s \cdot \bar{e}$. Hence $id_{E_{rB}} \cdot e_{rB} = s \cdot \bar{e} = s \cdot t \cdot e_{rB}$, i.e., $s \cdot t$ and $id_{E_{rB}}$ are both solutions to the diagram

$$\begin{array}{ccc} rB & \xrightarrow{e_{rB}} & E_{rB} \\ {\scriptstyle id_{rB}}\Big\| & & \\ rB & \xrightarrow[e_{rB}]{} & E_{rB} \end{array}$$

and so by the locally orthogonal extension property of e_{rB}, $s{\cdot}t = id_{E_{rB}}$. Further, we have $e_B = \bar{e} \cdot r_B = t \cdot e_{rB} \cdot r_B = t \cdot s \cdot e_B$, hence since $e_B \in E$ is in particular an epimorphism relative to $\mathcal{A} = E_\perp$, it follows that $t \cdot s = id_{E_B}$. Hence $E_B \simeq E_{rB}$, and $E_{rB} \in \mathcal{A}$; consequently, by our assumptions, e_{rB} is a $\mathcal{B}$-monomorphism, so that $\bar{e} = s \cdot e_{rB}$ is a $\mathcal{B}$-monomorphism. By the maximality of (B, e_B), either $\bar{e}$ is a $\mathcal{B}$-isomorphism, or r_B is a $\mathcal{B}$-isomorphism. If $\bar{e}$ was an isomorphism in $\mathcal{B}$, then e_B would be a $\mathcal{C}$-reflection for B, hence we would have $\mathcal{C} \subset \{e_B\}_\perp$ - a contradiction. So r_B is an isomorphism, and consequently $B \in \mathcal{C}$. □

2.6 Corollary. *Suppose that $\mathcal{A} = E_\perp$, where E has the locally orthogonal extension property, such that*

(1) $\{E_B \mid B \in \mathcal{B}\} \subset \mathcal{S}(\mathcal{A})$,

(2) for each $\mathcal{B}$-object B, $E_B \in \mathcal{A}$ implies e_B is monic, and

(3) there exists $D \in \mathcal{B}$ with $E_D \in \mathcal{A}$ such that (D, e_D) is a maximal subobject of E_D, and $\{e_D\}_\perp$ contains no reflective subcategory of $\mathcal{B}$ containing $\mathcal{A}$.

Then $\mathcal{A}$ is not an intersection of reflective subcategories of $\mathcal{B}$.

PROOF. By 2.5, the domain D of the morphism e_D in condition (3) belongs to every reflective subcategory of $\mathcal{B}$ containing $\mathcal{A}$, hence to the intersection of all such, but is not an $\mathcal{A}$-object, since otherwise $e_D : D \to E_D$ would not be a proper subobject of E_D. □

2.7 Remark. Thus 2.6 above is an appropriate generalisation of 1.5, in the following way : with $\mathcal{B} =$ **Frm**, $\mathcal{A} =$ **CBool**, we note that $\mathcal{B} = \mathcal{S}(\mathcal{A})$; take $E = \{c_L : L \to \mathcal{C}L \mid L \in \mathbf{Frm}\}$ (see Section 1). Then E has the locally orthogonal extension property, and the embedding from **3** into **4** is a locally orthogonal E-extension with codomain in $\mathcal{A}$ which has the maximal subobject property.

Concerning prereflective subcategories and intersections of reflective subcategories, we have :

2.8 Corollary. *Suppose that* $\mathcal{A} = \{B \in \mathcal{B} \mid \eta_B$ *is an isomorphism*$\}$ *for some prereflection* (T, η) *on* $\mathcal{B}$*, such that*
(1) $\{TB \mid B \in \mathcal{B}\} \subset \mathcal{S}(\mathcal{A})$,
(2) for each $B \in \mathcal{B}$, $TB \in \mathcal{A}$ *implies that* η_B *is monic, and*
(3) there exists $D \in \mathcal{B}$ *with* $TD \in \mathcal{A}$ *such that* (D, η_D) *is a maximal subobject of* TD *and* $\{\eta_D\}_\perp$ *contains no reflective subcategory of* $\mathcal{B}$ *containing* $\mathcal{A}$.

Then $\mathcal{A}$ *is not an intersection of reflective subcategories of* $\mathcal{B}$.

PROOF. From the prereflection property of (T, η) it follows that the class $\{\eta_B \mid B \in \mathcal{B}\}$ has the locally orthogonal extension property. In addition, note that $\mathcal{A} = \{\eta_B \mid B \in \mathcal{B}\}_\perp$. The initial conditions of 2.6 are thus satisfied, and 2.6 can be applied. □

3 Reflective Hulls of Almost Reflective Subcategories.

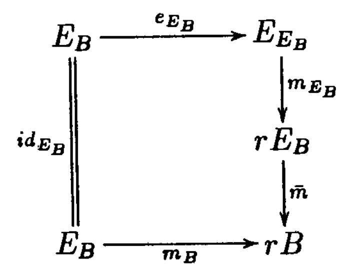
We provide some new information concerning almost reflective subcategories of **Frm** which contain **CBool**. First we deduce a general result concerning E-reflective hulls of almost reflective subcategories (which also holds for weakly reflective subcategories).

3.1 Proposition. *Let* $(E, \mathbf{M})$ *be a morphism factorisation structure on* $\mathcal{B}$ *such that* E *is contained in the class of all* $\mathcal{B}$*-epimorphisms. Then every almost reflective subcategory of* $\mathcal{B}$ *has an* E*-reflective hull in* $\mathcal{B}$.

PROOF. Suppose that $\mathcal{A}$ is almost reflective in $\mathcal{B}$; let, for each $\mathcal{B}$-object B, $r_B : B \to rB$ denote an almost $\mathcal{A}$-reflection for B. For each $\mathcal{B}$-object B, let $m_B \cdot e_B : B \to E_B \to rB$ denote the $(E, \mathbf{M})$-factorisation of r_B. We shall show that the subcategory $\mathbf{ER}(\mathcal{A}) = \{B \in \mathcal{B} \mid e_B$ is an isomorphism$\}$ of $\mathcal{B}$ is the E-reflective hull of $\mathcal{A}$ in $\mathcal{B}$.

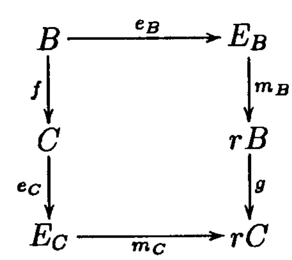
(a) $\mathcal{A} \subset \mathbf{ER}(\mathcal{A})$: let $A \in \mathcal{A}$. Then A is a retract of its almost $\mathcal{A}$-reflection, i.e., there exists $s : rA \to A$ in $\mathcal{B}$ such that $s \cdot r_A = id_A$. So $s \cdot m_A \cdot e_A = id_A$, i.e., e_A is an epimorphic section, hence an isomorphism.

(b) For each $B \in \mathcal{B}$, $E_B \in \mathbf{ER}(\mathcal{A})$: by the almost reflectivity of $\mathcal{A}$ in $\mathcal{B}$ there exists $\bar{m} : rE_B \to rB$ such that $m_B = \bar{m} \cdot r_{E_B}$. Hence the diagram



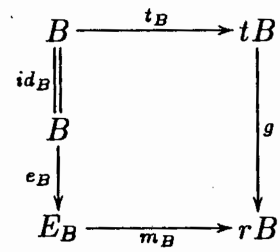
commutes, where $m_{E_B} \cdot e_{E_B}$ is the $(E, \mathbf{M})$-factorisation of r_{E_B}. So by the $(E, \mathbf{M})$-diagonalisation property there exists a unique $d : E_{E_B} \to E_B$ such that $d \cdot e_{E_B} = id_{E_B}$ and $m_B \cdot d = \bar{m} \cdot m_{E_B}$. In particular, e_{E_B} is an epimorphic section, hence an isomorphism.

(c) $\mathbf{ER}(\mathcal{A})$ is reflective in $\mathcal{B}$: let $B \in \mathcal{B}$; we wish to show that e_B is the $\mathbf{ER}(\mathcal{A})$-reflection for B. Let $f : B \to C$ be a morphism in $\mathcal{B}$ from B to an $\mathbf{ER}(\mathcal{A})$-object C. By the almost reflectivity of $\mathcal{A}$ in $\mathcal{B}$, there exists $g : rB \to rC$ with the property that $g \cdot r_B = r_C \cdot f$; in other words, $g \cdot m_B \cdot e_B = m_C \cdot e_C \cdot f$, i.e., the diagram



commutes. Hence by the $(E, \mathbf{M})$-diagonalisation property there exists a unique $d : E_B \to E_C$ such that $e_C \cdot f = d \cdot e_B$ and $m_C \cdot d = g \cdot m_B$. But by assumption e_C is a $\mathcal{B}$-isomorphism, so we have $f = (e_C)^{-1} \cdot d \cdot e_B$, and the uniqueness of $(e_C)^{-1} \cdot d$ with respect to this property follows since e_B is an epimorphism in $\mathcal{B}$.

(d) $\mathbf{ER}(\mathcal{A})$ is the E-reflective hull of $\mathcal{A}$ in $\mathcal{B}$: let $\mathcal{C}$ be E-reflective in $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{C}$. Let $B \in \mathbf{ER}(\mathcal{A})$, and denote by $t_B : B \to tB$ the $\mathcal{C}$-reflection for B. Since $\mathcal{A} \subset \mathcal{C}$, and $rB \in \mathcal{A}$, there exists a (unique) $g : tB \to rB$ such that $g \cdot t_B = r_B$, i.e., such that the diagram



commutes. Now $t_B \in E$, hence by the $(E, \mathbf{M})$-diagonalisation property there exists a unique $d : tB \to E_B$ such that $e_B = d \cdot t_B$ and $m_B \cdot d = g$. But by assumption e_B is a $\mathcal{B}$-isomorphism, hence $(e_B)^{-1} \cdot d \cdot t_B = id_B$, i.e., t_B is an epimorphic section, hence a $\mathcal{B}$-isomorphism. □

3.2 Lemma. *Let $\mathcal{A}$ be a subcategory of $\mathcal{B}$ such that every $\mathcal{B}$-object is a subobject of some $\mathcal{A}$-object. If $\mathcal{C}$ is any reflective subcategory of $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{C}$, then $\mathcal{C}$ is epireflective in $\mathcal{B}$.*

PROOF. If $\mathcal{C}$ is reflective in $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{C}$, and $B \in \mathcal{B}$, then there exists a monomorphism $m : B \to A$ with $A \in \mathcal{A}$. Hence, since $\mathcal{A} \subset \mathcal{C}$, m factorises via the $\mathcal{C}$-reflection for B, hence the $\mathcal{C}$-reflection of B, being the first factor in a factorisation of a monomorphism, is itself a monomorphism. So $\mathcal{C}$ is monoreflective in $\mathcal{B}$, and consequently epireflective in $\mathcal{B}$. □

3.3 Corollary. *Every almost monoreflective subcategory of an (Epi, Extremal Mono)-structured category $\mathcal{B}$ has a reflective hull in $\mathcal{B}$.*

PROOF. By 3.2 and 3.1. □

We digress to apply 3.3 above to two concrete examples : let **Pos** denote the category of partially-ordered sets and order-preserving maps, and let **CPos** denote the full subcategory of **Pos** consisting of the complete posets.

3.4 Corollary. **Pos** *is the reflective hull of* **CPos** *in* **Pos**.

PROOF. It is well-known (see, e.g., [Herrlich 1993a]) that **CPos** is almost reflective in **Pos**, an almost **CPos**-reflection of a poset P given by the embedding of $m_P : P \hookrightarrow P^*$ of P into its Mac Neille completion. Further, since **Pos** is complete and wellpowered we can, for each poset P, form the (Epi, Extremal Mono)-factorisation $n_P \cdot e_P : P \to C \to P^*$ of m_P ([Adámek, Herrlich and Strecker 1990] 14.19). By the argument used in 3.1 (c), and by 3.3, e_P is the reflection into the reflective hull of **CPos** in **Pos**. But by [Adámek, Herrlich and Strecker 1990] 8.8 (8) m_P is an extremal mono in **Pos**, hence e_P must be an isomorphism in **Pos**, consequently **Pos** coincides with the reflective hull of **CPos** in **Pos**. □

In [Herrlich 1993a] the injective T_0-spaces (i.e., retracts of powers of the Sierpinski space) are presented as an almost reflective subcategory of $\mathbf{T_0}$**-Top**, an almost reflection of a given T_0-space being an appropriate embedding (which is an extremal monomorphism) into a power of the Sierpinski space. It is well-known that $\mathbf{T_0}$**-Top** has (Epi, Extremal Mono)-factorisations for morphisms. Hence an argument analogous to that given in 3.4 above yields :

3.5 Corollary. $\mathbf{T_0}$**-Top** *is the reflective hull of the injective T_0-spaces in* $\mathbf{T_0}$**-Top**. □

Now, applying 3.3 above to the concrete situation **CBool** ⊂ **Frm**, we obtain :

3.6 Corollary. *Every almost reflective subcategory of* **Frm** *which contains* **CBool** *has a reflective hull in* **Frm**.

PROOF. The statement follows from 3.3 : **Frm** is complete and wellpowered, hence (Epi, Extremal Mono)-structured by [Adámek, Herrlich and Strecker 1990] 14.19, and evey almost reflective subcategory of **Frm** which contains **CBool** is almost monoreflective (by [Johnstone 1982] p. 53, every frame L is embeddable into a complete Boolean algebra; this embedding must factorise through any almost $\mathcal{A}$-reflection r_L of L for an almost reflective $\mathcal{A}$ in **Frm** containing **CBool**, hence r_L, being the first factor of a frame monomorphism, is itself a frame monomorphism). □

Finally, we are able to use 3.3 to give some new information concerning almost reflective subcategories of **Frm** which contain **CBool** :

3.7 Proposition. *Let $\mathcal{A}$ be an almost reflective subcategory of* **Frm** *containing* **CBool**. *Then any almost $\mathcal{A}$-reflection of the 3-element chain is an extremal monomorphism.*

PROOF. First note that, since **Frm** is complete and wellpowered, it is (Epi, Extremal Mono)-structured (see [Adámek, Herrlich and Strecker 1990] 14.19). Let $r_3 : \mathbf{3} \to r\mathbf{3}$ denote an almost $\mathcal{A}$-reflection for the 3-element chain **3**, and let $m \cdot e$ be the (Epi, Extremal Mono)-factorisation of r_3. From the proof of 3.1 (c) it follows that e is the $\mathbf{ER}(\mathcal{A})$-reflection for **3**. Hence by 1.4 and since $\mathbf{CBool} \subset \mathcal{A} \subset \mathbf{ER}(\mathcal{A})$, **3** belongs to $\mathbf{ER}(\mathcal{A})$, i.e., e is a frame isomorphism. Consequently r_3 is an extremal monomorphism. □

3.8 Remarks. (1) Note that since **Frm** has pushouts, the extremal monos in **Frm** coincide with the strong monos in **Frm** (see [Adámek, Herrlich and Strecker 1990] 14C).
(2) Whether the above considerations can be used to deduce that **CBool** is/is not an intersection of almost reflective subcategories of **Frm** may depend on finding a suitable characterisation of extremal (strong) monomorphisms in **Frm**; such a characterisation does not seem to be available at present.

Chapter 3

Relative Factorisations and Reflective Hulls

In this Chapter we consider a generalisation of the usual notion of factorisation structure for sources, by introducing the concept of factorisation structure *relative* to a given subcategory. Our motivation for studying this concept is two-fold : firstly, factorisation structures have been used extensively in the study of reflective subcategories (see for example [Cassidy, Hébert and Kelly 1985], [Tholen 1987], [Kelly 1987] and [Adámek, Herrlich and Strecker 1990]). It can be argued that one drawback of the theory of factorisation structures $(E, \mathbf{M})$ for sources is that the existence of an $(E, \mathbf{M})$-factorisation structure for sources on a category $\mathcal{B}$ implies, inter alia, that the class E consists only of $\mathcal{B}$-epimorphisms. Hence the usual notion of factorisation structure for sources is really applicable only to the study of epireflective subcategories, or more generally E-reflective subcategories, where the class E consists of epimorphisms only. In the theory of relative factorisation structures which we shall present, this disadvantage does not occur, hence relative factorisation structures are more readily applicable to the study of reflectivity in general.

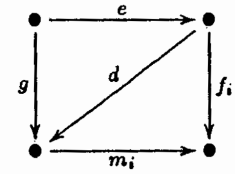
Secondly, the following well-known and important example leads to a motivating example for the study of relative factorisation structures : recall that in **Top** every continuous map can be factorised via a dense map followed by a closed embedding. This factorisation structure for morphisms does not extend to a factorisation structure ({dense maps}, $\mathbf{M}$) for sources on **Top**, but on the subcategory **Haus** of Hausdorff spaces can be extended to a factorisation structure ({dense maps}, $\mathbf{M}$) for sources (see, e.g. [Adámek, Herrlich and Strecker 1990] 15.22). In fact, it is well-known that every source

$(f_i : B \to C_i)_I$ in **Top** with the property that for each $i \in I$, C_i is a Hausdorff space, can be factorised via a dense map followed by a *closed* source (a source $(m_i : B \to C_i)_I$ in **Top** is called closed provided that there exists a sub*set* $J \subset I$ such that the induced canonical morphism $\prod_J m_i : B \to \prod_J C_i$ is a closed embedding) (see [Herrlich 1986] 8.4.12). Hence ({dense maps}, {closed sources with codomains in **Haus**}) can be considered in some sense (to be made precise below) as a factorisation structure on **Top** relative to **Haus**.

1 Basic Properties.

For the purposes of this Section we assume $\mathcal{A}$ to be a fixed subcategory of $\mathcal{B}$. The second part of the following definition generalises the usual notion of $(E, \mathbf{M})$-factorisation structure for all $\mathcal{B}$-sources to a notion of factorisation structure for all $\mathcal{A}$-valued $\mathcal{B}$-sources (see [Adámek, Herrlich and Strecker 1990], pp. 239) :

1.1 Definition. (1) A source $(f_i : B \to A_i)_I$ in $\mathcal{B}$ is said to be *$\mathcal{A}$-structured* if A_i belongs to $\mathcal{A}$ for each $i \in I$.
(2) Given a class E of $\mathcal{B}$-morphisms, and a conglomerate $\mathbf{M}$ of $\mathcal{A}$-structured sources, the pair $(E, \mathbf{M})$ is called a *factorisation structure relative to $\mathcal{A}$* (or, *$\mathcal{A}$-relative factorisation structure*) if the following conditions are satisfied :
(a) E is closed under postcomposition with $\mathcal{B}$-isomorphisms, and $\mathbf{M}$ is closed under precomposition with $\mathcal{B}$-isomorphisms,
(b) $\mathcal{B}$ has $\mathcal{A}$-relative $(E, \mathbf{M})$-factorisations, i.e., every $\mathcal{A}$-structured source $(f_i : B \to A_i)_I$ has a factorisation $(m_i \cdot e)_I$, with $e \in E$ and $(m_i)_I \in \mathbf{M}$,
(c) $\mathcal{B}$ has the $(E, \mathbf{M})$-diagonalisation property, i.e., $E \subset \mathbf{M}^{\uparrow}$ and $\mathbf{M} \subset E^{\downarrow}$, that is, given $e \in E$ and $(m_i)_I \in \mathbf{M}$, along with a $\mathcal{B}$-morphism g and an $\mathcal{A}$-structured source $(f_i)_I$ such that the outer rectangle of the diagram



commutes for each $i \in I$, there exists a unique diagonal d such that $d \cdot e = g$ and $f_i = m_i \cdot d$ for each $i \in I$.

1.2 Examples. (1) If $\mathcal{A}$ is taken to be the entire category $\mathcal{B}$ in 1.1 above, we obtain the usual definition of (E, M)-factorisation structure for $\mathcal{B}$-sources (henceforth we shall refer to such factorisation structures as *standard* factorisation structures).
(2) If (E, M) is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$, then (E, M) is a standard factorisation on $\mathcal{A}$.
(3) If (E, M) is a standard factorisation structure on $\mathcal{B}$, then (E, M') is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$, where M' consists of all those sources in M which are $\mathcal{A}$-structured.
(4) From the introductory remarks to this Chapter it follows that ({dense maps}, {closed **Haus**-structured sources}) is a factorisation structure on **Top** relative to **Haus**.
(5) Let $\mathcal{A}$ be the full subcategory of **Top** consisting of the object [0,1]. Then ({dense maps}, {closed $\mathcal{A}$-structured sources}) is an $\mathcal{A}$-relative factorisation structure on **Top**. Here, closed $\mathcal{A}$-structured sources need not belong to $\mathcal{A}$.

1.3 Remark. As for standard (E, M)-factorisations, the requirement that the diagonal d in 1.1 (2)(c) above be unique is redundant; the uniqueness of d follows from the observation below that the class E is contained in the class of all $\mathcal{B}$-morphisms which are epimorphisms relative to $\mathcal{A}$.

Henceforth we shall denote the class of epimorphisms relative to $\mathcal{A}$ by $\mathcal{A}$-**Epi**.

1.4 Definition. Given an $\mathcal{A}$-relative factorisation structure (E, M) on $\mathcal{B}$, define $\mathrm{M}(\mathcal{A})$ to be the subcategory of $\mathcal{B}$ obtained by adding to $\mathcal{A}$ all $\mathcal{B}$-objects which are domains of sources belonging to M. If $\mathrm{M}(\mathcal{A}) = \mathcal{A}$, then $\mathcal{A}$ is said to be *closed under the formation of* M*-sources.*

1.5 Example. In the situation of 1.2 (5), $\mathrm{M}(\mathcal{A})$ is the category of compact Hausdorff spaces.

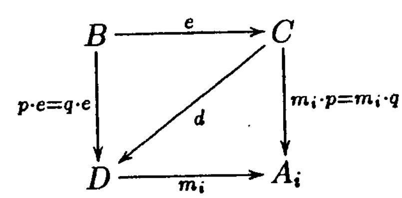
The proof of the following useful observation is a direct analogue of [Adámek, Herrlich and Strecker 1990], 15.4 :

1.6 Proposition. *If (E, M) is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$, then $E \subset$* $\mathcal{A}$-**Epi**. □

Observe that since $\mathcal{A} \subset \mathrm{M}(\mathcal{A})$, we have $\mathrm{M}(\mathcal{A})$-**Epi** $\subset$ $\mathcal{A}$-**Epi**. In fact, we can obtain a result which is stronger than 1.6 (see also 1.11) :

1.7 Proposition. *If $(E, \mathbf{M})$ is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$, then $E \subset$* $\mathbf{M}(\mathcal{A})$**-Epi**.

PROOF. Let $e : B \to C$ be a member of E, and consider $\mathcal{B}$-morphisms $p, q : C \rightrightarrows D$, where $D \in \mathbf{M}(\mathcal{A})$, such that $p \cdot e = q \cdot e$. So, D is the domain of some source $(m_i : D \to A_i)_I$ in $\mathbf{M}$. Now, we have that, for each $i \in I$, $m_i \cdot p \cdot e = m_i \cdot q \cdot e$, hence, since $e \in \mathcal{A}$**-Epi** by 1.6, $m_i \cdot p = m_i \cdot q$ for each $i \in I$. Then, for each $i \in I$, the diagram



commutes for $d = q$ and $d = p$; hence, by the uniqueness of diagonals, $p = q$. □

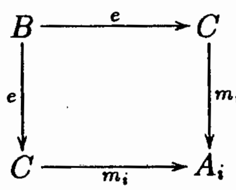
We list some further basic properties of $\mathcal{A}$-relative factorisation structures, which are analogues of those given for standard $(E, \mathbf{M})$-factorisation structures (see [Adámek, Herrlich and Strecker 1990] 15.5).

1.8 Proposition. *For $\mathcal{A}$-relative factorisation structures $(E, \mathbf{M})$, the following conditions hold :*

(1) $\mathcal{A}$-relative factorisations are unique up to isomorphism,
(2) $E \cap \mathbf{M} = \mathbf{Iso}(\mathcal{A})$,
(3) $\mathbf{M}$ is closed with respect to composition,
(4) if $f \in E$, $g \in E$, $g \cdot f$ is defined and cod$(g) \in \mathcal{A}$, then $g \cdot f \in E$,
*(5) if $g \cdot f \in E$, $f \in \mathcal{A}$-**Epi** and cod$(g) \in \mathcal{A}$, then $g \in E$,*
(6) if $g \cdot f \in E$, g is a section and cod$(f) \in \mathcal{A}$, then $f \in E$,
(7) if $\mathcal{S}$ an $\mathcal{A}$-structured source, $(\mathcal{S}_i)_I$ a family of $\mathcal{A}$-structured sources, and $(\mathcal{S}_i \cdot \mathcal{S})_I$ belongs to $\mathbf{M}$, then $\mathcal{S}$ belongs to $\mathbf{M}$,
(8) if a subsource of an $\mathcal{A}$-structured source $\mathcal{S}$ belongs to $\mathbf{M}$, then $\mathcal{S}$ belongs to $\mathbf{M}$,
(9) $\mathbf{M} = E^{\downarrow} \cap \{\mathcal{S} \mid \mathcal{S}$ is an $\mathcal{A}$-structured source$\}$.

PROOF. The proofs are the same as the proofs of the corresponding properties for standard factorisation structures, relative to the obvious modifications. The proofs of these facts use 1.6 and always the construction of an $\mathcal{A}$-relative $(E, \mathbf{M})$-factorisation

$(m_i \cdot e : B \to C \to A_i)_I$, together with a (composite) morphism $h : C \to C$ such that h and id_C are diagonals for the diagram



Hence, by the uniqueness of diagonals, $h = id_C$. □

1.9 Remarks. (1) There does not seem to be a convenient description for the class E, as given for $\mathbf{M}$ in (9) above. This stems from the fact that the class E need not be determined by $\mathbf{M}$ via the diagonalistion property. However, there are two natural classes E^0 and E^1 such that $(E^0, \mathbf{M})$ and $(E^1, \mathbf{M})$ are $\mathcal{A}$-relative factorisation structures on $\mathcal{B}$, namely, $E^0 = \{e \in E \mid cod(e) \in \mathbf{M}(\mathcal{A})\}$ and $E^1 = \mathbf{M}^\dagger$, and for any class E' of $\mathcal{B}$-morphisms it is obvious that $(E', \mathbf{M})$ is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$ if and only if $E^0 \subset E' \subset E^1$.
(2) By 1.8 (3) the conglomerate $\mathbf{M}$ is always closed with respect to composition. In contrast to the situation for standard $(E, \mathbf{M})$-factorisation structures, the class E need not be. Note that for any $\mathcal{A}$-relative factorisation structure $(E, \mathbf{M})$, E^0 and E^1, as defined above, are closed under composition.

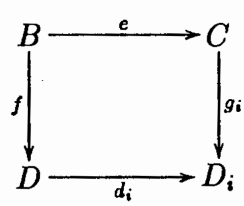
We exhibit a concrete situation illustrating the two points mentioned above :

1.10 Example. Let $\mathcal{B} = \mathbf{Top}$, $\mathcal{A} = \mathbf{Haus}$, $(E, \mathbf{M}) = (\{\text{dense maps}\}, \{\text{closed } \mathbf{Haus}\text{-structured sources}\})$. Consider the identical embedding e from the one-point space $\{1\}$ into the Sierpinski space S. Then it can be checked that e is an epimorphism relative to **Haus**, but that e is not dense. Concerning remark (1) above, it can be shown that $e \in \{\text{closed } \mathbf{Haus}\text{-structured sources}\}^\dagger$ (given a closed **Haus**-structured source $(m_i : X \to A_i)_I$, a source $(g_i : S \to A_i)_I$ and a continuous map $f : \{1\} \to X$ such that $g_i \cdot e = m_i \cdot f$ for each $i \in I$, define $d : S \to X$ to be the constant map sending 0 and 1 to $f(1)$; then d is the required diagonal, since e is an epimorphism relative to **Haus**, and $(m_i)_I$ is a mono-source). So $\{\text{dense maps}\}$ is properly contained in $\{\text{closed } \mathbf{Haus}\text{-structured sources}\}^\dagger$. Concerning remark (2) above, let $E' = E \cup \{e\}$. Then $(E', \mathbf{M})$ is a **Haus**-relative factorisation structure on **Top**, but E' is not closed with respect to composition : let T denote the space with underlying set $\{0,1,2\}$ and topology $\{\emptyset,$

{0,1}, {0,2}, {0,1,2}}, and let d be the identical embedding of S into T. Then d is dense, but $d \cdot e$ is not dense, consequently $d \cdot e$ does not belong to E'.

1.11 Proposition. *Let (E, M) be an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$. Then $(E, \bar{\mathrm{M}})$ is an $\mathrm{M}(\mathcal{A})$-relative factorisation structure on $\mathcal{B}$, where $\bar{\mathrm{M}}$ is the conglomerate of all $\mathcal{B}$-sources $\mathcal{S}$ for which there exists a family $(\mathcal{S}_i)_I$ of M-sources with $(\mathcal{S}_i)_I \cdot \mathcal{S}$ in* M.

PROOF. Let $(f_i : B \to C_i)_I$ be an $\mathrm{M}(\mathcal{A})$-structured $\mathcal{B}$-source. For each $i \in I$, $C_i \in \mathrm{M}(\mathcal{A})$, hence is the domain of an M-source $(m_{ik} : C_i \to A_{ik})_{K_i}$, say. Let $(n_{ik} \cdot e : B \to D \to A_{ik})_{i \in I, k \in K_i}$ denote the $\mathcal{A}$-relative (E, M)-factorisation of the source $(m_{ik} \cdot f_i)_{i \in I, k \in K_i}$. Then by the (E, M)-diagonalisation property there exists for each $i \in I$ a morphism $d_i : C \to C_i$ such that $f_i = d_i \cdot e$ and for each $k \in K_i$, $m_{ik} = n_{ik} \cdot d_i$. Clearly $(d_i)_I$ belongs to $\bar{\mathrm{M}}$, hence $(d_i \cdot e)_I$ is an $\mathrm{M}(\mathcal{A})$-relative $(E, \bar{\mathrm{M}})$-factorisation of $(f_i)_I$. The $(E, \bar{\mathrm{M}})$-diagonalisation condition remains to be verified : for this, let $e : B \to C$ be in E, $(d_i : D \to D_i)_I$ be in $\bar{\mathrm{M}}$, and consider a $\mathcal{B}$-morphism $f : B \to D$ and a $\mathcal{B}$-source $(g_i : C \to D_i)_I$ such that the diagram



commutes. Since $(d_i)_I$ belongs to $\bar{\mathrm{M}}$, there exists for each $i \in I$ a source $(m_{ik} : D_i \to A_{ik})_{K_i}$ in M such that the composite $(m_{ik} \cdot d_i)_{i \in I, k \in K_i}$ belongs to M. Applying the (E, M)-diagonalisation property to e and $(m_{ik} \cdot d_i)_{i \in I, k \in K_i}$, there exists a $\mathcal{B}$-morphism $d : C \to D$ such that $f = d \cdot e$ and $(m_{ik} \cdot d_i \cdot d)_{i \in I, k \in K_i} = (m_{ik} \cdot g_i)_{i \in I, k \in K_i}$. Now note that for each $i \in I$, $g_i \cdot e = d_i \cdot f = d_i \cdot d \cdot e$ (since $f = d \cdot e$); since $e \in \mathrm{M}(\mathcal{A})$-**Epi** by 1.7, $g_i = d_i \cdot d$ for each $i \in I$. So d is the required diagonal morphism. □

1.12 Remark. If (E, M) is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$, then, for the class $E' = \{e \in E \mid dom(e) \in \mathrm{M}(\mathcal{A})\}$ the inclusion functor $\mathrm{M}(\mathcal{A}) \hookrightarrow \mathcal{B}$ is an $(E', -)$-functor in the sense of [Adámek, Herrlich and Strecker 1990] 17.3, equivalently, an orthogonal $(E', -)$-functor in the sense of [Tholen 1979] 7.2 (see the proof of 1.11 above).

Next, for an $\mathcal{A}$-relative factorisation structure (E, M) on $\mathcal{B}$, we characterise the E-reflectivity of $\mathcal{A}$ in terms of injectivity and closedness under M-sources respectively. This extends a well-known result for standard factorisation structures (see [Adámek, Herrlich and Strecker 1990] 16.8 and 16.14).

1.13 Proposition. *Let (E, M) be an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$. Then the following conditions are equivalent :*

(1) $\mathcal{A}$ is E-reflective in $\mathcal{B}$;

(2) $\mathcal{A} = \mathcal{H}_{\perp} = Inj(\mathcal{H})$, for some class $\mathcal{H} \subset E$;

(3) $\mathcal{A}$ is closed in $\mathcal{B}$ under the formation of M*-sources, i.e.,* $\mathrm{M} \subset Sour(\mathcal{A})$.

PROOF. $(1) \Rightarrow (2)$: let $\mathcal{A}$ be E-reflective in $\mathcal{B}$. Set $\mathcal{H}$ to be the class of all $\mathcal{A}$-reflection arrows. Then $\mathcal{H} \subset E$. Further, the inclusion $\mathcal{A} \subset Inj(\mathcal{H})$ is clear. For the reverse inclusion, let $B \in Inj(\mathcal{H})$; if $r_B : B \to rB$ denotes the $\mathcal{A}$-reflection for B, then B is injective with respect to r_B, i.e., there exists $s : rB \to B$ such that $s \cdot r_B = id_B$. So, r_B is a section, hence a $\mathcal{B}$-isomorphism, and $B \in \mathcal{A}$.
$(2) \Rightarrow (3)$: suppose that $\mathcal{A} = Inj(\mathcal{H})$, for some $\mathcal{H} \subset E$. Consider a $\mathcal{B}$-object B which is the domain of some source $(m_i : B \to A_i)_I$ belonging to M. To show that $B \in \mathcal{A}$, it is sufficient by (2) to show that $B \in Inj(\mathcal{H})$. So, suppose that $h : C \to D$ is a member of $\mathcal{H}$, and let $g : C \to B$ be any $\mathcal{B}$-morphism. By (2), $\mathcal{A} = Inj(\mathcal{H})$, so for each $i \in I$ there exists (unique) $p_i : D \to A_i$ such that $p_i \cdot h = m_i \cdot g$. Since $h \in \mathcal{H} \subset E$, and $(m_i)_I \in \mathrm{M}$, there exists a unique diagonal $d : D \to B$ such that $g = d \cdot h$ and $p_i = m_i \cdot d$ for each $i \in I$, i.e., in particular, $B \in Inj(\mathcal{H}) = \mathcal{A}$.
$(3) \Rightarrow (1)$: given any $\mathcal{B}$-object B, the E-part of the $\mathcal{A}$-relative (E, M)-factorisation of the all-source $All(B, \mathcal{A})$ from B to $\mathcal{A}$ is the $\mathcal{A}$-reflection arrow for B, since by assumption $\mathrm{M}(\mathcal{A}) = \mathcal{A}$. □

2 Relative Factorisations and Reflectivity.

In this section we establish a correspondence between the collection of reflective subcategories of $\mathcal{B}$ and the collection of *relative factorisation structures* on $\mathcal{B}$ (in a slight generalisation of 1.1 (2), a relative factorisation structure on $\mathcal{B}$ is a factorisation structure relative to $\mathcal{C}$ for some $\mathcal{C} \subset \mathcal{B}$). Specifically, given an $\mathcal{A}$-relative factorisation structure (E, M) on $\mathcal{B}$, we show that $\mathrm{M}(\mathcal{A})$ is reflective in $\mathcal{B}$, i.e., there is an assignment $(E, \mathrm{M}) \mapsto \Psi(E, \mathrm{M}) = \mathrm{M}(\mathcal{A})$ from $\mathcal{A}$-relative factorisation structures on $\mathcal{B}$ to reflective subcategories of $\mathcal{B}$. Conversely, given a reflective subcategory $\mathcal{A}$ of $\mathcal{B}$, it shall be shown that $(\mathcal{A}^{\perp}, Sour(\mathcal{A}))$ (where $Sour(\mathcal{A})$ denotes the conglomerate of all $\mathcal{A}$-sources) is a factorisation structure on $\mathcal{B}$ relative to $\mathcal{A}$, i.e., there is an assignment $\mathcal{A} \mapsto \Phi(\mathcal{A}) = (\mathcal{A}^{\perp}, Sour(\mathcal{A}))$ from reflective subcategories of $\mathcal{B}$ to relative factorisation structures on $\mathcal{B}$. The correspondence described above will also be localised, or restricted, to a Galois correspondence, for a fixed subcategory $\mathcal{A}$ of $\mathcal{B}$, between ($\mathcal{A}$-relative) factorisation structures on $\mathcal{B}$ and reflective subcategories of $\mathcal{B}$ containing $\mathcal{A}$; via this localised correspondence, we will make some observations on the existence of reflective hulls in the presence of relative factorisation structures.

The above-mentioned correspondence has its roots in correspondences between factorisation structures for morphisms and reflective subcategories in suitable categories $\mathcal{B}$, studied in [Cassidy, Hébert and Kelly 1985] and [Kelly 1987]. More general results involving correspondences between suitable classes of morphisms and reflective subcategories have been obtained in [Korostenski and Tholen 1986], [Tholen 1986] and [Tholen 1987].

First, given a subcategory $\mathcal{A}$ of $\mathcal{B}$, we show how any factorisation structure relative to $\mathcal{A}$ induces an E-reflective subcategory of $\mathcal{B}$ which contains $\mathcal{A}$ (namely, $\mathrm{M}(\mathcal{A})$, the closure of $\mathcal{A}$ in $\mathcal{B}$ under all sources in M) :

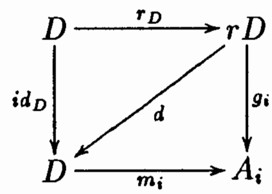
2.1 Proposition. *Let (E, M) be an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$. Then $\mathrm{M}(\mathcal{A})$ is E-reflective in $\mathcal{B}$.*

PROOF. By 1.11, $(E, \bar{\mathrm{M}})$ is an $\mathrm{M}(\mathcal{A})$-relative factorisation structure on $\mathcal{B}$ and $\mathrm{M}(\mathcal{A})$ is closed in $\mathcal{B}$ under the formation of $\bar{\mathrm{M}}$-sources. Hence by 1.13 $\mathrm{M}(\mathcal{A})$ is E-reflective in $\mathcal{B}$. □

In the presence of an $\mathcal{A}$-relative factorisation structure (E, M) on $\mathcal{B}$, we are able to make some observations concerning (E-)reflective hulls :

2.2 Proposition. *If (E, M) is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$, then* $\mathrm{M}(\mathcal{A})$ *is the E-reflective hull of $\mathcal{A}$ in $\mathcal{B}$.*

PROOF. Let $\mathcal{C}$ be E-reflective in $\mathcal{B}$, with $\mathcal{A} \subset \mathcal{C}$. Given an $\mathbf{M}(\mathcal{A})$-object D, let $r_D : D \to rD$ denote the $\mathcal{C}$-reflection arrow for D. By assumption, D is the domain of some $\mathcal{A}$-structured source $(m_i : D \to A_i)_I$, say. Since $\mathcal{A} \subset \mathcal{C}$, there exists, by the universality of r_D, for each $i \in I$ a unique $g_i : rD \to A_i$ such that $m_i = g_i \cdot r_D$. Since $r_D \in E$, and $(m_i)_I \in \mathbf{M}$, there exists a unique $d : rD \to D$ making the diagram

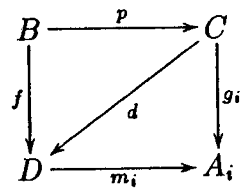


commute for each $i \in I$. In particular, r_D is a section, hence an isomorphism, and so $D \in \mathcal{C}$. Consequently, we have shown that $\mathbf{M}(\mathcal{A}) \subset \mathcal{C}$. □

Recall that for any subcategory $\mathcal{A}$ of a category $\mathcal{B}$ the inclusion $\mathcal{A}^{\perp} \subset \mathcal{A}$-**Epi** always holds, but is not always reversible : the identity map from the discrete space D_2 on $\{0,1\}$ to the Sierpinski space S is a **Haus**-epi which is not orthogonal with respect to D_2 itself (consider the map id_{D_2}). We show that, given any $\mathcal{A}$-relative factorisation structure $(E, \mathbf{M})$ for which $\mathcal{A}^{\perp} \subset E$, $\mathbf{M}(\mathcal{A})$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$ (in particular, if $E = \mathcal{A}$-**Epi**, then $\mathbf{M}(\mathcal{A})$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$) :

2.3 Proposition. *Let $(E, \mathbf{M})$ be an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$. If $\mathcal{A}^{\perp} \subset E$, then $\mathbf{M}(\mathcal{A})$ coincides with the orthogonal closure $(\mathcal{A}^{\perp})_{\perp}$ of $\mathcal{A}$ in $\mathcal{B}$, and hence is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$.*

PROOF. The inclusion $(\mathcal{A}^{\perp})_{\perp} \subset \mathbf{M}(\mathcal{A})$ is clear since by 2.1 $\mathbf{M}(\mathcal{A})$ is reflective (hence orthogonal) in $\mathcal{B}$ and contains $\mathcal{A}$. For the reverse inclusion, let $p : B \to C$ be in $\mathcal{A}^{\perp}$, $D \in \mathbf{M}(\mathcal{A})$, and consider a $\mathcal{B}$-morphism $f : B \to D$. Then D is the domain of an $\mathbf{M}$-source $(m_i : D \to A_i)_I$, say. Since $p \in \mathcal{A}^{\perp}$, there exists for each $i \in I$ a $\mathcal{B}$-morphism $g_i : C \to A_i$ such that $m_i \cdot f = g_i \cdot p$. Now $p \in \mathcal{A}^{\perp} \subset E$, so we can apply the $(E, \mathbf{M})$-diagonalisaton property to p and $(m_i)_I$ to conclude that there exists a unique $d : C \to D$ such that



$f = d \cdot p$ and $g_i = m_i \cdot d$ for each $i \in I$. The uniqueness of d with respect to the property $f = d \cdot p$ follows from the fact that p, being an element of E, belongs to the class $\mathbf{M}(\mathcal{A})$-**Epi** by 1.7. So $\mathbf{M}(\mathcal{A}) \subset (\mathcal{A}^{\perp})_{\perp}$. □

Next, we show that every reflective subcategory $\mathcal{A}$ of $\mathcal{B}$ induces an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$ in a natural way :

2.4 Proposition. *The following conditions are equivalent :*

(1) $\mathcal{A}$ is reflective in $\mathcal{B}$;

(2) $(\mathcal{A}^{\perp}, Sour(\mathcal{A}))$ is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$;

(3) there exists an $\mathcal{A}$-relative factorisation structure $(E, \mathbf{M})$ on $\mathcal{B}$ with $\mathbf{M} \subset Sour(\mathcal{A})$.

PROOF. The implication (2) $\Rightarrow$ (3) is clear, while (3) $\Rightarrow$ (1) follows from 1.13. It remains to verify the implication (1) $\Rightarrow$ (2) : since $\mathcal{A}$ is reflective in $\mathcal{B}$, and since every $\mathcal{A}$-reflection is an element of $\mathcal{A}^{\perp}$ with codomain in $\mathcal{A}$, every $\mathcal{A}$-structured source $(f_i : B \to A_i)_I$ in $\mathcal{B}$ has the appropriate factorisation, with first factor the $\mathcal{A}$-reflection morphism for B. Note also that $Sour(\mathcal{A}) = (\mathcal{A}^{\perp})^{\downarrow} \cap \{\mathcal{A}\text{-structured } \mathcal{B}\text{-sources}\}$ in the present context. Hence $(\mathcal{A}^{\perp}, Sour(\mathcal{A}))$ is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$. □

2.4 above can be generalised to characterise the reflective subcategories of $\mathcal{B}$ containing a given subcategory $\mathcal{A}$ of $\mathcal{B}$:

2.5 Proposition. *For any subcategory $\mathcal{C}$ of $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{C}$, the following conditions are equivalent :*

(1) $\mathcal{C}$ is reflective in $\mathcal{B}$;

(2) $(\mathcal{C}^{\perp}, \{\mathcal{A}\text{-structured } \mathcal{C}\text{-sources}\})$ is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$;

(3) there exists an $\mathcal{A}$-relative factorisation structure (E, M) on $\mathcal{B}$ with $\mathrm{M} \subset \{\mathcal{A}$-structured $\mathcal{C}$-sources$\}$ and $E \subset \mathcal{C}^{\perp}$.

PROOF. (2) $\Rightarrow$ (3) : clear.
(1) $\Rightarrow$ (2) : apply the argument of 2.4 (1) $\Rightarrow$ (2) to $\mathcal{C}$, noting that under the assumption of (1) the collection of $\mathcal{A}$-structured $\mathcal{C}$-sources coincides with the conglomerate of all $\mathcal{A}$-structured sources which belong to $(\mathcal{C}^{\perp})^{\downarrow}$.
(3) $\Rightarrow$ (1) : let (E, M) be an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$ satisfying the given properties. From 2.1 it follows that $\mathrm{M}(\mathcal{A})$ is reflective in $\mathcal{B}$. We shall show that $\mathrm{M}(\mathcal{A}) = \mathcal{C}$: the forward inclusion is clear, since $\mathrm{M} \subset \{\mathcal{A}$-structured $\mathcal{C}$-sources$\}$. For the reverse inclusion, let $C \in \mathcal{C}$. If $e : C \to D$ denotes the E-part in the (E, M)-factorisation of the empty $\mathcal{A}$-structured source with domain C, then by assumption $e \in \mathcal{C}^{\perp}$, so there exists a unique $t : D \to C$ such that $t \cdot e = id_C$. Further, $e \cdot t \cdot e = e$, so since $D \in \mathrm{M}(\mathcal{A})$ and $e \in \mathrm{M}(\mathcal{A})$-**Epi** by 1.7, we have $e \cdot t = id_D$. So e is an isomorphism, and the empty $\mathcal{A}$-structured source with domain C belongs to M. Consequently, $C \in \mathrm{M}(\mathcal{A})$. $\square$

An ordering can be imposed on the collection of $\mathcal{A}$-relative factorisation structures on $\mathcal{B}$, as has been done for other kinds of factorisations (see, e.g., [Kelly 1987] and [Cassidy, Hébert and Kelly 1985]), as follows : $(E, \mathrm{M}) \leq (E', \mathrm{M}')$ (to be read as : (E, M) is *finer than* (E', M')) if and only if $\mathrm{M} \subset \mathrm{M}'$, where (E, M) and (E', M') are $\mathcal{A}$-relative factorisation structures on $\mathcal{B}$. Note that in the present context, since E, M and E', M$'$ respectively need not determine each other via the diagonalisation property, $E' \subset E$ implies $\mathrm{M} \subset \mathrm{M}'$, but $\mathrm{M} \subset \mathrm{M}'$ need not imply $E' \subset E$ (see, e.g., 1.10), hence the finer than relation is only a preorder. We shall, however, consider $\mathcal{A}$-relative factorisation structures (E, M) and (E', M') on $\mathcal{B}$ to be essentially the same if the collections M and M$'$ coincide. Hence by a *finest* $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$ shall be meant a finest $\mathcal{A}$-relative factorisation structure up to the canonical equivalence induced by the finer than relation.

2.6 Proposition. *The assignment $(E, \mathrm{M}) \mapsto \Psi(E, \mathrm{M}) = \mathrm{M}(\mathcal{A})$, where (E, M) is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$, defines an order-preserving map from the collection of $\mathcal{A}$-relative factorisation structures on $\mathcal{B}$ to the collection of reflective subcategories of $\mathcal{B}$ which contain $\mathcal{A}$.*

PROOF. In view of 2.1, we need only to check that Ψ preserves order. But for $(E, \mathrm{M}) \leq (E', \mathrm{M}')$ we have $\mathrm{M} \subset \mathrm{M}'$, hence $\mathrm{M}(\mathcal{A}) \subset \mathrm{M}'(\mathcal{A})$. $\square$

2.7 Proposition. *The assignment $\mathcal{C} \mapsto \Phi(\mathcal{C}) = (\mathcal{C}^{\perp}, \{\mathcal{A}$-structured $\mathcal{C}$-sources$\})$, $\mathcal{C}$ reflective in $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{C}$, defines an order-preserving map from the collection of reflective subcategories of $\mathcal{B}$ which contain $\mathcal{A}$ to the collection of $\mathcal{A}$-relative factorisation structures on $\mathcal{B}$.*

PROOF. That Φ is an assignment follows from 2.5, and it is clear that Φ preserves order. □

2.8 Proposition. *Let $\mathcal{C}$ be reflective in $\mathcal{B}$, with $\mathcal{A} \subset \mathcal{C}$. Then $\Psi(\Phi(\mathcal{C})) = \mathcal{C}$.*

PROOF. From the respective definitions of Φ and Ψ it follows that $\Psi(\Phi(\mathcal{C})) \subset \mathcal{C}$. For the reverse inclusion, note that every $\mathcal{C}$-object C is the domain of the empty $\mathcal{A}$-structured $\mathcal{C}$-source. □

In contrast, it need not be the case that $\Phi(\Psi(E, \mathbf{M})) \simeq (E, \mathbf{M})$ for an $\mathcal{A}$-relative factorisation structure $(E, \mathbf{M})$ on $\mathcal{B}$ (where $\simeq$ denotes the equivalence relation induced by the finer than relation). This is seen from the following characterisation of a distinguished subcollection of $\mathcal{A}$-relative factorisation structures by noting that for the **Haus**-relative factorisation structure ({dense maps}, {closed **Haus**-structured sources}) on **Top**, dense maps need not be **Haus**-orthogonal :

2.9 Proposition. *For an $\mathcal{A}$-relative factorisation structure $(E, \mathbf{M})$ on $\mathcal{B}$, the following conditions are equivalent :*

(1) $\Phi(\Psi(E, \mathbf{M})) \simeq (E, \mathbf{M})$, i.e., $\mathbf{M} = \{\mathcal{A}$-structured $\mathbf{M}(\mathcal{A})$-sources$\}$;

(2) $E \subset \mathbf{M}(\mathcal{A})^{\perp}$;

(3) $E \subset \mathcal{A}^{\perp}$.

PROOF. (1) $\Rightarrow$ (2) : suppose that $e : B \to C$ belongs to E; let $f : B \to D$ be a $\mathcal{B}$-morphism from B to an $\mathbf{M}(\mathcal{A})$-object D. By assumption, $\mathbf{M} = \{\mathcal{A}$-structured $\mathbf{M}(\mathcal{A})$-sources$\}$, so in particular the empty $\mathcal{A}$-structured $\mathbf{M}(\mathcal{A})$-source with domain D belongs to $\mathbf{M}$. Hence by the $(E, \mathbf{M})$-diagonalisation property there exists a unique $d : C \to D$ such that $f = d \cdot e$. Consequently, $e \in \mathbf{M}(\mathcal{A})^{\perp}$.
(2) $\Rightarrow$ (3) : clear, since $\mathcal{A} \subset \mathbf{M}(\mathcal{A})$.
(3) $\Rightarrow$ (1) : the inclusion $\mathbf{M} \subset \{\mathcal{A}$-structured $\mathbf{M}(\mathcal{A})$-sources$\}$ trivially holds. For

the reverse inclusion, let $(f_i : D \to A_i)_I$ be an $\mathcal{A}$-structured source in $\mathbf{M}(\mathcal{A})$, i.e., $D \in \mathbf{M}(\mathcal{A})$. Let $(m_i \cdot e : D \to C \to A_i)_I$ be the $\mathcal{A}$-relative $(E, \mathbf{M})$-factorisation of $(f_i)_I$. Since $D \in \mathbf{M}(\mathcal{A})$, D is the domain of an $\mathbf{M}$-source $(n_j : D \to \bar{A}_j)_J$, say. By assumption $E \subset \mathcal{A}^{\perp}$, so there exists for each $j \in J$ a unique $\mathcal{B}$-morphism $h_j : C \to \bar{A}_j$ such that $h_j \cdot e = n_j$. From the $(E, \mathbf{M})$-diagonalisation property it follows that there exists a unique $d : C \to D$ with the property that $d \cdot e = id_D$ and $n_j \cdot d = h_j$ for each $j \in J$. Hence e is a section, and this, together with the fact that $e \in \mathbf{M}(\mathcal{A})$-**Epi** by 1.7 implies that e is an isomorphism. Hence $(f_i)_I$ belongs to $\mathbf{M}$. □

From 2.8 and 2.9 we obtain :

2.10 Proposition. *For a fixed subcategory $\mathcal{A}$ of $\mathcal{B}$, the operators Ψ and Φ define a Galois correspondence between the collection of reflective subcategories of $\mathcal{B}$ containing $\mathcal{A}$ and the collection of $\mathcal{A}$-relative factorisation structures on $\mathcal{B}$.* □

The $\mathcal{A}$-relative factorisation structures satisfying the conditions in 2.9 above will play an important role in our subsequent characterisations of the existence of reflective hulls. To this effect,

2.11 Definition. Call an $\mathcal{A}$-relative factorisation structure $(E, \mathbf{M})$ on $\mathcal{B}$ an $\mathcal{A}$-*orthogonal* factorisation structure if and only if the equivalent conditions of 2.9 above are satisfied.

Given $\mathcal{A} \subset \mathcal{B}$, and a class $E \subset \mathcal{A}^{\perp}$, there is a convenient characterisation of the E-reflectivity of $\mathcal{A}$ in terms of $\mathcal{A}$-orthogonal factorisation structures :

2.12 Proposition. *Let $E \subset \mathcal{A}^{\perp}$. Then the following conditions are equivalent :*

(1) $\mathcal{A}$ is E-reflective in $\mathcal{B}$;

(2) $(E, Sour(\mathcal{A}))$ is an $\mathcal{A}$-orthogonal factorisation structure on $\mathcal{B}$.

PROOF. (1) $\Rightarrow$ (2) : for any $\mathcal{A}$-structured source $(f_i : B \to A_i)_I$, the $\mathcal{A}$-reflection of B is the first part in the $(E, Sour(\mathcal{A}))$-factorisation of $(f_i)_I$. The $(E, Sour(\mathcal{A}))$-diagonalisation property follows from the inclusion $E \subset \mathcal{A}^{\perp}$.
(2) $\Rightarrow$ (1) : suppose (2) holds; then since $\mathcal{A}$ is closed under the formation of sources in $Sour(\mathcal{A})$, it follows from 1.13 that $\mathcal{A}$ is E-reflective in $\mathcal{B}$. □

We are now prepared to characterise the existence of reflective hulls in the language of relative factorisation structures :

2.13 Proposition. *The following conditions are equivalent, for a subcategory $\mathcal{R}$ of $\mathcal{B}$ which contains $\mathcal{A}$:*

(1) $\mathcal{R}$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$;

(2) $(\mathcal{R}^{\perp}, \{\mathcal{A}\text{-structured } \mathcal{R}\text{-sources}\})$ is a finest $\mathcal{A}$-orthogonal factorisation structure on $\mathcal{B}$.

PROOF. (1) $\Rightarrow$ (2) : By 2.5, $(\mathcal{R}^{\perp}, \{\mathcal{A}\text{-structured } \mathcal{R}\text{-sources}\})$ is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$, and is moreover an $\mathcal{A}$-orthogonal factorisation structure on $\mathcal{B}$ since $\mathcal{A} \subset \mathcal{R}$ implies $\mathcal{R}^{\perp} \subset \mathcal{A}^{\perp}$. Let $(E, \mathbf{M})$ be an $\mathcal{A}$-orthogonal factorisation structure on $\mathcal{B}$. By 2.1, $\mathbf{M}(\mathcal{A})$ is reflective in $\mathcal{B}$, and $\mathcal{A} \subset \mathbf{M}(\mathcal{A})$. By the reflective hull property of $\mathcal{R}$, $\mathcal{R} \subset \mathbf{M}(\mathcal{A})$. We need to show that every $\mathcal{A}$-structured source in $\mathcal{R}$ belongs to $\mathbf{M}$. So, let $(f_i : R \to A_i)_I$ be an $\mathcal{A}$-structured source in $\mathcal{R}$. Let $(m_i \cdot e : R \to C \to A_i)_I$ be the $\mathcal{A}$-relative $(E, \mathbf{M})$-factorisation of $(f_i)_I$. Since $\mathcal{R} \subset \mathbf{M}(\mathcal{A})$, $R \in \mathbf{M}(\mathcal{A})$, i.e., R is the domain of an $\mathbf{M}$-source $(n_j : R \to \bar{A}_j)_J$, say. Now, $(E, \mathbf{M})$ is an $\mathcal{A}$-orthogonal factorisation structure, i.e., $E \subset \mathcal{A}^{\perp}$, hence for each $j \in J$ there exists a unique $g_j : C \to \bar{A}_j$ such that $g_j \cdot e = n_j$. So by the $(E, \mathbf{M})$-diagonalisation property there exists a unique $d : C \to R$ such that $d \cdot e = id_R$ and $n_j \cdot d = g_j$ for each $j \in J$. Hence e is a section. By 1.7 we also have $e \in \mathbf{M}(\mathcal{A})$-**Epi**, so since $R \in \mathbf{M}(\mathcal{A})$, e is an isomorphism. Consequently, $(f_i)_I$ belongs to $\mathbf{M}$.
(2) $\Rightarrow$ (1) : let $\mathcal{C}$ be a reflective subcategory of $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{C}$. By 2.5, $(\mathcal{C}^{\perp}, \{\mathcal{A}\text{-structured } \mathcal{C}\text{-sources}\})$ is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$, and moreover is an $\mathcal{A}$-orthogonal factorisation structure on $\mathcal{B}$. Since $(\mathcal{R}^{\perp}, \{\mathcal{A}\text{-structured } \mathcal{R}\text{-sources}\})$ is a finest such, every $\mathcal{A}$-structured $\mathcal{R}$-source is a source in $\mathcal{C}$. In particular, every empty $\mathcal{A}$-structured $\mathcal{R}$-source is a source in $\mathcal{C}$. Hence $\mathcal{R} \subset \mathcal{C}$. $\square$

It is possible to delete the occurrence of $\mathcal{R}$ in 2.13 so that a pure existence criterion for reflective hulls is obtained :

2.14 Theorem. *The following conditions are equivalent :*

(1) $\mathcal{A}$ has a reflective hull in $\mathcal{B}$;

(2) there exists a finest $\mathcal{A}$-orthogonal factorisation structure on $\mathcal{B}$.

PROOF. (1) $\Rightarrow$ (2) follows from 2.13.
(2) $\Rightarrow$ (1) : let $(E, \mathbf{M})$ be a finest $\mathcal{A}$-orthogonal factorisation structure on $\mathcal{B}$. We shall show that $\mathbf{M}(\mathcal{A})$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$. So, let $\mathcal{C}$ be reflective in $\mathcal{B}$, with $\mathcal{A} \subset \mathcal{C}$. By 2.5, $(\mathcal{C}^{\perp}$, {$\mathcal{A}$-structured $\mathcal{C}$-sources}) is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$, and moreover is an $\mathcal{A}$-orthogonal factorisation structure on $\mathcal{B}$. Then by assumption $\mathbf{M} \subset$ {$\mathcal{A}$-structured $\mathcal{C}$-sources}. Hence every $\mathbf{M}(\mathcal{A})$-object belongs to $\mathcal{C}$, i.e., $\mathbf{M}(\mathcal{A}) \subset \mathcal{C}$. □

From 2.14 it can be deduced that the existence of a finest $\mathcal{A}$-relative factorisation structure implies the existence of a reflective hull :

2.15 Corollary. *If there exists a finest $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$ (e.g., if ($\mathcal{A}$-**Epi**, $\mathbf{M}$) is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$), then $\mathcal{A}$ has a reflective hull in $\mathcal{B}$.*

PROOF. If $(E, \mathbf{M})$ is a finest $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$, then by 2.1 $\mathbf{M}(\mathcal{A})$ is reflective in $\mathcal{B}$. Now apply the argument used in 2.14 (2) $\Rightarrow$ (1). □

2.16 Remarks. (1) The converse to 2.15 does not hold : let $\mathcal{B}$ be the poset of natural numbers, considered as a category, and put $\mathcal{A} = \emptyset$.
(2) If there exists a reflective hull of $\mathcal{A}$ in $\mathcal{B}$, then the reflective hull coincides with $\bigcap\{\mathbf{M}(\mathcal{A}) \mid (-, \mathbf{M})$ is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}\}$. If the aforementioned intersection is reflective in $\mathcal{B}$, then it is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$.

The situation when $\mathcal{B}$ itself is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$ can be characterised :

2.17 Corollary. *$\mathcal{B}$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$ if and only if there exists essentially only one $\mathcal{A}$-orthogonal factorisation structure on $\mathcal{B}$.*

PROOF. "only if" : suppose that $\mathcal{B}$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$. Then by the argument used in 2.14 (1) $\Rightarrow$ (2), $(\mathcal{B}^{\perp} = \mathbf{Iso}(\mathcal{B})$, {$\mathcal{A}$-structured $\mathcal{B}$-sources}) is a finest $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$ with first part contained in $\mathcal{A}^{\perp}$. So for any $\mathcal{A}$-orthogonal factorisation structure $(E, \mathbf{M})$ on $\mathcal{B}$, we have {$\mathcal{A}$-structured $\mathcal{B}$-sources} $= \mathbf{M}$.
"if" : note that $(E, \mathbf{M}) = (\mathbf{Iso}(\mathcal{B})$, {$\mathcal{A}$-structured $\mathcal{B}$-sources}) is always an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$, with first part contained in $\mathcal{A}^{\perp}$, so by our assumption this must be a finest such. But then $\mathbf{M}(\mathcal{A}) = \mathcal{B}$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$ by 2.14. □

The reflectivity of the orthogonal closure of $\mathcal{A}$ in $\mathcal{B}$ can also be characterised in terms of $\mathcal{A}$-orthogonal factorisation structures :

2.18 Proposition. *The following conditions are equivalent :*

(1) The orthogonal closure of $\mathcal{A}$ in $\mathcal{B}$ is reflective in $\mathcal{B}$;

(2) the orthogonal closure of $\mathcal{A}$ in $\mathcal{B}$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$;

(3) $(\mathcal{A}^{\perp}, \{\mathcal{A}\text{-structured } (\mathcal{A}^{\perp})_{\perp}\text{-sources}\})$ *is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$.*

PROOF. (1) $\Leftrightarrow$ (2) : clear.
(2) $\Rightarrow$ (3) : suppose that $(\mathcal{A}^{\perp})_{\perp}$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$. Note first that $((\mathcal{A}^{\perp})_{\perp})^{\perp} = \mathcal{A}^{\perp}$. So by 2.5 $(\mathcal{A}^{\perp}, \{\mathcal{A}\text{-structured } (\mathcal{A}^{\perp})_{\perp}\text{-sources}\})$ is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$.
(3) $\Rightarrow$ (2) : let (E, M) be an $\mathcal{A}$-orthogonal factorisation structure on $\mathcal{B}$. Then since $E \subset \mathcal{A}^{\perp}$, it follows that every $\mathcal{A}$-structured $(\mathcal{A}^{\perp})_{\perp}$-source belongs to M. Hence $(\mathcal{A}^{\perp}, \{\mathcal{A}\text{-structured } (\mathcal{A}^{\perp})_{\perp}\text{-sources}\})$ is a finest $\mathcal{A}$-orthogonal factorisation structure on $\mathcal{B}$, and since $((\mathcal{A}^{\perp})_{\perp})^{\perp} = \mathcal{A}^{\perp}$, it follows by 2.13 (2) $\Rightarrow$ (1) that $(\mathcal{A}^{\perp})_{\perp}$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$. □

Concerning intersections of reflective subcategories, we have :

2.19 Proposition. *For any collection $(\mathcal{R}_i)_I$ of reflective subcategories of $\mathcal{B}$, put $\mathcal{R} = \bigcap_I \mathcal{R}_i$, $\mathrm{M} = \{\mathcal{R}\text{-sources}\}$ and $\mathrm{M}_i = \{\mathcal{R}\text{-structured } \mathcal{R}_i\text{-sources}\}$ $(i \in I)$. Then the following conditions are equivalent :*

(1) $\mathcal{R}$ is reflective in $\mathcal{B}$, i.e., $\mathcal{R} = \bigwedge_I \mathcal{R}_i$;

(2) $(\mathcal{R}^{\perp}, \mathrm{M})$ is an infimum of $\{(\mathcal{R}_i^{\perp}, \mathrm{M}_i) \mid i \in I\}$;

(3) an infimum of $\{(\mathcal{R}_i^{\perp}, \mathrm{M}_i) \mid i \in I\}$ exists.

PROOF. (1) $\Rightarrow$ (2) : this implication follows from 2.4 and 2.14 (applied on $\mathcal{A} = \mathcal{R}$).
(2) $\Rightarrow$ (3) : clear.
(3) $\Rightarrow$ (1) : let (E, M) be an infimum of $\{(\mathcal{R}_i^{\perp}, \mathrm{M}_i) \mid i \in I\}$. By 2.1, $\mathrm{M}(\mathcal{R})$ is reflective in $\mathcal{B}$. We show that $\mathcal{R} = \mathrm{M}(\mathcal{R})$. The forward inclusion is clear. For the reverse inclusion, let $D \in \mathrm{M}(\mathcal{R})$. Then D is the domain of an M-source $(m_j : D \to R_j)_J$, say. Given $i \in I$, let $r_D^i : D \to r^i D$ denote the $\mathcal{R}_i$-reflection for D. Since $\mathcal{R} \subset \mathcal{R}_i$, we have

$\mathcal{R}_i^\perp \subset \mathcal{R}^\perp$, hence since $(m_j)_J$ is an $\mathcal{R}$-structured source, there exists for each $j \in J$ a unique $g_j : r^i D \to R_j$ such that $m_j = g_j \cdot r^i_D$. By assumption $\mathbf{M} \subset \mathbf{M}_i$, so $(m_j)_J$ belongs to $\mathbf{M}_i$, hence by the $(\mathcal{R}_i^\perp, \mathbf{M}_i)$-diagonlisation property there exists a unique $d : r^i_D \to D$ such that $d \cdot r^i_D = id_D$ and $m_j \cdot d = g_j$ for each $j \in J$. In particular, r^i_D is a section, hence an isomorphism, and so $D \in \mathcal{R}_i$. Hence, since $D \in \mathcal{R}_i$ for each $i \in I$, $D \in \mathcal{R}$. $\square$

Chapter 4

Reflective Hulls Via Relative Multiple Pushouts

We further develop some of the general theory of relative factorisation structures which were defined in Chapter 4, and apply these considerations to the reflective hull problem. The question of existence of relative factorisation structures is investigated in some detail, the primary tool being a modification of the concept of a relative multiple pushout, a special type of colimit *relative* to a given subcategory. Using results obtained in Chapter 4, further characterisations for the existence of reflective hulls are obtained.

The primary goal in Section 1 below is to obtain a relativised version of the existence theorem for standard factorisations ([Adámek, Herrlich and Strecker 1990] 15.14), which states that a class E of $\mathcal{B}$-morphisms induces a (standard) factorisation structure (E, M) on $\mathcal{B}$ if and only if E satisfies the following conditions :

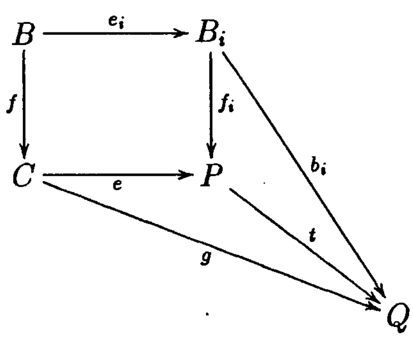
(1) E contains the class of $\mathcal{B}$-isomorphisms and consists only of $\mathcal{B}$-epimorphisms,
(2) E is closed under composition, and
(3) pushouts of E-morphisms and cointersections of E-sources exist in $\mathcal{B}$ and E is stable under the formation of these constructions.

In anticipation of the concept of relative E-multiple pushout to be defined in Section 1 below, note that condition (3) above can be replaced by the following (equivalent) condition :

(3*) $\mathcal{B}$ has, and E is stable under the formation of, multiple pushouts,

where the *multiple pushout* of a $\mathcal{B}$-source $(e_i : B \to B_i)_I$ along a $\mathcal{B}$-morphism $f : B \to C$

consists of a $\mathcal{B}$-morphism $e : C \to P$ and a $\mathcal{B}$-sink $(f_i : B_i \to P)_I$, with the property that $e \cdot f = f_i \cdot e_i$ for each $i \in I$, and for any pair $(g, (b_i)_I)$ consisting of a $\mathcal{B}$-morphism $g : C \to Q$ and a $\mathcal{B}$-sink $(b_i : B_i \to Q)_I$ such that $g \cdot f = b_i \cdot e_i$ for each $i \in I$, there exists a unique $t : P \to Q$ such that



$g = t \cdot e$ and $t \cdot f_i = b_i$ for each $i \in I$. Stability of E under the formation of the multiple pushout $(e, (f_i)_I)$ of $(f, (e_i)_I)$ means that whenever $(e_i)_I$ is an E-source, then e belongs to E. Multiple pushouts in this sense have previously been used in [Tholen and Wolff 1981].

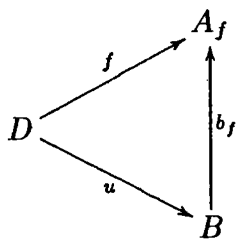
1 Relative Factorisations and Multiple Pushouts.

A fundamental tool in this and the next Section will be, for a subcategory $\mathcal{A}$ of $\mathcal{B}$ and a given class E of $\mathcal{B}$-morphisms, the concept of an E-multiple pushout *relative* to $\mathcal{A}$. $\mathcal{A}$-relative E-multiple pushouts (resp. $\mathcal{A}$-relative E-cointersections) are instances of the concept of an *$\mathcal{A}$-relative colimit* (to be defined) due to H. W. Bargenda, which has its roots in [Adámek, Herrlich and Strecker 1990] 12.8 . We first consider the special case of partially ordered classes :

1.1 Example. Let $\mathcal{B}$ be a partially ordered class, D a subclass of $\mathcal{B}$. Further, let $D^\dagger$ be the class of all upper bounds of D. Then it follows that, if it exists, $\bigvee D$ is an upper bound for D and a lower bound for $D^\dagger$. Now, given a subclass $\mathcal{A}$ of $\mathcal{B}$, consider the class $D^\dagger \cap \{C \in \mathcal{B} \mid C \text{ is a lower bound for } D^\dagger \cap \mathcal{A}\}$. If this class has a maximum element, then we may call it the *$\mathcal{A}$-relative supremum of D* (or, *supremum of D over $\mathcal{A}$*). Note that for a one-element subclass $D = \{B\}$, the $\mathcal{A}$-relative supremum of D is exactly $\bigvee\{C \in \mathcal{B} \mid B \leq C \text{ and for all } A \in \mathcal{A}\ (B \leq A \Rightarrow C \leq A)\}$ (cf. 2.5 of Chapter 1).

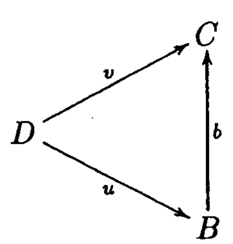
Generalising 1.1 above to arbitrary categories, we have the following :

1.2 Definition. Let $\mathcal{A} \subset \mathcal{B}$, and suppose that $D : I \to \mathcal{B}$ is any diagram.
(1) Let $u = (u_i)_I : D \to B$ be an upper bound (i.e., a natural sink) for D in $\mathcal{B}$, and let $(D \uparrow \mathcal{A})$ denote the collection of all upper bounds $f : D \to A_f$ which are $\mathcal{A}$-valued, i.e., $A_f \in \mathcal{A}$. Then u is called *$\mathcal{A}$-orthogonal* if there exists exactly one $\mathcal{B}$-source $(b_f : B \to A_f)_{(D\uparrow\mathcal{A})}$ with



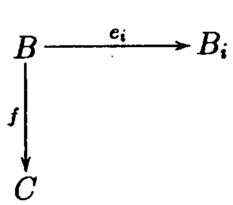
$f = b_f \cdot u$ for each $f \in (D \uparrow \mathcal{A})$ (i.e., $f_i = b_f \cdot u_i$ for each $f \in (D \uparrow \mathcal{A})$ and $i \in I$).

(2) Let $\mathcal{E}$ be a collection of $\mathcal{A}$-orthogonal upper bounds for D. An element $v : D \to C$ of $\mathcal{E}$ is called an *$\mathcal{A}$-relative $\mathcal{E}$-colimit* (of D in $\mathcal{B}$) if for each $u : D \to B$ in $\mathcal{E}$ there exists exactly one $\mathcal{B}$-morphism $b : B \to C$ with

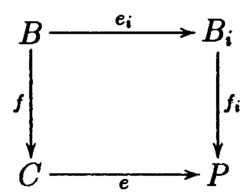


$v = b{\cdot}u$. If $\mathcal{E}$ is the collection of all $\mathcal{A}$-orthogonal upper bounds for D, then an $\mathcal{A}$-relative $\mathcal{E}$-colimit of D will simply be called an *$\mathcal{A}$-relative colimit* of D.

Now, let $(e_i : B \to B_i)_I$ be a source of $\mathcal{B}$-morphisms and $f : B \to C$ any $\mathcal{B}$-morphism. Consider the diagram with scheme

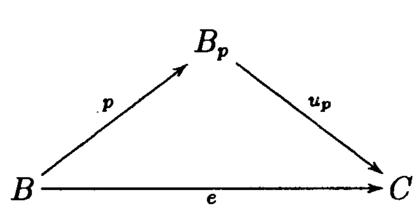


denoted by $(f,(e_i)_I)$. Upper bounds of $(f,(e_i)_I)$ are pairs $(e,(f_i)_I)$ such that



commutes for each $i \in I$.
(3) For any class E of $\mathcal{B}$-morphisms, let $\mathcal{E}$ be the collection of all $\mathcal{A}$-orthogonal upper bounds $(e,(f_i)_I)$ of a given diagram $(f,(e_i)_I)$ with $(e_i)_I$ an E-source, such that $e \in E$. Then an $\mathcal{A}$-relative $\mathcal{E}$-colimit of $(f,(e_i)_I)$ is called an *$\mathcal{A}$-relative E-multiple pushout* (of $(f,(e_i)_I)$ in $\mathcal{B}$).
(4) An *$\mathcal{A}$-relative E-cointersection* of a source $(e_i : B \to B_i)_I$ of E-morphisms is an $\mathcal{A}$-relative E-multiple pushout of $(id_B,(e_i)_I)$.

1.3 Remarks. (1) Let $D : I \to \mathcal{B}$ be any diagram. If $\mathcal{A} = \mathcal{B}$, then any $\mathcal{A}$-orthogonal upper bound for D coincides with the colimit of D.
(2) Let D be a diagram in $\mathcal{B}$ of the form $(f : B \to C, (e_i : B \to B_i)_I)$, and suppose that E is a class of $\mathcal{B}$-morphisms. If $\mathcal{A} = \mathcal{B}$, then the $\mathcal{A}$-relative E-multiple pushout $(e,(f_i)_I)$ of $(f,(e_i)_I)$ is exactly the usual multiple pushout of $(f,(e_i)_I)$ in $\mathcal{B}$, with e and all e_i in E.
Note that the usual two conditions "$\mathcal{B}$ has $\mathcal{A}$-relative multiple pushouts of diagrams of form $(f,(e_i)_I)$ with all e_i in E" and "E is stable under the formation of $\mathcal{A}$-relative multiple pushouts in $\mathcal{B}$" are both incorporated in our more general condition "$\mathcal{B}$ has $\mathcal{A}$-relative E-multiple pushouts".
(3) Consider the following special case of an $\mathcal{A}$-relative colimit : given a $\mathcal{B}$-object B, suppose that the $\mathcal{A}$-relative $\mathcal{A}^{\perp}$-cointersection of the empty source with domain B exists. Thus we have an $\mathcal{A}^{\perp}$-morphism $e : B \to C$, such that for every $p : B \to B_p$ in $\mathcal{A}^{\perp}$ there exists a unique $u_p : B_p \to C$ making the diagram



commute. In this sense, C can be considered as a "closest $\mathcal{A}$-orthogonal approximation" of the $\mathcal{B}$-object B to the subcategory $\mathcal{A}$.

In fact, the $\mathcal{A}$-approaching colimits of Chapter 1 occur as "closest approximations" in the sense of 1.3 (3) above :

1.4 Proposition. *Let $e : B \to C$ be a $\mathcal{B}$-morphism. Then the following conditions are equivalent :*

(1) The $\mathcal{A}$-approaching colimit $(u_p : B_p \to C)_{\mathcal{A}^\perp_B}$ of $D^\perp_B$ exists in $\mathcal{B}$, with codiagonal morphism e;

(2) $(e, (\emptyset, C))$ is the $\mathcal{A}$-relative $\mathcal{A}^\perp$-cointersection of the empty source $(B, \emptyset)$.

PROOF. (1) $\Rightarrow$ (2) : suppose that $((u_p)_{\mathcal{A}^\perp_B}, e)$ is the $\mathcal{A}$-approaching colimit of $D^\perp_B$. Then $e \in \mathcal{A}^\perp_B$ by 2.1 of Chapter 1. We must show that for each $p \in \mathcal{A}^\perp_B$ there exists a unique $x : B_p \to C$ such that $x \cdot p = e$. So, let $p : B \to B_p$ be in $\mathcal{A}^\perp$. Note first that $e = u_p \cdot p$. Now consider $x : B_p \to C$ such that $x \cdot p = e$. Since $x \cdot p = e$, $x : (p, B_p) \to (e, C)$ is a morphism in $\mathcal{A}^\perp_B$, hence (since $(u_p)_{\mathcal{A}^\perp_B}$ is an upper bound for $D^\perp_B$) $u_e \cdot x = u_p$. For each $p \in \mathcal{A}^\perp_B$, $u_p : (p, B_p) \to (e, C)$ is a morphism in $\mathcal{A}^\perp_B$, hence since $(u_p)_{\mathcal{A}^\perp_B}$ is an upper bound for $D^\perp_B$, $u_e \cdot u_p = u_p$ (and $u_e \cdot e = e$). Thus, by the universal property of $\mathcal{A}$-approaching colimits, $u_e = id_C$. So $x = u_p$.
(2) $\Rightarrow$ (1) : by the $\mathcal{A}$-relative $\mathcal{A}^\perp$-cointersection property of $(e, (\emptyset, C))$ there exists for each $p : B \to B_p$ in $\mathcal{A}^\perp_B$ a unique $u_p : B_p \to C$ such that $e = u_p \cdot p$. Note first that $(u_p)_{\mathcal{A}^\perp_B}$ is an upper bound for $D^\perp_B$, since for each $p \in \mathcal{A}^\perp_B$, $u_p \in \mathcal{A}$-**Epi**. Suppose that $(v_p : B_p \to D)_{\mathcal{A}^\perp_B}$ is an upper bound for $D^\perp_B$. Since $e \in \mathcal{A}^\perp_B$, there exists $v_e : C \to D$. For each $p \in \mathcal{A}^\perp_B$, $u_p : (p, B_p) \to (e, C)$ is a morphism in $\mathcal{A}^\perp_B$, hence $v_e \cdot u_p = v_p$ for each $p \in \mathcal{A}^\perp_B$. Of course, $u_e = id_C$. Hence, if $x : C \to D$ is a $\mathcal{B}$-morphism with $x \cdot u_p = v_p$ for each $p \in \mathcal{A}^\perp_B$, then, in particular, $x = x \cdot u_e = v_e$. ☐

Before we characterise relative factorisation structures in terms of relative multiple pushouts, we need two Lemmata :

1.5 Lemma. *If $\mathcal{B}$ has $\mathcal{A}$-relative E-cointersections, then $E \subset \mathcal{A}$-**Epi**.*

PROOF. Suppose that $e : B \to C$ belongs to E, and let I denote the class of all $\mathcal{A}$-structured morphisms. For each $f \in I$ put $(e_f : B \to C_f) = e$. Let $(\bar{e} : B \to P, (g_f)_I)$

denote the $\mathcal{A}$-relative E-cointersection of $(e_f)_I$. Now let $r, s : C \rightrightarrows A$ be $\mathcal{A}$-structured $\mathcal{B}$-morphisms with $r \cdot e = s \cdot e$; define the sink $(a_f : C_f = C \to A)_I$ by

$$a_f = \begin{cases} r & \text{if } f \cdot g_f = s \\ s & \text{otherwise.} \end{cases}$$

Then $(r \cdot e, (a_f)_I)$ is clearly an $\mathcal{A}$-valued upper bound for $(e_f)_I$. Since $(\bar{e}, (g_f)_I)$ is $\mathcal{A}$-orthogonal, there exists a $\mathcal{B}$-morphism $d : P \to A$ with $a_f = d \cdot g_f$ for each $f \in I$. Since $d \in I$, we obtain $d \cdot g_d = r$ if and only if $d \cdot g_d = s$, and since $d \cdot g_d = r$ or $d \cdot g_d = s$, it follows that $r = s$. □

1.6 Lemma. *Suppose that $(e : C \to P, (f_i : B_i \to P)_I)$ is an $\mathcal{A}$-relative E-multiple pushout of $(f : B \to C, (e_i : B \to B_i)_I)$, with $E \subset \mathcal{A}$-**Epi**. If $x : P \to P$ is a $\mathcal{B}$-morphism with $x \cdot e = e$, then x is an isomorphism in $\mathcal{B}$.*

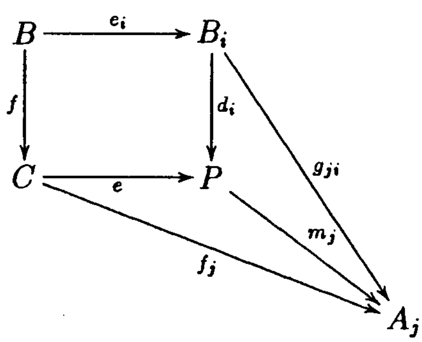
PROOF. Note that $(e, (x \cdot f_i)_I)$ is an upper bound for $(f, (e_i)_I)$. We show that $(e, (x \cdot f_i)_I)$ is $\mathcal{A}$-orthogonal : let $(a : C \to A, (a_i)_I)$ be an $\mathcal{A}$-valued upper bound of $(f, (e_i)_I)$. Because $(e, (f_i)_I)$ is $\mathcal{A}$-orthogonal, there exists $\bar{a} : P \to A$ with $\bar{a} \cdot e = a$ and $\bar{a} \cdot f_i = a_i$ for each $i \in I$. Hence $\bar{a} \cdot (x \cdot f_i) \cdot e_i = \bar{a} \cdot x \cdot e \cdot f = \bar{a} \cdot e \cdot f = a \cdot f = a_i \cdot e_i$, so $\bar{a} \cdot (x \cdot f_i) = a_i$ for each $i \in I$, since $e_i \in \mathcal{A}$-**Epi** for each $i \in I$. Now, since $(e, (f_i)_I)$ is an $\mathcal{A}$-relative E-multiple pushout, there exists (a unique) $t : P \to P$ with $t \cdot e = e$ and $t \cdot x \cdot f_i = f_i$ for each $i \in I$. Hence, the endomorphism $t \cdot x : P \to P$ satisfies $(t \cdot x) \cdot e = t \cdot e = e$ and $(t \cdot x) \cdot f_i = f_i$ for each $i \in I$. By the $\mathcal{A}$-relative E-multiple pushout property of $(e, (f_i)_I)$, $t \cdot x = id_P$. Further, $(e, (t \cdot f_i)_I)$ is an upper bound for $(f, (e_i)_I)$, and, as above, one shows that $(e, (t \cdot f_i)_I)$ is $\mathcal{A}$-orthogonal. Hence, there exists (a unique) $s : P \to P$ with $s \cdot e = e$ and $s \cdot t \cdot f_i = f_i$ for each $i \in I$. So, the endomorphism $s \cdot t : P \to P$ satisfies $s \cdot x \cdot e = s \cdot e = e$ and $(s \cdot t) \cdot f_i = f_i$ for each $i \in I$, thus $s \cdot t = id_P$, which together with $t \cdot x = id_P$ implies that t is an isomorphism. Therefore, x is an isomorphism. □

1.7 Theorem. *Let E be a class of $\mathcal{B}$-morphisms which is closed under composition. Then the following conditions are equivalent :*

(1) E induces an $\mathcal{A}$-relative factorisation structure $(E, \mathbf{M})$ on $\mathcal{B}$;

(2) $\mathcal{B}$ has $\mathcal{A}$-relative E-multiple pushouts.

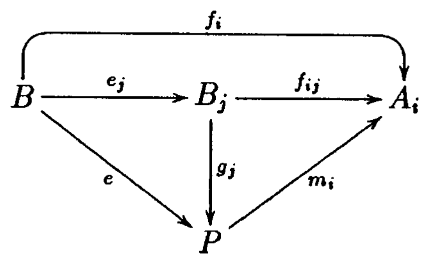
PROOF. (1) ⇒ (2) : suppose that there exists a conglomerate $\mathbf{M}$ such that $(E, \mathbf{M})$ is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$. By 1.7 of Chapter 4, $E \subset \mathbf{M}(\mathcal{A})$-**Epi**. Let D be a diagram of the form $(f, (e_i)_I)$, where $(e_i : B \to B_i)_I$ is a source of E-morphisms, and $f : B \to C$ is a $\mathcal{B}$-morphism. Let $((f_j : C \to A_j)_J, (g_{ji} : B_i \to A_j)_I)$ denote the collection of all $\mathcal{A}$-valued upper bounds for D. If $(m_j \cdot e : C \to P \to A_j)_J$ denotes the $\mathcal{A}$- relative $(E, \mathbf{M})$-factorisation of $(f_j)_J$, then we have $m_j \cdot e \cdot f = g_{ji} \cdot e_i$ for each $j \in J$ and $i \in I$, so by the $(E, \mathbf{M})$-diagonalisation property there exists for each $i \in I$ a unique $d_i : B_i \to P$ such that $e \cdot f = d_i \cdot e_i$ and for each $j \in J$, $m_j \cdot d_i = g_{ji}$.



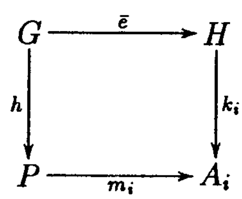
Hence $(e, (d_i)_I)$ is an upper bound for D, and $\mathcal{A}$-orthogonal since $P \in \mathbf{M}(\mathcal{A})$ and $e \in \mathbf{M}(\mathcal{A})$-**Epi**. It remains to be shown that $(e, (d_i)_I)$ is the E-multiple pushout of D. So, let $(\bar{e} : C \to Q, (h_i : B_i \to Q)_I)$ be an $\mathcal{A}$-orthogonal upper bound for D, with $\bar{e} \in E$. Hence there exists a $\mathcal{B}$-source $(p_j : Q \to A_j)_J$ with $(f_j)_J = (p_j \cdot \bar{e})_J$ and $g_{ji} = p_j \cdot h_i$ for all $j \in J$ and $i \in I$. Thus, $p_j \cdot \bar{e} = m_j \cdot e$ for each $j \in J$, hence, since $\bar{e} \in E$ and $(m_j)_J \in \mathbf{M}$, there exists a unique diagonal $d : Q \to P$ such that $e = d \cdot \bar{e}$ and $p_j = m_j \cdot d$ for each $j \in J$. Further, we have, for each $i \in I$, $d \cdot h_i \cdot e_i = d \cdot \bar{e} \cdot f = e \cdot f = d_i \cdot e_i$, hence $d \cdot h_i = d_i$ since $e_i \in \mathbf{M}(\mathcal{A})$-**Epi** and $P \in \mathbf{M}(\mathcal{A})$. Finally, if $c : Q \to P$ is such that $c \cdot \bar{e} = e$ and $d_i = c \cdot h_i$ for each $i \in I$, we have $d \cdot \bar{e} = e = c \cdot \bar{e}$, hence $d = c$, since $P \in \mathbf{M}(\mathcal{A})$ and $\bar{e} \in \mathbf{M}(\mathcal{A})$-**Epi**.

(2) ⇒ (1) : by 1.5 above, $E \subset \mathcal{A}$-**Epi**. Let $\mathbf{M}$ be the conglomerate of all $\mathcal{A}$-structured sources which belong to the collection $E^{\downarrow}$. We need to show that every $\mathcal{A}$-structured source in $\mathcal{B}$ has an $(E, \mathbf{M})$-factorisation. So, consider an $\mathcal{A}$-structured source $(f_i : B \to A_i)_I$. Let $(e_j : B \to B_j)_J$ be the source of all morphisms $e_j \in E$ with the property that for every $i \in I$ there exists a (unique) $\mathcal{B}$-morphism $f_{ij} : B_j \to A_i$ satisfying $f_i = f_{ij} \cdot e_j$. Let D be the diagram given by the source $(e_j)_J$, and let $(e : B \to P, (g_j)_J)$ denote the $\mathcal{A}$-relative E-cointersection of D. For each $i \in I$, $(f_i, (f_{ij})_J)$ is an $\mathcal{A}$-valued upper

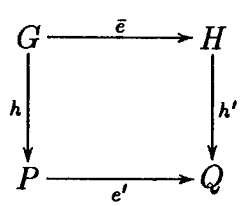
bound for D, so there exists a $\mathcal{B}$-morphism $m_i : P \to A_i$ such that $m_i \cdot g_j = f_{ij}$ for each $j \in J$; in particular, we have $f_i = m_i \cdot e$.



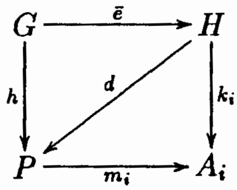
So $(m_i \cdot e : B \to P \to A_i)_I$ is a factorisation of $(f_i)_I$, and since $e \in E$, it remains to show that the source $(m_i)_I$ belongs to M. For this, consider a commutative diagram of the form



where $\bar{e} \in E$. Form the $\mathcal{A}$-relative E-pushout $(e' : P \to Q, h' : H \to Q)$ of $(h, \bar{e})$:



Let $i \in I$; then since (m_i, k_i) is an $\mathcal{A}$-valued upper bound for $(h, \bar{e})$, there exists $p_i : Q \to A_i$ such that $m_i = p_i \cdot e'$ and $k_i = p_i \cdot h'$. Since $e \in E$ and $e' \in E$, the composition $e' \cdot e$ belongs to E, and for each $i \in I$ we have $p_i \cdot e' \cdot e = m_i \cdot e = f_i$, i.e., $e' \cdot e$ is an E-morphism which factorises $(f_i)_I$. Thus $e' \cdot e = e_{j_0}$ for some $j_0 \in J$, so for $x = g_{j_0} \cdot e'$, $x \cdot e = g_{j_0} \cdot e' \cdot e = g_{j_0} \cdot e_{j_0} = e$. Hence, by 1.6 applied on the $\mathcal{A}$-relative E-cointersection $(e, (g_j)_J)$ of $(e_j)_J$, x is an isomorphism, i.e., e' is a section. For $x' = e' \cdot x^{-1} \cdot g_{j_0}$, $x' \cdot e' = e'$. Hence, by 1.6 applied on the $\mathcal{A}$-relative E-pushout (e', h') of $(h, \bar{e})$, x' is an isomorphism, i.e., e' is a retraction, thus an isomorphism. It follows that $d = (e')^{-1} \cdot h'$ is a diagonal in



since $d \cdot \bar{e} = (e')^{-1} \cdot h' \cdot \bar{e} = (e')^{-1} \cdot e' \cdot h = h$, and for each $i \in I$, $m_i \cdot d \cdot \bar{e} = m_i \cdot h = k_i \cdot \bar{e}$ implies that $m_i \cdot d = k_i$ because $\bar{e} \in \mathcal{A}$-**Epi**. Now, let $d' : H \to P$ be any $\mathcal{B}$-morphism with $d' \cdot \bar{e} = h$ and $m_i \cdot d' = k_i$ for each $i \in I$. Then, of course, $(e', e' \cdot d')$ is an upper bound for $(h, \bar{e})$, and $(e', e' \cdot d')$ is moreover $\mathcal{A}$-orthogonal, since if $(b, a : H \to A)$ is any $\mathcal{A}$-valued upper bound for $(h, \bar{e})$, there exists $f : Q \to A$ in $\mathcal{B}$ with $b = f \cdot e'$ and $a = f \cdot h'$ (since (e', h') is an $\mathcal{A}$-relative E-pushout of $(h, \bar{e})$), hence $f \cdot e' \cdot d' \cdot \bar{e} = f \cdot e' \cdot h = b \cdot h = a \cdot \bar{e}$, thus $f \cdot e' \cdot d' = a$ since $\bar{e} \in \mathcal{A}$-**Epi**, and any $f' : Q \to A$ in $\mathcal{B}$ is already uniquely determined by $b = f' \cdot e'$ because $e' \in \mathcal{A}$-**Epi**. By the $\mathcal{A}$-relative E-pushout property of (e', h'), there exists a unique $t : Q \to Q$ such that $e' = t \cdot e'$ and $h' = t \cdot e' \cdot d'$. So $d = (e')^{-1} \cdot h' = (e')^{-1} \cdot t \cdot e' \cdot d' = (e')^{-1} \cdot e' \cdot d' = d'$. Hence $(m_i)_I$ belongs to **M**. □

As an immediate consequence of 1.7 above is the following :

1.8 Corollary. *($\mathcal{A}$-**Epi**, **M**) is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$ for some* **M** *if and only if $\mathcal{B}$ has $\mathcal{A}$-relative $\mathcal{A}$-***Epi***-multiple pushouts.* □

1.9 Remark. Applying 2.3 of Chapter 4 to 1.8 above, we obtain that **M**($\mathcal{A}$) is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$, so by 2.9 and 2.13 of Chapter 4 **M** induces a finest $\mathcal{A}$-orthogonal factorisation structure on $\mathcal{B}$.

2 Relative Multiple Pushouts and Reflectivity.

In this section we apply 1.7 above to give alternative characterisations of (E-)reflective hulls of subcategories. We begin by characterising reflectivity in terms of relative cointersections, in the same spirit as the characterisations of reflectivity given in terms of canonical limits and approaching colimits (see 1.2 and 2.2 of Chapter 1). Via this characterisation, an $\mathcal{A}$-reflection arrow for a $\mathcal{B}$-object B can be considered as a "best approximation" of B to the subcategory $\mathcal{A}$, in the sense of 1.3 (3) :

2.1 Proposition. *Let $\mathcal{A} \subset \mathcal{B}$ and let $B \in \mathcal{B}$. Then the following are equivalent :*

(1) B has an $\mathcal{A}$-reflection;

(2) the $\mathcal{A}$-relative $\mathcal{A}^{\perp}$-cointersection of the empty $\mathcal{A}^{\perp}$-source with domain B exists in $\mathcal{B}$, with cointersection object in $\mathcal{A}$;

(3) there exists a source $(e_i : B \to B_i)_I$ of $\mathcal{A}^{\perp}$-morphisms such that the $\mathcal{A}$-relative $\mathcal{A}^{\perp}$-cointersection of $(e_i)_I$ exists in $\mathcal{B}$, with cointersection object in $\mathcal{A}$.

PROOF. (1) $\Rightarrow$ (2) : let $r_B : B \to rB$ denote the $\mathcal{A}$-reflection morphism for B. Then $(r_B, (\emptyset, rB))$ is an $\mathcal{A}$-orthogonal upper bound for $(B, \emptyset)$, since $r_B \in \mathcal{A}^{\perp}$. Let $(p : B \to C, (\emptyset, C))$ be an $\mathcal{A}$-orthogonal upper bound for $(B, \emptyset)$. Then since $(r_B, (\emptyset, rB))$ is in particular an $\mathcal{A}$-valued upper bound for $(B, \emptyset)$, there exists a unique $\mathcal{B}$-morphism $t : C \to rB$ such that $r_B = t \cdot p$. So $(r_B, (\emptyset, rB))$ is the $\mathcal{A}$-relative $\mathcal{A}^{\perp}$-cointersection of $(B, \emptyset)$.
(2) $\Rightarrow$ (3) : clear.
(3) $\Rightarrow$ (1) : suppose that $(e_i : B \to B_i)_I$ is a source of $\mathcal{A}^{\perp}$-morphisms for which the $\mathcal{A}$-relative $\mathcal{A}^{\perp}$-cointersection $(c : B \to P, (g_i)_I)$ of $(e_i)_I$ exists in $\mathcal{B}$, with $P \in \mathcal{A}$. Then c is the $\mathcal{A}$-reflection arrow for B : let $f : B \to A$ be a $\mathcal{B}$-morphism from B to an $\mathcal{A}$-object A. Then for each $i \in I$, since $e_i \in \mathcal{A}^{\perp}$, there exists (a unique) $h_i : B_i \to A$ such that $f = h_i \cdot e_i$. Note that $(f, (h_i)_I)$ is an $\mathcal{A}$-valued upper bound for $(e_i)_I$; hence since $(c, (g_i)_I)$ is in particular an $\mathcal{A}$-orthogonal upper bound for $(e_i)_I$, there exists a unique $t : P \to A$ such that $f = t \cdot c$ and $h_i = t \cdot g_i$ for each $i \in I$. The uniqueness of this t with respect to the property $f = t \cdot c$ follows since $c \in \mathcal{A}^{\perp} \subset \mathcal{A}$-**Epi** by the $\mathcal{A}$-relative $\mathcal{A}^{\perp}$-cointersection property of $(c, (g_i)_I)$. □

We characterise, for an $\mathcal{A}$-relative factorisation structure $(E, \mathbf{M})$ on $\mathcal{B}$, the $\mathbf{M}(\mathcal{A})$-reflection arrows in terms of $\mathcal{A}$-relative E-cointersections, and then deduce a description of the $\mathbf{M}(\mathcal{A})$-objects using $\mathcal{A}$-relative E-cointersections.

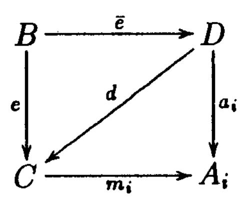
2.2 Proposition. *Let $(E, \mathbf{M})$ be an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$. Then the following conditions are equivalent, for any $\mathcal{B}$-morphism $e : B \to C$:*

(1) e is the $\mathbf{M}(\mathcal{A})$-reflection for B;

(2) $(e, (\emptyset, C))$ is the $\mathcal{A}$-relative E-cointersection of the empty source $(B, \emptyset)$;

(3) there exists a source $(e_i : B \to B_i)_I$ of $\mathcal{A}$-orthogonal E-morphisms and a sink $(g_i)_I$ such that $(e, (g_i)_I)$ is the $\mathcal{A}$-relative E-cointersection of $(e_i)_I$.

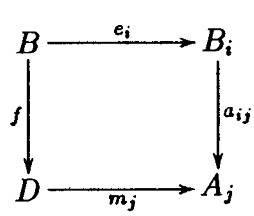
PROOF. (1) $\Rightarrow$ (2) : note first that $(e, (\emptyset, C))$ is an upper bound for $(B, \emptyset)$, and is moreover an $\mathcal{A}$-orthogonal upper bound since e is the $\mathbf{M}(\mathcal{A})$-reflection for B, and e belongs to E by 2.1 of Chapter 4. Any $\mathcal{A}$-orthogonal upper bound for $(B, \emptyset)$ has the form $(\bar{e}, (\emptyset, D))$; consider an $\mathcal{A}$-orthogonal upper bound $(\bar{e}, (\emptyset, D))$ for $(B, \emptyset)$ with $\bar{e} \in E$. Since $C \in \mathbf{M}(\mathcal{A})$, there exists an $\mathbf{M}$-source $(m_i : C \to A_i)_I$, say. So for each $i \in I$, $(m_i \cdot e, (\emptyset, A_i))$ is an $\mathcal{A}$-valued upper bound for $(B, \emptyset)$, hence there exists a $\mathcal{B}$-morphism $a_i : D \to A_i$ and thus a diagonal $d : D \to C$ in $\mathcal{B}$ such that



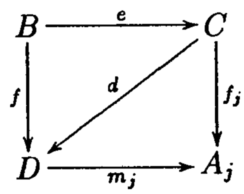
commutes in all parts, and d is uniquely determined with respect to $d \cdot \bar{e} = e$, since $C \in \mathbf{M}(\mathcal{A})$, and $\bar{e} \in \mathbf{M}(\mathcal{A})$-**Epi** by 1.7 of Chapter 4. Thus $(e, (\emptyset, C))$ is the $\mathcal{A}$-relative E-cointersection of $(e_i)_I$.

(2) $\Rightarrow$ (3) : clear.

(3) $\Rightarrow$ (1) : let $f : B \to D$ be a $\mathcal{B}$-morphism with $D \in \mathbf{M}(\mathcal{A})$. So, there exists an $\mathbf{M}$-source $(m_j : D \to A_j)_J$, say. Hence, for all $i \in I$ and $j \in J$ there exists a $\mathcal{B}$-morphism $a_{ij} : B_i \to A_j$ such that the diagram

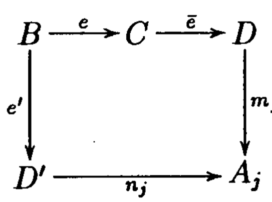


commutes, since $e_i \in \mathcal{A}^{\perp}$ for each $i \in I$. So $(m_j \cdot f, (a_{ij})_I)$ is an $\mathcal{A}$-valued upper bound for $(e_i)_I$ for each $j \in J$, hence there exists $f_i : C \to A_j$ with $g_i \cdot f_j = a_{ij}$ and $f_j \cdot e = m_j \cdot f$ for all $i \in I$ and $j \in J$. Consequently, there exists a unique diagonal $d : C \to D$ such that



$f = d \cdot e$ and $m_j \cdot d = f_j$ for each $j \in J$. Since $D \in \mathbf{M}(\mathcal{A})$ and $e \in \mathbf{M}(\mathcal{A})$-**Epi**, d is uniquely determined by $d \cdot e = f$.

It remains to prove that $C \in \mathbf{M}(\mathcal{A})$. We prove this without using the $\mathcal{A}$-orthogonality of the members of $(e_i)_I$. Let $(f_j : C \to A_j)_J$ denote the source of all $\mathcal{B}$-morphisms from C to $\mathcal{A}$; let $(m_j \cdot \bar{e} : C \to D \to A_j)_J$ be the $\mathcal{A}$-relative $(E, \mathbf{M})$-factorisation of $(f_j)_J$. Note first that $\bar{e} \cdot e$ belongs to E : if $(n_j \cdot e' : B \to D' \to A_j)$ is the $\mathcal{A}$-relative $(E, \mathbf{M})$-factorisation of the source $(m_j \cdot \bar{e} \cdot e)_J$, then we have the commutativity of the following diagram, for each $j \in J$:



Applying the $(E, \mathbf{M})$-diagonalisation property to e and $(n_j)_J$, there exists a unique $d : C \to D'$ such that $e' = d \cdot e$ and $m_j \cdot \bar{e} = n_j \cdot d$ for each $j \in J$. Hence, since $\bar{e} \in E$ and $(n_j)_J \in \mathbf{M}$, there exists a unique $s : D \to D'$ such that $d = s \cdot \bar{e}$ and $n_j \cdot s = m_j$ for each $j \in J$. Further, applying the $(E, \mathbf{M})$-diagonalisation property to e' and $(m_j)_J$, there exists a unique $t : D' \to D$ such that $\bar{e} \cdot e = t \cdot e'$ and $m_j \cdot t = n_j$ for each $j \in J$. Hence, $t \cdot s \cdot \bar{e} \cdot e = t \cdot d \cdot e = t \cdot e' = \bar{e} \cdot e$, so since $D \in \mathbf{M}(\mathcal{A})$ and $e \in \mathbf{M}(\mathcal{A})$-**Epi**, $t \cdot s \cdot \bar{e} = \bar{e}$, consequently, since $\bar{e} \in \mathbf{M}(\mathcal{A})$-**Epi**, $t \cdot s = id_D$. Further, $s \cdot t \cdot e' = s \cdot \bar{e} \cdot e = d \cdot e = e'$, hence since $D' \in \mathbf{M}(\mathcal{A})$ and $e' \in \mathbf{M}(\mathcal{A})$-**Epi**, $s \cdot t = id_{D'}$. So since E is closed under composition with respect to isomorphisms, $\bar{e} \cdot e \in E$. Now, note that $(\bar{e} \cdot e, (\bar{e} \cdot g_i)_I)$ is an upper bound for $(e_i)_I$ and also an $\mathcal{A}$-orthogonal upper bound for $(e_i)_I$ since for any $\mathcal{A}$-valued upper bound $(a : B \to A, (a_i)_I)$ for $(e_i)_I$ there exists a $\mathcal{B}$-morphism $f : C \to A$ with $a = f \cdot e$ and $a_i = f \cdot g_i$ for each $i \in I$. By definition of $(f_j)_J$ there exists $k \in J$ with $f = f_k$. Thus, $a = m_k \cdot \bar{e} \cdot e$ and $a_i = m_k \cdot \bar{e} \cdot g_i$ for each $i \in I$, and m_k is uniquely determined by $a = m_k \cdot \bar{e} \cdot e$, since $\bar{e} \cdot e \in \mathcal{A}$-**Epi**. Hence by the universal property of $(e, (g_i)_I)$ there exists a unique $s : D \to C$ such that $g_i = s \cdot \bar{e} \cdot g_i$ for each $i \in I$ and

$e = s \cdot \bar{e} \cdot e$. By 1.6 $s \cdot \bar{e} = id_C$. Thus $\bar{e} \cdot s \cdot \bar{e} = \bar{e}$, and so $\bar{e} \cdot s = id_D$, since $\bar{e} \in \mathbf{M}(\mathcal{A})$-**Epi**. Consequently, $\bar{e}$ is an isomorphism, and so $C \in \mathbf{M}(\mathcal{A})$. □

2.3 Corollary. *Let $(E, \mathbf{M})$ be an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$. Then the following conditions are equivalent, for a $\mathcal{B}$-object C :*

(1) $C \in \mathbf{M}(\mathcal{A})$;

(2) C is the $\mathcal{A}$-relative E-cointersection object of the empty source $(C, \emptyset)$;

(3) C is the $\mathcal{A}$-relative E-cointersection object of a source of E-morphisms.

PROOF. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) follow from 2.2. The proof of (3) $\Rightarrow$ (1) coincides with the second part of the proof of 2.2 (3) $\Rightarrow$ (1). □

Next, we characterise the reflectivity of a subcategory $\mathcal{A} \subset \mathcal{B}$ in terms of $\mathcal{A}$-relative $\mathcal{A}^{\perp}$-multiple pushouts :

2.4 Proposition. *The following conditions are equivalent :*

(1) $\mathcal{A}$ is reflective in $\mathcal{B}$;

(2) $\mathcal{A}$ is orthogonal in $\mathcal{B}$, and $\mathcal{B}$ has $\mathcal{A}$-relative $\mathcal{A}^{\perp}$-multiple pushouts.

PROOF. (1) $\Rightarrow$ (2) : suppose that $\mathcal{A}$ is reflective in $\mathcal{B}$. Then $\mathcal{A}$ is orthogonal in $\mathcal{B}$, and $\mathcal{B}$ has $\mathcal{A}$-relative $\mathcal{A}^{\perp}$-multiple pushouts by 2.4 of Chapter 4 and 1.7 above.
(2) $\Rightarrow$ (1) : since $\mathcal{A}^{\perp}$ is closed under composition, it follows from 1.7 that $\mathcal{A}^{\perp}$ induces an $\mathcal{A}$-relative factorisation structure $(\mathcal{A}^{\perp}, \mathbf{M})$ on $\mathcal{B}$. We show that $\mathcal{A} = \mathbf{M}(\mathcal{A})$. There is nothing to show for the forward inclusion. For the reverse inclusion, it is sufficient to show, since $\mathcal{A}$ is orthogonal in $\mathcal{B}$, that $\mathbf{M}(\mathcal{A}) \subset (\mathcal{A}^{\perp})_{\perp}$. Let $D \in \mathbf{M}(\mathcal{A})$. Then D is the domain of an $\mathbf{M}$-source $(m_i : D \to A_i)_I$, say. Let $p : B \to C$ belong to $\mathcal{A}^{\perp}$, and consider a $\mathcal{B}$-morphism $f : B \to D$. Since $p \in \mathcal{A}^{\perp}$, there exists for each $i \in I$ a unique $g_i : C \to A_i$ such that $g_i \cdot p = m_i \cdot f$. Hence by the $(\mathcal{A}^{\perp}, \mathbf{M})$-diagonalisation property there exists a unique $d : C \to D$ such that $f = d \cdot p$ and $m_i \cdot d = g_i$ for each $i \in I$. Hence, since $p \in \mathbf{M}(\mathcal{A})$-**Epi** by 1.7 of Chapter 4, d is the unique morphism with the property that $f = d \cdot p$. Hence $D \in (\mathcal{A}^{\perp})_{\perp}$. □

2.5 Corollary. *The following conditions are equivalent :*

(1) The orthogonal closure of $\mathcal{A}$ in $\mathcal{B}$ is reflective in $\mathcal{B}$;

(2) $\mathcal{B}$ has $\mathcal{A}$-relative $\mathcal{A}^\perp$-multiple pushouts.

PROOF. (1) $\Rightarrow$ (2) : since $((\mathcal{A}^\perp)_\perp)^\perp = \mathcal{A}^\perp$, and by 2.5 of Chapter 4, $(\mathcal{A}^\perp, \{\mathcal{A}$-structured $(\mathcal{A}^\perp)_\perp$ sources$\}$ is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$. Thus, (2) follows from 1.7.
(2) $\Rightarrow$ (1) : by 1.7, $(\mathcal{A}^\perp, \mathrm{M})$ is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$ for some M. Hence, by 2.3 of Chapter 4, $\mathrm{M}(\mathcal{A}) = (\mathcal{A}^\perp)_\perp$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$. □

Using 1.7 and some results of Chapter 4, an alternative characterisation for the existence of reflective hulls can be given :

2.6 Theorem. *The following conditions are equivalent, for a subcategory $\mathcal{A}$ of $\mathcal{B}$:*

(1) $\mathcal{A}$ has a reflective hull in $\mathcal{B}$;

(2) there exists a largest composition-closed class $E \subset \mathcal{A}^\perp$ such that $\mathcal{B}$ has $\mathcal{A}$-relative E-multiple pushouts.

PROOF. (1) $\Rightarrow$ (2) : suppose that $\mathcal{R}$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$. By 2.13 of Chapter 4 ($\mathcal{R}^\perp$,$\{\mathcal{A}$-structured $\mathcal{R}$-sources$\}$) is a (finest) $\mathcal{A}$-orthogonal factorisation structure on $\mathcal{B}$. So by 1.7 $\mathcal{B}$ has $\mathcal{A}$-relative $\mathcal{R}^\perp$-multiple pushouts, since $\mathcal{R}^\perp$ is closed under composition. Note also that $\mathcal{R}^\perp \subset \mathcal{A}^\perp$. Now, let $E \subset \mathcal{A}^\perp$ be closed under composition, and suppose that $\mathcal{B}$ has $\mathcal{A}$-relative E-multiple pushouts. Then by 1.7 there exists a conglomerate M such that (E, M) is an $\mathcal{A}$-relative (hence $\mathcal{A}$-orthogonal) factorisation structure on $\mathcal{B}$. Since ($\mathcal{R}^\perp$,$\{\mathcal{A}$-structured $\mathcal{R}$-sources$\}$) is a finest such, we have $\{\mathcal{A}$-structured $\mathcal{R}$-sources$\} \subset \mathrm{M}$. Let $e : B \to C$ be a morphism in E, and consider a $\mathcal{B}$-morphism $f : B \to R$, where $R \in \mathcal{R}$. Then the empty $\mathcal{A}$-structured $\mathcal{R}$-source with domain R belongs to M, hence by the (E, M)-diagonalisation property there exists a unique $d : C \to R$ such that $f = d \cdot e$. So $e \in \mathcal{R}^\perp$.
(2) $\Rightarrow$ (1) : let E be the assumed largest class. By 1.7 there exists a conglomerate M such that (E, M) is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$. By 2.1 of Chapter 4, $\mathrm{M}(\mathcal{A})$ is reflective in $\mathcal{B}$. We show that $\mathrm{M}(\mathcal{A})$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$. Suppose that $\mathcal{C}$ is reflective in $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{C}$; then by 2.5 of Chapter 4 ($\mathcal{C}^\perp$, $\{\mathcal{A}$-structured $\mathcal{C}$-sources$\}$) is an $\mathcal{A}$-relative factorisation structure on $\mathcal{B}$, hence by 1.7 $\mathcal{B}$ has $\mathcal{A}$-relative $\mathcal{C}^\perp$-multiple pushouts, and consequently $\mathcal{C}^\perp \subset E$. Now let $D \in \mathrm{M}(\mathcal{A})$, and denote by

$r_D : D \to rD$ the $\mathcal{C}$-reflection arrow for D. Since D is the domain of an $\mathcal{A}$-structured M-source $(m_i : D \to A_i)_I$, say, and $r_D \in \mathcal{A}^\perp$, there exists for each $i \in I$ a unique $g_i : rD \to A_i$ such that $m_i = g_i \cdot r_D$. Since $r_D \in \mathcal{C}^\perp \subset E$, and $(m_i)_I \in \mathbf{M}$, there exists by the $(E, \mathbf{M})$-diagonalisation property a unique $d : rD \to D$ such that $m_i \cdot d = g_i$ for each $i \in I$ and $d \cdot r_D = id_D$. So r_D is a section, hence an isomorphism, and consequently $D \in \mathcal{C}$. □

2.7 Corollary. *Let the conditions (1) or (2) of 2.6 above hold, and let E be according to (2). Then the objects of the reflective hull of $\mathcal{A}$ in $\mathcal{B}$ are precisely the $\mathcal{A}$-relative E-cointersections of E-sources.*

PROOF. Immediate from 2.3 and 2.6. □

2.8 Remark. Let $\mathcal{R}$ be a subcategory of $\mathcal{B}$, with $\mathcal{A} \subset \mathcal{R}$. If $\mathcal{R}$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$, then one obtains via the proof of 2.6 (1) $\Rightarrow$ (2) a concrete description of the class E in (2) of 2.6, namely, E is just the class $\mathcal{R}^\perp$. Note, however, that $\mathcal{R}$ need not coincide with the reflective hull of $\mathcal{A}$ in $\mathcal{B}$ under the assumption that $\mathcal{R}^\perp$ is the largest class $E \subset \mathcal{A}^\perp$ for which $\mathcal{B}$ has $\mathcal{A}$-relative E-multiple pushouts : consider a complete and wellpowered category $\mathcal{B}$, and let $\mathcal{A}$ be a small non-orthogonal subcategory of $\mathcal{B}$. Then from [Tholen 1987] 6.1 and a subsequent remark it follows that the limit closure (hence the orthogonal closure) of $\mathcal{A}$ in $\mathcal{B}$ is reflective in $\mathcal{B}$, i.e., by 2.5 $\mathcal{B}$ has $\mathcal{A}$-relative $\mathcal{A}^\perp$-multiple pushouts, so in this case $\mathcal{A}^\perp$ coincides with the largest class E described in 2.6 (2), but $\mathcal{A}$ is not reflective in $\mathcal{B}$.

Bibliography

[Adámek, Herrlich and Strecker 1990] J. Adámek, H. Herrlich and G.E. Strecker, *Abstract and Concrete Categories*, John Wiley and Sons Ltd., 1990.

[Adámek and Koubek 1980] J. Adámek and V. Koubek, "Completions of concrete categories", Cahiers Top. Géom. Diff. Cat. **22**(2) (1980), 209 - 228.

[Adámek and Rosický 1988] J. Adámek and J. Rosický, "Intersections of reflective subcategories", Proc. Amer. Math. Soc. **103** (1988), 710 - 712.

[Adámek and Rosický 1991] J. Adámek and J. Rosický, "What are locally generated categories ?", in : Proceedings of Conference on Category Theory (Como 1990), Lecture Notes in Mathematics 1488, Springer, Berlin, 1991, 14 - 19.

[Adámek and Rosický 1993] J. Adámek and J. Rosický, "On injectivity in locally presentable categories", Trans. Amer. Math. Soc. **336** (1993), 785 - 804.

[Adámek, Rosický and Trnková 1988] J. Adámek, J. Rosický and V. Trnková, "Are all limit-closed subcategories of locally presentable categories reflective?", in : Proc. Categ. Conf. Louvain-la-Neuve, Lecture Notes in Mathematics 1348, Springer-Verlag (1988), 1 - 18.

[Banaschewski, Frith and Gilmour 1987] B. Banaschewski, J.L. Frith and C.R.A. Gilmour, "On the congruence lattice of a frame", Pacific J. Math. **130** (1987), 209 - 213.

[Bénabou 1965] J. Bénabou, "Criteres de représentabilité des foncteurs", C. R. Acad. Sci. Paris **260** (1965), 752 - 755.

[Bentley and Herrlich 1992] H. L. Bentley and H. Herrlich, "Compactness = completeness ∩ total boundedness - a natural example of a non-reflective intersection of reflective subcategories", in : Recent Developments of General Topology and its Applications, Proceedings of International Conference in Memory of Felix Hausdorff (1868 - 1942), Mathematical Research **67**, Berlin 1992, 46 - 56.

[Börger and Tholen 1991] "Strong, regular and dense generators", Cahiers Top. Géom. Diff. Cat. **32**(3) (1991), 257 - 276.

[Bourbaki 1966] N. Bourbaki, *General Topology* (Part I), Addison-Wesley, 1966.

[Bousfield 1977] A. K. Bousfield, "Construction of factorisation systems in categories", J. Pure Applied Algebra **9** (1977), 207 - 220.

[Cassidy, Hébert and Kelly 1985] C. Cassidy, M. Hébert and G.M. Kelly, "Reflective subcategories, localizations and factorisation systems", J. Austral. Math. Soc. (Series A) **38** (1985), 287 - 319.

[Davey and Priestley 1990] B.A. Davey and H.A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, 1990.

[Dyckhoff 1972] R. Dyckhoff, "Factorisation theorems and projective spaces in Topology", Math. Z. **127** (1972), 256 - 264.

[Freyd 1960] P. Freyd, *Functor Theory*, Ph.D. Thesis, Princeton University, 1960.

[Freyd 1964] P. Freyd, *Abelian Categories*, Harper and Row, New York, 1964.

[Freyd and Kelly 1972] P.J. Freyd and G.M. Kelly, "Categories of continuous functors I", J. Pure Applied Algebra **2** (1972), 169 - 191.

[Freyd and Scedrov 1990] P. J. Freyd and A. Scedrov, *Categories, Allegories*, North-Holland Publishing Company, Amsterdam-New York-Oxford-Tokyo, 1990.

[Frith 1987] J. L. Frith, "Structured Frames", Ph.D. Thesis, University of Cape Town, 1987.

[Herrlich 1968] H. Herrlich, *Topologische Reflexionen und Coreflexionen*, Lecture Notes in Mathematics 78, Springer-Verlag, 1968.

[Herrlich 1971] H. Herrlich, "A generalization of perfect maps", in : Proceedings of the Third Prague Symposium on General Topology and its Relations to Modern Analysis and Algebra, 1971

[Herrlich 1972] H. Herrlich, "Perfect subcategories and factorizations", Colloquia Mathematica Societis János Bolyai 8. Topics in Topology, Keszthely (Hungary), 1972, 387 - 403.

[Herrlich 1975] H. Herrlich, "Epireflective subcategories of TOP need not be cowellpowered", Comment. Math. Univ. Carolinae **16**(4) (1975), 713 - 715.

[Herrlich 1980] H. Herrlich, "Universal completions of concrete categories", in : Proceedings of the Conference on Categorical aspects of Topology and Analysis, Ottawa, 1980.

[Herrlich 1986] H. Herrlich (unter Mitarbeit von H. Bargenda), *Topologie I : Topologische Räume*, Heldermann Verlag Berlin, 1986.

[Herrlich 1993a] H. Herrlich, "Almost reflective subcategories of Top", Top. Appl. **49** (1993), 251 - 264.

[Herrlich 1993b] H. Herrlich, "Compact T_0 spaces and T_0-compactifications", Applied Categorical Structures **1**(1) (1993), 111 - 132.

[Herrlich and Hušek 1992] H. Herrlich and M. Hušek, "Categorical Topology", in : Recent Progress in General Topology, Ed. M. Hušek and J. van Mill, Elsevier Science Publishers, 1993, 371 - 403.

[Herrlich and Hušek 1993] H. Herrlich and M. Hušek, "Some open categorical problems in **Top**", Applied Categorical Structures **1**(1) (1993), 1 - 19.

[Herrlich and Strecker 1979] H. Herrlich and G.E. Strecker, *Category Theory* (2nd ed.), Helderman-Verlag Berlin, 1979.

[Herrlich and Strecker 1994] H. Herrlich and G. E. Strecker, "Categorical topology - its origins", preprint, 1994.

[Herrlich, Salicrup and Strecker 1987] H.Herrlich, G. Salicrup and G.E. Strecker, "Factorizations, denseness, separation, and relatively compact objects", Top. Appl. **27** (1987), 157 - 169.

[Hušek 1992] M. Hušek, "Reflection and its generalizations from subcategorial and functorial points of view", Seminarberichte FernUniversität Hagen **44** (1992), Teil 1, 150 - 157.

[Isbell 1964] J. R. Isbell, *Uniform Spaces*, American Mathematical Society, Providence, Rhode Island, 1964.

[Johnstone 1982] P. T. Johnstone, *Stone Spaces*, Cambridge University Press, 1982.

[Kelly 1980] G.M. Kelly, "A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on", Bull. Austral. Math. Soc. **22** (1980), 1 - 84.

[Kelly 1987] G.M. Kelly, "On the ordered set of reflective subcategories", Bull. Austral. Math. Soc. **36** (1987), 137 - 152.

[Kennison 1967] J.F. Kennison, "A note on reflection maps", Illinois J. Math., **11** (1967), 404 - 409.

[Kennison 1968] J.F. Kennison, "Full reflective subcategories and generalized covering spaces", Illinois J. Math., **12** (1968), 353 - 365.

[Korostenski and Tholen 1986] M. Korostenski and W. Tholen, "On left-cancellable classes of morphisms", Comm. Algebra **14** (1986), 741 - 766.

[Korostenski and Tholen 1990] M. Korostenski and W. Tholen, "Prelocalizations and natural numbers", Quaestiones Mathematicae **13** (3 and 4) (1990), 301 - 320.

[Koubek 1975] V. Koubek, "Each concrete category has a representation by T_2 paracompact topological spaces", Comment. Math. Univ. Carolinae **15**(4) (1974), 655 - 665.

[Lord 1989] H. Lord, "(epi, **M**) factorisation structures on Top", in : Categorical Topology and its relation to Analysis, Algebra and Combinatorics, Prague, Czechoslovakia, 22 - 27 August 1988, Ed. J. Adámek and S. Mac Lane, World Scientific Publishing Company, 1989, 95 - 119.

[Mac Lane 1971] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, Berlin, 1971.

[Mac Lane 1989] S. Mac Lane, "The development of mathematical ideas by collision: the case of categories and topos theory", in : Categorical Topology and its relation to Analysis, Algebra and Combinatorics, Prague, Czechoslovakia, 22 - 27 August 1988, Ed. J. Adámek and S. Mac Lane, World Scientific Publishing Company, 1989, 1 - 9.

[Madden and Molitor 1991] J.J. Madden and A. Molitor, "Epimorphisms of frames", J. Pure Appl. Algebra **70** (1991), 129 - 132.

[Maranda 1964] J.M. Maranda, "Injective Structures", Trans. Amer. Math. Soc. **110** (1964), 98 - 135.

[Pultr and Trnková 1980] A. Pultr and V. Trnková, *Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories*, North-Holland Publishing Company, Amsterdam-New York-Oxford, 1980.

[Ringel 1970] C.M. Ringel, "Diagonalisierungspaare I", Math. Z. **117** (1970), 248 - 266.

[Ringel 1971] C.M. Ringel, "Monofunctors as reflectors", Trans. Amer. Math. Soc. **161** (1971), 293 - 306.

[Rosický and Tholen 1988] J. Rosický and W. Tholen, "Orthogonal and prereflective subcategories", Cahiers Top. Géom. Diff. Cat. **29**(3) (1988), 203 - 215.

[Tholen 1979] W. Tholen, "Semitopological functors I", J. Pure Applied Algebra **15** (1979), 53 - 73.

[Tholen 1983] W. Tholen, "Factorisations, localizations, and the orthogonal subcategory problem", Math. Nachr. **114** (1983), 63 - 85.

[Tholen 1986] W. Tholen, "Prereflections and reflections", Comm. Algebra **14** (1986), 717 - 740.

[Tholen 1987] W. Tholen, "Reflective Subcategories", Top. Appl. **27** (1987), 201 - 212.

[Tholen and Wolff 1981] W. Tholen and H. Wolff, "Extensions of factorization systems", Cahiers Top. Géom. Diff. Cat. **22**(2) (1981), 175 - 190.

[Trnková, Adámek and Rosický 1990] V. Trnková, J. Adámek and J. Rosický, "Topological reflections revisited", Proc. Amer. Math. Soc. **108** (1990), 605 - 612.

[Vajner 1993] V. Vajner, "The category of complete Boolean algebras is not an intersection of reflective subcategories of the category of frames", Applied Categorical Structures **1**(2) (1993), 151 - 156.