



# REALCOMPACTIFICATIONS OF FRAMES

By

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A thesis prepared under the supervision  
of Dr C.R.A. Gilmour for the degree of  
Master of Science in Mathematics.

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# Synopsis

The theory of frames can be traced back to Stone [1937] and Wallman [1938], who were the first to study topological concepts from a lattice-theoretic viewpoint. It was only in the late 1950's that Ehressman and his student Ben-abou considered certain complete lattices, called 'local lattices' as generalised topological spaces. These ideas were pursued by Dowker and Strauss, who, in a series of papers in the 1960's and 1970's studied topological properties of these lattices. It was Dowker who introduced the term 'frame'. A full account of the history of these ideas may be found in the book of Peter Johnstone 'Stone Spaces' which is the standard reference on frames for this thesis.

Sigma frames are generalisations of frames, where only the existence of countable joins is required. Regular  $\sigma$ -frames were introduced by Charalambous [1974] formulated as the perfectly normal frames and were later studied by Reynolds [1979] formulated as the Alexandroff Algebras. Reynolds showed that the Alexandroff Algebras are exactly the cozero-set lattices of completely regular frames. Their simplest description, as those  $\sigma$ -frames which are regular, was first observed by Gilmour [1981] using the nice observation of Banaschewski that every regular  $\sigma$ -frame is normal. Madden and Vermeer [1986] showed that the frame of  $\sigma$ -ideals of a regular  $\sigma$ -frame is regular Lindelöf, thus giving rise to an equivalence between the category of regular  $\sigma$ -frames and the category of regular Lindelöf frames.

Alexandroff spaces were introduced by Alexandroff [1940], and were later studied in greater detail by Gordon [1971], under the guise of zero-set spaces. Gordon showed these spaces to be a natural setting for the study of realcompactness and pseudocompactness. Gilmour [1981] gave a dual adjunction between the Alexandroff spaces and regular  $\sigma$ -frames, and showed that the realcompact Alexandroff spaces are precisely the fixed objects for the duality. As a consequence, alternative descriptions for  $v$  and  $\beta$ , the realcompact epireflector and the compact epireflector in the category of Tychonoff spaces were obtained as well as their analogues for Alexandroff spaces.

The first notion of realcompactness in frames was introduced by Reynolds [1979], and it was shown by Madden and Vermeer [1986] that this coincides with the Lindelöf property. My thesis advisor suggested that more general realcompactifications of a frame  $L$  could be constructed by considering regular sub  $\sigma$ -frames which join generate  $L$ . This was motivated by the fact that the Alexandroff bases, which are used to construct the Wallman realcompactifications of a space  $X$ , are, as shown by Gilmour, simply the regular sub  $\sigma$ -frames of the frame of open sets of  $X$ . The key definition of realcompactness needed here is due to Schlitt [1990] and it is his construction of the universal realcompactification that we modify in order to obtain the Wallman realcompactifications.

We give an outline of the thesis:

**Chapter 0** : This chapter contains the background to the material which is used in subsequent chapters.

**Chapter 1 :** In this chapter the notion of realcompactness of frames, as given by Schlitt [1990] is discussed. We construct Wallman realcompactifications of a frame  $L$  and it is shown that the universal realcompactification given by Schlitt is one such realcompactification. The notion of a Alexandroff frame is introduced, as a frame-theoretic analogue of Alexandroff spaces. Thus, the Wallman realcompactifications are delivered by a functor from the category of Alexandroff frames to the category of completely regular frames.

**Chapter 2 :** Johnstone's [1984b] construction of the Wallman compactifications for frames is discussed here. Following a suggestion by Bernhard Banaschewski Johnstone's construction is used to obtain a generalisation of the key lemma needed for our characterisation of realcompactness of frames in Chapter 1. As a consequence we can characterise the Stone-Čech compactification of a completely regular frame using Johnstone's method applied to its cozero part. Johnstone's construction is easily generalised to give the Wallman compactifications for  $\sigma$ -frames. Pseudocompact frames are considered, and it is shown, using a characterisation of pseudocompactness given by Gilmour, that a pseudocompact frame is compact if and only if it is realcompact. Furthermore, given a pseudocompact frame  $L$  our construction of the Wallman realcompactification of  $L$  is shown to coincide with Johnstone's Wallman compactification of  $L$ . Compactifications of Alexandroff frames are considered using the Wallman compactifications for frames and  $\sigma$ -frames. It is shown that pseudocompact Alexandroff frames admit unique compactifications. The counterpart of this last result is known to fail to hold for

Tychonoff spaces and consequently for frames.

# Chapter 0

## Preliminaries

### 0.1 Introduction

We give definitions and introduce notions which will be used later on. Background and proofs of the introductory material concerning frames can be found in Johnstone's "Stone Spaces" [1980], and further details on  $\sigma$ -frames can be found in the thesis of Gilmour [1981], as well as the joint paper by Banaschewski and Gilmour [1989] and Madden's " $\kappa$ -frames" [1989].

Hager's paper entitled "Cozero Fields" [1980], and the thesis of Gilmour [1981] provide a survey of the theory of Alexandroff spaces.

## 0.2 Frames

A frame  $L$  is a complete lattice which satisfies the following distributive law:  $a \wedge \bigvee_L S = \bigvee_L \{a \wedge s \mid s \in S\}$ , where  $S \subseteq L$ . We denote the bottom of  $L$  by  $0_L$  and the top of  $L$  by  $1_L$ . A map  $L \xrightarrow{h} M$  is called a frame homomorphism if  $h$  preserves finite meets and arbitrary joins (and hence also the top and bottom elements). The category of all frames and frame homomorphisms is denoted **Frm**. A typical example of a frame is the lattice  $\mathcal{O}X$  of open sets of a topological space  $X$ . Frames isomorphic to such are called spatial frames.

Given a bounded distributive lattice  $L$ , and supposing  $a, b \in L$ , then  $a$  is said to be rather below  $b$ , written  $a \prec b$ , if there exists  $s \in L$ , called a separating element, such that  $a \wedge s = 0_L$ , and  $b \vee s = 1_L$ . In the case of open sets of a topological space  $X$ ,  $A \prec B$  iff  $Cl_X A \subseteq B$  (where  $Cl_X A$  is the closure of  $A$  in  $X$ ). We say  $a$  is completely below  $b$ , written  $a \prec\prec b$ , if there is a family  $\{s_q \mid q \in \mathbf{Q} \cap [0, 1]\}$  such that  $s_0 = a$ ,  $s_1 = b$ , and  $i < j$  implies  $s_i \prec s_j$ . An element  $a \in L$  is called a regular element (respectively, completely regular element) if  $a$  is a join of elements rather below (respectively, completely below)  $a$ . A frame  $L$  is called (completely) regular if each  $a \in L$  is a (completely) regular element. The full subcategories of **Frm** consisting of all regular and completely regular frames are denoted **RegFrm** and **CregFrm** respectively.

An element  $p \in L$  is called prime if  $a \wedge b = p$  implies  $a = p$  or  $b = p$ . Given a frame  $L$ , the spectrum  $\Sigma L$  of  $L$  is the topological space consisting

of all prime elements of  $L$  and basic open sets of the form  $\{p \text{ prime} \mid u \not\leq p\}$  for  $u \in L$ . We note the following simple but important result:

**Lemma 0.2.1** *If  $L$  is a regular frame, then the prime elements are precisely the maximal elements of  $L$ .*

The spectrum of  $L$  can equivalently be described in terms of the completely prime filters of  $L$ . A filter  $F$  is a subset of  $L$  satisfying:

(i)  $a \in F, b \geq a \Rightarrow b \in F$ .

(ii)  $a, b \in F \Rightarrow a \wedge b \in F$ .

Furthermore,  $F$  is said to be completely prime if:

(iii) Given  $S \subseteq F$  with  $\bigvee_L S \in F$ , then  $S \cap F \neq \emptyset$ .

The space  $\Sigma L$  consists of all completely prime filters on  $L$ , with open sets of the form  $\{F \in \Sigma L \mid u \in F\}$  for  $u \in L$ .

A frame  $L$  is said to be compact if, given a subset  $S \subseteq L$  with  $\bigvee_L S = 1_L$ , then there exists a finite subset  $T \subseteq S$  with  $\bigvee_L T = 1_L$ . The frame  $IdlB$  of ideals on a bounded distributive lattice  $B$  is compact. An ideal  $I$  is (completely) regular if it is a (completely) regular element in the frame  $IdlB$ . It is easily seen that an ideal  $I$  is completely regular iff for any  $a \in I$ , there exists  $b \in I$  with  $a \prec\prec b$ . The corresponding characterisation of regular ideals also holds. It was shown by Banaschewski and Mulvey [1980] that the subframe  $CrgIdlL$  of  $IdlL$ , consisting of all completely regular ideals on a frame  $L$  is the universal completely regular compactification of  $L$ , and is therefore called the Stone-Ćech compactification of  $L$ , and is denoted by  $\beta L$ .

In the subsequent chapters we shall be concerned with compactifications and realcompactifications of frames. A (real)compactification  $(Y, f)$  of a space  $X$  is a dense embedding  $f : X \hookrightarrow Y$ , where  $Y$  is a (real)compact space. The corresponding frame homomorphism  $h : \mathcal{O}Y \rightarrow \mathcal{O}X$  is a surjective map with the property:  $h(a) = 0_{\mathcal{O}X} \Rightarrow a = 0_{\mathcal{O}Y}$ . Frame homomorphisms with this property are called dense. Likewise, a frame homomorphism  $h : L \rightarrow M$  is called codense if  $h(a) = 1_M$  implies  $a = 1_L$ . We note the following results concerning dense and codense maps:

**Lemma 0.2.2** *Let  $L \xrightarrow{h} M$  be a morphism in the category **RegFrm**.*

- (i) *If  $h$  is dense, then it is monic in **RegFrm**.*
- (ii)  *$h$  is injective iff it is codense.*
- (iii) *If  $M$  is compact and  $h$  is dense, then  $h$  is injective.*

In the setting of compactifications and realcompactifications, the frame  $\mathcal{O}Y$  is a quotient of  $\mathcal{O}X$ . One way of forming frame quotients is via certain closure operators, called nuclei. This approach was initiated by Simmons [1978]. A nucleus  $n$  on a frame  $L$  is a map  $n : L \rightarrow L$  satisfying: For all  $a, b \in L$

- (i)  $a \leq n(a)$
- (ii)  $n(a) \wedge n(b) = n(a \wedge b)$
- (iii)  $n^2(a) = n(a)$

The quotient frame of  $L$  with respect to  $n$  is  $\text{Fix } n = \{a \in L \mid n(a) = a\}$  and is denoted  $(L)_n$ . Finite meets in  $(L)_n$  are formed as in  $L$ , and  $\bigvee_{(L)_n} S = n \bigvee_L S$  for  $S \subseteq (L)_n$ . Further details are given by Johnstone [1982].

### 0.3 Sigma Frames

A  $\sigma$ -frame is a lattice  $A$  which is closed under countable joins and finite meets (and thus possesses a greatest element  $1_A$  and a least element  $0_A$ ) and satisfies the following distribution law:  $a \wedge \bigvee_A S = \bigvee_A \{a \wedge s \mid s \in S\}$ , where  $S$  is a countable subset of  $A$ . If  $A$  and  $B$  are  $\sigma$ -frames, then a map  $A \xrightarrow{h} B$  is called a  $\sigma$ -frame homomorphism if  $h$  preserves finite meets and countable joins (and hence also  $1_A$  and  $0_A$ ). The category of all  $\sigma$ -frames and  $\sigma$ -frame homomorphisms is denoted  $\sigma\mathbf{Frm}$ . A  $\sigma$ -frame  $A$  is said to be regular if each  $a \in A$  is a countable join of elements rather below it. The full subcategory of  $\sigma\mathbf{Frm}$  consisting of all regular  $\sigma$ -frames is denoted  $\mathbf{Reg}\sigma\mathbf{Frm}$ . A bounded distributive lattice  $B$  is called normal if for each pair  $a, b$  of elements of  $L$  with  $a \vee b = 1_B$ , there exists  $u, v$  in  $B$  such that  $a \vee u = 1_B = b \vee v$  and  $u \wedge v = 0_B$ . The following result, which is due to Banaschewski [1980] and appears in Banaschewski and Gilmour [1989], has some far-reaching consequences.

**Lemma 0.3.1** *Every regular  $\sigma$ -frame is normal.*

Using the above result, it can be shown that the relation  $\prec$  interpolates in regular  $\sigma$ -frames. Thus, in the category  $\mathbf{Reg}\sigma\mathbf{Frm}$ , the rather below and completely below relations coincide.

One important notion, particularly in the study of realcompact frames, is that of the cozero part of a frame. Given a completely regular frame  $L$ , an element  $a \in L$  is called a cozero element of  $L$  if  $a = h(\mathbf{R} \setminus \{0\})$  for some frame

homomorphism  $\mathcal{OR} \xrightarrow{h} L$ . The set of all cozero elements of  $L$  is denoted  $CozL$ . We give some of the important characteristics of  $CozL$ , which we will be using. These results are due to Banaschewski and Reynolds, and can be extracted from Johnstone's "Stone Spaces" [1982].

**Lemma 0.3.2** *Given a completely regular frame  $L$ . Then:*

- (i)  $CozL$  is a regular sub  $\sigma$ -frame of  $L$ .
- (ii) For each  $a \in L$ ,  $a = \bigvee_L S$ , where  $S \subseteq CozL$ , ie  $CozL$  join generates  $L$ .
- (iii) An element  $a \in CozL$  iff  $a = \bigvee_L a_n$  for some sequence  $(a_n)$  in  $L$  with  $a_i \prec\prec a_{i+1}$  for all  $i \in \mathbb{N}$ .
- (iv) If  $A$  is a regular sub  $\sigma$ -frame of  $L$ , then  $A \subseteq CozL$ .

It is immediate from the definition that frame homomorphisms preserve cozero elements. Thus,  $Coz$  is a functor from  $\mathbf{CrgFrm}$  to  $\mathbf{Reg}\sigma\mathbf{Frm}$ .

A  $\sigma$ -frame  $A$  is said to be compact iff whenever there is a countable subset  $S \subseteq A$  satisfying  $\bigvee_A S = 1_A$ , then there is a finite subset  $T \subseteq S$  with  $\bigvee_A T = 1_A$ .

All the results of Lemma 0.2.2 apply equally to regular  $\sigma$ -frames:

**Lemma 0.3.3** *Let  $A$  be a regular  $\sigma$ -frame, and suppose  $A \xrightarrow{h} B$  is a  $\sigma$ -frame homomorphism.*

- (i) *If  $h$  is dense then it is monic in the category of regular  $\sigma$ -frames.*
- (ii)  *$h$  is injective iff it is codense.*
- (iii) *If  $B$  is compact and  $h$  is dense, then  $h$  is injective.*

## 0.4 Adjunction between Frames and Sigma Frames

A  $\sigma$ -ideal on a  $\sigma$ -frame  $A$  is an ideal which is closed under countable joins. The frame of  $\sigma$ -ideals on a  $\sigma$ -frame  $A$  is denoted  $\mathcal{H}A$ . Given a  $\sigma$ -frame homomorphism  $A \xrightarrow{h} B$  there is a frame homomorphism  $\mathcal{H}A \xrightarrow{\mathcal{H}h} \mathcal{H}B$ , where  $\mathcal{H}h(J)$  is the  $\sigma$ -ideal generated by  $h[J]$ . A frame  $L$  is said to be Lindelöf if whenever there is a subset  $S \subseteq L$  with  $\bigvee_L S = 1_L$ , then there is a countable subset  $T \subseteq S$  with  $\bigvee_L T = 1_L$ . If  $A$  is a regular  $\sigma$ -frame, then  $\mathcal{H}A$  is a regular Lindelöf frame. Reynolds [1979] showed that the functor  $\mathcal{H}$  from  $\mathbf{Reg}\sigma\mathbf{Frm}$  to  $\mathbf{CregFrm}$  is left adjoint to  $\mathit{Coz}$ , and the unit of the adjunction is an isomorphism. Thus, the functors  $\mathcal{H}$  and  $\mathit{Coz}$  induce an equivalence between the category  $\mathbf{Reg}\sigma\mathbf{Frm}$  and the category  $\mathbf{RegLindFrm}$ , of regular Lindelöf frames.

## 0.5 Alexandroff Spaces

Let  $X$  be a set and let  $\mathcal{A}$  be a collection of subsets of  $X$  satisfying:

- (i) Each pair of distinct points of  $X$  are contained in disjoint members of  $\mathcal{A}$ .
- (ii)  $\mathcal{A}$  is closed under finite intersections and countable unions; in particular  $\emptyset$  and  $X$  are in  $\mathcal{A}$ .
- (iii) If  $A_1$  and  $A_2$  are in  $\mathcal{A}$ , and  $A_1 \cup A_2 = X$ , then there are sets  $B_1$  and  $B_2$  in  $\mathcal{A}$  such that  $B_1 \cup A_1 = X = B_2 \cup A_2$  and  $B_1 \cap B_2 = \emptyset$ .
- (iv) If  $A \in \mathcal{A}$  then there is a sequence  $(A_n)$  in  $\mathcal{A}$  such that  $A = \bigcup(X \setminus A_n)$ .

$\mathcal{A}$  is called an Alexandroff structure on  $X$ , and the elements of  $\mathcal{A}$  are called cozero-sets. Complements of cozero-sets are called zero-sets. The space  $(X, \mathcal{A})$  is called an Alexandroff space. Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be Alexandroff spaces, then a function  $(X, \mathcal{A}) \xrightarrow{f} (Y, \mathcal{B})$  is called a cozero-map if preimages of cozero-sets are cozero-sets. The category of all Alexandroff spaces and cozero-maps is denoted by **Alex**.

Let  $X$  be a Tychonoff space, and let  $\mathcal{A}$  be the collection of cozero-sets of  $X$ , then  $(X, \mathcal{A})$  is an Alexandroff space.

The axioms (i)-(iv) are well-known properties of cozero-sets of a Tychonoff space, and, as shown by Gordon [1971], such a family is precisely the collection of cozero-sets of cozero-maps from  $(X, \mathcal{A})$  to  $\mathbf{R}$  with its usual cozero-sets. The cozero-sets of an Alexandroff space  $(X, \mathcal{A})$  form a base for a Tychonoff topology on  $X$ ; such a base is called an Alexandroff base for the underlying topology.

The notions of realcompactness and pseudocompactness are generalised in the setting of Alexandroff spaces. These notions were introduced by Gordon [1971], in analogy with the corresponding notions in topology.

Let  $(X, \mathcal{A})$  be an Alexandroff space, and let  $\mathcal{Z}$  be the collection of all zero-sets of  $X$ , then  $X$  is said to be realcompact if every  $\mathcal{Z}$ -ultrafilter with the countable intersection property has non-empty intersection.

The underlying topology of a realcompact Alexandroff space is always realcompact, and it is shown in Gilmour [1983] that the Alexandroff bases

are precisely those bases giving rise to Wallman realcompactifications for the underlying topology. However, the following is an example of an Alexandroff space which is not realcompact, but has underlying topology that is. Consider the Alexandroff space  $(\mathbf{R}, \mathcal{A})$ , where  $\mathcal{A}$  is the collection of all countable and cocountable subsets of  $\mathbf{R}$ . Then  $(\mathbf{R}, \mathcal{A})$  is not realcompact, since the  $\mathcal{A}$ -ultrafilter consisting of all cocountable subsets of  $\mathbf{R}$  has the countable intersection property, and is not fixed. But the underlying topology of  $(\mathbf{R}, \mathcal{A})$  is discrete, and hence realcompact.

An Alexandroff space  $(X, \mathcal{A})$  is called pseudocompact if it has no hyperreal  $\mathcal{Z}$ -ultrafilters, ie if every  $\mathcal{Z}$ -ultrafilter has the countable intersection property, where  $\mathcal{Z}$  is the collection of all zero-sets. Gordon [1971] shows that an Alexandroff space  $X$  is pseudocompact iff  $\beta X \cong vX$ , where  $\beta X$  and  $vX$  denote the Stone-Ćech compactification and the Hewitt realcompactification in Alex respectively.

## 0.6 Adjunction between Alexandroff Spaces and Regular Sigma Frames

Given an Alexandroff space  $X$ , denote by  $\mathcal{A}X$ , the lattice of cozero-sets of  $X$ . If  $A, B \in \mathcal{A}X$  then  $A \prec B$  iff there exists  $C \in \mathcal{A}X$  such that  $C \cap A = \emptyset$ , and  $C \cup B = X$ . But then  $A \subseteq X \setminus C \subseteq B$ . Thus, each  $A \in \mathcal{A}X$  is a join of elements rather below it, ie  $\mathcal{A}X$  is a regular  $\sigma$ -frame. More can be said. As pointed out by Gilmour [1981], the Alexandroff structures on a set  $X$  are precisely the regular sub  $\sigma$ -frames of  $\mathcal{P}X$ . If  $X$  and  $Y$  are Alexandroff spaces

and  $f : X \rightarrow Y$  is a cozero-map, then  $\mathcal{A}f = f^{-1} : \mathcal{A}Y \rightarrow \mathcal{A}X$  is a  $\sigma$ -frame homomorphism. This defines a contravariant functor  $\mathcal{A} : \mathbf{Alex} \rightarrow \mathbf{Reg}\sigma\mathbf{Frm}$ .

A  $\sigma$ -prime filter  $F$  of a  $\sigma$ -frame  $A$  is a filter satisfying :

$$\bigvee_A S \in F \ (S \subseteq L \text{ countable}) \Rightarrow S \cap F \neq \emptyset$$

A regular  $\sigma$ -frame  $A$  is called an Alexandroff  $\sigma$ -frame if it has enough  $\sigma$ -prime filters, ie if  $a$  and  $b$  are distinct elements of  $A$ , then there is a  $\sigma$ -prime filter  $F$ , with  $a \in F$  and  $b \notin F$ , or  $a \notin F$  and  $b \in F$ .

The prime spectrum  $\Psi A$  of a regular  $\sigma$ -frame  $A$  is an Alexandroff space consisting of all the  $\sigma$ -prime filters on  $A$ , with cozero-sets of the form  $\Psi_a = \{F \in \Psi A \mid a \in F\}$ , where  $a \in A$ . Given a  $\sigma$ -frame homomorphism  $f : A \rightarrow B$ , then  $\Psi f = f^{-1} : \Psi B \rightarrow \Psi A$  is a cozero-map. Thus  $\Psi : \mathbf{Reg}\sigma\mathbf{frm} \rightarrow \mathbf{Alex}$  is a contravariant functor. It was shown by Gilmour [1981] that the functors  $\Psi$  and  $\mathcal{A}$  are adjoint on the right, and that they induce a dual equivalence between the categories  $\mathbf{RlcmpAlex}$ , of realcompact Alexandroff spaces, and  $\mathbf{Alex}\sigma\mathbf{Frm}$ , of Alexandroff  $\sigma$ -frames. Full details of the above results are given by Gilmour [1984].

# Chapter 1

## Realcompact frames

### 1.1 Introduction

Reynolds [1979] in a paper entitled “On the spectrum of a real representable ring” showed that the category  $\mathbf{RegLindFrm}$ , of regular Lindelöf frames is coreflective in the category of completely regular frames. Using a standard categorical argument, it follows that  $\mathbf{RegLindFrm}$  is closed under colimits, and in particular coproducts. This property was first shown directly by Dowker and Strauss [1976]. It is well known that the product of even two Lindelöf spaces need not be Lindelöf. This nice behaviour of regular Lindelöf frames has as a noticeable consequence that the relationship between the Lindelöf property and realcompactness is in some sense more intimate in the category  $\mathbf{Frm}$  than in  $\mathbf{Top}$ . This is illustrated by the following result, which is due to Madden and Vermeer, [1986]:

**Theorem 1.1.1** *For a completely regular frame  $L$ , the following are equivalent:*

(i)  $L$  is Lindelöf.

(ii)  $L$  is a closed quotient of  $\bigoplus_I \mathcal{O}\mathbf{R}$ , for some index set  $I$ .

(iii)  $L \cong \mathcal{H}\text{Coz}L$

Property (ii) above is suggestive of the well-known characterisation of realcompactness. That is, a completely regular space  $X$  is realcompact iff  $X$  can be embedded as a closed subspace of  $\mathbf{R}^I$ , for some index set  $I$ . For this reason Schlitt, [1990] refers to this notion as Stone-realcompactness. However, the frame of open sets of a realcompact topological space need not be Stone-realcompact. Consider an uncountable discrete space  $X$  with a non-measurable cardinality. Then  $X$  is realcompact, but  $\mathcal{O}X$  is not Lindelöf, and therefore not Stone-realcompact. Schlitt formulated a definition of realcompactness, for which a space  $X$  is realcompact iff  $\mathcal{O}X$  is realcompact (which he refers to as Herrlich-realcompactness, or H-realcompactness); and it is this definition which we adopt below.

## 1.2 Realcompact frames

**Definition 1.2.1** For any frame  $L$ , an ideal  $I \subseteq L$  is  $\sigma$ -proper iff  $\bigvee_L S \neq 1_L$  for any countable  $S \subseteq I$ .  $I$  is said to be completely proper iff  $\bigvee_L I \neq 1_L$ .

**Definition 1.2.2 (Schlitt)** A completely regular frame  $L$  is realcompact iff every  $\sigma$ -proper maximal completely regular ideal is completely proper.

The definition given above differs from the original definition given by Schlitt, which he chose for the reason of avoiding choice principles. On the assumption of the axiom of choice, the two definitions are equivalent, as

pointed out by Schlitt.

Given a bounded distributive lattice  $L$ ,  $MaxL$  denotes the topological space consisting of all maximal ideals on  $L$  with a base for open sets consisting of the sets  $\{I \in MaxL \mid a \notin I\}$ , where  $a \in L$ . We denote by  $Max_cL$  the topological space consisting of all maximal completely regular ideals with basic open sets of the form  $\{I \in Max_cL \mid a \notin I\}$ , where  $a \in L$ .

The following lemma is a generalisation of Theorem 0.0.2 of Schlitt [1990], and allows for our characterisation of realcompactness in Proposition 1.2.4.

**Lemma 1.2.3** *For any completely regular frame  $L$ ,  $Max_cL \cong MaxCozL$ .*

**Proof** Consider the maps

$$\phi : Max_cL \longrightarrow MaxCozL$$

$$\psi : MaxCozL \longrightarrow Max_cL$$

defined by

$$\phi(I) = \{a \in CozL \mid (\downarrow a) \vee (I \cap CozL) \neq 1_{IdlCozL}\}$$

$$\psi(I) = \{u \in L \mid u \prec\prec v, \text{ for some } v \in I\}$$

We show that the maps  $\psi$  and  $\phi$  are well-defined. Let  $I \in Max_cL$ . Let  $a, b \in \phi(I)$ , ie  $(\downarrow a) \vee (I \cap CozL) \neq 1_{IdlCozL}$  and  $(\downarrow b) \vee (I \cap CozL) \neq 1_{IdlCozL}$ . Suppose  $a \vee b \vee c = 1_{CozL}$ , for some  $c \in I \cap CozL$ . Now,  $a \vee c \vee d \neq 1_{CozL}$  for any  $d \in I \cap CozL$ , since  $(\downarrow a) \vee (I \cap CozL) \neq 1_{IdlCozL}$ . Also, since  $I$  is

completely regular, it follows that for any  $u \in I$ , there exists  $v \in I \cap \text{Coz}L$  such that  $u \leq v$ . Hence  $a \vee c \vee u \neq 1_{\text{Coz}L}$  for any  $u \in I$ . Let  $a \in L$ , and define  $k_L = \{b \in L \mid b \prec\prec a\}$ . Then  $k_L(a \vee c) \vee I \neq 1_{\text{Id}L}$ . Since  $I$  is maximal, it follows that  $k_L(a \vee c) \subseteq I$ . Now,  $\text{Coz}L$  is a regular  $\sigma$ -frame, and hence normal. Thus there exists  $w \in \text{Coz}L$  such that  $w \prec\prec a \vee c$ , ie  $w \in I \cap \text{Coz}L$ , and  $w \vee b = 1_L$ . But this contradicts the fact that  $(\downarrow b) \vee (I \cap \text{Coz}L) \neq 1_{\text{Id}\text{Coz}L}$ . Hence  $a \vee b \vee c \neq 1_{\text{Id}\text{Coz}L}$  and thus  $a \vee b \in \phi(I)$ . Also, if  $a \in \phi(I)$ , and  $b \leq a$ , then  $(\downarrow b) \vee (I \cap \text{Coz}L) \subseteq (\downarrow a) \vee (I \cap \text{Coz}L)$ , and hence  $b \in \phi(I)$ . Thus  $\phi(I)$  is an ideal in  $\text{Coz}L$ . If  $a \in \text{Coz}L \setminus \phi(I)$ , then  $(\downarrow a) \vee (I \cap \text{Coz}L) = 1_{\text{Id}\text{Coz}L}$ . Hence  $a \vee i = 1_L$ , for some  $i \in I \cap \text{Coz}L$ , and since  $I \cap \text{Coz}L \subseteq \phi(I)$ , it follows that  $\phi(I)$  is a maximal ideal in  $\text{Coz}L$ . Thus  $\phi$  is well-defined.

To see that  $\psi$  is well-defined, let  $J$  be a maximal ideal in  $\text{Coz}L$ . Since the relation  $\prec\prec$  interpolates, it follows that  $\psi(J)$  is a completely regular ideal. Suppose that  $K$  is a completely regular ideal in  $L$ , properly containing  $\psi(J)$ . Let  $u \in K \setminus \psi(J)$ . Then there exists  $v \in K$  such that  $u \prec\prec v$ . From Lemma 0.3.2 there exists  $w \in \text{Coz}L$  such that  $u \prec\prec w \prec\prec v$ , ie  $w \notin J$ . Now,  $J$  is maximal, so there exists  $a \in J$  such that  $a \vee w = 1_L$ . Since  $\text{Coz}L$  is normal, there exists  $s, t \in \text{Coz}L$  such that  $s \vee a = 1_L = t \vee w$  and  $s \wedge t = 0_L$ . Note that  $t \prec a$ , since  $t \wedge s = 0_L$  and  $s \vee a = 1_L$ . Since  $\text{Coz}L$  is a regular  $\sigma$ -frame, it follows that  $t \prec\prec a$ , and consequently  $t \in \psi(J) \subseteq K$ . But then  $K$  is not a proper ideal, since  $w \vee t = 1_L$ . Thus,  $\psi(J)$  is a maximal completely regular ideal in  $L$ , ie  $\psi$  is well-defined.

Let  $I$  be a maximal completely regular ideal in  $L$ . Then,

$$\begin{aligned}
u \in \psi\phi(I) &\Rightarrow u \prec\prec v, \text{ for some } v \in \phi(I) \\
&\Rightarrow u \prec\prec v, \text{ where } v \vee a \neq 1_{\text{Coz}L}, \text{ for any } a \in I \cap \text{Coz}L \\
&\Rightarrow u \in I
\end{aligned}$$

Hence  $\psi\phi(I) \subseteq I$ , and since  $\psi\phi(I)$  is a maximal completely regular ideal, it follows that  $\psi\phi(I) = I$ .

Let  $J$  be a maximal ideal in  $\text{Coz}L$ . Then,

$$\begin{aligned}
a \in J &\Rightarrow a \vee k \neq 1_{\text{Coz}L} \text{ for each } k \in J \\
&\Rightarrow a \vee k \neq 1_{\text{Coz}L} \text{ for each } k \in \psi(J) \cap \text{Coz}L \\
&\Rightarrow (\downarrow a) \vee (\psi(J) \cap \text{Coz}L) \neq 1_{\text{IdlCoz}L} \\
&\Rightarrow a \in \phi\psi(J)
\end{aligned}$$

Hence  $J \subseteq \phi\psi(J)$ , and since  $J$  is a maximal ideal in  $\text{Coz}L$ , it follows that  $J = \phi\psi(J)$

Let  $\mathcal{U}$  be a basic open set in  $\text{MaxCoz}L$ , ie  $\mathcal{U} = \{I \in \text{MaxCoz}L \mid u \notin I\}$  for some  $u \in L$ . Now,  $\phi^{-1}(\mathcal{U}) = \{\psi(I) \in \text{Max}_cL \mid \psi(u) \notin \psi(I)\}$ . Hence  $\phi$  is continuous. Similarly,  $\psi$  is continuous.

**Proposition 1.2.4** *A completely regular frame  $L$  is realcompact iff every  $\sigma$ -proper maximal ideal in  $\text{Coz}L$  is completely proper.*

**Proof** Let  $I$  be a  $\sigma$ -proper maximal ideal in  $\text{Coz}L$ . Then  $\psi(I) = \{a \in L \mid a \prec\prec i \text{ for some } i \in I\}$  is clearly  $\sigma$ -proper. Also,  $\psi(I)$  is a maximal completely regular ideal in  $\text{Coz}L$ , by Lemma 1.2.3 above. Conversely, suppose

$I \in \text{MaxCoz}L$  is not  $\sigma$ -proper. Then there is a set  $S = \{a_i \mid i \in \mathbf{N}\}$  with  $S \subseteq I$  and  $\bigvee_L S = 1_L$ . Now,  $a_i \in \text{Coz}L$ , for each  $i \in \mathbf{N}$ , so  $a_i = \bigvee_j b_{ij}$ , where  $b_{ij} \prec a_i$  for each  $j \in \mathbf{N}$ . Thus,  $b_{ij} \in \psi(I)$ , for each  $i \in \mathbf{N}$  and each  $j \in \mathbf{N}$ . But  $\{b_{ij} \mid i \in \mathbf{N}, j \in \mathbf{N}\}$  is a countable set, and  $\bigvee_L \{b_{ij} \mid i \in \mathbf{N}, j \in \mathbf{N}\} = \bigvee_L \{a_i \mid i \in \mathbf{N}\} = 1_L$ , so that  $\psi(I)$  is not  $\sigma$ -proper. Thus,  $\psi$  induces a one-to-one correspondence between the  $\sigma$ -proper maximal ideals in  $\text{Coz}L$  and the  $\sigma$ -proper maximal completely regular ideals in  $L$ . Suppose  $L$  is realcompact. Let  $I$  be a  $\sigma$ -proper maximal ideal in  $\text{Coz}L$ . Then  $\psi(I)$  is a  $\sigma$ -proper maximal completely regular ideal in  $L$ . Thus,  $\bigvee_L \psi(I) \neq 1_L$ . But

$$\begin{aligned} \bigvee_L I &= \bigvee_L \{a \in L \mid a \prec \prec i, \text{ for some } i \in I\} \\ &= \bigvee_L \psi(I) \end{aligned}$$

Hence  $I$  is completely proper.

Conversely, let  $I$  be a  $\sigma$ -proper maximal completely regular ideal in  $L$ . Then  $\phi(I)$  is a  $\sigma$ -proper maximal ideal in  $\text{Coz}L$ , so  $\bigvee_L \phi(I) \neq 1_L$ . But  $\bigvee_L \phi(I) = \bigvee_L \psi\phi(I) = \bigvee_L I$ , so  $I$  is completely proper.

**Remark** A filter  $\mathcal{F}$  in a completely regular space  $X$  is a  $z$ -ultrafilter iff  $\text{C}\mathcal{F} = \{I \mid X \setminus I \in \mathcal{F}\}$  is a maximal ideal in  $\text{Coz}X$ .  $\mathcal{F}$  has the countable intersection property (c.i.p) iff  $\text{C}\mathcal{F}$  is  $\sigma$ -proper. Hence,  $X$  is realcompact iff every  $\sigma$ -proper maximal ideal in  $\text{Coz}X$  is completely proper. Thus, a completely regular space  $X$  is realcompact iff  $\mathcal{O}X$  is realcompact.

**Definition 1.2.5** *The full subcategory of realcompact frames is denoted  $\mathbf{RlcmpFrm}$ .*

**Proposition 1.2.6** *A completely regular frame  $L$  is Lindelöf iff every  $\sigma$ -proper ideal in  $\mathit{Coz}L$  is completely proper.*

**Proof** “ $\Rightarrow$ ” Suppose  $L$  is Lindelöf. Then  $L \cong \mathcal{H}\mathit{Coz}L$ , ie every proper  $\sigma$ -ideal in  $\mathit{Coz}L$  is completely proper. Let  $I$  be a  $\sigma$ -proper ideal in  $\mathit{Coz}L$ . Then  $\langle I \rangle$ , the  $\sigma$ -ideal generated by  $I$  in  $\mathit{Coz}L$  is completely proper, and hence  $I$  is completely proper.

“ $\Leftarrow$ ” Every  $\sigma$ -ideal in  $\mathit{Coz}L$  is a  $\sigma$ -proper ideal in  $\mathit{Coz}L$ . Thus, the map  $\mathcal{H}\mathit{Coz}L \xrightarrow{j_L} L$ , given by join is codense, and hence injective. On the other hand,  $\mathcal{H}\mathit{Coz}L \xrightarrow{j_L} L$  is surjective, since for each  $a \in L$ ,  $(\downarrow a) \cap \mathit{Coz}L \in \mathcal{H}\mathit{Coz}L$ , and  $j_L((\downarrow a) \cap \mathit{Coz}L) = a$  since  $\mathit{Coz}L$  join generates  $L$ . Hence  $L \cong \mathcal{H}\mathit{Coz}L$ , ie  $L$  is Lindelöf.

**Corollary 1.2.7** *Every Lindelöf frame is realcompact.*

### 1.3 Realcompactifications of frames

**Definition 1.3.1** *Let  $L$  be a completely regular frame. Then  $(M, h)$  is a realcompactification of  $L$  iff  $M$  is a realcompact frame, and  $M \xrightarrow{h} L$  is a dense surjection.*

Recall that the functor  $\mathcal{H}$  assigns to each regular  $\sigma$ -frame,  $A$ , a Lindelöf frame,  $\mathcal{H}A$ . We construct a realcompactification of a completely regular

frame  $L$  by forming a suitable quotient of  $\mathcal{H}A$ , where  $A$  is a regular sub  $\sigma$ -frame of  $L$ , with the property that each  $a \in L$  can be written as a join of elements in  $A$ , ie  $A$  join generates  $L$ . The technique used here is essentially an adaptation of that used by Shlitt, [1990] in his construction of the Hewitt realcompactification for frames, and is motivated by the construction of realcompactifications of spaces using Alexandroff bases, and the adjunction of Section 0.6. As this construction is akin to that of Wallman for compactifications, we will call the realcompactification obtained the Wallman realcompactification.

Let  $L$  be a completely regular frame, and let  $A$  be a regular sub  $\sigma$ -frame join generating  $L$ . Define  $\mathcal{H}A \xrightarrow{h_L} \mathcal{H}A$  by

$$h_L I = \downarrow(\bigvee_L I) \cap \bigcap \{J \in \sigma PMax A \mid I \subseteq J\},$$

where  $\sigma PMax A$  is the collection of all  $\sigma$ -proper maximal ideals in  $A$ .

**Lemma 1.3.2** *The map  $h_L$ , given above, is a nucleus.*

**Proof** We firstly show that  $h_L$  is well-defined. Let  $I \in \mathcal{H}A$ . Suppose  $S$  is a countable subset of  $h_L I$ , then  $\bigvee_L S \in \downarrow(\bigvee_L I)$ . Let  $J$  be a  $\sigma$ -proper maximal ideal containing  $I$ . Then  $J$  is a  $\sigma$ -ideal, and since  $S \subseteq J$ , it follows that  $\bigvee_L S \subseteq J$ . Let  $u \in h_L I$ , and suppose  $v \leq u$ . Then  $v \in \downarrow(\bigvee_L I)$ , since  $u \in \downarrow(\bigvee_L I)$ . Given any  $\sigma$ -proper maximal ideal  $J \supseteq I$ . Then  $u \in J$ , and hence  $v \in J$ , ie  $v \in h_L I$ . Thus  $h_L I \in \mathcal{H}A$  and hence  $h_L$  is well-defined. We now show that  $h_L$  is a nucleus:

- (i) It is clear that  $I \subseteq h_L I$
- (ii) Since  $h_L$  is order-preserving, it follows that  $h_L(I \cap K) \subseteq h_L I \cap h_L K$ .

Now, suppose  $u \in h_L I \cap h_L K$ . Then,  $u \leq \bigvee_L I \wedge \bigvee_L K = \bigvee_L (I \cap K)$ . Suppose  $J \supseteq I \cap K$  is a  $\sigma$ -proper maximal ideal in  $A$ . Since  $J$  is maximal and hence prime,  $J \supseteq I$ , or  $J \supseteq K$ . But then  $u \in J$  and so  $u \in h_L (I \cap K)$ . ie  $h_L (I \cap K) = h_L I \cap h_L K$ .

(iii) Let  $u \in h_L^2 I$ . Then  $u \leq \bigvee_L h_L I \leq \bigvee_L I$ . Now, suppose  $I \subseteq J$ , where  $J$  is a  $\sigma$ -proper maximal ideal in  $A$ . Then  $h_L I \subseteq J$ , by definition of  $h_L$ . Thus  $u \in J$  and hence  $u \in h_L I$ , ie  $h_L^2 I \subseteq h_L I$

Let  $(\mathcal{H}A)_{h_L} = \{I \in \mathcal{H}A \mid h_L I = I\}$  be the quotient frame corresponding to the nucleus  $h_L$ . It is shown below that  $(\mathcal{H}A)_{h_L}$  with the join map is a realcompactification of  $L$ .

**Lemma 1.3.3** *Let  $A$  be a regular  $\sigma$ -frame, then  $\text{Coz}(\mathcal{H}A)_{h_L} \cong A$ .*

**Proof** We show that the frame  $\text{Coz}(\mathcal{H}A)_{h_L}$  is precisely the frame of principal ideals in  $A$ . Firstly, suppose  $I \in \text{Coz}(\mathcal{H}A)_{h_L}$ . Then there exists a sequence  $(J_n)$  in  $(\mathcal{H}A)_{h_L}$  with  $J_1 \prec\prec J_2 \prec\prec J_3 \prec\prec \dots$ , and  $I = \bigvee_{(\mathcal{H}A)_{h_L}} J_i$ . Now, for each  $n \in \mathbb{N}$ ,  $J_n \prec I$ . Thus, for each  $n \in \mathbb{N}$ , there exists  $S_n \in (\mathcal{H}A)_{h_L}$  such that  $J_n \wedge S_n = 0_{\text{Idl}A}$  and  $I \vee S_n = 1_{\text{Idl}A}$ . For each  $n \in \mathbb{N}$ , take  $s_n \in S_n$ . Then  $j_{n_i} \wedge s_n = 0_L$  for each  $j_{n_i} \in J_n$ , and there exists  $k_n \in I$  such that  $k_n \vee s_n = 1_L$ , ie  $j_{n_i} \prec k_n$  for each  $j_{n_i} \in J_n$ . Let  $k = \bigvee_L k_n$ . Then  $k \in I$ , since  $I \in (\mathcal{H}A)_{h_L}$ . Also,  $j_{n_i} \prec k$ , for each  $j_{n_i} \in J_n$  and each  $n \in \mathbb{N}$ . But then  $J_n \subseteq \downarrow k$  for each  $n \in \mathbb{N}$ . hence  $\bigvee_{\mathcal{H}A} J_i \subseteq \downarrow k$ . Now,  $h_L$  is order preserving, and therefore  $h_L \bigvee_{\mathcal{H}A} J_i = \bigvee_{(\mathcal{H}A)_{h_L}} J_i \subseteq h_L \downarrow k = \downarrow k$ . Since  $k \in I$ , it follows that  $I = \downarrow k$ .

On the other hand, let  $a \in A$ . Then, since  $A$  is a regular  $\sigma$ -frame,  $a = \bigvee_A S$ , where  $S = \{a_i \mid i \in \mathbb{N}\}$  and  $a_i \prec a$  for each  $i \in \mathbb{N}$ . Now, the relation  $\prec$  interpolates in regular  $\sigma$ -frames, so  $\downarrow a_i \prec \prec \downarrow a$  in  $\mathcal{H}A$ . But then there is an  $I_i \in \text{Coz}\mathcal{H}A$  such that  $\downarrow a_i \prec \prec I_i \prec \prec \downarrow a$ . Hence  $\bigvee_{\mathcal{H}A} I_i = \downarrow a$ , and thus  $\downarrow a \in \text{Coz}\mathcal{H}A$  since it is a countable join of cozero elements.

**Lemma 1.3.4** *The frame  $(\mathcal{H}A)_{h_L}$ , is realcompact.*

**Proof** Let  $\mathcal{J}$  be a  $\sigma$ -proper maximal ideal in  $\text{Coz}(\mathcal{H}A)_{h_L}$ . Then  $\mathcal{J} = \{\downarrow a \mid a \in J\}$ , where  $J$  is a  $\sigma$ -proper maximal ideal in  $A$ . But then  $\bigvee_{(\mathcal{H}A)_{h_L}} \mathcal{J} = h_L(\bigvee_{\mathcal{H}A} \mathcal{J}) = h_L(J) = J$ . Thus  $\bigvee_{(\mathcal{H}A)_{h_L}} \mathcal{J}$  is a  $\sigma$ -proper maximal ideal in  $A$ , so that  $\mathcal{J}$  is completely proper in  $(\mathcal{H}A)_{h_L}$ .

**Proposition 1.3.5** *The map  $(\mathcal{H}A)_{h_L} \xrightarrow{j_L} L$ , given by join is a dense surjection.*

**Proof** It is clear that for each  $a \in L$ ,  $h_L(\downarrow a \cap A) = (\downarrow a \cap A)$ , and hence  $(\downarrow a \cap A) \in (\mathcal{H}A)_{h_L}$ . Also  $j_L(\downarrow a \cap A) = a$  for each  $a \in L$ , since  $A$  join generates  $L$ , so  $j_L$  is surjective. Suppose  $j_L(I) = 0_L$ , then  $I = \{0_L\}$ , ie  $j_L$  is dense.

Hence  $((\mathcal{H}A)_{h_L}, j_L)$  is a realcompactification of  $L$ .

**Definition 1.3.6** *Let  $L$  be a completely regular frame, and let  $A$  be a regular sub  $\sigma$ -frame of  $L$ , join generating  $L$ , then  $((\mathcal{H}A)_{h_L}, j_L)$  is called the Wallman realcompactification of  $L$  with respect to  $A$ .  $(\mathcal{H}A)_{h_L}$  is denoted  $v_A L$ .*

**Lemma 1.3.7** *If  $L$  is realcompact, then  $v_{CozL}L \cong L$ .*

**Proof** It suffices to show that  $v_{CozL}L \xrightarrow{j_L} L$  is codense. Suppose  $j_L(I) = 1_L$ , i.e.  $I$  is not completely proper. Now, if  $I \neq 1_{IdlCozL}$ , then since  $I \in v_{CozL}L$ , there is a  $\sigma$ -proper maximal ideal  $J \supseteq I$ . But then  $J$  is completely proper, since  $L$  is realcompact. This contradicts the fact that  $I$  is not completely proper. Hence  $I = 1_{IdlCozL}$ .

**Remark** The above result cannot be generalised to arbitrary regular sub  $\sigma$ -frames of  $L$ . As a counterexample, let  $L = \mathcal{P}\mathbf{R}$ , the power set of  $\mathbf{R}$  and let  $A$  be the collection of all countable and cocountable subsets of  $\mathbf{R}$ . Then  $A$  is a regular sub  $\sigma$ -frame join generating  $L$ , but  $v_AL \not\cong L$ . For if  $v_AL \cong L$ , then  $Cozv_AL$  and  $CozL$  would be isomorphic as  $\sigma$ -frames. But  $Cozv_AL \cong A$ , and  $CozL = L$ , from which it would follow that  $L$  and  $A$  are isomorphic as  $\sigma$ -frames.

We now proceed to prove that  $v_{CozL}L$  is the universal realcompactification of  $L$ . Thus, the H-realcompactification of Schlitt is a special case of the Wallman realcompactification constructed above.

**Lemma 1.3.8** *Let  $L$  be a completely regular frame, and let  $\{K_\alpha \mid \alpha \in \Lambda\}$  be a collection of  $\sigma$ -ideals in  $CozL$ . Then  $h_L \bigvee_{\mathcal{H}CozL} h_L K_\alpha = h_L \bigvee_{\mathcal{H}CozL} K_\alpha$ .*

**Proof** So as not to complicate notation, we shall suppress mention of the index set  $\Lambda$ .

$$h_L \bigvee_{\mathcal{H}CozL} h_L K_\alpha$$

$$\begin{aligned}
&= h_L \bigvee_{\mathcal{H}\text{Coz}L} (\downarrow (\bigvee_L K_\alpha) \cap \bigcap \{K \in \sigma\text{PMaxCoz}L \mid K \supseteq K_\alpha\}) \\
&= h_L (\bigvee_{\mathcal{H}\text{Coz}L} \downarrow (\bigvee_L K_\alpha) \cap \bigvee_{\mathcal{H}\text{Coz}L} \bigcap \{K \in \sigma\text{PMaxCoz}L \mid K \supseteq K_\alpha\}) \\
&= h_L J, \text{ say} \\
&= \downarrow (\bigvee_L \bigvee_{\mathcal{H}\text{Coz}L} \downarrow (\bigvee_L K_\alpha)) \cap \downarrow (\bigvee_L \bigvee_{\mathcal{H}\text{Coz}L} \bigcap \{K \in \sigma\text{PMaxCoz}L \mid K \supseteq K_\alpha\}) \\
&\quad \cap \bigcap \{I \in \sigma\text{PMaxCoz}L \mid I \supseteq J\} \\
&= \downarrow (\bigvee_L \bigvee_{\mathcal{H}\text{Coz}L} K_\alpha) \cap \bigcap \{I \in \sigma\text{PMaxCoz}L \mid I \supseteq J\}
\end{aligned}$$

Let  $I$  be a  $\sigma$ -proper maximal ideal containing  $\bigvee_{\mathcal{H}\text{Coz}L} K_\alpha$ , then  $I$  contains  $\bigcap \{K \in \sigma\text{PMaxCoz}L \mid K \supseteq K_\alpha\}$ , for each  $\alpha$ , and thus  $I$  contains  $\bigvee_{\mathcal{H}\text{Coz}L} \bigcap \{K \in \sigma\text{PMaxCoz}L \mid K \supseteq K_\alpha\}$ . Consequently,  $I \supseteq \bigvee_{\mathcal{H}\text{Coz}L} \downarrow \bigvee_L K_\alpha \cap \bigvee_{\mathcal{H}\text{Coz}L} \bigcap \{K \in \sigma\text{PMaxCoz}L \mid K \supseteq K_\alpha\} = J$ .

Hence,

$$\bigcap \{I \in \sigma\text{PMaxCoz}L \mid I \supseteq J\} \subseteq \bigcap \{K \in \sigma\text{PMaxCoz}L \mid K \supseteq \bigvee_{\mathcal{H}\text{Coz}L} K_\alpha\}$$

from which it follows that

$$\begin{aligned}
h_L \bigvee_{\mathcal{H}\text{Coz}L} h_L K_\alpha &\subseteq \downarrow (\bigvee_L \bigvee_{\mathcal{H}\text{Coz}L} K_\alpha) \cap \\
&\quad \bigcap \{K \in \sigma\text{PMaxCoz}L \mid K \supseteq \bigvee_{\mathcal{H}\text{Coz}L} K_\alpha\} \\
&= h_L \bigvee_{\mathcal{H}\text{Coz}L} K_\alpha
\end{aligned}$$

For the reverse inclusion, note that  $K_\alpha \subseteq h_L K_\alpha$  for each  $\alpha$ . Since  $h_L$  is order-preserving, it follows that  $h_L \bigvee_{\mathcal{H}\text{Coz}L} K_\alpha \subseteq h_L \bigvee_{\mathcal{H}\text{Coz}L} h_L K_\alpha$ .

**Lemma 1.3.9** *Let  $L$  and  $M$  be completely regular frames, and suppose  $L \xrightarrow{\phi} M$  is a frame homomorphism. Then the map  $(\mathcal{H}\text{Coz}L)_{h_L} \xrightarrow{\bar{\phi}} (\mathcal{H}\text{Coz}M)_{h_M}$ , given by  $\bar{\phi}(J) = h_M \mathcal{H}\text{Coz}\phi(J)$ , is frame homomorphism. Furthermore,*

$j_L \cdot \bar{\phi} = \phi \cdot j_M$ , where  $j_L$  and  $j_M$  are both join maps in the corresponding frames.

**Proof** Firstly, note that both  $h_M$  and  $\mathcal{H}Coz\phi$  preserve intersection, so  $\bar{\phi}$  preserves intersection. We show that  $\bar{\phi}$  preserves arbitrary joins. Let  $\{J_\alpha \mid \alpha \in A\}$  be a collection of  $\sigma$ -ideals in  $(\mathcal{H}CozL)_{h_L}$ . Then

$$\begin{aligned}
& \bar{\phi}( \bigvee_{(\mathcal{H}CozL)_{h_L}} J_\alpha ) \\
&= \bar{\phi}(h_L \bigvee_{\mathcal{H}CozL} J_\alpha) \\
&= h_M \mathcal{H}Coz\phi \cdot h_L \bigvee_{\mathcal{H}CozL} J_\alpha \\
&= \downarrow( \bigvee_M \mathcal{H}Coz\phi \cdot h_L \bigvee_{\mathcal{H}CozL} J_\alpha ) \cap \\
&\quad \bigcap \{J \in \sigma PMaxCozM \mid J \supseteq \mathcal{H}Coz\phi \cdot h_L \bigvee_{\mathcal{H}CozL} J_\alpha\} \\
&= \downarrow(\phi( \bigvee_L h_L \bigvee_{\mathcal{H}CozL} J_\alpha )) \cap \\
&\quad \bigcap \{J \in \sigma PMaxCozM \mid J \supseteq \mathcal{H}Coz\phi \cdot h_L \bigvee_{\mathcal{H}CozL} J_\alpha\}
\end{aligned}$$

Now, for any  $J \in \mathcal{H}CozL$ ,  $h_L J \subseteq \downarrow( \bigvee_L J )$ , and hence  $\bigvee_L h_L J \leq \bigvee_L J$ . Also,  $J \subseteq h_L J$ , so that  $\bigvee_L J \subseteq \bigvee_L h_L J$ . Thus,  $\bigvee_L J = \bigvee_L h_L J$ . So, we get:

$$\begin{aligned}
& \bar{\phi}( \bigvee_{(\mathcal{H}CozL)_{h_L}} J_\alpha ) \\
&= \downarrow(\phi( \bigvee_L \bigvee_{\mathcal{H}CozL} J_\alpha )) \cap \\
&\quad \bigcap \{J \in \sigma PMaxCozM \mid J \supseteq \mathcal{H}Coz\phi \cdot h_L \bigvee_{\mathcal{H}CozL} J_\alpha\} \\
&= \downarrow( \bigvee_M \mathcal{H}Coz\phi( \bigvee_{\mathcal{H}CozL} J_\alpha )) \cap \\
&\quad \bigcap \{J \in \sigma PMaxCozM \mid J \supseteq \mathcal{H}Coz\phi \cdot h_L \bigvee_{\mathcal{H}CozL} J_\alpha\}
\end{aligned}$$

On the other hand,

$$\bigvee_{(\mathcal{H}CozM)_{h_M}} \bar{\phi}(J_\alpha)$$

$$\begin{aligned}
&= h_M \bigvee_{\mathcal{H}CozM} h_M \mathcal{H}Coz\phi(J_\alpha) \\
&= h_M \bigvee_{\mathcal{H}CozM} \mathcal{H}Coz\phi(J_\alpha), \text{ from Lemma 1.3.8 above} \\
&= \downarrow(\bigvee_M \bigvee_{\mathcal{H}CozM} \mathcal{H}Coz\phi(J_\alpha)) \cap \\
&\quad \bigcap \{J \in \sigma PMaxCozM \mid J \supseteq \mathcal{H}Coz\phi \bigvee_{\mathcal{H}CozL} J_\alpha\}
\end{aligned}$$

We now show that

$$\begin{aligned}
&\{J \in \sigma PMaxCozM \mid J \supseteq \mathcal{H}Coz\phi \cdot \bigvee_{\mathcal{H}CozL} J_\alpha\} \\
&= \{J \in \sigma PMaxCozM \mid J \supseteq \mathcal{H}Coz\phi \cdot h_L \bigvee_{\mathcal{H}CozL} J_\alpha\}
\end{aligned}$$

It is clear that

$$\begin{aligned}
&\{J \in \sigma PMaxCozM \mid J \supseteq \mathcal{H}Coz\phi \cdot h_L \bigvee_{\mathcal{H}CozL} J_\alpha\} \\
&\subseteq \{J \in \sigma PMaxCozM \mid J \supseteq \mathcal{H}Coz\phi \cdot \bigvee_{\mathcal{H}CozL} J_\alpha\}
\end{aligned}$$

Conversely, suppose  $J$  is a  $\sigma$ -proper maximal ideal in  $CozM$ , containing  $\mathcal{H}Coz\phi \bigvee_{\mathcal{H}CozL} J_\alpha$ . Let  $K = \bigvee_{\mathcal{H}CozL} \{I \in \mathcal{H}CozL \mid \mathcal{H}Coz\phi(I) \subseteq J\}$ . Suppose  $K'$  is an ideal properly containing  $K$ . Then  $\mathcal{H}Coz\phi(K') = 1_{IdCozM}$ , ie there exists  $k' \in K'$  such that  $\phi(k') = 1_M$ . Now, since  $k' \in CozL$ , and  $CozL$  is a regular  $\sigma$ -frame, there exists a set  $S = \{k_i \mid i \in \mathbb{N}\}$  with  $k_i \prec k'$  for each  $i \in \mathbb{N}$ , and  $k' = \bigvee_L S$ . But, since  $K$  is  $\sigma$ -proper, there exists  $j \in \mathbb{N}$  such that  $k_j \notin K$ , ie  $\phi(k_j) \vee p = 1_M$ , for some  $p \in J$ . But then  $\phi(k_j^*) \leq (\phi(k_j))^* \leq p$ , (where  $k_j^*$  denotes the pseudocomplement of  $k_j$  in  $L$ ) Since  $k_j \prec k'$ , there exists  $s \in CozL$  such that  $k_j \wedge s = 0_L$  and  $k' \vee s = 1_L$ . But then  $s \leq k_j^*$ , and therefore  $\phi(s) \leq \phi(k_j^*) \leq p$ . Hence,  $s \in K'$ , from which it follows that  $K' = 1_{IdCozL}$ . Thus,  $K$  is a  $\sigma$ -proper maximal ideal in  $CozL$ , and hence  $h_L K = K$ . Now,  $\bigvee_{\mathcal{H}CozL} J_\alpha \subseteq K$  so that

$\mathcal{H}\text{Coz}\phi h_L \bigvee_{\mathcal{H}\text{Coz}L} J_\alpha \subseteq \mathcal{H}\text{Coz}\phi h_L K = \mathcal{H}\text{Coz}\phi K \subseteq J$ , and hence

$$\begin{aligned} & \{J \in \sigma P\text{Max}\text{Coz}M \mid J \supseteq \mathcal{H}\text{Coz}\phi \cdot \bigvee_{\mathcal{H}\text{Coz}L} J_\alpha\} \\ & \subseteq \{J \in \sigma P\text{Max}\text{Coz}M \mid J \supseteq \mathcal{H}\text{Coz}\phi \cdot h_L \bigvee_{\mathcal{H}\text{Coz}L} J_\alpha\} \end{aligned}$$

This gives equality, and therefore  $\bar{\phi}(\bigvee_{(\mathcal{H}\text{Coz}L)_{h_L}} J_\alpha) = \bigvee_{(\mathcal{H}\text{Coz}M)_{h_M}} \bar{\phi}(J_\alpha)$ .

Finally,

$$\begin{aligned} j_L \cdot \bar{\phi}(J) &= j_L \cdot h_L \mathcal{H}\text{Coz}\phi(J) \\ &= \bigvee_L \mathcal{H}\text{C}\hat{\text{oz}}\phi(J) \\ &= \phi(\bigvee_M J) \\ &= \phi \cdot j_M(J) \end{aligned}$$

Thus,  $j_L \bar{\phi} = \phi j_M$ , which concludes the proof.

**Proposition 1.3.10** *The map  $(\mathcal{H}\text{Coz}L)_{h_L} \xrightarrow{j_L} L$ , given by join is universal as a map from realcompact frames to  $L$ .*

**Proof** Let  $K$  be a realcompact frame and  $K \xrightarrow{\phi} L$  a frame homomorphism.

$$\begin{array}{ccc} (\mathcal{H}\text{Coz}L)_{h_L} & \xrightarrow{j_L} & L \\ \uparrow h_L \mathcal{H}\text{Coz}\phi & & \uparrow \phi \\ (\mathcal{H}\text{Coz}K)_{h_K} & \xrightarrow{j_K} & K \end{array}$$

Then by Lemma 1.3.7,  $K \cong (\mathcal{H}C_{oz}K)_{h_K}$ . Now, from Lemma 1.3.9,  $h_L \mathcal{H}C_{oz}\phi$  is a frame homomorphism, with  $j_L \cdot h_L \mathcal{H}C_{oz}\phi = \phi \cdot j_K$ . Thus,  $j_L \cdot h_L \mathcal{H}C_{oz}\phi \cdot j_K^{-1} = \phi$ . Uniqueness of the map  $h_L \mathcal{H}C_{oz}\phi \cdot j_K^{-1}$  follows from the fact that  $j_K$  is dense and hence monic.

The category **RlcmpFrm** is therefore a coreflective subcategory of **CrgFrm**. The coreflection  $v_{C_{oz}L}$  of a completely regular frame  $L$  shall be denoted simply by  $vL$ .

## 1.4 Alexandroff Frames

We introduce the notion of an Alexandroff frame, which is the frame analogue of Alexandroff spaces. Realcompactness of Alexandroff frames is defined in a natural way, so that the Wallman realcompactification can be viewed as a functor from the category of Alexandroff frames to the category **RlcmpFrm**. Wallman realcompactifications are not functorial in spaces but were shown by Gilmour to be delivered by a functor on **Alex**.

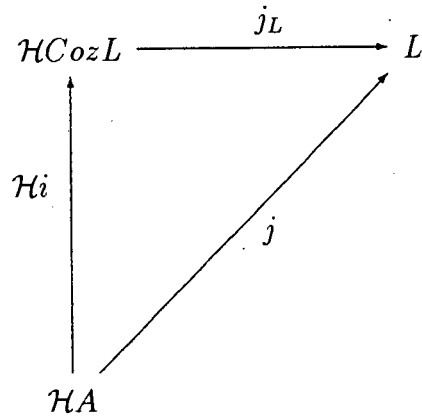
**Definition 1.4.1** *Let  $L$  be a completely regular frame, and let  $A$  be a regular sub  $\sigma$ -frame join generating  $L$ , then  $(L, A)$  is called an Alexandroff frame. A map  $(L, A) \xrightarrow{h} (M, B)$  is called an Alexandroff frame homomorphism iff  $h$  is a frame homomorphism and  $A \xrightarrow{h_A} B$  (the restriction of  $h$  to  $A$ ) is a  $\sigma$ -frame homomorphism.*

**Definition 1.4.2** *The category of Alexandroff frames and Alexandroff frame homomorphisms is denoted **AlexFrm**.*

**Definition 1.4.3** An Alexandroff frame  $(L, A)$  is said to be Lindelöf iff  $L$  is a Lindelöf frame.

**Proposition 1.4.4** Let  $L$  be a completely regular frame. If  $L$  is Lindelöf, then there is a unique regular sub  $\sigma$ -frame which join generates  $L$ . On the other hand, a regular  $\sigma$ -frame  $A$  admits a unique Lindelöf frame.

**Proof** Let  $A$  be a regular sub  $\sigma$ -frame join generating  $L$ .



Then the join map  $\mathcal{H}A \xrightarrow{j} L$  is surjective. Now,  $A$  is a regular sub  $\sigma$ -frame of  $L$ , and therefore  $A \subseteq CozL$ . Thus, the map  $\mathcal{H}A \xrightarrow{\mathcal{H}i} \mathcal{H}CozL$ , defined by  $\mathcal{H}i(I) = (\downarrow I) \cap CozL$  is an injective map. Since  $j_L$  is a coreflective map, it follows that  $j_L \mathcal{H}i = j$ . Now,  $j_L$  is an isomorphism, and hence  $j_L \mathcal{H}i$  is injective. Thus  $j$  is an isomorphism, ie  $\mathcal{H}A \cong \mathcal{H}CozL \cong L$ , but  $Coz\mathcal{H}A \cong A$ , so that  $A \cong CozL$ .

Suppose  $A$  is a regular  $\sigma$ -frame. Let  $L$  be a Lindelöf frame such that  $A$  join generates  $L$ , and  $L \not\cong \mathcal{H}A$ . Then  $CozL \not\cong A$ , which contradicts the fact that  $L$  is Lindelöf.

**Definition 1.4.5** An Alexandroff frame  $(L, A)$  is said to be realcompact iff every  $\sigma$ -proper maximal ideal in  $A$  is completely proper.

**Proposition 1.4.6** An Alexandroff frame  $(L, A)$  is realcompact iff  $L \cong (\mathcal{H}A)_{h_L}$ .

**Proof** By Proposition 1.3.5, the map  $(\mathcal{H}A)_{h_L} \xrightarrow{j_L} L$  is surjective. Using a similar argument to that in Lemma 1.3.7 it can be shown that  $j_L$  is codense, and hence injective, from which it follows that  $L \cong (\mathcal{H}A)_{h_L}$ .

**Example** Let  $L = \mathcal{P}\mathbf{R}$  and let  $A$  be the collection of all countable and cocountable subsets of  $\mathbf{R}$ . Then  $L$  is realcompact, but  $(L, A)$  is not a realcompact Alexandroff frame. To see this, let  $I = \mathcal{P}_c\mathbf{R}$ , the collection of all countable subsets of  $\mathbf{R}$ . Then  $I$  is a  $\sigma$ -proper maximal ideal in  $A$  which is not completely proper.

Using an adaptation of Lemma 1.3.9, it is easily seen that, given an Alexandroff frame homomorphism  $(L, A) \xrightarrow{\phi} (M, B)$ , there is a frame homomorphism  $(\mathcal{H}A)_{h_L} \xrightarrow{\bar{\phi}} (\mathcal{H}B)_{h_M}$ , where  $\bar{\phi} = h_M \mathcal{H}\phi|_A$ , and the following square commutes:

$$\begin{array}{ccc}
(\mathcal{H}A)_{h_L} & \xrightarrow{j_L} & (L, A) \\
\downarrow \bar{\phi} & & \downarrow \phi \\
(\mathcal{H}B)_{h_M} & \xrightarrow{j_M} & (M, B)
\end{array}$$

So, we obtain a functor  $\bar{\mathcal{H}} : \mathbf{AlexFrm} \rightarrow \mathbf{CrgFrm}$  with

$$\bar{\mathcal{H}}((L, A) \xrightarrow{\circ} (M, B)) = (\mathcal{H}A)_{h_M} \xrightarrow{\bar{\phi}} (\mathcal{H}B)_{h_M}$$

Note that  $\bar{\mathcal{H}}(L, A) = v_A L$

We define the functor  $\mathit{Coz} : \mathbf{CrgFrm} \rightarrow \mathbf{AlexFrm}$  where

$$\mathit{Coz}(L \xrightarrow{h} M) = (L, \mathit{Coz}L) \xrightarrow{h} (M, \mathit{Coz}M)$$

The above, together with Proposition 1.4.6 can be used in exactly the same way as Proposition 1.3.10 to show that the map  $(v_A L, \mathit{Coz}v_A L) \xrightarrow{j_L} (L, A)$ , given by join is universal as a map from realcompact Alexandroff frames to  $(L, A)$ .

**Lemma 1.4.7** *The functor  $\bar{\mathcal{H}}$  is left adjoint to  $\mathit{Coz}$ .*

**Proof** Let  $(L, A)$  be an Alexandroff frame, and let the unit  $\eta_L : (L, A) \rightarrow \mathit{Coz}\bar{\mathcal{H}}(L, A)$  be the map defined by  $\eta_L(a) = (\downarrow a) \cap A$ . From Proposition 1.3.5,  $(\downarrow a) \in \mathit{Coz}\bar{\mathcal{H}}(L, A) = (v_A L, \mathit{Coz}v_A L)$ , for each  $a \in (L, A)$ . Hence  $\eta_L$  is

well-defined. For naturality of  $\eta_L$ , let  $\phi : (L, A) \longrightarrow (M, B)$  be an Alexandroff frame homomorphism, and let  $a \in L$ . Then

$$\begin{array}{ccc}
 (L, A) & \xrightarrow{\eta_L} & \text{Coz}\bar{\mathcal{H}}(L, A) \\
 \downarrow \phi & & \downarrow \text{Coz}\bar{\mathcal{H}}\phi \\
 (M, B) & \xrightarrow{\eta_B} & \text{Coz}\bar{\mathcal{H}}(M, B)
 \end{array}$$

$$\begin{aligned}
 \text{Coz}\bar{\mathcal{H}}\phi \cdot \eta_L(a) &= \text{Coz}\bar{\mathcal{H}}\phi(\downarrow a \cap A) \\
 &= \downarrow \phi(a) \cap B \\
 &= \eta_M \cdot \phi(a)
 \end{aligned}$$

Let  $L$  be a completely regular frame, and let the counit  $\epsilon_L : \bar{\mathcal{H}}\text{Coz}L \longrightarrow L$  be the map defined by join. It is clear that  $\epsilon_L$  is a frame homomorphism. For naturality of  $\epsilon$ , let  $I \in \bar{\mathcal{H}}\text{Coz}L$ , and suppose  $\phi : L \rightarrow M$  is a frame homomorphism, then

$$\begin{array}{ccc}
\bar{\mathcal{H}}\text{Coz}L & \xrightarrow{\epsilon_L} & L \\
\bar{\mathcal{H}}\text{Coz}\phi \downarrow & & \downarrow \phi \\
\bar{\mathcal{H}}\text{Coz}M & \xrightarrow{\epsilon_M} & M
\end{array}$$

$$\begin{aligned}
\epsilon_M \cdot h_M \bar{\mathcal{H}}\text{Coz}\phi(I) &= \bigvee_M h_M \bar{\mathcal{H}}\text{Coz}\phi(I) \\
&= \phi(\bigvee_L I) \\
&= \phi \cdot \epsilon_L(I)
\end{aligned}$$

For adjointness it only remains to verify the following identities:

$\epsilon_{\bar{\mathcal{H}}(L,A)} \cdot \bar{\mathcal{H}}\eta_L = id_{\bar{\mathcal{H}}(L,A)}$ , and  $\text{Coz}\epsilon_M \cdot \eta_{\text{Coz}M} = id_{\text{Coz}M}$ . Let  $J \in \bar{\mathcal{H}}(L,A) = v_AL$ . Then  $\bar{\mathcal{H}}\eta_L(J) = \{I \in v_AL \mid I \subseteq J\}$ . Now,  $\bigvee_{v_AL} \{I \in v_AL \mid I \subseteq J\} = J$ , and therefore  $\epsilon_{\bar{\mathcal{H}}(L,A)} \cdot \bar{\mathcal{H}}\eta_L(J) = J$ .

Let  $M$  be a completely regular frame, and  $a \in \text{Coz}M$ . Then  $\eta_{\text{Coz}M}(a) = \{b \in \text{Coz}M \mid b \leq a\}$ . Now,  $\bigvee_L \{b \in \text{Coz}M \mid b \leq a\} = a$ , so  $\text{Coz}\epsilon_L \cdot \eta_{\text{Coz}M}(a) = a$ .

# Chapter 2

## Compactifications of frames

### 2.1 Introduction

In this chapter we discuss compactifications of frames and their relations with the realcompactifications given in Chapter 1. Banaschewski and Mulvey [1980], had shown that the frame  $CregIdlL$ , of all completely regular ideals of a completely regular frame  $L$  is its coreflection in the category  $\mathbf{KCregFrm}$  of compact regular frames. Thus,  $CregIdlL$  is the frame analogue of the Stone-Čech compactification of  $L$ . The frame  $CregIdlL$  is therefore denoted  $\beta L$ . Using a particular type of base, Johnstone [1984] constructed certain compactifications, which he referred to as Wallman compactifications, since they exhibit much the same characteristics as Wallman's [1938] compactifications for spaces. It turns out, as is to be expected, that the Stone-Čech compactification as given by Banaschewski and Mulvey, is in fact a Wallman compactification with respect to a particular base. All lattices discussed in this chapter are assumed to be bounded distributive lattices.

## 2.2 The Wallman Compactification of Frames

Banaschewski [1963] used certain families of closed sets, called Wallman bases, to construct Hausdorff compactifications of Tychonoff spaces. (See also Frink [1964]). By considering the frame analogue of such bases, and by forming suitable quotients of the frame of ideals on these bases, Johnstone [1984b] constructed the frame theoretic analogue of these Wallman compactifications. In this section we give a description of these compactifications, as well as some of their properties.

**Definition 2.2.1** *A lattice  $B$  is said to be conjunctive iff for each  $a$  and  $b$  in  $B$  with  $a \not\leq b$ , there exists  $c \in B$  such that  $c \vee a = 1_B$  and  $c \vee b \neq 1_B$ .*

Let  $B$  be a normal conjunctive lattice. Define  $s : IdlB \longrightarrow IdlB$  by

$$sI = \{a \in B \mid a \vee b = 1_B \Rightarrow b \vee c = 1_B, \text{ for some } c \in I\}$$

It was shown by Johnstone [1984a] that the map  $s$  given above is in fact a nucleus. We refer to this nucleus as the saturation nucleus, and ideals fixed under this nucleus are called saturated ideals.

The following proposition was proved by Johnstone [1984b].

**Proposition 2.2.2** *Let  $B$  be a normal conjunctive lattice, then the frame  $(IdlB)_s$ , of all saturated ideals is compact regular.*

**Proof** Towards showing that  $(IdlB)_s$  is compact, let  $\mathcal{I}$  be a collection of ideals in  $(IdlB)_s$ , with  $\bigvee_{(IdlB)_s} \mathcal{I} = 1_{IdlB}$ . Then  $s \bigvee_{IdlB} \mathcal{I} = 1_{IdlB}$ , ie  $1_B \in s \bigvee_{IdlB} \mathcal{I}$ . Hence, there exists  $c \in \bigvee_{IdlB} \mathcal{I}$  with  $c \vee 0_B = 1_B$ , from

which it follows that  $1_B \in \bigvee_{IdlB} \mathcal{I}$ , so  $\bigvee_{IdlB} \mathcal{I} = 1_{IdlB}$ . But  $IdlB$  is compact, so that  $\bigvee_{IdlB} \mathcal{J} = 1_{IdlB}$ , for some finite subset  $\mathcal{J}$  of  $\mathcal{I}$ . Now,  $\bigvee_{IdlB} \mathcal{J} \subseteq s \bigvee_{IdlB} \mathcal{J}$ , so  $\bigvee_{(IdlB)_s} \mathcal{J} = 1_{IdlB}$ .

For regularity, we firstly observe that  $\downarrow a \in (IdlB)_s$ , for each  $a \in B$ . To see this, suppose  $c \notin \downarrow a$ , ie  $c \not\leq a$ . Then, since  $B$  is conjunctive, there exists  $d \in B$  such that  $c \vee d = 1_B$ , and  $a \vee d \neq 1_B$ . But then  $c \notin s(\downarrow a)$ , so  $s(\downarrow a) \subseteq \downarrow a$ , ie  $\downarrow a$  is a saturated ideal. Secondly,  $\downarrow x \prec \downarrow a$  in  $(IdlB)_s$  iff  $x \prec a$  in  $B$ . Suppose  $\downarrow x \prec \downarrow a$  in  $(IdlB)_s$ . Then there exists  $I \in (IdlB)_s$  such that  $I \wedge \downarrow x = 0_{IdlB}$  and  $I \vee \downarrow a = 1_{IdlB}$ . Hence  $i \wedge x = 0_B$ , for each  $i \in I$ , and  $j \vee a = 1_B$ , for some  $j \in I$ . Thus,  $x \prec a$  in  $B$ . Conversely, suppose  $x \prec a$  in  $B$ . Then there exists  $j \in B$  such that  $x \wedge j = 0_B$  and  $a \vee j = 1_B$ . But then  $\downarrow x \wedge \downarrow j = 0_{IdlB}$  and  $\downarrow a \vee \downarrow j = 1_{IdlB}$ . And, since  $\downarrow j \in (IdlB)_s$ , it follows that  $\downarrow x \prec \downarrow a$ . So, it suffices to show that  $\downarrow a = \bigvee_{(IdlB)_s} \{\downarrow x \mid x \prec a\}$ , for each  $a \in B$ . Let  $a \in B$ , and let  $z \leq a$ . Suppose that  $z \vee b = 1_B$ . Then  $a \vee b = 1_B$ , and hence  $c \vee b = 1_B$ , for some  $c \prec a$ , since  $B$  is normal. But then  $z \in s\{x \mid x \prec a\} = s \bigvee_{IdlB} \{\downarrow x \mid x \prec a\}$ . Thus  $\downarrow a \subseteq \bigvee_{(IdlB)_s} \{\downarrow x \mid x \prec a\}$ . The other inclusion follows trivially from the fact that  $\bigvee_{IdlB} \{\downarrow x \mid x \prec a\} \subseteq \downarrow a$  and that  $\downarrow a$  is a saturated ideal.

**Definition 2.2.3** *Let  $L$  be a frame. A sublattice  $A$  of  $L$  is called a base of  $L$  iff for each  $a \in L$ ,  $a = \bigvee_L S_a$ , where  $S_a \subseteq A$ .*

By considering normal conjunctive bases of a completely regular frame,  $L$ , Johnstone's results are used to form compactifications of  $L$ .

**Proposition 2.2.4** *Let  $L$  be a completely regular frame, and let  $B$  be a normal conjunctive base of  $L$ . Let  $(IdlB)_s \xrightarrow{j_L} L$  be the map given by join. Then  $((IdlB)_s, j_L)$  is a compactification of  $L$ .*

**Proof** From Proposition 2.2.3 above,  $(IdlB)_s$  is a compact regular frame. Let  $a \in L$ . Then there exists a subset  $W$  of  $B$  such that  $\bigvee_L W = a$ . Now,  $\bigvee_{(IdlB)_s} \{\downarrow b \mid b \in W\} \in (IdlB)_s$ , and  $j_L(\bigvee_{(IdlB)_s} \{\downarrow b \mid b \in W\}) = \bigvee_L \bigvee_{(IdlB)_s} \{\downarrow b \mid b \in W\} = a$ . So,  $j_L$  is surjective. Suppose  $j_L I = 0_B$ , then  $I = \{0_B\} = 0_{IdlB}$ , so  $j_L$  is dense.

**Definition 2.2.5** *Let  $L$  be a completely regular frame, and  $B$  a normal conjunctive base of  $L$ , then the compactification  $((IdlB)_s, j_L)$  is called the Wallman compactification of  $L$  with respect to  $B$ . The frame  $(IdlB)_s$  is denoted  $\beta_B L$ .*

The following sequence of results culminating in Corollary 2.2.9, and the remark which follows Corollary 2.2.9 arose from suggestions made to me by Bernhard Banaschewski.

**Lemma 2.2.6** *Let  $L$  be a completely regular frame. Then  $\beta L \cong RegIdlCozL$ , the frame of all regular ideals of  $CozL$ .*

**Proof** Consider the map  $\phi : \beta L \longrightarrow RegIdlCozL$  defined by  $\phi(I) = I \cap CozL$ . Now,  $\phi$  is well-defined, since, if  $I$  is an ideal in  $L$  then clearly  $I \cap CozL$  is an ideal in  $CozL$ . Also,  $I \cap CozL$  is regular, since, if  $a \in I \cap CozL$ , then  $a \prec\prec b$ , for some  $b \in I$ . But then  $a \prec\prec x \prec\prec b$ , for some  $x \in CozL$ , so  $a \prec x$  for some  $x \in I \cap CozL$ . It is easily seen that  $\phi$  is a frame homomorphism. Suppose  $\phi(J) = 1_{RegIdlCozL}$ . Then  $J \cap CozL = CozL$ , from which

it follows that  $J = L$ . Thus,  $\phi$  is codense, and hence injective. Let  $K$  be a regular ideal in  $\text{Coz}L$ . Consider  $I = \{a \in L \mid a \leq b, \text{ for some } b \in K\}$ . Now,  $a \in I \Rightarrow a \leq b$ , for some  $b \in K$ . But  $K$  is regular, so  $b \prec c$ , for some  $c \in K$ , and since the relation  $\prec$  interpolates in  $\mathbf{Reg}\sigma\mathbf{Frm}$ , it follows that  $a \leq b \prec c$ , ie  $a \prec c$ , for some  $c \in K \subseteq I$ . Thus,  $I \in \beta L$ . Furthermore,  $\phi(I) = I \cap \text{Coz}L = \{a \in \text{Coz}L \mid a \leq b, \text{ for some } b \in K\} = K$ . Hence  $\phi$  is surjective, which completes the proof.

**Lemma 2.2.7** *Let  $B$  be a normal conjunctive lattice. Then there is a one-to-one correspondence between the saturated ideals on  $B$  and the regular ideals on  $B$ .*

**Proof** Firstly, suppose  $I$  is a saturated ideal. Then, since  $(\text{Idl}B)_s$  is regular,

$$\begin{aligned} I &= \bigvee_{(\text{Idl}B)_s} \{J \in (\text{Idl}B)_s \mid J \prec I\} \\ &= s \bigvee_{\text{Idl}B} \{J \in (\text{Idl}B)_s \mid J \prec I\} \end{aligned}$$

But, an ideal is regular iff it can be written as a join (in  $\text{Idl}B$ ) of ideals rather below it. Thus,  $\bigvee_{\text{Idl}B} \{J \in (\text{Idl}B)_s \mid J \prec I\}$  is a regular ideal. Hence, every saturated ideal is the saturation of a regular ideal.

On the other hand, let  $J$  and  $K$  be distinct regular ideals on  $B$ . Suppose, without loss of generality, that  $J \not\subseteq K$ . Choose  $a \in J \setminus K$ . Then  $a \prec b$ , for some  $b \in J$ . So, there exists  $d \in B$  such that  $b \vee d = 1_B$ , and  $a \wedge d = 0_B$ . We claim that  $b \notin sK$ . For, suppose  $b \in sK$ , then there exists  $k \in K$  such that  $k \vee d = 1_B$ . Then  $a \prec k$  and thus  $a \in K$ .

**Corollary 2.2.8** *For any normal conjunctive lattice  $B$ ,  $(\text{Idl}B)_s \cong \text{RegIdl}B$ , the frame of all regular ideals in  $B$ .*

**Proof** We show that the map  $\bar{s} : \text{RegIdl}B \rightarrow (\text{Idl}B)_s$ , defined by  $\bar{s}(I) = sI$ , is a frame isomorphism. From Lemma 2.2.7 above,  $\bar{s}$  is bijective, so it suffices to show that  $\bar{s}$  preserves arbitrary joins. Let  $\{I_\alpha \mid \alpha \in A\}$  be a collection of regular ideals in  $B$ . Then,

$$a \in s \bigvee_{\text{Idl}B} sI_\alpha \iff a \vee b = 1_B \Rightarrow b \vee c = 1_B, \text{ for some } c \in \bigvee_{\text{Idl}B} sI_\alpha$$

Now,  $c \in \bigvee_{\text{Idl}B} sI_\alpha$  iff  $c = c_{\alpha_1} \vee c_{\alpha_2} \vee \cdots \vee c_{\alpha_n}$ , where  $c_{\alpha_i} \in sI_{\alpha_i}$ , for each  $i \in \{1, 2, \dots, n\}$ . Thus,  $b \vee c = 1_B$  iff  $b \vee c_{\alpha_1} \vee \cdots \vee c_{\alpha_n} = 1_B$ , where  $c_{\alpha_i} \in sI_{\alpha_i}$ , for each  $i \in \{1, 2, \dots, n\}$ . But then  $b \vee c_{\alpha_1} \vee \cdots \vee c_{\alpha_{n-1}} \vee d_{\alpha_n} = 1_B$ , for some  $d_{\alpha_n} \in I_{\alpha_n}$ . Continuing in this way, we obtain,  $b \vee d_{\alpha_1} \vee \cdots \vee d_{\alpha_n} = 1_B$ , where  $d_{\alpha_i} \in I_{\alpha_i}$ , for each  $i \in \{1, 2, \dots, n\}$ . Thus,  $b \vee c = 1_B$ , for some  $c \in \bigvee_{\text{Idl}B} sI_\alpha$  iff  $b \vee d = 1_B$ , for some  $d \in \bigvee_{\text{Idl}B} I_\alpha$ . Consequently,

$$\begin{aligned} a \in s \bigvee_{\text{Idl}B} sI_\alpha &\iff a \vee b = 1_B \Rightarrow b \vee d = 1_B, \text{ for some } c \in \bigvee_{\text{Idl}B} I_\alpha \\ &\iff a \in s \bigvee_{\text{Idl}B} I_\alpha \end{aligned}$$

Thus,  $s \bigvee_{\text{Idl}B} sI_\alpha = s \bigvee_{\text{Idl}B} I_\alpha = s \bigvee_{\text{RegIdl}B} I_\alpha$ , and hence  $\bar{s}(\bigvee_{\text{RegIdl}B} I_\alpha) = \bigvee_{(\text{Idl}B)_s} \bar{s}(I_\alpha)$ .

**Corollary 2.2.9** *Let  $L$  be a completely regular frame. Then  $\beta L \cong (\text{Idl}CozL)_s$ .*

**Proof** It suffices to show that  $CozL$  is a normal conjunctive base for  $L$ , and the result would follow from Lemma 2.2.6 and Corollary 2.2.8. All regular  $\sigma$ -frames are normal, and thus  $CozL$  is normal. Suppose  $a \not\leq b$ . Then, since  $CozL$  is regular, there exists  $c \in CozL$  with  $c \prec a$  and  $c \not\leq b$ . Thus, there exists  $s \in CozL$  such that  $c \wedge s = 0_L$  and  $a \vee s = 1_L$ . Now,  $b \vee s \neq 1_L$ ,

since

$$\begin{aligned}
b \vee s = 1_L &\Rightarrow (b \vee s) \wedge c = c \\
&\Rightarrow (b \wedge c) \vee (s \wedge c) = c \\
&\Rightarrow b \wedge c = c \\
&\Rightarrow c \leq b
\end{aligned}$$

which contradicts the fact that  $c \not\leq b$ . Hence,  $CozL$  is conjunctive, since  $a \not\leq b \Rightarrow \exists s \in CozL$  such that  $a \vee s = 1_L$  and  $b \vee s \neq 1_L$ .

**Remark** The above corollary is really a generalisation of Lemma 1.2.3. Since  $(IdlCozL)_s$  is completely regular,  $\Sigma(IdlCozL)_s$  is the space consisting of all the maximal saturated ideals in  $CozL$ . But the maximal saturated ideals in  $CozL$  are precisely the maximal ideals in  $CozL$ . To see this, let  $P$  be a maximal saturated ideal in  $CozL$ , and let  $J$  be any ideal in  $CozL$  properly containing  $P$ . Then  $sJ \supset P$ , ie  $sJ = 1_{(IdlCozL)_s}$ . But the nucleus  $s$  is codense, thus  $J = 1_{(IdlCozL)_s}$ . On the other hand, all maximal ideals in  $CozL$  are saturated (since  $J \subseteq sJ$ , and  $s$  is codense). Thus,  $\Sigma(IdlCozL)_s = MaxCozL$ . But  $\Sigma\beta L = Max_c L$ , so the result follows.

Gilmour [1981] showed that the Wallman realcompactifications of a topological space  $X$  can be obtained via the  $\sigma$ -prime spectra of Alexandroff bases of  $X$ . In contrast to this, Wallman compactifications of a space  $X$  are obtained via the minimal prime filters of normal conjunctive bases of  $\mathcal{O}X$ . These minimal prime filters are discussed in Johnstone [1980].

A prime filter  $P$  on a lattice  $B$  is a filter satisfying the following condition: If  $S$  is a finite subset of  $B$  with  $\bigvee_B S \in P$ , then  $S \cap P \neq \emptyset$ .

Consider a normal lattice  $B$ . Let  $MinB$  be the topological space consisting of all minimal prime filters on  $B$ , and basic open sets of the form  $P_a = \{P \in MinB \mid a \in P\}$  for  $a \in B$ . It is easily seen that  $P_a \cap P_b = P_{a \wedge b}$ ,  $P_a \cup P_b = P_{a \vee b}$ ,  $P_{1_B} = MinB$ , and  $P_{0_B} = \emptyset$ .

**Proposition 2.2.10** *Let  $B$  be a normal lattice, then  $MinB \cong \Sigma(IdlB)_s$ .*

**Proof** Since  $B$  is normal  $(IdlB)_s$  is a completely regular frame, and hence  $\Sigma(IdlB)_s = MaxB$ . Every maximal ideal in  $B$  is prime, and hence  $B \setminus I$  is a minimal prime filter. On the other hand, if  $F$  is a minimal prime filter, then  $B \setminus F$  is a maximal (prime) ideal. Thus, the map  $\phi: MaxB \rightarrow MinB$ , defined by  $\phi(I) = B \setminus I$  is a bijective map. Let  $a \in B$ , then  $\phi^{-1}(P_a) = \{I \in MaxB \mid a \notin I\}$ , and  $\phi(\{I \in MaxB \mid a \notin I\}) = \{F \in MinB \mid a \in F\} = P_a$ . Thus,  $MinB \cong \Sigma(IdlB)_s$ .

**Remark** From the above result, it follows that, for any normal distributive lattice  $B$ , the space  $MinB$  is compact Hausdorff, and can be used to form the Wallman compactification for spaces. This is contrasted with the coherent space  $\Sigma IdlB$ , which is homeomorphic to  $\Pi B$ , the topological space, consisting of all prime filters, and basic open sets of the form  $\Pi_a = \{P \in \Pi B \mid a \in P\}$ . (See Johnstone [1980]).

**Corollary 2.2.11** *Let  $X$  be a completely regular space, then  $MinCozX \cong \Sigma(IdlCozX)_s \cong \beta X$ .*

## 2.3 Compact Sigma Frames

In this section we construct the Wallman compactification for  $\sigma$ -frames. This construction follows that of Johnstone for the case of frames. As in the case of frames, quotients of  $\sigma$ -frames can be formed via nuclei. Given a nucleus  $n$  on a  $\sigma$ -frame  $A$ , the quotient of  $A$  with respect to  $n$  is the  $\sigma$ -frame  $\text{Fix } n = \{a \in A \mid n(a) = a\}$ , and is denoted by  $(A)_n$ .

**Definition 2.3.1** *A  $\sigma$ -frame  $A$  is said to be compact iff whenever there is a countable subset  $S \subseteq A$  with  $\bigvee_A S = 1_A$ , then  $\bigvee_A T = 1_A$ , for some finite subset  $T \subseteq S$ .  $(B, f)$  is a compactification of  $A$  iff  $B$  is a compact  $\sigma$ -frame and  $A \xrightarrow{f} B$  is a dense surjection.*

**Definition 2.3.2** *The full subcategory of  $\text{Reg}\sigma\text{Frm}$  consisting of all compact regular  $\sigma$ -frames is denoted  $\mathbf{KReg}\sigma\text{Frm}$ .*

**Proposition 2.3.3** *The functors  $\mathcal{H}$  and  $\text{Coz}$  both preserve compactness.*

**Proof** It is clear that  $\text{Coz}L$  is compact for any compact frame  $L$ . On the other hand, suppose  $A$  is a compact regular  $\sigma$ -frame. Let  $S = \{J_\lambda \mid \lambda \in \Lambda\}$  be a collection of  $\sigma$ -ideals in  $A$ , with  $\bigvee_{\mathcal{H}A} S = 1_{\mathcal{H}A}$ . Then there is a countable subset  $T$  of  $\bigcup S$ , with  $\bigvee_A T = 1_A$ . Since  $A$  is compact,  $1_A$  can be written as a join of finitely many elements of  $T$ , ie  $a_1 \vee \cdots \vee a_n = 1_A$ , with  $a_i \in J_{\alpha_i}$  for  $i \in \{1, \dots, n\}$ . Then  $A = J_{\alpha_1} \vee \cdots \vee J_{\alpha_n}$ , and hence  $\mathcal{H}A$  is a compact frame.

**Definition 2.3.4** Let  $B$  be any lattice. An ideal  $I \in \text{Idl}B$  is said to be countably generated if  $I$  is the join of countably many principal ideals on  $B$ . The collection of all countably generated ideals on  $B$  is denoted  $\text{Idl}_\sigma B$ .

**Proposition 2.3.5** Let  $B$  be a normal conjunctive lattice. Then  $\text{Coz}(\text{Idl}B)_s = (\text{Idl}_\sigma B)_s$ .

**Proof** Suppose  $I \in \text{Coz}(\text{Idl}B)_s$ . Then there exists a sequence  $(J_n)$  in  $(\text{Idl}B)_s$  with  $J_0 \prec\prec J_1 \prec\prec J_2 \prec\prec \dots$ , and  $I = \bigvee_{(\text{Idl}B)_s} J_i$ . For each  $n \in \mathbb{N}$ ,  $J_n \prec I$ . Thus, for each  $n \in \mathbb{N}$ , there exists  $S_n \in (\text{Idl}B)_s$  such that  $J_n \wedge S_n = 0_{(\text{Idl}B)_s}$ , and  $I \vee S_n = 1_{(\text{Idl}B)_s}$ . For each  $n \in \mathbb{N}$ , choose  $s_n \in S_n$ . Then  $j_{n_i} \wedge s_n = 0_B$  for each  $j_{n_i} \in J_n$ , and  $k_n \vee s_n = 1_B$ , for some  $k_n \in I$ . Thus,  $j_{n_i} \prec k_n$ , for each  $j_{n_i} \in J_n$ . Hence,  $J_n \subseteq \downarrow k_n$  for each  $n \in \mathbb{N}$ . Consequently,  $I = \bigvee_{(\text{Idl}B)_s} J_i \subseteq \bigvee_{(\text{Idl}B)_s} \{\downarrow k_n \mid n \in \mathbb{N}\}$ . On the other hand,  $\downarrow k_n \subseteq I$ , for each  $n \in \mathbb{N}$ , and hence  $\bigvee_{(\text{Idl}B)_s} \{\downarrow k_n \mid n \in \mathbb{N}\} \subseteq I$ . Thus,  $I = \bigvee_{(\text{Idl}B)_s} \{\downarrow k_n \mid n \in \mathbb{N}\}$ , so  $I$  is a countably generated saturated ideal, ie  $I \in (\text{Idl}_\sigma B)_s$ . Conversely, suppose  $I \in (\text{Idl}_\sigma B)_s$ . Since  $(\text{Idl}B)_s$  is completely regular,  $I$  is a join of saturated ideals  $I_\alpha$  where each  $I_\alpha$  is completely below  $I$ . Now,  $I$  is countably generated, so only countably many of the  $I_\alpha$  will do. Since for each  $I_\alpha$  there is a  $J_\alpha \in \text{Coz}(\text{Idl}B)_s$  with  $I_\alpha \prec\prec J_\alpha \prec\prec I$ . Thus,  $I$  is a join of countably many elements of  $\text{Coz}(\text{Idl}B)_s$ , so  $I \in \text{Coz}(\text{Idl}B)_s$ .

**Proposition 2.3.6** Let  $A$  be a regular  $\sigma$ -frame, and let  $B$  be a normal conjunctive base for  $A$ . Then  $((\text{Idl}_\sigma B)_s, j_A)$  is a compactification of  $A$ , where the map  $(\text{Idl}_\sigma B)_s \xrightarrow{j_A} A$  is given by join.

**Proof** From Proposition 2.3.5 above,  $(Idl_\sigma B)_s = Coz(IdlB)_s$ , and is hence a compact regular  $\sigma$ -frame. Suppose  $a \in A$ . Then there is a countable subset  $W$  of  $B$  such that  $\bigvee_A W = a$ . Thus,  $\bigvee_{(IdlB)_s} \{\downarrow b \mid b \in W\}$  is a countably generated saturated ideal, and  $j_A(\bigvee_{(IdlB)_s} \{\downarrow b \mid b \in W\}) = \bigvee_A \bigvee_{(IdlB)_s} \{\downarrow b \mid b \in W\} = a$ . Suppose  $j_A I = 0_B$ , then  $I = \{0_B\} = 0_{IdlB}$ . Hence,  $j_A$  is a dense surjection.

**Definition 2.3.7** Let  $A$  be a regular  $\sigma$ -frame, and  $B$  a normal conjunctive base for  $A$ , then  $((Idl_\sigma B)_s, j_A)$  is called a Wallman compactification of  $A$ . The frame  $(Idl_\sigma B)_s$  is denoted  $\beta_B A$ .

As a corollary we obtain a result due to Walters [1990].

**Corollary 2.3.8** Let  $L$  be a completely regular frame. Then  $Coz\beta L \cong \beta_{CozL} CozL$ .

**Proof**

$$\begin{aligned} \beta_{CozL} CozL &= (Idl_\sigma CozL)_s \\ &= Coz(Idl CozL)_s \text{ by 2.3.5} \\ &= Coz\beta L \text{ by 2.2.9} \end{aligned}$$

**Proposition 2.3.9** Every regular  $\sigma$ -frame  $A$  is conjunctive.

**Proof** Let  $a, b \in A$  with  $a \not\leq b$ . Then there exists  $c \in A$  such that  $c \prec a$  and  $c \not\leq b$ . Since  $c \prec a$ , there exists  $s \in A$  such that  $s \wedge c = 0_A$  and

$s \vee a 1_A$ . We claim that  $s \vee b \neq 1_A$ , for if  $s \vee b = 1_A$ , then  $b = (s \wedge c) \vee b = (s \vee b) \wedge (c \vee b) = c \vee b$ , which contradicts the fact that  $c \not\leq b$ .

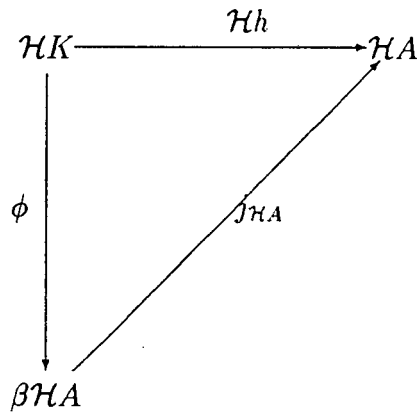
**Lemma 2.3.10** *Let  $A$  be a regular  $\sigma$ -frame, then  $\beta\mathcal{H}A \cong \mathcal{H}\beta_A A$ .*

**Proof**

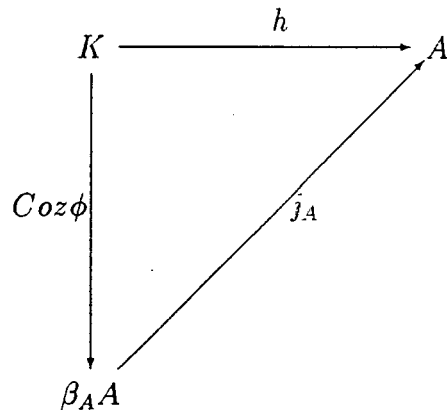
$$\begin{aligned}
\beta\mathcal{H}A &= (\text{Idl}Coz\mathcal{H}A)_s \text{ by 2.2.9} \\
&\cong (\text{Idl}A)_s \\
&\cong \mathcal{H}Coz(\text{Idl}A)_s \text{ since } (\text{Idl}A)_s \text{ is compact} \\
&\cong \mathcal{H}(\text{Idl}_\sigma A)_s \text{ by 2.3.5} \\
&= \mathcal{H}\beta_A A
\end{aligned}$$

**Proposition 2.3.11** *Let  $A$  be a regular  $\sigma$ -frame. Then the map  $\beta_A A \xrightarrow{j_A} A$  is universal with respect to maps from compact regular  $\sigma$ -frames to  $A$ .*

**Proof** Let  $K$  be a compact regular  $\sigma$ -frame, and suppose  $K \xrightarrow{h} A$  is a  $\sigma$ -frame homomorphism. Then  $\mathcal{H}K \xrightarrow{\mathcal{H}h} \mathcal{H}A$  is a frame homomorphism, and  $\mathcal{H}K$  is compact, by Proposition 2.3.3 Thus, there exists a map  $\phi : \mathcal{H}K \rightarrow \beta\mathcal{H}A$  such that  $j_{\mathcal{H}A} \cdot \phi = \mathcal{H}h$ , where  $j_{\mathcal{H}A}\mathcal{I} = \bigvee_{\mathcal{H}A}\mathcal{I}$ .



Applying the functor  $Coz$  to the above diagram, and using the fact that  $\beta\mathcal{H}A \cong \mathcal{H}\beta_A A$ , we obtain the following commutative triangle:



Uniqueness of the map  $Coz\phi : K \rightarrow \beta_A A$  follows from the fact that  $j_A$  is dense, and hence monic.

**Remark** The compactification  $(\beta_A A, j_A)$  shall be called the Stone-Čech compactification for regular  $\sigma$ -frames. We denote  $\beta_A A$  simply by  $\beta A$ . From

Proposition 2.3.11 above,  $\beta A$  is an alternative construction of the universal compactification, as the  $\sigma$ -frame of all countably generated regular ideals of  $A$ , given by Banaschewski and Gilmour [1989].

## 2.4 Pseudocompact Frames

It is well known that a pseudocompact space is realcompact iff it is compact. We investigate the relationship between pseudocompactness, realcompactness and compactness in frames, as well as the Wallman compactification given by Johnstone, and the Wallman realcompactification given above.

**Definition 2.4.1** *A sequence  $(a_n)$  in a frame  $L$  is said to be completely regular iff  $a_1 \prec\prec a_2 \prec\prec a_3 \prec\prec \dots$ . We say  $(a_n)$  is regular iff  $a_1 \prec a_2 \prec a_3 \dots$ .*

**Definition 2.4.2** *A completely regular frame  $L$  is called pseudocompact iff every completely regular sequence  $(a_n)$  in  $L$  with  $\bigvee_L a_n = 1_L$  is eventually constant; that is,  $a_k = 1_L$ , for some  $k \in \mathbf{N}$ .*

**Remark** Pseudocompactness is usually defined in the following way:

A frame  $L$  is pseudocompact iff every frame homomorphism  $\phi : \mathcal{O}\mathbf{R} \rightarrow L$  is bounded, ie  $\phi((-\infty, -a) \vee (a, \infty)) = 0_L$ , for some  $n \in \mathbf{N}$ . Using the methods of Urysohn's lemma, Gilmour has shown that this is equivalent to saying that every completely regular sequence  $(a_n)$ , with  $\bigvee_L a_n = 1_L$ , is eventually constant. The latter description was used by Baboolal and Banaschewski [1991] as a definition of pseudocompactness, since it eliminates reference to

the reals, and also because it seems a much more natural definition in the setting of frames.

**Lemma 2.4.3** *A completely regular frame  $L$  is pseudocompact iff every regular sequence  $(a_n)$  in  $\text{Coz}L$  with  $\bigvee_L a_n = 1_L$  is eventually constant.*

**Proof** Every regular sequence in  $\text{Coz}L$  is a completely regular sequence in  $L$ , so the forward implication is trivial. Conversely, let  $(a_n)$  be a completely regular sequence in  $L$ . Then for each  $i \in \mathbb{N}$ , there exists  $b_i \in \text{Coz}L$  with  $a_i \prec\prec b_i \prec\prec a_{i+1}$ . But then  $(b_n)$  is a regular sequence in  $\text{Coz}L$ , and is therefore eventually constant. Consequently,  $(a_n)$  is eventually constant.

**Lemma 2.4.4** *A completely regular frame  $L$  is pseudocompact iff every countably generated regular ideal in  $\text{Coz}L$  is completely proper.*

**Proof** Suppose  $L$  is pseudocompact. Let  $I$  be a countable generated regular ideal in  $\text{Coz}L$ . Then  $I$  is generated by some regular sequence  $(a_n)$  (Banaschewski and Gilmour [1989]). But then  $I$  is completely proper, since otherwise  $\bigvee_L a_n = 1_L$ , from which  $a_k = 1_L$ , for some  $k \in \mathbb{N}$ , which would contradict the fact that  $I$  is a proper ideal.

Conversely, suppose  $L$  is not pseudocompact. Then there exists a regular sequence  $(a_n)$  in  $\text{Coz}L$  with  $\bigvee_L a_n = 1_L$ , and  $a_j \neq 1_L$ , for any  $j \in \mathbb{N}$ . Let  $I$  be generated by  $(a_n)$ . Then  $I$  is a countably generated regular ideal in  $\text{Coz}L$ , but  $I$  is not completely proper.

As a corollary, we obtain the following well-known result, which is due to Gilmour [1981].

**Corollary 2.4.5** *A completely regular frame  $L$  is pseudocompact iff  $\text{Coz}L$  is a compact  $\sigma$ -frame.*

**Proof** Recall that  $\text{RegIdl}_\sigma \text{Coz}L = \beta \text{Coz}L \cong \text{Coz}\beta L$ . Now, every ideal  $I \in \text{RegIdl}_\sigma \text{Coz}L$  is completely proper iff the map  $\beta \text{Coz}L \xrightarrow{j_{\text{Coz}L}} \text{Coz}L$  given by join is codense, and hence injective. But  $\beta \text{Coz}L \xrightarrow{j_{\text{Coz}L}} \text{Coz}L$  is surjective. Thus,  $L$  is pseudocompact iff  $\beta \text{Coz}L \cong \text{Coz}L$ , ie iff  $\text{Coz}L$  is a compact  $\sigma$ -frame.

**Lemma 2.4.6** *A completely regular frame  $L$  is pseudocompact iff every maximal ideal in  $\text{Coz}L$  is  $\sigma$ -proper.*

**Proof** Suppose  $L$  is pseudocompact. Let  $I$  be a maximal ideal in  $\text{Coz}L$  which is not  $\sigma$ -proper. But then there is a finite subset  $S \subseteq I$  with  $\bigvee_L S = 1_L$ , which contradicts the fact that  $L$  is proper.

Conversely, suppose  $L$  is not pseudocompact. Then there exist a countable subset  $S \subseteq \text{Coz}L$ , with  $\bigvee_L S = 1_L$ , and  $\bigvee_L F \neq 1_L$ , for any finite subset  $F \subseteq S$ . Thus, the ideal  $I$  generated by  $S$  is proper. Let  $J$  be a maximal ideal containing  $I$ , then  $S \subseteq J$ , so  $J$  is not  $\sigma$ -proper.

**Corollary 2.4.7** *Let  $L$  be pseudocompact. Then  $L$  is realcompact iff  $L$  is compact.*

**Proof** Every compact frame is Lindelöf, and thus, by Corollary 1.2.7, every compact frame is realcompact.

Conversely, suppose  $L$  is realcompact. Then every  $\sigma$ -proper maximal ideal in  $\text{Coz}L$  is completely proper. But by Lemma 2.4.6 above, every maximal ideal in  $\text{Coz}L$  is  $\sigma$ -proper. Thus,  $L$  is realcompact iff every maximal ideal

in  $\text{Coz}L$  is completely proper. But this means precisely that  $L$  is compact. Thus,  $L$  is realcompact iff  $L$  is compact.

**Proposition 2.4.8** *Let  $L$  be a completely regular frame, and let  $A$  be a compact regular sub  $\sigma$ -frame join generating  $L$ . Then  $\beta_A L \cong v_A L$ .*

**Proof** Since  $\text{Coz}v_A L \cong A$ , it follows that  $v_A L$  is pseudocompact. But  $v_A L$  is realcompact, and so by Lemma 2.4.7 above,  $v_A L$  is compact. Note that  $\beta v_A L = (\text{IdlCoz}v_A L)_s \cong (\text{Idl}A)_s = \beta_A L$ . Thus  $v_A L \cong \beta v_A L \cong \beta_A L$ .

**Corollary 2.4.9** *Let  $L$  be a pseudocompact frame, and let  $A$  be a regular sub  $\sigma$ -frame join generating  $L$ . Then  $v_A L \cong \beta_A L$ .*

**Proof** Since  $L$  is pseudocompact, by Corollary 2.4.5, every regular sub  $\sigma$ -frame of  $L$  is compact. The result follows from Proposition 2.4.8 above.

**Corollary 2.4.10** *A completely regular frame  $L$  is pseudocompact iff  $vL \cong \beta L$ .*

**Poof** The forward implication follows from Corollary 2.4.9 above. For the reverse implication, let  $vL \cong \beta L$ . Then  $\text{Coz}vL \cong \text{Coz}\beta L$ . But  $\text{Coz}vL \cong \text{Coz}L$ , and  $\text{Coz}\beta L$  is compact, so  $\text{Coz}L$  is compact, and therefore  $L$  is pseudocompact.

## 2.5 Compactifications of Alexandroff Frames

Quotient maps in the category of Alexandroff frames are maps of the form  $(L, A) \xrightarrow{f} (M, B)$ , where both  $L \xrightarrow{f} M$  and  $A \xrightarrow{f_A} B$  are surjective maps.

**Definition 2.5.1** *An Alexandroff frame  $(L, A)$  is compact iff  $L$  is a compact frame.  $((M, B), h)$  is a compactification of  $(L, A)$  iff  $(M, B)$  is a compact Alexandroff frame, and  $(L, A) \xrightarrow{h} (M, B)$  is a dense surjection which restricts to a surjective map on  $A$ .*

**Definition 2.5.2** *The full subcategory of  $\mathbf{AlexFrm}$  consisting of all compact Alexandroff frames is denoted  $\mathbf{KAlexFrm}$ .*

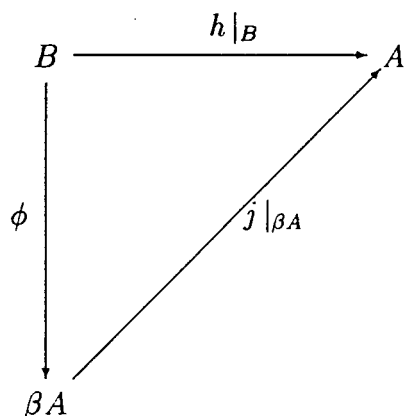
**Remark** It is easily seen that  $((M, B), h)$  is a compactification of  $(L, A)$  iff  $(M, B)$  is a compact Alexandroff frame, and  $(B, h|_B)$  is a compactification of  $A$ . Also, from Propositions 1.4.4 and 2.3.4, compact  $\sigma$ -frames admit unique compact frames. Thus, all compactifications of an Alexandroff frame  $(L, A)$  arise from compactifications of the  $\sigma$ -frame  $A$ .

Wallman compactifications of an Alexandroff frame  $(L, A)$  are formed by considering normal conjunctive bases of  $A$ . Let  $B$  be a normal conjunctive base for  $A$ , then it is easily seen that  $(\beta_B L, \beta_B A)$  is a compactification of  $(L, A)$ .

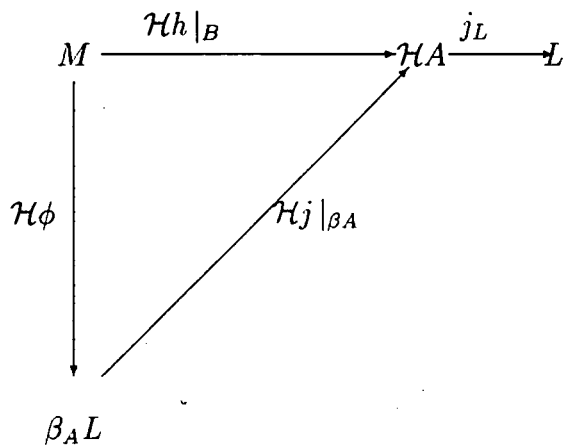
**Proposition 2.5.3** *Let  $(L, A)$  be an Alexandroff frame. Then the map  $(\beta_A L, \beta_A A) \xrightarrow{j} (L, A)$ , given by join is universal as a map from  $\mathbf{KAlexFrm}$  to  $L$ .*

**Proof** Let  $(M, B)$  be a compact Alexandroff frame, and suppose  $(M, B) \xrightarrow{h} (L, A)$  is an Alexandroff frame homomorphism. Then there exists a  $\sigma$ -frame

homomorphism  $\phi$  such that the following triangle commutes:



Applying the functor  $\mathcal{H}$ , and using the fact that  $M = \mathcal{H}B$ , we obtain:  
 $j \cdot \mathcal{H}\phi = h$ .



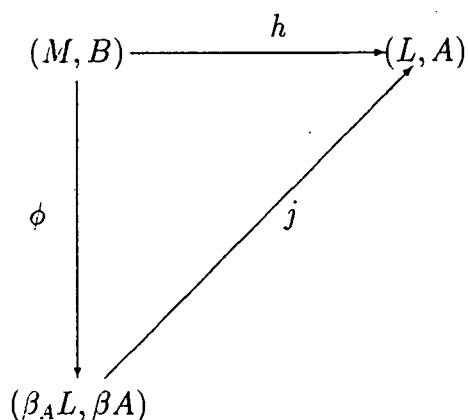
Let  $\mathcal{H}A \xrightarrow{j_L} L$  be the map given by join, then, for  $m \in M$ ,  $j_L \cdot \mathcal{H}h(m) = \bigvee_L \{a \in A \mid a \leq h(m)\} = h(m)$ , since  $A$  join generates  $L$ . Similarly, for  $I \in \beta_A L$ ,  $j_L \cdot \mathcal{H}j|_{\beta_A}(I) = j(I)$ . Thus,  $(M, B) \xrightarrow{\mathcal{H}\phi} (L, A)$  is an Alexandroff frame homomorphism with  $j \cdot \mathcal{H}\phi = h$ .

**Definition 2.5.4** An Alexandroff frame  $(L, A)$  is said to be pseudocompact iff  $A$  is a compact  $\sigma$ -frame.

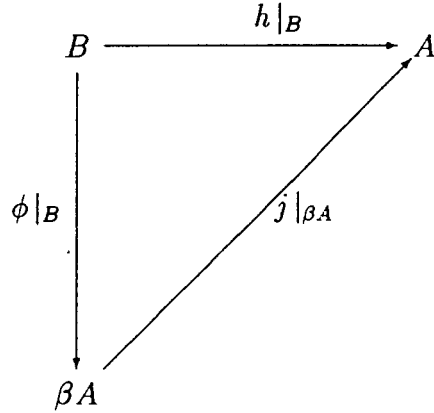
In contrast with the situation for topological spaces, we obtain the following result for Alexandroff frames.

**Proposition 2.5.5** Let  $(L, A)$  be an Alexandroff frame. If  $(L, A)$  is pseudocompact then it admits a unique compactification.

**Proof** Suppose  $(L, A)$  is pseudocompact. Let  $((M, B), h)$  be a compactification of  $(L, A)$ . Then the following triangle commutes:



where  $j$  is the map given by join. Since these are Alexandroff frame homomorphisms, we obtain the following commutative diagram:



Since  $(L, A)$  is pseudocompact,  $A \cong \beta A$ , ie  $j|_{\beta A}$  is an isomorphism. Both  $h|_B$  and  $j|_{\beta A}$  are dense maps, so it follows that  $\phi|_B$  is also dense. Now,  $\beta A$  is compact, and therefore by Lemma 0.3.3,  $\phi|_B$  is injective. But then  $j|_{\beta A} \cdot \phi|_B = h|_B$  is bijective, and thus  $A \cong \beta A \cong B$ . Thus,  $(\beta_A L, \beta A) \cong (M, B)$ .

Gordon [1971] showed that a zero-set space is pseudocompact iff it admits a unique compactification. Pseudocompact topological spaces do not generally admit unique compactifications. The topological spaces with this property are called almost compact spaces (terminology of Gillman and Jerison [1960]). We have been unable to characterise those frames which admit unique compactifications.

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