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UNIFORM SIGMA FRAMES

AND

THE COZERO PART OF UNIFORM FRAMES

by

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CONTENTS

SYNOPSIS

The study of topological concepts from a lattice theoretical viewpoint was first initiated by Wallman (1938) ; inspired by the work of Stone (1937). The actual term 'frame' was introduced by Dowker and Papert in the middle sixties to describe a 'local lattice', a concept which was studied in the seminar of Ehresmann in Paris, 1959, attended by Seymour and Dona Papert (Strauss). A local lattice is defined to be a complete lattice with the appropriate distributivity property (ie. finite meets distribute over arbitrary joins) and so corresponds to the notion of a generalized topological space. These concepts were pursued by Benabou (1957), Banaschewski (1969), Isbell (1972) and others. Johnstone's book 'Stone Spaces' [1982] gives a detailed background on the influence of Stone's work on mathematics in general, and is also a basic reference for the theory of frames (or locales). The 'Compendium of Continuous Lattices' [1980] is a useful reference for many standard lattice theoretical results.

The generalizations of frames to those lattices closed under, at most, countable joins and satisfying the property that finite meets distribute over countable joins, namely sigma frames, was considered by Charalambous [1974] and has been explored in detail by Reynolds [1979] and Banaschewski [1980]. Gilmour [1985] shows that there exists a dual adjunction between the regular sigma frames (a full subcategory of sigma frames) and the Alexandroff (or zero set) spaces. Madden and Vermeer [1985] obtain a countable version of the Stone duality for regular sigma frames and regular Lindelöf frames.

Isbell [1972] first considered the notion of a 'uniformity' on a frame by means of covers. This was further explored by Pultr [1975] who also defined metric diameters, the analogs of pseudometrics in the spatial setting. Pultr shows that a frame is uniformizable if and only if it is completely regular. Frith [1987] further develops results on uniform frames from a covering perspective, obtaining the same characterization as Pultr and an adjoint situation between this category and the category of uniform spaces using a structured version of the spectrum and 'open' functors.

It is this characterization of completely regular frames, together with the result of Reynolds [1979] that the completely regular frames are those frames which are 'generated' by regular sigma frames, which gave additional motivation for considering the concept of a uniformity on a sigma frame. The idea arises quite naturally as the generalisation of a uniformity on an Alexandroff space.

In this thesis some general results on uniform frames are established and then, after defining a 'uniform sigma frame', the correspondence between the two is explored via the 'uniform cozero part' of a uniform frame. It is shown that the Lindelöf uniform frames and the uniform sigma frames are in fact equivalent as categories, and that properties of, and constructions using separable uniform frames can be obtained by considering the uniform cozero part. For example, the Samuel compactification of a separable uniform frame can be obtained via the Samuel compactification (in the sigma frame sense) of the underlying cozero part of the uniform frame. Throughout the thesis, choice principles such as the axioms of choice and countably dependent choice, are used, and generally without mention.

An outline of the thesis:

Chapter 0 : This chapter gives some basic definitions and results on frames and sigma frames which will be used in subsequent chapters. Proofs of the results will usually be omitted. Most of these appear in Johnstone's book 'Stone Spaces' but other references are given for proofs which use techniques more appropriate in this context.

Chapter 1 : The relationship between completely regular frames and regular sigma frames, as explored by Banaschewski [1980b], is dicussed and the notion of continuous and coherent σ–frames, the latter being a non–full subcategory of the former is introduced. Regular continuous σ–frames are shown to be 'Alexandroff' σ–frames (ie. fixed objects of the adjunction between Alexandroff spaces and regular sigma frames) which generalizes the result of Banaschewski [1980b]. The full subcategory consisting of Stone σ–frames is also introduced.

Chapter 2 : Basic definitions of uniform frames (and the equivalence of the different definitions given by Pultr [1884 I] and Frith [1986]) are given together with the description of some structures and coreflections in the category of uniform frames. This category is shown to be co–complete and to have products. The main result in this section shows that the uniform cozero part of a uniform frame is a regular σ–frame which 'generates' the underlying frame. Moreover the uniform cozero elements are characterized in such a way that (as shown in the next chapter) it is possible to consider the uniformity restricted to the regular σ–frame of its uniform cozero part. This is accomplished once it is shown that every uniformity has a basis of uniform cozero covers (that is, uniform covers consisting of uniform cozero elements). The simple proof of this latter result was pointed out by Professor Banaschewski.

Chapter 3 : The category of uniform σ–frames is defined and some basic results are given, including the description of the Samuel compactification of a uniform σ–frame (which is, in some ways analogous to that of a uniform frame, but requires techniques introduced by Banaschewski and Gilmour [1989] for a construction of the Stone–Cech compactification of a regular σ–frame). Many of the results correspond to those in the frame setting, but often the proofs differ because of the possible lack of pseudo–complements in the setting of σ–frames. Full use is made of the abundance of separating elements for regular σ–frames.

A structured version of the adjoint situation existing between frames and σ–frames is given for uniform frames and uniform σ–frames. This in fact leads to an equivalence between the subcategory of Lindelöf uniform frames and uniform σ–frames. This interaction and the results in chapter 2 combine in such a way that the given canonical definition of a uniform σ–frame is shown to be appropriate. It is also shown that the Samuel compactification of a separable uniform frame is in fact determined completely by its uniform cozero part.

Finally, proximity σ–frames are introduced via the notion of 'strong inclusions'. These are the natural generalizations of proximity frames, which Frith [1986] showed to be dual to the proximity spaces, and equivalent to the totally bounded (precompact) uniform frames. Similar results are obtained for σ–frames and also for the correspondence between compactifications of σ–frames and strong inlusions on them, as occurs in the setting of frames. (Banaschewski [1980a]).

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Selected categories.

Selected functors.

Binary relations.

Chapter 0. BACKGROUND.

Frames.

A *frame* (= locale) is a complete lattice L satisfying the (arbitrary) distribution law : $x \wedge \bigvee x_i = \bigvee x \wedge x_i$ for binary meet $\wedge$, and arbitrary join $\bigvee$, and $x, x_i \in L$.

A *frame morphism* $h: L \longrightarrow M$ is a map between frames preserving finite meets (including the unit, or top, e) and arbitrary joins (including the zero, or bottom, 0). The resulting category is denoted **Frm**.

On any complete lattice L, a is *rather below* b, written $a \prec b$, if there exists $s \in L$, called a separating element, such that $a \wedge s = 0$ and $b \vee s = e$.

a is *completely below* b, written $a \prec\prec b$, if there is a family $\{ x_i \mid i \in \mathbb{Q} \cap [0,1] \}$ of elements in L satisfying $x_0 = a$, $x_1 = b$, $i \leq j \Rightarrow x_i \prec x_j$.

A frame L is *regular* if the relation $\prec$ is approximating. That is, for each $a \in L$, $a = \bigvee \{ x \mid x \prec a \}$, and L is *completely regular* if the relation $\prec\prec$ is approximating, that is, for each $a \in L$, $a = \bigvee \{ x \mid x \prec\prec a \}$. Since frame morphisms preserve these relations, this gives the full subcategories denoted **RegFrm** and **CrgFrm** respectively.

An element $c \in L$, a frame, is *compact* if for any $X \subseteq L$ with $c \leq \bigvee X$ there exists finite $E \subseteq X$ with $c \leq \bigvee E$. L is *compact* as a frame if e is a compact element. (This is precisely saying that every arbitrary cover has a finite subcover, where $C \subseteq L$ is a *cover* if $\bigvee C = e$.) Let kL denote the collection of all compact elements of L. L is *coherent* if kL is a sublattice join–generating L. That is, each $a \in L$ is the join of compact elements below it. A frame morphism between coherent frames is *proper* if it preserves compact elements. These frames and special frame maps form the non–full subcategory denoted **CohFrm**.

Associated with any distributive lattice with zero and unit, there is the frame $\mathfrak{J}L$ of all the ideals of L, that is, all non–empty $J \subseteq L$ such that $a, b \in J$ implies $a \vee b \in J$, and $x \leq a$ implies $x \in J$. This correspondence is functorial, any lattice morphism

σ–**frames.**

A *σ–frame* is a lattice L which has countable joins, finite meets, a greatest element e, a least element 0 and satisfies the (countable) distribution law : $x \wedge \bigvee x_n = \bigvee x \wedge x_n$ ($n \in I$, countable) for binary meet $\wedge$, countable join $\bigvee$ and $x, x_n \in L$.

A *σ–frame morphism* $h: L \longrightarrow M$ is a map between σ–frames preserving countable joins, finite meets, the least element 0 and the greatest element e. The resulting category is denoted **σFrm**.

A σ–frame L is *regular* if the rather below relation is σ–approximating, that is, for each $a \in L$, there exists a sequence (a_n) in L with $a_n \prec a$ and $a = \bigvee a_n$. L is *normal* if for each pair a, b of elements of L with $a \vee b = e$, there exists u, v in L such that $a \vee u = e = b \vee v$ and $u \wedge v = 0$. Banaschewski [1980b] shows that every regular σ–frame is normal, and hence the rather below relation $\prec$ interpolates. The full subcategory of regular σ–frames is denoted **RegσFrm**, and is coreflective in **σFrm** with coreflection functor $\mathcal{R}$ which assigns to any σ–frame it's largest regular sub–σ–frame.

Any morphism $h: L \longrightarrow M$ in **Frm** or **σFrm** is *dense* (respectively *codense*) if $h(x) = 0$ implies $x = 0$ (respectively $h(x) = e$ implies $x = e$). In **RegFrm** and **RegσFrm**, if $h: L \longrightarrow M$ is dense then h is monic, and if h is codense then it is injective. The proof given here is for the result in **RegσFrm** :

> Assume h is dense. Consider any $f, g: N \longrightarrow L$ such that $h.f = h.g$. Let $a \in N$, then by regularity, $a = \bigvee \{ x_n \mid x_n \prec a \}$. Take $x \prec a$ with $x \wedge y = 0$ and $y \vee a = e$. Then $0 = h.f(x) \wedge h.f(y) = h.f(x) \wedge h.g(y) = h(f(x) \wedge g(y))$, and since h is dense, it follows that $f(x) \wedge g(y) = 0$. Now $g(a) \vee g(y) = e$, and consequently $f(x) = f(x) \wedge (g(a) \vee g(y)) = f(x) \wedge g(a)$. Hence $f(x) \leq g(a)$. This holds for any $x \prec a$ and thus $f(a) = \bigvee \{ f(x_n) \mid x_n \prec a \} \leq g(a)$. By symmetry, $g(a) \leq f(a)$, and so $f = g$.

Assume h is codense. Consider $h(a) = h(b)$ for $a, b \in L$. Since L is regular, $a = \bigvee \{ a_n \mid a_n \prec a \}$ and $b = \{ b_n \mid b_n \prec b \}$. This implies that $h(a) = \bigvee \{ h(a_n) \mid h(a_n) \prec h(a) \} = \bigvee \{ h(b_n) \mid h(b_n) \prec h(b) \} = h(b)$. Take $b_n \prec b$ with separating element x_n, then $h(x_n)$ separates $h(b_n) \prec h(b)$. Since $h(a) = h(b)$, $h(x_n)$ separates $h(b_n) \prec h(a)$. This implies that $e = h(x_n) \vee h(a) = h(x_n \vee a)$. Since h is codense, $x_n \vee a = e$ and therefore $b_n \prec a$. This holds for all the b_n, thus $b \leq a$ and then the result follows by symmetry.

The notion of compact elements and compact σ–frames is defined as for frames restricted to considering countable joins, that is, an element $c \in L$ is *compact* if for any countable $X \subseteq L$ with $c \leq \bigvee X$, there exists finite $E \subseteq X$ with $c \leq \bigvee E$.

An ideal $J \subseteq L$ is *regular* if for each $x \in J$ there exists $y \in J$ with $x \prec y$, and J is *countably generated* if there exists a sequence (x_n) in J such that for each $a \in J$, $a \leq x_n$ for some n.

The full subcategory of compact regular σ–frames, denoted **KRegσFrm**, is coreflective in **RegσFrm** with the coreflection functor given by $\mathscr{K}_\sigma L$, the σ–frame consisting of all countably generated regular ideals, and the coreflection map given by join. This gives the Stone–Cech compactification of a σ–frame as shown by Banaschewski and Gilmour [1989].

The adjoint situation between regular σ–frames and Alexandroff spaces.

An *Alexandroff space* is a pair (X,Z) where X is a set and Z, the *Alexandroff structure* on X, is a collection of subsets of X satisfying the following conditions :

(Z1) Z is closed under finite unions and countable intersections.

(Z2) If $A, B \in Z$ and $A \cap B = \phi$, then there exists $C, D \in Z$ with $A \cap C = \phi = B \cap D$ and $C \cup D = X$.

(Z3) If $A \in Z$ then there exists a sequence (A_n) in Z with $X - A = \bigcup A_n$.

(Z4) For each pair of distinct points in X there is an $A \in Z$ containing just one of them.

The sets in Z are called *zero–sets* and their complements with respect to X are *cozero–sets.* A map $f : (X,Z) \longrightarrow (Y,Z')$ is called a *coz–map* if $f^{-1}(A) \in Z$ for each $A \in Z'$. **Alex** denotes the category of Alexandroff spaces and coz–maps. The zero–sets of the continuous functions on a Tychonoff space satisfy the conditions (Z1) – (Z4), and this defines a functor $\mathrm{coz} : \mathbf{Tych} \longrightarrow \mathbf{Alex}$. (Gilmour [1974]).

A filter P of a σ–frame L is *σ–prime* if for each countable $X \subseteq L$ with $\bigvee X \in P$ then $X \cap P \neq \phi$. The contravariant *spectrum* functor Ψ: **RegσFrm** $\longrightarrow$ **Alex** may be defined by letting ΨL consist of all σ–prime filters of L. The cozero sets of ΨL are the sets of the form $\Psi_a = \{ P \in \Psi L \mid a \in P \}$ for each $a \in L$. If $h: L \longrightarrow M$ is a σ–frame morphism then $\Psi h: \Psi M \longrightarrow \Psi L$ is given by $\Psi h(P) = h^{-1}(P)$ for each $P \in \Psi M$.

The contravariant *'cozero'* functor $\mathfrak{A}$: **Alex** $\longrightarrow$ **RegσFrm** associates with an Alexandroff space the lattice of cozero sets which is a σ–frame, and the cozero maps are taken to the associated inverse image lattice maps.

Ψ and $\mathfrak{A}$ are adjoint on the right (Gilmour [1984]) with the fixed subcategories consisting of the Alexandroff σ–frames, that is, those σ–frames for which $\epsilon_L : L \longrightarrow \mathfrak{A}\,\Psi L$ is an isomorphism, and the realcompact Alexandroff spaces.

The adjoint situation between Frm and σFrm.

An ideal $J \subseteq L$ is a *σ-ideal* if it is closed under countable joins, that is, if for countable $X \subseteq J$ then $\bigvee X \in J$.

The covariant functor $\mathcal{H}$: σ**Frm** $\longrightarrow$ **Frm** is defined by letting $\mathcal{H}L$ be the frame of all the σ-ideals of L, and for $h: L \longrightarrow M$ in σ**Frm**, $\mathcal{H}h: \mathcal{H}L \longrightarrow \mathcal{H}M$ takes a σ-ideal J in L to the σ-ideal $[h(J)]$ generated by $h(J)$. Since every frame is a σ-frame, this gives a forgetful functor $\mathcal{U}$: **Frm** $\longrightarrow$ σ**Frm** taking a frame L to its underlying σ-frame and, moreover $\mathcal{H}$ is left adjoint to $\mathcal{U}$ (Banaschewski [1980b]).

Chapter 1 FRAMES AND SIGMA FRAMES.

Some relations between completely regular frames and regular σ–frames.

In this section it will be shown that a completely regular frame is in fact join generated by by a regular σ–frame, namely its 'cozero part'. This was shown by Reynolds [1979]. The description given below is due to Banaschewski [1980]. This result is useful when considering certain 'topological' conditions in the frame setting, for example pseudocompactness and realcompactness.

1.1. **Definition**: For $L \in \mathbf{Frm}$, $a \in L$ is a *cozero element* if $a = h(\mathbb{R} - \{0\})$ for some $h: \mathfrak{O}\mathbb{R} \longrightarrow L$ in **Frm** .

Let $\mathrm{Coz}\, L$ denote the set of all cozero elements of L .

If L is spatial then $\mathrm{Frm}(\mathfrak{O}\mathbb{R}, L) \cong \mathrm{Top}(\Sigma L, \mathbb{R})$, and therefore $\mathrm{Coz}\, L \cong \mathfrak{A}.\mathrm{coz}\, \Sigma L$.

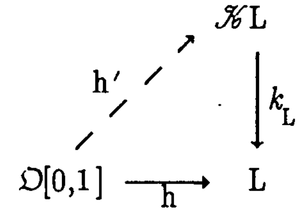
1.2. **Proposition**: *$a \in \mathrm{Coz}\, L \Leftrightarrow a = h((0,1])$ for some $h: \mathfrak{O}[0,1] \longrightarrow L$ in **Frm**.*

Proof: ($\Longrightarrow$) Define $g: \mathbb{R} \longrightarrow [0,1]$ by $g(\alpha) = \alpha^2 \wedge 1$. Then $g(\alpha) > 0$ iff $\alpha \neq 0$. Applying the functor $\mathfrak{O}$ gives $\mathfrak{O}g: \mathfrak{O}[0,1] \longrightarrow \mathfrak{O}\mathbb{R}$ with $\mathfrak{O}g((0,1]) = \mathbb{R} - \{0\}$. Now take $a \in \mathrm{Coz}\, L$, then $a = h(\mathbb{R} - \{0\})$ for some frame morphism $h: \mathfrak{O}\mathbb{R} \longrightarrow L$. Hence $a = h.\mathfrak{O}g((0,1])$ and $h.\mathfrak{O}g: \mathfrak{O}[0,1] \longrightarrow L$.

($\Longleftarrow$) Let $a = h((0,1])$ for some $h: \mathfrak{O}[0,1] \longrightarrow L$. Applying the functor $\mathfrak{O}$ to the inclusion map $e: [0,1] \longrightarrow \mathbb{R}$ gives the frame morphism $\mathfrak{O}e: \mathfrak{O}\mathbb{R} \longrightarrow \mathfrak{O}[0,1]$ with $\mathfrak{O}(\mathbb{R} - \{0\}) = (0,1]$. Hence $a = h.\mathfrak{O}e(\mathbb{R} - \{0\})$ and $h.\mathfrak{O}e: \mathfrak{O}\mathbb{R} \longrightarrow L$.

1.3. Lemma: *For $L \in$ **CrgFrm**, if $\mathscr{K}L$ denotes the compact regular coreflection of L with coreflection map k_L given by join, then $Coz\ L = k_L(Coz\ \mathscr{K}L)$.*

Proof: Since k_L is a frame morphism, and the image of a cozero elements are cozero elements, $k_L(\text{Coz } \mathscr{K}\text{L}) \subseteq \text{Coz L}$. Take $a \in \text{Coz L}$, say $a = h((0,1])$ for $h: \mathfrak{O}[0,1] \longrightarrow L$. Since $\mathfrak{O}[0,1]$ is compact, there exists a unique $h': \mathfrak{O}[0,1] \longrightarrow \mathscr{K}L$ with $k_L.h' = h$:



Hence $a = k_L.h'((0,1])$ and since $h'((0,1]) \in \text{Coz } \mathscr{K}\text{L}$, $a \in k_L(\text{Coz } \mathscr{K}\text{L})$.

1.4. Corollary: *Coz L is a regular sub–σ–frame of L as a σ–frame.*

Proof: $\mathscr{K}$L is compact regular, and hence spatial. Therefore $\text{Coz } \mathscr{K}\text{L} \cong \mathfrak{A}.\text{coz}.\Sigma\, \mathscr{K}\text{L}$. Since $\mathfrak{A}$ is a functor to **RegσFrm**, it follows that Coz $\mathscr{K}$L is a regular σ–frame. Now k_L is a surjective frame morphism, so $k_L(\text{Coz } \mathscr{K}\text{L})$ is a regular σ–frame, and thus by the previous lemma so is Coz L.

1.5. Lemma: *J is a cozero element of $\mathscr{K}L$ iff J is a countably generated completely regular ideal.*

Proof: ($\Longrightarrow$) Since $\mathscr{K}$L is spatial, each cozero element is a countable join of 'zero sets', which are closed and hence compact. Thus cozero elements have the

Lindelöf property. Let $J \in \mathrm{Coz}\ \mathcal{K}L$. For each $b \in J$, choose a sequence $b = b_1 \prec\!\prec b_2 \prec\!\prec b_3 \ldots$ in J . Let J_b be the completely regular ideal generated by this sequence. Then $J = \bigvee\{ J_b \mid b \in J \}$. Since J has the Lindelöf property, J is a countable join of such elements, and thus is countably generated.

(⟸) Let $J \in \mathcal{K}L$ and be countably generated. By regularity of $\mathcal{K}L$, J is the join of elements rather below it, and since countably generated, a countable join of such elements. Hence J corresponds to an open F_σ in a compact regular space, and therefore to a cozero set. Hence $J \in \mathrm{Coz}\ \mathcal{K}L$.

1.6. Corollary: *For any frame L, $a \in \mathrm{Coz}\, L$ iff $a = \bigvee a_n$ for some sequence (a_n) with $a_i \prec\!\prec a_{i+1}$* for all i .

Proof: (⟹) Let $a \in \mathrm{Coz}\ L$, then by lemma 1.3 , $a = k_L(J)$ for some $J \in \mathrm{Coz}\ \mathcal{K}L$. But by lemma 1.5 , J is countably generated, say by (c_n) . Now define a sequence (a_n) by letting $a_1 = c_1$, and choosing a_{n+1} such that $a_n \vee c_{n+1} \prec\!\prec a_{n+1}$. Then $a = k_L(J) = \bigvee a_n$ and $a_i \prec\!\prec a_{i+1}$.

(⟸) Suppose $a = \bigvee a_n$ with $a_i \prec\!\prec a_{i+1}$. Then $J = \{ x \in L \mid x \leq a_n ,\ \text{some } n\}$, is a countably generated completely regular ideal, and so $J \in \mathrm{Coz}\ \mathcal{K}L$. But $a = k_L(J)$ and hence $a \in k_L(\mathrm{Coz}\ \mathcal{K}L) = \mathrm{Coz}\ L$.

1.7. Corollary: *If $\mathcal{R}$ denotes the regular coreflection of a σ–frame and $\mathcal{U}$ the forgetful functor from **Frm** to **σFrm**, then for $L \in$ **CrgFrm**, $\mathrm{Coz}\, L \cong \mathcal{R}\mathcal{U}L$.*
(ie. Coz L is the largest regular sub–σ–frame of L as a σ–frame.)

Proof: Since Coz L is regular, Coz L $\subseteq \mathscr{R}\mathscr{U}$L. Take $a \in \mathscr{R}\mathscr{U}$L, then $a = \bigvee a_n$ with $a_i \prec a_{i+1}$ in $\mathscr{R}\mathscr{U}$L, since $\mathscr{R}\mathscr{U}$L is regular. But $\prec$ interpolates in regular σ–frames, so $a_i \prec\prec a_{i+1}$, and therefore $a \in$ Coz L. (Corollary 1.6.)

1.8. **Proposition:** *For $L \in \sigma$**Frm**, the following are equivalent :*

(1) L is regular.

*(2) $L \cong$ Coz M for some $M \in$ **Frm**.*

(3) L is the image of some compact regular σ–frame.

Proof: (1) ⇒ (2) Consider the unit $\eta_L: L \to \mathscr{U}\mathscr{H}L$ defined by $\eta_L(x) = \downarrow x$. Since L is regular, $\eta_L(L) \subseteq \mathscr{R}\mathscr{U}\mathscr{H}L \cong$ Coz $\mathscr{H}$L. Take J in $\mathscr{R}\mathscr{U}\mathscr{H}$L, then by regularity, $J = \bigvee\{ I_n \mid I_i \prec I_{i+1} \}$. Hence for each n there exists K_n such that $I_n \cap K_n = \{0\}$, and $I_{n+1} \vee K_n = L$. Thus for every $x \in I_n$, there exists $k \in K_n$ and $x_n \in I_{n+1}$ such that $x \wedge k = 0$ and $k \vee x_n = e$. This implies that $I_n \subseteq \downarrow x_n \subseteq J$. Let $a = \bigvee x_n$, then $a \in J$ and in fact $\downarrow a = J$, since $I_n \subseteq \downarrow a$ for all n. Hence $\eta_L: L \to \mathscr{R}\mathscr{U}\mathscr{H}L$ is surjective. Since η_L is always injective, $L \cong \mathscr{R}\mathscr{U}\mathscr{H}L \cong$ Coz $\mathscr{H}$L.

(2) ⇒ (3) Suppose $L \cong$ Coz M for some frame M. Then since k_L: Coz $\mathscr{K}$M $\to$ Coz M is surjective and Coz $\mathscr{K}$M is compact as a σ–frame, L is the image of a compact regular σ–frame.

(3) ⇒ (1) This is trivial since the image of a regular σ–frame is regular.

1.9. **Proposition:** *For $L \in$ **CrgFrm**, Coz L generates L as a frame.*

(ie. each element of L is a join of cozero elements.)

Proof: Let $a \in L$, then $a = k_L(J)$ for some $J \in \mathscr{K}$L, since k_L is surjective. But

$J = \bigvee\{ J_b \mid b \in J \}$ with $J_b \in \text{Coz } \mathcal{H} L$ (as in lemma 1.5.). Hence $a = k_L(\bigvee\{ J_b \mid b \in J \}) = \bigvee\{ k_L(J_b) \mid b \in J \}$ since k_L is a frame morphism. The result follows since $k_L(J_b) \in \text{Coz } L$ for each $b \in J$.

A Countable analogue of Stone duality

The definitions and results 1.10 – 1.14 of this section are due to Madden and Vermeer [1985] , the proofs have been slightly adapted using observations of Reynolds [1979] and Johnstone [1982].

1.10. Definition: Let $L \in \mathbf{Frm}$, $a \in L$ is *countable* if for any set $B \subseteq L$ with $a \leq \bigvee B$, there exists a countable $X \subseteq B$ such that $a \leq \bigvee X$.

L is *Lindelof* if e is countable.

Let $\sigma(L)$ denote the collection of all countable elements of L.

L is *σ–coherent* if $\sigma(L)$ is a sub–lattice of L which generates L.

For L, M σ–coherent frames, $h: L \to M$ in **Frm** is *σ–coherent* if $h[\sigma(L)] \subseteq \sigma(M)$.

σCohFrm is the category consisting of σ–coherent frames and σ–coherent frame morphisms.

1.11. Proposition: *σFrm and σCohFrm are equivalent as categories.*

Proof: Consider $\mathcal{H}: \sigma\mathbf{Frm} \to \mathbf{Frm}$ which sends a σ–frame L to its frame of σ–ideals, $\mathcal{H}L$. The countable elements of $\mathcal{H}L$ are precisely the principal ideals :

take $\downarrow b \subseteq \bigvee\{ J_i \mid i \in I \}$ where J_i are σ–ideals of L and $b \in L$. $b \in \bigvee J_i$, and hence is a countable join of elements in the J_i, say $b = \bigvee k_{i_n}$. It follows that $\downarrow b \subseteq \bigvee J_{i_n}$. The converse follows since a countable element of $\mathscr{H}L$ is a countable join of principal ideals and $\bigvee(\downarrow b_n) = \downarrow(\bigvee b_n)$.

Since $L = \downarrow e$, $\downarrow a \cap \downarrow b = \downarrow(a \wedge b)$ and for each σ–ideal J, $J = \bigvee\{ \downarrow a \mid a \in J \}$, $\mathscr{H}L$ is σ–coherent. Now for each σ–frame morphism f, $\mathscr{H}f$ is σ–coherent and hence $\mathscr{H}$: **σFrm** ⟶ **σCohFrm**.

Now if $L \in$ **σCohFrm**, then $L \cong \mathscr{H}\sigma(L)$:

define $\Phi: L \longrightarrow \mathscr{H}\sigma(L)$ by $\Phi(a) = \{ s \in \sigma(L) \mid s \leq a \}$ and $\bigvee: \mathscr{H}\sigma(L) \longrightarrow L$ by taking joins. Then for $I \in \mathscr{H}\sigma(L)$, if $c \in \sigma(L)$ with $c \leq \bigvee I$, then there exists a countable $X \subseteq I$ with $c \leq \bigvee X$ and hence $c \in I$ since I is a σ–ideal. Hence $\Phi(\bigvee I) = I$. Conversely, for $a \in L$, $\bigvee(\Phi(a)) = a$ since L is σ–coherent. This isomorphism at object level extends to an equivalence of categories because Φ is clearly order preserving and functorial.

1.12. Proposition: *For a regular frame L, L is Lindelof iff L is σ–coherent.*

Proof: If L is σ–coherent then, since $\sigma(L)$ is a sublattice, e is countable. Conversely, every regular Lindelöf frame is normal, and hence $\prec$ interpolates. Thus L is completely regular. As sets $\sigma(L) = \mathrm{Coz}\, L$: $\sigma(L) \subseteq \mathrm{Coz}\, L$ since $\sigma(L)$ is a regular sub–σ–frame of L. Now take $a \in \mathrm{Coz}\, L$, then $a = \bigvee a_n$ with $a_i \prec\!\prec a_{i+1}$. Suppose that $a \leq \bigvee B$ for some $B \subseteq L$. Then $a_n \prec \bigvee B$ for each n, and so for each n, there exists a seperating element s_n with $s_n \vee \bigvee B = e$. Since L is Lindelöf, there exists a countable $X_n \subseteq B$ with $s_n \vee \bigvee X_n = e$ and hence $a_n \prec \bigvee X_n$, for each n. It follows that $a \leq \bigvee X$ where X is the union of the X_n and hence is countable. Therefore $a \in \sigma(L)$.

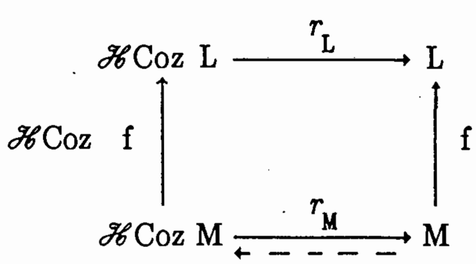
Since Coz L generates L so does $\sigma(L)$, and hence L is σ–coherent.

1.13. **Proposition:** *RegLindFrm and RegσFrm are equivalent as categories.*

Proof: Since frame maps preserve cozero elements, and hence countable elements in a regular Lindelöf setting, **RegLindFrm**, or equivalently **RegσCohFrm**, is a full subcategory of **Frm**. Thus σ acting on this subcategory is the same as Coz . To show that the functors $\mathcal{H}$ and Coz give this equivalence, it suffices to show that $\mathcal{H}$L is regular as a frame if and only if L is regular as a σ–frame : take $a \in L$, then $\downarrow a$ is a countable, hence cozero, element of $\mathcal{H}$L. Thus $\downarrow a$ is a countable join of principal ideals rather below it, say $\downarrow a = \bigvee \downarrow a_n$. It follows that $a = \bigvee a_n$ and $a_n \prec a$. Conversely for J in $\mathcal{H}$L , $J = \bigvee\{ \downarrow a \mid a \in J \}$ and $\downarrow a = \downarrow \bigvee a_n = \bigvee \downarrow a_n$ with $a_n \prec a$, hence $\downarrow a_n \prec \downarrow a$ and thus J is the join of elements rather below it.

1.14. **Proposition:** *RegLindFrm is coreflective in CrgFrm with coreflection functor $\mathcal{H}$ Coz and coreflection map r_L given by join.*

Proof: For $L \in$ **CrgFrm**, Coz L is a regular σ–frame and thus $\mathcal{H}$ Coz L is a regular Lindelöf frame. Now consider any $f : M \longrightarrow L$ in **Frm**, with $M \in$ **RegLindFrm**.



Since M is regular Lindelöf, $M \cong \mathcal{H}$ Coz M and the diagram commutes because $r_L . \mathcal{H}\mathrm{Coz}\, f . \Phi_M = f . r_M . \Phi_M = f$. Hence f factors via r_L .

Moreover this factorisation is unique : take any $g: M \longrightarrow \mathscr{H}\mathrm{Coz}\, L$ such that $r_L \cdot g = f$. For $a \in \mathrm{Coz}\, M$, $g(a)$ is a cozero element of $\mathscr{H}\mathrm{Coz}\, L$, and hence a principal ideal, say $g(a) = \downarrow b$ for some $b \in \mathrm{Coz}\, L$. But then $f(a) = r_L \cdot g(a) = r_L(\downarrow b) = b$, hence $g(a) = \downarrow f(a) = \mathscr{H}\mathrm{Coz}\, f . \Phi_L(a)$. Since $\mathrm{Coz}\, M$ generates M, $g = \mathscr{H}\mathrm{Coz}\, f . \Phi_M$.

This corresponds to the Hewitt realcompactification in the spatial setting.

1.15. Lemma: *The functor* $\mathscr{H}: \sigma\mathbf{Frm} \longrightarrow \mathbf{Frm}$ *preserves compactness.*

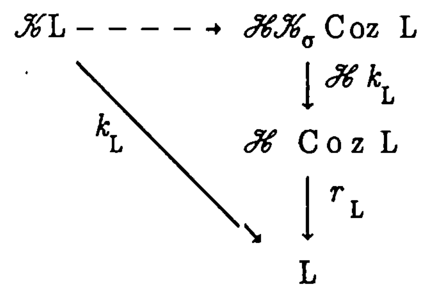
Proof: Let L be a compact σ–frame and suppose $\bigvee\{ J_i \mid i \in I \} = L$ for $J_i \in \mathscr{H}L$. This implies that $e \in \bigvee J_i$, and hence e is a countable join of elements in the J_i. Since L is compact as a σ–frame, e is a finite join of such elements, say $e = a_1 \vee \ldots \vee a_n$ with $a_j \in J_{i_j}$ for $j = 1, \ldots n$. Then $L = J_{i_1} \vee \ldots \vee J_{i_n}$ and hence $\mathscr{H}L$ is compact as a frame.

Using this lemma and the previous results showing that a completely regular frame is determined by its cozero elements allows for an alternate description of the compact regular coreflection in **Frm**, via the compact regular coreflection in **RegσFrm**.

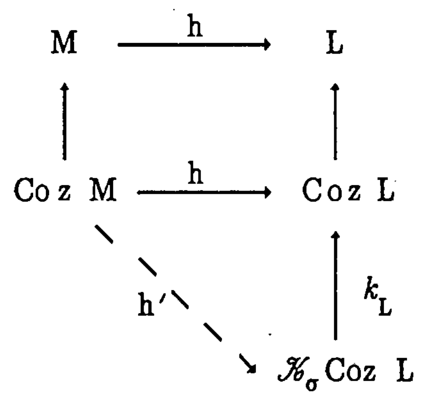
1.16. Proposition: *For a completely regular frame* L, $\mathscr{H}\mathscr{K}_\sigma \mathrm{Coz}\, L \cong \mathscr{K}L$.

Proof: Let $L \in$ **CrgFrm**, then $\mathrm{Coz}\, L \in$ **RegσFrm** and generates L as a frame. Take the compact regular coreflection of $\mathrm{Coz}\, L$, namely $k_L: \mathscr{K}_\sigma \mathrm{Coz}\, L \longrightarrow \mathrm{Coz}\, L$, and apply the functor $\mathscr{H}$ to get $\mathscr{H}k_L: \mathscr{H}\mathscr{K}_\sigma \mathrm{Coz}\, L \longrightarrow \mathscr{H}\mathrm{Coz}\, L$. Composing this map with the regular Lindelöf coreflection map $r_L: \mathscr{H}\mathrm{Coz}\, L \longrightarrow L$, gives a morphism from $\mathscr{H}\mathscr{K}_\sigma \mathrm{Coz}\, L$ to L.

Since $\mathcal{H}$ preserves compactness (lemma 1.15), $\mathcal{H}\mathcal{K}_\sigma$Coz L is compact and hence this map factors via k_L :



Consider h: M ⟶ L in **Frm** with M ∈ **KRegFrm** , then the corresponding situation in σ**Frm** is as follows :



Since M is compact, Coz M is compact as a σ–frame, and hence h restricted to Coz M which maps to Coz L , factors uniquely via the compact regular coreflection $\mathcal{K}_\sigma$Coz L. Applying $\mathcal{H}$ gives $\mathcal{H}$h′: $\mathcal{H}$Coz M ⟶ $\mathcal{H}\mathcal{K}_\sigma$Coz L . Now since M is compact and hence Lindelöf, $\mathcal{H}$Coz M ≅ M . Together this gives the required unique factorization of h via $\mathcal{H}\mathcal{K}_\sigma$Coz L ⟶ L .

Gilmour [1974] shows that a Tychonoff space is pseudocompact iff its σ–frame of cozero sets, is compact. This leads to a natural definition of pseudocompactness for completely regular frames in terms of the cozero elements.

1.17. **Definition:** L ∈ **CrgFrm** is *pseudocompact* if Coz L is compact as a σ–frame.

Obviously if L is compact, then L is pseudocompact.

1.18. **Proposition:** *L ∈ **RegLindFrm** is pseudocompact iff L is compact.*

Proof: If L is pseudocompact then Coz L is compact and hence, by lemma 1.15 , so is $\mathscr{H}$Coz L . But since L is regular Lindelöf, L $\cong$ $\mathscr{H}$Coz L .

This is precisely the analogue to the spatial result that any realcompact space which is pseudocompact is compact.

Coherent and continuous σ–frames.

1.19. DEFINITION: For $L \in \sigma\mathbf{Frm}$, an element $a \in L$ is *compact* if for every countable $X \subseteq L$ such that $a \leq \bigvee X$, there exists a finite $E \subseteq X$ with $a \leq \bigvee E$. (Since L is a σ–frame, only countable joins need exist, and hence only countable sets are considered.)

Let kL be the set of all compact elements of L.

L is *coherent* if kL is a sublattice generating L, that is, kL is closed under finite meets and finite joins and each element of L is a countable join of compact elements.

$h: L \to M$ is *coherent* if h preserves compact elements, that is, if $c \in kL$ then $h(c) \in kM$.

L is *Stone* if L is compact (ie. e is a compact element) and the complemented elements of L generate L. (ie. L is zero–dimensional.) Let CL denote all complemented elements of L.

CohσFrm is the category consisting of coherent σ–frames and coherent maps. Since every σ–frame morphism between coherent σ–frames is not necessarily coherent, this is not a full subcategory of σ**Frm**.

StσFrm is the full subcategory of σ**Frm** consisting of all Stone σ–frames.

1.20. **PROPOSITION:** *In a Stone σ–frame, an element is compact iff it is complemented.*

Proof: (⇐) Let a be complemented, say by $\bar{a}$. Take countable $X \subseteq L$ with $a \leq \bigvee X$, then $e = \bar{a} \vee \bigvee X$. Since L is Stone, hence compact, there exists finite $E \subseteq X$ with $e = \bar{a} \vee \bigvee E$. Therefore $a = a \wedge e \leq \bigvee E$, so a is compact.

(⇒) Let a be compact. Since L is Stone, a is a countable join of complemented elements, hence a finite join, say $a = x_1 \vee .. \vee x_n$ with x_i complemented by $\bar{x}_i$. Then $a \vee (\bar{x}_1 \vee ... \vee \bar{x}_n) = e$ and $a \wedge (\bar{x}_1 \vee ... \vee \bar{x}_n) = 0$, so a is complemented.

1.21. **COROLLARY:** *StσFrm is a full subcategory of CohσFrm.*

Proof: By proposition 1.20. every Stone σ–frame is coherent. Since σ–frame morphisms preserve complemented, hence compact elements in a Stone setting, every morphism between Stone σ–frames is coherent.

1.22. **PROPOSITION:** *A coherent σ–frame is Stone iff it is regular.*

Proof: (⇒) If y is complemented then $y \prec y$, thus $y \prec x$ for any $x \geq y$. Since each element of a Stone frame is a countable join of complemented elements, each element is a countable join of elements rather below it.

(⇐) Take any compact element c. Since L is regular, c is a countable and hence finite join of elements rather below it. The rather–below relation is preserved by finite joins so $c \prec c$, hence c is complemented. Since L is coherent, each element is a countable join of compact, and hence in this setting, complemented elements.

The functor $\mathfrak{J}_\sigma$: $\mathfrak{D} \longrightarrow \sigma\mathbf{Frm}$ assigns to any distributive lattice A the σ–frame consisting of all countably generated ideals of A , denoted $\mathfrak{J}_\sigma(A)$, and any lattice morphism h: A $\longrightarrow$ B to $\mathfrak{J}_\sigma h$: $\mathfrak{J}_\sigma A \longrightarrow \mathfrak{J}_\sigma B$ which takes any ideal in $\mathfrak{J}_\sigma A$ to the ideal generated by its image under h .

1.23. **Lemma**: *For $A \in \mathfrak{D}$, the principal ideals in $\mathfrak{J}_\sigma(A)$ are precisely the compact elements of $\mathfrak{J}_\sigma(A)$.*

Proof: Let $\downarrow a \subseteq \bigvee J_n$ for some sequence (J_n) of countably generated ideals of A , then $a \in \bigvee J_n$, so $a \leq x_{i_1} \vee \ldots \vee x_{i_n}$ for $x_{i_j} \in J_j$. Therefore $\downarrow a \subseteq J_{i_1} \vee \ldots \vee J_{i_n}$, and thus $\downarrow a$ is compact.

Suppose J is compact . Since J is a countable join of principal elements, J is a finite join of principal elements and hence is itself principal.

1.24. **Proposition**: *$\mathfrak{J}_\sigma(A)$ is coherent for each $A \in \mathfrak{D}$.*

Proof: Since each J in $\mathfrak{J}_\sigma(A)$ is countably generated, say by (a_n) , $J = \bigvee \downarrow a_n$, thus each σ–ideal is a countable join of principal, and hence compact elements. (Lemma 1.23.) Since $\downarrow a \cap \downarrow b = \downarrow(a \wedge b)$, $\downarrow a \cup \downarrow b = \downarrow(a \vee b)$ and $A = \downarrow e$ is principal, hence compact, it follows that $k\mathfrak{J}_\sigma(A)$ is a sublattice generating $\mathfrak{J}_\sigma(A)$.

Madden [1988] shows that a σ–frame is coherent iff it is the free σ–frame over a distributive lattice.

1.25. Proposition: *If $h: A \longrightarrow B$ in $\mathfrak{D}$ then $\mathfrak{J}_\sigma h: \mathfrak{J}_\sigma A \longrightarrow \mathfrak{J}_\sigma B$ is coherent. Hence $\mathfrak{J}_\sigma : \mathfrak{D} \longrightarrow$ **CohσFrm** is functorial.*

Proof: $\mathfrak{J}_\sigma h$ is a σ–frame morphism since $\mathfrak{J}_\sigma$ is functorial, and $h(\downarrow a) = \downarrow h(a)$ so $\mathfrak{J}_\sigma h$ is coherent.

1.26. Lemma: *If $M \in$ **CohσFrm** then $\bigvee : \mathfrak{J}_\sigma M \longrightarrow M$ has a right inverse.*

Proof: Define $\gamma : M \longrightarrow \mathfrak{J}_\sigma M$ by letting $\gamma(x)$ = ideal generated by $(kM) \cap \downarrow x$. Now $\gamma(0) = 0$, $\gamma(e) = kM$ and $\gamma(x \wedge y) = \gamma(x) \wedge \gamma(y)$. Also $\gamma(x \vee y) = \gamma(x) \vee \gamma(y)$: take any compact $c \leq x \vee y$, then $c \leq a \vee b$ for some $a \leq x$, $b \leq y$ with $a, b \in kM$, and thus $c = (c \wedge a) \vee (c \wedge b) \in \gamma(x) \vee \gamma(y)$. And for any countable up–directed set $X \subseteq M$, $\gamma(\bigvee X) \subseteq \bigvee \gamma(X) \subseteq \gamma(\bigvee X)$. Hence γ is a σ–frame morphism. For any $c \in kM$, $\gamma(c) = \downarrow c$, which is compact, so γ is coherent. Further, $\bigvee(\gamma(x)) = \bigvee$(ideal generated by $kM \cap \downarrow x) = x$ since x is a countable join of compact elements. Thus γ is right inverse to $\bigvee$.

1.27. Proposition: ***CohσFrm** is coreflective in **σFrm**, with coreflection functor $\mathfrak{J}_\sigma$, and coreflection map $\bigvee$.*

Proof: Consider $h: M \longrightarrow L$ in σFrm, with $M \in$ CohσFrm.

$$
\begin{array}{ccc}
\mathfrak{I}_\sigma L & \xrightarrow{\ \bigvee\ } & L \\
{\scriptstyle \mathfrak{I}_\sigma h}\uparrow & & \uparrow{\scriptstyle h} \\
\mathfrak{I}_\sigma M & \underset{\gamma}{\overset{\bigvee}{\rightleftarrows}} & M
\end{array}
$$

For $c \in kM$, $\bigvee.\mathfrak{I}_\sigma h.\gamma(c) = \bigvee.\mathfrak{I}_\sigma h(\downarrow c) = \bigvee\downarrow h(c) = h(c)$. Since kM generates M, the diagram commutes.

Uniqueness: take any coherent $g : M \longrightarrow \mathfrak{I}_\sigma L$ such that $\bigvee.g = h$. For $c \in kM$, $g(c) = \downarrow a$ for some $a \in L$, since g is coherent and the compact elements of $\mathfrak{I}_\sigma L$ are precisely the principal ideals. Therefore $h(c) = \bigvee\downarrow a = a$, but then $g(c) = \downarrow h(c) = \mathfrak{I}_\sigma h.\gamma(c)$, so g and $\mathfrak{I}_\sigma h.\gamma$ agree on the generating set, and hence are equal.

1.28. PROPOSITION: *For any $A \in \mathfrak{D}$, $\mathfrak{I}_\sigma(A)$ is regular iff A is a boolean algebra.*

Proof: ($\Longrightarrow$) $\mathfrak{I}_\sigma(A)$ is regular hence a Stone σ–frame, so an element is complemented iff it is compact. Take $a \in A$, then $\downarrow a$ is compact, hence complemented. Moreover the complement is compact and hence a principal ideal, so there exists $b \in A$ such that $\downarrow a \cap \downarrow b = 0$ and $\downarrow a \vee \downarrow b = A = \downarrow e$. This implies that $a \wedge b = 0$ and $a \vee b = e$, and so every element of A is complemented.

($\Longleftarrow$) A is boolean so every element is complemented. The compact elements of $\mathfrak{I}_\sigma(A)$, which generate it, are the principal ideals which are thus complemented, hence $\mathfrak{I}_\sigma(A)$ is in fact a Stone σ–frame and therefore regular.

The functor C: σ**Frm** $\longrightarrow$ **Boo** assigns to any σ–frame L the collection, denoted C(L) , of all the complemented elements of L , and any σ–frame morphism is restricted to this subset.

1.29. **Corollary:** *For* $L \in \sigma\mathbf{Frm}$, $\mathfrak{J}_\sigma CL$ *is a stone* σ*–frame.*

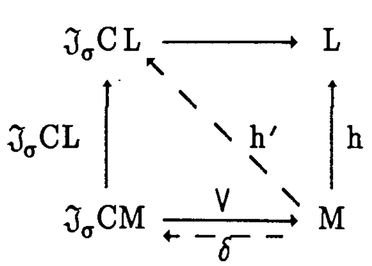
Proof: This follows immediately from proposition 1.28. and proposition 1.22 .

1.30. **Lemma:** *For a stone* σ*–frame* M, $\bigvee : \mathfrak{J}_\sigma CM \longrightarrow M$ *is an isomorphism.*

Proof: $\bigvee$ is onto since CM generates M and since $\bigvee$ is codense and M is regular, $\bigvee$ is injective. In fact the map $\delta : M \longrightarrow \mathfrak{J}_\sigma M$ defined by $\delta(x) = CM \cap {\downarrow}x$ is (right) inverse to $\bigvee$. (The proof is as in lemma 1.26.)

1.31. **Proposition:** *St*σ***Frm*** *is coreflective in* σ***Frm***, *with coreflection functor* $\mathfrak{J}_\sigma C$ *and coreflection map* $\bigvee$.

Proof: Consider h: M $\longrightarrow$ L in σ**Frm** with M $\in$ **St**σ**Frm**.



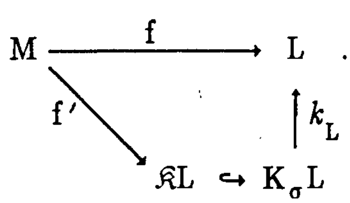
In fact $h'(a) = \mathfrak{J}_\sigma C\, h(CM \cap {\downarrow}a) = \bigcup\{CL \cap {\downarrow}h(x) \mid x \in CM \cap {\downarrow}a\}$.

Uniqueness : Suppose $f : M \longrightarrow \mathfrak{J}_\sigma L$ such that $\bigvee.f = h$. For $a \in CM$, $f(a)$ is complemented in $\mathfrak{J}_\sigma L$ hence $f(a) = CL \cap \downarrow b$ for some $b \in CL$. But $h(a) = \bigvee f(a) = b$, so $f(a) = CL \cap \downarrow h(a)$, thus $f(a) = h'(a)$. Since CM generates M, $f = h'$.

It is also possible to obtain the coreflection to **StσFrm** from the compact regular coreflection or the coherent coreflection, as in the frame setting.

1.32. Proposition: *If $\mathcal{K}_\sigma L$ is the compact regular coreflection of L, and let $\mathfrak{K}L$ be the sub–σ–frame generated by $C(\mathcal{K}_\sigma L)$, then $\mathfrak{K}L$ gives the stone σ–frame coreflection. (ie. $\mathfrak{K}L \cong \mathfrak{J}_\sigma CL$.)*

Proof: $\mathfrak{K}L$ is a sub–σ–frame of a compact frame, hence is compact, and is generated by complemented elements, thus $\mathfrak{K}L$ is a Stone σ–frame. Consider $f : M \longrightarrow L$ in **σFrm** with $M \in$ **StσFrm**. Since M is Stone, it is compact and regular and thus factors uniquely via $k_L : \mathcal{K}_\sigma L \longrightarrow L$, say by f'.
Take $a \in CM$, then $f'(a) \in C(\mathcal{K}_\sigma L) \subseteq \mathfrak{K}L$ and since CM generates M; $f'[M] \subseteq \mathfrak{K}L$.



1.33. PROPOSITION: *Let $\mathfrak{C}L$ be the sub–σ–frame of $\mathfrak{J}_\sigma L$ generated by the complemented elements of $\mathfrak{J}_\sigma L$, then $\mathfrak{C}L$ is the Stone σ–frame coreflection. (ie. $\mathfrak{C}L \cong \mathfrak{J}_\sigma CL$).*

Proof: Since $\mathfrak{J}_\sigma L$ is coherent, the compact elements form a sublattice and hence the top element, $\downarrow e$, is compact. Thus $\mathfrak{J}_\sigma L$ is compact as a σ–frame, and then so is $\mathfrak{C}L$ compact since it is a sub–σ–frame of $\mathfrak{J}_\sigma L$. Consequently $\mathfrak{C}L$ is a Stone σ–frame since it it generated by the complemented elements of $\mathfrak{J}_\sigma L$. Consider any $f : M \longrightarrow L$ in **σFrm** with $M \in$ **StσFrm**. Since **StσFrm** $\subseteq$ **CohσFrm**, M is coherent and therefore, by the coreflection property of coherent σ–frames, f factors uniquely via $\mathfrak{J}_\sigma L$, say by f'. By the same argument as used in the previous proposition, $f'[M] \subseteq \mathfrak{C}L$ and hence $\mathfrak{C}L$ has the universal mapping property.

1.34. DEFINITION: For x, y in $L \in$ **σFrm**, x is *σ–way below* y, written $x \ll_\sigma y$, if for some countable $X \subseteq L$ with $y \leq \bigvee X$ there exists finite $E \subseteq X$ with $x \leq \bigvee E$. In particular this means if $y \leq \bigvee J$, for some countably generated ideal, then $x \in J$.

L is *continuous* if $\ll_\sigma$ is a σ–approximating relation. That is, for each $a \in L$ there exists a sequence (a_n) such that $a_n \ll_\sigma a$ and $a = \bigvee a_n$.

L is *stably continuous* if L is continuous and the relation $\ll_\sigma$ is closed under finite meets in $L \times L$.

$h: L \longrightarrow M$ is a *proper* σ–frame morphism if h preserves the σ–way below relation. That is, $x \ll_\sigma y \Rightarrow h(x) \ll_\sigma h(y)$.

ContσFrm is the full subcategory of **σFrm** consisting of all continuous σ–frames.

StContσFrm is the subcategory of stably continuous σ–frames and proper σ–frame morphisms.

This terminology corresponds to that used in the more general setting of continuous lattices. (Johnstone [1982] or the Compendium [1980].) In Madden [1988], σ–way below corresponds to ω–inside, and stably continuous to ω–suitable. Banaschewski [1979] shows that **RegContσFrm** is dual to the σ–compact locally compact Hausdorff spaces.

1.35. **Definition:** A subset $U \subseteq L$ is *σ–Scott open* if U is an upset and if for some countable $X \subseteq L$, $\bigvee X \in U$ then there exists finite $E \subseteq X$ with $\bigvee E \in U$.

The following propositions and proofs about σ–Scott open sets, some of which have been obtained by Banaschewski [1979], correspond to those about Scott open sets in the setting of frames.

1.36. **Proposition:** *In a continuous σ–frame, an upset U is σ–Scott open iff for each x in U, there exists $y \ll_\sigma x$ in U.*

Proof: ($\Rightarrow$) Assume an upset U is σ–Scott open in a continuous σ–frame L. Let $x \in U$. Since L is continuous, x is a countable join of elements σ–way below it and hence some finite join of elements σ–way below x is in U. Let y be this finite join.

($\Leftarrow$) Assume the condition above holds for all x in some upset U of L. Suppose that $\bigvee X \in U$ for some countable $X \subseteq L$, then there exists $y \ll_\sigma \bigvee X$ in U. Hence there exists finite $E \subseteq X$ with $y \leq \bigvee E$, but then $\bigvee E \in U$ since U is an upset.

1.37. Proposition: *In a continuous σ-frame, a filter is prime and σ-Scott open iff it is σ-prime.*

Proof: ($\Leftarrow$) Let P be a σ-prime filter in a continuous σ-frame L. Then P is prime and since L is continuous, each $a \in P$ is a countable join of elements σ-way below it, and hence one such element is in P. (ie. there exists an $x \ll_\sigma a$ in P.) By the proposition above, P is σ-Scott open.

($\Rightarrow$) Let P be σ-Scott open and a prime filter of any σ-frame L. Suppose $X \subseteq L$ is countable with $\bigvee X \in P$, then since P is σ-Scott open, there exists finite $E \subseteq X$ with $\bigvee E \in P$. But then by the primeness of P, there exists $x \in E \subseteq X$ with $x \in P$. Hence P is σ-prime.

1.38. Lemma: *In a continuous σ-frame, any σ-Scott open subset is the union of σ-Scott open filters.*

Proof: Let U be σ-Scott open. For each a in U, choose $a = a_1 \gg_\sigma a_2 \gg_\sigma a_3 \ldots$ in U. Let $F_a = \{ x \in L \mid x \geq a_k \text{ for some } k \}$, a σ-Scott open filter, then $a \in F_a \subseteq U$.

1.39. Proposition: *Any σ-Scott open filter is the intersection of σ-Scott open prime filters.*

Proof: Let F be a σ-Scott open filter, and $a \notin F$. Using the axiom of choice, there exists a σ-Scott open filter G, maximal such that $G \supseteq F$ and $a \notin G$. Take any $b, c \notin G$ with $b \vee c \in G$, and let $H = \{ x \in L \mid b \vee x \in G \}$. Now H is a filter and H is σ-Scott open : suppose $\bigvee X \in H$, for some countable set X, then $b \vee \bigvee X \in G$, hence $\bigvee\{ b \vee x \mid x \in X \} \in G$, and so, since G is σ–Scott open, there exists finite $E \subseteq X$ such that $\bigvee\{ b \vee x \mid x \in E \} \in G$. Hence $\bigvee E \in H$. Since $c \in H$, and $c \notin G$, G is a proper subset of H . By maximality of G with respect to not containing a , it follows that $a \in H$. Now let $K = \{ x \in L \mid x \vee a \in G \}$. By repeating the argument above, it can be shown that K is σ–Scott open with $G \subset K$, $G \neq K$ and hence $a \in K$. This implies that $a \vee a = a \in G$, which is a contradiction. Hence G is a prime filter.

1.40. Corollary: *Every regular continuous σ–frame is an Alexandroff σ–frame. ie. $L \cong \mathfrak{A}\Psi L$.*

Proof: It suffices to show that σ–prime filters separate elements of L. (Gilmour [1984]) Let $a, b \in L$ and without loss of generality, assume $a \leq b$ and $a \neq b$. This implies that $b \in \{ x \in L \mid x \notin \downarrow a \}$ which is σ–Scott open. By lemma 1.38, there exists a σ–Scott open filter F with $b \in F \subseteq \{ x \in L \mid x \notin \downarrow a \}$, hence $a \notin F$. Now by the previous proposition, there exists a σ–Scott open prime filter P with $a \notin P$ and $F \subseteq P$. So P is σ–prime and $a \notin P$, $b \in P$.

1.41. Lemma: *Every compact regular σ–frame is continuous.*

Proof: This follows since in a compact regular σ–frame, $x \ll_\sigma y$ iff $x \prec y$: Suppose $x \ll_\sigma y$. By regularity $y = \bigvee \{ y_n \mid y_n \prec y \}$, and thus x is less than a finite join of such y_i . From properties of the rather below relation, it follows that $x \prec y$.

Now suppose $x \prec y$, and let X be a countable set such that $y \leq \bigvee X$. Since $x \prec y$, there exists a separating element s . Since $y \leq \bigvee X$ and $y \vee s = e$, $\bigvee X \vee s = e$ and so by compactness, there exists a finite $E \subseteq X$ with $\bigvee E \vee s = e$. Hence $x \leq \bigvee E$.

This lemma in conjunction with the result in corollary 1.40. immediately give the following result obtained by Gilmour [1984].

1.42. **Corollary:** *For every compact regular σ–frame L, $L \cong \mathfrak{A}\Psi L$.*

1.43. **Proposition:** *Every coherent σ–frame is continuous.*

Proof: In a coherent σ–frame, $x \ll_\sigma y$ iff there exists a compact element z with $x \leq z \leq y$: Suppose $x \ll_\sigma y$. Since the σ–frame is coherent, y is a countable join of compact elements and hence x is less than a finite join of such elements. Put z equal to this finite join, then z is compact and z lies between x and y . Now take $y \leq \bigvee X$, for some countable set X . Since z is compact and $z \leq y$, there exists finite $E \subseteq X$ with $z \leq \bigvee E$, hence $x \leq \bigvee E$.

1.44. **Proposition:** *If M is a stably continuous σ–frame, then M is a retract of a coherent σ–frame, namely $\mathfrak{J}_\sigma M$.*

Proof: Define $\tau : M \longrightarrow \mathfrak{J}_\sigma M$ by letting $\tau(a)$ be $\{ x \in M \mid x \ll_\sigma a \}$. This is a countably generated ideal of M since M is stably continuous. τ is a morphism: since $0 \ll_\sigma 0$ and $e \ll_\sigma e$, $\tau(0) = \{0\}$ and $\tau(e) = M$. If $x \in \tau(a) \cap \tau(b)$ then $x \ll_\sigma a$ and $x \ll_\sigma b$, and hence $x \ll_\sigma a \wedge b$ since M is continuous. Thus $\tau(a \wedge b) = \tau(a) \cap \tau(b)$.

Now if $x \ll_\sigma a \vee b$, $x \leq u \vee v$ for some $u \ll_\sigma a$, $v \ll_\sigma b$. (Since M is continuous a and b are countable joins of elements σ–way below them, and this relation is preserved by finite joins.) But then $x = (x \wedge u) \vee (x \wedge v)$, and so $x \in \tau(a) \cup \tau(b)$. Also, for a countable updirected set $\{a_n\}$ in M, $\tau(\bigvee\{a_n\}) = \bigcup\{\tau(a_n)\}$. Since M is continuous, $\bigvee.\tau(a) = a$, and hence M is a retract of $\mathfrak{J}_\sigma M$. Note that τ is in fact proper: if $a \ll_\sigma b$, then $\tau(a) \subseteq {\downarrow}a \subseteq \tau(b)$. Since ${\downarrow}a$ is compact, ${\downarrow}a \ll_\sigma {\downarrow}a$ and therefore $\tau(a) \ll_\sigma \tau(b)$.

1.45. **Lemma:** *Any retract of a stably continuous σ–frame is stably continuous.*

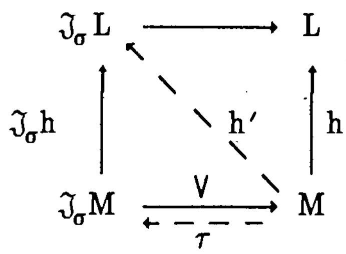
Proof: See Madden [1988].

From the above results it follows that as subcategories of σ**Frm**:

CohσFrm $\subseteq$ **StContσFrm** $\subseteq$ **ContσFrm**.

1.46. **Proposition:** *StContσFrm is coreflective in σFrm with coreflection functor $\mathfrak{J}_\sigma L$ and coreflection map $\bigvee$.*

Proof: Consider $h: M \longrightarrow L$ in σ**Frm** with M stably continuous.



Let $h' = \mathfrak{I}_\sigma h.\tau$. This map is proper since both τ and $\mathfrak{I}_\sigma h$ are. Moreover the diagram commutes since $V_L.h' = V_L.\mathfrak{I}_\sigma h.\tau = h.V_M.\tau = h.id_M = h$.

h' is unique : take any proper morphism $f : M \longrightarrow \mathfrak{I}_\sigma L$ such that $V.f = h$. For a given $a \in M$, if $x \ll_\sigma a$ then $f(x) \ll_\sigma f(a)$ and hence there exists $b \in L$ with $f(x) \subseteq \downarrow b \subseteq f(a)$. Thus $h(x) = V.f(x) \leq b \in f(a)$. This is true for all $x \ll_\sigma a$, and therefore $h'(a) \subseteq f(a)$. Now consider $t \in f(a)$, then $\downarrow t \ll_\sigma f(a)$. Since $f(a)$ is a countable join of elements of the form $f(x)$ where $x \ll_\sigma a$, and hence $f(x) \ll_\sigma f(a)$, it follows that $\downarrow t \subseteq f(x)$ for some $x \ll_\sigma a$. Thus $t \leq V.f(x) = h(x)$, and so $t \in h'(a)$. This gives $f = h'$.

Chapter 2 UNIFORM FRAMES.

Definition of a Uniform Frame.

2.1. DEFINITION: For $L \in \mathbf{Frm}$, $A \subseteq L$ is a *cover* of L if $\bigvee A = e$.

Let $\mathrm{Cov}(L)$ denote the collection of all covers of L.

For $C, D \in \mathrm{Cov}(L)$,

C *refines* D, written $C \leq D$, if for each $c \in C$, there exists $d \in D$ such that $c \leq d$.

C *meet* D, written $C \wedge D$, is $\{ c \wedge d \mid c \in C,\ d \in D \}$.

For $x \in L$,

the *star of* x with respect to C is $Cx = \bigvee \{ c \in C \mid c \wedge x \neq 0 \}$.

C *star refines* D, written $C \leq^* D$, if $C^* = \{ Cc \mid c \in C \} \leq D$.

(ie. for each $c \in C$ there exists $d \in D$ such that $Cc \leq d$.)

2.2. DEFINITION: Let $L \in \mathbf{Frm}$, μ a non-empty subset of $\mathrm{Cov}(L)$ such that

(U1) μ is a filter with respect to $\wedge$ and $\leq$.

(U2) For each $A \in \mu$, there exists $B \in \mu$ such that $B \leq^* A$.

then μ is a *uniformity*.

For $a, b \in L$, a is *uniformly below* b, written $a \triangleleft b$, if there exists $A \in \mu$ such that $Aa \leq b$.

μ is *compatible* on L if

(U3) For each $a \in L$, $a = \bigvee \{ b \in L \mid b \triangleleft a \}$.

If μ is compatible on L then (L,μ) is called a *uniform frame.*

If μ satisfies (U2) then μ is a *subbasis* for some uniformity, said to be generated by μ, which consists of all covers refined by a finite meet of covers in μ. If μ satisfies (U2) and is closed under finite meets, then μ is a *basis* for some uniformity, said to be generated by μ, which consists of all covers refined by some cover in μ.

A frame morphism $h: (L,\mu) \longrightarrow (M,\nu)$ is *uniform* if h preserves uniform covers. That is, for each $A \in \mu$, $h(A) \in \nu$.

These are the objects and maps of the category **UniFrm**.

The above definition is given by Frith [1986], A. Pultr [1984 I] gives a different definition for a uniformity : if $A^{\diamond} = \{ \bigvee B \mid B \subseteq A, (a, b \in B \Longrightarrow a \wedge b \neq 0) \}$ (ie. elements of B have pairwise non–zero meet), then Pultr replaces the star–refinement condition (U2) with the condition (U2)′ : For each $A \in \mu$, there exists $B \in \mu$ with $B^{\diamond} \leq A$.

The following sequence of lemmas shows the two definitions to be equivalent :

2.3. Lemma: *For any cover A, $A^* \leq A^{\diamond\diamond}$.*

Proof: Let $a \in A$, then $Aa = \bigvee \{ x \in A \mid x \wedge a \neq 0 \}$. For each $x \in A$ such that $x \wedge a \neq 0$, $\{x,a\} \subseteq A$ and the elements have pairwise non–zero meet, hence $\bigvee\{x,a\} \in A^{\diamond}$. Thus $\{ \bigvee\{x,a\} \mid x \in A, x \wedge a \neq 0 \} \subseteq A^{\diamond}$, and since

$\vee\{x,a\} \wedge \vee\{y,a\} \geq a$, elements of this subset have pairwise non–zero meet, therefore $\{ \vee\{x,a\} \mid x \in A, x \wedge a \neq 0 \} \in A^{\diamond\diamond}$.

Since $Aa = \vee \{ \vee\{x,a\} \mid x \in A, x \wedge a \neq 0 \}$, it follows that $A^* \leq A^{\diamond\diamond}$.

2.4. Lemma: *If $X \leq Y$ then $X^\diamond \leq Y^\diamond$.*

Proof: Take $\vee B \in X^\diamond$, (ie. $B \subseteq X$ such that the elements of B have pairwise non–zero meet.) For each b in B , $b \in X$, so there exists $y_b \in Y$ with $b \leq y_b$. Thus $\vee B \leq \vee\{ y_b \mid b \in B \}$. Now consider $\{ y_b \mid b \in B \} \subseteq Y$:

$y_b \wedge y_a \geq b \wedge a \neq 0$ since $a, b \in B$, hence the elements of this subset have pairwise non–zero meets, thus $\vee\{ y_b \mid b \in B \} \in Y^\diamond$.

2.5. Proposition: *If μ satisfies (U1) and (U3) then μ satisfies (U2)′ if and only if μ satisfies (U2) .*

Proof: Assume μ satisfies (U2)′ : let $A \in \mu$, then there exist B and D in μ such that $B^\diamond \leq A$ and $D^\diamond \leq B$. (ie. $D^\diamond \leq B \leq B^\diamond \leq A$.) But $D^{\diamond\diamond} \leq B^\diamond$ (lemma 2.4) and $D^* \leq D^{\diamond\diamond}$ (lemma 2.3), hence $D^* \leq A$.

Conversely, assume μ satisfies (U2) : let $B \leq^* A$, take $D \subseteq B$ such that $(x,y \in D \Longrightarrow x \wedge y \neq 0)$. If $D = \phi$ then $\vee D = 0 \leq a$ for each $a \in A$, else fix $d \in D$, then $\vee D \leq Dd \leq Bd \leq a$, for some $a \in A$, since $B \leq^* A$. Hence $B^\diamond = \{ \vee D \mid D \subseteq B, (x, y \in D \Longrightarrow x \wedge y \neq 0) \} \leq A$.

2.6. PROPOSITION: *For $(L,\mu) \in$ **UniFrm**, the uniformly below relation $\triangleleft$ is a sublattice of $L \times L$ satisfying the following properties :*

i) for any a, b, x, y in L, $x \leq a \triangleleft b \leq y \implies x \triangleleft y$.

ii) $a \triangleleft b \implies a \prec b$.

iii) $\triangleleft$ interpolates.

iv) $a \triangleleft b$ then $b^* \triangleleft a^*$ where $a^* = \bigvee \{ x \in L \mid x \wedge a = 0 \}$.

v) $\triangleleft$ is approximating. (ie. each element is a join of elements uniformly below it.)

vi) $\triangleleft$ is preserved by uniform frame morphisms.

Proof: The proofs of these properties, except (iv), is as for the σ–frame case in proposition 3.4, considering arbitrary and not countable covers.

iv) Let $a \triangleleft b$, with $A \in \mu$ such that $Aa \leq b$. Now consider $Ab^* = \bigvee \{ x \in A \mid x \wedge b^* \neq 0 \}$: take $x \in A$ with $x \wedge b^* \neq 0$, then $x \nleq b$ hence $x \wedge a = 0$, and so $x \leq a^*$.

Structures in the category of Uniform frames.

Product in UniFrm :

Consider a family $(L_i,\mu_i)_{i \in I}$ in **UniFrm**.

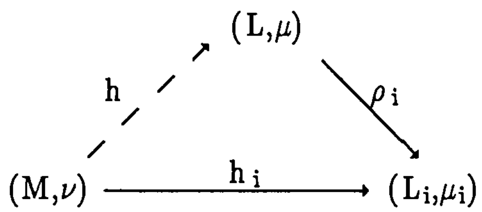
Let L be the cartesian product of the underlying sets $|L_i|$ of the L_i with $\leq$ defined componentwise. Then $L \in$ **Frm** and the projections ρ_i: $L \longrightarrow L_i$ are frame homomorphisms.

Let μ be the uniformity generated by those $A \in Cov(L)$ with $\rho_i(A) \in \mu_i$ $\forall i \in I$, (ie. $A_i \in \mu_i$ $\forall i \in I$) then μ is compatible with L :

Take $x \in L$, then $x = (x_i)_I$ with $x_i \in L_i$. Let $x^i \in L$ be defined by $x^i = (z_j)$ where $z_j = 0$ if $j \neq i$ and $z_i = x_i$, then $x = \bigvee \{ x^i \mid i \in I \}$. Now $x_i \in L_i$

implies that $x = \bigvee \{ y_r \mid y_r \triangleleft x_i \text{ wrt } \mu_i ,\ r \in J \}$, for some index set J . For each $r \in J$, $y_r \triangleleft x_i$ wrt μ_i implies that there exists $A_r \in \mu_i$ such that $A_r y_r \leq x_i$. Define A by $(A)_j = e$ if $j \neq i$ and $(A)_i = A_r$, then A is a cover of L and $A \in \mu$ (by definition of μ), and define $y^r \in L$ as x^i is defined above. Also, $(Ay^r)_j = 0$ for all $j \neq i$, and $(Ay^r)_i = A_r y_r \leq x_i$, hence $Ay^r \leq x^i$. That is, $y^r \triangleleft x^i$ wrt μ . Thus $x^i = \bigvee \{ y^r \mid y^r \triangleleft x^i \}$, and hence each element of L is a join of elements uniformly below it.

Consider an arbitrary family of maps $h_i: (M,\nu) \longrightarrow (L_i,\mu_i)$, $i \in I$ in **UniFrm**. Since L is the product of the L_i in **Frm**, there exists a unique frame homomorphism $h: M \longrightarrow L$ such that $h_i = \rho_i . h$,



In fact h is uniform : Let $V \in \nu$, then $h_i(V) \in \mu_i$ for all $i \in I$. Hence $\rho_i . h\,(V) \in \mu_i$ for all i , which implies that $h\,(V) \in \mu$.

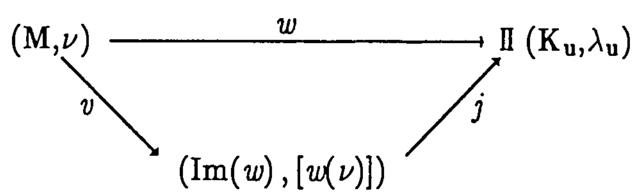
Hence (L,μ) is the uniform product of $(L_i,\mu_i)_{i \in I}$. By construction, the underlying frame of the uniform product is the product of the underlying frames. This says the forgetful functor **UniFrm** $\longrightarrow$ **Frm** preserves products.

Coequalizers in UniFrm :

The method of proof of the existence of coequalisers and coproducts in **UniFrm** correspond to those used by Banaschewski [1980a] to show their existence in **Frm**.

Consider two maps $f,g : (L,\mu) \longrightarrow (M,\nu)$ in **UniFrm**.

Let S be a representative set of all surjections $u: (M,\nu) \longrightarrow (K_u,\lambda_u)$ in **UniFrm** (ie $u: M \longrightarrow K_u$ is onto and $u(\nu)$ generates λ_u.) such that $u.f = u.g$. Take the uniform product of $\{ (K_u,\lambda_u) \mid u \in S \}$, then, by the universality of products there exists a map $w : (M,\nu) \longrightarrow \Pi\, (K_u,\lambda_u)$. Consider the factorization of w as follows :



where $[w(\nu)]$ is the uniformity generated by the image of ν (which is easily seen to be compatible on $\mathrm{Im}(w)$ since uniform maps preserve the uniformly below relation.) and j is the inclusion map. Then v is the coequaliser :

> take any $h: (M,\nu) \longrightarrow (N,\omega)$ in **UniFrm** with $h.f = h.g$, and let h' be h with codomain retricted to $\mathrm{Im}(h)$ and uniformity generated by $h(\nu)$. Since $h'.f = h'.g$, there exists $u_0 \in S$ with $\ell: (\mathrm{Im}(h),[h(\nu)]) \longrightarrow (K_{u_0},\lambda_{u_0})$ an isomorphism. Let q be the restriction of the projection ρ_{u_0} to $(\mathrm{Im}(w),[w(\nu)])$, then $h = i \,.\, \ell \,.\, q \,.\, v$ where $i :(\mathrm{Im}(h),[h(\nu)]) \longrightarrow (M,\nu)$ is the inclusion map. Thus h factors via v , and uniqueness follows since v is surjective.

Coproduct in UniFrm :

Consider a family $(L_i,\mu_i)_{i \in I}$ in **UniFrm**.

Let R be a represetative set for the family of maps $u = (u_i)_I$, where $u_i: (L_i,\mu_i) \longrightarrow (K_u,\nu_u)$ are uniform frame maps, such that K_u is generated by $\bigcup \{ \mathrm{Im}(u_i) \mid i \in I \}$ and ν_u is generated by $\bigcup \{ u_i(\mu_i) \mid i \in I \}$.

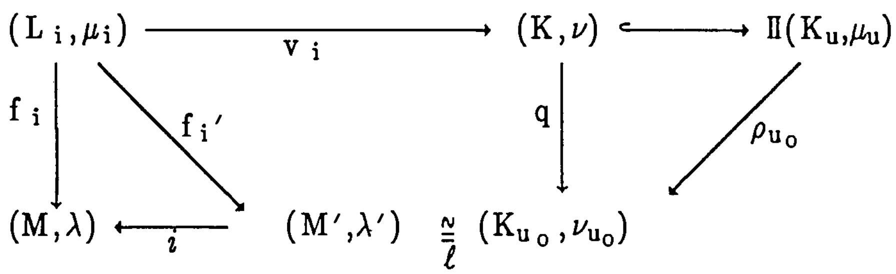
[ν_u is compatible on K_u :

> If $x \in K_u$ then $x = \bigvee \{ u_i(x_i) \mid i \in I \}$ with $x_i \in L_i$. Now $x_i \in L_i$ implies that $x_i = \bigvee \{ y_r \mid y_r \triangleleft x_i \text{ wrt } \mu_i ,\ r \in J \}$ for some index set J .

Since u_i is uniform and ν_u is generated by $\bigcup u_i(\mu_i)$, it follows that $u_i(x_i) = \bigvee\{ u_i(y_r) \mid u_i(y_r) \triangleleft u_i(x_i) \text{ wrt } \nu_u \ , \ r \in J \}$ and consequently $x = \bigvee\{ a \mid a \triangleleft x \text{ wrt } \nu_u\}$]

Take the uniform product of $\{ (K_u,\nu_u) \mid u \in R \}$, and define $w_i: (L_i,\mu_i) \longrightarrow \Pi (K_u,\nu_u)$ by $w_i(x) = (u_i(x))_{u\in R}$, then w_i is a uniform frame map (by the definition of the uniform product). Let K be the subframe of ΠK_u which is generated by $\bigcup \mathrm{Im}(w_i)$ with uniformity ν generated by $\bigcup w_i(\mu_i)$ (ν is compatible on K : the proof is as for ν_u above), and let the uniform frame maps w_i restricted to (K,ν) be v_i .

Consider an arbitrary family of maps $f_i: (L_i,\mu_i) \longrightarrow (M,\lambda)$ in **UniFrm**. Let M' be the subframe of M generated by $\bigcup \mathrm{Im}(f_i)$ with uniformity $\lambda' \subseteq \lambda$ generated by $\bigcup f_i(\mu_i)$, then $(M',\lambda') \cong (K_{u_0},\nu_{u_0})$ for some $u_0 \in R$,



hence $f_i = i \, . \, \ell \, . \, q \, . \, v_i$, thus f_i factors via v_i .

Suppose that $f_i = g \, . \, v_i$ for all i , then $i \, . \, \ell \, . \, q$ and g coincide on $\bigcup \mathrm{Im}(v_i)$, which generates K . Since two homomorphisms coinciding on generating sets are equal, $g = i \, . \, \ell \, . \, q$. Hence the factorization is unique.

The 'Fine' uniformity :

It has been shown that a frame is uniformizable iff it is completely regular. (Pultr [1984 I] and Frith [1986]) As in the spatial situation, it is natural to consider under what conditions the collection of all covers of a given frame form a uniformity, and if not, what the 'finest' uniformity on the frame is.

Let $L \in$ **CrgFrm**, and consider all uniformities $\mu \subseteq Cov(L)$ compatible on L.

Let μ_F be the uniformity generated by $\bar{\mu} = \cup \{ \mu \mid (L,\mu) \in$ **UniFrm** $\}$ (ie. $\mu_F = \{ A \in Cov(L) \mid \exists A_1, \dots A_k \in \bar{\mu}$ with $A_1 \wedge \dots \wedge A_k \leq A \}$) This is a uniformity since $\bar{\mu}$ is a subbasis : for each $A \in \bar{\mu}$, $A \in \mu$ for some $(L,\mu) \in$ **UniFrm** , hence there exists $B \stackrel{*}{\leq} A$ in μ (by (U2)), but then $B \in \bar{\mu}$.)

Since each μ is compatible on L , so is μ_F , hence $(L,\mu_F) \in$ **UniFrm**. Obviously any uniformity μ compatible on L is contained in $\bar{\mu}$, hence $(L,\mu) \xrightarrow{id_L} (L,\mu_F)$ is a uniform frame morphism.

2.7. Definition: A sequence of covers $A_1, A_2, \dots$ is a *normal sequence* if $A_{n+1} \leq^* A_n$.
$A \in Cov(L)$ is a *normal cover* if $A = A_1$ in some normal sequence.
A family $\mu \subseteq Cov(L)$ is a *normal family* if for each $A \in \mu$, there exists $B \in \mu$, with $B \leq^* A$. (ie. μ satisfies (U2), hence is a subbase for a uniformity .)

In **Frm** all covers correspond to 'open' covers in the spatial setting, hence a normal cover is 'normally open' in the sense of Willard [1968] .

2.8. Proposition: *The 'fine' uniformity* μ_F *on* L *is the uniformity on* L *consisting of all normal covers of* L .

Proof: Let μ be any uniformity on L and take any normal cover A , say with $A = A_1$ in some normal sequence $\{A_n\}$, then $\mu \cup \{A_n\}$ is a normal family, hence a subbase for some uniformity which is compatible with L since μ is. Hence $A \in \mu_F$. On the otherhand, by the star–refinement property for uniformities, each $A \in \mu_F$ is a normal cover.

2.9 Lemma: *Let $(M,\nu) \in$ **UniFrm** and $h: M \longrightarrow L$ in **CrgFrm**, then $h: (M,\nu) \longrightarrow (L,\mu_F)$ is uniform.*

Proof: It suffices to show that $h[\nu]$ is a normal family, (ie. that $h[A]$ is a normal cover for each $A \in \nu$.) Let $A \in \nu$, that is $h[A] \in h[\nu]$. Since ν is a uniformity, there exists $B \in \nu$ with $B \leq^* A$. Now take $b \in B$, then there exists $a \in A$ with $Bb \leq a$,

so $h[B]h(b) = \bigvee \{ h(x) \mid h(x) \wedge h(b) \neq 0, x \in B \}$

$\leq \bigvee \{ h(x) \mid x \wedge b \neq 0, x \in B \}$

$= h [\bigvee \{ x \mid x \wedge b \neq 0, x \in B \}$ (h is a frame map)

$= h [Bb] \leq h(a)$

Hence $h[B] \leq^* h[A]$.

Banaschewski [1988] shows directly that a compact regular frame has a unique uniformity, and the analogous result holds for the appropriate σ–frames as will be shown in lemma 3.15. This fact, in conjunction with the above lemma, yields the following corollary :

2.10. Corollary: *If $h: L \longrightarrow M$ in **CrgFrm** and $M \in$ **KRegFrm**, then for any uniformity μ on L, $h: (L,\mu) \longrightarrow M$ is uniform.*

This is precisely the frame analogue to the result that any continuous function from a compact space to a uniform space is uniformly continuous. (Willard [1968])

2.11. Definition: L ∈ **Frm** is *paracompact* if each cover has a locally finite refinement, where a cover B is locally finite if there is a cover C such that each element of B has non–zero meet with only finitely many elements of C .

L ∈ **Frm** is *fully normal* if each cover has a star refinement (ie. Cov (L) is a normal family.)

Dowker and Strauss [1975] show that a frame is fully normal iff it is paracompact and normal. Since paracompact and normal imply complete regularity, (Dowker and Strauss [1972]) every fully normal frame is uniformizable.

2.12. Proposition: *In a paracompact, normal frame L (ie. fully normal), the fine uniformity consists of all covers.* (ie. $\mu_F = Cov\ (L)$)

Proof: By proposition 2.8. μ_F consists of all normal covers of L . Since L is paracompact and normal, hence fully normal, every cover of L is normal, hence every cover is in μ_F .

In the spatial case, paracompact implies normal in a Hausdorff setting, thus giving the classical result that in paracompact spaces, the fine uniformity is generated by all open covers. (Willard [1968]).

Some coreflections in UniFrm :

2.13. DEFINITION: (L,μ) is *precompact* if μ has a basis of finite covers.

separable if μ has a basis of countable covers.

2.14. LEMMA: *a) In a uniform frame, every finite uniform cover has a finite uniform star–refinement.*

b) In a uniform frame, every countable uniform cover has a countable uniform star–refinement.

Proof: a) For $(L,\mu) \in$ **UniFrm**, let $A \in \mu$ be any finite cover. Then there exists $C \leq^* A$ in μ (by the star–refinement property (U2).) Define an equivalence relation on C as follows : $x \sim y$ if and only if i) $x \leq a \Leftrightarrow y \leq a$ for all $a \in A$ and ii) $Cx \leq a \Leftrightarrow Cy \leq a$ for all $a \in A$.

Since A is finite there are only finitely many equivalence classes. Let D be the cover consisting of the joins of the equivalence classes of C wrt $\sim$. (ie. $D = \{ \bar{x} \mid \bar{x} = \bigvee\{ z \in C \mid z \sim x \} \}$). Since $C \leq D$, $D \in \mu$. Take any $\bar{x} \in D$ and $a \in A$ such that $Cx \leq a$ (since $C \leq^* A$). Suppose that $\bar{y} \wedge \bar{x} \neq 0$ for $\bar{y} \in D$, then there exists $z_x, z_y \in C$ with $z_y \wedge z_x \neq 0$ and $z_x \sim x$, $z_y \sim y$. Since $z_y \sim x$ and $Cx \leq a$, $Cz_x \leq a$, but $z_y \leq Cz_x$, hence $y \leq Cz_x$ (by definition of the equivalence). Hence $y \leq a$ and moreover for each $z \sim y$ in C, $z \leq a$ and so it follows that $\bar{y} \leq a$. Since this holds for any element of D having non–zero meet with $\bar{x}$, $D\bar{x} \leq a$, hence $D \leq^* A$.

b) For $(L\ \mu)$ a uniform frame, let A be a countable uniform cover. There exist covers $B, D \in \mu$ such that $B \leq^* D \leq^* A$. Suppose that $A = \{a_i\}$,

$B = \{b_\alpha\}$. Define three functions m, n and q as follows:

$m(\alpha)$ = least i such that $b_\alpha^* \leq a_i$

$n(\alpha)$ = least i such that $b_\alpha^{**} \leq a_i$

$q(\alpha)$ = set of all i such that $b_\alpha \leq a_i$ and $i \leq n(\alpha)$.

Then $m(\alpha) \leq n(\alpha)$ and $m(\alpha), n(\alpha) \in q(\alpha)$.

For $j \leq k$ and ϕ a set of integers $\leq k$, including both j and k, let

$$c_{jk\phi} = \bigvee\{b_\alpha \mid m(\alpha) = j,\ n(\alpha) = k,\ q(\alpha) = \phi\}.$$

Let the b_α's forming this join be called "parts of $c_{jk\phi}$". Clearly $C = \{c_{jk\phi}\}$ is countable and uniform since $B \leq C$. Take b_α part of $c_{jk\phi}$, b_β part of $c_{rs\psi}$ with $b_\alpha \wedge b_\beta \neq 0$. It follows that $b_\alpha \leq b_\beta^*$, so $b_\alpha^* \leq b_\beta^{**} \leq a_s$, whence $j \leq s$. Moreover $b_\beta \leq b_\alpha^* \leq a_j$, hence $j \in \psi$. This implies that $c_{rs\psi} \leq a_j$. Since this holds for each such $c_{rs\psi}$ which has non–zero meet with $c_{jk\phi}$, $c_{jk\phi} \leq^* a_j$. Therefore $C \leq^* A$.

The above proofs for cases a) and b) are essentially those given by Ginsburg and Isbell [1959 (Theorems 1.1 and 1.2)] for the analogous results for spaces.

2.15. Lemma: *For $(L,\mu) \in$ **UniFrm**, let μ_p be the uniformity generated by all finite uniform covers of μ. Then (L,μ_p) is a precompact uniform frame.*

Proof: By lemma 2.14, the finite covers form a uniform base, hence μ_p is a uniformity and μ_p is precompact by definition. Suppose $b \triangleleft a$ wrt μ, then there exists $A \in \mu$ with $Ab \leq a$. Let $a_1 = \bigvee\{x \in A \mid x \wedge b \neq 0\}$ (ie. $a_1 = Ab$) and $a_2 = \bigvee\{x \in A \mid x \wedge b = 0\}$, then $A' = \{a_1, a_2\} \geq A$, hence $A' \in \mu_p$ and $A'b \leq a$. Therefore $b \triangleleft a$ wrt μ_p. Since $(L,\mu) \in$ **UniFrm**, for each $a \in L$, $a = \bigvee\{b \mid b \triangleleft a \text{ wrt } \mu\}$ hence $a = \bigvee\{b \mid b \triangleleft a \text{ wrt } \mu_p\}$. This shows that μ_p is compatible on L.

2.16. **Proposition**: *Precompact uniform frames are coreflective in* ***UniFrm****, with coreflection map* $id_L : (L,\mu_P) \longrightarrow (L,\mu)$.

Proof: Let $(L,\mu) \in$ **UniFrm**, then (L,μ_P) is a precompact uniform frame. Consider any uniform frame homomorphism $h: (M,\nu) \longrightarrow (L,\mu)$ with (M,ν) a precompact uniform frame. Since (M,ν) is precompact, ν has a basis of finite covers. Let $V \in \nu$, then there exists a finite $V' \leq V$ in ν, which implies that $h[V'] \in \mu$ is finite and hence in μ_P . Therefore $h[V] \in \mu_P$ since $h[V'] \leq h[V]$, hence h factors via the identity on L :

$$\begin{array}{ccc} (L,\mu_P) & \xrightarrow{id_L} & (L,\mu) \\ & \nwarrow & \uparrow h \\ & & (M,\nu) \end{array}$$

Uniqueness of the factorisation follows since id_L is monic.

2.17. **Lemma**: *For* $(L,\mu) \in$ ***UniFrm****, let* μ_S *be the uniformity generated by all countable uniform covers of* μ *, then* (L,μ_S) *is a separable uniform frame.*

Proof: As for lemma 2.15. above.

2.18. **Proposition**: *Separable uniform frames are coreflective in* ***UniFrm****, with coreflection map* $id_L : (L.\mu_S) \longrightarrow (L,\mu)$.

Proof: As for proposition 2.16. above.

2.19. PROPOSITION: *For any cardinal κ and uniform structure μ, the following are equivalent:*

1) Every uniform cover has a subcover of cardinality κ.

2) Every uniform cover is refined by a uniform cover of cardinality κ.

Proof: 2) $\Rightarrow$ 1) is trivial. Assume condition 1) : let $A \in \mu$, and since μ is a uniformity take $E \leq^* D \leq^* A$ in μ. By assumption there exists a subcover $F \subseteq E$ of cardinality κ. Put $B = \{ Df \mid f \in F \}$, a cover of cardinality κ, and now take $f \in F$, then there exists $d \in D$ with $f \leq d$, hence $Df \leq Dd \leq a$ for some $a \in A$, since $D \leq^* A$. Thus B refines A. Moreover B is a uniform cover since it is refined by E: if $f \in F \subseteq E$, then $f \leq Df \in B$ (since D is a cover), else $e \in E$ and $e \notin F$, then there exists $g \in F$ with $g \wedge e \neq 0$ (since F is a cover). This implies that $e \leq Eg \leq Dg \in B$.

2.20. COROLLARY: *A Lindelof uniform frame is separable.*

The cozero part of a uniform frame.

2.21. DEFINITION: An ideal J of L is *uniformly regular* if for each $x \in J$, there exists $y \in J$ with $x \triangleleft y$.

Let $\mathfrak{R}L$ be the collection of all uniformly regular ideals of (L,μ).

2.22. PROPOSITION: ***KRegFrm*** *is coreflective in* ***UniFrm*** *with coreflection functor $\mathfrak{R}$, and coreflection map $\rho_L : \mathfrak{R}L \longrightarrow L$ given by join.*

2.25. COROLLARY: *Coz_u L is a regular sub−σ−frame of L as a σ−frame.*

Proof: $\mathfrak{R}L$ is compact regular and hence spatial. Thus $\mathrm{Coz}\,\mathfrak{R}L \cong \mathfrak{A}.\mathrm{coz}\,\Sigma\mathfrak{R}L \in$ **RegσFrm** . Since the image of a regular σ–frame is a regular σ–frame, $\rho_L(\mathrm{Coz}\,\mathfrak{R}L)$ is a regular σ–frame, and thus so is $\mathrm{Coz}_u\,L$.

2.26. LEMMA: *Coz $\mathfrak{R}L$ consists of precisely the countably generated uniformly regular ideals.*

Proof: Let $J \in \mathrm{Coz}\,\mathfrak{R}L$. For each $b \in J$ choose a sequence $b \triangleleft b_1 \triangleleft b_2 \triangleleft \ldots$ in J. Let J_b be the uniformly regular ideal generated by this sequence, then J is the join of all the J_b's . By the same argument as lemma 1.5. since $\mathfrak{R}L$ is compact, J is countably generated. Conversely, let $J \in \mathfrak{R}L$ be a countably generated . Since $\mathfrak{R}L$ is regular, J is a join of ideals rather below it, and again using the same argument as lemma 1.5. it follows that J is a cozero element.

2.27. COROLLARY: *For $(L,\mu) \in$ **UniFrm**, the uniform cozero elements are precisely those elements which are the join of a sequence of elements uniformly below each other.*

Proof: Let $a \in \mathrm{Coz}_u\,L$, then $a = \rho_L(J)$ for some $J \in \mathrm{Coz}\,\mathfrak{R}L$. Now J is countably generated, say by $c_1, c_2, c_3, \ldots$. Define a sequence (a_n) by $a_1 = c_1$ and a_{n+1} such that $a_n \vee c_{n+1} \triangleleft a_{n+1}$. Then $a = \bigvee J = \bigvee a_n$ and $a_n \triangleleft a_{n+1}$. Conversely, suppose $a = \bigvee a_n$, with $a_n \triangleleft a_{n+1}$. Let $J = \{ x \in L \mid x \leq a_n ,\ \text{for some } n \}$. Then J is countably generated uniformly regular ideal and so is in $\mathrm{Coz}\,\mathfrak{R}L$. Since $a = \bigvee J$, $a \in \mathrm{Coz}_u\,L$.

2.28. Corollary: *For $(L,\mu) \in$ **UniFrm**, each uniform cozero element is a countable join of uniform cozero elements uniformly below it.*

Proof: As above, let $a \in Coz_u L$ with $a = \rho_L(J)$ and J generated by $a_1 \lhd a_2 \lhd a_3 \lhd \ldots$ Then let J_n be the ideal generated by $a_n = a_{n_1} \lhd a_{n_2} \lhd \ldots\ldots a_{n+1}$. (ie. repeated interpolation of $a_n \lhd a_{n+1}$). Then J is the join of these J_n's and $J_n \in Coz\, \mathfrak{R}L$ for each n. Let $a_n = \rho_L(J_n)$, then $a_n \in Coz_u L$. Moreover $a_n \lhd a$ for each n. Now $a = \rho_L(J) = \rho_L(\bigvee J_n) = \bigvee \rho_L(J_n) = \bigvee a_n$.

2.29. Lemma: *$Coz_u L$ generates L as a frame.*

Proof: In lemma 2.26. it was shown that each J in $\mathfrak{R}L$ is the join of cozero elements of $\mathfrak{R}L$, thus $Coz\, \mathfrak{R}L$ generates $\mathfrak{R}L$ as a frame. Now $\rho_L : \mathfrak{R}L \longrightarrow L$ is onto : consider $\mathfrak{r}(a) = \{ x \in L \mid x \lhd a \}$ for $a \in L$. By the properties of the uniformly below relation this is a uniformly regular ideal, and since each element of L is the join of elements uniformly below it, $a = \rho_L(\mathfrak{r}(a))$. But $\mathfrak{r}(a)$ is the join of cozero elements below it, say $\mathfrak{r}(a) = \bigvee J_i$ with $J_i \in Coz\, \mathfrak{R}L$. Then $a = \rho_L(\bigvee J_i) = \bigvee \rho_L(J_i)$ and $\rho_L(J_i) \in Coz_u L$ for each i. Hence each element of L is the join of uniform cozero elements.

2.30. Proposition: *Every uniform frame has a basis of cozero covers, that is, covers consisting of uniform cozero elements.*

Proof: This follows easily from the observation that if $a \lhd b$, then there exists $c \in Coz_u L$ such that $a \leq c \leq b$: Let $a \lhd b$ in L, then by repeated interpolation of the uniformly below relation, there exists a sequence

$a = x_1 \triangleleft x_2 \triangleleft \ldots \triangleleft b$. Put $c = \bigvee x_n$, then by corollary 2.27. c is a uniform cozero element, and $a \leq c \leq b$.

Note that the result in lemma 2.29. follows directly from this proposition and corollary 2.27.

Chapter 3 UNIFORM σ–FRAMES.

Definition of a uniform σ–frame.

3.1. Definition: For $L \in \sigma\mathbf{Frm}$, a *cover* of L is a sequence (a_n) in L such that $\bigvee a_n = e$.

Let Cov (L) denote all the covers of L.

A cover $A = (a_n)$ *refines* $B = (b_n)$, written $A \leq B$, if for each n, there exists m such that $a_n \leq b_m$.

Two covers A and B are *equivalent* if $A \leq B$ and $B \leq A$.

For $x \in L$ and $A \in Cov(L)$,

the *star of x* with respect to A is $Ax = \bigvee \{ a \in A \mid a \wedge x \neq 0 \}$,

and the *star of A* is $A^* = \{ Aa \mid a \in A \}$, which is a cover of L since it is countable and refined by A.

For $A, B \in Cov(L)$,

the *meet* of A and B is defined by $A \wedge B = \{ a \wedge b \mid a \in A, b \in B \}$, which is a countable set and a cover (by the distributivity property of L.)

A *star refines* B, written $A \leq^* B$, if $A^* \leq B$. That is, for each n, there exists m such that $Aa_n \leq b_m$.

The statements of the following lemmas correspond to similar properties for frames (Pultr [1984 I] and Banaschewski [1988])

3.2. Lemma: For A, B, C, D $\in$ Cov(L) ,

i) $x \prec Ax$ for each $x \in L$.

ii) $A \leq^* B \implies A(Ax) \leq Bx$ for each $x \in L$.

iii) $A \leq^* B,\ C \leq^* D \implies A \wedge C \leq^* B \wedge D$.

Proof: i) Let $A \in$ Cov (L) with $A = (a_n)$ and $x \in L$, then

$e = \vee a_n = \vee \{ a_i \mid a_i \wedge x = 0 \} \vee \vee \{ a_i \mid a_i \wedge x \neq 0) = s \vee Ax$ where $s = \vee \{ a_i \mid a_i \wedge x = 0 \}$ and since $s \wedge x = 0$, s separates x and Ax .

ii) Let $A \leq^* B$ and $x \in L$, then $A(Ax) = \vee \{ a_n \in A \mid a_n \wedge Ax \neq 0 \}$. Take $a_n \in A$ such that $a_n \wedge Ax \neq 0$, then $a_n \wedge \vee \{ a_i \in A \mid a_i \wedge x \neq 0 \} \neq 0$, and hence there exists a_i such that $a_n \wedge a_i \neq 0$ and $a_i \wedge x \neq 0$. It follows that $a_n \leq Aa_i$, and $a_i \wedge x \neq 0$. Now since $A \leq^* B$, there exists $b_{m_i} \in B$ with $Aa_i \leq b_{m_i}$ and thus $b_{m_i} \leq Bx$. Moreover $a_n \leq b_{m_i}$, so $a_n \leq Bx$. This holds for all a_n which have non–zero meet with Ax , hence $A(Ax) \leq Bx$.

iii) Let $A \leq^* B$ and $C \leq^* D$. Take $a_n \in A$, $c_m \in C$, then $(A \wedge C)(a_n \wedge c_m) = \{ a_i \wedge c_j \mid (a_i \wedge c_j) \wedge (a_n \wedge c_m) \neq 0 \} \leq Aa_n \wedge Cc_m$. Since $A \leq^* B,\ C \leq^* D$ there exists b_{n_0} and d_{m_0} such that $Aa_n \leq b_{n_0}$ and $Cc_m \leq d_{m_0}$. Hence $(A \wedge C)(a_n \wedge c_m) \leq b_{n_0} \wedge d_{m_0}$.

3.3. Definition: A *uniformity* is a subset μ of Cov (L) such that

(U1) μ is a filter (with respect to $\leq$).

(U2) For each $A \in \mu$, there exists $B \in \mu$ such that $B \leq^* A$.

For $a, b \in L$, a is *uniformly below* b, written $a \triangleleft b$, if there exists $A \in \mu$ such that $Aa \leq b$.

μ is *compatible* on L if

(U3) For each $a \in L$, there exists a sequence (a_n) in L with

$a = \bigvee a_n$ and $a_n \triangleleft a$, for all n.

If μ is compatible on L then (L,μ) is called a *uniform σ-frame.*

If μ satisfies (U2) then μ is a *subbasis* for some uniformity, said to be generated by μ, which consists of all covers refined by a finite meet of covers in μ. If μ satisfies (U2) and is closed under finite meets, then μ is a *basis* for some uniformity, said to be generated by μ, which consists of all covers refined by some cover in μ.

$h: (L,\mu) \longrightarrow (M,\nu)$ in **σFrm** is *uniform* if for every $A \in \mu$, $h[A] \in \nu$. That is, h preserves uniform covers.

UniσFrm is the category consisting of all uniform σ–frames and uniform σ–frame morphisms.

3.4. PROPOSITION: *For $(L,\mu) \in$ UniσFrm, the uniformly below relation $\triangleleft$ is a sublattice of $L \times L$ satisfying the following properties :*

i) for any a, b, x, y in L, $x \leq a \triangleleft b \leq y \implies x \triangleleft y$.

ii) $a \triangleleft b \implies a \prec b$.

iii) $\triangleleft$ interpolates.

iv) $\triangleleft$ is σ–approximating.

v) $\triangleleft$ is preserved by uniform σ–frame morphisms.

Proof: $\triangleleft$ is a sublattice of $L \times L$: Take $a \triangleleft b$, $c \triangleleft d$ in (L,μ), then there exists $A, C \in \mu$ with $Aa \leq b$ and $Cc \leq d$. Now consider $A \wedge C \in \mu$, since $(A \wedge C)(a \vee c) \leq Aa \vee Cc \leq b \vee d$, and $(A \wedge C)(a \wedge c) \leq Aa \wedge Cc \leq b \wedge d$, it follows that $a \vee c \triangleleft b \vee d$ and $a \wedge c \triangleleft b \wedge d$. For any cover A, $A0 \leq a$ and $Aa \leq e$ for every $a \in L$, hence $0 \triangleleft a$ and $a \triangleleft e$. Thus $0 \triangleleft 0$ and $e \triangleleft e$ so $\triangleleft$ is a bounded sublattice of $L \times L$.

i) If $x \leq a \triangleleft b \leq y$ with $Aa \leq b$, then $Ax \leq Aa \leq b \leq y$, thus $x \triangleleft y$.

ii) By lemma 3.2, $a \prec Aa \leq b$, hence $a \prec b$.

iii) If $a \triangleleft b$ with $Ba \leq b$ then, since μ is a uniformity, there exists $A \overset{*}{\leq} B$ in μ, and then by lemma 3.2, $A(Aa) \leq Ba$, and so $a \triangleleft Aa \triangleleft Ba \leq b$. This shows that $\triangleleft$ interpolates.

iv) The compatibility of μ with L immediately gives that $\triangleleft$ is σ–approximating.

v) Consider $h: (L,\mu) \longrightarrow (M,\nu)$ in **UniσFrm**, and $a \triangleleft b$ in L with $Aa \leq b$, then $A \in \mu$ implies that $h[A] \in \nu$ (since h is uniform). Now

$$\begin{aligned} h[A]h(a) &= \bigvee \{ h(x) \in h[A] \mid h(a) \wedge h(x) \neq 0 \} \\ &= \bigvee \{ h(x) \mid h(x \wedge a) \neq 0 \} \\ &\leq \bigvee \{ h(x) \mid x \wedge a \neq 0 \} \\ &= h(\bigvee \{ x \in A \mid x \wedge a \neq 0 \} \\ &= h[Aa] \leq h(b). \end{aligned}$$

Hence $h(a) \triangleleft h(b)$.

That every uniform frame is regular follow as a corollary to 3.4 . This is the σ–frame counterpart of the result for spaces that every uniformizable space is completely regular.

3.5. Lemma: *If $a \prec b$ in $L \in$ RegσFrm with separating element x, then there exists c in L such that $a \prec c \prec b$ and x separates $a \prec c$. Moreover there exists y separating $c \prec b$ such that $y \prec x$.*

Proof: In **σFrm**, regular implies normal (Banaschewski [1980]) and hence the rather below relation interpolates : take $a \prec b$ with separating element x . Since $x \vee b = e$, by normality there exists c, y in L such that $c \vee x = e = y \vee b$ and $c \wedge y = 0$. Hence x separates a and c , y separates c and b , and c separates y and x .

3.6. Proposition: *Every regular σ–frame has a compatible uniformity.*

Proof: Let L be a regular σ–frame. For each $a \prec b$ in L , there exists a separating element s_{ab} , and thus $\{s_{ab},b\} \in \mathrm{Cov}(L)$. Let $\bar{\mu}$ be the collection of all covers of the form $\{s_{ab},b\}$ for $a \prec b$ in L , ranging over all possible separating elements. Let μ consist of all those covers which are finite meets of covers of $\bar{\mu}$. Then to show that μ is a base for some uniformity it suffices, by 3.2 iii) to show that μ satisfies (U2) : take $\{s_{ab},b\} \in \mu$, so $a \prec b$ in L with separating element s_{ab} . By the lemma above there exists c, y in L such that $a \prec c \prec b$, s_{ab} separates $a \prec c$, y separates $c \prec b$ and $y \prec s_{ab}$.

Applying this lemma again gives d, z in L such that $c \prec d \prec b$, y separates $c \prec d$, z separates $d \prec b$ and $z \prec y$. Let $y = s_{cd}$, $z = s_{db}$. Then $a \prec c \prec d \prec b$ and $s_{db} \prec s_{cd} \prec s_{ab}$. Now let

$$C = \{s_{ab}, c\} \wedge \{s_{cd}, d\} \wedge \{s_{db}, b\} \in \mu$$

$$= \{s_{db}, s_{cd} \wedge b, s_{ab} \wedge d, c\}.$$

Then $C(s_{db}) \leq s_{db} \vee (s_{cd} \wedge b) \leq s_{cd} \leq s_{ab}$,

$$C(s_{cd} \wedge b) \leq (s_{cd} \wedge b) \vee s_{db} \vee (s_{ab} \wedge d)$$

$$\leq s_{cd} \vee (s_{ab} \wedge d)$$

$$\leq s_{ab},$$

$$C(s_{ab} \wedge d) \leq (s_{ab} \wedge d) \vee (s_{cd} \wedge b) \vee c$$

$$\leq (s_{ab} \wedge b) \vee c$$

$$\leq b,$$

$$Cc \leq c \vee (s_{ab} \wedge d) \leq d \leq b.$$

Hence $C \leq^* \{s_{ab},b\}$. Also if $a \prec b$ then $\{s_{ab},b\} \in \mu$ for some separating element s_{ab}, and $\{s_{ab},b\}(a) = b$, hence $a \triangleleft b$. Thus the compatibility of the uniformity generated by μ follows as a consequence of the regularity of L.

For any regular σ–frame L, the uniformity generated as above will be denoted by μ_L. Since regularity implies normality in σ**Frm**, and normality is equivalent to having each finite cover shrinkable (Banaschewski and Gilmour [1987]), the above construction of μ gives the same uniformity as the one generated by all the finite covers. (see proposition 3.43.)

3.7. PROPOSITION: *If $f: L \longrightarrow M$ in RegσFrm, then $f: (L,\mu_L) \longrightarrow (M,\mu_M)$ is uniform.*

Proof: If $a \prec b$ in L then $f(a) \prec f(b)$ in M. Moreover, if s_{ab} separates $a \prec b$, then $f(s_{ab})$ separates $f(a) \prec f(b)$, so if $\{s_{ab},b\} \in \mu_L$ then $f[\{s_{ab},b\}] = \{f(s_{ab}),f(b)\} \in \mu_M$. Hence f is uniform.

3.8. Corollary: *There is a functor from* **RegσFrm** *to* **UniσFrm** *which assigns to any regular σ–frame L, the uniformity* μ_L.

The technicalities here are different to those employed in the proof of the corresponding result in **Frm** (ie. every completely regular frame is shown to have a compatible uniformity. (Frith [1986])) where pseudocomplements exist and these obviate the necessity of considering all separating elements. Thus in the frame setting a coarser uniformity suffices.

Structured version of the spectrum functor Ψ: RegσFrm ⟶ Alex.

For $(L,\mu) \in$ **UniσFrm**, let ΨL be the spectrum of all σ–prime filters on L with cozero sets $\Psi_a \subseteq \Psi L$ where $\Psi_a = \{ P \in \Psi L \mid a \in P \}$. For each $A \in \mu$, let $\Psi_A = \{ \Psi_a \mid a \in A \}$.

3.9. Proposition: $\{ \Psi_A \mid A \in \mu \}$ *is a basis for a separable uniformity* $\Psi\mu$ *on* ΨL.

Proof: For each $A \in \mu$, $\bigcup\{ \Psi_{a_n} \mid a_n \in A \} = \Psi_{\vee a_n} = \Psi_e = L$, hence Ψ_A is a cover of ΨL. If $A \leq B$ in μ then $\Psi_A \leq \Psi_B$ and if $A, B \in \mu$ then $\Psi_{A \wedge B} = \Psi_A \wedge \Psi_B$, so since $A \wedge B \in \mu$, this collection of covers is closed under finite meets. For each $B \in \mu$, there exists $A \leq^* B$ in μ. Now take $a \in A$, then

$\Psi_A(\Psi_a) = \bigcup \Psi_x \in \Psi_A \mid \Psi_a \wedge \Psi_x \neq \phi \} = \bigcup \{ \Psi_x \mid \Psi_{a \wedge x} \neq \phi \}$

$= \bigcup \{ \Psi_x \mid a \wedge x \neq 0 \} = \Psi_{Aa}$, and since $Aa \leq b$ for some $b \in B$, it follows that $\Psi_{Aa} \leq \Psi_b$. Hence $\Psi_A \leq^* \Psi_B$.

3.10. Proposition: *The topology induced by* $\Psi\mu$ *on* ΨL *is the topology associated with* ΨL *as Alexandroff space.*

Proof: Take any open set $X \subseteq \Psi L$, then X is an arbitrary union of cozero sets, say $X = \bigcup \{ \Psi_{a_i} \mid i \in I \}$ for some index set I. For each $P \in X$, there exists $i \in I$ such that $a_i \in P$. Now a_i is the countable join of elements uniformly below it, and since P is σ–prime, there exists $b \triangleleft a_i$ in L with $b \in P$. Hence $P \in \Psi_b \triangleleft \Psi_{a_i}$. It follows that X is open with respect to the topology induced by the uniformity. Now consider $X \subseteq \Psi L$, open with respect to the topology induced by the uniformity. For each $P \in X$ there exists $\Psi_A \in \Psi\mu$ such that $\Psi_A(\{P\}) \subseteq X$. Choose any $a_p \in A$ such that $P \in \Psi_{a_p} \subseteq X$, then $X = \bigcup \{ \Psi_{a_p} \mid P \in X \}$.

3.11. Proposition: *If* $h: (L,\mu) \longrightarrow (M,\nu)$ *in* ***UniσFrm****, then* $\Psi h: (\Psi M,\Psi\nu) \longrightarrow (\Psi L,\Psi\mu)$ *defined by* $\Psi h(P) = h^{-1}(P)$ *for* $P \in \Psi M$, *is a uniformly continuous map.*

Proof: Since $\Psi : \sigma\mathbf{Frm} \longrightarrow \mathbf{Alex}$ is a contravariant functor, it is only necessary to check that the map is uniformly continuous : take $\Psi_A \in \Psi\mu$, with $\Psi_A = \{ \Psi_{a_n} \mid a_n \in A \}$. Since $(\Psi h)^{-1}(\Psi_a) = \Psi_{h(a)}$, $(\Psi h)^{-1}[\Psi_A] = \Psi_{h[A]}$. Now since h is uniform, $h[A] \in \nu$, and hence $\Psi_{h[A]} \in \Psi\nu$.

The Samuel compactification of a uniform σ–frame.

The details below, 3.12 – 3.19, are adapted in part from those presented by Banaschewski [1988] for the analogous construction for frames.

3.12. Definition: For $(L,\mu) \in \mathbf{Uni\sigma Frm}$, an ideal $J \subseteq L$ is *uniformly regular* if for each $x \in J$ there exists $y \in J$ with $x \triangleleft y$. Let $\mathfrak{R}_\sigma L$ be the set of all countably generated uniformly regular ideals. (Then each such ideal is generated by a sequence $a_1 \triangleleft a_2 \triangleleft \ldots$)

3.13. Lemma: For $a \triangleleft b \triangleleft c$ in (L,μ), there exists $x, y \in L$ such that x separates $a \prec b$, $y \vee c = e$ and $y \triangleleft x$.

Proof: For $a \triangleleft b$ in (L,μ), there exists $A \in \mu$ with $Aa \leq b$. Take $B \leq^* A$ in μ, then $a \triangleleft Ba \triangleleft Aa \leq b \triangleleft c$. Now let $x = \bigvee \{ z \in B \mid z \wedge a = 0 \}$, then x separates $a \prec Ba$ and hence also $a \prec b$. And let $y = \bigvee \{ z \in B \mid z \wedge Ba = 0 \}$ then $y \vee Ba = e$ and hence $y \vee c = e$. Suppose that $d \in B$ and $d \wedge y \neq 0$, then for some $z \in B$ with $z \wedge Ba = 0$, $d \wedge z \neq 0$. Since $z \wedge Ba = 0$ implies that $z \wedge z' = 0$ for all $z' \wedge a \neq 0$ with $z' \in B$, it follows that $d \wedge a = 0$, and so $d \leq x$, and hence $By \leq x$.

3.14. Proposition: *$\mathfrak{R}_\sigma L$ is a compact regular σ–frame.*

Proof: By the properties of the uniformly below relation, it follows that if $I, J \in \mathfrak{R}_\sigma L$ then $I \cap J \in \mathfrak{R}_\sigma L$ and $I \vee J \in \mathfrak{R}_\sigma L$. Any updirected countable join of ideals in $\mathfrak{R}_\sigma L$ is clearly in $\mathfrak{R}_\sigma L$, hence $\mathfrak{R}_\sigma L$ is closed under finite meets and joins.

Moreover $\mathfrak{R}_\sigma L \subseteq \mathscr{K}_\sigma L$, since uniformly below implies rather below, and hence $\mathfrak{R}_\sigma L$ is a compact σ–frame. Now take any $J \in \mathfrak{R}_\sigma L$ with generating sequence $a_1 \triangleleft a_2 \triangleleft \dots$. For each n , let J_n be the ideal generated by the sequence $a_n = a_{n_0} \triangleleft a_{n_2} \triangleleft \dots\dots \triangleleft a_{n+1}$, obtained by repeated interpolation of the uniformly below relation. Now $J_n \in \mathfrak{R}_\sigma L$ and $J = \bigvee J_n$. For each n , $a_n \triangleleft a_{n+1} \triangleleft a_{n+2}$, and hence by the preceeding lemma, there exists b_n, c_n such that b_n separates $a_n \prec a_{n+1}$, $c_n \vee a_{n+2} = e$ and $c_n \triangleleft b_n$. Define I_n as the ideal generated by the sequence $c_n = c_{n_0} \triangleleft c_{n_1} \triangleleft c_{n_2} \triangleleft \dots \triangleleft b_n$. Then $I_n \in \mathfrak{R}_\sigma L$. Now $I_n \cap J_{n+1} = 0$ since $a_n \wedge b_n = 0$ and $I_n \vee J_{n+2} = L$ since $a_{n+2} \vee c_n = e$. Thus $J_{n+1} \prec J_{n+2} \subseteq J$ and hence $\mathfrak{R}_\sigma L$ is regular.

(This proof of regularity uses the same method as used by Banaschewski and Gilmour [1989] to prove the regularity of $\mathscr{K}_\sigma L$.)

3.15. Lemma: *A compact regular σ–frame has a unique compatible uniformity generated by all the finite covers.*

Proof: The proof that each compact regular frame has such a unique compatible uniformity carries over to the σ–frame setting. Let A be a finite cover, then for each $a \in A$, a is a countable join of elements rather below it, and hence $\bigvee A = \bigvee \{ \bigvee \{x_n \mid x_n \prec a \} \mid a \in A \} = e$. By compactness there exists a finite subcover and by the finiteness of A , for each a in A take $x_a \prec a$ such that $\bigvee \{x_a \mid a \in A \} = e$. Let z_a separate $x_a \prec a$, then $\bigwedge z_a = 0$ and $B = \bigwedge \{ \{a,z_a\} \mid a \in A \} \leq A$. Thus in order to show that each finite cover has a finite star–refinement, it suffices to prove that each cover consisting of two elements has a finite star–refinement.

Consider a cover $\{a, b\}$. By normality there exists $x \prec a$, $y \prec b$ such that $x \vee y = e$. Let s and t be the separating elements of $x \prec a$ and $y \prec b$. Then $\{s, a\}$, $\{t, b\}$ and $\{x, y\}$ are covers. Now let D be the meet of these three covers, then $D = \{ s \wedge b \wedge y, a \wedge b \wedge x, a \wedge b \wedge y, a \wedge t \wedge x \}$. Since $D(s \wedge b \wedge y) \leq b$, $D(a \wedge b \wedge x) \leq a$, $D(a \wedge b \wedge y) \leq b$ and $D(a \wedge t \wedge x) \leq a$, $D \leq^* \{a, b\}$. Now $x \prec a$ implies that there exists a separating element s, and hence $\{s, a\}$ is a finite cover and $\{s, a\}(x) = a$, thus it follows that $x \triangleleft a$ with respect to finite covers, and hence the compatibility of the uniformity generated by the finite covers comes directly from the regularity.

Uniqueness : Let μ be any compatible uniformity on $L \in$ **KRegσFrm**, and take any finite $A \in \mathrm{Cov}(L)$. For each $a \in A$, take $x_a \triangleleft a$ such that $\bigvee \{ x_a \mid a \in A \} = e$. Since A is finite and $x_a \triangleleft a$, there exists $B \in \mu$ with $Bx_a \leq a$ for all $a \in A$. Take $t \in B$, then $t \wedge x_a \neq 0$ for some $a \in A$, since the x_a's form a cover. This implies that $t \leq Bx_a \leq a$, hence $B \leq A$, and so $A \in \mu$.

3.16. Corollary: *KRegσFrm is a full subcategory of UniσFrm.*

3.17. Lemma: $\mathfrak{R}_\sigma$ *: UniσFrm $\longrightarrow$ KRegσFrm is a functor.*

Proof: Let $h: (L,\mu) \longrightarrow (M,\nu)$ in **UniσFrm**. Since h preserves the uniformly below relation, if $J \in \mathfrak{R}_\sigma L$ then $[h(J)] \in \mathfrak{R}_\sigma M$ where $[h(J)]$ is the uniformly regular ideal generated by the sequence $h(a_1) \triangleleft h(a_2) \triangleleft \ldots$ if $a_1 \triangleleft a_2 \triangleleft \ldots$ generates J.

Note that $[h(J)]$ does not depend on the particular choice of the sequence (a_n) : Let (b_n) be any other such sequence generating J then $\bigvee a_n = \bigvee J = \bigvee b_n$. Now take any $a_i \triangleleft a_{i+1}$ with separating element s_i . Then $s_i \vee \bigvee b_n = e$ and hence by compactness $s_i \vee b_j = e$ for some j , and therefore $a_i \leq b_j$. This shows that the ideal generated by the a_n is contained in the ideal generated by the b_n . By symmetry, equality is obtained.

Therefore, since h is a σ–frame morphism, $\mathfrak{R}_\sigma h: \mathfrak{R}_\sigma L \longrightarrow \mathfrak{R}_\sigma M$ defined by sending J to $[h(J)]$, is a σ–frame morphism.

3.18. Lemma: *For $(L,\mu) \in$ **UniσFrm**, $\rho_L: \mathfrak{R}_\sigma L \longrightarrow (L,\mu)$ given by join is a uniform σ–frame morphism.*

Proof: ρ_L is the restriction of k_L to the countably generated uniformly regular ideals, a sub–σ–frame of $\mathscr{K}_\sigma L$, and hence ρ_L is a σ–frame morphism. Take any finite cover $J_1, \ldots J_n$ of $\mathfrak{R}_\sigma L$, then there exists $a_i \in J_i$ such that $a_1 \vee \ldots \vee a_n = e$. Let $c_i = \rho_L(J_i)$, then $a_i \triangleleft c_i$ (since the J_i are uniformly regular.) Since there are only finitely many relations, there exists $B \in \mu$ such that $Ba_i \leq c_i$ for all i . For any $t \in B$, $t \wedge a_i \neq 0$ for some i , since the a_i's form a cover. Hence $t \leq Ba_i \leq c_i$, and thus it follows that $B \leq \{ c_1, \ldots c_n \}$. Therefore $\rho_L [\{ J_1, \ldots J_n \}] = \{c_1, \ldots c_n\} \in \mu$.

3.19. Lemma: *For $M \in$ KRegσFrm, $\rho_M \colon \mathfrak{R}_\sigma M \longrightarrow M$ is an isomorphism.*

Proof: ρ_M is always surjective (since each element of the uniform σ–frame is a countable join of elements uniformly below it.) and dense, hence injective. (see Ch.0) Since both M and $\mathfrak{R}_\sigma$M are compact regular, each have a unique compatible uniformity generated by all the finite covers, and hence ρ_M is an isomorphism.

The following proposition establishes the existence of a Samuel compactification for uniform σ–frames.

3.20. Proposition: *KRegσFrm is coreflective in UniσFrm with coreflection functor $\mathfrak{R}_\sigma$ and coreflection map $\rho_L \colon \mathfrak{R}_\sigma L \longrightarrow (L,\mu)$.*

Proof: Consider any h: M $\longrightarrow$ (L,μ) in UniσFrm with M $\in$ KRegσFrm.

$$\begin{array}{ccc} \mathfrak{R}_\sigma L & \xrightarrow{\ \rho_L\ } & (L,\mu) \\ {\scriptstyle \mathfrak{R}_\sigma h}\uparrow & & \uparrow{\scriptstyle h} \\ \mathfrak{R}_\sigma M & \xrightarrow[\rho_M]{\ \cong\ } & M \end{array}$$

The diagram commutes since h preserves countable joins. Since M is compact regular, ρ_M is an isomorphism, hence $\rho_L.\mathfrak{R}_\sigma h.(\rho_M)^{-1} = h$, that is, h factors via ρ_L . The uniqueness of this factorisation follows since ρ_L is dense and hence monic.

The following results are readily established, with only the necessary small changes to the proofs of 2.14 – 2.16 :

3.21. Lemma: *Every finite uniform cover of a uniform σ–frame is star–refined by a finite uniform cover.*

3.22. Definition: A uniform σ–frame is *precompact* if it has a uniform base consisting of finite covers.

3.23. Lemma: *For $(L,\mu) \in$ **UniσFrm** , if μ_P denotes the uniformity generated by the finite uniform covers of μ , then (L,μ_P) is a precompact uniform σ–frame.*

3.24. Proposition: *Precompact uniform σ–frames are coreflective in **UniσFrm** with $id_L : (L,\mu_P) \longrightarrow (L,\mu)$ as the coreflection map.*

Structured versions of functors $\mathscr{H}$ and Coz.

The same notation is used as for the unstructured functors, the context should prevent any confusion.

For $(L,\mu) \in$ **UniσFrm** , let $\mathscr{H}L$ be the frame of all σ–ideals of L , and let $\mathscr{H}\mu$ be generated by $\{ \downarrow(A) \mid A \in \mu \}$ where $\downarrow(A) = \{ \downarrow a_n \mid a_n \in A \}$.

3.25. LEMMA: *$\mathscr{H}\mu$ is a uniformity compatible with $\mathscr{H}L$.*

Proof: For $A \in \mu$, $\{\downarrow a_n \mid a_n \in A\}$ is a cover of $\mathscr{H}L$ since $\bigvee(\downarrow a_n) = \downarrow(\bigvee a_n) = \downarrow e = L$. The set $\{\downarrow(A) \mid A \in \mu\}$ satisfies the star refinement property (U2): For $A \in \mu$, there exists $B \leq^* A$ in μ. Take $b_i \in B$, then there is $a_{m_i} \in A$ with $Bb_i \leq a_{m_i}$. Now $(\downarrow B)(\downarrow b_i) = \bigvee\{\downarrow b_j \mid (\downarrow b_i) \cap (\downarrow b_j) \neq \phi, b_j \in B\}$ $= \bigvee\{\downarrow b_j \mid \downarrow(b_j \wedge b_i) \neq \phi, b_j \in B\} = \bigvee\{\downarrow b_j \mid b_i \wedge b_j \neq 0, b_j \in B\}$ $= \downarrow\bigvee\{b_j \in B \mid b_j \wedge b_i \neq 0\} = \downarrow(Bb_i) \subseteq \downarrow a_{m_i}$. Hence $(\downarrow B) \leq^* (\downarrow A)$. For $J \in \mathscr{H}L$, $J = \bigvee\{\downarrow a \mid a \in J\}$. For each $a \in J$, a is a countable join of elements uniformly below it, say $\{a_n\}$, since (L,μ) is a uniform σ–frame. Hence $J = \bigvee\downarrow(\bigvee a_n) = \bigvee\downarrow a_n$ with $a_n \triangleleft a$ in J. Now $a_n \triangleleft a$ implies that $\downarrow a_n \triangleleft \downarrow a \subseteq J$. Thus each element of $\mathscr{H}L$ is the join of elements uniformly below it.

Hence $(\mathscr{H}L, \mathscr{H}\mu)$ is a uniform frame and since $\mathscr{H}\mu$ has a basis of countable covers, in fact countable cozero covers, it is separable. This also follows immediately from the fact that $\mathscr{H}L$ is Lindelöf.

3.26. PROPOSITION: *$\mathscr{H}$: UniσFrm ⟶ UniFrm is functorial.*

Proof: Consider $f : (L,\mu) \longrightarrow (M,\nu)$ in **UniσFrm**. Then $\mathscr{H}f : (\mathscr{H}L, \mathscr{H}\mu) \longrightarrow (\mathscr{H}M, \mathscr{H}\nu)$ is defined by sending any ideal J to the ideal generated by $f(J)$, denoted $[f(J)]$, and a basis cover $\downarrow A$ to $\downarrow f(A)$. $\mathscr{H}f$ is a frame homomorphism and since f is uniform, $f(A)$ is a uniform cover of M, and thus $\downarrow f(A)$ is a uniform cover of $\mathscr{H}M$, so $\mathscr{H}f$ is a uniform frame morphism.

The underlying σ–frame of a completely regular frame need not be regular (This can be seen by considering the topology of any completely regular space which is not perfectly normal.) and in fact, since the cozero part of a completely regular frame is the largest regular sub–σ–frame, the underlying σ–frame is only regular if it equals the cozero part of the frame. Thus it is appropriate to consider a structured version of the functor Coz .

For $(L,\mu) \in$ **UniFrm**, let $Coz_u\, L$ consist of all uniformly cozero elements and let $Coz_u\, \mu$ be the collection of all the countable uniform covers consisting of uniformly cozero elements.

3.27. PROPOSITION: *$Coz_u\, \mu$ is a uniformity compatible with $Coz_u\, L$.*

Proof: For any uniformity the collection of countable uniform covers is a uniform basis (see lemma 2.14) and any uniformity has a basis of uniformly cozero covers, thus it follows that $Coz_u\, \mu$ contains a countable cozero basis and hence is a uniformity on a σ–frame. By Corollary 2.27 each uniform cozero element of L is a countable join of uniform cozero elements uniform relative to μ and hence relative to $Coz_u\, \mu$. Thus $Coz_u\, \mu$ is compatible with $Coz_u\, L$.

Hence $(Coz_u\, L, Coz_u\, \mu)$ is a uniform σ–frame.

3.28. PROPOSITION: *$(Coz_u\, L, Coz_u\, \mu)$ generates the separable coreflection of (L,μ).*

Proof: $Coz_u\, L$ generates L as a frame. (Lemma 2.29) Taking $Coz_u\, \mu$ as the basis for a uniformity on L gives a separable uniformity which is compatible with L since each element of L is a join of uniform cozero elements, each of which is the countable join of elements uniformly below it. By the definition of $Coz_u\, \mu$, it follows that the frame uniformity generated by $Coz_u\, \mu$ is precisely μ_s .

3.29. PROPOSITION: *Coz_u: UniFrm ⟶ UniσFrm is functorial.*

Proof: Consider $f : (L,\mu) \longrightarrow (M,\nu)$ in **UniFrm**. Since frame morphisms preserve cozero elements, f restricted to $Coz_u\, L$ maps into $Coz_u\, M$, so $Coz_u\, f = f$ obviously preserves composition and identities.

3.30. LEMMA: *If $(L,\mu) \in$ UniFrm with $L \in$ RegLindFrm ≅ RegσCohFrm then*

$\sigma(L) = Coz\, L = Coz_u L$.

Proof: For all regular Lindelöf frames $\sigma(L) = Coz\ L$, and for all uniform frames $Coz_u L \subseteq Coz\ L$. Take $a \in \sigma(L)$, since a is countable and L is uniform, $a = \bigvee \{ a_n \mid a_n \triangleleft a \}$. By Corollary 2.27 it follows that a is a uniformly cozero element.

3.31. PROPOSITION: *$\mathscr{H}$ is left adjoint to Coz_u.*

Proof: For $(L,\mu) \in$ **UniσFrm** , let the unit $\eta_L : (L,\mu) \longrightarrow Coz_u\, (\mathscr{H}L, \mathscr{H}\mu)$ be the map defined by $\eta_L(x) = \downarrow x$. By lemma 3.30 , since $\mathscr{H}L$ is regular Lindelöf, the countable elements of $\mathscr{H}L$ are precisely the uniformly cozero ones, and thus η_L is a well–defined frame map. A uniform cover $\{a_n\}$ of L is taken to $\{\downarrow a_n\}$ which is a countable uniform cover of $\mathscr{H}L$, and consists of uniformly cozero elements and thus belongs to $Coz_u\, \mathscr{H}L$. Thus η_L is uniform.

Naturality of η : If $f : (L,\mu) \longrightarrow (M,\nu)$ is any uniform σ–frame morphism and $x \in L$,

$$
\begin{array}{ccc}
(L,\mu) & \xrightarrow{\ \eta_L\ } & \mathrm{Coz}_u(\mathscr{H}L, \mathscr{H}\mu) \\
f \downarrow & & \downarrow \mathrm{Coz}_u \mathscr{H}f \\
(M,\nu) & \xrightarrow{\ \eta_M\ } & \mathrm{Coz}_u(\mathscr{H}M, \mathscr{H}\nu)
\end{array}
\qquad\qquad
\begin{array}{ccc}
x & \longmapsto & \downarrow x \\
\downarrow & & \\
f(x) & \longmapsto & \downarrow f(x)
\end{array}
$$

then $\mathrm{Coz}_u\ \mathscr{H}f(\downarrow x) = \mathrm{Coz}_u(\downarrow f(x)) = \downarrow f(x)$.

For $(L,\mu) \in$ **UniFrm**, the counit $\varepsilon_L\colon \mathscr{H}\mathrm{Coz}_u(L,\mu) \longrightarrow (L,\mu)$ is the map given by join. Any basic uniform cover of $\mathscr{H}\mathrm{Coz}_u\ L$ is of the form $\{\downarrow a_n\}$, with $a_n \in \mathrm{Coz}_u\ L$, which is taken by ε_L to $\{a_n\}$ which is a uniform cover of L. Hence ε_L is uniform.

Naturality of ε_L : if $g\colon (L,\mu) \longrightarrow (M,\nu)$ is a uniform frame morphism and $J \in \mathscr{H}\mathrm{Coz}_u\ L$,

$$
\begin{array}{ccc}
\mathscr{H}\mathrm{Coz}_u(L,\mu) & \xrightarrow{\ \varepsilon_L\ } & (L,\mu) \\
\mathscr{H}\mathrm{Coz}_u\ g \downarrow & & \downarrow g \\
\mathscr{H}\mathrm{Coz}_u(M,\nu) & \xrightarrow{\ \varepsilon_M\ } & (M,\nu)
\end{array}
\qquad\qquad
\begin{array}{ccc}
J & \longmapsto & \bigvee J \\
\downarrow & & \\
[g(J)] & \longmapsto & \bigvee [g(J)]
\end{array}
$$

then $g(\bigvee J) = \bigvee g(J) = \bigvee [g(J)]$, since g is a frame map.

For adjointness it only remains to verify the identities :

$$\varepsilon_{\mathscr{H}(L,\mu)} \cdot \mathscr{H}\eta_L = \mathrm{id}_{\mathscr{H}(L,\mu)} \quad \text{and} \quad \mathrm{Coz}_u . \varepsilon_{(M,\nu)} \cdot \eta_{\mathrm{Coz}_u(M,\nu)} = \mathrm{id}_{\mathrm{Coz}_u(M,\nu)}.$$

Take $J \in \mathscr{H}(L,\mu)$ where $(L,\mu) \in$ **UniσFrm**,

$\mathscr{H}\eta_L(J) = \{\ I \in \mathrm{Coz}_u\ \mathscr{H}(L,\mu) \mid I \subseteq J\ \}$ and then since $\mathscr{H}L$ is a completely regular frame, the join of this set is J ie. $\varepsilon_{\mathscr{H}(L,\mu)} \cdot \mathscr{H}\eta_{(L,\mu)}(J) = J$.

Take $a \in \mathrm{Coz}_u(M,\nu)$ where $(M,\nu) \in$ **UniFrm**,

$\eta_{\mathrm{Coz}_u(M,\nu)}(a) = \{\ x \in \mathrm{Coz}_u\ M \mid x \leq a\ \}$ and the join of this set is a, hence $\mathrm{Coz}_u\ \varepsilon_{(M,\nu)} \cdot \eta_{\mathrm{Coz}_u(M,\nu)}(a) = a$.

3.32. LEMMA: *For a Lindelof uniform frame* $\mathscr{H}\mathrm{Coz_u}\,(L,\mu) \cong (L,\mu)$.

Proof: Consider maps $\bigvee\colon \mathscr{H}\mathrm{Coz_u}\, L \longrightarrow L$ (ie. the counit ε_L.) and $\varphi : L \longrightarrow \mathscr{H}\mathrm{Coz_u}\, L$ defined by $\varphi\,(a) = \{\, x \in \mathrm{Coz_u}\, L \mid x \leq a \,\}$. These are well–defined frame morphisms. Since $\mathrm{Coz_u}\, L$ generates L, $\bigvee(\varphi\,(a)) = a$. Now let $J \in \mathscr{H}\mathrm{Coz_u}\, L$, and take $x \in \mathrm{Coz_u}\, L$ with $x \leq \bigvee J$, ie. $x \in \varphi\,(\bigvee J)$. Since L is regular and Lindelöf, $\mathrm{Coz_u}\, L = \sigma(L)$, and thus x is countable, so $x \leq \bigvee b_n$ with $b_n \in J$, and hence $x \in J$ since J is a σ–ideal. Therefore $\varphi\,(\bigvee J) = J$. This shows that $\mathscr{H}\mathrm{Coz_u}\, L \cong L$ as frames. By the definition of $\mathscr{H}$, $\bigvee$ extends to a uniform frame map $\bigvee\colon \mathscr{H}\mathrm{Coz_u}\,(L,\mu) \longrightarrow (L,\mu)$. Since (L,μ) is Lindelöf, hence separable, it has a basis of countable uniformly cozero covers. Take any such basic cover A, then $A \in \mathrm{Cov}\,(\mathrm{Coz_u}\, L)$ and so $\varphi\,(A) = \{\, \varphi\,(a) \mid a \in A \,\}$ is in $\mathscr{H}\mathrm{Coz_u}\, \mu$. Hence φ is also a uniform frame map, and the isomorphism extends to the objects considered as uniform frames.

3.33. LEMMA: *For any* $(L,\mu) \in$ ***UniσFrm***, $(L,\mu) \cong \mathrm{Coz_u}\, \mathscr{H}(L,\mu)$.

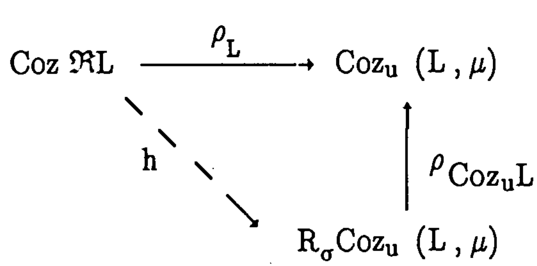
Proof: Let $(L,\mu) \in$ **UniσFrm**. Consider the uniform σ–frame maps $\bigvee\colon \mathrm{Coz_u}\, \mathscr{H}(L,\mu) \longrightarrow (L,\mu)$ and $\eta_L\colon (L,\mu) \longrightarrow \mathrm{Coz_u}\, \mathscr{H}(L,\mu)$. Since $\mathscr{H}L$ is regular Lindelöf, $\mathrm{Coz_u}\, \mathscr{H}L = \sigma\,(\mathscr{H}L)$ and the countable elements are precisely the principal ideals. (Proposition 1.11.) From this follows that the two maps are inverse to each other, and so $(L,\mu) \cong \mathrm{Coz_u}\, \mathscr{H}(L,\mu)$. That the uniformities are isomorphic follows easily since $\mathscr{H}(L,\mu)$ is separable.

3.34. PROPOSITION: *Lindelof uniform frames and uniform σ–frames are the fixed objects of the adjunction described above.*

These interactions between uniform σ–frames and uniform frames allow for an alternate description of the Samuel compactification $\mathfrak{R}$ L of a separable uniform frame L via the corresponding compactification of a uniform σ–frame, similar to that done for the Stone–Cech compactification. Thus the following propositions shows that $\mathfrak{R}$ L is completely determined by the cozero part of L.

3.35. PROPOSITION: *For any separable uniform frame (L,μ), $Coz\,\mathfrak{R}L \cong \mathfrak{R}_\sigma\, Coz_u\, L$.*

Proof: $(Coz_u\, L, Coz_u\, \mu) \in$ **UniσFrm** and since $\mathfrak{R}L \in$ **KRegFrm**, Coz $\mathfrak{R}$L is a compact regular σ–frame and hence has a unique uniformity. Now $\rho_L : \mathfrak{R}L \longrightarrow (L,\mu)$ restricts to a σ–frame morphism, say ρ_L: Coz $\mathfrak{R}L \longrightarrow Coz_u$ L. Take any finite, and thus uniform, cover A of Coz $\mathfrak{R}$L, then A is a finite, hence uniform cover of $\mathfrak{R}$L, and thus $\rho_L(A) \in \mu$. But $\rho_L(A)$ is countable and consists of uniformly cozero elements, so $\rho_L(A) \in Coz_u\, \mu$. Thus the restriction ρ_L: Coz $\mathfrak{R}L \longrightarrow (Coz_u\, L, Coz_u\, \mu)$ is a uniform σ–frame morphism. Thus by the coreflection property of **KRegσFrm**, this map factors via the Samuel compactification of the codomain :



Now define f : Coz_u L $\longrightarrow$ Coz $\mathfrak{R}$L by f (a) = { x $\in$ Coz_u L | x $\triangleleft$ a }. Since each element of Coz_u L is a countable join of elements of Coz_u L uniformly below it, this is a countably generated uniformly regular ideal, hence this is a well–defined σ–frame morphism and moreover, since Coz $\mathfrak{R}$L is compact,

$f : (Coz_u\ L, Coz_u\ \mu) \longrightarrow Coz\ \mathfrak{R}L$ is uniform. Also $f.\rho_L = id_{Coz_u\ L}$ and so it follows that $f.\rho_{Coz_u L} : \mathfrak{R}_\sigma Coz_u\ L \longrightarrow Coz\ \mathfrak{R}L$ is inverse to h. Hence $\mathfrak{R}_\sigma Coz_u\ L \cong Coz\ \mathfrak{R}L$.

3.36. PROPOSITION: *For a separable uniform frame (L,μ), $\mathscr{H}\mathfrak{R}_\sigma Coz_u\ (L,\mu) \cong \mathfrak{R}\ (L,\mu)$.*

Proof: Since $\mathfrak{R}L$ is compact, it is obviously Lindelöf and hence, by Lemma 3.32, $\mathscr{H}Coz\ \mathfrak{R}L \cong \mathfrak{R}L$. Now applying the functor $\mathscr{H}$ to the result in Proposition 3.35 gives that $\mathscr{H}Coz\ \mathfrak{R}\ (L,\mu) \cong \mathscr{H}\mathfrak{R}_\sigma Coz_u\ (L,\mu)$. These combine to give $\mathscr{H}\mathfrak{R}_\sigma Coz_u\ (L,\mu) \cong \mathfrak{R}\ (L,\mu)$.

3.37. PROPOSITION: *For a separable uniform frame, pseudocompactness implies precompactness.*

Proof: Let (L,μ) be a separable uniform frame, which is pseudocompact. Take any $A \in \mu$, then, by separability, there exists $B \leq A$ in μ such that B is a countable cozero cover. Each cover of $Coz_u\ L$ has a finite subcover, since $Coz_u\ L \subseteq Coz\ L$, and $Coz\ L$ is compact, and thus by lemma 3.15 each uniform cover is refined by a finite uniform cover. From this it follows that there exists a finite uniform cover of $Coz_u\ (L,\mu)$ refining B and hence A. Thus (L,μ) is precompact.

Proximity σ–frames and strong inclusions.

3.38. Definition: For $L \in \sigma\mathbf{Frm}$, a binary relation $\triangleleft$ on L is a *strong inclusion* if

(SI1) $x \leq a \triangleleft b \leq y \implies x \triangleleft y$.

(SI2) $\triangleleft \subseteq L \times L$ is a sublattice

ie. $0 \triangleleft 0$, $e \triangleleft e$, $x, y \triangleleft a, b \implies x \triangleleft a \wedge b$, $x \vee y \triangleleft a$.

(SI3) $x \triangleleft a \implies x \prec a$.

(SI4) $\triangleleft$ interpolates.

(SI5) $x \triangleleft y \implies$ there exists $a, b \in L$ with $b \triangleleft a$, $b \vee y = e$ and $a \wedge x = 0$.

$(L,\triangleleft)$ is called a *proximal σ–frame* if

(SI6) for each $a \in L$, there exists a sequence (a_n) with $a = \bigvee a_n$ and $a_n \triangleleft a$. That is, $\triangleleft$ is a σ–approximating relation.

$h : (M,\triangleleft') \longrightarrow (L,\triangleleft)$ in $\sigma\mathbf{Frm}$ is *proximal* if $h \times h\,[\triangleleft'] \subseteq \triangleleft$. That is, h preserves strong inclusions.

These are the objects and morphisms of the category **ProxσFrm**.

This is analogous to the definition of proximal frames and **ProxFrm** in the setting of frames. Condition (SI5) is the appropriate replacement for the frame condition '$x \triangleleft y \implies y^* \triangleleft x^*$'. Frith [1986] shows that a duality exists between the categories of proximal frames and proximity spaces. He also shows that the category of proximity frames is equivalent to the subcategory of **UniFrm** consisting of the totally bounded (precompact) uniform frames.

3.39. Proposition: *Every proximal σ–frame is regular.*

Proof: This follows easily since $\triangleleft$ implies $\prec$ and then (SI6) gives the required condition for regularity.

3.40. Proposition: *For $L \in Reg\sigma Frm$, the rather below relation $\prec$ is a strong inclusion, and hence $(L,\prec)$ is a proximal σ–frame.*

Proof: It is well known that the rather below relation always satisfies the conditions (SI1) to (SI3). (SI4) and (SI5) follow immediately from lemma 3.5. and the compatibility condition (SI6) is just the definition of a regular σ–frame.

3.41. Corollary: *The rather below relation is the 'finest' strong inclusion compatible with a σ–frame.*

Proof: As sublattices of $L \times L$, since $\triangleleft$ implies $\prec$ for any strong inclusion $\triangleleft$, $\triangleleft \subseteq \prec$.

3.42. Proposition: *For $(L,\mu) \in Uni\sigma Frm$, the uniformly below relation $\triangleleft$ is a strong inclusion such that $(L,\triangleleft) \in Prox\sigma Frm$.*

Proof: The properties of the uniformly below relation dicussed in propositon 3.4. give precisely conditions (SI1) to (SI4). Lemma 3.13. gives (SI5), and the compatability follows immediately from the definition of a uniform σ–frame.

Moreover, since uniform σ–frame morphisms preserve the uniformly below relation (proposition 3.4.) this assignment is in fact functorial.

3.43. Proposition: *If $(L,\triangleleft) \in ProxoFrm$ then there exists a uniformity compatible on L such that the uniformly below relation is precisely $\triangleleft$.*

Proof: Let $\mu_\triangleleft$ be the uniformity generated by the collection of all those finite covers of L which are $\triangleleft$–refined by some finite cover, say $\bar{\mu}$. Then $\bar{\mu} = \{ A \in Cov(L) \mid A$ is finite and there exists finite $B \in Cov(L)$ with $B \triangleleft A \}$ is a uniform basis : it suffices to check that any cover in $\bar{\mu}$ consisting of two elements is star–refined by a member of $\bar{\mu}$. Take $\{a,b\} \in \bar{\mu}$ with $\{c,d\} \triangleleft \{a,b\}$ ie. $c \triangleleft a$ and $d \triangleleft b$. By (SI4) , this relation interpolates to give $c \triangleleft c_1 \triangleleft c_2 \triangleleft a$ and $d \triangleleft d_1 \triangleleft d_2 \triangleleft b$. Now applying (SI5) to $c_1 \triangleleft c_2$ and $d_1 \triangleleft d_2$ there exists $y \triangleleft x$ with $y \vee c_2 = e$ and $x \wedge c_1 = 0$ and there exists $t \triangleleft s$ with $t \vee d_2 = e$ and $s \wedge d_1 = 0$. Hence $\{y,c_2\} \triangleleft \{x,a\}$ and $\{t,d_2\} \triangleleft \{s,b\}$ also $\{c,d\} \triangleleft \{c_1,d_1\}$. Then $C = \{x,a\} \wedge \{s,b\} \wedge \{c_1,d_1\} = \{x \wedge d_1 , c_1 \wedge s , c_1 \wedge b , a \wedge d_1\}$. Moreover $C(x \wedge d_1) \leq b$, $C(c_1 \wedge s) \leq a$, $C(c_1 \wedge b) \leq a \wedge b$ and $C(a \wedge d_1) \leq a \wedge b$ and hence $C \leq^* \{a,b\}$.

If $a \triangleleft b$ in $(L,\triangleleft)$ then for $a \triangleleft c \triangleleft b$ there exists $y \triangleleft x$ with $y \vee c = e$ and $x \wedge a = 0$, hence $\{x,b\} \in \bar{\mu}$ and $\{x,b\}a = b$, thus a is uniformly below b and consequently $\mu_\triangleleft$ is compatible on L . Conversely if a is uniformly below b then there exists A in $\bar{\mu}$ with $Aa \leq b$. Take a finite cover $B \triangleleft A$, then since B is a cover and by (SI2) , $a \leq Ba \triangleleft Aa \leq b$ hence by (SI1) , $a \triangleleft b$.

3.44. Corollary: *Let $(L,\triangleleft\,) \in$ **ProxσFrm**. Suppose that ν is a precompact uniformity compatible on L which induces $\triangleleft$, then $\nu \subseteq \mu_\triangleleft$.*

3.45. Proposition: *Let $h: (M,\triangleleft') \longrightarrow (L,\triangleleft\,)$ in **ProxσFrm** then $h: (M,\nu_{\triangleleft'}) \longrightarrow (L,\mu_\triangleleft)$ is uniform.*

Proof: Take any finite $A \in \nu_{\triangleleft'}$, then there exists a finite cover $B \triangleleft' A$. Since h preserves strong inclusions, $h[B] \triangleleft h[A]$ and hence $h[A] \in \mu_\triangleleft$.

3.46. Corollary: *The assignment of a uniformity to a proximity σ−frame described above is a functor from **ProxσFrm** to **UniσFrm**.*

3.47. Proposition: *Suppose $(L,\triangleleft\,) \in$ **ProxσFrm** and ν is any compatible uniform structure on L which induces $\triangleleft$. Then $\mu_\triangleleft \subseteq \nu$.*

Proof: It suffices to show that each two element cover of $\mu_\triangleleft$ is in ν. Take $\{a,b\} \in \mu_\triangleleft$, then there exists a cover $\{c,d\} \triangleleft \{a,b\}$. Since ν induces $\triangleleft$, there exists covers A, B in ν with $Ac \leq a$ and $Bd \leq b$, moreover if $C = A \wedge B$ then $Cc \leq a$, $Cd \leq b$ and $C \in \nu$. For any $x \in C$, either $x \wedge c \neq 0$ and then $x \leq Cc \leq a$, or $x \wedge c = 0$ and then since $\{c,d\}$ is a cover, $x \wedge d \neq 0$ so $x \leq Cd \leq b$. Therefore $C \leq \{a,b\}$ and so $\{a,b\} \in \nu$.

3.52. PROPOSITION: *Given any compactification* $h: M \longrightarrow L$ *in* σ*Frm, then the binary relation defined by* $x \lhd y$ *if and only if there exists* $a,b \in M$ *with* $a \prec b$ *and* $h(a) = x$, $h(b) = y$, *is a strong inclusion compatible on* L .

Proof: (SI1) Take $x' \leq x \lhd y \leq y'$ in L , then since h is surjective and by the definition of $\lhd$, there exists $a, a', b, b' \in M$ with $a \prec b$ and $h(a) = x$, $h(a') = x'$, $h(b) = y$ and $h(b') = y'$. Now $x' = x' \wedge x = h(a') \wedge h(a) = h(a' \wedge a)$ and similarly $y' = h(b' \vee b)$. Since $a' \wedge a \leq a \prec b \leq b \vee b'$, it follows that $x' \lhd y'$.

(SI2) $0 \lhd 0$ and $e \lhd e$ since $0 \prec 0$, $e \prec e$ and h preserves the top and bottom. Take $x, x' \lhd y$, then there exists $a, a', b \in M$ with $a, a' \prec b$ and $h(a) = x$, $h(a') = x'$ and $h(b) = y$. Now $x \vee x' = h(a \vee a')$ and $a \vee a' \prec b$, hence $x \vee x' \lhd y$. A similar argument shows that if $x \lhd y, y'$ then $x \lhd y \wedge y'$.

(SI3) If $x \lhd y$ with $x = h(a)$, $y = h(b)$ and $a \prec b$ in M , then since h preserves the rather below relation, $x \prec y$.

(SI4) Take $x \lhd y$ as above. Since the rather below relation interpolates in **RegσFrm** there exists c with $a \prec c \prec b$ and hence $x \lhd h(c) \lhd y$.

(SI5) Take $x \lhd y$ as above. Since $a \prec b$ in M , there exists $c \prec d$ with $d \wedge a = 0$ and $c \vee b = e$. Now $h(c) \lhd h(d)$ and since h is dense $h(d) \wedge x = h(d) \wedge h(a) = 0$. Also $h(c) \vee y = h(c) \vee h(b) = e$.

(SI6) Since h is surjective, for each $a \in L$, $a = h(x)$ for some $x \in M$. By the regularity of M , $x = \bigvee x_n$ with $x_n \prec x$. Hence $h(x_n) \lhd h(x)$ and so $a = \bigvee h(x_n)$.

These two results show that there is a one–one correspondence between compactifications and strong inclusions.

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