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H-CLOSED SPACES AND ALMOST REALCOMPACT SPACES

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CONTENTS

INTRODUCTION

It was P. Alexandroff and P. Urysohn who first introduced the class of H-closed spaces. Since then it has evoked the interest of many topologists. The concept of H-closedness is closely related to that of compactness, and it is for this reason that attempts have been made to see what theorems about compactness hold for H-closedness. Miroslav Katětov constructed an analogue to the Stone-Čech compactification, which is the Katětov H-closed extension. He, and many others, then attempted to see under what circumstances this could be regarded as an (epi)-reflection. It was Douglas Harris who finally settled this question. Other H-closed extensions were also investigated, but have not been afforded as much attention as the Katětov H-closed extension.

G.E. Strecker made an extensive study of H-closed spaces and their cospaces in his Ph.D. thesis. He presented several characterizations of H-closed spaces in terms of their cospaces. In addition he examined H-closed spaces with various separation properties. We will however discuss H-closed spaces which are Hausdorff only.

Once H-closed spaces had been investigated to some extent, it did not take long for the idea of an almost realcompact space to develop. It was Z. Frolik who initiated the interest in this class of spaces. Regarding such spaces as an analogue to realcompact spaces, he characterised almost realcompact spaces in terms of the manner in which they are embedded in a particular product of the non-negative reals with the upper topology. It is hoped that this result can be improved upon.

Finally P. Fletcher and W.F. Lindgren became interested in almost realcompact spaces from the point of view of quasi-uniformities, and characterised such spaces in terms of the existence of a certain compatible transitive quasi-uniformity. A corresponding result for H-closed spaces does not appear to exist.

We now give a summary of the material covered in this thesis.

In chapter one, the concept of a cospace and an H-closed space are introduced. Some elementary results regarding H-closed spaces which have analogues for compact spaces are given. A characterization of H-closed spaces in terms of filters is presented, and finally the relationship between H-closed spaces and their cospaces is discussed.

We examine subspaces of H-closed spaces in the first and second sections of chapter two. However we are not able to establish a necessary and sufficient condition for a subset of an H-closed space to be H-closed. The productivity of H-closed spaces is discussed in the third section. Section four deals with C.T. Liu's version of the construction of the Katětov H-closed extension of an arbitrary Hausdorff space.

In chapter three, another H-closed extension is investigated. Properties of this extension are given. In section two, we get a correspondence between the points of this extension and the points of a certain subset of a product of two-point spaces with the upper topology. Consequently we show how the extension is embedded in the product.

Chapter four contains a characterization due to Frolik and Liu of H-closed spaces obtained by embedding in a certain way in a product of the unit interval with the upper topology. However, we modify their argument, noting that, in fact, only a two-point space with the upper topology is required.

The functorial aspects of the Katětov extension are presented in chapter five. Douglas Harris characterised the type of maps which he calls p-maps that make the category of H-closed spaces and continuous functions an epi-reflective subcategory of Hausdorff spaces and p-maps. This is the main theorem of this chapter.

In chapter six, the notion of an H-closed space is extended to that of an almost realcompact space in a manner analogous to the way the notion of compactness is extended to realcompactness. A similar embedding characterization, due again to Frolik and Liu, to that for H-closed spaces is given. They use the non-negative reals with the upper topology, but as before, the situation is really simpler, and the non-negative integers with the upper topology is employed.

Finally chapter seven deals with a characterization of almost realcompactness in terms of the existence of a certain compatible quasi-uniformity. We give some basic theory on quasi-uniformities in the first section, devoting the second section to the characterization, which is due to P. Fletcher and W.F. Lindgren.

We conclude with a bibliographical note on each chapter.

NOTATION

We denote by $\overline{A}$ the closure of a set A. Whenever a new topology S is constructed on a space (X,T), we denote the closure of A in (X,T) by $\overline{A}$ (although occasionally to avoid ambiguity $\mathrm{cl}_T A$ is used), and the closure in (X,S) by $\mathrm{cl}_S A$. Whenever we have an extension Y of a space X, the closure of A in the extension is denoted by $\overline{A}$, and in the subspace X of Y, it is denoted by $\mathrm{cl}_X A$.

A^0 denotes the interior of A. We use $\mathrm{int}_S A$, $\mathrm{int}_X A$ in the same way as explained above.

When we omit to specify the topology on a space, by writing X instead of (X,T), no confusion should result.

*** denotes the end of a proof.

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CHAPTER ONE

COSPACES AND H-CLOSED SPACES:

1. COSPACES:

The concept of a cotopology was first introduced by J. de Groot. He defined it roughly as being that part of topology in which cospaces are used to study the properties of a given topological space. He wished to find a topological characterization of metrizability, and his investigations of cospaces enabled him to show that a metrizable space is topologically complete if and only if it is cocompact. In [24], G. E. Strecker discussed cotopologies and generalized compactness. In particular, he considered what type of cospaces a topological space should possess in order that it be H-closed. It is this that will occupy most of the discussion in this chapter.

1.1.1 Definition: For each base $\mathcal{B}$ for a topological space (X,T), let $T(\mathcal{B})$ denote the topology on X that is generated by the sub-base $\{X - \overline{B} \mid B \in \mathcal{B}\}$. $(X,T(\mathcal{B}))$ is called the cospace of (X,T) with respect to $\mathcal{B}$.

$(X,T(\mathcal{B}))$ is clearly weaker than (X,T), so that every cospace $(X,T(\mathcal{B}))$ is the continuous image of its generating space (X,T).

1.1.2 Definition: If P is a topological property, then a space is co-P if and only if it has some cospace with the property P . If every cospace has the property P , then the space is said to be totally co-P .

1.1.3 Example: Consider the reals with the usual topology. The base $\mathcal{B}$ of open intervals yields a subbase for a cotopology with respect to $\mathcal{B}$ which has elements of the form $(-\infty, a)\cup(b, \infty)$. Every pair of basic open sets in the cotopology has non empty intersection, and the complement of every basic open set consists of finitely many closed intervals. Since every closed interval is compact in the generating space and every open set in the cospace is open in the generating space, any open cover of a closed interval by open sets of the cospace will have a finite subcover. Therefore the cospace of the reals with respect to $\mathcal{B}$ is compact. The reals with the usual topology is thus co-compact.

Remark: Since every cospace of a space (X,T) is the continuous image of its generating space, every compact (connected) space is totally co-compact (co-connected).

2. H-CLOSED SPACES:

Because compactness has many pleasant properties, topologists have been interested in studying properties related to it in order to see how they compare. For instance, it is well known that if a compact Hausdorff space X is embedded in a Hausdorff space Y , then the image of X is a closed subspace of Y .

A property related to this is the following:

<u>1.2.1 Definition</u>: A Hausdorff space (X,T) is H-closed (Hausdorff closed) provided that X is a closed set in every Hausdorff space in which it can be embedded.

This property was first investigated by P. Alexandroff and P. Urysohn in 1924. The former showed that the H-closed property is equivalent to the property of every open cover having a finite "proximate" subcover. Bourbaki observed that a space is H-closed if and only if every open filter base on X has non-empty adherence.

<u>1.2.2 Definition</u>: A non-empty family $\mathcal{F}$ of non-empty open subsets of a space (X,T) is an open filter base provided that for each $F_1, F_2 \in \mathcal{F}$, there exists $F_3 \in \mathcal{F}$ such that $F_1 \cap F_2 \supset F_3$.

An open filter $\mathcal{G}$ is a family of open sets generated by some open filter base as follows:

$$\mathcal{G} = \{G \in T \mid G \supset F \text{ some } F \in \mathcal{F}\} .$$

An open ultrafilter is a maximal element in the set of open filters of (X,T).

An open filter-base $\mathcal{F}$ is said to be fixed provided that $\cap[\overline{F} \mid F \in \mathcal{F}]$ is non-empty, i.e. $\mathcal{F}$ has an adherent point.

<u>1.2.3 Definition</u>: A family $\mathcal{C}$ of open sets in a space (X,T) is a proximate cover of (X,T) if and only if

$$\cup[C \mid C \in \mathcal{C}] \text{ is dense in } X .$$

The following characterization of H-closed spaces stresses the analogy of H-closed spaces and compact Hausdorff spaces.

1.2.4 Theorem: For a Hausdorff space (X,T), the following are equivalent:

(i) (X,T) is H-closed.

(ii) Every open cover contains a finite proximate subcover.

(iii) Every open filter base is fixed.

(iv) Every open ultrafilter converges.

Proof: (i) $\Longrightarrow$ (ii) Suppose $\mathscr{C}$ is an open cover of (X,T) such that for all finite subcollections $C_1, C_2, \ldots, C_n$,

$$\bigcup_{i=1}^{n} \overline{C}_i \neq X$$

For each such finite subcollection, $X - \bigcup_{i=1}^{n} \overline{C}_i$ is open and non-empty.

Let $Y = X \cup \{p\}$, and let the basic open neighbourhoods of p be the sets of the form

$$\{p\} \cup (X - \bigcup_{i=1}^{n} \overline{C}_i) \text{ , for all finite subcollections of } \mathscr{C} .$$

Y is Hausdorff. For, if $x \neq p$, there exists $C \in \mathscr{C}$ such that $x \in C$ and $p \in \{p\} \cup (X - \overline{C})$, and these two neighbourhoods are disjoint.

Furthermore X is dense in Y, thus not H-closed.

(ii) $\Longrightarrow$ (iii) Let $\mathscr{F}$ be an open filter base on (X,T), and suppose that

$$\cap[\overline{F} \mid \overline{F} \in \mathscr{F}] = \cap\overline{F} = \emptyset \text{ , then}$$

$\cup(X - \overline{F}) = X$, so

$\{X - \overline{F} \mid F \in \mathcal{F}\}$ is an open cover of (X,T) ; thus there exist $F_1, \ldots, F_n$ such that

$$X = \bigcup_{i=1}^{n} \overline{(X - \overline{F}_i)}$$

therefore $$\emptyset = \bigcap_{i=1}^{n} X - \overline{(X - \overline{F}_i)}$$

But $$\bigcap_{i=1}^{n} X - \overline{(X - \overline{F}_i)} \supset \bigcap_{i=1}^{n} F_i$$

so $$\bigcap_{i=1}^{n} F_i = \emptyset$$

This is a contradiction, since $\mathcal{F}$ is an open filter base.

(iii) $\Longrightarrow$ (iv) If $\mathcal{U}$ is an open ultrafilter, then $\mathcal{U}$ is an open filter base so there exists $x \in X$ such that

$x \in \cap[\overline{U} \mid U \in \mathcal{U}]$,

thus $\mathcal{U}$ converges to x .

(iv) $\Longrightarrow$ (i) Suppose (X,T) is not H-closed. Then there exists a Hausdorff space Y , $Y \supset X$ such that X is not a closed subspace of Y .

Let $q \in \overline{X} - X$. The family

$\mathcal{F} = \{V \cap X \mid V$ an open neighbourhood of q in $Y\}$

is an open filter base on X . By Zorn's Lemma, there exists an open ultrafilter $\mathcal{U}$ on X containing $\mathcal{F}$.

Since $\cap\overline{V} = q$ $q \notin X$, for all open V containing q

$\cap \mathrm{cl}_X U = \emptyset$ for all $U \in \mathcal{U}$.

Thus $\mathcal{U}$ fails to converge on X . ***

Another property related to compactness, is that of a space being minimal Hausdorff. Such a space is also H-closed.

1.2.5 Definition: A space (X,T) is minimal Hausdorff if T is a Hausdorff topology and there exists no Hausdorff topology on X strictly weaker than T.

A characterization of minimal Hausdorff spaces in terms of open filter bases is given by the following:

1.2.6 Theorem: A Hausdorff space (X,T) is minimal Hausdorff if and only if every open filter base which has a unique adherent point, converges to that point.

Proof: Suppose (X,T) is not minimal Hausdorff, then there exists a Hausdorff topology S on X such that S is strictly weaker than T. Thus there exists an $x \in X$ with

$$\mathcal{N}_S(x) \subsetneqq \mathcal{N}_T(x) .$$

($\mathcal{N}_S(x)$ denotes the neighbourhood system of x in S.)

Since (X,S) is Hausdorff,

$$\{x\} = \bigcap \mathrm{cl}_S\, N_S(x) \supseteq \bigcap \mathrm{cl}_T\, N_S(x) \supseteq \bigcap \mathrm{cl}_T\, N_T(x) = \{x\} .$$

So "$\supseteq$" may be replaced by "$=$".

Now let $\mathcal{F}$ be the S-open neighbourhoods of x; then $\mathcal{F}$ is an open filter base on (X,T) with the unique adherent point x. Because there is a T neighbourhood $N_T(x)$ which is not an S-neighbourhood,

$$N_T(x) \not\supset N_S(x) \quad \text{for all} \quad N_S(x) \in \mathcal{N}_S(x) .$$

Thus $\mathcal{F}$ does not converge to x.

Conversely: Suppose that $\mathcal{F}$ is an open filter base on (X,T) with the unique adherent point x_0, but that $\mathcal{F}$ does not converge to x_0. There is thus a neighbourhood $M_T(x_0)$ of x_0 which does not contain any of the members of $\mathcal{F}$.

Define another topology S on X as follows: Each $x \neq x_0$ has the same neighbourhood system in S as it had in T, but for x_0, define the S-neighbourhood system as follows:

$$\mathcal{N}_S(x_0) = \{F \cup N_T(x_0) \mid F \in \mathcal{F}, \; N_T(x_0) \in \mathcal{N}_T(x_0)\} .$$

S is a Hausdorff topology;

For if $x \neq x_0$, since (X,T) is Hausdorff, there exist neighbourhoods $N_T(x)$ and $N_T(x_0)$ such that

$$N_T(x) \cap N_T(x_0) = \emptyset .$$

Since $x \notin \cap\overline{F}$, there exist $F \in \mathcal{F}$, and $M_T(x)$ a neighbourhood of x for which

$$F \cap M_T(x) = \emptyset .$$

Thus $N_T(x) \cap M_T(x) \cap (F \cup N_T(x_0)) = \emptyset$

and $N_T(x) \cap M_T(x)$ is a neighbourhood of x in S.

Thus S is Hausdorff.

S is a strictly weaker topology on X than T since

$$\mathcal{N}_S(x_0) \subsetneq \mathcal{N}_T(x_0)$$

Since $M_T(x_0) \subsetneq F \cup M_T(x_0)$ for each $F \in \mathcal{F}$

and $M_T(x_0)$ is not an S-neighbourhood of x_0.

Thus (X,T) is not minimal Hausdorff. ***

A minimal Hausdorff space is H-closed, for suppose there is an open filter base $\mathcal{F}$ which has empty adherence. Take any point $x \in X$, and let $\mathcal{V}$ be the filter base of open neighbourhoods of x.

Let $\mathcal{G} = \{V \cup F \mid V \in \mathcal{V}, F \in \mathcal{F}\}$;

then x is the unique adherence point of the open filter $\mathcal{G}$.

Thus $\mathcal{G}$ converges to x.

However, for each $V \cup F \in \mathcal{G}$, $F \subset V \cup F$, and it follows that $\mathcal{F}$ converges to x, and thus x is an adherent point of $\mathcal{F}$. This is a contradiction. Therefore every minimal Hausdorff space is H-closed.

3. COSPACES AND H-CLOSED SPACES

Several characterizations of H-closed spaces in terms of cospaces will now be given. The following fact regarding the continuous images of H-closed spaces, which is an analogue to the compact situation, will be required.

1.3.1 Proposition: If (X,T) and (Y,S) are Hausdorff spaces, and $f : (X,T) \longrightarrow (Y,S)$ is continuous, and onto, and if (X,T) is H-closed, then (Y,S) is H-closed.

Proof: Let $\mathcal{C}$ be an open cover of (Y,S), then

$$f^{\leftarrow}(\mathcal{C}) = \{f^{\leftarrow}(C) \mid C \in \mathcal{C}\}$$

is an open cover of (X,T), so there exist $C_1, \dots, C_n$ belonging to $\mathcal{C}$ such that

$$X = \bigcup_{i=1}^{n} \overline{f^{\leftarrow}(C_i)}$$

then $Y = f(x) = f\left(\overline{\bigcup_{i=1}^{n} f^{\leftarrow} (C_i)}\right) \subseteq \bigcup_{i=1}^{n} \overline{C}_i$ ***

The concept of regular-openness is frequently required when investigating H-closed spaces. A set U is regular open if it is equal to the interior of its closure, i.e. $U = U^{-0}$.

A space (X,T) is said to be semi-regular if it has a base of regular open sets. Clearly every regular space is semi-regular, but not conversely. Regular open sets are closed under finite intersections.

Given a space (X,T) , the regular open sets form a base for a semi-regular topology T_S on X , since for each $U \in T$,

$$U^{-0} = \mathrm{int}_{T_S}\, \mathrm{cl}_{T_S}\, U .$$

For, if $U \in T$ then $\bar{U} \subset \mathrm{cl}_{T_S} U$, since T_S is weaker than T .

If $x \in \mathrm{cl}_{T_S} U$,

let O be a T-open neighbourhood of x , then $O^{-0} \in T_S$,

so $O^{-0} \cap U \neq \emptyset$, so $O \cap U \neq \emptyset$

therefore $x \in \bar{U}$, so $\bar{U} = \mathrm{cl}_{T_S} U$,

and since $U^{-0-0} = U^{-0}$, the regular open sets of (X,T) are regular open in (X,T_S) . T_S is called the semi-regularization of T .

<u>1.3.2 Theorem</u>: A Hausdorff space (X,T) is H-closed if and only if it is totally co-Hausdorff.

<u>Proof</u>: For the sufficiency, suppose (X,T) is not H-closed. Then there is a Hausdorff space Y containing X as a dense subspace.

Let $y \in Y - X$, and let $\{V_a \mid a \in A\}$ be the family of all neighbourhoods of y open in Y .

For each $a \in A$, let

$O_a = X - \overline{V_a \cap X}$

Since Y is Hausdorff, $\{O_a \mid a \in A\}$ is an open cover of X .

Let $\mathcal{B} = \{U \cap O_a \mid U \in T \text{ , } a \in A\}$

Then $\mathcal{B}$ is a base for T .

It must be shown that the cospace $(X,T(\mathcal{B}))$ of (X,T) is extremely non-Hausdorff, i.e. that no pair of points have disjoint neighbourhoods in $(X,T(\mathcal{B}))$. In order to do this, it is sufficient to show that every pair of basic non-empty open sets has non-empty intersection.

Take B_1, B_2, basic open sets in $(X,T(\mathcal{B}))$.

They are of the form

$$B_1 = X - \overline{\bigcup_{i=1}^{n} U_i \cap O_{a_i}} \qquad B_2 = X - \overline{\bigcup_{i=1}^{m} W_i \cap O_{b_i}}$$

$U_i \in T \quad i=1 , \ldots , n$; $W_i \in T \quad i=1 , \ldots , m$;

$b_i \in A \quad i=1 , \ldots , m$.

$$B_1 \cap B_2 = X - \overline{\bigcup_{i=1}^{n} U_i \cap O_{a_i}} \cap X - \overline{\bigcup_{i=1}^{m} W_i \cap O_{b_i}}$$

$$\supset X - \overline{\bigcup_{i=1}^{n} O_{a_i}} \cap X - \overline{\bigcup_{i=1}^{m} O_{b_i}}$$

$$\supset \bigcap_{i=1}^{n} V_{a_i} \cap \bigcap_{i=1}^{m} V_{b_i} \cap X \neq \emptyset .$$

Since X is dense in Y and V_{a_i}'s , V_{b_i}'s are open.

Thus (X,T) is not totally co-Hausdorff.

To prove the necessity, suppose that (X,T) is H-closed, and that $\mathcal{B}$ is a base for T such that $(X,T(\mathcal{B}))$ is not a Hausdorff cospace. Then there exists some filter base $\mathcal{F}$ which converges to two points x and y with respect to $T(\mathcal{B})$. Let

$$\mathcal{G} = \{G \in T \mid F \subset G \text{ , some } F \in \mathcal{F}\}$$

$\mathcal{G}$ is an open filter base on (X,T), and since (X,T) is H-closed, $\mathcal{G}$ has an adherent point z with respect to (X,T).

Either $z \neq x$ or $z \neq y$. Without loss of generality, assume $z \neq x$. Since T is a Hausdorff topology, there are disjoint members U and V of $\mathcal{B}$ such that

$$x \in U \quad \text{and} \quad z \in V .$$

Therefore $x \in X - \overline{V} \in T(\mathcal{B})$.

But since $\mathcal{F}$ converges to x with respect to $T(\mathcal{B})$, there exists $F \in \mathcal{F}$ such that

$$F \subset X - \overline{V}$$

Thus $(X - \overline{V}) \in \mathcal{G}$

However $(X - \overline{V}) \cap V = \emptyset$ and $z \in V$

which implies that z is not an adherent point of $\mathcal{G}$ with respect to T. This is a contradiction, therefore every cospace is Hausdorff. ***

<u>1.3.3 Lemma</u>: If $\mathcal{B}$ is a base for (X,T) such that for each $U \in \mathcal{B}$, $X - \overline{U} \in \mathcal{B}$, then

$$\overline{U} = \mathrm{cl}_{T(\mathcal{B})}U \quad \text{for each} \quad U \in \mathcal{B} .$$

<u>Proof</u>: Clearly $\overline{U} \subset \mathrm{cl}_{T(\mathcal{B})}U$ since $T(\mathcal{B})$ is weaker than T.

Let $x \in cl_{T(\mathcal{B})}U$, and let O be a T-open neighbourhood of x, then there exists $B \in \mathcal{B}$ such that

$$x \in B \subset O .$$

But $X - \overline{B} \in \mathcal{B}$ so $X - \overline{X - \overline{B}} \in T(\mathcal{B})$ and $x \in X - \overline{X - \overline{B}}$

thus $X - \overline{X - \overline{B}} \cap U \neq \emptyset$ so $B \cap U \neq \emptyset$

thus $x \in \overline{U}$.

***

1.3.4 Theorem: If a Hausdorff space (X,T) is H-closed, then it is totally co-minimal Hausdorff.

Proof: It will be shown that the cospace $(X,T(T))$, where a base for the open sets of this cospace is

$$\mathcal{B} = \{X - \overline{U} \mid U \in T\} ,$$

is minimal Hausdorff. It will then follow from the fact, proved above, that every cospace is Hausdorff and that any other cospace of (X,T) is weaker than the cospace $(X,T(T))$, that every cospace is minimal Hausdorff.

Since T, as a base for (X,T) has the property that if $U \in T$ then $X - \overline{U} \in T$, for each $U \in T$,

$$\overline{U} = cl_{T(T)}U .$$

Furthermore, for $X - \overline{U} \in \mathcal{B}$

$$X - \overline{U} = X - \overline{(X - \overline{(X - \overline{U})})}$$
$$= X - cl_{T(T)}(X - cl_{T(T)}(X - \overline{U})) .$$

Thus $(X,T(T))$ has a base of regular open sets and is thus semi-regular. By the previous theorem $(X,T(T))$ is Hausdorff, and since it is the continuous image of its generating space (X,T) which is H-closed, it is H-closed too.

Let $\mathcal{F}$ be an open filter base consisting of elements of $T(T)$, and suppose that x is the unique adherent point of $\mathcal{F}$. $\mathcal{F}$ must be shown to converge to x .

Take a $T(T)$-open neighbourhood 0 of x , then there exists $X - \overline{U} \in \mathcal{B}$ such that

$$x \in X - \overline{U} \subset 0 .$$

Let

$\mathcal{G} = \{G \in T(T) \mid \text{there exists } F \in \mathcal{F} \text{ such that } F - \overline{X - \overline{U}} \subset G\}$.

If $G_1, G_2 \in \mathcal{G}$, there exist $F_1, F_2 \in \mathcal{F}$ such that

$$F_1 - \overline{X - \overline{U}} \subset G_1 \quad \text{and} \quad F_2 - \overline{X - \overline{U}} \subset G_2 .$$

But $\mathcal{F}$ is an open filter base, so there exists $F_3 \in$, such that

$$F_1 \cap F_2 \supset F_3$$

so $F_3 - \overline{X - \overline{U}} \subset G_1 \cap G_2$

Thus $G_1 \cap G_2 \in \mathcal{G}$.

Therefore if $\emptyset \notin \mathcal{G}$, $\mathcal{G}$ is a filter base of open sets in $(X,T(T))$, and since $(X,T(T))$ is H-closed, $\mathcal{G}$ will have an adherent point y with respect to $T(T)$.

(i) Consider the case $x = y$

Now $X - \overline{X - \overline{U}} \in T(T)$

For any $F \in \mathcal{F}$,

$$X - \overline{X - \overline{U}} \supset F - \overline{X - \overline{U}}$$

so $X - \overline{X - \overline{U}} \in \mathcal{G}$.

But $y = x \in X - \overline{U}$,

and $(X - \overline{U}) \cap (X - \overline{X - \overline{U}}) = \emptyset$

This contradicts the fact that y is an adherent point of $\mathcal{S}$ with respect to $T(T)$.

(ii) If $x \neq y$, since x is the unique adherent point of $\mathcal{F}$, $y \notin \cap\overline{F}$

Thus there exists $F \in \mathcal{F}$ such that $y \notin \overline{F}$, and there exists $W \in T$ such that $y \in W$

and $W \cap F = \emptyset$

thus $F \subset X - \overline{W}$

so $F - \overline{X - \overline{U}} \subset X - \overline{W} \in T(T)$

hence $X - \overline{W} \in \mathcal{S}$.

But $X - \overline{X - \overline{W}} \cap X - \overline{W} = \emptyset$

which again contradicts the fact that y is an adherent point of $\mathcal{S}$ with respect to $T(T)$.

These contradictions lead to the conclusion that $\mathcal{S}$ is not a filter base, so that $\emptyset \in \mathcal{S}$.

Hence for some $F \in \mathcal{F}$

$$F - \overline{X - \overline{U}} = \emptyset$$

ie $F \subset \overline{X - \overline{U}}$

Therefore $F \subset X - \overline{(X - (\overline{X - \overline{U}}))} = X - \overline{U} \subset 0$

So every $T(T)$ neighbourhood of x contains a member of $\mathcal{F}$, ie $\mathcal{F}$ converges to x . ***

It has thus been shown that an H-closed space (X,T) has $(X,T(T))$ as a minimal Hausdorff cospace. It is clear that if $\mathcal{A}$ and $\mathcal{B}$ are bases for T , and $\mathcal{A} \subset \mathcal{B}$, then

$T(\mathcal{A}) \subset T(\mathcal{B})$

So in particular, for all bases $\mathcal{B}$ of T ,

$$T(\mathcal{B}) \subset T(T)$$

Thus if T is H-closed, then every cospace of (X,T) is Hausdorff. But $(X,T(T))$ is minimal Hausdorff, therefore every cospace is homeomorphic to it; and so every cospace is minimal Hausdorff and semi-regular too.

<u>1.3.5 Theorem</u>: A Hausdorff space (X,T) is H-closed if and only if it is totally co-H-closed.

<u>Proof</u>: The necessity follows easily from the fact that each cospace is the continuous image of its generating space, and that the continuous image of an H-closed space is H-closed.

Conversely, if (X,T) is totally co-H-closed, it is totally co-Hausdorff, hence H-closed. ***

The above three theorems give the following characterization of H-closed spaces in terms of the cospaces.

<u>1.3.6 Theorem</u>: For a Hausdorff space (X,T) the following are equivalent:

(i) (X,T) is H-closed.

(ii) (X,T) is totally co-minimal Hausdorff.

(iii) All the cospaces of (X,T) are homeomorphic.

(iv) (X,T) is totally co-semi-regular.

(v) (X,T) is totally co-Hausdorff.

(vi) (X,T) is totally co-H-closed.

CHAPTER TWO

SUBSPACES AND PRODUCTS OF H-CLOSED SPACES AND H-CLOSED EXTENSIONS

2.1 H-CLOSURE AND SUBSPACES:

H-closure is not an hereditary property in that a subspace A of an H-closed space (X,T) need not necessarily be H-closed (in the relative topology of A).

H-closure is not an absolute property, for a subspace A may not be H-closed in its relative topology, and yet be H-closed in the topology T (i.e. every T-open cover $\mathscr{C}$ of A has a finite subcover satisfying $A \subset cl_T(\bigcup_{i=1}^{n} C_i))$.

This is illustrated by the following example of Urysohn [25] .

2.1.1 Example: Let

$$X = \{a_{ij}, b_{ij}, c_i, a, b \mid i=1, 2, \dots \} .$$

All elements are assumed to be distinct.

Define the following neighbourhood system on X :

Each a_{ij} and each b_{ij} is isolated:

$$N(c_i) = \{V_n(C_i) = \bigcup_{j=n}^{\infty} \{a_{ij}, b_{ij}, c_i\} \mid n=1, 2, \dots \} .$$

$$N(a) = \{V_n(a) = \bigcup_{j=1}^{\infty} \bigcup_{i=n}^{\infty} \{a_{ij}, a\} \mid n=1, 2, \dots \} .$$

$$N(b) = \{V_n(b) = \bigcup_{j=1}^{\infty} \bigcup_{i=n}^{\infty} \{b_{ij}, b\} \mid n=1, 2, \dots \} .$$

Denote X with this topology by (X,T) .

$$\overline{V_k(a)} = V_k(a) \cup \{c_j \mid j=k, k+1, \ldots\} .$$

and $\overline{V_k(b)} = V_k(b) \cup \{c_j \mid j=k, k+1, \ldots\}$.

(X,T) is H-closed, the subspace $(A, T|A)$ where

$$A = \{a, c_i \mid i=1, 2, \ldots\} .$$

$T|A$, the relative topology on A , is the discrete topology on A , is not H-closed in its relative topology, but is H-closed relative to the topology on X .

If a subspace is dense in X and is H-closed relative to X , then it is H-closed.

Clearly if a subspace of an H-closed space X is H-closed, then it is H-closed relative to the topology on X .

2.2 H-closedness and Regular closed sets:

A set B is regular closed if it is the same as the closure of its interior i.e. $B = B^{0-}$. Regular closed sets are closed under finite unions.

With regard to subspaces of H-closed spaces, and H-closed subspaces, we have the following:

2.2.1 Proposition: If (X,T) is an H-closed space, and $B \subset X$ is regularly closed, then B is H-closed.

Proof: Suppose $\mathscr{C}$ is an open covering of B , then for each $C \in \mathscr{C}$, there exists C^* open in X such that

$$C^* \cap B = C .$$

Then $\mathscr{C}^* = \{C^* \mid C^* \cap B = C \, ; \, C \in \mathscr{C}\} \cup \{X - B\}$ is an open cover of X. Thus there exist $C_1^*, \ldots , C_n^*$ such that

$$X = \bigcup_{i=1}^{n} \overline{C_i^*} \cup \overline{X - B}$$

Now $B = B^{0-}$, and $B^0 = X - (\overline{X - B})$

thus

$$B^0 \subset \bigcup_{i=1}^{n} \overline{C_i^*}$$

Since

$$\overline{B^0 \cap \bigcup_{i=1}^{n} \overline{C_i^*}} = \overline{B^0 \cap \bigcup_{i=1}^{n} C_i^*} ,$$

it follows that

$$B \subset \bigcup_{i=1}^{n} \overline{C_i}$$

Thus B is H-closed. ***

<u>Note</u>: The converse of the above is not true. However it is true that if A is a closed subspace of an H-closed space, and the boundary of A is H-closed, then A is H-closed. For

$$A = A^{0-} \cup A - A^0$$

and A^{0-} is regularly closed, hence H-closed, and $A - A^0$ is H-closed, hence A is H-closed. Also the closure of any open subset of an H-closed space is H-closed, since such sets are regularly closed. It is well known that compact subsets of Hausdorff spaces are closed, and the analogous result holds for H-closed spaces, since H-closed subsets of Hausdorff spaces are closed.

2.3 Products of H-closed Spaces:

Regarding the productivity of H-closed spaces, the following theorem was proved by

C. Chevelly and O. Frink in [6]. It uses the characterization of H-closedness in terms of the convergence of every open ultrafilter. The following fact regarding ultrafilters is also required.

<u>2.3.1 Lemma</u>: If $\mathcal{U}$ is an open ultrafilter on X, and 0 is an open subset of X, then $0 \cap U \neq \emptyset$ for all $U \in \mathcal{U}$ if and only if $0 \in \mathcal{U}$.

<u>Proof</u>: Suppose 0 is open and $0 \cap U \neq \emptyset$ for all $U \in \mathcal{U}$, but that $0 \notin \mathcal{U}$. Then $\mathcal{U} \cup 0$ is an open filter larger than $\mathcal{U}$, which is not possible.
The converse is obvious. ***

<u>2.3.2 Theorem</u>: A product H of non-empty Hausdorff spaces H_a, $a \in A$ is H-closed if and only if each co-ordinate space is H-closed.

<u>Proof</u>: The necessity is evident from the fact that the continuous image of an H-closed space is H-closed.

To prove the sufficiency, it will be shown that every open ultrafilter converges.
Let $\mathcal{U}$ be an ultrafilter of open sets of

$$H = \prod_{a\in A} H_a$$

For each a, let

$$\pi_a(\mathcal{U}) = \{\pi_a(U) \in H_a \mid \text{for each } U \in \mathcal{U}\} .$$

where π_a is the a'th projection onto H_a.

Since π_a is an open mapping, each $\pi_a(U)$ is open in H_a. Thus $\pi_a(\mathcal{U})$ is an open filter base on H_a, for each a. Since H_a is H-closed, there exists $p_a \in H_a$ such that

$$p_a \in \cap[\overline{\pi_a(U)} \mid U \in \mathcal{U}]$$

The point $p \in H$ which has p_a in the a'th coordinate position, for each a, will be shown to be the point to which $\mathcal{U}$ converges.

Each p_a was such that for each open neighbourhood N_a of p_a,

$$N_a \cap \pi_a(U) \neq \emptyset \qquad \text{for each } U \in \mathcal{U}$$

thus

$$W_a = \pi_a^{\leftarrow}(N_a) \times \prod_{b \neq a} H_b \qquad b \in A$$

is an open neighbourhood in H of p, and is such that

$$W_a \cap U \neq \emptyset \qquad \text{for each } U \in \mathcal{U}$$

Since $\mathcal{U}$ is maximal, and W_a is open in H,

$W_a \in \mathcal{U}$ by the above Lemma.

Furthermore $\bigcap_{i=1}^{n} W_{a_i} \in \mathcal{U}$.

Thus every open neighbourhood of p is in $\mathcal{U}$.

Therefore for any neighbourhood N of p, there exists an open neighbourhood W of p such that

$$p \in W \subset N .$$

But $W \in \mathcal{U}$; thus $\mathcal{U}$ converges to p. ***

2.4 H-closed Extensions: Given an arbitrary Hausdorff space (X,T), it is possible to construct an H-closed space in which X is embedded as a dense subspace. Two methods will be discussed.

The first is due to Miroslav Katětov.

In [17], Katětov proved that for any Hausdorff space (X,T), there exists a Hausdorff H-closed space kX in which X is densely embedded, and has the property that if f is a continuous mapping from X to a Hausdorff space T such that $f(X)$ is dense in T, then there exists a subspace M of kX, and a continuous mapping F from M onto T such that

$X \subset M$ and $F|_X = f$

In [20], C-T. Liu discusses Katětov's method of construction, and extension properties of certain continuous function on X which show that kX is the largest H-closure of X in a sense which will be defined.

<u>2.3.1 Lemma [12, p. 5]</u>: Let X be dense in each of the Hausdorff spaces S and T. If the identity mapping on X has continuous extensions σ from S into T, and τ from T into S, then σ is a homeomorphism onto, and $\sigma^{\leftarrow} = \tau$.

<u>2.3.2 Definition [20, 0.7]</u>: Let X be dense in each of the spaces S and T. If f is a homeomorphism from S to T such that f is the identity on X, we say that S is essentially the same as T relative to X.

<u>2.3.3 Definition [20, 1.6]</u>: Let X and Y be Hausdorff spaces such that

(a) X is dense in Y.

(b) Y is H-closed.

Y is called an H-closure of X.

The words absolutely closed and absolute closure are also used to mean H-closed and H-closure.

2.3.4 Definition [20 , 1.7]: Let Y be an H-closure of X. Y is a largest H-closure of X, if for any other H-closure, say T of X, and if

$i : X \longrightarrow T$ is the inclusion, then there exists

$\bar{\imath} : Y \longrightarrow T$, which is onto, and such that

$$\bar{\imath}|_X = i$$

Such a Y will be shown to be essentially unique, provided that it exists.

Liu constructs kX similarly to Katětov although the description and proof is simpler.

The following properties of ultrafilters are required.

2.3.5 Lemma [20 , 0.2]: If $\mathcal{U}$ is an open ultrafilter on X, the following hold.

(a) If G_1, G_2 are open subsets of X, and $G_1 \cup G_2 \in \mathcal{U}$, then $G_1 \in \mathcal{U}$ or $G_2 \in \mathcal{U}$.

(b) If $G_1 \notin \mathcal{U}$, and G_1 is open, then $G_2 = (X - \overline{G}_1) \in \mathcal{U}$.

(c) If p is a cluster point of $\mathcal{U}$, then $\mathcal{U}$ converges to p.

Proof: (a) and (c) are straightforward.

(b) If $G_1 \notin \mathcal{U}$, then there exists $U \in \mathcal{U}$ such that

$G_1 \cap U = \emptyset$ thus $\overline{G}_1 \cap U = \emptyset$

Thus $U \subset X - \overline{G}_1 = G_2$, so $G_2 \in \mathcal{U}$.

2.3.6 Lemma [20, 0.4]: Suppose X is dense subspace of a space Y, and $\mathcal{U}$ is an open ultrafilter on X.
Let $\mathcal{U}^1 = \{G \mid G$ is open in Y, $G \cap X \in \mathcal{U}\}$
Then $\mathcal{U}^1$ is an open ultrafilter on Y.
Furthermore $\mathcal{U}$ converges to p if and only if $\mathcal{U}^1$ converges to p.

Proof: Clearly $\emptyset \notin \mathcal{U}^1$, and if $G_1, G_2 \in \mathcal{U}^1$, then $G_1 \cap G_2 \in \mathcal{U}^1$. If $G \in \mathcal{U}^1$, 0 open in Y with $G \subset 0$,
then $\emptyset \neq G \cap X \subset 0 \cap X$ and $G \cap X \in \mathcal{U}$,
so $0 \cap X \in \mathcal{U}$
Therefore $0 \in \mathcal{U}^1$, and $\mathcal{U}^1$ is an open filter on Y.

If G_0 is open,
and $G_0 \cap G \neq \emptyset$ for all $G \in \mathcal{U}^1$
then $G_0 \cap U \neq \emptyset$ for all $U \in \mathcal{U}$
since $U = G_1 \cap X$, some G_1,
and $G_0 \cap G_1 \cap X \neq \emptyset$.
Hence $G_0 \cap X \in \mathcal{U}$
Hence $G_0 \in \mathcal{U}^1$
Thus $\mathcal{U}^1$ is an open ultrafilter on Y.
If $\mathcal{U}$ converges to p,
then $p \in [\overline{U} \cap X \mid U \in \mathcal{U}]$
so $p \in \cap[\overline{U^1 \cap X} \mid U^1 \in \mathcal{U}^1]$.
i.e. $p \in \cap[\overline{U^1} \mid U^1 \in \mathcal{U}^1]$.
Therefore p is an adherent point of $\mathcal{U}^1$ and thus $\mathcal{U}^1$ converges to p.

Conversely $\mathcal{U}^1$ convergent to p implies $\mathcal{U}$ convergent to p.

2.3.7 Lemma [20, 0.5]. Let X be a dense subspace of a space Y . If $\mathcal{U}^1$ is an open ultrafilter on Y , and

$$\mathcal{U} = \mathcal{U}^1 \cap X = \{U^1 \cap X \mid U^1 \in \mathcal{U}\} ,$$

then $\mathcal{U}$ is an open ultrafilter on X . Also $\mathcal{U}$ converges to p if and only if $\mathcal{U}^1$ converges to p .

Proof: Procedure similar to the proof of the above Lemma.

Note: If in the above Lemma X is also open in Y , then $X \in \mathcal{U}^1$ (by Lemma 2.3.5 (b)), and

$$= \{U \in \mathcal{U}^1 : U \subset X\} .$$

Given an arbitrary Hausdorff space (X,T) , let

$X^\vee = \{\mathcal{U} \mid \mathcal{U}$ is an open nonconvergent ultrafilter on $X\}$.

Put $kX = X \cup X^\vee$, the disjoint union.

Define a topology on kX as follows:

Let $\mathcal{B}$ be a basis for the topology, where

$B \in \mathcal{B}$ if and only if

(i) B is open in X , or

(ii) $B = U \cup \{\mathcal{U}\}$ where $\mathcal{U} \in X^\vee$, and $U \in \mathcal{U}$.

2.3.8 Theorem. For any Hausdorff space X , there exists a Hausdorff space kX which is a largest H-closure of X .

Furthermore kX is essentially unique.

Proof: Construct kX and the topology on kX as described above.

(a) kX with this topology is a Hausdorff space.

It is clear that distinct points in X have disjoint neighbourhoods in kX .

If $x \in X$, and $\mathcal{U} \in X^{\vee}$, since $\mathcal{U}$ is a nonconvergent open ultrafilter, there exists an open neighbourhood 0 of x in X such that $0 \notin \mathcal{U}$. Thus there exists $U \in \mathcal{U}$ such that $0 \cap U = \emptyset$.

Therefore 0 and $U \cup \{\mathcal{U}\}$ are disjoint neighbourhoods in kX of x and $\mathcal{U}$ respectively.

If $\mathcal{U} \neq \mathcal{V}$ and $\mathcal{U}, \mathcal{V} \in X^{\vee}$, then there exist $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $U \cap V = \emptyset$.

So $U \cup \{\mathcal{U}\}$ and $V \cup \{\mathcal{V}\}$ are disjoint neighbourhoods in kX of $\mathcal{U}$ and $\mathcal{V}$ respectively.

(b) kX is H-closed.

Every open ultrafilter will be shown to converge.

Let $\mathcal{U}$ be an open ultrafilter on kX , and set

$$\mathcal{P} = \{U \cap X \mid U \in \mathcal{U}\} .$$

By Lemma 2.3.7, $\mathcal{P}$ is an open ultrafilter on X .

If $\mathcal{P}$ converges to x in X , then x is an adherent point of $\mathcal{U}$ and $\mathcal{U}$ converges to x (in the topology of kX).

If $\mathcal{P}$ fails to converge on X , then $\mathcal{P} \in X^{\vee}$.

Let $P \cup \{\mathcal{P}\}$ be a neighbourhood of $\mathcal{P}$ in kX , then $P \in \mathcal{P}$.

Since X is open in kX , and dense in kX , $X \in \mathcal{U}$, and so

$$\mathcal{P} = \{U \in \mathcal{U} \mid U \subset X\} .$$

Thus $P \in \mathcal{U}$. Therefore

$$P \cap U \neq \emptyset \quad \text{for each } U \in \mathcal{U}$$

and $(P \cup \{\wp\}) \cap U \neq \emptyset$ for each $U \in \mathcal{U}$

Thus $\wp$ is an adherent point of $\mathcal{U}$ in kX, so converges to $\wp$.

(c) X is a largest H-closure of X.

Let T be an H-closure of X, and

$$i : X \longrightarrow T$$

be the inclusion. We wish to extend i to

$$\overline{\imath} : kX \longrightarrow T \text{ continuously.}$$

If such a continuous $\overline{\imath}$ exists, it will be onto. For $\overline{\imath}(kX)$ is H-closed and hence is closed in T. But it contains a dense subset X of T, so $\overline{\imath}(kX) = T$.

Let $\wp \in X^{\vee}$, and let

$$\mathcal{U} = \{U \mid U \text{ open in } T \text{ and } U \cap X \in \wp\} .$$

By Lemma 2.3.6, $\mathcal{U}$ is an open ultrafilter on T. T is H-closed, thus there exists $p \in T$ such that $\mathcal{U}$ converges to p, and so $\wp$ converges to p in T.

Now define for each $\wp \in X^{\vee}$

$$\overline{\imath}(\wp) = p \text{, and for } x \in X ,$$

$$\overline{\imath}(x) = x .$$

$\overline{\imath}$ is continuous:

(i) For $x \in X$,

Let W be an open neighbourhood of x in T, then

$$\overline{\imath}^{\leftarrow}(W) \cap X = i^{\leftarrow}(W) = G$$

where G is an open neighbourhood of X in X, hence open in kX, and $\overline{\imath}(G) \subset W$.

(ii) For $\mathscr{P} \in X^{\vee}$,

$\bar{\imath}(\mathscr{P}) = p$, where $\mathscr{P}$ converges to p in T .

Let W be an open neighbourhood of p in T , then there exists $P \in \mathscr{P}$ such that

$P \subset W$.

Thus $P \cup \{\mathscr{P}\}$ is an open neighbourhood of $\mathscr{P}$ in kX such that

$$\begin{aligned} \bar{\imath}(P \cup \{\mathscr{P}\}) &= \bar{\imath}(P) \cup \bar{\imath}(\{\mathscr{P}\}) \\ &= i(P) \cup \{p\} \\ &= P \cup \{p\} \subset W . \end{aligned}$$

(d) X is essentially unique.

If Y is an H-closure of X such that Y is largest in the sense of Definition 2.3.4, then there exists

$\bar{\imath} : Y \longrightarrow kX$,

where $\bar{\imath}$ is a continuous extension of

$i : X \longrightarrow kX$.

Also there exists

$\bar{\jmath} : kX \longrightarrow Y$,

such that $\bar{\jmath}$ is also a continuous extension of the inclusion map j . $\bar{\imath}$ is a homeomorphism by Lemma 2.3.1, so that kX is essentially the same as Y .

CHAPTER THREE

THE EMBEDDING OF AN H-CLOSURE IN ΠD_f

3.1 THE H-CLOSURE OF F. OBREANU:

In the previous chapter, the Katětov H-closure of a Hausdorff space was discussed. Other methods of constructing an H-closed extension of a Hausdorff space have been investigated. In particular, a method due to F. Obreanu [20] will be discussed.

Given an arbitrary Hausdorff space (X,T), define $X^{\vee}$ as before, and let

$\tilde{X} = X \cup X^{\vee}$ as before.

Define the following topology on $\tilde{X}$, which will be denoted by $O(T)$. The basic open sets of $\tilde{X}$ are those of the form

$U^* = U \cup \{\mathcal{U} \in X^{\vee} \mid U \in \mathcal{U}\}$. U open in (X,T) .

It is clear that $(\tilde{X},O(T))$ is Hausdorff once it has been observed that for $U, V \in T$,

$U \cap V = \phi$ in X if and only if

$U^* \cap V^* = (U \cap V)^* = \phi$.

(X,T) is clearly dense, but not open, in $(\tilde{X},O(T))$. Furthermore $(\tilde{X},O(T))$ is H-closed.

It has been shown that kX has the property that if Y was any other H-closure of X, $\iota : X \longrightarrow Y$ the inclusion, then ι extended to $\bar{\iota} : kX \longrightarrow Y$ continuously.

$(\tilde{X},O(T))$ does not have this property, since $\bar{\iota} : (\tilde{X},O(T)) \longrightarrow Y$ may not be continuous. However $\bar{\iota}$ will be shown to be "θ-continuous".

3.1.1 Definition [16 pg 42]: A mapping f of a space X into a space Y is θ-continuous at x if for every neighbourhood U of $y = f(x)$, there exists a neighbourhood V of x such that

$$f(\overline{V}) \subset \overline{U} .$$

The mapping is θ-continuous if it is θ-continuous at every point of X. The mapping is a θ-homeomorphism if it is $1 - 1$, θ-continuous, and has θ-continuous inverse.

3.1.2 Proposition: The θ-continuous image of an H-closed space is H-closed.

Proof: Suppose $f : (X,T) \longrightarrow (Y,S)$ is θ-continuous and onto, and $\mathcal{C}$ an open cover of Y.

For each $x \in X$, there exists $U_x \in \mathcal{U}$ such that

$$f(x) \in U_x .$$

Then there exists V_x open in X, $x \in V_x$ such that

$$f(\overline{V}_x) \subset \overline{U}_x .$$

$\mathcal{V} = \{V_x \mid x \in X\}$ is an open cover of X, so there exist $x_1 , \ldots , x_n$ such that

$$X = \bigcup_{i=1}^{n} \overline{V}_{x_i}$$

thus $Y = \bigcup_{i=1}^{n} f(\overline{V}_{x_i}) \subset \bigcup_{i=1}^{n} \overline{U}_{x_i}$

and (Y,S) is therefore H-closed. ***

3.1.3 Proposition: Let (Y,S) be an H-closure of (X,T) and $i : X \longrightarrow Y$ the inclusion. Then i can be extended to

$$\bar{\iota} : (\tilde{X},O(T)) \longrightarrow (Y,S) \qquad \theta\text{-continuously.}$$

Proof: For $\mathcal{U} \in X^{\vee}$, define

$$\bar{\iota}(\mathcal{U}) = y \ ,$$

where $\mathcal{U}$ converges to y in (Y,S), as before.

$$\bar{\iota}(x) = x \qquad \text{for } x \in X \ .$$

$\bar{\iota}$ is θ-continuous:

Firstly, it is clear that

$$\begin{aligned}\overline{U^*} &= \overline{U \cup \{\mathcal{U} \in X^{\vee} \mid U \in \mathcal{U}\}} \ . \\ &= cl_X U \cup \{\mathcal{U} \in X^{\vee} \mid U \in \mathcal{U}\} \ .\end{aligned}$$

(i) For $x \in X$, let W be an open neighbourhood of x in (Y,S) then $\bar{\iota}^{\leftarrow}(W) \cap X = i^{\leftarrow}(W) = G$ where G is an open neighbourhood of x in X. Thus G^* is an open neighbourhood of x in $(\tilde{X},O(T))$, and

$$\bar{\iota}(\overline{G^*}) \subset \overline{W} \qquad .$$

For
$$\begin{aligned}\bar{\iota}(\overline{G^*}) &= \bar{\iota}(cl_X G \cup \{\mathcal{G} \in X^{\vee} \mid G \in \mathcal{G}\}) \\ &= cl_X G \cup \bar{\iota}(\{\mathcal{G} \in X^{\vee} \mid G \in \mathcal{G}\}) \\ &= cl_X G \cup \{y \in Y \mid \mathcal{G} \rightarrow y \text{ , each } \mathcal{G} \text{ such that } G \in \mathcal{G}\} \\ &\subset \overline{W}\end{aligned}$$

Since each y for which $\mathcal{G}$ converges to y, each $\mathcal{G}$ with $G \in \mathcal{G}$, is such that $y \in \overline{W}$

For each neighbourhood N of y in (Y,S),

$$N \cap G \neq \emptyset \qquad \text{each } G \in \mathcal{G}$$

so $N \cap W \neq \emptyset$ each neighbourhood N of y

thus $y \in \overline{W}$.

(ii) For $\mathcal{U} \in X^{\vee}$, $\mathcal{U}$ converges to y in (Y,S) .
Let W be an open neighbourhood of y in (Y,S) . Then there exists $U \in \mathcal{U}$ such that $U \subset W$.
So U^* is an open neighbourhood of y in $(\tilde{X},O(T))$ such that

$\bar{\iota}(\overline{U^*}) \subset \overline{W}$ as before . ***

Thus if (Y,S) is an H-closure of (X,T) , then
$\bar{\iota} : (\tilde{X},O(T)) \longrightarrow (Y,S)$ is a θ-continuous extension of
$i : (X,T) \longrightarrow (Y,S)$. Furthermore $\bar{\iota}$ is onto, for by Proposition 3.1.2, $\bar{\iota}(\tilde{X})$ is H-closed, thus closed in (Y,S) , and as it contains the dense subset X of T ,

$\bar{\iota}(\tilde{X}) = Y$.

The following indicates how (X,T) is embedded in $(\tilde{X},O(T))$.

<u>3.1.4 Proposition</u>: The Hausdorff space (X,T) , as a dense subspace of $(\tilde{X},O(T))$, has the following four properties.

(i) If F_1 , $F_2 \subset X$, are relatively closed in X , not nowhere dense, in X , then

$$F_1 \cap F_2 = \emptyset \quad \text{implies} \quad \overline{F}_1 \cap \overline{F}_2 = \emptyset$$

(ii) If F_1 , F_2 are relatively closed, not nowhere dense, in X , then

$$\overline{F_1 \cap F_2} = \overline{F}_1 \cap \overline{F}_2$$

(iii) If F_1 , F_2 are relatively closed, not nowhere dense, in X , then

$F_1 \cap F_2$ nowhere dense in X implies

$F_1 \cap F_2 = \overline{F}_1 \cap \overline{F}_2$

(iv) If G_1 , G_2 are relatively open and

$G_1 \cap G_2 = \emptyset$ then $\overline{G}_1 \cap \overline{G}_2 \subset X$

<u>Proof</u>: (i) Let F_1 , F_2 be closed in X , and

$F_1 \cap F_2 = \emptyset$

Suppose $\overline{F}_1 \cap \overline{F}_2 \neq \emptyset$

then there exists $p \in \tilde{X}$ such that $p \in \overline{F}_i$ $i=1, 2$.

so $N(p) \cap F_i \neq \emptyset$ $i=1, 2$.

for each neighbourhood $N(p)$ of p in $(\tilde{X}, O(T))$.

If $p = x \in X$, then

$U \cap F_i \neq \emptyset$ for each U open in (X,T) with $p \in U$.

so $p \in F_1 \cap F_2$ contradiction.

If $p = \mathcal{U} \in X^{\vee}$, then

$U^* \cap F_i \neq \emptyset$ for each $U \in \mathcal{U}$ and $i=1, 2$

thus $U \cap F_i \neq \emptyset$ " "

and $U \cap \text{int}_X F_i \neq \emptyset$ " "

therefore $\text{int}_X F_i \in \mathcal{U}$ $i=1, 2$

so $\text{int}_X F_1 \cap \text{int}_X F_2 \in \mathcal{U}$

and $F_1 \cap F_2 \neq \emptyset$ contradiction.

(ii) To show that $\overline{F_1 \cap F_2} = \overline{F}_1 \cap \overline{F}_2$, it is only necessary to show that if $\mathcal{U} \in X^{\vee}$ and $\mathcal{U} \in \overline{F}_1 \cap \overline{F}_2$, then

$\mathcal{U} \in \overline{F_1 \cap F_2}$,

and this is clear.

(iii) We show $F_1 \cap F_2 = \overline{F}_1 \cap \overline{F}_2$

If $x \in \overline{F}_1 \cap \overline{F}_2$, then clearly $x \in F_1 \cap F_2$.

If $\mathcal{U} \in \overline{F}_1 \cap \overline{F}_2$, as before, we conclude that

$\mathrm{int}_X F_i \in \mathcal{U}$ $\quad i=1,2$

so $\mathrm{int}_X(F_1 \cap F_2) \neq \emptyset$ $\quad$ contradiction

thus $\overline{F}_1 \cap \overline{F}_2 = F_1 \cap F_2$.

(iv) If G_1 , G_2 are relatively open in X and $G_1 \cap G_2 = \emptyset$.

Suppose there exists $\mathcal{U} \in X^{\vee}$ such that

$\mathcal{U} \in \overline{G}_1 \cap \overline{G}_2$

then $U^* \cap G_i \neq \emptyset$ $\quad i=1, 2$, each $U \in \mathcal{U}$.

thus $U \cap G_i \neq \emptyset$ $\quad$ " $\quad$ "

and $G_i \in \mathcal{U}$ $\quad i=1, 2$

so $G_1 \cap G_2 \neq \emptyset$ $\quad$ contradiction

thus $\overline{G}_1 \cap \overline{G}_2 \subset X$. ***

3.2 AN EMBEDDING OF $(X,O(T))$ IN $\Pi D_{\mathcal{F}}$

It is well known that the Stone Čech compactification of a completely regular space X may be constructed by embedding the space in a product of copies of the unit interval with the usual topology, and then taking the closure. This yields βX .

The question arises as to whether a similar method can be used to construct an H-closure of a Hausdorff space.

Observe firstly that every topological space is determined by its continuous maps into (D,u) where $D = \{0,1\}$ and u is the upper topology on D , i.e. $\emptyset$, $\{0,1\}$, $\{0\}$; (or by continuous maps into (D,l) where l is the lower topology on D, i.e. ϕ, $\{0,1\}$, $\{1\}$.).

For an arbitrary Hausdorff space (X,T), let $F(X)$ be the family of all continuous functions on (X,T) to (D,u) . Let

$$\Pi D_f = \Pi[(D,u)_f \mid f \in F(X)]$$

(the product of (D,u) taken $F(X)$ times).

ΠD_f is a T_0 space.

The evaluation map

$$e : X \longrightarrow \Pi D_f$$

where $e(x)$ is such that its f-th coordinate is $f(x)$ for each $f \in F[X]$, is continuous. In fact e is an embedding since $F(X)$ separates points and closed sets.

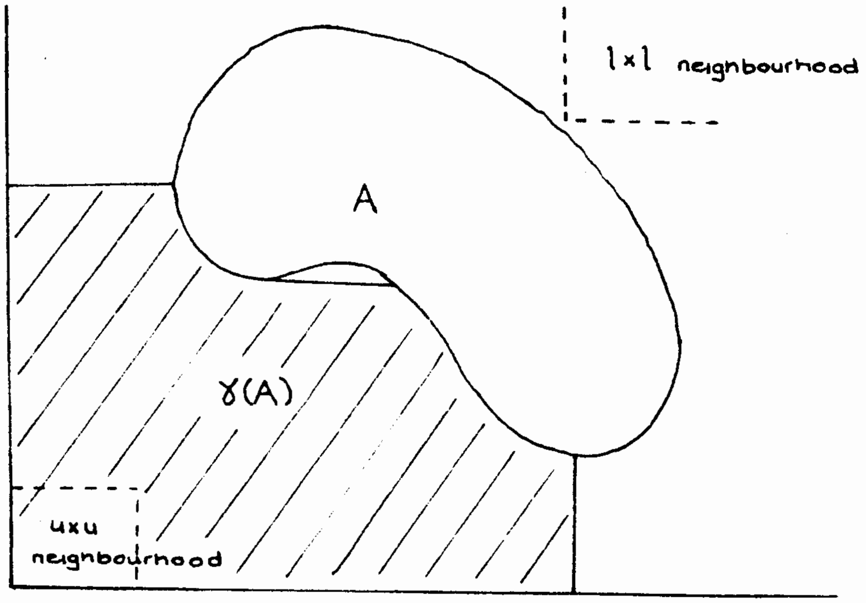
The following definition is required for the purpose of investigating how the H-closure $(X,0(T))$ of (X,T) is embedded in ΠD_f .

<u>3.2.1 Definition</u>: Let (X,P,Q) be a bitopological space. For $A \subset X$, define $\gamma(A)$ as follows:

$x \in \gamma(A)$ if for every Q-neighbourhood V of x ; there exists a P-neighbourhood W of x such that

$$\phi \neq W \cap A \subset V .$$

<u>3.2.2 Example</u>: Let $(R^+,u,l) \times (R^+,u,l) = (R^+ \times R^+, u \times u , l \times l)$ R^+ the positive reals, u the upper and l the lower topology on R^+ .



Let $\delta(e[X]) = \Pi(u \vee l)\text{-closure}\ (e[X]) \cap \gamma(e[X]) \subset \Pi D_f$.

where

$\alpha \in \Pi(u \vee l)\text{-closure}\ (e[X])$

if for every upper product neighbourhood U of α , and every lower product neighbourhood V of α ,

$U \cap V \cap e[X] \neq \emptyset$.

For $\gamma(e[X])$, $e[X]$ takes the place of A in definition 3.2.1 and Πu , Πl (the upper product and lower product topologies respectively on ΠD_f) , take the place of P and Q respectively.

It is well known that the continuous real-valued functions on a topological space form a ring.

Consider $F(X)$ with $+$ and $\cdot$ defined as follows.

$1 + 0 = 1$ $\qquad$ $1 \cdot 1 = 1$

$0 + 1 = 1$ $\qquad$ $1 \cdot 0 = 0$

$1 + 1 = 1$ $\qquad$ $0 \cdot 1 = 0$

$0 + 0 = 0$ $\qquad$ $0 \cdot 0 = 0$

$(F(X) , + , \cdot)$ is a semiring.

We now wish to show there is a 1 - 1 correspondence between the open ultrafilters on (X,T), and $\delta(e[X]) \subset \Pi D_f$. Later we will show to what subset of $\delta(e[X])$ the non-convergent ultrafilters correspond.

3.2.2 Lemma: Each $\alpha \in \delta(e[X])$ satisfies the following.

(i) $\alpha_f + \alpha_g = \alpha_{f+g}$

(ii) $\alpha_{f\cdot g} = \alpha_f \cdot \alpha_g$

(iii) $\alpha_0 = 0$ and $\alpha_1 = 1$

(iv) For each $\alpha_f = 1$, there exists $g : \alpha_g = 0$ and $f + g = 1$.

Proof: (i) If $\alpha_f, \alpha_g = 0$, then

$\alpha_f \times \Pi D_{f'}$ and $\alpha_g \times \Pi D_{g'}$

where $\Pi D_{f'} = \Pi[(D,u)_{f'} \mid f' \in F(X)$ and $f' \neq f]$

are upper product neighbourhoods of α, so

$$\alpha_f \times \Pi D_{f'} \cap \alpha_g \times \Pi D_{g'} \cap e[X] \neq \emptyset .$$

Thus for each x such that $e(x) \in \alpha_f \times \Pi D_{f'} \cap \alpha_g \times \Pi D'_g$,

$$f(x) = 0 \text{ and } g(x) = 0$$

so $(f + g)(x) = 0$ i.e. $(e(x))_{f+g} = 0$.

If $\alpha_{f+g} = 1$,

then $\alpha_{f+g} \times \Pi D_{(f+g)'}$ is a lower neighbourhood of α, and

$$\alpha_f \times \Pi D_{f'} \cap \alpha_g \times \Pi D_{g'} \cap \alpha_{f+g} \times \Pi D_{(f+g)'} \cap e[X] \neq \emptyset$$

since $\alpha \in \Pi(u \vee l)$-closure $(e[X])$

so there exists $e(x) \in \alpha_f \times \Pi D_{f'} \cap \alpha_g \times \Pi D_{g'}$ such that

$(f + g)(x) = 1$ contradiction.

If $\alpha_f, \alpha_g = 1$, a similar argument applies.

If $\alpha_f = 0$, $\alpha_g = 1$, then

$\alpha_f \times \Pi D_{f'}$, $\alpha_g \times \Pi D_{g'}$ are respectively upper and lower neighbourhoods of α . Thus

$$\emptyset \neq \alpha_f \times \Pi D_{f'} \cap \alpha_g \times \Pi D_{g'} \cap e[X]$$

Thus for each x such that

$$e(x) \in \alpha_f \times \Pi D_{f'} \cap \alpha_g \times \Pi D_{g'} ,$$

$$f(x) = 0 \quad \text{and} \quad g(x) = 1$$

so $(f + g)(x) = 1$ i.e. $(e(x))_{f+g} = 1$.

As before it can be shown that

$$\alpha_{f+g} = 1$$

Thus $\alpha_f + \alpha_g = \alpha_{f+g}$

(ii) $\alpha_{f \cdot g} = \alpha_f \cdot \alpha_g$

If $\alpha_f = 1$ and $\alpha_g = 1$ then $\alpha_f \cdot \alpha_g = 1$ and an argument similar to the above shows that

$$\alpha_{f \cdot g} = 1 .$$

The same argument can be applied for all the other possibilities.

(iii) $\alpha_0 = 0$.

If $\alpha_0 = 1$ then $\alpha_0 \times \Pi D_{0'}$ is a lower neighbourhood of α , so

$$\emptyset \neq \alpha_0 \times \Pi D_{0'} \cap e[X]$$

thus there exists x such that

$$(e(x))_0 = 1 \quad \text{i.e.} \quad 0(x) = 1$$

and this is not possible.

Similarly it can be shown that

$$\alpha_1 = 1 .$$

(iv) For each $\alpha_f = 1$, there exists g such that $\alpha_g = 0$ and $f + g = 1$. Since $\alpha \in \gamma(e[X])$, for every lower neighbourhood V of α , there exists an upper neighbourhood W of α such that

$$\phi \neq W \cap e[X] \subset V .$$

Let $\alpha_f = 1$, then $\alpha_f \times \Pi D_{f'}$ is a lower neighbourhood of α , so there exists an upper neighbourhood

$\alpha_g \times \Pi D_{g'}$ (see Note below)

$\alpha_g = 0$, such that

$$\phi = \alpha_g \times \Pi D_{g'} \cap e[X] \subset \alpha_f \times \Pi D_{f'}$$

So for each x such that

$$e(x) \in \alpha_g \times \Pi D_{g'} ,$$

$(e(x))_g = 0$ i.e. $g(x) = 0$, and

$(e(x))_f = 1$ i.e. $f(x) = 1$

therefore $(f + g)(x) = 1$ for $e(x) \in \alpha_g \times \Pi D_{g'}$

If $e(x) \notin \alpha_g \times \Pi D_{g'}$, then

$$(e(x))_g = 1$$

Thus $(f + g)(x) = 1$ for each $x \in X$

i.e. $f + g = 1$, and (iv) holds. ***

Note: An upper neighbourhood of α is of the form

$$\alpha f_1 \times \cdots \times \alpha f_n \times \Pi D_{f'_1} , \ldots , f'_n$$

with $\alpha f_i = 0$ $i=1, \ldots , n$.

thus by property (i) of the above lemma,

$$\alpha f_1 + \cdots + \alpha f_n = \alpha f_1 + \cdots + f_n = 0 .$$

Thus we need only consider upper neighbourhoods of the form $\alpha_f \times \Pi D_{f'}$.

Similarly for the lower neighbourhoods, since

$$\alpha_{f_1} \cdots\cdots \alpha_{f_n} = \alpha_{f_1} \cdots\cdots \alpha_{f_n} \qquad \alpha_{f_i} = 1 \quad i=1, \ldots , n .$$

<u>3.2.3 Proposition</u>: Each $\alpha = \{\alpha_f\}$ in $\delta(e[X])$ gives rise to an open ultrafilter $\mathcal{U}$ on X .

<u>Proof</u>: We show that $\alpha = \{\alpha_f\}$ in $\delta(e[X])$ gives rise to an open ultrafilter $\mathcal{U}$, where

$$\mathcal{U} = \{f^{\leftarrow}[0] \mid \alpha_f = 0\} .$$

$\mathcal{U}$ is an open filter.

(i) If $f_1^{\leftarrow}[0]$, $f_2^{\leftarrow}[0] \in \mathcal{U}$, then

$$\alpha_{f_1} = 0 \quad \text{and} \quad \alpha_{f_2} = 0$$

thus $\alpha_{f_1} + \alpha_{f_2} = \alpha_{f_1 + f_2} = 0$

and $(f_1 + f_2)^{\leftarrow}[0] = f_1^{\leftarrow}[0] \cap f_2^{\leftarrow}[0] \in \mathcal{U}$.

Also $(f_1 + f_2)^{\leftarrow}[0] \neq \emptyset$ since $\alpha \in \delta(e[X])$.

(ii) If $f^{\leftarrow}[0] \in \mathcal{U}$ and $g^{\leftarrow}[0] \subset f^{\leftarrow}[0]$, then

$$\alpha_f = 0 \quad \text{therefore} \quad \alpha_f \cdot \alpha_g = \alpha_{f\cdot g} = 0$$

Hence $(f \cdot g)^{\leftarrow}[0] = f^{\leftarrow}[0] \cup g^{\leftarrow}[0]$.

$$= g^{\leftarrow}[0] \in \mathcal{U} .$$

$\mathcal{U}$ is an open ultrafilter.

Suppose that $\mathcal{S}$ is a family of open subsets of X such that

$$\mathcal{U} \subsetneqq \mathcal{S} .$$

there exists $G \in \mathcal{S}$ such that $G \notin \mathcal{U}$.

Now $\quad G = g^{\leftarrow}[0]$ some $g \in F(X)$, and $\alpha_g = 1$

so by (iv) of Lemma 3.2.2, there exists $f \in F(X)$ such that

$$\alpha_f = 0 \quad \text{and} \quad f + g = 1$$

since $\alpha_f = 0$, $f^{\leftarrow}[0] \in \mathcal{U} \subsetneqq \mathcal{S}$

and $f^{\leftarrow}[0] \in \mathcal{S}$

but $f^{\leftarrow}[0] \cap g^{\leftarrow}[0] = \emptyset$

thus $\mathcal{S}$ is not a filter.

Therefore $\mathcal{U}$ is an open ultrafilter on (X,T) . ***

Conversely we have:

<u>3.2.4 Proposition</u>: Every open ultrafilter $\mathcal{U}$ on (X,T) gives rise to an element $\alpha \in \delta(e[X]) \subset \Pi D_f$.

<u>Proof</u>: Suppose $\mathcal{U}$ is an open ultrafilter on (X,T) .

Then let $\alpha = \{\alpha_f\}$ be the element of ΠD_f for which

$$\alpha_f = \begin{cases} 0 & \text{if} \quad f^{\leftarrow}[0] \in \mathcal{U} \\ 1 & \text{if} \quad f^{\leftarrow}[0] \notin \mathcal{U} \end{cases}$$

We show $\alpha \in \delta(e[X]$.

(i) $\alpha \in \Pi(u \vee 1)$-closure $(e[X])$.

Let V be a $\Pi(u \vee 1)$-neighbourhood of α . We need only consider those of the form

$$V = (\alpha_f \times \Pi D_{f'}) \cap (\alpha_g \times \Pi D_{g'})$$

where $\alpha_f = 0$, $\alpha_g = 1$.

Thus $f^{\leftarrow}[0] \in \mathcal{U}$ and $g^{\leftarrow}[0] \notin \mathcal{U}$.

Hence $f^{\leftarrow}[0] \cap g^{\leftarrow}[0] \subsetneqq f^{\leftarrow}[0]$,

and there exists $x \in X$ such that

$x \in f^{\leftarrow}[0]$, $x \notin g^{\leftarrow}[0]$

so $f(x) = 0$ and $g(x) = 1$

i.e. $(e(x))_f = 0$, $(e(x))_g = 1$

thus $e(x) \in V$

and $V \cap e[X] \neq \phi$.

(ii) $\alpha \in \gamma(e[X])$.

Let $U = \alpha_g \times \Pi D_{g'}$ be a lower neighbourhood of α

where $\alpha_g = 1$

Thus $g^{\leftarrow}[0] \notin \mathcal{U}$

therefore there exists $f \in F(X)$ such that

$f^{\leftarrow}[0] \in \mathcal{U}$, and

$f^{\leftarrow}[0] \cap g^{\leftarrow}[0] = \phi$

and $W = \alpha_f \times \Pi D_{f'}$ is an upper neighbourhood of α because $\alpha_f = 0$.

Clearly

$W \cap e[X] \neq \phi$.

For each x such that $e(x) \in W$,

$x \in f^{\leftarrow}[0]$ and $x \notin g^{\leftarrow}[0]$

so $g(x) = 1$ and $(e(x))_g = 1$

therefore $\phi \neq W \cap e[X] \subset U$.

Thus $\mathcal{U}$ gives rise to an element $\alpha \in \delta(e[X])$. ***

3.2.5 Proposition: The mapping between the open ultrafilters $\mathcal{U}$ of (X,T) and elements $\alpha \in \delta(e[X]) \subset \Pi D_f$ is 1 - 1 and onto.

Proof: Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be distinct open ultrafilters and α^1 and α^2 their images in $\delta(e[X])$. There exists

$$f^{\leftarrow}[0] \in \mathcal{U}_1 \text{ and } g^{\leftarrow}[0] \in \mathcal{U}_2 \text{ such that}$$

$$f^{\leftarrow}[0] \cap g^{\leftarrow}[0] = \emptyset .$$

Thus $\alpha^1_f = 0$, $\alpha^2_g = 0$

and since $g^{\leftarrow}[0] \notin \mathcal{U}_1$, $f^{\leftarrow}[0] \notin \mathcal{U}_2$,

$$\alpha_g^{\,1} = 1 \quad , \quad \alpha_f^{\,2} = 1$$

thus $\alpha^1 \neq \alpha^2$ ***

We have shown that there is a 1 - 1 correspondence between the open ultrafilters on (X,T) , and the points of the subset $\delta(e[X])$ of ΠD_f . We now wish to see how $X \cup X^{\vee}$ is situated in ΠD_f . Therefore it is required to examine what subset of $\delta(e[X])$ the non-convergent open ultrafilters on (X,T) correspond to.

Define the subset $\delta^*(e[X])$ of $\delta(e[X])$ as follows:

$\alpha \in \delta^*(e[X])$ if and only if

for each $x \in X$, there exist $f , g \in F(x)$ such that

$$(e(x))_f = 0 \quad , \quad (e(x))_g = 1$$

and $\alpha_f = 1$, $\alpha_g = 0$.

The open non convergent ultrafilters on (X,T) correspond to $\delta^*(e[X])$.

For if $\mathcal{U}$, an open ultrafilter, does not converge, then for each $x \in X$, there exists an open neighbourhood V of x , and $U \in \mathcal{U}$ such that

$$U \cap V = \emptyset$$

Now $V = f^{\leftarrow}[0]$ some $f \in F(X)$

and $U = g^{\leftarrow}[0]$ some $g \in F(X)$

Thus $(e(x))_f = 0$ $(e(x))_g = 1$, $\alpha_f = 1$ $\alpha_g = 0$

The converse is clear.

Note: $e[X] \cup \delta(e[X])$ is not a Hausdorff subspace of ΠD_f . For if $\alpha \in \delta(e[X])$, and α corresponds to an open ultrafilter which converges to x in (X,T) , then for every neighbourhood V of x ,

$$V = f^{\leftarrow}[0] \quad \text{some } f \in F(X) ,$$

there exists $U = g^{\leftarrow}[0] \in \mathcal{U}$ such that

$$g^{\leftarrow}[0] \subset f^{\leftarrow}[0] , \quad \text{and } f^{\leftarrow}[0] \in \mathcal{U} ,$$

We thus have that

$$(e(x))_f = 0 \text{ implies } \alpha_f = 0$$

Therefore $e(x)$ and α do not have disjoint neighbourhoods in $e[X] \cup \delta(e[X])$.

Clearly $e[X] \cup \delta^*(e[X])$ is a Hausdorff subspace of ΠD_f .

Having established a 1 - 1 correspondence, call it φ , between $X \cup X^\vee$ and $e[X] \cup \delta^*(e[X])$, we now show that φ is an embedding.

3.2.6 Theorem: The mapping

$$\varphi : (X \cup X^\vee , O(T)) \longrightarrow (e[X] \cup \delta^*(e[X]) , R) ,$$

where R is the relative product topology on $e[X] \cup \delta^*(e[X])$ is a homeomorphism.

Proof: The following diagram commutes for each $f \in F(X)$.

$$\begin{tikzcd}
(X,T) \arrow[r, "e"] \arrow[dr, "f"'] & \prod D_f \arrow[d, "\pi_f"] \\
 & (D,u)_f
\end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd}
(X,T) \arrow[r, "e"] \arrow[d, "i"'] \arrow[dr, "f"] & \prod D_f \arrow[d, "\pi_f"] \\
(X \cup X^{\vee}, o(T)) \arrow[r, dashed, "\varphi_f"] \arrow[ur, dashed, "\varphi"] & (D,u)_f
\end{tikzcd}$$

where φ_f is defined as follows:

For $x \in X$,

$\varphi_f(x) = f(x)$

For $\mathcal{U} \in X^{\vee}$,

$$\varphi_f(\mathcal{U}) = \begin{cases} 0 & \text{if} \quad f^{\leftarrow}[0] \in \mathcal{U} \\ 1 & \text{if} \quad f^{\leftarrow}[0] \notin \mathcal{U} \end{cases}$$

φ_f is continuous.

If $x \in X$ and $\varphi_f(x) = f(x) = 1$, continuity is clear.

If $f(x) = 0$ then $f^{\leftarrow}[0]$ is an open neighbourhood of x in (X,T) . Thus $(f^{\leftarrow}[0])^*$ is an open neighbourhood of x in $(X \cup X^{\vee}, O(T))$

and $\varphi_f((f^{\leftarrow}[0]^*) = \varphi_f(f^{\leftarrow}[0] \cup \{\mathcal{V} \in X^{\vee} \mid f^{\leftarrow}[0] \in \mathcal{V}\} = \{0\}$

thus φ_f is continuous at each $x \in X$.

Similarly for $\mathcal{U} \in X^{\vee}$.

Thus the lower left hand triangle commutes.

φ is continuous:

Let $\alpha \in e[X] \cup \delta^*(e[X])$, and let

$$\pi_f^{\leftarrow}[0] \cap (e[X] \cup \delta^*(e[X]))$$
$$= \{\alpha^1 \in e[X] \cup \delta^*(e[X]) \mid \alpha^1{}_f = 0\}$$

be a basic open neighbourhood of α in $e[X] \cup \delta^*(e[X])$.

Now $\alpha = \varphi(x)$ or $\alpha = \varphi(\mathcal{U})$ $\quad \mathcal{U} \in X^{\vee}$.

In either case,

$(f^{\leftarrow}[0])^*$ is an open neighbourhood in $(X \cup X^{\vee} , 0(T))$

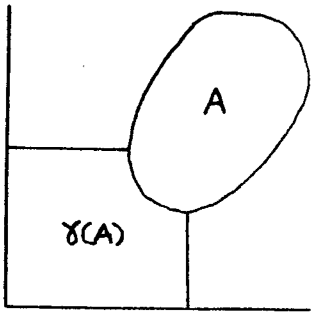
of x or $\mathcal{U}$ such that

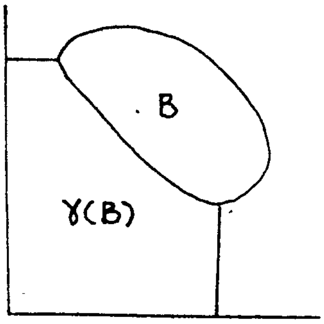
$$\varphi((f^{\leftarrow}[0])^*) = \varphi(f^{\leftarrow}[0] \cup \{\mathcal{V} \in X^{\vee} \mid f^{\leftarrow}[0] \in \mathcal{V}\})$$
$$= \{e(x) \in e[X] \mid (e(x))_f = 0\} \cup \{\alpha^1 \in \delta^*(e[X]) \mid \alpha^1{}_f = 0\}$$
$$= \{\alpha^1 \in e[X] \cup \delta^*(e[X]) \mid \alpha^1{}_f = 0\} .$$

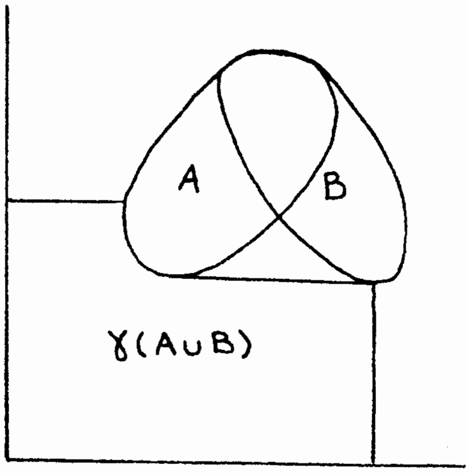
Thus φ is a homeomorphism, and the diagram commutes. ***

Remark: The operator $\gamma : A \longrightarrow \overset{\cdot}{(A)} \cup A$ is not a closure operator. For $A \cup \gamma(A) \cup B \cup \gamma(B) \neq A \cup B \cup \gamma(A \cup B)$ as the following example shows.

A and B are subsets of $(R^+ \times R^+ , u \times u , 1 \times 1)$.






A
B
γ(A∪B)

CHAPTER FOUR

AN EMBEDDING CHARACTERISATION OF H-CLOSED SPACES

It is well known that compact spaces are homeomorphic to a closed subspace of a product of the unit interval. In [11], Z. Frolik and C.T. Liu gave a similar characterisation for H-closed spaces. They showed that every H-closed space is a maximal Hausdorff subspace in its closure in a product of the unit interval with the upper topology. The same result holds if $D = (\{0\},\{1\})$ with the upper topology is used, indicating that in fact a simpler space may by used to characterise H-closed spaces in this way.

It was mentioned in the previous chapter that the topology T on any space X is determined by its continuous functions into (D,u), and that the evaluation map

$$e : (X,T) \longrightarrow \Pi D_f$$

is an embedding; where as before

$$\Pi D_f = \Pi[(D,u)_f \mid f \in F(X)]$$

and $F(X)$ is the family of all continuous functions on X into (D,u).

If $\mathcal{U}$ is an open nonconvergent ultrafilter on X, then any $f : (X,T) \longrightarrow (D,u)$ can be extended to a continuous function $\overline{f} : X \cup \{\mathcal{U}\} \longrightarrow (D,u)$ as follows: (the topology on $X \cup \{\mathcal{U}\}$ has base $\mathcal{B}$ where $B \in \mathcal{B}$ if and only if B is open in X or $B = U \cup \{\mathcal{U}\}$ where $U \in \mathcal{U}$ and U open in X)

for $x \in X$, $\bar{f}(x) = f(x)$

and $\bar{f}(\mathcal{U}) = 0$ if $f^{\leftarrow}[0] \in \mathcal{U}$

$= 1$ if $f^{\leftarrow}[0] \notin \mathcal{U}$.

Let $\bar{e}$ be the mapping from $X \cup \{\mathcal{U}\}$ into ΠD_f defined by $(\bar{e}(x))_f = \bar{f}(x) = f(x)$

$(\bar{e}(\mathcal{U}))_f = \bar{f}(\mathcal{U})$.

We show that $\bar{e}$ is an embedding.

The family $\{\bar{f} \mid \bar{f}$ a continuous extension of $f \in F(X)\}$

(i) separates the points of $X \cup \{\mathcal{U}\}$.

Let $x \in X$; since $\mathcal{U}$ does not converge, there exists an open neighbourhood $g^{\leftarrow}[0]$ of x and $f^{\leftarrow}[0] \in \mathcal{U}$ such that

$g^{\leftarrow}[0] \cap f^{\leftarrow}[0] = \emptyset$.

Thus $\bar{g}$ is such that

$\bar{g}(x) = 0$ and $\bar{g}(\mathcal{U}) = 1$.

(ii) Separates points and closed sets.

If A is closed in $X \cup \{\mathcal{U}\}$, and $\mathcal{U} \notin A$, then there exists an open neighbourhood $f^{\leftarrow}[0] \cup \{\mathcal{U}\}$ of $\mathcal{U}$,

$f^{\leftarrow}[0] \in \mathcal{U}$ suchthat

$f^{\leftarrow}[0] \cup \{\mathcal{U}\} \subset X \cup \{\mathcal{U}\} - A$

then $\bar{f}$ has the property that

$\bar{f}(\mathcal{U}) = 0$ and $\bar{f}(A) = 1$.

Consequently $\bar{e}$ is an embedding, and $\bar{e}|_X = e$.

<u>4.1.1 Theorem</u>: A space (X,T) is H-closed if and only if $e[X]$ is a maximal Hausdorff subspace of $\overline{e[X]}$, where the

closure is taken in ΠD_f .

Proof: For the necessity, suppose that there exists $p \in \overline{e[X]} - e[X]$ such that $F = e[X] \cup \{p\}$ is a Hausdorff subspace of $\overline{e[X]}$.

$e[X]$ is dense in F , thus X is embedded as a subpsace of the Hausdorff space F , and is not a closed subspace of F . Thus (X,T) is not H-closed.

For the sufficiency, suppose that (X,T) is not H-closed. Then there exists an open non-convergent ultrafilter $\mathcal{U}$ on X . By the above,

$$e : X \longrightarrow \Pi D_f$$

can be extended to an embedding

$$\overline{e} : X \cup \{\mathcal{U}\} \longrightarrow \Pi D_f .$$

Thus $\overline{e}(X \cup \{\mathcal{U}\}) = e[X] \cup \{\overline{e}(\mathcal{U})\}$ is a Hausdorff subspace of $\overline{e[X]}$ that properly contains $e[X]$.

***

Note: $e[X]$ is not a closed subspace of ΠD_f , since closed subspaces of T_0 spaces are not Hausdorff..

Question: (i) In the above, can one replace ΠD_f by an arbitrary product?

CHAPTER FIVE

THE KATETOV EXTENSION AS AN EPIREFLECTION

It is an established fact that the Stone-Čech compactification, the Hewitt realcompactification and the Uniform Space completion are epireflections in certain appropriate categories. The corresponding problem of when the Katětov H-closed extension may be regarded as a reflection or epireflection, has been investigated by Katětov [17], Bourbaki [5], Herrlich and Strecker [13], Liu [20] and Porter and Thomas [23].

It was Douglas Harris [13] who finally characterised the type of maps between Hausdorff Spaces which could be extended to maps between their Katětov extensions. Consequently he was able to specify in what category the category of H-closed spaces is epi-reflective.

<u>5.1.1 Definition [13]</u>: If $\mathcal{U}$ is a full subcategory of a category $\mathcal{B}$ and if for each object X in $\mathcal{B}$, there exists an object $X_{\mathcal{U}}$ in $\mathcal{U}$ and a morphism (respectively epi-morphism)

$$r : X \longrightarrow X_{\mathcal{U}}$$

such that for each object Y in $\mathcal{U}$ and each morphism

$$f : X \longrightarrow Y$$

there exists a unique morphism

$$\bar{f} : X_{\mathcal{U}} \longrightarrow Y$$

such that the following diagram

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[dr, "f"'] & X_{\mathcal{U}} \arrow[d, dashed, "\bar{f}"] \\
 & Y
\end{tikzcd}$$

is commutative, then $\mathcal{U}$ is a reflective (respectively epi-reflective) sub-category of $\mathcal{B}$ and r is a reflective morphism (respectively epi-morphism) from X to $X_{\mathcal{U}}$.

Note: In the category of Hausdorff spaces and continuous functions the epi-morphisms are precisely the continuous functions with dense images.

5.1.2 Definition: A p-cover of a Hausdorff space is an open cover such that the union of some finite subcollection is dense in the space.

A p-map is a map such that the inverse image of a p-cover of the co-domain is a p-cover of the domain.

By Theorem 1.1.7 (ii), a space is H-closed if and only if every open cover is a p-cover.

By Proposition 1.1.10, every map on an H-closed space is a p-map.

5.1.3 Lemma: A non-convergent open filter is a maximal open filter if and only if it contains a member of every p-cover.

Proof: Let $\mathcal{U}$ be a nonconvergent open filter that contains a member of every p-cover. Since $\mathcal{U}$ does not converge, for each $x \in X$, there exists an open neighbourhood V_x of x and $U \in \mathcal{U}$ such that

$$V_x \cap U = \emptyset .$$

Thus $\mathcal{V} = \{V_x \mid V_x \cap U = \emptyset \text{ some } U \in \mathcal{U} , x \in X\}$ is an open cover such that $\mathcal{U}$ contains no member of $\mathcal{V}$.

For any open set W, the cover $\mathcal{V} \cup \{W , X - \overline{W}\}$ is a p-cover. Thus for each open set W, either

$$W \in \mathcal{U} , \text{ or } X - \overline{W} \in \mathcal{U} \text{ i.e. } W \cap U = \emptyset \text{ some } U \in \mathcal{U} .$$

This means that $\mathcal{U}$ is an open ultrafilter.

Conversely, suppose $\mathcal{U}$ is a non-convergent open filter that contains no member of the open cover $\mathcal{C}$. Thus for each $C \in \mathcal{C}$, there exists $U \in \mathcal{U}$ such that

$$U \cap C = \emptyset .$$

Thus for arbitrary $C_1 , \ldots , C_n$, and the corresponding $U_1 , \ldots U_n$

$$\bigcup_{i=1}^{n} C_i \cap \bigcap_{i=1}^{n} U_i = \emptyset$$

Thus $\bigcup_{i=1}^{n} C_i$ is not dense, since $\bigcap_{i=1}^{n} U_i$ is open. ***

We now give the main result of Harris.

5.1.4 Theorem: A continuous map between two spaces can be extended to a continuous map between their Katětov extensions if and only if it is a p-map.

Proof: Suppose $f : X \longrightarrow Y$ is a p-map.

Let $\mathcal{U} \in X^{\vee} = kX - X$.

If $f(\mathcal{U})$ does not converge, define $\overline{f}(\mathcal{U})$ to be the open filter generated by the open subsets of the image filter $f(\mathcal{U})$ i.e.

$$\overline{f}(\mathcal{U}) = \mathcal{V} = \{V \mid f(U) \subset V \text{ some open } f(U) \in f(\mathcal{U}) \text{ , } V \text{ open}\} .$$

If $f(\mathcal{U})$ converges, say to y_0 , let

$$\overline{f}(\mathcal{U}) = y_0 .$$

When $\overline{f}(\mathcal{U}) = \mathcal{V}$, $\mathcal{V}$ is an open non-convergent ultrafilter on Y . For, let $\mathcal{C}$ be a p-cover of Y , then $f^{\leftarrow}(\mathcal{C})$ is a p-cover of X , thus $\mathcal{U}$ contains a member of $f^{\leftarrow}(\mathcal{C})$ and so $f(\mathcal{U})$ contains a member of $\mathcal{C}$, thus $\overline{f}(\mathcal{U})$ is an open non-convergent ultrafilter on Y .

We can now define

$$kf : X \longrightarrow Y \text{ as follows:}$$

$kf(x) = f(x)$ for each $x \in X$

$kf(\mathcal{U}) = \overline{f}(\mathcal{U})$ for each $\mathcal{U} \in X^{\vee}$

kf is continuous:

Continuity at each $x \in X$ is clear.

Let $\mathcal{U} \in X^{\vee}$.

If $kf(\mathcal{U}) = \overline{f}(\mathcal{U}) = \mathcal{V} \in Y^{\vee} \in kY - Y$

Let $V \cup \{\mathcal{V}\}$ be a neighbourhood of $\mathcal{V}$ in kY

then $f(U) \subset V$ for some $U \in \mathcal{U}$

Thus $U \cup \{\mathcal{U}\}$ is a neighbourhood of $\mathcal{U}$ in kX satisfying

$$kf(U \cup \{\mathcal{U}\}) \subset V \cup \{\mathcal{V}\}$$

If $\bar{f}(\mathcal{U}) = y_0$,

then $f(\mathcal{U})$ converges to y_0 .

Thus any neighbourhood W of y_0 contains a member $f(U)$ of $f(\mathcal{U})$. As before, $U \cup \{\mathcal{U}\}$ is such that

$$kf(U \cup \{\mathcal{U}\}) \subset W$$

Therefore $kf : X \longrightarrow kY$ is continuous.

Conversely;

Suppose $f : X \longrightarrow Y$ extends to a map

$kf : kX \longrightarrow kY$ continuously.

Let $\mathcal{C}$ be a cover of Y , and let $\mathcal{D}$ be the inverse image of $\mathcal{C}$ under f . If $\mathcal{D}$ is not a p-cover of X , then any finite union of members of $\mathcal{D}$ is not dense.

Thus $\{X - \bar{D} \mid D \in \mathcal{D}\}$ is contained in an open ultrafilter $\mathcal{U}$ on X . Since $\mathcal{D}$ covers X , and $\mathcal{U}$ contains no member of $\mathcal{D}$, $\mathcal{U}$ is non-convergent. Since $\mathcal{C}$ covers Y , the point $kf(\mathcal{U})$ is an open non-convergent. Since $\mathcal{C}$ covers Y , the point $kf(\mathcal{U})$ is an open non-convergent ultrafilter on Y , and since $f(\mathcal{U})$ contains no member of $\mathcal{C}$, by the above lemma, $\mathcal{C}$ is not a p-cover of Y .

Thus if $f : X \longrightarrow Y$ extends continuously to

$kf : kX \longrightarrow kY$

then f is a p-map. ***

Note: The composition of p-maps is a p-map, and there is thus a category of Hausdorff spaces and p-maps. Furthermore the category of H-closed spaces and continuous functions is a full subcategory of the category of Hausdorff spaces and p-maps, continuous functions on H-closed spaces being p-maps.

We thus have the following:

5.1.5 Theorem: The category of H-closed spaces and continuous functions is an epi-reflective subcategory of the category of Hausdorff spaces and p-maps, and the epi-reflection of a space is its Katětov extension.

Remark: Continuous functions on uniform spaces are uniformly continuous with respect to their Samuel compact reflections. Are p-maps uniformly continuous with respect to some generalised Samuel (precompact) reflection?

CHAPTER SIX

AN EMBEDDING CHARACTERIZATION OF ALMOST REALCOMPACT SPACES

Recall that a realcompact space is homeomorphic to a closed subpsace of a product of the real line. In [11], Frolik and Lui gave a similar characterization of almost realcompact spaces. Their main theorem was that every almost realcompact space is a maximal Hausdorff subspace in its closure in some product of the non-negative reals with the upper topology.

6.1.1 Definition: A space (X,T) is said to be almost real-compact if every open ultrafilter with the closed countable intersection property, (C.C.I.P.) i.e. every countable subcollection has a cluster point, has a cluster point.

C.T. Liu and G.E. Strecker [21] showed that the subspace pX of kX consisting of all $x \in X$ and all $\mathcal{U} \in X^{\vee} = kX - X$ with the C.C.I.P. in X is the smallest almost realcompact subspace of kX that contains X .

We will discuss Frolik and Liu's characterization of almost realcompact spaces using Z^+ instead of R^+ , where Z^+ is the set of all non-negative integers, with the topology consisting of Z^+ and all sets of the form

$\{n \in Z^+ \mid n < k\}$ for $k=1, 2, \ldots$

The use of the more sophisticated space R^+ appears to be unnecessary for the purpose of examining almost realcompact spaces.

Denote by $C_+(X)$ the set of all continuous functions on X to Z^+.

Let $\alpha = \{\mathscr{C} \mid \mathscr{C}$ a countable open covering of $X\}$, and

$$\beta = \{B(f) = \{\{x \mid f(x) < n\} \mid n=1, 2, \ldots\} \mid f \in C_+(X)\} .$$

6.1.2 Definition [11]: Let $\mathcal{U}$ be an open ultrafilter on X. Then $\mathcal{U}$ is said to be α-Cauchy if for each $\mathscr{C} \in \alpha$, there exist $C \in \mathscr{C}$ and $U \in \mathcal{U}$ such that $U \subset C$. $\mathcal{U}$ is said to be β-Cauchy if for each $f \in C_+(X)$, there exists $n \in N$ and $U \in \mathcal{U}$ such that

$$f(x) < n \qquad \text{for each} \quad x \in U .$$

Note: Every α-Cauchy family $\mathcal{U}$ is β-Cauchy.
For suppose there exists $f \in C_+(X)$ such that for each $U \in \mathcal{U}$ and each n,

$$f(U) \nless n$$

Let $\mathscr{C} = \{f^{\leftarrow}(\{0, 1, \ldots, n\}) \mid n=1, 2, \ldots\}$ be an open cover of X. Then $\mathscr{C} \in \alpha$, but for each $U \in \mathcal{U}$ and $C \in \mathscr{C}$,

$$U \not\subset C .$$

6.1.3 Lemma [11]: An open ultrafilter is α-Cauchy if and only if $\mathcal{U}$ has the C.C.I.P. on X.

6.1.4 Lemma [11]: If an open ultrafilter $\mathcal{U}$ on X is β-Cauchy, then $\mathcal{U}$ is α-Cauchy.

Proof: Suppose there exists $\mathscr{C} \in \alpha$, $\mathscr{C} = \{C_n \mid n=1, 2, \ldots\}$ such that $U \not\subset C_n$ for each $U \in \mathcal{U}$ and each n .
It may be assumed that $\mathscr{C}$ is nested, i.e. $C_1 \subset C_2 \subset \ldots \subset C_n \subset \ldots$

For each $x \in X$, let

$$f(x) = \min\{k \mid x \in C_k\} = n .$$

$f \in C_+(X)$: Any neighbourhood of n contains the set $\{0 , \ldots , n\}$ and C_n is a neighbourhood of x such that

$$f(C_n) \subset \{0 , \ldots , n\} ;$$

for if $y \in C_n$, $f(y) \leqslant n$.
Thus $f \in C_+(X)$ and is unbounded on each $U \in \mathcal{U}$. ***

6.1.5 Theorem [11]: An open ultrafilter $\mathcal{U}$ has C.C.I.P. if and only if for each $f \in C_+(X)$, f is bounded on some $U \in \mathcal{U}$.

Proof: Follows from the above two Lemmas and the note.

6.1.6 Lemma: If $f \in C_+(X)$ and if $\mathcal{U}$ is a maximal open filter in X such that f is bounded on some element of $\mathcal{U}$, then $f(\mathcal{U})$ converges to some $k \in Z^+$ in the usual topology on Z^+ .

Proof: Since f is bounded on some $U \in \mathcal{U}$, there exist $m , n \in Z^+$ such that $f(U) \subset \{m , m + 1 , \ldots , n\}$.
Let k be such that

$$f^{\leftarrow}(\{m , \ldots , k - 1\}) \notin \mathcal{U} \text{ but } f^{\leftarrow}(\{m , \ldots , k\}) \in \mathcal{U} .$$

Therefore there exists $V \in \mathcal{U}$ with

$V \cap f^{\leftarrow}\{m , \dots , k - 1\} = \emptyset$ but

$V \cap f^{\leftarrow}\{m , \dots , k\} \neq \emptyset$.

Now $f(\mathcal{U})$ converges to k , since

$f(V \cap f^{\leftarrow}\{m , \dots , k\}) = k$, and

$f(V \cap f^{\leftarrow}\{m , \dots , k\}) \in f(\mathcal{U})$. ***

<u>6.1.7 Theorem</u>: For any space X , $\rho X - X$ is the set A of all $\mathcal{U} \in kX - X$ such that every $f \in C_+(X)$ extends to $\bar{f} \in C_+(X \cup \{\mathcal{U}\})$ (where the topology on $X \cup \{\mathcal{U}\}$ is that described in Chapter 4).

<u>Proof</u>: If $\mathcal{U} \in \rho X - X$, then $\mathcal{U}$ has the C.C.I.P. By the preceding Lemma and Theorem $f(\mathcal{U})$ converges to $k \in Z^+$ (with the usual topology) for each $f \in C_+(X)$.

Define $\bar{f}(\mathcal{U}) = k$, $\bar{f}(x) = f(x)$ for $x \in X$.

$\bar{f}$ is continuous:

Any neighbourhood of k in Z^+ contains the set $\{0 , \dots , k\}$.

Since $f(\mathcal{U}) \longrightarrow k$, there exists $W \in \mathcal{U}$ such that

$f(W) = k$.

Thus $W \cup \{\mathcal{U}\}$ is a neighbourhood of $\mathcal{U}$ such that

$\bar{f}(W \cup \{\mathcal{U}\}) = \{k\} \subset \{0 , \dots , k\}$.

Hence $\bar{f} \in C_+(X \cup \{\mathcal{U}\})$, and

$\rho X - X \subset A$.

Conversely: Suppose $\mathcal{U} \in kX - \rho X$, then $\mathcal{U}$ is an open non-convergent ultrafilter on X without the C.C.I.P. By Theorem 4.1.5, there exists $f \in C_+(X)$ which is unbounded on each $U \in \mathcal{U}$. Thus f cannot be extended to $\bar{f} \in C_+(X \cup \{\mathcal{U}\})$. ***

<u>6.1.8 Corollary</u>: A space X is almost realcompact if and only if for each $\mathcal{U} \in kX - X$, there exists $f \in C_+(X)$ such that f cannot be extended to $\overline{f} \in C_+(X \cup \{\mathcal{U}\})$.

<u>6.1.9 Corollary</u>: X is almost realcompact if and only if for each $\mathcal{U} \in kX - X$, the continuous map

$$\psi : X \longrightarrow \Pi Z_f^+ = \Pi[Z_f^+ \mid f \in C_+(X)]$$

where $(\psi(x))_f = f(x)$, has no continuous extension from $X \cup \{\mathcal{U}\}$ into ΠZ_f^+.

It is clear that the characteristic functions on closed sets are continuous; in fact $\chi_A \in C_+(X)$ if and only if A is closed. Thus $C_+(X)$ distinguishes points and closed sets and ψ is therefore an embedding.

$\psi[X]$, being Hausdorff, is not closed in ΠZ_f^+.

Let $\mathcal{U} \in \rho X - X$, and for each $f \in C_+(X)$, let $\overline{f}$ be the extension on $X \cup \{\mathcal{U}\}$.

Let $\overline{\Psi}$ be the map from $X \cup \{\mathcal{U}\}$ to ΠZ_f^+ defined by

$(\overline{\Psi}(\mathcal{U}))_f = \overline{f}(\mathcal{U})$; $(\overline{\Psi}(x))_f = f(x)$ $x \in X$.

$\overline{\Psi}$ is an embedding:

If A is closed in $X \cup \{\mathcal{U}\}$, $\mathcal{U} \notin A$, then there exists $U \cup \{\mathcal{U}\}$ an open neighbourhood of $\mathcal{U}$ such that

$$U \cup \{\mathcal{U}\} \subset X \cup \{\mathcal{U}\} - A .$$

Thus the characteristic function χ_A on X has extension $\overline{\chi}_A$ on $X \cup \{\mathcal{U}\}$ and satisfies

$$\overline{\chi}_A(U \cup \{\mathcal{U}\}) = 0$$

$$\overline{\chi}_A(A) = 1 .$$

$\overline{\Psi}$ is therefore an embedding.

6.1.10 Theorem: A Hausdorff space X is almost realcompact if and only if $\psi[X]$ is a maximal Hausdorff subspace of $\overline{\psi[X]}$, where the closure is taken in ΠZ_f^+.

Proof: For the necessity; If there exists $p \in \overline{\psi[X]} - \psi[X]$ such that $\psi[X] \cup \{p\}$ is Hausdorff, let $\mathcal{U}(p)$ be the collection of all open neighbourhoods of p in ΠZ_f^+.
Then

$$\mathcal{G} = \mathcal{U}(p) \cap \psi[X]$$

is an open filter in $\psi[X]$. So

$$\mathcal{U} = \{\psi^{\leftarrow}(G) \mid G \in \mathcal{G}\}$$

is an open filter on X. Let $\mathcal{U}'$ be an open ultrafilter on X containing $\mathcal{U}$. Now $\cap\overline{\mathcal{U}'} = \emptyset$, and $\mathcal{U}' \in kX - X$.
Define $\bar{\psi}(\mathcal{U}') = p$, $\bar{\psi}(x) = \psi(x)$ $x \in X$.
$\bar{\psi}$ is a continuous extension of ψ from $X \cup \{\mathcal{U}'\}$ to ΠZ_f^+.
$\bar{\psi}$ is clearly continuous at each $x \in X$.
Let G be an open neighbourhood of p in ΠZ_f^+. Then

$$G \cap \psi[X] \in \mathcal{G},$$

so $W = \psi^{\leftarrow}(G \cap \psi[X]) \in \mathcal{U}'$,

and $W \cup \{\mathcal{U}'\}$ is an open neighbourhood of $\mathcal{U}'$ in $X \cup \{\mathcal{U}'\}$ such that

$$\bar{\psi}(W \cup \{\mathcal{U}\}) = \psi(W) \cup \bar{\psi}(\mathcal{U}') \subset (G \cap \psi[X]) \cup \{p\}.$$

Hence $\bar{\psi}$ is a continuous extension of ψ, and this contradicts Corollary 6.1.9,

For the sufficiency; Suppose X is not almost realcompact. Then there exists $\mathcal{U} \in kX - X$ such that each $f \in C_+(X)$ can be extended. The map $\bar{\psi} : X \cup \{\mathcal{U}\} \longrightarrow \Pi Z_f^+$ is an embedding, and $\overline{\psi}(X \cup \{\mathcal{U}\}) = \psi(X) \cup \{\overline{\psi}(\mathcal{U})\}$ is Hausdorff, and strictly contains $\psi[X]$.

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CHAPTER SEVEN

CHARACTERIZATION OF ALMOST REALCOMPACTNESS IN TERMS OF QUASI UNIFORMITIES

7.1 Preliminaries: P. Fletcher and W.F. Lindgren [9] showed that every topological space admits at least one compatible transitive quasi-uniformity which is the Pervin quasi-uniformity, and that a topological space is compact if and only if its finest transitive quasi-uniformity is precompact. They gave an analogous result for almost realcompact spaces, namely that a space is almost realcompact if and only if there is a compatible almost complete countably precompact transitive quasi-uniformity. In this chapter we present this characterization of almost realcompactness.

7.1.1 Definition [22]: Let X be a non-empty set, and $\mathcal{U}$ a family of subsets of $X \times X$; $\mathcal{U}$ is a quasi-uniformity on X if and only if

(i) $\Delta \subset U$ for each $U \in \mathcal{U}$ $\Delta = \{(x,x) \in X \times X \mid x \in X\}$.

(ii) $U, V \in \mathcal{U}$ implies $U \cap V \in \mathcal{U}$.

(iii) For each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$.

(iv) $U \in \mathcal{U}$ and $U \subset V$ implies $V \in \mathcal{U}$.

$(X,\mathcal{U})$ is a quasi-uniform space, and the members of $\mathcal{U}$ are called entourages.

7.1.2 Definition [22]: Let $\mathcal{U}$ be a quasi-uniformity on X. The sub-family $\mathcal{B}$ of $\mathcal{U}$ is said to be a base for $\mathcal{U}$ if and only if every entourage contains a member of $\mathcal{B}$.

A family $\mathcal{S}$ of subsets of $X \times X$ is a subbase for a quasi-uniformity if and only if the family $\mathcal{B}$ of finite intersections of members of $\mathcal{S}$ is a base for the quasi-uniformity.

A (sub)base $\mathcal{B}$ for a quasi-uniformity $\mathcal{U}$ is transitive provided that for each $B \in \mathcal{B}$, $B \circ B = B$.

A quasi-uniformity with a transitive base is called a transitive quasi-uniformity.

Let $\mathcal{C}$ denote an open cover of a topological space (X,T); and for each $x \in X$, denote $\cap[C \in \mathcal{C} \mid x \in C]$ by $A_x^{\mathcal{C}}$.

We discuss two general methods for constructing a compatible quasi- uniformity for an arbitrary topological space, discussed in [9].

(i) Covering quasi-uniformities.

A Q-cover of a topological space (X,T) is an open cover $\mathcal{C}$ of X such that for each $x \in X$, $A_x^{\mathcal{C}} \in T$.

Let $\mathcal{A}$ be a family of Q-covers of (X,T) such that if $x \in A \in T$, then there exists $\mathcal{C} \in \mathcal{A}$ such that $A_x^{\mathcal{C}} \subset A$. For each $\mathcal{C} \in \mathcal{A}$, let

$$U_{\mathcal{C}} = \cup\{\{x\} \times A_x^{\mathcal{C}} \mid x \in X\},$$

and $\mathcal{S} = \{U_{\mathcal{C}} \mid \mathcal{C} \in \mathcal{A}\}$,

then $\mathcal{S}$ is a transitive subbase for a compatible quasi-uniformity

$\mathcal{U}_{\mathcal{A}}$ for (X,T) [7, Theorem 1]. $\mathcal{U}_{\mathcal{A}}$ is called the covering quasi-uniformity for (X,T) with respect to $\mathcal{A}$.

(ii) Upper semi-continuous quasi-uniformities.

Let (X,T) be a topological space, and let $F_+(X)$ be a collection fo continuous functions from X to R^+ (reals with the upper topology) which includes all the continuous characteristic functions. For each $f \in F_+(X)$ and each $\varepsilon > 0$, let

$$U_{(f,\varepsilon)} = \{(x,y) \in X \times X \mid f(x) - f(y) < \varepsilon\} ,$$

Then $\mathcal{B} = \{U_{(f,\varepsilon)} \mid f \in F_+(X) , \varepsilon > 0\}$
is a subbase for a compatible quasi-uniformity $\mathcal{U}_{F_+(X)}$ for (X,T) [8].

In the case when $D_+(X)$ is the collection of all continuous functions on X to R^+ , $\mathcal{U}_{D_+(X)}$ is called the upper semi-continuous quasi-uniformity, denoted by UCS .

7.1.3 Definition [9]: An open spectrum α in X is a sequence $\{A_n\}$ of open subsets of X indexed by the integers, Z , such that for each $n \in Z$, $A_n \subset A_{n+1}$,

$$\bigcap_{n \in Z} A_n = \emptyset \quad , \quad \text{and} \quad \bigcup_{n \in Z} A_n = X .$$

Each open spectrum is a Q-cover.

7.1.4 Theorem [9]: Let $\mathcal{A}$ be the collection of all open spectra in a space X . Then $\mathcal{U}_{\mathcal{A}}$ = USC .

7.2 Almost complete and countably precompact quasi-uniformities.

Let $(X,\mathcal{U})$ be a quasi-uniform space, and let $\mathcal{F}$ be a filter on X. Then $\mathcal{F}$ is Cauchy if for each $V \in \mathcal{U}$ there exists $x \in X$ such that

$$V[x] \in \mathcal{F} .$$

$(X,\mathcal{U})$ is almost complete if every open Cauchy filter has a cluster point.

$(X,\mathcal{U})$ is countably precompact if for each $U \in \mathcal{U}$, there exists a countable subset F of X such that

$$U[F] = X .$$

7.2.1 Lemma: Let $\mathcal{A}$ be the collection of all open spectra in a space S. Then $\mathcal{U}_{\mathcal{A}}$ is countably precompact.

Proof: Let U be a basic entourage.

then $U = \bigcap_{i=1}^{n} \mathcal{C}_i$ $\quad \mathcal{C}_i$ open spectra $\quad i=1, \ldots, n$

$$= \bigcap_{i=1}^{n} [\cup\{\{x\} \times A_x^{\mathcal{C}_i} \mid x \in X\}] .$$

Let $a_k \in \bigcap_{i=1}^{n} A_{i,k} - \bigcap_{i=1}^{n} A_{i,k-1}$ where $A_{i,k}, A_{i,k-1} \in \mathcal{C}_i$

then $F = \{a_k \mid a_k \in \bigcap_{i=1}^{n} A_{i,k} - \bigcap_{i=1}^{n} A_{i,k-1}\}$

is a countable set. We show that $U[F] \in X$.

$$U[F] = \bigcup_{a\in F} U[a]$$

$$= \bigcup_{a\in F} \left(\bigcap_{i=1}^{n} U_{\mathcal{C}_i}\right)[a]$$

Let $x \in X$, then there exists j such that

$$x \in \bigcap_{i=1}^{n} A_{i,j} - \bigcap_{i=1}^{n} A_{i,j-1}$$

We show $x \in \left(\bigcap_{i=1}^{n} U_{\mathscr{C}_i}\right)[a_j] = \bigcap_{i=1}^{n} (U_{\mathscr{C}_i}[a_j])$.

$$U_{\mathscr{C}_i}[a_j] = \{y \in X \mid (a_j,y) \in \cup\{\{x\} \times A_x^{\mathscr{C}_i} \mid x \in X\}\}$$

But $(a_j,x) \in \cup\{\{x\} \times A_x^{\mathscr{C}_i} \mid x \in X\}$ if and only if

$$x \in A_{a_j}^{\mathscr{C}_i}$$

and $A_{a_j}^{\mathscr{C}_i} = \cap[A_{i,j} \in \mathscr{C}_i \mid a_j \in A_{i,j}]$

$$= A_{i,j}$$

and $x \in A_{i,j}$ $i=1, \ldots , n$

therefore $(a_j,x) \in \cup\{\{x\} \times A_x^{\mathscr{C}_i} \mid x \in X\}\}$ $i=1, \ldots , n$

thus $U[F] = X$, and $\mathcal{U}_{\mathcal{A}}$ is countably precompact.

7.2.2 Corollary [9]: For any topological space, USC is a countably precompact transitive quasi-uniformity.

7.2.3 Theorem [9]: A topological space is almost realcompact if and only if there is a compatible almost complete countably precompact transitive quasi-uniformity.

Proof: Suppose (X,T) is realcompact; it is sufficient to show that USC is almost complete. Let $\mathcal{F}$ be a USC-Cauchy open ultrafilter. $\mathcal{F}$ has a C.C.I.P. if and only if each continuous function into the non-negative reals with the upper topology is bounded on some $F \in \mathcal{F}$ [11, Theorem 1]. Let f be such a function, and $\varepsilon > 0$. Since $\mathcal{F}$ in USC-Cauchy, for the entourage $U_{(f,\varepsilon)}$ there exists $p \in X$ such that

$$U_{(f,\varepsilon)}[p] \in \mathcal{F} .$$

f is bounded on $U_{(f,\varepsilon)}[p]$, and $\mathcal{F}$ has C.C.I.P. on X and thus converges.

Conversely let $\mathcal{U}$ be an almost complete countably precompact transitive quasi-uniformity which is compatible with the topology on X.

Let $\mathcal{F}$ be an open ultrafilter with the C.C.I.P. We show that $\mathcal{F}$ is $\mathcal{U}$-Cauchy.

Let $V \in \mathcal{U}$, since $\mathcal{U}$ is transitive, there exists $U \in \mathcal{U}$ with $U \circ U = U$, and $U \subset V$.

Furthermore, since $\mathcal{U}$ is countably precompact, there exists a countable set A such that $U[A] = X$.

We show that for some $a \in A$,

$$U[a] \in \mathcal{F} .$$

For suppose $U[a] \notin \mathcal{F}$, for each $a \in A$, $U[a]$ is open, since $\mathcal{U}$ is transitive then $X - \overline{U[a]} \in \mathcal{F}$. This holds for each $a \in A$.

Now $\bigcap\left[\overline{X - \overline{U[a]}} \mid a \in A\right] \neq \emptyset$, since $\mathcal{F}$ has C.C.I.P.

So there exists $b \in \bigcap[\overline{X - \overline{U[a]}}]$

But there exists $a \in A$ such that

$$b \in U[a]$$

Thus $U[b] \subset U[a]$. This contradicts the fact that

$b \in \overline{X - \overline{U[a]}}$

Therefore $\mathcal{F}$ is a Cauchy filter, and since $\mathcal{U}$ is almost complete, $\mathcal{F}$ converges.

<u>Question</u>: What type of compatible quasi-uniformity does an H-closed space have? What type of compatible quasi-uniformity must a topological space have in order that it be H-closed?

BIBLIOGRAPHICAL NOTES

CHAPTER 1: The definitions and results on cospaces were extracted from the Ph.D. Dissertation of G.E. Strecker [24], although they do not appear there as stated here. Theorem 1.3.2 may be found in [25]. The definitions and elementary results on H-closed spaces appear in the survey by Berri, Porter and Stephenson. Theorem 1.2.4 and Theorem 1.3.6 are also stated there.

CHAPTER 2: The results on subspaces of H-closed spaces were obtained from [23]. The proof by Chevelley and Frink [6] for the productivity of compact spaces was modified to suit the H-closed situation. General properties of open ultrafilters were mainly encountered in [20]. Section 2.4 follows C.T. Liu's [20] exposition of the Katětov H-closure.

CHAPTER 3: The H-closed extension due to F. Obreanu appears in [20] Definition 3.1.1 of θ-continuous functions comes from [16]. The content of section 3.2 was made possible by several suggestions on the part of my supervisor.

CHAPTER 4: The characterization of H-closed spaces presented here may be found in the paper of Frolik and Liu [11]. We give a modification of their argument, as pointed out in the chapter.

CHAPTER 5: The exposition follows that of D. Harris [13], where he discusses the Katětov extension as a functor in the first part of his paper. The proofs of the results have to be adjusted to suit the way we have discussed the Katětov extension.

CHAPTER 6: As in chapter 4, the discussion on almost realcompactness follows that of Frolik and Liu [11].

CHAPTER 7: The basic results may be found in "Quasi-uniform topological spaces" by Murdeshwar and Naimpally [22]. The discussion on the methods of constructing a compatible quasi-uniformity for a topological space is in [9], as are the results that follow. We had to include Lemma 7.2.1 however.

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