Multi-Curve Frameworks and Information-Based Models

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy specialising in Quantitative Finance at the University of Cape Town. It has not been submitted before for any degree or examination at any other university.

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Abstract

The distinction between bank funding cash and derivative markets were magnified in the aftermath of the 2008 global financial crisis, and further fortified by the need for reference rate reform post the Financial Stability Board's review of major interest benchmarks in 2014. The cognisance of previously negligible liquidity and credit risks has had various implications for market microstructure and risk management. Accordingly, this has created the need for new interest rate modelling frameworks.

Part I proposes one such framework, referred to as the "*market-based approach*", which is a multi-curve generalisation of the single-curve pricing kernel approach, and is motivated by material differences that emerge due to term-related risks when executing compounding strategies at different frequencies. In this framework, a distinct stochastic discount factor is assigned to each tradable term within a given market. This term-cognisant approach is first applied to the deposit market, where a novel argument based on *funding-swap duality* and a constructed stylised systemic and symmetric setting enables the derivation of a system of arbitrage-free discrete-time calibrated pricing kernels. It is then shown how one may construct an *exchange of risk* mechanism to transfer risks across terms in a fair manner, which in turn enables economically meaningful and theoretically consistent pricing and valuation of financial instruments with features that span across terms. Finally, it is shown that the repo and bank funding markets are also compatible with the market-based approach, which paves the way for the development of derivative pricing and valuation.

In Part II, the exchange of risk mechanism is generalised using a system of continuous-time pricing kernels and an FX analogy which results in the creation of the *curve-conversion factor process*. This process is then used to derive the *across-curve pricing formula*, which is a generalisation of the fundamental single pricing kernel formula, and defines the arbitrage-free mechanics of the "*xy-approach*" — a continuous-time reduced-form abstraction of the market-based approach. As a natural application, consistent multi-curve frameworks are formulated for bank funding cash and derivative markets within emerging and developed economies. Given the xy-approach, existing multi-curve frameworks based on HJM and rational pricing kernel models are recovered, reviewed, and generalised; and single-curve models are extended to a multi-curve setting. In a final application, it is shown how the xy-approach offers a flexible framework for solving pricing problems involving financial instruments with floating nominal rate, inflation and foreign exchange exposures, in a consistent manner.

Part III presents a reformulation of the information-based asset pricing framework, introduced by Macrina (2006), within a general non-linear stochastic filtering framework founded upon Markov observation and signal processes, in order to enhance tractability for model development. A general framework for modelling short, instantaneous forward, and discrete forward rates using pricing kernels is derived, which enables the creation of information-based pricing kernel and forward rate models using novel *information-driven martingale processes*.

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Contents

Chapter 1

Introduction

1.1 Background and Overview

The interest rate centric financial markets that form part of the financial system of a modern local economy, and contribute to the operation of the global economy, generally constitute the *debt* (government and corporate) and *money* markets. Moreover, the existence of the *foreign exchange* market enables local entities to participate in all of the foreign versions of the aforementioned markets. While all of these interest rate markets are considered to be *primitive* in nature, supporting assets that are generally in *positive net supply*[1], they each have associated *derivatives* markets, supporting financial derivatives that are in *zero net supply*[2]. These primitive markets are generally sources of *capital* and *funding*, while the associated financial derivatives markets generally offer the opportunity to *hedge* risks, *speculate* on the performance of underlying primitive assets or exploit *arbitrage* opportunities. Excluding foreign exchange derivatives and funded structured products, the associated derivative securities are *capital-* and *funding-neutral*[3]. Fundamental differences between capital and funding, and primitive assets versus derivative securities will have a major bearing on the technical direction of the research undertaken in this thesis, and will be elaborated upon in section 1.3.

Primitive markets generally offer participants exposure to assets that bear complex composite risks which are *inseparable* and *non-fungible*. The former means that constituent risks may not be separated or compartmentalised while the latter means that risks may not be replicated. When viewed from the guise of modern derivative pricing and valuation, following theoretical developments since the seminal work of Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983), primitive markets are quintessential examples of *incomplete markets*. Primitive assets are therefore idiosyncratically and fundamentally valued through an absolute assessment of risk and reward via a preference- or risk premia-adjusted discounted value of potential future cash flows (income and capital appreciation). The risk premia-related approach has been formalised via the pricing kernel construct, which has been comprehensively introduced and developed, for general asset pricing, by Cochrane (2009), Back

[1] Within the context of financial securities, this means that an any time an entity has an issuance of a finite quantity of a non-negative valued asset (in the primary market) that has been bought by participants for investment purposes, or speculative trading in the secondary market.

[2] Financial derivatives are real-valued contingent-securities, and a derivative transaction only exists once there is a willing buyer and seller of the same contract for one of the purposes of hedging, speculation or arbitrage.

[3] This is not to be confused with funding required by derivative market-makers for hedge portfolios or margin and collateral commitments, for exchange-traded and over-the-counter derivatives respectively.

(2010), Duffie (2001) and Macrina (2014). For the pricing and valuation of assets that are interest rate sensitive, Constantinides (1992), Rogers (1997), Jin and Glasserman (2001), Hughston and Rafailidis (2005), Akahori *et al.* (2014) and Filipović *et al.* (2017) are excellent references. The preference-centric approach will be discussed and referenced in the next paragraph. Market prices for these assets are then formed via demand and supply equilibrium, with participants' idiosyncratic valuations informing demand and supply schedules that filter into respective *limit-order books*.

While one may conceive of theoretical market models for derivative security pricing and valuation that are *complete*, all real-world derivative markets are necessarily *incomplete*. Being contingent claims, usually contingent on some feature of a primitive asset, derivative securities also bear complex composite risks. However some elements of risk are *separable* and *fungible*, in general[4]. Therefore, market-makers of such derivative securities generally find themselves managing net portfolios with hedged, partially hedged and unhedged risk exposures. The process of pricing and valuation is therefore similar to that of primitive assets, except that future cash flows are replaced by future payoffs and subject to model choice and assumptions it may be possible to derive hedging strategies that are optimal in a model sense. This, in turn, enables the determination of fair values for the respective derivative securities. This process has been broadly formalised into two broad approaches: (i) the derivation of optimal equivalent (local) martingale measures, or equivalently optimal hedging strategies, in incomplete market models based on specific quantitative criteria related to the profitability of the respective hedging strategies; and (ii) the utility maximisation and indifference pricing approaches, which use utility functions as an abstract representation of preferences in the valuation process.

While not an exhaustive list, the aforementioned approaches have been pioneered by the work of Schweizer (1988, 1990, 1991, 1992, 1995a,b, 1996), Föllmer and Schweizer (1989, 1991), Duffie and Richardson (1991), Lamberton and Lapeyre (1993), Delbaen and Schachermayer (1994, 1996, 1998), Gouriéroux *et al.* (1998), Heath *et al.* (1998), Pham *et al.* (1998), Davis (1997, 1999, 2000), Frittelli (2000), Becherer (2001), Zariphopoulou (2001), Henderson (2002) and Henderson and Hobson (2004). Given these approaches for market-making, market prices of derivative securities are again determined via demand and supply equilibrium, as with primitive assets. One caveat to take note of is that all of the above, regarding derivative security pricing, assumes a backdrop that is void or agnostic of liquidity and counterparty credit risks. Section 1.3 updates this perspective by describing modern derivative security pricing within these risky contexts.

The preceding paragraphs have offered a high-level overview of primitive asset and derivative security pricing from the perspective of a *market-maker*, i.e., an entity that has adequate risk appetite and resources to create liquidity[5] in the primary and secondary markets for a primitive asset or derivative security. The alternate perspective is that of a *market-taker*, the end-user of the financial instrument and a consumer of liquidity, i.e., an entity that may want to utilise that financial instrument for investment, speculation, hedging or arbitrage activities. Of course, every market-maker may also assume the

[4] For example, consider a derivative security written on a tradable liquid primitive asset, such as the equity of a blue-chip company. The ability to trade the underlying shares would enable a market-maker to effect a static or dynamic hedge against the derivative and eliminate equity risk (at least locally, in the dynamic continuous case).

[5] Liquidity here refers to the ability to buy/sell the financial instrument, at all trading times, at a price that is commensurate with the market-making entity's risk appetite and assessment of related and relevant risks.

role of market-taker but not necessarily vice versa. At the formative stages of a market for a new financial instrument, the market-maker plays a critical role by using proprietary information and models to derive optimal bid and offer values (along with associated hedging strategies) commensurate with their risk-appetite, within an incomplete market setting, to stimulate trading activity. Once the market has matured, with many market-makers competing to create liquidity, one can also be assured that any opportunities for arbitrage will be readily exploited (by market-makers or -takers, where possible). Both market-making competition and the elimination of arbitrage will lead to a fair and efficient market for the financial instrument. At this stage, market-takers will have access to a rich source of market information that enables: (i) the development of models that best capture empirical stylised market features; and (ii) the estimation and/or calibration of the models from (i). In general then, the role of a market-taker may be considered to be far more reactive and passive when compared to that of a market-maker who has to be proactive and dynamic. Both of these roles and perspectives will be important in all that follows — for further information, please refer to Appendix A.1.

Having provided a very general overview of primitive asset and derivative security pricing (and valuation) as well as the perspectives of market-makers versus -takers, the rest of this introductory chapter and the next is devoted to describing and contextualising the research that has been undertaken in this thesis, which constitutes three parts: Part I — Market-Based Multi-Curve Frameworks; Part II — Reduced-Form Multi-Curve Frameworks; and III — Information-Based Interest Rate Models.

The sections that follow contextualises the research undertaken from the following perspectives:

(i) *the historical context*, i.e., fundamental changes in the interest rate centric financial markets under consideration during the twenty-first century; and

(ii) *the practical context*, i.e., a description of relevant financial markets and products along with important characterisations of risk related thereto;

while the chapter that follows describes:

(iii) *the modelling context*, which first considers the applicability of the widely accepted *reduced-form interest rate modelling approach* to the respective financial markets presented in (ii); and then introduces a novel *market-based approach* which is conjectured to be more suitable for certain financial markets, and forms the basis for the analysis that is undertaken in Part I.

As it will be shown, the market-based approach is essentially a generalisation of the reduced-form approach, but a method of modelling that should be more natural to the perspective of market-takers.

In addition to the above, the first two chapters, which are both introductory in nature, provide a high-level description of the novel ideas, conjectures, assumptions, perspectives and interpretations that are adopted for the theoretical analysis of the aforementioned interest rate centric markets, and applied in the development of the frameworks and models that constitute this thesis.

Remark 1.1.1 (A note on the usage of conjectures within this thesis)
Classically, a conjecture is uitlised in mathematics in order to purport that a result holds true, without any proof thereof. Such a conjecture may then be refuted through a counterexample or validated via the

establishment of a valid formal proof. In this thesis, the author extends the use of the notion of a conjecture beyond that which is standard for the purpose of emphasis. In general, any fundamental opinion, subjective assumption, or subjective definition that the author considers to be vital for the advancement and progression of the theory that is being developed is also recorded herein as a conjecture.

The chapter concludes with further detail on each of the three parts listed above, along with an attempt to contextualise the results and findings of the constituent parts within the existing body of related academic literature.

1.2 Historical Context

The twenty-first century has already seen numerous financial crises that have had adverse impacts across the global landscape[6], such as the *dot-com crash* in 2001, the *Global Financial Crisis* (GFC) in 2007/08, the *European debt crisis* in 2009/10, the *COVID-19 pandemic induced crash* of 2020, and the ongoing (at the time of writing) global impact of the *Russian invasion of Ukraine* which began in 2022. There are various systemic and idiosyncratic implications of each of these crises; however, within the context of this thesis, the GFC's structural impact on bank market-made interest rate markets is the most relevant event during this period. These structural changes are summarised in the next subsection, which also serves as the main motivation for Part I of this thesis.

1.2.1 Before the Global Financial Crisis

Before the GFC, commercial banks were able to source term funding at *near risk-free term rates.* This may be evidenced by the following:

(a) short-term interbank reference rates[7] moved in lockstep with central bank policy rates, with subdued spreads and the 3-month term being the key funding rate[8], in general;

(b) medium- to long-term deposit rates coincided with those implied from forward rate agreements (FRAs) and interest rate swaps (IRSs), modulo insignificant spreads; and

(c) spreads between IRSs that referenced interbank reference rates with different tenors, or tenor basis swap (TBSs) spreads, were negligible or non-material.

These stylised features of the bank funding and linear derivatives markets are described and corroborated in, for instance, the research undertaken by Collin-Dufresne and Solnik (2001), Mercurio (2009), Morini (2009), Bianchetti and Carlicchi (2011), and Beau *et al.* (2014). Two important implications of these features, within these markets, were the following:

(i) *Cross-Sectional Term Agnosticism*, i.e., non-material liquidity and credit risks implied that all term funding-related forward rates were tradable and replicable, at least synthetically[9] via the linear derivatives market, i.e., using FRAs and/or IRSs; and

[6] For instance, see the Financial Crisis Wikipedia web page for a repository of major global financial crises throughout history.

[7] For example, the set of London Interbank Offered Rates (LIBORs).

[8] Thereby, also the main reference rate for derivatives and transmission tool for the central bank's monetary policy.

[9] It would have also been possible to trade and replicate term funding-related forward rates via the classical textbook "*cash-and-carry*" type of strategy. However, the respective market-maker would have had to consider the sourcing and placement of funding carefully, for this to have been practically and economically viable.

(ii) *Funding-Swap Duality*, i.e., consistency and coherence between the *market-making* of banks' term funding and linear interest rate derivative transactions.

Both of these implications manifested a *single-curve framework* for the *primary market* pricing and *secondary market* valuation of funding and interest rate derivative transactions.

1.2.2 After the Global Financial Crisis and Reference Rate Reform

Post the GFC, cognisance of significant liquidity and credit risks has persisted within bank funding markets. As a result, only feature (a) from the previous sub-section remains, albeit reduced in form, with the bulk of interbank funding transactions transitioning from term-based to overnight rates. Interbank markets have all transitioned, or are in the process of transitioning, from indicative *term-based reference rates* (TBRRs) to transaction-based *overnight reference rates* (ONRRs), with the latter considered as being *near risk-free* while the former (in particular the 3-month term) now incorporates material elements of liquidity and credit risk.

These paradigm shifts in funding and derivatives markets were motivated both by the vulnerabilities of banking entities that were revealed by the GFC and the findings of the fundamental review of major interest rate benchmarks conducted by the Financial Stability Board in 2014 — see FSB (2014) for further detail on the latter. For a review of *reference rate reform*, in general, one may refer to Schrimpf and Sushko (2019). Also, see Guggenheim (2020), Klingler and Syrstad (2021), Nelson (2020) and Skov and Skovmand (2021) for further discussion and specific details regarding the potential creation and replacement of bank funding-related term rates.

Therefore, the pre-GFC interbank market microstructure constructed upon key 3-month funding and reference rates has now, post-GFC, been '*replaced*' by a similar one based on overnight rates, such that the interbank funding and its associated linear derivatives market[10] remains the source of *near risk-free term rates*. General bank funding that originates within the non-interbank funding market will not transition into the overnight rate regime in general, primarily due to the fact that non-bank entities will seek to earn funding yields in excess of risk-free rates. In other words, these entities would want to be exposed to *term risk*, which naturally manifests due to liquidity and credit-related risks.

From the perspective of implication (ii), the discussion above reveals that funding-swap duality should be maintained within the interbank market, conditional on funding transactions that reference overnight rates. However, there is now a dislocation between the interbank funding and derivatives market and the non-interbank funding market, conditional on funding transactions that reference term rates and yield term risk premia. This dislocation arises as a result of implication (i), cross-sectional term agnosticism, no longer holding in the presence of term risk. While it is impossible to recover implication (i) within the post-GFC market microstructure, it is possible to recover implication (ii) from a systemic perspective, which is the primary objective of Part I of this thesis.

The analysis undertaken in Part I enables the development of a *market-based approach* to multi-curve

[10] This market consists of Interest Rate Futures (IRFs) and IRSs that reference the respective overnight reference rate, with the latter IRS referred to as an Overnight Indexed Swap (OIS).

interest rate modelling — the exact definition and interpretation of "*market-based*" is provided in chapter 2, while the theoretical exposition is based on the work done in Macrina and Mahomed (2023), which is titled: "*A Systemic Perspective on Term Risk in Bank Funding Markets*". More detail on results and contributions are provided in section 1.4.

1.3 Practical Context

This section defines and describes the markets and financial instruments that are considered either directly or indirectly, along with a characterisation of relevant risk characteristics related thereto. The first sub-section classifies the set of interest rate centric financial markets and instruments that are relevant. These are used later, in section 1.4, to define a set of economies with varying degrees of financial market sophistication. The reason for this is to enable the reader to further understand real-world scenarios under which the frameworks presented in Part I and II are practically applicable.

1.3.1 Financial Markets

While not exhaustive within the realm of interest rate centric financial markets, the following subset of local and global markets are relevant and important to describe:

(i) **Government Debt Market**: This market generally constitutes short-term treasury bills (TBs), medium- and long-term treasury notes (TNs), fixed coupon bonds (FCBs) and inflation-linked bonds (ILBs), all of which are issued by the local government and denominated in local currency.

(ii) **Corporate Debt Market**: This market generally constitutes short-term commercial paper (CP), medium- and long-term corporate fixed coupon bonds (FCBs) and floating rate notes (FRNs), all of which are issued by private sector entities and denominated in local currency.

(iii) **Deposit Market**: This is an *over-the-counter* (OTC) market that is created by commercial banks for the purpose of accessing finance in an unsecured bilateral fashion in local currency in the form of fixed-term deposits (FTDs). Technically, this is not a financial market since it is bilateral in nature and therefore does not have an organised secondary market. Practically, this means that FTDs may not be liquidated prior to maturity — an important feature to take note of.

(iv) **Money Market**: This is a market that is enabled by commercial banks for the main purpose of sourcing short- to medium-term funding in local currency. This market is also generally *unsecured* in nature and may be characterised by short-term negotiable certificates of deposit (NCDs), medium-term fixed coupon deposits (FCDs) and floating rate-linked deposits (FRLDs), all of which are issued and/or structured by commercial banks. Since all entities participate in this market by depositing funds with the set of commercial banks, including interbank activity, via one of the aforementioned instruments, this is the market within which the key unsecured bank funding reference rate (BFRR) is determined[11].

(v) **Repo Market**: This *secured* market is also generally enabled by commercial banks and may be characterised by bilateral repo or buy/sell-back (BSB) transactions, which are short-term in nature and collateralised with government-issued debt securities. Being fundamentally bilateral,

[11] Prior to reference rate reform, BFRRs were TBRRs, in particular 3-month term rates. Post reference rate reform, BFRRs are now either secured or unsecured ONRRs — unsecured ONRRs are determined via deposit and money market activity.

like the deposit market, this market does not constitute a secondary market. In economies like the United States of America, where bank funding has transitioned to a secured overnight standard post the GFC and reference rate reform, the secured BFRR is determined via this market.

(vi) **Interest Rate Derivatives Market**: This is another market that is created by commercial banks for the primary and predominant purpose of market-making linear and non-linear derivatives that enable hedging, speculation and arbitrage activity in relation to the key BFRR. In terms of linear derivatives, which are the main focus of this thesis, this market generally constitutes FRAs, IRFs, IRSs (which include OISs) and TBSs. Non-linear derivatives are most commonly interest rate caplets, floorlets, caps, floors and swaptions.

(vii) **Foreign Exchange Derivatives Market**: This is a global financial market that is again market-made by commercial banks and enables the usual activity relating to hedging, speculation and arbitrage, but here in relation to cross-country currency and funding risks. The most important linear derivatives are forward exchange contracts (FECs) and cross-currency basis swaps (CCBSs). Spot exchange rates (SERs) are a subset of the FEC market, and therefore the foreign exchange spot market is a natural constituent of the foreign exchange derivatives market.

(viii) **Sovereign and Foreign Debt Markets**: Foreign currency denominated FCBs issued by local public and private sector entities constitute the foreign debt market, while foreign currency denominated FCBs issued by local government treasuries constitute the sovereign debt market.

Other interest rate-related derivatives such as forwards, futures and options that are written on government and corporate bonds are not mentioned nor formally categorised, since they are not the focal point of the research that is undertaken in this thesis.

All of the debt markets are of course markets for primitive assets, i.e., markets which enable the issuers of assets to raise *capital*. The deposit, money and repo markets serve a similar function or role, enabling banking entities to raise *funding*. At the surface level, capital and funding may appear to be one and the same, and indeed are often used as synonyms in many academic- and industry-related contexts. However, a key conjecture in this thesis is that these activities are fundamentally different; so much so that this difference has significant implications for general risk characterisation and the mathematical modelling thereof — this idea is explained further in the next sub-section.

1.3.2 Capital versus Funding Assets

In section 1.1, the distinction between primitive assets and derivative securities was explained from a financial economic and mathematical perspective. Here, primitive assets are further categorised into capital and funding assets, which enables a deeper understanding of the liquidity risks associated with primitive assets. The qualitative distinction between the two sub-classes, from a liquidity risk perspective, is a critical assumption that is recorded in the next conjecture.

Conjecture 1.3.1 (Capital, funding assets, market-making and secondary market liquidity)
Both capital and funding assets provide their respective issuers finance over a specific tenor or horizon, but the key distinguishing feature between the two is that:

- *issuers of funding assets play the role of market-maker in the asset's primary and secondary markets, thereby guaranteeing secondary market liquidity; while*
- *issuers of capital assets may play the role of market-maker in the primary market but provide no secondary market or outsource these market-making functions to relevant financial intermediaries, thereby offering no guarantee for secondary market liquidity.*

Practically, this means that issuers of capital assets have guaranteed use of the finance over the entire tenor of the issued asset, while issuers of funding assets do not have the same guarantee.

Within the context of the markets presented in sub-section 1.3.1, financial instruments that constitute the set of debt markets are all examples of capital assets. FTDs and repos, or BSBs, which constitute the deposit and repo markets, respectively, are also examples of capital assets. The money market provides examples of funding assets, with banking entities being the primary issuers of these instruments. The implications of Conjecture 1.3.1 from a risk perspective are discussed in the content that follows.

1.3.3 Derivative Securities Before and After the GFC

The emergence of significant liquidity and credit risks during and post the GFC has changed the practical financial and economic nature; market microstructures; primary market pricing or market-making processes; secondary market valuation processes; and risk management frameworks associated with derivative securities. In this sub-section, the practical nature of interest rate derivative securities will be the main focus. Latter sections will deal with issues related to pricing, valuation and risk management.

A market-maker at a bank implementing a classic academic "*cash-and-carry*" FRA replication strategy, using their own *internal treasury*, would require borrowing (lending) for the full-term of the FRA and depositing (borrowing) back the same amount for the same term but agreeing to roll-over at the interim reset/settlement date associated with the FRA. From the treasury's perspective, this is clearly a funding-neutral strategy and one that essentially transfers the burden of price discovery for the implied forward rate unto them, or the primitive funding market. Since there isn't any tangible economic benefit for the treasury, particularly in relation to funding, it would not make any rational sense for a banking entity to support such a business model. On the other hand, allowing market-makers to transact with *external treasuries* as well, may be beneficial from the perspective of funding, but will come at the cost of liquidity and credit risk exposures — hence, this is another unviable business model.

Rather, a viable business model is to treat linear interest rate derivatives, such as FRAs and IRSs, as somewhat primitive in nature with price discovery of the relevant key forward rates occurring within the derivatives market. Technically, this is equivalent to acknowledging that the bank funding market is an *incomplete market* — a feature that has emerged prominently post the GFC, and the main reason for the price discovery dislocation between the primitive and derivative bank funding markets[12]. Practically, this means that complete hedging of a derivative security using relevant primitive financial instruments is not possible. Therefore, market-makers would have to be afforded sufficient resources and risk appetite to bear residual risk in order to create liquid primary and secondary markets for such

[12] This dislocation is related but subtly different to that described in section 1.2, which highlights the dislocation between price discovery in interbank and non-interbank funding markets.

derivative securities. These incomplete market implications are not unique or localised to interest rate markets, but pervasive across all asset classes and associated financial markets.

Section 1.2 highlights the fact that the interest rate derivatives market experienced a paradigm shift post the GFC. Before the GFC, "*Funding-Swap Duality*" prevailed — this feature, in a practical sense, refers to the coherence between forward rates derived from BFRRs and corresponding FRA rates. In other words, notwithstanding the discussion in the preceding couple of paragraphs, it was at least theoretically possible to replicate a FRA using the classic academic "*cash-and-carry*" type strategy, due to non-material liquidity and credit risks. This served as a useful reference and driver of coherence for market-making across bank funding primitive and derivative markets. Post the GFC, the processes of price discovery across these markets has bifurcated due to the reasons outlined above, along with the additional need to incorporate *valuation adjustments* into derivative transactions in order to account for funding- and counterparty credit-related risks. The analysis of valuation adjustments, and thereby the risky valuation of derivatives, is beyond the scope of this thesis. Rather, the primary objective is to develop multi-curve frameworks for primitive interest rate assets that are mathematically consistent, and to recover coherent risk-free values of relevant derivatives as a secondary objective.

While local interest rate derivative markets enable trading in *pure* or *classical* derivative securities that are capital- and funding-neutral, the same cannot be claimed of all derivative securities. There is a subset of derivatives that breach this feature. These will be referred to as *quasi-primitive securities* and are formally defined below.

Conjecture 1.3.2 (Quasi-primitive securities)
Any linear derivative security that enables the exchange or swapping of capital or funding assets is a quasi-primitive security. The pricing and valuation of such instruments are amenable to classical replication strategies but requires careful consideration due to their innate hybrid primitive and derivative nature.

All derivative securities that originate in the foreign exchange derivatives market are examples of quasi-primitive securities. This is due to the fact that all of the foreign exchange derivatives mentioned in the previous sub-section enable the exchange or transfer of bank funding across economies. In addition, there are bespoke foreign exchange swaps (FXSs), not listed above, which enable the exchange of capital across economies. An example of a local currency denominated derivative that may be classified as a quasi-primitive security is a par-par asset swap (PPAS), which generally enables the exchange of a capital asset for a corresponding bank funding asset.

1.3.4 Risk Categories and Characteristics

Having qualitative descriptions and segments of relevant financial markets and instruments, it is now possible to characterise these in relation to their risk exposures. As has become standard with modern, or post GFC, interest rate modelling, three broad categories of risk is generally considered: (i) *risk-free floating interest rate risk*; (ii) *liquidity risk*; and (iii) *credit risk*. Given a framework that sufficiently recovers the features and dynamics of all three of these risk categories, it is theoretically possible to develop models for *market rates* that are key to each of the financial markets described above — this is the primary objective of Parts I and II.

Risk-Free Floating Interest Rate Risk

The most important interest rate within any economy is the *policy rate* that is set by the central bank, which essentially determines the "price" of *central bank reserves*, and is the main tool that is used to effect monetary policy within the respective economy under consideration. The respective monetary policy *implementation framework* or *operating procedures* dictates the exact practical nature of the policy rate. In general, there are scarce- and surplus-reserves systems with the following variations: (1) a shortage or classical cash reserve system; (2) a scarce-reserves or mid-corridor system; (3) a surplus-reserves or floor system; and (4) a tiered-floor system. For more information on monetary policy implementation frameworks, one may refer to SARB (2022) and references therein.

Under a *surplus-reserves* system, as is the case with the new framework adopted by the South African Reserve Bank (SARB), the policy rate represents:

> "*...the return on a maximally safe and liquid asset: a Rand deposit held overnight at the SARB...*",

as mentioned in SARB (2022). In a *scarce-reserves* system, the policy rate essentially represents the "*cost of a maximally safe and liquid overnight collateralised loan offered by a central bank to a relevant regulated institution*". The type of collateral that is generally accepted is local government-issued debt securities. These frameworks and their monetary policy transmission processes are therefore structurally different, the details of which are beyond the scope of this thesis. Despite the mechanics of the specific implementation framework, the policy rate generally plays a major role in:

(i) pinning the short-end of relevant yield curves; as well as

(ii) driving the dynamics of risk-free rates within an economy.

Once these two features are achieved and effective, transmission of the policy rate is achieved naturally via the pricing of all financial instruments within the economy.

Central bank reserves and thereby the policy rate are only accessible to relevant regulated banking entities. Therefore, transmission of this effect to the greater economy is primarily enabled by the commercial banking entities by way of interest rates paid (charged) on deposits (loans), as well as through their market-making, or pricing, processes for all of the interest rate-sensitive financial instruments that were listed in sub-section 1.3.1. The BFRR plays a major role in this process since it provides a benchmark for commercial banks' cost of funding, which in turn enables the quantification of capital and funding risks and the securitisation thereof via the interest rate derivatives market.

Before the GFC, the main BFRR in each global economy was generally the 3-month *unsecured* interbank term rate[13]. These were described in section 1.2 as TBRRs, along with the observation that these rates were considered to be the best proxies for risk-free rates within their respective economies, due at the time to the perceived high credit quality of systemically important banks and the lack of significant liquidity risks. Post the GFC and reference rate reform, BFRRs have now transitioned, or are

[13] Such as the set of London Interbank Offered Rates (LIBORs), the Euro Interbank Offered Rate (EURIBOR), the Tokyo Interbank Offered Rate (TIBOR), and the Johannesburg Interbank Average Rate (JIBAR).

in the process of transitioning, to either *secured*[14] or *unsecured*[15] ONRRs, which are the new proxies for risk-free rates within global economies[16]. As a result, the genesis of fundamental risk-free *floating interest rate risk* has shifted from risk that is borne out of rolling over a deposit linked to a TBRR to risk that emanates from rolling over a deposit that is linked to an ONRR.

The securitsation of risk-free floating interest rate risk, and the ability to exchange risk-free fixed-for-floating interest rate risk, has therefore also shifted from FRAs, IRFs and IRSs that reference TBRRs to IRF and OIS contracts that reference ONRRs. Therefore, while the policy rate is the true maximally safe and liquid deposit rate within an economy, the corresponding ONRR represents the best approximation of the policy rate that is accessible to all participants within an economy. These comments are succinctly captured in the conjecture below.

Conjecture 1.3.3 (Risk-free term rate market-making)
The market for linear derivatives, i.e., IRF and OIS contracts, referencing the main ONRR within a given economy provides the platform for market-making and price discovery of risk-free term rates.

Liquidity Risk

The severity of liquidity risk exposure during and after the GFC has arguably had the most significant impact on risk management and prudential financial regulation. A major objective of Parts I and II is to develop robust and consistent interest rate modelling frameworks that are cognisant of liquidity risk. The chosen characterisation of liquidity risk is the same as that provided in Acerbi and Scandolo (2008), Bianchetti and Carlicchi (2011) and Morini (2009), and presented in the next conjecture.

Conjecture 1.3.4 (Liquidity risk for primitive assets)
Within the context of primitive assets, the following three categories are conjectured to encapsulate all forms of liquidity risk that impact capital and funding transactions:

(i) funding-liquidity risk[17] *— a primary market risk which refers to the uncertainty associated with the general availability of capital or funding at the initiation and all interim, or roll-over, times during the life of a capital or funding transaction;*

(ii) market-liquidity risk[18] *— a secondary market risk which refers to the uncertainty related to the ability and cost of liquidating an existing capital or funding financial instrument beyond fair and expected market frictions such as transaction costs, taxes and profit margins; and*

(iii) systemic-liquidity risk[19] *— the unexpected realisation of risks (i) and/or (ii) as a result of systemic risks or vulnerabilities, and may therefore impact both the primary and secondary markets.*

It is hopefully clear that categories (i), (ii) and (iii) are all intricately linked and are phrases that are used synonymously in practice, as reported in Acerbi and Scandolo (2008). Here, an important relation

[14] For example, the Secured Overnight Financing Rate (SOFR) in the United States of America.
[15] For example, the Sterling Overnight Index Average (SONIA) in the United Kingdom, the Euro Short-Term Rate (€STR) in the European Union, and the South African Overnight Index Average (ZARONIA) in South Africa.
[16] This is due to the practical assumption that the overnight tenor poses minimal bank credit and liquidity risk.
[17] The work by Eisenschmidt and Tapking (2009) supports the existence of this type of liquidity risk within money markets.
[18] For a slightly different yet in-depth analysis on funding- and market -liquidity, see Brunnermeier and Pedersen (2009).
[19] For a more practical macroeconomic, perspective on systemic liquidity risk within the banking context, see Acharya *et al.* (2011) and Acharya and Skeie (2011)

between categories (i) and (ii) is enabled and posited due to the implications of Conjecture 1.3.1. These relations are presented in the next two conjectures.

Conjecture 1.3.5 (Capital assets and liquidity risk)
Based on Conjecture 1.3.1, issuers of capital assets are immunised from funding-liquidity risk over the tenor of the asset but expose the buyers of the asset to market-liquidity risk.

If issuers of capital assets are primary market-makers, i.e., they have full discretion over price discovery in the primary market, and are acting rationally then these issuers must compensate buyers for the protection from funding-liquidity risk that they are inadvertently offering — an example of such an asset is an FTD. One may also interpret this as the issuer offering the buyer compensation for the market-liquidity risk that the buyer is forced to bear.

However, capital asset issuers are primary market-takers in general. This is indeed the case, for example, for issuers of debt securities in the debt markets defined in sub-section 1.3.1. Therefore, based on the above observations and implications, one may infer that capital asset buyers will demand, in the primary market, compensation for the market-liquidity risk that they expect to be exposed to over the tenor of the respective asset under consideration.

In relation to the systemic-liquidity risk category, it is assumed that this realises unexpectedly only in times of market distress and turmoil. Therefore, it is further assumed that both issuers and buyers of capital assets account for and price this risk reactively and not proactively.

Conjecture 1.3.6 (Funding assets and liquidity risk)
Based on Conjecture 1.3.1, issuers of funding assets protect buyers of the asset from market-liquidity risk at the cost of exposing themselves to funding-liquidity risk over the tenor of the respective funding asset.

Issuers of funding assets, such as NCDs, FCDs and FRLDs, generally play the role of primary and secondary market-maker. In particular, and significantly, it is assumed that the issuer protects buyers against market-liquidity risk in the secondary market. These market dynamics have two outcomes:

(i) issuers are fully exposed to funding-liquidity risk by enabling secondary market liquidity and are therefore not required to compensate buyers for funding-liquidity risk; and

(ii) buyers are immunised from market-liquidity risk and cannot demand compensation for exposure to market-liquidity risk.

Issuers and buyers of funding assets will remain exposed to potential systemic-liquidity risks, which may exacerbate funding-liquidity risks for issuers and create market-liquidity risks for buyers.

The underlying market microstructure used in the descriptions of Conjectures 1.3.5 and 1.3.6 obfuscate some of the intuitive insights, which is the motivation for the next remark.

Remark 1.3.1 (The dual nature of funding- and market-liquidity risks for primitive assets)
Adopting a stylised theoretical bilateral perspective for a primitive asset transaction, and ignoring the practical roles and features of market-maker and -taker, then in the most extreme case:

(a) an issuer's funding-liquidity risk will be fully realised if they're not able to fully replace the finance obtained when the buyer demands that the asset be liquidated prior to maturity; and

(b) the buyer's market-liquidity risk will also be fully realised in this scenario since the issuer will not be capable of enabling liquidation of the asset prior to maturity.

The partial case may also be explained by way of liquidation prior to maturity as follows:

(c) an issuer's funding-liquidity risk will only be partially realised if they are able to replace the finance obtained at a cost above the prevailing fair value upon request of liquidation by the buyer; and

(d) the buyer's market-liquidity risk will also only be partially realised in this scenario if the marginal cost above fair value is passed on to the buyer by way of a lower liquidation value.

From (a), (b), (c) and (d), it should be clear that both funding- and market-liquidity risks are triggered by the buyer's decision to liquidate the asset, while the issuer's ability to source replacement funding-liquidity, or a replacement "buyer" for the asset, determines whether the risks are actually realised. These observations demonstrate the dual nature of funding- and market-liquidity risks, as well as the 'duality' between liquidation and funding-liquidity within the context of primitive assets.

The insights from Remark 1.3.1 prove to be useful for theoretical model development, and are utilised extensively in Part I when modelling potential market illiquidity.

The remaining aspects to consider in this section, is how derivative (and quasi-primitive) securities are impacted by liquidity risk. In keeping with the changes described in section 1.2 and the insights from sub-section 1.3.3, there has also been a paradigm shift in relation to the effect of liquidity risk on the pricing and valuation of derivative securities post the GFC — see, for instance, PWC (2015) or Deloitte (2017) for a high-level qualitative summary. Appendix A.2.1 describes the main liquidity risk-related valuation adjustments that one has to consider for derivative securities, which includes the funding valuation adjustment (FVA), collateral valuation adjustment (COLVA) and the margin valuation adjustment (MVA). These valuation adjustments are not researched any further, but are mentioned here merely for completeness.

Credit Risk

While the fundamental nature, understanding and cognisance of credit risk has not changed significantly before and after the GFC, one may argue that the materiality thereof, particularly in relation to the banking sector, has increased substantially. The general market-making process for derivative transactions has been impacted the most by these events, requiring one to now include *valuation adjustments* for bilateral counterparty credit-related risks, which have taken the form of *credit valuation adjustments* (CVAs) and *debt valuation adjustments* (DVAs). First though, the characterisation of credit risk exposures for primitive assets will be considered.

As with the case of liquidity risk exposures that differed between capital and funding assets, as articulated in Conjectures 1.3.5 and 1.3.6, credit risk exposures are slightly different for capital versus funding assets. These risk characteristics are recorded in the next couple of conjectures.

Conjecture 1.3.7 (Capital assets and credit risk)
The issuer of a capital asset must compensate buyers for been exposed to their credit risk over the entire tenor of the respective asset, assuming that buyers will not be able to fully recover a commensurate fair value upon default of the issuer.

Conjecture 1.3.8 (Funding assets and credit risk)
Conjecture 1.3.7 applies again for funding assets, in addition to the issuer being exposed to funding-liquidity contingent credit risks.

It is important to take note of the asymmetric or unilateral nature of credit risk exposure in relation to primitive assets, i.e., only the buyer of the asset is exposed to the credit risk of the issuer. This is simply due to the fact that a primitive asset will always have non-negative (non-positive) value to the buyer (issuer). Conjecture 1.3.8 captures the interplay between liquidity and solvency risks (arguably most relevant for banking entities), which results due to the implications of Conjecture 1.3.6, viz., the funding asset issuer's exposure to funding-liquidity risk.

Further details on CVA and DVA are provided in Appendix A.2.2, along with their relations to the liquidity risk-related valuation adjustments for uncollateralised, collateralised and cleared derivative securities. As mentioned before, the analysis of valuation adjustments is beyond the scope of the research undertaken in this thesis — for further details on this subject, the interested reader may refer to, for example, Green (2015), Zeitsch (2017), Gregory (2020), and references therein.

This sub-section is concluded with Table 1.1 that summarises the main risk categories that feature in each of the interest rate centric financial markets that were defined at the beginning of section 1.3, along with the general valuation mechanism or methodology that is applicable in each of these markets. In some instances, it is necessary to highlight the primary market (PM) versus the secondary market (SM) and risk exposures relevant thereto. The valuation mechanisms will be expanded upon in the next chapter, which deals with the mathematical modelling context.

Market	Risk Categories	Valuation Mechanism
Government Debt Market	Risk-Free, Funding-Liquidity (PM), Market-Liquidity (SM) & Inflation	Risk-Adjusted Discounting
Corporate Debt Market	Risk-Free, Funding-Liquidity (PM), Market-Liquidity (SM) & Corporate Credit	Risk-Adjusted Discounting
Deposit Market	Risk-Free, Funding-Liquidity & Bank Credit	Risk-Adjusted Discounting, but no Secondary Market
Money Market	Risk-Free & Bank Credit	Risk-Adjusted Discounting
Repo Market	Risk-Free & Funding-Liquidity	Risk-Adjusted Discounting, but no Secondary Market
Interest Rate Derivatives Market	Risk-Free & Bank Credit + Liquidity & Credit Valuation Adjustments	Risk-Free Discounting + Valuation Adjustments
Foreign Exchange Derivatives Market	Risk-Free, Bank Credit & Foreign Exchange + Liquidity & Credit Valuation Adjustments	Risk-Free Discounting + Valuation Adjustments
Sovereign & Foreign Debt Market	Risk-Free, Funding-Liquidity (PM), Market-Liquidity (SM), Government/Corporate Credit & Foreign Exchange	Risk-Adjusted Discounting

Tab. 1.1: Financial markets, risk categories and valuation mechanisms.

1.4 Structure and Contributions

In order to consolidate the content from the previous sections and to further contextualise the research problems that are considered, three types of economies are theoretically postulated which are characterised by varying degrees of financial market sophistication and liquidity.

Definition 1.4.1 (Frontier, Emerging and Developed Economies)
*A **frontier economy** is assumed to have satisfactorily liquid debt and deposit markets, an illiquid money market and extremely illiquid or non-existent derivative markets.*
*An **emerging economy** is assumed to have satisfactorily liquid versions of all financial markets, but still maintains the pre-GFC market microstructure*[20].
*A **developed economy** is assumed to have liquid versions of all financial markets, and has completed the process of reference rate reform.*

A general approach that will be taken is to assume the perspective of a *market-taker* and then develop theory that enables the role of a *market-maker* and the creation of new financial instruments and markets. The economies defined above provide useful practical starting contexts for the market-taker and end goals for the market-maker. In fact, the three parts of this thesis may be described as follows:

- In Part I, the starting context is that of a frontier economy and the objective is to use the liquid deposit market as a basis and develop models to market-make the money and its associated linear derivatives market, using a market- and first principles replication-based approach, that would first exist within an emerging and then a developed economy context. The framework developed herein adheres strongly to the practical structure of market rates, resulting in models that naturally admit discrete or term-linked rates and term structures thereof.

- In Part II, the framework from Part I is formalised and generalised within a reduced-form context. The developed approach is referred to as the xy-formalism, which basically enables all of the results from Part I within a more tractable reduced-form setting. The emerging economy context is considered and the modelling required to evolve into a developed economy is demonstrated and analysed. Developed economy pricing and valuation problems are then also considered, particularly those that involve multiple curves across different markets.

- In Part III, an information-based modelling framework is presented that enables one to develop specific models that are applicable for specific classes of financial instruments across markets that constitute frontier, emerging and developed economies. The multi-curve frameworks from Parts I and II may then be overlayed on these models for various real-world applications.

The next three sub-sections further contextualise the research that has been undertaken in each of the aforementioned parts within the existing body of academic literature. This also offers the opportunity to provide a high-level overview on the structure within each of the constituent parts.

[20] South Africa is a real-world example of such an economy, with 3-month JIBAR still (at the time of writing) playing the role of key BFRR and derivative market reference rate. However, plans are currently in motion for reference rate reform — the interested reader may refer to the following link for more information: SARB Market Practitioners Group.

1.4.1 Market-Based Multi-Curve Frameworks

The theory that is developed in Part I is based on the work done in Macrina and Mahomed (2023), which is titled: "*A Systemic Perspective on Term Risk in Bank Funding Markets*". While this research paper focuses on the same problem that is considered in Part I, the specific contexts are subtly but materially different. In Macrina and Mahomed (2023), no distinction is made between deposit and money markets, with the latter constituting bank funding markets, whereas here there are significant differences between the two — these differences have been articulated and explained in section 1.3 in a qualitative manner. Chapter 2 contributes further quantitative evidence to this discussion, which serves as the motivation for the introduction of the "*market-based approach*" as a framework for constructing models for markets such as the deposit and bank funding markets. The market-based approach is a generalisation of the classical single-curve reduced-form approach and is innately a multi-curve modelling method that is based on a system of stochastic discount factors and pricing kernels.

While the arbitrage-free mechanics of the single-curve reduced-form setting is well understood and established, it is not immediately and directly applicable to the multi-curve market-based setting. Therefore, a substantial portion of Part I is devoted to deriving the arbitrage-free mechanisms that should underpin the market-based approach. This analysis is achieved in a constructed stylised systemic and symmetric setting, which is presented and described in Chapter 3. The problem that is considered within this setting is the application of the market-based approach to model the deposit market, from the perspective of a systemic entity that is a market-taker within this market. A natural feature of the deposit market is that it is a capital asset market, as defined in sub-section 1.3.2, that does not have a secondary market. Therefore, the practical end goal for the systemic market-taker is the creation of a secondary market for FTDs. In other words, the market-taker aims to utilise the theorised framework in order to evolve into a market-maker in the secondary market.

Chapter 4 presents the derivation of the theory that leads to the development of systemic multi-curve market-based frameworks for the deposit market, within risk-free, liquidity risky and liquidity and credit risky contexts — an argument based on "*funding-swap duality*", described in Chapter 3, proves to be the novel idea and tool that enables the construction of calibrated pricing kernels in an arbitrage-free manner. Chapter 5 formally presents the market-based multi-curve framework for the deposit market, and motivates the need for a reduced-form version thereof. The reduced-form version of the framework enables the development of an exchange of risk mechanism, which: (i) allows for the transfer or transformation of composite risks across terms; (ii) emphasises the inability to hedge term-related risks; and (iii) facilitates the consistent pricing and valuation of financial instruments with cash flows that are forecasted and discounted using different curves. The chapter concludes with a description of how one may develop multi-curve market-based frameworks for bank funding and repo markets, along with links and relations to Part II.

Part I should be considered as the prequel to the framework developed in Macrina and Mahomed (2018) — this will be explained further in the next sub-section. The research undertaken by Filipović and Trolle (2013), Galliitschke *et al.* (2017) and Gefang *et al.* (2011) is also related, but primarily empirical compared to the theoretical work developed here — in their work tractable models are developed that capture multiple term structures of interbank risk post the GFC, and then they attempt to untangle credit and liquidity risks by calibrating these models to market data[21]. Also related is the research undertaken in Alfeus *et al.* (2020) and Backwell *et al.* (2023), where a reduced-form modelling framework is developed to model the dynamics of ONRRs and TBRRs, with the use of a novel modelling quantity that captures liquidity risk in the guise of *refinancing* or *roll-over risk*. However, these avenues of research are focussed on pricing and valuation within multi-curve interest rate derivatives markets, while the work here focuses on developing a systemic framework for pricing and valuation within primitive deposit and bank funding markets.

1.4.2 Reduced-Form Multi-Curve Frameworks

A high-level description of the research that is undertaken in Part II is provided at the beginning of this section; however, this has been contextualised within the narrative of this thesis. In fact, the work that has been done in Part II is based on Macrina and Mahomed (2018), which is titled: "*Consistent Valuation Across Curves using Pricing Kernels*", where the almost self-contained objective is to develop a theoretically consistent and tractable framework for pricing and valuing financial instruments with cash flows that are accrued at an interest rate which differs from the rate that is relevant for discounting. This problem contributes to the general class of problems in mathematical finance that are solved through *convexity adjustments* or *corrections*. In general, other problems that contribute to this class arise due to currency conversion, unnatural cash flow timing, fixing adjustment[22], margining and collateralisation features. For more information on convexity adjustments and corrections, one may refer to Baxter and Rennie (1996), Hunt and Kennedy (2000), Pelsser (2003), Brigo and Mercurio (2006), Lesniewski (2008), Andersen and Piterbarg (2011) and Burgess (2019), as well as references therein.

All of these references essentially advocate for a solution that requires direct computation under an appropriately adjusted measure that recovers the martingale feature; however, this generally requires a tractable joint distribution of the payoff/cash flow that is under consideration and the numeraire that supports and enables the adjusted measure. These methods of solution are therefore highly model dependent, and generally ignore any financial economic intuition. Here, the exchange of risk mechanism developed in Part I, and presented in section 5.3, is generalised to offer an alternative solution to the specific problem, and direct method described above. Since the exchange of risk mechanism is constructed using pricing kernels, the economic and technical underpinnings of the applicable pricing kernels offers both financial and mathematical rationale, respectively, to the eventual solution.

Chapter 6 introduces the *curve-conversion factor process* which is the generalisation of the exchange of risk mechanism. In turn, the curve-conversion factor process enables the generalisation of the fundamental pricing formula based on a pricing kernel — this new formula is referred to as the *across-curve pricing formula*. In turn, this enables the creation of a general modelling framework referred to as the xy-formalism or xy-approach, which is developed and applied to various practical problems throughout the remainder of Part II. As a heuristic, the 'x' moniker is used to identify the curve that is used for discounting (and thereby the numeraire), while the 'y' moniker identifies the curve that defines the characteristics of the interest rate at which future cash flows accrue or are forecasted. Most impor-

[21] The untangling of credit and liquidity risk is also considered by Schwarz (2019) using an econometric approach.

[22] Such as the lockout, lookback, observation period shift and payment lag adjustments used in the specification of financial instruments that reference ONRRs — see, for instance, ARRC (2021) for more information.

tantly, Remark 6.1.3 explains how the development of the xy-formalism completes the definition of the market-based approach, that was initiated in Chapter 2, by specifying the arbitrage-free mechanics for pricing and valuation within a market setting characterised by multiple pricing kernels. The chapter concludes by demonstrating how the xy-formalism may be utilised to model the bank funding derivative markets associated with emerging and developed economies, before reference rate reform. It is further demonstrated how the dual nature of the curve-conversion factor process enables the creation of a consistent multi-curve discounting system for an emerging economy. Other interesting comments and insights are offered regarding convexity corrections, curve-conversions and other financial engineering aspects in this regard.

One of the objectives for Chapter 7 is to demonstrate how the xy-approach allows for popular single-curve reduced-form models and frameworks to be repurposed and utilised within a multi-curve setting, in a theoretically consistent manner. To this end, the first part of this chapter reviews and reformulates existing Heath-Jarrow-Morton (HJM) multi-curve modelling approaches within the context of the xy-approach, and introduces a new multi-curve HJM framework called the xy-HJM framework. The second part introduces a generic class of rational multi-curve models, those devised by Macrina (2014), showing how those may be easily implemented with the xy-formalism. Then, recent rational multi-curve approaches based on pricing kernels are revisited and reviewed — in particular, it is shown how the rational multi-curve models of Crépey *et al.* (2016) and Nguyen and Seifried (2015) may be recovered within the xy-formalism. Finally, it is shown how the linear-rational term structure models of Filipović *et al.* (2017) belongs to the general class of pricing kernel-based rational models, presented by Macrina (2014), and are therefore also easily applicable within the xy-formalism. Chapter 8 concludes Part II by demonstrating how the xy-formalism offers a flexible yet rigorous framework which may be utilised to solve pricing problems involving financial instruments with floating nominal rate, inflation and foreign exchange exposures, in a consistent manner.

The *curve-conversion factor process* is essentially a generalisation of the spot and forward foreign exchange (FX) rate processes, when modelling FX markets using pricing kernels. Therefore, the xy-approach is related to the work done by Bianchetti (2009), who uses an FX analogy to model interest rate derivatives (or the bank funding derivatives market in a developed economy) within a LIBOR Market Model setting. As already mentioned, the range of applications here are much broader than just bank funding markets. Therefore, other related work is Flesaker and Hughston (1996a,b), who develop an arbitrage-free approach for the pricing of FX securities using pricing kernels; Frey and Sommer (1999) who consider extending classical short rate models, based on diffusions with deterministic coefficients, to model FX markets; and the approach by Jarrow and Yildirim (2003) who consider modelling inflation-linked bonds and derivatives related thereto using the HJM framework. The early work in 1998 by Hughston (1998) is also worth noting, who produced a general arbitrage-free approach for the pricing of inflation derivatives, in which — to the author's knowledge — an FX analogy was used in such a context for the first time. In Hughston (1998), the *consumer price index* is treated like a foreign exchange rate that links the nominal and the real interest rate systems as if they were FX basis curves related to domestic and foreign currencies, respectively. The work by Pilz and Schlögl (2013) on modelling commodities reinterprets a multi-currency LIBOR Market Model approach and therefore also has similarities to the work done in Chapter 8.

1.4.3 Information-Based Interest Rate Models

The third and final part of this thesis may be considered in isolation; however, there is at least one material common thread that links back to Parts I and II — the focus is still on the construction of stochastic discount factors and pricing kernels for the specific purpose of interest rate modelling. Therefore, specific pricing kernel models that are constructed within the derived information-based framework presented in this part may be utilised within the frameworks constructed in Parts I and II. Otherwise, Part III of this thesis has three major objectives:

(i) to recast the "*information-based asset pricing framework*" introduced by Macrina (2006) within the classical theory of stochastic filtering;

(ii) to establish a general framework for modelling short, instantaneous forward, and discrete forward rates using pricing kernels; and

(iii) to develop a framework that enables one to construct models for the bank funding market that is explicitly cognisant of monetary policy decision dates in a consistent manner.

Chapter 9 achieves objective (i) by considering the general problem of non-linear filtering within the context of Brownian motion-driven Markov signal and observation processes — this is a fairly standard and well researched problem, one may refer to any one of Jazwinski (2007), Xiong (2008), Bain and Crisan (2009), or Kallianpur (2013) for a thorough account thereof. While the information-based asset pricing framework has been significantly developed and generalised — see, for example, the work of Rutkowski and Yu (2007), Brody *et al.* (2008), Macrina and Parbhoo (2010), Hoyle *et al.* (2011), Filipović *et al.* (2012), Hughston and Macrina (2012) and Parbhoo (2013) — it has primarily been done so within the realm of bridge-based information (observation) processes with market factors (signals) which are random variables that are observable at the terminal times associated with the respective bridge-based processes. This is not the path taken in this work; rather, here it is revealed how the original information-based asset pricing framework based on Brownian-bridge processes may be recovered within the classical stochastic filtering framework, which in turn enables one to have access to a vast array of results, which serves as a useful toolkit that may aid in improving tractability when developing models. Moreover, in doing so, one also has access to a wider class of potential information (observation) processes, which may prove to be useful within different model development contexts.

Section 9.1 defines the general mathematical context within which the filtering problem is considered, with section 9.2 presenting the derivation of the classical theory that produces the aforementioned toolkit, which consists of: (i) the *reference measure*; (ii) the *Kallianpur-Striebel formula*; (iii) the *Zakai equation*; (iv) the *Kushner-Stratonovich* or *Fujisaki-Kallianpur-Kunita equation*; and (v) *Kushner's Theorem*. Section 9.3 concludes Chapter 9 by revealing how the constructed filtering framework recovers the canonical information-based asset pricing framework, while also allowing for generalised copula-based dependence amongst market factors (signals), correlations amongst information (observation) processes, and a broader class of potential information processes.

Objective (ii) is achieved in section 10.1 with Theorem 10.1.1 offering the necessary relations that enables one to model short, instantaneous forward and discrete forward rates using pricing kernels. It turns out that a family of martingale processes indexed by maturity time is the common technical modelling thread that binds the four different representations[23] of interest rate models. This finding also offers the necessary inspiration to generate pricing kernel, instantaneous and discrete forward rate models within the constructed non-linear filtering framework. Since the optimal filter of a time-homogenous market factor (signal) functional is a martingale, this tool offers a natural way to generate martingale processes, and thereby interest rate models based on the results from Theorem 10.1.1. These martingales are referred to as *information-driven martingales* and are critical for the development of *information-based forward rate models*, which is achieved in section 10.3. The development of *information-based pricing kernels* also makes use of information-driven martingales; however, a bit more structure is required to ensure that the resultant models are arbitrage-free and admit interest rates that are realistic — the required theory for such development is derived in section 10.2.

Finally, objective (iii) is demonstrated by way of an example, which is developed in an incremental fashion in Appendix D. The general filtering framework enables this result, with an infinite horizon information process, like the one presented in Appendix D.1, assumed to model the long-term interest rate dynamics, while finite horizon Brownian-bridge information processes, like the one presented in Appendix D.2, are utilised to model the impact of interest rate changes on monetary policy decision dates. Appendix D.3 offers a simple prototype for such a model, which is sufficient to demonstrate the idea and the potential for model development with this practical objective in mind.

[23] These being: (i) pricing kernel models; (ii) short rate models; (iii) instantaneous forward rate models; and (iv) discrete forward rate models.

Chapter 2

Modelling Context

In this chapter, the mathematical modelling choices and approaches that are utilised for analysis are introduced and explained. At a high-level, a "*top-down*" approach to interest rate modelling is adopted. An approach that begins with a postulated model for a set of market rates, with composite risk-free, liquidity and credit risk features, and then one seeks to incrementally develop models for rates without liquidity and credit risk by exogenously modelling the effect of these features — in this work, this method is referred to as a "*market-based approach*". This contrasts with the traditional "*reduced-form approach*" which is a "*bottom-up*" approach, where one begins with a model for risk-free interest rates and endeavours to construct risky rates by endogenously incorporating risky features. A description of these methods is the main objective of this chapter, but first the key interest rates that are to be modelled are presented in a qualitative axiomatic fashion.

2.1 Axiomatic Interest Rate Structures

In this section a set of key interest rates are introduced in an axiomatic way. The purpose of this is to add further context to the qualitative risk-based descriptions that were provided in the previous chapter, in order to inform the risk features that are required in the modelling process for each of these rates. All key interest rates will be specified using the *simple interest* convention, since such rates are generally ubiquitous across financial markets and a market-based approach has been chosen.

Notation 2.1.1 (Simple rates)
Let $A(u, u+n\delta)$ denote an arbitrary simple rate with accrual period $[u, u+n\delta]$, where $\delta > 0$ is representative of the length of an overnight tenor (unless stated otherwise), $n \in \mathbb{N}$ and $u \in \mathbb{R}_{\geq 0}$. In all that follows, the shorthand notation A_u^n will be used for such rates, such that an investment of one unit of currency at this rate is expected to yield $(1 + n\delta A_u^n)$ at the maturity time $u + n\delta$.

Based on section 1.3, the first rate to characterise in any economy is the central bank policy rate.

Axiom 2.1.1 (Central bank policy rate)
Central banks enable commercial banks to earn interest on reserve account surpluses or to settle reserve account deficits via a short-term repo facility, offering government bonds as collateral. The respective rate for these facilities is the policy rate set by the monetary policy committee periodically in response to changing macroeconomic conditions. Assuming that δ is representative of the tenor (generally overnight[1]) of these transactions, then r_u^1 denotes the policy rate and is a pure risk-free rate.

[1] Repo facilities may offer longer tenors, but usually no longer than one week.

As expressed in section 1.3, the publicly accessible risk-free rate proxy is a secured rate borne out of an overnight government bond repo or BSB transaction — this is captured in the next axiom.

Axiom 2.1.2 (Government bond repos — primitive capital assets)
The secondary government bond market enables participants to engage in bilateral repo, or buy/sell-back, transactions. These are short-term collateralised loans, where the borrower offers government bonds as collateral. Suitably aggregating many such transactions, the effective simple rate for an $n\delta$-term transaction is conjectured to be

$$S_u^n := x_{u,\mathrm{r}}^n + \ell_{u,\mathrm{r}}^n + c_{u,\mathrm{r}}^n \ , \tag{2.1.1}$$

*where the second subscript '*r*' is used to identify the repo market. This rate is also referred to as a secured financing rate*[2]*, where*

- $x_{u,\mathrm{r}}^n := \frac{1}{n\delta}\left(\mathbb{E}_u\left[\prod_{i=0}^{n-1}\left(1+\delta r_{u+i\delta}^1\right)\right]-1\right)$ *is an $n\delta$-term risk-free rate based on lenders' expectations for the evolution of the policy rate over this period given information available at time u, expressed here via the operator $\mathbb{E}_u[\cdot]$;*
- $\ell_{u,\mathrm{r}}^n$ *is a funding-liquidity spread, which is term-dependent and may be less than or equal to zero when lenders have significant surplus funds but is generally positive and increases with term;*
- $c_{u,\mathrm{r}}^n$ *is a residual credit spread which arises due to inadequate collateral, is non-negative and dependent on term, the levels of $x_{u,\mathrm{r}}^n$ and $\ell_{u,\mathrm{r}}^n$, loan-to-collateral value ratios and the volatility thereof.*

For overnight repos, i.e., $n = 1$, one can expect $c_{u,\mathrm{r}}^1$ to be negligible but $\ell_{u,\mathrm{r}}^1$ may still be material.

Remark 2.1.1 (Arbitrary conjectured additive rate/spread structure)
The conjectured additive structure of the repo rate, equation (2.1.1), is arbitrary and not meant to posit or imply any specific modelling choice. For example, a multiplicative structure such as

$$1 + n\delta S_u^n := \left(1 + n\delta x_{u,\mathrm{r}}^n\right)\left(1 + n\delta \ell_{u,\mathrm{r}}^n\right)\left(1 + n\delta c_{u,\mathrm{r}}^n\right) \ , \tag{2.1.2}$$

may have also been used. These equations merely add some quantitative context to the qualitative descriptions. The same applies to the rest of the rates that are described in this section.

The next rate to consider is one that is default-free but still exposed to liquidity risk, and these rates emerge within the government bond market.

Axiom 2.1.3 (Government bonds — primitive capital assets)
The local currency government bond market is a source for default-free term rates[3]*. While cash flow structures and quoted yield conventions may be non-homogeneous, a set of consistent effective simple term rates*

$$G_u^n := x_{u,\mathrm{g}}^n + \ell_{u,\mathrm{g}}^n \ , \tag{2.1.3}$$

*may be recovered from traded government bonds, where the second subscript '*g*' identifies the government bond market. This effective rate is conjectured to constitute a risk-free component, $x_{u,\mathrm{g}}^n$, as defined in Axiom 2.1.2 but now based on bond market participants' expectations of the policy rate*[4]*. It also constitutes a funding-liquidity spread, $\ell_{u,\mathrm{g}}^n$, which is motivated by Conjecture 1.3.5.*

[2] SOFR, the benchmark rate for USD-denominated derivatives and loans, is derived from the US Treasury repo market.
[3] Based on the assumption that a government will not default on debt issued in its local currency.
[4] If there is coherence between the government bond and its associated repo market, in terms of information and participants, then it would be plausible to assert that $x_{u,g}^n = x_{u,r}^n$, for all $u \in \mathbb{R}_{\geq 0}$ and $n \in \mathbb{N}$.

The natural next step is to consider a credit-risky rate that is associated with a capital asset, and these are fixed-term deposit reference rates (FTDRR) borne out of the banking sector.

Axiom 2.1.4 (Fixed term deposit rates — primitive capital assets)
Suitably aggregating fixed term deposit rates quoted by all relevant banking entities, an effective simple rate for an $n\delta$-term transaction is conjectured to be

$$R_u^n := x_u^n + \ell_u^n + d_u^n \ , \tag{2.1.4}$$

which is a representation of an unsecured reference rate associated with a bank capital asset. The banking sector's aggregate expectation of the evolution of the policy rate is encoded in x_u^n, the $n\delta$-term risk-free component. The funding-liquidity spread component, ℓ_u^n, exists due to Conjecture 1.3.5 and results from aggregation across the banking sector, as does the credit spread component, d_u^n.

The final rate that needs to be characterised for the purpose of completing this introduction is the set of bank funding rates, which, as an important reminder, are related to funding assets.

Axiom 2.1.5 (Bank funding rates — primitive funding assets)
Suitably aggregating bank funding rates quoted by relevant banking entities, or having access to a published reference rate, an effective simple rate for an $n\delta$-term transaction is conjectured to be

$$J_u^n := x_u^n + d_u^n + f_u^n \ , \tag{2.1.5}$$

which is a representation of an unsecured rate associated with a bank funding asset. As before, the component x_u^n is the risk-free component according to the bank sector's policy rate expectations, and d_u^n is the aggregated credit spread component. According to Conjecture 1.3.6, there is no funding-liquidity spread component; however, by Conjecture 1.3.8 there may be a component attributable to funding-liquidity contingent credit risk which is denoted here by f_u^n.

Take note that x_u^n and d_u^n refer to the same quantities in Axioms 2.1.4 and 2.1.5, since it is assumed that both the deposit and money markets are market-made by the same set of banking entities.

Remark 2.1.2 (Funding-liquidity contingent credit risk)
Conjecture 1.3.6 advocates that the issuers of funding assets should offer no compensation for funding-liquidity risk; however, equation (2.1.5) has a component f_u^n which is a spread attributable to funding-liquidity contingent credit risk. Theoretically, the idea here is that a bank may choose, on an ad hoc and reactive basis, to incorporate a positive spread for funding-liquidity if this risk poses potential solvency or default risk, consistent with Conjecture 1.3.8. Practically, in a modelling context, one may extract f_u^n as the residual component that remains after calibration/estimation using risk-free[5] and credit[6] data. This idea and spread component therefore aligns with the approach undertaken by Filipović and Trolle (2013), expressed in their work as a "non-default (liquidity) component". If this spread/component emerges to be negative, one may interpret this as a scenario where excess funding is available for the specific term, thereby reducing any potential liquidity-contingent solvency or default risk.

With the set of axioms above, it is now possible to postulate the structure of BFRRs before the GFC, after the GFC and post reference rate reform — these are presented in the conjectures below.

[5] For the risk-free component, one may use data of the ONRR and derivative security contracts written thereon.
[6] For the credit spread component, one may make use of data of credit default swaps (CDSs) that reference the respective bank(s) under consideration.

Conjecture 2.1.1 (Term-based reference rates before the GFC)
Based on observations from section 1.2, 1.3 and Axiom 2.1.5, the key local BFRR is conjectured to be

$$J_u^n = x_u^n + f_u^n \ , \tag{2.1.6}$$

where $n \approx 91$, since the 3-month tenor was fundamental, and $d_u^n \approx 0$, i.e., considered to be non-material due to the perceived high credit quality of relevant banking entities that contributed to this suitably aggregated reference rate. While not standardly recorded, the corresponding FTDRR is conjectured to be

$$R_u^n = x_u^n + \ell_u^n \ , \tag{2.1.7}$$

based on the result of Axiom 2.1.4, with $f_u^n \in [0, \ell_u^n]$ in a normal market with limited availability of capital/funding, or $f_u^n \in [\ell_u^n, 0)$ in a market with excess capital/funding.

Therefore, it is conjectured that there was still cognisance of liquidity risks; however, one may have also assumed that the funding-liquidity spread component, ℓ_u^n, was also non-material under normal market conditions[7] resulting in *near risk-free* reference rates and *single-curve frameworks.*

Conjecture 2.1.2 (Term-based reference rates after the GFC)
Based on observations from section 1.2, 1.3 and Axiom 2.1.5, the set of local BFRRs are conjectured to be

$$J_u^n = x_u^n + d_u^n + f_u^n \ , \tag{2.1.8}$$

where $n \in \{1, 30, 91, 182, 273, 365\}$ approximately, which represents the overnight, 1-, 3-, 6-, 9- and 12-month terms. Using Axiom 2.1.5, the expression

$$R_u^n = x_u^n + \ell_u^n + d_u^n \ , \tag{2.1.9}$$

defines the corresponding set of FTDRRs, with $f_u^n \in [0, \ell_u^n]$ or $f_u^n \in [\ell_u^n, 0)$.

Importantly, in this setting, even if funding-liquidity contingent credit risks are negligible, i.e., $f_u^n \approx 0$, the materiality of each BFRR's credit risk exposure means that each term encodes risk that is *non-fungible* or *unhedgeable.* This set of *risky* BFRRs manifests a natural *multi-curve framework.* The emergence of significant liquidity risks also places extra significance on the set of FTDRRs, each of which encode liquidity risk exposure over a specific term. These rates will play a major role in the theory that is developed in Part I of this thesis.

Conjecture 2.1.3 (Overnight reference rates after reference rate reform)
The unsecured overnight BFRR, or ONRR, is

$$J_u^1 = r_u^1 + d_u^1 + f_u^1 \ , \tag{2.1.10}$$

which is equal to the overnight FTDRR

$$R_u^1 = r_u^1 + \ell_u^1 + d_u^1 \ , \tag{2.1.11}$$

since the overnight term represents the shortest possible accrual period, and therefore early liquidation or a secondary market is not possible. This also means that $f_u^1 = \ell_u^1 \in \mathbb{R}$. The secured overnight BFRR is

$$S_u^1 = r_u^1 + \ell_{u,\mathrm{r}}^1 \ , \tag{2.1.12}$$

which follows from Axiom 2.1.2, with $\ell_{u,\mathrm{r}}^1 \in \mathbb{R}$ and $\ell_{u,\mathrm{r}}^1 \neq \ell_u^1$ in general.

[7] In South Africa, the 3-month NCD rates are utilised in the construction of 3-month JIBAR — see SARB-MMR (2021).

An important modelling implication of these conjectures is that the FTDRRs and the unsecured BFRRs are significantly related structurally, while the secured ONRR is fundamentally different. Practically, this may be attributable to the difference between the secured and unsecured market structures, with $\ell^1_{u,\mathrm{r}}$ versus ℓ^1_u capturing this effect theoretically. The same can be said of a theoretically non-defaultable FTD versus a corresponding government bond rate, from Axiom 2.1.3, which will differ due to differences in x^n_u versus $x^n_{u,\mathrm{g}}$ and ℓ^n_u versus $\ell^n_{u,\mathrm{g}}$, respectively.

2.2 The Reduced-Form Approach

While full mathematical rigour will not be adhered to for the purpose of this introductory chapter, some probabilistic structure is required to present the ideas in this section. It is assumed that all stochastic variables are supported by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_u)_{u\geq 0}, \mathbb{P})$ satisfying the *usual conditions*, where $\mathbb{P}$ denotes the real-world probability measure.

Any method whereby a structural market feature is modelled using an abstract mathematical construct that reproduces the desired effect of the feature without any functional relation to any financial economic variables may be ascribed the moniker of being a reduced-form model. All three categories of risk described in section 1.3 may be modelled in a reduced-form way. The classical method for modelling risk-free rates is to use an instantaneous short rate which is continuously compounded; while potential defaults and credit spreads are modelled using abstract stopping times and the probabilities of stopping events not occurring over respective intervals, respectively. To the beset of the author's knowledge, a canonical method for the reduced-form modelling of funding- or market-liquidity risks does not exist. Therefore, a practically intuitive approach is postulated and presented in this section which makes use of counting processes.

All of these reduced-form approaches lead to the creation of *stochastic discount factors* (SDFs) which may be combined to create market rates with composite risk features. *Pricing kernels* (PKs) provide a universal and flexible framework for modelling SDFs, and will therefore be used as the fundamental modelling objects. Before these approaches are introduced within the context of the primitive markets associated with Axioms 2.1.1 to 2.1.5, a general distinction and description of the processes of model estimation and calibration is provided.

2.2.1 Estimation versus Calibration

A general SDF denoted by $(Y_u)_{u\geq 0}$ is assumed to be a suitable $\mathcal{F}_u$-adapted process. Moreover, it is assumed that $\mathcal{F}_u := \mathcal{G}_u \vee \mathcal{L}_u \vee \mathcal{H}_u$, for all $u \geq 0$. Economically, an SDF encodes the potential contemporaneous present value of a future cash flow with the respective discounting rate encoding the composite economic risks faced by a holder with a claim to the cash flow. In a continuous-time setting, the SDF (and implied rate) is determined through the continuous compounding of instantaneous rates and/or spreads, which quantify the set of risk exposures. Each of the constituent σ-algebras are related to the three broad categories of risk defined in section 1.3, and have the following definitions:

- $\mathcal{G}_u$ models information related to all tradable variables, in particular events that determine the realisation of bond prices (and thereby implied interest rates) which in turn requires the realisation of risk-free rates and the probabilities associated with default- and liquidity-related events;

- $\mathcal{L}_u$ models information connected with the realisation of all liquidity-related events, which are assumed to be independent of all events related to $\mathcal{G}_u$ and $\mathcal{H}_u$; and
- $\mathcal{H}_u$ models information pertaining to all default-related events, which are also assumed to be independent of all events related to $\mathcal{G}_u$ and $\mathcal{L}_u$.

This filtration specification is simple and minimal in order to ease this introductory exposition, but also because all modelling in this thesis is considered at a market segment, market or systemic level. Therefore it is deemed less important to consider dependencies between liquidity- and default-related events. This would not be the case if one were constructing idiosyncratic models related to a specific entity, which conforms with observations and conjectures from section 1.3. Nonetheless, from the perspective of the prices of tradable financial instruments, dependencies amongst the probabilities of liquidity- and credit-related events, which effectively define interest rate spreads, and other tradable quantities may be incorporated via variables that are $\mathcal{G}_u$-adapted.

Since $(\mathcal{G}_u)_{u\geq 0}$ is the filtration related to tradable variables, the SDF that is appropriate for the purposes of arbitrage-free pricing and valuation should also be $\mathcal{G}_u$-adapted. A general way to transform the $\mathcal{F}_u$-adapted SDF into an equivalent $\mathcal{G}_u$-adapted process is to define

$$D_u := \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[Y_u \,|\, \mathcal{F}_u\right] \,|\, \mathcal{G}_u\right] = \mathbb{E}^{\mathbb{P}}\left[Y_u \,|\, \mathcal{G}_u\right] \ , \tag{2.2.1}$$

for $u \geq 0$, which occurs naturally in applications, as will be revealed later in this chapter. An alternate, more common, and simple approach is to specify $(D_u)_{u\geq 0}$ directly as a suitable $\mathcal{G}_u$-adapted process. This is generally done so using a positive supermartingale process as the SDF. This is suitable for almost all interest rate applications, unless one requires a model that permits negative interest rates[8]. Directly specifying a $\mathcal{G}_u$-adapted SDF in this way generally focuses on modelling interest rates at a composite level, while an $\mathcal{F}_u$-adapted specification enables a reduced-form "structural" construction of the individual interest rate risk exposures. The former provides the pathway to the top-down market-based approach that will be described later in this chapter, while the latter is synonymous with the classical bottom-up reduced-form approach that will be described first.

Once defined, the SDF may be used to define an arbitrary $n\delta$-term ZCB as follows

$$\widehat{P}_{u,u+n\delta} = \frac{1}{Y_u}\mathbb{E}^{\mathbb{P}}\left[Y_{u+n\delta} \,\middle|\, \mathcal{F}_u\right] \ , \tag{2.2.2}$$

or as follows

$$\widehat{P}_{u,u+n\delta} = \frac{1}{D_u}\mathbb{E}^{\mathbb{P}}\left[D_{u+n\delta} \,\middle|\, \mathcal{G}_u\right] \ , \tag{2.2.3}$$

depending on one's choice of approach and model specification. For the rest of this sub-section on estimation versus calibration, the $\mathcal{G}_u$-adapted SDF will be utilised. Assuming that $(D_u)_{u\geq 0}$ is a Markov supermartingale that is time-homogenous and driven by a suitable k-dimensional continuous semimartingale factor process, then one can expect the $n\delta$-term ZCB to have the functional form

$$\widehat{P}_{u,u+n\delta} = \frac{1}{D_u}\mathbb{E}^{\mathbb{P}}\left[D_{u+n\delta} \,\middle|\, \mathcal{G}_u\right] = f(z_u, p, n\delta) \ , \tag{2.2.4}$$

[8] A strict positive supermartingale, i.e., $\mathbb{E}^{\mathbb{P}}\left[D_v \,|\, \mathcal{G}_u\right] < D_u$ for $u < v$, will be required to ensure that implied interest rates are positive, while a general positive square-integrable process will enable implied interest rates that are non-positive.

where $f : \mathbb{R}^k \times \mathbb{R}^\ell \times \mathbb{R}_{>0} \to (0, 1]$ is a deterministic function, with z_u and p denoting the states, at time u, and the set of parameters associated with the k-dimensional factor process, respectively. This time-homogeneous bond price expression enables statistical estimation using longitudinal time-series data. There are two types of estimation that are relevant and worth distinguishing between.

The government debt, deposit and money markets are all naturally characterised by a term structure of rates. Given a historical set of market rates from one of these markets, say for example:

$$\{A_u^n\,; n \in \{1, 2, \ldots, m\}, u \in \{0, \delta, 2\delta, \ldots, t-\delta, t\}\} \ , \tag{2.2.5}$$

where t is the current time, it will be possible to use this set of data and the following model for each $n\delta$-term market rate, viz., $\widehat{A}_u^n := \frac{1}{n\delta}\left(1/\widehat{P}_{u,u+n\delta} - 1\right) = \frac{1}{n\delta}\left(1/f(z_u, p, n\delta) - 1\right)$, to estimate the latent state variables z_u and the set of parameters p using a viable method. This may be achieved, for example, by suitably linearising the above model and performing a joint state and parameter estimation process using a Kalman filter. For ease of reference, this type of estimation is recorded in the next definition.

Definition 2.2.1 (Term structure-based estimation)
The process of estimating a suitable $\mathbb{P}$-model for an SDF using historical time-series data of the entire term structure, from a specific market, is referred to as term structure-based estimation.

Markets post reference rate reform, which are characterised by ONRRs, will only admit a historical time-series for a single overnight rate, say for example:

$$\left\{A_u^1\,; u \in \{0, \delta, 2\delta, \ldots, t-\delta, t\}\right\} \ , \tag{2.2.6}$$

at the current time t. Again, it will be possible to estimate the model above, but this time one would only require the model for the δ-term rate, i.e., $\widehat{A}_u^1$. As before, this type of estimation is recorded below.

Definition 2.2.2 (Single rate-based estimation)
The process of estimating a suitable $\mathbb{P}$-model for an SDF using historical time-series data of a single interest rate, from a specific market, is referred to as single rate-based estimation.

The type of estimation that is applicable depends on the specific practical context and the modelling objectives. However with both types of estimation, once the estimation process has been completed, one may compute the following estimates for the current term structure:

$$\left\{\widehat{A}_t^1, \widehat{A}_t^2, \ldots, \widehat{A}_t^m\right\} \ , \tag{2.2.7}$$

which is commonly referred to as the endogenously specified initial term structure. This is of course enabled by computing the values of the corresponding set of ZCBs:

$$\left\{\widehat{P}_{t,t+\delta}, \widehat{P}_{t,t+2\delta}, \ldots, \widehat{P}_{t,t+m\delta}\right\} \ . \tag{2.2.8}$$

In a market characterised by a single overnight rate, the set of term rates or ZCBs above may be used to inform the market-making process for term rates that are constructed by compounding the overnight rate over the respective terms under consideration — for e.g., models such as these may contribute to the genesis of a new OIS market that references the overnight rate under consideration. When the single rate has a term longer than overnight, similar implications are possible but this will be covered in more detail in the section that describes the market-based approach.

For a market characterised by a natural term structure, one may compare the model's endogenously estimated term structure to that which is observed at the current time, viz., $\{A_t^1, A_t^2, \ldots, A_t^m\}$. If these two align and there is no market for non-linear derivatives that reference these market rates, then one would have a perfectly specified arbitrage-free model, which may then be using for pricing, valuation and risk management. However, this is a highly improbable outcome, hence the need for calibration.

Within the modelling context just described, calibration to an observable initial term structure is enabled by the specification of a PK, defined here by

$$\pi_u := \Lambda_u D_u \ , \tag{2.2.9}$$

for $u \in [t, t+m\delta]$, where $(\Lambda_u)_{t \le u \le t+m\delta}$ is assumed to be a time-inhomogeneous $\{\mathcal{G}_u, \mathbb{P}\}$-density martingale process which enables a change-of-measure from $\mathbb{P}$ to $\mathbb{A}$ on $(\mathcal{G}_u)_{t \le u \le t+m\delta}$ such that

$$P_{t,t+n\delta} = \frac{1}{\Lambda_t D_t} \mathbb{E}^{\mathbb{P}} \left[\Lambda_{t+n\delta} D_{t+n\delta} \,|\, \mathcal{G}_t \right] = \frac{1}{D_t} \mathbb{E}^{\mathbb{A}} \left[D_{t+n\delta} \,|\, \mathcal{G}_t \right] \ , \tag{2.2.10}$$

is the price of the calibrated $n\delta$-term ZCB, for $n \in \{1, 2, \ldots, m\}$. One can expect this calibrated $n\delta$-term ZCB to have the functional form

$$P_{t,t+n\delta} = g\big(z_t, p, n\delta, \lambda(t, t+n\delta)\big) \ , \tag{2.2.11}$$

where $g : \mathbb{R}^k \times \mathbb{R}^\ell \times \mathbb{R}_{>0} \times \mathbb{R} \to (0, 1]$ is a deterministic function, similar to the function postulated in the case of estimation, and $\lambda : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}$ is another deterministic function which captures the effect of the assumed time-inhomogeneity imparted by the change-of-measure. The estimates for z_t and p are as before, with the time-inhomogeneous function $\lambda(t, t+n\delta)$ assumed to provide sufficient degrees of freedom to ensure that

$$P_{t,t+n\delta} = g\big(z_t, p, n\delta, \lambda(t, t+n\delta)\big) = \frac{1}{1 + n\delta A_t^n} \ , \tag{2.2.12}$$

i.e., ensuring that the model now recovers the observed initial term structure at the current time t. Intuitively, this sequence of modelling makes use of: (i) longitudinal data to estimate states and parameters associated with the general variance or volatility dynamics of the model; and (ii) the cross-sectional observable data to calibrate forward rates (or expectations of the respective market rates) at the current time. For reference purposes, this is recorded in the next definition.

Definition 2.2.3 (Term structure-based calibration)
The process of applying a single rate- or term structure-based estimation, and then calibrating this model to an initial term structure using a time-inhomogeneous change-of-measure (or, equivalently, a PK) is referred to as term structure-based calibration.

The term structure-based calibration procedure is also applicable to linear interest rate derivatives markets, such as the OIS market. However, the existence of an associated non-linear derivatives market allows one to consider the complete calibration of a suitable PK model without any need for statistical estimation. Since the OIS market has become the new proxy for near risk-free term rates, the existence of liquid prices for non-linear derivatives (such as caps, floors and swaptions) that also reference the relevant ONRR allows for one to calibrate all of the parameters associated with a PK using cross-sectional data, and thereby fix a risk-neutral measure. With regard to the example model used thus far, this means that cross-sectional data that is observable at the current time t may be used to calibrate z_t, p and all parameters associated with the function $\lambda(t,u)$, for $u \in [t, t+m\delta]$. This method of calibration enables an alternative to the change-of-measure specification of the PK — one may now specify a time-inhomogeneous SDF directly under the risk-neutral measure that is amenable to complete cross-sectional calibration.

For interest rate derivative markets other than the OIS market, the opportunity for complete cross-sectional calibration is not as straightforward. Take for example the government bond forward and options markets, and consider risk-free valuation only, i.e., ignore the effects of potential valuation adjustments. Here, the problem of "two curves" becomes prevalent, since one requires the relevant OIS term structure for risk-free discounting and the risk-neutral dynamics of the government bond term structure in order to "forecast" bond prices that are arbitrage-free. Intuitively then, within the context of PKs and based on the distinct risk characteristics of the two markets, it is reasonable to conject that two distinct PKs are required.

Assume that π_u^{ois} and π_u^{g} are the PKs for the OIS and government debt markets, for $u \in [t, t+m\delta]$, respectively. If these have been constructed using term structure-based calibration, then one has the opportunity to price and market-make derivatives immediately. Consider a long forward contract with strike price K which expires at $t+i\delta$ and is written on a government bond that maturers at $t+j\delta$, viz., $P^{\text{g}}_{t+i\delta,t+j\delta}$, where $1 \leq i < j \leq m$. Then using standard risk-neutral valuation, the value of the forward at the current time t is

$$V_t := \frac{1}{\pi_t^{\text{ois}}}\mathbb{E}^{\mathbb{P}}\left[\pi^{\text{ois}}_{t+i\delta}\left(P^{\text{g}}_{t+i\delta,t+j\delta} - K\right)\,\middle|\,\mathcal{G}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{\pi^{\text{ois}}_{t+i\delta}}{\pi^{\text{ois}}_t}P^{\text{g}}_{t+i\delta,t+j\delta}\,\middle|\,\mathcal{G}_t\right] - KP^{\text{ois}}_{t,t+i\delta}\,, \tag{2.2.13}$$

which reveals an inconsistency with this approach. The fair bond forward price, from first principles, should be $K^{\text{f}} := P^{\text{g}}_{t,t+j\delta}/P^{\text{ois}}_{t,t+i\delta}$, which means that the first expectation after the second equality in the equation above should resolve to $P^{\text{g}}_{t,t+j\delta}$. But this is not the case, in general, since the process $(\pi^{\text{ois}}_{t+i\delta}P^{\text{g}}_{t+i\delta,t+j\delta})_{i\leq j}$ is not a $\{\mathcal{G}_u, \mathbb{P}\}$-martingale. Or, put differently, the government bond price is not a martingale under the risk-neutral measure. In order to resolve this issue one requires a *convexity adjustment* technique — this general issue is considered in both Parts I and II of this thesis. Nonetheless, notwithstanding the convexity adjustment issue, this approach is only applicable for market-making and price discovery for an illiquid or a newly created derivatives market.

A viable method for an existing derivatives market is to use an OIS market PK, π_u^{ois}, that is suitably calibrated and a government debt market PK, say $\pi_u^{\text{g}} := \Lambda_u^{\text{g}} D_u^{\text{g}}$, which is completely free to specify. Then, one may make use of all of the cross-sectional linear and non-linear government bond-related derivative data to calibrate the free parameters associated with the SDF and the change-of-measure. In this way, the convexity adjustment issue may be bypassed; however, this is achieved at the cost of having a misspecified government debt market PK, i.e., π_u^{g} will not recover the initial term structure in the government debt market, in general. Therefore, this approach yields a PK that is specialised for use in the derivatives market and may not be applicable in the associated primitive market. This process is captured in the next definition.

Definition 2.2.4 (Non-linear derivatives-based calibration)
The process of specifying all of the free parameters associated with a PK or a time-inhomogeneous SDF using only cross-sectional linear and non-linear derivative data is referred to as non-linear derivatives-based calibration.

Another potential complexity in this process is the effect of valuation adjustments in the observable market prices of derivative securities. These valuation adjustments represent another source of risk that is unique to the derivatives market — along with convexity adjustments, this is another idiosyncrasy that hinders consistency with primitive markets. Definitions 2.2.1, 2.2.2, 2.2.3 and 2.2.4 will be used in the sub-sections that follow, which aim to introduce reduced-form modelling approaches that are viable for the main categories of risk.

2.2.2 Near Risk-Free Term Rates before the GFC

The first market that is considered is the pre-GFC bank funding market and the key rate being the $n\delta$-term near risk-free term rate J_u^n, as described in Conjecture 2.1.1. Using a $\mathcal{G}_u$-adapted instantaneous short rate process $(r_u)_{u\geq 0}$[9], the SDF is defined as $D_u = \exp\left[-\int_0^u r_s\,\mathrm{d}s\right]$, and an $n\delta$-term near risk-free zero coupon bond (ZCB) by

$$\widehat{P}^{\mathrm{rf}}_{u,u+n\delta} = \frac{1}{D_u}\mathbb{E}^{\mathbb{P}}\left[D_{u+n\delta}\,\middle|\,\mathcal{G}_u\right] \ , \tag{2.2.14}$$

so that a model for J_u^n may be specified as $\widehat{J}_u^n := \frac{1}{n\delta}\left(1/\widehat{P}^{\mathrm{rf}}_{u,u+n\delta} - 1\right)$. Since cross-sectional data existed for this particular market in the form of fair rates for linear derivatives and corresponding term funding rates, which aligned due to the prevalence of funding-swap duality and cross-sectional term agnosticism, a single term structure of rates (and associated ZCBs) was available at any point in time. This model was therefore amenable to term structure-based estimation.

Accordingly, this model is also amenable to term structure-based calibration. As explained in the previous sub-section, this may be achieved through a PK defined by $\pi_u^{\mathrm{rf}} := \Lambda_u^{\mathrm{rf}} D_u$, where $(\Lambda_u^{\mathrm{rf}})_{t\leq u\leq t+m\delta}$ is a $\{\mathcal{G}_u, \mathbb{P}\}$-density martingale that enables a change-of-measure from $\mathbb{P}$ to $\mathbb{Q}$ on $(\mathcal{G}_u)_{t\leq u\leq t+m\delta}$. Within the context of this market, $\mathbb{Q}$ may be interpreted as the risk-neutral measure and the set of calibrated bonds $\left\{P^{\mathrm{rf}}_{t,t+\delta}, P^{\mathrm{rf}}_{t,t+2\delta}, \ldots, P^{\mathrm{rf}}_{t,t+m\delta}\right\}$ encodes the set of near risk-free term rates observed at time t. In applications where cross-sectional non-linear derivative data is also readily available, one may proceed with non-linear derivatives-based calibration with a SDF specified under $\mathbb{Q}$ directly. This usually requires a time-inhomogeneous model specification which impedes potential paths to statistical estimation. If the $\mathbb{Q}$-model happens to be or have a time-homogeneous representation, one may then essentially reverse the process of building a PK from a SDF, i.e., construct a change-of-measure from $\mathbb{Q}$ to $\mathbb{P}$, to create a model under $\mathbb{P}$ that is amenable for statistical estimation.

2.2.3 Default-Free Term Rates

Local currency denominated FCBs (or nominal bonds) from the government debt market are a natural source of default-free term rates within any economy. Based on Axiom 2.1.3, the two distinct risk

[9] Assumed to be an abstract latent stochastic process that models the key pre-GFC bank funding market term rate, viz., J_u^n.

exposures that requires modelling are the risk-free and the funding- or market-liquidity components. Retaining the approach for risk-free rates from the previous scenario but using a superscript 'g' to distinguish between the markets under consideration, another process $(\Theta_u^{\mathrm{g}})_{u\geq 0}$ is introduced here to capture the liquidity risks. This process may be specified in numerous ways — the next remark offers some practical insights in relation to the financial economic role of the process.

Remark 2.2.1 (Interpretation of the liquidity risk process)
Regardless of the specification, the process is meant to be a reduced-form representation of the marginal cost an issuer may have to pay over and above the risk-free cost to source term finance if they opted for a (continuous) roll-over strategy. From a secondary market perspective, the buyer will not be willing to pay the risk-free value of the bond due to potential market-liquidity risks, the costs of which are captured by this process. The seller, on the other hand will be very willing to accept the risk-free value, but will have to offer the buyer compensation for the transfer of future market-liquidity risks. Therefore, in the secondary market, the set of mid rates will capture the equilibrium values of perceived future market-liquidity risks.

For introductory purposes here, the following practically intuitive specification is proposed:

$$\Theta_u^{\mathrm{g}} := \prod_{i=0}^{M_u^{\mathrm{g}}} \theta_i^{\mathrm{g}} := \theta_{0,M_u}^{\mathrm{g}} = \sum_{i=0}^{\infty} \mathbb{I}_{\{M_u^{\mathrm{g}}=i\}} \theta_{0,i}^{\mathrm{g}} \,, \tag{2.2.15}$$

for $u \geq 0$, where $(M_u^{\mathrm{g}})_{u\geq 0}$ is a Cox process and $\{\theta_i^{\mathrm{g}}\,;\, i \in \mathbb{N}\}$ is a set of *independent and identically distributed* strictly positive real-valued random variables, with $\Theta_0^{\mathrm{g}} = \theta_0^{\mathrm{g}} := 1$ and $\theta_{0,i}^{\mathrm{g}} := \prod_{j=0}^{i} \theta_j^{\mathrm{g}}$. The time of each increment of the counting process is the only indicator of a potential liquidity-related event, while the accompanied realisation of the independent random variable quantifies the cost/benefit[10] thereof. Therefore, the filtration

$$\mathcal{L}_u := \sigma\big(\big\{(M_s^{\mathrm{g}})_{s\leq u}\,,\, \{\theta_i^{\mathrm{g}}\,;\, i \in \{1,2,\ldots,M_u^{\mathrm{g}}\}\}\,\big\}\big)\,, \tag{2.2.16}$$

is solely responsible for modelling the state and cost of liquidity. The stochastic intensity of the Cox process, which governs the probability of an illiquid event, is assumed to be $\mathcal{G}_u$-adapted. Further, the set of random variables $\{\theta_i^{\mathrm{g}}\,;\, i \in \mathbb{N}\}$ are assumed to be independent of all other stochastic variables. The total model filtration is then $\mathcal{D}_u := \mathcal{G}_u \vee \mathcal{L}_u$ at each time u, and the SDF may be defined as the $\mathcal{D}_u$-adapted process $Y_u^{\mathrm{g}} := D_u^{\mathrm{g}} \Theta_u^{\mathrm{g}}$, for all $u \geq 0$. In order to define an $n\delta$-term ZCB at some time u, the following ratio is required:

$$\begin{aligned}\frac{\Theta_{u+n\delta}^{\mathrm{g}}}{\Theta_u^{\mathrm{g}}} &= \mathbb{I}_{\{M_u^{\mathrm{g}}=M_{u+n\delta}^{\mathrm{g}}\}} + \mathbb{I}_{\{M_u^{\mathrm{g}}<M_{u+n\delta}^{\mathrm{g}}\}} \prod_{i=M_u^{\mathrm{g}}+1}^{M_{u+n\delta}^{\mathrm{g}}} \theta_i^{\mathrm{g}} \\ &= \mathbb{I}_{\{\Delta M_{u,n}^{\mathrm{g}}=0\}} + \sum_{i=1}^{\infty} \mathbb{I}_{\{\Delta M_{u,n}^{\mathrm{g}}=i\}} \frac{\theta_{0,M_u^{\mathrm{g}}+i}^{\mathrm{g}}}{\theta_{0,M_u^{\mathrm{g}}}^{\mathrm{g}}}\,,\end{aligned} \tag{2.2.17}$$

[10] If the realised value is less (more) than 1, then this indicates a marginal liquidity cost (benefit). A benefit may arise in a scenario when excessive liquidity is available, which is of course not a feature of any normally functioning market.

where $\Delta M^{\mathrm{g}}_{u,n} := M^{\mathrm{g}}_{u+n\delta} - M^{\mathrm{g}}_u$, then a model for the relevant $n\delta$-term ZCB is given by

$$\begin{aligned}
\widehat{P}^{\mathrm{g}}_{u,u+n\delta} &= \frac{1}{Y^{\mathrm{g}}_u}\mathbb{E}^{\mathbb{P}}\left[Y^{\mathrm{g}}_{u+n\delta}\,\middle|\,\mathcal{D}_u\right] \\
&= \frac{1}{D^{\mathrm{g}}_u\Theta^{\mathrm{g}}_u}\mathbb{E}^{\mathbb{P}}\left[D^{\mathrm{g}}_{u+n\delta}\Theta^{\mathrm{g}}_{u+n\delta}\,\middle|\,\mathcal{D}_u\right] \\
&= \frac{1}{D^{\mathrm{g}}_u}\mathbb{E}^{\mathbb{P}}\left[D^{\mathrm{g}}_{u+n\delta}\mathbb{I}_{\{\Delta M^{\mathrm{g}}_{u,n}=0\}}\,\middle|\,\mathcal{D}_u\right] + \sum_{i=1}^{\infty}\frac{1}{D^{\mathrm{g}}_u\theta^{\mathrm{g}}_{0,M^{\mathrm{g}}_u}}\mathbb{E}^{\mathbb{P}}\left[D^{\mathrm{g}}_{u+n\delta}\mathbb{I}_{\{\Delta M^{\mathrm{g}}_{u,n}=i\}}\theta^{\mathrm{g}}_{0,M^{\mathrm{g}}_u+i}\,\middle|\,\mathcal{D}_u\right] \\
&= \frac{1}{D^{\mathrm{g}}_u}\mathbb{E}^{\mathbb{P}}\left[D^{\mathrm{g}}_{u+n\delta}\mathbb{P}\big[\Delta M^{\mathrm{g}}_{u,n}=0\,\big|\,\mathcal{G}_{u+n\delta}\big]\,\middle|\,\mathcal{G}_u\right] \\
&\quad + \sum_{i=1}^{\infty}\frac{1}{D^{\mathrm{g}}_u}\mathbb{E}^{\mathbb{P}}\left[D^{\mathrm{g}}_{u+n\delta}\mathbb{P}\big[\Delta M^{\mathrm{g}}_{u,n}=i\,\big|\,\mathcal{G}_{u+n\delta}\big]\,\middle|\,\mathcal{G}_u\right]\mathbb{E}^{\mathbb{P}}\left[\frac{\theta^{\mathrm{g}}_{0,M^{\mathrm{g}}_u+i}}{\theta^{\mathrm{g}}_{0,M^{\mathrm{g}}_u}}\right], \qquad (2.2.18)
\end{aligned}$$

where the second equality follows by substituting equation (2.2.17), and the third equality follows due to: (i) $\Delta M^{\mathrm{g}}_{u,n}$ being independent of $\mathcal{L}_u$; (ii) $\frac{\theta^{\mathrm{g}}_{0,M^{\mathrm{g}}_u+i}}{\theta^{\mathrm{g}}_{0,M^{\mathrm{g}}_u}}$ being independent of $\mathcal{L}_u$ and all other variables; and (iii) an application of the tower property of conditional expectations to resolve the probability of increments in the Cox process, since the stochastic intensity thereof is assumed to be $\mathcal{G}_u$-adapted. A model for the $n\delta$-term rate is then $\widehat{G}^n_u := \frac{1}{n\delta}\left(1/\widehat{P}^{\mathrm{g}}_{u,u+n\delta} - 1\right)$, the parameters of which may be statistically estimated using term structure-based estimation.

Defining $\pi^{\mathrm{g}}_u := \Lambda^{\mathrm{g}}_u Y^{\mathrm{g}}_u = \Lambda^{\mathrm{g}}_u D^{\mathrm{g}}_u \Theta^{\mathrm{g}}_u$ as the applicable PK, where $(\Lambda^{\mathrm{g}}_u)_{t\leq u\leq t+m\delta}$ is a $\{\mathcal{G}_u,\mathbb{P}\}$-density martingale that enables a change-of-measure from $\mathbb{P}$ to $\mathbb{G}$ on $(\mathcal{G}_u)_{t\leq u\leq t+m\delta}$, it is possible to define

$$\begin{aligned}
P^{\mathrm{g}}_{t,t+j\delta} &= \frac{1}{\pi^{\mathrm{g}}_t}\mathbb{E}^{\mathbb{P}}\left[\pi^{\mathrm{g}}_{t+j\delta}\,\middle|\,\mathcal{D}_t\right] \\
&= \frac{1}{\Lambda^{\mathrm{g}}_t D^{\mathrm{g}}_t\Theta^{\mathrm{g}}_t}\mathbb{E}^{\mathbb{P}}\left[\Lambda^{\mathrm{g}}_{t+j\delta}D^{\mathrm{g}}_{t+j\delta}\Theta^{\mathrm{g}}_{t+j\delta}\,\middle|\,\mathcal{D}_t\right] \\
&= \frac{1}{D^{\mathrm{g}}_t}\mathbb{E}^{\mathbb{G}}\left[D^{\mathrm{g}}_{t+j\delta}\,\mathbb{G}\big[\Delta M^{\mathrm{g}}_{t,j}=0\,\big|\,\mathcal{G}_{t+j\delta}\big]\,\middle|\,\mathcal{G}_t\right] \\
&\quad + \sum_{i=1}^{\infty}\frac{1}{D^{\mathrm{g}}_t}\mathbb{E}^{\mathbb{G}}\left[D^{\mathrm{g}}_{t+j\delta}\,\mathbb{G}\big[\Delta M^{\mathrm{g}}_{t,j}=i\,\big|\,\mathcal{G}_{t+j\delta}\big]\,\middle|\,\mathcal{G}_t\right]\mathbb{E}^{\mathbb{P}}\left[\frac{\theta^{\mathrm{g}}_{0,M^{\mathrm{g}}_t+i}}{\theta^{\mathrm{g}}_{0,M^{\mathrm{g}}_t}}\right], \qquad (2.2.19)
\end{aligned}$$

which is the ZCB value that recovers the observed $j\delta$-term rate, denoted by G^j_t, for $j \in \{1, 2, \ldots, m\}$, using term structure-based calibration and assuming that these rates are observable at time t. The $\mathbb{G}$ measure may be interpreted as a pricing measure that is appropriate for all government debt market financial instruments. Moreover, this pricing measure only impacts variables that are $\mathcal{G}_u$-adapted, for $u \in [t, t + m\delta]$. As discussed in the sub-section on estimation versus calibration, the availability of government bond derivative data does not enable a simple non-linear derivatives-based calibration opportunity as was the situation with the previous case. In general, one would require a calibrated model for the risk-free rate and thereby the risk-neutral measure[11], and a joint model that recovers the prices of vanilla government bonds. Then, ignoring convexity adjustment issues[12], enough free parameters would have to be available to ensure that bond forward and option prices are recovered.

[11] The process $(D^{\mathrm{g}}_u)_{u\geq 0}$ may serve as the risk-free SDF, but here it is assumed to be estimated/calibrated using data solely from the government debt market.

[12] These will arise since the prices of government bonds will not be martingales under the risk-neutral measure, in general.

2.2.4 Risky Term Rates after the GFC

The local currency denominated market that offers exposure to term rates that are both credit and liquidity risky is the FTD market, with rates characterised in Axiom 2.1.4. These rates, constructed via suitable aggregation, will therefore exhibit credit risk that emanates from a subset of the banking sector[13], one that will change and evolve through time. While individual constituent banks may default, the likelihood of the entire subset defaulting should be fairly benign in comparison. Nonetheless, in order to model the credit risk component it is assumed that the likelihood of a banking sub-sector default is non-negligible. Then, the following filtration

$$\mathcal{H}_u := \sigma\big(\{\tau_s > v\}; v \leq u, s \in \{0, \delta, 2\delta, \dots, \lfloor u/\delta \rfloor\}\big) \ , \tag{2.2.20}$$

for $u \geq 0$, captures the information related to the default of the aforementioned banking sub-sector, where τ_s denotes the default time for the respective banking sub-sector that is constituted at time s. While practically intuitive, the time-inhomogeneity of this approach poses substantial modelling challenges particularly when attempting to specify a model that is viable for estimation using longitudinal time-series data. Therefore, to enable the creation of a viable $\mathbb{P}$-model, a static defaultable single-entity that is representative of the dynamic banking sub-sector throughout all time is assumed to exist. Then the filtration given by equation (2.2.20) reduces to the simpler form:

$$\mathcal{H}_u := \sigma\left(\{\tau > v\}; v \leq u\right) \ , \tag{2.2.21}$$

for $u \geq 0$, where τ is now the single default time associated with the representative entity, and conforms with the canonical approach developed by Bielecki and Rutkowski (2001) and Jamshidian (2004)[14]. Building upon the general approach for default-free term rates, the default event is assumed to be independent of all liquidity-related events. However, factors that determine the probability of default are assumed to be $\mathcal{G}_u$-adapted and may therefore be dependent on factors that drive the risk-free rate dynamics as well as the probability of liquidity-related events. The total model filtration in this scenario is therefore $\mathcal{F}_u := \mathcal{D}_u \vee \mathcal{H}_u$, and the $\mathcal{F}_u$-adapted SDF is $Y_u^{\mathrm{d}} := D_u \Theta_u^{\mathrm{d}} \mathbb{I}_{\{\tau > u\}}$, for $u \geq 0$. Using this setup, the $n\delta$-term ZCB associated with an $n\delta$-term FTD may be then be modelled as

$$\begin{aligned}
\widehat{P}^{\mathrm{d}}_{u,u+n\delta} &= \frac{1}{Y_u^{\mathrm{d}}} \mathbb{E}^{\mathbb{P}}\left[Y^{\mathrm{d}}_{u+n\delta} \,\middle|\, \mathcal{F}_u\right] \\
&= \frac{1}{D_u \Theta_u^{\mathrm{d}} \mathbb{I}_{\{\tau > u\}}} \mathbb{E}^{\mathbb{P}}\left[D_{u+n\delta} \Theta^{\mathrm{d}}_{u+n\delta} \mathbb{I}_{\{\tau > u+n\delta\}} \,\middle|\, \mathcal{F}_u\right] \\
&= \frac{1}{D_u \Theta_u^{\mathrm{d}} \mathbb{P}\left[\tau > u \,|\, \mathcal{G}_u\right]} \mathbb{E}^{\mathbb{P}}\left[D_{u+n\delta} \Theta^{\mathrm{d}}_{u+n\delta} \mathbb{P}\left[\tau > u+n\delta \,|\, \mathcal{G}_{u+n\delta}\right] \,\middle|\, \mathcal{D}_u\right] \ ,
\end{aligned} \tag{2.2.22}$$

where the second equality is an application of the *filtration switching formula* shown by Brigo and Mercurio (2006), which in turn is based on the work of Dellacherie (1972) and Bielecki and Rutkowski (2001); along with an application of the tower property of conditional expectations, recalling the relation $\mathcal{D}_{u+n\delta} = \mathcal{G}_{u+n\delta} \vee \mathcal{L}_{u+n\delta}$, and the fact that

$$\mathbb{P}\left[\tau > u+n\delta \,|\, \mathcal{D}_{u+n\delta}\right] = \mathbb{P}\left[\tau > u+n\delta \,|\, \mathcal{G}_{u+n\delta}\right] \ , \tag{2.2.23}$$

[13] The underlying assumption here is that only banks with a satisfactory level of solvency, or credit worthiness, will contribute to the price discovery process for the rates under consideration.

[14] Issues related to recovery upon default are not considered at all in this thesis, the interested reader may refer to Altman *et al.* (2004) and references therein.

since this default time and probability of default is not dependent on $\mathcal{L}_{u+n\delta}$, by assumption. As with the previous case, it is possible to resolve the expression for the $n\delta$-term ZCB further, which yields

$$\widehat{P}^{\mathrm{d}}_{u,v} = \mathbb{E}^{\mathbb{P}}\left[\frac{D_v\mathbb{P}^{\tau}_v(v)}{D_u\mathbb{P}^{\tau}_u(u)}\mathbb{P}^{M}_v(0)\,\middle|\,\mathcal{G}_u\right] + \sum_{i=1}^{\infty}\mathbb{E}^{\mathbb{P}}\left[\frac{D_v\mathbb{P}^{\tau}_v(v)}{D_u\mathbb{P}^{\tau}_u(u)}\mathbb{P}^{M}_v(i)\,\middle|\,\mathcal{G}_u\right]\mathbb{E}^{\mathbb{P}}\left[\frac{\theta_{0,M^{\mathrm{d}}_u+i}}{\theta_{0,M^{\mathrm{d}}_u}}\right], \tag{2.2.24}$$

where, for the sake of brevity, $v := u+n\delta$, $\mathbb{P}^{\tau}_x(x) := \mathbb{P}\left[\tau > x \,|\, \mathcal{G}_x\right]$ and $\mathbb{P}^{M}_x(i) := \mathbb{P}\left[\Delta M^{\mathrm{d}}_{u,n} = i \,|\, \mathcal{G}_x\right]$. A $\mathbb{P}$-model for R^n_u may then be specified as $\widehat{R}^n_u := \frac{1}{n\delta}\left(1/\widehat{P}^{\mathrm{d}}_{u,u+n\delta} - 1\right)$, the parameters of which may be statistically estimated using either single rate- or term structure-based estimation.

As with the previous scenarios, having access to a set of FTD rates $\{R^1_t, R^2_t, \ldots, R^m_t\}$ at the current time t, it is possible to apply term structure-based calibration. The appropriate PK now takes the form of $\pi^{\mathrm{d}}_u := \Lambda^{\mathrm{d}}_u D_u \Theta^{\mathrm{d}}_u \mathbb{I}_{\{\tau>u\}}$, where $(\Lambda^{\mathrm{d}}_u)_{t\le u\le t+m\delta}$ is a $\{\mathcal{G}_u, \mathbb{P}\}$-density martingale that enables a change-of measure from $\mathbb{P}$ to $\mathbb{D}$ on $(\mathcal{G}_u)_{t\le u\le t+m\delta}$. Then the model for the calibrated $j\delta$-term ZCB is

$$\begin{aligned}
P^{\mathrm{d}}_{t,t+j\delta} &= \frac{1}{\pi^{\mathrm{d}}_t}\mathbb{E}^{\mathbb{P}}\left[\pi^{\mathrm{d}}_{t+j\delta}\,\middle|\,\mathcal{F}_t\right] \\
&= \frac{1}{\Lambda^{\mathrm{d}}_t D_t \Theta^{\mathrm{d}}_t \mathbb{I}_{\{\tau>t\}}}\mathbb{E}^{\mathbb{P}}\left[\Lambda^{\mathrm{d}}_{t+j\delta} D_{t+j\delta}\Theta^{\mathrm{d}}_{t+j\delta}\mathbb{I}_{\{\tau>t+j\delta\}}\,\middle|\,\mathcal{F}_t\right] \\
&= \frac{1}{D_t \Theta^{\mathrm{d}}_t \mathbb{I}_{\{\tau>t\}}}\mathbb{E}^{\mathbb{D}}\left[D_{t+j\delta}\Theta^{\mathrm{d}}_{t+j\delta}\mathbb{I}_{\{\tau>t+j\delta\}}\,\middle|\,\mathcal{F}_t\right] \\
&= \frac{1}{D_t \Theta^{\mathrm{d}}_t \mathbb{D}\left[\tau > t \,|\, \mathcal{G}_t\right]}\mathbb{E}^{\mathbb{D}}\left[D_{t+j\delta}\Theta^{\mathrm{d}}_{t+j\delta}\mathbb{D}\left[\tau > t+j\delta \,|\, \mathcal{G}_{t+j\delta}\right]\,\middle|\,\mathcal{D}_t\right] \\
&= \mathbb{E}^{\mathbb{D}}\left[\frac{D_v\mathbb{D}^{\tau}_v(v)}{D_t\mathbb{D}^{\tau}_t(t)}\mathbb{D}^{M}_v(0)\,\middle|\,\mathcal{G}_t\right] + \sum_{i=1}^{\infty}\mathbb{E}^{\mathbb{D}}\left[\frac{D_v\mathbb{D}^{\tau}_v(v)}{D_t\mathbb{D}^{\tau}_t(t)}\mathbb{D}^{M}_v(i)\,\middle|\,\mathcal{G}_t\right]\mathbb{E}^{\mathbb{P}}\left[\frac{\theta_{0,M^{\mathrm{d}}_t+i}}{\theta_{0,M^{\mathrm{d}}_t}}\right],
\end{aligned} \tag{2.2.25}$$

where, similar to before, $v := t+j\delta$, $\mathbb{D}^{\tau}_x(x) := \mathbb{D}\left[\tau > x \,|\, \mathcal{G}_x\right]$ and $\mathbb{D}^{M}_x(i) := \mathbb{D}\left[\Delta M^{\mathrm{d}}_{t,j} = i \,|\, \mathcal{G}_x\right]$. This derivation follows in the same way as before, except that the second equality follows from changing measure on $(\mathcal{G}_u)_{t\le u\le t+m\delta}$ from $\mathbb{P}$ to $\mathbb{D}$, which does not impact the probabilities and expectations associated with any variables that are $\mathcal{L}_u$-adapted or $\mathcal{H}_u$-adapted.

Recall that the $\{\mathcal{F}_u, \mathbb{P}\}$-model, which is $(Y^{\mathrm{d}}_u)_{u\ge 0}$, is assumed to be time-homogeneous with the density martingale process $(\Lambda^{\mathrm{d}}_u)_{t\le u\le t+m\delta}$ assumed to introduce time-inhomogeneity on $(\mathcal{G}_u)_{t\le u\le t+m\delta}$ to enable calibration to the initial term structure observable at time t. Given the discussion at the beginning of this sub-section regarding the time-inhomogeneity of the relevant banking sub-sector, one may offer an economic interpretation to the change-of-measure. The economic role of the change-of-measure, as it relates to the default time for the representative single-entity, is to adjust the historically estimated probabilities of default to reflect the market-implied probabilities of default for the prevailing banking sub-sector at the current time t. Derivative securities are not written on FTD rates in general, therefore a complete non-linear derivatives-based calibration is not possible.

Having introduced the reduced-form models that may be utilised to model the risk-free, liquidity and credit risk components, Table 2.1 below summarises the form of the PKs that may be applied in each of the primitive markets associated with Axioms 2.1.2 to 2.1.5, post the GFC and reference rate reform.

The following features are worth taking note of:

(i) practically, the dynamics of the risk-free rate component differs amongst some markets[15], which may be attributed to differences in risk-free term rate expectations and the general role thereof in the processes of price discovery across these markets;

(ii) each market is distinct and has its own unique liquidity characteristics, as evidenced by the distinct set of liquidity-related processes;

(iii) the funding-liquidity contingent credit risk exposure in the money market, as described in Remark 2.1.2, is captured by a liquidity-related process but classified as a credit risk;

(iv) the deposit and money markets offer exposure to banking sector credit risk, while the government debt and repo markets are assumed to be default-free — in particular, all repo transactions for all terms are assumed to be sufficiently collateralised; and

(v) the changes-of-measure that enable term structure-based calibration are all assumed to be distinct processes, due to the unique sets and combinations of component risk processes that features within each market.

Market	Risk-Free	Liquidity	Credit	Pricing Kernel	Calibrated Rates & ZCBs
Gov Debt Market	D^{g}_u	Θ^{g}_u		$\pi^{\mathrm{g}}_u := \Lambda^{\mathrm{g}}_u D^{\mathrm{g}}_u \Theta^{\mathrm{g}}_u$	$G^n_u = \frac{1}{n\delta}\left(1/P^{\mathrm{g}}_{u,u+n\delta} - 1\right)$
Deposit Market	D_u	Θ^{d}_u	$\mathbb{I}_{\{\tau>u\}}$	$\pi^{\mathrm{d}}_u := \Lambda^{\mathrm{d}}_u D_u \Theta^{\mathrm{d}}_u \mathbb{I}_{\{\tau>u\}}$	$R^n_u = \frac{1}{n\delta}\left(1/P^{\mathrm{d}}_{u,u+n\delta} - 1\right)$
Money Market	D_u		$\Theta^{\mathrm{m}}_u \mathbb{I}_{\{\tau>u\}}$	$\pi^{\mathrm{m}}_u := \Lambda^{\mathrm{m}}_u D_u \Theta^{\mathrm{m}}_u \mathbb{I}_{\{\tau>u\}}$	$J^n_u = \frac{1}{n\delta}\left(1/P^{\mathrm{m}}_{u,u+n\delta} - 1\right)$
Repo Market	D_u	Θ^{r}_u		$\pi^{\mathrm{r}}_u := \Lambda^{\mathrm{r}}_u D_u \Theta^{\mathrm{r}}_u$	$S^n_u = \frac{1}{n\delta}\left(1/P^{\mathrm{r}}_{u,u+n\delta} - 1\right)$

Tab. 2.1: Primitive capital and funding markets from Axioms 2.1.2 to 2.1.5, their reduced-form pricing kernel models and their associated term rates and ZCBs.

Remark 2.2.2 (The deposit and money markets)
The commercial bank created deposit and money markets, and therefore FTD and BFRRs, are highly related, as described in section 1.3 and Conjecture 2.1.2. Therefore, aggregated credit risk and price discovery for the risk-free component is assumed to be the same in both these markets. Under stable market conditions, one can expect funding-liquidity contingent credit risks to be negligible, i.e., $\Theta^{\mathrm{m}}_u \to 1$*. However, under stressed market conditions, one should expect the levels of BFRRs to tend toward the levels of corresponding FTDRRs, since banks will attempt to incentivise all funders and depositors to hold their positions until maturity. Heuristically, this means that* $\Theta^{\mathrm{m}}_u \to \Theta^{\mathrm{d}}_u$ *and* $\pi^{\mathrm{m}}_u \to \pi^{\mathrm{d}}_u$ *under this scenario.*

Remark 2.2.3 (Arbitrage considerations)
Each of the models defined in Table 2.1 is arbitrage-free by virtue of their PK specifications. Moreover,

[15] The deposit, money and repo markets have the same risk-free component, since they are all assumed to be enabled by the same set of commercial banks. Therefore, this component models the aggregate of their risk-free rate expectations.

individual component risks are not prone to arbitrage across markets, since one may only gain exposure to these risk exposures via composite rates, with components which are not separable. However, one may construct an arbitrage across the models associated with the repo and government debt markets, at least theoretically from the perspective of a market-taker. If the repo market's term structure is lower (higher) than the government debt market's term structure, then one could potentially access cheap (offer expensive) term finance in the repo market and purchase higher (sell lower) yielding bonds for the exact same term in the government debt market, which are in turn used as collateral for the repo market transaction. Practically, this arbitrage is not easily exploited because:

(a) repo transactions are generally very short-term and would therefore only match tenor with government bonds that are close to maturity, which precludes the arbitrage opportunity across the entire government debt market term structure, in general; and

(b) bonds that are used as collateral generally have remaining tenors that are longer than the term of the respective repo transaction.

Both of these issues introduce floating interest rate risk into the potential arbitrage strategy, thereby precluding the cross market arbitrage from a practical perspective.

Table 2.1 summarises the general structure of reduced-form PK models for four important primitive markets post the GFC, with the repo market offering a potential source for risk-free rates. Unfortunately, as mentioned in Remark 2.2.3, repo transactions are generally very short-term which hinders the creation of a sufficiently long term structure for the purposes of risk-neutral valuation of derivative securities. Following Conjectures 1.3.3 and 2.1.3, the source for risk-free or near risk-free term rates are linear derivatives that reference secured or unsecured ONRRs, respectively. These derivative markets are considered in the next sub-section.

2.2.5 Near Risk-Free Term Rates after the GFC

Beginning with Conjecture 2.1.3, the case of the market for linear derivatives that reference a secured ONRR is considered first. A viable model for this market coincides with that proposed for the repo market in Table 2.1, i.e., a $\mathbb{P}$-model for the secured ONRR may be specified as $\widehat{S}^1_u := \frac{1}{\delta}\left(1/\widehat{P}^{\mathrm{r}}_{u,u+\delta} - 1\right)$, where the δ-term ZCB is defined as $\widehat{P}^{\mathrm{r}}_{u,u+\delta} = \frac{1}{D_u\Theta^{\mathrm{r}}_u}\mathbb{E}^{\mathbb{P}}\left[D_{u+\delta}\Theta^{\mathrm{r}}_{u+\delta}\,|\,\mathcal{D}_u\right]$. This model must then be estimated using single rate-based estimation, i.e., using the available time-series of δ-term repo rates $\{S^1_0, S^1_\delta, S^1_{2\delta}, \ldots, S^1_t\}$, in order to infer the values of parameters that will encode the correct dynamics. The floating leg of an $n\delta$-term OIS contract, with unit nominal, that references the secured ONRR may then be defined as

$$\widehat{C}^{\mathrm{sois}}_{u,u+n\delta} := \prod_{i=0}^{n-1}\left(1+\delta\widehat{S}^1_{u+i\delta}\right) = \prod_{i=0}^{n-1}\frac{1}{\widehat{P}^{\mathrm{r}}_{u+i\delta,u+(i+1)\delta}}\,, \tag{2.2.26}$$

and since it can be shown that $\frac{1}{D_u\Theta^{\mathrm{r}}_u}\mathbb{E}^{\mathbb{P}}\left[D_{u+n\delta}\Theta^{\mathrm{r}}_{u+n\delta}\widehat{C}^{\mathrm{sois}}_{u,u+n\delta}\,\middle|\,\mathcal{D}_u\right] = 1$, it follows that

$$\widehat{P}^{\mathrm{r}}_{u,u+n\delta} = \frac{1}{D_u\Theta^{\mathrm{r}}_u}\mathbb{E}^{\mathbb{P}}\left[D_{u+n\delta}\Theta^{\mathrm{r}}_{u+n\delta}\,|\,\mathcal{D}_u\right]\,, \tag{2.2.27}$$

provides a market-making model for the fixed rate associated with the $n\delta$-term OIS contract, which is given by the following expression: $\widehat{K}^{\mathrm{sois}}_{u,u+n\delta} := \frac{1}{n\delta}\left(1/\widehat{P}^{\mathrm{r}}_{u,u+n\delta} - 1\right)$.

Consider an existing OIS market with an initial term structure $\{K^{\text{sois}}_{t,t+\delta}, K^{\text{sois}}_{t,t+2\delta}, \ldots, K^{\text{sois}}_{t,t+m\delta}\}$, where the first rate $K^{\text{sois}}_{t,t+\delta} = S^1_t$, as defined in Axiom 2.1.2 and Conjecture 2.1.3. Then it is possible to implement term structure-based calibration using the following PK:

$$\pi^{\text{sois}}_u := \Lambda^{\text{sois}}_u D_u \Theta^{\text{r}}_u \ , \tag{2.2.28}$$

where the $\{\mathcal{G}_u, \mathbb{P}\}$-density martingale $(\Lambda^{\text{sois}}_u)_{t \leq u \leq t+m\delta}$ enables a change-of-measure from $\mathbb{P}$ to $\mathbb{Q}$ on $(\mathcal{G}_u)_{t \leq u \leq t+m\delta}$, where $\mathbb{Q}$ may now be interpreted as a risk-neutral measure. If one also has access to non-linear derivatives that reference the respective secured ONRR then, as was described in subsection 2.2.1, it is possible to specify a time-inhomogeneous model, say $(D^{\text{sois}}_u \Theta^{\text{sois}}_u)_{t \leq u \leq t+m\delta}$, directly under a risk-neutral measure $\mathbb{Q}$ that may be calibrated completely by using all of the cross-sectional derivative data available at the current time t, i.e., non-linear derivatives-based calibration.

Next, back to Conjecture 2.1.3, the case of linear derivatives that reference an unsecured ONRR is considered. Based on Conjecture 2.1.3, both the δ-term segments of the deposit and money markets should provide viable base models for this derivatives market; however, this is not the case since these models are naturally specified for term rates. In particular, the static single-entity that is assumed to be representative for the dynamic banking sub-sector poses a practical modelling issue within the context of unsecured ONRRs. This issue is described in the next remark.

Remark 2.2.4 (Term versus compounded overnight rates)
A key feature of the new paradigm of ONRRs is the requirement to compound in order to create a term rate. The mechanics of overnight compounding offers the depositor (borrower) the flexibility to choose the entity to deposit with (borrow from) at each overnight roll-over point, over the respective term under consideration. This has an important implication on the appropriate modelling approach.

Using the $\mathbb{P}$-model associated with the money market from Table 2.1, which for the sake of brevity shall be denoted here by $M_u := D_u \Theta^{\text{m}}_u \mathbb{I}_{\{\tau > u\}}$, a roll-over deposit strategy that compounds at the δ-term frequency over the interval $[t, t+m\delta]$ may be defined as

$$\widehat{C}^{\text{m}}_{t,t+m\delta} := \left(\frac{1}{\widehat{P}^{\text{m}}_{t,t+\delta}}\right)\left(\frac{1}{\widehat{P}^{\text{m}}_{t+\delta,t+2\delta}}\right)\cdots\left(\frac{1}{\widehat{P}^{\text{m}}_{t+(m-1)\delta,t+m\delta}}\right) = \prod_{i=0}^{m-1}\left(1+\delta \widehat{J}^{1}_{t+i\delta}\right) \ , \tag{2.2.29}$$

where $\widehat{P}^{\text{m}}_{t+i\delta,t+j\delta} := \frac{1}{M_{t+i\delta}}\mathbb{E}^{\mathbb{P}}\left[M_{t+j\delta} \,\middle|\, \mathcal{F}_{t+i\delta}\right] = 1/(1+\delta \widehat{J}^1_{t+i\delta})$, for $j = i+1$ and $i \in \{0, 1, \ldots, m-1\}$. Then, if one were to define a swap contract with payoff equal to $\left(\widehat{C}^{\text{m}}_{t,t+m\delta} - K\right)$, it is fairly straightforward to show that the fair swap rate is $K = 1/\widehat{P}^{\text{m}}_{t,t+m\delta}$, since $\widehat{P}^{\text{m}}_{t,t+m\delta} := \frac{1}{M_t}\mathbb{E}^{\mathbb{P}}\left[M_{t+m\delta} \,\middle|\, \mathcal{F}_t\right]$ and it can also be shown that $\frac{1}{M_t}\mathbb{E}^{\mathbb{P}}\left[M_{t+m\delta}\widehat{C}^{\text{m}}_{t,t+m\delta} \,\middle|\, \mathcal{F}_t\right] = 1$. Therefore, using this approach to model an OIS market that references unsecured ONRRs will just recover the dynamics of equivalent money market deposit term rates[16]. Fixing an entity to represent the dynamic banking sub-sector therefore encapsulates the incorrect structural dynamics for the modelling of a compounded rate.

[16] A corresponding roll-over loan strategy may not be symmetric, since the idiosyncratic credit risk of the borrower may be substantially different to the representative banking sub-sector. However, if the borrower is assumed to be the representative single-entity, or at least a stable and significant member of the banking sub-sector, then the symmetry may be recovered.

Rather, the following specification $M_u^{(s)} := D_u \Theta_u^{(s)} \mathbb{I}_{\{\tau_s > u\}}$, for $u \geq 0$ and $s \leq u$, is more appropriate where $\tau_s > s$ and $(\Theta_u^{(s)})_{u \geq s}$ is the default time and funding-liquidity contingent credit risk process, respectively, associated with the prevailing banking sub-sector that is constituted at time s. Then the δ-term roll-over deposit strategy[17] *over $[t, t + m\delta]$ becomes*

$$\widehat{C}^{\text{dyn}}_{t,t+m\delta} := \left(\frac{1}{\widehat{P}^{(t)}_{t,t+\delta}} \right) \left(\frac{1}{\widehat{P}^{(t+\delta)}_{t+\delta,t+2\delta}} \right) \cdots \left(\frac{1}{\widehat{P}^{(t+(m-1)\delta)}_{t+(m-1)\delta,t+m\delta}} \right) = \prod_{i=0}^{m-1} \left(1 + \delta \widehat{J}^1_{t+i\delta} \right) , \qquad (2.2.30)$$

where $\widehat{P}^{(t+i\delta)}_{t+i\delta,t+j\delta} := \frac{1}{M^{(t+i\delta)}_{t+i\delta}} \mathbb{E}^{\mathbb{P}} \left[M^{(t+i\delta)}_{t+j\delta} \,\middle|\, \mathcal{F}_{t+i\delta} \right]$, for $j = i + 1$ and $i \in \{0, 1, \ldots, m - 1\}$. As before, a model for the δ-term rate may be derived from the relation $\widehat{P}^{(t+i\delta)}_{t+i\delta,t+j\delta} = 1/(1 + \delta \widehat{J}^1_{t+i\delta})$. The idea here is that at an aggregated market-level, at each roll-over point market participants will choose the set of banking entities that they would prefer to deposit funds with for the next δ-term. Therefore, credit risk exposures are distinct for each δ-term and non-cumulative over the entire $m\delta$-term, in general. While this is instructive and compatible for the OIS market under consideration, it is a highly complex time- and structurally-inhomogeneous approach which requires information about future states of the banking sector, and may therefore not enable any practically viable models.

The next couple of conjectures offers a potential path to the development of a reduced-form model that is practically viable, parsimonious and tractable. This is achieved by making a rational observation and implication in relation to the differences between static term and dynamic compounding strategies.

Conjecture 2.2.1 (Compounding strategies and risk minimisation)
The rationale for market participants to opt for dynamic compounding over comparable static strategies over the same term is to minimise risk. This, in turn, means that the objective would be to reduce the cost of loan strategies, or reduce the yield on corresponding deposit strategies (by symmetry).

Conjecture 2.2.2 (Static term versus dynamic compounding strategies)
Using the time- and structurally-inhomogeneous approach from Remark 2.2.4, a δ-term compounding deposit strategy over $[t, t + m\delta]$ executed in a static fashion with the banking sub-sector as constituted at time t will stochastically yield

$$\widehat{C}^{\text{stat}}_{t,t+m\delta} := \left(\frac{1}{\widehat{P}^{(t)}_{t,t+\delta}} \right) \left(\frac{1}{\widehat{P}^{(t)}_{t+\delta,t+2\delta}} \right) \cdots \left(\frac{1}{\widehat{P}^{(t)}_{t+(m-1)\delta,t+m\delta}} \right) , \qquad (2.2.31)$$

which will, on average, dominate the stochastic capitalisation factor derived from the corresponding dynamic compounding strategy, viz., $\widehat{C}^{\text{dyn}}_{t,t+m\delta}$, based on Conjecture 2.2.1.

The following qualitative rationale supports the statement in Conjecture 2.2.2. Considering Conjecture 2.2.1 and assuming that market participants act rationally in aggregate, it is expected that they will prefer to deposit with the set of banks that exhibit the best risk characteristics at the prevailing time when implementing a compounding strategy. Having the banking sub-sector at time t as a reference point, they have, in aggregate, one of two options at each roll-over point:

(i) if the banking sub-sector from time t exhibits the best risk characteristics, then revert back to the same deposit activity that was undertaken at time t; otherwise

[17] Once again a symmetric roll-over loan strategy may be devised if the borrower is assumed to be a stable and significant member of the banking sub-sector throughout the entire term under consideration.

(ii) if another banking sub-sector demonstrates better risk characteristics than that from time t, then transition deposit activity to the new set of constituent banks.

In the case of (ii), the expectation is that the δ-term composite yield offered by the new banking sub-sector will be less than that offered by the original sub-sector constituted at time t.

Based on Conjecture 2.2.2, a simple way to develop a reduced-form model is to postulate a single-entity that is representative of the dynamic banking sub-sector. Then, the form of the model remains the same, for e.g., $M_u^{\text{dyn}} := D_u \Theta_u^{\text{dyn}} \mathbb{I}_{\{\tau_{\text{dyn}} > u\}}$ is an appropriate $\mathbb{P}$-model, for $u \geq 0$, where τ_{dyn} and $(\Theta_u^{\text{dyn}})_{u \geq 0}$ is the default time and funding-liquidity contingent credit risk process, respectively, associated with the representative dynamic entity. This model may then be statistically estimated using single rate-based estimation, i.e., using the available time-series data for the δ-term unsecured ONRR only, i.e., $\{J_0^1, J_\delta^1, J_{2\delta}^1, \dots, J_t^1\}$. Then, if specified and estimated correctly, one should expect

$$\frac{1}{M_u^{\text{dyn}}} \mathbb{E}^{\mathbb{P}} \left[M_{u+n\delta}^{\text{dyn}} \,|\, \mathcal{F}_u \right] = \widehat{P}_{u,u+n\delta}^{\text{dyn}} \geq \widehat{P}_{u,u+n\delta}^{\text{m}} = \frac{1}{M_u} \mathbb{E}^{\mathbb{P}} \left[M_{u+n\delta} \,|\, \mathcal{F}_u \right] \,, \tag{2.2.32}$$

since the money market model is assumed to be estimated using term structure-based estimation, and will therefore encode the endogenous average term risk associated with the average static banking sub-sector over the arbitrary $n\delta$-term.

Now, as was the case with OIS contracts referencing secured ONRRs, having an estimated $\mathbb{P}$-model using the unsecured δ-term ONRRs enables one to market-make fixed rates associated with $n\delta$-term OIS contracts, which is given here by the following expression: $\widehat{K}_{u,u+n\delta}^{\text{uois}} := \frac{1}{n\delta} \left(1 / \widehat{P}_{u,u+n\delta}^{\text{dyn}} - 1 \right)$. Also as before, the existence of an OIS market with an initial term structure $\{K_{t,t+\delta}^{\text{uois}}, K_{t,t+2\delta}^{\text{uois}}, \dots, K_{t,t+m\delta}^{\text{uois}}\}$, where the first rate $K_{t,t+\delta}^{\text{uois}} = J_t^1$, as defined in Axiom 2.1.5 and Conjecture 2.1.3, allows for term structure-based calibration using

$$\pi_u^{\text{uois}} := \Lambda_u^{\text{uois}} D_u \Theta_u^{\text{dyn}} \mathbb{I}_{\{\tau_{\text{dyn}} > u\}} \,, \tag{2.2.33}$$

where the $\{\mathcal{G}_u, \mathbb{P}\}$-density martingale $(\Lambda_u^{\text{uois}})_{t \leq u \leq t+m\delta}$ enables a change-of-measure from $\mathbb{P}$ to $\mathbb{P}^1$ on $(\mathcal{G}_u)_{t \leq u \leq t+m\delta}$. The measure $\mathbb{P}^1$ may be interpreted as a *proxy for a risk-neutral measure*. Again, as before, the availability of non-linear derivative data allows one to specify a time-inhomogeneous model, say $(D_u^{\text{uois}} \Theta_u^{\text{uois}} \mathbb{I}_{\{\tau_{\text{uois}} > u\}})_{t \leq u \leq t+m\delta}$, directly under a proxy risk-neutral measure $\mathbb{P}^1$ that may be calibrated using non-linear derivatives-based calibration at the current time t.

2.3 The Market-Based Approach

The previous sub-section showed that the reduced-form approach may be adapted to develop models for all markets under consideration, however there is an issue in a risky context when compounding is a key feature. This issue is the reason and motivation for Remark 2.2.4, and Conjectures 2.2.1 and 2.2.2. In turn, the observations made therein are the primary motivation for the market-based approach that is introduced in this sub-section. The defining feature of this approach is the cognisance of the difference across terms due to the effect of compounding. This contrasts with the reduced-form approach where the entire term structure is effectively generated through compounding, i.e., different term rates are agnostic of any of the practical effects of compounding over different frequencies.

Consider an arbitrary liquidity and credit risky market that is characterised by a set of m term rates with the following history:

$$\{A_u^n\,; n \in \{1,2,\dots,m\}, u \in \{0,\delta,2\delta,\dots,t-\delta,t\}\} \ , \tag{2.3.1}$$

and the current time being t. Under the reduced-form approach, an SDF, say $(D_u)_{u\geq 0}$ which is $\mathcal{G}_u$-adapted, may be specified and estimated using term structure-based estimation. This single SDF is therefore assumed to encode the dynamics of m distinct term rates. If one were to consider the estimated models for a δ- and $n\delta$-term rate, then it is straightforward to show that

$$\mathbb{E}^{\mathbb{P}}\left[\prod_{i=0}^{n-1}\left(\frac{1}{1+\delta\widehat{A}^1_{u+i\delta}}\right)\,\Big|\,\mathcal{G}_u\right] = \frac{1}{D_u}\mathbb{E}^{\mathbb{P}}\left[D_{u+n\delta}\,|\,\mathcal{G}_u\right] = \frac{1}{1+n\delta\widehat{A}^n_u} \ , \tag{2.3.2}$$

i.e., the expected value of compounding the δ-term rate model over $[u, u+n\delta]$ is equal to the model for the $n\delta$-term rate that is applicable over the same interval. The relation above is a manifestation of the *expectations hypothesis*, a theory about the term structure that has been widely researched since the introduction of the concept by Macaulay (1938). For an excellent review on the expectations hypothesis, one may refer to Sangvinatsos (2010). A significant contribution to this specific branch of research is beyond the scope of this thesis. Rather, as with Remark 2.2.4 and Conjecture 2.2.2, qualitative reasoning is used to motivate why the expectations hypothesis-inspired relation is egregious within certain modelling contexts, which in turn serves as motivation for the market-based approach — the next conjecture summarises the assertion.

Conjecture 2.3.1 (The expectations hypothesis within a modelling context)
When developing interest rate models, the expectations hypothesis must hold if the market being modelled is risk-free or a primitive market linked to a single entity/issuer.

The case for a *pure* risk-free market is shown in detail in Part I, from first principles. The rationale for a primitive market linked to a single entity follows from the discussion in Remark 2.2.4. When a single entity is responsible for issuing a primitive asset, then it is logical to assert that a strategy that commits to compound an investment at a δ-term frequency over an $n\delta$-term is the same as a commitment to transact in *one-shot* over the same $n\delta$-term. The government debt market is an example of such a primitive market. If the corporate debt, deposit and money markets are segmented by issuer, then each of these segments should also have models that obey the expectations hypothesis. The same applies to the repo market when viewed from the perspective of a single and specific borrower (since the activity of borrowing is economically equivalent to issuing a primitive asset).

When viewed at an aggregate level, the corporate debt, deposit, money and repo markets will offer market participants the flexibility to choose the most favourable entity, or set of entities, at each roll-over point in a compounding strategy. Term rates on the other hand are priced in isolation per primitive asset issuer, at a given time, and then suitably aggregated to create reference rates at that time. Therefore, it is conjectured here that

$$\mathbb{E}^{\mathbb{P}}\left[\prod_{i=0}^{n-1}\left(\frac{1}{1+\delta\widehat{A}^1_{u+i\delta}}\right)\,\Big|\,\mathcal{G}_u\right] \neq \frac{1}{1+n\delta\widehat{A}^n_u} \ , \tag{2.3.3}$$

i.e., the expectations hypothesis should not hold in general at an aggregated or market level. Again, Remark 2.2.4 provides further insight into how the modelling of the $n\delta$-term rate, using a standard reduced-form model, assumes a static entity throughout the entire term. The next definition offers a potential method to achieve the inequality in equation (2.3.3), which is the market-based approach.

Definition 2.3.1 (The market-based approach)
To adapt the reduced-form approach to incorporate the effect of a dynamic entity, enabled by the flexibility of compounding, a set of $\mathcal{G}_u$-adapted SDFs are introduced:

$$\left\{(D^i_u)_{u\geq 0}\,;\, i\in\{1,2,\ldots,m\}\right\}\ , \tag{2.3.4}$$

one for each distinct term. The i-th SDF, $(D^i_u)_{u\geq 0}$, is assumed to encode the dynamics associated with a compounding strategy effected at the $i\delta$-term frequency, and is estimated using single rate-based estimation, i.e., using the set of rates $\{A^i_0, A^i_\delta, \ldots, A^i_{t-\delta}, A^i_t\}$. Then, an estimated $i\delta$-term ZCB may be calculated as follows

$$\widehat{P}^i_{u,u+ji\delta} = \frac{1}{D^i_u}\mathbb{E}^{\mathbb{P}}\left[D^i_{u+ji\delta}\,\middle|\,\mathcal{G}_u\right]\ , \tag{2.3.5}$$

where $j\in\mathbb{N}$, which is actually a ZCB with a tenor equal to $ji\delta$ and defined by compounding at the $i\delta$-term over $[u, u+ji\delta]$. The expectations hypothesis relation now becomes

$$\mathbb{E}^{\mathbb{P}}\left[\prod_{i=0}^{n-1}\left(\frac{1}{1+\delta\widehat{A}^1_{u+i\delta}}\right)\,\middle|\,\mathcal{G}_u\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{D^1_{u+n\delta}}{D^1_u}\,\middle|\,\mathcal{G}_u\right] \neq \mathbb{E}^{\mathbb{P}}\left[\frac{D^n_{u+n\delta}}{D^n_u}\,\middle|\,\mathcal{G}_u\right] = \frac{1}{1+n\delta\widehat{A}^n_u}\ , \tag{2.3.6}$$

and one can generally expect the expression on the left, i.e., $\widehat{P}^1_{u,u+n\delta}$, to be greater than or equal to the expression on the right, i.e., $\widehat{P}^n_{u,u+n\delta}$, based on Conjectures 2.2.1 and 2.2.2.

In summary, equation (2.3.4) provides the defining feature of the market-based approach, i.e., a distinct SDF for each compounding term. A corollary or update to Conjecture 2.3.1 may be that the market-based approach is viable for any interest rate market that is not risk-free or a primitive market created by a single issuer. Therefore, applicable markets would be characterised by a set of risky term rates, in general. In the case of the reduced-form approach, the endogenous specification of the different categories of risk enable the modelling of multiple term rates that have composite risk exposures. Moreover, these rates are structurally linked in a natural and consistent way since they are modelled with a single SDF. Also, the transition from statistical modelling to pricing and valuation applications via a PK enables no arbitrage considerations in an innate manner. This is, of course, not the case with the market-based approach, since each SDF encodes the composite set of risks that drive the dynamics of the rates associated with each respective term. Dependencies between the set of SDFs may be incorporated in order to create dependencies amongst the set of term rates for the purposes of statistical modelling. Since each term rate is risky, the arbitrage-related constraints are not particularly restrictive. Nonetheless, it is not immediately clear how an arbitrage-free system of PKs may be specified and utilised — this problem is solved in Parts I and II.

All of the reduced-form approach estimation results regarding SDFs are still applicable. Single rate-based estimation is now the only logical and viable estimation process per SDF. Given a set of single rate-based estimated SDFs, one may then make use of these for the purpose of price discovery and market-making modulo issues relating to convexity adjustments. The derivation or construction of PKs is not as straightforward as was the case with the reduced-form approach. In particular, the notion of a tradable term structure associated with each term is by no means a standard market construct, in general, therefore term structure-based calibration is not as clearly applicable if at all. Following on from the discussion in section 1.3, the bank funding segment of the money market offers derivatives that reference BFRRs with various terms, at least post the GFC and prior to reference rate reform. This market segment therefore offers the opportunity for term structure-based calibration for the overnight term and non-linear derivative calibration for all terms. For all terms beyond overnight, convexity and valuation adjustments would have to be considered.

In the case of the reduced-form approach, it was possible to provide a description of the process required to transition from an estimated SDF to a PK using term structure-based calibration. Part I analyses this problem for the market-based approach, within the context of the deposit and money markets. These markets are vulnerable to all three categories of risk, that were defined in section 1.3, and it is shown that the replication of FRAs, which are quasi-primitive in nature as defined in Conjecture 1.3.2, enables the exchange of these risks and allows for the creation of an arbitrage-free market-based system of PKs. The resulting framework remains true and cognisant of term-related risks, with discrete rates of differing terms forming the basis of the resultant models. Since a discrete rate setting can be rather restrictive for practical applications, Part II generalises all of the results from Part I, making all of the results available in a continuous (or short) rate setting and extending them to be applicable across markets and economies — this general approach that is developed in Part II is referred to as the xy-formalism.

Part I

Market-Based Multi-Curve Frameworks

Chapter 3

A Systemic & Symmetric Perpective

The main objective for Part I and II is to develop frameworks that are capable of modelling markets associated with bank financing (both capital and funding) before and after the GFC, and after reference rate reform. Conjectures 2.1.1, 2.1.2 and 2.1.3 describe these markets and their associated rates. For Part I, the specific economic background is that of a frontier economy, described in section 1.4, and therefore the aim is to produce market-making models for the money and derivatives markets. Since the deposit and money markets are materially structurally related, based on the observations following Conjecture 2.1.3 and those from section 2.2, these are of primary focus. The results derived for the aforementioned markets are then used to postulate suitable models for the associated derivatives markets. Since the deposit and money markets are fundamentally unsecured, the modelling of secured repo markets are not considered directly in Part I — however, the results obtained offers important intuition to inform the setup of frameworks for modelling secured interest rate markets.

In this chapter a stylised version of the financial system is conjectured which enables the analysis of the deposit market at a systemic level. Further assumptions also enable the consideration of activity within the deposit market in a symmetric way, which greatly simplifies the method of analysis. All of this allows for an intuitive introduction to the notion of *funding-swap duality*, an important feature which linked financing and associated linear derivatives market through no-arbitrage replication considerations before the GFC. Post the GFC, it is argued that such markets now exhibit *funding-swap dislocations*, all of which have manifested due to significant levels of liquidity and credit risks which have contributed to the breakdown of no-arbitrage replication-based relations. Furthermore, the rationale and logic of such replication strategies are further exacerbated by asymmetric activity and idiosyncratic risk considerations. The systemic and symmetric liquidity and credit risky setting constructed here overcomes these issues to once again reveal the practical utility of such replication strategies. In particular, these linear derivative replication strategies reveal quantitative mechanisms for pricing and valuing funding or financing transactions with various liquidity and credit risky characteristics — this is the most significant result from Part I, and is presented in Chapter 5. This result may therefore be interpreted as one method for recovering an economically meaningful version of the *funding-swap duality* relation within a liquidity and credit risky setting.

3.1 A Stylised Financial System

Figure 3.1 provides a depiction of a stylised financial system within a generic economy, along with interactions between the constituent entities at a specific point in time. The key financial services entities are a set of *systemically important banks* (SIBs), which are aggregated and referred to collectively as a SIFI, and a set of *conventional banking entities* (CBEs), viz., Bank i, Bank j and Bank k. The CBEs are representative of banks that contribute materially to the constitution and functioning of the banking sector, but expose counterparties to significant liquidity and credit risks. On the other hand, SIBs are assumed to be fundamental to the constitution and functioning of the banking sector, that still expose counterparties to liquidity risks, but low levels of credit risk in comparison to CBEs. While Figure 3.1 only shows three CBEs, this is meant to be an arbitrary yet dynamic representation of this sub-sector, i.e., through time CBEs may default and new entities may enter this sub-sector. The same may occur for SIBs, however the likelihood of default and new entry into this sub-sector is substantially less.

The idea behind this stylised system is that, at any point in time, the SIFI and the set of CBEs form the banking sector, but the active part of the sector will be the SIFI combined with only a subset of the CBEs — in-line with the discussion in sub-section 2.2.5, the active part will be referred to as the banking sub-sector. This idea is formalised in the next definition.

Definition 3.1.1 (Active banking sub-sector)
The active or prevailing banking sub-sector, as referenced in Remark 2.2.4 and Conjecture 2.2.2, at any point in time will constitute the SIFI along with a subset of CBEs that are deemed to be risk managed satisfactorily, from an enterprise-wide liquidity and solvency standpoint.

Observe from Figure 3.1 that each banking entity is split into components based on activity, and these are a *treasury* (TR) operation and a *sales and trading unit* (ST). The TR is solely responsible for: (i) the sourcing of finance in the form of capital or funding; and (ii) the extension of lending. The ST is responsible for financial engineering and product creation. For each banking entity, the ST is an internal client of the TR and vice versa, therefore these components are assumed to exhibit the same risks as the parent entity. For the SIFI, these components are aggregated across SIBs and are referred to as the *SIB treasuries* (STR) and *SIB sales & trading units* (SST), respectively. To be clear, the STR and SST are collections of SIB TRs and SIB STs, respectively. Interactions amongst the STR and TRs, depicted by the green arrows, define the *interbank cash market*. This market constitutes interbank segments of the deposit, money and repo markets and thereby enable the transfer of surplus funds within the banking system. Each ST will borrow (deposit) funds required for (generated through) product creation, trading and hedging processes with their respective TR — the black arrows reflect these interactions. The SST and STs interact amongst each other, depicted by the blue arrows, to create the *interbank derivatives market* which enables hedging, arbitrage and speculative strategies within the banking system. This market of course includes the interbank interest rate derivatives market as a segment.

Individuals, corporates and government entities constitute *external clients* (ECs) of the banking system. Functionally this group may be categorised further as *depositors* (EDs), *borrowers* (EBs) and *end product users* (EPUs). The cyan coloured interaction between EDs and the set of TRs defines the *non-interbank finance market* which also contributes to activity in the deposit, money and repo markets. The red coloured interaction between the EBs and the TRs defines bilateral loan activity. The inter-

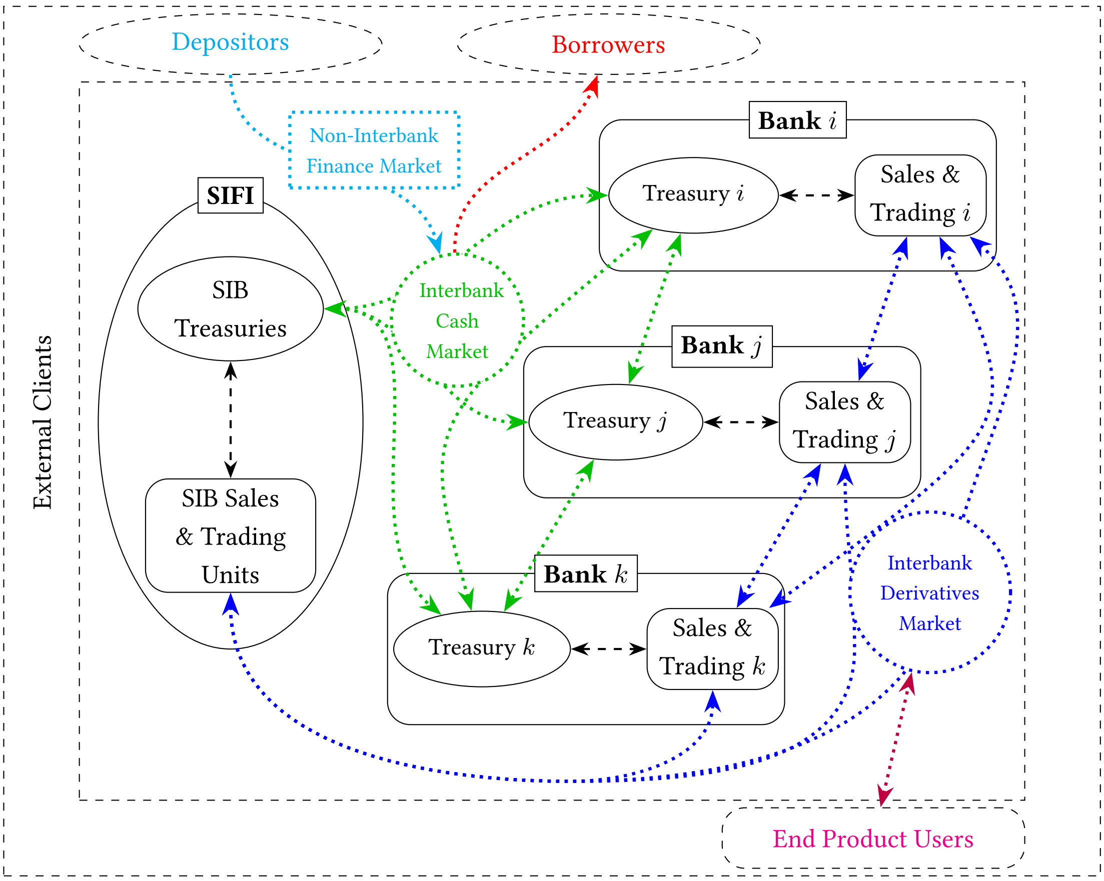

Fig. 3.1: Stylised version of the financial system under consideration.

action between EPUs and the set of STs define the *non-interbank derivatives market*. Of course these categorisations of ECs are not mutually exclusive, in general.

One important entity that is not explicitly depicted in Figure 3.1 is the *central bank* (CB), which is the banker and regulator for all banking activities and entities. Banks may engage in repo transactions or unsecured facilities with the CB in order to settle accounts that are in overdraft, therefore the CB is generally considered as the *lender of last resort*. This facility will be utilised later on to argue against the possibility of a banking system default.

3.2 Adopting a Systemic Perspective

Assuming the perspective of an arbitrary but specific banking entity, both the deposit and money markets are difficult to model and analyse for the following reasons:

(i) *Asymmetric Activity*: For a bank, these markets are a standardised source of finance, or borrowing, only — an activity that is characterised by an assessment of the respective bank's risks. The symmetric and complementary activity of lending is not a part of these markets, and is completely characterised by the risk profile of the borrower.

(ii) *Idiosyncratic Risks*: Since the financing activity is dependent on the specific bank's risk exposures, the relevant rates may be substantially different to the market's reference rates.

Both (i) and (ii) pose various practical issues and complexities, especially if one seeks to model at the market level, from first principles by making use of replication arguments. The perspective of the SIFI, from Figure 3.1, and its components, which are the STR and the SST, offers a viable systemic alternative. The next conjecture summarises key features of the SIFI, and its constituent components, in relation to points (i), (ii) and other risk considerations.

Conjecture 3.2.1 (The SIFI perspective)
The STR will be the dominant market-maker in the non-interbank finance, bilateral lending and interbank cash markets, and in so doing will be exposed to:

— *general floating interest rate risk;*

— *liquidity risks, as defined in sub-section 1.3.4;*

— *credit risks associated with bilateral lending; and*

— *credit risks associated with lending to CBEs.*

Despite these risk exposures, the STR's default risk will be negligible. The STR will also be the only market-maker for the SST, and may choose to either expose or immunise the SST from liquidiy and credit risks.

The risk exposures faced by the STR are self-explanatory given the discussion thus far. Remark 3.2.1 explains why the STR has negligible default risk. The STR playing the role of market-maker for the SST follows naturally from the roles of the individual SIB TRs and SIB STs.

Remark 3.2.1 (Systemic credit spread and default paradox)
SIBs are credit risky, and therefore the rates that are market-made by each SIB's TR for fixed-term deposit and bank funding transactions will include a credit spread component. Since the STR is simply an aggregation of SIB TRs, the corresponding rates for this systemic entity will also reflect a credit spread component, indicating a chance for systemic default. However, in reality, the CB will preclude the possibility of a systemic default; or equivalently, the possibility of a SIFI default. Hence the paradox, which is a natural artefact of the systemic perspective[1]*.*

For the purposes of modelling, the implications of Conjecture 3.2.1 are now the following:

(a) *Symmetric Activity*: The SST will be able to deposit with and borrow from the STR at the same rates (ignoring all market frictions), since the aggregated STR and SST entities have the same risk characteristics as the SIFI, by construction.

(b) *Systemic Risks*: Being a permanent member of the *active banking sub-sector*, as per Definition 3.1.1, and the dominant market-maker, the SIFI will, on average, be a reasonable representative of the banking system as a whole. Therefore, market reference rates will also, on average, be a fair reflection of the SIFI's composite risk exposures.

[1] An alternate way to rationalise and interpret the credit spread component associated with the SIFI is the additional cost that must be borne by the SIFI through distressed borrowing from the CB during adverse economic events. This cost and spread will therefore vary based on relevant market participants' perceived likelihoods of such adverse systemic events.

Therefore, the systemic perspective of the SIFI, and its component entities, resolves the specific complexities posed by the idiosyncratic perspective.

In what follows, this systemic perspective will be utilised to model a frontier economy which first requires a market-taker model for the existing deposit market, and then market-making models for the money, repo and the associated derivatives markets. The role of the STR as a market-maker to the SST is a fundamental one in this process, and is recorded in the next remark.

Remark 3.2.2 (The STR's market-making process for the SST)
As indicated in Conjecture 3.2.1, the STR is exposed to various risks, however it is assumed that the STR may immunise the SST against certain risks. In particular, three cases are considered in this regard:

(i) immunisation from liquidity and credit risks;

(ii) pass-through of liquidity risks but immunisation from credit risks only; and

(iii) pass-through of both liquidity and credit risks.

The first case essentially creates a proxy for a risk-free scenario for the SST, the second a default-free scenario, and the third a full risk scenario[2]*.*

In the next chapter, the market-based approach is developed under all three of the scenarios mentioned in Remark 3.2.2, with the ultimate aim of creating a market-making model for the money market. This is analogous to section 2.2, where the classical reduced-form approach is presented under the same scenarios. In the sections that follow in this chapter, further insights are provided regarding the nature of liquidity and credit risks within the contexts of a frontier economy, which has been described in Definition 1.4.1, and the systemic perspective described in this section.

3.3 Systemic & Symmetric Risks within a Frontier Economy

Firstly, recall from Definition 1.4.1 that a frontier economy is defined as having a satisfactorily liquid deposit market — this market will therefore be used as the basis in the process of developing market-making models for the money and its associated derivatives markets.

Axiom 2.1.4 describes the conjectured risk characteristics of FTDRRs and importantly, these features are contingent upon the depositor foregoing their liquidity over the full term of the respective FTD transaction. Also, there is no secondary market for the deposit market. Conjecture 1.3.5 describes these liquidity risk features in relation to capital assets, with an FTD being an example of such an asset. To emphasise the importance of these features, they are recorded for reference in the next remark.

Remark 3.3.1 (FTDs and liquidity risks)
Defined as a capital asset, issuers of $n\delta$-term FTDs are immunised from funding-liquidity risk over the $n\delta$-term, meaning that the issuer has uninterrupted use of the finance over this term. In return, the depositor earns a funding-liquidity spread, ℓ_u^n, as compensation for foregoing their own liquidity over the $n\delta$-term.

[2] One may also consider the case whereby the STR passes-through credit risks but immunises the SST from liquidity risks. However, this is considered to be an illogical scenario, since credit risks are contingent on the availability of liquidity — put differently, credit risks can only manifest once liquidity prevails for the underlying borrowing and lending transactions.

Practically, this means that if one were to secure, at time t, a 2δ-term FTD over $[t, t+2\delta]$ at the market rate R_t^2, then one does not have the option to liquidate at $t+\delta$, even if there is a market for the δ-term FTD rate, $R_{t+\delta}^1$, prevailing at that time. Therefore, even though one can compute a fair liquidation value at time $t+\delta$, this is a purely theoretical endeavour. If one required this option for liquidation, then one would have to forego or suitably adjust the funding-liquidity spread component, ℓ_t^2, which would fundamentally alter the nature of the deposit transaction.

Equally as important to emphasise are the credit risk features of an FTD, and the time-inhomogeneous nature of FTDRR's liquidity and credit risk exposures. This is done in the next couple of remarks, with insights based on Conjecture 1.3.7, Axiom 2.1.4 and observations from sub-section 2.2.5.

Remark 3.3.2 (FTDs and credit risks)
Being a capital asset, issuers of $n\delta$-term FTDs expose holders thereof, or depositors, to the risk of the issuer's default over the $n\delta$-term. In return, the depositors earn a credit spread component, d_u^n, as compensation for bearing this term-related credit risk.

Remark 3.3.3 (FTDRRs and time-inhomogeneous structural risks)
Inspired by the observations in Remark 2.2.4, the risk characteristics of an FTDRR will be absolutely dependent on the constituency of the active banking sub-sector at the time of calculation. Therefore, if one were to fix a specific time t, then liquidity and credit risk may be thought of as being cumulative over all FTD transaction terms that are spanned from this point in time. However, if one were considering a compounding strategy that makes use of FTDRRs, then the aforementioned risks can no longer be thought of as cumulative — for the same reasons articulated in Remark 2.2.4.

Based on the features highlighted above, the non-fungible nature and distinct integrity of each FTDRR indicates that the market-based modelling approach would be more suitable than the reduced-form approach for the deposit market, at the systemic or market level. Since each FTDRR encodes information regarding the premium that the market offers for funding-liquidity and credit risk, the idea is to exploit this information across the different tradable terms in order to develop a mechanism that enables an exchange of liquidity and credit risks across terms. The market-based approach turns out to be the framework that enables such a mechanism, while the market-making of forward rate agreements that are written on FTD rates turns out to be the modelling problem that must be solved in order to complete the practical specification of the framework. The sections that follow introduce the concept of "*funding-swap duality*", which reveal how the pricing of the aforementioned forward rate agreement is equivalent to the dual problem of pricing a structured financing transaction that offers a fixed rate over an $n\delta$-term but offers liquidity at every δ-term increment. These issues are dealt with in detail in Chapters 4 and 5. For what remains of this section, the focus is on how liquidity and credit risks manifest from and impact interactions between the STR and SST in the systemic and symmetric deposit market.

Following on from Remark 3.2.2, the next three assumptions provide more detail on the STR's market-making process for the SST. Qualitative reasoning supporting these assumptions is provided thereafter.

Assumption 3.3.1 (STR market-making & immunisation of all risk)
In scenario (i) from Remark 3.2.2, the STR will guarantee that the SST will be:

(a) able to deposit and borrow at any of the available $n\delta$-term FTDRRs;

(b) *able to deposit and borrow freely at any point in time;*

(c) *immunised from any counterparty defaults.*

Assumption 3.3.2 (STR market-making & cognisance of liquidity risk)
In scenario (ii) from Remark 3.2.2, the STR will not be able to guarantee that the SST will be able to deposit and borrow freely at any point in time; but when possible the SST will be:

(a) *able to deposit and borrow at the relevant $n\delta$-term FTDRR;*

(c) *immunised from any counterparty defaults related thereto.*

Assumption 3.3.3 (STR market-making & cognisance of liquidity and credit risk)
In scenario (iii) from Remark 3.2.2, the STR will not be able to guarantee that the SST will be able to deposit and borrow freely at any point in time; but when possible the SST will be:

(a) *able to deposit and borrow at the relevant $n\delta$-term FTDRR;*

(d) *exposed to any counterparty defaults related thereto.*

Four properties are highlighted, enumerated (a) to (d), across the three assumptions above, with (d) being the negation of (c). Property (a) follows from the *symmetric activity* feature described in the previous sub-section, and is based on the assumption that the STR will immunise the SST from any basis risk related to market reference or mid rates, and will not transfer any expected market frictions such as transaction costs, taxes and profit margins.

Property (b) relates to systemic funding-liquidity risks as it pertains to the deposit market, which involves deposit activity from the non-interbank finance market, and deposit and lending activity within the interbank cash markets. The case of the SST seeking to borrow over an $n\delta$-term offers an intuitive interpretation. In the scenario where the STR does not have surplus funds to meet such a request, it would have to enable this transaction by attracting a deposit over the same term. This poses two potential funding-liquidity risks for the STR:

1. *total funding-liquidity risk*, i.e., not being able to attract a deposit at all; or
2. *partial funding-liquidity risk*, i.e., not being able to attract the full notional of the deposit required.

A third risk may realise if the STR is not able to attract a deposit for the required full term, which will lead to a potential combination of *floating interest rate risk* and the aforementioned risks.

The case of the SST seeking to deposit over an $n\delta$-term is slightly less intuitive. In the scenario where the STR has no risk appetite or capacity to consume $n\delta$-term finance, it would have to enable this transaction by lending over the same term. The same risks as those described in the previous case may arise if the STR's confidence in the credit worthiness of potential borrowers (CBEs and eligible EBs[3]) has deteriorated at the relevant time of request from the SST. While seemingly distinct and different from the case of the SST seeking finance, these scenarios are indeed *two sides of the same coin*, with the *coin* being the general deterioration of confidence in the banking sector, which generally arises due to

[3] By eligible, it is meant that these EBs have credit characteristics that are similar or better to the active CBEs. This is to ensure that the lending rate offered to the EB is, at least, of the same order of magnitude as the deposit rate offered to the SST.

enterprise-related liquidity and solvency issues. Further, the realisation of such will result in a general: (i) decrease in the demand for saving; and (ii) increase in the demand for borrowing, which are exactly the underlying financial economic conditions that are required in each of the cases, respectively.

Based on the reasoning above, the modelling of systemic liquidity risks in relation to Assumptions 3.3.2 and 3.3.3 is done so in a symmetric manner — this will be achieved through the use of exogenous *systemic liquidity indicators*, which will be introduced in section 4.4 of Chapter 4. The remaining sections in this chapter provide a complementary view to the systemic liquidity risks discussed here, one that demonstrates a duality between illiquidity and liquidation risks. This view is recorded in Remark 3.7.2, and is related to the insights recorded in Remark 1.3.1.

Properties (c) and (d) are an articulation of the default risks related to the activity undertaken by the STR when satisfying the requirements of the SST. In the case of Assumption 3.3.3, any financial impact realising due to default will be passed from the STR onto the SST. Since both the STR and SST are essentially default-free, by construction and as explained in Remark 3.2.1, only the credit worthiness of counterparties are considered, i.e., a unilateral consideration of default risk.

Property (b) has illustrated the types of interactions that may occur between the STR and the SST within the context of the symmetrised deposit market. It is possible for the STR to create credit-related exposures to: (i) CBEs and EDs when sourcing deposits to satisfy the borrowing requests from the SST; and (ii) CBEs and EBs when lending in order to consume finance offered by the SST. Default risks associated with case (ii) are far more intuitive to consider than those associated with case (i). However, in order to maintain symmetry, two simplifying assumptions will be made:

1. *Defaultable CBEs*: To standardise and localise the analysis to the active banking sub-sector, only the set of constituent CBEs will be considered as being defaultable at each point in time.

2. *Symmetric Costs and Benefits*: Loan transactions from case (ii) will realise costs for the SST, while deposit transactions from case (i) will yield benefits to the SST in the event of defaults; and will be equivalent in magnitude if the term and notional of transactions are the same.

While these assumptions greatly simplify the theoretical exposition, the dynamic nature of the constituency of the active banking sub-sector, defined in Definition 3.1.1, offers an added complexity. This systemic credit-related risk will be modelled via exogenously specified *systemic default indicators*, which will be introduced in section 4.5 of Chapter 4.

3.4 Funding-Swap Duality

At the current time t, consider an ED that offers finance to the STR over $[t, t + 2\delta]$ at the δ-term FTDRR[4]. Moreover, the ED would like to fix the δ-term rate that is applicable over $[t + \delta, t + 2\delta]$. In other words, the ED would like to deposit over the 2δ-term but have the option to liquidate after the first δ-term. The STR therefore has one of two financing transaction options:

[4] The choice of δ and 2δ as the terms under consideration is arbitrary. The implications and intuitions that apply for this two-period transaction persists for multiple-period versions, in general.

(i) **Actual Fixed Deposit:** For one unit worth of finance, the STR will pay the ED

$$\left(1+\delta R_t^1\right)\left(1+\delta K_t^{\mathrm{STR}}\right) ,$$

at time $t+2\delta$, with the STR assuming the responsibility of market-making the fixed rate K_t^{STR} at the current time t, which has an accrual period equal to $[t+\delta, t+2\delta]$.

(ii) **Synthetic Fixed Deposit:** For one unit worth of finance, the STR can passively pay

$$\left(1+\delta R_t^1\right)\left(1+\delta R_{t+\delta}^1\right) .$$

If the STR enters a long position in a fair $\delta \times 2\delta$ FRA at time t with nominal equal to $\left(1+\delta R_t^1\right)$, offered by the SST, its net terminal payoff will be

$$-\left(1+\delta R_t^1\right)\left(1+\delta R_{t+\delta}^1\right)+\left(1+\delta R_t^1\right)\left[R_{t+\delta}^1-K_t^{\mathrm{SST}}\right]\delta=-\left(1+\delta R_t^1\right)\left(1+\delta K_t^{\mathrm{SST}}\right) ,$$

at time $t+2\delta$, where K_t^{SST} is the fair $\delta \times 2\delta$ FRA rate, which must be market-made by the SST. This is a negative cash flow for the STR, since it must be paid to the ED.

Options (i) and (ii) have the same payoff apart from K_t^{STR} and K_t^{SST}. However, option (ii) eliminates all floating rate risk for the STR, at the cost of a residual short position in a $\delta \times 2\delta$ FRA for the SST. Therefore, by comparison, this implies that option (i) must hide a residual synthetic short $\delta \times 2\delta$ FRA position for the STR. As is standard in mathematical finance, devising hedging strategies for these residual risks will provide an objective path to pricing the fixed rates. Both options require the replication of a long position in a fair $\delta \times 2\delta$ FRA contract in order to eliminate all of the residual risk.

From the above, it is conjectured that the pricing of such a financing transaction is equivalent to the dual process of pricing a long position in the associated FRA, which may also be thought of as a single-period IRS, hence the moniker "*funding-swap duality*".

Remark 3.4.1 (STR lending)
By similar logic, it is possible to show that the pricing of a similarly structured lending (or offering of finance) transaction is equivalent to the dual process of pricing a short position in the associated FRA. In this scenario, the borrower would like the option to terminate the loan transaction after the δ-term. A lending transaction would expose the STR to the idiosyncratic credit risk of the borrowing entity. Therefore, to preclude such risk one would have to postulate and assume a lending transaction to another systemic banking entity — this could, for example, be an aggregation of the CBEs.

The word '*funding*' in the phrase "*funding-swap*", is slightly misleading here since the financing transaction is based on an FTD, which is a capital and not a funding asset, as described in section 1.3.2. However, this phrase has its genesis in a pre-GFC near risk-free market, when the distinction between fixed-term deposit and bank funding transactions was less material. Also, the term '*funding*' is widely used within the context of banking to generally refer to the process of sourcing finance.

Post the GFC, it is conjectured that the "*funding-swap*" relation dislocates in two ways: (i) as per Conjecture 2.1.2, there are significant differences between FTDRRs and corresponding BFRRs; and (ii) there are significant differences between funding and corresponding linear derivative transactions that reference the same BFRR, as described in Conjecture 3.6.1.

3.5 Fixed-Term Deposit Forward Rate Agreements

The FRA required in this risky setting is not the same as a standardly traded FRA because the reference rate in a standard FRA is a BFRR, such as $J^1_{t+\delta}$, and not an FTDRR, viz., $R^1_{t+\delta}$. This distinction is redundant in a near risk-free setting, based on Conjecture 2.1.1, and the assumption of non-material funding-liquidity risks. However, this risky setting warrants the definition of a fixed-term deposit forward rate agreement (FTDFRA) in order to emphasise the difference between such an instrument and a standardly traded FRA. Apart from the reference rate, the contractual specification of an FTDFRA is the same as that of a standard FRA. For completeness, a $\delta \times 2\delta$ FTDFRA is formally defined below.

Definition 3.5.1 ($\delta \times 2\delta$ Fixed-term deposit forward rate agreement)
The terminal payoff for a $\delta \times 2\delta$ FTDFRA that is initiated at time t is given by

$$V_{t+2\delta} = \alpha N \left(R^1_{t+\delta} - K\right) \delta = \alpha N \left([1 + \delta R^1_{t+\delta}] - [1 + \delta K]\right) \ , \tag{3.5.1}$$

where α equals $1(-1)$ for a long (short) position, N denotes the nominal, and $R^1_{t+\delta}$ is the reference rate.

The discussion in sub-section 1.3.3 offers some motivation as to why the pricing and valuation of standardly traded FRAs are no longer conducive to classical replication strategies. However, given the background context of the frontier economy, the nature of FTDs as described in Remark 3.3.1, and the systemic and symmetric perspective that has been developed in this chapter, replication is the method that is utilised and developed from hereon for the pricing and valuation of FTDFRAs.

In order to eliminate risk in options (i) and (ii) above, a long position in a replicated $\delta \times 2\delta$ FTDFRA, with nominal equal to $(1 + \delta R^1_t)$ is advocated. This is achieved by depositing one unit of currency at the δ-term FTD rate over the interval $[t, t+2\delta]$, which yields $\left(1 + \delta R^1_t\right)\left(1 + \delta R^1_{t+\delta}\right)$, i.e., the floating leg of the long $\delta \times 2\delta$ FTDFRA payoff. Then, the fixed leg of the FTDFRA, which is the source of the finance for the aforementioned deposit, is created as follows:

- In option (ii), the deposit is financed by the SST borrowing one unit at the 2δ-term FTD rate.
- In option (i), the STR has already acquired finance of one unit for the period $[t, t + 2\delta]$.

Considering the risks associated with these transactions in combination with the short FTDFRA exposure[5] (i.e., the actual or implied exposure from options (i) and (ii)) will ultimately enable the pricing of K^{STR}_t and K^{SST}_t, respectively, and the hedging of the residual risk. In principle, the floating leg for both options will be exposed to the same liquidity and credit risks. However, the fixed legs are fundamentally different, since the finance acquired in option (i) by the STR offers the provider with liquidity at the δ-term frequency, while that acquired by the SST in option (ii) does not. This highlights a few subtle yet insightful observations regarding liquidity risks and requirements, and these are:

(1) for the floating legs, both the SST and STR will be concerned with the availability of δ-term FTD liquidity at time $t + \delta$, in order to roll-over at the δ-term frequency;

(2) the STR will be concerned with the possibility of the ED liquidating the fixed leg at time $t + \delta$, which will force the liquidation of their floating leg; and

[5] This is merely a semantic update to the result from the previous section, changing terminology from FRA to FTDFRA.

(3) the SST will be concerned with the possibility of the STR liquidating their long FTDFRA position at time $t + \delta$, which will force the liquidation of both their floating and fixed legs.

As mentioned in section 3.3, observation (1) is not as intuitive as concerns relating to sourcing finance or borrowing. However, observations (2) and (3) reveal how a liquidation event may force the SST and STR to renege on rolling-over their deposits at time $t + \delta$. Take note that (3) will be triggered by the same event as (2), i.e., the ED deciding to liquidate their deposit at time $t + \delta$. This liquidation event is a far more intuitive reason, albeit an idiosyncratic one, for the inability to roll-over a deposit transaction — this feature is recorded in Remark 3.7.2. Observation (3) reveals a critical insight and constraint regarding the SST's fixed leg — the SST cannot borrow at the 2δ-term FTD rate if liquidity is required at time $t + \delta$. Therefore, the SST has to source finance with the same characteristics as that sourced by the STR, which is another manifestation of funding-swap duality. The process for pricing such FTDFRAs and financing transactions (represented by ZCBs) is developed in the next chapter.

It is important to note that the replication approach is theoretically appealing since it is instructive of the fundamental nature of the financing risks at play. This approach will be developed further in Chapter 4, and will be shown to be a novel tool for modelling and analysis.

Remark 3.5.1 (Replicated FTDFRAs and exchanges of term risk)
The funding-swap duality example has revealed the need for fixed-term deposit/loan instruments that offer liquidity at the δ-term. Such instruments do not exist within the frontier economy context, and will therefore have to be market-made. This will be done via replication with available instruments, i.e., FTDs. Recall that replicating a long FTDFRA position, from the perspectives of the SST, required finance acquired over the 2δ-term to be exchanged for a deposit over the same tenor but at the δ-term frequency. Such an exchange of term risk maintains the risk characteristics of the underlying transactions. Within the systemic perspective constructed here, the residual risks in such a strategy will be attributed to:

— *floating interest rate risk, under Assumption 3.3.1;*

— *floating interest rate and liquidity risk, under Assumption 3.3.2; or*

— *floating interest rate, liquidity and credit risk, under Assumption 3.3.3.*

3.6 Funding-Swap Dislocation

This section attempts to describe the dislocation between financing transactions and associated linear derivative securities in general, post the GFC and reference rate reform.

As described in sub-section 1.3.3, the replication approach is not viable for the market-making of a standardly traded FRA due to its punitive use of funding, and the associated liquidity and credit risks. Traded FRAs are *incomplete market* derivative securities with the FRA rate treated as the key pricing variable, along with *collateral and clearing* market microstructure to mitigate *counterparty credit risk*. The FRA rate is practically determined via demand and supply, but may be theoretically justified via classical risk-neutral valuation, in addition to *valuation adjustments* (most notably, the MVA, for centrally cleared traded FRAs which are the most liquid market segment). For more information on valuation adjustments, please refer to sub-section 1.3.4 and Appendices A.2.1 and A.2.2.

Remark 3.6.1 (Traded FRAs and exchanges of floating-for-fixed interest rate risk)
Traded FRAs treat the FRA rate as a tradable variable, and may be liquidated at any time in the secondary market. Therefore, apart from enabling the exchange of floating-for-fixed interest rate risk, none of the other features of related financing transactions are captured. Moreover, this is further exacerbated by the fact that the respective market-maker must also consider idiosyncratic valuation adjustments.

Conjecture 3.6.1 (Funding-swap dislocation post the GFC and reference rate reform)
Based on the discussion leading up to this point, it is conjectured here that the reason for the dislocation between the funding and its associated linear derivatives market is three-fold:

(a) traded FRAs are not directly related to underlying financing transactions;

(b) the market-making process for traded FRAs requires idiosyncratic valuation adjustments thereby exacerbating the issue from (a); and

(c) the transition from TBRRs to ONRRs implies that associated modern linear derivatives only enable the exchange of floating-for-fixed interest rate risk related to the overnight term, and are thereby only applicable to interbank funding transactions[6].

In what follows in Part I, it is shown how funding-swap duality may be recovered in the modern risky market, or post GFC, context. This is shown to be achievable with the exogenous modelling of FTDFRAs within the constructed systemic and symmetric setting, supported by the market-based approach. Moreover, the differences between FTDRRs and BFRRs, as articulated in Conjecture 2.1.2, is shown to be reconcilable using a novel exchange-of-risk mechanism based on a foreign exchange analogy. Some of the key ideas are articulated in the next sub-section.

3.7 Systemic Funding-Swap Duality

If the systemic setting is risk-free, i.e., neither liquidity nor credit risks, the following will apply:

(1) the STR will be indifferent between sourcing finance at the δ- or the 2δ-term FTD rate over $[t, t+2\delta]$, since there are no liquidity risks; and

(2) the STR will be indifferent between FTDFRAs and traded FRAs, since the traded FRA is replicable, at least theoretically, which also means that $K_t^{\mathrm{STR}} = K_t^{\mathrm{SST}}$.

In such a risk-free scenario, one can expect Assumption 3.3.1 to define the market-making relation between the STR and the SST. This is the backdrop for sections 4.2 and 4.3, where the characteristics of the pre-GFC bank funding and associated derivatives market are recovered.

If the systemic setting is risky, the following applies:

(1) the STR will rationally prefer 2δ- over δ-term FTD finance over $[t, t+2\delta]$, since the latter implies that the depositor is only willing to bear risk over the δ-term, or put differently, they require the availability of liquidity at the δ-term frequency over the interval $[t, t+2\delta]$; and

[6] The assumption here, which is commensurate with current market activity, is that the majority of interbank funding will reference overnight rates, in order to limit exposure to credit risk.

(2) the STR will rationally prefer the FTDFRA over the traded FRA, based on the discussion in the previous section. The SST will endeavour to market-make the FTDFRA via replication and compute K_t^{SST} by taking into account the effect of potential systemic liquidity and default risks. Moreover, when the STR is the counterparty to the FTDFRA, the SST is precluded from any counterparty default risk and thereby any need for collateralisation and valuation adjustments.

In such a risky scenario, one can expect either Assumption 3.3.2 or 3.3.3 to define the market-making relation between the STR and the SST. Section 4.4 models the illiquid setting with no credit-related risks, while section 4.5 models the setting with illiquidity and default risks. In all of the scenarios presented in the next chapter, the perspective of the SST is assumed, and its role in the market-making of FTDFRAs is detailed and analysed.

Remark 3.7.1 (The practicality of the SST market-making FTDFRAs)
The purpose of sections 3.4, 3.5, 3.6 and the current one is to provide more practical insight into the seemingly theoretical process of the SST market-making FTDFRAs via replication, within the systemic setting.

The funding-swap duality feature reveals the dual nature of market-making FTDFRAs and related financing transactions between the STR and the SST. Therefore, once a market for either of these instruments has been established, the other is immediately enabled by no-arbitrage considerations. In the chapter that follows, it is assumed that the SST creates the market for FTDFRAs by adopting the vantage point of a market-taker that executes the required replication strategies with the STR. Once this derivative market has been established by the SST, its dual financing market will be automatically enabled by the STR. This new financing market may then be utilised by the SST to market-make new financial instruments.

Remark 3.7.2 (Funding-swap duality $\Rightarrow$ Liquidation-illiquidity duality)
A key quantity in the FTDFRA replication or market-making process is the systemic liquidity indicator, described in Definition 4.4.1. When market-making the $\delta \times 2\delta$ FTDFRA, one specific version of this abstract stylised quantity, viz., $L^1_{t+\delta}$, defines the availability of symmetric[7] systemic liquidity in the δ-term FTD rate at time $t+\delta$ from the SST's perspective.

For the market-making of a long position in a $\delta \times 2\delta$ FTDFRA, the lack of systemic liquidity for a δ-term FTD at time $t+\delta$ may seem counter-intuitive; however, viewed from the perspective of the STR's dual process of market-making the corresponding financing transaction, i.e., the actual fixed deposit, or option (i), this event coincides with the ED choosing to liquidate their funding position at $t+\delta$ and thereby forcing the STR to renege on rolling-over the δ-term FTD at this time. Similar logic may be applied to the market-making process for a short position in a FTDFRA; however, the lack of funds to borrow at the δ-term FTD rate at time $t+\delta$ is an intuitive event in the realisation of systemic illiquidity.

Therefore, an implication of funding-swap duality is liquidation-illiquidity duality.

[7] Symmetric here refers to the SST's ability to deposit and borrow at the δ-term FTD rate at time $t+\delta$.

Chapter 4

Systemic Multi-Curve Frameworks

In this chapter, the deposit market within a frontier economy is modelled using the market-based approach, within the systemic and symmetric perspective developed in the previous chapter. The starting point is a primary market for FTDs, which is enabled by the STR. At a high-level, the main practical objective for this chapter is to develop a theory for the secondary market for FTDs in the presence of systemic and symmetric liquidity and credit risks. The first section presents the mathematical framework that supports the existing primary market for FTDs within the frontier economy.

4.1 Mathematical Framework for the Frontier Economy

It is assumed that this economy is *incomplete*, *arbitrage-free* and supported by a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_u)_{u\geq 0}, \mathbb{P}\right)$ satisfying the *usual conditions*, where $\mathbb{P}$ denotes the real-world probability measure, and where

$$\mathcal{F}_u := \mathcal{G}_u \vee \mathcal{L}_u \vee \mathcal{H}_u, \tag{4.1.1}$$

for $u \geq 0$. This filtration structure mirrors that presented in subsection 2.2.1, where $\mathcal{G}_u$ models information related to all tradable variables, $\mathcal{L}_u$ models information about all liquidity-related events and variables associated thereto, and $\mathcal{H}_u$ models all events related to features of credit risk and default.

The deposit market is assumed to constitute m FTD instruments at any time $u \geq 0$, which enables the creation of the following set of reference rates:

$$\left\{R^n_u \,; n \in \{1, 2, \ldots, m\}\right\}, \tag{4.1.2}$$

which forms the basis for fair valuation at the systemic level. Based on the motivation from section 2.3 and 3.3, this market is suitable for the application of the market-based approach. As described in Definition 2.3.1, the key modelling quantities are a set of SDFs which are defined next.

Definition 4.1.1 (Estimated FTDSDFs and FTDRRs)
The estimated $n\delta$-term FTD-linked SDF (FTDSDF), $(\widehat{D}^n_u)_{u\geq 0}$, is assumed to be a $\{\mathcal{G}_u, \mathbb{P}\}$-continuous semimartingale that is estimated using single rate-based estimation, as per Definition 2.2.2. At any time u, it is possible to calculate an estimate for the $n\delta$-term FTD-linked ZCB (FTDZCB) and associated FTDRR as

$$\widehat{P}^n_{u,u+n\delta} := \frac{1}{\widehat{D}^n_u}\mathbb{E}^{\mathbb{P}}\left[\widehat{D}^n_{u+n\delta} \,\middle|\, \mathcal{G}_u\right] \quad \textit{and} \quad \widehat{R}^n_u := \frac{1}{n\delta}\left(\frac{1}{\widehat{P}^n_{u,u+n\delta}} - 1\right), \tag{4.1.3}$$

respectively, by making use of the estimated $n\delta$-term FTDSDF, for $n \in \{1, 2, \ldots, m\}$.

For a practical and explicit example of an $n\delta$-term FTDSDF, refer to Appendix B.1. Notice that the notation that is used for the estimated SDFs is different to that used in Chapter 2 — the hat notation is used to emphasise the estimated feature, and to distinguish this SDF from a calibrated version that will be introduced in the next section.

Having this basic setup, the objective is to now model the deposit market from the perspective of the SST in a systematic manner, based on Remark 3.2.2 and Assumptions 3.3.1, 3.3.2 and 3.3.3. The following market scenarios are considered:

(i) *a single FTD with no liquidity or credit risk*, i.e., based on Assumption 3.3.1 with the STR only market-making a single FTD;

(ii) *multiple FTDs with no liquidity or credit risk*, i.e., based on Assumption 3.3.1 with the STR market-making all m FTDs;

(iii) *multiple FTDs with liquidity but no credit risk*, i.e., based on Assumption 3.3.2 with the STR market-making all m FTDs; and

(iv) *multiple FTDs with liquidity and credit risk*, i.e., based on Assumption 3.3.3 with the STR market-making all m FTDs.

These scenarios are considered in the next four sections, with the overarching objective in each being the creation of a secondary market for FTDs. For ease of reference, the properties of the primary market for FTDs, from the perspective of the SST, are summarised in the next remark.

Remark 4.1.1 (Primary market features of the deposit market for the SST)
While described in section 3.3, there are two features that are worth highlighting here:

(i) ***Initial liquidity is not guaranteed****: The SST may not be able to satisfy their full lending and borrowing requirements at each $n\delta$-term FTDRR, at any time u, for $n \in \{1, 2, \ldots, m\}$.*

(ii) ***Only terminal liquidation is guaranteed****: Given a successful deposit or borrow transaction at an $n\delta$-term FTDRR at some time u, the SST is guaranteed the ability to liquidate this position at the maturity time $u + n\delta$, for $n \in \{1, 2, \ldots, m\}$.*

Feature (ii) emphasises that liquidation is not possible over $(u, u + n\delta)$. Also, take note that feature (ii) may be impacted by the realisation of default risks over $(u, u + n\delta]$.

4.2 A Single FTD with No Risk

The first market scenario that is considered is one that is based on Assumption 3.3.1, with property (a) adjusted so that the STR only market-makes a single FTD rate, which is assumed to be the δ-term rate[1]. Such a market-making scenario may exist in the genesis phase of a post GFC and reference rate reform market. However, it is unlikely that such a scenario poses no liquidity and credit-related risks. Nonetheless, this is still a useful theoretical scenario to analyse.

[1] Technically, the FTD rates are not tradable. However, the systemic and symmetric setting allows the SST to deposit with and borrow from the STR at these rates, which is analogous to the SST buying and selling corresponding FTD-linked ZCBs, respectively. Therefore, for all practical purposes, one may consider the set of FTDs to be tradable financial instruments.

Now, in accordance with the progression presented in the previous chapter, it is assumed that the SST's objective is to market-make interest rate derivatives that reference the δ-term FTD rate. With an estimated δ-term FTDSDF, from Definition 4.1.1, the SST may now estimate a set of δ-term FTDZCB prices for various tenors by considering the process of compounding the δ-term FTDRR. For ease of reference, these FTD-linked ZCBs will be referred to as FTDZCBs. This procedure is described next.

Definition 4.2.1 (Estimated δ-term FTDZCB prices)
Assuming that the current time is t, then for $i, j \in \mathbb{N}_0$ with $i \leq j$, the expression

$$\widehat{P}^1_{t+i\delta,t+j\delta} := \frac{1}{\widehat{D}^1_{t+i\delta}} \mathbb{E}^{\mathbb{P}} \left[\widehat{D}^1_{t+j\delta} \,\middle|\, \mathcal{G}_{t+i\delta} \right] , \tag{4.2.1}$$

is the estimated price at time $t+i\delta$ for a δ-term FTDZCB, with unit nominal, that matures at time $t+j\delta$, i.e., a ZCB with $(j-i)\delta$-tenor that accrues interest via compounding $(j-i)$ δ-term FTD rates that are implied by the estimated δ-term FTDSDF and information available at time $t+i\delta$.
***Shorthand notation**: $\widehat{P}^1_{t,i,j} := \widehat{P}^1_{t+i\delta,t+j\delta}$ for all $i, j \in \mathbb{N}_0$, with $i \leq j$ and $t \in \mathbb{R}_{\geq 0}$.*

The SST's estimated δ-term FTDZCB term structure at time t is then given by $\{\widehat{P}^1_{t,0,j}\,;\, j \in \mathbb{N}_0\}$. These ZCB prices are completely model-dependent and are therefore not tradable, in general. Rather they may be utilised by the SST in the market-making process for related products. These ZCBs are equivalent to synthetic FTD rates with $j\delta$-tenors, which accrue interest in a compounded fashion at the δ-term frequency. Later it will be shown how such ZCBs may be structured with FRAs or, equivalently, IRSs that reference FTD rates. There is however one ZCB that is directly linked to the δ-term FTDRR, and therefore tradable — this ZCB is introduced next.

Definition 4.2.2 (Tradable δ-term FTDZCB)
Assuming that the current time is t then the δ-term FTDRR, R^1_t, would be tradable by the SST based on Assumption 3.3.1. The price at time t of a tradable δ-term FTDZCB, with unit nominal and δ-tenor, is

$$P^1_{t,t+\delta} := \frac{1}{1+\delta R^1_t} . \tag{4.2.2}$$

Shorthand notation: $P^1_{t,i,1} := P^1_{t+i\delta,t+\delta}$ for each $i \in \{0,1\}$ and $t \in \mathbb{R}_{\geq 0}$.

The SST's estimated price $\widehat{P}^1_{t,0,1}$ for this ZCB will not be equal to $P^1_{t,0,1}$, in general. This discrepancy would expose the SST to potential arbitrage if their estimated model were used for pricing and valuation. Therefore, their estimated model must be adjusted to recover the price of the tradable ZCB. In the next lemma, the *δ-term FTD-linked pricing measure*, denoted here by $\mathbb{D}_1$, the calibrated δ-term FTDSDF, and the FTD-linked pricing kernel (FTDPK) are introduced.

Lemma 4.2.1 (Calibrated δ-term FTDSDF and FTDPK)
At time $t+\delta$, the $\mathcal{G}_t$-measurable FTDSDF associated with the δ-term FTD-linked pricing measure $\mathbb{D}_1$ is given by

$$D^1_{t+\delta} := \frac{1}{\Lambda^1_t \, \widehat{D}^1_t} \mathbb{E}^{\mathbb{P}} \left[\Lambda^1_{t+\delta} \, \widehat{D}^1_{t+\delta} \,\middle|\, \mathcal{G}_t \right] = \frac{1}{\widehat{D}^1_t} \mathbb{E}^{\mathbb{D}_1} \left[\widehat{D}^1_{t+\delta} \,\middle|\, \mathcal{G}_t \right] , \tag{4.2.3}$$

with $D^1_{t+\delta} = P^1_{t,0,1}$. The $\{\mathcal{G}_u, \mathbb{P}\}$-density martingale $(\Lambda^1_u)_{t \leq u \leq t+\delta}$, with $\Lambda^1_t := 1$, induces a measure change $\mathbb{P} \to \mathbb{D}_1$, on $(\mathcal{G}_u)_{t \leq u \leq t+\delta}$, via the Radon-Nikodym derivative $\Lambda^1_{t+\delta}/\Lambda^1_t := (\mathrm{d}\mathbb{D}_1/\mathrm{d}\mathbb{P})|_{\mathcal{G}_{t+\delta}}$.
For $j \in \{0,1\}$, the $\mathcal{G}_{t+j\delta}$-measurable δ-term FTDPK is thus given by

$$\pi^1_{t+j\delta} := \Lambda^1_{t+j\delta} D^1_{t+j\delta} , \tag{4.2.4}$$

where $D^1_t := 1$, and where the FTDPK is calibrated to the tradable δ-term FTDZCB.

Proof. The estimated δ-term FTDSDF $\{\widehat{D}^1_{t+j\delta}\,;j\in\{0,1\}\}$ is considered as an initial candidate for the calibrated δ-term FTDSDF, which must be defined in discrete-time on the set $\{t,t+\delta\}$. Since $\widehat{D}^1_t\,\widehat{P}^1_{t,0,1} = \mathbb{E}^{\mathbb{P}}[\widehat{D}^1_{t+\delta}\,|\,\mathcal{G}_t]$ and $\widehat{P}^1_{t,0,1}\neq P^1_{t,0,1}$ in general, the estimated δ-term FTDSDF and $\mathbb{P}$ are not viable candidates for the calibrated FTDSDF and pricing measure, respectively. Constructing and calibrating the change-of-measure $\{\mathcal{G}_u,\mathbb{P}\}$-martingale $(\Lambda^1_u)_{t\le u\le t+\delta}$ such that equation (4.2.3) holds, with $D^1_{t+\delta} = P^1_{t,0,1}$, yields the correct calibrated FTDSDF specification. For $i\in\{0,1\}$, the δ-term FTDPK specification, equation (4.2.4), follows trivially, from where it may be verified that $\frac{1}{\pi^1_{t+i\delta}}\mathbb{E}^{\mathbb{P}}\left[\pi^1_{t+\delta}\,\middle|\,\mathcal{G}_{t+i\delta}\right] = \frac{1}{D^1_{t+i\delta}}\mathbb{E}^{\mathbb{D}_1}\left[D^1_{t+\delta}\,\middle|\,\mathcal{G}_{t+i\delta}\right] = P^1_{t,i,1}$, which concludes the proof. □

Observe that Lemma 4.2.1 invokes and exploits a version of term structure-based calibration, the concept that was first introduced in Definition 2.2.3. In addition, while the estimated δ-term FTDSDF is defined in continuous-time over the interval $[t,t+\delta]$, and thereby a reduced-form way, the calibrated δ-term FTDSDF and FTDPK are defined in discrete-time on the set $\{t,t+\delta\}$ only, thereby reinforcing the idea of a market-based approach. For a more practical perspective on the calibrated δ-term FTDSDF, refer to Appendix B.2 for a specific example.

Remark 4.2.1 (Single-period arbitrage-free model)
The availability of a tradable δ-term FTD rate only, enables the SST to construct a single-period arbitrage-free model only, over $[t,t+\delta]$, with volatility estimated statistically via single rate-based estimation, since no derivatives market exist for non-linear derivatives-based calibration.

Remark 4.2.2 (Market price of δ-term FTD-linked risk)
Under the real-world measure $\mathbb{P}$ and with respect to the traded information filtration $(\mathcal{G}_u)_{t\le u\le t+\delta}$, the martingale $\{\Lambda^1_{t+j\delta}\,;j\in\{0,1\}\}$ adjusts the real-world estimated δ-term FTDZCB price to the arbitrage-free tradable price, and therefore encodes the market price of δ-term FTD-linked risk over $[t,t+\delta]$.

Remark 4.2.3 (The SST's market-making opportunities)
In this scenario, the SST cannot contribute any further to the FTD market created by the STR, i.e., there is no scope for a secondary market when the δ-term, or overnight, FTD is the only traded instrument.
If afforded the required risk appetite, the SST may make use of the estimated δ-term FTDZCBs to create their own primary and secondary markets in synthetic FTDs.

4.3 Multiple FTDs with No Risk

While the first market scenario offered the opportunity to introduce various fundamental results and concepts, the availability of a single δ-term, or overnight, FTD does not allow the scope for the creation of a secondary market. This second market scenario is an extension of the first, where property (a) of Assumption 3.3.1 is now reinstated in full. First though, the effect of introducing the 2δ-term FTD into the market setup from the previous section is analysed.

Using the estimated 2δ-term FTDSDF, $(\widehat{D}^2_u)_{u\ge 0}$, the SST may estimate the 2δ-term FTDZCB-system $\{\widehat{P}^2_{t+2i\delta,t+2j\delta}\,;i,j\in\mathbb{N}_0\,,i\le j\}$ at the current time t by directly replicating all of the results from the previous section. The shorthand notation $\widehat{P}^2_{t,2i,2j} := \widehat{P}^2_{t+2i\delta,t+2j\delta}$ is adopted, for all $i,j\in\mathbb{N}_0$, where

$i \leq j$ and $t \in \mathbb{R}_{\geq 0}$. Using an analog of Lemma 4.2.1, the $\mathcal{G}_{t+2j\delta}$-measurable 2δ-term FTDPK:

$$\pi^2_{t+2j\delta} := \Lambda^2_{t+2j\delta} D^2_{t+2j\delta} \ , \tag{4.3.1}$$

may be obtained, where $j \in \{0,1\}$. The $\{\mathcal{G}_u, \mathbb{P}\}$-martingale $\left(\Lambda^2_u\right)_{t\leq u\leq t+2\delta}$, with $\Lambda^2_t := 1$, enables the change-of-measure from $\mathbb{P} \to \mathbb{D}_2$, the 2δ-term FTD-linked pricing measure, on $(\mathcal{G}_u)_{t\leq u\leq t+2\delta}$, via the Radon-Nikodym derivative $\Lambda^2_{t+2\delta}/\Lambda^2_t := (\mathrm{d}\mathbb{D}_2/\mathrm{d}\mathbb{P})|_{\mathcal{G}_{t+2\delta}}$. This FTDPK is calibrated to

$$P^2_{t,t+2\delta} := \frac{1}{1+2\delta R^2_t} = \frac{1}{\pi^2_t}\mathbb{E}^{\mathbb{P}}\left[\pi^2_{t+2\delta} \,\middle|\, \mathcal{G}_t\right] \ , \tag{4.3.2}$$

which is the price of the tradable 2δ-term FTDZCB at the current time t. Similar shorthand notation is employed as before, viz., $P^2_{t,2i,2} := P^2_{t+2i\delta,t+2\delta}$ for each $i \in \{0,1\}$ and $t \in \mathbb{R}_{\geq 0}$.

In general, the estimated δ- and 2δ-term FTDSDFs are not equal, in an almost sure sense. Even if they are specified as the same model, their single rate-based estimation processes rely upon the historical time series of two non-homogeneous FTDRRs. Each of these FTDSDFs encode different sources of floating composite interest rate risk. Accordingly, the estimated prices of δ- and 2δ-term FTDZCBs with tenor equal to 2δ will also not be equal in general, i.e.,

$$\widehat{P}^2_{t,0,2} = \frac{1}{\widehat{D}^2_t}\mathbb{E}^{\mathbb{P}}\left[\widehat{D}^2_{t+2\delta} \,\middle|\, \mathcal{G}_t\right] \neq \frac{1}{\widehat{D}^1_t}\mathbb{E}^{\mathbb{P}}\left[\widehat{D}^1_{t+2\delta} \,\middle|\, \mathcal{G}_t\right] = \widehat{P}^1_{t,0,2} \ . \tag{4.3.3}$$

However, there are two relations between the tradable δ- and 2δ-term FTDZCBs that must hold to preclude arbitrage from the perspective of the SST. These relations, described in the lemmas below, allow the definition of the δ-term FTDPK to be extended from $t+\delta$ to $t+2\delta$.

Lemma 4.3.1 (Early liquidation enforced by replacement)
At time $t+\delta$, the fair early liquidation value of the tradable 2δ-term FTDZCB, issued at time t, is equal to

$$P^1_{t,1,2} := \frac{1}{1+\delta R^1_{t+\delta}} = \frac{1}{\pi^1_{t+\delta}}\mathbb{E}^{\mathbb{P}}\left[\pi^1_{t+2\delta} \,\middle|\, \mathcal{G}_{t+\delta}\right] = \frac{1}{D^1_{t+\delta}}\mathbb{E}^{\mathbb{D}_1}\left[D^1_{t+2\delta} \,\middle|\, \mathcal{G}_{t+\delta}\right] \ , \tag{4.3.4}$$

which is the initial value of the tradable δ-term FTDZCB. Moreover, the calibrated δ-term FTDSDF, defined in Lemma 4.2.1, may be specified at time $t+2\delta$ as

$$D^1_{t+2\delta} := \frac{D^1_{t+\delta}}{\Lambda^1_{t+\delta}\widehat{D}^1_{t+\delta}}\mathbb{E}^{\mathbb{P}}\left[\Lambda^1_{t+2\delta}\,\widehat{D}^1_{t+2\delta} \,\middle|\, \mathcal{G}_{t+\delta}\right] = \frac{D^1_{t+\delta}}{\widehat{D}^1_{t+\delta}}\mathbb{E}^{\mathbb{D}_1}\left[\widehat{D}^1_{t+2\delta} \,\middle|\, \mathcal{G}_{t+\delta}\right] \ , \tag{4.3.5}$$

with the definition of the $\{\mathcal{G}_u, \mathbb{P}\}$-martingale $\left(\Lambda^1_u\right)_{t\leq u\leq t+2\delta}$ extended to $t+2\delta$ with time-inhomogeneous parameters, such that $\Lambda^1_{t+2\delta}/\Lambda^1_{t+\delta} = (\mathrm{d}\mathbb{D}_1/\mathrm{d}\mathbb{P})\,|_{\mathcal{G}_{t+2\delta}}$ and $D^1_{t+2\delta} = D^1_{t+\delta}P^1_{t,1,2}$.

Proof. While the STR will not allow for the early liquidation of a deposit/loan transaction based on a 2δ-term FTD rate, Assumption 3.3.1 allows the SST to enable early liquidation via replacement. The SST, as a market-taker, could easily terminate (redeem) the loan (deposit) at time $t+\delta$, by taking an opposite position using the tradable δ-term FTDRR. Therefore, to preclude arbitrage, equation (4.3.4) must be the fair liquidation value of the tradable 2δ-term FTDZCB at time $t+\delta$. At time $t+2\delta$, the calibrated δ-term FTDSDF must have the representation

$$D^1_{t+2\delta} := \frac{1}{\left(1+\delta R^1_t\right)}\frac{1}{\left(1+\delta R^1_{t+\delta}\right)} \ , \tag{4.3.6}$$

to preclude arbitrage. Since R^1_u is $\mathcal{G}_u$-measurable respectively, it should be clear that $D^1_{t+2\delta}$ is $\mathcal{G}_{t+\delta}$-measurable. From equation (4.3.4), it then follows that $D^1_{t+2\delta} = D^1_{t+\delta}P^1_{t,1,2}$. Finally, equation (4.3.5) follows in a similar manner to Lemma 4.2.1 with the assumed free time-dependent parameter associated with $(\Lambda^1_u)_{t+\delta<u\leq t+2\delta}$ enabling the calibration, which concludes the proof. □

The previous lemma enabled the specification and calibration of the δ-term FTDPK up to time $t+2\delta$, from the vantage point of time $t+\delta$. The next result will provide information that will enable calibration up to the same time, but from the vantage point of time t.

Lemma 4.3.2 (Synthetic δ-term FTDZCB with 2δ-tenor)
At the current time t, the tradable δ- and 2δ-term FTDZCBs allow the SST to market-make or create a fair FTDFRA, i.e., at zero cost, with payoff

$$V_{t+2\delta} = \alpha N \left[\left(1 + \delta R^1_{t+\delta}\right) - \frac{P^1_{t,0,1}}{P^2_{t,0,2}} \right] , \tag{4.3.7}$$

at time $t+2\delta$, where α is equal to 1 (-1) for a long (short) position and N denotes the nominal amount. This in turn enables the creation of $P^1_{t,t+2\delta}$, a tradable synthetic δ-term FTDZCB at time t with maturity time equal to $t+2\delta$, with $P^1_{t,t+2\delta} = P^2_{t,0,2}$.
***Shorthand notation**: $P^1_{t,i,2} := P^1_{t+i\delta,t+2\delta}$ for $i \in \{0,1,2\}$ and $t \in \mathbb{R}_{\geq 0}$.*

Proof. The long (short) FTDFRA payoff, equation (4.3.7), may be replicated, at zero cost (i.e., $V_t = 0$), by borrowing (depositing) $NP^1_{t,0,1}$ at the 2δ-term FTDRR and simultaneously depositing (borrowing) $NP^1_{t,0,1}$ at the δ-term FTDRR at time t, and depositing (borrowing) the proceeds thereof, which is the nominal amount N, again at the δ-term FTDRR at time $t+\delta$.[2] Using Lemma 4.3.1, it is straightforward to show that the fair value of the FTDFRA at time $t+\delta$ is

$$V_{t+\delta} = \alpha N \left[1 - \frac{P^1_{t,0,1}}{P^2_{t,0,2}} P^1_{t,1,2} \right] . \tag{4.3.8}$$

Borrowing (Depositing) $P^2_{t,0,2}$ units of currency at the δ-term FTDRR at time t and refinancing (redepositing) the total cost (proceeds) thereof at time $t+\delta$ would cumulatively cost (yield) $P^2_{t,0,2}/D^1_{t+2\delta}$ at time $t+2\delta$. Combining this position with a long (short) FTDFRA, with $N = P^2_{t,0,2}/D^1_{t+\delta} = P^2_{t,0,2}/P^1_{t,0,1}$, would result in a net cost (yield) of 1 unit of currency at time $t+2\delta$, and a net cost (yield) of $P^1_{t,1,2}$ at time $t+\delta$, using equation (4.3.8). The combined strategy therefore has a value of $P^2_{t,0,2}$ at time t and replicates the value of the 2δ-term FTDZCB at times $t+\delta$ and $t+2\delta$. Having the δ-term FTDSDF as the key building block, this strategy creates a synthetic δ-term FTDZCB, $\{P^1_{t,i,2}\,; i \in \{0,1,2\}\}$, at time t with maturity time $t+2\delta$, such that $P^1_{t,0,2} = P^2_{t,0,2}$, $P^1_{t,2,2} = P^2_{t,2,2} = 1$ and $P^1_{t,1,2}$ is the interim fair value at time $t+\delta$, which concludes the proof. □

With the result of Lemma 4.3.2 at hand, it is now possible to consider the calibration of the δ-term FTDPK up to time $t+2\delta$, using market information that is available at time t.

Theorem 4.3.1 (Initial calibration of the δ-term FTDPK to $t+2\delta$)
At the current time t, the tradable synthetic δ-term FTDZCB with 2δ-tenor may be modelled as

$$P^1_{t,0,2} := D^1_{t+\delta}\mathbb{E}^{\mathbb{D}_1}\left[\mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda^1_{t+2\delta}\widehat{D}^1_{t+2\delta}}{\Lambda^1_{t+\delta}\widehat{D}^1_{t+\delta}} \,\middle|\, \mathcal{G}_{t+\delta} \right] \,\middle|\, \mathcal{G}_t \right] = D^1_{t+\delta}\,\mathbb{E}^{\mathbb{D}_1}\left[\frac{\widehat{D}^1_{t+2\delta}}{\widehat{D}^1_{t+\delta}} \,\middle|\, \mathcal{G}_t \right] , \tag{4.3.9}$$

[2] Depositing (borrowing) at one of the FTDRRs is equivalent to buying (selling) the associated FTDZCB.

where the time-inhomogeneous $\{\mathcal{G}_u, \mathbb{P}\}$-martingale $(\Lambda_u^1)_{t \le u \le t+2\delta}$ induces the change-of-measure from $\mathbb{P} \to \mathbb{D}_1$ on $(\mathcal{G}_u)_{t \le u \le t+2\delta}$, via the Radon-Nikodym derivative $\Lambda_{t+2\delta}^1 / \Lambda_t^1 = (\mathrm{d}\mathbb{D}_1/\mathrm{d}\mathbb{P})\big|_{\mathcal{G}_{t+2\delta}}$, and the parameters associated with $(\Lambda_u^1)_{t+\delta < u \le t+2\delta}$ is calibrated such that $\mathbb{E}^{\mathbb{D}_1}[\widehat{D}_{t+2\delta}^1 / \widehat{D}_{t+\delta}^1 \,|\, \mathcal{G}_t] = P_{t,0,2}^2 / P_{t,0,1}^1$.

Proof. The definitions of the calibrated δ-term FTDSDF and FTDPK are specified for maturity time $t + 2\delta$ with information at time $t + \delta$ in Lemma 4.3.1, viz., $\pi_{t+2\delta}^1 := \Lambda_{t+2\delta}^1 D_{t+2\delta}^1$. That structure is maintained here along with the observation that the no-arbitrage initial value of the tradable synthetic δ-term FTDZCB with tenor equal to 2δ must be

$$P_{t,0,2}^1 := \frac{1}{\pi_t^1} \mathbb{E}^{\mathbb{P}} \left[\pi_{t+2\delta}^1 \,\middle|\, \mathcal{G}_t \right] = \frac{1}{D_t^1} \mathbb{E}^{\mathbb{D}_1} \left[D_{t+2\delta}^1 \,\middle|\, \mathcal{G}_t \right] \ , \tag{4.3.10}$$

where $D_t^1 := 1$ and $\Lambda_t^1 := 1 \Rightarrow \pi_t^1 = 1$. Substituting expression (4.3.5) into the right-hand-side of the above equation yields

$$\mathbb{E}^{\mathbb{D}_1} \left[D_{t+2\delta}^1 \,\middle|\, \mathcal{G}_t \right] = \mathbb{E}^{\mathbb{D}_1} \left[\frac{D_{t+\delta}^1}{\widehat{D}_{t+\delta}^1} \, \mathbb{E}^{\mathbb{D}_1} \left[\widehat{D}_{t+2\delta}^1 \,\middle|\, \mathcal{G}_{t+\delta} \right] \,\middle|\, \mathcal{G}_t \right] = D_{t+\delta}^1 \, \mathbb{E}^{\mathbb{D}_1} \left[\frac{\widehat{D}_{t+2\delta}^1}{\widehat{D}_{t+\delta}^1} \,\middle|\, \mathcal{G}_t \right] \ , \tag{4.3.11}$$

where the second equality follows by the tower property of conditional expectations, and since $D_{t+\delta}^1$, defined in equation (4.2.3), is $\mathcal{G}_t$-measurable. Now, working with the expression after the second equality, observe that

$$\mathbb{E}^{\mathbb{D}_1} \left[\frac{\widehat{D}_{t+2\delta}^1}{\widehat{D}_{t+\delta}^1} \,\middle|\, \mathcal{G}_t \right] = \mathbb{E}^{\mathbb{D}_1} \left[\mathbb{E}^{\mathbb{P}} \left[\frac{\Lambda_{t+2\delta}^1 \widehat{D}_{t+2\delta}^1}{\Lambda_{t+\delta}^1 \widehat{D}_{t+\delta}^1} \,\middle|\, \mathcal{G}_{t+\delta} \right] \,\middle|\, \mathcal{G}_t \right] = \mathbb{E}^{\mathbb{D}_1} \left[P_{t,1,2}^1 \,\middle|\, \mathcal{G}_t \right] \ , \tag{4.3.12}$$

i.e., the inner expectation computed with information up to $\mathcal{G}_{t+\delta}$ must equal $P_{t,1,2}^1$, such that

$$\mathbb{E}^{\mathbb{D}_1} \left[D_{t+2\delta}^1 \,\middle|\, \mathcal{G}_t \right] = D_{t+\delta}^1 \mathbb{E}^{\mathbb{D}_1} \left[P_{t,1,2}^1 \,\middle|\, \mathcal{G}_t \right] \ , \tag{4.3.13}$$

which follows after substituting equation (4.3.12) back into equation (4.3.11). Lemma 4.3.2 advocates that $P_{t,0,2}^1$ must equal $P_{t,0,2}^2$, and therefore it follows from equations (4.3.10) and (4.3.13) that

$$\mathbb{E}^{\mathbb{D}_1} \left[P_{t,1,2}^1 \,\middle|\, \mathcal{G}_t \right] = \frac{P_{t,0,2}^2}{D_{t+\delta}^1} = \frac{P_{t,0,2}^2}{P_{t,0,1}^1} \ , \tag{4.3.14}$$

i.e., this enforces a numerical value for the expectation from equation (4.3.12). Since the time-dependent parameter associated with $(\Lambda_u^1)_{t+\delta < u \le t+2\delta}$ is free to specify at time t, it is possible to calibrate this quantity such that equation (4.3.13) holds, which concludes the proof. □

Remark 4.3.1 (Term-dependent market price of FTD-linked risk)
Since the estimated δ-term FTDSDF is used to model the tradable 2δ-term FTDZCB, it is conjectured that a time-inhomogeneous market price of risk structure is necessary. There are two notions of time in this framework: (i) universal calendar time defined by the variable u and a current time denoted by t; and (ii) term and tenor times determined by natural number multiples of δ. Therefore, the time-inhomogeneous process $\{\Lambda_{t+j\delta}^1 \,;\, j \in \{0,1,2\}\}$, shown here in discrete-time, actually implies that the framework advocates for term-dependent parameters, or term-dependent market prices of FTD-linked risk.

Remark 4.3.2 (The availability of multiple FTDs)
Theorem 4.3.1 may be iteratively repeated out to $t+j\delta$*, for* $j \in \{3,4,5,\ldots,m\}$ *if the corresponding FTDs and FTDRRs,* R_t^j*, are market-made by the STR. The calibrated* δ*-term FTDSDF may then be extended as*

$$D^1_{t+j\delta} := \frac{D^1_{t+(j-1)\delta}}{\Lambda^1_{t+(j-1)\delta}\widehat{D}^1_{t+(j-1)\delta}}\mathbb{E}^{\mathbb{P}}\left[\Lambda^1_{t+j\delta}\,\widehat{D}^1_{t+j\delta}\,\middle|\,\mathcal{G}_{t+(j-1)\delta}\right] = \frac{D^1_{t+(j-1)\delta}}{\widehat{D}^1_{t+(j-1)\delta}}\mathbb{E}^{\mathbb{D}_1}\left[\widehat{D}^1_{t+j\delta}\,\middle|\,\mathcal{G}_{t+(j-1)\delta}\right] .$$

If one or a subset of these FTDs are not market-made or quoted, the result of the theorem still applies but there will now be a range of viable values for the missing rates for arbitrage-free calibration.

Remark 4.3.3 (Term-specific FTDSDFs, but a unique term structure)
Each tradable FTD term has a distinct estimated FTDSDF, which encodes floating composite interest rate risk, along with a unique single-period FTDPK. The ability to replicate all tradable FTDZCBs via a system of FTDFRAs leads to a single-curve interest rate term structure, and enables multi-period calibration for all FTDPKs. This is nothing more than a manifestation of "funding-swap duality" within a near risk-free market context that has been enabled by the STR. Since the δ*-term FTD bears the least composite risk relative to all other terms, its pricing measure is the closest to the classical risk-neutral measure.*

Theorem 4.3.1 relied on the SST's ability to create a FTDFRA, which was achieved in Lemma 4.3.2. The resultant δ-term FTDPK therefore encodes the arbitrage-free mechanics to price and value such a product. Pricing this FTDFRA, formalised in the corollary below, will reveal the fair FTDFRA rate to be the *simple forward rate* that is constructed from the δ- and 2δ-term FTDRRs, which is defined next.

Definition 4.3.1 ($\delta \times 2\delta$ FTD-linked forward rate)
At time $u \in \mathbb{R}_{\geq 0}$ *, the* $\delta \times 2\delta$ *FTD-linked forward rate is a simple rate, denoted by* $F(u; u+\delta, u+2\delta)$ *, at which one may deposit (borrow) money over the future* δ*-term* $[u+\delta, u+2\delta]$*. The net capital plus interest yield (cost) at time* $u+2\delta$ *is*

$$1 + \delta F(u; u+\delta, u+2\delta) := \left(1 + 2\delta R_u^2\right) / \left(1 + \delta R_u^1\right) \ , \tag{4.3.15}$$

with $F(u; u+\delta, u+2\delta)$ *being* $\mathcal{G}_u$*-measurable.*
Shorthand notation*: In general, the* $j\delta \times (j+1)\delta$ *forward rate will be denoted as:*

$$F^1_{u,i,j} := F\left(u + i\delta; u + j\delta, u + (j+1)\delta\right) \ , \tag{4.3.16}$$

for all $i, j \in \mathbb{N}_0$ *, with* $i \leq j$ *and* $u \in \mathbb{R}_{\geq 0}$ *.*

Remark 4.3.4 ($\delta \times 2\delta$ forward rate construction)
Depositing (borrowing) one unit of currency at $F^1_{u,01}$ *is achieved by:*

(i) borrowing (depositing) $P^1_{u,0,1}$ *units of currency at the* δ*-term FTDRR at time* u*;*

(ii) simultaneously depositing (borrowing) the same amount at the 2δ*-term FTDRR; and*

(iii) depositing (borrowing) one unit of currency to settle transaction (i) at time $u+\delta$*.*

Transaction (ii) offers the general strategy to follow to deposit/borrow at $F^1_{u,0,1}$*.*

Corollary 4.3.1 ($\delta \times 2\delta$ FTDFRA pricing)
The fair strike rate process for the general version of the $\delta \times 2\delta$ FTDFRA, described in Definition 3.5.1, is given by the $\mathcal{G}_{t+i\delta}$-measurable process

$$F^1_{t,i,1} = \frac{1}{\delta}\left(\frac{P^1_{t,i,1}}{P^1_{t,i,2}} - 1\right) \tag{4.3.17}$$

for $i \in \{0,1\}$, with $F^1_{t,1,1} = R^1_{t+\delta}$, the δ-term FTDRR at time $t+\delta$.

Proof. The general version of the $\delta \times 2\delta$ FTDFRA has the following terminal payoff

$$V_{t+2\delta} = \alpha N\left[\left(1+\delta R^1_{t+\delta}\right) - (1+\delta K)\right] = \alpha N\left[\frac{1}{P^1_{t,1,2}} - (1+\delta K)\right], \tag{4.3.18}$$

where K is an arbitrary fixed rate specified at initiation of the contract, at time t. Using the calibrated δ-term FTDPK from Theorem 4.3.1, the initial arbitrage-free value of the FTDFRA is

$$\begin{aligned}
V_t &= \frac{1}{\pi^1_t}\,\mathbb{E}^{\mathbb{P}}\left[\pi^1_{t+2\delta}V_{t+2\delta}\,\middle|\,\mathcal{G}_t\right] \\
&= \frac{1}{D^1_t}\,\mathbb{E}^{\mathbb{D}_1}\left[D^1_{t+2\delta}V_{t+2\delta}\,\middle|\,\mathcal{G}_t\right] \\
&= \alpha\mathbb{E}^{\mathbb{D}_1}\left[D^1_{t+\delta}\,\middle|\,\mathcal{G}_t\right] - \alpha(1+\delta K)\,\mathbb{E}^{\mathbb{D}_1}\left[D^1_{t+2\delta}\,\middle|\,\mathcal{G}_t\right] \\
&= \alpha P^1_{t,0,1} - \alpha(1+\delta K)\,P^1_{t,0,2}\,,
\end{aligned} \tag{4.3.19}$$

where the third line follows from Lemma 4.3.1, viz., $D^1_{t+2\delta} = D^1_{t+\delta}P^1_{t,1,2}$, and the last line by definition of the δ-term FTDPK. Setting $V_t = 0$ and solving for the fair strike rate yields $K = \left(P^1_{t,0,1}/P^1_{t,0,2} - 1\right)/\delta$. Repeating this process at time $t+\delta$, for exactly the same contract, yields $K = \left(P^1_{t,1,1}/P^1_{t,1,2} - 1\right)/\delta$, as required, which concludes the proof. □

Remark 4.3.5 (Multi-period arbitrage-free models)
Within this near risk-free setting for the SST, the availability of multiple tradable FTDs enables the construction of term-dependent multi-period arbitrage-free models:

$$\pi^n_u := \Lambda^n_u D^n_u\,, \tag{4.3.20}$$

for $n \in \{1,2,\ldots,m\}$ and $u \subset \{t, t+\delta, t+2\delta, \ldots, t+m\delta\}$, in the same way that has been shown for the δ-term. These term-dependent FTDPKs will serve as key modelling tools in the two sections that follow. The underlying volatility dynamics for each of these models will still be determined through single rate-based estimation. Therefore, while the SST may use the multi-period δ-term FTDPK for general pricing and valuation, it would be inconsistent to use this FTDPK for financial instruments that reference rates other than the δ-term FTDRR.

Remark 4.3.6 (The SST's market-making opportunities)
In this scenario, Lemmas 4.3.1 and 4.3.2, Theorem 4.3.1, as well as comments from Remarks 4.3.3 and 4.3.5, show that the SST can easily enable a secondary market for FTDs. Moreover, the near risk-free market created by the STR for the SST, by way of Assumption 3.3.1, means that the SST does not need to take on any residual risk in order to enable a secondary market for FTDs and create a new market in synthetic FTDs. Put differently, the SST requires no risk appetite to create both a secondary market for FTDs and a linear derivatives market that references FTDRRs.

The previous remark summarises the market-making opportunities for the SST, which in this scenario is essentially unlimited and can be undertaken with minimal or no risk. However, this is only achievable because of Assumption 3.3.1, and thereby the STR immunising the SST from all risks that were articulated in section 3.3. This is unrealistic, therefore the next two sections will attempt to reproduce the same results but in the presence of liquidity and credit risks.

4.4 Multiple FTDs with Liquidity Risk

In this third scenario, Assumption 3.3.2 is assumed to govern the STR's market-making process for the SST within the FTD market. This scenario is the same as the previous one, except that property (b) from Assumption 3.3.1 no longer holds, which means that the SST will now be exposed to general liquidity risks, viz., illiquidity related to lending/borrowing and liquidation risks related thereto. These features have to be explicitly modelled into the system that has been setup in the previous two sections.

In Appendix B.3, a comprehensive model is proposed for a general $n\delta$-term FTD rate that may be quoted by the STR to the SST, in Definition B.3.1. This model attempts to capture features of the limit-order book within the systemic setting beyond that which has been described, advocated and assumed in section 3.3. In particular, this model includes spread effects for term, nominal size and type of transaction (loan or deposit), and caters for the case of specific SST and systemic illiquidity. Recall that the systemic and symmetric risk setting, from section 3.3, assumes that the STR will immunise the SST from all basis risks related to the set of FTDRRs. Through suitable aggregation and fair valuation at the systemic level, Proposition B.3.1 and Corollary B.3.1 provide the necessary justification for a simpler symmetric model specification based on a *systemic liquidity indicator* (SLI), which confirms again with the setup advocated in section 3.3 and Assumption 3.3.2. The SLI is formalised in the next definition, followed by the definition of a potentially illiquid $n\delta$-term FTD rate, from the perspective of the SST.

Definition 4.4.1 (Systemic liquidity indicators)
At some time $u \in \mathbb{R}_{\geq 0}$, the SLI associated with the $n\delta$-term FTD is the binary random variable

$$L_u^n := \begin{cases} 1 \ , & \text{if perfect systemic liquidity prevails, with probability } \ p_u^n \ , \\ 0 \ , & \text{if otherwise, with probability } \ 1 - p_u^n \ . \end{cases} \tag{4.4.1}$$

If the current time is t, then the natural filtration associated with systemic liquidity is

$$\mathcal{L}_t^{\text{SLI}} := \sigma\left(\left\{\left\{L_u^1, L_u^2, \ldots, L_u^m\right\}; u \in \{0, \delta, 2\delta, \ldots, t-\delta, t\}\right\}\right) \subset \mathcal{L}_t \ , \tag{4.4.2}$$

where $\{0, \delta, 2\delta, \ldots, t-\delta, t\}$ denotes the set of trading days that lie within the interval $[0, t]$. The SLIs are assumed to exhibit both serial and cross-sectional independence, and are independent of all tradable information, or more formally:

$$\mathbb{E}^{\mathbb{P}}\left[L_u^n \,\middle|\, \mathcal{G}_t \vee \mathcal{L}_t \cap \sigma\left(\{L_u^n \notin \{0,1\}\}\right)\right] = \mathbb{E}^{\mathbb{P}}\left[L_u^n\right] = \mathbb{P}\left[L_u^n = 1\right] := p_u^n \ , \tag{4.4.3}$$

for all $t \leq u$, with $p_u^n := p(u, u+n\delta)$ being a deterministic function for the probability of perfect systemic $n\delta$-term liquidity at time u.

A more general 3-state costly SLI is also presented in Appendix B.3, in Definition B.3.2, along with comparable results for the important Lemma 4.4.1 that is derived later in this section. The simpler 2-state specification, defined above, is considered here, which also conforms with the assumptions from section 3.3 regarding the STR immunising the SST from all basis risks. Nonetheless, all of the results derived here still hold within the more general setting, modulo minor adjustments and assumptions.

Remark 4.4.1 (SLI independence is supported by a fragmented active banking sub-sector)
The serial independence of the SLIs may be qualitatively justified by acknowledging the dynamic nature of the active banking sub-sector through time, as per Definition 3.1.1. Cross-sectional independence may be justified by assuming that the active banking sub-sector, at any point in time, fragments in relation to their offering of liquidity across FTD terms. Therefore, to support the assumption of serial and cross-sectional SLI independence, one may update Definition 3.1.1 to include the feature of the active banking sub-sector fragmenting into sub-groups that offer varying degrees of FTD liquidity across term and time.

Remark 4.4.2 (More realistic structure is possible but superfluous)
It is, of course, possible to impose more realistic structure — for e.g., one may model the set of SLIs at time u in a reduced-form way as

$$L_u^n := \mathbb{I}_{\{\tau_u^{\mathrm{L}} > u + n\delta\}} \ , \tag{4.4.4}$$

for $n \in \{1, 2, \dots, m\}$ and $u \in \{0, \delta, 2\delta, \dots, t-\delta, t\}$, where τ_u^{L} is a stopping time that is assumed to be associated with the active banking sub-sector at time u that determines in an abstract fashion the availability of liquidity across terms. If the set of stopping times, $\{\tau_u^{\mathrm{L}} ; u \in \{0, \delta, 2\delta, \dots, t-\delta, t\}\}$, are assumed to be serially independent, this model would still exhibit cross-sectional dependence. Since the overarching objective here is to theoretically develop and justify the market-based approach, which focuses on modelling at a composite market rate level, this level of structural complexity is deemed to be superfluous. The serially and cross-sectionally independent SLIs offer a satisfactory level of structure to reveal the intuition and justify the way forward for the market-based approach within this market context.

To be clear, by "*perfect systemic liquidity*" associated with the $n\delta$-term, it is meant that the SST is able to deposit and borrow at the $n\delta$-term FTDRR freely at the respective point in time that is under consideration. The "*otherwise*" case constitutes all of the events articulated in section 3.3, as well as the liquidation-illiquidity duality feature that is described in sections 3.5 and 3.7.

Definition 4.4.2 (Potentially illiquid $n\delta$-term FTD rate)
At some arbitrary time $u < t$, the $n\delta$-term FTD rate

$$\widetilde{R}_t^n := R_t^n L_t^n \ , \tag{4.4.5}$$

is potentially illiquid from the vantage point of u if $L_t^n = 0$, i.e., it will not be possible for the SST to borrow from (or deposit with) the STR, at time t, for a tenor equal to $n\delta$. For further insight on the definition and interpretation of this quantity, refer to Appendix B.3.

These liquidity indicators enable the definition of various liquidity regimes, which in turn enables the definition of a set of *term-* and *liquidity-dependent FTD pricing kernels* (LDFTDPKs), all from the perspective of the SST. First, the various regimes of liquidity are defined.

Definition 4.4.3 (Liquidity regimes)
Let $i, j \in \mathbb{N}_0$, with $i < j$, and define the counting sets

$$\mathbb{N}_{i,j} := \{i, i+1, \dots, j-1, j\} \ \text{ and}$$
$$\mathbb{N}_{i,j}^n := \{i, i+n, \dots, i+(k-1)n, i+kn\} \ ,$$

where $k := \lfloor (j-i)/n \rfloor - 1$ *,* $\mathbb{N}_{i,i} := \emptyset$ *and* $\mathbb{N}^n_{i,i} := \emptyset$*. At time* u*, the following liquidity regimes are possible over the interval* $[u, u+m\delta]$*:*

(i) ***NPFL*** *- No present nor future liquidity exists on the set*

$$\mathcal{L}^{\mathrm{NPFL}}_{u,u+m\delta} := \sigma\left(\left\{L^n_{u+i\delta} = 0\,; n \in \mathbb{N}_{1,m}\,, i \in \mathbb{N}^n_{0,m}\right\}\right)\,. \tag{4.4.6}$$

(ii) ***NPL*** *- No present liquidity only exists on the set*

$$\mathcal{L}^{\mathrm{NPL}}_{u,u+m\delta} := \sigma\left(\{L^n_u = 0\,; n \in \mathbb{N}_{1,m}\}\right)\,. \tag{4.4.7}$$

(iii) ***PPL*** *- Only partial present liquidity exists on the set*

$$\mathcal{L}^{\mathrm{PPL}}_{u,u+m\delta} := \sigma\left(\{L^n_u = 1\,; n \subset \mathbb{N}_{1,m}\}\right)\,. \tag{4.4.8}$$

(iv) ***CPL*** *- Complete present liquidity only exists on the set*

$$\mathcal{L}^{\mathrm{CPL}}_{u,u+m\delta} := \sigma\left(\{L^n_u = 1\,; n \in \mathbb{N}_{1,m}\}\right)\,. \tag{4.4.9}$$

(v) ***CPFL****: Complete present and future liquidity exists on the set*

$$\begin{aligned}\mathcal{L}^{\mathrm{CPFL}}_{u,u+m\delta} &:= \mathcal{L}^1_{u,u+m\delta} \vee \mathcal{L}^2_{u,u+m\delta} \vee \ldots \vee \mathcal{L}^m_{u,u+m\delta} \\ &= \mathcal{L}^{\mathrm{CPL}}_{u,u+m\delta} \vee \mathcal{L}^1_{u+\delta,u+m\delta} \vee \mathcal{L}^2_{u+2\delta,u+m\delta} \vee \ldots \vee \mathcal{L}^m_{u+m\delta,u+m\delta}\,,\end{aligned} \tag{4.4.10}$$

where

$$\mathcal{L}^n_{u+i\delta,u+m\delta} := \sigma\left(\left\{L^n_{u+j\delta} = 1\,;\, j \in \mathbb{N}^n_{i,m}\right\}\right) \tag{4.4.11}$$

models liquidity in the $n\delta$*-term FTD over* $[u+i\delta, u+m\delta]$ *if* $(m-i) \bmod n = 0$ *, or* $[u+i\delta, u+m\delta)$ *otherwise, for* $i \in \mathbb{N}_{0,m}$ *.*

Regimes (i) and (ii) are complements of (iv) and (v), respectively. Regimes (ii), (iii) and (iv) pose uncertain future liquidity, with (iii) also posing uncertain present liquidity for some terms.
Shorthand Notation*:* $\mathcal{L}^{\mathrm{X}}_{u,i,j} := \mathcal{L}^{\mathrm{X}}_{u+i\delta,u+j\delta}$ *for all* $i, j \in \mathbb{N}_0$ *, with* $i \leq j$ *and* $u \in \mathbb{R}_{\geq 0}$ *.*

Please take note of equation (4.4.11) and the shorthand notation related thereto, since this liquidity regime definition will be used extensively in all that follows.

Remark 4.4.3 (Implications of the CPFL regime)
Under the CPFL regime all of the results from Sections 4.2 and 4.3 may be recovered, i.e., multi-period arbitrage-free term-dependent FTDPKs with associated term-dependent FTD-linked pricing measures.

These liquidity regimes enable the definition of an $n\delta$-term LDFTDPK over a single period $[t, t+n\delta]$, and then over multiple periods $[t, t+i\delta]$, for $i \in \mathbb{N}^n_{0,m+n}$. These definitions are provided next.

Definition 4.4.4 ($n\delta$-Term LDFTDPK over a single period)
At the current time t *, under the CPL or CPFL liquidity regime the* $n\delta$*-term LDFTDPK is defined by*

$$\widetilde{\pi}^n_{t+in\delta} := \pi^n_{t+in\delta}\,, \tag{4.4.12}$$

for $i \in \{0, 1\}$*, i.e., perfect liquidity enables the definition of the* $n\delta$*-term FTDPK. This scenario therefore allows a market-taker, such as the SST, to access* $n\delta$*-term liquidity.*

If the NPFL or NPL liquidity regime prevails then the $n\delta$-term LDFTDPK is given by:

$$\widetilde{\pi}^n_{t+in\delta} := \widehat{D}^n_{t+in\delta} \,, \tag{4.4.13}$$

for $i \in \{0, 1\}$, i.e., in this case it is assumed that the SST will resort to estimation of an $n\delta$-term FTDRR using the respective estimated FTDSDF.

Under the PPL liquidity regime with the $n\delta$-term being illiquid but the $i\delta$- and $j\delta$-terms being liquid such that $j < n < k$, the SST may define the $n\delta$-term LDFTDPK by

$$\widetilde{\pi}^n_{t+in\delta} := \Lambda^n_{t+in\delta} D^n_{t+in\delta} \,, \tag{4.4.14}$$

where the $\{\mathcal{G}_u, \mathbb{P}\}$-martingale $(\Lambda^n_u)_{t \le u \le t+n\delta}$ must be chosen so that

$$\mathbb{E}^{\mathbb{D}_n}\left[\widehat{D}^n_{t+n\delta} \,\middle|\, \mathcal{G}_t\right] = D^n_{t+n\delta} \in \left(D^k_{t+k\delta} \,,\, D^j_{t+j\delta}\right)$$

in order to ensure positive forward rates over $[t + j\delta, t + k\delta]$, i.e., in this case it is assumed that the SST will again resort to using the estimated $n\delta$-term FTDSDF, but will have bounds for the calibrated version based on the FTD terms that are liquid.

Remark 4.4.4 (Illiquid regimes and SST market-making)
Observe that under the NPFL, NPL and PPL regimes, it is assumed that the SST resorts to utilising the estimated FTDSDFs. The assumption here is that even though the illiquid FTDs may be tradable for other economic agents, the fact that they are not liquid for the SST means that the SST will resort to its own market-making process to enable liquidity for itself in relation to these specific FTDs. This will of course require the SST to bear residual risk, and therefore be afforded the resources to be exposed to such risk.

Definition 4.4.5 ($n\delta$-Term LDFTDPK over multiple periods)
At the current time t, under the CPFL liquidity regime the $n\delta$-term LDFTDPK is defined by

$$\widetilde{\pi}^n_{t+i\delta} := \pi^n_{t+i\delta} \,, \tag{4.4.15}$$

for $i \in \mathbb{N}^n_{0,m+n}$, i.e., perfect systemic liquidity enables the definition of the $n\delta$-term FTDPK over the interval $[t, t + m\delta]$ if $m \bmod n = 0$, or $[t, t + m\delta)$ otherwise.

If the CPL or PPL (as defined in Definition 4.4.4) liquidity regime prevails then

$$\widetilde{\pi}^n_{t+i\delta} = \begin{cases} \pi^n_{t+i\delta} \,, & i \in \mathbb{N}^n_{0,2n} \,, \\ \widehat{D}^n_{t+i\delta} \,, & i \in \mathbb{N}^n_{2n,m+n} \,, \end{cases} \tag{4.4.16}$$

i.e., potential future illiquidity requires the SST to market-make beyond the first period.

If the NPL or NPFL liquidity regime prevails then the $n\delta$-term LDFTDPK is defined by

$$\widetilde{\pi}^n_{t+i\delta} := \widehat{D}^n_{t+i\delta} \,, \tag{4.4.17}$$

for $i \in \mathbb{N}^n_{0,m+n}$, i.e., no present liquidity and no/uncertain future liquidity requires the SST to market-make and create liquidity for all periods.

Both Definitions 4.4.4 and 4.4.5 clearly reveal that the regime of liquidity has a significant impact on the form of the pricing kernel associated with each tradable term. The impact of present liquidity or illiquidity is fundamental, with the latter requiring the subjective process of market-making, as described in Remark 4.4.4. Modelling the FTD market-making process of the STR is beyond the scope of the research undertaken in this part of the thesis, therefore the prevalence of the CPL regime will be a minimal assumption in all that follows. Then, from a practical perspective, in order to deal with potential future illiquidity, presently available liquidity must be fully exploited. This is achieved through the definition of the following *hybrid-term LDFTDPK.*

Definition 4.4.6 (Hybrid-term LDFTDPK)
At the current time t, the PK defined by

$$\widetilde{\pi}_{t+i\delta} = \begin{cases} \pi^1_{t+i\delta} \ , & \text{conditional on } \mathcal{L}^{\mathrm{CPL}}_{t,0,i} \vee \mathcal{L}^1_{t,1,i} \ , \ \text{for } i \in \mathbb{N}^1_{1,m+1} \ , \\ \pi^2_{t+i\delta} \ , & \text{conditional on } \mathcal{L}^{\mathrm{CPL}}_{t,0,i} \vee \mathcal{L}^2_{t,2,i} \ , \ \text{for } i \in \mathbb{N}^2_{2,m+2} \ , \\ \quad \vdots \quad , & \qquad \vdots \qquad\qquad\qquad \vdots \qquad , \qquad \vdots \qquad\qquad , \\ \pi^m_{t+i\delta} \ , & \text{conditional on } \mathcal{L}^{\mathrm{CPL}}_{t,0,i} \vee \mathcal{L}^m_{t,m,i} \ , \ \text{for } i \in \mathbb{N}^m_{m,2m} \ , \end{cases} \tag{4.4.18}$$

with $\widetilde{\pi}_t := 1$ *, is called the hybrid-term liquidity-dependent FTD-linked pricing kernel.*

The hybrid-term LDFTDPK is only well-defined at time t, and may be used to present value future cash flows due at time $t+i\delta$ back to time t only, for $i \in \mathbb{N}_{1,m}$. For the intertemporal valuation of the same cash flows back to a future time $t+h\delta$, for $h \in \mathbb{N}_{1,i-1}$, but from the vantage point of time t, one would require the prevalence of the CPFL regime at time t in order to have a well-defined PK over $[t+h\delta, t+i\delta]$. However, invoking the CPFL regime would recover the setup of the previous section, the PK would coincide with one of the term-dependent FTDPKs defined in that section, and there'd be no need for the hybrid-term LDFTDPK.

With the hybrid-term LDFTDPK it is now possible to consider again the pricing (or market-making) of a $\delta \times 2\delta$ FTDFRA, from the perspective of the SST, under potential future illiquidity. Assuming that the CPL regime prevails at the time of pricing, the SST will have to subjectively specify the probability of future liquidity, articulated here via the SLIs.

Lemma 4.4.1 ($\delta \times 2\delta$ FTDFRA pricing under potential future illiquidity)
The fair strike rate process for the general version of the $\delta \times 2\delta$ *FTDFRA defined in Definition 3.5.1 is*

$$\overline{F}(t+i\delta; t+\delta, t+2\delta) = \begin{cases} p^1_{t+\delta} F^1_{t,0,1} \ , & i=0 \ \textit{and conditional on } \ \mathcal{L}^{\mathrm{CPL}}_{t,0,2} \ , \\ F^1_{t,1,1} \ , & i=1 \ \textit{and conditional on } \ \mathcal{L}^{\mathrm{CPL}}_{t,1,2} \ , \end{cases} \tag{4.4.19}$$

which is also $\mathcal{G}_{t+i\delta}$*-measurable.*
Shorthand Notation*: In general the* $j\delta \times (j+1)\delta$ *FTDFRA rate will be denoted by:*

$$\overline{F}^1_{u,i,j} := \overline{F}\left(u+i\delta; u+j\delta, u+(j+1)\delta\right) \ , \tag{4.4.20}$$

for all $i, j \in \mathbb{N}_0$ *, with* $i \leq j$ *and* $u \in \mathbb{R}_{\geq 0}$ *.*

Proof. Assuming that $\mathcal{L}_t \supset \mathcal{L}^{\mathrm{CPL}}_{t,0,2}$, the standard FTDFRA replication strategy yields

$$\begin{aligned} \widetilde{V}_{t+2\delta} &= \alpha N \left[\left(1+\delta \widetilde{R}^1_{t+\delta}\right) - \left(1+\delta F^1_{t,0,1}\right) \right] \\ &= V_{t+2\delta} - \alpha N \delta \left(1 - L^1_{t+\delta}\right) R^1_{t+\delta} \ , \end{aligned}$$

where $F^1_{t,0,1}$ is defined in Definition 4.3.1 and $V_{t+2\delta}$ is the payoff of a fair $\delta \times 2\delta$ FTDFRA that is not exposed to liquidity risk. Let $\mathcal{M}_t := \mathcal{G}_t \vee \mathcal{L}_t$, then using the hybrid-term LDFTDPK from Definition 4.4.6, the current value of the above payoff is

$$
\begin{aligned}
\widetilde{\pi}_t \widetilde{V}_t &= \mathbb{E}^{\mathbb{P}}\left[\widetilde{\pi}_{t+2\delta}\widetilde{V}_{t+2\delta} \,\middle|\, \mathcal{M}_t\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\widetilde{\pi}_{t+2\delta}\widetilde{V}_{t+2\delta} \,\middle|\, \mathcal{M}_t\,, L^1_{t+\delta}\right] \,\middle|\, \mathcal{M}_t\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\widetilde{\pi}_{t+2\delta}\widetilde{V}_{t+2\delta} \,\middle|\, \mathcal{M}_t\,, L^1_{t+\delta}=1\right] \mathbb{P}\left[L^1_{t+\delta}=1\right] \,\middle|\, \mathcal{M}_t\right] \\
&\quad + \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\widetilde{\pi}_{t+2\delta}\widetilde{V}_{t+2\delta} \,\middle|\, \mathcal{M}_t\,, L^1_{t+\delta}=0\right] \mathbb{P}\left[L^1_{t+\delta}=0\right] \,\middle|\, \mathcal{M}_t\right]
\end{aligned}
$$

which follows by the tower property of conditional expectations. Since $\widetilde{\pi}_t := 1$ and observing that $\mathcal{L}^1_{t,0,2} = \sigma\left(\{L^1_{t+\delta}=1\}\right)$ and $\mathcal{L}^2_{t,0,2} = \emptyset$, it follows that

$$
\begin{aligned}
\widetilde{V}_t &= \mathbb{E}^{\mathbb{P}}\left[\pi^1_{t+2\delta}V_{t+2\delta} \,\middle|\, \mathcal{M}_t\right] p^1_{t+\delta} - \mathbb{E}^{\mathbb{P}}\left[\pi^2_{t+2\delta}\alpha N \delta F^1_{t,0,1} \,\middle|\, \mathcal{M}_t\right]\left(1-p^1_{t+\delta}\right) \\
&= p^1_{t+\delta}V_t - \alpha N\left(1-p^1_{t+\delta}\right)\delta F^1_{t,0,1}P^2_{t,0,2}\ ,
\end{aligned}
$$

using the definition of the hybrid-term LDFTDPK. V_t is the fair value of the FTDFRA under perfect liquidity, and therefore equal to 0. Trading this FTDFRA with the strike rate equal to the fair FTDFRA rate defined in the perfect liquidity setting therefore leads to an initial loss (gain) if the market-maker is long (short). The assumption here is that $V_{t+2\delta}$ will still be the FTDFRA payoff when $L^1_{t+\delta} = 0$ and even when the strong case of no systemic liquidity is in effect. In reality, there will still be a δ-term FTDRR that is contractually specified for such a case by relevant derivative trading regulation.
Setting the FTDFRA strike rate to an arbitrary value, $\overline{F}^1_{t,0,1}$, and pricing via the same process gives

$$
\widetilde{V}_t = \alpha p^1_{t+\delta}N\left[P^1_{t,0,1} - \left(1+\delta\overline{F}^1_{t,0,1}\right)P^1_{t,0,2}\right] - \alpha N\left(1-p^1_{t+\delta}\right)\delta\overline{F}^1_{t,0,1}P^2_{t,0,2}\ ,
$$

while setting $\widetilde{V}_t = 0$, recalling that $P^2_{t,0,2} = P^1_{t,0,2}$, and solving for the fair FTDFRA strike rate yields $\overline{F}^1_{t,0,1} = p^1_{t+\delta}F^1_{t,0,1}$, as required. Repeating this pricing process at time $t+\delta$, for exactly the same contract and assuming that the CPL liquidity regime, $\mathcal{L}^{\mathrm{CPL}}_{t,1,2}$, prevails at this time, it is trivial to show that $\overline{F}^1_{t,1,1} = F^1_{t,1,1}$, which completes the proof. □

Remark 4.4.5 (The dual liquidation problem)
The standard replication strategy requires the SST to:

(i) borrow (deposit) $P^1_{t,0,1}$ units of currency at R^2_t, i.e., the 2δ-term FTDRR at time t;

(ii) deposit (borrow) $P^1_{t,0,1}$ units of currency at R^1_t, i.e., the δ-term FTDRR at time t; and

(iii) deposit (borrow) 1 unit of currency at $R^1_{t+\delta}$, i.e., the δ-term FTDRR at time $t+\delta$;

in order to replicate a long (short) position in a $\delta \times 2\delta$ FTDFRA.
In the proof above, the probability of liquidity for transaction (iii) is the key subjective pricing variable. A second way to view this is to consider the probability of the counterparty demanding to liquidate the FTDFRA at time $t+\delta$. This poses and results in the same risk exposure for the SST, since transaction (i) cannot be liquidated, by definition of an FTD.
A third way to view this is via funding-swap duality, as articulated in Remark 3.7.2. The key subjective pricing variable from this perspective would be the probability of the depositor/borrower liquidating their financing transaction at time $t+\delta$.

Contingent on present liquidity, the pricing of an FTDFRA still requires a subjective view on future liquidity. Therefore, the SST must be afforded some level of *risk appetite* in order to market-make such derivatives. This is in stark contrast with the perfect liquidity setting where the SST could replicate FTDFRAs perfectly, and thereby did not require nor did it deserve any risk appetite. In general then, it is assumed that the SST has the capacity for exposure to residual risk, which in turn allows for subjectivity in their market-making processes.

Remark 4.4.6 (FTDFRA liquidity is not completely contingent on the CPL regime)
Even if the NPL regime were to prevail, the SST's capacity to carry residual risk and potentially hedge in the future, through offsetting positions, will still enable the pricing of FTDFRAs. Practically, this decouples the theoretical contingency of the SST on the STR's market-making of FTD-linked deposits and loans. Equivalently, one may argue that the pricing of interest rate derivatives that reference FTD rates is therefore also somewhat decoupled from the set of underlying FTDs. However, Lemma 4.4.1 reveals structure to this decoupling, with FTD-implied forward rates required to dominate corresponding fair FTDFRA rates.

Lemma 4.3.2, presented under Assumption 3.3.1, enforced the early liquidation value of a 2δ-term FTD (or FTDZCB) by replacement. This result was used in conjunction with the replication of a $\delta \times 2\delta$ FTDFRA to create a synthetic δ-term FTDZCB with 2δ-tenor. An analogous result is possible here, however it is contingent on $\delta \times 2\delta$ FTDFRA and δ-term FTD liquidity.

Therefore, based on the discussion leading up to and including Remark 4.4.6, it is now assumed that an FTDFRA market has been established within the interbank derivatives market. While the individual STs that constitute the SST would be responsible for the establishment of the FTDFRA market through active market-making via model creation; here the SST is considered to be a separate entity that is observing this market at a systemic level and considering the problem of passive market-making via model calibration. Only the stylised problem from Lemma 4.3.2 and Theorem 4.3.1 is considered again here — the general version of this problem is considered in the next chapter.

Assumption 4.4.1 (δ-Term FTDFRA market-making)
The individual STs have sufficient risk appetite to market-make and enable the liquidity of the full set of $(m-1)$ δ-term FTDFRAs at time t, which constitutes the $j\delta \times (j+1)\delta$ FTDFRA for $j \in \mathbb{N}_{1,m-1}$.
At this time, the fair or mid market $j\delta \times (j+1)\delta$ FTDFRA rate, denoted here by $\widehat{F}^1_{t,0,j}$, is used by the SST together with the generalised result from Lemma 4.4.1 for the purpose of calibration.

Remark 4.4.7 (Calibration of $p^1_{t+\delta}$ using a market $\delta \times 2\delta$ FTDFRA)
Setting $\overline{F}^1_{t,0,1} = \widehat{F}^1_{t,0,1}$ and assuming that $\mathcal{L}^{\mathrm{CPL}}_{t,0,2} \subset \mathcal{L}_t$ enables the SST to compute

$$p^1_{t+\delta} = \frac{\overline{F}^1_{t,0,1}}{F^1_{t,0,1}}\ , \tag{4.4.21}$$

using equation (4.4.19), which is now the market-implied probability of perfect δ-term liquidity at time $t+\delta$ using information available at the current time t.

Definition 4.4.7 (δ-term FTDFRA systemic liquidity indicators)
At time $t+i\delta$, for each $i \in \mathbb{N}_{0,j}$, the binary random variable $\overline{L}^1_{t,i,j}$ assumes a value of 1 if perfect systemic liquidity exists for the $j\delta \times (j+1)\delta$ FTDFRA, or 0 otherwise, for $j \in \mathbb{N}_{1,m-1}$.

When Assumption 4.4.1 holds, it is also assumed that

$$\sigma\left(\left\{\overline{L}^1_{t,0,j} = 1\,;\, j \in \mathbb{N}_{1,m-1}\right\}\right) \supset \mathcal{L}^{\mathrm{CPL}}_{t,0,m}\ , \tag{4.4.22}$$

i.e., the CPL liquidity regime prevails to assist market-making of δ-term FTDFRAs at time t. If the current time is t, then the natural filtration associated with δ-term FTDFRA systemic liquidity is

$$\overline{\mathcal{L}}^1_t := \sigma\left(\left\{\overline{L}^1_{u,0,j}\,;\, j \in \mathbb{N}_{1,m-1}\,,\, u \in \{0, \delta, 2\delta, \ldots, t-\delta, t\}\right\}\right)\ , \tag{4.4.23}$$

such that $\mathcal{L}^{\mathrm{SLI}}_t \vee \overline{\mathcal{L}}^1_t \subset \mathcal{L}_t$, where $\mathcal{L}^{\mathrm{SLI}}_t$ is defined in Definition 4.4.1, equation (4.4.2). Since the FTDFRA systemic liquidity indicators will only be used to indicate regimes of liquidity and will not be used for pricing, the probabilistic structure of these are left unspecified.

Using the $\delta \times 2\delta$ FTDFRA along with the δ-term FTD, it is now possible to formulate the analog to Lemma 4.3.2 within this setting of potential illiquidity. The synthetic δ-term FTDZCB that is constructed here is referred to as a *liquidity-contingent FTD-linked zero coupon bond* (LCFTDZCB), since its definition relies on the availability of liquidity in the aforementioned instruments.

Lemma 4.4.2 (Synthetic δ-term LCFTDZCB with 2δ-tenor)
Assuming that $\overline{L}^1_{t,0,1} = 1$, and setting $\overline{F}^1_{t,0,1} = \widehat{F}^1_{t,0,1}$, it is possible to replicate the following ZCB:

$$\overline{P}^1_{t,i,2} := \begin{cases} D^1_{t+\delta} \Big/ \left(1 + \delta \overline{F}^1_{t,0,1}\right)\ , & i = 0\ , \\ P^1_{t,1,2}\ , & i = 1\ , \\ 1\ , & i = 2\ , \end{cases} \tag{4.4.24}$$

provided that $L^1_t = L^1_{t+\delta} = 1$, or equivalently that $\mathcal{L}^1_{t,0,2}$ holds.

Proof. Assuming that $L^1_t = L^1_{t+\delta} = 1$, it is possible to borrow (deposit) M units of currency at the δ-term FTDRR at time t and refinance (re-deposit) the total cost (proceeds) thereof at time $t+\delta$ at the prevailing FTDRR, $R^1_{t+\delta}$, such that the cumulative cost (yield) is $M/D^1_{t+2\delta}$ at time $t+2\delta$.
Combining this loan (deposit) with a long (short) position in a fair market $\delta \times 2\delta$ FTDFRA with strike rate $\overline{F}_{t,0,1}$ and $N = M/D^1_{t+\delta}$ will enable the conversion of the floating cost (yield) to a fixed cost (yield) equal to $M(1+\delta\overline{F}^1_{t,0,1})/D^1_{t+\delta}$ at time $t+2\delta$. Setting $M = D^1_{t+\delta}/(1+\delta\overline{F}^1_{t,0,1})$, enables the creation of the synthetic δ-term LCFTDZCB, with 2δ-tenor, given by equation (4.5.13). Since it is assumed that $L^1_{t+\delta} = 1$, it is clear that $\overline{P}^1_{t,1,2} = P^1_{t,1,2}$, as required. □

Lemma 4.4.1 and 4.4.2 provides the basis for the construction of a δ-term LCFTDZCB system, one that is created by exchanging δ-term FTD floating-for-fixed interest rate risk. It is possible to model this system via the definition of a *liquidity-contingent FTD-linked pricing kernel* (LCFTDPK). The δ-term LCFTDPK is defined over the interval $[t, t+2\delta]$ in the next theorem.

Theorem 4.4.1 (δ-Term LCFTDPK)
Contingent on $\overline{L}^1_{t,0,1} = L^1_t = L^1_{t+\delta} = 1$, or equivalently

$$\sigma\left(\left\{\overline{L}^1_{t,0,1} = 1\right\}\right) \vee \mathcal{L}^1_{t,0,2} \subset \mathcal{L}_t\ , \tag{4.4.25}$$

using the result from Lemma 4.4.2 and defining $\mathcal{M}_{t+j\delta} := \mathcal{G}_{t+j\delta} \vee \mathcal{L}_{t+j\delta}$, the δ-term LCFTDPK may be defined as

$$\overline{\pi}^1_{t+j\delta} := \pi^1_{t+j\delta}\Theta^1_{t+j\delta}\ , \tag{4.4.26}$$

for $j \in \{0,1,2\}$, where the time-inhomogeneous $\{\mathcal{G}_u, \mathbb{D}_1\}$-martingale $\left(\Theta^1_u\right)_{t \le u \le t+2\delta}$, with $\Theta^1_t := 1$, enables the change-of-measure from $\mathbb{D}_1 \to \mathbb{L}_1$ on $(\mathcal{G}_u)_{t \le u \le t+2\delta}$, via the Radon-Nikodym derivative $\Theta^1_{t+2\delta}/\Theta^1_t = (\mathrm{d}\mathbb{L}_1/\mathrm{d}\mathbb{D}_1)\big|_{\mathcal{G}_{t+2\delta}}$, such that $\mathbb{E}^{\mathbb{P}}\left[\overline{\pi}^1_{t+\delta} \,\middle|\, \mathcal{M}_t\right] = P^1_{t,0,1}$ and $\mathbb{E}^{\mathbb{P}}\left[\overline{\pi}^1_{t+2\delta} \,\middle|\, \mathcal{M}_t\right] = \overline{P}^1_{t,0,2}$.

Proof. Since $\mathcal{L}^{\mathrm{CPL}}_{t,0,2} \subset \sigma(\{\overline{L}^1_{t,0,1} = 1\})$, by Definition 4.4.7, it follows that $\mathcal{L}^{\mathrm{CPFL}}_{t,0,2} \subset \mathcal{L}_t$ and that the δ-term FTDPK is well defined over $[t, t+2\delta]$. Therefore, $\{\pi^1_{t+j\delta}, j \in \{0,1,2\}\}$ is a good initial candidate for the LCFTDPK, however it does not recover the initial price of the synthetic δ-term LCFTDZCB with 2δ-tenor. The $\{\mathcal{G}_u, \mathbb{D}_1\}$-martingale $(\Theta^1_u)_{t \le u \le t+2\delta}$ enables a change-of-measure such that

$$\mathbb{E}^{\mathbb{P}}\left[\Lambda^1_{t+2\delta}\Theta^1_{t+2\delta}D^1_{t+2\delta} \,\middle|\, \mathcal{M}_t\right] = \mathbb{E}^{\mathbb{D}_1}\left[\Theta^1_{t+2\delta}D^1_{t+2\delta} \,\middle|\, \mathcal{M}_t\right] = \mathbb{E}^{\mathbb{L}_1}\left[D^1_{t+2\delta} \,\middle|\, \mathcal{M}_t\right] := \overline{P}^1_{t,0,2},$$

as required, recalling that $\Lambda^1_t = \Theta^1_t = 1$, i.e., the free time-dependent parameters associated with $(\Theta^1_u)_{t+\delta < u \le t+2\delta}$ is free to specify at time t such that the expectation equals $\overline{P}^1_{t,0,2}$. Also, at the future time $t+\delta$, since $\mathcal{L}_{t+\delta} \supset \mathcal{L}_t \supset \mathcal{L}^1_{t,0,2}$ it follows that $L^1_{t+\delta} = 1$ and recalling that $D^1_{t+2\delta}$ is $\mathcal{G}_{t+\delta}$-measurable, then

$$\frac{1}{\overline{\pi}^1_{t+\delta}}\mathbb{E}^{\mathbb{P}}\left[\Lambda^1_{t+2\delta}\Theta^1_{t+2\delta}D^1_{t+2\delta} \,\middle|\, \mathcal{M}_{t+\delta}\right] = \frac{D^1_{t+2\delta}}{D^1_{t+\delta}\Theta^1_{t+\delta}}\mathbb{E}^{\mathbb{D}_1}\left[\Theta^1_{t+2\delta} \,\middle|\, \mathcal{M}_{t+\delta}\right] = \frac{D^1_{t+2\delta}}{D^1_{t+\delta}} = P^1_{t,1,2},$$

which shows that the value of the synthetic δ-term LCFTDZCB, given by equation (4.5.13), is recovered by the δ-term LCFTDPK. Since $D^1_{t+\delta} = P^1_{t,0,1}$ is $\mathcal{G}_t$-measurable, it follows straightforwardly that

$$\mathbb{E}^{\mathbb{P}}\left[\Lambda^1_{t+\delta}\Theta^1_{t+\delta}D^1_{t+\delta} \,\middle|\, \mathcal{F}_t\right] = P^1_{t,0,1}\mathbb{E}^{\mathbb{D}_1}\left[\Theta^1_{t+\delta} \,\middle|\, \mathcal{F}_t\right] = P^1_{t,0,1},$$

which completes the proof, showing that: (i) the free time-dependent parameters associated with $(\Theta^1_u)_{t<u\le t+\delta}$ may be specified freely at time t; and (ii) the δ-term LCFTDPK is calibrated to the δ-term FTDZCB and the synthetic δ-term LCFTDZCB with 2δ-tenor. ☐

It may not be apparent but the definitions of the synthetic δ-term LCFTDZCB and its associated LCFTDPK, from Lemma 4.4.2 and Theorem 4.4.1, had two steps and associated contingencies:

(i) the interval $[t, t+2\delta]$, or more specifically the set $\{t, t+\delta, t+2\delta\}$, requires that $\mathcal{L}^1_{t,0,2}$ holds, or equivalently that $L^1_t = L^1_{t+\delta} = 1$, and that $\overline{L}^1_{t,0,1} = 1$; and

(ii) the future interval $[t+\delta, t+2\delta]$, or more specifically the set $\{t+\delta, t+2\delta\}$, requires that $\mathcal{L}^1_{t,1,2}$ holds, or equivalently that $L^1_{t+\delta} = 1$.

In general, to extend these definitions over the interval $[t, t+m\delta]$, would require m steps:

- at the current time t and over the set $\{t, t+\delta, \ldots, t+m\delta\}$, one would require that $\mathcal{L}^1_{t,0,m}$ holds and that $\overline{L}^1_{t,0,j} = 1$ for $j \in \mathbb{N}_{1,m-1}$;
- at each future time $t+i\delta$ and over the set $\{t+i\delta, t+(i+1)\delta, \ldots, t+m\delta\}$, for $i \in \mathbb{N}_{1,m-2}$, one would require $\mathcal{L}^1_{t,i,m}$ and that $\overline{L}^1_{t,i,j} = 1$ for $j \in \mathbb{N}_{i+1,m-1}$; and
- at the future time $t+(m-1)\delta$ and over the set $\{t+(m-1)\delta, t+m\delta\}$, one would require that $\mathcal{L}^1_{t,m-1,m}$ holds.

This will form the basis for the reduced-form modelling approach that is developed in the next chapter. This section is concluded with a few remarks that aim to assist the reader to build intuition in relation to all of the theory that has been presented thus far.

Remark 4.4.8 (LCFTDZCBs are not tradable, in general)
The δ-term FTDZCB-system that was introduced in section 4.3, under Assumption 3.3.1, viz.,

$$\left\{ P^1_{t,i,j} \,; i \in \mathbb{N}_{0,j} \,, j \in \mathbb{N}_{0,m} \right\} \ , \tag{4.4.27}$$

denotes a set of tradable FTDZCBs whose tenors span the interval $[t, t+m\delta]$. Moreover, recall that under these assumptions all term FTDZCBs may be replicated via the δ-term system, i.e., $P^1_{t,i,j} = P^{j-i}_{t,i,j}$.

Under Assumption 3.3.2, the definition and tradability of the above set of FTDZCBs requires $\mathcal{L}^{\mathrm{CPFL}}_{t,0,m}$ to hold. When $\mathcal{L}^{\mathrm{CPL}}_{t,0,m}$ holds, this set reduces to

$$\left\{ P^n_{t,0,n} \,; n \in \mathbb{N}_{0,m} \right\} \ , \tag{4.4.28}$$

i.e., the set of FTDZCBs derived from the set of m FTD instruments, assumed to be tradable in section 4.1. Then, the δ-term LCFTDZCB-system, that was introduced in this section, is

$$\left\{ \overline{P}^1_{t,i,j} \,; i \in \mathbb{N}_{0,j} \,, j \in \mathbb{N}_{0,m} \right\} \ , \tag{4.4.29}$$

and is contingent upon $\overline{L}^1_{t,i,j} = 1$ for all $j \in \mathbb{N}_{i+1,m-1}$ and $\mathcal{L}^1_{t,i,m}$ holding for each $i \in \mathbb{N}_{0,m}$. Apart from $\overline{P}^1_{t,i,i+1} = P^1_{t,i,i+1}$, which only requires present liquidity at $t+i\delta$, i.e., $L^1_{t+i\delta} = 1$, for each $i \in \mathbb{N}_{0,m-1}$, the remainder of the set of LCZCBs are contingent on the availability of future liquidity and are therefore not tradable, in general.

Remark 4.4.9 (Interpretation of the δ-term LCFTDPK)
Following Definition 4.4.2, an intuitive approach might have been to:

(i) specify a δ-term liquidity-cognisant FTDSDF under $\mathbb{D}_1$ and over $[t, t+j\delta]$ as

$$\widetilde{D}^1_{t+j\delta} := \prod_{i=0}^{j-1} \frac{1}{1+\delta \widetilde{R}^1_{t+i\delta}} = \prod_{i=0}^{j-1} \frac{1}{1+\delta L^1_{t+i\delta} R^1_{t+i\delta}} \ , \tag{4.4.30}$$

for $j \in \mathbb{N}_{1,m}$, with $\widetilde{D}^1_t := 1$; and

(ii) proceed to directly calculate FTDZCB prices using the δ-term liquidity-cognisant FTDSDF.

The FTDSDF, from (i), and its associated bank account, $\widetilde{B}^1_{t+j\delta} := 1/\widetilde{D}^1_{t+j\delta}$, are relevant from a practical perspective. The latter directly models the total proceeds (costs) of a deposit (loan) strategy that rolls over at the δ-term frequency. The unavailability of liquidity at any roll-over time will translate into an interest rate loss (gain) for the deposit (loan) strategy[3]*. It is also assumed that the depositor (borrower) will attempt again to re-invest (refinance) the total value of their investment (liability) at the next roll-over time.*

While the FTDZCB prices, from (ii), may be calculated theoretically (with plausible and tractable model specifications), such FTDZCBs do not exist in reality. If these FTDZCBs existed, they would ensure multi-period δ-term funding at fixed rates while also ensuring early liquidation at the δ-term frequency, by definition. In other words, such FTDZCBs would immunise long (depositors) and short (borrowers) holders

[3] The gain for the loan strategy comes at the cost of the borrower having to settle their total liability at the roll-over time, as opposed to deferring payment by refinancing at the δ-term rate at this time.

from all liquidity risks. In practice, there are no financial instruments that offer protection against liquidity risks, hence the approach that has been taken in this section which has culminated in the definition of LCFTDZCBs and LCFTDPKs.

The δ-term LCFTDPK has the following form:

$$\overline{\pi}^1_{t+j\delta} = \begin{cases} \Lambda^1_{t+j\delta} D^1_{t+j\delta} \Theta^1_{t+j\delta} \ , & \text{under } \mathbb{P} \ , \\ D^1_{t+j\delta} \Theta^1_{t+j\delta} \ , & \text{under } \mathbb{D}_1 \ , \\ D^1_{t+j\delta} \ , & \text{under } \mathbb{L}_1 \ , \end{cases}$$

is $\mathcal{G}_{t+j\delta}$-measurable, for $j \in \mathbb{N}_{0,m}$, and is calibrated at the current time t such that it recovers the fair market $j\delta \times (j+1)\delta$ FTDFRA rates, for $j \in \mathbb{N}_{1,m-1}$. In other words, unlike the intuitive liquidity-cognisant δ-term FTDSDF presented above, the δ-term LCFTDPK is designed for the purpose of FTDFRA pricing and valuation. The calibration process is enabled by the definition of the set of LCFTDZCBs $\{\overline{P}^1_{t,0,j}\,;\, j \in \mathbb{N}_{0,m}\}$, and the $\{\mathcal{G}_u, \mathbb{D}_1\}$-martingale $(\Theta^1_{t+v\delta})_{0\leq v\leq m}$ which, via the fair market FTDFRA rates, encodes the likelihood of δ-term liquidity ex-ante. In particular, this martingale process inflates (deflates) the δ-term FTDSDF (bank account) based on the lack of liquidity. Hence $D^1_{t+j\delta}\Theta^1_{t+j\delta}$, which is the δ-term LCFTDPK under $\mathbb{D}_1$, is an abstract representation of the δ-term liquidity-cognisant FTDSDF, equation (4.4.30), both ex-ante and ex-post.

Remark 4.4.10 (Liquidity-contingent term-dependent market price of FTD-linked risk)
Under Assumption 3.3.2, the δ-term market price of FTD-linked risk modelled by $\{\Lambda^1_{t+j\delta}\,;\, j \in \mathbb{N}_{0,m}\}$ is effectively adjusted for the potential cost of illiquidity incurred when market-making FTDFRAs through the process $\{\Theta^1_{t+j\delta}\,;\, j \in \mathbb{N}_{0,m}\}$. However, this is all strictly contingent on the availability of δ-term FTD and FTDFRA liquidity, as described above. Therefore, the product of the aforementioned processes $\{\Lambda^1_{t+j\delta}\Theta^1_{t+j\delta}\,;\, j \in \mathbb{N}_{0,m}\}$ models the liquidity-contingent term-dependent market price of FTD-linked risk associated with the δ-term LCFTDPK.

Remark 4.4.11 (Multiple LCFTDPKs and liquidity-contingent term structures)
Each $n\delta$-term has a distinct LCFTDPK, $\{\overline{\pi}^n_{t+j\delta}\,;\, j \in \mathbb{N}^n_{0,m}\}$, with a liquidity-contingent FTD-linked pricing measure, $\mathbb{L}_n$, modelled upon its near risk-free counterparts, $\{\pi^n_{t+j\delta}\,;\, j \in \mathbb{N}^n_{0,m}\}$ and $\mathbb{P}_n$, respectively. This will be further illustrated in the next chapter. The inability to replicate all tradable FTDZCBs via the system of FTDFRAs, along with the contingencies on future FTD and FTDFRA liquidity, leads to liquidity-contingent multi-period calibration for each tradable term. This in turn leads to multiple liquidity-contingent term structures.

4.5 Multiple FTDs with Liquidity and Credit Risks

This fourth and final scenario is the one that encapsulates *full risk* for the SST, with the cognisance of systemic liquidity and credit risks, and is based on market-making Assumption 3.3.3. Since credit-related risks can only arise post the conclusion of a transaction, there is a natural contingency of these risks on the availability of liquidity. Therefore, the framework for liquidity that has been developed in the previous section is augmented with exogenous *systemic default indicators* (SDIs) in this section in order to create new framework that is cognisant of all risk categories.

Section 3.3 offers the qualitative context and reasoning behind credit-related risks and how these may arise within the constructed systemic and symmetric setting. Importantly, it is assumed that default risks may emanate from CBEs that constitute the dynamic active banking sub-sector, and the financial impact related thereto is further assumed to be symmetric for the SST, i.e., default costs when the SST lends/deposits equate to default benefits when the SST borrows, all else being equal. The next definition introduces the SDI, the fundamental exogenous modelling quantity that is augmented onto the framework that has been developed in the previous section for modelling illiquidity.

Definition 4.5.1 (Systemic default indicators)
Each SLI, L_u^n, defined in Definition 4.4.1, has an associated SDI if $L_u^n = 1$, which is defined by the binary random variable

$$C_{u+n\delta}^n := \begin{cases} 1 \ , & \text{if no default realises over } (u, u+n\delta], \text{ with probability } \ s_u^n \ , \\ 0 \ , & \text{if otherwise, with probability } \ 1 - s_u^n \ , \end{cases} \tag{4.5.1}$$

where the probability $s_u^n := s(u, u+n\delta)$ is a deterministic function, with $u \in \mathbb{R}_{\geq 0}$ and $n \in \mathbb{N}_{1,m}$. Note that while L_u^n is measurable at time u, its default counterpart $C_{u+n\delta}^n$ is only measurable at time $u+n\delta$. If the current time is t, then the natural filtration associated with systemic default is

$$\mathcal{H}_t^{\text{SDI}} := \sigma\left(\left\{C_{u_n+n\delta}^n \, ; \, u_n \in \mathcal{T}_{[0,t-n\delta]}^n \, , n \in \mathbb{N}_{1,m}\right\}\right) \subset \mathcal{H}_t \ , \tag{4.5.2}$$

where $\mathcal{T}_{[0,t-n\delta]}^n$ denotes the subset of trading days within the interval $[0, t-n\delta]$ when perfect systemic liquidity prevailed, i.e., $L_u^n = 1$ for all $u \in \mathcal{T}_{[0,t-n\delta]}^n$ and $L_u^n = 0$ for all $u \notin \mathcal{T}_{[0,t-n\delta]}^n$.
The SDIs are also assumed to exhibit both serial and cross-sectional independence, are independent of all traded information and are statistically independent of all SLIs, or more formally:

$$\mathbb{E}^{\mathbb{P}}\left[C_{u+n\delta}^n \,\middle|\, \mathcal{G}_t \vee \mathcal{L}_t^* \vee \mathcal{H}_t^*\right] = \mathbb{E}^{\mathbb{P}}\left[C_{u+n\delta}^n\right] = \mathbb{P}\left[C_{u+n\delta}^n = 1\right] := s_u^n \ , \tag{4.5.3}$$

where $\mathcal{L}_t^* := \mathcal{L}_t \cap \sigma\left(\{L_u^n \neq 0\}\right)$ and $\mathcal{H}_t^* := \mathcal{H}_t \cap \sigma\left(\{C_u^n \notin \{0,1\}\}\right)$, for all $t \leq u$.

Remark 4.5.1 (SDI independence is supported by a fragmented active banking sub-sector)
The assumed serial and cross-sectional independence of the set of SDIs are naturally inherited from the corresponding set of associated SLIs, and the qualitative justification provided in Remark 4.4.1.

Remark 4.5.2 (More realistic structure is possible but superfluous)
Similar to Remark 4.4.4, it is possible for the set of SDIs at time u to be modelled in a reduced-form way as

$$C_{u+n\delta}^n = \mathbb{I}_{\{\tau_u > u+n\delta\}} \ , \tag{4.5.4}$$

for $n \in \mathbb{N}_{1,m}$ and $u \in \{0, \delta, 2\delta, \ldots, t-\delta, t\}$, where τ_u is a default time associated with the active banking sub-sector that is constituted at time u. Again, if the set of default times, $\{\tau_u \, ; u \in \{0, \delta, 2\delta, \ldots, t-\delta, t\}\}$, are assumed to be serially independent, this model would still exhibit cross-sectional dependence. One may also incorporate statistical dependence amongst the liquidity-related stopping times, from Remark 4.4.4 and these default times. However, for the same reasons as those provided in Remark 4.4.4, this level of structure is considered to be superfluous considering the market-based approach.

Having the definition of the SDI, and remarks related thereto, the rest of this section has a structure that is very similar to the previous section except that the development of a new pricing kernel structure is not required — the hybrid-term LDFTDPK will again be utilised as the key tool for pricing purposes. Next, using both the SLI and the SDI, it is possible to introduce the notion of a potentially illiquid and defaultable $n\delta$-term FTD-linked deposit/loan, from the perspective of the SST.

Definition 4.5.2 (Potentially illiquid and defaultable $n\delta$-term FTD-linked deposit/loan)
At some arbitrary time $u < t$, the $n\delta$-term FTD-linked quantity

$$\alpha\left(1 + n\delta R_t^n L_t^n\right) C_{t+n\delta}^n \ , \tag{4.5.5}$$

is a potentially illiquid and defaultable deposit if $\alpha = 1$, or loan if $\alpha = -1$, from the perspective of the SST transacting with the STR, and the STR passing on default risks from externalising this activity as articulated in section 3.3. As described in Definition 4.5.1, the existence of $C_{t+n\delta}^n$ is dependent on $L_t^n = 1$, therefore the above quantity does not exist if $L_t^n = 0$.

Having the notion of an illiquid and defaultable $n\delta$-term FTD-linked deposit/loan, it is possible to once again consider the pricing of a $\delta \times 2\delta$ FTDFRA from the perspective of the SST, but now under potential future illiquidity and default. Once again the CPL regime is assumed to prevail at the time of pricing, however the SST will now have to exogenously and subjectively specify the probability of future liquidity and survival via the effects of the SLI and SDIs.

Lemma 4.5.1 ($\delta \times 2\delta$ FTDFRA pricing under potential future illiquidity and default)
The fair strike rate, at the current time t, for the general version of the $\delta \times 2\delta$ FTDFRA defined in Definition 3.5.1 is given in the expression

$$1 + \delta \ddot{F}_{t,0,1}^1 = \frac{s_{t+\delta}^1}{s_t^2 / s_t^1} \left[\frac{\left(1 - p_{t+\delta}^1\right) + s_{t+\delta}^1 p_{t+\delta}^1}{s_{t+\delta}^1} + \delta p_{t+\delta}^1 F_{t,0,1}^1 \right] \ , \tag{4.5.6}$$

where $\ddot{F}_{t,0,1}^1$ denotes the fair strike rate, is $\mathcal{G}_t$-measurable, and is conditional on the liquidity regime $\mathcal{L}_{t,0,2}^{\mathrm{CPL}}$. The fair strike rate for the same FTDFRA at time $t + \delta$ is $\ddot{F}_{t,1,1} = F_{t,1,1}^1$, which is $\mathcal{G}_{t+\delta}$-measurable and conditional on the liquidity regime $\mathcal{L}_{t,1,2}^{\mathrm{CPL}}$.
***Shorthand Notation**: In general the $j\delta \times (j+1)\delta$ FTDFRA rate will be denoted by:*

$$\ddot{F}_{u,i,j}^1 := \ddot{F}\left(u + i\delta; u + j\delta, u + (j+1)\delta\right) \ , \tag{4.5.7}$$

for all $i, j \in \mathbb{N}_0$, with $i \leq j$ and $u \in \mathbb{R}_{\geq 0}$.

Proof. Assuming that $\mathcal{L}_t \supset \mathcal{L}_{t,0,2}^{\mathrm{CPL}}$, the standard FTDFRA replication strategy will yield

$$\ddot{V}_{t+2\delta} = \alpha N \left[C_{t+\delta}^1 \left(1 + \delta \widetilde{R}_{t+\delta}^1\right) C_{t+2\delta}^1 - \left(1 + \delta F_{t,0,1}^1\right) C_{t+2\delta}^2 \right] \ ,$$

where $F_{t,0,1}^1$, defined in Definition 4.3.1, is the $\delta \times 2\delta$ forward rate and the fair FTDFRA. As has been shown in Lemma 4.4.1, attempting to value the above payoff in a risky setting using the hybrid-term LDFTDPK will result in an initial gain/loss, since $F_{t,0,1}^1$ is not the correct fair FTDFRA rate. Setting the FTDFRA strike rate to an arbitrary value, $\ddot{F}_{t,0,1}^1$, will yield the following payoff scenarios

$$\ddot{V}_{t+2\delta} = \begin{cases} \alpha N \left[C_{t+\delta}^1 - \left(1 + \delta \ddot{F}_{t,0,1}^1\right) C_{t+2\delta}^2 \right] \ , & \text{if } \ L_{t+\delta}^1 = 0 \ , \\ \alpha N \left[C_{t+\delta}^1 \left(1 + \delta R_{t+\delta}^1\right) C_{t+2\delta}^1 - \left(1 + \delta \ddot{F}_{t,0,1}^1\right) C_{t+2\delta}^2 \right] \ , & \text{if } \ L_{t+\delta}^1 = 1 \ , \end{cases}$$

based on the availability of future liquidity at time $t+\delta$. Then, spanning out cases based on potential future defaults yields the following scenarios

$$\ddot{V}_{t+2\delta} = \begin{cases} -\alpha N \delta \ddot{F}^1_{t,0,1} \ , & \text{if } \ L^1_{t+\delta}=0, C^1_{t+\delta}=1, C^2_{t+2\delta}=1 \ , \\ \alpha N \ , & \text{if } \ L^1_{t+\delta}=0, C^1_{t+\delta}=1, C^2_{t+2\delta}=0 \ , \\ -\alpha N \left(1+\delta \ddot{F}^1_{t,0,1}\right) \ , & \text{if } \ L^1_{t+\delta}=0, C^1_{t+\delta}=0, C^2_{t+2\delta}=1 \ , \\ 0 \ , & \text{if } \ L^1_{t+\delta}=0, C^1_{t+\delta}=0, C^2_{t+2\delta}=0 \ , \\ \alpha N[(1+\delta R^1_{t+\delta})-(1+\delta \ddot{F}^1_{t,0,1})] \ , & \text{if } \ L^1_{t+\delta}=1, C^1_{t+\delta}=1, C^1_{t+2\delta}=1, C^2_{t+2\delta}=1 \ , \\ \alpha N \left(1+\delta R^1_{t+\delta}\right) \ , & \text{if } \ L^1_{t+\delta}=1, C^1_{t+\delta}=1, C^1_{t+2\delta}=1, C^2_{t+2\delta}=0 \ , \\ 0 \ , & \text{if } \ L^1_{t+\delta}=1, C^1_{t+\delta}=1, C^1_{t+2\delta}=0, C^2_{t+2\delta}=0 \ , \\ 0 \ , & \text{if } \ L^1_{t+\delta}=1, C^1_{t+\delta}=0, C^1_{t+2\delta}=0, C^2_{t+2\delta}=0 \ , \\ -\alpha N \left(1+\delta \ddot{F}^1_{t,0,1}\right) \ , & \text{if } \ L^1_{t+\delta}=1, C^1_{t+\delta}=0, C^1_{t+2\delta}=0, C^2_{t+2\delta}=1 \ , \\ -\alpha N \left(1+\delta \ddot{F}^1_{t,0,1}\right) \ , & \text{if } \ L^1_{t+\delta}=1, C^1_{t+\delta}=0, C^1_{t+2\delta}=1, C^2_{t+2\delta}=1 \ , \\ -\alpha N \left(1+\delta \ddot{F}^1_{t,0,1}\right) \ , & \text{if } \ L^1_{t+\delta}=1, C^1_{t+\delta}=1, C^1_{t+2\delta}=0, C^2_{t+2\delta}=1 \ , \\ 0 \ , & \text{if } \ L^1_{t+\delta}=1, C^1_{t+\delta}=0, C^1_{t+2\delta}=1, C^2_{t+2\delta}=0 \ , \end{cases}$$

which may now be priced with the hybrid-term LDFTDPK, along with careful conditioning based on future liquidity. Let $\mathcal{M}_t := \mathcal{G}_t \vee \mathcal{L}_t \vee \mathcal{H}_t$, then using the hybrid-term LDFTDPK from Definition 4.4.6, the current value of the above payoff is

$$\begin{aligned} \ddot{V}_t &= \frac{1}{\widetilde{\pi}_t}\mathbb{E}^{\mathbb{P}}\left[\widetilde{\pi}_{t+2\delta}\ddot{V}_{t+2\delta} \,\middle|\, \mathcal{M}_t\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\widetilde{\pi}_{t+2\delta}\ddot{V}_{t+2\delta} \,\middle|\, \mathcal{M}_t\,, L^1_{t+\delta}\right] \,\middle|\, \mathcal{M}_t\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\widetilde{\pi}_{t+2\delta}\ddot{V}_{t+2\delta} \,\middle|\, \mathcal{M}_t\,, L^1_{t+\delta}=1\right]\mathbb{P}\left[L^1_{t+\delta}=1\right] \,\middle|\, \mathcal{M}_t\right] \\ &\quad + \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\widetilde{\pi}_{t+2\delta}\ddot{V}_{t+2\delta} \,\middle|\, \mathcal{M}_t\,, L^1_{t+\delta}=0\right]\mathbb{P}\left[L^1_{t+\delta}=0\right] \,\middle|\, \mathcal{M}_t\right] \ , \end{aligned} \tag{4.5.8}$$

which follows by the tower property of conditional expectations, and since $\widetilde{\pi}_t = 1$. Then, it is easier to deal with each expectation conditioned on the paths of future liquidity and illiquidity, respectively. Let $\mathcal{M}^0_t := \mathcal{M}_t \vee \sigma\left(\left\{L^1_{t+\delta}=0\right\}\right)$, then the expectation conditioned on the path of future illiquidity resolves as follows

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}\left[\widetilde{\pi}_{t+2\delta}\ddot{V}_{t+2\delta} \,\middle|\, \mathcal{M}^0_t\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\pi^2_{t+2\delta}\ddot{V}_{t+2\delta} \,\middle|\, \mathcal{M}^0_t\,, C^1_{t+\delta}, C^2_{t+2\delta}\right] \,\middle|\, \mathcal{M}^0_t\right] \\ &= -\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\pi^2_{t+2\delta}(\alpha N\delta \ddot{F}^1_{t,0,1}) \,\middle|\, \mathcal{M}^0_t\,, C^1_{t+\delta}=1, C^2_{t+2\delta}=1\right]\mathbb{P}\left[C^1_{t+\delta}=1, C^2_{t+2\delta}=1\right] \,\middle|\, \mathcal{M}^0_t\right] \\ &\quad + \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\pi^2_{t+2\delta}(\alpha N) \,\middle|\, \mathcal{M}^0_t\,, C^1_{t+\delta}=1, C^2_{t+2\delta}=0\right]\mathbb{P}\left[C^1_{t+\delta}=1, C^2_{t+2\delta}=0\right] \,\middle|\, \mathcal{M}^0_t\right] \\ &\quad - \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\pi^2_{t+2\delta}(\alpha N(1+\delta \ddot{F}^1_{t,0,1})) \,\middle|\, \mathcal{M}^0_t\,, C^1_{t+\delta}=0, C^2_{t+2\delta}=1\right]\mathbb{P}\left[C^1_{t+\delta}=0, C^2_{t+2\delta}=1\right] \,\middle|\, \mathcal{M}^0_t\right] \\ &= \alpha N P^2_{t,0,2}\left[-\delta \ddot{F}^1_{t,0,1}s^1_t s^2_t + s^1_t(1-s^2_t) - (1+\delta \ddot{F}^1_{t,0,1})(1-s^1_t)s^2_t\right] \\ &= \alpha N P^2_{t,0,2}\left[s^1_t - (1+\delta \ddot{F}^1_{t,0,1})s^2_t\right] \ , \end{aligned}$$

which follows by the definition of the hybrid-term LDFTDPK, which takes the form of the 2δ-term FTDPK, and the tower property of conditional expectations, after conditioning on future information regarding default on this path of future illiquidity. Setting $\mathcal{M}_t^1 := \mathcal{M}_t \vee \sigma\left(\left\{L_{t+\delta}^1 = 1\right\}\right)$, the second expectation conditioned on the path of future liquidity resolves in a similar way as follows

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\widetilde{\pi}_{t+2\delta}\ddot{V}_{t+2\delta} \,\middle|\, \mathcal{M}_t^1\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\pi_{t+2\delta}^1\ddot{V}_{t+2\delta} \,\middle|\, \mathcal{M}_t^1, C_{t+\delta}^1, C_{t+2\delta}^1, C_{t+2\delta}^2\right] \,\middle|\, \mathcal{M}_t^1\right] \\
&= \alpha N P_{t,0,2}^1\left[(1+\delta F_{t,0,1}^1) - (1+\delta\ddot{F}_{t,0,1}^1)\right] s_t^1 s_{t+\delta}^1 s_t^2 \\
&\quad + \alpha N P_{t,0,2}^1\left[(1+\delta F_{t,0,1}^1)\right] s_t^1 s_{t+\delta}^1(1-s_t^2) \\
&\quad - \alpha N P_{t,0,2}^1\left[(1+\delta\ddot{F}_{t,0,1}^1)\right]\left[(1-s_t^1)(1-s_{t+\delta}^1)s_t^2 + (1-s_t^1)s_{t+\delta}^1 s_t^2 + s_t^1(1-s_{t+\delta}^1)s_t^2\right] \\
&= \alpha N P_{t,0,2}^1\left[(1+\delta F_{t,0,1}^1)s_t^1 s_{t+\delta}^1 - (1+\delta\ddot{F}_{t,0,1}^1)s_t^2\right] ,
\end{aligned}$$

where the conditioning on future default information has been suppressed, but take note that the hybrid-term LDFTDPK now takes the form of the δ-term FTDPK.

Using the previous two results, equation 4.5.8 may then be resolved as follows

$$\begin{aligned}
\ddot{V}_t &= (1-p_{t+\delta}^1)\alpha N P_{t,0,2}^2\left[s_t^1 - (1+\delta\ddot{F}_{t,0,1}^1)s_t^2\right] \\
&\quad + p_{t+\delta}^1\alpha N P_{t,0,2}^1\left[(1+\delta F_{t,0,1}^1)s_t^1 s_{t+\delta}^1 - (1+\delta\ddot{F}_{t,0,1}^1)s_t^2\right] ,
\end{aligned}$$

using the definition of the SLI, from Definition 4.4.1. Setting $\ddot{V}_t = 0$, recalling that $P_{t,0,2}^2 = P_{t,0,2}^1$, and solving for the expression $(1+\delta\ddot{F}_{t,0,1}^1)$ yields the required result. Repeating this pricing process at time $t+\delta$, for exactly the same contract and assuming that the CPL liquidity regime, $\mathcal{L}_{t,1,2}^{\mathrm{CPL}}$, prevails at this time, it is trivial to show that $\ddot{F}_{t,1,1}^1 = F_{t,1,1}^1$, which completes the proof. □

Remark 4.5.3 (Interpretation of the fair $\delta \times 2\delta$ FTDFRA rate, $\ddot{F}_{t,0,1}^1$)

If one were to consider the scenario of perfect systemic liquidity prevailing, i.e., $\mathcal{L}_{t,0,2}^{\mathrm{CPFL}}$, then the fair $\delta \times 2\delta$ FTDFRA rate would resolve to

$$1+\delta\ddot{F}_{t,0,1}^1 = \frac{s_{t+\delta}^1}{s_t^2/s_t^1}\left[1+\delta F_{t,0,1}^1\right] ,$$

which highlights the ratio $\frac{s_{t+\delta}^1}{s_t^2/s_t^1}$, where

- *the denominator s_t^2/s_t^1 may be interpreted as a forward survival probability that is applicable over the interval $[t+\delta, t+2\delta]$, given information at the current time t, also assuming that the same active banking sub-sector has enabled both the related δ- and 2δ-term transactions; and*

- *the numerator $s_{t+\delta}^1$ is the future survival probability also applicable over the interval $[t+\delta, t+2\delta]$, given information at the current time t, but now related to the active banking sub-sector that enables the δ-term transaction at time $t+\delta$.*

If the market-maker perceives the forward probability, in the denominator, to be a fair estimate for the future probability, in the numerator, then the fair FTDFRA rate coincides with the corresponding forward rate, thereby recovering the result from the near risk-free setting in section 4.3.

This may be plausible when viewed from the perspective of the FTDFRA replication strategy, which involves lending (borrowing) over $[t, t+2\delta]$ at the δ-term frequency and borrowing (lending) over $[t, t+2\delta[$ at the 2δ-term frequency. In a symmetric and unilateral risk setting, it is reasonable to expect that costs related to counterparty defaults on deposits are offset by benefits related to corresponding defaults on loans, on average, thereby enabling a recovery of the standard forward rate.

From the perspective of funding-swap duality, as described in sections 3.4 and 3.5, the fact that the equivalent and associated financing transaction to a long $\delta \times 2\delta$ FTDFRA position is a fixed deposit over $[t, t+2\delta]$ that allows for liquidation at time $t+\delta$, it would be egregious for a market-maker to offer a depositor full compensation for bearing credit risk over the full 2δ-term. Similar logic applies for a corresponding short FTDFRA and loan transaction — it would be egregious for the lender to charge the borrower the 2δ-term credit risk spread, if they have the option to terminate the loan at time $t+\delta$. It is therefore conjectured that a rational market-maker will price

$$s^1_{t+\delta} < s^2_t / s^1_t \ , \tag{4.5.9}$$

which may may be interpreted as adjusting the implicit 2δ-term credit risk spread/premium embedded in the standard forward rate to reflect the appropriate δ-term spread/premium. Qualitatively, this may also be interpreted as a general risk-averse view of the creditworthiness of the future active banking sub-sector relative to the expected state of the current one.

Moreover, the role of $p^1_{t+\delta}$ in the fair FTDFRA rate, $\overline{F}^1_{t,0,1}$, derived in Lemma 4.4.1 under Assumption 3.3.2, plays the analogous role of adjusting the implicit 2δ-term funding-liquidity risk spread/premium in the standard forward rate to reflect the appropriate δ-term spread/premium. This effect features again in equation 4.5.6, i.e., the expression involving $\ddot{F}^1_{t,0,1}$, along with the interaction term:

$$\frac{(1 - p^1_{t+\delta}) + s^1_{t+\delta} p^1_{t+\delta}}{s^1_{t+\delta}} > 1 \ , \tag{4.5.10}$$

which will be greater than 1, as long as $p^1_{t+\delta} < 1$ and $s^1_{t+\delta} < 1$. This term may be interpreted as the effect of the interaction between future states of liquidity/illiquidity and survival.

To summarise then, three effects feature in equation 4.5.6, the expression for $\ddot{F}^1_{t,0,1}$ and these are

(i) a credit risk/spread adjustment $\frac{s^1_{t+\delta}}{s^2_t / s^1_t} < 1$;

(ii) a funding-liquidity risk/spread adjustment $p^1_{t+\delta} < 1$; and

(iii) an interaction effect, equation (4.5.10), which is greater than 1.

In general, since effect (i) deflates both (ii) and (iii), it is conjectured that

$$\ddot{F}^1_{t,0,1} < \overline{F}^1_{t,0,1} < F^1_{t,0,1} \ , \tag{4.5.11}$$

under markets with non-negligible liquidity and credit risks.

It is now possible to essentially repeat Assumption 4.4.1 and Definition 4.4.7 for δ-term FTDFRA market-making and systemic liquidity indicators, respectively, in order to create the corresponding natural filtration associated with δ-term FTDFRA systemic liquidity within this liquidity and credit risky setting. This filtration takes the form of

$$\ddot{\mathcal{L}}_t^1 := \sigma\left(\left\{\ddot{L}_{u,0,j}^1 \,;\, j \in \mathbb{N}_{1,m-1} \,,\, u \in \{0, \delta, 2\delta, \ldots, t-\delta, t\}\right\}\right) \,, \tag{4.5.12}$$

where the *overline* notation, used for the key modelling quantities in the potentially illiquid setting, has essentially been replaced by the *double dot* notation for the corresponding quantities in this potentially illiquid and credit risky setting. In a similar fashion, it is also possible to reproduce a comparable version of Lemma 4.4.2, which enables here the creation of a synthetic δ-term *survival and liquidity-contingent FTD-linked zero coupon bond* (SLCFTDZCB). This is a ZCB that is contingent on liquidity and survival in the δ-term FTD over $[t, t+2\delta]$, as well as liquidity in the $\delta \times 2\delta$ FTDFRA — counterparty credit-related risks for derivatives are not considered or assumed to be collateralised away.

Lemma 4.5.2 (Synthetic δ-term SLCFTDZCB with 2δ-tenor)
Assuming that $\ddot{L}_{t,0,1}^1 = 1$, and setting $\ddot{F}_{t,0,1}^1 = \widehat{F}_{t,0,1}^1$, it is possible to replicate the following ZCB:

$$\ddot{P}_{t,i,2}^1 := \begin{cases} D_{t+\delta}^1 \Big/ \left(1 + \delta \ddot{F}_{t,0,1}^1\right) \,, & i = 0 \,, \\ P_{t,1,2}^1 \,, & i = 1 \,, \\ 1 \,, & i = 2 \,, \end{cases} \tag{4.5.13}$$

provided that $L_t^1 = L_{t+\delta}^1 = 1$, or equivalently that $\mathcal{L}_{t,0,2}^1$ holds, and that $C_{t+\delta}^1 = C_{t+2\delta}^1 = 1$.

Proof. The proof is almost identical to that for Lemma 4.4.2, apart from replacing the critical fair FTD-FRA quantities and values. □

Then, it is also possible to create the analog of Theorem 4.4.1 within this setting, using the results of Lemma 4.5.1 and 4.5.2, which enables the definition of the δ-term *survival and liquidity-contingent FTD-linked pricing kernel* (SLCFTDPK) over the interval $[t, t+2\delta]$.

Theorem 4.5.1 (δ-Term SLCFTDPK)
Contingent on $\ddot{L}_{t,0,1}^1 = L_t^1 = L_{t+\delta}^1 = 1$, or equivalently

$$\sigma\left(\left\{\ddot{L}_{t,0,1}^1 = 1\right\}\right) \vee \mathcal{L}_{t,0,2}^1 \subset \mathcal{L}_t \,, \tag{4.5.14}$$

and $C_{t+\delta}^1 = C_{t+2\delta}^1 = 1$, or equivalently

$$\sigma\left(\left\{C_{t+\delta}^1 = 1, C_{t+2\delta}^1 = 1\right\}\right) \subset \mathcal{H}_t \,, \tag{4.5.15}$$

then using the result from Lemma 4.5.2 and defining $\mathcal{F}_{t+j\delta} := \mathcal{G}_{t+j\delta} \vee \mathcal{L}_{t+j\delta} \vee \mathcal{H}_{t+j\delta}$, the δ-term SLCFTDPK may be defined as

$$\ddot{\pi}_{t+j\delta}^1 := \overline{\pi}_{t+j\delta}^1 X_{t+j\delta}^1 = \pi_{t+j\delta}^1 \Theta_{t+j\delta}^1 X_{t+j\delta}^1 \,, \tag{4.5.16}$$

for $j \in \{0,1,2\}$, where the time-inhomogeneous $\{\mathcal{G}_u, \mathbb{L}_1\}$-martingale $\left(X_u^1\right)_{t \le u \le t+2\delta}$, with $X_t^1 := 1$, enables the change-of-measure from $\mathbb{L}_1 \to \mathbb{Q}_1$ on $(\mathcal{G}_u)_{t \le u \le t+2\delta}$, via the Radon-Nikodym derivative $X_{t+2\delta}^1 / X_t^1 = (\mathrm{d}\mathbb{Q}_1 / \mathrm{d}\mathbb{L}_1)\big|_{\mathcal{G}_{t+2\delta}}$, such that $\mathbb{E}^{\mathbb{P}}\left[\ddot{\pi}_{t+\delta}^1 \,\middle|\, \mathcal{F}_t\right] = P_{t,0,1}^1$ and $\mathbb{E}^{\mathbb{P}}\left[\ddot{\pi}_{t+2\delta}^1 \,\middle|\, \mathcal{F}_t\right] = P_{t,0,2}^1$.

Proof. This proof follows in a similar manner to that for Theorem 4.4.1, if one uses the δ-term LCFT-DPK, $(\overline{\pi}_u^1)_{t \le u \le t+2\delta}$, in the same role as that played by the δ-term FTDPK, $(\pi_u^1)_{t \le u \le t+2\delta}$, in the proof for Theorem 4.4.1. □

As noted, after the proof of Theorem 4.4.1 and at the end of section 4.4, it is also possible to extend the definitions of the δ-term SLCFTDZCB-system and SLCFTDPK over the interval $[t, t+m\delta]$, which will be shown in the next chapter. The insights offered by Remark 4.4.8 are again valid within this liquidity and credit risky context. Most notably, SLCFTDZCBs are also not tradable, in general.

Remark 4.5.4 (Interpretation of the δ-term SLCFTDPK)
The δ-term SLCFTDPK has the following form:

$$\ddot{\pi}^1_{t+j\delta} = \begin{cases} \Lambda^1_{t+j\delta} D^1_{t+j\delta} \Theta^1_{t+j\delta} X^1_{t+j\delta} \ , & \text{under } \mathbb{P} \ , \\ D^1_{t+j\delta} \Theta^1_{t+j\delta} X^1_{t+j\delta} \ , & \text{under } \mathbb{D}_1 \ , \\ D^1_{t+j\delta} X^1_{t+j\delta} \ , & \text{under } \mathbb{L}_1 \ , \\ D^1_{t+j\delta} \ , & \text{under } \mathbb{Q}_1 \ , \end{cases}$$

is $\mathcal{G}_{t+j\delta}$-measurable, for $j \in \mathbb{N}_{0,m}$, and is calibrated at the current time t such that it recovers the fair market $j\delta \times (j+1)\delta$ FTDFRA rates, for $j \in \mathbb{N}_{1,m-1}$.
Intuitively and heuristically, the $\{\mathcal{G}_u, \mathbb{D}_1\}$-martingale $(\Theta^1_u)_{t \leq u \leq t+m\delta}$ encodes the effect of the series of funding-liquidity risk/spread adjustments $\{p^1_{t+\delta}, p^1_{t+2\delta}, \ldots, p^1_{t+(m-1)\delta}\}$, while the $\{\mathcal{G}_u, \mathbb{L}_1\}$-martingale $(X^1_u)_{t \leq u \leq t+m\delta}$ encodes the effect of the series of credit spread adjustments $\left\{ \frac{s^1_{t+\delta}}{s^2_t / s^1_t}, \frac{s^1_{t+2\delta}}{s^3_t / s^2_t}, \ldots, \frac{s^1_{t+(m-1)\delta}}{s^m_t / s^{m-1}_t} \right\}$, ignoring the impact of the series of interaction effects.
Within this context, $\mathbb{Q}_1$ may be thought of as the best proxy for the risk-neutral measure, with this system encoding δ-term floating interest rate, liquidity and credit risk.

As a final concluding remark in this section, these results are not restricted nor confined to the δ-term only. Therefore, the analogs of Remarks 4.4.10 and 4.4.11 are applicable.

Chapter 5

Exchanges of Risk, Pricing and Valuation

The last two sections of the previous chapter culminated in the creation of systems of LCFTDZCBs and SLCFTDZCBs, and associated LCFTDPKs and SLCFTDPKs, respectively. While the construction of the δ-term system was the focal point, the same strategies may be employed in order to create comparable quantities and systems for any $n\delta$-term, for $n \in \{1, 2, \ldots, m\}$, in both the liquidity risky and the liquidity and credit risky settings. This is first formalised in the next section, section 5.1, and then the rigidity of this framework for ZCBs and PKs is used as motivation for the development of an equivalent reduced-form framework, which is done in section 5.2. The reduced-form framework is conducive to practical pricing and valuation applications; however, before this is considered it is shown how the system of PKs enables a novel exchange of risk mechanism — this is the main purpose of section 5.3. Thereafter, in section 5.4, it is shown how the exchange of risk mechanism enables the pricing and valuation of tradable versions of FTDFRAs. This chapter and Part I then concludes with section 5.5 which describes how unsecured and secured BFRRs may be modelled within the framework that has been developed, along with a discussion on the links between Parts I and II.

5.1 A Market-Based Multi-Curve Framework for FTDs

The objective of this section is to consolidate the results from the previous chapter, and present a market-based multi-curve framework for the FTD market. More specifically, given the existence of a primary market, a framework that may be developed to establish a secondary market, and that can be utilised for various transactions therein, from a market or systemic perspective. As demonstrated in the full risk scenario in section 4.5, the term-dependent SLCFTDZCB-system and its associated SLCFTDPKs were the key financial and modelling objects, respectively. However, the LCFTDZCB-system and the associated set of LCFTDPKs prove to be useful from both a financial economic and quantitative modelling perspective. Therefore, frameworks for both are considered in all that follows.

In order to formalise the construction of an arbitrary $n\delta$-term LCFTDZCB- and SLCFTDZCB-system and corresponding LCFTDPK and SLCFTDPK over an arbitrary horizon $[t, t + pn\delta]$, for $p \in \mathbb{N}$, it is useful to provide a definition for FTDFRA liquidity regimes, as well as FTD survival regimes akin to the FTD liquidity regimes from Definition 4.4.3.

Definition 5.1.1 (FTDFRA liquidity regimes)
At an arbitrary time $u+i\delta$, complete $n\delta$-term FTDFRA liquidity over the interval $[u+i\delta, u+m\delta]$, under a liquidity risky setting like that in section 4.4, exists on the set

$$\overline{\mathcal{L}}^n_{u+i\delta,u+m\delta} := \sigma\big(\{\overline{L}^n_{u,i,i+j} = 1\,;\, j \in \mathbb{N}^n_{n,m-i}\}\big) \ , \tag{5.1.1}$$

and complete $n\delta$-term FTDFRA liquidity over the interval $[u+i\delta, u+m\delta]$, under a liquidity and credit risky setting like that in section 4.5, exists on the set

$$\ddot{\mathcal{L}}^n_{u+i\delta,u+m\delta} := \sigma\big(\{\ddot{L}^n_{u,i,i+j} = 1\,;\, j \in \mathbb{N}^n_{n,m-i}\}\big) \ , \tag{5.1.2}$$

where $i, m \in \mathbb{N}_0$ with $i \leq m$ and, as in Definition 4.4.7, the binary random variable $\overline{L}^n_{u,i,i+j}$, and the corresponding random variable $\ddot{L}^n_{u,i,i+j}$, is equal to 1 if perfect systemic liquidity exists for the $j\delta\times(j+n)\delta$ FTDFRA, or is equal to 0 otherwise. Also, it is assumed that

$$\sigma\left(\left\{\overline{L}^n_{u,i,i+j} = 1\right\}\right) \supset \sigma\left(\left\{L^j_{u+i\delta} = 1, L^{j+n}_{u+i\delta} = 1\right\}\right) \ , \tag{5.1.3}$$

and that

$$\sigma\left(\left\{\ddot{L}^n_{u,i,i+j} = 1\right\}\right) \supset \sigma\left(\left\{L^j_{u+i\delta} = 1, L^{j+n}_{u+i\delta} = 1\right\}\right) \ , \tag{5.1.4}$$

which is the analogous assumption to equation (4.4.22) from Definition 4.4.7.
***Shorthand Notation**: For the liquidity risky setting, the following shorthand notation is used:*

$$\overline{\mathcal{L}}^n_{u,i,j} := \overline{\mathcal{L}}^n_{u+i\delta,u+j\delta} \ ,$$

for all $i, j \in \mathbb{N}_0$, with $i \leq j$ and $u \in \mathbb{R}_{\geq 0}$. For the credit and liquidity risky setting, the following shorthand notation is used:

$$\ddot{\mathcal{L}}^n_{u,i,j} := \ddot{\mathcal{L}}^n_{u+i\delta,u+j\delta}$$

for all $i, j \in \mathbb{N}_0$, with $i \leq j$ and $u \in \mathbb{R}_{\geq 0}$.

Definition 5.1.2 (Survival regimes)
Future survival linked to the $n\delta$-term FTD over $[u+i\delta, u+m\delta]$ exists on the set

$$\mathcal{H}^n_{u+i\delta,u+m\delta} := \sigma\big(\{C^n_{u+(j+n)\delta} = 1\ ;\ j \in \mathbb{N}^n_{i,m}\}\big) \tag{5.1.5}$$

if $(m-i) \bmod n = 0$, or $[u+i\delta, u+m\delta)$ otherwise, for $i \in \mathbb{N}_{0,m}$.
***Shorthand Notation**: $\mathcal{H}^n_{u,i,j} := \mathcal{H}^n_{u+i\delta,u+j\delta}$ for all $i, j \in \mathbb{N}_0$, with $i \leq j$ and $u \in \mathbb{R}_{\geq 0}$.*

Analogous to the construction process for:

(a) the δ-term LCFTDZCB-system and its associated LCFTDPK, described in section 4.4; and

(b) the δ-term SLCFTDZCB-system and its associated SLCFTDPK that was described in section 4.5;

the construction of the comparable $n\delta$-term quantities:

(a) $\{\overline{P}^n_{t,in,jn}\,; i, j \in \mathbb{N}_{0,p}\,, i \leq j\}$ and $\{\overline{\pi}^n_{t+jn\delta}\,; j \in \mathbb{N}_{0,p}\}$; and

(b) $\{\ddot{P}^n_{t,in,jn}\,; i, j \in \mathbb{N}_{0,p}\,, i \leq j\}$ and $\{\ddot{\pi}^n_{t+jn\delta}\,; j \in \mathbb{N}_{0,p}\}$, over the interval $[t, t+pn\delta]$;

would require p steps:

- at the current time t and over the set $\{t, t + n\delta, t + 2n\delta, \ldots, t + pn\delta\}$, one would require that $\mathcal{L}^n_{t,0,pn}$ and $\overline{\mathcal{L}}^n_{t,0,pn}$ holds for (a), and $\mathcal{H}^n_{t,0,pn}$ holds for (b);
- at each future time $t + in\delta$ and over the set $\{t + in\delta, t + (i+1)n\delta, \ldots, t + pn\delta\}$, for $i \in \mathbb{N}_{1,p-2}$, one would require that $\mathcal{L}^n_{t,in,pn}$ and $\overline{\mathcal{L}}^n_{t,in,pn}$ holds for (a), and $\mathcal{H}^n_{t,in,pn}$ holds for (b); and
- at the future time $t + (p-1)n\delta$ and over the set $\{t + (p-1)n\delta, t + pn\delta\}$, one would require that $\mathcal{L}^n_{t,(p-1)n,pn}$ holds for (a), and $\mathcal{H}^n_{t,(p-1)n,pn}$ holds for (b).

While the construction of such term-dependent and -consistent quantities is theoretically appealing, maintains the integrity of market rates, and thereby clearly manifests the *market-based approach*, it is far too rigid for real-world pricing, valuation and risk management. This is practically demonstrated by the inability of the $n\delta$-term LCFTDPK or the $n\delta$-term SLCFTDPK to model the natural tenor, or maturity, transformation associated with FTDs through the passage of time, as well as the asynchronicity between calendar (t), term ($n\delta$) and tenor ($pn\delta$) time. The objective from hereon is to adapt this rigid framework to cater for the aforementioned considerations, and this is achieved in a practical manner which culminates in a reduced-form version for this rigid market-based framework or approach.

5.2 Reduced-Form Framework Development

To construct reduced-form versions of the $n\delta$-term LCFTDZCB- and SLCFTDZCB-system, and corresponding LCFTDPK and SLCFTDPK, over an arbitrary time interval $[t + i\delta, t + m\delta]$, where $n \le m$ and $i \in \mathbb{N}_{0,m}$, the following assumptions are required.

Assumption 5.2.1 (STR enables CPL)
At each time $t + i\delta$, the STR enables the CPL regime $\mathcal{L}^{\mathrm{CPL}}_{t,i,m}$.

Assumption 5.2.2 (FTDFRA market-making)
At each time $t+i\delta$, the individual STs have sufficient risk appetite to market-make and enable the liquidity of each $k\delta \times (k+n)\delta$ FTDFRA with fair or mid market FRA rate $\widehat{F}^n_{t,i,i+k}$, for $k \in \mathbb{N}_{1,m-n-i}$. This FTDFRA liquidity regime exists on the set

$$\overline{\mathcal{L}}^{(n)}_{u+i\delta,u+m\delta} := \sigma\big(\{\overline{L}^n_{u,i,i+k} = 1\,;\,, k \in \mathbb{N}_{1,m-n-i}\}\big) \supset \overline{\mathcal{L}}^n_{u+i\delta,u+m\delta}\ ,$$

under a liquidity risk setting, such as that in section 4.4, and on the set

$$\ddot{\mathcal{L}}^{(n)}_{u+i\delta,u+m\delta} := \sigma\big(\{\ddot{L}^n_{u,i,i+k} = 1\,;\,, k \in \mathbb{N}_{1,m-n-i}\}\big) \supset \ddot{\mathcal{L}}^n_{u+i\delta,u+m\delta}\ ,$$

under a liquidity and credit risky setting, such as that in section 4.5, and is therefore a richer set than that defined in Definition 5.1.1. Using Lemma 4.4.1 within this context and the liquidity risky setting, the model fair $k\delta \times (k+n)\delta$ FTDFRA rate is

$$\overline{F}^n_{t,i,i+k} := p^n_{t+(i+k)\delta} F^n_{t,i,i+k}\ ,$$

and under the liquidity and credit risky setting, the fair model rate is

$$1 + n\delta \ddot{F}^n_{t,i,i+k} = \frac{s^n_{t+(i+k)\delta}}{s^{k+n}_{t+i\delta}/s^k_{t+i\delta}} \left[\frac{(1 - p^n_{t+(i+k)\delta}) + s^n_{t+(i+k)\delta} p^n_{t+(i+k)\delta}}{s^n_{t+(i+k)\delta}} + n\delta p^n_{t+(i+k)\delta} F^n_{t,i,i+k} \right]\ ,$$

the form of which the SST may utilise, together with the fair market rate $\widehat{F}^n_{t,i,i+k}$, for the purposes of calibration, which is assumed to be achievable but not detailed any further. Assumption 5.2.1 enables the computation of $F^n_{t,i,i+k}$.

Assumption 5.2.3 (Future $n\delta$-term liquidity and survival)
At each time $t + i\delta$, future $n\delta$-term liquidity according to the set

$$\mathcal{L}^{(n)}_{t,i,m} := \sigma\left(\left\{L^n_{t+(i+k)\delta} = 1\,;k \in \mathbb{N}_{1,m-n-i}\right\}\right) \supset \mathcal{L}^n_{t,i,m}\ ,$$

is assumed to exist, with the definitions of the reduced-form $n\delta$-term LCFTDZCB-system and LCFTDPK being contingent upon thus assumption. Similarly, future survival linked to the $n\delta$-term is defined according to the set

$$\mathcal{H}^{(n)}_{t,i,m} := \sigma\left(\left\{C^n_{t+(i+k+n)\delta} = 1\,;k \in \mathbb{N}_{1,m-n-i}\right\}\right) \supset \mathcal{H}^n_{t,i,m}\ ,$$

which is also assumed to exist, in order to enable the definitions of the reduced-form $n\delta$-term SLCFTDZCB-system and SLCFTDPK.

Assumption 5.2.4 (Reduced-form $n\delta$-term FTDPK)
Using the estimated $n\delta$-term FTDSDF from Definition 4.1.1, and contingent on the CPFL regime, the calibrated reduced-form $n\delta$-term FTDSDF is given by

$$\begin{aligned} D^{(n)}_{t+j\delta} &:= \frac{D^{(n)}_{t+(j-1)\delta}}{\Lambda^{(n)}_{t+(j-1)\delta}\widehat{D}^{(n)}_{t+(j-1)\delta}}\mathbb{E}^{\mathbb{P}}\left[\Lambda^{(n)}_{t+j\delta}\,\widehat{D}^n_{t+j\delta}\,\middle|\,\mathcal{G}_{t+(j-1)\delta}\right] \\ &= \frac{D^{(n)}_{t+(j-1)\delta}}{\widehat{D}^{(n)}_{t+(j-1)\delta}}\mathbb{E}^{\mathbb{D}_{(n)}}\left[\widehat{D}^n_{t+j\delta}\,\middle|\,\mathcal{G}_{t+(j-1)\delta}\right]\ , \end{aligned}$$

which follows from Remark 4.3.2, where the time-inhomogeneous process $(\Lambda^{(n)}_u)_{t\le u\le t+m\delta}$ is a $\{\mathcal{G}_u, \mathbb{P}\}$-martingale, with $\Lambda^{(n)}_t := 1$, that enables a change-of-measure from $\mathbb{P} \to \mathbb{D}_{(n)}$, the reduced-form $n\delta$-term FTD-linked pricing measure. Then, commensurate with the δ-term, the calibrated reduced-form $n\delta$-term FTDPK is defined by $\pi^{(n)}_{t+j\delta} := \Lambda^{(n)}_{t+j\delta}D^{(n)}_{t+j\delta}$, with the time-inhomogeneous parameters associated with $\Lambda^{(n)}_{t+j\delta}$ chosen such that

$$P^{(n)}_{t,i,j} := \frac{1}{\pi^{(n)}_{t+i\delta}}\mathbb{E}^{\mathbb{P}}\left[\pi^{(n)}_{t+j\delta}\,\middle|\,\mathcal{G}_{t+i\delta}\right] = \frac{1}{D^{(n)}_{t+i\delta}}\mathbb{E}^{\mathbb{D}_{(n)}}\left[D^{(n)}_{t+j\delta}\,\middle|\,\mathcal{G}_{t+i\delta}\right] = P^{j-i}_{t,i,j}\ ,$$

which defines the reduced-form $n\delta$-term FTDZCB-system, for $i, j \in \mathbb{N}_{0,m}$ with $i \le j$. Finally, the reduced-form $n\delta$-term rate is defined by

$$R^{(n)}_{t+i\delta} := \frac{1}{n\delta}\left(\frac{1}{P^{(n)}_{t,i,j}} - 1\right)\ ,$$

when $j - i = n$. For $n = 1$, the reduced-form δ-term FTDPK is identical to its counterpart.

It is now possible to define reduced-form versions of the synthetic $n\delta$-term LCFTDZCB- and the SLCFTDZCB-system, the intertemporal values of which will be used in the definition of the corresponding reduced-form versions of the $n\delta$-term LCFTDPK and SLCFTDPK.

Lemma 5.2.1 (Reduced-form synthetic $n\delta$-term LCFTDZCB- and SLCFTDZCB-system)
Given assumptions 5.2.1, 5.2.2, 5.2.3 and 5.2.4, the reduced-form synthetic $n\delta$-term LCFTDZCB-system is defined by

$$\overline{P}^{(n)}_{t,i,j} := \begin{cases} \frac{D^1_{t+(i+1)\delta}}{D^1_{t+i\delta}} \prod_{k=0}^{(j-i-n-1)/n} \left(1+n\delta\overline{F}^n_{t,i,i+nk+1}\right)^{-1} \ , & \operatorname{mod}(j-i,n)=1 \ , \\ \frac{D^2_{t+(i+2)\delta}}{D^2_{t+i\delta}} \prod_{k=0}^{(j-i-n-2)/n} \left(1+n\delta\overline{F}^n_{t,i,i+nk+2}\right)^{-1} \ , & \operatorname{mod}(j-i,n)=2 \ , \\ \vdots \ , & \vdots \ , \\ \frac{D^{n-1}_{t+(i+n-1)\delta}}{D^{n-1}_{t+i\delta}} \prod_{k=0}^{(j-i-2n+1)/n} \left(1+n\delta\overline{F}^n_{t,i,i+nk+n-1}\right)^{-1} \ , & \operatorname{mod}(j-i,n)=n-1 \ , \\ \frac{D^n_{t+(i+n)\delta}}{D^n_{t+i\delta}} \prod_{k=0}^{(j-i-2n)/n} \left(1+n\delta\overline{F}^n_{t,i,i+n(k+1)}\right)^{-1} \ , & \operatorname{mod}(j-i,n)=0 \ , \end{cases} \tag{5.2.1}$$

for $n < (j-i) \le m$, while for $0 \le (j-i) \le n$ the definition resolves to

$$\overline{P}^{(n)}_{t,i,j} := \begin{cases} P^n_{t,j-n,j} \ , & i = j-n \ , \\ P^{n-1}_{t,j-n+1,j} \ , & i = j-(n-1) \ , \\ \vdots \ , & \vdots \ , \\ P^1_{t,j-1,j} \ , & i = j-1 \ , \\ 1 \ , & i = j \ , \end{cases} \tag{5.2.2}$$

with $i,j \in \mathbb{N}_{0,m}$ and $i \le j$. The same definition applies to the reduced-form synthetic $n\delta$-term SLCFT-DZCB-system, except that the fair FTDFRA rates must be replaced with the liquidity and credit risky versions — aesthetically, the overline notation is replaced with the double dot notation.

Proof. At time $t+i\delta$, the present value of 1 unit of currency due at time $t+j\delta$ is equal to

$$P^{j-i}_{t,i,j} = \frac{1}{1+(j-i)\delta R^{j-i}_{t+i\delta}} \ ,$$

provided that $i \le j \le m$, according to Assumption 5.2.1. Therefore, considering a synthetic $n\delta$-term LCFTDZCB with tenor less than or equal to $n\delta$, i.e. $(j-i) \le n$, it follows that $\overline{P}^{(n)}_{t,i,j} = P^{j-i}_{t,i,j}$, which is the result shown in equation (5.2.2).

For the case of $n < (j-i) \le m$ and $\operatorname{mod}(j-i,n) = h$, with $h \in \mathbb{N}_{0,n-1}$, a synthetic $n\delta$-term LCFTDZCB with $(j-i)\delta$-tenor may be constructed using Assumptions 5.2.1, 5.2.2 and 5.2.3 as follows. At time $t+i\delta$, if $h > 0$:

(i) Borrow (Deposit) M units of currency at the $h\delta$-term rate.

(0) Long (Short) the $h\delta \times (h+n)\delta$ fair FRA with nominal equal to $M\frac{D^h_{t+i\delta}}{D^h_{t+(i+h)\delta}}$.

(1) Long (Short) the $(h+n)\delta \times (h+2n)\delta$ fair FRA with nominal equal to

$$M\frac{D^h_{t+i\delta}}{D^h_{t+(i+h)\delta}}\left(1+n\delta\overline{F}^n_{t,i,i+h}\right) \ .$$

$\vdots \qquad\qquad\qquad \vdots$

(N) Long (Short) the $(j-n)\delta \times j\delta$ fair FRA with nominal equal to

$$M\frac{D^h_{t+i\delta}}{D^h_{t+(i+h)\delta}}\left(1+n\delta\overline{F}^n_{t,i,i+h}\right)\left(1+n\delta\overline{F}^n_{t,i,i+h+n}\right)\cdots\left(1+n\delta\overline{F}^n_{t,i,j-2n}\right) .$$

At time $t+(i+h)\delta$, using Assumption 5.2.3:

(i) The loan (deposit) matures which costs (yields): $M\frac{D^h_{t+i\delta}}{D^h_{t+(i+h)\delta}}$.

(ii) Refinance (Re-deposit) the costs (proceeds) from (i) at the $n\delta$-term rate.

At time $t+(i+h+n)\delta$, using Assumption 5.2.3:

(ii) The loan (deposit) matures which costs (yields): $M\frac{D^h_{t+i\delta}}{D^h_{t+(i+h)\delta}}\frac{D^n_{t+(i+h)\delta}}{D^n_{t+(i+h+n)\delta}}$.

(0) The long (short) FRA payoff: $(-)M\frac{D^h_{t+i\delta}}{D^h_{t+(i+h)\delta}}\left[\frac{D^n_{t+(i+h)\delta}}{D^n_{t+(i+h+n)\delta}}-\left(1+n\delta\overline{F}^n_{t,i,i+h}\right)\right]$.

(iii) Add (ii) and (0), and refinance (re-deposit) the costs (proceeds) at the $n\delta$-term rate.

Repeating this process at each time $t+(i+h+nk)\delta$, for $k=2,3,\ldots,(j-i-n-h)/n$, will eventually result in a total cost (yield) equal to

$$M\frac{D^h_{t+i\delta}}{D^h_{t+(i+h)\delta}}\left(1+n\delta\overline{F}^n_{t,i,i+h}\right)\left(1+n\delta\overline{F}^n_{t,i,i+h+n}\right)\cdots\left(1+n\delta\overline{F}^n_{t,i,j-n}\right) .$$

at time $t+j\delta$, which is measurable at time $t+i\delta$. This strategy may then be used to create the synthetic $n\delta$-term LCFTDZCB with $(j-i)\delta$-tenor by setting:

$$M:=\frac{D^h_{t+(i+h)\delta}}{D^h_{t+i\delta}}\left(1+n\delta\overline{F}^n_{t,i,i+h}\right)^{-1}\left(1+n\delta\overline{F}^n_{t,i,i+h+n}\right)^{-1}\cdots\left(1+n\delta\overline{F}^n_{t,i,j-n}\right)^{-1} ,$$

and therefore $\overline{P}^{(n)}_{t,i,j}=M$, which is consistent with equation (5.2.1) for $h>0$. If $h=0$ then the relevant contracts are the $(k+n)\delta\times(k+2n)\delta$ FRA contracts for $k=0,n,2n,\ldots,(j-i-2n)$. The same strategy may be employed for $h=0$, and the corresponding synthetic $n\delta$-term SLCFTDZCB, which completes the proof. □

Using the results from the preceding lemma, along with the reduced-form version of the $n\delta$-term FTDPK, it is possible to define a reduced-form version for the $n\delta$-term LCFTDPK.

Theorem 5.2.1 (Reduced-form $n\delta$-term LCFTDPK)
Maintaining the setup of Lemma 5.2.1, as well as the result thereof, a reduced-form $n\delta$-term LCFTDPK may be defined as

$$\overline{\pi}^{(n)}_{t+j\delta}:=\pi^{(n)}_{t+j\delta}\Theta^{(n)}_{t+j\delta} , \tag{5.2.3}$$

for $j\in\mathbb{N}_{0,m}$, where $\Theta^{(n)}_t:=1$ and

$$\frac{\Theta^{(n)}_{t+j\delta}}{\Theta^{(n)}_{t+i\delta}}=\begin{cases}1 , & 0\le j-i\le n ,\\ \dfrac{A^n_{t+j\delta}}{A^n_{t+i\delta}} , & n<j-i\le m .\end{cases}$$

The process $(A^n_u)_{t\leq u\leq t+m\delta}$ is a time-inhomogeneous $\{\mathcal{G}_u, \mathbb{D}_{(n)}\}$-martingale, with $A^n_t := 1$, that enables a change-of-measure from $\mathbb{D}_{(n)} \to \mathbb{L}_{(n)}$ on $(\mathcal{G}_u)_{t\leq u\leq t+m\delta}$, such that

$$\overline{\pi}^{(n)}_{t+i\delta}\overline{P}^{(n)}_{t,i,j} = \mathbb{E}^{\mathbb{P}}\left[\overline{\pi}^{(n)}_{t+j\delta}\,\middle|\,\mathcal{G}_{t+i\delta}\right] \ ,$$

for all $i, j \in \mathbb{N}_{0,m}$ with $i \leq j$.

Proof. By Assumption 5.2.4 and construction, the reduced-form $n\delta$-term FTDPK recovers the reduced-form synthetic $n\delta$-term LCFTDZCB value for $0 \leq (j-i) \leq n$, i.e., $\overline{P}^{(n)}_{t,i,j} = P^{j-i}_{t,i,j} = P^{(n)}_{t,i,j}$. Therefore, the reduced-form $n\delta$-term FTDPK $\{\pi^{(n)}_{t+j\delta}\,; j \in \mathbb{N}_{0,m}\}$ is a good initial candidate for the reduced-form $n\delta$-term LCFTDPK. However, when $n < (j-i) \leq m$ then $P^{(n)}_{t,i,j} = P^{j-i}_{t,i,j} \neq \overline{P}^{n}_{t,i,j}$. The definition of the $\{\mathcal{G}_u, \mathbb{D}_{(n)}\}$-martingale $\left(A^n_u\right)_{t\leq u\leq t+m\delta}$ enables a change-of-measure such that

$$\mathbb{E}^{\mathbb{P}}\left[\frac{\Lambda^{(n)}_{t+j\delta}}{\Lambda^{(n)}_{t+i\delta}}\frac{\Theta^{n}_{t+j\delta}}{\Theta^{n}_{t+i\delta}}\frac{D^{(n)}_{t+j\delta}}{D^{(n)}_{t+i\delta}}\,\middle|\,\mathcal{G}_{t+i\delta}\right] = \mathbb{E}^{\mathbb{D}_{(n)}}\left[\frac{A^{n}_{t+j\delta}}{A^{n}_{t+i\delta}}\frac{D^{(n)}_{t+j\delta}}{D^{(n)}_{t+i\delta}}\,\middle|\,\mathcal{G}_{t+i\delta}\right] = \mathbb{E}^{\mathbb{L}_{(n)}}\left[\frac{D^{(n)}_{t+j\delta}}{D^{(n)}_{t+i\delta}}\,\middle|\,\mathcal{G}_{t+i\delta}\right] \ ,$$

may be set to the value of $\overline{P}^{n}_{t,i,j}$ by calibrating the free time-dependent parameters associated with $A^n_{t+j\delta}$. Finally, observe that

$$\frac{1}{\overline{\pi}^{(n)}_{t+i\delta}}\mathbb{E}^{\mathbb{P}}\left[\overline{\pi}^{(n)}_{t+j\delta}\,\middle|\,\mathcal{G}_{t+i\delta}\right] = \begin{cases} 1 \ , & j-i=0 \ , \\ P^{j-i}_{t,i,j} \ , & j-i \leq n \ , \\ \dfrac{1}{D^{(n)}_{t+i\delta}}\mathbb{E}^{\mathbb{L}_{(n)}}\left[D^{(n)}_{t+j\delta}\,\middle|\,\mathcal{G}_{t+i\delta}\right] \ , & j-i > n \ , \end{cases}$$

for all $i, j \in \mathbb{N}_{0,m}$ with $i \leq j$, which is the required dynamics and completes the proof. ☐

Using the reduced-form version of the $n\delta$-term LCFTDPK as a based model, it is possible to once again use the results from Lemma 5.2.1 to derive a reduced-form version of the $n\delta$-term SLCFTDPK in a similar fashion to Theorem 5.2.1.

Theorem 5.2.2 (Reduced-form $n\delta$-term SLCFTDPK)
Maintaining the setup of Lemma 5.2.1 and Theorem 5.2.1, as well as the results thereof, a reduced-form $n\delta$-term SLCFTDPK may be defined as

$$\ddot{\pi}^{(n)}_{t+j\delta} := \overline{\pi}^{(n)}_{t+j\delta}X^{(n)}_{t+j\delta} \ , \tag{5.2.4}$$

for $j \in \mathbb{N}_{0,m}$, where $X^{(n)}_t := 1$ and

$$\frac{X^{(n)}_{t+j\delta}}{X^{(n)}_{t+i\delta}} = \begin{cases} 1 \ , & 0 \leq j-i \leq n \ , \\ \dfrac{B^n_{t+j\delta}}{B^n_{t+i\delta}} \ , & n < j-i \leq m \ . \end{cases}$$

The process $(B^n_u)_{t\leq u\leq t+m\delta}$ is a time-inhomogeneous $\{\mathcal{G}_u, \mathbb{L}_{(n)}\}$-martingale, with $B^n_t := 1$, that enables a change-of-measure from $\mathbb{L}_{(n)} \to \mathbb{Q}_{(n)}$ on $(\mathcal{G}_u)_{t\leq u\leq t+m\delta}$, such that

$$\ddot{\pi}^{(n)}_{t+i\delta}\ddot{P}^{(n)}_{t,i,j} = \mathbb{E}^{\mathbb{P}}\left[\ddot{\pi}^{(n)}_{t+j\delta}\,\middle|\,\mathcal{G}_{t+i\delta}\right] \ ,$$

for all $i, j \in \mathbb{N}_{0,m}$ with $i \leq j$.

Proof. The method of proof is almost identical to that for Theorem 5.2.1, with the reduced-form $n\delta$-term LCFTDPK playing the role of the reduced-form $n\delta$-term FTDPK. ☐

5.3 Exchanges of Risk

Consider an EC, with negligible credit risk[1], that has initiated an agreement, at some past time u, for a forward-starting $n\delta$-term FTD-linked loan/deposit with the STR, should there be liquidity at this future time. At the current time $t \geq u$, the remaining tenor until initiation and maturity equals $i\delta$ and $(n+i)\delta$, respectively. Importantly, this agreement does not guarantee the EC liquidity in the respective loan/deposit, nor does it guarantee a fixed rate. Let the cash flow that the EC will pay/receive at maturity time $t+(n+i)\delta$ be denoted by N. Then, at initiation time $t+i\delta$, the value of this cash flow will be

$$NP^n_{t,i,n+i} = N\frac{1}{\ddot{\pi}^n_{t+i\delta}}\mathbb{E}^{\mathbb{P}}\left[\ddot{\pi}^n_{t+(n+i)\delta}\,\middle|\,\mathcal{G}_{t+i\delta}\right] , \tag{5.3.1}$$

since $P^n_{t,i,n+i} = \ddot{P}^n_{t,i,n+i}$ by Theorem 5.2.2 and Lemma 5.2.1, which is the effective nominal or principal value of the underlying transaction. At the current time, the theoretical value of the overall agreement may be calculated with the reduced-form version of the $n\delta$-term SLCFTDPK as follows:

$$N\ddot{P}^{(n)}_{t,0,n+i} = N\frac{1}{\ddot{\pi}^{(n)}_t}\mathbb{E}^{\mathbb{P}}\left[\ddot{\pi}^{(n)}_{t+i\delta}P^n_{t,i,n+i}\,\middle|\,\mathcal{G}_t\right] , \tag{5.3.2}$$

again since $P^n_{t,i,n+i} = \ddot{P}^{(n)}_{t,i,n+i}$, by construction. Since there is no guarantee in this agreement, the value above is theoretical and need not be exchanged between the EC and STR. If the future transaction is a deposit (loan), then the first equation above is the amount that the EC must pay to (receive from) the STR. At any time prior to initiation, the EC or the STR may wish to change the nature of the future transaction. They have one of the following options:

(i) early terminate the agreement prior to initiation;

(ii) restructure interest and capital cash flows while preserving the effective term rate;

(iii) restructure liquidity characteristics while preserving credit characteristics;

(iv) restructure both liquidity and credit characteristics;

(v) restructure the remaining tenor until maturity, from the initiation time; or

(vi) perform (v) and then (ii), (iii) or (iv).

Option (i) bears no current risk, nor any exchange of cash flows, but the STR potentially loses future: (a) term finance in the case of a deposit; or (b) interest income in the case of a loan.

In the case of a loan, option (ii) is beneficial to the STR since it may finance the loan via a corresponding FTD and then demand for periodic loan repayments (capital plus interest) at an *internal rate of return* that matches the effective cost of the aforementioned term finance, and hence the corresponding term rate as well. In reality, such a mechanism reduces credit risk exposure for the STR. In the case of a deposit, this option allows the STR to offer periodic interest payments to the EC that matches the effective term rate. The EC will therefore have improved liquidity, while effectively securing a term rate. It should be noted however, that restructured interest rate and periodic capital payments in an

[1] This assumption or caveat is required in order to avoid the need for any idiosyncratic credit risk assessment, thereby maintaining symmetry in the pricing and valuation of FTD-linked deposits and loans.

amortising capital structure, will affect the *effective duration* of the term loan/deposit, which will in turn impact the effective term rate offered by the STR. Such a restructure will therefore overlap with option (v). The restructured cash flow for option (ii) at time $t+(n+i)\delta$ may be generally represented as

$$N\frac{P^{n}_{t,i,n+i}}{P^{(n)}_{t,i,n+i}} = N \ , \tag{5.3.3}$$

since $P^{(n)}_{t,i,n+i}$, which is equivalent in value to $P^{n}_{t,i,n+i}$ by construction, is the reduced-form version of the $n\delta$-term FTDZCB from Assumption 5.2.4, and thereby encodes potential interest rate structures that yield the same effective $n\delta$-term FTD rate. The reduced-form $n\delta$-term SLCFTDPK is still valid in this option, hence the current and initiation time valuation remains unchanged. Apart from the restructured interest cash flows, this option is therefore equivalent to the original agreement.

Option (iii) leaves the effective nominal of the transaction unchanged, at initiation time, but seeks to change the liquidity characteristics from the $n\delta$-term to a $j\delta$-term, where $j<n$. From the vantage point of the initiation time, the restructured cash flow at time $t+(n+i)\delta$ then becomes

$$N\frac{P^{n}_{t,i,n+i}}{\overline{P}^{(j)}_{t,i,n+i}} < N \ , \tag{5.3.4}$$

which should now be less than N in value, due to the improved liquidity or liquidation characteristics offered by the $j\delta$-term. Similarly, option (iv) seeks a restructure of both liquidity and credit characteristics from the $n\delta$-term to a $j\delta$-term, where $j<n$, with the restructured cash flow at time $t+(n+i)\delta$ now becoming

$$N\frac{P^{n}_{t,i,n+i}}{\ddot{P}^{(j)}_{t,i,n+i}} < N\frac{P^{n}_{t,i,n+i}}{\overline{P}^{(j)}_{t,i,n+i}} < N \ , \tag{5.3.5}$$

which should be less than the corresponding cash flow above, i.e., equation (5.3.3), since this conversion alters both the liquidity and credit risk characteristics from the $n\delta$- to the $j\delta$-term. Option (iv) is therefore more natural, since credit risks are naturally linked to the availability of liquidity, as articulated in section 4.5. However, option (iii) offers a novel mechanism for the conversion of a capital asset/transaction/rate into a corresponding funding asset/transaction/rate — recall the distinction made between capital and funding assets in section 1.3.2. The value of these terminal cash flows from options (iii) and (iv) at the initiation time is given by

$$NP^{n}_{t,i,n+i} = \frac{1}{\overline{\pi}^{j}_{t+i\delta}}\mathbb{E}^{\mathbb{P}}\left[\overline{\pi}^{j}_{t+(n+i)\delta}\, N\frac{P^{n}_{t,i,n+i}}{\overline{P}^{(j)}_{t,i,n+i}}\,\middle|\,\mathcal{G}_{t+i\delta}\right] \ , \tag{5.3.6}$$

and

$$NP^{n}_{t,i,n+i} = \frac{1}{\ddot{\pi}^{j}_{t+i\delta}}\mathbb{E}^{\mathbb{P}}\left[\ddot{\pi}^{j}_{t+(n+i)\delta}\, N\frac{P^{n}_{t,i,n+i}}{\ddot{P}^{(j)}_{t,i,n+i}}\,\middle|\,\mathcal{G}_{t+i\delta}\right] \ , \tag{5.3.7}$$

respectively, and is therefore unchanged from the original agreement in both cases. The choice to change the risk characteristics of the future transaction at the time of initiation implies a change of the underlying numeraire and PK from the $n\delta$- to the $j\delta$-term, when viewed from the vantage point of the current time. This is enabled by an exchange of units between the respective numeraires at time

$t + i\delta$, when viewed from the vantage point of time t, which yields

$$\frac{\ddot{\pi}^{(n)}_{t+i\delta}}{\bar{\pi}^{(j)}_{t+i\delta}} NP^{(n)}_{t,i,n+i} \quad \text{and} \quad \frac{\ddot{\pi}^{(n)}_{t+i\delta}}{\ddot{\pi}^{(j)}_{t+i\delta}} NP^{(n)}_{t,i,n+i} \,, \tag{5.3.8}$$

in the case of option (iii) and (iv), respectively, with the reduced-form quantities ensuring time synchronicity between the $j\delta$- and $n\delta$-terms. The theoretical value of the new agreement at the current time is then

$$N\ddot{P}^{(n)}_{t,0,n+i} = \frac{1}{\bar{\pi}^{(j)}_t} \mathbb{E}^{\mathbb{P}} \left[\bar{\pi}^{(j)}_{t+i\delta} \frac{\ddot{\pi}^{(n)}_{t+i\delta}}{\bar{\pi}^{(j)}_{t+i\delta}} NP^{(n)}_{t,i,n+i} \,\middle|\, \mathcal{G}_t \right] \,, \tag{5.3.9}$$

in the case of option (iii), since $\ddot{\pi}^{(n)}_t = \bar{\pi}^{(j)}_t := 1$, and

$$N\ddot{P}^{(n)}_{t,0,n+i} = \frac{1}{\ddot{\pi}^{(j)}_t} \mathbb{E}^{\mathbb{P}} \left[\ddot{\pi}^{(j)}_{t+i\delta} \frac{\ddot{\pi}^{(n)}_{t+i\delta}}{\ddot{\pi}^{(j)}_{t+i\delta}} NP^{(n)}_{t,i,n+i} \,\middle|\, \mathcal{G}_t \right] \,, \tag{5.3.10}$$

in the case of option (iv), since $\ddot{\pi}^{(n)}_t = \ddot{\pi}^{(j)}_t := 1$. Again, this initial theoretical value is unchanged from the original agreement, as required.

Option (v) yields similar intuitions and results as that for options (iii) and (iv), assuming a conversion from a $n\delta$ to a $j\delta$ tenor for the future FTD-linked deposit/loan, then the terminal cash flow at time $t + (j+i)\delta$ becomes

$$N \frac{P^n_{t,i,n+i}}{P^j_{t,i,j+i}} \,, \tag{5.3.11}$$

the value of this cash flow at initiation time is then given by

$$NP^n_{t,i,n+i} = \frac{1}{\ddot{\pi}^j_{t+i\delta}} \mathbb{E}^{\mathbb{P}} \left[\ddot{\pi}^j_{t+(j+i)\delta} N \frac{P^n_{t,i,n+i}}{P^j_{t,i,j+i}} \,\middle|\, \mathcal{G}_{t+i\delta} \right] \,, \tag{5.3.12}$$

the change of numeraire at the future time $t + i\delta$ is given by

$$\frac{\ddot{\pi}^{(n)}_{t+i\delta}}{\ddot{\pi}^{(j)}_{t+i\delta}} NP^{(n)}_{t,i,n+i} \,, \tag{5.3.13}$$

so that the theoretical value of this new agreement at time t is

$$N\ddot{P}^{(n)}_{t,0,n+i} = \frac{1}{\ddot{\pi}^{(j)}_t} \mathbb{E}^{\mathbb{P}} \left[\ddot{\pi}^{(j)}_{t+i\delta} \frac{\ddot{\pi}^{(n)}_{t+i\delta}}{\ddot{\pi}^{(j)}_{t+i\delta}} NP^{(n)}_{t,i,n+i} \,\middle|\, \mathcal{G}_t \right] \,, \tag{5.3.14}$$

since $\ddot{\pi}^{(n)}_t = \ddot{\pi}^{(j)}_t := 1$. Being a combination of previous options, option (vi) follows in a similar manner to that already shown for (v), (ii), (iii) and (iv). One may also draw similar inferences as was made for option (i) regarding the effect on the STR, recorded in points (a) and (b). Theses effects across each of the options are rather intuitive, once one grasps a financial economic understanding of each of the FTDZCB systems that are at play.

Each of the restructuring options (ii)-(vi) actually enables an *exchange of risk*, in relation to the agreement between the EC and the STR. The next lemma generalises this concept, using the reduced-form quantities defined in Lemma 5.2.1, and Theorems 5.2.1 and 5.2.2.

Lemma 5.3.1 (Exchanges of risk)
Assuming that the current time is t, a fixed cash flow N that is due at a future time $t+(i+j_1)\delta$ with a fixed accrual rate based on the $n_1\delta$-term FTD-linked rate that is determined at time $t+i\delta$ may be exchanged for an equivalent cash flow

$$N\frac{\ddot{\pi}^{(n_1)}_{t+i\delta}\,\ddot{P}^{(n_1)}_{t,i,i+j_1}}{\ddot{\pi}^{(n_2)}_{t+i\delta}\,\ddot{P}^{(n_2)}_{t,i,i+j_2}}\ , \tag{5.3.15}$$

if the $n_1\delta$- and $n_2\delta$-term FTD-linked rates are both liquidity and credit risky; or

$$N\frac{\ddot{\pi}^{(n_1)}_{t+i\delta}\,\ddot{P}^{(n_1)}_{t,i,i+j_1}}{\overline{\pi}^{(n_2)}_{t+i\delta}\,\overline{P}^{(n_2)}_{t,i,i+j_2}}\ , \tag{5.3.16}$$

if the $n_1\delta$-term FTD-linked rate is both liquidity and credit risky and the $n2\delta$-term FTD-linked rate is only liquidity risky; or

$$N\frac{\overline{\pi}^{(n_1)}_{t+i\delta}\,\overline{P}^{(n_1)}_{t,i,i+j_1}}{\ddot{\pi}^{(n_2)}_{t+i\delta}\,\ddot{P}^{(n_2)}_{t,i,i+j_2}}\ , \tag{5.3.17}$$

if the $n_1\delta$-term FTD-linked rate is only liquidity risky and the $n2\delta$-term FTD-linked rate is both liquidity and credit risky; or

$$N\frac{\overline{\pi}^{(n_1)}_{t+i\delta}\,\overline{P}^{(n_1)}_{t,i,i+j_1}}{\overline{\pi}^{(n_2)}_{t+i\delta}\,\overline{P}^{(n_2)}_{t,i,i+j_2}}\ , \tag{5.3.18}$$

if the $n_1\delta$- and $n_2\delta$-term FTD-linked rates are both liquidity risky, which is payable at time $t+(i+j_2)\delta$, where $i, j_1, j_2 \in \mathbb{N}_0$ and $n_1, n_2 \in \mathbb{N}$.

Proof. Using the reduced-form version of the $n_1\delta$-term SLCFTDPK or LCFTDPK, the original cash flow has a value of $N\ddot{P}^{(n_1)}_{t,0,i+j_1}$ in relation to the first two cases, and $N\overline{P}^{(n_1)}_{t,0,i+j_1}$ in relation to the last two cases, respectively, at the current time t. At the interim time $t+i\delta$, the values of these cash flows are $N\ddot{P}^{(n_1)}_{t,i,i+j_1}$ and $N\overline{P}^{(n_1)}_{t,i,i+j_1}$, respectively. Exchanging the underlying FTD-linked term rate requires an exchange of units across the respective numeraires at the determination time of the respective term-dependent accrual rates. This is enabled by equations (5.3.15), (5.3.16), (5.3.17) and (5.3.18).

Assuming that the current time is $t+i\delta$, the value of the exchanged cash flow in case one is

$$N\ddot{P}^{(n_1)}_{t,i,i+j_1} = \frac{1}{\ddot{\pi}^{(n_2)}_{t+i\delta}}\mathbb{E}^{\mathbb{P}}\left[\ddot{\pi}^{(n_2)}_{t+(i+j_2)\delta}\,N\frac{\ddot{\pi}^{(n_1)}_{t+i\delta}\,\ddot{P}^{(n_1)}_{t,i,i+j_1}}{\ddot{\pi}^{(n_2)}_{t+i\delta}\,\ddot{P}^{(n_2)}_{t,i,i+j_2}}\,\middle|\,\mathcal{G}_{t+i\delta}\right]\ , \tag{5.3.19}$$

since $\ddot{\pi}^{(n_1)}_{t+i\delta} = \ddot{\pi}^{(n_2)}_{t+i\delta} = 1$, standing at time $t+i\delta$, and since $\ddot{P}^{(n_2)}_{t,i,i+j_2} = \mathbb{E}^{\mathbb{P}}[\ddot{\pi}^{(n_2)}_{t+(i+j_2)\delta}\,|\,\mathcal{G}_{t+i\delta}]/\ddot{\pi}^{(n_2)}_{t+i\delta}$. The same result is applicable for the other three cases.

Assuming that the current time is t, the value of the exchanged cash flow in case one is

$$N\ddot{P}^{(n_1)}_{t,0,i+j_1} = \frac{1}{\ddot{\pi}^{(n_2)}_{t}}\mathbb{E}^{\mathbb{P}}\left[\ddot{\pi}^{(n_2)}_{t+(i+j_2)\delta}\,N\frac{\ddot{\pi}^{(n_1)}_{t+i\delta}\,\ddot{P}^{(n_1)}_{t,i,i+j_1}}{\ddot{\pi}^{(n_2)}_{t+i\delta}\,\ddot{P}^{(n_2)}_{t,i,i+j_2}}\,\middle|\,\mathcal{G}_{t}\right]\ , \tag{5.3.20}$$

since the tower property of conditional expectations and $\ddot{\pi}^{(n_2)}_{t+i\delta}\ddot{P}^{(n_2)}_{t,i,i+j_2} = \mathbb{E}^{\mathbb{P}}[\ddot{\pi}^{(n_2)}_{t+(i+j_2)\delta}\,|\,\mathcal{G}_{t+i\delta}]$ may be utilised to resolve the expectation to

$$N\mathbb{E}^{\mathbb{P}}\left[\ddot{\pi}^{(n_1)}_{t+i\delta}\,\ddot{P}^{(n_1)}_{t,i,i+j_1}\,\middle|\,\mathcal{G}_t\right] = N\ddot{\pi}^{(n_1)}_{t}\,\ddot{P}^{(n_1)}_{t,0,i+j_1}\ , \tag{5.3.21}$$

and since $\ddot{\pi}_t^{(n_1)} = \ddot{\pi}_t^{(n_2)} = 1$. The same approach may be utilised for the other three cases to show that values of the original and exchanged cash flows align at times t and $t+i\delta$, respectively, as required. $\square$

5.4 Pricing and Valuation

The next result shows how the stochastic cash flow given by equation (5.3.15) may be exchanged for an equivalent fixed cash flow through the creation of a forward contract.

Corollary 5.4.1 (Fair forward pricing)
The equivalent exchanged cash flow, given by equation (5.3.15), may be considered as a fixed cash flow H that is due at time $t+(i+j_2)\delta$ with a fixed accrual rate derived from the reduced-form version of the $n_2\delta$-term SLCFTDPK over the interval $[t, t+(i+j_2)\delta]$.

Proof. Consider a forward contract with the following terminal payoff:

$$V_{t+(i+j_2)\delta} := N \frac{\ddot{\pi}_{t+i\delta}^{(n_1)} \ddot{P}_{t,i,i+j_1}^{(n_1)}}{\ddot{\pi}_{t+i\delta}^{(n_2)} \ddot{P}_{t,i,i+j_2}^{(n_2)}} - H \ , \tag{5.4.1}$$

where H is a fixed strike price. Such a contract enables the exchange of the floating/uncertain cash flow (5.3.15), measurable at time $t+i\delta$ and payable at time $t+(i+j_2)\delta$, for a fixed cash flow H at time $t+(i+j_2)\delta$. Since the underlying cash flow naturally accrues interest that is defined by the $n_2\delta$-term SLCFTDPK, the term, survival and liquidity consistent current value of such a forward is given by

$$\begin{aligned} \ddot{\pi}_t^{(n_2)} V_t &= \mathbb{E}^{\mathbb{P}}\left[\ddot{\pi}_{t+(i+j_2)\delta}^{(n_2)} V_{t+(i+j_2)\delta} \,\middle|\, \mathcal{G}_t\right] \\ V_t &= \frac{N}{\ddot{\pi}_t^{(n_2)}} \mathbb{E}^{\mathbb{P}}\left[\ddot{\pi}_{t+i\delta}^{(n_1)} \ddot{P}_{t,i,i+j_1}^{(n_1)} \,\middle|\, \mathcal{G}_t\right] - \frac{H}{\ddot{\pi}_t^{(n_2)}} \mathbb{E}^{\mathbb{P}}\left[\ddot{\pi}_{t+(i+j_2)\delta}^{(n_2)} \,\middle|\, \mathcal{G}_t\right] \\ &= N \frac{\ddot{\pi}_t^{(n_1)}}{\ddot{\pi}_t^{(n_2)}} \ddot{P}_{t,0,i+j_1}^{(n_1)} - H \ddot{P}_{t,0,i+j_1}^{(n_2)} \ , \end{aligned}$$

where the first term on the right hand side of the second line follows by the tower property of conditional expectation, since $V_{t+(i+j_2)\delta}$ is $\mathcal{G}_{t+i\delta}$-measurable. Setting $V_t = 0$, and solving for H reveals the fair forward price to be

$$H = N \frac{\ddot{\pi}_t^{(n_1)} \ddot{P}_{t,0,i+j_1}^{(n_1)}}{\ddot{\pi}_t^{(n_2)} \ddot{P}_{t,0,i+j_2}^{(n_2)}} = N \frac{\ddot{P}_{t,0,i+j_1}^{(n_1)}}{\ddot{P}_{t,0,i+j_2}^{(n_2)}} \ , \tag{5.4.2}$$

recalling that $\ddot{\pi}_t^{(n_1)} = \ddot{\pi}_t^{(n_2)} = 1$. Lastly, observe that the fair forward price is $\mathcal{G}_t$-measurable, due at time $t+(i+j_2)\delta$ and accrues interest commensurate with the reduced-form version of the $n_2\delta$-term SLCFTDZCB over the interval $[t, t+(i+j_2)\delta]$, as required. $\square$

Corollary 5.4.1 may be repeated for the exchanged cash flows given by equations (5.3.16), (5.3.17) and (5.3.18), which will yield the following fair forward prices: $N\ddot{P}_{t,0,i+j_1}^{(n_1)} \big/ \overline{P}_{t,0,i+j_2}^{(n_2)}$, $N\overline{P}_{t,0,i+j_1}^{(n_1)} \big/ \ddot{P}_{t,0,i+j_2}^{(n_2)}$, and $N\overline{P}_{t,0,i+j_1}^{(n_1)} \big/ \overline{P}_{t,0,i+j_2}^{(n_2)}$, respectively. Thus far, a general $i\delta \times (i+n)\delta$ FTDFRA, considered from the vantage point of time t, inherited its potential liquidity characteristics from the associated FTDs, which are the $i\delta$-, $(i+n)\delta$- and $n\delta$-term FTDs, and is therefore naturally potentially tradable on the set $\{t, t+i\delta, t+(i+n)\delta\}$. The result of Corollary 5.4.1 provides a mechanism to extend the potential tradability of a general $i\delta \times (i+n)\delta$ FTDFRA to the set:

$$\{t, t+\delta, t+2\delta, \ldots, t+(i+n-1)\delta, t+(i+n)\delta\} \ ,$$

i.e., to the δ-term frequency over the interval $[t, t+(i+n)\delta]$. Such an FTDFRA is more likely to be traded as a derivative security in a real-world setting — the market-making mechanics of which is summarised in the next theorem.

Theorem 5.4.1 (Tradable $i\delta \times (i+n)\delta$ FTDFRA)
At time $t < t+i\delta$, the tradable version of the $i\delta \times (i+n)\delta$ FTDFRA has a fair strike rate given by

$$F^{1,n}_{t,0,i} := \frac{\ddot{P}^{(n)}_{t,0,i+n}}{\ddot{P}^{1}_{t,0,i+n}} \ddot{F}^{n}_{t,0,i} \, , \tag{5.4.3}$$

so that the fair value of such a FTDFRA with arbitrary strike rate H is

$$V_t = \alpha N \ddot{P}^{1}_{t,0,i+n} \left[F^{1,n}_{t,0,i} - H \right] n\delta \, , \tag{5.4.4}$$

where the δ-term FTD-linked rate is representative of an ONRR, or is an overnight FTDRR.

Proof. Firstly, an exchange of risk is required to convert the standard FTDFRA payoff

$$\alpha N \left[R^{n}_{t+i\delta} - K \right] n\delta \, ,$$

into an equivalent traded version that may be liquidated at the highest frequency, the δ-term. This is achieved by applying the result from Lemma 5.3.1, with $n_1 = n$, $n_2 = 1$ and $j_1 = j_2 = n$, which yields the equivalent exchanged payoff

$$V_{t+(i+n)\delta} = \alpha N \frac{\ddot{\pi}^{(n)}_{t+i\delta} \ddot{P}^{(n)}_{t,i,i+n}}{\ddot{\pi}^{1}_{t+i\delta} \ddot{P}^{1}_{t,i,i+n}} \left[R^{n}_{t+i\delta} - K \right] n\delta \, , \tag{5.4.5}$$

while an application of Corollary 5.4.1 enables the equivalent representation of the exchanged fixed strike rate as a new fixed strike rate $H := K \, \ddot{P}^{(n)}_{t,0,i+n} / \ddot{P}^{1}_{t,0,i+n}$. The current value of the tradable version of the FTDFRA is then given by

$$\begin{aligned} V_t &= \frac{1}{\ddot{\pi}^{1}_{t}} \mathbb{E}^{\mathbb{P}} \left[\ddot{\pi}^{1}_{t+(i+n)\delta} V_{t+(i+n)\delta} \,\middle|\, \mathcal{G}_t \right] \\ &= \alpha N \left(\mathbb{E}^{\mathbb{P}} \left[\ddot{\pi}^{(n)}_{t+i\delta} \ddot{P}^{(n)}_{t,i,i+n} R^{n}_{t+i\delta} \,\middle|\, \mathcal{G}_t \right] - H \ddot{P}^{1}_{t,0,i+n} \right) n\delta \\ &= \alpha N \ddot{P}^{1}_{t,0,i+n} \left[\frac{\ddot{P}^{(n)}_{t,0,i+n}}{\ddot{P}^{1}_{t,0,i+n}} \ddot{F}^{n}_{t,0,i} - H \right] n\delta \, , \end{aligned}$$

where the second line follows by the use of the tower property and from the fact that

$$\mathbb{E}^{\mathbb{P}} \left[\ddot{\pi}^{1}_{t+(i+n)\delta} \,\middle|\, \mathcal{G}_{t+i\delta} \right] = \ddot{\pi}^{1}_{t+i\delta} \ddot{P}^{1}_{t,i,i+n} \, ,$$

and that

$$\mathbb{E}^{\mathbb{P}} \left[\ddot{\pi}^{(n)}_{t+i\delta} \ddot{P}^{(n)}_{t,i,i+n} R^{n}_{t+i\delta} \,\middle|\, \mathcal{G}_t \right] = \ddot{\pi}^{(n)}_{t} \ddot{P}^{(n)}_{t,0,i+n} \ddot{F}^{n}_{t,0,i} \, .$$

Setting V_t equal to 0 and solving for H yields the fair FTDFRA strike rate, given by equation (5.4.3). Then, for any tradable version of an FTDFRA that references the $n\delta$-term rate with arbitrary strike rate H, the value of such a FTDFRA at time t is given by equation (5.4.4). □

Remark 5.4.1 (Alternative exchanges of risk and tradable fair FTDFRA rates)
While the version of the tradable FTDFRA presented in Theorem 5.4.1 is the most natural, encapsulating both liquidity and credit risk and exchanges thereof across the $n\delta$- and δ-terms, it is also possible to create other versions of the tradable FTDFRA. This is possible using the alternate versions of the exchanges of risk from Lemma 5.3.1, viz., those given by equations (5.3.16), (5.3.17) and (5.3.18). Then, through similar versions of Corollary 5.4.1, it is possible to define the following fair FTDFRA rates:

$$F^{1,n}_{t,0,i} := \frac{\ddot{P}^{(n)}_{t,0,i+n}}{\overline{P}^{1}_{t,0,i+n}}\ \ddot{F}^{n}_{t,0,i}\ , \tag{5.4.6}$$

when the tradable FTDFRA and exchange of risk is based on equation (5.3.16), or

$$F^{1,n}_{t,0,i} := \frac{\overline{P}^{(n)}_{t,0,i+n}}{\ddot{P}^{1}_{t,0,i+n}}\ \overline{F}^{n}_{t,0,i}\ , \tag{5.4.7}$$

when the tradable FTDFRA and exchange of risk is based on equation (5.3.17), or

$$F^{1,n}_{t,0,i} := \frac{\overline{P}^{(n)}_{t,0,i+n}}{\overline{P}^{1}_{t,0,i+n}}\ \overline{F}^{n}_{t,0,i}\ , \tag{5.4.8}$$

when the tradable FTDFRA and exchange of risk is based on equation (5.3.18).

From Theorem 5.4.1, Remark 5.4.1, and understanding that the values of the set of $n\delta$-term FTDZCBs will be less than that of the corresponding δ-term FTDZCBs with $n\delta$ tenor, in general, it should be clear that the fair strike rate associated with the tradable version of an FTDFRA is less than that associated with the corresponding generic FTDFRAs that were considered in Chapter 4. This means that

$$F^{1,n}_{t,0,i} \leq \ddot{F}^{n}_{t,0,i} \leq \overline{F}^{n}_{t,0,i} \leq F^{n}_{t,0,i}\ ,$$

is the expected ordering of fair $i\delta \times (i+n)\delta$ FTDFRA rates, in the natural case of considering both liquidity and credit risks in the pricing of the tradable version of the FTDFRA, i.e., Theorem 5.4.1.

One unique feature about the postulated tradable version of the FTDFRA is that the mechanism utilised to exchange risk across term and risk characteristics results in an FTDFRA that incorporates a floating rate that resets at the exchanged version of the underlying reference rate. This is expected, given the theory that has been developed in this and the previous chapter. However, this is not the standard convention for FRA products that are defined and traded in practice — these and other related features and observations are recorded in the next couple of remarks.

Remark 5.4.2 (Tradable version of the FTDFRA on and after the reset time)
At expiry time $t+(i+n)\delta$, using equation (5.4.4), the tradable FTDFRA has the following payoff:

$$\alpha N \left[\frac{P^{(n)}_{t,i,i+n}}{\ddot{P}^{1}_{t,i,i+n}} R^{n}_{t+i\delta} - H \right] n\delta\ ,$$

since $P^{(n)}_{t,i,i+n} = \ddot{P}^{(n)}_{t,i,i+n}$ by definition and construction, and since $\ddot{\pi}^{(n)}_{t+i\delta} = \ddot{\pi}^{1}_{t+i\delta} = 1$ from the vantage point at time $t+i\delta$. Therefore, at any interim time between reset and expiry the FTDFRA's fair value is

$$\ddot{P}^{1}_{t,j,i+n} \alpha N \left[\frac{P^{(n)}_{t,i,i+n}}{\ddot{P}^{1}_{t,i,i+n}} R^{n}_{t+i\delta} - H \right] n\delta\ ,$$

where $j \in \{i, i+1, \ldots, i+n\}$, which does not conform with the standard market definition for a FRA.

Remark 5.4.3 (A possible standard market-defined FTDFRA)
If the current time is t, a possible definition for an $i\delta \times (i+n)\delta$ FTDFRA that conforms with the standard market specification of a FRA is a terminal payoff equal to $\alpha N \left[R^n_{t+i\delta} - K\right] n\delta$, with the current value computed as

$$\begin{aligned} V_t &= \alpha N \ddot{P}^1_{t,0,i+n} \left[\mathbb{E}^{\mathbb{P}} \left[\frac{\ddot{\pi}^1_{t+(i+n)\delta}}{\ddot{\pi}^1_t \ddot{P}^1_{t,0,i+n}} R^n_{t+i\delta} \,\middle|\, \mathcal{G}_t \right] - K \right] n\delta \\ &= \alpha N \ddot{P}^1_{t,0,i+n} \left[\mathbb{E}^{\mathbb{Q}^1_{t,0,i+n}} \left[R^n_{t+i\delta} \,\middle|\, \mathcal{G}_t \right] - K \right] n\delta \ , \end{aligned}$$

where $\mathbb{Q}^1_{t,0,i+n}$ is the relevant $t+(i+n)\delta$-forward measure that is associated with the δ-term FTD-linked pricing measure. Therefore, the fair FTDFRA rate becomes

$$F^{1,n}_{t,0,i} := \mathbb{E}^{\mathbb{Q}^1_{t,0,i+n}} \left[R^n_{t+i\delta} \,\middle|\, \mathcal{G}_t \right] \ ,$$

which generally requires a convexity adjustment or an exogenous model to be specified, in order to enable this quantity to be a martingale under the $t+(i+n)\delta$-forward measure. Intuitively then, the market approach is to consider the underlying reference rate merely as a reference indicator, without any cognisance of the associated primitive or fundamental FTD-linked deposit and loan characteristics.

Remark 5.4.3 reveals that standardly market-defined FRAs generally require the specification of an exogenous model that drives the FRA rate process. Ignoring the financial economic interpretation and underpinnings of the fair FTDFRA rate process that was derived in Theorem 5.4.1, this process may be used as the exogenous model for the fair FRA rate process associated with a standardly market-defined FTDFRA. While this offers a potentially highly tractable approach, this would come at the cost of losing the financial economic interpretation of the mechanism for exchanging risk.

This issue of the standard market definition of a FRA not matching that derived from first principles, in the presence of liquidity and credit risks, is yet another ramification of the GFC and a feature of the post GFC market context. This structural difference between practically traded FRAs and theoretically consistent FRAs, like the FTDFRAs defined in this work, may be considered as yet another contributor to *funding-swap dislocation* post the GFC. Therefore, when attempting to model post GFC bank-related interest rate markets, it is conjectured that one has to generally develop models for primitive and derivative markets separately. Put differently, the fact that market participants are agnostic of the fundamental nature of interest rates and the impact thereof on their structural arbitrage-free interactions when defining derivative instruments, implies that one has to generally develop models for primitive and derivative markets separately — these findings will feature again in Part II.

5.5 Models for BFRRs and the Segue to Part II

Part I has evolved from the development of a market-based systemic multi-curve framework for the FTD market, to the development of an equivalent reduced-form version of the same framework. Careful consideration and attention was ascribed to the nature of liquidity and credit risks within the systemic and symmetric FTD market setting, which enabled the development of theory from first principles. Below, Table 5.1 summarises the main results from the last two chapters, which is the construction of a set of pricing kernels, measures and associated ZCB-systems under a risk-free, liquidity risky, and liquidity and credit risky setting. Also, the progression from strict market-based to reduced-form versions of the aforementioned quantities is also recorded in this table. In particular, take note of the indices that support the respective pricing kernel and ZCB processes, which are $n \in \{1, 2, \ldots, m\}$; $i \leq j \in \{0, 1, 2, \ldots, \lfloor m/n \rfloor\}$; and $h \leq k \in \{0, 1, 2, \ldots, m-1, m\}$.

Framework	Context	Pricing Kernel	Measure	ZCB-System
Market-Based	Risk-Free	$\pi^n_{t+jn\delta}$	$\mathbb{D}_n$	$P^n_{t,in,jn}$
	Liquidity Risky	$\overline{\pi}^n_{t+jn\delta}$	$\mathbb{L}_n$	$\overline{P}^n_{t,in,jn}$
	Liquidity & Credit Risky	$\ddot{\pi}^n_{t+jn\delta}$	$\mathbb{Q}_n$	$\ddot{P}^n_{t,in,jn}$
Reduced-Form	Risk-Free	$\pi^{(n)}_{t+k\delta}$	$\mathbb{D}_{(n)}$	$P^{(n)}_{t,h,k}$
	Liquidity Risky	$\overline{\pi}^{(n)}_{t+k\delta}$	$\mathbb{L}_{(n)}$	$\overline{P}^{(n)}_{t,h,k}$
	Liquidity & Credit Risky	$\ddot{\pi}^{(n)}_{t+k\delta}$	$\mathbb{Q}_{(n)}$	$\ddot{P}^{(n)}_{t,h,k}$

Tab. 5.1: A summary of the key results obtained for the FTD market.

The respective FTDZCB-, LCFTDZCB- and SLCFTDZCB-systems may be market-made by the SST under Assumptions 3.3.1, 3.3.2 and 3.3.3, respectively, along with the assumptions outlined in section 5.2. Therefore, the market-making process underpinning each system is not void of risk, requiring the relevant market-makers to have an adequate amount of risk appetite in order to bear such residual risks. Based on these observations, it is conjectured that such ZCB-systems may co-exist, with the relevant market-makers precluding any arbitrage opportunities.

While the theory that has been derived in Part I is intricately linked to the nature of the FTD market, or a frontier economy as described in Definition 1.4.1, the multi-curve and exchange of risk principles may be theoretically generalised and made applicable to any economy with multiple interest rate markets. This is the main objective for Part II, and is achieved within the contexts of emerging and developed economies, again as described in Definition 1.4.1. In particular, the bank funding and its associated linear derivatives market will be the main focus in Part II. The next couple of remarks describes how BFRRs may be modelled within the framework that has been developed for FTDRRs.

Remark 5.5.1 (Models for unsecured BFRRs)
One interesting outcome from the theory developed in section 4.4 is that the δ-term LCFTDPK and LCFTDZCB-system may be interpreted and considered as an appropriate model for systemic bank funding rates or BFRRs. This means that

$$J^n_t := \frac{1}{n\delta}\left(\frac{1}{\overline{P}^1_{t,0,n}} - 1\right) , \tag{5.5.1}$$

may be utilised as a model for the $n\delta$-term BFRR, for $n \in \{1, 2, \ldots, m\}$. This follows from the characterisation of funding assets provided in section 1.3 and Axiom 2.1.5, as well as the understanding that the δ-term LCFTDPK and LCFTDZCB-system encodes the term-related credit risk inherited from the set of FTDs, with the market-making of the set of δ-term FTDFRAs enabling the potential liquidation of the aforementioned FTDs at the δ-term frequency and thereby removing the implicit funding-liquidity premia.

The δ-term LCFTDPK therefore enables the modelling of the initial term structure in the banking funding market at each point in time, and plays the same role as the δ-term FTDPK in the deposit market. With the time-series for the set of BFRRs, it is possible to repeat the analysis undertaken in Part I for the bank *funding market, which will yield the key results presented in Table 5.2. The analog to Assumptions 3.3.1 and 3.3.3 are applicable within this scenario and enable the "Risk-Free" and "Credit Risky" contexts. An analog of Assumption 3.3.2 will not be applicable here, since the STR will always immunise the SST from liquidity risk, by Conjecture 1.3.6. As before with Table 5.1, the indices that support the respective pricing kernel and ZCB processes in Table 5.2 are* $n \in \{1,2,\ldots,m\}$*;* $i \leq j \in \{0,1,2,\ldots,\lfloor m/n \rfloor\}$*; and* $h \leq k \in \{0,1,2,\ldots,m-1,m\}$*.*

Framework	Context	Pricing Kernel	Measure	ZCB-System
Market-Based	Risk-Free	$\phi^n_{t+jn\delta}$	$\ddot{\mathbb{L}}_n$	$A^n_{t,in,jn}$
	Credit Risky	$\ddot{\phi}^n_{t+jn\delta}$	$\ddot{\mathbb{Q}}_n$	$\ddot{A}^n_{t,in,jn}$
Reduced-Form	Risk-Free	$\phi^{(n)}_{t+k\delta}$	$\ddot{\mathbb{L}}_{(n)}$	$A^{(n)}_{t,h,k}$
	Credit Risky	$\ddot{\phi}^{(n)}_{t+k\delta}$	$\ddot{\mathbb{Q}}_{(n)}$	$\ddot{A}^{(n)}_{t,h,k}$

Tab. 5.2: A summary of the key results obtained for the bank funding market.

Within the risk-free context, the analog of Assumption 3.3.1 means that the STR will immunise the SST from all risks which would result in a system of PKs and ZCBs that recovers the set of reference rates, as was the case in sections 4.2 and 4.3 — the term- and bank funding-linked pricing measures are denoted here by $\ddot{\mathbb{L}}_n$*, for* $n \in \{1,2,\ldots,m\}$*. The analogs of Assumption 3.3.3 and section 4.5 will enable the creation of survival-contingent term- and bank funding-linked PKs and ZCBs, again by exploiting the funding-swap duality concept. These term-linked quantities will encode the default risk premia associated with each distinct term, with the term- and bank funding-linked survival-contingent pricing measure denoted here by* $\ddot{\mathbb{Q}}_n$*. The* δ*-term LCFTDPK coincides with* $(\phi^1_{t+j\delta})$ *and therefore* $\ddot{\mathbb{L}}_1$ *coincides with* $\mathbb{L}_1$*, while the* δ*-term SLCFTDPK should theoretically coincide with* $(\ddot{\phi}^1_{t+j\delta})$ *and therefore* $\ddot{\mathbb{Q}}_1$ *should be equivalent to* $\mathbb{Q}_1$*.*

Using the exchange of risk mechanism from Lemma 5.3.1 in the way that it has been applied in Theorem 5.4.1, but fixing the vantage point of the current time t*, it is possible to show that*

$$F^{1,n}_{t,j,i} := \frac{\ddot{\pi}^{(n)}_{t+j\delta}\ddot{P}^{(n)}_{t,j,i+n}}{\overline{\pi}^{1}_{t+j\delta}\overline{P}^{1}_{t,j,i+n}}\ \ddot{F}^{n}_{t,j,i}\ , \tag{5.5.2}$$

for $j \in \{0,1,2,\ldots,i-1\}$*, with*

$$J^n_{t+i\delta} := F^{1,n}_{t,i,i} = \frac{\ddot{\pi}^{(n)}_{t+i\delta}\ddot{P}^{(n)}_{t,i,i+n}}{\overline{\pi}^{1}_{t+i\delta}\overline{P}^{1}_{t,i,i+n}}\ \ddot{F}^{n}_{t,i,i} = \frac{\ddot{\pi}^{(n)}_{t+i\delta}\ddot{P}^{(n)}_{t,i,i+n}}{\overline{\pi}^{1}_{t+i\delta}\overline{P}^{1}_{t,i,i+n}}\ R^{n}_{t+i\delta}\ , \tag{5.5.3}$$

for $j = i$*, may be interpreted as the fair strike rate process associated with a generic* $i\delta \times (i+n)\delta$ *FRA that references the* $n\delta$*-term BFRR, i.e.,* $J^n_{t+i\delta}$ *defined above. Allowing the vantage point to progress through time, then equation (5.5.2) will become*

$$F^{1,n}_{t,j,i} := \frac{\ddot{P}^{(n)}_{t,j,i+n}}{\overline{P}^{1}_{t,j,i+n}}\ \ddot{F}^{n}_{t,j,i}\ , \tag{5.5.4}$$

at the current time $t + j\delta$*, for* $j \in \{0,1,2,\ldots,i-1\}$*, with equation (5.5.3) becoming*

$$J^n_{t+i\delta} := F^{1,n}_{t,i,i} = \frac{\ddot{P}^{(n)}_{t,i,i+n}}{\overline{P}^{1}_{t,i,i+n}}\ \ddot{F}^{n}_{t,i,i} = \frac{\ddot{P}^{(n)}_{t,i,i+n}}{\overline{P}^{1}_{t,i,i+n}}\ R^{n}_{t+i\delta}\ , \tag{5.5.5}$$

at the current time $t + i\delta$. The exchange of risk mechanism, which here enables the exchange of liquidity risk associated with the $n\delta$-term for that associated with the δ-term, therefore enables the development of a forward or FRA rate modelling framework for the set of BFRRs.

Remark 5.5.2 (Models for secured BFRRs)
From Conjecture 2.1.3 and the comments made thereafter, the secured overnight BFRR is fundamentally different to its unsecured counterparts, viz., the overnight BFRR and FTDRR. The same applies for the term-based versions, as evidenced by Axioms 2.1.2, 2.1.4 and 2.1.5. As described in Remark 5.5.1, it is possible to construct models for unsecured BFRRs from corresponding models for FTDRRs; however, the same is not possible for secured BFRRs since a mechanism to exchange term-related credit risk for no credit risk has not been constructed. In fact such an exchange of risk is not technically possible within the FTD market.

Assume that a repo market exists and constitutes a set of m reference rates:

$$\left\{ S_u^n \, ; n \in \{1, 2, \ldots, m\} \right\} , \tag{5.5.6}$$

at any time $u \geq 0$, each of which characterise and enable both repo (collateralised borrowing) and reverse repo (collateralised lending) activity. Then, the assumption of an adequate level of available high quality collateral at all times enables a systemic and symmetric setting similar to that constructed for the FTD market in Chapter 3. With this setup it is possible to repeat the analysis undertaken in Part I for this repo market, which will yield the key results presented in Table 5.3. The analog to Assumptions 3.3.1 and 3.3.2 are applicable within this scenario and enable the "Risk-Free" and "Liquidity Risky" contexts. An analog of Assumption 3.3.3 will not be applicable here, by definition, due to the collateralised nature of the repo market combined with the assumption of an adequate level of available high quality collateral. As before with Table 5.1, the indices that support the respective pricing kernel and ZCB processes in Table 5.3 are $n \in \{1, 2, \ldots, m\}$; $i \leq j \in \{0, 1, 2, \ldots, \lfloor m/n \rfloor\}$; and $h \leq k \in \{0, 1, 2, \ldots, m-1, m\}$.

Framework	**Context**	**Pricing Kernel**	**Measure**	**ZCB-System**
Market-Based	Risk-Free	$\zeta^n_{t+jn\delta}$	$\overline{\mathbb{D}}_n$	$Z^n_{t,in,jn}$
	Liquidity Risky	$\overline{\zeta}^n_{t+jn\delta}$	$\overline{\mathbb{Q}}_n$	$\overline{Z}^n_{t,in,jn}$
Reduced-Form	Risk-Free	$\zeta^{(n)}_{t+k\delta}$	$\overline{\mathbb{D}}_{(n)}$	$Z^{(n)}_{t,h,k}$
	Liquidity Risky	$\overline{\zeta}^{(n)}_{t+k\delta}$	$\overline{\mathbb{Q}}_{(n)}$	$\overline{Z}^{(n)}_{t,h,k}$

Tab. 5.3: A summary of the key results that may be obtained for the repo market defined above.

Within the risk-free context, the analog of Assumption 3.3.1 means that the STR will immunise the SST from all risks which would result in a system of PKs and ZCBs that recovers the set of reference rates, as was the case in sections 4.2 and 4.3 — the term- and repo-linked pricing measures are denoted here by $\overline{\mathbb{D}}_n$, for $n \in \{1, 2, \ldots, m\}$. The analogs of Assumption 3.3.2 and section 4.4 will enable the creation of liquidity-contingent term- and repo-linked PKs and ZCBs, again by exploiting the funding-swap duality concept. These term-linked quantities will encode the funding-liquidity premia associated with each distinct term, with the term- and repo-linked liquidity-contingent pricing measure denoted here by $\overline{\mathbb{Q}}_n$.

Then, within this repo market context, the best proxy for the risk-neutral measure will be $\overline{\mathbb{Q}}_1$, i.e., the measure associated with the δ-term, along with contingency on the availability of liquidity at the δ-term frequency. Recall that $\mathbb{Q}_1$ was the best proxy for the risk-neutral measure within the FTD market, which *was also associated with the δ-term but contingent on both liquidity and survival at the δ-term frequency. Accordingly, since $\overline{\mathbb{Q}}_1$ is associated with an underlying strategy that is immunised from credit risks, at least theoretically, it should be the preferred choice for the risk-neutral measure within an economy.*

The models for BFRRs presented above are recovered within a more general setting in Part II, and shown to be applicable within both the contexts of an emerging and a developed economy. Moreover, fair strike rates associated with tradable versions of FRAs that reference BFRRs are also recovered in a similar manner to Theorem 5.4.1. This is enabled by the introduction of a construct called a *curve-conversion factor process*, which generalises the exchange of risk mechanism presented in Lemma 5.3.1 and in turn allows for the definition of an *across-curve pricing formula*, which is essentially a generalisation of Corollary 5.4.1 and Theorem 5.4.1. The curve-conversion factor process is shown to play a dual role, enabling an exchange of risk at either the cash flow or curve level which ultimately allows for the definition of models for BFRRs for both emerging and developed economies.

Part II

Reduced-Form Multi-Curve Frameworks

Chapter 6

Across-Curve Pricing and Valuation

A general motivation for the research that is undertaken in Part II is provided in section 1.4, and subsection 1.4.2. As described in those sections, the almost self-contained objective for Part II is to develop a theoretically consistent and tractable framework for pricing and valuing financial instruments with cash flows that are accrued at an interest rate which differs from the interest rate that is relevant for discounting. This problem contributes to the general class of problems in mathematical finance that are solved through *convexity adjustments* or *corrections*. Here, the exchange of risk mechanism developed in Part I, and presented in section 5.3, is generalised to offer a different type of solution to the one offered by convexity corrections. The benefit of this is that the economic and technical underpinnings of the associated and applicable pricing kernels offers both financial and mathematical rationale, respectively, for the exchange of risk mechanism that leads to the eventual solution.

Section 6.1 introduces the *curve-conversion factor process* which is the generalisation of the exchange of risk mechanism. In turn, the curve-conversion factor process enables the generalisation of the fundamental pricing formula based on a pricing kernel — this new formula is referred to as the *across-curve pricing formula*. In turn, this enables the creation of a general modelling framework referred to as the xy-formalism or xy-approach, which is developed and applied to various practical problems throughout the remainder of Part II. As a heuristic, the 'x' moniker is used to identify the curve that is used for discounting (and thereby the numeraire), while the 'y' moniker identifies the curve that defines the characteristics of the interest rate at which future cash flows accrue or are forecasted. Most importantly, Remark 6.1.3 explains how the development of the xy-formalism completes the definition of the market-based approach, that was initiated in Chapter 2, by specifying the arbitrage-free mechanics for pricing and valuation within a market setting characterised by multiple pricing kernels.

The economic context for this chapter is first and foremost that of an emerging economy, followed by that of a developed economy before the process of reference rate reform, based on Definition 1.4.1. Section 6.2 describes how the xy-approach may be applied to model the bank funding markets within the aforementioned economies, followed by sections 6.3 and 6.4 which offer more specific modelling detail for each of the respective economic contexts under consideration. The chapter concludes with section 6.5, which describes how an emerging economy market participant may model the transition from a pre-GFC (single-curve) to a post-GFC (multi-curve) banking funding market microstructure, using the xy-approach. Where possible and relevant, the links and relations back to Part I will be highlighted and emphasised.

6.1 The Across-Curve Pricing Formula

In this section the *curve-conversion factor process* is defined which enables the development of the *across-curve pricing formula.* At the basis of the curve-conversion factor process lies the assumption that, within a given economy, there is a distinct market or sub-market associated with each curve. Each of these markets are characterised by its own set of market, liquidity and credit risk factors. In turn, each set of market, liquidity and credit risk factors may be systemic or idiosyncratic in nature. The curve-conversion factor process plays a dual role:

(i) it provides a mechanism — akin to a ladder — that enables one to transit consistently from one discount curve system to another; and

(ii) it facilitates the equivalent representation of cash flows across markets (or curves), no matter what financial instrument is implicitly being priced or interest rate system being modelled.

This feature enables consistent valuation across different curves (or markets). As already indicated, the paradigm that is adopted for the development of the across-curve pricing approach is the one based on pricing kernels. Previous research developing and applying the pricing kernel paradigm includes the work done by Constantinides (1992), Flesaker and Hughston (1996a,b, 1997), Rogers (1997), Jin and Glasserman (2001), Hughston and Rafailidis (2005), Akahori *et al.* (2014), Macrina (2014), and Filipović *et al.* (2017); however, this is by no means an exhaustive list of references. Next, the stochastic basis that supports the fundamental pricing kernel system is introduced.

A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions is assumed to exist, where $(\mathcal{F}_t)_{t\geq 0}$ denotes the filtration and $\mathbb{P}$ the real-world probability measure. An $(\mathcal{F}_t)$-adapted pricing kernel process $(h_t)_{t\geq 0}$ is introduced, which is assumed to govern the intertemporal relation between asset values at different times within a financial market. This pricing kernel is a fundamental ingredient in the so-called standard no-arbitrage pricing formula, for a non-dividend-paying financial asset H, given by

$$H_t = \frac{1}{h_t}\mathbb{E}\left[h_T\, H_T \,|\, \mathcal{F}_t\right] \ . \tag{6.1.1}$$

The no-arbitrage asset price process $(H_t)_{t\geq 0}$ is obtained by taking the conditional expectation of the random cash flow H_T, occurring at the fixed future date $T \geq t \geq 0$, that is discounted by the pricing kernel. Standard references, in which asset pricing using pricing kernels is discussed include, for e.g., Hunt and Kennedy (2000), Duffie (2001), Cochrane (2009), and Grbac and Runggaldier (2015).

In order to deduce the across-curve pricing formula — seen as an extension to the pricing formula given by equation (6.1.1) — the existence of a set of (continuous-time) $(\mathcal{F}_t)$-adapted pricing kernel processes $(h^y_t)_{t\geq 0}$ is assumed, where $y \in \{0, 1, 2, \dots, n\}$, each linked to a distinct y-market. The price H^y_t at $t \in [0, T]$ of a non-dividend-paying financial asset H, with (random) cash flow H^y_T at the fixed future date T, is then given by

$$H^y_t = \frac{1}{h^y_t}\mathbb{E}\left[h^y_T\, H^y_T \,|\, \mathcal{F}_t\right] \ . \tag{6.1.2}$$

The superscript y emphasises that the pricing formula given by equation (6.1.2) holds for the valuation of assets in the y-market (or y-sub market). In fact, the pricing kernel process (h^y_t) governs the intertemporal relation between the present value of financial assets and their future cash flows in the associated y-market. It then follows in a straightforward manner, that the price process $(P^y_{tT})_{0\leq t\leq T}$ of a ZCB, with payoff $H^y_T = P^y_{TT} = 1$ at the fixed maturity T and quoted in the y-market, is given by

$$P^y_{tT} = \frac{1}{h^y_t}\mathbb{E}\left[h^y_T \,|\, \mathcal{F}_t\right] \ .$$

The discount bond system — spanned in theory by a continuum, but in practice a finite number of maturities $T \in \{T_1, T_2, \ldots, T_n\}$ — generates a term structure curve. Since this curve is indexed by the particular market y, it is referred to as the *y-curve*. In all that follows, one of the set of the y-markets (and thereby its associated y-curve) is singled out and referred to as the x-market (and its associated term structure curve as the x-curve). Of course then, this market also has an associated $(\mathcal{F}_t)$-adapted pricing kernel (h^x_t). The x-market is the market within which pricing and valuation (or discounting) occurs, while the y-market denotes the market within which the cash flows of financial instruments are forecasted (or accrued).

As described in the introduction to this chapter, the fundamental pricing problem that is considered in Part II is one where a financial instrument's cash flows accruse at a rate of interest that differs from that used for discounting. First, the problem of cash flow forecasting and equivalent representations under different curves (or markets) is considered, before the problem of pricing and valuation (or discounting) is considered. An equivalent cash flow representation across curves (or markets) is justified in Appendix C.1 using no-arbitrage portfolio-based strategies. These findings are formalised in the next definition that introduces the *curve-conversion factor process*, which is the generalisation of the exchange of risk mechanism that was introduced in Lemma 5.3.1.

Definition 6.1.1 (The curve-conversion factor process)
Consider an economy with n distinct markets characterised by a set of pricing kernel processes (h^y_t) and associated discount bond systems (P^y_{tT}), where $y \in \{0, 1, 2, \ldots, n\}$ and $t \in [0, T]$. The converted value C^x_t in the x-market at time t of any spot cash flow C^y_t determined in the y-market is given by

$$C^x_t = \frac{h^y_t}{h^x_t} C^y_t \ ,$$

where $x, y \in \{0, 1, \ldots, n\}$. The converted value $C^x_t(t, T)$ at time t in the x-market of any forward cash flow C^y_t (t,T), measurable at time t but payable at time T, determined in the y-market is given by

$$C^x_t(t, T) = \frac{h^y_t P^y_{tT}}{h^x_t P^x_{tT}} C^y_t(t, T) \ ,$$

where $x, y \in \{0, 1, \ldots, n\}$. These two relations are combined by the definition of the $(\mathcal{F}_t)$-adapted curve-conversion factor process

$$Q^{xy}_{tT} = \frac{\mathbb{E}\left[h^y_T \,|\, \mathcal{F}_t\right]}{\mathbb{E}\left[h^x_T \,|\, \mathcal{F}_t\right]} = \frac{h^y_t P^y_{tT}}{h^x_t P^x_{tT}} \ , \tag{6.1.3}$$

where $t \in [0, T]$ is the time until which the cash flow being converted is measurable and $T > 0$ is the cash flow payment date.

Take note that the cash flows $C^x_t(t, T)$ and $C^y_t(t, T)$ are linked by the identity $C^x_t(t, T) = Q^{xy}_{tT} C^y_t(t, T)$, for $t \in [0, T]$. This definition provides the necessary tool to resolve the fundamental pricing problem that is under consideration in Part II, i.e., valuing a generic financial instrument that accrues cash flows under one curve, the y-curve, but is priced under another curve, the x-curve. This approach is consistent with the FX analogy proposed by Bianchetti (2009), but formalised in an economy modelled by a set of pricing kernels — the xy-formalism is the terminology used to describe this approach. At the heart of this formalism is the pricing formula that is presented next, which is referred to as the *across-curve pricing formula.* The relation of this novel formula to the fundamental pricing formula, given by equation (6.1.1), is demonstrated in the proof of the next proposition.

Proposition 6.1.1 (The across-curve pricing formula)
Let $s \le t \in [0,T]$, and consider a generic financial asset H that has a single $\mathcal{F}_t$-measurable cash flow $H_t^y(t,T)$ occurring or being paid at the fixed time $T \ge t$ and determined by the y-curve (or the y-market). It is noted that in the time interval $[t,T]$, the quantity $H_t^y(t,T)$ is fixed at the value observed at time $t \ge 0$. Within the xy-approach, the price process $(H_{sT}^{xy})_{0\le s\le T}$ of a financial instrument, determined by the x-curve (or x-market) and contingent on the asset H, is given by

$$H_{sT}^{xy} = \begin{cases} \dfrac{1}{h_s^x}\mathbb{E}\left[h_t^x\, P_{tT}^x\, Q_{tT}^{xy}\, H_t^y(t,T)\,|\,\mathcal{F}_s\right] \ , & 0 \le s < t, \\ P_{sT}^x\, Q_{tT}^{xy}\, H_t^y(t,T) \ , & t \le s \le T. \end{cases} \tag{6.1.4}$$

The curve-conversion factor process $(Q_{tT}^{xy})_{0\le t\le T}$ is as described in Definition 6.1.1.

Proof. For information and comparison, take note that a direct application of the relation (6.1.2) will recover the price process $(H_t^y)_{0\le t\le T}$ for the financial asset H, which takes the following form

$$H_t^y = \frac{1}{h_t^y}\mathbb{E}\left[h_T^y\, H_t^y(t,T)\,|\,\mathcal{F}_t\right] = H_t^y(t,T)P_{tT}^y \ , \tag{6.1.5}$$

since the cash flow $H_t^y(t,T)$ is $\mathcal{F}_t$-measurable and it occurs at the future time T. At time $t \in [0,T]$, the first step is to convert $H_t^y(t,T)$ to the corresponding value $H_t^x(t,T)$ in the x-market by use of the conversion factor Q_{tT}^{xy}, which yields

$$H_t^x(t,T) = Q_{tT}^{xy} H_t^y(t,T) \ . \tag{6.1.6}$$

Next, insert the converted cash flow $H_t^x(t,T)$ in the standard no-arbitrage formula, equation (6.1.1) (or equation (6.1.2), where $y = 0$ is taken to be the x-curve), where $h_t = h_t^x$ is assumed, which gives

$$H_{sT}^x = \frac{1}{h_s^x}\mathbb{E}\left[h_t^x\, H_t^x(t,T)\,|\,\mathcal{F}_s\right] \ . \tag{6.1.7}$$

Given that $H_t^x(t,T)$ is $\mathcal{F}_t$-measurable, by the tower property of conditional expectation:

$$H_{sT}^x = \frac{1}{h_s^x}\mathbb{E}\left[\mathbb{E}\left[h_T^x\,|\,\mathcal{F}_t\right] H_t^x(t,T)\,|\,\mathcal{F}_s\right] = \frac{1}{h_s^x}\mathbb{E}\left[h_t^x\, P_{tT}^x\, H_t^x(t,T)\,|\,\mathcal{F}_s\right] \ , \tag{6.1.8}$$

for $s \in [0,t)$. In addition, for $s \in [t,T]$, it follows that $H_{sT}^x = P_{sT}^x\, H_t^x(t,T)$.
Recalling that $H_t^x(t,T) = Q_{tT}^{xy}\, H_t^y(t,T)$, and by choosing to write H_{sT}^{xy} for H_{sT}^x in order to emphasise the interaction between the x- and the y-curves, the proof is complete. Take note that the one-to-one across-curve extension to the standard pricing formula, given by equation (6.1.1), is recovered by setting $t = T$ in the relation (6.1.4). □

Remark 6.1.1 (Quanto or xy-ZCBs)
When $t = T$ and $H^y_{TT} = 1$, using Proposition 6.1.1, one may define the following ZCB:

$$P^{xy}_{sT} = \frac{1}{h^x_s}\mathbb{E}\left[h^y_T \,|\, \mathcal{F}_s\right] = Q^{xy}_{ss} P^y_{sT} = P^x_{sT} Q^{xy}_{sT} \ , \tag{6.1.9}$$

for $s \in [0, T]$, which has two representations using the definition of the conversion factor (6.1.3).

Given Proposition 6.1.1, the dual role that is played by the *curve-conversion factor process* within the xy-formalism can now be presented, which is described in the following corollary.

Corollary 6.1.1 (The dual role of the curve-conversion factor process)
Within the xy-formalism, if the cash flow $H^y_t(t, T)$ is directly observable in the economy, then the curve-conversion factor process enables valuation by acting at the level of the discounting curve as follows:

$$H^{xy}_{sT} = \frac{1}{h^x_s}\mathbb{E}\left[h^x_t P^x_{tT} Q^{xy}_{tT} H^y_t(t, T) \,|\, \mathcal{F}_s\right] = \frac{1}{h^x_s}\mathbb{E}\left[h^y_t P^y_{tT} H^y_t(t, T) \,|\, \mathcal{F}_s\right] \ . \tag{6.1.10}$$

However, if the curve-converted cash flow H^{xy}_{tT} is directly observable in the economy, then the curve-conversion factor process enables valuation by acting at the level of the cash flow as follows:

$$H^{xy}_{sT} = \frac{1}{h^x_s}\mathbb{E}\left[h^x_t P^x_{tT} Q^{xy}_{tT} H^y_t(t, T) \,|\, \mathcal{F}_s\right] = \frac{1}{h^x_s}\mathbb{E}\left[h^x_t H^{xy}_{tT} \,|\, \mathcal{F}_s\right] \ , \tag{6.1.11}$$

where $(H^{xy}_{sT})_{0 \le s \le t}$ is the x-market value of $H^y_t(t, T)$, for $s \le t \le T$.

Proof. If $H^y_t(t, T)$ is determined in the y-market and directly observable (i.e., quoted or market-made) within the economy, then according to Proposition 6.1.1 the value of such a payoff within the x-market, at the future terminal time T, is given by

$$H^{xy}_{TT} = Q^{xy}_{tT} H^y_t(t, T) \ , \tag{6.1.12}$$

which is model-implied, since Q^{xy}_{tT} is determined by the specific forms of the pricing kernels (h^x_t) and (h^y_t), respectively. Therefore, since H^{xy}_{TT} is not directly observable in the economy due to Q^{xy}_{tT}, the curve-conversion factor process is subsumed into the discounting process in equation (6.1.4) for $s \in [0, t)$, by observing that $h^x_t P^x_{tT} Q^{xy}_{tT} = h^y_t P^y_{tT}$, which yields equation (6.1.10).
Conversely, if $H^y_t(t, T)$ is determined in the y-market but the converted quantity H^{xy}_{TT} is directly observable within the economy, then the following quantity is model-implied:

$$H^y_t(t, T) = \frac{H^{xy}_{TT}}{Q^{xy}_{tT}} \ , \tag{6.1.13}$$

which is subsumed into the cash flow process by observing that $H^{xy}_{sT} = P^x_{sT} Q^{xy}_{tT} H^y_t(t, T)$ for $s \in [t, T]$, from equation (6.1.4), which yields equation (6.1.11). □

In Appendix C.2, a consistent set of change of numeraire assets and associated equivalent probability measures are provided, which ensure that no arbitrage is produced when the across-pricing formula is applied using an equivalent martingale measure.

Remark 6.1.2 (The dual role of Q^{xy}_{tT} in emerging and developed economies)
Corollary 6.1.1 proves to be critical in section 6.2, where consistent multi-curve systems are derived for emerging and developed economy's bank funding markets. Prior to reference rate reform, key BFRRs were

the set of Interbank Offered Rates (IBORs) with varying terms or tenors, with FRAs (the fundamental bank funding market linear derivative security) having IBORs as their underlyings. It turns out that the y-market determined IBOR process is directly observable in the emerging economy, but its curve-converted equivalent is directly observable in the developed economy. In this instance, the dual nature of the curve-conversion factor process caters for this apparent cross-economy market inconsistency, enabling the use of one consistent modelling framework.

Remark 6.1.3 (The evolution of the market-based approach and connections to Part I)
In Chapter 2, the market-based approach was introduced as a generalisation of the reduced-form approach which was required in order to capture the stylised feature of inhomogeneous effects that result due to compounding at different frequencies within a risky context. However, the exact arbitrage-free mechanics for such an approach were yet to be resolved and specified.

The analysis undertaken in Part I, within the context of the frontier economy, was the first step to developing the arbitrage-free pricing and valuation mechanics for the market-based approach. This was ultimately achieved in Chapter 5; however, this was all done within the specific backdrop of the FTD market and under the assumption that no derivatives market was in existence — these results were all summarised in section 5.5. Moreover, while the set of estimated SDFs were continuous-time processes, the range of constructed and calibrated PKs were all discrete-time quantities — even the reduced-form versions that were constructed in section 5.2 were defined on a time grid that incremented at the δ-term frequency.

The xy-approach recovers all of the results that were derived in Part I — in particular, notwithstanding the fact that the set of pricing kernels (h_t^y) are assumed to be continuous-time processes, these may be directly assigned to be the set of FTDPKs $(\pi_t^{(n)})$, LCFTDPKs $(\overline{\pi}_t^{(n)})$, and SLCFTDPKs $(\ddot{\pi}_t^{(n)})$, which would then characterise the FTD market. One may also include in the assignment, the set of pricing kernels that were introduced for the bank funding and repo markets in Remarks 5.5.1 and 5.5.2, viz., $(\phi_t^{(n)})$, $(\ddot{\phi}_t^{(n)})$, $(\zeta_t^{(n)})$ and $(\overline{\zeta}_t^{(n)})$. Then, the exchange of risk mechanism and pricing and valuation results will follow through Definition 6.1.1 and Proposition 6.1.1, and one will be able to price and value financial instruments defined across the deposit, banking funding and repo markets.

Therefore, the xy-formalism completes the definition and specification of the market-based approach that was initiated in Chapter 2. This completed definition now enables one to consider the direct modelling of derivatives markets, since it is now possible to specify the system of pricing kernels in an a priori fashion, without the need for careful construction of a derivative market-making model using the primitive market's system of stochastic discount factors as a basis or foundation, as was the case with Part I. This is the standard approach that is considered throughout Part II, i.e., derivatives markets are assumed to exist and the direct modelling of these markets are the default context.

6.2 The xy-Formalism applied to Bank Funding Markets

First, the definition of a spot IBOR is considered here, i.e., a deposit rate[1] that is offered at a fixed time $t \geq 0$ by a set of suitably credit-rated banks within a given economy. This IBOR conforms to the

[1] This may also be considered as a loan rate, since the activity is confined to the interbank sector. Bank A borrowing from (depositing with) Bank B is equivalent to Bank B depositing with (lending to) Bank A.

characterisation provided in Axiom 2.1.5 and Conjecture 2.1.2. It is assumed that the maturity of said IBOR is $t + \delta > t$, so that the associated tenor is given by $\delta > 0$. Then, one may define (or represent) the spot IBOR process via ZCB instruments by

$$J_t(t, t+\delta) = \frac{1}{\delta}\left(\frac{1}{P_{tt+\delta}} - 1\right), \tag{6.2.1}$$

where $t \geq 0$, $\delta > 0$, and where $P_{tt+\delta}$ is the price at time t of a ZCB, with tenor δ, that matures at time $t + \delta$. In the classical single-curve framework, where IBORs are considered an appropriate proxy for risk-free rates and where a tradable discount bond system is assumed, one can then proceed to define the forward IBOR process via the canonical no-arbitrage pricing relation

$$J_t(T_{i-1}, T_i) = \frac{1}{h_t P_{tT_i}} \mathbb{E}\left[h_{T_{i-1}} P_{T_{i-1}T_i} J_{T_{i-1}}(T_{i-1}, T_i) \,\middle|\, \mathcal{F}_t\right], \tag{6.2.2}$$

for $0 \leq t \leq T_{i-1}$, and where $\delta_i = T_i - T_{i-1}$ is the IBOR tenor and $(h_t)_{t\geq 0}$ is the pricing kernel process. By use of the relation (6.2.1) with $t = T_{i-1}$ and $\delta = \delta_i$, and the ZCB pricing relation $h_t P_{tT_i} = \mathbb{E}[h_{T_{i-1}} P_{T_{i-1}T_i} \,|\, \mathcal{F}_t]$, one obtains the forward IBOR process

$$J_t(T_{i-1}, T_i) = \frac{1}{\delta_i}\left(\frac{P_{tT_{i-1}}}{P_{tT_i}} - 1\right), \tag{6.2.3}$$

for $0 \leq t \leq T_{i-1}$. Take note that the product of the pricing kernel and the discounted forward IBOR $(h_t P_{tT_i} J_t(T_{i-1}, T_i))_{0\leq t\leq T_{i-1}}$ is an $((\mathcal{F}_t), \mathbb{P})$-martingale, which is analogous to the forward IBOR process being a martingale under the T_i-forward measure in the classical single-curve theory.

The classical relation (6.2.3) states that the forward IBOR value at time t can be replicated by a linear combination of ZCBs, i.e., by one maturing at the IBOR reset date T_{i-1} and another ZCB maturing at the IBOR settlement date T_i. In a market where the spread between an OIS rate and the corresponding IBOR is non-zero, relation (6.2.3) is no longer acceptable. That is, the now *risky* IBOR can no longer be replicated using *risk-free* ZCBs. In other words, the IBOR market is exposed to risk factors which are not necessarily affecting the *risk-free* ZCB market, while the magnitude of the risk exposure also varies depending on the IBOR tenor $\delta_i = T_i - T_{i-1}$. Hence, one needs to assume that holding a financial contract written on a 3-month IBOR exposes a holder to a different risk profile than when holding an instrument written on a 6-month IBOR. It follows that assuming *risk-free* ZCBs can replicate the same risk exposures as contracts written on an IBOR is wrong because: (a) an IBOR may be subject to more risk sources than the *risk-free* ZCBs; and (b) the number of risk factors affecting an IBOR contract may depend on the IBOR tenor.

Here the following question is asked: If one insisted on keeping the relation (6.2.3), albeit subject to modifications, how would one need to adjust — *in a consistent and arbitrage-free manner* — the relation between an IBOR model and the associated ZCBs in a multi-curve setup? It turns out that the answer is an extension based on the xy-formalism introduced above.

First, consider a collection of interest rate curves indexed by $x, y \in \{0, 1, 2, \ldots, n\}$ where the x-curve is referred to as the *discounting curve* and the y-curve as the *forecasting curve*. An example for a pair of curves (x, y) may be the pair $(0, 1)$ where the 0-curve will generally be indicative of the OIS curve and the 1-curve is the 1-month IBOR curve. The case where $x = y$ is the (classical) single-curve economy. Next, the relationship (6.2.3) is made curve-dependent such that

$$J_t^y(T_{i-1}, T_i) = \frac{1}{\delta_i}\left(\frac{P^y_{tT_{i-1}}}{P^y_{tT_i}} - 1\right). \tag{6.2.4}$$

Thus, the y-ZCB system $P^y_{tT_i}$ has an associated y-tenored IBOR, which is subject to the same set of risk factors, i.e., the y-tenored IBOR defines the y-ZCB system. Moreover, the y-ZCB price process satisfies the martingale relation $h^y_t P^y_{tT_i} = \mathbb{E}[h^y_{T_i} \,|\, \mathcal{F}_t]$, which is to say that no-arbitrage is assumed within the self-consistent y-market. Next, the development of consistent multi-curve interest rate systems inspired by the xy-formalism is detailed for bank funding markets within an emerging and a developed economy, post the GFC but before reference rate reform.

6.3 Discounting Systems in Emerging Economies

In this section, the simpler case of an emerging economy is considered, the market microstructure of which conforms with Definition 1.4.1. To be precise, the spot ONRR is observable but there are no tradable and liquid OISs, i.e., there is no OIS derivative market to enable the construction of a yield curve. For more information on the specific nuances and issues relating to emerging economy's bank funding and associated derivatives markets, one may refer to Jakarasi *et al.* (2015), and references therein, which considers the problem of estimating an OIS zero-coupon yield curve in South Africa. In such an emerging economy, all forecasting and discounting of cash flows is done by one liquid, risky y-tenored IBOR zero-coupon yield curve, only.

The path to deriving a multi-curve discounting system within the xy-formalism begins with the pricing of standard FRAs, which enable price discovery for: (i) forward IBORs in a single-curve or risk-free setting; and (ii) fair FRA rates associated with IBORs in a multi-curve or risky setting. The FRA considered here has reset time $T_{i-1} > 0$ and maturity time $T_i > T_{i-1}$, which is also assumed to be the settlement time, and is therefore written on the future spot IBOR $J^y_{T_{i-1}}(T_{i-1}, T_i)$. The value at time $t \in [0, T_i]$ of this FRA is denoted by $V^{yy}_{tT_i}$, with the first character of the superscript indicating the discount curve, and the second character denoting the forecasting curve. For a unit nominal, the FRA's payoff at T_i is given by

$$V^{yy}_{T_iT_i} = \delta_i\left(J^y_{T_{i-1}}(T_{i-1}, T_i) - K^y\right), \tag{6.3.1}$$

where K^y is an arbitrary strike rate expressed in the y-market. Take note that the FRA's payoff is actually measurable at time T_{i-1}, however the actual cash flow is only paid at time T_i. As a consequence, it is also possible to define the in-advance FRA payoff $V^{yy}_{T_{i-1}T_i}$ at T_{i-1}, which is the value $V^{yy}_{T_iT_i}$ discounted by $P^y_{T_{i-1}T_i}$, by

$$V^{yy}_{T_{i-1}T_i} = P^y_{T_{i-1}T_i}\delta_i\left(J^y_{T_{i-1}}(T_{i-1}, T_i) - K^y\right). \tag{6.3.2}$$

Using the pricing formula (6.1.4) with $x = y$, along with relations (6.2.2), (6.2.3) and (6.2.4), the FRA price process is derived as

$$\begin{aligned} V^{yy}_{tT_i} &= \frac{\delta_i}{h^y_t}\mathbb{E}\left[h^y_{T_{i-1}} V^{yy}_{T_{i-1}T_i} \mid \mathcal{F}_t\right] \\ &= \frac{\delta_i}{h^y_t}\mathbb{E}\left[h^y_{T_{i-1}} P^y_{T_{i-1}T_i}\left(J^y_{T_{i-1}}(T_{i-1},T_i) - K^y\right) \mid \mathcal{F}_t\right] \\ &= \delta_i P^y_{tT_i}\left(J^y_t(T_{i-1},T_i) - K^y\right) \\ &= P^y_{tT_{i-1}} - (1+\delta_i K^y) P^y_{tT_i} \ . \end{aligned} \tag{6.3.3}$$

By setting $V^{yy}_{tT_i} = 0$, the fair FRA rate process is recovered and is given by

$$K^{yy}_t(T_{i-1},T_i) = J^y_t(T_{i-1},T_i) \ , \tag{6.3.4}$$

for $t \in [0, T_{i-1}]$. The notation $K^{yy}_t(T_{i-1},T_i)$ emphasises that this fair FRA strike rate applies when the y-curve is used for both, discounting and forecasting.

Next, consider a standard IRS with unit nominal, referencing the y-tenored IBOR with reset times $\{T_0, T_1, \ldots, T_{n-1}\}$, payment times $\{T_1, T_2, \ldots, T_n\}$, and arbitrary fixed swap rate under the y-market denoted by S^y. Again applying pricing relation (6.1.4) with $x = y$, together with relations (6.2.2), (6.2.3) and (6.2.4), the IRS price process is derived as

$$\begin{aligned} V^{yy}_{tT_n} &= \sum_{i=1}^{n}\frac{\delta_i}{h^y_t}\mathbb{E}\left[h^y_{T_i}\left(J^y_{T_{i-1}}(T_{i-1},T_i) - S^y\right) \mid \mathcal{F}_t\right] \\ &= P^y_{tT_0} - P^y_{tT_n} - S^y\sum_{i=1}^{n}\delta_i P^y_{tT_i} \ , \end{aligned} \tag{6.3.5}$$

for $t \leq T_0$. Using the same notation convention as with the FRA, the fair IRS rate process is given by

$$S^{yy}_t(T_0,T_n) = \frac{P^y_{tT_0} - P^y_{tT_n}}{\sum_{i=1}^{n}\delta_i P^y_{tT_i}} \ , \tag{6.3.6}$$

for $t \leq T_0$. For a brief treatment of bootstrapping in an emerging economy, refer to Appendix C.3.1.

6.4 Discounting Systems in Developed Economies

In this section, the more complex case of a developed economy is considered, the market microstructure of which conforms with Conjecture 2.1.2. In such an economy, it is assumed that the bank funding derivatives market is characterised by financial instruments with cash flows that are forecast using the y-tenored IBOR zero-coupon yield curve but discounted using the OIS zero-coupon yield curve. This is a natural feature of derivatives markets after the GFC, see section 1.3.3, particularly those where the natural underlying market risk factors (such as risky IBORs) cannot be hedged. Hedging these market risks is generally achieved by trading the exact opposite derivative position that is centrally cleared, or with a counterparty under a zero-threshold CSA. Therefore, the natural numeraire for such derivative securities becomes the rate at which margin or collateral is renumerated, which is usually the relevant ONRR that underlies the set of OIS contracts — for more information on the risky valuation of derivative securities, see Appendix A.2 and references therein.

The FRA from the emerging economy case is considered again; however, it is now assumed that discounting occurs under the x-curve (or the OIS curve, to be more specific). One now has to make use of relation (6.1.4) in order to define the FRA's payoff.

Proposition 6.4.1 (The developed economy FRA after the GFC)
The developed economy FRA with reset time T_{i-1}, expiry time T_i and unit nominal has a terminal payoff, within the x-market, given by

$$V^{xy}_{T_iT_i} = Q^{xy}_{T_{i-1}T_i}\delta_i\left(J^y_{T_{i-1}}(T_{i-1},T_i) - K^y\right) = Q^{xy}_{T_{i-1}T_i}V^{yy}_{T_iT_i}\ , \tag{6.4.1}$$

where, as before, $\delta_i = T_i - T_{i-1}$ and K^y is the strike rate within the y-market. The in-advance FRA payoff is then given by

$$V^{xy}_{T_{i-1}T_i} = P^x_{T_{i-1}T_i}Q^{xy}_{T_{i-1}T_i}V^{yy}_{T_iT_i}\ , \tag{6.4.2}$$

which is the discounted value of the terminal payoff within the x-market.

Proof. A direct application of relation (6.1.4) leads to the result in Proposition 6.4.1. Like the emerging economy FRA, notice that the *developed market* FRA's payoff is also measurable at T_{i-1} with the actual cash flow occurring at T_i. □

Before considering the derivation of the value of the developed economy FRA, the following Lemmas will prove to be useful in this regard.

Lemma 6.4.1 (The converted y-market forward IBOR process)
The converted y-tenored forward IBOR process

$$J^{xy}_t(T_{i-1},T_i) = Q^{xy}_{tT_i}J^y_t(T_{i-1},T_i)\ , \tag{6.4.3}$$

for $t \in [0, T_{i-1}]$, satisfies the martingale relation

$$J^{xy}_s(T_{i-1},T_i) = \frac{1}{h^x_s P^x_{sT_i}}\mathbb{E}\left[h^x_{T_i}J^{xy}_{T_{i-1}}(T_{i-1},T_i)\,\middle|\,\mathcal{F}_s\right], \tag{6.4.4}$$

for $0 \le s \le t \le T_{i-1}$.

Proof. This statement follows from equations (6.1.3), (6.4.3) and (6.2.2). □

The previous and the next Lemma are essentially generalisations of Theorem 5.4.1 and Corollary 5.4.1, both of which made use of the exchange of risk mechanism and results related thereto to solve the same problem. Theorem 5.4.1 is also the specific analog of the general result derived in Theorem 6.4.1.

Lemma 6.4.2 (The fair forward price for a y-market fixed cash flow)
The fair forward price K^x of a forward contract initiated at time t to exchange a cash flow K^y, determined in the y-market, for a cash flow of K^x, in the x-market, with K^y being converted at $Q^{xy}_{T_{i-1}T_i}$ but the final payoff occurring at expiry $T_i > T_{i-1} \ge t$ is given by

$$K^x = \frac{h^y_t P^y_{tT_i}}{h^x_t P^x_{tT_i}}K^y = Q^{xy}_{tT_i}K^y\ . \tag{6.4.5}$$

Proof. The value of such a forward contract is given by

$$\begin{aligned} V_{tT_i}^{xy} &= \frac{1}{h_t^x}\mathbb{E}\left[h_{T_i}^x\left(K^y Q_{T_{i-1}T_i}^{xy} - K^x\right) \mid \mathcal{F}_t\right] \\ &= K^y \frac{h_t^y}{h_t^x} P_{tT_i}^y - P_{tT_i}^x K^x \;, \end{aligned} \tag{6.4.6}$$

which follows from equation (6.1.3) and the tower property of conditional expectations, while setting $V_{tT_i}^{xy} = 0$ and solving for K^x yields the required result. □

With this, the necessary results are now available to derive the value of the developed economy FRA, which is presented in the next Theorem.

Theorem 6.4.1 (Fair value of a developed economy FRA)
The value of the developed economy FRA with reset time T_{i-1}, expiry time T_i and unit nominal, within the x-market, is given by

$$V_{tT_i}^{xy} = \delta_i P_{tT_i}^x \left(J_t^{xy}(T_{i-1}, T_i) - K^x\right), \tag{6.4.7}$$

for $t \in [0, T_{i-1}]$, where $\delta_i = T_i - T_{i-1}$ and K^x is the strike rate within the x-market.

Proof. Using Proposition 6.4.1, the value of the developed economy FRA, for $t \in [0, T_{i-1}]$, is given by

$$\begin{aligned} V_{tT_i}^{xy} &= \frac{1}{h_t^x}\mathbb{E}\left[h_{T_{i-1}}^x P_{T_{i-1}T_i}^x Q_{T_{i-1}T_i}^{xy} \delta_i \left(J_{T_{i-1}}^y(T_{i-1}, T_i) - K^y\right) \mid \mathcal{F}_t\right] \\ &= \delta_i P_{tT_i}^x J_t^{xy}(T_{i-1}, T_i) - \delta_i P_{tT_i}^x Q_{tT_i}^{xy} K^y \;, \end{aligned} \tag{6.4.8}$$

with the first term following from Lemma 6.4.1 and the second term from equation (6.1.3). Equation (6.4.7) follows by applying the result of Lemma 6.4.2 to the second term and factorising accordingly. □

The value of this FRA is commensurate with the value of a tradable FRA within a developed economy with a multi-curve market microstructure, i.e., the price dynamics are consistent with the standard FRA contract traded in developed economies post the GFC. This observation contrasts with those made in Remark 5.4.2, where it is noted that the tradable version of the $i\delta \times (1+n)\delta$ FTDFRA, constructed with the exchange of risk mechanism, required its reference rate $R_{t+i\delta}^n$ to be adjusted by the ratio of ZCBs $P_{t,i,i+n}^{(n)} \big/ \dot{P}_{t,i,i+n}^1$ at the reset time $t + i\delta$. However, the same effect is at play here, which can be seen through the result of Lemma 6.4.1, since the reference rate for the developed economy FRA is set to be the converted IBOR value

$$J_{T_{i-1}}^{xy}(T_{i-1}, T_i) = Q_{T_{i-1}T_i}^{xy} J_{T_{i-1}}^y(T_{i-1}, T_i) \;, \tag{6.4.9}$$

at the outset, or a priori. From the vantage point of T_{i-1}, the value of $Q_{T_{i-1}T_i}^{xy}$ will be $P_{T_{i-1}T_i}^y \big/ P_{T_{i-1}T_i}^x$ since $h_{T_{i-1}}^y = h_{T_{i-1}}^x := 1$, at this point in time. Therefore, if one insisted on $J_{T_{i-1}}^y(T_{i-1}, T_i)$ being the developed economy FRA's reference rate, then the same or similar observations as those made in Remarks 5.4.2 and 5.4.3 would again apply.

Remark 6.4.1 (Convexity corrections versus Curve-conversions)
In summary, there are two important inferences based on the discussion preceding this remark:

(i) a specification analogous to Remark 5.4.3, where $J_{T_{i-1}}^y(T_{i-1}, T_i)$ will be the FRA reference rate, comes at the cost of the explicit computation of a convexity adjustment or correction, which is model dependent, within the general market-based approach; while

(ii) the exchange of risk or curve-conversion mechanism enables one to specify a converted reference rate $J^{xy}_{T_{i-1}}(T_{i-1}, T_i)$, which is naturally tractable within the general market-based approach but comes at the potential cost of losing the intuitive interpretability of the y-curve.

Linking back to Part I, the x-market may be modelled with $(\ddot{\pi}^1_t)$ or $(\overline{\zeta}^1_t)$, depending on whether the ONRR is unsecured or secured, while the y-market may be modelled with $(\ddot{\phi}^{(n)}_t)$. Then, to echo the discussion at the end of section 5.4, the exchange of risk or curve-conversion mechanism should be the preferred approach since it is cognisant of fundamental characteristics of the relevant interest rates under consideration. These insights inform the next conjecture which formalises this thread of discussion.

Conjecture 6.4.1 (Theoretically consistent frameworks are inconsistent due to market features)
Since market participants' definitions of interest rate derivatives are agnostic of fundamental primitive market characteristics, theoretically consistent modelling frameworks for bank funding primitive and derivative markets cannot be practically consistent.

One piece of evidence in support of the preceding conjecture is the theory developed thus far. The market-based approach combined with the xy-formalism offers a theoretically consistent framework for multi-curve interest rate modelling. However, as revealed by the analysis undertaken for the developed economy FRA, one needs to carefully engineer the underlying reference rate to ensure that it conforms to the definition of the aforementioned FRA and the necessary arbitrage constraints.

Back to specific findings from Theorem 6.4.1, the form of the developed economy FRA's value within the xy-formalism leads to the following definition for the multi-curve forward IBOR process.

Definition 6.4.1 (The multi-curve market-implied forward IBOR process)
The multi-curve market-implied y-tenored forward IBOR process is given by

$$J^{xy}_t(T_{i-1}, T_i) = Q^{xy}_{tT_i} J^y_t(T_{i-1}, T_i) = \frac{P^{xy}_{tT_i}}{\delta_i P^x_{tT_i}} \left(\frac{P^{xy}_{tT_{i-1}}}{P^{xy}_{tT_i}} - 1 \right) , \tag{6.4.10}$$

for $t \in [0, T_{i-1}]$, where $(P^{xy}_{tT_i})$ is defined in Remark 6.1.1.

Moreover, it is also possible to derive the fair developed economy FRA rate given the value of the FRA provided by Theorem 6.4.1.

Corollary 6.4.1 (The fair developed economy FRA rate process)
The fair FRA rate process $K^{xy}_t(T_{i-1}, T_i)$ at time t for a developed economy FRA written on the market-implied y-tenored forward IBOR (6.4.3), with reset time T_{i-1} and settlement time T_i, is given by

$$K^{xy}_t(T_{i-1}, T_i) = J^{xy}_t(T_{i-1}, T_i) , \tag{6.4.11}$$

for $t \in [0, T_{i-1}]$.

Proof. Setting the value of the developed market FRA, given by equation (6.4.7), equal to zero, it follows that $K^x = J^{xy}_t(T_{i-1}, T_i)$ at time t. Then for any time $t \in [0, T_{i-1}]$, the result for the fair FRA rate process, $K^{xy}_t(T_{i-1}, T_i) = J^{xy}_t(T_{i-1}, T_i)$, follows accordingly. □

Remark 6.4.2 (Multi- and single-curve similarity)
Relation (6.4.11) is the direct multi-curve analogy to the single-curve relation (6.3.4). In fact, for $x = y$ one recovers the single-curve expressions (6.3.3) and (6.3.4).

Remark 6.4.3 (Recovering a familiar FRA valuation formula in the multi-curve setting)
Using Definition 6.4.1, one may restate the value of the developed economy FRA as

$$V^{xy}_{tT_i} = P^{xy}_{tT_{i-1}} - (1 + \delta_i K^y) P^{xy}_{tT_i} \ , \tag{6.4.12}$$

for $t \in [0, T_{i-1}]$*, which is the direct multi-curve analogy to the emerging economy FRA value (6.3.3) with the* y*-ZCBs replaced by the* xy*-ZCBs.*

Given all of these results, it is also important to consider some stylised features of the framework and check that they conform with practical intuition — one such feature is the relationship between $J^{xy}_t(T_{i-1}, T_i)$ and $J^x_t(T_{i-1}, T_i)$. In particular, one would want $J^{xy}_t(T_{i-1}, T_i) \geq J^x_t(T_{i-1}, T_i)$ due to the greater degree of risk associated with the multi-curve y-tenored forward IBOR process versus the corresponding x-tenored process, i.e., the OIS market and curve in this developed economy setting. The following corollary reveals the conditions under which this feature is achieved, by making use of the associated *forward capitalisation factor* (FCF) processes.

Corollary 6.4.2 (Intuitive behaviour of the multi-curve market implied FCF)
The multi-curve market-implied y*-tenored FCF process* $\overline{v}^{xy}_t(T_{i-1}, T_i)$*, observed at time* $t \leq T_{i-1}$ *and applying over the period* $[T_{i-1}, T_i]$*, defined by*

$$\overline{v}^{xy}_t(T_{i-1}, T_i) := 1 + \delta_i J^{xy}_t(T_{i-1}, T_i) \ , \tag{6.4.13}$$

is greater than or equal to the corresponding x*-tenored FCF process*

$$v^x_t(T_{i-1}, T_i) := 1 + \delta_i J^x_t(T_{i-1}, T_i) \ , \tag{6.4.14}$$

if interest rates are non-negative and $h^y_t \leq h^x_t$ *for all* $t \in [0, T_{i-1}]$ *where* $T_{i-1} \leq T_i$*.*

Proof. Using equation (6.4.13) and Definition 6.4.1, it can be shown that

$$\begin{aligned}\overline{v}^{xy}_t(T_{i-1}, T_i) &= 1 + \delta_i Q^{xy}_{tT_i} J^y_t(T_{i-1}, T_i) \\ &= 1 + Q^{xy}_{tT_i}\left(v^y_t(T_{i-1}, T_i) - 1\right) \\ &= 1 - Q^{xy}_{tT_i} + v^{xy}_t(T_{i-1}, T_i) \ ,\end{aligned}$$

where $v^y_t(T_{i-1}, T_i)$ is the y-tenored FCF and $v^{xy}_t(T_{i-1}, T_i) := Q^{xy}_{tT_i} v^y_t(T_{i-1}, T_i)$ is the y-tenored FCF represented equivalently in the x-market. Then, using Definition 6.1.1 and equation (6.1.9), it can be shown that

$$v^{xy}_t(T_{i-1}, T_i) = Q^{xy}_{tT_i} \frac{P^y_{tT_{i-1}}}{P^y_{tT_i}} = Q^{xy}_{tT_i} \frac{P^{xy}_{tT_{i-1}}}{P^{xy}_{tT_i}} = \frac{P^x_{tT_{i-1}}}{P^x_{tT_i}} Q^{xy}_{tT_{i-1}} = v^x_t(T_{i-1}, T_i) Q^{xy}_{tT_{i-1}} \ .$$

Now in order to have $v^x_t(T_{i-1}, T_i) \leq \overline{v}^{xy}_t(T_{i-1}, T_i)$, the following must hold:

$$\begin{aligned}v^x_t(T_{i-1}, T_i) &\leq 1 - Q^{xy}_{tT_i} + v^x_t(T_{i-1}, T_i) Q^{xy}_{tT_{i-1}} \\ v^x_t(T_{i-1}, T_i)\left(1 - Q^{xy}_{tT_{i-1}}\right) &\leq 1 - Q^{xy}_{tT_i} \\ 1 - Q^{xy}_{tT_{i-1}} &\leq 1 - Q^{xy}_{tT_i} \ ,\end{aligned}$$

where the last inequality holds if interest rates are non-negative, i.e., $v^x_t(T_{i-1}, T_i) \geq 1$. Finally, $Q^{xy}_{tT_{i-1}} \geq Q^{xy}_{tT_i}$ if interest rates are non-negative and $h^y_t \leq h^x_t$ for all $t \in [0, T_{i-1}]$ where $T_{i-1} \leq T_i$. This may be easily evidenced by setting $t = T_{i-1}$ and allowing T_i to vary, while also using the linear and monotonic properties of conditional expectations. □

This corollary proves that the xy-approach applied to a *developed economy* yields a y-market ZCB-system that is dominated by the x-market system, i.e., $P^y_{tT} \leq P^x_{tT}$ for $0 \leq t \leq T$. Furthermore, this y-market system offers a forward IBOR process, $J^y_t(T_{i-1}, T_i)$, and enables the construction of a conversion factor process $Q^{xy}_{tT_i}$, which facilitates the definition of the developed economy y-tenored forward IBOR process $J^{xy}_t(T_{i-1}, T_i)$. Therefore, while the y-market system is still fictitious, given that it cannot be directly observed, it is still considered to be a model-consistent system given the curve-conversion mechanism that is inspired by the exchange of risk mechanism and FX modelling.

Remark 6.4.4 (An alternative method using the FCF and a comparison to Nguyen and Seifried (2015)) *Using the FCF, one may also express the terminal payoff of the developed economy FRA as*

$$V^{xy}_{T_iT_i} = Q^{xy}_{T_{i-1}T_i}\left(v^y_{T_{i-1}}(T_{i-1}, T_i) - v^y_K\right) , \tag{6.4.15}$$

where $v^y_K := (1 + \delta_i K^y)$. Then, using the same method as before, the value of the FRA, for $t \in [0, T_{i-1}]$, is

$$V^{xy}_{tT_i} = P^x_{tT_i}\left(v^{xy}_t(T_{i-1}, T_i) - v^x_K\right) , \tag{6.4.16}$$

where $v^x_K = Q^{xy}_{tT_i} v^y_K$. If the multi-curve y-tenored forward IBOR process is defined by

$$\overline{J}^{xy}_t(T_{i-1}, T_i) := \frac{1}{\delta_i}\left(v^{xy}_t(T_{i-1}, T_i) - 1\right), \tag{6.4.17}$$

and the multi-curve x-market equivalent FRA strike rate by

$$\overline{K}^x := \frac{1}{\delta_i}\left(v^x_K - 1\right), \tag{6.4.18}$$

one may then recover the developed economy FRA price process:

$$V^{xy}_{tT_i} = P^x_{tT_i}\delta_i\left(\overline{J}^{xy}_t(T_{i-1}, T_i) - \overline{K}^x\right) . \tag{6.4.19}$$

Take note that in this model $v^{xy}_t(T_{i-1}, T_i) = v^x_t(T_{i-1}, T_i)Q^{xy}_{tT_{i-1}}$ so that

$$\frac{1 + \delta_i\overline{J}^{xy}_t(T_{i-1}, T_i)}{1 + \delta_i J^x_t(T_{i-1}, T_i)} = \frac{h^y_t P^y_{tT_{i-1}}}{h^x_t P^x_{tT_{i-1}}} , \tag{6.4.20}$$

and $\overline{J}^{xy}_t(T_{i-1}, T_i) \geq J^x_t(T_{i-1}, T_i)$ if interest rates are non-negative and $h^x_t \leq h^y_t$ for all $t \in [0, T_{i-1}]$ and for $T_{i-1} \leq T_i$. This is the approach adopted by Nguyen and Seifried (2015) and it shall be revisited in section 7.2. Two comments on their multi-curve model, given the context of the xy-approach, follow:

(i) The quantities $\overline{J}^{xy}_t(T_{i-1}, T_i)$ and $\overline{K}^x$ that determine the FRA's cash flows are derived from the curve-converted quantities $v^{xy}_t(T_{i-1}, T_i)$ and v^x_K respectively. This is in contrast with $J^{xy}_t(T_{i-1}, T_i)$ and K^x, the directly comparable curve-converted quantities used in the xy-formalism. Therefore, these derived quantities are no longer consistent with an FX modelling analogy, with each differing from the correctly converted quantities by an additive factor of $(Q^{xy}_{tT_i} - 1)/\delta_i$.

(ii) Observation (i) is further supported by equation (6.4.20) which shows that the conversion factor process effectively models the spread between the multi-curve y-tenored FCF and the corresponding x-tenored FCF, as opposed to the classical forward exchange rate. Moreover, the derived y-market system has almost no relation to the developed market y-tenored interest rate system, that one seeks to model, since the model derived y-market ZCB-system dominates the x-market ZCB-system, i.e., $P^x_{tT} \leq P^y_{tT}$ for $0 \leq t \leq T$.

Remark 6.4.5 (Revisiting the impact of the market's definition of a developed economy FRA)
The mathematical quantity that directly models the y*-tenored forward IBOR process is* $J^{xy}_{\cdot}(\cdot,\cdot)$ *and not* $J^{y}_{\cdot}(\cdot,\cdot)$*. As explained earlier, this is a consequence of standardly accepted market conventions in developed economies, that the product of the* x*-pricing kernel and the* x*-curve discounted* y*-tenored forward IBOR process is a martingale under the* $\mathbb{P}$*-measure. In the xy-approach, this implies that*

$$h^x_s P^x_{sT_i} J^y_s(T_{i-1},T_i) = \mathbb{E}\left[h^x_t P^x_{tT_i} J^y_t(T_{i-1},T_i) \mid \mathcal{F}_s\right], \tag{6.4.21}$$

for $0 \leq s \leq t \leq T_{i-1}$*. It is not possible to achieve this relationship within the xy-formalism, given the definition of the* y*-tenored forward IBOR process (6.2.4). However, this relationship is achieved if one replaces* $J^{y}_{\cdot}(\cdot,\cdot)$ *with* $J^{xy}_{\cdot}(\cdot,\cdot)$*. The market-implied* y*-tenored forward IBOR process,* $J^{xy}_{\cdot}(\cdot,\cdot)$*, reveals the convolution of a conversion factor (which facilitates the market's martingale assumption (6.4.21)) and the model* y*-tenored forward IBOR process,* $J^{y}_{\cdot}(\cdot,\cdot)$*. This result questions the utility of the* y*-market ZCB-system within the developed economy context. The* y*-market ZCB-system is a model construct, derived from the* y*-tenored model-consistent or model-implied forward IBOR process,* $J^{y}_{\cdot}(\cdot,\cdot)$*, which unravels the market's martingale adjustment from the observed* y*-tenored market-implied IBOR process,* $J^{xy}_{\cdot}(\cdot,\cdot)$*, via the conversion factor* $Q^{yx}_{\cdot\cdot}$*.*

Remark 6.4.6 (Curve-conversion at the level of the x-curve)
The xy-approach advocates the following price process for a multi-curve FRA:

$$V^{xy}_{tT_i} = \delta_i P^{xy}_{tT_i}\left(J^y_t(T_{i-1},T_i) - K^y\right), \tag{6.4.22}$$

for $t \in [0,T_i]$*. Take note that the conversion factor (or martingale adjustment) has been applied to the discounting* x*-market ZCB-system and not to the model for the* y*-tenored forward IBOR process. However, take note that the terminal FRA payoff would now be*

$$V^{xy}_{T_iT_i} = \delta_i \frac{h^y_{T_i}}{h^x_{T_i}}\left(J^y_{T_{i-1}}(T_{i-1},T_i) - K^y\right).$$

This allows one to disentangle the y*-market ZCB system from the* x*-market ZCB-system, which enables one to model* y*-curve discounting in a consistent, robust and rigorous fashion. From a financial economics perspective, if one compares the profit & loss generated from an xy-FRA to a yy-FRA, one can show that*

$$V^{yy}_{tT_i} - V^{yy}_{0T_i} > V^{xy}_{tT_i} - V^{xy}_{0T_i} = V^{xy}_{tT_i} - V^{yy}_{0T_i}, \tag{6.4.23}$$

on average, as required, since discounting at the x*-curve essentially represents a collateralised version of a FRA which should therefore return the holder less than an equivalent position in a non-collateralised FRA, represented here by* y*-curve discounting.*

Next a developed economy IRS is considered, i.e., one which forecasts cash flows under the y-curve but discounts under the x-curve, unlike the emerging economy version of the IRS.

Theorem 6.4.2 (Fair value of a developed economy IRS)
The value of a developed economy IRS, within the x*-market, with reset times* $\{T_0,T_1,\ldots,T_{n-1}\}$*, payment times* $\{T_1,T_2,\ldots,T_n\}$ *and unit nominal, referencing the* y*-tenored IBOR is given by*

$$V^{xy}_{tT_n} = \sum_{i=1}^{n} \delta_i P^x_{tT_i}\left(J^{xy}_t(T_{i-1},T_i) - S^x\right), \tag{6.4.24}$$

for $t \leq T_0$*, where* $\delta_i = T_i - T_{i-1}$ *and where* S^x *is the fixed swap rate within the* x*-market.*

Proof. Beginning with the emerging economy version of the IRS with fixed swap rate S^y within the y-market and applying pricing relation (6.1.4), analogous to Proposition 6.4.1, the developed economy IRS price process is given by

$$V^{xy}_{tT_n} = \sum_{i=1}^{n} \frac{\delta_i}{h^x_t} \mathbb{E}\left[h^x_{T_{i-1}} P^x_{T_{i-1}T_i} Q^{xy}_{T_{i-1}T_i} \left(J^y_{T_{i-1}}(T_{i-1}, T_i) - S^y\right) \big| \mathcal{F}_t\right] , \tag{6.4.25}$$

which, upon application of Lemma 6.4.1 and equation (6.1.3), simplifies to

$$V^{xy}_{tT_n} = \sum_{i=1}^{n} \delta_i P^x_{tT_i} \left(J^{xy}_t(T_{i-1}, T_i) - Q^{xy}_{tT_i} S^y\right) , \tag{6.4.26}$$

for $t \leq T_0$. The result follows by observing that the fixed IRS rate may be expressed in the x-market by $S^x = S^y(\sum_{i=1}^n \delta_i P^x_{tT_i} Q^{xy}_{tT_i})/(\sum_{i=1}^n \delta_i P^x_{tT_i})$. This may be justified in an analogous fashion to the fixed FRA rate, but this time making use of a fixed-for-fixed swap contract as opposed to a forward contract, as in Lemma 6.4.2. □

Remark 6.4.7 (Recovering a familiar IRS valuation formula in the multi-curve setting)
Using Definition 6.4.10, one may restate the value of the developed economy IRS as

$$V^{xy}_{tT_n} = \left[P^{xy}_{tT_0} - P^{xy}_{tT_n}\right] - S^y \sum_{i=1}^{n} \delta_i P^{xy}_{tT_i}, \tag{6.4.27}$$

for $t \leq T_0$, which is the direct multi-curve analogy to the emerging economy version of the IRS value (6.3.5) with the y-ZCBs replaced by the xy-ZCBs.

Corollary 6.4.3 (The fair developed economy IRS fixed rate process)
The fair fixed swap rate process $S^{xy}_t(T_0, T_n)$ for a developed economy IRS written on the market-implied y-tenored forward IBOR (6.4.3), with reset times $\{T_0, T_1, \ldots, T_{n-1}\}$, payment times $\{T_1, T_2, \ldots, T_n\}$ and unit nominal, is given by

$$S^{xy}_t(T_0, T_n) = \frac{P^{xy}_{tT_0} - P^{xy}_{tT_n}}{\sum_{i=1}^n \delta_i P^x_{tT_i}}, \tag{6.4.28}$$

for $t \leq T_0$.

Proof. Setting the value of the developed economy IRS equal to zero, given by equation (6.4.27), it follows that the y-market fair fixed IRS rate is $S^y = \left(P^{xy}_{tT_0} - P^{xy}_{tT_n}\right) / \sum_{i=1}^n \delta_i P^{xy}_{tT_i}$ at time t. Using the proof of Theorem 6.4.2 and Remark 6.1.1, the x-market fair fixed IRS rate (converting the y-market rate) is given by $S^x = S^y(\sum_{i=1}^n \delta_i P^x_{tT_i} Q^{xy}_{tT_i})/(\sum_{i=1}^n \delta_i P^x_{tT_i})$ at time t. Then for any time $t \leq T_0$, the result for the developed economy fair IRS rate follows accordingly by setting $S^{xy}_t(T_0, T_n) = S^x$. □

For a brief description of bootstrapping within a developed economy, refer to Appendix C.3.2. The final linear derivative security to consider within the pre-reference rate reform, or multi-curve, developed economy context is a TBS or floating-for-floating IRS. Quite simply, the utility of such derivatives increased significantly post the GFC since such securities allow market participants to hedge basis risks across IBOR tenors. Given Definition 6.4.1 for the multi-curve market-implied forward IBOR, or fair FRA rate, process, the valuation of TBSs are straightforward, as shown in the next Theorem.

Theorem 6.4.3 (Fair value of a developed economy TBS)
The value of a developed economy TBS, within the x-market, referencing the y- and z-tenored IBOR, with the z-tenor being longer than the y-tenor, is given by

$$V_{tT_n}^{x,yz} = \sum_{j=1}^{m} \Delta_j P_{tU_j}^{x} J_t^{xz}(U_{j-1}, U_j) - \sum_{i=1}^{n} \delta_i P_{tT_i}^{x} \left[J_t^{xy}(T_{i-1}, T_i) + b^x\right] , \tag{6.4.29}$$

where $t \leq T_0 = U_0$, b^x is the fixed TBS spread rate, with

- *$\{T_0, T_1, \dots, T_{n-1}\}$ and $\{T_1, T_2, \dots, T_n\}$ being the reset and payment times associated with the y-tenor, and $\delta_i = T_i - T_{i-1}$;*
- *$\{U_0, U_1, \dots, U_{m-1}\}$ and $\{U_1, U_2, \dots, U_m\}$ being the reset and payment times associated with the z-tenor, $\Delta_j = U_j - U_{j-1}$, and $U_m = T_n$.*

Proof. Being a derivative that only exists in a multi-curve market setting, the fixed TBS spread rate b^x is naturally an x-market quantity and therefore the valuation of the fixed basis-related cash flows are straightforward using the x-market pricing kernel. Then, repeating the process of valuing the floating leg of the developed economy IRS, from Theorem 6.4.2, for each of the floating legs associated with the TBS yields the desired result. □

Corollary 6.4.4 (The fair developed economy TBS fixed spread rate process)
The fair fixed TBS spread rate $b_t^{x,yz}(T_0, T_n)$ for the developed economy TBS defined in Theorem 6.4.3 is given by

$$b_t^{x,yz}(T_0, T_n) = \frac{P_{tU_0}^{xz} - P_{tU_m}^{xz} - \left[P_{tT_0}^{xy} - P_{tT_n}^{xy}\right]}{\sum_{i=1}^{n} \delta_i P_{tT_i}^{x}} , \tag{6.4.30}$$

for $t \leq T_0 = U_0$.

Proof. Using the result from Remark 6.4.7, it is possible to restate the fair value for the developed economy TBS, equation 6.4.29, as follows

$$V_{tT_n}^{x,yz} = P_{tU_0}^{xz} - P_{tU_m}^{xz} - \left[P_{tT_0}^{xy} - P_{tT_n}^{xy}\right] - b^x \sum_{i=1}^{n} \delta_i P_{tT_i}^{x} . \tag{6.4.31}$$

Setting the above value of the TBS equal to zero and solving for b^x then yields required result. □

6.5 Multi-Curve Discounting in Emerging Economies

Now that an understanding of how the xy-formalism enables the modelling of multi-curve systems within developed economies has been established, it is possible to consider resolving the same problem for the case of an emerging economy, as described in Definition 1.4.1. The first hurdle in regressing from the developed to an emerging economy setting is the non-existence of the OIS curve.

Recall that the existence of a collection of curves indexed by $x, y \in \{0, 1, 2, \dots, n\}$ is assumed where the x-curve is referred to as the *discounting curve* and the y-curve as the *forecasting curve*. In a common developed economy, $n = 4$ with 0 denoting the nominal OIS curve, 1 the 1-month IBOR curve, 2 the 3-month IBOR curve, 3 the 6-month IBOR curve, and 4 the 12-month IBOR curve. Moreover, the stochastic evolution of each of these curves is modelled via a pricing kernel process (h_t^y) which is in turn calibrated using liquid primitive, and linear and non-linear derivative instruments. In a common *emerging economy*, only one IBOR tenor is usually tradable and liquid, therefore it is not possible to calibrate the entire set of pricing kernel processes (h_t^y) which span the common developed economy. This leads one to the next conjecture.

Conjecture 6.5.1 (Estimating non-existent or illiquid OIS and IBOR curves in an emerging economy) *In the common emerging economy, only one IBOR tenor, y^*, is tradable and liquid thereby enabling the specification and calibration of a well-defined pricing kernel process $(h_t^{y^*})$. Pricing kernel processes for all other IBOR tenors $(\widehat{h}_t^y)$ have to be estimated statistically (or otherwise) as a suitable functional form of $(h_t^{y^*})$, conjectured to be*

$$\widehat{h}_t^y = f\left(h_t^{y^*}\right) , \tag{6.5.1}$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is measurable and adapted, such that the corresponding estimated y-market ZCB-systems, $(\widehat{P}_{tT}^y)$, may be constructed via

$$\widehat{P}_{tT}^y = \frac{1}{\widehat{h}_t^y}\mathbb{E}\left[\widehat{h}_T^y \,|\, \mathcal{F}_t\right] = \frac{1}{f\left(h_t^{y^*}\right)}\mathbb{E}\left[f\left(h_T^{y^*}\right) \,|\, \mathcal{F}_t\right] , \tag{6.5.2}$$

for $0 \leq t \leq T$.

In the above conjecture, if the function $f(\cdot)$ is linear, then the estimated y-ZCB is given by

$$\widehat{P}_{tT}^y = \frac{1}{f\left(h_t^{y^*}\right)} f\left(h_t^{y^*} P_{tT}^{y^*}\right) , \tag{6.5.3}$$

which implies that it is possible to directly replicate the estimated y-ZCB through either a static or dynamic replication strategy using the y^*-ZCB. However, this may not be possible, in general, if the function $f(\cdot)$ is convex (concave), as the estimated y-ZCB will be governed by the following inequality

$$\widehat{P}_{tT}^y \geq (\leq) \frac{1}{f\left(h_t^{y^*}\right)} f\left(h_t^{y^*} P_{tT}^{y^*}\right) , \tag{6.5.4}$$

which follows by the application of Jensen's inequality.

Assuming that one has resolved the estimation problem posed by Conjecture 6.5.1, the xy-formalism may be applied in the emerging economy setting. One would have a collection of interest rate curves, indexed by $x, y \in \{0, 1, \ldots, y^*, \ldots, n\}$, that are modelled by the calibrated pricing kernel process $(h_t^{y^*})$ and the set of estimated pricing kernel processes $\{(\widehat{h}_t^y)\,;\, y \neq y^*\}$. First, the developed economy or multi-curve FRA is considered within the emerging economy context, i.e., one where the payoff is forecasted by the y-curve and then discounted by the x-curve. The terminal and in-advance FRA payoffs remain unchanged and are identical to equations (6.4.1) and (6.4.2), respectively[2], with the FRA price process also assuming the familiar form

$$\begin{aligned} V_{tT_i}^{xy} &= \delta_i P_{tT_i}^{xy}\left(J_t^y(T_{i-1}, T_i) - K^y\right) \\ &= P_{tT_{i-1}}^{xy} - (1 + \delta_i K^y) P_{tT_i}^{xy} , \end{aligned} \tag{6.5.5}$$

for $t \in [0, T_{i-1}]$, while $J_t^y(T_{i-1}, T_i)$ continues to be the correct forward IBOR process. Notice that the derivation of equation (6.5.5) follows by a direct application of Corollary 6.1.1.

[2] Assuming that one insists on maintaining measurability of the payoff at the IBOR reset time T_{i-1}.

Definition 6.5.1 (The multi-curve emerging economy forward IBOR process)
The multi-curve emerging economy y-tenored forward IBOR process is given by

$$J_t^y(T_{i-1}, T_i) = \frac{1}{\delta_i} \left(\frac{P^y_{tT_{i-1}}}{P^y_{tT_i}} - 1 \right) , \tag{6.5.6}$$

for $t \in [0, T_{i-1}]$, unlike the developed economy which required the definition of the market-implied y-tenored forward IBOR process $J_t^{xy}(T_{i-1}, T_i) := Q^{xy}_{tT_i} J_t^y(T_{i-1}, T_i)$ for $t \in [0, T_{i-1}]$.

This is due to the assumption that there is no market standard for pricing an *emerging economy* FRA that is forecasted and discounted under different curves, with the only observable market quantity being the spot IBOR process $J_t^y(t, t+\delta)$ for $t \geq 0$. It is also possible, as is the case for the developed economy, to define a fair FRA rate process, $K_t^{xy}(T_{i-1}, T_i) = J_t^y(T_{i-1}, T_i)$, however one would not be able to observe this quantity in the market (since these FRAs are not traded, in general), therefore this would be a model-implied quantity[3] — this pricing process would therefore inform the general market-making procedure for such FRAs.

Similarly, one may consider the developed economy IRS within the context of an emerging economy. The value of the IRS at time $t \leq T_0$, making use of the same relations as before, is again given by

$$\begin{aligned} V^{xy}_{tT_n} &= \sum_{i=1}^{n} \delta_i P^{xy}_{tT_i} \left(J_t^y(T_{i-1}, T_i) - S^y \right) \\ &= P^{xy}_{tT_0} - P^{xy}_{tT_n} - S^y \sum_{i=1}^{n} \delta_i P^{xy}_{tT_i} , \end{aligned} \tag{6.5.7}$$

where $J_t^y(T_{i-1}, T_i)$, for $t \in [0, T_{i-1}]$, continues to be the correct forward IBOR process, analogous to the FRA result. As with the FRA, the fair IRS rate process is model-implied and given by

$$S_t^{xy}(T_0, T_n) = \frac{P^{xy}_{tT_0} - P^{xy}_{tT_n}}{\sum_{i=1}^{n} \delta_i P^{xy}_{tT_i}} , \tag{6.5.8}$$

for $t \leq T_0$, unless the y-tenored IBOR process is the tradable tenor and $t = T_0 = 0$.

In a multi-curve emerging economy modelled with the xy-approach, the initial (estimated) y-market ZCB-systems may be constructed in a completely analogous fashion to the single-curve relations, see Appendix C.3, since $K_0^{xy}(T_{i-1}, T_i) = J_0^y(T_{i-1}, T_i)$ and $S_0^{xy}(0, T_n) = (1 - P^y_{0T_n})/(\sum_{i=1}^{n} \delta_i P^y_{0T_i})$. That is, all initial model-implied quantities are only dependent on the y-curve or y-market ZCB-systems.

Consider a FRA and an IRS within this context with payoffs forecasted by the y^*-curve and discounted by one of the other curves, denoted by the x-curve, then the pricing formulae are given by

$$V^{xy^*}_{tT_i} = P^{xy^*}_{tT_{i-1}} - (1 + \delta_i K^{y^*}) P^{xy^*}_{tT_i} , \tag{6.5.9}$$

and

$$V^{xy^*}_{tT_n} = \left[P^{xy^*}_{tT_0} - P^{xy^*}_{tT_n} \right] - S^y \sum_{i=1}^{n} \delta_i P^{xy^*}_{tT_i} , \tag{6.5.10}$$

[3] If the y-tenored IBOR corresponds to the most liquid and tradable tenor, i.e., $y = y^*$, then one will also have access to the set of forward IBOR processes $J_t^y(T_{i-1}, T_i)$ for $0 \leq t \leq T_{i-1}$, from the standard and liquidly tradable set of single-curve emerging economy FRAs, and $K_t^{xy}(T_{i-1}, T_i) = K_t^{yy}(T_{i-1}, T_i) = J_t^y(T_{i-1}, T_i)$.

from equations (6.5.5) and (6.5.7), respectively. At this point, take note that the xy^*-ZCB, $(P_{tT}^{xy^*})$, plays the same role as the y^*-ZCB, $(P_{tT}^{y^*})$, does in the single-curve emerging economy setting in section 6.3. This leads to the following general definition for the xy-ZCB system.

Definition 6.5.2 (The quanto-bond)
In the multi-curve system derived within the xy-formalism, the xy-ZCB system, (P_{tT}^{xy}), defined by

$$P_{tT}^{xy} = \frac{1}{h_t^x}\mathbb{E}\left[h_T^x\left(\frac{h_T^y}{h_T^x}\right)(1)\,\middle|\,\mathcal{F}_t\right] = P_{tT}^x Q_{tT}^{xy} = Q_{tt}^{xy}P_{tT}^y ,$$

may be considered to be a quanto-bond assuming the existence and liquidity of:

(i) the x-curve with varying notional defined by the forward conversion factor Q_{tT}^{xy}; or

(ii) the y-curve with varying notional defined by the spot conversion factor Q_{tt}^{xy}.

Remark 6.5.1 (Dynamic replication of ZCBs and quanto-bonds)
Within the developed economy context — where the nominal OIS curve is considered to be the distinct, single-curve tradable system, which is denoted here as the x^*-curve — one may dynamically replicate y-ZCBs and x^*y-ZCBs, where $y \neq x^*$, via the following set of x^*-curve quanto-bonds:

$$P_{tT}^{y} = \frac{Q_{tT}^{x^*y}}{Q_{tt}^{x^*y}}P_{tT}^{x^*} \quad \text{and} \quad P_{tT}^{x^*y} = Q_{tT}^{x^*y}P_{tT}^{x^*} ,$$

whereas, within the emerging economy context — where one nominal IBOR swap curve is considered to be the distinct, single-curve tradable system, which is denoted here as the y^*-curve — one may dynamically replicate x-ZCBs and xy^*-ZCBs, where $x \neq y^*$, via the following set of y^*-curve quanto-bonds:

$$P_{tT}^{x} = \frac{Q_{tt}^{xy^*}}{Q_{tT}^{xy^*}}P_{tT}^{y^*} \quad \text{and} \quad P_{tT}^{xy^*} = Q_{tt}^{xy^*}P_{tT}^{y^*} .$$

Finally, one may consider the emerging economy version of the TBS that was introduced and defined in Theorem 6.4.3. In keeping with the results derived thus far and based on Definition 6.5.1, and the assumption of applying Conjecture 6.5.1, the curve-conversion factor process has to act at the level of the discounting curve in the valuation of the emerging economy TBS, which will yield:

$$V_{tT_n}^{x,yz} = \sum_{j=1}^{m}\Delta_j P_{tU_j}^{xz}J_t^z(U_{j-1},U_j) - \sum_{i=1}^{n}\delta_i P_{tT_i}^{xy}\left[J_t^y(T_{i-1},T_i) + b^y\right] , \tag{6.5.11}$$

for $t \le T_0 = U_0$. Observe that the fixed TBS spread rate is a naturally defined y-market quantity here, due to the curve-conversion factor process acting at the level of the discounting curve and the definition of the multi-curve y-tenored IBOR process. Then, as with the cases for the FRA and IRS contracts, the fair TBS spread rate is model-implied and given by

$$b_t^{x,yz}(T_0,T_n) = \frac{P_{tU_0}^{xz} - P_{tU_m}^{xz} - \left[P_{tT_0}^{xy} - P_{tT_n}^{xy}\right]}{\sum_{i=1}^{n}\delta_i P_{tT_i}^{xy}} , \tag{6.5.12}$$

for $t \le T_0 = U_0$. It is important to emphasise that Conjecture 6.5.1 has a significant impact on the results derived in this section, since it forces one to estimate explicit pricing kernel models for the non-existent and illiquid OIS and IBOR markets, with the related assumption that all of these become market observable quantities (despite being model-implied). This is considered to be the correct approach since the assumption is that the single liquid and observable IBOR market should not be converted/adjusted when defining multi-curve derivative securities. Rather, the discounting curves should be converted.

Chapter 7

Reduced-Form Multi-Curve Frameworks

Given the natural multi-curve nature of the xy-formalism, and as described in section 1.4.2, existing multi-curve approaches based on HJM and rational pricing kernel models are recovered, reviewed and generalised in this section. Moreover, some popular single-curve models are extended and evolved into a multi-curve setting using the xy-formalism. Before this is done, a brief account of the relevant academic literature is provided. This strand of literature is classified into four categories or modelling approaches, with a non-exhaustive list of references and a brief summary of the main contributions related thereto provided for each.

The first category is *short-rate models*. Kijima *et al.* (2009) propose a three-yield curve model (discount, swap and government bond curves) for an economy with short rates governed by Gaussian, exponentially quadratic models. Kenyon (2010) and Morino and Runggaldier (2014) consider Vasicek, Hull-White and Cox-Ingersoll-Ross (CIR) short-rate models for the OIS, IBOR and/or OIS-IBOR spread curves. Filipović and Trolle (2013) propose a Vasicek process with stochastic long-term mean as the OIS short-rate model with explicit models for default and liquidity risk. Alfeus *et al.* (2020) adopt a novel approach of modelling "roll-over risk" explicitly in a reduced-form setting and consider multi-factor CIR-type processes for this and the OIS short rate — this approach has been significantly evolved by the recent work of Backwell *et al.* (2023), which also caters for reference rate reform.

Heath-Jarrow-Morton (HJM) models constitute the second category. Pallavicini and Tarenghi (2010), Fujii *et al.* (2011), Moreni and Pallavicini (2014), Crépey *et al.* (2015) and Miglietta (2015) all focus on a *hybrid HJM-LMM (LIBOR Market Model) approach* where the OIS curve is modelled using the classical HJM model, while the IBOR forward rates are modelled in an ad hoc manner. Crépey *et al.* (2012) pioneered the use of the HJM framework via a credit risk analogy, while Miglietta (2015) and Grbac and Runggaldier (2015) do the same using a foreign exchange (FX) analogy. Pallavicini and Tarenghi (2010) focus on aspects of calibration, while Moreni and Pallavicini (2014) propose a specific Markovian factor representation which expedites calibration. Crépey *et al.* (2015) consider Lévy driven models, while Cuchiero *et al.* (2016) consider a general semimartingale setup with multiplicative OIS-IBOR spreads.

Category three is the class of *LIBOR Market Models (LMMs)*. Morini (2009), Mercurio (2009, 2010a,b,c) and Bianchetti Bianchetti (2009) were the first to extend the LMM to a multi-curve setting, with the latter doing so via an FX analogy. Mercurio (2010a) and Mercurio and Xie (2012) formalised the first approach utilising an additive spread between OIS and IBOR forward rates that were modelled as martingales under the classical T-forward measure, while Ametrano and Bianchetti (2013) formalised the associated multi-curve bootstrapping process. Grbac *et al.* (2014) provide an alternative to the aforementioned approach using a class of affine LIBOR models, first proposed by Keller-Ressel *et al.* (2013).

The fourth and final catergory are *pricing kernel models*. At the time of writing Macrina and Mahomed (2018), the authors were only aware of Crépey *et al.* (2016) and Nguyen and Seifried (2015) who had formulated multi-curve systems with pricing kernels. It is highlighted there, and again here, that both these papers adopt a hybrid pricing kernel-LMM approach since the OIS curve is modelled with a pricing kernel while the IBOR process is modelled in an ad hoc fashion — this has already been touched on in Chapter 6 and will be expanded on in section 7.2. In Macrina and Mahomed (2018), and thereby Part II of this thesis, a pure pricing kernel based approach to multi-curve modelling is developed, which is believed to be the first of such a modelling class at the time of writing.

For a detailed review of the post-GFC multi-curve interest rate paradigm from both a theoretical and practical perspective, the interested reader is referred to Bianchetti and Morini (2013), Henrard (2014), and Grbac and Runggaldier (2015).

7.1 xy-HJM Multi-Curve Models

In this section, the xy-formalism is utilised to develop a multi-curve modelling framework based on the classical Heath-Jarrow-Morton (HJM) instantaneous forward rate framework. The xy-HJM multi-curve system or framework will be derived using results from Chapter 6.

Consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ where $(\mathcal{F}_t)_{0\leq t}$ is the filtration generated by two sets of independent multi-dimensional $\mathbb{P}$-Brownian motions $(W_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$, respectively. Being synonymous with the xy-formalism, an economy with two distinct markets, x and y, is considered, where x may be interpreted as the market which provides the best proxy for risk-free interest rates, i.e., an OIS-based market, and y as a risky IBOR-based market. Furthermore, we assume that the x- and y-markets are driven by the multi-dimensional $\mathbb{P}$-Brownian motions $(W^x_t)_{t\geq 0} = (W_t)_{t\geq 0}$ and $(W^y_t)_{t\geq 0} = (W_t, Z_t)_{t\geq 0}$ respectively, where $(W_t)_{t\geq 0}$ is n-dimensional and $(Z_t)_{t\geq 0}$ is m-dimensional. This allows one to define the pricing kernel process associated with each market.

Definition 7.1.1 (Pricing kernel processes associated with the xy-HJM framework)
The $(\mathcal{F}_t)$-adapted x- and y-market pricing kernel processes $(h^x_t)_{0\leq t}$ and $(h^y_t)_{0\leq t}$ satisfy, respectively,

$$\frac{\mathrm{d}h^x_t}{h^x_t} = -r^x_t \mathrm{d}t - \lambda^x_t \mathrm{d}W^x_t \quad \textit{and} \quad \frac{\mathrm{d}h^y_t}{h^y_t} = -r^y_t \mathrm{d}t - \lambda^y_t \mathrm{d}W^y_t \,, \tag{7.1.1}$$

where $(r^x_t)_{t\geq 0}$ and $(r^y_t)_{t\geq 0}$ are the short rates; and $(\lambda^x_t)_{t\geq 0}$ and $(\lambda^y_t)_{t\geq 0} = (\lambda^x_t, \lambda^z_t)_{t\geq 0}$ are the n- and $(n+m)$-dimensional market price of risk processes associated with the x- and y-markets, respectively.

Next, let $(X_{tT})_{0\le t\le T}$ and $(Y_{tT})_{0\le t\le T}$ be (well-defined) processes, respectively satisfying

$$\frac{\mathrm{d}X_{tT}}{X_{tT}} = \left(-A^x_{tT} + \frac{1}{2}\,|\Sigma^x_{tT}|^2\right)\mathrm{d}t - \left(\Sigma^x_{tT} + \lambda^x_t\right)\mathrm{d}W^x_t,$$
$$\frac{\mathrm{d}Y_{tT}}{Y_{tT}} = \left(-A^y_{tT} + \frac{1}{2}\,|\Sigma^y_{tT}|^2\right)\mathrm{d}t - \left(\Sigma^y_{tT} + \lambda^y_t\right)\mathrm{d}W^y_t, \tag{7.1.2}$$

for $0 \le t \le T$, where $A^{(\cdot)}_{tT} = \int_t^T a^{(\cdot)}_{tu}\mathrm{d}u$ is 1-dimensional, $|\cdot|$ denotes the Euclidean norm, and $\Sigma^{(\cdot)}_{tT} = \int_t^T \sigma^{(\cdot)}_{tu}\,\mathrm{d}u$ with Σ^x_{tT} and Σ^y_{tT} being n- and $(n+m)$-dimensional, respectively. The processes (σ^x_{tT}), $(\sigma^y_{tT}) = (\sigma^w_{tT}, \sigma^z_{tT})$ and $(a^{(\cdot)}_{tT})$ are generic adapted processes satisfying the implicit integrability conditions, with σ^w_{tT} and σ^z_{tT} being n- and m-dimensional, respectively. The respective ZCB prices may then be defined as follows:

Definition 7.1.2 (ZCB prices associated with the xy-HJM framework)
Setting $P^x_{tT} := X_{tT}/h^x_t$ and $P^y_{tT} := Y_{tT}/h^y_t$, the $(\mathcal{F}_t)$-adapted x- and y-market ZCB-systems satisfy, respectively, the dynamical equations

$$\frac{\mathrm{d}P^x_{tT}}{P^x_{tT}} = \left(r^x_t - A^x_{tT} + \frac{1}{2}\,|\Sigma^x_{tT}|^2 - \lambda^x_t\Sigma^x_{tT}\right)\mathrm{d}t - \Sigma^x_{tT}\mathrm{d}W^x_t\ ,$$
$$\frac{\mathrm{d}P^y_{tT}}{P^y_{tT}} = \left(r^y_t - A^y_{tT} + \frac{1}{2}\,|\Sigma^y_{tT}|^2 - \lambda^y_t\Sigma^y_{tT}\right)\mathrm{d}t - \Sigma^y_{tT}\mathrm{d}W^y_t\ , \tag{7.1.3}$$

following the application of Ito's Lemma. Invoking the classical HJM drift condition, $A^{(\cdot)}_{tT} = \frac{1}{2}|\Sigma^{(\cdot)}_{tT}|^2$, results in $(h^{(\cdot)}_t P^{(\cdot)}_{tT})_{0\le t\le T}$ being a $\mathbb{P}$-(local) martingale which is a requirement for the xy-HJM framework.

Proposition 7.1.1 (The instantaneous forward rates associated with the xy-HJM framework)
Assuming that the x- and y-market ZCB-systems are differentiable in T, the instantaneous forward rate processes $(f^x_{tT})_{0\le t\le T}$ and $(f^y_{tT})_{0\le t\le T}$, defined by $f^x_{tT} = -\partial_T \ln(P^x_{tT})$ and $f^y_{tT} = -\partial_T \ln(P^y_{tT})$, respectively, satisfy

$$\mathrm{d}f^x_{tT} = \left(a^x_{tT} + \lambda^x_t\sigma^x_{tT}\right)\mathrm{d}t + \sigma^x_{tT}\mathrm{d}W^x_t\ ,$$
$$\mathrm{d}f^y_{tT} = \left(a^y_{tT} + \lambda^y_t\sigma^y_{tT}\right)\mathrm{d}t + \sigma^y_{tT}\mathrm{d}W^y_t\ ,$$

which are consistent with the classical HJM instantaneous forward rate modelling framework.

Proof. By direct application of Ito's Lemma, the logarithm of the ZCB price process is

$$\ln\left(P^{(\cdot)}_{tT}\right) = \ln\left(P^{(\cdot)}_{0T}\right) + \int_0^t \left(r_s - A^{(\cdot)}_{sT} - \lambda^{(\cdot)}_s\Sigma^{(\cdot)}_{sT}\right)\mathrm{d}s - \int_0^t \Sigma^{(\cdot)}_{sT}\mathrm{d}W^{(\cdot)}_s,$$

and therefore taking the negative and differentiating with respect to T gives

$$-\frac{\partial}{\partial T}\ln\left(P^{(\cdot)}_{tT}\right) = -\frac{\partial}{\partial T}\ln\left(P^{(\cdot)}_{0T}\right) + \int_0^t \left(a^{(\cdot)}_{sT} + \lambda^{(\cdot)}_s\sigma^{(\cdot)}_{sT}\right)\mathrm{d}s + \int_0^t \sigma^{(\cdot)}_{sT}\mathrm{d}W^{(\cdot)}_s,$$

which yields the required instantaneous forward rate result. □

Grbac and Runggaldier (2015) provide a thorough account of the approaches that have been adopted in modeling a developed economy multi-curve interest rate system with the HJM framework. Here the key results are reprised, given the economy that has already been introduced in this section, in order to contextualise the xy-HJM framework within the existing body of literature. Grbac and Runggaldier (2015) note that all approaches that have been adopted model the x-market ZCB-system (P^x_{tT}) with the classical HJM model while the multi-curve market-implied y-tenored forward IBOR process, which is denoted here by $\overline{J}^{xy}_t(T_{i-1}, T_i)$, is modelled in one of three ways:

(i) $\overline{J}^{xy}_t(T_{i-1}, T_i)$ is specified in an ad hoc fashion (usually) inspired by the LIBOR Market Model (LMM) such that this approach is referred to as a *hybrid HJM-LMM*;

(ii) $\overline{J}^{xy}_t(T_{i-1}, T_i) := (v^y_t(T_{i-1}, T_i) - 1)/\delta_i$ where, as before, $v^y_t(T_{i-1}, T_i) := P^y_{tT_{i-1}}/P^y_{tT_i}$ defines the FCF such that under certain parameter restrictions (see Proposition 7.1.2 below), the process $(h^x_t P^x_{tT_i} v^y_t(T_{i-1}, T_i))_{0\le t\le T_{i-1}}$ is a $\mathbb{P}$-(local) martingale; and

(iii) $\overline{J}^{xy}_t(T_{i-1}, T_i) := \mathbb{E}[h^x_{T_i} J^y_{T_{i-1}}(T_{i-1}, T_i) \,|\, \mathcal{F}_t]/(h^x_t P^x_{tT_i})$ assuming the classical HJM drift condition for the y-market ZCB system.

In each approach $(h^x_t P^x_{tT_i} \overline{J}^{xy}_t(T_{i-1}, T_i))_{0\le t\le T_{i-1}}$ is a $\mathbb{P}$-(local) martingale, as required. Model (i) is inconsistent with the xy-approach, since the xy-formalism focuses on modeling ZCB-systems directly and implying simple spot and forward rate models therefrom. Therefore, modelling approach (i) is noted for completeness. Models (ii) and (iii) are comparable to the xy-approach, therefore these are expanded upon below.

Proposition 7.1.2 (Parameter restrictions for model (ii))
If the following parameter restrictions hold:

$$A^y_{tT_i} - A^y_{tT_{i-1}} = -\frac{1}{2}\left|\Sigma^y_{tT_i} - \Sigma^y_{tT_{i-1}}\right|^2 + \Sigma^x_{tT_i}\left(\Sigma^w_{tT_i} - \Sigma^w_{tT_{i-1}}\right) - \lambda^z_t\left(\Sigma^z_{tT_i} - \Sigma^z_{tT_{i-1}}\right), \tag{7.1.4}$$

then $(h^x_t P^x_{tT_i} v^y_t(T_{i-1}, T_i))_{0\le t\le T_{i-1}}$ *a* $\mathbb{P}$*-(local) martingale, thereby enabling the use of model (ii).*

Proof. Applying Ito's Lemma to $h^x_t P^x_{tT_i} v^y_t(T_{i-1}, T_i)$, using equations (7.1.1) and (7.1.3), yields

$$\begin{aligned} &\frac{\mathrm{d}\left(h^x_t P^x_{tT_i} v^y_t(T_{i-1}, T_i)\right)}{h^x_t P^x_{tT_i} v^y_t(T_{i-1}, T_i)} \\ &= \left[A^y_{tT_i} - A^y_{tT_{i-1}} + \frac{1}{2}\left|\Sigma^y_{tT_i} - \Sigma^y_{tT_{i-1}}\right|^2 - \Sigma^x_{tT_i}\left(\Sigma^w_{tT_i} - \Sigma^w_{tT_{i-1}}\right) + \lambda^z_t\left(\Sigma^z_{tT_i} - \Sigma^z_{tT_{i-1}}\right)\right]\mathrm{d}t \\ &\quad + \left(\Sigma^w_{tT_i} - \Sigma^w_{tT_{i-1}} - \Sigma^x_{tT_i} - \lambda^x_t\right)\mathrm{d}W^x_t + \left(\Sigma^z_{tT_i} - \Sigma^z_{tT_{i-1}}\right)\mathrm{d}Z_t, \end{aligned}$$

from which it follows that the martingale condition is achieved only if equation (7.1.4) is enforced. □

Remark 7.1.1 (Parameter restrictions for model (ii) — only one set of risk factors)
In Grbac and Runggaldier (2015), both the x*- and* y*-markets have the same sources of risk, i.e., are driven by the same set of Brownian motions, which resolves the parameter restrictions to*

$$A^y_{tT_i} - A^y_{tT_{i-1}} = -\frac{1}{2}\left(\Sigma^y_{tT_i} - \Sigma^y_{tT_{i-1}}\right)^2 + \Sigma^x_{tT_i}\left(\Sigma^y_{tT_i} - \Sigma^y_{tT_{i-1}}\right),$$

for $0 \le t \le T_{i-1}$.

Model (iii) requires one to compute a conditional expectation, $\mathbb{E}[h^x_{T_i} J^y_{T_{i-1}}(T_{i-1}, T_i) \,|\, \mathcal{F}_t]$, which is possible given equations (7.1.1) and (7.1.3), along with the classical HJM drift condition applied to the y-market ZCB-system. Note that in Grbac and Runggaldier (2015), this model is justified by analogies to credit and foreign exchange modeling. Their model setup leads to two different parameter restrictions, depending on which analogy is assumed. Here, the pricing kernel-based HJM setup leads to a unique parameter restriction (the classical HJM drift condition for the y-market ZCB-system) which subsumes both analogies, since the xy-approach does not require one to specify an exchange rate process in an ad hoc exogenous manner. This may be seen in the next proposition, recalling Proposition 6.1.1.

Proposition 7.1.3 (The curve-conversion processes associated with the xy-HJM framework)
The xy-HJM framework's forward curve-conversion factor process $(Q^{xy}_{tT})_{0\leq t\leq T}$ *satisfies*

$$\frac{\mathrm{d}Q^{xy}_{tT}}{Q^{xy}_{tT}} = (\Sigma^x_{tT} - \Sigma^w_{tT})(\Sigma^x_{tT}\mathrm{d}t + \lambda^x_t\mathrm{d}t + \mathrm{d}W^x_t) - (\Sigma^z_{tT} + \lambda^z_t)\mathrm{d}Z_t \ ,$$

while the spot curve-conversion factor process $(Q^{xy}_{tt})_{t\geq 0}$ *satisfies*

$$\frac{\mathrm{d}Q^{xy}_{tt}}{Q^{xy}_{tt}} = (r^x_t - r^y_t)\,\mathrm{d}t + \lambda^x_t\mathrm{d}W^x_t - \lambda^y_t\mathrm{d}W^y_t \ ,$$

where $\lambda^x_t\mathrm{d}W^x_t - \lambda^y_t\mathrm{d}W^y_t = -\lambda^z_t\mathrm{d}Z_t$.

Proof. Using the definition of the conversion factor, equation (6.1.3), along with Definition 7.1.2, observe that $Q^{xy}_{tT} = Y_{tT}/X_{tT}$ while $Q^{xy}_{tt} = h^y_t/h^x_t$. The result then follows by a straightforward application of Ito's Lemma using equations (7.1.2) and (7.1.1). ☐

Remark 7.1.2
By the Girsanov Theorem, it is straightforward to show that there is a multi-dimensional $\mathbb{Q}_x$*-Brownian motion* $(W^{\mathbb{Q}_x}_t)$ *that satisfies* $\mathrm{d}W^{\mathbb{Q}_x}_t = \lambda^x_t\mathrm{d}t + \mathrm{d}W^x_t$ *upon changing measure from* $\mathbb{P}$ *to the* x*-market pricing measure* $\mathbb{Q}_x$*. Moreover there is also a multi-dimensional* $\mathbb{Q}^T_x$*-Brownian motion* $(W^{\mathbb{Q}^T_x}_t)$ *that satisfies* $\mathrm{d}W^{\mathbb{Q}^T_x}_t = \Sigma^x_{tT}\mathrm{d}t + \mathrm{d}W^{\mathbb{Q}_x}_t$ *upon changing measure from* $\mathbb{Q}_x$ *to the* x*-market* T*-forward measure* $\mathbb{Q}^T_x$*.*

In Part II, the xy-formalism has been developed for multi-curve interest rate modelling, and in turn advocated model structures for both multi-curve emerging and developed economy forward IBOR processes (see Definitions 6.5.1 and 6.4.1, respectively). The xy-HJM framework version of these multi-curve forward IBOR processes are provided in the next definition.

Definition 7.1.3 (Multi-curve emerging and developed economy forward IBOR processes)
Within the xy-HJM framework, the multi-curve emerging economy y*-tenored forward IBOR process is given by*

$$J^y_t(T_{i-1},T_i) = \frac{1}{\delta_i}\left(v^y_t(T_{i-1},T_i) - 1\right),$$

with the FCF process, $v^y_t(T_{i-1},T_i)$*, satisfying*

$$\frac{\mathrm{d}v^y_t(T_{i-1},T_i)}{v^y_t(T_{i-1},T_i)} = \left(\Sigma^y_{tT_i} - \Sigma^y_{tT_{i-1}}\right)\left(\Sigma^y_{tT_i}\mathrm{d}t + \lambda^y_t\mathrm{d}t + \mathrm{d}W^y_t\right),$$

for $0 \leq t \leq T_{i-1}$*, such that the process* $(h^y_t P^y_{tT_i} v^y_t(T_{i-1},T_i))_{0\leq t\leq T_{i-1}}$ *is a* $\mathbb{P}$*-(local) martingale. The multi-curve developed economy* y*-tenored forward IBOR process is given by*

$$J^{xy}_t(T_{i-1},T_i) = Q^{xy}_{tT_i}J^y_t(T_{i-1},T_i) = \frac{1}{\delta_i}\left(v^{xy}_t(T_{i-1},T_i) - Q^{xy}_{tT_i}\right),$$

with the converted FCF process, $v_t^{xy}(T_{i-1},T_i) := Q_{tT_i}^{xy} v_t^y(T_{i-1},T_i)$*, satisfying*

$$\frac{\mathrm{d}v_t^{xy}(T_{i-1},T_i)}{v_t^{xy}(T_{i-1},T_i)} = \left(\Sigma_{tT_i}^x - \Sigma_{tT_{i-1}}^w\right)\left(\Sigma_{tT_i}^x \mathrm{d}t + \lambda_t^x \mathrm{d}t + \mathrm{d}W_t^x\right) - (\Sigma_{tT_{i-1}}^z + \lambda_t^z)\mathrm{d}Z_t \ ,$$

for $0 \le t \le T_{i-1}$*, such that the process* $(h_t^x P_{tT_i}^x v_t^{xy}(T_{i-1},T_i))_{0\le t\le T_{i-1}}$ *is a* $\mathbb{P}$*-(local) martingale.*

Take note that $(h_t^x P_{tT_i}^x J_t^{xy}(T_{i-1},T_i))_{0\le t\le T_{i-1}}$ is also a $\mathbb{P}$-(local) martingale, however the multi-curve developed economy y-tenored forward IBOR process does not have an elegant differential representation as it is essentially the difference between two stochastic processes, these being the converted FCF process and the curve-conversion factor process.

Remark 7.1.3 (The xy-HJM framework compared to other approaches)
The only parameter restrictions required by the xy-HJM framework are the classical HJM drift conditions for both the x*- and* y*-market ZCB-systems. Therefore model (iii), as presented in Grbac and Runggaldier (2015), is also a viable model for the developed economy forward IBOR process, albeit an unnatural one given the incompatibility between the* x*-market pricing kernel* (h_t^x) *and the* y*-market forward IBOR process* $(J_t^y(T_{i-1},T_i))$*. Another viable model within the xy-HJM framework is that of Nguyen and Seifried (2015), given by equation (6.4.17), however recall the observations in Remark 6.4.4 regarding this model.*

In the next section, the class of rational multi-curve models are introduced. Such models, and in particular those produced in Section 7.2.2, provide a rich class of flexible and tractable specifications for xy-HJM multi-curve models and associated spread dynamics.

7.2 Rational Multi-Curve Models

As reported in Grbac and Runggaldier (2015), multi-curve rational interest rate models based on the pricing kernel approach have appeared in Crépey *et al.* (2016) and in Nguyen and Seifried (2015).

The multi-curve approach proposed by Crépey *et al.* (2016) assumes a discount bond system associated with an OIS market and introduces a (forward) IBOR process that has a built-in spread when compared to the OIS rate. The OIS-based discount bond price system, which in the xy-formalism corresponds to the x-curve ZCB-system, is generated by pricing kernel models driven by stochastic factors. The (forward) IBOR process is derived by pricing a FRA written on the IBOR. The factor-based model of the multi-curve (forward) IBOR process is then deduced from the no-arbitrage relation the FRA price process is required to satisfy.

The IBOR model turns out to be a rational function(al) of stochastic drivers that is given in units of the OIS pricing kernel proxy. Thus, whenever the IBOR dynamics depend on an idiosyncratic driving factor (not affecting the OIS pricing kernel proxy), an OIS-IBOR spread is generated that depends on a spread-idiosyncratic stochastic factor. The source of the spread can be readily read off from the expression of the IBOR model owing to the transparency of the multi-curve approach brought forward.

Given that the OIS-IBOR spread is obtained by focusing on how the IBOR is modelled, the approaches of Crépey *et al.* (2016) and Nguyen and Seifried (2015) may be thought of as *rate-based modelling approaches*. The next section classifies these models as *hybrid rational-LMM multi-curve models*, shows how they may be recovered within the xy-approach and analyses various features from the vantage point of the xy-formalism. Thereafter, the sections that follow reveal how the xy-formalism enables the specification of multi-curve interest rate models that are completely or purely rational, and thereby enables the use of certain popular sub-classes of rational pricing kernel models that have been applied with success in the single-curve setting.

7.2.1 Hybrid Rational-LMM Multi-Curve Models

A feature that is rather telling in understanding the structure of multi-curve models, and thus helps in their classification, is the nature of the discount and the forecasting curve, respectively. In the multi-curve models by Crépey *et al.* (2016), the term structure of the discount (OIS-based) curve is constructed by a rational model. However, the IBOR model is postulated in a rather ad-hoc manner and ensues directly from modelling the payoff of the FRA written on it. Similar to the hybrid HJM-LMM models in section 7.1, the forecasting curve (i.e., IBOR-based term structure) is constructed akin to LMMs. This is why the models of Crépey *et al.* (2016), and to some extent also those of Nguyen and Seifried (2015), are referred to as *rational-LMM hybrid models.* Next, the relations between these models and the xy-approach is established.

Proposition 7.2.1 (The fair FRA rate from Crépey *et al.* (2016) versus the xy-approach)
Let $K(t;T_{i-1},T_i)$ denote the fair FRA rate obtained in Crépey et al. (2016), section 2.1. Then, it holds that $K(t;T_{i-1},T_i) = K^{xy}_t(T_{i-1},T_i)$, where $K^{xy}_t(T_{i-1},T_i)$ is determined by equation (6.4.11).

Proof. By setting $P_{tT_i} = P^x_{tT_i}$, it follows that

$$J(t;T_{i-1},T_i) = P^x_{tT_i} J^{xy}_t(T_{i-1},T_i) = P^{xy}_{tT_i} J^y_t(T_{i-1},T_i)\ , \tag{7.2.1}$$

where $J(t;T_{i-1},T_i)$ is the IBOR specified in Crépey *et al.* (2016), equation (2.6). □

Furthermore, in section 2.2 of Crépey *et al.* (2016), a particular class of rational IBOR models is presented that becomes the workhorse, later in the paper. Next it is shown how such a class may be obtained within the xy-formalism.

Remark 7.2.1 (Recovering the approach of Crépey *et al.* (2016) within the xy-formalism)
From the relation (7.2.1) and by recalling that $P^y_{tT_i} = \mathbb{E}[h^y_{T_i}|\mathcal{F}_t]/h^y_t$, one may deduce that

$$J(t;T_{i-1},T_i) = \frac{1}{\delta_i\, h^x_t}\left(\mathbb{E}\left[h^y_{T_{i-1}}\,|\,\mathcal{F}_t\right] - \mathbb{E}\left[h^y_{T_i}\,|\,\mathcal{F}_t\right]\right). \tag{7.2.2}$$

Next, specify the discounting and forecasting kernels as follows:

$$h^x_t = P^x_{0t} + b_1(t)\,A^{(1)}_t,$$
$$h^y_t = P^y_{0t} + \bar{b}_2(t)\,A^{(2)}_t + \bar{b}_3(t)\,A^{(3)}_t,$$

where, for $i\in\{1,2,3\}$, the processes $(A^{(i)}_t)$ are martingales. The quantities P^x_{0t}, P^y_{0t}, $b_1(t)$ and $\bar{b}_i(t)$, $i\in\{2,3\}$, are suitably chosen deterministic functions. The correspondence to the rational multi-curve

IBOR models by Crépey et al. (2016), section 2.2, is found by setting

$$J(0;T_{i-1},T_i) = \frac{1}{\delta_i}\left(P^y_{0T_{i-1}} - P^y_{0T_i}\right),$$
$$b_2(T_i,T_{i-1}) = \frac{1}{\delta_i}\left[\bar{b}_2(T_{i-1}) - \bar{b}_2(T_i)\right],$$
$$b_3(T_i,T_{i-1}) = \frac{1}{\delta_i}\left[\bar{b}_3(T_{i-1}) - \bar{b}_3(T_i)\right]. \tag{7.2.3}$$

The specifications (7.2.3) cause a slight loss of generality. However, whether in practical terms such specifications are indeed restrictive can be decided once this model class is calibrated to actual market data.

Attention is now given to the rational multi-curve models presented in Nguyen and Seifried (2015). The authors propose to make use of the so-called FX-analogy to motivate pricing kernel models for the spread observed between the OIS rate and IBORs. In particular in section 4, Theorem 4.1, a multiplicative spread is considered. The spread is given by the ratio of a conditional expectation of the OIS-based pricing kernel (state-price deflator) and a conditional expectation of a hypothetical pricing kernel. The latter deflator may be associated with a foreign currency, although such an interpretation is dissuaded, c.f. section 4.2 of Nguyen and Seifried (2015).

Although the OIS-IBOR spread is interpreted as a kind of foreign exchange rate in their work, the deduced rational multi-curve IBOR models are of the kind derived by Crépey *et al.* (2016). This is especially so because the rational IBOR models developed in Nguyen and Seifried (2015) are *rate-based models* — just as those produced by Crépey *et al.* (2016) — which relate the OIS and IBOR forward rates, directly. Moreover, it can be shown that the approach of Nguyen and Seifried (2015) may be derived within a pricing kernel setup without any need for the application of an FX-analogy — this is the purpose of the next proposition.

Proposition 7.2.2 (The Nguyen and Seifried (2015) model recovered using Crépey *et al.* (2016))
In Nguyen and Seifried (2015), the multi-curve fair FRA rate $J^\Delta(t;T,T+\Delta)$ is given by

$$J^\Delta(t;T,T+\Delta) = \frac{1}{\Delta}\left(\frac{p(t,T)}{p(t,T+\Delta)}\frac{\mathbb{E}\left[D^\Delta_T|\,\mathcal{F}_t\right]}{\mathbb{E}\left[D_T|\,\mathcal{F}_t\right]} - 1\right), \tag{7.2.4}$$

for $t \in [0,T]$. This model may be obtained by specifying the IBOR process $(J(t;T_{i-1},T_i))_{0\le t\le T_{i-1}}$, for $i \in \{1,2,\ldots,n\}$, in Crépey et al. (2016), section 2.1, equation (2.7):

$$J(t;T_{i-1},T_i) = \frac{1}{\Delta D_t}\left(\mathbb{E}\left[D^\Delta_T\,|\,\mathcal{F}_t\right] - \mathbb{E}\left[D_{T+\Delta}\,|\,\mathcal{F}_t\right]\right), \tag{7.2.5}$$

where $T_{i-1} = T$ and $T_i = T+\Delta$.

Proof. Relation (7.2.5) is directly obtained by equating the fair FRA rate, equation (4.2), in Nguyen and Seifried (2015) with the fair FRA rate, euqation (2.7), in Crépey *et al.* (2016). This shows that the OIS-IBOR spread models, given in Theorem 4.1 in Nguyen and Seifried (2015), do not require the use of the FX-analogy in order to derive (rate-based) multi-curve models using pricing kernels. While (D_t) corresponds to the OIS-associated pricing kernel process (π_t) in Crépey *et al.* (2016), there is indeed no reason to identify the process (D^Δ_t) with a fictitious pricing kernel associated with a foreign currency/economy. It may just be viewed as an idiosyncratic component of the IBOR process. □

Remark 7.2.2
Comparing equation (7.2.5) with equation (7.2.2), a discrepancy in the way that the conversion to a multi-curve setup is obtained, is observed in Nguyen and Seifried (2015). The source of such incongruence is discussed in Remark 6.4.4, (i). The difference is resolved by the following adjustment in the multi-curve model (7.2.4):

$$J^{\Delta}(t;T,T+\Delta)=\frac{1}{\Delta}\left(\frac{p(t,T)}{p(t,T+\Delta)}\frac{\mathbb{E}\left[D_T^{\Delta}|\,\mathcal{F}_t\right]}{\mathbb{E}\left[D_T|\,\mathcal{F}_t\right]}-1\cdot Q(t,T+\Delta)\right),$$

where, based on to the xy-approach, the conversion factor $Q(t,T+\Delta)$, or spread process, is given by

$$Q(t,T+\Delta)=\frac{\mathbb{E}\left[D_{T+\Delta}^{\Delta}|\,\mathcal{F}_t\right]}{\mathbb{E}\left[D_{T+\Delta}|\,\mathcal{F}_t\right]}\ .$$

The adjustment allows the model to be derived by a consistent application of the FX-analogy in a pricing kernel setup, as produced in the xy-approach.

7.2.2 Pure-Rational Multi-Curve Models

Unlike the preceding rational-LMM hybrid multi-curve models, rational models for both the discounting curve and the forecasting curve is considered, i.e., for the ZCB price process $(P_{tT_i}^x)_{0\leq t\leq T_i}$ and $(P_{tT_i}^y)_{0\leq t\leq T_i}$, respectively. This is desirable for: (i) tractability; (ii) transparency of the dependence structure among the risk factors; and thus (iii) a good understanding of the resulting model for the spread dynamics between the x- (discounting) and the y- (forecasting) curves. The rational price models considered by Macrina (2014), and by Crépey *et al.* (2016) for multi-curve modelling in particular, offers the set of properties that are required. For the x- and y-ZCB, the following are postulated:

$$\begin{aligned} P_{tT_i}^x &= \frac{P_{0T_i}^x\prod_{k=1}^m Z_k^x(t,T_i)}{P_{0t}^x\prod_{k=1}^m Z_k^x(t)}\ , \quad \text{and} \\ P_{tT_i}^y &= \frac{P_{0T_i}^y\prod_{\ell=1}^n Z_\ell^y(t,T_i)}{P_{0t}^y\prod_{\ell=1}^n Z_\ell^y(t)}\ , \end{aligned} \tag{7.2.6}$$

where $Z_k^x(t,T_i)=(1+b_k^x(T_i)\,A_{t,\,k}^x)$ and $Z_\ell^y(t,T_i)=(1+b_\ell^y(T_i)\,A_{t,\,\ell}^y)$ are taken to be positive processes. The quantities P_{0t}^x and P_{0t}^y are the initial term structures of the x and y ZCBs, b_k and b_ℓ are deterministic functions, and $(A_{t,\,k}^x)$ and $(A_{t,\,\ell}^y)$ are martingales with respect to some ($\mathbb{P}$-equivalent) probability measure. For further technical details, please refer to Macrina (2014) and Crépey *et al.* (2016). A closer look at $(P_{tT_i}^y)$ is taken, although the structural properties of the model also apply to $(P_{tT_i}^x)$. The return process of the forecasting ZCB is given by

$$\ln\left(P_{tT_i}^y\right)=\ln\left(\frac{P_{0T_i}^y}{P_{0t}^y}\right)+\sum_{\ell=1}^n\ln\left(\frac{1+b_\ell^y(T_i)\,A_{t,\,\ell}^y}{1+b_\ell^y(t)\,A_{t,\,\ell}^y}\right).$$

The associated short rate process (r_t^y) is given by

$$r_t^y=-\left(\frac{\partial_t P_{0t}^y}{P_{0t}^y}+\sum_{\ell=1}^n\theta_{t,\,\ell}^y\right), \tag{7.2.7}$$

where the $(A_{t,\,\ell}^y)$-driven factor component $(\theta_{t,\,\ell}^y)$ is defined as

$$\theta_{t,\,\ell}^y=\frac{\partial_t b_\ell^y(t)\,A_{t,\,\ell}^y}{1+b_\ell^y(t)\,A_{t,\,\ell}^y}. \tag{7.2.8}$$

Now assume, for the sake of the explanation, that the number n of factor components is given by the particular tenor y. Let $a \in \{1,2,3,\ldots\}$; $y = 3a$ which denotes the 3-, 6-, 9-, 12-month, etc., IBOR tenor; and $n = a+2$ the number of factor components. For the 1-month IBOR tenor, assume that the short rate is driven by two factor components, i.e., $n = 2$. Then, the following additive structure for the short rate model associated with the corresponding forecasting ZCBs is obtained:

$$
\begin{aligned}
&y = 1\text{-month-tenored ZCB},\ (P^1_{tT_i}): && r^1_t = -\left(\frac{\partial_t P^1_{0t}}{P^1_{0t}} + \theta^1_1(t) + \theta^1_2(t)\right), \qquad (7.2.9)\\
&y = 3\text{-month-tenored ZCB},\ (P^3_{tT_i}): && r^3_t = -\left(\frac{\partial_t P^3_{0t}}{P^3_{0t}} + \theta^3_1(t) + \theta^3_2(t) + \theta^3_3(t)\right),\\
&\quad\vdots && \quad\vdots\\
&y = 3a\text{-month-tenored ZCB},\ (P^y_{tT_i}): && r^y_t = -\left(\frac{\partial_t P^y_{0t}}{P^y_{0t}} + \sum_{\ell=1}^{a+2}\theta^y_\ell(t)\right), \quad a \in \{1,2,3,\ldots\}.
\end{aligned}
$$

Depending on the IBOR market, one could envisage the situation where $\theta^i_\ell = \theta^j_\ell$ for all $i, j = 3a$. This would mean that the various y-curves only differ by the number of factor components driving the corresponding short rates (i.e., forecasting ZCBs). One would then have

$$
y\text{-month-tenored ZCB},\ (P^y_{tT_i}): \qquad r^y_t = -\left(\frac{\partial_t P^y_{0t}}{P^y_{0t}} + \sum_{\ell=1}^{a+2}\theta_\ell(t)\right), \quad a \in \{1,2,3,\ldots,\},
$$

where the short rate model for the 1-month tenor ZCB is recovered by setting $a = 0$. From the FRA price process (6.4.12), one sees that the quantity responsible for the consistent transfer from a single-curve to a multi-curve setting is the quanto-bond with price process $(P^{xy}_{tT_i})$.

Next, the multiplicative class of rational models for the quanto-bond price process $(P^{xy}_{tT_i})_{0\le t\le T_i}$ is introduced:

$$
P^{xy}_{tT_i} = \frac{1}{h^x_t}\mathbb{E}\left[h^y_{T_i}\,\middle|\,\mathcal{F}_t\right] = \frac{P^y_{0T_i}\prod_{\ell=1}^n Z^y_\ell(t,T_i)}{P^x_{0t}\prod_{k=1}^m Z^x_k(t)}.
$$

The model for the quanto-bond's short rate $(r^{xy}_t)_{0\le t}$ is obtained by $r^{xy}_t = -\partial_{T_i}\ln(P^{xy}_{tT_i})|_{T=t}$, assuming that the quanto-bond price function is differentiable in its maturity T_i. It follows that $r^{xy}_t = r^y_t$. One could argue that it is somewhat artificial to introduce the x-market ZCB-system because, after all, the x-curve may be a specific y-curve. However, to allow for more generality, there is no reason why the type of model ought to be the same for the x-ZCB and for the y-ZCB. It is only for convenience that the same type of pricing model for both types of bonds is considered here.

In any case, the discounting curve — identified by the 1-day IBOR — can be viewed as forecasting curve identified as $y = 0$ in the above setup (7.2.9):

$$
1\text{-day-tenored ZCB},\ P^x_{tT_i} = P^0_{tT_i}: \qquad r^x_t = r^0_t = -\left(\frac{\partial_t P^0_{0t}}{P^0_{0t}} + \theta^0_1(t)\right).
$$

A byproduct of the multi-curve modelling approach based on bonds as is the case with the xy-formalism, is the implicit, or rather emerging, spread models. Within the rational models, the process for the spread between the y and $y+3$m curves is given by

$$
s^{y,y+3m}_{tT_i} = \frac{P^{y+3m}_{tT_i}}{P^y_{tT_i}} = \frac{P^y_{0t}}{P^{y+3m}_{0t}}\frac{P^{y+3m}_{0T_i}}{P^y_{0T_i}}\,\Delta_{a+2}(t,T_i), \quad a \in \{1,2,3,\ldots\},
$$

where the stochastic spread process $(\Delta_{a+2}(t,T_i))_{0\le t\le T_i}$ is given by

$$\Delta_{a+2}(t,T_i) = \frac{1 + b^y_{a+2}(T_i)\,A^y_{t,\,a+2}}{1 + b^y_{a+2}(t)\,A^y_{t,\,a+2}} \; .$$

Take note that the stochastic spread is positive assuming that the rates underlying the tenors are non-negative, see Corollary 6.4.2.

7.2.3 Linear-Rational Term Structure Models

Filipović *et al.* (2017) introduce the so-called *linear-rational term structure* (LRTS) models. In this section it is shown how the multi-curve extension to the LRTS is produced by showing that the these models belong to the more general class introduced in the previous section. It is thus proved that: (a) the LRTS models belong to the class of models developed in Macrina (2014) when an infinite-time horizon is considered; and (b) that the pricing kernel generating the LRTS model is a *weighted heat kernel* (WHK). Pricing kernels generated by WHKs in an infinite time horizon setting are introduced in Akahori *et al.* (2014) and developed in Akahori and Macrina (2012) for the case of the WHK being driven by a time-inhomogeneous Markov process. In particular, it shall be shown that the LRTS models produce ZCB price processes $(P_{tT})_{0\le t\le T}$ of the form

$$P_{tT} = \frac{P_{0T} + b(T)\,A_t}{P_{0t} + b(t)\,A_t} \; , \tag{7.2.10}$$

for $0 \le t \le T$, which are identified as a class of Markov functionals. The function $b(t)$ is deterministic and $(A_t)_{0\le t}$ is a martingale process. The explicit construction of this class of term structure models is presented in Macrina (2014).

Definition 7.2.1 (Linear-Rational Term Structure Models, Filipović *et al.* (2017))
Let $(Z_t)_{0\le t}$ denote the multivariate process with state space $E \subset \mathbb{R}^m$ that satisfies the SDE:

$$\mathrm{d}Z_t = \kappa(\theta - Z_t)\mathrm{d}t + \mathrm{d}M_t \; , \tag{7.2.11}$$

where $\kappa \in \mathbb{R}^{m\times m}$ and $\theta \in \mathbb{R}^m$, and where $(M_t)_{0\le t}$ is an m-dimensional martingale. Let $(\zeta_t)_{0\le t}$ denote the pricing kernel process defined by

$$\zeta_t = \mathrm{e}^{-\alpha t}\left(\phi + \psi Z_t\right) \; , \tag{7.2.12}$$

where $\alpha \in \mathbb{R}$, $\phi \in \mathbb{R}$, and $\psi \in \mathbb{R}^m$ such that $\phi + \psi z > 0$ for all $z \in E$. The linear-rational term structure, generated by the linear pricing kernel process $(\zeta_t)_{0\le t}$, has a ZCB price process $(P_{tT})_{0\le t\le T}$ given by

$$P_{tT} = \mathrm{e}^{-\alpha(T-t)}\frac{\phi + \psi\theta + \psi \mathrm{e}^{(T-t)}\left(Z_t - \theta\right)}{\phi + \psi Z_t} \; , \tag{7.2.13}$$

where T is the bond maturity date.

Proposition 7.2.3 (Unique solution for the SDE 7.2.11)
The SDE (7.2.11) has the unique solution given by

$$Z_t = \mathrm{e}^{-\kappa t}\left(Z_0 + \kappa\int_0^t \mathrm{e}^{\kappa s}\mathrm{d}s\,\theta\right) + \mathrm{e}^{-\kappa t}A_t \; . \tag{7.2.14}$$

The process $(A_t)_{0\le t}$*, defined by*

$$A_t = \int_0^t \mathrm{e}^{\kappa s}\mathrm{d}M_s \ , \tag{7.2.15}$$

is a martingale.

Proof. That the mean-reverting process (7.2.14) is the unique solution to the SDE (7.2.11) follows from a straightforward application of Ito's Lemma. To show that $(A_t)_{0\le t}$ is a martingale, one remarks that $\mathbb{E}[|\,A_t|] < \infty$, for all $t \ge 0$, and that $\mathbb{E}[A_u|\mathcal{F}_s] = A_s$, for $0 \le s \le u$. The latter follows by calculating $\mathbb{E}[\int_s^u \mathrm{d}[\phi(t)M_t]\,|\,\mathcal{F}_s]$, where $0 \le s \le t \le u$, and by applying Fubini's Theorem. One then obtains

$$\mathbb{E}[A_u|\mathcal{F}_s] - A_s = \mathbb{E}[\phi(u)M_u - \phi(s)M_s\,|\,\mathcal{F}_s] - \mathbb{E}\left[\int_s^u M_t\partial_t\phi(t)\mathrm{d}t\,\middle|\,\mathcal{F}_s\right] = 0, \tag{7.2.16}$$

which completes the proof. ☐

Theorem 7.2.1 (Recovering the LRTS model within the class of models developed in Macrina (2014)) *The pricing kernel process* $(\zeta_t)_{0\le t}$ *that generates the LRTS models, specified in Definition 7.2.1, is given by*

$$\zeta_t = \zeta_0\left[P_{0t} + b(t)\,A_t\right],$$

where $\zeta_0 = \phi + \psi Z_0$*. The positive, deterministic function* $(P_{0t})_{0\le t\le T}$ *is the initial term structure of the associated* T*-maturity ZCB-system with price process*

$$P_{tT} = \frac{P_{0T} + b(T)\,A_t}{P_{0t} + b(t)\,A_t} \ , \tag{7.2.17}$$

where P_{0t}*, the deterministic function* $b(t)$ *and the martingale* (A_t) *are determined by*

$$\begin{aligned}
P_{0t} &= \frac{\mathrm{e}^{-\alpha t}}{\phi+\psi Z_0}\left[\phi + \psi\mathrm{e}^{-\kappa t}\left(Z_0 + \kappa\int_0^t \mathrm{e}^{\kappa s}\mathrm{d}s\,\theta\right)\right], \qquad 0\le t\le T,\\
b(t) &= \frac{\mathrm{e}^{-\alpha t}}{\phi+\psi Z_0}\,\psi\mathrm{e}^{-\kappa t}, \qquad 0\le t\le T,\\
A_t &= \int_0^t \mathrm{e}^{\kappa s}\mathrm{d}M_s, \qquad t\ge 0.
\end{aligned} \tag{7.2.18}$$

Proof. One direction is straightforward: it suffices to insert (7.2.18), (7.2.18) and (7.2.18) in (7.2.17) to obtain (7.2.13). The other direction, i.e., beginning from Definition 7.2.1, goes as follows: The solution (7.2.14) is inserted in (7.2.13) to obtain

$$P_{tT} = \frac{\mathrm{e}^{-\alpha T}\left[\phi + \psi\mathrm{e}^{-\kappa T}\left(Z_0 + \kappa\int_0^t \mathrm{e}^{\kappa s}\mathrm{d}s\,\theta\right)\right] + \mathrm{e}^{-\alpha T}\psi\kappa\int_0^t \mathrm{e}^{-\kappa(T-s)}\mathrm{d}s\,\theta + \mathrm{e}^{-\alpha T}\psi\mathrm{e}^{-\kappa T}A_t}{\mathrm{e}^{-\alpha t}\left[\phi + \psi\mathrm{e}^{-\kappa t}\left(Z_0 + \kappa\int_0^t \mathrm{e}^{\kappa s}\mathrm{d}s\,\theta\right)\right] + \mathrm{e}^{-\alpha t}\psi\mathrm{e}^{-\kappa t}A_t}\,.$$

Next, define the functions $\gamma(t,T)$, $\lambda(t,T)$ and $\tilde{b}(t)$ by

$$\begin{aligned}
\gamma(t,T) &= \mathrm{e}^{-\alpha T}\left[\phi + \psi\mathrm{e}^{-\kappa T}\left(Z_0 + \kappa\int_0^t \mathrm{e}^{\kappa s}\mathrm{d}s\,\theta\right)\right] \ ,\\
\lambda(t,T) &= \mathrm{e}^{-\alpha T}\psi\kappa\int_0^t \mathrm{e}^{-\kappa(T-s)}\mathrm{d}s\,\theta \ ,\\
\tilde{b}(t) &= \mathrm{e}^{-\alpha t}\psi\mathrm{e}^{-\kappa t} \ ,
\end{aligned}$$

for $t \in [0, T]$, and therewith express the bond price process in the form

$$P_{tT} = \frac{\gamma(t,T) + \lambda(t,T) + \tilde{b}(T)\,A_t}{\gamma(t,t) + \tilde{b}(t)\,A_t}\ .$$

The initial term structure $(P_{0t})_{0\le t\le T}$ satisfies the relation $\gamma(0,0)P_{0t} = \gamma(0,t) + \lambda(0,t) = \gamma(t,t)$. Furthermore, $\gamma(t,T) + \lambda(t,T) - [\gamma(0,T) + \lambda(0,T)] = 0$ holds. Then, write

$$P_{tT} = \frac{\gamma(t,T) + \lambda(t,T) - [\gamma(0,T) + \lambda(0,T)] + \gamma(0,0)P_{0T} + \tilde{b}(T)\,A_t}{\gamma(0,0)P_{0t} + \tilde{b}(t)\,A_t}\ ,$$

and immediately obtain (7.2.17) by observing that $b(t) = \tilde{b}(t)/\gamma(0,0)$ for $0 \le t \le T$. □

Corollary 7.2.1 (Enabling calibration of the LRTS model)
The LRTS models can be expressed in the form

$$P_{tT} = \frac{P_{0T}\left[1 + \bar{b}(T)\,A_t\right]}{P_{0t}\left[1 + \bar{b}(t)\,A_t\right]}\ , \tag{7.2.19}$$

for $0 \le t \le T$*, where* $\bar{b}(t) = b(t)/P_{0t}$*. This is the form (7.2.6) for* $m = 1$*, and thus the necessary basis for the extension to the multi-curve LRTS models via Theorem 6.4.1 and Definition 6.4.10, in a developed economy, and via Definitions 6.5.1 and 6.5.2 in the emerging economy.*

Proof. This follows directly from (7.2.17). □

Remark 7.2.3 (Endogenously specified initial term structure and unspanned stochastic volatility)
It is emphasised that the form (7.2.17), or equivalently (7.2.19), shows that the LRTS model has, by (7.2.18), a fully functionally specified initial term structure (P_{0t}) *of ZCB prices for* $t \in [0, T]$*. Also, the models (7.2.17) specified by (7.2.18)-(7.2.18) produce an example of the larger class (7.2.10), or equivalently (7.2.19), of term structure models that can accommodate unspanned stochastic volatility as considered in Filipović et al. (2017), section C.*

Next, WHK processes over an infinite-time horizon are considered, see Akahori *et al.* (2014), and in particular the case where the propagator is a conditional expectation, as in Akahori and Macrina (2012) and Macrina (2014). Such WHKs are used to generate (explicit) pricing kernel processes. The definition that follows provides WHKs in a multivariate setting.

Definition 7.2.2 (Pricing kernels generated from WHKs in a multivariate setting)
Let $(X_t)_{0\le t}$ *be an* m*-dimensional* $(\mathcal{F}_t)$*-adapted Markov process,* $F(t,x)$ *be a vector-valued and deterministic function in* $\mathbb{R}^m$*, and* $w(t,u)$ *a matrix-valued deterministic function in* $\mathbb{R}^{m\times m}$*. Furthermore, let the functions* $f_0(t) \in \mathbb{R}$ *and* $f_1(t) \in \mathbb{R}^m$ *be deterministic. The process* $(\pi_t)_{0\le t}$ *is a WHK defined by*

$$\pi_t = f_0(t) + f_1(t)\int_0^\infty w(t,u)\mathbb{E}\left[F(t+u, X_{t+u})\,|\,\mathcal{F}_t\right]\mathrm{d}u\ , \tag{7.2.20}$$

where $t \wedge u \ge 0$*, and* $f_0(t)$*,* $f_1(t)$*,* $F(t,x)$ *and* $w(t,u)$ *are chosen such that* (π_t) *is a positive and finite (scalar-valued) process.*

The next statement asserts that the pricing kernel process (ζ_t) in Filipović *et al.* (2017) is a WHK and it establishes the relation between (ζ_t) and the class (7.2.20).

Theorem 7.2.2 (Recovering the LRTS pricing kernel as a WHK)
The pricing kernel (7.2.12) that generates the LRTS models by Filipović et al. (2017), is a special case of the process (7.2.20) where the following holds:

1. *Let* (X_t) *be the Markov process* (Z_t) *that satisfies (7.2.11).*
2. $F(t, X_t) = Z_t$*, for all* $t \geq 0$*.*
3. $w(t, u) = \mathrm{e}^{-\beta(t+u)}$, $\beta \in \mathbb{R}^{m\times m}$ *invertible, where* $\beta\kappa = \kappa\beta$*, for* $\kappa \in \mathbb{R}^{m\times m}$ *invertible.*
4. *The functions* $f_0(t)$ *and* $f_1(t)$ *are given by*

$$f_0(t) = f_1(t)\left[(\beta+\kappa)^{-1} - \beta^{-1}\right]\mathrm{e}^{-\beta t}\theta + \mathrm{e}^{-\alpha t}\phi\,, \tag{7.2.21}$$

$$f_1(t) = \mathrm{e}^{-\alpha t}\psi\mathrm{e}^{\beta t}(\beta+\kappa)\,, \tag{7.2.22}$$

where $\phi \in \mathbb{R}$*,* $\alpha \in \mathbb{R}$*,* $\theta \in \mathbb{R}^m$*,* $\psi \in \mathbb{R}^m$ *and* $\beta \in \mathbb{R}^{m\times m}$ *with* $\beta\kappa = \kappa\beta$ *for* $\kappa \in \mathbb{R}^{m\times m}$*. It is assumed that* $(\beta+\kappa)$ *is invertible.*

Proof. One direction is straightforward: It suffices to insert items 1 - 4 into equation (7.2.20) to obtain the pricing kernel process (7.2.12). In the other direction, that is starting from (7.2.20), one makes the initial assumptions that the first and second items shall hold. This leads to

$$\mathbb{E}\left[Z_{t+u}) \,|\, \mathcal{F}_t\right] = \mathrm{e}^{-\kappa(t+u)}\left(Z_0 + \kappa\int_0^{t+u}\mathrm{e}^{\kappa s}\mathrm{d}s\,\theta\right) + \mathrm{e}^{-\kappa(t+u)}\int_0^t \mathrm{e}^{\kappa s}\mathrm{d}M_s\,.$$

Then, by choosing the *ansatz* given in the third item, one obtains

$$\int_0^\infty w(t,u)\mathbb{E}\left[F(t+u, X_{t+u}) \,|\, \mathcal{F}_t\right]\mathrm{d}u = (\beta+\kappa)^{-1}\mathrm{e}^{-\beta t}Z_t + \left[\beta^{-1} - (\beta+\kappa)^{-1}\right]\mathrm{e}^{-\beta t}\theta.$$

Thus, the functions $f_0(t)$ and $f_1(t)$ are selected such that the pricing kernel process (7.2.12) is obtained, that is (7.2.21) and (7.2.22). ☐

Chapter 8

Consistent Pricing and Valuation Across Curves

In this chapter, the utility and versatility of the across-curve pricing and valuation formula, presented in Proposition 6.1.1, is demonstrated by applying the xy-formalism to various practical financial instrument pricing problems. The curve-conversion factor process, presented in Definition 6.1.1, may be conveniently applied to the pricing and hedging of inflation-linked and foreign exchange securities, as well as hybrids thereof. In particular, the quanto-bond process $(P^{xy}_{tT})_{0\leq t\leq T}$, formally defined in Definition 6.5.2, plays a fundamental role in the pricing of hybrid securities, suchlike inflation-linked foreign exchange financial instruments, where consistent pricing and valuation may still prove to be a complex task that poses various challenges.

8.1 Inflation-Linked Financial Instruments

It is customary when modelling the pricing and hedging of *inflation-linked* (IL) financial instruments to consider an economy that constitutes a nominal segment, denoted here by n, and a real segment, denoted here by r. Such a viewpoint matches the xy-formalism so much so that the curve-conversion factor process associated with IL financial instrument pricing is obtained with little effort — this is indeed a significant natural advantage of this framework. The nominal, or cash-based segment of the economy is associated with the x-curve, and thereby discounting, while the real segment, or goods/services-based segment of the economy, is associated with the y-curve. Therefore, to be clear, for IL pricing one would set $x = \mathrm{n}$ and $y = \mathrm{r}$. Also, depending on the particular pricing and valuation application, the nominal- and real-curve may be represented by the: (i) nominal and real government bond curves; or (ii) the OIS and a real curve derived from linear inflation-linked derivative securities. Next, the theory developed in Chapters 6 is applied.

The nominal and real economic segments are assumed to be governed by the positive pricing kernel processes $(h^{\mathrm{n}}_t)_{0\leq t}$ and $(h^{\mathrm{r}}_t)_{0\leq t}$, respectively. Then, the process $(C_t)_{0\leq t}$ defined by

$$C_t := C_0 \frac{h^{\mathrm{r}}_t}{h^{\mathrm{n}}_t} = C_0 Q^{\mathrm{nr}}_{tt} \,, \tag{8.1.1}$$

links prices between the nominal and real segments and is commonly referred to as the *Consumer Price Index*, with C_0 being the base price level at time 0 (not necessarily normalised to 1).

The price P^{nr}_{tT} at some time $t \leq T$ of an IL ZCB, which has been initiated at time 0, is given by

$$P^{\mathrm{nr}}_{tT} = \frac{1}{h^{\mathrm{n}}_t} \, \mathbb{E}\left[h^{\mathrm{n}}_T \frac{C_T}{C_0} \,\middle|\, \mathcal{F}_t \right] = \frac{1}{h^{\mathrm{n}}_t} \, \mathbb{E}\left[h^{\mathrm{r}}_T \,|\, \mathcal{F}_t \right] \, . \tag{8.1.2}$$

In the xy-formalism, recalling that $x = \mathrm{n}$ and $y = \mathrm{r}$, it is possible to write the price process (P^{nr}_{tT}) in terms of the conversion formula

$$P^{\mathrm{nr}}_{tT} = P^{\mathrm{n}}_{tT} \, Q^{\mathrm{nr}}_{tT} \, , \tag{8.1.3}$$

where (P^{n}_{tT}) is the price process of the nominal ZCB, and where

$$Q^{\mathrm{nr}}_{tT} = \frac{\mathbb{E}\left[h^{\mathrm{r}}_T \,|\, \mathcal{F}_t \right]}{\mathbb{E}\left[h^{\mathrm{n}}_T \,|\, \mathcal{F}_t \right]} \, , \tag{8.1.4}$$

is the curve-conversion factor (spread) process, which actually models expected inflation over $[t, T]$, with $t \in [0, T]$. The expression for the quanto-bond, equation (8.1.2), can be obtained in a straightforward fashion from equation (6.1.4) by setting $t = T$, thereafter replacing the pricing time s with t, and further by setting $x = \mathrm{n}$, $y = \mathrm{r}$ and $H^{\mathrm{r}}_T = 1$. The nominal curve serves as the base-curve; hence the curve-conversion factor process (8.1.4), in the relation (8.1.3), quantifies the number of positions in the nominal T-maturity discount bond necessary to replicate the no-arbitrage value at $t \in [0, T]$ of the IL ZCB with value P^{nr}_{tT} at time t. Given that the nominal ZCB P^{n}_{tT} and the IL ZCB P^{nr}_{tT} are traded and sufficiently liquid, one would then be able to imply from the market the IL conversion factor via

$$Q^{\mathrm{nr}}_{tT} = \frac{P^{\mathrm{nr}}_{tT}}{P^{\mathrm{n}}_{tT}} \, .$$

The pricing formulae for an IL FRA, zero-coupon swap, or for a year-on-year swap contract can be expressed in terms of the conversion factor. The derivations of such pricing formulae follow in a similar manner to those presented for FRAs, IRSs and TBSs in Chapter 6. Pricing and valuation models for IL financial instruments, which are based on explicit pricing kernel models — hence, on explicit curve-conversion factor processes — have been developed by Dam *et al.* (2020). Such models feature a high degree of flexibility, tractability, and offer good calibration properties.

8.2 Foreign Exchange Financial Instruments

When modelling FX, it is standard to first clarify the domestic economy, denoted here by d, and the foreign economy, denoted here by f. Then, within the xy-formalism, one has to identify the relevant markets within these respective economies that are appropriate and applicable for the determination of FX related cash flows. The appropriate markets are the respective FX basis markets within each economy, which essentially constitute FEC and CCBS contracts that are specified with reference to the USD, since the USD is the global reserve currency with the economy of the United States of America thereby offering the basis/standard for comparison for the purpose of pricing. The fundamental curve that is market-made within the aforementioned markets are the *FX basis curves*, which characterise the cost of synthetic bank funding within an economy. To this end, it is possible to add a sixth axiom to section 2.1 in order to explain the qualitative underpinnings of such rates. First though, it is important to acknowledge another manifestation of *funding-swap duality* within this context.

Remark 8.2.1 (Synthetic bank funding and funding-swap duality)
In any economy, a bank may consider term funding activity (borrowing and lending) via the local money market denominated in local currency, or equivalent activity in a synthetic manner denominated in USD via the FX derivatives market. For instance, one may structure a synthetic loan by:

(i) borrowing USD over the δ-term; and simultaneously entering a

(ii) long position in a single-period δ-term FXS, with cash flows that match the initial USD nominal and final USD settlement.

Equivalently, such a long position in a single-period FXS may be replicated as a short position in an overnight FEC and a long position in a δ-term FEC. The net effect of (i) and (ii) is a synthetic loan in local currency, with the FX derivatives market playing the key role in price discovery for the implied synthetic funding rate. Of course, the USD funding rate is priced separately via the transaction in (i).

A similar approach may be taken for single-period synthetic deposit or lending transaction, as well as multi-period and floating rate versions of all of the above. Therefore, the dual nature and role played by FEC, FXS and CCBS pricing within the context of synthetic funding is hopefully clear.

The above remark enables the articulation of the next axiom, which attempts to describe the price discovery process for synthetic BFRRs.

Axiom 8.2.1 (Synthetic bank funding rates — quasi-primitive funding securities)
If one were able to aggregate synthetic bank funding rates quoted and/or traded across relevant local banking entities, an effective simple rate for an $n\delta$-term transaction is conjectured to be

$$\begin{aligned} \overline{J}^n_u &:= J^n_u + b^n_u \\ &= x^n_u + d^n_u + f^n_u + b^n_u \ , \end{aligned} \tag{8.2.1}$$

where J^n_u and components thereof are as described in Axiom 2.1.5, and b^n_u is the aggregated $n\delta$-term simple FX basis spread component — which may be interpreted as a USD (relative to local currency) funding-liquidity risk, i.e., if local currency term funding is as easily accessible[1] as USD term funding, one should expect b^n_u to be negligible[2]. Also, take note that these transactions are characterised as quasi-primitive in nature, due to the role of linear FX derivatives in the price discovery process.

As with BFRRs, the issue with the synthetic BFRR, given by equation 8.2.1, is the term-related credit and liquidity risks which hinders arbitrage-free modelling, replication and product creation. However, given such data, one could apply the market-based approach that was developed in Part I, with b^n_u playing a role similar to ℓ^n_u from Axiom 2.1.4, to construct a system of arbitrage-free survival- and liquidity-contingent pricing kernels. In order to avoid the need for such, the market microstructure post reference reference rate reform advocates for these synthetic BFRRs to be based and derived from funding transactions that reference ONRRs, which leads to the next analog of Conjecture 2.1.3.

[1] One may consider various market microstructure and financial economic effects to qualitatively explain ease of accessibility, such as capital/currency mobility and substitutability.

[2] If local currency funding is more difficult to access than corresponding USD funding, and the USD is preferred over local currency, then one may expect b^n_u to be materially positive, and vice versa.

Conjecture 8.2.1 (Synthetic bank funding reference rates after reference rate reform)
The unsecured synthetic BFRR, or ONRR, is

$$\overline{J}^1_u := J^1_u + b^1_u \ ,$$

where $J^1_u = r^1_u + d^1_u + f^1_u$ *, which should also be equal to*

$$\overline{R}^1_u := R^1_u + b^1_u \ ,$$

for similar reasons as those provided in Conjecture 2.1.3, with $R^1_u = r^1_u + \ell^1_u + d^1_u$*. The synthetic secured overnight BFRR is*

$$\overline{S}^1_u := S^1_u + b^1_u \ ,$$

with $S^1_u := r^1_u + \ell^1_{u,\mathrm{r}}$ *, where* b^1_u *is assumed to be the same spread that features across the secured and unsecured markets, since its genesis is the same, i.e., the FX derivatives market*[3]*.*

This conjecture simplifies the analysis that is required here, as well as in the actual market, as all focus may once again be cast upon the OIS market, with the synthetic bank funding market effectively modelled as a spread applied to the OIS-curve. To this end, two pricing kernels (h^{d}_t) and (h^{f}_t) are introduced here as models for the FX basis, or synthetic bank funding, markets within the domestic and foreign economies, respectively. If one of the domestic or foreign economies happens to be the United States of America, then the corresponding pricing kernel will coincide with that which applies for the OIS market. Assume that the current spot exchange rate is denoted by X^{df}_0, i.e., one unit of foreign currency is equivalent to X^{df}_0 units of the domestic currency at time 0. Then, one unit of foreign currency funding accessed over $[0, t]$ via an overnight roll-over strategy should theoretically yield/cost $(1/h^{\mathrm{f}}_t)$ at the future fixed time t, assuming that $h^{\mathrm{f}}_0 := 1$, while an equivalent strategy executed in the domestic economy should theoretically yield/cost $X^{\mathrm{df}}_0(1/h^{\mathrm{d}}_t)$ assuming that $h^{\mathrm{d}}_0 := 1$. Therefore, to avoid a fundamental arbitrage opportunity, it must hold that

$$X^{\mathrm{df}}_t \frac{1}{h^{\mathrm{f}}_t} = X^{\mathrm{df}}_0 \frac{1}{h^{\mathrm{d}}_t} \ ,$$

with $X^{\mathrm{df}}_t = X^{\mathrm{df}}_0 h^{\mathrm{f}}_t / h^{\mathrm{d}}_t$ being the fair overnight FEC rate at time t. Recasting this result within the guise of the xy-formalism, this fair rate becomes

$$X^{\mathrm{df}}_t := X^{\mathrm{df}}_0 Q^{\mathrm{df}}_{tt} \ ,$$

which accentuates the *curve-conversion* feature of the curve-conversion factor process. Now, consider the definition of an FEC, written on one unit of foreign currency, over $[0, T]$ and denominated in foreign currency:

$$H^{\mathrm{f}}_{TT} = 1 - \frac{K^{\mathrm{d}}}{X^{\mathrm{df}}_T} \ ,$$

then curve-converted from f to d, which yields:

$$H^{\mathrm{df}}_{TT} = Q^{\mathrm{df}}_{TT} H^{\mathrm{f}}_{TT} = Q^{\mathrm{df}}_{TT} - \frac{K^{\mathrm{d}}}{X^{\mathrm{df}}_0} \ ,$$

[3] Valuation adjustments due to liquidity and counterparty credit-related risks are assumed to be handled in an ad hoc and explicit manner, such that FX basis spreads are true reflections of market mid levels.

and finally exchange rate converted, to give:

$$X_0^{\mathrm{df}} H_{TT}^{\mathrm{df}} = X_T^{\mathrm{df}} H_{TT}^{\mathrm{f}} = X_T^{\mathrm{df}} - K^{\mathrm{d}} \ ,$$

which is the standard market definition for such an FEC, with K^{d} being the fixed strike rate denominated in domestic currency. With the above terminal payoff, it is now possible to price such an FEC using the domestic economy pricing kernel as follows:

$$\begin{aligned} H_{tT}^{\mathrm{df}} &= \frac{1}{h_t^{\mathrm{d}}} \mathbb{E}\left[h_T^{\mathrm{d}} X_0^{\mathrm{df}} H_{TT}^{\mathrm{df}} \,|\, \mathcal{F}_t\right] \\ &= \frac{1}{h_t^{\mathrm{d}}} \mathbb{E}\left[h_T^{\mathrm{d}} \left(X_0^{\mathrm{df}} Q_{TT}^{\mathrm{df}} - K^{\mathrm{d}}\right) \,|\, \mathcal{F}_t\right] \\ &= X_0^{\mathrm{df}} \frac{h_t^{\mathrm{f}}}{h_t^{\mathrm{d}}} P_{tT}^{\mathrm{f}} - K^{\mathrm{d}} P_{tT}^{\mathrm{d}} \ , \end{aligned}$$

from where it is straightforward to see that the fair level for K^{d} equals $X_t^{\mathrm{df}} P_{tT}^{\mathrm{f}} / P_{tT}^{\mathrm{d}}$, which also defines the fair FEC rate process, denoted here by

$$X_{tT}^{\mathrm{df}} := X_t^{\mathrm{df}} \frac{P_{tT}^{\mathrm{f}}}{P_{tT}^{\mathrm{d}}} = X_0^{\mathrm{df}} Q_{tT}^{\mathrm{df}} \ , \tag{8.2.2}$$

for $t \in [0, T]$ and $T > 0$. This result was enabled by an application of the across-curve pricing formula, equation (6.1.4); however, this was a slightly adjusted version since the payoff did not only require an adjustment for the relative difference between the accrual rates underlying each curve but also a level adjustment based on the initial FX rate.

8.3 Multi-Curve Interest Rate FX Hybrid Instruments

In this section, the problem of pricing the developed economy FRA, as defined in Proposition 6.4.1, is considered once again. However the underlying reference rate is assumed to be an IBOR in the foreign economy, originating in the y_{f}-market, while the pricing perspective is assumed to be that of a domestic economy market participant, with the appropriate discounting curve thereby originating in the domestic economy's synthetic bank funding of FX basis market, i.e., the d-market.

The terminal payoff for such a FRA within the y_{f}-market is given by

$$H_{T_i T_i}^{y_{\mathrm{f}}} = N^{\mathrm{f}} \delta_i \left(J_{T_{i-1}}^{y_{\mathrm{f}}}(T_{i-1}, T_i) - K^{y_{\mathrm{f}}}\right) \ ,$$

where N^{f} is the FRA nominal amount, denominated in foreign currency. Even though the nominal isn't one unit of currency, the approach from section 6.4 may once again be repeated, and one could then express the terminal payoff within the x_{f}-market, or the foreign OIS market, as

$$H_{T_i T_i}^{x_{\mathrm{f}} y_{\mathrm{f}}} = Q_{T_{i-1} T_i}^{x_{\mathrm{f}} y_{\mathrm{f}}} N^{\mathrm{f}} \delta_i \left(J_{T_{i-1}}^{y_{\mathrm{f}}}(T_{i-1}, T_i) - K^{y_{\mathrm{f}}}\right) \ ,$$

and the analysis to price the FRA would be identical to that undertaken in section 6.4, resulting in a fair FRA rate process equal to $J_t^{x_{\mathrm{f}} y_{\mathrm{f}}}(T_{i-1}, T_i)$, as defined in Lemma 6.4.1 and Definition 6.4.1. In general then, this specification of the FRA is not conducive to an FX-based conversion, since the f-curve, or foreign economy FX basis curve, does not feature. This is simple enough to remedy by curve-converting the payoff from x_f to f as follows:

$$Q^{\mathrm{f}x_\mathrm{f}}_{T_{i-1}T_i} H^{x_\mathrm{f}y_\mathrm{f}}_{T_iT_i} = Q^{\mathrm{f}x_\mathrm{f}}_{T_{i-1}T_i} Q^{x_\mathrm{f}y_\mathrm{f}}_{T_{i-1}T_i} H^{y_\mathrm{f}}_{T_iT_i} = Q^{\mathrm{f}y_\mathrm{f}}_{T_{i-1}T_i} H^{y_\mathrm{f}}_{T_iT_i} \ ,$$

which now fundamentally alters the definition of the FRA, with

$$J^{\mathrm{f}y_\mathrm{f}}_t(T_{i-1},T_i) = Q^{\mathrm{f}x_\mathrm{f}}_{tT_i} J^{x_\mathrm{f}y_\mathrm{f}}_t(T_{i-1},T_i) = Q^{\mathrm{f}y_\mathrm{f}}_{tT_i} J^{y_\mathrm{f}}_t(T_{i-1},T_i) \ ,$$

now emerging as the fair FRA rate process, for $t \in [0, T_{i-1}]$.

The final curve-conversion that is required is to convert the payoff from the f-market to the d-market. This may be achieved with the fair FEC rate, given by equation (8.2.2), over $[T_{i-1}, T_i]$ which therefore also embeds the required currency conversion. This may be achieved as follows:

$$\begin{aligned} X^{\mathrm{df}}_{T_{i-1}T_i} Q^{\mathrm{f}x_\mathrm{f}}_{T_{i-1}T_i} H^{x_\mathrm{f}y_\mathrm{f}}_{T_iT_i} &= X^{\mathrm{df}}_0 Q^{\mathrm{df}}_{T_{i-1}T_i} Q^{\mathrm{f}x_\mathrm{f}}_{T_{i-1}T_i} H^{x_\mathrm{f}y_\mathrm{f}}_{T_iT_i} \\ &= X^{\mathrm{df}}_0 Q^{\mathrm{d}y_\mathrm{f}}_{T_{i-1}T_i} H^{y_\mathrm{f}}_{T_iT_i} \\ &= N^\mathrm{d} Q^{\mathrm{d}y_\mathrm{f}}_{T_{i-1}T_i} \delta_i \left(J^{y_\mathrm{f}}_{T_{i-1}}(T_{i-1},T_i) - K^{y_\mathrm{f}} \right) \ , \end{aligned}$$

where $N^\mathrm{d} := X^{\mathrm{df}}_0 N^\mathrm{f}$ is the equivalent FRA nominal in domestic currency. Then, it can be shown that the interim value of such a FRA, for $t \in [0, T_{i-1}]$, is

$$X^{\mathrm{df}}_0 H^{\mathrm{d}y_\mathrm{f}}_{tT_i} = N^\mathrm{d} P^\mathrm{d}_{tT_i} \left(J^{\mathrm{d}y_\mathrm{f}}_t(T_{i-1},T_i) - K^\mathrm{d} \right) \delta_i \ ,$$

using the same approach as that undertaken in section 6.4, with K^d being the converted strike rate, and from where it follows that $J^{\mathrm{d}y_\mathrm{f}}_t(T_{i-1},T_i) = Q^{\mathrm{d}y_\mathrm{f}}_{tT_i} J^{y_\mathrm{f}}_t(T_{i-1},T_i)$ is the fair FRA rate process.

8.4 Inflation-Linked FX Hybrid Instruments

The problem of pricing a foreign economy IL forward contract, from the perspective of a domestic economy participant, is considered in this section. Within the foreign economy, this type of financial instrument would enable market participants to extract an expectation of inflationary growth over the life of the contract, assumed here to be $[0, T]$. The terminal payoff for such a forward contract in the real segment of the foreign economy is then

$$H^{\mathrm{r_f}}_T = N^\mathrm{f} \left(1 - \frac{K^{\mathrm{n_f}}}{Q^{\mathrm{n_f r_f}}_{TT}} \right) \ ,$$

where N^f denotes the nominal amount denominated in foreign currency, and $Q^{\mathrm{n_f r_f}}_{TT} = C^\mathrm{f}_T / C^\mathrm{f}_0$ is the inflationary growth in the foreign consumer price index over $[0, T]$, as defined in equation (8.1.1). Then, converting this payoff from the real to the nominal segment of the economy yields the equivalent payoff:

$$H^{\mathrm{n_f r_f}}_T = Q^{\mathrm{n_f r_f}}_{TT} H^{\mathrm{r_f}}_T = N^\mathrm{f} \left(Q^{\mathrm{n_f r_f}}_{TT} - K^{\mathrm{n_f}} \right) \ ,$$

such that the application of the across-curve pricing formula gives

$$H^{\mathrm{n_f r_f}}_t = \frac{1}{h^{\mathrm{n_f}}_t} \mathbb{E} \left[h^{\mathrm{n_f}}_T H^{\mathrm{n_f r_f}}_T \mid \mathcal{F}_t \right] = N^\mathrm{f} \left(P^{\mathrm{n_f r_f}}_{tT} - P^{\mathrm{n_f}}_{tT} K^{\mathrm{n_f}} \right) \ ,$$

as the interim value for the contract within the foreign economy, for $t \in [0,T]$. Therefore, the fair strike rate process turns out to be $K_t^{\mathrm{n_f}} := P_{tT}^{\mathrm{n_f r_f}}/P_{tT}^{\mathrm{n_f}} = Q_{tT}^{\mathrm{n_f r_f}}$, with the last equality following by equation (8.1.3). Converting the payoff from the $\mathrm{n_f}$-market to the foreign economy's FX basis market, or the f-market, is done as follows:

$$H_T^{\mathrm{fr_f}} = Q_{TT}^{\mathrm{fn_f}} H_T^{\mathrm{n_f r_f}} = N^{\mathrm{f}} \left(Q_{TT}^{\mathrm{fr_f}} - Q_{TT}^{\mathrm{fn_f}} K^{\mathrm{n_f}} \right) \ ,$$

which now enables the conversion to domestic currency as follows:

$$X_0^{\mathrm{df}} Q_{TT}^{\mathrm{df}} H_T^{\mathrm{fr_f}} = N^{\mathrm{d}} \left(Q_{TT}^{\mathrm{dr_f}} - Q_{TT}^{\mathrm{dn_f}} K^{\mathrm{n_f}} \right) \ ,$$

where $N^{\mathrm{d}} := X_0^{\mathrm{df}} N^{\mathrm{f}}$ is the equivalent nominal in domestic currency. Then, once again making use of the across-curve pricing formula, it follows that

$$\begin{aligned} &\frac{1}{h_t^{\mathrm{d}}} \mathbb{E} \left[h_T^{\mathrm{d}} N^{\mathrm{d}} \left(Q_{TT}^{\mathrm{dr_f}} - Q_{TT}^{\mathrm{dn_f}} K^{\mathrm{n_f}} \right) \,\middle|\, \mathcal{F}_t \right] \\ &= N^{\mathrm{d}} \left(P_{tT}^{\mathrm{dr_f}} - P_{tT}^{\mathrm{dn_f}} K^{\mathrm{n_f}} \right) \\ &= N^{\mathrm{d}} P_{tT}^{\mathrm{d}} \left(Q_{tT}^{\mathrm{dr_f}} - K^{\mathrm{d}} \right) \ , \end{aligned}$$

is the interim value of such an IL forward contract within the d-market, for $t \in [0,T]$. Observe that the new variable $K^{\mathrm{d}} := Q_{tT}^{\mathrm{dn_f}} K^{\mathrm{n_f}}$ is the converted strike level, now expressed and denominated in domestic currency. Finally, it should be clear that the fair strike level process resolves to

$$K_t^{\mathrm{d}} = Q_{tT}^{\mathrm{dr_f}} = Q_{tT}^{\mathrm{df}} Q_{tT}^{\mathrm{fn_f}} Q_{tT}^{\mathrm{n_f r_f}} \ ,$$

for $t \in [0,T)$, which reveals the relevant curve-conversions (or exchanges of risk).

One further conversion may be considered, if one were interested in considering pricing and valuation at the level of the nominal segment of the domestic economy, i.e., the $\mathrm{n_d}$-market. Then, the interim value of such a contract would be $N^{\mathrm{d}} P_{tT}^{\mathrm{n_d}} (Q_{tT}^{\mathrm{n_d r_f}} - K^{\mathrm{n_d}})$, and one would obtain

$$K_t^{\mathrm{n_d}} = Q_{tT}^{\mathrm{n_d r_f}} = Q_{tT}^{\mathrm{n_d d}} Q_{tT}^{\mathrm{df}} Q_{tT}^{\mathrm{fn_f}} Q_{tT}^{\mathrm{n_f r_f}} \ ,$$

as the fair strike level process, for $t \in [0,T)$, with $K^{\mathrm{n_d}} := Q_{tT}^{\mathrm{n_d n_f}} K^{\mathrm{n_f}}$ being the curve-converted strike level in this scenario.

Therefore, it should be clear that the xy-formalism enables a flexible and agile framework that is capable of modelling multiple interest rate centric financial markets in a rigorous, consistent, and compatible manner such that the same models may be combined to model hybrid financial instruments.

Part III

Information-Based Interest Rate Models

Chapter 9

Filtering and Filtration Modelling

The objective of the classical filtering problem is to estimate a so-called *signal process* that is causally correlated to, but "hidden" in, a noisy *observation process.* The problem's canonical (linear) form is the *signal plus white noise* structure:

$$i_t = X_t + w_t \ , \tag{9.0.1}$$

where $(i_t)_{t\geq 0}$ is the observation process, $(X_t)_{t\geq 0}$ is the signal process and $(w_t)_{t\geq 0}$ is *white noise* (or colloquially, the time derivative of the standard Wiener process). To avoid the problematic covariance structure of white noise [1], and obtain a more mathematically amenable form, the integrated version of equation (9.0.1) given by

$$\int_0^t i_s \,\mathrm{d}s = \int_0^t X_s \,\mathrm{d}s + \int_0^t w_s \,\mathrm{d}s \quad \Longleftrightarrow \quad I_t = \int_0^t X_s \,\mathrm{d}s + W_t \ ,$$

or in stochastic differential form:

$$\mathrm{d}I_t = X_t \,\mathrm{d}t + \mathrm{d}W_t \ ,$$

is preferred, where $(I_t)_{t\geq 0}$ is now the observation process and $(W_t)_{t\geq 0}$ is a standard Wiener process. Assuming that the signal process $(X_t)_{t\geq 0}$ satisfies the necessary integrability conditions; the best estimate thereof, given the history of the observation process, is given by

$$\widehat{X}_t := \mathbb{E}\left[X_t \,|\, \mathcal{I}_t\right] \ ,$$

for $t \geq 0$, in the *least mean square sense*, where $\mathcal{I}_t := \sigma\left\{(I_s)_{s\leq t}\right\}$, i.e., the σ-algebra generated by the observation process up to time t. In particular, take note that

$$\mathcal{I}_t \subseteq \mathcal{C}_t := \sigma\left\{(X_s)_{s\leq t}, (W_s)_{s\leq t}\right\} \ ,$$

for $t \geq 0$, where $(\mathcal{C}_t)_{t\geq 0}$ is the inaccessible filtration containing complete information.

This problem manifests itself in various forms across numerous fields (navigation, image processing, telecommunication, engineering, biology and finance, to name a few). As such, there exists a well-defined theory for the solution of a broad class of linear and non-linear stochastic filtering[2] problems with the canonical *signal plus white noise* structure — for example, the interested reader may refer to

[1] Given by Dirac's delta function — a *generalised function.*

[2] Linear (Non-linear) stochastic filtering refers to the case where the observation process is a linear (non-linear) functional of the signal process.

any one of Jazwinski (2007), Xiong (2008), Bain and Crisan (2009), or Kallianpur (2013).

The goal here is to propose another application of filtering within the field of mathematical finance, inspired by the work of Macrina (2006), Rutkowski and Yu (2007), Brody *et al.* (2008), Macrina and Parbhoo (2010), Hoyle *et al.* (2011), Filipović *et al.* (2012), Hughston and Macrina (2012) and Parbhoo (2013). In the aforementioned research, the filtering problem is reframed in the guise of incomplete financial market information. The signal process assumes the role of a financial market factor (such as a price, rate, financial variable, economic indicator, etc.), while the observation process is assumed to model noisy realisations thereof that is driven by, say, measurement error, market innuendo, rumours and speculation. Particular emphasis is placed on modelling the set of information processes that in turn model the (accessible) filtration $(\mathcal{I}_t)$, hence the authors refer to this application as an *information-based* or *filtration modelling* approach.

The sections that follow will:

(i) present a non-linear filtering framework (using Markov processes) that includes observation processes with time- and state-dependent instantaneous volatility and correlation structure;

(ii) reprise the main filtering results, within the context outlined in (i), leading to the general solution and its associated stochastic differential equation (SDE), viz., the *Kallianpur-Striebel formula*, the *Zakai equation* and the *Kushner-Stratonovich equation*, along with the conditional probability density function associated with the filtering process, i.e., *Kushner's Theorem*; and

(iii) describe the approach that will be taken to apply the filtering framework to model financial market information, and show that this approach recovers the canonical Brownian bridge-based information process approach initially proposed by Macrina (2006).

9.1 Non-Linear Filtering with Markov Processes

Assume the existence of a filtered probability space $(\Omega, \mathcal{C}, (\mathcal{C}_t)_{t\geq 0}, \mathbb{S})$, satisfying the *usual conditions*, with

$$\mathcal{C}_t = \sigma\left\{X_0, (Z_s)_{s\leq t}, (W_s)_{s\leq t}\right\} ,$$

for $t \geq 0$, where X_0 is an m-dimensional column vector of square integrable real-valued $\mathcal{C}_0$-measurable random variables, i.e., $X_0 \in \mathcal{L}^2(\Omega, \mathcal{C}_0, \mathbb{S})$ and $X_0 : \Omega \to \mathbb{R}^m$, while $(Z_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ are ℓ- and n-dimensional column vector standard Wiener processes respectively. Furthermore, X_0, $(Z_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ are all assumed to be independent of each other.

The set of signal processes are then defined as an m-dimensional column vector time-inhomogeneous Itô process as follows:

$$\mathrm{d}X_t = a(t, X_t)\,\mathrm{d}t + b(t, X_t)\,\mathrm{d}Z_t \ , \tag{9.1.1}$$

where $a : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}^m$ and $b : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}^{m\times\ell}$ are measurable functions, *uniformly Lipschitz continuous*, i.e.,

$$\|a(t,x) - a(t,y)\| + \|b(t,x) - b(t,y)\| \leq C_1 \|x - y\| \ ,$$

and satisfy the *linear growth condition*, viz.,

$$\|a(t,x)\| + \|b(t,x)\| \leq C_2\left(1 + \|x\|\right) ,$$

for all $t \in \mathbb{R}_{\geq 0}$, $x, y \in \mathbb{R}^m$ and constants C_1 and C_2[3].Then there exists a ($\mathbb{S}$-a.s.) unique solution $(X_t)_{t\geq 0}$ to the SDE (9.1.1) that is $\mathcal{C}_t$-adapted, with continuous trajectories and is Markov. Furthermore, $(X_t)_{t\leq T} \in \mathcal{L}^2\left([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{C}, \mathbb{L}_T\times\mathbb{S}\right)$ where $\mathbb{L}_T\times\mathbb{S}$ is the product measure on $[0,T]\times\Omega$ with $\mathbb{L}_T$ being the Lebesgue-measure on $[0,T]$, i.e.

$$\mathbb{E}^{\mathbb{L}_T\times\mathbb{S}}\left[\| X_{\cdot}\|^2\right] = \mathbb{E}^{\mathbb{S}}\left[\int_0^T \| X_t\|^2 \,\mathrm{d}t\right] < \infty ,$$

for some $T > 0$.[4] These technical conditions related to the existence and uniqueness of solutions to this class of SDEs are based on results from Björk (2004), Oksendal (2013) and van Handel (2007).

The set of observation processes are defined as an n-dimensional column vector time-inhomogeneous non-linear SDE:

$$\mathrm{d}I_t = u(t, X_t, I_t)\,\mathrm{d}t + v(t, I_t)\,\mathrm{d}W_t , \tag{9.1.2}$$

where $u : \mathbb{R}_{\geq 0}\times\mathbb{R}^m\times\mathbb{R}^n \to \mathbb{R}^n$ and $v : \mathbb{R}_{\geq 0}\times\mathbb{R}^n \to \mathbb{R}^{n\times n}$ are measurable functions, satisfying the Lipschitz condition

$$\|u(t,z,x) - u(t,z,y)\|^2 + \|v(t,x) - v(t,y)\|^2 \leq C_3\,\|x-y\|^2 ,$$

and the growth condition

$$\|u(t,z,x)\|^2 + \|v(t,x)\|^2 \leq C_4\left(1 + \|x\|^2 + \|L(t,z)\|^2\right) ,$$

for all $t \in \mathbb{R}_{\geq 0}$, $z \in \mathbb{R}^m$, $x, y \in \mathbb{R}^n$ and constants C_3 and C_4, and where $L : \mathbb{R}_{\geq 0}\times\mathbb{R}^m \to \mathbb{R}^n$ is a measurable function such that $\mathbb{E}^{\mathbb{S}}\left[\int_0^T \| L(t, X_t)\|^2 \,\mathrm{d}t\right] < \infty$, for some $T > 0$ (considering the properties of the solution to the SDE (9.1.1), one example of such a function is $L(t,z) = z$). Then there exists a ($\mathbb{S}$-a.s.) unique solution $(I_t)_{t\geq 0}$ to the SDE (9.1.2) with continuous trajectories that is Markov and $\mathcal{I}_t$-adapted, where $\mathcal{I}_t = \sigma\left\{(I_s)_{s\leq t}\right\}$ for $t \geq 0$. Naturally, it follows that $\mathcal{I}_t \subseteq \mathcal{C}_t$ for all $t \geq 0$. Moreover, $(I_t)_{t\leq T} \in \mathcal{L}^2\left([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{I}, \mathbb{L}_T\times\mathbb{S}\right)$.[5] These technical conditions are based on results from Chapters 5 and 8 from Kallianpur (2013).

For any vector $x \in \mathbb{R}^n$, let $D[x]$ denote the corresponding $(n\times n)$-diagonal matrix constructed therefrom. Now, more structure is posited onto the function which is the diffusion coefficient in (9.1.2), as follows:

$$v(t, I_t) := S(t, I_t)\,H(t, I_t) ,$$

where $S : \mathbb{R}_{\geq 0}\times\mathbb{R}^n \to D[\mathbb{R}^n_{>0}]$ denotes the time- and state-dependent instantaneous volatility structure, and $H : \mathbb{R}_{\geq 0}\times\mathbb{R}^n \to \mathbb{R}^{n\times n}$ denotes the Cholesky factorisation of the positive definite

[3] Recall that the Euclidean norm for $a \in \mathbb{R}^d$ is defined as $\|a\| = \sqrt{\sum_{i=1}^d a_i^2}$, while for a matrix $A \in \mathbb{R}^{d_1\times d_2}$ it is defined as $\|A\| = \sqrt{\sum_{i=1}^{d_1}\sum_{j=1}^{d_2} A_{ij}^2}$.

[4] Again, it is assumed that the product space $([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{C}, \mathbb{L}_T\times\mathbb{S})$ satisfies the *usual conditions*.

[5] Again, it is assumed that the product space $([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{I}, \mathbb{L}_T\times\mathbb{S})$ satisfies the *usual conditions*.

time- and state-dependent instantaneous correlation matrix $\rho(t, I_t) := H(t, I_t)H(t, I_t)^\intercal$. Defining $Y_t := \int_0^t H(s, I_s)\,\mathrm{d}W_s$, therefore implies that

$$\mathrm{d}Y_t^{(i)}\mathrm{d}Y_t^{(j)} = \begin{cases} \rho_{ii}(t, I_t)\,\mathrm{d}t = \mathrm{d}t \ , & 1 \leq i = j \leq n \ , \\ \rho_{ij}(t, I_t)\,\mathrm{d}t = \rho_{ji}(t, I_t)\,\mathrm{d}t \ , & 1 \leq i \neq j \leq n \ , \end{cases}$$

for $t \geq 0$. If an n-dimensional (local) martingale process is defined as $\mathrm{d}V_t := v(t, I_t)\,\mathrm{d}W_t = S(t, I_t)\,\mathrm{d}Y_t$, then its associated time- and state-dependent instantaneous variance-covariance structure is given by $\Sigma(t, I_t) := S(t, I_t)\,\rho(t, I_t)\,S(t, I_t)$, i.e.,

$$\mathrm{d}V_t^{(i)}\mathrm{d}V_t^{(j)} = \begin{cases} \Sigma_{ii}(t, I_t)\,\mathrm{d}t = [S_{ii}(t, I_t)]^2\,\mathrm{d}t \ , & 1 \leq i = j \leq n \ , \\ \Sigma_{ij}(t, I_t)\,\mathrm{d}t = S_{ii}(t, I_t)\,\rho_{ij}(t, I_t)\,S_{jj}(t, I_t)\,\mathrm{d}t \ , & 1 \leq i \neq j \leq n \ . \end{cases}$$

Remark 9.1.1 (Invertibility of $\Sigma(t, I_t)$)
Since $S : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to D[\mathbb{R}^n_{>0}]$, by assumption, the diagonal elements of the diagonal matrix $S(t, I_t)$ are strictly positive and therefore its' inverse is always well-defined. Also, the inverse of the instantaneous correlation matrix is always well-defined, since $\rho(t, I_t)$ is assumed to be positive definite. Therefore, by assumption, the inverse of $\Sigma(t, I_t)$ is also well-defined, for all $t \in \mathbb{R}_{\geq 0}$ and $I_t \in \mathbb{R}^n$.

Now that the structure of the signal and the observation processes have been finalised, *the optimal filter* may be formally defined.

Definition 9.1.1 (The optimal filter)
For each $t > 0$, the objective of the filtering problem is to establish the best estimate of some measurable and square integrable function of the signal process $g(X_t)$, given information about the observation process $(I_s)_{s\leq t}$ over the interval $[0, t]$. Mathematically, this is achieved via the optimal filter, or least mean square estimate, defined by the conditional expectation

$$\widehat{g}_t := \mathbb{E}^{\mathbb{S}}\left[g(X_t)\,\middle|\,\mathcal{I}_t\right] \ , \tag{9.1.3}$$

where $g(X_t) \in \mathcal{L}^2(\Omega, \mathcal{C}, \mathbb{S})$ and $\mathcal{I}_t = \sigma\left\{(I_s)_{s\leq t}\right\} \subseteq \mathcal{C}_t$ for each $t \geq 0$.

Remark 9.1.2 (Least mean square estimate)
Let $g_t := g(X_t)$, then it is easy to show that

$$\mathbb{E}^{\mathbb{S}}\left[(g_t - \widehat{g}_t)^2\right] = \min\left\{\mathbb{E}^{\mathbb{S}}\left[(g_t - \xi)^2\right]; \xi \in \mathcal{L}^2(\Omega, \mathcal{I}_t, \mathbb{S})\right\} \ ,$$

for each $t \geq 0$. Choose any $\xi \in \mathcal{L}^2(\Omega, \mathcal{I}_t, \mathbb{S})$, then the result follows by observing that

$$\mathbb{E}^{\mathbb{S}}\left[(g_t - \xi)^2\right] - \mathbb{E}^{\mathbb{S}}\left[(g_t - \widehat{g}_t)^2\right] = \mathbb{E}^{\mathbb{S}}\left[(\widehat{g}_t - \xi)^2\right] \geq 0 \,,$$

after application of the tower property of conditional expectations. The interested reader may refer to van Handel (2007) and Xiong (2008) for further information.

Two approaches exist in the academic literature for resolving the optimal filter into the so-called *filtering equations* — the set of SDEs that govern the dynamics of the optimal filter. The first is the *innovations or martingale approach* pioneered by Fujisaki *et al.* (1972). The central idea behind this approach is the derivation of an $\{(\mathcal{I}_t), \mathbb{S}\}$-Wiener process that essentially forms a stochastic basis for the observation filtration $(\mathcal{I}_t)_{t\geq 0}$, thereby enabling a stochastic integral representation for the martingale part of the optimal filter $(\widehat{g}_t)_{t\geq 0}$. The second, which is adopted here, is the *reference-* or *change-of-measure approach*. The objective here is to attempt direct computation of the optimal filter. This is achieved by:

(i) constructing a suitable Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{F}}{\mathrm{d}\mathbb{S}}$ on $\mathcal{C}_t$, for some $t > 0$, such that the new measure $\mathbb{F} \sim \mathbb{S}$ over the interval $[0,t]$, and $(I_s)_{s\leq t}$ is an n-dimensional $\{(\mathcal{C}_s), \mathbb{F}\}$-(local) martingale that is independent of X_0 and $(Z_s)_{s\leq t}$ (both of which have the same law under $\mathbb{F}$);

(ii) invoking a general Bayes formula to convert the optimal filter conditional expectation (9.1.3) under $\mathbb{S}$ into a ratio of conditional expectations under $\mathbb{F}$; and finally

(iii) applying Itô's formula to arrive at the required filtering equations.

For further details, one may refer to Bensoussan (1992), Elliott *et al.* (1995), and van Handel (2007).

9.2 The Filtering Equations

The set of filtering equations that determine the dynamics of the optimal filter for the filtering problem posed in equations (9.1.1), (9.1.2) and (9.1.3) is derived in this section. The objective of the change-of-measure approach is to construct a new (absolutely continuous) measure under which the observation process is a (local) martingale. This is achieved in the following Lemma. In order to lighten notation, when it is not necessary to explicitly emphasise function arguments all functions will be denoted by their respective names with a subscript denoting the time dimension, for e.g., $u_t := u(t, X_t, I_t)$, $v_t := v(t, I_t)$, and $g_t := g(X_t)$.

Lemma 9.2.1 (The reference measure)
The Radon-Nikodym derivative defined on $\mathcal{C}_T$ by

$$\Psi_T := \left.\frac{\mathrm{d}\mathbb{F}}{\mathrm{d}\mathbb{S}}\right|_{\mathcal{C}_T} = \exp\left(-\int_0^T \left[\Sigma_s^{-1} u_s\right]^\intercal \mathrm{d}V_s - \frac{1}{2}\int_0^T u_s^\intercal \Sigma_s^{-1} u_s \, \mathrm{d}s\right), \tag{9.2.1}$$

induces the measure $\mathbb{F} \sim \mathbb{S}$ over the interval $[0,T]$, for some $T > 0$, if the Novikov condition

$$\mathbb{E}^{\mathbb{S}}\left[\exp\left(\frac{1}{2}\int_0^T u_s^\intercal \Sigma_s^{-1} u_s \, \mathrm{d}s\right)\right] < \infty, \tag{9.2.2}$$

is satisfied. Then $(I_s)_{s\leq T}$ is a $\{(\mathcal{C}_t), \mathbb{F}\}$-(local) martingale defined by

$$\mathrm{d}I_t = u_t \, \mathrm{d}t + \mathrm{d}V_t = v_t \, \mathrm{d}\overline{W}_t, \tag{9.2.3}$$

where $\overline{W}_t = W_t + \int_0^t v_s^{-1} u_s \, \mathrm{d}s$ is a $\{(\mathcal{C}_t), \mathbb{F}\}$-Wiener process, for $t \in [0,T]$.

Proof. Recall that $\mathrm{d}V_t = v_t \, \mathrm{d}W_t$, and therefore

$$\left[\Sigma_t^{-1} u_t\right]^\intercal v_t \, \mathrm{d}W_t = u_t^\intercal \Sigma_t^{-1} v_t \, \mathrm{d}W_t = u_t^\intercal \left[v_t v_t^\intercal\right]^{-1} v_t \, \mathrm{d}W_t = \left[v_t^{-1} u_t\right]^\intercal \mathrm{d}W_t,$$

while

$$u_t^\intercal \Sigma_t^{-1} u_t = u_t^\intercal \left[v_t v_t^\intercal\right]^{-1} u_t = \left[v_t^{-1} u_t\right]^\intercal v_t^{-1} u_t = \left\|v_t^{-1} u_t\right\|^2.$$

Setting $\varphi_t := v_t^{-1} u_t$, which is an n-dimensional $\mathcal{C}_t$-adapted column vector process, it's clear that the Radon-Nikodym derivative (9.2.1) is actually $\Psi_T = \exp\left(-\int_0^T \varphi_s^\intercal \, \mathrm{d}W_s - \frac{1}{2}\int_0^T \left\|\varphi_s\right\|^2 \mathrm{d}s\right)$. Given the Novikov condition (9.2.2), $\Psi_t = \mathbb{E}^{\mathbb{S}}\left[\Psi_T \,|\, \mathcal{C}_t\right]$ for $t \in [0,T]$, with $\Psi_0 = 1$, is a positive $\{(\mathcal{C}_t, \mathbb{S}\}$-martingale which serves as a suitable likelihood process to induce the necessary change-of-measure by Girsanov's theorem. The other minor results follow accordingly. □

Remark 9.2.1 (Independence between X_0, Z_t and $\overline{W}_t$)
Set $B_t := (Z_t, W_t)$, i.e., define $(B_t)_{t\geq 0}$ to be an $(\ell + n)$-dimensional column vector $\{(\mathcal{C}_t), \mathbb{S}\}$-Wiener process. Set $\varphi_t := (\mathbf{0}_\ell, v_t^{-1} u_t)$, in the proof of Lemma 9.2.1, where $\mathbf{0}_\ell$ is an ℓ-dimensional zero column vector. Then $(\Psi_t)_{t\leq T}$ enables the same change-of-measure from $\mathbb{S}$ to $\mathbb{F}$, such that $\overline{B}_t = (Z_t, \overline{W}_t)$ is a $\{(\mathcal{C}_t), \mathbb{F}\}$-Wiener process, for $t \in [0,T]$. Therefore $(Z_t)_{t\leq T}$ is independent of $(\overline{W}_t)_{t\leq T}$, while both processes are also independent of X_0 (by definition, since X_0 is $\mathcal{C}_0$-measurable).

Remark 9.2.2 ($\overline{W}_t$ is also $\mathcal{I}_t$-adapted)
From equation (9.2.3), the observation process under $\mathbb{F}$, note that the Wiener process $(\overline{W}_t)_{t\leq T}$ may also be represented as

$$\overline{W}_t = \int_0^t \left[v(s, I_s)\right]^{-1} \mathrm{d}I_s \,,$$

and is therefore also $\mathcal{I}_t$-adapted, where $\mathcal{I}_t = \sigma\left\{(I_s)_{s\leq t}\right\} \subseteq \mathcal{C}_t$ for each $t \geq 0$.

Corollary 9.2.1 (From $\mathbb{F}$ to $\mathbb{S}$)
The Radon-Nikodym derivative defined by

$$\Gamma_T := \left.\frac{\mathrm{d}\mathbb{S}}{\mathrm{d}\mathbb{F}}\right|_{\mathcal{C}_T} = \exp\left(\int_0^T \left[\Sigma_s^{-1} u_s\right]^\intercal \mathrm{d}I_s - \frac{1}{2}\int_0^T u_s^\intercal \Sigma_s^{-1} u_s \,\mathrm{d}s\right) \,,$$

on $\mathcal{C}_T$ enables the change-of-measure from $\mathbb{F}$ to $\mathbb{S}$ over the interval $[0,T]$, for $T > 0$.

Proof. Being the reciprocal change-of-measure, the proof is identical to that of Lemma 9.2.1 with $\varphi_t = -v_t^{-1} u_t$; $(W_t)_{t\leq T}$ switching roles with $(\overline{W}_t)_{t\leq T}$, and $\mathbb{S}$ with $\mathbb{F}$. □

Now that the reference measure has been defined, it is possible to resolve the optimal filter (9.1.3) further using a general Bayes formula that describes how a conditional expectation transforms under a change-of-measure. The following theorem summarises this result, which was first shown by Kallianpur and Striebel (1968). Also see Lipster and Shiryaev (2001) for further detailed insights.

Theorem 9.2.1 (The Kallianpur-Striebel formula)
Given Definition 9.1.1, Lemma 9.2.1 and Corollary 9.2.1, the optimal filter may be resolved as

$$\widehat{g}_t = \mathbb{E}^{\mathbb{S}}\left[g_t \,\middle|\, \mathcal{I}_t\right] = \frac{\hat{o}_t(g)}{\hat{o}_t(1)} \qquad \mathbb{S} - a.s. \,, \tag{9.2.4}$$

where $\hat{o}_t(g) := \mathbb{E}^{\mathbb{F}}\left[g_t \Gamma_t \,\middle|\, \mathcal{I}_t\right]$ is the unnormalised filtered estimate, for each $t \geq 0$.

Proof. Since $g_t \in \mathcal{L}^2(\Omega, \mathcal{C}, \mathbb{S})$ and $(\Gamma_s)_{s\leq t}$ is a positive $\{(\mathcal{C}_t), \mathbb{F}\}$-martingale, by assumption and construction, all integrands in equation (9.2.4) are integrable and therefore all conditional expectations are well-defined.

Assume that $A \in \mathcal{I}_t$, with $\mathbb{I}_A$ the indicator function defined on subset A, then

$$\begin{aligned}\mathbb{E}^{\mathbb{F}}\left[\mathbb{I}_A\,\mathbb{E}^{\mathbb{F}}\left[g_t\Gamma_t \,|\, \mathcal{I}_t\right]\right] &= \mathbb{E}^{\mathbb{F}}\left[\mathbb{I}_A\, g_t\Gamma_t\right] \\ &= \mathbb{E}^{\mathbb{S}}\left[\mathbb{I}_A\, g_t\right] = \mathbb{E}^{\mathbb{S}}\left[\mathbb{I}_A\,\mathbb{E}^{\mathbb{S}}\left[g_t \,|\, \mathcal{I}_t\right]\right] \\ &= \mathbb{E}^{\mathbb{F}}\left[\mathbb{I}_A\,\Gamma_t\mathbb{E}^{\mathbb{S}}\left[g_t \,|\, \mathcal{I}_t\right]\right] = \mathbb{E}^{\mathbb{F}}\left[\mathbb{I}_A\,\mathbb{E}^{\mathbb{S}}\left[g_t \,|\, \mathcal{I}_t\right]\mathbb{E}^{\mathbb{F}}\left[\Gamma_t \,|\, \mathcal{I}_t\right]\right]\,,\end{aligned}$$

which shows that: $\hat{o}_t(g) = \widehat{g}_t\,\hat{o}_t(1) \quad \mathbb{F} - a.s.$, since A is any arbitrary set in $\mathcal{I}_t$. The result follows by observing that $(\Gamma_s)_{s\le t}$ is a strictly positive martingale hence $\mathbb{S} \sim \mathbb{F}$ and $\hat{o}_t(1) > 0$, which permits division by this quantity. ☐

The martingale property induced by the reference measure $\mathbb{F}$ combined with the added structure provided by Theorem 9.2.1 enables further tractability. This comes in the form of the *Zakai equation*, which is presented in the next Lemma — see Kunita (1990) for further information.

Lemma 9.2.2 (The Zakai equation)
Assuming that $g_t \in C^2$ and that all of its derivatives are bounded, the SDE that governs the unnormalised filtered estimate is given by

$$\mathrm{d}\hat{o}_t(g) = \left(\hat{o}_t(p) + \frac{1}{2}\hat{o}_t(q)\right)\,\mathrm{d}t + \hat{o}_t\,(gu)^{\intercal}\,\Sigma_t^{-1}\,\mathrm{d}I_t\,, \tag{9.2.5}$$

where the following measurable functions have been introduced:

$$\begin{aligned}p(t, X_t) &:= \nabla\,(g_t)^{\intercal}\,a_t\,, \quad \textit{and} \\ q(t, X_t) &:= \mathbf{1}_m^{\intercal}\left[b_t b_t^{\intercal} \circ \mathbf{H}\,(g_t)\right]\mathbf{1}_m\,,\end{aligned}$$

with $\nabla(\cdot)$ and $\mathbf{H}(\cdot)$ denoting the gradient vector and Hessian matrix partial derivative operators respectively, while $\mathbf{1}_m$ denotes an m-dimensional column unit vector, and $(A \circ B)$ denotes the Hadamard (or entry-wise) product of two matrices A and B with the same dimensions.

Proof. From Theorem 9.2.1 and Corollary 9.2.1, the change-of-measure density process $(\Gamma_s)_{s\le T}$ is governed by the following SDE:

$$\mathrm{d}\Gamma_t = \Gamma_t\left[\Sigma_t^{-1}u_t\right]^{\intercal}\mathrm{d}I_t = \Gamma_t\left[v_t^{-1}u_t\right]^{\intercal}\,\mathrm{d}\overline{W}_t\,,$$

while an application of Itô's formula to $g_t := g(X_t)$ yields

$$\begin{aligned}\mathrm{d}g_t &= \frac{1}{2}\left(\mathbf{1}_m^{\intercal}\left[b_t b_t^{\intercal} \circ \mathbf{H}(g_t)\right]\mathbf{1}_m\right)\,\mathrm{d}t + \nabla(g_t)^{\intercal}a_t\,\mathrm{d}t + \nabla(g_t)^{\intercal}b_t\,\mathrm{d}Z_t \\ &= \left(p_t + \frac{1}{2}q_t\right)\,\mathrm{d}t + \nabla(g_t)^{\intercal}b_t\,\mathrm{d}Z_t\,,\end{aligned}$$

where $p_t := p(t, X_t)$ and $q_t := q(t, X_t)$, as defined above. Applying Itô's product rule then leads to the following SDE:

$$\mathrm{d}(\Gamma_t g_t) = \Gamma_t\left(p_t + \frac{1}{2}q_t\right)\,\mathrm{d}t + \Gamma_t\nabla(g_t)^{\intercal}b_t\,\mathrm{d}Z_t + \Gamma_t g_t\left[\Sigma_t^{-1}u_t\right]^{\intercal}\,\mathrm{d}I_t\,.$$

Integrating (all integrands are integrable), taking expectations (conditional on $\mathcal{I}_t$) and applying Fubini's theorem (where necessary and appropriate) then produces

$$\begin{aligned}\mathbb{E}^{\mathbb{F}}\left[\Gamma_t g_t \,|\, \mathcal{I}_t\right] &= \mathbb{E}^{\mathbb{F}}\left[g_0 \,|\, \mathcal{I}_t\right] + \int_0^t \left(\mathbb{E}^{\mathbb{F}}\left[\Gamma_s p_s \,|\, \mathcal{I}_t\right] + \frac{1}{2}\mathbb{E}^{\mathbb{F}}\left[\Gamma_s q_s \,|\, \mathcal{I}_t\right]\right)\,\mathrm{d}s \\ &\quad + \int_0^t \mathbb{E}^{\mathbb{F}}\left[\Gamma_s g_s u_s^{\intercal} \,|\, \mathcal{I}_t\right]\Sigma_s^{-1}\,\mathrm{d}I_s + \mathbb{E}^{\mathbb{F}}\left[\int_0^t \Gamma_s\nabla(g_s)^{\intercal}b_s\,\mathrm{d}Z_s \,\middle|\, \mathcal{I}_t\right]\,.\end{aligned} \tag{9.2.6}$$

Remarks 9.2.1 and 9.2.2 show that X_0 (and hence g_0) is independent of $\mathcal{I}_t$ under $\mathbb{F}$, therefore

$$\mathbb{E}^{\mathbb{F}}\left[g_0 \,|\, \mathcal{I}_t\right] = \mathbb{E}^{\mathbb{F}}\left[g_0\right] = \mathbb{E}^{\mathbb{S}}\left[g_0\right] . \tag{9.2.7}$$

If $\mathcal{I}_{st} := \sigma\left\{(I_r - I_s)_{s\le r\le t}\right\}$, then $\mathcal{I}_s$ is independent of $\mathcal{I}_{st}$ and $\mathcal{I}_t = \mathcal{I}_s \vee \mathcal{I}_{st}$. Then, by standard properties of conditional expectations

$$\mathbb{E}^{\mathbb{F}}\left[h_s \,|\, \mathcal{I}_t\right] = \mathbb{E}^{\mathbb{F}}\left[h_s \,|\, \mathcal{I}_s\right] , \tag{9.2.8}$$

assuming that h_s is $\mathcal{C}_s$-measurable and therefore also independent of $\mathcal{I}_{st}$. Let $A \in \mathcal{I}_t$, then by Kolmogorov's definition for conditional expectations and Itô's representation theorem applied to $\mathbb{I}_A$, the indicator function defined on subset A, the relation

$$\begin{aligned}\mathbb{E}^{\mathbb{F}}\left[\mathbb{I}_A\, \mathbb{E}^{\mathbb{F}}\left[\int_0^t k_s\, \mathrm{d}Z_s \,\middle|\, \mathcal{I}_t\right]\right] &= \mathbb{E}^{\mathbb{F}}\left[\mathbb{I}_A \int_0^t k_s\, \mathrm{d}Z_s\right] \\ &= \mathbb{E}^{\mathbb{F}}\left[\left(\mathbb{F}(A) + \int_0^t K_s\, \mathrm{d}\overline{W}_s\right) \int_0^t k_s\, \mathrm{d}Z_s\right] = 0\,,\end{aligned} \tag{9.2.9}$$

where $(K_s)_{s\le t}$ is $\mathcal{I}_t$-adapted and square integrable, and $\mathbb{E}\left[\int_0^t k_s^2\, \mathrm{d}s\right] < \infty$. Equation (9.2.9) implies that $\mathbb{E}^{\mathbb{F}}\left[\int_0^t k_s\, \mathrm{d}Z_s \,\middle|\, \mathcal{I}_t\right] = 0$. This holds since A is an arbitrary subset of $\mathcal{I}_t$, $(Z_s)_{s\le t}$ and $(\overline{W}_s)_{s\le t}$ are independent and the expected value of the Itô integral with respect to $(Z_s)_{s\le t}$ equals zero. Applying results (9.2.7), (9.2.8) and (9.2.9) to equation (9.2.6), along with the shorthand notation, yields

$$\hat{o}_t(g) = \hat{o}_0(g) + \int_0^t \left(\hat{o}_s(p) + \frac{1}{2}\hat{o}_s(q)\right) \mathrm{d}s + \int_0^t \hat{o}_s(gu)^\intercal \Sigma_s^{-1}\, \mathrm{d}I_s\,,$$

which concludes the result. ☐

Remark 9.2.3 (SDE for $\hat{o}_t(1)$)
A straightforward application of Lemma 9.2.2 shows that the SDE which governs the denominator of the optimal filter (9.2.4) is given by

$$\mathrm{d}\hat{o}_t(1) = \hat{o}_t\,(u)^\intercal\, \Sigma_t^{-1}\, \mathrm{d}I_t\,, \tag{9.2.10}$$

since $g_t = 1$, hence $p_t = 0$ and $q_t = 0$.

With the SDE for the unnormalised filtered estimate at hand, another application of Itô's Lemma yields the SDE for the optimal filter, also known as the *Kushner-Stratonovich* (KS) or the *Fujisaki-Kallianpur-Kunita* (FKK) *equation.*

Theorem 9.2.2 (The KS or FKK equation)
For $g_t \in \mathcal{L}^2\left(\Omega, \mathcal{C}, \mathbb{S}\right)$ and $g_t \in C^2$, with bounded derivatives, the optimal filter satisfies

$$\mathrm{d}\widehat{g}_t = \left(\widehat{p}_t + \frac{1}{2}\widehat{q}_t\right) \mathrm{d}t + \left[\widehat{gu}_t - \widehat{g}_t\widehat{u}_t\right]^\intercal \Sigma_t^{-1}\left(\mathrm{d}I_t - \widehat{u}_t\, \mathrm{d}t\right) , \tag{9.2.11}$$

where $\widehat{gu}_t := \mathbb{E}^{\mathbb{S}}\left[g_t u_t \,|\, \mathcal{I}_t\right]$. Furthermore, the observation process resolves to

$$\mathrm{d}I_t = v_t\, \mathrm{d}\overline{W}_t = \widehat{u}_t\, \mathrm{d}t + v_t\, \mathrm{d}\widehat{W}_t\,,$$

where the innovations process defined by

$$\widehat{W}_t := \overline{W}_t - \int_0^t v_s^{-1}\, \widehat{u}_s\, \mathrm{d}s = W_t + \int_0^t v_s^{-1}\left[u_s - \widehat{u}_s\right]\, \mathrm{d}s\,, \tag{9.2.12}$$

is a $\{(\mathcal{I}_t), \mathbb{S}\}$*-Wiener process, provided that*

$$\mathbb{E}^{\mathbb{S}}\left[\int_0^t u_r^\intercal \Sigma_r^{-1} u_r \,\mathrm{d}r\right] < \infty \quad \textit{and} \quad \mathbb{E}^{\mathbb{S}}\left[\int_0^t \hat{u}_r^\intercal \Sigma_r^{-1} \hat{u}_r \,\mathrm{d}r\right] < \infty\,.$$

Proof. Applying Itô's quotient rule to the optimal filter $\widehat{g}_t = \hat{o}_t(g) \,/\, \hat{o}_t(1)$, the SDE that governs this process is given by

$$\begin{aligned}
\mathrm{d}\widehat{g}_t &= \frac{\mathrm{d}\hat{o}_t(g)}{\hat{o}_t(1)} - \widehat{g}_t \frac{\mathrm{d}\hat{o}_t(1)}{\hat{o}_t(1)} + \widehat{g}_t \frac{\mathrm{d}\big\langle \hat{o}(1)\big\rangle_t}{\left(\hat{o}_t(1)\right)^2} - \frac{\mathrm{d}\big\langle \hat{o}(g), \hat{o}(1)\big\rangle_t}{\left(\hat{o}_t(1)\right)^2} \\
&= \left(\widehat{p}_t + \frac{1}{2}\widehat{q}_t\right) \mathrm{d}t + \left[\widehat{gu_t^\intercal} - \widehat{g}_t \widehat{u}_t^\intercal\right] \Sigma_t^{-1} \left(\mathrm{d}I_t - \widehat{u}_t \,\mathrm{d}t\right)\,,
\end{aligned}$$

which follows after using equations (9.2.5), (9.2.10) and some simplification. To prove that $(\widehat{W}_t)_{t\geq 0}$ is an $\{(\mathcal{I}_t), \mathbb{S}\}$-Wiener process, Lévy's characterisation is used. First it is clear that $\widehat{W}_0 = 0$ a.s., from equation (9.2.12). Using equation (9.2.12), along with conditions (9.2.13), it is possible to show that

$$\mathbb{E}^{\mathbb{S}}\left[\big|\widehat{W}_t\big|\right] \leq \mathbb{E}^{\mathbb{S}}\left[|W_t|\right] + \mathbb{E}^{\mathbb{S}}\left[\int_0^t \left|v_r^{-1} u_r\right| \mathrm{d}r\right] + \mathbb{E}^{\mathbb{S}}\left[\int_0^t \left|v_r^{-1} \hat{u}_r\right| \mathrm{d}r\right] < \infty\,,$$

element-by-element, after a suitable application of Hölder's inequality. For $0 \leq s < t$, the martingale property is recovered as follows:

$$\begin{aligned}
\mathbb{E}^{\mathbb{S}}\left[\widehat{W}_t \,\middle|\, \mathcal{I}_s\right] &= \mathbb{E}^{\mathbb{S}}\left[\widehat{W}_t - \widehat{W}_s \,\middle|\, \mathcal{I}_s\right] + \widehat{W}_s \\
&= \mathbb{E}^{\mathbb{S}}\left[W_t - W_s + \int_s^t v_r^{-1}\left[u_r - \hat{u}_r\right] \mathrm{d}r \,\middle|\, \mathcal{I}_s\right] + \widehat{W}_s \\
&= \mathbb{E}^{\mathbb{S}}\left[W_t - W_s\right] + \mathbb{E}^{\mathbb{S}}\left[\int_s^t v_r^{-1}\left[u_r - \hat{u}_r\right] \mathrm{d}r \,\middle|\, \mathcal{I}_s\right] + \widehat{W}_s \\
&= \widehat{W}_s\,,
\end{aligned}$$

since the increment $W_t - W_s$ is independent of $\mathcal{I}_s$ and has zero expectation, while an application of Fubini's theorem and the tower property of conditional expectations shows that

$$\mathbb{E}^{\mathbb{S}}\left[\int_s^t v_r^{-1}\left[u_r - \hat{u}_r\right] \mathrm{d}r \,\middle|\, \mathcal{I}_s\right] = \mathbb{E}^{\mathbb{S}}\left[\int_s^t v_r^{-1}\left[\mathbb{E}^{\mathbb{S}}\left[u_r \,|\, \mathcal{I}_r\right] - \hat{u}_r\right] \mathrm{d}r \,\middle|\, \mathcal{I}_s\right] = 0\,,$$

because v_r^{-1} is $\mathcal{I}_r$-measurable and $\hat{u}_r := \mathbb{E}^{\mathbb{S}}\left[u_r \,|\, \mathcal{I}_r\right]$. Finally, from equation (9.2.12) it is clear that the quadratic variation of the innovations process is $\langle \widehat{W} \rangle_t = t$. Therefore the innovations process satisfies all of Lévy's criteria, which completes the proof. □

Remark 9.2.4 (Versions of the observation process)
In setting up the filtering problem and ultimately deriving the filtering equation in Theorem 9.2.2, the observation process has undergone the following transformations:

$$\mathrm{d}I_t = \begin{cases} u_t \,\mathrm{d}t + v_t \,\mathrm{d}W_t\,, & \textit{under } \mathbb{S} \textit{ on } (\mathcal{C}_t)_{t\geq 0}\,, \\ v_t \,\mathrm{d}\overline{W}_t\,, & \textit{under } \mathbb{F} \textit{ on } (\mathcal{C}_t)_{t\geq 0} \textit{ and } (\mathcal{I}_t)_{t\geq 0}\,, \\ \hat{u}_t \,\mathrm{d}t + v_t \,\mathrm{d}\widehat{W}_t\,, & \textit{under } \mathbb{S} \textit{ on } (\mathcal{I}_t)_{t\geq 0}\,, \end{cases}$$

where all of the components of the final version are $\mathcal{I}_t$*-measurable under* $\mathbb{S}$*.*

In many practical cases, it is much more convenient to work with the *conditional probability density function* admitted by the filtering process (assuming that one is available).

Definition 9.2.1 (The conditional probability density function)
The definition of the optimal filter, provided in Definition 9.1.1, may be extended as follows:

$$\widehat{g}_t = \mathbb{E}^{\mathbb{S}}\left[g(X_t)\,\middle|\,\mathcal{I}_t\right] = \int_{\mathbb{R}^m} g(x) f_t(x)\,\mathrm{d}x\,,$$

where the process $f_t : \mathbb{R}^m \to \mathbb{R}$ is the probability density function of X_t conditional on $\mathcal{I}_t$. Moreover, the initial condition $f_0(\cdot)$ is the unconditional probability density function of X_0 .

With the conditional probability density function process defined, what follows next is effectively the stochastic filtering equivalent of the Kolmogorov forward equations.

Lemma 9.2.3 (The Zakai equation in PDE form)
The definition of the unnormalised filtered estimate, introduced in Theorem 9.2.1, may also be extended as

$$\hat{o}_t(g) = \mathbb{E}^{\mathbb{F}}\left[\Gamma_t g(X_t)\,\middle|\,\mathcal{I}_t\right] = \int_{\mathbb{R}^m} g(x)\,h_t(x)\,\mathrm{d}x\,,$$

once again assuming that the process $h_t : \mathbb{R}^m \to \mathbb{R}$ is the unnormalised probability density function of X_t conditional on $\mathcal{I}_t$, under the $\mathbb{F}$-measure. Moreover, the conditional density process satisfies the following linear stochastic partial differential equation:

$$\mathrm{d}h_t(x) = \left(\overline{p}(t,x;h) + \frac{1}{2}\,\overline{q}(t,x;h)\right)\,\mathrm{d}t + h_t(x)\,u_t^{\intercal}\Sigma_t^{-1}\mathrm{d}I_t\,,$$

for $t \geq 0$, where

$$\overline{p}(t,X_t;h) := -\sum_{i=1}^{m}\frac{\partial a_i h_t}{\partial x_i}(t,X_t) \quad \textit{and} \quad \overline{q}(t,X_t;h) := \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial c_{ij} h_t}{\partial x_i}(t,X_t)\,,$$

with $c_{ij}(t,X_t) := \sum_{k=1}^{\ell} b_{ik}(t,X_t) b_{kj}(t,X_t)$, or equivalently $c(t,X_t) = b(t,X_t)\,[b(t,X_t)]^{\intercal}$.

Proof. The integrated form of the Zakai SDE (9.2.5) may be expanded as follows:

$$\begin{aligned}\int_{\mathbb{R}^m} g(x)\,h_t(x)\,\mathrm{d}x = \int_{\mathbb{R}^m} g(x)\,h_0(x)\,\mathrm{d}x + \int_0^t\int_{\mathbb{R}^m}\left(p(s,x) + \frac{1}{2}q(s,x)\right)h_s(x)\,\mathrm{d}x\,\mathrm{d}s \\ + \int_0^t\int_{\mathbb{R}^m} g(x) u(s,x,I_s)^{\intercal}\Sigma_s^{-1} h_s(x)\,\mathrm{d}x\,\mathrm{d}I_s\,.\end{aligned}$$

Furthermore, the functions $p(t,X_t)$ and $q(t,X_t)$ expand as

$$\begin{aligned}p(t,X_t) &:= \nabla\left(g_t\right)^{\intercal} a_t = \sum_{i=1}^{m} a_i(t,X_t)\frac{\partial g}{\partial x_i}(X_t)\,, \quad \text{and} \\ q(t,X_t) &:= \mathbf{1}_m^{\intercal}\left[b_t b_t^{\intercal} \circ \mathbf{H}\left(g_t\right)\right]\mathbf{1}_m = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}(t,X_t)\frac{\partial^2 g}{\partial x_i \partial x_j}(X_t)\,.\end{aligned}$$

Using integration by parts on the second integral above, it may be shown that

$$\int_{\mathbb{R}} a_i(s,x)\frac{\partial g}{\partial x_i}(x) h_t(x)\,\mathrm{d}x_i = -\int_{\mathbb{R}} g(x)\frac{\partial a_i h_t}{\partial x_i}(s,x)\,\mathrm{d}x_i\,, \quad \text{and}$$

$$\int_{\mathbb{R}^2} c_{ij}(s,x)\frac{\partial^2 g}{\partial x_i \partial x_j}(x) h_t(x)\,\mathrm{d}x_i\,\mathrm{d}x_j = \int_{\mathbb{R}} g(x)\frac{\partial^2 c_{ij} h_t}{\partial x_i \partial x_j}(s,x)\,\mathrm{d}x_i\,\mathrm{d}x_j\,,$$

for all $1 \leq i, j \leq m$. Using this result, and changing the orders of integration in the second and third integrals in the integrated Zakai equation gives

$$h_t(x) = h_0(x) - \int_0^t \sum_{i=1}^m \frac{\partial a_i h_s}{\partial x_i}(s,x) + \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m \frac{\partial c_{ij} h_s}{\partial x_i}(s,x)\,\mathrm{d}s + \int_0^t h_s(x) u(s,x,I_s)^\intercal \Sigma_s^{-1}\mathrm{d}I_s\,,$$

which concludes the proof. □

Remark 9.2.5 (Normalised and unnormalised conditional densities)
From Theorem 9.2.1, Definition 9.2.1 and Lemma 9.2.3, it should be clear that

$$\widehat{g}_t = \int_{\mathbb{R}^m} g(x) f_t(x)\,\mathrm{d}x = \frac{\int_{\mathbb{R}^m} g(x)\, h_t(x)\,\mathrm{d}x}{\int_{\mathbb{R}^m} h_t(x)\,\mathrm{d}x} = \frac{\hat{o}_t(g)}{\hat{o}_t(1)}\,,$$

which implies that $f_t(x) = h_t(x) / \int_{\mathbb{R}^m} h_t(x)\,\mathrm{d}x$ *, for all* $x \in \mathbb{R}^m$ *and* $t \geq 0$.

The next Theorem, known as *Kushner's Theorem*, shows the corresponding result for the conditional density process associated with the optimal filter.

Theorem 9.2.3 (Kushner's Theorem)
The conditional density process associated with the optimal filter satisfies the following non-linear stochastic partial integro-differential equation

$$\mathrm{d}f_t(x) = \left(\overline{p}(t,x;f) + \frac{1}{2}\,\overline{q}(t,x;f)\right)\,\mathrm{d}t + f_t(x)\,\left[u_t - \widehat{u}_t\right]^\intercal \Sigma_t^{-1} v_t \mathrm{d}\widehat{W}_t\,, \tag{9.2.13}$$

for $t \geq 0$, *where* $\overline{p}(t, X_t, f)$ *and* $\overline{q}(t, X_t, f)$ *are as defined in Lemma 9.2.3.*

Proof. The integrated form of the Kushner-Stratonovich equation (9.2.11), from Theorem 9.2.2, is given by

$$\begin{aligned}\int_{\mathbb{R}^m} g(x)\, f_t(x)\,\mathrm{d}x = & \int_{\mathbb{R}^m} g(x)\, f_0(x)\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^m} \left(p(s,x) + \frac{1}{2} q(s,x)\right) f_s(x)\,\mathrm{d}x\,\mathrm{d}s \\ & + \int_0^t \int_{\mathbb{R}^m} g(x)\left[u(s,x,I_s) - \widehat{u}(s,x,I_s)\right]^\intercal \Sigma_s^{-1} v_s f_s(x)\,\mathrm{d}x\,\mathrm{d}\widehat{W}_s\,.\end{aligned}$$

Then, by the same approach applied in Lemma 9.2.3, the result follows. □

9.3 Modelling Financial Market Information

In this section the non-linear filtering framework, presented in the previous sub-section, is applied to model financial market information *à la* Macrina (2006), Brody *et al.* (2008) and Parbhoo (2013). This is achieved in an abstract sense where, as mentioned earlier, the signal process represents some (potentially latent) set of financial market factors, while noisy realisations thereof are accessible via the observation process. The natural filtration generated by the observation process therefore models the accessible information, which gives rise to the *information-based* or *filtration modelling* nomenclature.

In consonance with the information-based approach, the full framework presented in section 9.1 is not applied here. In particular, the signal process, which models the market factors, is assumed to be an unobservable multi-dimensional vector of random variables — this is formally defined below. This is a special case of the wider framework, presented earlier, that is arguably more intuitive (for the objective here), enables greater tractability and provides a mechanism for constructing *information-driven martingales* — objects that will be fundamental for the application to interest rate modelling.

Definition 9.3.1 (Market factors)
The financial market under consideration is assumed to be governed by a set consisting of m market factors defined by the m-dimensional column vector signal process (9.1.1), $(X_t)_{t\geq 0}$, with $a(t,x) = \mathbf{0}_m$ and $b(t,x) = \mathbf{0}_{m\times\ell}$ for all $t \in \mathbb{R}_{\geq 0}$ and $x \in \mathbb{R}^m$, i.e.,

$$\mathrm{d}X_t = \mathbf{0}_m \Rightarrow X_t = X_0 \,, \tag{9.3.1}$$

for all $t \geq 0$. Therefore, in order to lighten notation the unobservable random vector X_0 will be denoted by X, in all that follows. The joint cumulative distribution function function of X will be defined and denoted by

$$F_X(x) := C_X(F(x)) \,, \tag{9.3.2}$$

for all $x \in \mathbb{R}^m$, where

- *$X = [X_1, X_2, \ldots, X_m]^\intercal$ is a $\mathcal{C}_0$-measurable, square-integrable random vector;*
- *$C_X : [0,1]^m \to [0,1]$ is a copula, i.e., a joint cumulative distribution function of an m-dimensional standard uniform random vector; and*
- *$F : \mathbb{R}^m \to [0,1]^m$ is an m-dimensional column vector of the marginal cumulative distribution functions, i.e., $F_i(x) = F_{X_i}(x_i)$ for $x_i \in \mathbb{R}$ and $i \in \{1, 2, \ldots, m\}$.*

In order to generalise the dependence structure amongst the market factors, their general joint distribution is characterised with a *copula*, which is possible by Sklar's theorem. For more information on copulas, see, for instance, Sklar (1973) and Nelsen (1999).

Remark 9.3.1 (Absolutely continuous distributions)
If the copula, in equation (9.3.2), has a distribution that is absolutely continuous (with respect to the respective Lebesgue measure) then it admits the density function

$$c_X(F(x)) = \frac{\partial}{\partial u} C_X(u) \bigg|_{u=F(x)} ,$$

for all $x \in \mathbb{R}^m$. Moreover, if all of the marginal distributions of the market factors are also absolutely continuous, each will also admit a density function, such that

$$\begin{aligned} f_X(x) &= \frac{\partial}{\partial v} C_X(F(v)) \bigg|_{v=x} \\ &= \frac{\partial C_X(u)}{\partial u} \frac{\partial F(v)}{\partial v} \bigg|_{u=F(v), v=x} \\ &= c_X(F(x)) \prod_{i=1}^{m} f_{X_i}(x_i) \,, \end{aligned}$$

is the joint probability density function of X, with $f_{X_i}(x)$ being the marginal probability density functions, for $x_i \in \mathbb{R}$ and $i \in \{1, 2, \dots, m\}$. If the marginal distributions are assumed to be independent, then $c_X(F(x)) = 1$ or $C_X(F(x)) = \prod_{i=1}^m F_i(x) = \prod_{i=1}^m F_{X_i}(x_i)$, for all $x \in \mathbb{R}^m$. Of course, if $m = 1$, then $c_X(F(x)) = 1$ or $C_X(F(x)) = F_1(x) = F_{X_1}(x_1)$, for all $x_1 \in \mathbb{R}$.

Definition 9.3.2 (Information processes)
Associated with the set of unobservable market factors is an observable set consisting of n information processes defined by the n-dimensional column vector observation process (9.1.2), $(I_t)_{t\geq 0}$. In general, the information processes considered hereafter will allow for feedback (or mean-reversion) effects as well as a time- and state-dependent instantaneous variance-covariance structure, as defined in equation (9.1.2)

Considering the narrower framework applied here, the next Lemma summarises the results of the previous section within this context.

Lemma 9.3.1 (Filtered time-homogeneous market factor functional)
Given Definitions 9.3.1 and 9.3.2, the $\mathcal{I}_t$-adapted set of observable information processes defined by

$$\mathrm{d}I_t = u(t, X, I_t)\,\mathrm{d}t + v(t, I_t)\,\mathrm{d}W_t\,, \tag{9.3.3}$$

for $t \geq 0$, with $\mathcal{C}_0$-measurable and unobservable set of market factors X, yields the optimal filtered process

$$\mathrm{d}\widehat{g}_t = \left[\widehat{gu}_t - \widehat{g}_t\widehat{u}_t\right]^\intercal \Sigma_t^{-1} v_t \mathrm{d}\widehat{W}_t\,, \tag{9.3.4}$$

for the measurable, real-valued and integrable time-homogeneous functional of the market factors $g(X)$, where $\widehat{g}_t = \mathbb{E}^{\mathbb{S}}\left[g(X)\,\middle|\,\mathcal{I}_t\right]$ and $(\widehat{W}_t)_{t\geq 0}$ is a $\{(\mathcal{I}_t), \mathbb{S}\}$-Wiener process. Moreover, if the marginal distributions of X are absolutely continuous then:

$$\widehat{g}_t = \frac{\mathbb{E}^{\mathbb{F}}\left[g(X)\,\Gamma_t\, c_X(Y)\,\middle|\,\mathcal{I}_t\right]}{\mathbb{E}^{\mathbb{F}}\left[\Gamma_t\, c_X(Y)\,\middle|\,\mathcal{I}_t\right]}\,, \tag{9.3.5}$$

where $Y := F(X) \in [0,1]^m$ and $c_X : [0,1]^m \to \mathbb{R}$ is the respective copula density function.

Proof. Equation (9.3.4) follows trivially from Theorem 9.2.2, while equation (9.3.5) follows from Theorem 9.2.1 and Remark 9.3.1, with absolutely continuous distributions leading to

$$\begin{aligned}\mathbb{E}^{\mathbb{F}}\left[g(X)\,\Gamma_t\,\middle|\,\mathcal{I}_t\right] &= \int_{\mathbb{R}^m} g(x)\,\Gamma_t\, c_X(F(x)) \prod_{i=1}^m f_{X_i}(x_i)\,\mathrm{d}x \\ &= \mathbb{E}^{\mathbb{F}}\left[g(X)\,\Gamma_t\, c_X(F(X))\,\middle|\,\mathcal{I}_t\right]\,,\end{aligned}$$

which portrays the role of the copula-implied dependence structure as a random variable and completes the result. ☐

Within the context of financial market modelling, equation (9.3.5) represents the best estimate of the functional of the set of market factors, given noisy observations thereof. Moreover, this estimate is also a (local) martingale, hence the term *information-driven martingales* used earlier. In order to use this information filtering framework for the general purpose of asset pricing, a minor adjustment to Lemma 9.3.1 is required. Instead of the time-homogeneous market factor functional $g : \mathbb{R}^m \to \mathbb{R}$, a time-inhomogeneous version is necessary — this is achieved via the following theorem.

Theorem 9.3.1 (Filtered time-inhomogeneous market factor functional)
Assume the setup of Lemma 9.3.1, but now consider a measurable and square integrable function $g : \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to \mathbb{R}$*, i.e.,* $g(X,t) \in \mathcal{L}^2(\Omega \times \mathbb{R}_{\geq 0}, \mathcal{C}_0 \otimes \mathcal{B}(\mathbb{R}_{\geq 0}), \mathbb{S} \times \mathbb{L})$*, that is (at least) twice differentiable in the first argument and once in the time argument, i.e.,* $g_{x,t} := g(X,t) \in C^{2,1}$*. For all* $t \geq 0$*, the optimal filtered process then satisfies*

$$\mathrm{d}\widehat{g}_{tt} = \widehat{\partial_t g}_{tt}\,\mathrm{d}t + \left[\widehat{gu}_{tt} - \widehat{g}_{tt}\widehat{u}_t\right]^\intercal \Sigma_t^{-1} v_t\,\mathrm{d}\widehat{W}_t\,, \tag{9.3.6}$$

where $\widehat{g}_{tt} := \mathbb{E}^{\mathbb{S}}\left[g_{x,t} \,\middle|\, \mathcal{I}_t\right]$, $\widehat{gu}_{tt} := \mathbb{E}^{\mathbb{S}}\left[g_{x,t}u_t \,|\, \mathcal{I}_t\right]$ *and* $\widehat{\partial_t g}_{tt} := \mathbb{E}^{\mathbb{S}}\left[\frac{\partial}{\partial t} g_{x,t} \,\middle|\, \mathcal{I}_t\right]$.

Proof. The Kallianpur-Striebel formula (9.2.4) remains unchanged. In the proof of the Zakai equation, the SDE for $(\Gamma_t)_{t\geq 0}$ is also unchanged, however

$$\mathrm{d}g_{x,t} = \frac{\partial}{\partial t} g_{x,t}\,\mathrm{d}t = \frac{\partial}{\partial t} g(X,t)\,\mathrm{d}t\,,$$

from whence the result follows, following the remaining steps in the proofs of the Zakai and the Kushner-Stratonovich equations. □

Remark 9.3.2 (Optimal filter when g is a function of X and a fixed quantity $T > t$)
Consider now $g_{x,T} := g(X,T)$*, where* $T > t$ *is fixed, then the optimal filtered process is governed by*

$$\mathrm{d}\widehat{g}_{tT} = \left[\widehat{gu}_{tT} - \widehat{g}_{tT}\widehat{u}_t\right]^\intercal \Sigma_t^{-1} v_t\,\mathrm{d}\widehat{W}_t\,, \tag{9.3.7}$$

for all $t \in [0,T]$*, which follows from Theorem 9.3.1 since* $\frac{\partial}{\partial t} g_{x,T} = 0$*, with* $\widehat{g}_{tT} := \mathbb{E}^{\mathbb{S}}\left[g_{x,T} \,\middle|\, \mathcal{I}_t\right]$ *and* $\widehat{gu}_{tT} := \mathbb{E}^{\mathbb{S}}\left[g_{x,T}u_t \,|\, \mathcal{I}_t\right]$.

Take note of the new shorthand notation used in Lemma 9.3.1 and Theorem 9.3.1 relating to the market factor functional. To summarise, thus far $g_t := g(X_t)$, $g := g(X)$ and $g_{x,t} := g(X,t)$.

Remark 9.3.3 (A conditioning argument)
Defining the sigma-algebra $\mathcal{A}_t := \sigma\{\mathcal{I}_t, c_X(Y)\}$*, where* $Y = F(X)$ *and* $t \geq 0$*, it is then possible to show that*

$$\begin{aligned}
\mathbb{E}^{\mathbb{F}}\left[g(X,t)\,\Gamma_t \,\middle|\, \mathcal{I}_t\right] &= \mathbb{E}^{\mathbb{F}}\left[g(X,t)\,\Gamma_t\, c_X(Y) \,\middle|\, \mathcal{I}_t\right] \\
&= \mathbb{E}^{\mathbb{F}}\left[c_X(Y)\,\mathbb{E}^{\mathbb{F}}\left[g(X,t)\,\Gamma_t \,\middle|\, \mathcal{A}_t\right] \,\middle|\, \mathcal{I}_t\right] \\
&= \int_{[0,1]^m} c_X(y)\,\mathbb{E}^{\mathbb{F}}\left[g(F^{-1}(y),t)\,\Gamma_t \,\middle|\, \mathcal{A}_t\right]\,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} c_X(F(x))\,\mathbb{E}^{\mathbb{F}}\left[g(x,t)\,\Gamma_t \,\middle|\, \mathcal{A}_t\right]\,\mathrm{d}x\,,
\end{aligned}$$

using the tower property of conditional expectations, where $F^{-1} : [0,1]^m \to \mathbb{R}^m$ *is the* m*-dimensional column vector of marginal inverse cumulative distribution functions, i.e.,* $F_i^{-1}(y) = F_{X_i}^{-1}(y_i) = x_i$ *for* $y_i \in [0,1]$ *and* $i \in \{1,2,\ldots,m\}$*. Take note that*

$$\mathbb{E}^{\mathbb{F}}\left[g(X,t)\,\Gamma_t \,\middle|\, \mathcal{A}_t\right] = \int_{\mathbb{R}^m} g(x,t)\,\Gamma_t \prod_{i=1}^m f_{X_i}(x_i)\,\mathrm{d}x\,,$$

which is a conditional expectation that only involves the independent marginal distributions. This conditioning argument is useful for computations in certain scenarios, especially when the functional $g(x,t)$ *is multiplicatively separable, i.e.,* $g(x,t) = \prod_{i=1}^m g_i(x_i,t)$ *for all* $x \in \mathbb{R}^m$.

The next lemma demonstrates how the canonical information-based approach that was first presented by Macrina (2006) may be recovered within the information filtering framework which has been developed and presented here.

Lemma 9.3.2 (The information-based approach)
Assuming that the set of market factors are independent, and setting $m = n$, $\alpha \in \mathbb{R}^n$, $T \in \mathbb{R}^n_{>0}$, $u(t,x,y) = D[T-t]^{-1}(D[\alpha]\,x - y)$ and $v(t,y) = \mathbf{I}_n = D[\mathbf{1}_n]$ in equations (9.3.1) and (9.3.3), where $T - t := [T_1 - t, T_2 - t, \ldots, T_n - t]^\intercal$, the information process becomes

$$\begin{aligned} \mathrm{d}I_t &= D[T-t]^{-1}\left(D[\alpha]\,X - I_t\right)\,\mathrm{d}t + \mathrm{d}W_t \\ \iff \qquad \mathrm{d}I_t^{(i)} &= \frac{1}{T_i - t}\left(\alpha_i\,X_i - I_t^{(i)}\right)\,\mathrm{d}t + \mathrm{d}W_t^{(i)}\,, \end{aligned} \tag{9.3.8}$$

for $t \in [0, T_i)$ and $i \in \{1, 2, \ldots, n\}$, with a unique solution given by

$$\begin{aligned} I_t &= D[T-t]D[T]^{-1}I_0 + D[\alpha]D[T]^{-1}\,Xt + D[T-t]\int_0^t D[T-s]^{-1}\,\mathrm{d}W_s \\ \iff I_t^{(i)} &= I_0^{(i)}\left(\frac{T_i - t}{T_i}\right) + \alpha_i\,X_i\left(\frac{t}{T_i}\right) + (T_i - t)\int_0^t \frac{1}{T_i - s}\,\mathrm{d}W_s^{(i)}\,. \end{aligned} \tag{9.3.9}$$

Setting $I_0 = \mathbf{0}_n$ and $\alpha = [\alpha_1 T_1, \alpha_2 T_2, \ldots, \alpha_n T_n]^\intercal$ then recovers the Brownian bridge-based information process, first presented by Macrina (2006).

Proof. For each $i \in \{1, 2, \ldots, n\}$, using the integrating factor $\exp\left(\int_0^t \frac{1}{T_i - s}\,\mathrm{d}s\right) = \frac{T_i}{T_i - t}$, it is straightforward to show, using equation (9.3.8) and Itô's rule, that

$$\mathrm{d}\left(\frac{T_i}{T_i - t} I_t^{(i)}\right) = \frac{T_i}{(T_i - t)^2}\,\alpha_i\,X_i\,\mathrm{d}t + \frac{T_i}{T_i - t}\,\mathrm{d}W_t^{(i)}\,,$$

while integrating both sides and simplifying yields equation (9.3.9). The result follows by setting parameters as specified, and observing that $B_{tT} := D[T-t]\int_0^t D[T-s]^{-1}\,\mathrm{d}W_s$ is an n-dimensional column vector standard Brownian bridge process, see Revuz and Yor (1999). Also take note that $I_0 := 0$ (by definition) $\Rightarrow \mathcal{I}_0 \subset \mathcal{C}_0$, while for each i the terminal value of the information process $I_{T_i}^{(i)} = \alpha_i X_i$ and therefore $\sigma\{X_i\} \subseteq \mathcal{I}_{T_i}$, even though $\sigma\{X_i\} \subset \mathcal{C}_0$. Therefore $\mathcal{I}_t \subset \mathcal{C}_t$ for all other $t \in (0, T_i)$. ☐

Remark 9.3.4 (Other information processes)
Another notable information process is the Brownian motion-based process

$$\mathrm{d}I_t = D[\alpha]\,X\,\mathrm{d}t + \mathrm{d}W_t\,, \tag{9.3.10}$$

along with the geometric Brownian motion-based process

$$\mathrm{d}I_t = D[\alpha]\,X\,I_t\,\mathrm{d}t + I_t\,\mathrm{d}W_t\,, \tag{9.3.11}$$

for $t \geq 0$, with all of the parameters as defined in Lemma 9.3.2 and Corollary 9.3.1.

Considering the slightly more general filtering framework that has been formulated, three extra features may be considered here:

(i) correlated noise and thereby information processes, i.e., when the diffusion coefficient $v(t, I_t) \neq \mathbf{I}_n$, where $\mathbf{I}_n$ denotes the $(n \times n)$-identity matrix;

(ii) jointly distributed market factors, i.e., when the copula density is not trivial; and

(iii) a wider class of potential information process, given the SDE specified by equation (9.3.3) — one example of such is presented in the next corollary.

Corollary 9.3.1 (Infinite horizon mean-reverting information)
Setting $m = n$, $\alpha \in \mathbb{R}^n$ and $u(t, x, y) = D[\alpha]\,(x - y)$ in equations (9.3.1) and (9.3.3), the information process becomes

$$\mathrm{d}I_t = D[\alpha]\,(X - I_t)\,\mathrm{d}t + v_t\,\mathrm{d}W_t\ , \tag{9.3.12}$$

for $t \in [0, \infty)$, with a unique solution given by

$$I_t = D[e^{-\alpha t}]\,I_0 + D[\mathbf{1}_n - e^{-\alpha t}]\,X + \int_0^t D[e^{-\alpha(t-s)}]\,v_s\,\mathrm{d}W_s\ , \tag{9.3.13}$$

where $e^{-\alpha x} := [e^{-\alpha_1 x}, e^{-\alpha_2 x}, \dots, e^{-\alpha_n x}]^\intercal$.

Proof. Using the integrating factor $D[e^{\alpha t}]$, it follows that

$$\mathrm{d}\left(D[e^{\alpha t}]I_t\right) = D[e^{\alpha t}]\,X\,\mathrm{d}t + D[e^{\alpha t}]\,v_t\,\mathrm{d}W_t\ ,$$

while integrating both sides and simplifying yields equation (9.3.13). □

Observe that for each $i \in \{1, 2, \dots, n\}$, this information process, equation (9.3.13), exhibits the following limiting behaviour:

$$\lim_{t\to\infty} I_t^{(i)} = \lim_{\alpha_i\to\infty} I_t^{(i)} = X_i\ ,$$

which, along with the mean-reverting feature, demonstrates why this process is considered to be the infinite horizon version of the standard Brownian bridge-based information process, given by equation (9.3.9). The results of Lemma 9.3.1 and Theorem 9.3.1 of course hold for all versions of the information processes presented above. While theoretically appealing, the infinite horizon mean-reverting information process may not be conducive for practical and tractable modelling applications, since it naturally manifests non-Markov filtered processes — for a practical perspective on this issue, one may refer to the example that has been created in Appendix D.2, after working through the theory that has been derived in sections 10.1 and 10.2.

Chapter 10

Information-Based Pricing Kernels

As stated in section 1.4.3, the main objective for section 10.1 is to derive a general framework that is capable of modelling short, instantaneous forward, and discrete forward rates using pricing kernels. This is achieved through the derivation of Theorem 10.1.1, the results of which inspire the approaches undertaken in sections 10.2 and 10.3. In both of these sections, it is shown how the information filtering framework that has been constructed in the previous chapter may be applied to generate pricing kernel (or short rate), instantaneous forward rate, and discrete forward rate models that are information-based. It turns out that the key modelling objects are *information-driven martingale processes*, as advocated by the result of Theorem 10.1.1, which are naturally generated within the developed information filtering framework if one considers the nature of the optimal filter based on time-homogeneous market factor functional, as shown in Lemma 9.3.1 and Remark 9.3.2.

10.1 Interest Rate Modelling Framework

The interest rate financial market under consideration is assumed to be *incomplete*, *arbitrage-free* and supported by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_u)_{u\geq 0}, \mathbb{P})$ satisfying the *usual conditions*, with $\mathbb{P}$ denoting the real-world probability measure and

$$\mathcal{F}_u \;:=\; \mathcal{G}_u \vee \mathcal{H}_u \;,$$

for $u \geq 0$, where the filtration $(\mathcal{G}_u)_{u\geq 0}$ models information about all *tradable* variables; while $(\mathcal{H}_u)_{u\geq 0}$ contains information about all *non-tradable* variables; such that $(\mathcal{F}_u)_{u\geq 0}$ models information about all *tradable* and *non-tradable* variables. The tradable variables under consideration are the market prices, implicit rates and latent factors associated with tradable instruments (both *primitive* and *derivative*). Non-tradable variables include economic statistics, benchmarks, indicators and latent factors therein.

Remark 10.1.1 (Modelling relations with the information filtering framework)
With respect to the information filtering framework developed in Chapter 9, the observation filtration, denoted by $\mathcal{I}$, and its adapted processes may be utilised to model either $\mathcal{G}$ or $\mathcal{H}$.

Assumption 10.1.1 (Estimation and calibration)
It is assumed that models specified with respect to information from $(\mathcal{F}_u)_{u\geq 0}$ will recover and fix $\mathbb{P}$ via statistical estimation. On the other hand, calibration and the construction of an equivalent risk-neutral measure $\mathbb{Q}$, and equivalent martingale measures related thereto, will require models that are specified with respect to information from $(\mathcal{G}_u)_{u\geq 0}$ only.

Definition 10.1.1 (Continuously and discretely compounded rates)
For $u \in \mathbb{R}_{\geq 0}$, the continuously compounded (or short) rate that applies over the time interval $[u, u + \mathrm{d}u]$ has an estimated version $\widehat{r}_u$ that is $\mathcal{F}_u$-adapted and a calibrated version r_u that is $\mathcal{G}_u$-adapted. For some arbitrary accrual period $\delta > 0$, the simple rate that applies over $[u, u+\delta]$ has an estimated version $\widehat{R}_u$ that is $\mathcal{F}_u$-adapted and a calibrated version R_u that is $\mathcal{G}_u$-adapted. This simple rate may be discretely compounded over multiple periods, as shown in the next definition.

Assumption 10.1.2 (Stochastic framework)
Both versions of the interest rate are assumed to be modelled by continuous semimartingales, while an equivalent change-of-measure from $\mathbb{P}$ to $\mathbb{Q}$ is assumed to be enabled by a Girsanov transformation using a time-inhomogeneous uniformly integrable $\{(\mathcal{G}_u), \mathbb{P}\}$-martingale process $(\Lambda_u)_{u \geq 0}$, with $\Lambda_0 := 1$.

Assumption 10.1.2 is a natural consequence of the information filtering framework, and the resultant models that may be specified and derived therein.

Definition 10.1.2 (Estimated and calibrated stochastic discount factors)
The corresponding estimated and calibrated stochastic discount factors that apply over the interval $[0, t]$ are

$$\widehat{D}_t := \exp\left(-\int_0^t \widehat{r}_u \, \mathrm{d}u\right) \quad \textit{and} \quad D_t := \exp\left(-\int_0^t r_u \, \mathrm{d}u\right) ,$$

respectively, with $\widehat{D}_t$ being $\mathcal{F}_t$-adapted and D_t being $\mathcal{G}_t$-adapted. For $n \in \mathbb{N}$, assuming that $t = n\delta$, the equivalent discrete compounding versions are

$$\widehat{D}_{t+\delta} := \prod_{i=1}^{n} \frac{1}{1 + \delta \widehat{R}_{i\delta}} \quad \textit{and} \quad D_{t+\delta} := \prod_{i=1}^{n} \frac{1}{1 + \delta R_{i\delta}} ,$$

respectively, with $\widehat{D}_{t+\delta}$ being $\mathcal{F}_t$-adapted, $D_{t+\delta}$ being $\mathcal{G}_t$-adapted, and $\widehat{D}_0 = D_0 := 1$.

Remark 10.1.2 (Estimation and calibration using $(\mathcal{G}_u)_{u \geq 0}$ only)
Estimation and calibration both undertaken on $(\mathcal{G}_u)_{u \geq 0}$ is the classical approach. Under this setting, it is assumed that $r_u = \widehat{r}_u$ a.s. and $R_u = \widehat{R}_u$ a.s. with both quantities being $\mathcal{G}_u$-measurable, for all $u \geq 0$. Therefore, $D_t = \widehat{D}_t$ a.s. and the pricing kernel assumes the standard form $\pi_t := \Lambda_t D_t$. Accordingly then

$$\widehat{P}_{tT} := \frac{1}{D_t} \mathbb{E}^{\mathbb{P}} \left[D_T \,|\, \mathcal{G}_t \right] \quad \textit{and} \quad P_{tT} := \frac{1}{D_t} \mathbb{E}^{\mathbb{Q}} \left[D_T \,|\, \mathcal{G}_t \right] ,$$

are the estimated and calibrated zero coupon bond systems, respectively, where $t \in [0, T]$ for continuous rates or $t \in \{0, \delta, \ldots, n\delta\}$ for discrete rates with $T = n\delta$ and $n \in \mathbb{N}$. For discretely compounded rates, $D_{t+\delta}$ is $\mathcal{G}_t$-measurable and therefore $\widehat{P}_{tt+\delta} = P_{tt+\delta}$ for all $t \in \{0, \delta, \ldots\}$.

A more general result, than the above remark, making use of $\mathcal{F}$ and $\mathcal{G}$ and the respective models formulated under $\mathbb{P}$ and $\mathbb{Q}$ is detailed in the next Lemma and its' associated corollary.

Lemma 10.1.1 (From $\mathbb{P}$ to $\mathbb{Q}$)
Assuming the existence of the estimated versions of the stochastic discount factors, the calibrated stochastic discount factors may be defined as

$$D_t := \mathbb{E}^{\mathbb{P}} \left[\widehat{D}_t \,\middle|\, \mathcal{G}_t \right] \quad \textit{and} \quad D_{t+\delta} := \mathbb{E}^{\mathbb{P}} \left[\widehat{D}_{t+\delta} \,\middle|\, \mathcal{G}_t \right] . \tag{10.1.1}$$

for the short and discretely compounded rate versions, respectively. The pricing kernel assumes the standard form $\pi_t := \Lambda_t D_t$ *, with*

$$\widehat{P}_{tT} := \frac{1}{\widehat{D}_t}\mathbb{E}^{\mathbb{P}}\left[\widehat{D}_T \mid \mathcal{F}_t\right] \quad \textit{and} \quad P_{tT} := \frac{1}{D_t}\mathbb{E}^{\mathbb{Q}}\left[D_T \mid \mathcal{G}_t\right] \ , \tag{10.1.2}$$

being the estimated and calibrated ZCB-systems, respectively, where $t \in [0, T]$ *for continuous rates or* $t \in \{0, \delta, 2\delta, \ldots, n\delta\}$ *for discrete rates with* $T = n\delta$ *and* $n \in \mathbb{N}$ *.*

Proof. If rates are continuous then $\widehat{D}_t$ is $\mathcal{F}_t$-adapted and $\widehat{P}_{tT} := \frac{1}{\widehat{D}_t}\mathbb{E}^{\mathbb{P}}\left[\widehat{D}_T \mid \mathcal{F}_t\right]$ is the estimated ZCB-system, for $t \in [0, T]$, by definition. By Assumption 10.1.1, a model specified with information from $(\mathcal{G}_u)_{u\geq 0}$ is required for the purposes of calibration, resulting in the definition of the calibrated stochastic discount factors in equation (10.1.1). If $(P_{0T})_{T\geq 0}$ denotes the initial term structure of zero coupon bond prices, then using Assumption 10.1.2 one may recover these prices via the relation $\mathbb{E}^{\mathbb{P}}\left[\Lambda_T D_T \mid \mathcal{G}_0\right]$, after calibrating the time-dependent parameters associated with Λ_T at each instant T, respectively. Moreover, since $(\Lambda_t)_{t\geq 0}$ enables an equivalent change-of-measure from $\mathbb{P}$ to $\mathbb{Q}$, the result for the calibrated ZCB-system follows accordingly, i.e., the second relation in equation (10.1.2). An analogous argument works for the case of discrete rates, which completes the proof. ☐

A corollary of Lemma 10.1.1 that considers modelling in the opposite direction, from $\mathbb{Q}$ to $\mathbb{P}$, is presented below without proof. The result follows trivially from Assumptions 10.1.1, 10.1.2 and the observations from Remark 10.1.2.

Corollary 10.1.1 (From $\mathbb{Q}$ to $\mathbb{P}$)
Assuming the existence of the $\{(\mathcal{G}_t), \mathbb{Q}\}$*-process* $(D_t)_{t\geq 0}$ *or* $\{D_t; t \in \{0, \delta, \ldots\}\}$ *in the case of continuous or discrete rates, respectively, the estimated stochastic discount factor may be modelled by the* $\mathcal{G}_t$*-adapted process*

$$\widehat{D}_t := \begin{cases} \zeta_t D_t \ , & \textit{under } \mathbb{Q} \ , \\ D_t \ , & \textit{under } \mathbb{P} \ . \end{cases}$$

The time-inhomogeneous uniformly integrable $\{(\mathcal{G}_t), \mathbb{Q}\}$*-martingale* $(\zeta_t)_{t\geq 0}$ *in the case of continuous rates or* $\{\zeta_t; t \in \{0, \delta, \ldots\}\}$ *in the case of discrete rates, enables the change-of-measure from* $\mathbb{Q}$ *to an equivalent measure* $\mathbb{P}$ *, with* $\zeta_0 := 1$*. However, if the pricing kernel is known then* $\pi_t := \Lambda_t D_t$ *under* $\mathbb{P}$ *, and* $\zeta_t = \Lambda_t^{-1}$ *so that* $\widehat{D}_t := \Lambda_t^{-1} D_t$ *under* $\mathbb{Q}$ *.*

Lemma 10.1.1 and Corollary 10.1.1 offer two alternative approaches to interest rate modelling, the choice of which depends on the particular application. An investment or risk manager would generally prefer to start with a $\mathbb{P}$-model while the derivatives market-maker or quantitative analyst would generally prefer to start with a $\mathbb{Q}$-model. Here, an intermediate perspective is taken, one that focuses on theoretical model development under an auxiliary measure $\mathbb{M}$ first, for the sake of tractability. Once a suitable $\mathbb{M}$-model has been developed, one may then adapt the model for practical applications under $\mathbb{P}$ and/or $\mathbb{Q}$ by constructing suitable changes-of-measure from $\mathbb{M}$ to $\mathbb{P}$ and $\mathbb{M}$ to $\mathbb{Q}$, respectively. This approach is inspired by the work of Macrina (2014).

Definition 10.1.3 (Auxiliary pricing kernel under $\mathbb{M}$)
The auxiliary pricing kernel $(\xi_t)_{t\geq 0}$ *in the case of continuous rates or* $\{\xi_t; t \in \{0, \delta, \ldots\}\}$ *for discrete rates, is the* $\mathcal{G}_t$*-adapted process*

$$\xi_t := \Phi_t D_t \ ,$$

where $(\Phi_t)_{t\geq 0}$ *or* $\{\Phi_t; t \in \{0, \delta, \ldots\}\}$, *with* $\Phi_0 := 1$, *is assumed to be a time-inhomogeneous uniformly integrable* $\{(\mathcal{G}_t), \mathbb{M}\}$*-martingale process that enables an equivalent change-of-measure from* $\mathbb{M}$ *to* $\mathbb{Q}$. *The standard pricing kernel, also under* $\mathbb{M}$, *may then be defined as*

$$\pi_t =: \Psi_t \xi_t = \Psi_t \Phi_t D_t \ ,$$

where $(\Psi_t)_{t\geq 0}$ *or* $\{\Psi_t; t \in \{0, \delta, \ldots\}\}$, *with* $\Psi_0 := 1$, *is assumed to be a time-inhomogeneous uniformly integrable* $\{(\mathcal{G}_t), \mathbb{M}\}$*-martingale process that enables an equivalent change-of-measure from* $\mathbb{M}$ *to* $\mathbb{P}$.

The next result offers a useful perspective on the equivalence between pricing kernel models, which are arguably synonymous with short rate models, associated ZCB-related martingale processes, instantaneous and discrete forward rate processes. These relations enable the construction of interest rate models within the information filtering framework in an intuitive manner by directing almost all focus to the construction of martingale processes.

Theorem 10.1.1 (Interest rate models under $\mathbb{M}$)
The following are equivalent specifications:

(i) $\xi_t P_{tT} = \mathbb{E}^{\mathbb{M}}\left[\xi_T \,|\, \mathcal{G}_t\right]$,

(ii) $m_{tt} P_{tT} = m_{tT}$,

(iii) $f_{tT} = -\frac{\partial_T m_{tT}}{m_{tT}}$, *and*

(iv) $F_t(T, T+\delta) = \frac{m_{tT} - m_{tT+\delta}}{\delta m_{tT+\delta}}$,

for $t \in [0, T]$, *in the case of continuous rates with* $T > 0$, *or* $t \in \{0, \delta, 2\delta, \ldots, n\delta\}$ *for discrete rates with* $T = n\delta$ *and* $n \in \mathbb{N}$. *For each fixed* T, *the process* $(m_{tT})_{t\leq T}$ *or* $\{m_{tT}; t \in \{0, \delta, \ldots, n\delta\}\}$ *is a* $\{(\mathcal{G}_t), \mathbb{M}\}$*-martingale with initial value* $m_{0T} := P_{0T}$. *The quantity* f_{tT} *is the instantaneous forward rate observed at time* t, *with future accrual period* $[T, T + \mathrm{d}t]$, *and* $F_t(T, T+\delta)$ *is a simple forward rate observed at time* t, *with future accrual period* $[T, T+\delta]$.

Proof. Relation (i) is a consequence of Definitions 10.1.1, 10.1.2 and Lemma 10.1.1. In the case of continuous rates, for a fixed $T \geq t$, setting $m_{tT} := \mathbb{E}^{\mathbb{M}}\left[\xi_T \,|\, \mathcal{G}_t\right]$, it is trivial to show that

$$\mathbb{E}^{\mathbb{M}}\left[m_{tT} \,|\, \mathcal{G}_s\right] = \mathbb{E}^{\mathbb{M}}\left[\mathbb{E}^{\mathbb{M}}\left[\xi_T \,|\, \mathcal{G}_t\right] \,\middle|\, \mathcal{G}_s\right] = \mathbb{E}^{\mathbb{M}}\left[\xi_T \,|\, \mathcal{G}_s\right] = m_{sT} \ ,$$

for all $s \leq t \leq T$, using the tower property of conditional expectation. Assuming that the auxiliary pricing kernel is integrable, i.e., $\mathbb{E}\left[|\xi_t|\right] < \infty$ for all $t \geq 0$, it follows that $(m_{tT})_{t\leq T}$ is a $\{(\mathcal{G}_t), \mathbb{M}\}$-martingale process. Since $\xi_0 := 1$, $m_{0T} = \mathbb{E}^{\mathbb{M}}\left[\xi_T \,|\, \mathcal{G}_0\right] = P_{0T}$, as required. A similar argument applies for discrete rates, hence (i) ⇒ (ii).
For the proof in the other direction, define a family of $\{(\mathcal{G}_t), \mathbb{M}\}$-martingale processes $(m_{tT})_{t\leq T}$ indexed by $T \geq 0$, with $m_{tT} := m_{tt} P_{tT}$. Then, setting $\xi_t := m_{tt}$ for all $t \geq 0$, it is straightforward to show that

$$\mathbb{E}^{\mathbb{M}}\left[\xi_T \,|\, \mathcal{G}_t\right] = \mathbb{E}^{\mathbb{M}}\left[m_{TT} \,|\, \mathcal{G}_t\right] = m_{tT} = m_{tt} P_{tT} \ ,$$

which recovers (i). A similar argument applies for discrete rates, hence (ii) ⇒ (i).
Since $P_{tT} := \exp\left(-\int_t^T f_{tu}\,\mathrm{d}u\right)$, relation (iii) follows easily by taking a partial derivative of (ii) with respect to T, and solving for f_{tT}. Lastly, since $1 + \delta F_t(T, T+\delta) = \frac{P_{tT}}{P_{tT+\delta}} = \frac{m_{tT}}{m_{tT+\delta}}$, relation (iv) follows trivially. ☐

Remark 10.1.3 (Supermartingales and non-negative rates)
The auxiliary pricing kernel $(\xi_t)_{t\geq 0}$ *in the case of continuous rates or* $\{\xi_t; t \in \{0, \delta, \ldots\}\}$ *in the case of discrete rates, may be specified to be a* $\{(\mathcal{G}_t), \mathbb{M}\}$*-supermartingale process. This means that*

$$\mathbb{E}^{\mathbb{M}}\left[\xi_U \,|\, \mathcal{G}_t\right] \leq \mathbb{E}^{\mathbb{M}}\left[\xi_T \,|\, \mathcal{G}_t\right] \quad \Longleftrightarrow \quad m_{tU} \leq m_{tT} \ ,$$

for all $t \leq T \leq U$ *, where the equivalence relation follows from Theorem 10.1.1.*

10.2 Information-Based Pricing Kernel Models

From Theorem 10.1.1, it should be clear that a viable model for the auxiliary pricing kernel, for both the cases of continuous and discrete rates, may be specified via a family of $\{(\mathcal{G}_t), \mathbb{M}\}$-martingale processes, i.e., $\xi_t = m_{tt}$. This connection enables one to use the information filtering framework to model the auxiliary pricing kernel. In particular, Theorem 9.3.1 and Remark 9.3.2 provide the necessary tools to propose what will be referred to as an *information-driven auxiliary pricing kernel.*

Proposition 10.2.1 (Information-driven auxiliary pricing kernel)
Assigning the filtration $\mathcal{I}$ *and measure* $\mathbb{S}$ *from Chapter 9 to the filtration* $\mathcal{G}$ *and measure* $\mathbb{M}$ *defined in this chapter, respectively, a suitable candidate for the auxiliary pricing kernel within the information filtering framework is*

$$\xi_t = m_{tt} := \widehat{g}_{tt} = \mathbb{E}^{\mathbb{M}}\left[g_{x,t} \,\middle|\, \mathcal{G}_t\right] \ , \tag{10.2.1}$$

where $\widehat{g}_{tt}$ *is defined in Theorem 9.3.1, with the ZCB price process then given by*

$$P_{tT} = \frac{\widehat{g}_{tT}}{\widehat{g}_{tt}} = \frac{\mathbb{E}^{\mathbb{M}}\left[g_{x,T} \,\middle|\, \mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{M}}\left[g_{x,t} \,\middle|\, \mathcal{G}_t\right]} \ , \tag{10.2.2}$$

for $t \in [0, T]$ *, in the case of continuous rates with* $T > 0$*, or* $t \in \{0, \delta, 2\delta, \ldots, n\delta\}$*, in the case of discrete rates with* $T = n\delta$ *and* $n \in \mathbb{N}$*.*

Proof. Using Theorem 10.1.1 and equation (10.2.1), it is clear that

$$\xi_t P_{tT} = \mathbb{E}^{\mathbb{M}}\left[\mathbb{E}^{\mathbb{P}}\left[g_{x,T} \,\middle|\, \mathcal{G}_T\right] \,\middle|\, \mathcal{G}_t\right] \ ,$$

with the result following by the tower property of conditional expectations, and Remark 9.3.2 which defines the quantity $\widehat{g}_{tT}$. □

The intuition here, quite simply, is that the pricing kernel, a fundamental modelling object, is an optimally filtered functional of the market factors, which ultimately determines the intertemporal transitions of ZCB prices within the interest rate market under consideration. Given the pivotal role of the functional $g(x, t)$, certain restrictions have to be imposed to achieve and maintain mathematical and financial economic integrity and consistency.

Corollary 10.2.1 (No arbitrage, deterministic, and positive interest rates)
In order to preclude:

(i) basic arbitrage due to negative or unbounded prices, the function $g : \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0} \wedge a$ *, where* a *is a fixed finite real number, i.e.,* $a \in (0, \infty)$*.*

(ii) deterministic interest rates, the function $g(x,t)$ must not be multiplicatively separable, i.e., $g(x,t) \neq g_1(x)g_2(t)$ for all $x \in \mathbb{R}^m$ and $t \in \mathbb{R}_{\geq 0}$.

(iii) negative rates, the function $g : \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to (0,1]$ and $\frac{\partial g}{\partial t} : \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\leq 0}$ when $g(x,\cdot)$ is a continuous function, or $g(x,t) \geq g(x,T)$ for all $t \leq T$ when $g(x,\cdot)$ is a discrete function.

(iv) non-positive rates, the function $g : \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to (0,1]$ and $\frac{\partial g}{\partial t} : \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{<0}$ when $g(x,\cdot)$ is a continuous function, or $g(x,t) > g(x,T)$ for all $t \leq T$ when $g(x,\cdot)$ is a discrete function.

Proof. Using condition (i) and the auxiliary pricing kernel defined by equation (10.2.1), it follows that $0 < \xi_t < \infty$ for all $t \geq 0$, by the positivity property of conditional expectations. Similarly, $0 < \mathbb{E}^{\mathbb{M}}\left[\xi_T \,\middle|\, \mathcal{G}_t\right] < \infty$ for all $t \in [0,T]$. Therefore, it is clear that $0 < P_{tT} < \infty$, using Theorem 10.1.1, which concludes the proof for (i).

Assuming condition (ii) holds and using equation (10.2.2), it follows that:

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{M}}\left[g_1(X)g_2(T) \,\middle|\, \mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{M}}\left[g_1(X)g_2(t) \,\middle|\, \mathcal{G}_t\right]} = \frac{g_2(T)}{g_2(t)}\,,$$

which concludes the result.

Including the upper bound condition from (iii) and (iv) in the proof for (i), it follows by the monotonicity property of conditional expectations that $0 < \xi_t \leq 1$ and $0 < \mathbb{E}^{\mathbb{M}}\left[\xi_T \,\middle|\, \mathcal{G}_t\right] \leq 1$ for all $0 \leq t \leq T$. Moreover, $g(X,t) \geq g(X,T)$ by the non-positive time-derivative condition, therefore $0 < P_{tT} \leq 1$ by equation (10.2.2), which guarantees non-negative rates. If the stricter negative time-derivative condition is imposed, as in condition (iv), then $g(X,t) > g(X,T)$. Using Proposition 10.2.1 it follows that $m_{tt} > m_{tT}$ and that rates are strictly positive using relations (iii) and (iv) from Theorem 10.1.1. □

Given the Wiener process (or Brownian motion) based foundation of the information filtering framework, in the case of continuous rates the short rate process and Girsanov kernel associated with the change-of-measure from the auxiliary modelling measure $\mathbb{M}$ to the equivalent risk-neutral measure $\mathbb{Q}$ may be easily derived. This Girsanov kernel is referred to as a set of *model prices of risk* in this instance, due to the interpretation of $\mathbb{M}$ as an auxiliary modelling measure.

Corollary 10.2.2 (Information-driven short rate and model prices of risk)
The short rate process associated with the information-driven auxiliary pricing kernel (10.2.1) is

$$r_t := -\frac{\widehat{\partial_t g_{tt}}}{\widehat{g}_{tt}} = -\frac{\mathbb{E}^{\mathbb{M}}\left[\frac{\partial}{\partial t} g_{x,t} \,\middle|\, \mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{M}}\left[g_{x,t} \,\middle|\, \mathcal{G}_t\right]}\,, \tag{10.2.3}$$

with associated model prices of risk given by

$$\phi_t := -\frac{1}{\widehat{g}_{tt}}\, v_t^{\intercal} \Sigma_t^{-1} \left[\widehat{gu}_{tt} - \widehat{g}_{tt}\widehat{u}_t\right]\,, \tag{10.2.4}$$

which is an n-dimensional column vector, for all $t \geq 0$. Further, assuming that the Novikov condition

$$\mathbb{E}^{\mathbb{M}}\left[\exp\left(\frac{1}{2}\int_0^T \phi_s^{\intercal}\phi_s \,\mathrm{d}s\right)\right] < \infty\,, \tag{10.2.5}$$

is satisfied, then the Radon-Nikodym derivative defined on $\mathcal{G}_T$ by

$$\Phi_T := \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{M}}\right|_{\mathcal{G}_T} = \exp\left(-\int_0^T \phi_s^{\intercal}\,\mathrm{d}\widehat{W}_s - \frac{1}{2}\int_0^T \phi_s^{\intercal}\phi_s \,\mathrm{d}s\right)\,, \tag{10.2.6}$$

induces the new measure $\mathbb{Q} \sim \mathbb{M}$ *over* $[0, T]$*, for some* $T > 0$*, along with the new* $\{(\mathcal{G}_t), \mathbb{Q}\}$*-Wiener process* $W_t^{\mathbb{Q}} := \widehat{W}_t + \int_0^t \phi_s \, \mathrm{d}s$ *for* $t \in [0, T]$.

Proof. Using Theorem 10.1.1 and Proposition 10.2.1, the instantaneous forward rate process is

$$f_{tT} := -\frac{\partial_T m_{tT}}{m_{tT}} = -\frac{\frac{\partial}{\partial T}\mathbb{E}^{\mathbb{M}}\left[g_{x,T} \,\middle|\, \mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{M}}\left[g_{x,T} \,\middle|\, \mathcal{G}_t\right]} \ ,$$

from which the short rate follows after using Fubini's Theorem and recalling that $r_t := f_{tt}$. From Definition 10.1.3, the auxiliary pricing kernel may be decomposed into the product of a stochastic discount factor, $D_t = \exp\left(-\int_0^t r_s \mathrm{d}s\right)$, and a change-of-measure density process, Φ_t, i.e., $\xi_t = D_t \Phi_t$ for all $t \geq 0$. Also, the SDE that governs the density process, in a Wiener process setting, has the form: $\mathrm{d}\Phi_t / \Phi_t = -\phi_t^{\intercal} \, \mathrm{d}\widehat{W}_t$, with ϕ_t denoting the model prices of risk — see, for example, Duffie (2001) or Björk (2004). Therefore, by Itô's quotient rule and results from Proposition 10.2.1:

$$\begin{aligned}
\frac{\mathrm{d}\Phi_t}{\Phi_t} &= \frac{\mathrm{d}\xi_t}{\xi_t} - \frac{\mathrm{d}D_t}{D_t} + \frac{\mathrm{d}\langle D \rangle_t}{D_t^2} - \frac{\mathrm{d}\langle \xi, D \rangle_t}{\xi_t D_t} \\
&= \frac{\widehat{\partial g}_{tt}}{\widehat{g}_{tt}} \, \mathrm{d}t + \frac{1}{\widehat{g}_{tt}} \left[\widehat{gu}_{tt} - \widehat{g}_{tt}\widehat{u}_t\right]^{\intercal} \Sigma_t^{-1} v_t \, \mathrm{d}\widehat{W}_t - \frac{\widehat{\partial g}_{tt}}{\widehat{g}_{tt}} \, \mathrm{d}t \\
&= \frac{1}{\widehat{g}_{tt}} \left[\widehat{gu}_{tt} - \widehat{g}_{tt}\widehat{u}_t\right]^{\intercal} \Sigma_t^{-1} v_t \, \mathrm{d}\widehat{W}_t := -\phi_t \, \mathrm{d}\widehat{W}_t \, ,
\end{aligned}$$

which shows equation (10.2.4). Equation (10.2.6) is a solution to the SDE for the density process. Given the Novikov condition (10.2.5), $\Phi_t = \mathbb{E}^{\mathbb{M}}\left[\Phi_T \,\middle|\, \mathcal{G}_t\right]$ for $t \in [0, T]$, with $\Phi_0 = 1$, is a positive $\{(\mathcal{G}_t), \mathbb{M}\}$-martingale which serves as a suitable likelihood process to induce the necessary change-of-measure by Girsanov's Theorem. ☐

Remark 10.2.1 (From $\mathbb{M}$ to $\mathbb{P}$)
The Radon-Nikodym derivative defined on $\mathcal{G}_T$ *by*

$$\Psi_T := \left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{M}}\right|_{\mathcal{G}_T} = \exp\left(\int_0^T \psi_s^{\intercal} \, \mathrm{d}\widehat{W}_s - \frac{1}{2}\int_0^T \psi_s^{\intercal}\psi_s \, \mathrm{d}s\right) \, , \tag{10.2.7}$$

induces the new measure $\mathbb{P} \sim \mathbb{M}$ *over* $[0, T]$*, for some* $T > 0$*, along with the new* $\{(\mathcal{G}_t), \mathbb{P}\}$*-Wiener process* $W_t^{\mathbb{P}} := \widehat{W}_t - \int_0^t \psi_s \, \mathrm{d}s$ *for* $t \in [0, T]$*. Then, using the result of Corollary 10.2.2, it follows that*

$$W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} + \int_0^t \left[\psi_s + \phi_s\right] \mathrm{d}s \ ,$$

for $t \in [0, T]$*, which may also be achieved via the Radon-Nikodym derivative defined on* $\mathcal{G}_T$ *by*

$$\Lambda_T := \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_T} = \exp\left(-\int_0^T \lambda_s^{\intercal} \, \mathrm{d}W_s^{\mathbb{P}} - \frac{1}{2}\int_0^T \lambda_s^{\intercal}\lambda_s \, \mathrm{d}s\right) \, , \tag{10.2.8}$$

where $\lambda_s := \psi_s + \phi_s$ *defines the set of market prices of risk. The Girsanov kernel* $(\psi_t)_{t \leq T}$ *is then the difference between the respective market and model prices of risk.*

Remark 10.2.2 (Forward model prices of risk)
From Proposition 10.2.1, it is clear that

$$\mathbb{E}^{\mathbb{M}}\left[\xi_T \,\middle|\, \mathcal{G}_t\right] = \widehat{g}_{tT} \, ,$$

while from Remark 9.3.2, the SDE that governs this process may now be written as

$$\mathrm{d}\widehat{g}_{tT} = -\widehat{g}_{tT}\phi_{tT}\,\mathrm{d}\widehat{W}_t\,,$$

for $t \in [0,T]$*, where* $\phi_{tT} := -v_t^\intercal \Sigma_t^{-1}\left[\widehat{gu}_{tT} - \widehat{g}_{tT}\widehat{u}_t\right] / \widehat{g}_{tT}$ *is interpreted as the set of forward model prices of risk at time* T*, as seen (expected) from time* t*. Observe also that* $\phi_t = \lim_{T\to t}\phi_{tT}$*.*

Corollary 10.2.3 (SDEs for the ZCB, short and forward rates)
Under the $\mathbb{M}$*-measure, the ZCB price process is governed by the SDE*

$$\frac{\mathrm{d}P_{tT}}{P_{tT}} = \left(r_t - \Theta_{tT}^\intercal\phi_t\right)\,\mathrm{d}t - \Theta_{tT}^\intercal \mathrm{d}\widehat{W}_t\,, \tag{10.2.9}$$

for $t \in [0,T]$*, the instantaneous forward rate follows*

$$\mathrm{d}f_{tT} = \theta_{tT}^\intercal\left[\Theta_{tT} + \phi_t\right]\mathrm{d}t + \theta_{tT}^\intercal \mathrm{d}\widehat{W}_t\,, \tag{10.2.10}$$

while the short rate has the following dynamics:

$$\mathrm{d}r_t = \left(\partial_T f_{tt} + \theta_{tt}^\intercal\phi_t\right)\,\mathrm{d}t + \theta_{tt}^\intercal \mathrm{d}\widehat{W}_t\,, \tag{10.2.11}$$

where $\partial_T f_{tt} := \frac{\partial}{\partial T} f_{tT}\big|_{T=t}$*,* $\theta_{tT} := \frac{\partial}{\partial T}\phi_{tT}$ *and* $\Theta_{tT} := \int_t^T \theta_{tu}\,\mathrm{d}u = \phi_{tT} - \phi_t$*, for* $t \in [0,T]$*.*

Proof. Applying Itô's quotient rule to equation (10.2.2), from Proposition 10.2.1, it follows that:

$$\begin{aligned}
\frac{\mathrm{d}P_{tT}}{P_{tT}} &= \frac{\mathrm{d}\widehat{g}_{tT}}{\widehat{g}_{tT}} - \frac{\mathrm{d}\widehat{g}_{tt}}{\widehat{g}_{tt}} + \frac{\mathrm{d}\langle \widehat{g}_{\cdot t}\rangle_t}{\widehat{g}_{tt}^2} - \frac{\mathrm{d}\langle \widehat{g}_{\cdot T}, \widehat{g}_{\cdot t}\rangle_t}{\widehat{g}_{tT}\widehat{g}_{tt}} \\
&= -\frac{\widehat{\partial_t g_{tt}}}{\widehat{g}_{tt}}\,\mathrm{d}t + \left[\phi_t - \phi_{tT}\right]^\intercal\,\mathrm{d}\widehat{W}_t + \phi_t^\intercal\phi_t\,\mathrm{d}t - \phi_{tT}^\intercal\phi_t\,\mathrm{d}t \\
&= \left(r_t - \left[\phi_{tT} - \phi_t\right]^\intercal\phi_t\right)\,\mathrm{d}t - \left[\phi_{tT} - \phi_t\right]^\intercal\,\mathrm{d}\widehat{W}_t\,,
\end{aligned}$$

using Theorem 9.3.1, Corollary 10.2.2 and Remark 10.2.2. The result for the ZCB's SDE follows by setting $\theta_{tT} = \frac{\partial}{\partial T}\phi_{tT}$ and observing that $\Theta_{tT} = \int_t^T \theta_{tu}\,\mathrm{d}u = \phi_{tT} - \phi_t$. Integrating the log of the ZCB price process as follows

$$\ln\left(P_{tT}\right) = \ln\left(P_{0T}\right) + \int_0^t \left(r_s - \frac{1}{2}\Theta_{sT}^\intercal\Theta_{sT} - \Theta_{sT}^\intercal\phi_s\right)\,\mathrm{d}s - \int_0^t \Theta_{sT}^\intercal\,\mathrm{d}\widehat{W}_s\,,$$

taking the partial derivative with respect to T[1] and multiplying by negative one yields the SDE for the instantaneous forward rate, equation (10.2.10).
Let $\sigma_{tT} := \theta_{tT}$ and $\alpha_{tT} := \theta_{tT}^\intercal\left[\Theta_{tT} + \phi_t\right]$, then

$$\mathrm{d}f_{tT} = \alpha_{tT}\,\mathrm{d}t + \sigma_{tT}^\intercal\,\mathrm{d}\widehat{W}_t\,.$$

Let $\partial_T f_{st}$ denote the time of maturity derivative of the instantaneous forward rate at time s evaluated at maturity time t, i.e. $\partial_T f_{st} = \frac{\partial}{\partial T} f_{sT}\big|_{T=t}$. Similarly, let $\partial_T\alpha_{st} = \frac{\partial}{\partial T}\alpha_{sT}\big|_{T=t}$, and $\partial_T\sigma_{st} = \frac{\partial}{\partial T}\sigma_{sT}\big|_{T=t}$, then the following relations hold:

$$\begin{aligned}
\alpha_{st} &= \alpha_{ss} + \int_s^t \partial_T\alpha_{su}\,du\,, \\
\sigma_{st} &= \sigma_{ss} + \int_s^t \partial_T\sigma_{su}\,du\,, \quad \text{and} \\
\mathrm{d}\left(\partial_T f_{st}\right) &= \partial_T\alpha_{st}\,\mathrm{d}t + \left[\partial_T\sigma_{st}\right]^\intercal\,\mathrm{d}\widehat{W}_t\,.
\end{aligned}$$

[1] Assuming that all processes are regular enough to allow differentiation under the integral sign.

With these relations at hand, it follows that the short rate is given by:

$$
\begin{aligned}
r_t &= f_{0t} + \int_0^t \alpha_{st}\,\mathrm{d}s + \int_0^t \sigma_{st}^{\intercal}\,\mathrm{d}\widehat{W}_s \\
&= f_{0t} + \int_0^t \alpha_{ss}\,\mathrm{d}s + \int_0^t \int_s^t \partial_T \alpha_{su}\,\mathrm{d}u\,\mathrm{d}s + \int_0^t \sigma_{ss}^{\intercal}\,\mathrm{d}\widehat{W}_s + \int_0^t \int_s^t \left[\partial_T \sigma_{su}\right]^{\intercal}\,\mathrm{d}u\,\mathrm{d}\widehat{W}_s \\
&= f_{0t} + \int_0^t \alpha_{ss}\,\mathrm{d}s + \sigma_{ss}^{\intercal}\,\mathrm{d}\widehat{W}_s + \int_0^t \int_0^u \left(\partial_T \alpha_{su}\,\mathrm{d}s + \left[\partial_T \sigma_{su}\right]^{\intercal}\,\mathrm{d}\widehat{W}_s\right)\mathrm{d}u \\
&= f_{0t} + \int_0^t \alpha_{ss}\,\mathrm{d}s + \sigma_{ss}^{\intercal}\,\mathrm{d}\widehat{W}_s + \int_0^t \int_0^u \mathrm{d}\left(\partial_T f_{su}\right)\mathrm{d}u\,,
\end{aligned}
$$

where the third equality follows by changing the order of integration. The short rate result follows by observing that $\int_0^u \mathrm{d}\left(\partial_T f_{su}\right) = \partial_T f_{uu} - \partial_T f_{0u}$ and $\int_0^t \partial_T f_{0u}\,\mathrm{d}u = f_{0t} - r_0$. □

Remark 10.2.3 (Dynamics under the risk-neutral measure)
From Corollary 10.2.2, the Radon-Nikodym derivative $(\Phi_t)_{t\geq 0}$ enables the change-of-measure from the auxiliary measure $\mathbb{M}$ to the equivalent risk-neutral measure $\mathbb{Q}$ such that $W_t^{\mathbb{Q}} := \widehat{W}_t + \int_0^t \phi_s\,\mathrm{d}s$ is an n-dimensional column vector $\{(\mathcal{G}_t), \mathbb{Q}\}$-Wiener process, for all $t \geq 0$. The short rate dynamics under $\mathbb{Q}$ is given by

$$
\mathrm{d}r_t = \partial_T f_{tt}\,\mathrm{d}t + \left[\frac{\partial}{\partial t}\phi_t\right]^{\intercal}\mathrm{d}W_t^{\mathbb{Q}}\,, \tag{10.2.12}
$$

with the drift quantity resolving to

$$
\partial_T f_{tt} = \frac{\partial}{\partial T} f_{tT}\bigg|_{T=t} = r_t^2 - \frac{1}{\xi_t}\mathbb{E}^{\mathbb{M}}\left[\frac{\partial^2}{\partial t^2} g_{x,t}\,\bigg|\,\mathcal{G}_t\right],
$$

which completely determines the term structure as well as the prices of all tradable financial instruments. One may refer to Björk (2004) for a similar result within the context of the HJM framework.

The information-based interest rate modelling framework presented thus far also admits models for discretely compounded interest rates — the next corollary presents such models for the simple rate defined in Definition 10.1.1 along with its associated forward rates.

Corollary 10.2.4 (Information-driven simple spot and forward rates)
The process followed by the simple spot rate, with accrual period equal to δ, that is associated with the information-driven auxiliary pricing kernel (10.2.1) is

$$
R_t := \frac{m_{tt} - m_{tt+\delta}}{\delta m_{tt+\delta}} = \frac{\widehat{g}_{tt} - \widehat{g}_{tt+\delta}}{\delta \widehat{g}_{tt+\delta}}\,, \tag{10.2.13}
$$

for $t \geq 0$, in the case of continuous rates, or $t \in \{t_0, t_1, t_2, \ldots\}$, in the case of discrete rates where $t_0 := 0$, $t_i := t_0 + i\delta$ for $i \in \mathbb{N}$. Each spot rate R_T has an associated forward rate process

$$
F_{tT}^{\delta} := F_t(T, T+\delta) := \frac{m_{tT} - m_{tT+\delta}}{\delta m_{tT+\delta}} = \frac{\widehat{g}_{tT} - \widehat{g}_{tT+\delta}}{\delta \widehat{g}_{tT+\delta}}\,, \tag{10.2.14}
$$

which is governed by the following SDE

$$
\mathrm{d}F_{tT}^{\delta} = \frac{1}{\delta}\left(1 + \delta F_{tT}^{\delta}\right)\left[\Theta_{tT}^{\delta}\right]^{\intercal}\left[\phi_{tT+\delta}\,\mathrm{d}t + \mathrm{d}\widehat{W}_t\right],
$$

for $t \in [0, T]$, where $\Theta_{tT}^{\delta} := \int_0^{\delta} \theta_{tT+v}\,\mathrm{d}v = \phi_{tT+\delta} - \phi_{tT}$.

Proof. Equations (10.2.13) and (10.2.14) follows easily from Proposition 10.2.1 and Theorem 10.1.1. Applying Itô's quotient rule to $\widehat{g}_{tT}/\widehat{g}_{tT+\delta}$ it follows that

$$\begin{aligned}\frac{\mathrm{d}\left(\widehat{g}_{tT}/\widehat{g}_{tT+\delta}\right)}{\widehat{g}_{tT}/\widehat{g}_{tT+\delta}} &= \frac{\mathrm{d}\widehat{g}_{tT}}{\widehat{g}_{tT}} - \frac{\mathrm{d}\widehat{g}_{tT+\delta}}{\widehat{g}_{tT+\delta}} + \frac{\mathrm{d}\langle \widehat{g}_{\cdot T+\delta}\rangle_t}{\widehat{g}^2_{tT+\delta}} - \frac{\mathrm{d}\langle \widehat{g}_{\cdot T}, \widehat{g}_{\cdot T+\delta}\rangle_t}{\widehat{g}_{tT}\widehat{g}_{tT+\delta}} \\ &= \left[\phi_{tT+\delta} - \phi_{tT}\right]^\intercal \mathrm{d}\widehat{W}_t + \phi^\intercal_{tT+\delta}\phi_{tT+\delta}\,\mathrm{d}t - \phi^\intercal_{tT+\delta}\phi_{tT}\,\mathrm{d}t \\ &= \left[\phi_{tT+\delta} - \phi_{tT}\right]^\intercal \left[\phi_{tT+\delta}\,\mathrm{d}t + \mathrm{d}\widehat{W}_t\right] ,\end{aligned}$$

using Remark 10.2.2. The SDE for the simple forward rate then follows since

$$\mathrm{d}F^\delta_{tT} = \frac{1}{\delta}\mathrm{d}\left(\frac{\widehat{g}_{tT}}{\widehat{g}_{tT+\delta}}\right) ,$$

and the definition of the quantity Θ^δ_{tT}, which completes the proof. □

Remark 10.2.4 (The T-forward measure)
Within the pricing kernel-based interest rate modelling framework, the process

$$\xi_t P_{tT} = \mathbb{E}^{\mathbb{M}}\left[\xi_T \,\middle|\, \mathcal{G}_t\right] = \widehat{g}_{tT} ,$$

for $t \in [0,T]$, *defines the Radon-Nikodym derivative on* $\mathcal{G}_t$ *given by*

$$\Phi_{tT} := \widehat{g}_{tT} = \left.\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{M}}\right|_{\mathcal{F}_t} = \exp\left(-\int_0^t \phi^\intercal_{sT}\mathrm{d}\widehat{W}_s - \frac{1}{2}\int_0^t \phi^\intercal_{sT}\phi_{sT}\,\mathrm{d}s\right) , \tag{10.2.15}$$

which induces the T*-forward measure* $\mathbb{Q}^T \sim \mathbb{M}$ *on* $[0,T]$, *provided that the Novikov condition*

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \phi^\intercal_{sT}\phi_{sT}\,\mathrm{d}s\right)\right] < \infty , \tag{10.2.16}$$

is satisfied. Under the $\mathbb{Q}^T$*-measure, the process* $W^{\mathbb{Q}}_{tT} := \widehat{W}_t + \int_0^t \phi_{sT}\,\mathrm{d}s = \int_0^t \left[\Theta_{sT} + \phi_s\right]\mathrm{d}s$ *is an* n*-dimensional column vector standard* $\{(\mathcal{G}_t), \mathbb{Q}^T\}$*-Wiener process. Similarly, the process*

$$\begin{aligned}\frac{\Phi_{tT}}{\Phi_t} &:= D_t P_{tT} = \left.\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{G}_t} \qquad (10.2.17)\\ &= \exp\left(-\int_0^t \left[\phi_{sT} - \phi_s\right]^\intercal \mathrm{d}W^{\mathbb{Q}}_s - \frac{1}{2}\int_0^t \left[\phi_{sT} - \phi_s\right]^\intercal \left[\phi_{sT} - \phi_s\right]\mathrm{d}s\right) \\ &= \exp\left(-\int_0^t \Theta^\intercal_{sT}\mathrm{d}W^{\mathbb{Q}}_s - \frac{1}{2}\int_0^t \Theta^\intercal_{sT}\Theta_{sT}\,\mathrm{d}s\right) ,\end{aligned}$$

induces the T*-forward measure* $\mathbb{Q}^T \sim \mathbb{Q}$ *on* $[0,T]$, *provided that the Novikov condition*

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\frac{1}{2}\int_0^T \Theta^\intercal_{sT}\Theta_{sT}\,\mathrm{d}s\right)\right] < \infty , \tag{10.2.18}$$

is satisfied. Under the $\mathbb{Q}^T$*-measure, the process* $W^{\mathbb{Q}}_{tT} := W^{\mathbb{Q}}_t + \int_0^t \Theta_{sT}\,\mathrm{d}s$ *is an* n*-dimensional column vector standard* $\{(\mathcal{G}_t), \mathbb{Q}^T\}$*-Wiener process.*

Market Factor Functional Specifications

Corollary 10.2.1 presented some generic features that the time-inhomogeneous market factor functional $g(x, t)$ must satisfy in order to preclude basic arbitrage, model degeneracy and negative interest rates. In this section, three specific classes of functionals are defined and proposed for use in specific model development. Note that this is by no means an exhaustive list of viable functional specifications.

Definition 10.2.1 (Nelson-Siegel-Svensson inspired functional)
The first class of functions, inspired by the term structure models proposed by Nelson and Siegel (1987) and the extension thereof proposed by Svensson (1994), takes the exponential form

$$g(x,t) := \exp\left[-h(x,t)t\right] \ ,$$

where $h : \mathbb{R}^m \times \mathbb{R}_{>0} \to \mathbb{R}$ *is defined by*

$$h(x,t) := h_1(x,t) + h_2(x)a(t) + h_3(x)b(t) + h_4(x)c(t) \ ,$$

with $h_1 : \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ *,* $h_2, h_3, h_4 : \mathbb{R}^m \to \mathbb{R}$ *and the deterministic functions defined by*

$$\begin{aligned} a(t) &= \frac{1}{\alpha t}\left[1 - e^{-\alpha t}\right] \ , \\ b(t) &= \frac{1}{\alpha t}\left[1 - e^{-\alpha t}\right] - e^{-\alpha t} \ , \ \textit{and} \\ c(t) &= \frac{1}{\beta t}\left[1 - e^{-\beta t}\right] - e^{-\beta t} \ , \end{aligned}$$

where the constants $\alpha, \beta > 0$ *and* $g(x, 0) := 1$*. Conditions (i) and (ii) of Corollary 10.2.1 are easily satisfied, however further functional output constraints are required to be imposed on* $h_1(x, t)$*,* $h_2(x)$*,* $h_3(x)$ *and* $h_4(x)$ *if one wishes to satisfy conditions (iii) and/or (iv).*

Definition 10.2.2 (Flesaker-Hughston inspired functional)
The second class of functions, inspired by the Flesaker and Hughston (1996a) martingale construction approach, has the general form

$$g(x,t) := \int_t^{\infty} h(x,u)\,\mathrm{d}u \ ,$$

where $h : \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ *. A discrete-time version of this functional is*

$$g(x,t) := \sum_{j=t/\delta}^{n} \delta h_j(x)$$

where $t \in \{0, \delta, 2\delta, \ldots, n\delta\}$*,* $n \in \mathbb{N}$ *and* $h_j : \mathbb{R}^m \to \mathbb{R}_{>0}$ *. The positivity constraints ensures that conditions (i), (ii) and (iv) of Corollary 10.2.1 are readily satisfied for both the continuous- and discrete-time versions of the functional.*

Definition 10.2.3 (LIBOR market model inspired functional)
The third class of functions, inspired by Brace et al. (1997)'s LIBOR market model, takes the form

$$g(x,t) = \prod_{j=0}^{t/\delta - 1} \frac{1}{1 + \delta h_j(x)} \ ,$$

where $t \in \{\delta, 2\delta, \ldots, n\delta\}$ *,* $n \in \mathbb{N}$ *and* $h_j : \mathbb{R}^m \to \mathbb{R}_{>0}$ *and* $g(x, 0) := 1$*. Conditions (i), (ii) and (iv) of Corollary 10.2.1 are readily satisfied again by the positivity output constraint imposed on each function* $h_j(x)$ *, for* $j \in \{0, 1, 2, \ldots, n-1\}$*. Of course, the major drawback here is tractability given the rational form with market factor functionals featuring in the denominator.*

As discussed in section 1.4.3, Appendix D offers a couple of specific examples of information-driven pricing kernel models based on the simplest version of the functional defined in Definition 10.2.1. The appendix concludes with an example of a prototype for a pricing kernel model that can adequately capture the effect of monetary policy decisions, along with the dynamics of long-term average interest rate behaviour, in a consistent manner, which is considered to be a novel application of the framework that has been constructed in this section.

10.3 Information-Based Forward Rate Models

Directly specifying the pricing kernel within the information filtering framework leads to a comprehensive interest rate model that yields implied forward rate models. However, the explicit specification of forward rate models is often useful for tractability in most practical applications. Analogous to Proposition 10.2.1, modulo modelling under the $\mathbb{P}$-measure here versus the $\mathbb{M}$-measure in the previous section, Theorem 9.3.1 and Remark 9.3.2 provide the necessary tools to propose what will be referred to as *information-driven forward rates.*

Proposition 10.3.1 (Information-driven instantaneous forward rates)
Assigning the filtration $\mathcal{I}$ *and measure* $\mathbb{S}$ *from Chapter 9 to the filtration* $\mathcal{G}$ *and measure* $\mathbb{P}$ *defined in this chapter, respectively, a suitable candidate for an instantaneous forward rate within the information filtering framework is*

$$f_{tT} := \mathbb{E}^{\mathbb{Q}^T}\left[g(X,T)\,\middle|\,\mathcal{G}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{\Phi_{TT}}{\Phi_{tT}}g(X,T)\,\middle|\,\mathcal{G}_t\right], \tag{10.3.1}$$

so that $f_{TT} = \mathbb{E}^{\mathbb{P}}\left[g(X,T)\,\middle|\,\mathcal{G}_T\right] = \widehat{g}_{TT}$ *, with* $\Phi_{tT} = \pi_t P_{tT} := \frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{G}_t}$ *, for* $t \in [0, T]$ *, such that*

$$\mathrm{d}f_{tT} = \phi_{tT}^{\intercal}\left[V_{tT} + \lambda_t\right]\mathrm{d}t + \phi_{tT}^{\intercal}\,\mathrm{d}\widehat{W}_t\,,$$

where $\phi_{tT} := v_t^{\intercal}\Sigma_t^{-1}\left[\widehat{gu}_{tT} - \widehat{g}_{tT}\widehat{u}_t\right]$ *and* $V_{tT} := \int_t^T \phi_{tu}\,\mathrm{d}u$*. The unspecified* n*-dimensional column vector process* $(\lambda_t)_{t\geq 0}$ *denotes the associated set of market prices of risk.*

Proof. From the definition of the instantaneous forward rate (10.3.1), it follows that

$$\Phi_{tT}f_{tT} = \mathbb{E}^{\mathbb{P}}\left[\Phi_{TT}\mathbb{E}^{\mathbb{P}}\left[g(X,T)\,\middle|\,\mathcal{G}_T\right]\,\middle|\,\mathcal{G}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\Phi_{TT}\widehat{g}_{TT}\,\middle|\,\mathcal{G}_t\right],$$

by the tower property of conditional expectations. Under the $\mathbb{P}$-measure, using Remark 9.3.2, each filtered market factor functional satisfies

$$\widehat{g}_{TT} = \widehat{g}_{tT} + \int_t^T \phi_{uT}^{\intercal}\,\mathrm{d}\widehat{W}_u\,,$$

where $(\widehat{W}_t)_{t\geq 0}$ is an n-dimensional standard $\{(\mathcal{G}_t), \mathbb{P}\}$-Wiener process. Considering the Wiener setting, the likelihood process $(\Phi_{tT})_{t\leq T}$ satisfies $\mathrm{d}\Phi_{tT} = -\,\Phi_{tT}\left[V_{tT} + \lambda_t\right]^{\intercal}\mathrm{d}\widehat{W}_t$, where V_{tT} is the n-dimensional column vector $\mathcal{G}_t$-adapted diffusion coefficient of the corresponding T-maturity ZCB, i.e.,

$\mathrm{d}P_{tT}/P_{tT} = (r_t - V_{tT}^{\intercal}\lambda_t)\,\mathrm{d}t - V_{tT}^{\intercal}\,\mathrm{d}\widehat{W}_t$. Then, under the $\mathbb{Q}^T$-measure

$$\widehat{g}_{TT} = \widehat{g}_{tT} - \int_t^T \phi_{uT}^{\intercal}\left[V_{uT} + \lambda_u\right]\mathrm{d}u + \int_t^T \phi_{uT}^{\intercal}\,\mathrm{d}W_{uT}^{\mathbb{Q}}\,,$$

where $W_{tT}^{\mathbb{Q}} = \widehat{W}_t + \int_0^t V_{uT} + \lambda_u\,\mathrm{d}u$ is an n-dimensional standard $\{(\mathcal{G}_t), \mathbb{Q}^T\}$-Wiener process. Then, taking expectations under $\mathbb{Q}^T$ conditional on $\mathcal{G}_t$, the instantaneous forward rate is given by

$$f_{tT} = \widehat{g}_{tT} - \mathbb{E}^{\mathbb{Q}^T}\left[\int_t^T \phi_{uT}^{\intercal}\left[V_{uT} + \lambda_u\right]\mathrm{d}u \,\middle|\, \mathcal{G}_t\right]$$

$$\Longleftrightarrow \int_0^t \mathrm{d}f_{sT} = \int_0^t \mathrm{d}\widehat{g}_{sT} - \int_0^t \mathrm{d}\left(\mathbb{E}^{\mathbb{Q}^T}\left[\int_s^T \phi_{uT}^{\intercal}\left[V_{uT} + \lambda_u\right]\mathrm{d}u \,\middle|\, \mathcal{G}_s\right]\right)\,,$$

since $\mathbb{E}^{\mathbb{Q}^T}\left[\int_t^T \phi_{uT}^{\intercal}\,\mathrm{d}W_{uT}^{\mathbb{Q}} \,\middle|\, \mathcal{G}_t\right] = 0$, with dynamics given by

$$\begin{aligned}
\mathrm{d}f_{tT} &= \mathrm{d}\widehat{g}_{tT} - \mathrm{d}\left(\mathbb{E}^{\mathbb{Q}^T}\left[\int_t^T \phi_{uT}^{\intercal}\left[V_{uT} + \lambda_u\right]\mathrm{d}u \,\middle|\, \mathcal{G}_t\right]\right) \\
&= \mathbb{E}^{\mathbb{Q}^T}\left[\phi_{tT}^{\intercal}\left[V_{tT} + \lambda_t\right] \,\middle|\, \mathcal{G}_t\right]\mathrm{d}t + \phi_{tT}^{\intercal}\,\mathrm{d}\widehat{W}_t \\
&\overset{d}{=} \phi_{tT}^{\intercal}\left[V_{tT} + \lambda_t\right]\mathrm{d}t + \phi_{tT}^{\intercal}\,\mathrm{d}\widehat{W}_t\,,
\end{aligned}$$

which follows after applying Fubini's Theorem. Finally, in order to avoid arbitrage the T-maturity ZCB's diffusion coefficient V_{tT} must be equal to $\int_t^T \phi_{tu}\,\mathrm{d}u$, by the classical HJM drift condition. □

It is now possible to articulate the form of the pricing kernel implied by the set of information-driven instantaneous forward rates. This is summarised in the following remark.

Remark 10.3.1 (Information-driven pricing kernel)
From equation (10.3.1), in Proposition 10.3.1, it should be clear that $r_t = \widehat{g}_{tt} = \mathbb{E}^{\mathbb{P}}\left[g(X,t) \,\middle|\, \mathcal{G}_t\right]$, *by definition, for all* $t \geq 0$. *Since the short rate* $r_t = f_{tt}$, *by definition, the stochastic discount factor is given by*

$$D_t = \exp\left(-\int_0^t r_s\,\mathrm{d}s\right) = \exp\left(-\int_0^t \widehat{g}_{ss}\,\mathrm{d}s\right)\,,$$

for $t \geq 0$. *Since the likelihood process* $(\Lambda_t)_{t\geq 0}$ *has not been explicitly specified but has the general form* $\mathrm{d}\Lambda_t/\Lambda_t = -\lambda_t\,\mathrm{d}\widehat{W}_t$, *it follows that*

$$\mathrm{d}\pi_t = -\widehat{g}_{tt}\pi_t\,\mathrm{d}t - \lambda_t\pi_t\,\mathrm{d}\widehat{W}_t\,,$$

is now the general form of the pricing kernel, with the short rate process satisfying

$$\mathrm{d}\widehat{g}_{tt} = \widehat{\partial_t g}_{tt}\,\mathrm{d}t + \left[\widehat{gu}_{tt} - \widehat{g}_{tt}\widehat{u}_t\right]^{\intercal}\Sigma_t^{-1}v_t\,\mathrm{d}\widehat{W}_t\,,$$

i.e., equation (9.3.6), for $t \geq 0$. *Therefore, the short rate is governed by a parametric stochastic process and interest rates are positive (non-negative) if* $g : \mathbb{R}^m \to \mathbb{R}_{>0}$ $(\mathbb{R}_{\geq 0})$.

In a manner similar to Proposition 10.3.1, it is possible to define a procedure for creating *information-driven simple forward rate* models — this is the objective for the next proposition.

Proposition 10.3.2 (Information-driven simple forward rates)
Assigning the filtration $\mathcal{I}$ and measure $\mathbb{S}$ from Chapter 9 to the filtration $\mathcal{G}$ and measure $\mathbb{P}$ defined in this chapter, respectively, a suitable candidate for a simple forward rate, with accrual period equal to δ, within the information filtering framework is

$$F^{\delta}_{tT} := \mathbb{E}^{\mathbb{Q}^{T+\delta}}\left[g(X,T)\,\middle|\,\mathcal{G}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{\Phi_{TT+\delta}}{\Phi_{tT+\delta}}g(X,T)\,\middle|\,\mathcal{G}_t\right], \tag{10.3.2}$$

for $T \in \{t_0, t_1, \dots\}$ as defined in Corollary 10.2.4, so that $F^{\delta}_{TT} = \mathbb{E}^{\mathbb{P}}\left[g(X,T)\,\middle|\,\mathcal{G}_T\right] = \widehat{g}_{TT}$, with $\Phi_{tT+\delta} = \pi_t P_{tT+\delta} := \frac{\mathrm{d}\mathbb{Q}^{T+\delta}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{G}_t}$, for $t \in [0,T]$, such that

$$\mathrm{d}F^{\delta}_{tT} = \phi^{\intercal}_{tT}\left[V_{tT+\delta} + \lambda_t\right]\mathrm{d}t + \phi^{\intercal}_{tT}\,\mathrm{d}\widehat{W}_t\,,$$

where $\phi_{tT} := v^{\intercal}_t \Sigma^{-1}_t\left[\widehat{gu}_{tT} - \widehat{g}_{tT}\widehat{u}_t\right]$ and

$$V_{tT+\delta} = \sum_{i=j}^{k}\frac{\delta\phi_{tt_i}}{1+\delta F^{\delta}_{tt_i}}\,,$$

is the n-dimensional column vector $\mathcal{G}_t$-adapted diffusion coefficient associated with $P_{tT+\delta}$, with $t \in [t_{j-1}, t_j)$ and $T = t_k$ where $1 \le j \le k$. The unspecified n-dimensional column vector process $(\lambda_t)_{t\ge0}$ denotes the associated set of market prices of risk.

Proof. The proof is almost identical to Proposition 10.3.1 except that the discrete rate setting precludes one from directly invoking the HJM condition and inferring the instantaneous volatility structure associated with the $(T+\delta)$-maturity ZCB, $P_{tT+\delta}$.
Given $t = t_{j-1}$ and $T = t_k$, then

$$P_{t_{j-1}T+\delta} = \frac{1}{\left(1+\delta R_{t_{j-1}}\right)}\prod_{i=j}^{k}\frac{1}{\left(1+\delta F^{\delta}_{tt_i}\right)}\,,$$

however for $t \in (t_{j-1}, t_j)$, one needs to articulate continuous interest accrual which is achieved in the continuous rate setting via the bank account $(B_t)_{t\ge0}$ (also the inverse of the stochastic discount factor $(D_t)_{t\ge0}$). Here, as in Brigo and Mercurio (2006), the discretely compounded bank account

$$B_t := \left[1 + (t - t_{j-1})R_{t_{j-1}}\right]\prod_{i=0}^{j-2}\left(1+\delta R_{t_i}\right)$$

for $t \in [t_{j-1}, t_j)$, with associated implied short rate $r_t := R_{t_{j-1}}\big/\left[1+(t-t_{j-1})R_{t_{j-1}}\right]$. Then, one may define the continuous time $(T+\delta)$-maturity ZCB price as $P_{tT+\delta} := B_t P_{t_{j-1}T+\delta}\big/B_{t_{j-1}}$. Following similar arguments to Proposition 10.3.1, the SDE that governs $\left(F^{\delta}_{tt_i}\right)_{t\le t_i}$ is

$$\mathrm{d}F^{\delta}_{tt_i} = \phi^{\intercal}_{tt_i}\left[V_{tt_{i+1}} + \lambda_t\right]\mathrm{d}t + \phi^{\intercal}_{tt_i}\mathrm{d}\widehat{W}_t\,,$$

under the $\mathbb{P}$-measure. The $(T+\delta)$-maturity ZCB's SDE then follows via an application of Itô's formula to $P_{tT+\delta} := B_t P_{t_{j-1}T+\delta}\big/B_{t_{j-1}}$, which yields

$$\begin{aligned}\frac{\mathrm{d}P_{tT+\delta}}{P_{tT+\delta}} = {} & \left(r_t - \delta\sum_{i=j}^{k}\frac{\phi^{\intercal}_{tt_i}\lambda_t}{1+\delta F_{tt_i}}\right)\mathrm{d}t - \delta\sum_{i=j}^{k}\frac{\phi^{\intercal}_{tt_i}\mathrm{d}\widehat{W}_t}{1+\delta F_{tt_i}} \\ & + \delta^2\sum_{i=j}^{k}\sum_{h=j}^{i}\frac{\phi^{\intercal}_{tt_i}\phi_{tt_h}}{\left(1+\delta F_{tt_i}\right)\left(1+\delta F_{tt_h}\right)}\mathrm{d}t - \delta\sum_{i=j}^{k}\frac{\phi^{\intercal}_{tt_i}V_{tt_{i+1}}}{1+\delta F_{tt_i}}\mathrm{d}t\,.\end{aligned}$$

In order to preclude arbitrage, i.e., ensure that $(D_t P_{tT+\delta})_{t\in[t_{j-1},t_j)}$ is a $\{(\mathcal{G}_t),\mathbb{Q}\}$-martingale process, the sum of the last two terms in the SDE above must equal zero. This is easily achieved by setting

$$V_{tt_{i+1}} := \sum_{h=j}^{i} \frac{\delta\phi_{tt_h}}{1+\delta F_{tt_h}} \ ,$$

for $i \in \{j, j+1, \ldots, k\}$, which completes the proof. □

Remark 10.3.2 (Information-driven pricing kernel)
From equation (10.3.2), in Proposition 10.3.2, it should be clear that $R_t = \widehat{g}_{tt} = \mathbb{E}^{\mathbb{P}}\left[g(X,t)\,\middle|\,\mathcal{G}_t\right]$ *, by definition, for all* $t \in \{t_0, t_1, \ldots\}$. *Since the short rate* $r_t = \partial_t \ln\left[1+(t-t_{i-1})R_{t_{i-1}}\right]$ *, for* $t \in [t_{i-1}, t_i)$ *by definition, the stochastic discount factor is given by*

$$D_t = \exp\left(-\int_0^t r_s \, \mathrm{d}s\right) = \frac{1}{\left(1+(t-t_{j-1})\widehat{g}_{t_{j-1}t_{j-1}}\right)} \prod_{i=0}^{j-2} \frac{1}{\left(1+\delta\widehat{g}_{t_it_i}\right)} \ ,$$

for $t \in [t_{j-1}, t_j)$ *and* $j \in \mathbb{N}$. *Since the likelihood process* $(\Lambda_t)_{t\geq 0}$ *has not been explicitly specified but has the general form* $\mathrm{d}\Lambda_t/\Lambda_t = -\lambda_t\,\mathrm{d}\widehat{W}_t$ *, it follows that*

$$\mathrm{d}\pi_t = -r_t\pi_t\,\mathrm{d}t - \lambda_t\pi_t\,\mathrm{d}\widehat{W}_t \ ,$$

is now the general form of the pricing kernel, with the simple rate process now being governed by

$$\mathrm{d}\widehat{g}_{tt} = \widehat{\partial_t g}_{tt}\,\mathrm{d}t + \left[\widehat{gu}_{tt} - \widehat{g}_{tt}\widehat{u}_t\right]^{\intercal} \Sigma_t^{-1} v_t\,\mathrm{d}\widehat{W}_t \ ,$$

i.e., equation (9.3.6), for $t \geq 0$. *Therefore, the simple rate is governed by a parametric stochastic process and interest rates are positive (non-negative) if* $g : \mathbb{R}^m \to \mathbb{R}_{>0}$ $(\mathbb{R}_{\geq 0})$.

Chapter 11

Conclusion

Market microstructure related to interest rate financial markets has undergone a paradigm shift in the aftermath of the GFC and reference rate reform. Of particular interest to the author has been the resultant dislocation between market-making processes within related primitive and derivative markets, with the main example of this being the bank funding market. Prior to the GFC, when IBORs were considered to be near risk-free rates, banks engaged in IBOR-based term funding transactions amongst themselves actively, while simultaneously market-making active and liquid derivatives markets related thereto. The high-level of activity within these markets and the prevalence of funding-swap duality meant that there was a large degree of coherence, consistency and corroboration between the market-making processes of traders associated with the TR and ST — in technical terms, both sets of traders were focused on constructing the same numeraire.

Post the GFC, interbank term funding activity declined substantially while overnight funding activity rose in prominence, due to the manifestation of material term-related liquidity and credit risks. With this change, the natural numeraire applicable to the derivatives market transitioned to one that is derived from the OIS market — the new source for near risk-free term rates. One may therefore argue that consistency in market-making once again prevailed within the interbank funding primitive and derivatives markets. Non-interbank funding activity, on the other hand, will always be dominated by term-related activity, since non-interbank market participants should always rationally seek exposure to term-related liquidity and credit risk premia. Therefore, traders in this segment of the bank funding market will be faced with a risky set of numeraires, and a completely different paradigm of risk considerations compared to those involved in the OIS-related interbank funding market. There are two scenarios where the set of non-interbank funding risks manifest in the derivative market-maker's purview, and these are: (i) when market-making derivatives that reference bank funding term rates; and (ii) when having to deal with liquidity and counterparty credit-related valuation adjustments. Since activity (ii) is highly bespoke, specialised, and idiosyncratic, it is difficult to compare; however, activity (i) may be compared to the role of a TR market-maker within the non-interbank funding market. This serves as a summary of the primary motivation for the research that has been undertaken in Parts I and II. For the most part, the theory that has been developed, i.e., the *market-based approach* and the *xy-formalism*, and the results applicable therefrom, i.e., the *exchange of risk mechanism* and the *curve-conversion factor process*, offers a deep understanding of the fundamental and technical underpinnings of liquidity and credit risky primitive interest rate markets.

While the replication of derivative instruments enables the development of the aforementioned theory, the direct translation and application of these results to the context of actual/practical derivative market-making is not possible — this is largely due to the fact that derivative instruments and markets still function within the realm of a single (risk-free) numeraire. For instance, consider a FRA that references a risky term rate, therefore the floating cash flow is effectively determined by a risky numeraire. However, the fair FRA rate associated with such a FRA is engineered to be a martingale process under the T-forward measure associated with the single (risk-free) numeraire. In other words, key derivative pricing variables/quantities are considered more like abstract benchmarks with no recourse to their true fundamental or primitive nature. These issues are discussed at length in Chapters 5 and 6.

To summarise, one may conjecture that the only way to achieve coherence between the market-making processes of TR and ST traders is if the classical replication of derivative securities were again viable. This is definitely not possible within the post-GFC risky market context. From a technical perspective, this also means that a TR market-maker who estimates a model built within the market-based approach will not be able to offer the same model to their ST counterpart for the purposes of derivative market-making. Therefore, as demonstrated and discussed in Part II, the ST trader may construct and calibrate their own model either within the market-based or xy-approach for the purpose of their particular market-making process. There is however one exception to this, where it is possible to use the same model for both applications — this was described in section 6.5, however this requires one to adjust the payoffs of fundamental interest rate derivatives, viz., FRAs, IRSs and TBSs.

As described in section 1.4.3, the main motivation for reformulating the classical information-based asset pricing framework within the stochastic filtering theory framework, in Part III, was to have access to the vast array of results that has been derived within this well established research field. In doing so though, it was also possible to consider a wider class of information processes, compared to the original version of the framework that was first introduced by Macrina (2006). This enables the incorporation of market factor functionals that are never revealed, which in turn enables the construction of infinite horizon interest rate models. Apart from this, the most important result is derived in Theorem 10.1.1, which is model and framework agnostic, and offers the blueprint through which the information filtering framework may be used to generate interest rate models. The resultant information-based pricing kernel and forward rate modelling frameworks appear to offer promising features — in particular, the ability to model monetary policy decisions in an explicit and intuitive manner advocates the need for further research and development, in this direction. For example, this framework may offer the necessary structure and functionality to model the dynamics of ONRRs in an effective yet simple manner.

The primary objectives for this thesis was the theoretical development of economically meaningful and theoretically consistent multi-curve frameworks, along with a flexible and tractable version of the information-based asset pricing framework for general interest rate model development. It is the author's opinion that these objectives have been satisfactorily achieved, with a fair level of technical depth and rigour. Therefore, it is again the author's opinion that further theoretical development of the presented frameworks would not be a worthwhile endeavour. Rather, and as a general comment on further research, the author believes that the constructed frameworks offer numerous avenues and opportunities for specific model development for a variety of practical and empirical applications.

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Glossary

BSB Buy/Sell-Back. 6, 8, 22

BFRR Bank Funding Reference Rate. v, 6, 7, 9, 10, 15, 23, 24, 25, 35, 42, 52, 53, 55, 84, 98, 99, 100, 101, 102, 108, 140, 141, 193

CB Central Bank. 46, 47

CBE Conventional Banking Entity. 45, 47, 50, 51, 52, 77

CCBS Cross-Currency Basis Swap. 7, 139, 140

CDS Credit Default Swap. 23

CIR Cox-Ingersoll-Ross. 124

COLVA Collateral Valuation Adjustment. 13, 193, 194

CP Commercial Paper. 6

CPFL Complete present and future liquidity. 68, 69, 70, 87

CPL Complete present liquidity. 68, 69, 70, 71, 72, 73, 78, 80, 86

CSA Credit Support Annex. 112, 192, 193, 194

CVA Credit Valuation Adjustment. 13, 14, 193, 194

DVA Debt Valuation Adjustment. 13, 14, 193, 194

€STR Euro Short-Term Rate. 11

EURIBOR Euro Interbank Offered Rate. 10

EB Borrowers. 45, 50, 51

EC External Clients. 45, 46, 91, 93

ED Depositors. 45, 51, 52, 53, 54, 56

EPU End Product User. 45, 46

FCB Fixed Coupon Bond. 6, 7, 30

FCD Fixed Coupon Deposit. 6, 12

FCF Forward Capitalisation Factor. 116, 117, 127, 128, 129

FEC Forward Exchange Contract. 7, 139, 140, 141, 142

FKK Fujisaki-Kallianpur-Kunita. 153

FRA Forward Rate Agreement. 4, 7, 8, 9, 11, 42, 52, 53, 54, 55, 56, 59, 97, 98, 100, 101, 102, 109, 111, 112, 113, 114, 115, 116, 117, 118, 119, 121, 122, 123, 129, 130, 131, 133, 139, 142, 143, 178, 203

FRLD Floating Rate-Linked Deposit. 6, 12

FRN Floating Rate Note. 6

FTD Fixed-Term Deposit. iv, v, 6, 8, 12, 16, 25, 33, 34, 35, 48, 49, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 91, 92, 93, 94, 95, 96, 98, 99, 101, 109, 195, 197, 198

FTDFRA Fixed-Term Deposit Forward Rate Agreement. 53, 54, 55, 56, 62, 64, 65, 70, 71, 72, 73, 76, 78, 80, 81, 82, 83, 84, 85, 86, 88, 95, 96, 97, 98, 99, 114, 199, 200

FTDPK Fixed-Term Deposit-Linked Pricing Kernel. vi, 59, 60, 61, 62, 63, 64, 65, 68, 69, 70, 74, 80, 82, 87, 89, 90, 99, 109, 195, 196, 200

FTDRR Fixed-Term Deposit Reference Rate. 23, 24, 25, 35, 48, 49, 50, 51, 52, 53, 55, 57, 58, 59, 61, 62, 64, 65, 66, 67, 69, 71, 73, 96, 99, 101, 198

FTDSDF Fixed-Term Deposit-Linked Stochastic Discount Factor. vi, 57, 58, 59, 60, 61, 62, 63, 64, 69, 75, 76, 87, 195, 198

FTDZCB Fixed-Term Deposit-Linked Zero Coupon Bond. 57, 59, 60, 61, 62, 63, 64, 72, 73, 74, 75, 76, 87, 92, 93, 97, 99, 195

FVA Funding Valuation Adjustment. 13, 193, 194

FX Foreign Exchange. v, 18, 107, 117, 124, 125, 131, 132, 139, 140, 141, 142, 143, 144

FXS Foreign Exchange Swap. 9, 140

GFC 2007/2008 Global Financial Crisis. iv, 4, 5, 7, 8, 9, 10, 11, 13, 15, 16, 23, 24, 30, 33, 34, 36, 42, 44, 52, 54, 55, 58, 98, 104, 111, 112, 113, 114, 119, 125, 177, 178, 192

HJM Heath-Jarrow-Morton. v, 18, 124, 125, 126, 127, 128, 129, 130, 170, 174, 175

IBOR Interbank Offered Rate. 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 127, 128, 129, 130, 131, 133, 142, 177, 203

IL Inflation-Linked. 138, 139, 143, 144

ILB Inflation-Linked Bond. 6

IRF Interest Rate Futures Contract. 5, 7, 11

IRS Interest Rate Swap. 4, 5, 7, 8, 11, 52, 59, 112, 118, 119, 120, 122, 123, 139, 178, 203, 204

ISDA International Swaps and Derivatives Association. 192

JIBAR Johannesburg Interbank Average Rate. 10, 15, 24

KS Kushner-Stratonovich. 153

LCFTDPK Liquidity-Contingent Fixed-Term Deposit-Linked Pricing Kernel. 73, 74, 75, 76, 82, 84, 85, 86, 87, 89, 90, 94, 99, 100, 109

LCFTDZCB Liquidity-Contingent Fixed-Term Deposit-Linked Zero Coupon Bond. 73, 74, 75, 76, 84, 85, 86, 87, 88, 89, 90, 99

LDFTDPK Liquidity-Dependent Fixed-Term Deposit-Linked Pricing Kernel. 67, 68, 69, 70, 71, 78, 79, 80, 200

LIBOR London Interbank Offered Rate. 4, 10, 18, 124, 125, 127, 172

LMM LIBOR Market Model. v, 124, 125, 127, 129, 130, 132

LRTS Linear-Rational Term Structure. 134, 135, 136, 137

MVA Margin Valuation Adjustment. 13, 54, 193, 194

NCD Negotiable Certificate of Deposit. 6, 12, 24

NPFL No present nor future liquidity. 68, 69

NPL No present liquidity. 68, 69, 72

ONRR Overnight Reference Rate. 5, 6, 11, 17, 23, 24, 25, 27, 29, 36, 37, 39, 55, 96, 111, 112, 115, 140, 141, 178, 193

OIS Overnight Indexed Swap. 5, 7, 11, 27, 28, 29, 36, 37, 38, 39, 110, 111, 112, 113, 116, 120, 121, 123, 124, 125, 129, 130, 131, 138, 141, 142, 177, 204, 208

OTC Over-the-Counter. 6, 192, 193

PDE Partial Differential Equation. 155

PK Pricing Kernel. 25, 28, 29, 30, 32, 34, 35, 36, 37, 41, 42, 70, 84, 92, 100, 101, 109

PM Primary Market. 14

PPL Partial present liquidity. 68, 69

PPAS Par-Par Asset Swap. 9

SARB South African Reserve Bank. 10

SDE Stochastic Differential Equation. 134, 135, 147, 148, 149, 152, 153, 154, 155, 159, 161, 168, 169, 170, 171, 175, 176

SDF Stochastic Discount Factor. 25, 26, 27, 29, 30, 31, 32, 33, 40, 41, 42, 57, 58, 109, 195

SDI Systemic Default Indicator. 76, 77, 78

SER Spot Exchange Rate. 7

SIB Systemically Important Bank. 45, 47

SIFI Systemically Important Financial Institution. 45, 47, 48

SLCFTDPK Survival and Liquidity-Contingent Fixed-Term Deposit-Linked Pricing Kernel. 82, 83, 84, 85, 86, 87, 90, 91, 92, 94, 95, 100, 109

SLCFTDZCB Survival and Liquidity-Contingent Fixed-Term Deposit-Linked Zero Coupon Bond. 82, 83, 84, 85, 86, 87, 88, 89, 95, 99

SLI Systemic Liquidity Indicator. 66, 67, 70, 77, 78, 80

SM Secondary Market. 14

SOFR Secured Overnight Financing Rate. 11, 22

SONIA Sterling Overnight Index Average. 11

SST Systemically Important Bank Sales & Trading Units. 45, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 58, 59, 60, 61, 62, 64, 65, 66, 67, 68, 69, 70, 71, 72, 76, 77, 78, 87, 99, 100, 101, 197, 198

ST Sales & Trading Unit. 45, 46, 47, 72, 86, 177, 178

STR Systemically Important Bank Treasuries. 45, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 60, 61, 64, 65, 66, 67, 70, 72, 78, 86, 91, 92, 93, 100, 101, 197, 198

TB Treasury Bill. 6

TBRR Term-Based Reference Rate. 5, 6, 10, 11, 17, 55

TBS Tenor Basis Swap. 4, 7, 119, 120, 123, 139, 178

TIBOR Tokyo Interbank Offered Rate. 10

TN Treasury Note. 6

TR Treasury. 45, 47, 177, 178

USD United States Dollar. 139, 140

WHK Weighted Heat Kernel. 134, 136, 137

ZARONIA South African Overnight Index Average. 11

ZCB Zero Coupon Bond. 26, 27, 28, 30, 31, 32, 33, 34, 35, 36, 41, 54, 57, 58, 59, 73, 82, 84, 98, 99, 100, 101, 106, 108, 110, 111, 114, 116, 117, 118, 119, 121, 122, 123, 126, 127, 128, 129, 132, 133, 134, 135, 136, 139, 164, 165, 166, 169, 173, 174, 175, 203, 204, 206, 208, 209

Appendix A

Appendix for the Introduction

A.1 Market-Maker versus Maket-Taker

The figure below, Figure A.1, provides a stylised depiction of the operation, function and interactions of a market-maker at a generic financial services entity. The market-making function usually comprises traders and quantitative analysts who are highly knowledgable of all the fundamental and technical aspects of the relevant assets classes that are under consideration. These individuals are also adept at all of the administrative and operational aspects that are involved in the trade execution, risk management and settlement processes.

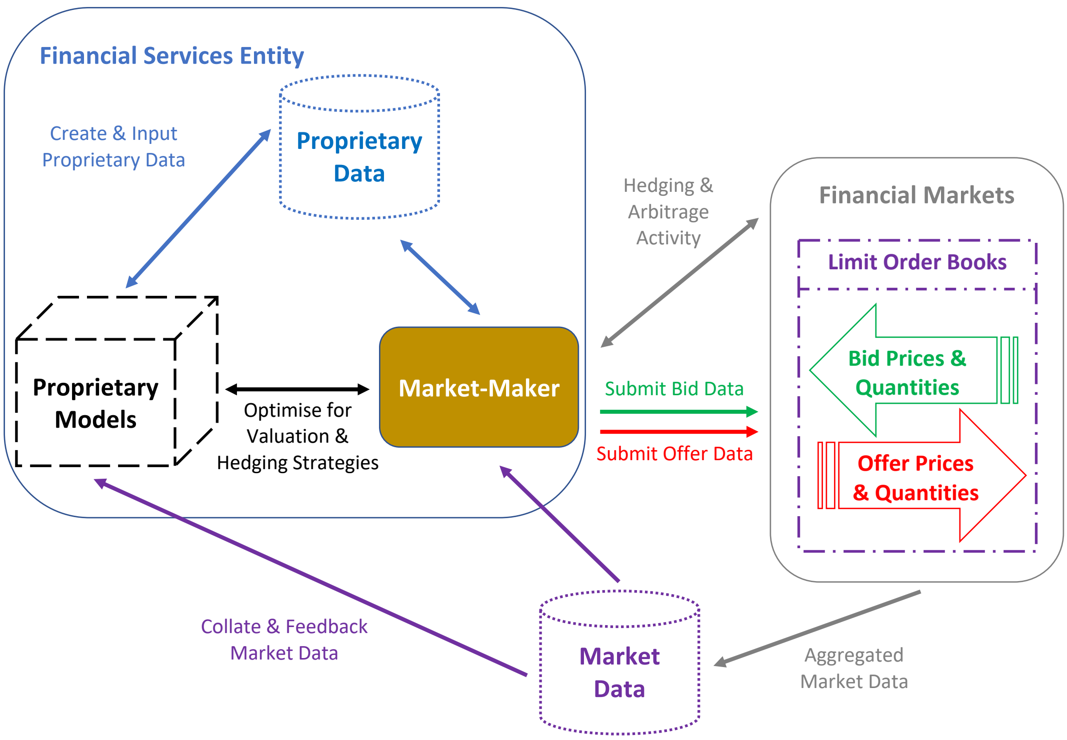

Fig. A.1: A stylised perspective of a market-maker at a generic financial services entity.

In the formative stage of a market for a new asset, the task of the market-making function is to utilise limited and incomplete information to conjecture and design proprietary models and related data that best describes the potential dynamics of the respective asset - this describes the interactions between the “Market-Maker”, “Proprietary Data” and Proprietary Models” objects in Figure A.1. Given the problem, the proprietary models developed here will generally be incomplete market models that require resolution methods which yield valuations, and associated hedging strategies, that are arbitrage-free and optimal with respect to the chosen model. Technically, for a given model, this is the same as finding an equivalent (local) martingale measure subject to an optimality condition, based on a utility function or a quantitative profitability criteria (all specified to be commensurate with risk appetite), being satisfied. Once the market-maker is satisfied with this modelling process, along with all of the theoretical and practical nuances, this enables the interactions between the “Market-Maker” and the respective “Financial Market”. In an electronic trading setting where the limit-order book is available digitally *on-screen*, the market-maker will submit bid and offer prices and associated quantities in real-time. In a *request-for-quote* setting where the limit-order book is only available upon request via a broker network, the market-maker will submit bid and offer prices upon request from end-users, or market-takers. Once this mechanism has been established, with an active group of market-makers and market-takers, one may assume that the financial market will mature and become efficient, with the aggregated market data providing a fair representation of equilibrium arbitrage-free price formation and dynamics. This market data then becomes an important input into each market makers' pricing process, which explains the interactions between the “Financial Markets”, “Market Data”, “Market-Maker” and “Proprietary Models”.

Ignoring the constraint of portrait presentation, it would have been best for Figure A.2 to be partially overlaid on the right hand side of Figure A.1 in order to provide a complete picture of the interactions between market-makers and -takers.

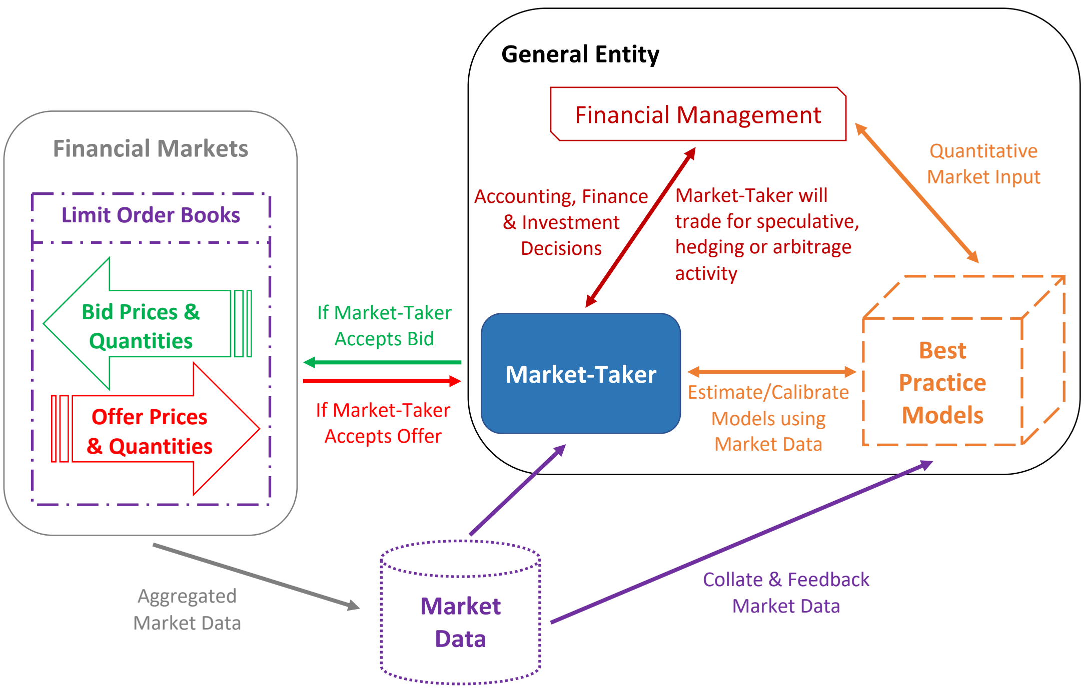

Fig. A.2: A stylised perspective of a market-taker at a generic entity.

Within a generic entity, the market-taker function usually forms part of the *treasury operation* and consists of individuals who are adequately skilled in quantitative finance, in order to understand pricing and valuation, and highly skilled in trade execution, settlement and enterprise wide risk management processes. If the generic entity happens to be an investment management firm, then market-takers are portfolio managers and traders who have a deep technical and fundamental understanding of all aspects of the assets under consideration.

With respect to a generic entity, the focus on enterprise wide risk management is fundamental, since the treasury operation will interact with senior management regarding financial risk management decisions strategies - this explains the interactions between the "Market-Taker" and "Financial Management" in Figure A.2. Once potential strategies and required assets for the implementation thereof have been decided upon, the market-taker may access available technical research to develop best practice models, and access available market data to estimate or calibrate these models. Technically, this will entail the construction of an arbitrage-free model, with the associated parameters estimated (calibrated) from market data if the asset under consideration is primitive (a derivative). This enables a deep quantitative understanding of the dynamics of the value of the assets under consideration, which is another valuable input into the financial and operational risk management processes. This explains the interactions between the "Market-Taker", "Market Data", "Best Practice Models" and "Financial Management".

Once the specific strategy and required transactions have been agreed upon, the market-taker will then participate directly via an electronic on-screen financial market enabled by market-makers, or interact off-screen with market-makers via a request-for-quote mechanism that is usually facilitated by a broker. This describes the interactions between the "Market-Taker" and the respective "Financial Market", i.e., if the market-taker would like to buy (sell) a particular asset and prevailing market-maker offer (bid) quotes are in-line with budgetary targets, then the market-taker will accept the respective best quote and a transaction will be concluded. There is some leeway for negotiation, i.e., the market-taker may choose not to participate, reject the quote or indicate an acceptable quote, however market-makers are not obligated to react.

A.2 Risk Categories and Characteristics

A.2.1 Liquidity Risk Valuation Adjustments for Derivatives

The general market-making process for derivative securities exposes traders to *market-liquidity risks* akin to those described for primitive assets. These risks are dependent on the risk position and appetite of the respective market-makers along with the state of relevant financial markets. This category of risk has not changed post the GFC. On the other hand, the transition of bank capital and funding costs from essentially risk-free levels prior the GFC to risk-free plus significant liquidity- and credit-risky levels post the GFC has had a substantial impact on the cost of hedging and replication strategies. This has in turn demanded a transition from risk-free to risky pricing and valuation of derivatives securities. Risky valuation frameworks now account for explicit liquidity- and credit-related hedging costs over and above the standard and classical risk-free cost of hedging a derivative security.

There are various manifestations of liquidity-related costs for banking entities that market-make derivative securities. The main features that determine the nature of these costs are whether transactions are dealt OTC and subject to *Credit Support Annexes*[1] (CSAs) or whether they are *cleared* (either *centrally* or *non-centrally*)[2]. Most client or end-user trades that are undertaken and enabled by banks are either

[1] A CSA forms part of an ISDA Master Agreeement and stipulates the specific terms of the collateral protocol that must be followed by both counterparties involved in a derivative security transaction.

[2] The process of "*clearing*" refers to a credit risk mitigation operational process whereby a third-party, the "*clearer*", is involved in a derivative transaction in order to play the role of an intermediary that minimises counterparty credit risk. The clearer

uncollateralised or subject to a CSA, but banks generally execute hedge trades with other financial services entities that generally require either a CSA or a clearing protocol. Table A.1 below summarises potential liquidity-related valuation adjustments that may arise when a client transaction is either uncollateralised, subject to a CSA, or cleared, and the corresponding hedge transaction is either subject to a CSA or cleared. For the sake of brevity, it is assumed that the terms of the transactions, CSAs and clearing protocols are perfectly matched for both client and hedge transactions in scenarios that match. Therefore, all risks are perfectly eliminated in the *CSA-CSA* and *Cleared-Cleared* scenarios. The specific valuation adjustments are briefly defined below.

The *funding valuation adjustment* (FVA[3]) is the expected marginal cost (benefit) that may be incurred (derived) from sourcing (offering) funding at a rate above the relevant ONRR, i.e., the benchmark BFRR that is also used in CSAs, when satisfying the CSA's variation in mark-to-market requirements. This process of settling the variation in mark-to-market is effectively equivalent to variation margin under a clearing protocol, which is why there is also an FVA under the *Uncollateralised-Cleared* pairing.

	Hedge Transaction	
Client Transaction	*CSA*	*Cleared*
Uncollateralised	FVA, COLVA	FVA, MVA
CSA		COLVA, MVA
Cleared	MVA, COLVA	

Tab. A.1: Types of liquidity-related valuation adjustments for different trade configurations.

The *collateral valuation adjustment* (COLVA)) mainly refers to the case when CSAs allow for multi-currency or -security collateral, which may also lead to an embedded type of optionality problem since one may determine the cheapest type of collateral to deliver at any point in time. Here, COLVA is also used as an adjustment that captures the expected marginal cost or benefit that arises due to differences in criteria to settle variations in market-to-market under the CSA and clearing protocols. Therefore, COLVA appears under the *Cleared-CSA* and *CSA-Cleared* pairings. The *margin valuation adjustment* (MVA) is similar to the FVA cost, except that it is the expected marginal cost that may be incurred from funding initial margin and variations thereof through the life of a cleared transaction, when funding is sourced at a rate above the relevant ONRR.

A.2.2 Credit Risk Valuation Adjustments for Derivatives

The case of primitive assets contrasts significantly with derivative securities where a market-maker (buyer or seller) and market-taker (seller or buyer) may hold a position with negative, zero or positive value. Hence, within the context of derivative securities, one needs to consider credit risk from a bilateral perspective. The quantification of these credit risks requires the notions of CVA and DVA, which are briefly defined in the next paragraph for completeness.

From the perspective of a market-maker at a bank, CVA quantifies an expectation of the current value that they could lose from their counterparty (market-taker) defaulting at some point during the life of the transaction (or set of transactions) — naturally, this computation depends on all future states of the relevant market where the transaction (or set of transactions) has positive value to the market-maker. Conversely, DVA quantifies an expectation of the current value that the counterparty (market-taker)

achieves this by stipulating, monitoring and maintaining collateral agreements with each counterparty in the form of trade or portfolio-level margining processes — both *initial* and *variation margin*. For standardised OTC transactions, it is possible for all counterparties to use a central clearer; however, bespoke OTC transactions typically require bespoke non-centralised clearers. All exchanged-traded derivative security transactions are also centrally cleared, by default.

[3] The application of FVA has been rather contentious from a practical perspective — the following timeline of events by *Risk.net* offers great insight into the matter: FVA: the story so far.

could lose if the market-maker were to default at some point during the life of the transaction (or set of transactions). Table A.1 may be updated to include these valuation adjustments, as shown below.

	Hedge Transaction	
Client Transaction	*CSA*	*Cleared*
Uncollateralised	FVA, COLVA, CVA, DVA	FVA, MVA, CVA, DVA
CSA		COLVA, MVA
Cleared	MVA, COLVA	

Tab. A.2: Types of credit- and liquidity-related valuation adjustments for various trade configurations.

In the above table, the applicable CSAs and clearing protocols are assumed to be the most stringent versions, i.e., specifications that preclude all counterparty credit-related risks. In reality, CSAs may still expose counterparties to some residual credit risks. Finally, it is worth noting that the inclusion of DVA is rather controversial for two reasons: (1) it is practically impossible to hedge DVA — from an accounting perspective, recognising DVA would require the market-maker to demonstrate that they are able to sell credit risk protection on their own financial services entity; and (2) recognising DVA and the FVA benefit may be double counting — from a technical perspective, under certain assumptions regarding the theoretical construction of funding rates, it can be shown that the FVA benefit coincides exactly with the DVA benefit.

Appendix B

Appendix for Part I

B.1 Example of an estimated $n\delta$-term FTDSDF

Assume that $\{(Z^i_u)_{u\geq 0}\,; i \in \{1, 2, \ldots, \ell\}\}$ is a set of continuous-time homogeneous $\{(\mathcal{G}_u), \mathbb{P}\}$-Markov processes, which represent latent factors underpinning all tradable market variables. Then, the classical notion of a short rate process $(r^n_u)_{u\geq 0}$ and estimated SDF $(\widehat{D}^n_u)_{u\geq 0}$, associated with the $n\delta$-term FTD rate, may be defined by

$$r^n_u := f_n(Z_u) \quad \text{and} \quad \widehat{D}^n_u := \exp\left(-\int_0^u r^n_v \,\mathrm{d}v\right) ,$$

respectively, for $n \in \{1, 2, \ldots, m\}$. Both quantities above are $\mathcal{G}_u$-measurable, and $f_n : \mathbb{R}^p \to \mathbb{R}$ is some suitably behaved and measurable function, with $Z_u := \left[Z^1_u, Z^2_u, \ldots, Z^p_u\right]^\intercal$, for some $p \in \{1, 2, \ldots, \ell\}$. As a practical example, assume that the set of time-homogeneous Markov processes are independent Ornstein-Uhlenbeck processes of the form

$$\mathrm{d}Z^i_u = a_i(b_i - Z^i_u)\,\mathrm{d}u + c_i\,\mathrm{d}W^i_u ,$$

for $u \geq 0$, with $(W^i_u)_{u\geq 0}$ being a standard $\{(\mathcal{G}_u), \mathbb{P}\}$-Wiener process, for $i \in \{1, 2, \ldots, \ell\}$. Next, assume that the set of short rate processes are defined by $r^n_u := \sum_{i=1}^h Z^i_u$, for $n \in \{1, 2, \ldots, m\}$ and $h \leq \ell$. Then, the estimated $n\delta$-term FTDSDF is given by $\widehat{D}^n_u = \prod_{i=1}^h \exp\left(-\int_0^u Z^i_v\,\mathrm{d}v\right)$, and

$$\widehat{P}^n_{u,u+n\delta} := \mathbb{E}^{\mathbb{P}}\left[\frac{\widehat{D}^n_{u+n\delta}}{\widehat{D}^n_u}\,\middle|\,\mathcal{G}_u\right] = \mathbb{E}^{\mathbb{P}}\left[\prod_{i=1}^h e^{-\int_0^{n\delta} Z^i_{u+v}\,\mathrm{d}v}\,\middle|\,\mathcal{G}_u\right] = \prod_{i=1}^h \exp\left[A_i(n\delta) - B_i(n\delta)Z^i_u\right] ,$$

where

$$B_i(x) := \frac{1}{a_i}\left(1 - e^{-a_i x}\right) \quad \text{and} \quad A_i(x) := \left(b_i - \frac{c_i^2}{2a_i^2}\right)\left[B_i(x) - x\right] - \frac{c_i^2}{4a_i}B_i(x)^2 ,$$

and $x \geq 0$. It follows that if the current time is t, the set of parameters

$$\left\{\{Z^i_0, a_i, b_i, c_i\}\,; i \in \{1, 2, \ldots, h\}\right\} , \tag{B.1.1}$$

may be estimated using the historical set of $n\delta$-term FTD rates $\{R^n_0, R^n_\delta, \ldots, R^n_t\}$, where $\{0, \delta, \ldots, t\}$ denotes the set of trading days in the interval $[0, t]$.

B.2 Example of a calibrated δ-term FTDPK

Consider an estimated δ-term FTDSDF as given in Appendix B.1, but only driven by a single Ornstein-Uhlenbeck process. Then, $\widehat{P}^1_{u,u+\delta} = \exp\left[A_1(\delta) - B_1(\delta)Z^1_u\right]$. Assuming that Assumption 3.3.1 prevails, the initial δ-term FTDZCB term structure is given by $\{P^1_{t,t+\delta}, P^1_{t,t+2\delta}, \ldots, P^1_{t,t+m\delta}\}$, for the current time t. A calibrated δ-term FTDPK may then be defined via a time-inhomogeneous change-of-measure from $\mathbb{P}$ to $\mathbb{D}_1$ through the density martingale:

$$\mathrm{d}\Lambda_u^1 = \phi_1(u)\Lambda_u^1\,\mathrm{d}W_u^1 \ ,$$

for $u \in [t, t+m\delta]$. Here, $\Lambda_t^1 := 1$, and $\phi_1(\cdot)$ is a real-valued deterministic function of time, so that

$$\mathrm{d}W_u^1 = \phi_1(u)\,\mathrm{d}u + \mathrm{d}W_u^{\mathbb{D}_1} \ ,$$

where $(W_u^{\mathbb{D}_1})_{u\geq t}$ is a standard $\{(\mathcal{G}_u), \mathbb{D}_1\}$-Wiener process. Setting $\phi_1(u) := \frac{a_1}{c_1} d_1(u)$, the factor process under the $\mathbb{D}_1$-measure becomes

$$\mathrm{d}Z_u^1 = a_1\left(b_1 + d_1(u) - Z_u^1\right)\,\mathrm{d}u + c_1\,\mathrm{d}W_u^{\mathbb{D}_1} \ .$$

The deterministic function $d_1(u)$ may be defined by

$$d_1(u) = \begin{cases} d_1^1 \ , & u \in [t, t+\delta] \ , \\ d_2^1 \ , & u \in (t+\delta, t+2\delta] \ , \\ \ \vdots \ , & \quad \vdots \ , \\ d_m^1 \ , & u \in (t+(m-1)\delta, t+m\delta] \ , \end{cases}$$

where $d_i^1 \in \mathbb{R}$ for $i \in \{1, 2, \ldots, m\}$. The following result is then required:

$$\mathbb{E}^{\mathbb{D}_1}\left[\exp\left(-\int_{i\delta}^{j\delta} Z_{t+v}^1 \mathrm{d}v\right)\,\middle|\,\mathcal{G}_{t+i\delta}\right] = \exp\left[A_1^*(t+i\delta, t+j\delta) - B_1((j-i)\delta)Z_{t+i\delta}^1\right] \ , \tag{B.2.1}$$

where $A_1^*(t+i\delta, t+j\delta) := A_1((j-i)\delta) + C_1(t+i\delta, t+j\delta)$ and

$$C_1(t+i\delta, t+j\delta) := \sum_{k=i+1}^{j} d_k^1\left[B_1((j-k+1)\delta) - B_1((j-k)\delta) - \delta\right] \ ,$$

for $i < j \in \{1, 2, \ldots, m\}$. Then, using this result and Lemma 4.2.1, it follows that

$$D_{t+\delta}^1 = \mathbb{E}^{\mathbb{D}_1}\left[\exp\left(-\int_0^{\delta} Z_{t+v}^1 \mathrm{d}v\right)\,\middle|\,\mathcal{G}_t\right] = \exp\left[A_1^*(t, t+\delta) - B_1(\delta)Z_t^1\right] \ ,$$

where $A_1^*(t, t+\delta) = A_1(\delta) + C_1(t, t+\delta)$ and $C_1(t, t+\delta) = d_1^1\left[B_1(\delta) - \delta\right]$, and d_1^1 may be set such that $D_{t+\delta}^1 = P_{t,t+\delta}^1 = P_{t,0,1}^1$. Next, using the above result and Lemma 4.3.1, it follows that

$$\frac{D_{t+2\delta}^1}{D_{t+\delta}^1} = \mathbb{E}^{\mathbb{D}_1}\left[\exp\left(-\int_{\delta}^{2\delta} Z_{t+v}^1 \mathrm{d}v\right)\,\middle|\,\mathcal{G}_{t+\delta}\right] = \exp\left[A_1^*(t+\delta, t+2\delta) - B_1(\delta)Z_{t+\delta}^1\right] \ ,$$

where $A_1^*(t+\delta, t+2\delta) = A_1(\delta) + C_1(t+\delta, t+2\delta)$ and $C_1(t+\delta, t+2\delta) = d_2^1\left[B_1(\delta) - \delta\right]$. Thus,

$$\begin{aligned} \mathbb{E}^{\mathbb{D}_1}\left[D_{t+2\delta}^1\,\middle|\,\mathcal{G}_t\right] &= \mathbb{E}^{\mathbb{D}_1}\left[D_{t+\delta}^1 \exp\left[A_1(\delta) + d_2^1[B_1(\delta) - \delta] - B_1(\delta)Z_{t+\delta}^1\right]\,\middle|\,\mathcal{G}_t\right] \\ &= D_{t+\delta}^1 \exp\left[A_1(\delta) + d_2^1[B_1(\delta) - \delta]\right]\mathbb{E}^{\mathbb{D}_1}\left[\exp\left[-B_1(\delta)Z_{t+\delta}^1\right]\,\middle|\,\mathcal{G}_t\right] \\ &= P_{t,0,1}^1 \exp\left[A_1(\delta) + d_2^1[B_1(\delta) - \delta]\right]\mathbb{E}^{\mathbb{D}_1}\left[\exp\left[-B_1(\delta)Z_{t+\delta}^1\right]\,\middle|\,\mathcal{G}_t\right] \ , \end{aligned}$$

which reveals that d_2^1 is a free parameter that may be set to ensure that $\mathbb{E}^{\mathbb{D}_1}\left[D_{t+2\delta}^1\,\middle|\,\mathcal{G}_t\right] = P_{t,0,2}^1$. In general, this construction may be repeated iteratively and it can be shown that

$$\mathbb{E}^{\mathbb{D}_1}\left[D_{t+j\delta}^1\,\middle|\,\mathcal{G}_t\right] = P_{t,0,j-1}^1 \exp\left[A_1(\delta) + d_j^1[B_1(\delta) - \delta]\right]\mathbb{E}^{\mathbb{D}_1}\left[\exp\left[-B_1(\delta)Z_{t+(j-1)\delta}^1\right]\,\middle|\,\mathcal{G}_t\right] \ ,$$

such that d_j^1 is a free parameter that enables $\mathbb{E}^{\mathbb{D}_1}\left[D_{t+j\delta}^1\,\middle|\,\mathcal{G}_t\right] = P_{t,0,j}^1$, for $j \in \{3, 4, \ldots, m\}$.

B.3 General Systemic Liquidity Indicators

Definition B.3.1 (General $n\delta$-term quoted FTD-linked rates)
Assume that u and t are quoting and trading times respectively, with $u < t$. For a nominal amount N, the SST may model a future $n\delta$-term FTD rate quoted by the STR as

$$R^{n,N}_{t,\alpha,\beta} := R^n_t L^{n,N}_{t,\alpha,\beta} \tag{B.3.1}$$

where α is equal to $\operatorname{sgn}(1)$ for a deposit, $\operatorname{sgn}(-1)$ for a loan; β is a state variable equal to 3 if the STR offers perfect liquidity, 2 if the STR offers costly liquidity, 1 if the STR can't offer liquidity to the SST only, and 0 if there is no systemic liquidity; and the $\mathcal{L}_t$-measurable liquidity indicator

$$L^{n,N}_{t,\alpha,\beta} := \begin{cases} 1 \ , & \text{if } \beta = 3, \text{ with probability } p^{n,N}_{t,\alpha,3} \ , \\ 1 - \alpha\Delta^{n,N}_{t,\alpha} \big/ \left(n\delta R^n_t P^n_{t,0,n}\right) \ , & \text{if } \beta = 2, \text{ with probability } p^{n,N}_{t,\alpha,2} \ , \\ 0 \ , & \text{if } \beta = 1, \text{ with probability } p^{n,N}_{t,\alpha,1} \ , \\ 0 \ , & \text{if } \beta = 0, \text{ with probability } p^{n,N}_{t,\alpha,0} \ ; \end{cases} \tag{B.3.2}$$

with $\Delta^{n,N}_{t,\alpha} := \Delta(t, t+n\delta, N, \alpha)$, a positive real-valued function which models the absolute future cost per unit nominal, and the probability $p^{n,N}_{t,\alpha,\beta} := p(t, t+n\delta, N, \alpha, \beta)$ both assumed to be deterministic functions. By the law of total probability it must follow that $\sum_{\beta=0}^{3} p^{n,N}_{t,\alpha,\beta} = 1$, while it must also hold that $L^{n,N}_{t,+,0}(\omega) = L^{n,N}_{t,-,0}(\omega)$ a.s., so that the likelihood of systemic illiquidity is equal for both FTD-linked loans and deposits, i.e., $p^{n,N}_{t,+,0} = p^{n,N}_{t,-,0}$.

Proposition B.3.1 (Expected future value and cost of $n\delta$-term liquidity)
Consider a set of nominals $\{N^n_{t,\alpha,1}, N^n_{t,\alpha,2}, \ldots, N^n_{t,\alpha,b}\}$ with weights $\{w^n_{t,\alpha,1}, w^n_{t,\alpha,2}, \ldots, w^n_{t,\alpha,b}\}$ that reflect the respective likelihood of the SST engaging in deposit and loan transactions at such nominals at time t for a term of $n\delta$, with $\sum_{i=1}^{b} w^n_{t,\alpha,i} = 1$, where $b \in \mathbb{N}$. To ease notation here, N_i and w_i are used to denote $N^n_{t,\alpha,i}$ and $w^n_{t,\alpha,i}$ respectively, for $i \in \{1, 2, \ldots, b\}$. From the vantage point of the SST at time u, the weighted average future value at time t of an $n\delta$-term FTD-linked deposit/loan with unit nominal is

$$V^n_{t,\alpha} = \alpha \left[p^n_{t,\alpha,0} \widehat{P}^n_{t,0,n} + p^n_{t,\alpha,1} P^n_{t,0,n} + p^n_{t,\alpha,2} \left(1 - \alpha\Delta^n_{t,\alpha}\right) + p^n_{t,\alpha,3} \right] \ , \tag{B.3.3}$$

where the aggregated probabilities and cost function are respectively defined by

$$p^n_{t,\alpha,\beta} := \left(\sum_{i=1}^{b} w_i N_i p^{n,N_i}_{t,\alpha,\beta} \right) \Bigg/ \left(\sum_{i=1}^{b} w_i N_i \right) \ , \text{ and} \tag{B.3.4}$$

$$\Delta^n_{t,\alpha} := \left(\sum_{i=1}^{b} w_i N_i p^{n,N_i}_{t,\alpha,2} \Delta^{n,N_i}_{t,\alpha} \right) \Bigg/ \left(p^n_{t,\alpha,2} \sum_{i=1}^{b} w_i N_i \right) \ . \tag{B.3.5}$$

The expected future cost of $n\delta$-term liquidity per unit nominal at time t is given by $\left(\alpha - V^n_{t,\alpha}\right)$.

Proof. At time t if $\beta = 3$ then perfect liquidity prevails, the reference $n\delta$-term FTD rate will exist and the fair value of the SST's deposit/loan will be

$$\frac{1}{\pi^n_t} \mathbb{E}^{\mathbb{P}} \left[\pi^n_{t+n\delta} \alpha N_i \left(1 + n\delta R^{n,N_i}_{t,\alpha,3}\right) \,\middle|\, \mathcal{G}_t \right] = \alpha N_i P^n_{t,0,n} \left(1 + n\delta R^{n,N_i}_{t,\alpha,3}\right) = \alpha N_i \ . \tag{B.3.6}$$

If $\beta = 2$, costly liquidity prevails, the reference $n\delta$-term FTD rate will exist and the fair value of the SST's deposit/loan will be

$$\frac{1}{\pi^n_t} \mathbb{E}^{\mathbb{P}} \left[\pi^n_{t+n\delta} \alpha N_i \left(1 + n\delta R^{n,N_i}_{t,\alpha,2}\right) \,\middle|\, \mathcal{G}_t \right] = \alpha N_i P^n_{t,0,n} \left(1 + n\delta R^{n,N_i}_{t,\alpha,2}\right) = \alpha N_i - N_i \Delta^{n,N_i}_{t,\alpha} \ . \tag{B.3.7}$$

If $\beta = 1$, only the SST can't access liquidity, the reference $n\delta$-term FTD rate will still exist and the fair value of the SST's position will now be

$$\frac{1}{\pi_t^n}\mathbb{E}^{\mathbb{P}}\left[\pi_{t+n\delta}^n \alpha N_i \left(1 + n\delta R_{t,\alpha,1}^{n,N_i}\right) \mid \mathcal{G}_t\right] = \alpha N_i P_{t,0,n}^n \ . \tag{B.3.8}$$

In the case of a deposit, this represents the value foregone by not being able to access the $n\delta$-term FTDRR. For a loan, this represents the value gained by having to settle a liability early as opposed to deferring payment by accessing funding through the $n\delta$-term FTDRR.
When $\beta = 0$, there is no systemic liquidity and therefore no reference $n\delta$-term FTD rate. In this scenario, the SST may estimate the fair value of their position as

$$\frac{1}{\widehat{D}_t^n}\mathbb{E}^{\mathbb{P}}\left[\widehat{D}_{t+n\delta}^n \alpha N_i \left(1 + n\delta R_{t,\alpha,0}^{n,N_i}\right) \mid \mathcal{G}_t\right] = \alpha N_i \widehat{P}_{t,0,n}^n \ , \tag{B.3.9}$$

by making use of the estimated $n\delta$-term FTDSDF. Then, the estimated value at time t is

$$V_{t,\alpha}^{n,N_i} = \alpha N_i \left[q_{t,\alpha,0}^{n,N_i} \widehat{P}_{t,0,n}^n + q_{t,\alpha,1}^{n,N_i} P_{t,0,n}^n + q_{t,\alpha,2}^{n,N_i}\left(1 - \alpha \Delta_{t,\alpha}^{n,N_i}\right) + q_{t,\alpha,3}^{n,N_i}\right] \ , \tag{B.3.10}$$

and the weighted average future value equation (B.3.3) is recovered by setting

$$V_{t,\alpha}^n := \left(\sum_{i=1}^{b} w_i V_{t,\alpha}^{n,N_i}\right) \Big/ \left(\sum_{i=1}^{b} w_i N_i\right) \ , \tag{B.3.11}$$

which holds for both $n\delta$-term FTD-linked loans or deposits, and concludes the proof. □

Corollary B.3.1 (Mid expected future value and the cost of $n\delta$-term liquidity)
From equation (B.3.3), it follows that the mid weighted average future value is

$$V_t^n = \alpha \left[p_{t,0}^n \widehat{P}_{t,0,n}^n + p_{t,1}^n P_{t,0,n}^n + p_{t,2}^n \left(1 + \epsilon_t^n\right) + p_{t,3}^n\right] \ , \tag{B.3.12}$$

where the mid probabilities and cost function are respectively defined by

$$p_{t,\beta}^n := \frac{1}{2}\left(p_{t,+,\beta}^n + p_{t,-,\beta}^n\right) \ , \ \textit{and} \tag{B.3.13}$$

$$\epsilon_t^n := \frac{1}{2p_{t,2}^n}\left(p_{t,-,2}^n \Delta_{t,-}^n - p_{t,+,2}^n \Delta_{t,+}^n\right) \ . \tag{B.3.14}$$

with the expected future cost of $n\delta$-term liquidity per unit nominal now given by $(\alpha - V_t^n)$.

Proof. Setting $V_t^n := \alpha\left(V_{t,+}^n - V_{t,-}^n\right)/2$ yields the mid future value equation (B.3.12), with the expected future cost of liquidity result then following trivially. □

Remark B.3.1 (The spread quantity: ϵ_t^n)
Having constructed a mid value in Corollary B.3.1, the quantity ϵ_t^n may be interpreted as the mid value of the bid-offer spread associated with $n\delta$-term deposit and loan liquidity at time t, suitably weighted by the probability of each transaction at specific nominals, from the perspective of the SST. The magnitude and sign of this mid spread depends on financial economic conditions and the state of the FTD market. In particular, one would expect:

- $\epsilon_t^n > 0$*, in a stressed market where the STR has difficulty sourcing term financing;*
- $\epsilon_t^n \approx 0$*, in a normal market where $\Delta_{t,+}^n$ and $\Delta_{t,-}^n$ may be attributed to profit margins;*
- $\epsilon_t^n < 0$*, in a stressed market where the STR has excess access to term finance, a scenario that is most likely to realise for near or shorter terms-to-maturity.*

Remark B.3.2 (Systemic liquidity indicators)
Proposition B.3.1 and Corollary B.3.1 have enabled the aggregation of the nominal effect in the general $n\delta$-term quoted rates, as well as the averaging of the spread asymmetry due to loans and deposits. The structure of the mid expected future value, equation B.3.12, indicates that a simpler symmetric and systemic specification for the liquidity indicator will suffice, especially under the assumption of a normal market, or $\epsilon_t^n \approx 0$. Therefore, in order to ease the exposition, the $\mathcal{L}_t$-measurable random variable

$$L_t^n := \begin{cases} 1 \ , & \textit{if perfect systemic liquidity prevails, with probability } p_t^n \ , \\ 0 \ , & \textit{if otherwise, with probability } 1 - p_t^n \ ; \end{cases} \tag{B.3.15}$$

is used to model systemic $n\delta$-term liquidity at time t, with $p_t^n := p(t, t + n\delta)$ being a deterministic function that determines the probability of perfect systemic liquidity or otherwise. With this indicator, states $\beta \in \{2, 3\}$ and $\beta \in \{0, 1\}$ of the general indicator are essentially combined, and provide a similar composite effect with $V_t^n = \alpha \left[(1 - p_t^n)\widehat{P}_{t,0,n}^n + p_t^n\right]$ now being the mid expected future value of $n\delta$-term liquidity per unit nominal at time t.

A more general version of the systemic liquidity indicator, defined in Remark B.3.2, which incorporates the state of costly liquidity is considered next. Using the definition of this new liquidity indicator, the lemma below reveals the impact of costly liquidity on the fair $\delta \times 2\delta$ FRA rate derived in Lemma 4.4.1.

Definition B.3.2 (Costly systemic liquidity indicators)
At time $u \in \mathbb{R}_{\geq 0}$, the random variable

$$X_u^n := \begin{cases} 0 \ , & \textit{no systemic liquidity, with probability } q_{u,0}^n \ , \\ 1 \ , & \textit{perfect systemic liquidity, with probability } q_{u,1}^n \ , \\ 1 + \epsilon_u^n \ , & \textit{costly systemic liquidity, with probability } q_{u,2}^n \ ; \end{cases} \tag{B.3.16}$$

models $n\delta$-term systemic liquidity, where $\epsilon_u^n \in \mathbb{R}$ is the deterministic spread quantity as defined in Corollary B.3.1 and described in Remark B.3.1. If the current time is t, then the natural filtration associated with liquidity is

$$\mathcal{L}_t^{\text{CSLI}} := \sigma\left(\left\{\{X_u^1, X_u^2, \ldots, X_u^m\} \, ; u \in \{0, \delta, 2\delta, \ldots, t - \delta, t\}\right\}\right) \subset \mathcal{L}_t \ , \tag{B.3.17}$$

where $\{0, \delta, 2\delta, \ldots, t - \delta, t\}$ denotes the set of trading days that lie within the interval $[0, t]$. These costly systemic liquidity indicators are assumed to exhibit both serial and cross-sectional independence, or more formally:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[X_u^n \,\middle|\, \mathcal{L}_t \cap \sigma\left(\{X_u^n \notin \{0, 1, 1 + \epsilon_u^n\}\}\right)\right] &= \mathbb{E}^{\mathbb{P}}\left[X_u^n\right] \\ &= \mathbb{P}\left[X_u^n = 1\right] + (1 + \epsilon_u^n)\,\mathbb{P}\left[X_u^n = 1 + \epsilon_u^n\right] \\ &= p_{u,1}^n + p_{u,2}^n\left(1 + \epsilon_u^n\right) \ , \end{aligned}$$

for all $t \leq u$, with $p_{u,i}^n := p_i(u, u + n\delta)$ being a deterministic function for $i \in \{0, 1, 2\}$.

Lemma B.3.1 ($\delta \times 2\delta$ FTDFRA pricing under potential costly liquidity and future illiquidity)
The fair strike rate process for the general version of the $\delta \times 2\delta$ FTDFRA defined in Definition 3.5.1 is

$$\overline{F}_{t,i,1}^1 = \begin{cases} F_{t,0,1}^1\left[p_{t+\delta,1}^1 + p_{t+\delta,2}^1\left(1 + \epsilon_{t+\delta}^1\right)\right] \ , & i = 0 \text{ and conditional on } X_t^1 = X_t^2 = 1 \ , \\ F_{t,1,1}^1 \ , & i = 1 \text{ and conditional on } X_{t+\delta}^1 = 1 \ , \end{cases}$$

which is also $\mathcal{G}_{t+i\delta}$-measurable.

Proof. Assuming that $\mathcal{L}_t = \mathcal{L}_{t^-} \vee \sigma\left(\left\{X_t^1 = 1, X_t^2 = 1\right\}\right)$, the standard FTDFRA replication strategy yields $\widetilde{V}_{t+2\delta} = V_{t+2\delta} - \alpha N\delta\left(1 - X_{t+\delta}^1\right)R_{t+\delta}^1$, as was the case in Lemma 4.4.1.

Let $\mathcal{M}_t := \mathcal{G}_t \vee \mathcal{L}_t$, then using the hybrid-term LDFTDPK from Definition 4.4.6, the current value of the above payoff is

$$\begin{aligned}
\widetilde{\pi}_t \widetilde{V}_t &= \mathbb{E}^{\mathbb{P}}\left[\widetilde{\pi}_{t+2\delta}\widetilde{V}_{t+2\delta} \,\middle|\, \mathcal{M}_t\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\widetilde{\pi}_{t+2\delta}\widetilde{V}_{t+2\delta} \,\middle|\, \mathcal{M}_t\,, X^1_{t+\delta}\right] \,\middle|\, \mathcal{M}_t\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\widetilde{\pi}_{t+2\delta}\widetilde{V}_{t+2\delta} \,\middle|\, \mathcal{M}_t\,, X^1_{t+\delta}=0\right] \mathbb{P}\left[X^1_{t+\delta}=0\right] \,\middle|\, \mathcal{M}_t\right] \\
&\quad + \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\widetilde{\pi}_{t+2\delta}\widetilde{V}_{t+2\delta} \,\middle|\, \mathcal{M}_t\,, X^1_{t+\delta}=1\right] \mathbb{P}\left[X^1_{t+\delta}=1\right] \,\middle|\, \mathcal{M}_t\right] \\
&\quad + \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\widetilde{\pi}_{t+2\delta}\widetilde{V}_{t+2\delta} \,\middle|\, \mathcal{M}_t\,, X^1_{t+\delta}=1+\epsilon^1_{t+\delta}\right] \mathbb{P}\left[X^1_{t+\delta}=1+\epsilon^1_{t+\delta}\right] \,\middle|\, \mathcal{M}_t\right]
\end{aligned}$$

which follows by the tower property of conditional expectations. Then, it follows that

$$\begin{aligned}
\widetilde{V}_t =\; & -\mathbb{E}^{\mathbb{P}}\left[\pi^2_{t+2\delta}\alpha N\delta F^1_{t,0,1} \,\middle|\, \mathcal{M}_t\right] p^1_{t+\delta,0} + \mathbb{E}^{\mathbb{P}}\left[\pi^1_{t+2\delta}V_{t+2\delta} \,\middle|\, \mathcal{M}_t\right] p^1_{t+\delta,1} \\
& + \mathbb{E}^{\mathbb{P}}\left[\pi^1_{t+2\delta}\left(V_{t+2\delta} + \alpha N\delta\epsilon^1_{t+\delta}R^1_{t+\delta}\right) \,\middle|\, \mathcal{M}_t\right] p^1_{t+\delta,2} \\
=\; & -\alpha N p^1_{t+\delta,0}\delta F^1_{t,0,1}P^2_{t,0,2} + p^1_{t+\delta,1}V_t + p^1_{t+\delta,2}V_t + \alpha N p^1_{t+\delta,2}\delta\epsilon^1_{t+\delta}F^1_{t,0,1}P^1_{t,0,2} \\
=\; & \left(p^1_{t+\delta,1} + p^1_{t+\delta,2}\right)V_t + \alpha N\delta F^1_{t,0,1}P^1_{t,0,2}\left(p^1_{t+\delta,2}\epsilon^1_{t+\delta} - p^1_{t+\delta,0}\right)
\end{aligned}$$

since $\widetilde{\pi}_t := 1$, using the definition of the hybrid-term LDFTDPK and taking note that the δ-term FTDPK is well-defined when $X^1_{t+\delta} \neq 0$. V_t is the fair value of the FTDFRA under perfect liquidity, and therefore equal to 0. Trading this FTDFRA with the strike rate equal to the fair FTDFRA rate defined in the perfect liquidity setting therefore leads to an initial loss (gain) if the market-maker is long (short). As with Lemma 4.4.1, the assumption here is that $V_{t+2\delta}$ will still be the FTDFRA payoff even when $X^1_{t+\delta} = 0$ and the strong case of no systemic liquidity is in effect. Setting the FTDFRA rate to an arbitrary value, $\overline{F}^1_{t,0,1}$, and pricing gives

$$\begin{aligned}
\widetilde{V}_t &= \alpha\left(p^1_{t+\delta,1} + p^1_{t+\delta,2}\right)N\left[P^1_{t,0,1} - \left(1+\delta\overline{F}^1_{t,0,1}\right)P^1_{t,0,2}\right] \\
&\quad - \alpha N p^1_{t+\delta,0}\delta\overline{F}^1_{t,0,1}P^2_{t,0,2} + \alpha N p^1_{t+\delta,2}\delta\epsilon^1_{t+\delta}F^1_{t,0,1}P^1_{t,0,2} \\
&= \alpha N\left(1 - p^1_{t+\delta,0}\right)\delta F^1_{t,0,1}P^1_{t,0,2} + \alpha N p^1_{t+\delta,2}\delta\epsilon^1_{t+\delta}F^1_{t,0,1}P^1_{t,0,2} \\
&\quad - \alpha N\left(1 - p^1_{t+\delta,0}\right)\delta\overline{F}^1_{t,0,1}P^1_{t,0,2} - \alpha N p^1_{t+\delta,0}\delta\overline{F}^1_{t,0,1}P^1_{t,0,2} \\
&= \alpha N\delta F^1_{t,0,1}P^1_{t,0,2}\left[\left(1 - p^1_{t+\delta,0}\right) + p^1_{t+\delta,2}\epsilon^1_{t+\delta}\right] - \alpha N\delta\overline{F}^1_{t,0,1}P^1_{t,0,2}\left[\left(1 - p^1_{t+\delta,0}\right) + p^1_{t+\delta,0}\right]
\end{aligned}$$

while setting $\widetilde{V}_t = 0$, recalling that $P^2_{t,0,2} = P^1_{t,0,2}$, and solving for the fair FTDFRA strike rate yields $\overline{F}^1_{t,0,1} = F^1_{t,0,1}\left[p^1_{t+\delta,1} + p^1_{t+\delta,2}\left(1+\epsilon^1_{t+\delta}\right)\right]$, as required. Repeating this pricing process at time $t+\delta$, for exactly the same contract and assuming that $X^1_{t+\delta} = 1$ at this time, it is trivial to show that $\overline{F}^1_{t,1,1} = F^1_{t,1,1}$, which completes the proof. □

Appendix C

Appendix for Part II

C.1 Arbitrage-Free Strategies for the Conversion of Cash Flows and Curves

First consider a simple arbitrage relationship for an economy with default-free and credit-risky interest rate curves, while assuming perfect market liquidity. Assume that the x-curve is the default-free curve while the y-curve is one of a potential set of credit-risky curves. Consider the following simple strategy, at time 0:

(i) Sell one unit of the numeraire asset in the x-market for $1/h_0^x$; and

(ii) Buy one unit of the numeraire asset in the y-market which costs $1/h_0^y$,

which costs zero to setup, i.e., $V_t = 0$, since $h_0^x = h_0^y = 1$. Transaction (i) is equivalent to borrowing money via the x-market's money market, while (ii) is equivalent to a deposit into the y-market's money market. Then at any time $t > 0$, the value of this strategy will be

$$V_t = \frac{1}{h_t^y} - \frac{1}{h_t^x},$$

which will be greater than zero if the risky entity that holds the investment has not defaulted by that time. Therefore, this strategy does not allow for arbitrage, in general. Now, assume that one is able to mitigate all of the default-risk associated with the entity offering the y-market investment via appropriate collateralisation. In such a circumstance, the value of the strategy at any time $t > 0$ must equal zero, if one were to preclude arbitrage, otherwise one would be ensured of earning a cash flow equal to V_t which would be greater than zero with certainty at any time $t > 0$. No arbitrage may be achieved by adjusting the y-market deposit by the ratio h_t^y/h_t^x. At time $t > 0$, this ratio is the realised multiplicative spread between the discount factors realised in the x- and y-markets respectively.

Remark C.1.1 (The relation to spot foreign exchange rate modelling)
When modelling foreign exchange, the ratio h_t^y/h_t^x is the appropriate model for the spot exchange rate between the x- and y-currencies. In particular, 1 unit of y-currency may be exchanged for h_t^y/h_t^x units of x-currency at time t.

Another relevant arbitrage relationship to consider involves a finite horizon loan and investment strategy. Maintaining the same assumptions as before, consider the same strategy as before at time 0, and then do the following at some time $t \in (0, T)$:

(i) Sell $1/h_t^x$ units of the x-market T-maturity bond for P_{tT}^x; and

(ii) Buy $1/h_t^y$ units of the y-market T-maturity bond for P_{tT}^y, which again costs zero to setup at time 0, as before, and terminates at time t when the money market loan and deposit is transferred to fixed horizon alternatives. Now, at any time $s \in [0, t)$, the same arguments apply as before while at time t the value of this strategy will be

$$V_t = \frac{1}{h_t^y P_{tT}^y} - \frac{1}{h_t^x P_{tT}^x},$$

which again does not permit an arbitrage opportunity, due to the credit risk associated with the investment leg of the strategy. If collateralisation of the investment leg of the strategy is again invoked, then arbitrage is precluded at: (a) all times $s \in [0, t)$ by adjusting the y-market deposit by the ratio h_t^y / h_t^x; and (b) at time t by adjusting the y-market fixed-term deposit by the ratio $h_t^y P_{tT}^y / h_t^x P_{tT}^x$. If these adjustments are not enforced post collateralisation, then one would be ensured of a risk-free profit equal to V_t for all times $t \in (0, T]$.

Remark C.1.2 (The relation to forward foreign exchange rate modelling)
In foreign exchange rate modelling, the ratio $h_t^y P_{tT}^y / h_t^x P_{tT}^x$ models the forward exchange rate between the x- and y-currencies. In particular, one can agree at time t to exchange 1 unit of y-currency for $h_t^y P_{tT}^y / h_t^x P_{tT}^x$ units of x-currency at time $T \geq t$.

C.2 Consistent Changes of Numeraire and Measure

Here changes-of-measure, numeraire assets, martingales and therefore no-arbitrage within the xy-formalism is discussed. The curve-conversion factor process (6.1.3) induces the changes-of-measure between all introduced y-markets (or y-curves). In particular, it governs no-arbitrage across all distinct markets associated with the economy under consideration. To demonstrate this, consider Proposition 6.1.1, along with an asset with a spot-defined future cash flow H_T^y, then the value of such an asset in the x-market is deduced to be

$$H_{tT}^{xy} = \frac{1}{h_t^x} \mathbb{E}\left[h_T^y H_T^y \,|\, \mathcal{F}_t\right], \tag{C.2.1}$$

for $t \in [0, T]$. For each of the markets $z \in \{x, y\}$, a change-of-measure density martingale $(m_t^z)_{0 \leq t \leq T}$ is introduced which changes measure from the real-world probability measure $\mathbb{P}$ to the equivalent pricing measure $\mathbb{P}_z$, along with the z-stochastic discount factor (D_t^z) such that $1/D_t^z$ is the natural numeraire under $\mathbb{P}_z$. This also means that the z-market's pricing kernel may be written as $h_t^z = D_t^z m_t^z$. The price process $(H_{tT}^{xy})_{0 \leq t \leq T}$ can now be expressed, equivalently, in terms of: (a) the $\mathbb{P}_x$ pricing measure; and (b) the $\mathbb{P}_y$ pricing measure:

$$\begin{aligned} H_{tT}^{xy} &= \frac{1}{D_t^x} \mathbb{E}^{\mathbb{P}_x}\left[D_T^x Q_{TT}^{xy} H_T^y \,|\, \mathcal{F}_t\right] \\ &= \frac{m_t^y}{m_t^x} \frac{1}{D_t^x} \mathbb{E}^{\mathbb{P}_y}\left[D_T^y H_T^y \,|\, \mathcal{F}_t\right] = Q_{tt}^{xy} H_t^y, \end{aligned} \tag{C.2.2}$$

where it is emphasised that $(D_t^x H_{tT}^{xy})_{0 \leq t \leq T}$ and $(D_t^y H_t^y)_{0 \leq t \leq T}$ are $\mathbb{P}_x$- and $\mathbb{P}_y$-martingales respectively, by construction. Moreover, it is possible to change measure from $\mathbb{P}_z$ to the T-forward measure $\mathbb{P}_z^T$ via the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{P}_z^T}{\mathrm{d}\mathbb{P}_z} = \frac{D_T^z P_{TT}^z}{D_t^z P_{tT}^z}, \tag{C.2.3}$$

which acts on $\mathcal{F}_T$ given information up until time t, i.e., $\mathcal{F}_t$, and therefore it is now possible to express the price process $(H_{tT}^{xy})_{0 \leq t \leq T}$ equivalently, in terms of: (a) the x-market's T-forward measure; and

(b) the y-market's T-forward measure:

$$\begin{aligned} H_{tT}^{xy} &= P_{tT}^{x}\mathbb{E}^{\mathbb{P}_x^T}\left[Q_{TT}^{xy}H_T^y \,|\, \mathcal{F}_t\right] \\ &= \frac{m_t^y D_t^y P_{tT}^y}{m_t^x D_t^x}\mathbb{E}^{\mathbb{P}_y^T}\left[H_T^y \,|\, \mathcal{F}_t\right] \\ &= P_{tT}^{x}Q_{tT}^{xy}\mathbb{E}^{\mathbb{P}_y^T}\left[H_T^y \,|\, \mathcal{F}_t\right] = P_{tT}^{x}Q_{tT}^{xy}\frac{H_t^y}{P_{tT}^y}, \end{aligned} \tag{C.2.4}$$

where $(H_{tT}^{xy}/P_{tT}^{x})_{0\leq t\leq T}$ and $(H_t^y/P_{tT}^y)_{0\leq t\leq T}$ are $\mathbb{P}_x^T$- and $\mathbb{P}_y^T$-martingales respectively. Equations (C.2.2) and (C.2.4) clearly demonstrate the role of the curve-conversion factor process in changing measure within and across markets. Furthermore, the price process' martingale property is preserved across markets (and curves), with the curve-conversion factor process again enabling this property. Therefore, the xy-approach precludes arbitrage within and across different markets (and curves).

C.3 Bootstrapping of Initial Term Structures

C.3.1 Emerging Economies

In an emerging economy, one would have the following initial data: (a) the y-tenored spot IBOR $J_0^y(0,T_1)$; (b) a set of fair FRA rates $\{K_0^{yy}(T_1,T_2), K_0^{yy}(T_2,T_3), \ldots, K_0^{yy}(T_{n-1},T_n)\}$; and (c) a set of fair IRS rates $\{S_0^{yy}(0,T_{n+1}), S_0^{yy}(0,T_{n+2}), \ldots, S_0^{yy}(0,T_{n+m})\}$. Using this data, one may construct the initial y-market ZCB-system by the relations

$$\begin{aligned} P_{0T_1}^y &= \frac{1}{1+J_0^y(0,T_1)\delta_1} \,, \\ P_{0T_i}^y &= \frac{P_{0T_{i-1}}^y}{1+K_0^{yy}(T_{i-1},T_i)\delta_i} \,, \\ P_{0T_{n+j}}^y &= \frac{1-S_0^{yy}(0,T_{n+j})\sum_{k=1}^{n+j-1}\delta_k P_{0T_k}^y}{1+\delta_{n+j}S_0^{yy}(0,T_{n+j})} \,, \end{aligned} \tag{C.3.1}$$

for $i \in \{2,3,\ldots,n\}$ and $j \in \{1,2,\ldots,m\}$. In general, one will have to make use of a suitable numerical bootstrapping technique to extend the y-market ZCB-system from the longest FRA maturity to the set of IRS maturities. These results are all consistent with a classical single-curve framework.

C.3.2 Developed Economies

In a developed economy, one would have the following initial data: (a) the y-tenored spot IBOR $J_0^{xy}(0,T_1)$; (b) a set of fair FRA rates $\{K_0^{xy}(T_1,T_2), K_0^{xy}(T_2,T_3), \ldots, K_0^{xy}(T_{n-1},T_n)\}$; and (c) a set of fair IRS rates $\{S_0^{xy}(0,T_{n+1}), S_0^{xy}(0,T_{n+2}), \ldots, S_0^{xy}(0,T_{n+m})\}$. Using this data, one may construct the initial y-market ZCB system by the relations

$$\begin{aligned} P_{0T_1}^y &= 1-\delta_i P_{0T_1}^x J_0^{xy}(0,T_1) \,, \\ P_{0T_i}^y &= P_{0T_{i-1}}^y - \delta_i P_{0T_i}^x K_0^{xy}(T_{i-1},T_i) \,, \\ P_{0T_{n+j}}^y &= 1-S_0^{xy}(0,T_{n+j})\sum_{k=1}^{n+j}\delta_k P_{0T_k}^x \,, \end{aligned} \tag{C.3.2}$$

for $i \in \{2,3,\ldots,n\}$ and $j \in \{1,2,\ldots,m\}$. In general, one will have to make use of a suitable numerical bootstrapping technique to extend the y-market ZCB-system from the longest FRA maturity to the set of IRS maturities. Interestingly, but not surprisingly, since $P^{xy}_{st} = (h^y_s/h^x_s)P^y_{st}$ for $0 \leq s \leq t$, it follows that

$$J^y_t(T_{i-1}, T_i) = \frac{1}{\delta_i}\left(\frac{P^y_{tT_{i-1}}}{P^y_{tT_i}} - 1\right) = \frac{1}{\delta_i}\left(\frac{P^{xy}_{tT_{i-1}}}{P^{xy}_{tT_i}} - 1\right), \tag{C.3.3}$$

and therefore $P^{xy}_{0t} = P^y_{0t}$ for $t \geq 0$.

Remark C.3.1 (Dual- or multi-curve bootstrapping)
Market practitioners may choose to construct a market-implied $\overline{y}$-market ZCB-system, which will be denoted by $\{P^{\overline{y}}_{0t}\}_{t\geq 0}$, as follows

$$\begin{aligned} P^{\overline{y}}_{0T_1} &= \frac{1}{1 + J^{xy}_0(0, T_1)\delta_1} \ , \\ P^{\overline{y}}_{0T_i} &= \frac{P^{\overline{y}}_{0T_{i-1}}}{1 + J^{xy}_0(T_{i-1}, T_i)\delta_i} \ , \\ P^{\overline{y}}_{0T_{n+j}} &= \frac{P^{\overline{y}}_{0T_{n+j-1}}}{1 + J^{xy}_0(T_{n+j-1}, T_{n+j})\delta_{n+j}} \ , \end{aligned} \tag{C.3.4}$$

where

$$J^{xy}_0(T_{n+j-1}, T_{n+j}) = \frac{S^{xy}_0(0, T_{n+j})\sum_{k=1}^{n+j}\delta_k P^x_{0T_k} - \sum_{k=1}^{n+j-1}\delta_k P^x_{0T_k} J^{xy}_0(T_{k-1}, T_k)}{\delta_{n+j}P^x_{0T_{n+j}}} \ , \tag{C.3.5}$$

for $i \in \{2, 3, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$, using the same initial data, available in a developed economy, as before. This is indeed what is currently done in practice, also referred to as multi-curve bootstrapping — the x-curve or OIS-curve is first bootstrapped and then used in the bootstrapping process for the other tenor-linked curves. The $\overline{y}$-market ZCB-system recovers the correct initial term structure while being completely agnostic of any convexity correction or martingale adjustment issues. This is the fundamental difference between the $\overline{y}$- and y-market ZCB systems.

Appendix D

Appendix for Part III

In this appendix, the general problem of modelling some interest rate centric financial market for the purposes of risk management is considered. Therefore, statistical estimation under the real-world probability measure is the ultimate and primary practical objective. In terms of the general modelling quantities defined in Chapter 9, this means assigning the filtration $\mathcal{I}$ and measure $\mathbb{S}$ to the filtration $\mathcal{F}$ and measure $\mathbb{M}$, respectively.

D.1 Example 1 — Arithmetic Brownian Motion Information

Consider the arithmetic Brownian motion information process, as defined in Remark 9.3.4, but a one-dimensional version:

$$\mathrm{d}I_t = \alpha X \,\mathrm{d}t + \beta \,\mathrm{d}W_t \ ,$$

incorporating one market factor $X \sim N(\mu, \sigma^2)$, with $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_{>0}$. Then, using the simplest version of the Nelson-Siegel-Svensson inspired functional, from Definition 10.2.1, with $h(x,t) = x$ and therefore $g(x,t) = \exp(-xt)$, the auxiliary pricing kernel becomes

$$\xi_t = \mathbb{E}^{\mathbb{M}}\left[\exp(-Xt) \,|\, \mathcal{F}_t\right] = \frac{\mathbb{E}^{\mathbb{F}}\left[\exp(-Xt)\Gamma_t \,|\, \mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{F}}\left[\Gamma_t \,|\, \mathcal{F}_t\right]} \ , \tag{D.1.1}$$

where the change-of-measure density martingale, see Corollary 9.2.1, is

$$\Gamma_t = \exp\left(\frac{\alpha}{\beta^2} X \Delta I_{0t} - \frac{\alpha^2}{2\beta^2} X^2 t\right) \ ,$$

where $\Delta I_{0t} := I_t - I_0$, with $I_0 \in \mathbb{R}$. Defining the following useful functions:

$$a_{i,t} = a(I_t, t) := \frac{\alpha}{\beta^2} \Delta I_{0t} - t \ ,$$

$$b_t = b(t) := \frac{\alpha^2}{2\beta^2} t \ ,$$

it is possible to write the expectation in the numerator of equation (D.1.1) in the compact form:

$$\mathbb{E}^{\mathbb{F}}\left[\exp(-Xt)\Gamma_t \,|\, \mathcal{F}_t\right] = \int_{-\infty}^{\infty} \exp\left(a_{i,t} x - b_t x^2\right) \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \mathrm{d}x \ ,$$

and then combining the exponentials and adopting the strategy to create a new normal probability density function by creating a new quadratic (via completing the square) yields:

$$\mathbb{E}^{\mathbb{F}}\left[\exp(-Xt)\Gamma_t \,|\, \mathcal{F}_t\right] = \frac{1}{\sqrt{v_t}} \exp\left(\frac{n_{i,t}}{2v_t}\right) \int_{-\infty}^{\infty} \frac{\sqrt{v_t}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-m_{i,t})^2}{2\sigma^2/v_t}\right) \mathrm{d}x \ , \tag{D.1.2}$$

where

$$\begin{aligned} m_{i,t} = m(I_t, t) &:= \frac{\mu + \sigma^2 a_{i,t}}{1 + 2\sigma^2 b_t} \,, \\ v_t = v(t) &:= 1 + 2\sigma^2 b_t \,, \\ n_{i,t} = n(I_t, t) &:= \sigma^2 a_{i,t}^2 + 2\mu a_{i,t} - 2\mu^2 b_t \,, \end{aligned}$$

one finally obtains

$$\mathbb{E}^{\mathbb{F}} \left[\exp(-Xt)\Gamma_t \,|\, \mathcal{F}_t \right] = \frac{1}{\sqrt{1 + 2\sigma^2 b_t}} \exp\left(\frac{\sigma^2 a_{i,t}^2 + 2\mu a_{i,t} - 2\mu^2 b_t}{2(1 + 2\sigma^2 b_t)} \right) \,, \tag{D.1.3}$$

since the integral in equation (D.1.2) equals one, being the integral over the entire domain of a normal probability density function with mean $m_{i,t}$ and variance σ^2/v_t. In a similar manner, it is possible to show that

$$\mathbb{E}^{\mathbb{F}} \left[\Gamma_t \,|\, \mathcal{F}_t \right] = \frac{1}{\sqrt{1 + 2\sigma^2 b_t}} \exp\left(\frac{\sigma^2 a_{i,0}^2 + 2\mu a_{i,0} - 2\mu^2 b_t}{2(1 + 2\sigma^2 b_t)} \right) \,, \tag{D.1.4}$$

such that the auxiliary pricing kernel resolves to

$$\begin{aligned} \xi_t = \mathbb{E}^{\mathbb{M}} \left[\exp(-Xt) \,|\, \mathcal{F}_t \right] &= \exp\left(\frac{\sigma^2 (a_{i,t}^2 - a_{i,0}^2) + 2\mu(a_{i,t} - a_{i,0})}{2(1 + 2\sigma^2 b_t)} \right) \\ &= \exp\left(\frac{\beta^2 \sigma^2 t^2 - 2\alpha\sigma^2 t[I_t - I_0] - 2\mu\beta^2 t}{2(\beta^2 + \alpha^2\sigma^2 t)} \right) \,, \end{aligned}$$

after cancelling terms that are common across equations (D.1.3) and (D.1.4). Also, it can be shown that

$$\mathbb{E}^{\mathbb{M}} \left[\exp(-XT) \,|\, \mathcal{F}_t \right] = \exp\left(\frac{\beta^2 \sigma^2 T^2 - 2\alpha\sigma^2 T[I_t - I_0] - 2\mu\beta^2 T}{2(\beta^2 + \alpha^2\sigma^2 t)} \right) \,,$$

so that the value of a T-maturity ZCB becomes

$$P_{tT} = \exp\left(\frac{\beta^2 \sigma^2 (T^2 - t^2) - 2\alpha\sigma^2 (T - t)[I_t - I_0] - 2\mu\beta^2 (T - t)}{2(\beta^2 + \alpha^2\sigma^2 t)} \right) \,,$$

for $t \in [0, T]$, which follows by Proposition 10.2.1. Constructing a suitable change-of-measure from $\mathbb{M}$ to $\mathbb{P}$, which may be achieved as in Remark 10.2.1, will then enable one to consider the task of statistical estimation using real-world time-series data for ZCB prices or yields.

D.2 Example 2 — Mean-Reverting Information

This example considers the mean-reverting information process presented in Lemma 9.3.2 and Corollary 9.3.1, but in one-dimension only. The general information process considered is

$$\mathrm{d}I_t = f(t)(\alpha X - I_t)\,\mathrm{d}t + \beta\,\mathrm{d}W_t \,,$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_{>0}$ and $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is just a deterministic function of time. The random variable X is again a single market factor, which in this case is assumed to be a mixture of n normal distributions, each with mean μ_j, variance σ_j, and component weight p_j, where $\mu_j \in \mathbb{R}$, $\sigma \in \mathbb{R}_{>0}$, and $p_j \in \mathbb{R}_{\geq 0}$ for $j \in \{1, 2, \ldots, n\}$, with $\sum_{j=1}^{n} p_j = 1$.

Then, using the simplest version of the Nelson-Siegel-Svensson inspired functional, from Definition 10.2.1, with $h(x,t) = x$ and therefore $g(x,t) = \exp(-xt)$, the auxiliary pricing kernel becomes

$$\xi_t = \mathbb{E}^{\mathbb{M}}\left[\exp(-Xt) \,|\, \mathcal{F}_t\right] = \frac{\mathbb{E}^{\mathbb{F}}\left[\exp(-Xt)\Gamma_t \,|\, \mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{F}}\left[\Gamma_t \,|\, \mathcal{F}_t\right]} \,, \tag{D.2.1}$$

where the change-of-measure density martingale, see Corollary 9.2.1, is

$$\begin{aligned}
\Gamma_t =\; & \exp\left(\frac{1}{\beta^2}\int_0^t f(s)\left(\alpha X - I_s\right)\mathrm{d}I_s - \frac{1}{2\beta^2}\int_0^t f(s)^2\left(\alpha X - I_s\right)^2\mathrm{d}s\right) \\
=\; & \exp\left(\frac{\alpha}{\beta^2}X\left[\int_0^t f(s)\mathrm{d}I_s + \int_0^t f(s)^2 I_s\mathrm{d}s\right]\right) && \text{(D.2.2)} \\
& \times \exp\left(-\frac{\alpha^2}{2\beta^2}X^2\int_0^t f(s)^2\mathrm{d}s\right) && \text{(D.2.3)} \\
& \times \exp\left(-\frac{1}{\beta^2}\int_0^t f(s)I_s\mathrm{d}I_s - \frac{1}{2\beta^2}\int_0^t f(s)^2 I_s^2\mathrm{d}s\right) \,, && \text{(D.2.4)}
\end{aligned}$$

which reveals that the tractability of this particular setup depends on the nature of the deterministic function $f(\cdot)$. First, take note that the third exponential D.2.4 may be thought of as a constant since all quantities are $\mathcal{F}_t$-measurable, and this exponential will be eliminated from the model since it will appear in both the numerator and denominator of equation (D.2.1). The second exponential (D.2.3) is generally straightforward to deal with; however, the first exponential (D.2.2) and the nature of the function $f(\cdot)$ plays a significant role in determining whether the eventual pricing kernel model is a Markov process or not. For instance, if $f(s) = c$, where $c \in \mathbb{R}$, as is the case with the infinite horizon mean-reverting process, then the exponent in (D.2.2) becomes $\frac{\alpha}{\beta^2}X[cI_t - cI_0 + c^2\int_0^t I_s\mathrm{d}s]$, which reveals the non-Markov structure.

To avoid this, one may demand that $\frac{\mathrm{d}}{\mathrm{d}t}f(t) = f(t)^2$, which is a straightforward ordinary differential equation that admits the solution $f(t) = 1/(c-t)$, for all $t \in \mathbb{R}\setminus\{c\}$. This choice of constraint therefore also recovers the classical finite horizon mean-reverting process, given in Lemma 9.3.2, and the change-of-measure density martingale process now becomes

$$\begin{aligned}
\Gamma_t =\; & \exp\left(\frac{\alpha}{\beta^2}X\left[f(t)I_t + f(0)I_0\right] - \frac{\alpha^2}{2\beta^2}X^2[f(t) - f(0)]\right) \\
& \times \exp\left(-\frac{1}{\beta^2}\int_0^t f(s)I_s\mathrm{d}I_s - \frac{1}{2\beta^2}\int_0^t f(s)^2 I_s^2\mathrm{d}s\right) \,,
\end{aligned}$$

revealing a structure that will now enable a Markov process. Defining the following useful functions:

$$\begin{aligned}
a_{i,t} = a(I_t,t) &:= \frac{\alpha}{\beta^2}\left[f(t)I_t - f(0)I_0\right] - t \,, \\
b_t = b(t) &:= \frac{\alpha^2}{2\beta^2}\left[f(t) - f(0)\right] \,, \\
c_{i,t} = c(I_t,t) &:= -\frac{1}{\beta^2}\int_0^t f(s)I_s\mathrm{d}I_s - \frac{1}{2\beta^2}\int_0^t f(s)^2 I_s^2\mathrm{d}s \,,
\end{aligned}$$

it is possible to write the expectation in the numerator of equation (D.1.1) in the compact form:

$$\mathbb{E}^{\mathbb{F}}\left[\exp(-Xt)\Gamma_t \,|\, \mathcal{F}_t\right] = \exp(c_{i,t})\sum_{j=1}^n p_j \int_{-\infty}^{\infty}\exp\left(a_{i,t}x - b_t x^2\right)\frac{1}{\sqrt{2\pi}\sigma_j}\exp\left(-\frac{(x-\mu_j)^2}{2\sigma_j^2}\right)\mathrm{d}x \,,$$

and then following a similar strategy as that adopted in Example 1, see section D.1, it follows that

$$\mathbb{E}^{\mathbb{F}}\left[\exp(-Xt)\Gamma_t \,|\, \mathcal{F}_t\right] = \exp(c_{i,t}) \sum_{j=1}^{n} \frac{p_j}{\sqrt{v_{j,t}}} \exp\left(\frac{n_{j,i,t}}{2v_{j,t}}\right) \int_{-\infty}^{\infty} \frac{\sqrt{v_{j,t}}}{\sqrt{2\pi}\sigma_j} \exp\left(-\frac{(x-m_{j,i,t})^2}{2\sigma_j^2/v_{j,t}}\right) \mathrm{d}x \ ,$$

where

$$\begin{aligned} m_{j,i,t} = m_j(I_t,t) &:= \frac{\mu_j + \sigma_j^2 a_{i,t}}{1+2\sigma_j^2 b_t} \ , \\ v_{j,t} = v_j(t) &:= 1 + 2\sigma_j^2 b_t \ , \\ n_{j,i,t} = n_j(I_t,t) &:= \sigma_j^2 a_{i,t}^2 + 2\mu_j a_{i,t} - 2\mu_j^2 b_t \ , \end{aligned}$$

so that one may finally obtain

$$\mathbb{E}^{\mathbb{F}}\left[\exp(-Xt)\Gamma_t \,|\, \mathcal{F}_t\right] = \exp(c_{i,t}) \sum_{j=1}^{n} \frac{p_j}{\sqrt{1+2\sigma_j^2 b_t}} \exp\left(\frac{\sigma_j^2 a_{i,t}^2 + 2\mu_j a_{i,t} - 2\mu_j^2 b_t}{2(1+2\sigma_j^2 b_t)}\right) . \tag{D.2.5}$$

In a similar manner it is possible to show that

$$\mathbb{E}^{\mathbb{F}}\left[\Gamma_t \,|\, \mathcal{F}_t\right] = \exp(c_{i,t}) \sum_{j=1}^{n} \frac{p_j}{\sqrt{1+2\sigma_j^2 b_t}} \exp\left(\frac{\sigma_j^2 d_{i,t}^2 + 2\mu_j d_{i,t} - 2\mu_j^2 b_t}{2(1+2\sigma_j^2 b_t)}\right) , \tag{D.2.6}$$

with the only difference being the functional $a_{i,t}$ from equation (D.2.5), which is now replaced with $d_{i,t} := \frac{\alpha}{\beta^2}[f(t)I_t - f(0)I_0]$, as shown in equation (D.2.6). Unfortunately, it is not possible to present the auxiliary pricing kernel in a more compact form because of the nature of the mixture distribution. However, assuming just one component, i.e., $n = 1$, it is possible to show that

$$\begin{aligned} \xi_t = \mathbb{E}^{\mathbb{M}}\left[\exp(-Xt) \,|\, \mathcal{F}_t\right] &= \exp\left(\frac{\sigma_1^2(-2d_{i,t}t + t^2) - 2\mu_1 t}{2(1+2\sigma_1^2 b_t)}\right) \\ &= \exp\left(\frac{\beta^2\sigma_1^2 t^2 - 2\alpha\sigma_1^2 t[f(t)I_t - f(0)I_0] - 2\mu_1\beta^2 t}{2(\beta^2 + \alpha_1^2\sigma_1^2[f(t)-f(0)])}\right) , \end{aligned}$$

and following similar reasoning to that applied in Example 1, see section D.1, the value of a T_1-maturity ZCB becomes

$$P_{tT_1} = \exp\left(\frac{\beta^2\sigma_1^2(T_1^2 - t^2) - 2\alpha\sigma_1^2(T_1 - t)[f(t)I_t - f(0)I_0] - 2\mu_1\beta^2(T_1 - t)}{2(\beta^2 + \alpha_1^2\sigma_1^2[f(t)-f(0)])}\right) ,$$

for $t \in [0, T_1]$ and assuming that $f(t) = 1/(T-t)$, with $T_1 \leq T$. As was the case with Example 1, constructing a suitable change-of-measure from $\mathbb{M}$ to $\mathbb{P}$, which may be achieved as in Remark 10.2.1, will then enable one to consider the task of statistical estimation using real-world time-series data for ZCB prices or yields.

D.3 Example 3 — Modelling Monetary Policy Decisions

The final example that is considered essentially combines the models that have been presented in sections D.1 and D.2. The idea is to present a potential practical application of the information filtering framework — the problem is modelling a bank funding primitive and derivative market, such as the OIS market, in a manner that is explicitly cognisant of monetary policy activity and interest rate changes related thereto. The model from Example 1, see section D.1, may be utilised to capture the long-term dynamics of the market under consideration, while the model from Example 2, see section D.2, may be utilised to model individual monetary policy decisions at known fixed future dates or times.

The following is an example of how one may construct a simple model that is cognisant of a single monetary policy decision date. Consider the following pair of information processes:

$$\begin{aligned} \mathrm{d}I_t^{(0)} &= X_0 \,\mathrm{d}t + \mathrm{d}W_t^{(0)} \,, \\ \mathrm{d}I_t^{(1)} &= \frac{1}{T-t}\left[X_1 - I_t^{(1)}\right] \mathrm{d}t + \mathrm{d}W_t^{(1)} \,, \end{aligned}$$

where $X_0 \sim N(\mu_0, \sigma_0^2)$ and $X_1 \sim N(\mu_1, \sigma_1^2)$, with X_0 being independent of X_1. The first process is assumed to model long-term dynamics of the interest rate market, while the second process is assumed to model a monetary policy decision that takes place at time T. Therefore, the actual decision is modelled by the market factor X_1, which is assumed to be normally distributed however one may choose a more appropriate distribution such as a discrete multinomial type of distribution.

Again, assuming the simplest version of the Nelson-Siegel-Svensson inspired function, from Definition 10.2.1, the auxiliary pricing kernel may be modelled as

$$\begin{aligned} \xi_t &= \mathbb{E}^{\mathbb{M}}\left[\exp(-X_0 t) \,|\, \mathcal{F}_t\right] \\ &= \exp\left(\frac{\sigma_0^2 t^2 - 2\sigma_0^2 t[I_t^{(0)} - I_0^{(0)}] - 2\mu_0 t}{2(1+\sigma_0^2 t)}\right) \,, \end{aligned}$$

for $t \in [0, T)$, which follows from the results derived in section D.1, and

$$\begin{aligned} \xi_t &= \mathbb{E}^{\mathbb{M}}\left[\exp[-X_0 t - X_1(t-T)] \,|\, \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{M}}\left[\exp(-X_0 t) \,|\, \mathcal{F}_t\right] \exp(-X_1(t-T)) \\ &= \exp\left(\frac{\sigma_0^2 t^2 - 2\sigma_0^2 t[I_t^{(0)} - I_0^{(0)}] - 2\mu_0 t}{2(1+\sigma_0^2 t)}\right) \exp(-X_1(t-T)) \,, \end{aligned}$$

for $t \in [T, \infty)$, since X_1 is $\mathcal{F}_t$-measurable. Therefore, the auxiliary pricing kernel captures the impact of the monetary policy decision in an *ex post* manner. However, the pricing of a ZCB will capture this potential change in interest rates in an *ex ante* manner, which is shown next.

Consider a U-maturity ZCB, with $U > T$, then for $t \in [0, T)$ the price process is

$$\begin{aligned} P_{tU} &= \frac{\mathbb{E}^{\mathbb{M}}\left[\exp(-X_0 U - X_1(U-T)) \,|\, \mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{M}}\left[\exp(-X_0 t) \,|\, \mathcal{F}_t\right]} \\ &= \exp\left(\frac{\sigma_0^2(U^2 - t^2) - 2\sigma_0^2(U-t)[I_t^{(0)} - I_0^{(0)}] - 2\mu_0(U-t)}{2(1+\sigma_0^2 t)}\right) \\ &\quad \times \exp\left(\frac{\sigma_1^2(U-T)^2 - 2\sigma_1^2(U-T)[I_t^{(1)}/(T-t) - I_0^{(1)}/T] - 2\mu_1(U-T)}{2(1+\sigma_1^2 t/(T-t))}\right) \,, \end{aligned}$$

and for $t \in [T, U]$ the price process is

$$\begin{aligned} P_{tU} &= \frac{\exp(-X_1(U-T))\mathbb{E}^{\mathbb{M}}\left[\exp(-X_0 U) \,|\, \mathcal{F}_t\right]}{\exp(-X_1(t-T))\mathbb{E}^{\mathbb{M}}\left[\exp(-X_0 t) \,|\, \mathcal{F}_t\right]} \\ &= \exp\left(-X_1(U-t)\right) \times \exp\left(\frac{\sigma_0^2(U^2 - t^2) - 2\sigma_0^2(U-t)[I_t^{(0)} - I_0^{(0)}] - 2\mu_0(U-t)}{2(1+\sigma_0^2 t)}\right) \,, \end{aligned}$$

which reveals the fact that X_1 is $\mathcal{F}_T$-measurable. The results for both of these scenarios are based on results derived in sections D.1 and D.2. Of course, this model may be extended to cater for multiple monetary policy decision dates, and adapted to feature far more realistic model dynamics.