

LONGITUDINAL STABILITY AND CONTROL ANALYSIS AND PARAMETER
SENSITIVITY INVESTIGATION OF FIXED WING AIRCRAFT IN GROUND
PROXIMITY USING VARIOUS AERODYNAMIC APPROACHES

by

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In loving memory of Oubaas, Ounooi and Saar

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SUMMARY

The investigation of the stability of longitudinal motion of fixed wing aircraft in ground proximity presented here focuses on rectangular wing and tail configurations similar to that of the Piper Cherokee PA-28-180. The majority of our results pertain to the height range down to one quarter of the wing span, and the main objectives of the investigation were the following:

- to reveal the physical nature and causes of the instability encountered in close ground proximity
- to assess the influence that design parameters such as centre of mass placement, tail length, tail height and wing aspect ratio have on the stability of longitudinal motion
- to suggest a feedback control strategy that would alleviate the instability

The first objective was accomplished by solving the trim problem at various heights above the ground, and subsequently computing, for each height, the stability and control derivatives of perturbed longitudinal motion relative to the trim state of flight. The trim problem was solved numerically by a multivariable Newton-Raphson iteration scheme that simultaneously finds the pitch angle that yields the required lift coefficient and the tail angle that zeroes the pitch moment coefficient. The lift, drag and pitch moment coefficients required to this end were obtained by the Vortex Lattice Method, which was implemented in a Matlab program. The program computes the induced drag coefficient by three methods: firstly, by using the conventional vortex lattice approach based on the computation of the induced downwash at the bound vortex midpoints, secondly, by using the leading edge suction approach based on the solution of the strength distribution of the leading edge singularity of the bound vortex distribution, and thirdly, by the far-field approach, which is essentially a method based on a lifting line approximation of the wing and tail and their images below the ground plane. Excellent agreement of these results was obtained for a rectangular wing of aspect ratio 2, but a 4 % discrepancy was found for the Warren 12 planform. This discrepancy, however, does not affect our final conclusions, because we concentrated on rectangular planform shapes. The vortex lattice results obtained for a steadily pitching wing of very high aspect ratio and with non-zero

angle of attack were found to agree very closely with the exact results obtained by two-dimensional thin wing theory. The drag reduction factor due to ground proximity was found to agree very well with published experimental results in the height range that was of interest here, and it was thus concluded that a stability and control analysis based on aerodynamic derivatives obtained from our vortex lattice algorithm should be quite reliable.

The second objective was accomplished by application of Routh's stability criteria to and root locus analysis of the characteristic equation of longitudinal motion, augmented by a state space solution of the response to a unit impulsive input to the all-movable tail. This revealed that the instability encountered in close ground proximity is due either to the characteristic root of the heave mode of motion becoming unstable, or the complex conjugate pair of phugoid roots becoming unstable. For some aircraft configurations the phugoid roots remain stable while the heave root becomes unstable, and vice versa for other configurations. If the phugoid remains stable, then its damping ratio is found to be less than critical initially, increasing sharply with closer ground proximity, and eventually yielding two monotonically convergent roots. The phugoid mode of motion then remains stable in ground proximity, as does the short period mode of motion. (The latter experiences some increase in frequency and a small decrease in damping in closer ground proximity, but is not affected drastically at all in the height range considered here.) If the phugoid remains stable, then the eventual instability of the heave root is found to be caused by C_{Lz} exceeding a certain critical value. If, on the other hand, the heave root remains stable, then the damping ratio of the phugoid mode decreases with closer ground proximity, eventually becoming an oscillatory divergence that is due to C_{Dz} exceeding another critical value. It was found that not only the unstable heave root, but also the phugoid roots can be approximated with remarkable accuracy (compared to the roots obtained from the full fifth order characteristic equation) by approximate factorisation of a third order polynomial judiciously extracted from the fifth order characteristic polynomial of longitudinal motion. The critical values of C_{Lz} and C_{Dz} mentioned above are found by application of Routh's stability criteria to this third order factor, a polynomial that is essentially a phugoid-and-heave characteristic expression. The extraction of this polynomial was done on the quite reasonable assumption that the pitch

moment of inertia offers negligible resistance to the very low pitch angular acceleration encountered in the slow modes of motion.

The results of the sensitivity analysis, as judged by the critical ratio b_w/H_{crit} of wing span to CM height, led to the following conclusions regarding longitudinal stability in ground proximity:

- a rearward shift in centre of mass is mildly destabilising in the CM range of 17.5 % to 25 % of chord behind the wing's leading edge, reducing b_w/H_{crit} from 1.30 to 0.98, respectively
- an increase in load is strongly stabilising, increasing b_w/H_{crit} from 0.69 to 1.71 when the load is increased from 0.75 to 1.25 times the nominal load
- an increase in tail length is mildly destabilising, as evidenced by a decrease in b_w/H_{crit} from 1.36 to 0.99 when the distance between aerodynamic centres of the wing and tail is increased from 0.75 to 1.25 times the nominal design value
- an increase in tail height h_t is strongly stabilising, as seen in the increase in b_w/H_{crit} from 1.14 to around 2 when the height of the aerodynamic centre of the tail above that of the wing is increased from 0 to 1.5 times the nominal design value. And our high tail configuration displayed a phugoid instability due to $C_{Dz} > C_{Dzcrit}$, rather than the heave instability encountered in most of our other analyses
- an increase in wing aspect ratio is mildly destabilising, as reflected by a decrease in b_w/H_{crit} from 1.30 to 1.06 when the wing aspect ratio is increased from 4 to 10. This influence becomes weaker and weaker with increasing aspect ratio
- none of these design changes on their own will stabilise the aircraft completely over the whole of the height range down to $\frac{1}{4}$ of the wing span, but an increase in tail height has the strongest stabilising effect of all.

Our final objective of alleviating the longitudinal instability encountered in ground proximity was achieved by feedback of the height perturbation to the control input of the all-movable tail. This strategy was motivated on physical grounds by the observation that not only the critical value C_{Lzcrit} below which the height derivative C_{Lz} of the lift coefficient must remain for stability, but also the critical value C_{Dzcrit} below which the

height derivative C_{Dz} of the drag coefficient must remain for stability, primarily depend on the height derivative C_{Mz} of the pitch moment coefficient. It was found, as can be expected naturally, that C_{Mz} is zero out-of-ground-effect, but positive in ground proximity, increasing as the height decreases. This increase reduces the critical limit C_{Lzcrit} (which is negative) of C_{Lz} (which is also negative) more rapidly than C_{Lz} is reduced by ground proximity, until C_{Lz} exceeds C_{Lzcrit} and instability sets in. It was therefore argued that one could reasonably expect design changes that primarily reduce C_{Mz} to improve longitudinal stability in ground proximity, irrespective of whether longitudinal instability occurs due to $C_{Lz} > C_{Lzcrit}$ or to $C_{Dz} > C_{Dzcrit}$. This argument led us to expect that negative feedback of height should improve stability. The reason for this is that such feedback will decrease the tail angle (nose down with respect to the wing chord is positive) when a positive (upward) height perturbation occurs, producing a reduction of pitch moment that thus reduces C_{Mz} . A root locus analysis with values of feedback gain that reduces the value of C_{Mz} by 25%, 50% and 75%, followed up by a state space analysis that allows for feedback in the computation of the impulse response, showed that this control strategy is indeed very effective in stabilising the aircraft. One finds that, in the height range down to $\frac{1}{4}$ the wing span, the instability is removed altogether. The characteristics of not only the phugoid mode, furthermore, but those of the short period mode too, becomes essentially insensitive to ground height when a scheduled feedback gain that consistently reduces C_{Mz} by 50% is used. Close scrutiny of stability results obtained for the closed loop state space system showed that, even though this feedback of height to the tail naturally affects not only C_{Mz} , but C_{Lz} and C_{Dz} too, the percentage change of the latter two stability derivatives is much less than that of C_{Mz} . Thus, by sound physical reasoning, one arrives at a very simple control strategy, one that essentially moves the stability boundary C_{Lzcrit} further away from C_{Lz} , and C_{Dzcrit} further away from C_{Dz} , without affecting anything else much. It was found, finally, that the effect of height feedback as suggested here is very similar to that of increasing the tail height, probably because the main influence of the latter is also a pronounced reduction in C_{Mz} .

The feedback strategy employed by Staufenbiel, which relies on perfect regulation of air speed, and feeds back not only the height, but also its time rate of change, was reported to produce a strong reduction in the damping of the short period mode, which must be

regarded as highly undesirable from a handling qualities perspective. We therefore conclude that feeding back the height perturbation to the all-movable tail, using the simple gain scheduling control strategy suggested here, offers a dramatic improvement over the control characteristics of the strategy employed by Staufenberg.

NOMENCLATURE

- A = aircraft reference point; coefficient of Fourier series expansion of vortex distribution
 b = span width
 c = chord length
 c_l = sectional lift coefficient
 c_m = sectional pitch moment coefficient
 C = aerodynamic coefficient
 C_{Di} = far field induced drag coefficient (from Trefftz plane analysis)
 C_{Dii} = near field induced drag coefficient (by leading edge singularity analysis)
 C_{DiiV} = near field induced drag coefficient (by conventional vortex lattice method)
 D = drag
 $\bar{e}$ = unit vector
 h_t = height of leading edge of tail above that of wing (along vertical)
 h_{to} = height of leading edge of tail above that of wing (normal to wing chord)
 $h_N, \Delta h$ = $-C_{M\alpha} / C_{L\alpha}$, the static stick-fixed longitudinal stability margin
 $h_{Nz}, \Delta h_z$ = $-C_{Mz} / C_{Lz}$, static height stability margin
 H = height of CM above ground plane
 H_{LEW} = height of leading edge of wing above ground plane
 i_{yy} = dimensionless pitch moment of inertia $8I_{yy} / (\rho S c^3)$
 k_D = propulsion parameter (2 for jets, 3 for propeller aircraft)
 $\vec{K}$ = resultant force acting on the air in a bound vortex field
 ℓ_t = distance of leading edge of tail aft of that of wing (along horizontal)
 ℓ_{to} = distance of leading edge of tail aft of that of wing (along wing chord)
 L = lift
 M = pitch moment
 m, n = maximum values of indices
 N_c = number of panels on chord
 N_s = number of panels on span

- N_v = total number of horseshoe vortices
 q = dimensionless perturbation of pitch rate $\Delta Q c / (2V_{\infty 0})$
 $\vec{r}$ = position vector, relative to A, of an arbitrary point in the flow field
 Q = pitch rate
 $\vec{s}$ = position, relative to A, of an *unspecified* point on aerodynamic surface
 $\vec{s}_0$ = position vector relative to A of a *specific* point on aerodynamic surface
 S = planform area; vortex sheet
 u = dimensionless air speed perturbation $\Delta V_{\infty} / V_{\infty 0}$
 u_i = local rearward induced velocity
 v_i = local sideward induced velocity (toward starboard wing tip)
 $\vec{V}$ = resultant air velocity of free stream and induced flow field
 $\vec{V}_A$ = translational velocity vector of aircraft reference point A
 $\vec{V}_i$ = induced velocity vector
 $\vec{V}_r$ = local wind velocity component due to angular velocity of aircraft
 $\vec{V}_R^C$ = resultant *relative wind* velocity, *excluding discontinuous component*
 $\vec{V}_R$ = resultant relative wind velocity, *including discontinuous component*
 $\vec{V}_{\infty}$ = free stream velocity vector
 w_i = local *upward* induced velocity
 W = weight
 x, ξ = chordwise co-ordinates on vortex sheet
 y, η = spanwise co-ordinates on vortex sheet
 z = dimensionless height perturbation $\Delta H/c$ of CM
 z, v = normal co-ordinates on vortex sheet
 X, Y, Z = co-ordinate differences between vortex corner points and induction point
 X_{AC} = distance between midspan $1/4$ -chord and mean aerodynamic $1/4$ -chord

 α = angle of attack; perturbation of angle of attack
 α_d = angle of descent
 α_i = local induced downwash angle

- α_Q = local increase in angle of attack due to pitch rate
 β_d = dihedral angle
 β_i = dimensionless local induced sideslip angle
 γ = spanwise component of vortex strength; angle of climb
 $\bar{\gamma}, \bar{\Omega}$ = vorticity (following Prandtl³ and Karamcheti¹², respectively)
 Γ = circulation
 δ_t = tail angle, leading edge down relative to wing chord
 δ_P = power control input
 δ_T = thrust control input
 θ = perturbation of pitch angle
 θ_c = slope of camber line
 Θ = nose up pitch angle of wing chord relative to horizontal
 τ = dimensionless time $2V_{\infty} t / c$
 μ = dimensionless mass $2m / (\rho S c)$
 $\bar{\zeta}$ = vortex strength (following Karamcheti¹²)
 χ_i = dimensionless local rearward induced velocity
 χ_Q = dimensionless local rearward velocity due to pitch rate
 $\bar{\omega}$ = angular velocity of aircraft

Subscripts

- A = aircraft reference point
 a = port corner point of horseshoe vortex; aerodynamic
 av = average
 B = bound vortex
 b = starboard corner point of horseshoe vortex
 BI = bound vortex image
 c = control point of horseshoe vortex
 $c/4$ = evaluated at quarterchord point
 $crit$ = critical value obtained from Routh's stability criteria
 D = drag

F	=	free (trailing) vortex
FI	=	free (trailing) vortex image
g	=	resolved in geodetic reference frame
i	=	induced; index
j, k	=	indices
ℓ	=	lower surface
$\ell e, LE$	=	leading edge
L	=	lift
m	=	maximum index value
M	=	pitch moment
MC	=	evaluated at midchord point
n	=	normal component; maximum index value
Q	=	due to pitch rate
t	=	component tangential to aerodynamic surface; tail
te, TE	=	trailing edge
tl	=	tail image
u	=	upper surface
w	=	wing
wI	=	wing image
x	=	component along chordwise direction
y	=	component along spanwise direction
α	=	aerodynamic derivative with respect to angle of attack
q	=	aerodynamic derivative with respect to dimensionless pitch rate
θ	=	aerodynamic derivative with respect to angle of attack pitch angle
z	=	aerodynamic derivative with respect to dimensionless height change
δ_t	=	aerodynamic derivative with respect to angle of elevator angle
δ	=	pertaining to aerodynamic control inputs

Superscripts

- C = Cauchy principal value (induced by Complementary vortex system, *excluding local vortex element*)
- g = resolved in geodetic reference frame
- L = induced by Local vortex element alone
- LE = measured relative to leading tip of midspan chord

Abbreviations

- CVLM = conventional vortex lattice method
- PLLM = Prandtl's lifting line method
- QVLM = quasi-vortex lattice method
- VLM = vortex lattice method
- WLLM = Weissinger's lifting line method

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1 INTRODUCTION

The main influences of ground proximity on the aerodynamic behaviour of an aircraft are the well-known reduction of drag, an increase in lift, a rearward shift of the neutral point and deterioration of longitudinal stability. The focus of this study is the extent and the nature of, and the physical causes for the deterioration of the longitudinal stability characteristics of a fixed wing aircraft in ground effect. The stability characteristics of an aircraft are *aeromechanical characteristics* that describe the nature of its flight, as opposed to the *aerodynamic characteristics* that describe the aerodynamic forces and moments it experiences in flight. We take the liberty of interpreting the *main problem of fixed wing aerodynamics* to be that of finding these aerodynamic characteristics for an aircraft of known geometry (basic aircraft data, angle of attack, elevator angle, etc.) in a known state of flight (air speed, descent rate, height, etc.). The *main problems of fixed wing flight mechanics*, on the other hand, are interpreted to be those of a) finding the geometry, particularly the angle of attack and elevator angle (that is, in longitudinal motion) that would yield equilibrium of forces and moments if the equilibrium flight condition is specified in terms of load, airspeed, angle of descent and height – *this is the trim problem*, and b) finding the motion that follows a disturbance from this equilibrium (e.g. due to a gust or a control input) – *these are the dynamic stability and control problems*.

The study of aeromechanical behaviour thus involves the consideration of unsteady aircraft motion, along with the aerodynamic forces to which it is due. Strictly speaking, one should therefore consider unsteady aerodynamic effects from the outset. One finds, however, that valuable perspectives can be gained by considering just the quasi-static aerodynamic effects. We will argue, furthermore, that the inclusion of unsteady aerodynamic effects should yield a longitudinal stability assessment that will probably not differ *essentially* from one obtained by a quasi-static approach, at least insofar as the stability of the long period motion is concerned.

The aeromechanical characteristics of fixed wing aircraft in ground effect has received much less attention over the years than their aerodynamic characteristics in ground effect. It is hoped that this study will contribute towards a better understanding of the nature and causes of the aeromechanical instabilities encountered in close ground proximity, namely the instability of the phugoid and heave modes of motion. In the conclusion of the present investigation we suggest a strategy for alleviating this instability by feedback control, one that is based on gain scheduled feedback of a measured height perturbation to the elevator control input. This gain scheduling is aimed at artificially reducing the observed pitch moment perturbation (per unit height perturbation) by a constant ratio, irrespective of height. Such a strategy may be of practical use in both manned and unmanned flight vehicles that operate in close ground proximity for extended periods of time, as it can stabilise the aircraft in the latter case, and reduce the pilot work load considerably in the former.

The investigation of aircraft stability in ground effect requires a solid background of not only the aerodynamics and flight mechanics of fixed wing aircraft, but preferably also of the state space analysis of their dynamic motion and control. The vast amount of available literature on aerodynamics and flight mechanics makes it impossible to provide a really comprehensive survey in the short introduction that we give here. We therefore limit ourselves to a necessarily brief overview that concentrates on:

- some of the pioneering aerodynamic theories developed nearly a century ago
- a few novel applications of and some highly instructive perspectives on these theories contributed by other authors up to the 1960's
- some milestone contributions, up to the late 1970's, towards the computational algorithms that are available today, specifically the Vortex Lattice Method (VLM)
- further refinements of the VLM, such as the treatment of free wake and unsteady models, to which many authors have devoted themselves over the past two decades
- some aspects of the longitudinal dynamics and control of fixed wing aircraft in ground effect

Some of the major aerodynamic developments of the past century are presented in the approximate chronological order of their publication here, but we have made no attempt to make absolutely sure of dates of *first* publication of key developments, nor to attribute these with absolute confidence to specific authors, even though we do quote other authors in these respects. The subsequent overview of longitudinal dynamics and control theory is intended specifically to provide a brief sketch of the background against which the present investigation was undertaken, rather than to trace the history of flight mechanics.

We have consulted most of our references first hand, but not all of them. Wherever we used second hand information, however, we have made a point of implying so by employing phrases such as: "... was reported by X to have proved Y." Even though, in these cases, we do supply bibliographic details in our list of references for the benefit of readers that may have access to such sources, we can obviously not guarantee the correctness of such details. Finally, the literature survey presented below is augmented later, where the need arises, by additional references and discussions more appropriately included in the subsequent chapters.

1.1 Aerodynamics

The methods and algorithms for aerodynamic analysis available to us today could not have been developed, had it not been for the groundbreaking theories developed by the early pioneers of modern aerodynamics at and around the end of the 19th century. Among these men the best known are surely Lanchester¹, Kutta², Joukowski and Prandtl³, the latter crediting Betz⁴ (see for example Prandtl³, pp.343), Lagally⁵ and Munck⁶ with having made notable contributions too. Extensions of the new theories appeared even while Prandtl's work was still in progress, and soon after, for example the ground effect work of Betz⁴ and Wieselsberger^{7, 8}, and the alternative lifting line method and the lifting surface method proposed in the 1940's by Weissinger⁹. Scholars such as Von Mises¹⁰, Milne-Thomson¹¹ and Karamcheti¹², to name but a few, contributed a range of keen perspectives in text books that eventually became classics too. These works paved a difficult stretch of the road for investigators in the 1960's and '70's, such as Wagner¹³, Lan¹⁴ and Miranda¹⁵, to bring aerodynamics to the point where previously untenable or

computationally daunting problems could be addressed. The state of the art today is that the free wake, unsteady analysis of the aerodynamic and/or aeromechanical behaviour of a wing or aircraft in ground effect yields results that agree well with wind tunnel and/or test flight measurements. Some reports of the 1980's and '90's, reports that each treat one or more of these aspects (i.e., the free wake and/or unsteady and/or ground effect modelling of aerodynamic and/or aeromechanical behaviour) are those of Konstadinopoulos¹⁶ et al., Du Plessis¹⁷, Chang and Muirhead¹⁸, Staufenbiel and Schlichting¹⁹, Suh and Ostowari²⁰, Nuhait and Mook²¹, Nuhait and Zedan²², Elzebda²³ et al., Coulliette and Plotkin²⁴, and Devinant²⁵.

In some of the discussions that follow now, and in the subsequent chapters, we will make use of the following definitions and symbols regarding vortex concepts:

- *Vorticity* is defined as the curl of velocity (m/s per m). Following Karamcheti¹² we use the symbol $\vec{\Omega}$ for the vorticity vector, even though Prandtl used $\vec{\gamma}$.
- *Circulation* Γ is defined as the line integral of velocity (m²/s).
- The *vortex strength* is denoted here by $\vec{\zeta}$, again in accordance with Karamcheti, even though other authors prefer $\vec{\gamma}$. It is a vector quantity that describes the strength of a vortex sheet, and its SI units are m/s. We interpret vortex strength from Karamcheti to be defined by $\vec{\zeta} = \lim_{\varepsilon \rightarrow 0} \varepsilon \vec{\Omega}$. This limit is envisaged by imagining that the region in which non-zero vorticity occurs is shrunk into a thin layer of which one then allows the thickness ε to approach zero. The vortex sheet thus obtained is an idealisation of the boundary layer and the trailing vortex layer of a wing in a real air stream. This definition of vortex strength, and the definition of vorticity in terms of spatial derivatives of the velocity vector, yields the result that the strength of a vortex sheet is finite everywhere, except perhaps at a sharp leading or side edge, despite the fact that the vorticity *inside* the vortex sheet is of infinite magnitude. (The vortex sheet *can* be thought of as having an inside, because even though its thickness *approaches* zero, it is not zero.)
- The *vortex distribution*, denoted by γ in many aerodynamic texts, is defined such that it represents the spanwise component ζ_y of the vortex strength defined above,

whereas ζ_x is the chordwise component of vortex strength. At places we will probably slip up and speak of vortex distribution when we mean vortex strength, and vice versa, but even when this occurs, the intended meaning should be clear in context. The components ζ_x and ζ_y of vortex strength (the notation γ_x and γ_y is perhaps more conventional, judging by Devinant²⁵) are quite useful in the integral formulas for the induced velocity components.

1.1.1 Pioneers of Modern Aerodynamics

Lanchester¹ is credited, for example by Prandtl³, as the first to suggest the close connection between the generation of wing lift and the circulation of flow around the wing. Lanchester also saw that, in the case of a finite wing, the vortex to which this circulation is attributed must be shed rearward into the free stream in the form of tip vortices, as shown schematically in Figure 1.1. (These “tip vortices” are actually created progressively by the trailing vortex sheet that rolls up with increasing distance behind the wing, as shown more accurately in Figure 1.2.) He realised furthermore, as mentioned by Milne-Thomson¹¹, that the shed vortices induce drag. For these two insights Milne-Thomson regards Lanchester as “in truth the pioneer and founder” of modern aerodynamics. Prandtl states in addition that, even though Lanchester was the first to formulate these and other perspectives, he did not proceed to the formulation of a quantitative relationship between circulation and lift. This breakthrough was accomplished independently by Kutta² (1902) and Joukowski (1905), who found that the lift distribution L' of a wing, i.e., its aerodynamic force component normal to the free stream, expressed per unit spanwise increment, is given by the product $L' = \rho V \Gamma$ of the fluid density ρ , the speed V of the free stream and the circulation Γ around the local wing section.

This relationship alone between circulation and lift, however, does not complete the picture, because the ideal-fluid model of a wing, finite or infinite, and even at a prescribed orientation in a uniform air stream, leaves the circulation around it entirely undetermined, which means that ideal-fluid aerodynamic theory could not yet predict the lift of a wing at all. The last missing piece of the puzzle was provided by Kutta's

hypothesis, also attributed to Joukowski (Von Mises¹⁰, pp. 178, 225), but now known as the Kutta condition, which states that the air flows smoothly off the trailing edge, not around it. This condition collapses the infinity of possible solutions for the flow field around a wing into a single solution that yields a unique circulation value, and therefore yields the unique lift solution that one expects intuitively, *but only if the wing has a sharp trailing edge*. Not only does a sharp trailing edge produce minimum drag, but to quote Von Mises¹⁰ again, wings have been designed with sharp trailing edges since the very beginning of flight, because this produces a well-defined lift. (Ideal-fluid theory leaves the circulation around a wing, and therefore the lift it produces, completely undetermined if the trailing edge is well-rounded.)

We now turn to Prandtl³, who was the first to formulate a quantitative theory of finite wings. In the general fundamentals (“allgemeine Grundlegung”) of his famous first exposition of lifting wing theory (“Tragflügeltheorie”) Prandtl explained, among many other ideas, that:

- the generation of lift requires, on average, a higher pressure below than above the wing, requiring by the Bernoulli equation that, on average, the tangential velocity on the lower wing surface must be lower than on the upper surface. This, in turn, is possible only if an additional forward velocity component occurs on the lower surface (over and above the free stream velocity), and a rearward component on the upper surface, thus creating the circulation of flow that generates the lift.
- a finite wing in steady flow must shed vortices into the free stream
- vortices are shed into a trailing vortex sheet, a surface across which the velocity is discontinuous, and such a vortex sheet will originate at a confluence line (“Zusammenflußlinie”), i.e. a line where the flows from the two sides of a finite wing meet
- the velocity at a sharp trailing edge is finite (as demanded by the Kutta condition)
- rigid bodies may be replaced by vortex sheets, provided that the condition of no penetration is satisfied everywhere on their surfaces. On the idea that the solid boundaries of aerodynamic bodies may be replaced by vortex sheets Prandtl refers to the work of Lagally⁵.

Prandtl formulated a general theory of steady motion, a theory that eventually yields a formula expressing the resultant force acting on a lifting volume as the sum of two volume integrals that depend on the spatial vorticity distribution. The first integral accounts for the component of the resultant force that is due to the *free stream velocity field* and the vorticity field, whereas the second accounts for the resultant induced force, i.e. the force component due to the *induced velocity field* and the vorticity field. Up to this point Prandtl made no simplifying assumptions other than steady incompressible flow in either of these integrals. He gave credit to his co-worker Munk⁶ for the insight that the self-induced force resultant arising from the induced velocity field of a bound vortex system is identically zero. (Munk's result stems from the facts that a) the mutually induced incremental forces of any two infinitesimal vortex filaments that are parallel cancel identically, even if they are of unequal length and unequal circulation strength, and b) that an arbitrary pair of vortex filaments, regardless of whether they are parallel and/or of equal strength, can be represented by a set of six vortex filaments consisting of three pairs of parallel vortex filaments. This leads to the conclusion that the induced force integral contains a term that is identically zero, namely the term that represents the resultant self-induced force of the whole bound vortex system.)

Introducing the concept of a lifting line (a line carrying a finite amount of circulation, not necessarily distributed uniformly), *but one of which he does not approximate the position, shape or local downwash velocity yet, not by any assumptions regarding the spatial vortex distribution*, Prandtl³ showed that the resultant force *acting on the fluid in a bound vortex field* is given by:

$$\vec{K} = \rho \iiint (\mathrm{d}\vec{s} \times \vec{v}) \vec{\gamma} \cdot \mathrm{d}\vec{F} \quad (1.1)$$

in which he used $\mathrm{d}\vec{s}$, $\mathrm{d}\vec{F}$, $\vec{v}$ and $\vec{\gamma}$ to denote a position increment along the local (infinitesimal) vortex tube, its cross-sectional area, the local velocity resultant, and the local vorticity vector, respectively. From this Prandtl then obtained a generalised form of

the Kutta-Joukowski Theorem. The lifting line to which it applies is an exact representation, as far as both the resultant force and moment are concerned, of the spatial vortex distribution. The resultant force is:

$$\vec{K} = \rho \int d\vec{s} \times \vec{v}_{av} \Gamma \quad (1.2)$$

in which Γ is the circulation around the whole of the *bound vortex* field considered. (Note that the force per unit volume of a *free vortex* field, on the other hand, is identically zero everywhere.) The force $\vec{K}$, furthermore, is the force acting on the fluid, and is therefore the negative of the force acting on the body to which the vortex field is bound. But if the direction of the vortex distribution as indicated by the direction of the position increment $d\vec{s}$ along a local vortex tube is not uniform, we interpret Prandtl as implying that the whole of the cross product $d\vec{s} \times \vec{v}_{av}$ is to be averaged. The formulation in equation (1.2), simple as it appears, is not suitable for direct practical application as is, because $\vec{v}_{av}$ includes the averaged induced velocity (which Prandtl indicated by $\vec{v}^*$) produced by the vortex field. An expression for this induced velocity, however, must be found in terms of the vorticity distribution by using the Biot-Savart Law, which is an integral formula that generally does not yield an *explicit* solution easily, if at all.

Stating precisely this difficulty, Prandtl finally made a number of simplifying assumptions, which included:

- restricting the analysis to induced velocities that are very small compared to that of the free stream
- departing from the exact spatial distribution of the lifting (bound) vortex system, and replacing each lifting surface with a single straight lifting line, as illustrated in Figure 1.3 a)
- replacing the trailing (free) vortex sheet of the wing (a vortex sheet of which the actual shape is such that its vortex lines are aligned along the *resultant velocity vector* everywhere, in order that its force distribution is zero everywhere) with a flat vortex sheet, one that emanates from the lifting line along the direction of the free stream

velocity vector. Von Mises¹⁰ (pp. 226) credits Lanchester for this idea of a continuously distributed trailing vortex sheet, too, but Milne-Thomson¹¹ (pp. 186) connects Lanchester's name only with the idea of tip vortices. The latter idea, which is a simplified horseshoe vortex model of the wing, as shown in Figure 1.3 b), seems to be the one with which Lanchester is most frequently credited.

The results of these simplifications were that the lift and drag of a wing, represented to first order approximation by a lifting line, could be estimated using the integral formulas

$$L = \rho V \int_{-b/2}^{b/2} \Gamma dy \quad (1.3)$$

$$\text{and} \quad D = \rho \int_{-b/2}^{b/2} \Gamma w dy \quad (1.4)$$

in the latter of which the induced downwash w is to be found, using the Biot-Savart Law, from the integral expression

$$w(y) = \frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{d\Gamma}{dy'} \frac{dy'}{y - y'} \quad (1.5)$$

This expression is much simpler than that which would have been obtained for the real trailing vortex sheet, and can be evaluated readily if a suitable expression for the circulation is used. The circulation distribution along the lifting line, however, is yet to be found or formulated. To this end Prandtl stated the hypothesis that

$$\Gamma = V c (c_1 \alpha' + c_2) \quad (1.6)$$

in which V is the free stream velocity, c is the chord length of the local wing section, c_1 is half the local lift curve slope, c_2 accounts for the local camber and α' is the effective value of the local angle of attack (angle of attack less induced angle of attack). Prandtl's

argument here amounts to the assertion that the local wing section can be assumed to behave aerodynamically as if it were part of an infinite wing impinged upon by a flow composed of the free stream flow and an *assumedly* uniform chordwise distribution of the induced flow field. This allows the assumption that the lift per spanwise increment obtained from 2D theory defines the magnitude of the force obtained from the Kutta-Joukowski Theorem, and that the direction of that force is normal to the *local relative velocity*, which is the *resultant of the local free stream velocity vector and the induced velocity vector*.

This formulation of the circulation around the local wing section leads to the now famous *integral equation of Prandtl's lifting line theory*, also known as the *monoplane equation*. We take the liberty here to use y for spanwise position and $\pm b/2$ for the wing tip positions. Except for this minor difference, it was given by Prandtl³ (pp. 340) in the form

$$\frac{1}{4\pi V} \int_{-b/2}^{b/2} \frac{d\Gamma}{dy'} \frac{dy'}{y-y'} + \frac{\Gamma}{c_1 V c(y)} - \frac{c_2}{c_1} = \alpha(y) \quad (1.7)$$

The first term on the left is the local downwash angle, the next represent the local sectional lift coefficient per unit lift curve slope, the third accounts for the lift obtained from a cambered airfoil at zero angle of attack, and the term on the right is the angle of attack of the local chord relative to the free stream velocity vector.

A solution of this equation for the case of rectangular wings is reported by Prandtl²⁶, pp. 342, to have been obtained in 1913. This was accomplished by using a circulation distribution, expressed in terms of the dimensionless spanwise co-ordinate ξ , of the form

$$\Gamma = \sqrt{1-\xi^2} (\Gamma_0 + \Gamma_2 \xi^2 + \Gamma_4 \xi^4 + \dots) \quad (1.8)$$

Note that ξ is defined such that the port and starboard wing tips are at $\xi = -1$ and $\xi = 1$, respectively. An alternative would have been to transform the linear co-ordinate y to an

angular co-ordinate θ , using $\xi = -\cos \theta$ (which places the port and starboard wing tips at $\theta = 0$ and $\theta = \pi$, respectively). One then finds the square root term to be $\sin \theta$, whereas the polynomial in ξ , now in even powers of $\cos \theta$, can be expressed as a weighted sum of cosines of multiples of θ . The $\sin \theta$ term guarantees zero circulation at the wing tips, on which Prandtl³ (pp. 338) insisted emphatically, and the remaining sum of cosines represent the circulation distribution in between.

1.1.2 Some Applications of and Perspectives on Vortex Theory

Wieselsberger^{7, 8} is credited by a number of authors (e.g., Chang¹⁸, Nuhait^{21, 22}, Coulliette and Plotkin²⁴) for having developed, in 1922, the first basic theory of a wing in ground effect by placing a mirror image of the real wing vortices below the ground in order to satisfy the flow tangency condition on the ground plane. Judging by Prandtl³, however, Betz⁴ must have obtained some success working on this problem as early as 1912, and Wieselsberger⁷ concerned himself with the reasons why migrating birds fly in an angular formation as early as 1914. This leads one to suspect that the mutual induction of two wings (or a wing and its image) may have been analysed well before 1920, but it could be that circumstances before the First World War prevented translation of these reports. (Incidentally, Prandtl²⁶ himself used the idea of an image below the ground plane in 1920 to calculate the pressure distribution on the ground below an aircraft of which the height is large compared to its span.) Be that as it may, Wieselsberger reportedly used Prandtl's lifting line theory in order to calculate lift and drag corrections, but he neglected the influence of the forward induced velocity due to the image of the bound vortex. (This is quite allowable in a first order approximation of a finite wing, because the downwash of the trailing vortices produces a decrease in the effective angle of attack proportional to the circulation, which is proportional to the angle of attack, and the influence of the forward induced velocity, as mentioned by Coulliette and Plotkin²⁴, is therefore on the order of the square of the circulation.)

A more recent study of drag reduction in ground effect was performed by Suh and Ostawari²⁰, who used a simplified horseshoe vortex model of the wing, with its trailing vortex segments assumed to emanate from the $\pi/4 \times b/2$ points. (This representation of a

wing by a horseshoe vortex is derived by assuming a lifting line with an elliptical circulation distribution, and then determining the width that a lifting line of uniform circulation equal to the maximum circulation of the elliptical distribution must have in order to produce the same lift.) They obtained good agreement with an empirical correlation of Prandtl, and fair agreement with experimental data for wings of aspect ratio 1 and 4.

The analysis of ground effect by placing image vortices below the ground plane is still the most popular method of treating the problem today, although care must be taken when the ground surface happens to be very uneven. This case, according to Nuhait²¹, can be treated by omitting the vortex images in favour of a vortex sheet placed on the ground. This is clearly allowable, because *any* impenetrable surface, not only those of aerodynamic bodies themselves, may be replaced by vortex sheets, provided that some condition of no penetration is satisfied. The vortex distribution of the ground sheet is then found, along with the distribution of the real vortex sheets, by *demanding flow tangency at the (uneven) ground plane explicitly*, whereas the *vortex images used below a flat ground plane satisfy this condition by implication*. A problem that we see in this approach, but one that is actually inescapable, is that even a large number of ground vortex panels cannot satisfy flow tangency everywhere on an uneven ground surface, whereas this is accomplished identically by the vortex images below an even ground plane.

In 1942 Weissinger⁹ presented two additional methods for predicting the lift distribution along the span of a finite wing, namely a lifting surface method and an alternative lifting line method, which he called the F-method and the L-method, respectively. The F-method replaces the wing and its trailing vortex sheet with a flat vortex sheet in the plane of the free stream, the forward part of which represents the bound part of the actual vortex system (instead of which Prandtl used a single straight lifting line), while the trailing part models the trailing vortex sheet in the same way that Prandtl did. The F-method, however, is based on finding the vortex distribution (from which the circulation and lift is obtained afterwards) by satisfying a flow tangency condition, whereas Prandtl's

theory equates the local Kutta-Joukowski lift per unit spanwise increment to that obtained from 2D airfoil theory instead. In addition, the treatment of the downwash velocity in the F-method makes provision, over and above the downwash term that Prandtl's equation contains, for the downwash contribution due to the chordwise distribution of the spanwise component of vortex strength, a contribution that Prandtl's theory neglects by taking the downwash value on the lifting line. The L-method that Weissinger⁹ proposed is a simplified version of the F-method in that it approximates the additional downwash contribution by its value at the $\frac{3}{4}$ -chord line, and then demands flow tangency on this line. This is the main difference between Weissinger's L-method and Prandtl's lifting line theory. Weissinger concluded that:

- a) lift distributions obtained by the F-method and the L-method are in very good agreement with one another for a rectangular wing of aspect ratio 5 and sweepback angles up to 45 degrees
- b) lift curve slopes obtained by the F-method (and therefore the L-method too) agree very well with experimental results obtained for thin rectangular wings with aspect ratios of 1, 2, 3, 4 and 5.
- c) the F-method (and therefore the L-method too) yields a lift result for a thin, straight rectangular wing with a NACA0006 profile (maximum thickness equal to 6 % of chord), and of aspect ratio 5, that is some 8 % lower than the lift predicted by Prandtl's lifting line theory. The theoretical lift curve slope of 2π is assumed in the latter, an assumption that is totally reasonable, because it agrees very well indeed with 2D wind tunnel measurements on an airfoil as thick as 9 % quoted by Bertin and Smith²⁷, pp. 212.
- d) the over-estimation of lift obtained by Prandtl's lifting line theory is due not to boundary layer effects, but to the fact that the theory neglects the influence of the chordwise distribution of circulation.

We conclude the discussion of Weissinger's work by mentioning that the selection of the $\frac{3}{4}$ -chord line as the place where flow tangency is required is based on the fact that the lifting line of an infinite wing, when placed at the $\frac{1}{4}$ -chord line, satisfies the condition of flow tangency at the $\frac{3}{4}$ -chord point exactly, as explained again in the recent report of

Coulliette and Plotkin²⁴. This, incidentally, is also the motivation for the selection of the discretisation scheme and the control point placement of what Lan¹⁴ calls the conventional vortex lattice method (CVLM), and even of the more recent generalised vortex lattice method of Konstadinopoulos¹⁶.

1.1.3 The Vortex Lattice Method

The vortex lattice method, as the term is used today, is based on the assumption that a vortex system composed of a number of discrete vortices will approximate the aerodynamic behaviour of a wing, or rather the continuous vortex sheet by which it can be replaced according to Lagally⁵, with increasing accuracy as the number of discrete vortices is increased. One can take various views of how this discrete system of vortices is related to the continuous vortex sheet it models. By this we mean that the discrete vortices can be regarded as vortex strips of finite dimensions and finite vortex strength, as Lan¹⁴ does, or they can be regarded as horseshoe type vortices, as a number of modern aerodynamic text books do, or they can be regarded as quadrangular vortex rings, as Konstadinopolous¹⁶ and Devinant²⁵ do. (The unsteady trailing vortex sheet model used by Devinant employed rectangular vortex rings, but its parent bound vortex sheet was modelled with a lifting line.) These are just different perspectives of the same vortex discretisation problem. Irrespective of the view taken, one should obtain the same result if the same corner points of the surface discretisation and the same control points at which flow tangency is sought are used, provided of course that the same version of the vortex lattice method is used, meaning that the results of a free wake version will obviously differ from that of a prescribed wake version.

The older versions of the vortex lattice method were steady state, prescribed wake methods, and favoured a linear distribution of both the vortex corner points and of the control points at which flow tangency is to be obtained. Another option for the vortex distribution is the cosine type that Lan¹⁴ described and motivated with such admirable mathematical rigour. A slightly different version of Lan's cosine distribution was used by Miranda¹⁵, and later by Du Plessis¹⁷. They placed their control points where Lan's vortex corner points are, and vice versa, with an additional vortex corner point right at each

wing tip in order that the number of horseshoe vortices matches the number of selected control points. The differences in results obtained by this version of the cosine distribution and that of Lan are not really significant.

The more recent versions of the vortex lattice method are unsteady, free wake types of approaches. They include provision for not only the time rate of change of circulation arising from the time rate of potential change in the unsteady version of the Bernoulli equation, but they also solve for the shape of the trailing vortex sheet by stipulating it to be force free, as reported by Konstadinopolous¹⁶ et al., and by Nuhait and Mook²¹. For reasons that we motivate later, however, we have used the steady, prescribed wake version of the method for our analyses.

The prescribed wake version of the method amounts to:

- a) selecting a suitable layout of the selected set of discrete vortices chosen to model the aerodynamic surface(s), and mirroring them about the ground plane, when in ground proximity
- b) formulating a flow tangency condition to be solved for the circulation values
- c) computing the distribution of induced velocity at points situated at suitable positions between the corner points of each bound horseshoe vortex leg, and
- d) computing the resultant aerodynamic force and moment components (rather their coefficients) using what we prefer to call either i) a Kutta-Joukowski type approach, or ii) a Blasius-Bernoulli type approach. The former is based essentially on reducing the volume integrals (of cross products of vorticity and velocity) of Prandtl's general theory of motion (see par. 1.1.1 above) to a Kutta-Joukowski type of integral form, similar to that in equation (1.2). (The integral expression obtained for the moment of aerodynamic force will contain a cross product of vortex strength with the local position vector.) One then approximates these integrals by a suitable numerical summation scheme, rather than attempting to perform the analytical integration. This approach yields the resultant aerodynamic force and moment directly. What we called the Blasius-Bernoulli type of approach above, however, amounts to the computation of the distribution of the normal force increments on the inboard surface segments. To

these must then be added the edge suction force components, the computation of which requires the distribution of the singularity strength at the leading edge(s). Our wish to couple Blasius's name to this approach stems from the analytical method of leading edge suction analysis by the use of the Blasius integrals, as described by Milne-Thomson¹¹. And it is probably not unreasonable to regard the leading edge singularity method, as described by Wagner¹³ and implemented in the quasi-vortex lattice method of Lan¹⁴, as the vortex lattice counterpart of the application of the Blasius integrals at a singularity line.

1.1.4 Recent Ground Effect Studies

A number of recent studies, such as those of Chang and Muirhead¹⁸, Nuhait and Mook²¹, and of Nuhait and Zedan²², investigated the unsteady ground effects of aircraft at a steady rate of descent. The steadily decreasing distance between the aircraft and its mirror image below the ground causes the shape of the wake to change not only continually, but also unsteadily, which yields a discrepancy between the aerodynamic characteristics computed by a quasi-steady analysis and that of an unsteady analysis, as can be expected. Chang and Muirhead¹⁸ concluded from a comparison of the static and dynamic wind tunnel tests (the latter carried out at steady sink rate of the wind tunnel model) with flight test data and vortex lattice calculations, however, that these dynamic effects have a relatively small influence on ground effect at a height down to 0.4 times the wing span in the case of the F-104 at 6.9 degrees angle of attack, and that the influence becomes smaller as the angle of attack increases.

Two particularly interesting results from the study of Nuhait and Mook²¹ are illustrated in Figure 1.4, in which one sees that the geometry of the trailing vortex sheet in ground effect, as compared to its shape out of ground effect, displays two distinct characteristics. The first is that the trailing vortex sheet seems to be "pushed up" by the ground effects in the downstream vicinity of the trailing edge. This, however, occurs not in the *immediate* downstream vicinity of the trailing edge: it is only at a distance of a *few chord lengths behind the wing* that the influence is seen to become rather pronounced. The direction of the flow off the trailing edge of the wing, after all, must be along the bisecting line of the

included trailing edge angle, which dictates that the flow direction immediately behind the trailing edge must be essentially the same out of ground effect as in ground effect. That is, if the equilibrium pitch angles for a given load are the same, which was found in the present study to be the case within a few tenths of a degree in the height range down to $\frac{1}{4}$ of the span. Thus the slope of the trailing vortex sheet, measured at the trailing edge relative to the ground plane, will be essentially the same in ground effect as out of ground effect. This observation leads one to suspect that *ground effects are due rather more to the direct reduction of downwash at the wing, specifically due to the upwash caused by the trailing image vortices, than to the indirect influence that the changed shape of the trailing vortex sheet in ground proximity exerts.*

And, secondly, the trailing vortex sheet is clearly seen to flatten out on the ground in the receding distance. This, however, will probably not affect the aerodynamic behaviour of the wing much, because it just happens too far away. But it is a result that should not surprise any of us smokers in the aerodynamic fraternity, or rather those among us that blow smoke rings. *We* ring-blowing smokers all know, which I bet the non-smokers don't, that a smoke ring blown at a mild angle of descent towards a desk top will make a "soft landing" on the desk, upon which it will quickly spread out, at a quickly decreasing rate, to a diameter of around 30 centimetres. And then it just lies down, perhaps still spinning a little, but staying put. Until some rude little gust disturbs it.

1.2 Concluding Remarks

Returning, on a more serious note, to the emphasis of the present investigation, we concentrated on the influence that ground proximity exerts on the longitudinal stability characteristics of *straight and level flight*. This focus, and the results obtained by Chang and Muirhead¹⁸, allows us to neglect the unsteady part of the influence that descent rate has on ground effects to first approximation, especially, furthermore, because it is the *stability of small perturbations* that we investigated, rather than the highly accurate prediction of the critical height. (Some influence of descent rate, at least, was taken into account in our vortex lattice calculations, namely the influence obtained from the quasi-static prescribed wake approach, in which the trailing vortex sheet is assumed to be in the

wind plane, thus changing the ground effect when the angle of descent changes.) The results of Nuhait and Mook²¹ showed for a delta wing of aspect ratio 1.456 that the height must go well below one chord length before the difference between aerodynamic characteristics at 0 and 5 degrees angle of descent becomes significant. In a state of small perturbational motion about a straight and level reference state of flight one may therefore expect a quasi-static analysis to give a good indication of stability trends.

It was found in the present study, furthermore, that it is the *long period modes of motion* that dominate the essential nature of longitudinal stability in ground effect. The results of Devinant²⁵, turning to the transient development of circulation, showed that the settling time of the circulation development following a step change in angle of attack is on the order of 10 dimensionless time units, whereas the damped period of the *short period mode of motion* encountered in our studies were typically 70 of Devinant's dimensionless time units, and that of our long period mode around 10 times longer still. One can therefore conclude that even the short period mode of rigid aircraft motion should be represented well by the results of a properly executed quasi-static analysis. This expectation is motivated by the general acceptance in dynamics and control theory that *transient dynamics of 5 to 10 times faster than the dominant dynamics can be neglected* with fair accuracy in a first analysis. And one can therefore conclude with confidence that this approximation will be even more accurate in its representation of the *long period modes*, because they are *on the order of 70 times slower than the transient aerodynamic effects*. A study of unsteady dynamic interference of a canard-and-wing configuration undertaken by Elzebda et al.²³, if one assumes that their dimensionless time unit is either based on the half chord or the full chord length, confirms the tendency for transient build-up of the circulation distribution to settle down in a time that is short enough (compared to the typical phugoid period) for unsteady aerodynamic effects to be neglected in an approximate analysis of longitudinal stability in ground proximity.

As we have mentioned at the outset, the study of aircraft stability in ground proximity seems to have received very little attention in the literature. The most recent study that presented a fairly comprehensive treatment of the subject was, to our knowledge, that of

Staufenbiel and Schlichting¹⁹. They considered an aircraft with a wing aspect ratio of 10, at the fairly high lift coefficient of $C_L = 1.8$. They came to the conclusion that it is its phugoid mode of motion that goes unstable, and that it happens at a height of around 0.6 times the span width. They managed to alleviate this instability by assuming the availability of a perfect air speed regulation system, and then feeding back the height perturbation and the descent rate to the elevator control input. This alleviation, however, came at the price of a pronounced decrease in the damping ratio of the short period mode of motion.

In the chapters that follow we present an overview of the required aerodynamic theory, a description of our aerodynamic computations, and an in depth analysis of longitudinal stability in ground effect. This is followed up by a sensitivity analysis aimed at revealing what changes, if any, in aircraft configuration may be made to alleviate the instability encountered in ground proximity, after which the study is concluded with a closed looped control analysis. The aircraft configuration that we have used as an example is based on the data of the Piper Cherokee PA-28-180 as obtained from McCormick²⁸.

2 VORTEX THEORY

It was decided, as motivated in Chapter 1, par. 1.2, to follow a quasi-steady approach in our analysis of the dynamic stability of a fixed wing aircraft in ground proximity. The study of dynamic stability requires that the influence of the angular velocity of an aircraft on the aerodynamic forces and moments it experiences must be accounted for. However, Prandtl's "general" theory of steady motion, which we discussed briefly in Chapter 1, does not include provision for the steady angular velocity of an aircraft. In the discussions that follow we outline the fundamental aerodynamic principles according to which such provision is made, mainly for fear of missing some detail that may be relevant to our eventual conclusions. Following Prandtl's basic lines of reasoning, we found it unavoidable to consider vortex theory in some depth before proceeding to the formulation of the required aerodynamic force and moment resultants. A truly illuminating discussion of vortex sheets is given by Von Mises¹⁰, pp. 211 - 231, especially pp. 219 - 231.

2.1 Flow Field and its Vortex Sheets

The velocity vector of the flow field around an aircraft, according to Von Mises¹⁰, can be expressed as the vector sum of the free stream velocity and the induced velocity field

$$\vec{V}(\vec{r}) = \vec{V}_{\infty} + \vec{V}_i(\vec{r}) \quad (2.1)$$

The *free stream velocity* is merely the negative of the translational velocity of the aircraft reference point

$$\vec{V}_{\infty} = -\vec{V}_A \quad (2.2)$$

This component of the velocity field is clearly one with a uniform distribution over space. We define the *local Relative free stream velocity vector* at an unspecified point on a solid surface by

$$\vec{V}_{Rf}(\vec{s}) = \vec{V}_{\infty} + \vec{V}_r(\vec{s}) \quad (2.3)$$

as illustrated in Figure 2.1. The velocity component due to the angular velocity of the aircraft

$$\vec{V}_r(\vec{s}) = -\vec{\omega} \times \vec{s} \quad (2.4)$$

has a cylindrical distribution. Note that it is the normal component of $\vec{V}_{Rf}(\vec{s})$ that must, in order to satisfy the flow tangency condition, be cancelled identically by the normal component of induced velocity everywhere on the bound part S_B of the vortex sheet S by which the aircraft is represented.

The induced velocity field may be expressed in terms of the strength distribution $\vec{\zeta}$ of the vortex sheet S by

$$\vec{V}_i(\vec{r}) = \frac{1}{4\pi} \iint_S \frac{(\vec{s} - \vec{r}) \times \vec{\zeta}(\vec{s})}{|\vec{s} - \vec{r}|^3} dS \quad (2.5)$$

This expression is obtained from the Biot-Savart Law, as given in terms of a volume integral over a vorticity field $\vec{\Omega}(\vec{s})$ by Prandtl³, pp. 333. By contracting the vorticity field to a vortex layer, taking the limit of zero thickness ε of this layer, and making use of the definition $\vec{\zeta} = \lim_{\varepsilon \rightarrow 0} \varepsilon \vec{\Omega}$ of vortex strength, the volume integral form of the Biot-Savart Law reduces to the surface integral form given here.

Of primary interest in aerodynamics and flight mechanics is the induced velocity not at an arbitrary point $\vec{r}$ in the flow field, but at a specified point $\vec{r} = \vec{s}_0$ on the aerodynamic surface S under consideration. This is because one needs an expression for the induced velocity on the surface in order to find the vortex distribution that would satisfy the flow tangency condition everywhere on the surface. It is clear, however, that the integrand in

equation (2.5) is singular at all $\bar{r} = \bar{s}$ on S . This implies that the integral in equation (2.5) is an *improper integral*, of which the value is its *principal* or *Cauchy principal* value, if it exists. If $\bar{r} = \bar{s}_0$ is a specific point on S , then equation (2.5) becomes

$$\bar{V}_i^C(\bar{s}_0) = \frac{1}{4\pi} \iint_S \frac{(\bar{s} - \bar{s}_0) \times \bar{\zeta}(\bar{s})}{|\bar{s} - \bar{s}_0|^3} dS \quad (2.6)$$

in which the superscript C indicates that the principal or Cauchy principal value of the integral (it must be found by excluding from the integration interval the infinitesimal in-plane interval on S that contains the singularity, as explained by Karamcheti¹², pp. 504) is applicable here.

It is to be noted that the principal value of the integral thus obtained yields the *velocity induced at $\bar{s}_0$ by the whole of the vortex system S , excluding the contribution of the local vortex filament situated at $\bar{s}_0$* . The induced velocity found from equation (2.6) for a single flat vortex sheet S will be normal to the surface everywhere. However, if a curved vortex sheet, or any set of more than one flat but non-coplanar vortex sheets is considered, then the principal value of the induced velocity vector will generally contain a tangential component too.

In preparation for our later discussion of thin wing theory we consider a single infinitesimally thin aerodynamic surface. One finds that the limits of $\bar{V}_i(\bar{r})$ as $\bar{r}$ approaches any $\bar{s}_0$ are not the same when approaching from the normal direction on the one side, say from

$$\bar{r}_+ = \bar{s}_0 + \bar{n}|\varepsilon| \quad (2.7)$$

than when approaching from the other side, i.e. from

$$\vec{r}_- = \vec{s}_o - \vec{n}|\varepsilon| \quad (2.8)$$

as shown in Figure 2.2. These limits are

$$\vec{V}_{i+} = \lim_{\varepsilon \rightarrow 0} \vec{V}_i(\vec{r}_+) = \frac{1}{4\pi} \lim_{\varepsilon \rightarrow 0} \iint_S \frac{(\vec{s} - \vec{s}_o - \vec{n}|\varepsilon|) \times \vec{\zeta}(\vec{s})}{|\vec{s} - \vec{s}_o - \vec{n}|\varepsilon|^3} dS \quad (2.9)$$

and

$$\vec{V}_{i-} = \lim_{\varepsilon \rightarrow 0} \vec{V}_i(\vec{r}_-) = \frac{1}{4\pi} \lim_{\varepsilon \rightarrow 0} \iint_S \frac{(\vec{s} - \vec{s}_o + \vec{n}|\varepsilon|) \times \vec{\zeta}(\vec{s})}{|\vec{s} - \vec{s}_o + \vec{n}|\varepsilon|^3} dS \quad (2.10)$$

The normal components of these two limits on a two sided surface, such as an infinitesimally thin wing, are found to be equal

$$\vec{V}_{in+}(\vec{s}_o) = \vec{V}_{in-}(\vec{s}_o) \quad (2.11)$$

because each of these must cancel the normal component of $\vec{V}_{Rf}$, of which the limits on the two sides of a thin wing are the same, as both $\vec{V}_\infty$ and $\vec{V}_r(\vec{s})$ are distributed continuously. The tangential components of the two limits, however, are not the same, producing a discontinuity in tangential velocity related to the local vortex strength by

$$\vec{V}_{it+}(\vec{s}_o) - \vec{V}_{it-}(\vec{s}_o) = -\vec{n}(\vec{s}_o) \times \vec{\zeta}(\vec{s}_o) \quad (2.12)$$

in which $\vec{n}(\vec{s}_o)$ is the outward unit normal vector on the side denoted by +. Karamcheti¹², pp. 532, states that a “vortex sheet *is* a surface of tangential discontinuity, and, conversely, a tangential discontinuity *is* a vortex sheet” (the italics are our own). This relationship between vortex strength and the discontinuity of velocity is illustrated in

Figure 2.3. Note that the vortex strength in equation (2.12) is defined inversely in terms of the tangential discontinuity by

$$\vec{\zeta}(\vec{s}) = \vec{n}(\vec{s}) \times [\vec{V}_{it+}(\vec{s}) - \vec{V}_{it-}(\vec{s})] \quad (2.13)$$

(We omit the subscript o from here on.) Noting furthermore that the normal induced velocity component is continuous across the surface, and the free stream velocity too, we formulate equations (2.12) and (2.13) in the form

$$\vec{V}_+(\vec{s}) - \vec{V}_-(\vec{s}) = -\vec{n}(\vec{s}) \times \vec{\zeta}(\vec{s}) \quad (2.14)$$

and

$$\vec{\zeta}(\vec{s}) = \vec{n}(\vec{s}) \times [\vec{V}_+(\vec{s}) - \vec{V}_-(\vec{s})] \quad (2.15)$$

Note that the implied inclusion of $\vec{V}_\infty$ here (it is contained in $\vec{V}$), and of the normal components of induced velocity, in equations (2.14) and (2.15) is allowable, because the cross products of the normal components with the normal unit vector are zero, and $\vec{V}_\infty$ cancels when forming the indicated vector difference. One can conclude therefore that a vortex sheet of non-zero strength must be the result of a finite discontinuity in velocity, or a finite discontinuity in velocity must occur across a vortex sheet of non-zero strength.

It is noted furthermore that, across a surface of infinitesimal thickness, *and on which the flow tangency condition is satisfied*, it holds true everywhere, by the geometric definition of any discontinuity in velocity, that

$$\vec{V}_{it+}(\vec{s}) = -\vec{V}_{it-}(\vec{s}) \quad (2.16)$$

whereas $\vec{V}_\infty$ and $\vec{V}_{in}$ are both continuous across such a surface. It follows from equations (2.1) and (2.15), therefore, that

$$\vec{\zeta}(\vec{s}) = 2 \vec{n}(\vec{s}) \times \vec{V}_+(\vec{s}) \quad (2.17)$$

and

$$\vec{V}_+(\vec{s}) = -\frac{1}{2} \vec{n}(\vec{s}) \times \vec{\zeta}(\vec{s}) \quad (2.18)$$

The subscripts i and t may be added to the velocity vectors in equations (2.17) and (2.18) if one wishes to emphasise that it is *only* the tangential component $\vec{V}_{it}$ of induced velocity that is discontinuous. Note that it is *specifically the component* $\vec{V}_{it}^L$ *due to (or that defines) the local vortex filament that is discontinuous, not the other components of the velocity field.* (One may call the latter component of induced velocity the Local complement of the principal value $\vec{V}_{it}^C$ of the induced velocity field, because the induced velocity field comprises both of these velocity components, and only these.)

One may therefore say quite reasonably that

$$\vec{\zeta}(\vec{s}) = 2 \vec{n}(\vec{s}) \times \vec{V}_{it+}^L(\vec{s}) \quad (2.19)$$

and

$$\vec{V}_{it+}^L(\vec{s}) = -\frac{1}{2} \vec{n}(\vec{s}) \times \vec{\zeta}(\vec{s}) \quad (2.20)$$

the superscript L indicating in equation (2.20) that the Local vortex filament at $\vec{s}$ can either be regarded as producing a discontinuous tangential component of induced velocity locally (the rest of the vortex system does not), or in equation (2.19) that a Local tangential discontinuity produces a vortex filament locally.

Despite our emphasis on thin wing theory thus far, we note in passing that the strength of a vortex sheet on a single sided surface, such as the external surface of a body of finite thickness, must be denoted by

$$\vec{\zeta}(\vec{s}) = \vec{n}(\vec{s}) \times \vec{V}_{it+}^L(\vec{s}) \quad (2.21)$$

(note the omission of the factor 2), because it has no other side. If the body is finite, then there is also a trailing vortex sheet, but one that is formed by the superposition of the upper and lower parts of the bound vortex sheet, and formed at the confluence line on the body surface, from where it is shed into the free stream in the form of a single sheet, as shown in Figure 1.2. (A sharp trailing edge is such a confluence line, and others are found wherever flow separation occurs.) Be this as it may, we terminate our afterthought concerning bodies of finite thickness here in favour of attending, *specifically*, to the flow tangency problem, the Kutta condition and the force and moment resultants of *thin wing theory*.

2.2 Mass Conservation and Infinity Condition

We emphasise at this point that the free stream velocity field satisfies the condition of mass conservation everywhere, because its distribution is uniform. The induced velocity field of equation (2.5) meets the requirement of mass conservation everywhere in the flow field too, but this time because it can be expressed as the curl of another vector, as shown by Karamcheti¹², pp. 526, and because of the fact that the divergence of the curl of any vector is identically zero. The resultant velocity vector of the external flow field therefore satisfies the continuity condition, as it must, by the linearity of the divergence operator in its application to the vector sum of equation (2.1).

The infinity condition requires that the external flow field be equal to the free stream flow field infinitely far from all aerodynamic surfaces and wakes. Close scrutiny of equation (2.5) reveals that the surface over which one integrates has, at its worst, one infinite dimension and one finite dimension (this holds true for an infinite wing, and for

the trailing vortex sheet of a finite wing), whereas the integrand reduces with the square of the magnitude of $\bar{r}$ as one approaches infinity. The induced flow field therefore satisfies the infinity condition identically, even if the vortex strength is singular along some edges of the vortex sheets considered. This is because a vortex filament, even one of finite circulation and infinite length, produces an induced velocity field that vanishes infinitely far away. The vortex filament at a leading edge or side edge singularity, according to Hassenpflug²⁹, has zero circulation in the limit of zero filament width, even though the inverse square root nature of such singularities, as discussed by Wagner¹³, determine that the vortex strength at such a singularity is of infinite magnitude.

Having thus established that the continuity condition and the infinity condition are satisfied by the induced velocity distribution attributed to a vortex sheet, we may now proceed with the formulation of the flow tangency problem, which is the problem of finding the strength distribution of the vortex sheet(s) of an aircraft such that the normal component of the resultant induced velocity vector cancels that of the *local relative free stream velocity* of equations (2.3) and (2.4) everywhere on its aerodynamic surface(s).

2.3 Flow Tangency Problem

The flow tangency condition demands that the flow be tangential everywhere on the bound part of a vortex sheet S . On the free part of a vortex sheet the flow must, in addition, be directed along the vortex lines everywhere in order that it experiences zero force everywhere. The former condition is that of no penetration, and is satisfied when

$$\bar{n} \cdot \bar{V}_R = 0 \quad \text{everywhere on } S \quad (2.22)$$

in which $\bar{n}$ is the local normal unit vector of S , and the *resultant local relative velocity vector* is

$$\bar{V}_R(\bar{s}) = \bar{V}_\infty - \bar{\omega} \times \bar{s} + \bar{V}_i(\bar{s}) \quad (2.23)$$

It follows that flow tangency requires

$$\bar{\mathbf{n}} \cdot \bar{\mathbf{V}}_i^C(\bar{\mathbf{s}}) = -\bar{\mathbf{n}} \cdot (\bar{\mathbf{V}}_\infty - \bar{\boldsymbol{\omega}} \times \bar{\mathbf{s}}) \quad \text{everywhere on } S \quad (2.24)$$

in which $\bar{\mathbf{V}}_i^L$ does not feature, because its dot product with the normal unit vector is zero. The satisfaction of the flow tangency condition can be seen, on inspection of equations (2.24) and (2.5), to require the solution of an integral equation for the vortex distribution. This can be seen by noting firstly that the small angle approximation of the local normal unit vector resolved in wind axes is

$$\bar{\mathbf{n}} = \bar{\mathbf{e}}_x(\alpha - \theta_c) - \bar{\mathbf{e}}_y\beta_d + \bar{\mathbf{e}}_z \quad (2.25)$$

in which the slope of the camber line is approximated by

$$\theta_c = \frac{\partial z_c}{\partial x} \quad (2.26)$$

The induced velocity vector (in the components of which we drop the superscript C for the sake of brevity, even though this is a bit dangerous), resolved in wind axes, has the forms

$$\bar{\mathbf{V}}_i^C = \bar{\mathbf{e}}_x u_i + \bar{\mathbf{e}}_y v_i + \bar{\mathbf{e}}_z w_i \approx V_\infty (\bar{\mathbf{e}}_x \chi_i + \bar{\mathbf{e}}_y \beta_i - \bar{\mathbf{e}}_z \alpha_i) \quad (2.27)$$

The wind axes used here have an x-z-plane that coincides with the longitudinal plane of symmetry, and an x-y-plane that contains the free stream velocity vector, as shown in Figure 2.4, which illustrates the flow tangency requirement. Note the implication that the induced rearward, sideward and upward velocity components u_i , v_i and w_i are divided by the air speed in the implied definitions of their respective dimensionless counterparts χ_i , β_i and α_i . Noting furthermore that χ_i , β_i and α_i are usually small quantities, it subsequently follows from equations (2.24) to (2.27) that the induced velocity contribution in equation (2.24) is given to first order approximation by

$$\hat{n} \cdot \vec{V}_i^C \approx -V_\infty \alpha_i = w_i \quad (2.28)$$

In addition, the relative wind velocity component due to angular velocity is, in the case of longitudinal motion, due to the pitch rate Q alone, so that

$$-\vec{\omega} \times \vec{s} = \bar{e}_x u_Q + \bar{e}_z w_Q \approx V_\infty (\bar{e}_x \chi_Q + \bar{e}_z \alpha_Q) \quad (2.29)$$

and the free stream velocity vector is

$$\vec{V}_\infty \approx V_\infty (\bar{e}_x + \bar{e}_y \beta)$$

The local resultant wind velocity due to translation and rotation in longitudinal motion is therefore

$$\vec{V}_\infty - \vec{\omega} \times \vec{s} = \bar{e}_x (V_\infty + u_Q) + \bar{e}_y V_\infty \beta + \bar{e}_z w_Q \approx V_\infty [\bar{e}_x (1 + \chi_Q) + \bar{e}_y \beta + \bar{e}_z \alpha_Q] \quad (2.30)$$

so that the first order approximation of this contribution in equation (2.24) becomes

$$-\hat{n} \cdot (\vec{V}_\infty - \vec{\omega} \times \vec{s}) = -V_\infty (\alpha - \theta_c + \alpha_Q) \quad (2.31)$$

It follows from these formulations, on substitution from equation (2.6) into equation (2.24), that the integral equation from which the distribution of vortex strength must be found is given in, as yet, unresolved vector form by

$$\frac{\hat{n}}{4\pi} \cdot \iint_S \frac{(\vec{s} - \vec{s}_0) \times \vec{\zeta}(\vec{s})}{|\vec{s} - \vec{s}_0|^3} dS = -V_\infty (\alpha - \theta_c + \alpha_Q) \quad (2.32)$$

The left hand side represents the local normal component of induced velocity, yet to be expanded in terms of the components of vortex strength. Even though $\bar{\zeta}(\bar{s})$ is the strength of a vortex sheet that is generally not flat, because the aerodynamic surfaces on which it is distributed may not be flat, and the trailing vortex sheets that these shed are not flat either, it can be argued that the flow tangency problem of thin wings will be approximated to fair accuracy by assuming all vortex sheets, bound and unbound, to be flat at small angle of attack. This is partly because the normal component of the induced velocity does not change by much in the immediate (normal) vicinity of the wing surface, and partly because the rolling up of the trailing vortex sheets occurs fairly gradually, i.e., gradually in the sense that the length of the near part of the vortex sheet behind a wing, the part that exerts the dominant influence on the downwash onto the wing surface, is at least *fairly* flat. Another simplification is made by assuming the trailing vortex sheet to be aligned with the free stream wind plane, an assumption that is motivated primarily by the success of this simplification in lifting line theories. Some aspects of Prandtl's lifting line theory were discussed in Chapter 1, and more details are given by Milne-Thomson¹¹. Milne-Thomson indicated that Prandtl's theory works well for large aspect ratios, and that an aspect ratio of 4 can be considered large to this end. The alternative lifting line theory of Weissinger⁹, on the other hand, agrees very well indeed with his quoted experimental data down to an aspect ratio of 1. Both of these lifting line theories assume a flat trailing vortex sheet aligned with the free stream. In addition to these motivations one can argue quite reasonably that, having already assumed a flat trailing vortex sheet, the small non-zero angle (relative to the trailing edge plane) at which it is assumed to be shed, rather than the zero angle at which it is shed in reality, will have only a second order effect on the downwash velocity it induces on the wing surface. Going one step further, one can just as well argue then that the bound part of the vortex sheet may be assumed, to first order approximation, to be in the wind plane too.

The results of the vector operations in equation (2.32) are, for the sake of brevity, presented in terms of the relative co-ordinates X, Y and Z of an arbitrary vortex filament at $\bar{s}$ with respect to the point $\bar{s}_0$ at which its induced velocity increment is considered. Defining, in general

$$\vec{s} = \vec{e}_x \xi + \vec{e}_y \eta + \vec{e}_z v \quad (2.34)$$

$$X = \xi - x \quad (2.35)$$

$$Y = \eta - y \quad (2.36)$$

$$\text{and } Z = v - z \quad (2.37)$$

it follows that

$$\vec{s} - \vec{s}_0 = \vec{e}_x X + \vec{e}_y Y + \vec{e}_z Z \quad \text{and} \quad |\vec{s} - \vec{s}_0| = X^2 + Y^2 + Z^2 \quad (2.38)$$

The assumption of flat vortex sheets allows the convenient simplification of expressing the local vortex strength in terms of two in-plane components alone, so that

$$\vec{\zeta} = \vec{e}_x \zeta_x + \vec{e}_y \zeta_y \quad (2.39)$$

in which $\zeta_y = \gamma$ is the component that features in two-dimensional (infinite) wing problems, whereas ζ_x is the component directed along the free stream, a component that is identically zero in two-dimensional flow. Expansion of the vector operations in the integrand of equation (2.32) thus yields

$$\vec{V}_i^c(x, y, z) = \frac{1}{4\pi} \iint_S \frac{-\vec{e}_x Z \zeta_y + \vec{e}_y Z \zeta_x + \vec{e}_z (X \zeta_y - Y \zeta_x)}{(X^2 + Y^2 + Z^2)^{3/2}} dS \quad (2.40)$$

in which the normal relative co-ordinate Z is retained for application to non-coplanar vortex sheets, such as those of the wing and tail, and their image vortex sheets in ground proximity. It is to be noted that

$$dS = d\eta d\xi \quad (2.41)$$

which indicates that the double integral is to be taken over the dummy arguments η and ξ of the spanwise and chordwise co-ordinates, respectively. The order in which these integrations are performed, if they can be found analytically, is that which is most convenient for the problem at hand. It follows from equations (2.27), (2.40) and (2.41) that

$$u_i^C(x, y, z) = -\frac{1}{4\pi} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{Z\zeta_y}{(X^2 + Y^2 + Z^2)^{3/2}} d\eta d\xi \quad (2.42)$$

$$v_i^C(x, y, z) = \frac{1}{4\pi} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{Z\zeta_x}{(X^2 + Y^2 + Z^2)^{3/2}} d\eta d\xi \quad (2.43)$$

$$\text{and } w_i^C(x, y, z) = \frac{1}{4\pi} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{X\zeta_y - Y\zeta_x}{(X^2 + Y^2 + Z^2)^{3/2}} d\eta d\xi \quad (2.44)$$

Note that the superscript C is retained in order to emphasise that the Cauchy principal value is to be taken whenever the integral is improper.

A first order approximation of the flow tangency requirement of an aerodynamic surface modelled by a flat vortex sheet is finally obtained from equations (2.28), (2.32) and (2.44), yielding

$$w_i^C(x, y, z) = \frac{1}{4\pi} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{X\zeta_y - Y\zeta_x}{(X^2 + Y^2 + Z^2)^{3/2}} d\eta d\xi = -V_\infty (\alpha - \theta_c + \alpha_Q) \quad (2.45)$$

Explicit solutions of the flow tangency problem for thin infinite wings can be found quite readily, as discussed in Appendix B.

The vortex distribution of a finite wing problem, however, requires solution by numerical methods, such as those provided by the respective lifting line theories of Prandtl³ and Weissinger⁹, and the quasi-vortex lattice method of Lan¹⁴. Two alternative formulations of equation (2.45) that are useful in the formulation of the components of which the resultant normal induced velocity due to a vortex lattice is composed can be found in terms of the circulation distribution, rather than the distribution of vortex strength, by noting at this point that

$$\zeta_x(\xi, \eta) = -\frac{\partial \Gamma}{\partial \eta} = -\Gamma_\eta(\xi, \eta) \quad (2.46)$$

$$\text{and } \gamma(\xi, \eta) = \zeta_y(\xi, \eta) = \frac{\partial \Gamma}{\partial \xi} = \Gamma_\xi(\xi, \eta) \quad (2.47)$$

in which $\Gamma(\xi, \eta)$ is the *circulation function* of the wing. The circulation function is defined as the circulation around any closed curve that passes through an arbitrary point (ξ, η) on the wing, and through which only the bound vortex lines between (ξ, η) and the leading edge passes, as illustrated in Figure 2.5.

Having formulated the flow tangency problem in sufficient detail for now, we turn to the Kutta condition at the trailing edge, after which the problem of the aerodynamic force and moment resultants will be formulated before addressing specific applications of these formulations.

2.4 Kutta Condition

Satisfaction of the flow tangency condition does not yield a unique solution for the vortex distribution, but an infinite family of solutions. The Kutta condition, however, requires smooth flow off all trailing edges. Because no pressure discontinuity is allowed just behind the trailing edge (or anywhere else, for that matter, except across an impermeable surface) the *velocities at the upper and lower surfaces of a cusped trailing edge must be equal in magnitude*. This requires that they differ at most in direction, which implies that

all vortex filaments (if any) shed at the trailing edge in steady flow must be aligned with the *tangential component of the local relative free stream* velocity vector at the trailing edge. We have assumed thin aerodynamic surfaces with small camber to be adequately represented by flat vortex sheets in the plane of the free stream. This plane is normal to the aircraft's plane of symmetry, and will be assumed to pass through the quarterchord point of the mean aerodynamic chord. (One does not know beforehand where the true neutral point is.) The mathematical formulation of the Kutta condition at the trailing edge, expressed in terms of vortex strength, requires strictly speaking that

$$\left(\vec{V}_\infty - \vec{\omega} \times \vec{s}\right)_t \times \vec{\zeta}(\vec{s}) = \vec{0} \quad \text{at} \quad \vec{s} = \vec{s}_{te} \quad (2.48)$$

in which $\vec{s}_{te}$ is the local position vector of the trailing edge, and the *subscript t on the velocity term means tangential to the trailing edge plane*. If we replace the wing by a flat bound vortex sheet in the free stream plane, as a first approximation, then its Kutta condition becomes

$$\vec{V}_\infty \times \vec{\zeta}(\vec{s}) = \vec{0} \quad \text{at} \quad \vec{s} = \vec{s}_{te} \quad (2.49)$$

This requires a zero component of the vortex strength normal to the free stream at the trailing edge, implying that *trailing vortex filaments, if any, must be aligned with the free stream velocity vector itself* in this approximate model of a thin wing, as opposed to the alignment required by equation (2.48).

2.5 Force and Moment Resultants

The force and moment resultants due to the air flow around the aerodynamic surfaces may be obtained either by application of the Kutta-Joukowski theorem, or by integrating the pressure forces obtained from the Bernoulli equation over the surfaces.

If a singularity is encountered, such as a sharp leading edge, then the Blasius integral, or a numerical equivalent, must be applied to find the leading edge suction force that acts in addition to the normal force increments obtained from the Bernoulli equation. The latter method may therefore be called the Blasius-Bernoulli approach. We favour the Kutta-Joukowski approach, because it yields the separate aerodynamic force components more directly, but the Blasius-Bernoulli approach will be discussed briefly, mainly in order to motivate the form in which we apply the Kutta-Joukowski theorem to an infinitesimal vortex filament. These two approaches are illustrated schematically in Figure 2.6 and discussed at length in the paragraphs that follow.

2.5.1 Kutta-Joukowski Approach

The Kutta-Joukowski theorem can be interpreted as stating for a straight infinite cylinder of arbitrary but uniform cross-sectional shape (not necessarily circular) in a uniform free stream that the aerodynamic force per unit length is given in vector form by

$$\vec{F}' = \rho \vec{V}_\infty \times \vec{e}_v \Gamma \quad (2.50)$$

in which $\vec{e}_v$ is the unit vector along the cylinder. The theorem is applicable not only to a cylinder carrying finite circulation, but also to an infinitesimal vortex filament, for which it provides the aerodynamic force increment acting on the filament. In application of the Kutta-Joukowski theorem to a vortex filament, $\vec{V}_\infty$ is to be replaced by the local resultant relative velocity, *excluding its discontinuous component*, and $\vec{e}_v \Gamma$ is to be replaced by either side of the equation

$$\vec{\zeta} dS = d\vec{\ell} d\Gamma \quad (2.51)$$

which relates the local vortex strength to the incremental circulation of the local vortex filament. Note that $d\vec{\ell}$ is a position increment parallel to $\vec{\zeta}$. This relationship can be explained in terms of the relationship between the vorticity and the vortex strength at

some point on the vortex filament, which can be written as $\vec{\Omega} dV = \epsilon \vec{\Omega} dS = \vec{\zeta} dS = d\vec{\ell} d\Gamma$. (A vortex filament is an infinitesimal vortex tube, which may be defined as a cylinder of infinitesimal cross-sectional area, again not necessarily circular, around which the circulation obtained for a closed curve that coincides with the boundary of an arbitrary tube section is the same, regardless of the shape or position of that section, as illustrated in Figure 2.7.) The relationship in equation (2.51) may alternatively be obtained by noting that, for a flat vortex filament of differential width dc and length $d\ell$, the circulation increment is given by

$$d\Gamma = (V_+ - V_-)dc \quad (2.52)$$

whereas the vortex strength is related to the velocity components on the upper and lower surfaces of a thin wing by

$$\vec{\zeta} dS = \vec{n} \times (\vec{V}_+ - \vec{V}_-)dc d\ell \quad (2.53)$$

The surface increment dS can be shown by consideration of the geometry of the vortex filament to be expressible either in terms of the differential dimensions dc and $d\ell$ (both in-plane) of the vortex filament, taken in the normal and tangential directions *of the vortex filament*, or in terms of the corresponding windwise and spanwise dimensions $d\xi$ and $d\eta$, so that

$$dS = dc d\ell = d\xi d\eta \quad (2.54)$$

Along these lines of reasoning one finds that the appropriate form of the Kutta-Joukowski theorem for application to a vortex filament is either

$$d\vec{F} = \rho \vec{V}_R^c \times d\vec{\ell} d\Gamma \quad (2.55)$$

or, expressed in terms of the local vortex strength

$$d\vec{F} = \rho \vec{V}_R^c \times \vec{\zeta} dS \quad (2.56)$$

in which the velocity vector to be used, rather than that of equation (2.23), is

$$\vec{V}_R^c(\vec{s}) = \vec{V}_\infty - \vec{\omega} \times \vec{s} + \vec{V}_i^c(\vec{s}) \quad (2.57)$$

We emphasise again that this *excludes the discontinuous velocity component*, because that component is already accounted for in the vortex strength.

The aerodynamic force and moment resultants are then found by integration over the bound vortex sheets of the vortex system to yield

$$\vec{F} = \rho \iint_S \vec{V}_R^c(\vec{s}) \times \vec{\zeta}(\vec{s}) dS \quad (2.58)$$

$$\text{and} \quad \vec{M} = \rho \iint_S \vec{s} \times [\vec{V}_R^c(\vec{s}) \times \vec{\zeta}(\vec{s})] dS \quad (2.59)$$

Resolved according to the three constituent components of $\vec{V}_R^c(\vec{s})$ in equation (2.57), if so required, three corresponding components of the force resultant are obtained, namely a component due to the free stream velocity

$$\vec{F}_\infty = \rho \vec{V}_\infty \times \iint_S \vec{\zeta}(\vec{s}) dS \quad (2.60)$$

a component due to the angular velocity

$$\vec{F}_r = -\rho \iint_S (\vec{\omega} \times \vec{s}) \times \vec{\zeta}(\vec{s}) dS \quad (2.61)$$

and a component due to the induced velocity

$$\vec{F}_i = \rho \iint_S \vec{V}_i^c(\vec{s}) \times \vec{\zeta}(\vec{s}) dS \quad (2.62)$$

It is emphasised, however, that $\vec{\zeta}$ depends on $\vec{\omega}$ via the flow tangency condition, so that not only $\vec{F}_r$, but $\vec{F}_\infty$ and $\vec{F}_i$ too, may depend on $\vec{\omega}$! (The subscripts on the forces in these equations merely indicate which velocity component in equation (2.57) were used in the Kutta-Joukowski formulation of that force.) The moment resultant of equation (2.59) can clearly be resolved in exactly the same way, if necessary.

2.5.2 Blasius-Bernoulli Approach

The Bernoulli equation applied to a lifting surface yields the normal force per unit area, but it has to be augmented by the Blasius integral if the resultant aerodynamic force is to include the suction force acting at a sharp leading edge, because the Bernoulli equation cannot be integrated through the singularity that occurs there. The perspective of the Bernoulli equation presented here is perhaps unusual. Our discussion of the Bernoulli equation, however, is aimed at the verifying the Kutta-Joukowski approach outlined above, verification that is obtained easier by a vector formulation of the force per unit area obtained from the Bernoulli equation, rather than the usual scalar formulation.

Satisfaction of the flow tangency condition implies that the normal component of $\vec{V}_{Rf}(\vec{s}) = \vec{V}_\infty + \vec{V}_r(\vec{s})$ is identically cancelled by that of $\vec{V}_i(\vec{s})$, so that

$$\vec{V}_{Rn}^c(\vec{s}) = \vec{V}_{\infty n} - (\vec{\omega} \times \vec{s})_n + \vec{V}_{in}(\vec{s}) = \vec{0} \quad (2.63)$$

This implies that the relative velocity

$$\vec{V}_R^c(\vec{s}) = \vec{V}_\infty - \vec{\omega} \times \vec{s} + \vec{V}_i^c(\vec{s}) \quad (2.64)$$

reduces to a form that contains only the remaining tangential components

$$\bar{V}_R^C(\bar{s}) = \bar{V}_{Rt}^C(\bar{s}) = \bar{V}_{\infty t} - (\bar{\omega} \times \bar{s})_t + \bar{V}_{it}^C(\bar{s}) \quad (2.65)$$

It can then be shown readily by incurring the definition of vortex strength in equation (2.19), expanding the triple cross product in equation (2.66) below, and rearranging the result that

$$\bar{V}_R^C(\bar{s}) \times \bar{\zeta}(\bar{s}) = \frac{1}{2} [\bar{V}_{Rt+}(\bar{s}) \cdot \bar{V}_{Rt+}(\bar{s}) - \bar{V}_{Rt-}(\bar{s}) \cdot \bar{V}_{Rt-}(\bar{s})] \bar{n}(\bar{s}) \quad (2.66)$$

Note that the tangential component $\bar{V}_{Rt}(\bar{s})$ of $\bar{V}_R(\bar{s})$ in equation (2.23), as opposed to $\bar{V}_R^C(\bar{s})$, *includes the discontinuous component of the induced velocity vector*. Equation (2.58) can thus be formulated alternatively to yield

$$\bar{F} = \frac{1}{2} \rho \iint_S [\bar{V}_{Rt+}(\bar{s}) \cdot \bar{V}_{Rt+}(\bar{s}) - \bar{V}_{Rt-}(\bar{s}) \cdot \bar{V}_{Rt-}(\bar{s})] \bar{n}(\bar{s}) dS \quad (2.67)$$

or

$$\bar{F} = \frac{1}{2} \rho \iint_S [V_{Rt+}^2(\bar{s}) - V_{Rt-}^2(\bar{s})] \bar{n}(\bar{s}) dS \quad (2.68)$$

This formulation is clearly the same as the conventional form

$$\bar{F} = \iint_S [p_-(\bar{s}) - p_+(\bar{s})] \bar{n}(\bar{s}) dS \quad (2.69)$$

obtained from the Bernoulli equation

$$p_{\infty+} = p_+ + \frac{1}{2}\rho V_+^2 = p_- + \frac{1}{2}\rho V_-^2 = p_{\infty-} \quad (2.70)$$

The velocities to be used here are found from equations (2.18) and (2.23), the former supplying the induced velocity component to be used in the latter, of which the tangential component is then used to evaluate V_{Rt+} and V_{Rt-} required in equation (2.68). Note that equation (2.70) implies (specifically for the streamlines on the upper and lower surfaces) that $p_{\infty+} = p_{\infty-}$, in other words, that their stagnation pressures are equal. This is not entirely obvious, because *the constant in the Bernoulli equation along a streamline to a may vary from streamline to streamline*, as explained by Karamcheti¹², pp. 224. If one considers a pair of streamlines that part at the leading edge stagnation line, however, their stagnation pressures *are* equal, and remain equal and constant on their separate trajectories over the upper and lower wing surfaces.

2.5.3 Comparison of Kutta-Joukowski and Blasius-Bernoulli Approaches

It must be stated emphatically that the force increment

$$d\vec{F}_n = \frac{1}{2}\rho[V_{Rt+}^2(\vec{s}) - V_{Rt-}^2(\vec{s})]\vec{n}(\vec{s})dS \quad (2.71)$$

that is integrated in equation (2.68), or (2.69), of the Blasius-Bernoulli approach, is a *force increment normal to the local surface element*. The *suction force at a sharp leading edge*, which is *directed into the wind and tangential to the leading edge surface*, must be obtained separately by application of the Blasius integral theorem to the singularity at the leading edge, or a numerical algorithm that accomplishes this, if no analytical solution is available. (The singularity approach followed in the quasi-vortex lattice method of Lan¹⁴ is such an algorithm.)

The striking feature of the Kutta-Joukowski approach formulated above is that, even though equation (2.58) starts out with a velocity vector that is tangential to the surface

everywhere (provided of course that the vortex distribution satisfies the flow tangency condition), it does not make explicit use of the flow tangency condition. In other words, it does not, as it could by the flow tangency condition of cancellation of normal velocity components, replace $\vec{V}_R^C$ in equation (2.57) with its local tangential component $\vec{V}_{\infty t} = (\vec{\omega} \times \vec{s})_t + \vec{V}_R^C(\vec{s})$. If it did, then it would reduce to the Blasius-Bernoulli approach! It rather resolves the velocity vector into the components indicated in equation (2.57), thus yielding the induced force of equation (2.62) separately from the aerodynamic force components of equations (2.60) and 2.61).

The effect of this separation of velocity components by the Kutta-Joukowski approach, components containing sub-components (the normal ones) that actually cancel identically, is that *the leading edge suction force at a sharp edge is replaced by one that is distributed over the surface*. The Blasius-Bernoulli approach, on the other hand, *does* make explicit use of the flow tangency condition, which is the reason why it reveals the reality of *a concentrated suction force at a sharp leading edge*, as illustrated in Figure 2.6.

The aerodynamic load distribution obtained from the Blasius-Bernoulli approach is, in theory, rigorously correct, and is therefore the one to be used in the structural design problem of a wing. The Kutta-Joukowski approach, as reflected by equations (2.60) to (2.62), while misrepresenting the aerodynamic load distribution to an extent that may render it too inaccurate for wing design (but then again, it may well be adequate for this purpose too), yields, in theory, exactly the same aerodynamic force resultant as the Blasius-Bernoulli approach, which means that it should be adequate for flight mechanical purposes.

We note in conclusion of this discussion, however, that the Kutta-Joukowski approach, by its effective assumption of a chordwise distribution of the suction force, rather than its concentration at a sharp leading edge, *may* lead to a *slight* inaccuracy in its prediction of the resultant aerodynamic moment, *if the camber is not zero*. But even its aerodynamic moment resultant agrees exactly with that of the Blasius-Bernoulli approach if the camber

is zero, or if the wing is replaced by a flat bound vortex sheet in both approaches. (We have not checked whether the Kutta-Joukowski approach actually *is* inaccurate in its prediction of the aerodynamic moment resultant of a *cambered wing*, but we expect that any such inaccuracy will not be significant, especially if the camber is small.)

2.6 Resolution of Force and Moment Resultants in Wind Axes

The aerodynamic force components of equations (2.60) to (2.62) are recombined here and expressed in terms of the components of the vortex strength, a vector that generally has a spanwise component and, in the case of a finite wing, a non-zero windwise component too. (Recall that we have assumed the vortex sheet to be in the wind plane that passes through the $1/4$ -chord point of the mean aerodynamic chord of the surface being considered. This wind plane is perpendicular to the longitudinal plane of symmetry of the aircraft.) Expressing these components of vortex strength explicitly as functions of the windwise and spanwise position co-ordinates ξ and η , merely in order to define their symmetry characteristics, equation (2.39) is rewritten in the form

$$\vec{\zeta}(\vec{s}) = \vec{e}_x \zeta_x(\xi, \eta) + \vec{e}_y \zeta_y(\xi, \eta) \quad (2.72)$$

in which $\vec{e}_x$ and $\vec{e}_y$ are unit vectors along the free stream velocity vector and in the starboard direction, respectively. We restrict the rest of our discussion to the aerodynamic force and moment resultants encountered in longitudinal aircraft motion. The aerodynamics of lateral-directional motion, however, can be analysed by reasoning along the same lines.

One finds that an aircraft in longitudinal motion, i.e. the state of zero lateral velocity, and zero rates of both roll and directional change, has a vortex distribution of which $\zeta_x(\xi, \eta)$ and $\zeta_y(\xi, \eta)$ is, respectively, skew-symmetric and symmetric about the vertical plane of symmetry, which means that

$$\zeta_x(\xi, \eta) = -\zeta_x(\xi, -\eta) \quad (2.73)$$

$$\text{and } \zeta_y(\xi, \eta) = \zeta_y(\xi, -\eta) \quad (2.74)$$

The free stream velocity vector, the angular velocity vector, the position vector of the local vortex filament and the induced velocity vector, all of them resolved in wind axes, are given, respectively, by

$$\vec{V}_\infty = \vec{e}_x V_\infty \quad (2.75)$$

$$\vec{\omega} = \vec{e}_y Q \quad (2.76)$$

$$\vec{s} = \vec{e}_x \xi + \vec{e}_y \eta + \vec{e}_z v \quad (2.77)$$

$$\text{and } \vec{V}_i^C = \vec{e}_x u_i + \vec{e}_y v_i + \vec{e}_z w_i \quad (2.78)$$

Our selection of wind axes for the resolution of these vectors is motivated by the custom to resolve the resultant aerodynamic force and moment vectors of equations (2.58) and (2.59) in wind axes to yield

$$\vec{F} = \vec{e}_x D + \vec{e}_z F_s + \vec{e}_z L \quad (2.79)$$

and

$$\vec{M} = \vec{e}_x M_x + \vec{e}_y M_y + \vec{e}_z M_z \quad (2.80)$$

in which D , F_s and L are the aerodynamic drag, lateral force and lift components, and M_x , M_y and M_z are the roll, pitch and yaw components of the resultant aerodynamic moment. Of these F_s , M_x and M_z are identically zero for a bilaterally symmetric aircraft in longitudinal motion, which is partly due to the symmetry of the spanwise distribution of

lift, as reflected by ζ_y , and partly due to the skew-symmetry of ζ_x . Substitution from equation (2.72) and equations (2.75) to (2.78) into equation (2.57), and subsequent substitution of the principal value $\bar{V}_R^C$ of the local relative velocity into the cross product terms of equations (2.58) and (2.59), yield the results that

$$D = -\rho \iint_S (w_i + Q \xi) \zeta_y dS \quad (2.81)$$

$$L = \rho \iint_S [(V_\infty + u_i - Q v) \zeta_y - v_i \zeta_x] dS \quad (2.82)$$

$$\text{and} \quad M = -\rho \iint_S \{ [(V_\infty + u_i) \xi + w_i v] \zeta_y - v_i \xi \zeta_x \} dS \quad (2.83)$$

in which dependence on ξ and/or η are implied, were applicable, but, for the sake of brevity, not indicated explicitly. The contributions of v_i and ζ_x to the lift and the pitch moment coefficients, contributions that are, admittedly, expected to be very small, are not identically zero if non-coplanar vortex sheets are considered, because v_i and ζ_x are both skew-symmetric. These two contributions, even though we retain them here for the sake of completeness, were both ignored in our later analyses. In our vortex lattice algorithm we did, however, take the direct contribution of the induced velocity component u_i to the lift and the pitch moment into account, as well as the contribution of w_i to M , even though these influences proved to be either negligible altogether, or of second order magnitude. Equation (2.83) shows the pitch moment to be apparently independent of the pitch rate, which seems surprising at first, because we know that a nose-up pitch rate produces a nose-down restoring moment. The pitch rate, however, *does* produce a pitch moment in equation (2.83), but indirectly so, and via the influence it exerts on the distribution of ζ_y due to the flow tangency condition. The most surprising aspect of the results in equations (2.81) to (2.83) is the unavoidable conclusion that positive pitch rate can produce thrust (negative drag)! Closer scrutiny of this result in a later analysis of a pitching 2D wing, however, showed that the (nose-up) moment of this (very real!) thrust,

taken about the centre of the pitching motion, cancels the nose-down moment produced by the change in circulation distribution exactly, thus ensuring that the power requirement *is* zero for a 2D pitching wing, as one would expect intuitively.

The dimensionless counterparts of equations (2.81) to (2.83), that is, the dimensionless drag, lift and pitch moment coefficients, are found to be

$$C_D = \frac{2}{S} \iint_S (\alpha_i - 2q \xi / c) \zeta_y^* dS \quad (2.84)$$

$$C_L = \frac{2}{S} \iint_S [(1 + \chi_i - 2q v / c) \zeta_y^* - \beta_i \zeta_x^*] dS \quad (2.85)$$

$$\text{and} \quad C_M = -\frac{2}{S} \iint_S \{ [(1 + \chi_i) \xi / c - \alpha_i v / c] \zeta_y^* - \beta_i \xi \zeta_x^* / c \} dS \quad (2.86)$$

On comparison with equations (2.81) to (2.83), one can see that the drag D and the lift L have been normalised, as usual, with respect to $\frac{1}{2} \rho V_\infty^2 S$, and the pitch moment M has been normalised with respect to $\frac{1}{2} \rho V_\infty^2 S c$, where c is the mean aerodynamic chord. The dimensionless induced velocity components χ_i , β_i and α_i , on the other hand, are respectively defined as u_i / V_∞ , v_i / V_∞ and $-w_i / V_\infty$, the latter being the first order approximation of the induced downwash angle, whereas the dimensionless components ζ_x^* and ζ_y^* of vortex strength are defined by $\zeta_x^* = \zeta_x / V_\infty$ and $\zeta_y^* = \zeta_y / V_\infty$, respectively.

It must be emphasised that it is the respective induced velocity components u_i and w_i *in wind axes* on which χ_i and α_i must be based, and the contribution of pitch rate in C_L and C_D must be based on the respective windwise and normal position co-ordinates, ξ and v , also resolved in wind axes, because the *derivation of equations (2.84) to (2.86) was based on vector products formed in the wind base*. Different components of the induced

velocity vector and the local position vector are obtained when resolving them in a *geodetic base*, of which the *x*-axis points horizontally to the rear, and the *z*-axis points upward. The selection of a geodetic base is more convenient for describing the geometry of the vortex sheets in ground proximity than a wind base, because the symmetry about the ground plane allows easier formulation of the position co-ordinates of the vortex images. It then comes natural to compute the induced velocity components in the geodetic base too. The wind base, however, is more convenient for resolving the aerodynamic force components, because these are customarily expressed as drag *D* along the wind and lift *L* normal to the wind. If it is the *geodetic components* of the induced velocity, u_{ig} and w_{ig} , and of the local position vector, x_g and z_g , that are available, rather than components in the wind base, then the required components in the wind base are obtained using the transformations:

$$\xi = x_g \cos \alpha_d + z_g \sin \alpha_d \quad \text{and} \quad v = -x_g \sin \alpha_d + z_g \cos \alpha_d \quad (2.87)$$

for the position co-ordinates, and similarly

$$u_i = u_{ig} \cos \alpha_d + w_{ig} \sin \alpha_d \quad \text{and} \quad w_i = -u_{ig} \sin \alpha_d + w_{ig} \cos \alpha_d \quad (2.88)$$

for the induced velocity components.

This concludes our discussion of vortex theory and its application to the problem of finding the lift and moment resultants of an aircraft in a given state of longitudinal motion. Even though we emphasised the aerodynamics of a single surface in our treatment here, the formulations above are quite sufficient to account for more than one surface, or to analyse the influence of ground proximity, by judicious application of these results to the various surfaces, after having accounted properly for the image vortex system below the ground plane in solving the flow tangency problem. The problem of solving these integral equations by numerical approximation in the investigation of a wing and tail configuration in ground proximity is treated in the next chapter. The basis of our numerical approximations of the lift, drag and pitch moment coefficients in the

following chapter, by the conventional vortex lattice method, will be the representation of the assumedly flat vortex sheets (and their images below the ground, when in ground proximity), by sets of horseshoe vortices. If one places one horseshoe alone on the $\frac{1}{4}$ -chord point of each of the selected number of wing sections, demanding flow tangency at the $\frac{3}{4}$ -chord point, then this approach reduces naturally to an approximate equivalent of the lifting line method of Weissinger⁹. If one replaces the flow tangency condition with the circulation condition implied by the lifting line method of Prandtl³ in equation (1.7), then the method is the approximate equivalent of Prandtl's method, *approximately equivalent* because one uses a set of horseshoe vortices for estimating the downwash instead of the customary Fourier series expansion.

3 DISCRETE VORTEX MODELS

The aerodynamic considerations of the preceding Chapter led to a theoretical formulation of the flow tangency problem, and subsequently, to expressions for the aerodynamic force and moment coefficients. The solution of the former yields the vortex distribution required in the computation of the aerodynamic force and moment coefficients on which the stability characteristics of an aircraft depends. These solutions, in the case of a finite wing, even a thin one, can generally not be found in closed form. Three numerical methods for obtaining approximate solutions were considered in this study. The first of these are the two lifting line method proposed by Prandtl³ and Weissinger⁹ (we call them the PLLM and the WLLM, for the sake of brevity in the subsequent discussion), of which we give a brief overview. The third method is the quasi-vortex-lattice method (QVLM) of Lan¹⁴, the one used as the basis of the present analysis, and therefore, the one to which we devote most of our attention here.

Steady flow and flat trailing vortex sheets aligned with the wind are assumed in all three of these methods. The trailing vortex sheet is not actually flat, as shown in Figures 1.2 and 1.4, and the flow is not steady in transient aircraft motion. However, we will not pursue the next logical alternative, namely to consider the more recent unsteady, free wake vortex lattice method of Konstadinopoulos¹⁶, even though the use of this approach is reported more often currently. In this approach the trailing vortex sheets are allowed to take the shape that produce zero aerodynamic force on them, as is strictly required, in addition to which it allows for unsteady aerodynamic effects due to the time rate of change of circulation that occurs in unsteady flow.

The basic assumption that we make is that a quasi-static QVLM analysis should reveal the essential nature of the longitudinal instability encountered in ground proximity, despite the fact that it neglects the influence of a) unsteady aerodynamic effects, and b) the curvature of the trailing vortex sheet. The numerical accuracy with which the stability characteristics is quantified by a QVLM analysis can, admittedly, be expected to be

inferior to that of an unsteady free wake analysis, but perhaps not so inferior that the essential trends are misrepresented.

3.1 Lifting Line Theory

In the discussion that follows we compare the lifting line methods of Prandtl and Weissinger, *specifically as applied to thin rectangular wings of finite aspect ratio*. The PLLM finds the circulation by assuming the local airfoil section to behave as if it were part of an infinite wing, and as if it were subject to a downwash velocity distribution that is uniformly distributed along the chord, but not along the span. (It assumes, furthermore, a local lift curve slope of 2π , that of a thin 2D wing.) The downwash velocity used in the PLLM is the downwash on the end of a semi-infinite flat rectangular trailing vortex sheet, as shown in Figure 1.3. This means that, *in the PLLM, the lifting line is tacitly assumed to be straight and situated on a straight trailing edge, but only for the purpose of the circulation and drag computations*. The assumption that each local wing section behaves as if it were in a two-dimensional flow field implies that one must also assume the neutral point of the local wing section to be on the $1/4$ -chord point, which means that, *for the purpose of estimating the pitch moment, the PLLM places the lifting line on the midspan $1/4$ -chord, if the wing is rectangular, but on the mean aerodynamic $1/4$ -chord, if not*.

The WLLM, on the other hand, finds the circulation by assuming the lifting line *is* on the $1/4$ -chord line (not only for the purpose of estimating the moment coefficient, but also when computing the circulation). The circulation is then computed, as opposed to the approach of the PLLM, by demanding flow tangency at the $3/4$ -chord point, for no other reason but that it works out exactly for a 2D wing. *But the $3/4$ -chord point flow tangency requirement works for the 2D wing only because the 2D lift curve slope is 2π , a fact on which also the WLLM (not only the PLLM) is thus seen to rely, albeit by implication*.

The fact that both the PLLM and the WLLM, when applied to a thin rectangular wing, rely essentially on a 2D lift curve slope of 2π , leads one to expect that they should yield exactly the same result. However, this is not the case, because Weissinger⁹ reported the PLLM to yield a 8% higher lift curve slope for a thin rectangular wing of aspect ratio 5

than the WLLM (refer to our discussion in Chapter 1), whereas the WLLM agreed essentially with a more sophisticated lifting surface approach proposed by Weissinger in the same paper.

The conventional method of solution by the PLLM is to use a Fourier series of sine terms for the circulation distribution when solving equation (1.7). But nothing prevents one from solving this equation by using a set of horseshoe vortices placed on the $\frac{1}{4}$ -chord line to estimate the required downwash term, rather than a Fourier series expansion of the spanwise distribution of circulation. Once our vortex lattice algorithm was up and running it was a trivial matter to modify the code so as to use the requirement stipulated by equation (1.7) in the computation of this circulation distribution. The results thus obtained could then be compared with those of a VLM analysis, performed beforehand with the same number and placement of horseshoe vortices, of course, but with the requirement of flow tangency at the $\frac{3}{4}$ -chord points, rather than the PLLM requirement of equation (1.7).

These two approaches outlined above are similar in all respects to those of the PLLM and the WLLM, even though the shape of the circulation distribution along the span used in our VLM approximation of these methods relies on the piecewise uniform circulation distribution of a set of horseshoe vortices, instead of using the truncated summation of continuous (harmonic) functions of the conventional LLM approaches. And they led to the same conclusion that Weissinger drew (not only in trend, but also quantitatively), namely that the PLLM yields a significantly larger lift curve slope than the WLLM, and that the discrepancy increases with decreasing aspect ratio.

Concerned by our observation above that one is not entirely unreasonable in expecting the two methods to give the same results we formulated both the PLLM and the WLLM analytically for the case of a rectangular wing of aspect ratio 5 represented by just one horseshoe vortex (one of width $\pi b/4$, in order to yield the lift of an elliptical circulation distribution). It was found that the two methods does not agree exactly, except when the aspect ratio approaches infinity, in which case both methods give a result of $\pi/4$ times 2π

(The factor $\pi/4$ obviously arises from the selection of horseshoe width mentioned above.) And it was found that the PLLM does indeed overestimate the lift curve slope, because it agrees exactly (in this case of the single, equivalent horseshoe vortex) with that of an elliptic distribution, whereas the WLLM result is some 2% less. The smaller lift curve slope obtained for a rectangular wing compared to an elliptic wing is to be expected, because the tip losses of the former can reasonably be expected to be larger than that of the latter. This discrepancy was found to increase rapidly to the 8% reported by Weissinger as the number of horseshoe vortices of our model was increased.

3.2 Vortex Lattice Theory

The conventional vortex-lattice method (CVLM), models the vortex sheet of a wing using a set of horseshoe vortices, discretising the wing linearly in both the spanwise and chordwise directions. *Both the spanwise and the chordwise components of vortex strength of the bound vortex sheet are thus assumed to be impulsively distributed.* The spanwise segment of each horseshoe vortex is placed at the $1/4$ -chord point of its parent wing panel, and a control point, at which flow tangency is required, is placed at the $3/4$ -chord point of the panel. The flow tangency condition of equation (2.45) is then formulated as a set of linear equations in terms of the unknown circulation values of the set of horseshoe vortices and the known values of the required downwash velocity at each control point. Simultaneous solution of these equations then yields the required circulation values, which are subsequently used to compute the total lift and pitch moment coefficients. The CVLM approach to the computation of the near-field induced drag coefficient is to compute the induced downwash at the centre of each bound horseshoe leg, and then to apply the Kutta-Joukowski theorem. This method of induced drag computation has been criticised by Hancock³¹ for its implied assumption of a chordwise distribution of the suction force, rather than a concentrated suction force at the leading edge. Referring to our discussion in paragraph 2.5, however, one may argue that this objection is not theoretically justified, because addition of the force components obtained separately in equations (2.60) to (2.62) yields exactly the same result as the addition of the force components obtained by a) applying the Bernoulli equation to the wing surface using equation (2.71), and then b) applying the Blasius integral formulas at its leading edge to

obtain the leading edge suction force. The Kutta-Joukowski approach thus yields, in theory, exactly the same resultant aerodynamic force, merely resolved in other components, and obtained from a perspective that differs from the Blasius-Bernoulli approach. The other objection of Hancock, the one based on the singularity of the downwash encountered at the corners of a bound horseshoe segment due to the trailing segment emanating from there, on the other hand, does seem to be justified. But is this singularity perhaps due only to the use of horseshoe vortices with sharp corners? Would a smooth but "flat-nosed" horseshoe suffer from the same singularity? We believe that this is a point well worth pondering, albeit only for the sake of academic curiosity.

The quasi-vortex-lattice method (QVLM) of Lan¹⁴, as opposed to the CVLM, seems at first glance to approximate the vortex density function γ by a distribution that is continuous in the chordwise direction, but piecewise uniform in the spanwise direction. Careful scrutiny of the QVLM formulation of the flow tangency condition, however, shows that it is equivalent in all respects to that of the CVLM, except for the fact that it uses a cosine type distribution of the vortex points and the control points. By equivalence we mean that the flow tangency condition of the QVLM is merely formulated in terms of vortex density values times the chordwise lengths of the respective vortex strips. If the vortex strips of the QVLM are centred according to the CVLM's linear discretisation scheme, then it yields vortex density values, that, on multiplication with the chordwise increments, give the same circulation as that obtained by the CVLM for its horseshoe vortices.

Besides the use of a cosine type discretisation scheme the QVLM differs from the CVLM not in its solution of the flow tangency problem, but in its subsequent computation of the induced drag coefficient. The QVLM uses namely an approximation of the leading edge singularity strength for finding the local leading edge thrust coefficient, and an approximation of the normal force coefficient found from the Bernoulli equation. The local sectional induced drag coefficient is then obtained from the leading edge thrust coefficient and the normal force coefficient, the latter of which is approximated by the sectional lift coefficient, if the angle of attack is small. This approximation of the leading

edge thrust coefficient can be regarded as the numerical counterpart of applying the Blasius method at the leading edge, in which the Bernoulli equation is integrated around an infinitesimal circle around the leading edge, as explained by Milne-Thomson¹¹. The CVLM, on the other hand, uses the Kutta-Joukowski theorem for both the lift and the induced drag computation. The singularity approach of the QVLM is numerically more effective than the CVLM, because it requires additional computation of the induced velocity only at the leading edge control points, whereas the Kutta-Joukowski approach of the CVLM, on the other hand, requires the computation of the induced velocity at all of the inboard bound vortex midpoints.

3.2.1 Induced Velocity of Rectangular Horseshoe Vortex

The main objective of this study is to investigate the essential nature of longitudinal stability in ground proximity, to which end we focus on the simplest possible configuration of a wing and tail, namely two flat rectangular surfaces, and we model their vortex sheets with a set of rectangular horseshoe vortices. The induced velocity of a rectangular horseshoe vortex with circulation strength Γ , port corner point a and starboard corner point b is expressed in the form

$$\vec{V}_i = \vec{V}_{iB} + \vec{V}_{iFa} + \vec{V}_{iFb} \quad (3.1)$$

in which the subscripts B, Fa and Fb refer to the contributions of the bound leg and trailing legs a and b of the vortex. These contributions can be expressed with the help of the results of Appendix A in the form

$$\vec{V}_{iB} = \frac{\Gamma}{4\pi} \frac{-\vec{e}_x Z + \vec{e}_z X}{(X^2 + Z^2)} \left(\frac{Y_b}{R_b} - \frac{Y_a}{R_a} \right) \quad (3.2)$$

$$\vec{V}_{iFa} = -\frac{\Gamma}{4\pi} \frac{\vec{e}_x \alpha_v Y_a + \vec{e}_y (Z - \alpha_v X) - \vec{e}_z Y_a}{(1 + \alpha_v^2) Y_a^2 + (Z - \alpha_v X)^2} \left(1 - \frac{X + \alpha_v Z}{R_a} \right) \quad (3.3)$$

$$\text{and } \bar{V}_{iFb} = \frac{\Gamma}{4\pi} \frac{\bar{e}_x \alpha_v Y_b + \bar{e}_y (Z - \alpha_v X) - \bar{e}_z Y_b}{(1 + \alpha_v^2) Y_b^2 + (Z - \alpha_v X)^2} \left(1 - \frac{X + \alpha_v Z}{R_b} \right) \quad (3.4)$$

in which

$$X = x_a - x \quad Y_a = y_a - y \quad Y_b = y_b - y \quad Z = z_a - z \quad (3.5)$$

$$R_a = \sqrt{X^2 + Y_a^2 + Z^2} \quad \text{and} \quad R_b = \sqrt{X^2 + Y_b^2 + Z^2} \quad (3.6)$$

The resultant induced velocity components at point (x, y, z) due to a single horseshoe vortex are expressed in terms of so-called influence coefficients denoted by H , and are given by

$$u_i = (H_{Bx} + H_{Fx}) \Gamma \quad (3.7)$$

$$v_i = (H_{By} + H_{Fy}) \Gamma \quad (3.8)$$

$$\text{and } w_i = (H_{Bz} + H_{Fz}) \Gamma \quad (3.9)$$

The contributions of the bound leg to these influence coefficients are given by

$$H_{Bx} = -\frac{Z}{4\pi(X^2 + Z^2)} \left(\frac{Y_b}{R_b} - \frac{Y_a}{R_a} \right) \quad (3.10)$$

$$H_{By} = 0 \quad (3.11)$$

$$\text{and } H_{Bz} = \frac{X}{4\pi(X^2 + Z^2)} \left(\frac{Y_b}{R_b} - \frac{Y_a}{R_a} \right) \quad (3.12)$$

The influence contributions of the trailing legs are found to be given in terms of

$$F_a = \frac{R_a - X - \alpha_v Z}{R_a \left[(1 + \alpha_v^2) Y_a^2 + (Z - \alpha_v X)^2 \right]} \quad (3.13)$$

$$F_b = \frac{R_b - X - \alpha_v Z}{R_b \left[(1 + \alpha_v^2) Y_b^2 + (Z - \alpha_v X)^2 \right]} \quad (3.14)$$

by

$$H_{Fz} = -\frac{1}{4\pi} (Y_b F_b - Y_a F_a) \quad (3.15)$$

$$H_{Fy} = \frac{1}{4\pi} (Z - \alpha_v X) (Y_b F_b - Y_a F_a) \quad (3.16)$$

$$\text{and} \quad H_{Fx} = -\alpha_v H_{Fz} \quad (3.17)$$

3.2.2 Approximation of Flow Tangency Condition

The resultant induced velocity components at a set of control points (x_j, y_j, z_j) due to a set of horseshoe vortices with circulation strengths Γ_k are given in terms of influence matrices $\underline{\underline{H}}_x$, $\underline{\underline{H}}_y$ and $\underline{\underline{H}}_z$ by

$$\bar{u}_i = \underline{\underline{H}}_x \bar{\Gamma} \quad \bar{v}_i = \underline{\underline{H}}_y \bar{\Gamma} \quad \text{and} \quad \bar{w}_i = \underline{\underline{H}}_z \bar{\Gamma} \quad (3.18)$$

The elements of $\underline{\underline{H}}_x$, $\underline{\underline{H}}_y$ and $\underline{\underline{H}}_z$ are computed using equations (3.10) to (3.17). The small angle approximation of the flow tangency condition, formulated in index notation, is then found to be

$$\sum_{k=1}^{N_v} \left[\left(\alpha - \frac{dz_{cj}}{dx} \right) H_{xj,k} - \beta_{dj} H_{yj,k} + H_{zj,k} \right] \Gamma_k = - \left(\alpha - \frac{dz_{cj}}{dx} \right) (V_\infty + u_{Qj}) - w_{Qj} \quad (3.19)$$

This is the vortex lattice counterpart of the flow tangency problem in equation (2.45), and its first order approximation is

$$\sum_{k=1}^{N_v} H_{zj,k} \Gamma_k^* = -\alpha + \frac{dz_{cj}}{dx} - \alpha_{Qj} \quad (3.20)$$

The notation Γ_k^* for the circulation strength (per unit air speed) of horseshoe vortex k is adopted here for the sake of brevity, and will be used later in the formulation of the vortex lattice approximation of the aerodynamic coefficients.

Once the influence coefficients of all N_v horseshoe vortices Γ_k^* at all N_v control points j have been computed, equation (3.20) can be solved for the circulation values (per unit air speed) of all the horseshoe vortices. Not only a set of real horseshoe vortices, but also their mirror images (placed below the ground to satisfy flow tangency on the ground plane) are to be considered when computing the influence coefficients in ground proximity. In this case the influence coefficients $H_{Izj,k}$ of the image vortices must be subtracted from the influence coefficients $H_{Rzj,k}$ of the real vortices, because the circulation values of the image vortices are equal in magnitude but opposed in direction to that of their real counterparts. Thus

$$H_{zj,k} = H_{Rzj,k} - H_{Izj,k} \quad (3.21)$$

in which the subscripts R and I refer to real and image vortices, respectively.

Three additional influence matrices are computed, namely the influence matrices $\bar{H}_{\ell e}$, $\bar{H}_{LL}$ and $\bar{H}_v$. Note that $\bar{H}_{\ell e}$ is not square, but of dimension $(N_{sw} + N_{st}) \times N_v$, if the wing

and tail are divided spanwise in N_{sw} and N_{st} panels, respectively, and a total of N_v horseshoe vortices are used, whereas $\bar{H}_{LL}$ is square and of dimension $(N_{sw} + N_{st})$. The former of these influence matrices is used to find the strength distribution of the leading edge singularities of the wing and tail for the leading edge suction computation, whereas $\bar{H}_{LL}$ is used in the far-field estimation of the induced drag coefficient. The third additional influence matrix, $\bar{H}_v$, has the same dimension as the influence matrix of the flow tangency problem, and is used in the induced drag estimation by the conventional CVLM. These three matrices therefore allow three different methods of induced drag computation, the comparison of which provides an indication of the accuracy of the vortex lattice analysis.

3.2.3 Matrix Vector Formulation of Flow Tangency Problem

The physical meaning of the circulation values $\Gamma_{j,k}^*$ per unit air speed obtained from the solution of the flow tangency problem above is that it represents the product of the locally estimated y-component of vortex strength $\gamma_{j,k}^*$ (the asterisk denotes normalisation with respect to air speed) and the incremental chordwise length $\Delta x_{j,k}$ of the local vortex panel given in index notation by

$$\Gamma_{j,k}^* = \gamma_{j,k}^* \Delta x_{j,k} \quad (3.22)$$

Here j and k represents row and column indices that respectively correspond to chordwise and spanwise indices of the surface panels to which these circulation increments $\Gamma_{j,k}^*$ belong. Because $\Gamma_{j,k}^*$ is actually a *finite increment* one should use $\Delta\Gamma_{j,k}^*$ rather than $\Gamma_{j,k}^*$, but we select the latter notation for the sake of brevity. Another option is the use of *matrix* notation to denote the circulation per unit air speed by $\bar{\Gamma}_w^*$ and $\bar{\Gamma}_t^*$ for the wing and tail, respectively. Yet another notation that we employ here is the use of *column vectors* $\bar{\Gamma}_w^*$ and $\bar{\Gamma}_t^*$ that contain all the elements extracted row by row from $\bar{\Gamma}_w^*$, and $\bar{\Gamma}_t^*$, respectively. The vector notation is more convenient in some cases, but the matrix

notation is perhaps more easily interpreted when displayed in a table of numbers. Using this vector notation for the circulation increments and for the required upwash per unit air speed the solution of the flow tangency problem stated in equation (3.20) is found after inversion of the matrix of influence coefficients to be given simply by

$$\bar{\Gamma}^* = -\bar{H}_z^{-1} \bar{\alpha} \quad (3.23)$$

in which

$$\bar{\alpha} = \begin{bmatrix} \bar{\alpha}_w \\ \bar{\alpha}_t \end{bmatrix} \quad (3.24)$$

where

$$\bar{\alpha}_w = \alpha + \frac{2q}{c_w} (\bar{x}_{cw} - \ell_{CM/LEw}) - \overline{\left(\frac{dz_{cw}}{dx} \right)} \quad (3.25)$$

$$\text{and } \bar{\alpha}_t = \alpha - \delta_t + \frac{2q}{c_w} (\bar{x}_{ct} - \ell_{CM/LEw}) - \overline{\left(\frac{dz_{ct}}{dx} \right)} \quad (3.26)$$

Note that $\overline{\left(\frac{dz_{cw}}{dx} \right)}$ indicates the column vector of slopes of the camber line of the wing, and that this *notation for the column vector of the values of some expression in brackets* will be used in subsequent formulations too. The terms $2q(\bar{x}_{cw} - \ell_{CM/LEw})/c_w$ and $2q(\bar{x}_{ct} - \ell_{CM/LEw})/c_w$ represent the contributions of pitch rate to the local incidence angle at the control points of the wing and tail, respectively. The dimensionless pitch rate q is defined in terms of the dimensional pitch rate Q by $q = Qc_w/2V_\infty$, in which multiplication with the aerodynamic time unit $c_w/2V_\infty$, which is the time it would take the free stream to traverse half the mean aerodynamic wing chord, accomplishes this

normalisation. Note furthermore that addition of a scalar to a vector as indicated in these expressions is, strictly speaking according to Greenberg³², *not mathematically allowable*, because the dimensions are not compatible. However, one can ignore this objection if it is agreed that the scalar is to be added to every element of the vector, because such an addition *is numerically allowable in Matlab*, and we wish to present formulations that are as compatible with the computational syntax of Matlab as possible.

The partitioned form of the vector of circulation values thus found is

$$\bar{\Gamma}^* = \begin{bmatrix} \bar{\Gamma}_w^* \\ \bar{\Gamma}_t^* \end{bmatrix} \quad (3.27)$$

The use of these circulation values in the computation of the aerodynamic force and moment resultant is described next.

3.2.4 Aerodynamic Coefficients of a Wing and Tail

Expressions for the aerodynamic coefficients of a single aerodynamic surface were presented in integral form in paragraph 2.6. The solution of the flow tangency problem as formulated above, however, yields not the strength components $\zeta_x(\xi, \eta)$ and $\zeta_y(\xi, \eta)$ of the vortex system in the functional form required there, but rather a column vector $\bar{\Gamma}^*$ of the circulation strengths (per unit air speed) of the set horseshoe vortices by which the vortex sheets of the wing and tail are approximated. The *integral expressions* for the aerodynamic coefficients are therefore *approximated by summations over the vortex lattices* of the wing and tail. Intended as a first approximation we used the simplest possible numerical approach in these approximations, namely the straightforward Euler integration algorithm. This method turned out to yield very accurate results compared to those of Lan¹⁴, so that further experiments with more sophisticated numerical integration schemes were not deemed necessary.

3.2.4.1 Lift Coefficient

Using equations (2.41), (2.47) and (2.85) it is not difficult to show that, if the induced lift C_{Li} and the contribution C_{LQ} due to pitch rate are negligible, then the combined lift coefficient (referred to the wing area S_w) of a wing and tail is approximately

$$C_L = \frac{2}{V_\infty S_w} \left[\int_{-b_w/2}^{b_w/2} \Gamma_w(\eta_w) d\eta_w + \int_{-b_t/2}^{b_t/2} \Gamma_t(\eta_t) d\eta_t \right] \quad (3.28)$$

It was found that even though C_{Li} is not entirely insignificant, it is quite negligible (on the order of 1 % of C_L) for the purpose of a first order analysis, except perhaps very close to the ground (less than one wing chord) and for aspect ratios that are very large indeed. Noting that $\Gamma(\eta)$ is the integral over the chord of $\zeta_y(\xi, \eta)$, reverting to the more popular notation $\gamma(\xi, \eta) = \zeta_y(\xi, \eta)$ and defining

$$c_\ell c = \frac{2}{V_\infty} \int_{x_{LE}}^{x_{TE}} \gamma(\xi, \eta) d\xi = \frac{2\Gamma(\eta)}{V_\infty} = 2\Gamma^*(\eta) \quad (3.29)$$

it follows that an alternative formulation of the lift coefficient is

$$C_L = \frac{1}{S_w} \left[\int_{-b_w/2}^{b_w/2} (c_\ell c)_w d\eta_w + \int_{-b_t/2}^{b_t/2} (c_\ell c)_t d\eta_t \right] \quad (3.30)$$

Recalling the relationship $\Gamma_{j,k}^* = \gamma_{j,k}^* \Delta x_{j,k}$ in equation (3.22) above it is not unreasonable to approximate equation (3.29) with

$$\overline{(c_\ell c)} = 2 \sum \bar{\Gamma}^* \quad (3.31)$$

in which *the indexless summation operator used in $\sum \bar{\Gamma}^*$ very specifically denotes the summation of elements down each column to produce a row vector of sums, as would be produced by the command **sum(Gamma)** in Matlab.* This row vector then contains, in each of its elements, the total bound circulation of the chordstrip of which the circulation increments are contained in the corresponding column of $\bar{\Gamma}^*$.

Having thus introduced a rather transparent notation, and one that is closely related to the computational syntax of Matlab, one can see that it is not unreasonable to approximate the integral expression for the lift coefficient in equation (3.30) with the summation

$$C_L = \frac{1}{S_w} \left[\left(\overline{c_\ell c} \right)_w^T \Delta \bar{y}_w + \left(\overline{c_\ell c} \right)_t^T \Delta \bar{y}_t \right] \quad (3.32)$$

The Matlab statements that would perform these numerical calculations could typically be

```
cl_c_w = 2*sum(Gam_mat_w);
cl_c_t = 2*sum(Gam_mat_t);
CL = 1/Sw * (cl_c_w * dy_w + cl_c_t * dy_t);
```

Note that cl_c_w , for example, must be a row vector, and dy_w must be an equally long column vector (of spanwise increments) in order to obtain the required inner product. (If dy_w happened to be a row vector resulting from prior computations, then dy_w' , merely transposed by the prime, would be the required column vector.)

3.2.4.2 Pitch Moment Coefficient

The pitch moment coefficient of a single surface can be inferred from equation (2.86) to be given with reference to the planform area and the mean aerodynamic chord by the first order approximation

$$C_M = -\frac{2}{V_\infty S_w c_w} \int_{-b/2}^{b/2} \int_{x_{LE}}^{x_{TE}} \xi \gamma(\xi, \eta) d\xi d\eta \quad (3.33)$$

If the pitch moment coefficient *about a given reference point*, for example the leading tip of the midspan chord, is required, then the position ξ of a local vortex element must be measured relative to this reference point. An alternative formulation in terms of the product $c_m c^2$ (note that c_m is the sectional pitch moment coefficient *about the selected reference point*) is obtained by defining

$$c_m c^2 = -\frac{2}{V_\infty} \int_{x_{LE}}^{x_{TE}} \xi \gamma(\xi, \eta) d\xi = -2 \int_{x_{LE}}^{x_{TE}} \xi \gamma^*(\xi, \eta) d\xi \quad (3.34)$$

It then follows from equations (3.33) and (3.34) that the pitch moment coefficient is expressed in terms of $c_m c^2$ by

$$C_M = -\frac{2}{S_w c_w} \int_{-b/2}^{b/2} \int_{x_{LE}}^{x_{TE}} \xi \gamma^*(\xi, \eta) d\xi d\eta = \frac{1}{S_w c_w} \int_{-b/2}^{b/2} c_m c^2 d\eta \quad (3.35)$$

If $C_M = C_{MLE}$ is the pitch moment taken about the leading tip of the midspan chord, and if the center of mass CM is situated at a distance $x_{CM/LE}$ behind this point, then the pitch moment coefficient about the CM is found, by translation of the lift force and compensation for the additional moment thus introduced, to be given by

$$C_M = \frac{1}{S_w c_w} \int_{-b/2}^{b/2} c_m c^2 d\eta + \frac{x_{CM/LE}}{c_w} C_L \quad (3.36)$$

Using the Euler integration scheme, as before in equation (3.31), the integral in equation (3.34) can be approximated by a finite sum. Application of the result to the wing and tail then yields sums over the respective chords given in the notation introduced above by

$$\left(\overline{c_m c^2}\right)_w = -2 \sum \underline{\bar{x}}_w \bullet \underline{\bar{\Gamma}}_w^* \quad \text{and} \quad \left(\overline{c_m c^2}\right)_t = -2 \sum \underline{\bar{x}}_t \bullet \underline{\bar{\Gamma}}_t^* \quad (3.37)$$

In this case, however, the *moment of force exerted by each bound vortex element is to be computed before summing over the chord* in order to yield the values of the local sectional pitch moment parameter $c_m c^2$. The bullets in equations (3.37) indicate the element by element multiplication of $\underline{\bar{x}}$ and $\underline{\bar{\Gamma}}^*$ that are to be performed before evaluating the sums over chord. Note that $\underline{\bar{x}}$ is a matrix that contains the centre points of the bound vortices, of which the circulation per unit air speed are contained in the corresponding matrix $\underline{\bar{\Gamma}}^*$. The total pitch moment coefficient about whichever point the elements of $\underline{\bar{x}}_w$ and $\underline{\bar{x}}_t$ were referred to is thus found to be approximated by

$$C_M = \frac{1}{S_w c_w} \left[\left(\overline{c_m c^2}\right)_w^T \Delta \bar{y}_w + \left(\overline{c_m c^2}\right)_t^T \Delta \bar{y}_t \right] \quad (3.38)$$

If the reference point happened to be the leading tip of the midspan chord of the wing, and if one then requires the moment coefficient C_{MCM} about the centre of mass, then C_{MCM} is merely

$$C_{MCM} = C_{MLEw} + \frac{x_{CM/LE}}{c_w} C_L \quad (3.39)$$

3.2.4.3 Induced Drag Coefficient by Conventional Vortex Lattice Method

The induced drag coefficient C_{Di} of a single surface, on which the distribution of induced downwash angle is given as α_i , is inferred from equation (2.84) to be

$$C_{Di} = \frac{2}{V_\infty S_w} \int_{-b/2}^{b/2} \int_0^c \alpha_i \gamma(\xi, \eta) d\xi d\eta \quad (3.40)$$

Note that the (negative) influence of pitch rate indicated in equation (2.84) is yet to be added in order to obtain the total drag. An alternative formulation of equation (3.40) in terms of the product $c_{di} c$ (note that c_{di} is the sectional induced drag coefficient) is obtained by defining

$$c_{di} c = \frac{2}{V_\infty} \int_{x_{LE}}^{x_{TE}} \alpha_i \gamma(\xi, \eta) d\xi = 2 \int_{x_{LE}}^{x_{TE}} \alpha_i \gamma^*(\xi, \eta) d\xi \quad (3.41)$$

It then follows from equations (3.40) and (3.41) that the induced drag coefficient is expressed in terms of $c_{di} c$ by

$$C_{Di} = \frac{2}{S_w} \int_{-b/2}^{b/2} \int_{x_{LE}}^{x_{TE}} \alpha_i \gamma^*(\xi, \eta) d\xi d\eta = \frac{1}{S_w} \int_{-b/2}^{b/2} c_{di} c d\eta \quad (3.42)$$

Proceeding along the same lines of reasoning that were followed for the pitch moment coefficient of the wing and tail above, one finds the spanwise distribution of induced drag on the wing and tail to be approximated by the chordwise sums

$$(\overline{c_{div} c})_w = 2 \sum \underline{\alpha}_{ivw} \bullet \underline{\Gamma}_w^* \quad \text{and} \quad (\overline{c_{div} c})_t = 2 \sum \underline{\alpha}_{ivt} \bullet \underline{\Gamma}_t^* \quad (3.43)$$

Note that the subscript v is introduced here in order to emphasise that equations (3.43) are based on the conventional vortex lattice method (CVLM) of induced drag computation. The total induced drag coefficient of the wing and tail is then obtained, similarly to the lift coefficient, by forming the weighted sum of inner products

$$C_{Div} = \frac{1}{S_w} \left[(\overline{c_{div} c})_w^T \Delta \bar{y}_w + (\overline{c_{div} c})_t^T \Delta \bar{y}_t \right] \quad (3.44)$$

This is the CVLM approximation of the integral expression

$$C_{Di} = \frac{1}{S_w} \left[\int_{-b_w/2}^{b_w/2} \alpha_i (c_\ell c)_w d\eta_w + \int_{-b_t/2}^{b_t/2} \alpha_i (c_\ell c)_t d\eta_t \right] \quad (3.45)$$

3.2.4.4 Induced Drag Coefficient by Far Field Approach

The so-called *far-field induced drag coefficient* is obtained by using straight lifting lines with lift distributions

$$\overline{(c_\ell c)}_w = 2 \sum \bar{\Gamma}_w^* \quad \text{and} \quad \overline{(c_\ell c)}_t = 2 \sum \bar{\Gamma}_t^* \quad (3.46)$$

passing through the leading tips of the midspan chords of the wing and tail, respectively. (One may prefer to have them passing through the ¼-chord points of the respective mean aerodynamic chords, but the difference in distance will be quite small if the sweep angles are moderate, and if the distance between the leading edges is a few wing chords.) The induced downwash distribution on these lifting lines are found using

$$\bar{\alpha}_{iLL} = -\bar{H}_{FzLL} \bar{\Gamma}_{LL}^* \quad (3.47)$$

from which the distributions $\bar{\alpha}_{iLLw}$ and $\bar{\alpha}_{iLLt}$ on the lifting lines of the wing and tail, respectively, are then extracted. Note that $\bar{\Gamma}_{LL}^*$ contains the circulation strengths of the horseshoe vortices that constitute the lifting lines. In other words, $\bar{\Gamma}_{LL}^*$ is the vector

$$\bar{\Gamma}_{LL}^* = \begin{bmatrix} \bar{\Gamma}_{LLw}^* \\ \bar{\Gamma}_{LLt}^* \end{bmatrix} \quad (3.48)$$

in which, according to equation (3.31) in rearranged form, we have

$$\bar{\Gamma}_{LLw}^* = \frac{1}{2} \overline{(c_\ell c)}_w = \sum \bar{\Gamma}_w^* \quad (3.49)$$

$$\text{and} \quad \bar{\Gamma}_{LLt}^* = \frac{1}{2} \overline{(c_\ell c)}_t = \sum \bar{\Gamma}_t^* \quad (3.50)$$

Note that $\bar{H}_{FzLL}$ to be used in equation (3.47) is the z-component of the influence matrix (evaluated at the lifting lines) of the free (trailing) legs of the horseshoe vortices, represented by $\bar{\Gamma}_{LLw}^*$ and $\bar{\Gamma}_{LLt}^*$, and their images. *The combination of these horseshoe vortices, which result from the chordwise contraction of the vortex lattices of the wing and the tail onto straight spanlines, can be regarded as a vortex lattice approximation of their respective continuous lifting lines.*

The corresponding far-field induced drag distribution on these approximate lifting lines are found from

$$\overline{(c_{di} c)}_w = \sum \bar{\alpha}_{iLLw} \bullet \overline{(c_\ell c)}_w \quad \text{and} \quad \overline{(c_{di} c)}_t = \sum \bar{\alpha}_{iLLt} \bullet \overline{(c_\ell c)}_t \quad (3.51a)$$

or, alternatively, using the first members of equations (3.49) and (3.50), from

$$\overline{(c_{di} c)}_w = 2 \sum \bar{\alpha}_{iLLw} \bullet \bar{\Gamma}_{LLw}^* \quad \text{and} \quad \overline{(c_{di} c)}_t = 2 \sum \bar{\alpha}_{iLLt} \bullet \bar{\Gamma}_{LLt}^* \quad (3.51b)$$

The latter formulation may be preferred for its similarity to equation (3.43)

The total far-field drag coefficient is finally given in terms of the distributions obtained from either of equations (3.51) by

$$C_{Di} = \frac{1}{S_w} \left[\overline{(c_{di} c)}_w^T \Delta \bar{y}_w + \overline{(c_{di} c)}_t^T \Delta \bar{y}_t \right] \quad (3.52)$$

as in equation (3.44), except that, in order to avoid confusion with the CVLM approximation of induced drag in equation (3.44), the subscript v is omitted in equation (3.52) here.

3.2.4.5 Induced Drag Coefficient by Leading Edge Singularity Approach

A final approach to induced drag estimation, one which is based on the near field induced velocity, as was the approach of the CVLM outlined in paragraph 2.6.4.3, is the leading edge singularity method (LESM) of Lan¹⁴. This method is applied to the case of *incompressible flow around a wing and tail in ground proximity* in this study, as opposed to its application to a *wing alone, with provision for compressibility*, as reported by Lan¹⁴. A modified version of this method, one that allows for pitch rate, ground proximity and non-zero tail angle, is therefore presented here. The three methods of drag estimation were found to be in very close agreement, within a fraction of 1 % for a wing alone far from the ground, and less than about 5 % for a wing and tail not too close to the ground. The close correspondence can be regarded as an indication that the vortex lattice computations yield results of acceptable accuracy.

The induced upwash $\bar{w}_{iLE}^*$ distribution on the leading edge control points of the wing and tail are found simultaneously using

$$\bar{w}_{iLE}^* = \bar{H}_{zLE} \bar{\Gamma}^* \quad (3.53)$$

from which the distributions $\bar{w}_{iLEw}^*$ and $\bar{w}_{iLEt}^*$ on the wing and tail may be extracted as required. The distribution of the leading edge singularity strength $\bar{C}^*$ of a single surface at zero pitch rate can be interpreted from Lan¹⁴ to be given by

$$\bar{C}^* = \frac{\cos \Lambda_{LE}}{N_c} \left[\bar{w}_{iLE}^* + \alpha - \left(\frac{dz_c}{dx} \right)_{LE} \right] \quad (3.54)$$

in which Λ_{LE} is the sweep angle of the leading edge (included for the sake of completeness here, even though we focussed on straight leading edges), and N_c is the

number of horseshoe vortices on a given chord line. The modification of the singularity method to allow for a wing and tail at non-zero pitch rate was found to yield

$$\overline{C}_w^* = \frac{\cos \Lambda_{LEw}}{N_{cw}} \left[\overline{w}_{iLEw}^* + \alpha + \frac{2q}{c_w} (\overline{x}_{cLEw} - \ell_{CM/LEw}) - \left(\overline{\frac{dz_c}{dx}} \right)_{LEw} \right] \quad (3.55)$$

$$\text{and} \quad \overline{C}_t^* = \frac{\cos \Lambda_{LEt}}{N_{ct}} \left[\overline{w}_{iLEt}^* + \alpha - \delta_t + \frac{2q}{c_w} (\overline{x}_{cLEt} - \ell_{CM/LEw}) - \left(\overline{\frac{dz_c}{dx}} \right)_{LEt} \right] \quad (3.56)$$

Note that the (column vectors of) respective position co-ordinates, $\overline{x}_{cLEw}$ and $\overline{x}_{cLEt}$, of the leading edge control points of both the wing and the tail are measured with respect to the leading tip of the wing's midspan chord, as is the distance $\ell_{CM/LEw}$ of the centre of mass (CM), which is also the centre of pitching motion.

These results are then used to obtain the distributions of the sectional leading edge thrust parameters c_{iLE} of the wing and tail from

$$(\overline{c_{iLE}c})_w = \frac{\pi c_w \overline{C}_w^* \bullet \overline{C}_w^*}{2 \cos \Lambda_{LEw}} \quad \text{and} \quad (\overline{c_{iLE}c})_t = \frac{\pi c_t \overline{C}_t^* \bullet \overline{C}_t^*}{2 \cos \Lambda_{LEt}} \quad (3.57)$$

(The sectional leading edge thrust coefficient c_{iLE} and the tail chord c_t are not to be confused here!) The small angle approximation of the total leading edge thrust coefficient C_T of the wing and tail, *not to be confused with the propulsion coefficient used elsewhere*, is then found from

$$C_T = \frac{1}{S_w} \left[(\overline{c_t c})_w^T \Delta \bar{y}_w + (\overline{c_t c})_t^T \Delta \bar{y}_t \right] \quad (3.58)$$

The leading edge singularity approximation of the induced drag coefficient is finally obtained from

$$C_{Dii} = \frac{1}{S_w} \left[\alpha (\overline{c_\ell c})_w^T \Delta \bar{y}_w + (\alpha - \delta_t) (\overline{c_\ell c})_t^T \Delta \bar{y}_t \right] - C_T \quad (3.59)$$

The results presented above were used in the penultimate version of our Matlab algorithms. In the process of producing our final results a few changes were made in order to check the accuracy of some first order approximations made along the way. For example, we made provision for the influence of the horizontal component u_i of induced velocity in the lift and moment computation, and for the (direct) influence of pitch rate as indicated in equations (2.84) and (2.85), and we included the term $\theta H_{zj,k}$ in equation (3.20). We finally reverted, furthermore, to a computational procedure that reflects equations (2.84) to (2.86) more directly (except for the necessary transformations between the geodetic base and the wind base) than the formulations above do. These changes, however, (or shall we rather say fortunately!), were all found to make no more than a few percent difference in the results. The program code listed in Appendix F, must thus be regarded as more reliable than the formulations above, because the latter obviously have a better chance of still containing some typing error than is likely in computer code of which the results have been checked and cross-checked numerous times.

In conclusion, for the benefit of the reader that wishes to inspect the Matlab code in Appendix F we must mention the following:

- all position co-ordinates required in the description of the vortex lattice geometry were resolved in the geodetic base, with its x-axis pointing horizontally to the rear, and the z-axis pointing vertically upwards
- these position co-ordinates were used in the computation of all of the required influence matrices. This implies that, solving the flow tangency problem for the required circulation distribution, and subsequently *computing the induced velocity at*

the bound vortex midpoints by using an influence matrix resolved in the geodetic base, yields induced velocity components in the geodetic base.

- the final computation of the lift, drag and pitch moment coefficients was done by first obtaining the aerodynamic force coefficients C_{Fx} and C_{Fz} in the geodetic base, and also the pitch moment coefficient C_M . (The latter is the same in the wind base and the geodetic base.) This was accomplished by using the discrete counterparts of equations (2.84) to (2.86), except that geodetic components of the local position vector and the local induced velocity vector were used, so that the geodetic force coefficients and C_{Fx} and C_{Fz} were thus obtained, rather than C_D and C_L .
- in order to obtain C_D and C_L , the geodetic force components were then transformed to the wind base, using the same transformations as those indicated by equations (2.87) and (2.88).
- as mentioned in Chapter 2, the contribution of the chordwise component ζ_x of bound vorticity was neglected in all computations.

4. STABILITY, CONTROL AND SENSITIVITY ANALYSIS

The longitudinal motion of a fixed wing aircraft is a state of motion in which the centre of mass is confined to the vertical plane that contains the longitudinal plane of bilateral symmetry of the aircraft. Under this restriction the velocity of the centre of mass is uniquely determined by its air speed and the angle of climb, and the angular velocity by the pitch rate. The resultant aerodynamic force and moment, however, depend on the angle of attack (and on the height above the ground too, when in ground proximity). Their computation thus requires that the *pitch angle (from which the angle of climb is subtracted to yield the angle of attack) and the height must be known in addition to the airspeed and rate of climb.*

Any state of free longitudinal motion far from the ground can be expressed as a combination of the two natural modes of longitudinal motion. These are the short period mode of motion (also known as the α -mode), which is essentially a pitch oscillation of the aircraft with little change in airspeed or angle of climb, and the so-called phugoid mode, which essentially describes the translational motion of the centre of mass. The well-known α -mode is characterised by large changes in pitch angle and angle of attack compared to the changes in air speed and height, as described by McCormick²⁸, Ashley³⁰ and others. The equally well-known long period mode of motion (the phugoid or P-mode) is a kind of roller coaster motion, and is characterised by large changes in air speed, pitch angle and height compared to the changes in angle of attack.

Von Mises¹⁰ and Milne-Thomson¹¹ both refer to the long period mode of motion as Lanchester's phugoid. (Lanchester, according to Von Mises, sought a Greek term for flight, and made a linguistic slip in selecting the word phugoid, because it actually means flight in the sense of fleeing, not flying!) Lanchester's phugoids referred originally to the different trajectories along which a particle can move under the influence (in addition to the weight) of a force that is *normal to the trajectory and proportional in magnitude to the square of the speed*, such as aerodynamic lift. These trajectories, illustrated in Figure 4.1, remind strongly of the three characteristics of a symmetric top, i.e. the

cuspidal motion, the looping precession and the direct precession! The term phugoid is interpreted these days, however, as referring to an aircraft's natural tendency to exhibit a slightly damped oscillation between climbing and descending flight, which, after allowed to damp out on its own, will settle down to a new line of straight and level flight (a line that is slightly above the initial line of flight if the initial angle of climb was positive). The damping is due to *aerodynamic drag*, which is also proportional to the square of the speed in reality, but it is *approximately proportional to the speed perturbation, if small perturbations from the equilibrium state of flight are considered*. The lift, too, varies in approximate proportion to a small speed perturbation, with the result that the perturbed state of longitudinal motion to which the term phugoid is applied in flight mechanics currently is a lightly damped oscillation obtained as the solution of a linear second order differential equation of motion, rather than one of a family of solutions of a non-linear second order equation.

In addition to the phugoid and α -mode modes of motion there is an aperiodic heave mode (the z-mode). This mode is of trivial concern far from the ground, where it merely reflects the insensitivity of aerodynamic forces to small height changes, an insensitivity that is due to the fact that the change in the local air density with a small change in height (one that is comparable to the aircraft's dimensions) is minute indeed. Close to the ground, however, the heave mode is of no trivial concern, because, for certain aircraft configurations, it can become unstable when the height is in the order of magnitude of the wing span. For other aircraft configurations, even though their heave modes remain stable, the phugoid mode can become unstable, as reported by Staufenberg¹⁹.

The purpose of the present investigation is to shed light on the nature, the causes and the implications of these instabilities, to assess the influence of various design parameters on this phenomenon, and to alleviate its effect by feedback control. The example on which we focus our attention is a wing-and-tail configuration that approximates that of the Piper Cherokee PA-28-180, of which the data obtained from McCormick²⁸ are given in Table 2. The methods of aerodynamic analysis by which we obtained the stability derivatives are:

- a) the quasi-vortex lattice method (QVLM) described by Lan¹⁴, which we restricted here to rectangular surfaces in incompressible flow,
- b) a modified version of the lifting line methods of Prandtl³ (PLLM) and Weissinger⁹ (WLLM), as discussed in Chapter 1, and mentioned briefly again in Chapter 3.

In both cases empirical provision was made for parasitic drag, and for such added mass terms as may be significant in the short period mode, based on the data in Table 2. The QVLM is preferred, primarily for its inherently more accurate estimation of the pitch moment derivatives.

The PLLM and the WLLM yielded aerodynamic derivatives that differ considerably from those obtained from the QVLM, the PLLM yielding lift curve slopes *and* pitch moment derivatives that are markedly larger in magnitude than those of the QVLM, whereas it is primarily the pitch moment derivatives of the WLLM that differ from the QVLM. The stability assessments based on PLLM and WLLM results, however, turned out to be essentially the same as that based on the QVLM results. These observations are just made in passing here: we present no corroborating results to this effect later, mainly because of our belief in the reliability of the QVLM results.

4.1 Equations of Motion

The equations of translational motion are obtained from Newton's Second Law in a particularly convenient form by summing the *force components resolved in the relative wind direction, on the one hand, and normal to the wind direction, on the other hand*. The force resultants thus obtained yield the acceleration components in the wind base, a reference frame of which the x-axis remains tangential to the flight path. The longitudinal equation of rotational motion merely equates the pitch inertia times the angular acceleration *in pitch* to the pitch moment M , taken about the centre of mass. These equations of translational and rotational motion must be augmented by three kinematic equations. Two of these are differential equations, of which one merely relates the pitch angle to the pitch rate, while the other relates the climb rate to the angle of climb. The

third of the kinematic equations, however, is an algebraic equation that relates the angle of climb to the pitch angle and the angle of attack, i.e., $\gamma = \Theta - \alpha$.

The physical vector form of the equation of translational motion is shown in Appendix D to be

$$m(\mathbf{D}_r \vec{V}_{CM} + \vec{\omega}_r \times \vec{V}_{CM}) = \vec{F}_a + \vec{F}_g \quad (4.1)$$

in which $\mathbf{D}_r$ is an operator that signifies the time derivative with respect to a rotating reference frame. The components of the aerodynamic force are the lift L , by definition normal to the flight path, and the total drag D opposed to the direction of flight. The equations of translational motion in the longitudinal mode are shown in Appendix D to be expressed in terms of airspeed, angle of climb, lift, drag, thrust and weight mg by

$$m \frac{dV_\infty}{dt} = T - D - mg \sin \gamma \quad (4.2)$$

$$\text{and} \quad m V_\infty \frac{d\gamma}{dt} = L - mg \cos \gamma \quad (4.3)$$

The time derivative of the angle of climb γ in equation (4.3) arises from (and is equal to) the angular velocity of the rotating reference frame used here. This is the reason why the reference x-axis remains aligned with the relative wind, and it ensures that only a single velocity component, namely the airspeed V_∞ , features in this formulation. Elimination of γ from equations (4.2) and (4.3) using $\gamma = \Theta - \alpha$ yields

$$m \frac{dV_\infty}{dt} = T - D - mg \sin(\Theta - \alpha) \quad (4.4)$$

$$\text{and} \quad m V_\infty \frac{d\alpha}{dt} = m V_\infty \frac{d\Theta}{dt} - L + mg \cos(\Theta - \alpha) \quad (4.5)$$

The physical vector-dyadic form of the equation of rotational motion is

$$\bar{\mathbf{J}} \cdot \mathbf{D}_b \bar{\boldsymbol{\omega}}_b + \bar{\boldsymbol{\omega}}_b \times (\bar{\mathbf{J}} \cdot \bar{\boldsymbol{\omega}}_b) = \bar{\mathbf{M}}_a \quad (4.6)$$

in which $\bar{\mathbf{J}}$ signifies the *physical inertia tensor*, a notation that represents a *physical dyadic quantity*, as opposed to the *component inertia tensor* expressed in Einstein's notation by J_{ij} . Note that $\bar{\mathbf{J}}$ in equation (4.6) is taken about the CM, and its *dyadic product* with the angular velocity vector in $\bar{\mathbf{H}}_{\text{CM}} = \bar{\mathbf{J}} \cdot \bar{\boldsymbol{\omega}}_b$ yields the absolute angular momentum about the CM. In the absence of rolling or yawing motion the equation of pitching motion is found from equation (4.6) to take the simple form

$$I_{yy} \frac{dQ}{dt} = M_a \quad (4.7)$$

Equations (4.4), (4.5) and (4.7), along with the two kinematic equations

$$\frac{d\Theta}{dt} = Q \quad (4.8)$$

$$\text{and} \quad \frac{dH}{dt} = V_\infty \sin \gamma \quad (4.9)$$

constitute the five *non-linear state equations of longitudinal motion in dimensional form*. Normalising and perturbing these differential equations, and noting that $T = D$ and $L = mg$ in equilibrium, the *linear state equations of longitudinal motion in dimensional form*, valid for small perturbations with respect to straight and level flight, are shown in Appendix D to be

$$2\mu \frac{du}{d\tau} = C_{D0} \delta_T - \Delta C_D - 2C_{D0} u - C_W (\theta - \alpha) \quad (\text{for jets and gliders}) \quad (4.10a)$$

$$\text{or} \quad 2\mu \frac{du}{d\tau} = C_{D0}\delta_P - \Delta C_D - 3C_{D0}u - C_W(\theta - \alpha) \quad (\text{for propeller aircraft}) \quad (4.10b)$$

$$2\mu \frac{d\alpha}{d\tau} = 2\mu q - 2C_{L0}u - \Delta C_L \quad (4.11)$$

$$i_{yy} \frac{dq}{d\tau} = \Delta C_M \quad (4.12)$$

$$\frac{d\theta}{d\tau} = q \quad (4.13)$$

$$\text{and} \quad \frac{dz}{d\tau} = \frac{1}{2}(\theta - \alpha) \quad (4.14)$$

The small perturbations of C_L , C_D and C_M required above are given by

$$\Delta C_L = C_{L\alpha}\alpha + C_{Lq}q + C_{L\theta}\theta + C_{Lz}z + C_{L\dot{\alpha}}\dot{\alpha} + C_{L\dot{q}}\dot{q} + C_{L\delta t}\delta_t \quad (4.15)$$

$$\Delta C_D = C_{D\alpha}\alpha + C_{Dq}q + C_{D\theta}\theta + C_{Dz}z + C_{D\dot{\alpha}}\dot{\alpha} + C_{D\dot{q}}\dot{q} + C_{D\delta t}\delta_t \quad (4.16)$$

$$\Delta C_M = C_{M\alpha}\alpha + C_{Mq}q + C_{M\theta}\theta + C_{Mz}z + C_{M\dot{\alpha}}\dot{\alpha} + C_{M\dot{q}}\dot{q} + C_{M\delta t}\delta_t \quad (4.17)$$

We note that the derivatives with respect to θ only come into play in close ground proximity, and their effect is quite small even then. The derivatives with respect to $\dot{\alpha}$ and $\dot{q}$ are found to affect the damping ratio of the short period mode of motion significantly. They exert, however, a very small influence on the phugoid mode, a mode of motion in which even the q -derivatives play but a small role, albeit not entirely negligible.

4.2 State Space Formulation

The perturbational equations of longitudinal motion with fixed controls, including the kinematical equations, are found from equations (4.10) to (4.17) to be given in matrix notation by the state space form

$$\underline{\bar{B}} \dot{\bar{x}} = \underline{\bar{A}} \bar{x} + \underline{\bar{A}}_\delta \bar{\delta} \quad (4.18)$$

in which $\bar{x} = [u \quad \alpha \quad q \quad \theta \quad z]^T$ is a small perturbation of the state vector, whereas $\bar{\delta} = [\delta_p \quad \delta_e]^T$ represents the vector of control inputs to the throttle and the elevator. The $\underline{\bar{B}}$ -matrix is given by

$$\underline{\bar{B}} = \begin{bmatrix} 2\mu & 0 & 0 & 0 & 0 \\ 0 & 2\mu + C_{L\dot{\alpha}} & 0 & 0 & 0 \\ 0 & -C_{M\dot{\alpha}} & i_{yy} - C_{Mq} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (4.19)$$

in which the upper left hand 3x3 diagonal block represents the mass matrix of the equations of motion, whereas the lower right hand 2x2 block arises from the kinematic equations.

The matrix of stability derivatives is found from equations (4.10) to (4.17) to be

$$\underline{\bar{A}} = \begin{bmatrix} -k_D C_{D0} & C_{L0} - C_{D\alpha} & -C_{Dq} & -C_{D\theta} - C_{L0} & -C_{Dz} \\ -2C_{L0} & -C_{L\alpha} & -C_{Lq} + 2\mu & -C_{L\theta} & -C_{Lz} \\ 0 & C_{M\alpha} & C_{Mq} & C_{M\theta} & C_{Mz} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} \quad (4.20)$$

in which the influence of the θ -derivatives of C_L , C_D and C_M is usually negligible. Note furthermore that $k_D = 2$ for jets and gliders, or $k_D = 3$ for propeller aircraft, according to the propulsion analysis in Appendix D. The matrix of aerodynamic control gains is

$$\underline{\underline{A}}_\delta = \begin{bmatrix} C_{D0} & -C_{D\delta t} \\ 0 & -C_{L\delta t} \\ 0 & C_{M\delta t} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (4.21)$$

in which one expects the effect of $C_{D\delta t}$ and $C_{L\delta t}$ to be quite small, because an elevator control input δ_t primarily causes an unbalanced pitch moment, but does not change the lift and drag by much *immediately*. The *immediate response to an increase in δ_t* , after all, *is nose up pitch acceleration*, followed by an *increase in angle of attack afterwards*. This eventual increase in angle of attack *does* change the lift and drag, especially the lift, but the *eventual lift increase thus obtained is much more than the initial lift reduction caused by an increase in δ_t* .

The eigenvalues of equation (4.18) are the values of λ that allow non-trivial solutions for $\bar{x}$ to be found from

$$(\underline{\underline{A}} - \lambda \underline{\underline{B}}) \bar{x} = \bar{0} \quad (4.22a)$$

Non-trivial solutions for $\bar{x}$ exist if and only if the values of λ satisfy

$$|\underline{\underline{A}} - \lambda \underline{\underline{B}}| = 0 \quad (4.22b)$$

which is a polynomial equation in λ (one of fifth order in the case of longitudinal motion) known as the characteristic equation. The roots of this equation are the required eigenvalues, also known as the characteristic roots of the system. *If the eigenvalues are*

all different, which they usually are, then, for a specific eigenvalue, an arbitrary value may be assigned to any one of the five state variables, after which the values of the other four elements in the eigenvector under consideration can be found by solving four of the five equations in (4.22a). Repetition of this procedure for each of the other four eigenvalues then yields four more eigenvectors. The ratios of the magnitudes of the elements of a specific eigenvector, relative to any selected element, are the same as the ratios of magnitudes of a set of initial conditions that would excite a motion that is characterised temporally by the eigenvalue to which the eigenvector under consideration belongs, provided that such initial values are phased according to the eigenphases. (This interpretation holds for complex conjugate pairs of characteristic roots and their complex conjugate pairs of eigenvectors too.)

The main objective of the present investigation is to quantify and explain the influence of ground proximity on the various eigenvalues and eigenvectors of a fixed wing aircraft in its longitudinal modes of motion in order to explain the nature of the longitudinal instabilities encountered in close ground proximity, such as the phugoid instability reported by Staufienbiel¹⁹.

4.3 Stability Analysis

The aerodynamic derivatives required in the solution of the eigenvalue problem were obtained using the QVLM of Lan¹⁴. Fair agreement was found between the eigenvalues computed with stability derivatives obtained from the QVLM and those obtained with empirically estimated stability derivatives by McCormick²⁸ for a Piper Cherokee at 1500 m altitude and $C_{L0} = 0.543$, as shown in Figures R3 a) and b).

Taking the Laplace transform of equation (4.18) with zero control inputs yields the same result as equation (4.22b), except in terms of s rather than λ . Using the definitions in equations (4.19) and (4.20), the characteristic equation of longitudinal motion (for negligible θ -derivatives) is thus found to be

$$\begin{vmatrix}
2\mu s + k_D C_{D0} & C_{D\alpha} - C_{L0} & C_{Dq} & C_{L0} & C_{Dz} \\
2C_{L0} & (2\mu + C_{L\dot{\alpha}})s + C_{L\alpha} & C_{Lq} - 2\mu & 0 & C_{Lz} \\
0 & -C_{M\dot{\alpha}}s - C_{M\alpha} & i_{yy}s - C_{Mq} & 0 & -C_{Mz} \\
0 & 0 & -1 & s & 0 \\
0 & \frac{1}{2} & 0 & -\frac{1}{2} & s
\end{vmatrix} = 0 \quad (4.23)$$

Expansion of the characteristic determinant yields the 5th order characteristic polynomial of longitudinal motion. The stability of the α -mode, on the one hand, and the phugoid-and-heave mode of motion, on the other hand, is subsequently analysed by approximate factorisation of this polynomial.

4.3.1 Decoupled Short Period Mode

The short period mode of motion is a rapid, fairly well-damped pitch oscillation characterised by changes in angle of attack of approximately equal magnitude to that of the pitch angle, while the horizontal speed perturbation and the height changes are very small indeed. Its characteristic roots are found with surprising accuracy by assuming the speed and height perturbations to produce negligible lift and moment perturbations in the heave and pitch equations of motion, equations (4.11) and (4.12). (A change in angle of attack at given pitch angle is due entirely to a change in the vertical velocity component, for which reason it is not unreasonable to refer to equation (4.11) as the equation of heave motion.) This results in the decoupling of the pitch and heave equations of motion from the surge equation, equation (4.10), and the kinematic equations (4.13) and (4.14). Careful scrutiny of equation (4.23) reveals that this decoupling yields a *reduced second order characteristic equation of short period motion* given by

$$\begin{vmatrix}
2\mu s + C_{L\alpha} & -2\mu + C_{Lq} \\
-C_{M\dot{\alpha}}s - C_{M\alpha} & (i_{yy} - C_{Mq})s - C_{Mq}
\end{vmatrix} = 0 \quad (4.24)$$

Expanding the determinant and normalising the resulting second order polynomial equation with respect to the coefficient of s^2 one finds this approximation of the characteristic equation of the short period mode to be of the form

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad (4.25)$$

The necessary and sufficient stability requirements of a second order system are merely that all coefficients be of the same sign, which, using equation (4.24), can be shown to require in this case that

$$\omega_n^2 = -\frac{C_{Mq}C_{L\alpha} - C_{M\alpha}(2\mu - C_{Lq})}{2\mu(i_{yy} - C_{M\dot{q}})} > 0 \quad (4.26)$$

$$\text{and } 2\zeta\omega_n = \frac{(i_{yy} - C_{M\dot{q}})C_{L\alpha} - 2\mu(C_{Mq} + C_{M\dot{\alpha}}) + C_{M\ddot{\alpha}}C_{Lq}}{2\mu(i_{yy} - C_{M\dot{q}})} > 0 \quad (4.27)$$

The results obtained for ω_n and $\sigma = \zeta\omega_n$ from equations (4.26) and (4.27) agree with those of Ashley³⁰ if both $C_{M\dot{q}}$ and $C_{L\dot{\alpha}}$ are negligible, which they usually are. The magnitudes of C_{Lq} and C_{Mq} are usually small compared to 2μ and i_{yy} , respectively. By the usual definition of the stick-fixed static margin Δh , the distance of the neutral point aft of the CM, the first of these stability criteria reduces to the requirement that

$$\Delta h = -\frac{C_{M\alpha}}{C_{L\alpha}} > \frac{C_{Mq}}{2\mu} \quad (4.28)$$

If $C_{Mq} < 0$, as usual, then a positive Δh , which is usually required from a handling qualities perspective, will satisfy this dynamic stability requirement of the α -mode automatically. Its second dynamic stability requirement reduces to

$$C_{L\alpha} > \frac{2\mu}{i_{yy}}(C_{Mq} + C_{M\dot{\alpha}}) \quad (4.29)$$

which is also automatically satisfied, because $C_{L\alpha}$ is always positive, while C_{Mq} and $C_{M\alpha}$ are always negative.

The short period roots computed using equations (4.25), (4.26) and (4.27) was found, as seen in Table 4 j) and illustrated in Figure R3 a), to differ in ω_n by less than 2 % from those obtained using the fifth order characteristic equation (4.23), and their rate of decay $\sigma = \zeta\omega_n$ was found to deviate by less than 6 % in the height range $0 < b/H < 0.25$.

The increase in ω_n , which is represented by the radial distance from the origin in Figure R3 a), and the increase in σ , which is represented by the horizontal distance to the left of the imaginary axis, may be explained by approximating equations (4.26) and (4.27). Approximation of equation (4.26) yields, after rearrangement

$$\omega_n^2 \approx -\frac{C_{M\alpha}}{i_{yy}} \left(1 - \frac{C_{Mq}}{2\mu \Delta h} \right) \quad (4.30)$$

One finds that the magnitude of $C_{M\alpha}$ increases appreciably with decreasing height, as shown in Table 3 d), while the changing magnitudes of C_{Mq} and Δh , as shown in Tables 3 e) and 4 b), are masked somewhat by the division through 2μ in equation (4.30). (Note that both $C_{M\alpha}$ and C_{Mq} are negative.) Even though the second term in the brackets of equation (4.30) is of appreciable magnitude, its magnitude does not *change* as much with decreasing height as that of $C_{M\alpha}$. The *increase in the undamped natural frequency ω_n of the short period mode with decreasing height* shown in Figure R3 a) is therefore *primarily due to the fact that $C_{M\alpha}$ increases in magnitude with decreasing height. And $C_{M\alpha}$ increases more rapidly than does $C_{L\alpha}$ with the result that the neutral point shifts aft with decreasing height*, as shown in the second column of Table 4 b).

Equation (4.27) yields, after approximation and rearrangement

$$\sigma \approx -\frac{C_{Mq}}{2i_{yy}} \left(1 + \frac{C_{M\dot{\alpha}}}{C_{Mq}} \right) \left(1 - \frac{i_{yy}}{2\mu} \frac{C_{L\alpha}}{C_{Mq} + C_{M\dot{\alpha}}} \right) \quad (4.31)$$

Equation (4.31) shows that σ is directly proportional to C_{Mq} , on the one hand, but one finds that it is enhanced quite strongly by the two bracketed terms. The increase in σ with decreasing height is partly due the direct influence of the increasing magnitude of C_{Mq} in the leading term of equation (4.31), and partly due to the fact that $C_{L\alpha}$ increases more rapidly with decreasing height than does the magnitude of C_{Mq} in the second bracketed term, as shown in Tables 3 d) and 3 e).

It was shown above how judicious simplifications allow a second order α -factor of equations (4.25) to (4.27) to be extracted with fair accuracy from the fifth order characteristic polynomial of longitudinal motion. The α -mode of motion in ground proximity was found to be influenced *indirectly* by ground height, primarily via the effect of height on $C_{L\alpha}$, $C_{D\alpha}$ and $C_{M\dot{\alpha}}$, *and not directly* via C_{Lz} , C_{Dz} and the C_{Mz} . The latter three do not even feature in this approximation, which explains the comparative insensitivity of the α -mode.

4.3.2 Approximation of Long Period Roots

An equally illuminating perspective of the characteristic roots of the P- and z-modes of longitudinal motion in ground proximity was found, this time by extraction of a third order characteristic determinant from the fifth order one, based on physical reasoning regarding the long period motion.

The long period P-mode of motion was said above to be a slow, lightly damped oscillation characterised by large perturbations of the air speed, large perturbations of the pitch angle and large perturbations of the dimensionless height compared to the perturbations of the angle of attack and the pitch rate.

The preceding statement cannot make any sense without defining different scales for these different variables, because one cannot, for example, compare a change in pitch angle with a height change directly, even if the latter is measured on a dimensionless scale. To this end we needed to scale the eigenvectors of the P- and z-modes of motion in a way that makes physical sense. (We recall that an eigenvector is a set of initial conditions that would excite only the motion characterised by the eigenvalue of the particular eigenvector being considered.) A reasonable basis of comparison, by which we mean a scale for which one has an intuitive feel, seems to be the expression of all air speed perturbations as a percentage of the reference air speed rather than a fraction of the reference air speed, and the expression of perturbations of angle of attack and pitch angle in degrees rather than radians. Having established these scales, we can judge whether the magnitudes of a particular perturbation associated with a given height perturbation is large or small. On this basis one finds that the phugoid mode out of ground proximity produces a *percentage* change in air speed and a pitch angle *in degrees* that are comparable to the height change per unit wing chord, but the *changes in angle of attack and pitch rate are quite small compared to the trimmed angle of attack*. It remains difficult to judge whether the change in pitch rate is really negligible. But if the maximum change in pitch *angle* is fairly small and the period of motion is fairly long, then the influence of changes in pitch rate *may* be negligible.) And *the fact that the dimensionless pitch rate q represents the increase in the local angle of attack per distance of half a wing chord advanced rearward provides a basis on which a given value of q can be regarded as large or not*, especially if it is expressed in degrees, as in Table 5.

We assume that there exists another mode of motion, hoping that it turns out to be a partially decoupled phugoid-and-heave state of motion, in which not only the perturbations of the lift and drag forces, but also the pitch moment produced by the perturbation of pitch rate are negligible. (By *partially* decoupled we mean that the motion is independent, for all practical purposes, of the short period state of motion, but that it is still partially coupled in the sense that the z-motion cannot be considered separately from the phugoid motion.) We assume furthermore that the moment of inertia in pitch is

negligible, which implies that the pitch moments produced by changes in angle of attack and the height changes in the P- and z-states of motion are approximately equal in magnitude and opposed in direction. And we assume finally that the *aerodynamic* forces and moments produced by the time rates of change of angle of attack and pitch rate are negligible. (The analysis of the stability of the long period modes that follows, in which all q-derivatives are neglected, was repeated later with inclusion of all the q-derivatives, which led to essentially the same stability assessments.) Referring to equation (4.12) and (4.17) these assumptions are seen to imply that

$$C_{M\alpha}\alpha + C_{Mz}z \approx 0 \quad (4.32)$$

from which it follows that

$$\alpha \approx -\frac{C_{Mz}z}{C_{M\alpha}} \quad (4.33)$$

Equations (4.32) and (4.33) therefore allow equation (4.12) and the angle of attack to be eliminated from the equations of longitudinal motion. The characteristic determinant of the resulting reduced system of equations is found by first reducing equation (4.23) to

$$\begin{vmatrix} 2\mu s + k_D C_{D0} & C_{D\alpha} - C_{L0} & 0 & C_{L0} & C_{Dz} \\ 2C_{L0} & 2\mu s + C_{L\alpha} & -2\mu & 0 & C_{Lz} \\ 0 & -C_{M\alpha} & 0 & 0 & -C_{Mz} \\ 0 & 0 & -1 & s & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & s \end{vmatrix} = 0 \quad (4.34)$$

The assumption of negligible pitch moment of inertia reduces equation (4.12) to an algebraic equation, after which we may eliminate it from the set of state equations, along with either the α or the z state variable. The elimination of equation (4.12) means that we can merely remove the third row from the determinant in equation (4.34). If we choose to eliminate α , then the reduced characteristic determinant may be found by replacing the

fifth column of the determinant in equation (4.34) with $C_{M\alpha}$ times the original fifth column less C_{Mz} times the second column, after which the second column may be removed from the determinant, because its parent state variable has been effectively eliminated from the state equations now. This yields a reduced characteristic equation given by

$$\begin{vmatrix} 2\mu s + k_D C_{D0} & 0 & C_{L0} & C_{Dz} C_{M\alpha} - C_{Mz} (C_{D\alpha} - C_{L0}) \\ 2C_{L0} & -2\mu & 0 & C_{Lz} C_{M\alpha} - C_{Mz} (2\mu s + C_{L\alpha}) \\ 0 & -1 & s & 0 \\ 0 & 0 & -\frac{1}{2} & C_{M\alpha} s - \frac{1}{2} C_{Mz} \end{vmatrix} = 0 \quad (4.35)$$

from which the characteristic polynomial is found to be of third order, even though the determinant is of dimension 4. The coefficients of the characteristic equation

$$As^3 + Bs^2 + Cs + D = 0 \quad (4.36)$$

are found to be

$$A = -4\mu^2 C_{M\alpha} \quad (4.37)$$

$$B = -2\mu k_D C_{D0} C_{M\alpha} \quad (4.38)$$

$$C = \mu (C_{M\alpha} C_{Lz} - C_{L\alpha} C_{Mz}) - 2 C_{M\alpha} C_{L0}^2 \quad (4.39)$$

and

$$D = \frac{1}{2} k_D C_{D0} (C_{M\alpha} C_{Lz} - C_{L\alpha} C_{Mz}) + C_{L0} (C_{Mz} C_{D\alpha} - C_{Dz} C_{M\alpha}) \quad (4.40)$$

The Routh-Hurwitz stability criteria demand that, if $A > 0$, then one must have $B > 0$, $D > 0$ and $BC - AD > 0$ for the system to be asymptotically stable. The first two of these criteria reduce to the requirements that $C_{M\alpha} < 0$ and $C_{D0} > 0$, the former of which is automatically met if the short period mode of motion is stable, as it has been shown to be, and the latter is obviously met by the fact that the equilibrium drag coefficient will always be positive. We note in passing that the z-derivatives out of ground effect are zero, so that the coefficients of the characteristic equation reduce to

$$A = -4\mu^2 C_{M\alpha} \quad (4.41)$$

$$B = -2\mu k_D C_{D0} C_{M\alpha} \quad (4.42)$$

$$C = -2 C_{M\alpha} C_{L0}^2 \quad (4.43)$$

and $D = 0$

It follows that the *longitudinal motion represented by equation (4.36), apart from its slow oscillating nature, is actually neutrally stable out of ground effect if $C_{M\alpha} < 0$ and $C_{D0} > 0$, the neutral stability being due to the root at $s = 0$ that reflects insensitivity to height changes*, as mentioned in the introductory discussion of this chapter. This root, however, will have a small negative value if the influence of the air density derivatives on the force and moment derivatives is taken into account, which will have the effect that, if disturbed from its initial steady flight condition, the aircraft will drift slowly back to the height at which it was trimmed, instead of leveling out at a slightly different height. The rate of decay of this drift will be slow indeed, very much slower even than the rate of decay of the phugoid mode, which is the reason why the density derivatives are usually ignored. *It is not unreasonable to call the root at $s = 0$ the height root, or even the heave root*. One must not, however, step into the trap of associating this heave root with the equation of heave motion, equation (4.11)! It is often totally meaningless to associate any characteristic root with any one of a set of coupled linear equations, unless such an

association is motivated really well with reference to a root locus, such as the locus that start at the origin in Figure 3 b), which really does trace this height root (or heave root).

The characteristic values and stability criteria of the phugoid roots far from the ground are thus found from equations (4.41) to (4.43) to be

$$\omega_n^2 = \frac{C_{L0}^2}{2\mu^2} > 0 \quad (4.44)$$

and

$$2\zeta\omega_n = \frac{k_D C_{D0}}{2\mu} > 0 \quad (4.45)$$

which agrees with the results of Ashley³⁰. The *phugoid mode of motion is thus found to be unconditionally stable far from the ground, if the α -mode is stable.*

Returning to equations (4.37) to (4.40) one finds that the stability criterion $BC - AD > 0$ can be solved for C_{Dz} to yield the requirement that

$$C_{Dz} < \frac{k_D C_{D0} C_{L0}}{\mu} + \frac{C_{Mz} C_{D\alpha}}{C_{M\alpha}} \quad (4.46)$$

and that $D > 0$ can be solved for C_{Lz} to yield the requirement that

$$C_{Lz} < \frac{C_{Mz} C_{L\alpha}}{C_{M\alpha}} + 2 \frac{C_{L0}}{k_D C_{D0}} \left(C_{Dz} - \frac{C_{Mz} C_{D\alpha}}{C_{M\alpha}} \right) \quad (4.47)$$

The maximum real value of all five roots of the fifth order characteristic equation, which was calculated *without making the assumptions* on which the stability conditions of equations (4.46) and (4.47) were based, is plotted against height in Figure R3 c), which

shows that this aircraft becomes unstable when it goes below a height of approximately 0.85 % of its wing span.

Equation (4.46) shows that there is a critical value

$$C_{Dzcrit1} < \frac{k_D C_{D0} C_{L0}}{\mu} + \frac{C_{Mz} C_{D\alpha}}{C_{M\alpha}} \quad (4.48)$$

below which C_{Dz} must remain for stability according to the approximate stability criterion $BC - AD > 0$ derived above. We emphasize that this criterion is based on the simplifying assumptions that allowed the approximate extraction of a third order characteristic polynomial of phugoid-and-heave motion from the full fifth order characteristic polynomial. The value of C_{Dz} is plotted against height in Figure R3 d), which shows, along with Table 4 a), that C_{Dz} exceeds its estimated critical value C_{Dzcrit} obtained above at a height of approximately 62.5 % of the span, well below the height of 80 % of span where the maximum real part of the characteristic roots have shown the instability in Figure R3 c) and Table 4 a) to have already occurred. *The longitudinal instability of this configuration is therefore not due to C_{Dz} exceeding C_{Dzcrit} .*

Equation (4.47) shows, in conclusion of the formulation of longitudinal stability criteria, that there is yet one more critical value, namely

$$C_{Lzcrit} = \frac{C_{Mz} C_{L\alpha}}{C_{M\alpha}} + 2 \frac{C_{L0}}{k_D C_{D0}} \left(C_{Dz} - \frac{C_{Mz} C_{D\alpha}}{C_{M\alpha}} \right) \quad (4.49)$$

below which C_{Lz} must remain for stability. This boundary follows from the *stability criterion $D > 0$* , the one that is also the *criterion of static phugoid and heave stability, as opposed to the static pitch stability criterion of equation (4.28)*. Plotting C_{Lz} against this critical value, as shown in Figure R3 e) produced with data obtained from Table 4 a), one finds C_{Lz} to exceed this critical value C_{Lzcrit} (obtained from the 3rd order reduced system) at a height of approximately 80 % of the wing span, i.e. where $b/H = 1.2$. Figures R3 d)

and R3 e), along with Table 4 a) thus show conclusively that, *for this aircraft configuration, the instability encountered in ground proximity is due to C_{Lz} exceeding C_{Lzcrit} .*

4.4 Typical Open Loop Stability Assessment

The following preliminary conclusions were drawn from a nominal stability analysis based on the criteria formulated above:

- a) the accuracy of the analysis of the α -mode of motion by decoupling it from the P- and z-modes is very good, as shown in Figure R3 a)
- b) the accuracy of the analysis of the P- and z-modes of motion by a reduced 3rd order characteristic polynomial is quite acceptable. This judgement is motivated by the good agreement between the eigenvalues estimated using the reduced third order characteristic determinant of equations (4.35) to (4.40) as compared to those obtained from the 5th order determinant in equation (4.23), especially in the height range down to where the longitudinal stability first occurs, as shown in Figure R3 c) and Tables 4 c) and 4 d).
- c) the stability criteria of equations (4.48) and (4.49) can be expected to be quite reliable, because they are based on the coefficients of the same characteristic equation (that of the reduced 3rd order characteristic polynomial of phugoid-and-heave motion) that were used to obtain the good estimation of eigenvalues in Figure R3 c). The conclusion drawn from Figure R3 e) that the onset of instability occurs at the point where $C_{Lz} > C_{Lzcrit}$ is therefore considered to be well motivated.
- d) the influence of positive and increasing C_{Mz} with decreasing height was found to be destabilising. Numerical results showed that equation (4.49) yields a *higher stability boundary C_{Lzcrit} the lower C_{Mz} becomes*, which *shifts the onset of longitudinal instability to a lower height*. It was this observation that led to the idea that any physical design change (or artificial stabilisation by feedback control) that tends to reduce the magnitude of C_{Mz} , either directly or effectively, should be stabilising.

4.5 Closed Loop Control

Our final objective of alleviating the longitudinal instability encountered in ground proximity was pursued by investigating the possibilities offered by feedback of the height perturbation to the control input of the all-movable tail. This strategy was motivated on physical grounds, the basis of which was the observation that not only the critical value $C_{Lz_{crit}}$ below which the height derivative C_{Lz} of the lift coefficient must remain for stability, but also the critical value $C_{Dz_{crit}}$ below which the height derivative C_{Dz} of the drag coefficient must remain for stability, primarily depend on the height derivative C_{Mz} of the pitch moment coefficient. It was found, as can be expected naturally, that C_{Mz} is zero out-of-ground-effect, but positive in ground proximity, increasing as the height decreases. This increase reduces the critical limit $C_{Lz_{crit}}$ (which is negative) of C_{Lz} (which is also negative) more rapidly than C_{Lz} is reduced by ground proximity, until C_{Lz} exceeds $C_{Lz_{crit}}$ and instability sets in. It was therefore argued that one could reasonably expect design changes that primarily reduce C_{Mz} to improve longitudinal stability in ground proximity, irrespective of whether longitudinal instability occurs due to $C_{Lz} > C_{Lz_{crit}}$ or to $C_{Dz} > C_{Dz_{crit}}$. This argument led us to expect that negative feedback of height should improve stability. The reason for this is that such feedback will decrease the tail angle δ_t (nose down with respect to the wing chord is positive) when a positive (upward) height perturbation z occurs, which will produce a reduction of pitch moment and thus reduce C_{Mz} too.

Careful consideration of equations (4.18), (4.20) and (4.21) reveals that, if:

- a) the derivatives C_{Dz} , C_{Lz} and C_{Mz} in equation (4.20) are denoted alternatively by C_{DzOL} , C_{LzOL} and C_{MzOL} in order to emphasise that they reflect the inherent Open Loop characteristics of the aircraft, and if
- b) the perturbation of elevator angle δ_t is replaced $\delta_{tSF} + \delta_{tFB}$ to indicate that it is in part due the (Stick-Fixed) input of the pilot, while FeedBack control (typically limited authority control by adjustment of the length of an active link) provides the remaining part, then if
- c) the feedback of height to the elevator is reflected mathematically by a so-called control law (its just a name encountered often in control theory, not really a law),

which merely states that, in this case, $\delta_{tFB} = -K_z z$, then finally the closed loop values C_{DzCL} , C_{LzCL} and C_{MzCL} of C_{Dz} , C_{Lz} and C_{Mz} will be:

$$C_{DzCL} = C_{DzOL} + K_z C_{D\delta t} \quad (4.50)$$

$$C_{LzCL} = C_{LzOL} + K_z C_{L\delta t} \quad (4.51)$$

$$C_{MzCL} = C_{MzOL} - K_z C_{M\delta t} \quad (4.52)$$

If one demands, furthermore, that the closed loop value of C_{Mz} be some specified reduction factor K_{CMz} times its open loop value, then substitution of $C_{MzCL} = K_{CMz} C_{MzOL}$ into equation (4.52) yields the required feedback gain:

$$K_z = (1 - K_{CMz}) C_{MzOL} / C_{M\delta t} \quad (4.53)$$

Equations (4.50) and (4.51) above show that, because $C_{D\delta t} < 0$ and $C_{L\delta t} < 0$, as seen in Table 3 h), this feedback strategy will, for positive K_z , increase the magnitude of C_{Lz} , which is negative, while reducing the value of C_{Dz} , which is positive. These changes, however, were found to be much smaller percentages than the (prescribed) percentage change that K_z causes in C_{Mz} .

It is to be noted that the strong dependence of C_{MzOL} on height, as seen in Table 7 l), requires that an increasing value of K_z must be used as the height decreases if C_{Mz} is to be reduced by a constant factor, which seems to be required in this strategy if stability is to be ensured over a range of heights. This means that it is a kind of gain scheduled feedback control that we propose here. The results that are obtained look very promising, and are discussed in more detail in the next (our final) chapter.

5 RESULTS AND DISCUSSION

The vortex lattice analysis of the longitudinal stability of fixed wing aircraft in ground proximity was the focus of the present investigation. It was therefore imperative that the reliability of our vortex lattice algorithm be assessed before the stability problem could be addressed with confidence. The next step was to perform a stability analysis of a suitable wing and tail configuration, to which end the Piper Cherokee PA-28-180 was selected as example. A parameter sensitivity analysis, in which the influence of some design parameters such as the CM position, the wing aspect ratio, the tail height, et cetera, was performed subsequently in order to draw conclusions that are applicable to other configurations too. The final step was to investigate briefly the possibility of alleviating, by feedback control, the instability encountered in ground proximity.

The following numerical results are presented and discussed here with reference to the section **TABLES OF NUMERICAL RESULTS**:

- an assessment of the accuracy of our computations, in Tables 1 a) to 1 c).
- the data of our nominal wing and tail configuration, in Table 2.
- the results of the nominal trim calculations, and the aerodynamic derivatives of the nominal configuration for these trim states, in Tables 3 a) to h).
- the results of the nominal open loop stability analysis, in Tables 4 a) to j) and Table 5.
- the results of the sensitivity analysis with respect to CM position, in Table 6
- the results of a closed loop stability analysis, performed for the nominal case with feedback of height perturbation to the elevator, in Table 7.
- Some stability results for a high tail configuration in Table 8

Tables 3 to 8 form a small, but a quite representative set of examples of the data on which the graphs in the section **GRAPHICAL RESULTS** are based. The following graphs are included in said section, again serving only as examples that illustrate the most important of our results, and form the basis of our final conclusions:

- illustrations of typical vortex lattice configurations, shown in Figures R1 a) to d) and discussed below.

- a comparison of some of the present aerodynamic results to those obtained by other investigators, in Figures R2 a) to c).
- results of the nominal open loop stability analysis, in Figures R3 a) to g). The first two of these include the stability roots of the short and long period modes of motion out of ground effect obtained by McCormick²⁸ using empirical methods.
- results of a sensitivity analysis with respect to CM position, in Figures R4 a) to f). The first of these representing the influence of CM position on the critical height at which instability sets in.
- results of sensitivity analyses with respect to other design parameters, in Figures R5 a) and b) to Figures R8 a) and b), grouped in pairs of identical format.
- an illustration of the success with which gain scheduled feedback of the height perturbation to the elevator control input can alleviate the instability of the long period motion in ground proximity, in Figures R9 a) and c), as compared to Figures R3 b) and g), respectively. This feedback was found to leave the short period motion unaffected.

The essential conclusions that we drew from these results are the following:

- the longitudinal instability encountered in ground proximity, in many different wing and tail configurations, is due to the *heave mode of motion* of the CM motion (a mode with a single real characteristic root) becoming *monotonically divergent*, while the so-called *phugoid mode, initially oscillatory and lightly damped, becomes monotonically convergent*. This instability, at least in all the occurrences we encountered, was found to be *due to the value of C_{Lz} exceeding a critical value C_{Lzcrit}* , a stability boundary that depends on C_{Mz} . It can equivalently be regarded as being due to h_{Nz} , the *static pitch moment stability margin with respect to height perturbation*, exceeding a critical value h_{Nzcrit} .
- in some cases, however, such as a high tail, the *phugoid mode* of CM motion, *changes from a stable, decaying oscillation to an unstable, growing oscillation*, while the *heave mode remains monotonically convergent*. This instability, again in all the occurrences we encountered, was found to be *due to the value of C_{Dz} exceeding a critical value C_{Dzcrit}* , a stability boundary that also depends on C_{Mz} .

- irrespective of which kind of instability occurs, it can be alleviated successfully by *feedback of the height perturbation to the elevator control input*, because such feedback *primarily decreases C_{Mz} , thus raising the stability boundary*. This control strategy leaves the short period mode of motion virtually unaffected.

The rest of this Chapter is devoted to motivating these conclusions.

5.1 Reliability of Vortex Lattice Results

Our vortex lattice results were compared to those obtained for:

- a rectangular wing of aspect ratio $AR = 2$ out of ground effect by Lan¹⁴
- a wing with a Warren 12 planform out of ground effect by Lan¹⁴. (The Warren 12 planform is defined by a leading edge sweep angle of 53.54 deg and a taper ratio, i.e. ratio of tip chord to midspan chord, of $\lambda = 1/3$.)
- a flat rectangular wing of infinite aspect ratio at non-zero angle of attack and non-zero pitch rate out of ground effect analysed by thin wing theory
- flat rectangular wings of aspect ratios 1, 2 and 4 in ground proximity by Du Plessis¹⁷ (who quoted Saunders³³) and Suh²⁰.

Examples of the lattice geometry we used are illustrated in Figures R1 a) to R1 d). These were produced by the Matlab program we developed for our aerodynamic analyses, and illustrate the typical vortex lattice geometry of a wing and tail combined. Wing and tail surfaces of $AR = 2$ are shown for illustrative purposes in Figure R1 a) and R1 b), using the cosine distributions of Lan¹⁴ for the vortex corner points, control points and vortex midpoints. (We have tried the similar distribution used by Miranda¹⁵ et al., and later by Du Plessis¹⁷ too, and found that the differences in the results they produce were not really significant. They placed their vortex corner points where our control points are, adding corner points on the wing tips, and they used control points where our vortex corner points are.) The corner points and the control points are respectively indicated by x's, o's and light dots, the centre of mass (CM) by a heavy dot, and the $1/4$ -chord points of the mean aerodynamic chords are represented by squares. The side view in Figure R1 b) shows a wing with a high tail in ground proximity. A fairly large pitch angle and a fairly

large angle of descent was used for illustrative purposes. (Note that we use a geodetic reference frame centred at the wing's leading edge – its lateral symmetry simplifies the formulation of the image lattice co-ordinates in ground proximity.) The ground plane is shown in the heavy solid line at approximately -2 m. The slanted solid lines represent the wing and tail chords, and the long dashes represent the wing and tail vortex sheets, which are both assumed to be flat, aligned with the free stream, and passing through the $\frac{1}{4}$ chord points of the respective mean aerodynamic chords. The lattice geometry of two swept and tapered surfaces, both with Warren 12 planform shapes, are illustrated in Figures R1 c) and R1 d).

The results we obtained using wing vortex lattices such as those shown in Figures R1 a) to R1 d), are compared to the results of previous investigators in Tables 1 a) to 1 c), and in Figures R2 a) to R2 c). It is seen in these Tables that:

- the predictions of $C_{L\alpha}$, $C_{M\alpha}$ and C_{Di}/C_L^2 by the present analysis for a wing with $AR = 2$ in Table 1 a) agree within less than 0.18 % with those of Lan¹⁴.
- our predictions of $C_{L\alpha}$ and $C_{M\alpha}$ for a wing with a Warren 12 planform in Table 1 b) agree within less than 0.4 % with those of Lan¹⁴. The deviation of 4.4 % in the value of C_{Di}/C_L^2 for the Warren 12 planform is not of serious consequence in this study, because we concentrated on wings of rectangular planform in the conclusions that we eventually drew concerning longitudinal stability in ground proximity.
- the predictions of C_L , C_M , C_D and the ratio of $C_D/(2 q C_M)$ by the present analysis of a flat, rectangular wing of very large aspect ratio at non-zero angle of attack and pitch rate are shown in Table 1 c) to agree very well indeed with the exact 2D results from Appendix B.

On inspection of Table 1 c) it seems, at first glance, that the negative C_D must obviously be wrong. A vortex lattice result of non-zero at very high aspect ratio, one that approaches the 2D result for infinite aspect ratio so closely, is surprising indeed, and the implied *positive thrust* is even more surprising. This unexpected result, however, stems from the fact that the wing motion considered here is pure pitch motion about a fixed centre of pitch (CP). The leading edge (LE) rotates about the CP at a pitch rate Q

expressed in terms of the normalised pitch rate q by $Q = 2q V_{LE} / c$, where $V_{LE} = QR$ is the speed of the leading edge, and R is the distance between the CP and the LE. The angle of attack α is defined as the direction of LE motion (i.e. the incidence angle of air flow) with respect to the chord line at the LE. Downstream of the LE the incidence angle increases, to first order approximation, at q radians per downstream distance of $\frac{1}{2}$ the wing chord. The 2D analysis of Appendix B applied to this case show that $c_l = 2\pi(\alpha + 3q/2)$, $c_d = -2\pi q(\alpha/2 + q)$ and $c_{mLE} = -(\pi q + c_l)/4$. The pitching motion is clearly resisted by the nose down pitch moment, but this moment turns out to cancel the moment of thrust (taken about the centre of rotation) exactly, thus resolving the paradox of negative C_D . Regarded in terms of energy principles, the lift does no work, because it is by definition a force normal to the velocity vector of the LE, while the positive rate of work $D V_{LE}$ done by the drag is cancelled exactly by the negative rate of work $M_{LE} Q$ done by the pitching moment.

Concerning the accuracy of our vortex lattice model at a decreased number of chordwise and spanwise horseshoe vortices we concluded furthermore that:

- a reduction of the number of chordwise and spanwise wing segments from 8 and 15 to 3 and 6, respectively, does not reduce the accuracy significantly in the case of a rectangular wing of $AR = 2$.
- a further reduction of the number of chordwise and spanwise wing segments to 2 and 4, respectively, was later found to still represent the essential trends of the longitudinal stability characteristics very well, even though the accuracy of results such as the critical height at which instability sets in deteriorates by a few percent.

Figures R2 a) to R2 c) show furthermore that our results for $C_{L\alpha}$, $C_{M\alpha}$ and $C_{Di} / C_{Di\infty}$ of a rectangular wing in ground proximity agree very well with those of Du Plessis¹⁷ (who quoted Saunders³³) and Suh²⁰, at least down to a height of $\frac{1}{4}$ of the wing span.

We therefore conclude that our vortex lattice algorithm computes the lift, pitch moment and drag coefficients with good accuracy. And one can assume safely that its use in the solution of the trim problem should yield good results, as should its subsequent use in the

computation of the required aerodynamic derivatives of C_L , C_M and C_D with respect to α , q , θ and z . Additional confidence in our vortex lattice results is obtained when considering the induced drag comparisons between the leading edge singularity (LES) approach (C_{Di} based on the downwash at the leading edges), the far field or Trefftz plane approach (C_{Di} based on the downwash at the lifting lines), and the conventional vortex lattice (CVLM) method (C_{Diiv} based on the downwash at the midpoints of the bound horseshoe vortex segments). We prefer the CVLM over the LES approach, because it was found to be less sensitive to the various selections of the chordwise and spanwise number of horseshoe vortices than the LES approach, and it accounts for the influence of the induced backwash on C_L and C_M easily, small as that influence is. The Trefftz plane approach, however, is just as accurate, and is computationally much more effective. Table 3 contains the trim solutions and the aerodynamic derivatives obtained as function of height for the aircraft data defined in Table 2. The trim solutions of Table 3 show that the CVLM drag in close ground proximity corresponds much better with the Trefftz plane drag than does the LES drag, which is another reason for our stated preference.

5.2 Discussion of Nominal Open Loop Results

The data of our nominal configuration are presented in Table 2 as obtained from McCormick²⁸ for the Piper Cherokee PA-28-180. Our stability and control analysis was based primarily on these data, but sensitivity analyses were performed in order to assess the influence that changes of some design parameters have on longitudinal stability. The results of these analyses are presented and discussed next, with reference to the selection of the tables and figures mentioned above. The nominal open loop results discussed here were obtained using the input data listed in Table 2, with a vortex lattice geometry defined by:

Number of Chordwise Segments on Wing:	2
Number of Spanwise Segments on Wing:	6
Number of Chordwise Segments on Tail:	2
Number of Spanwise Segments on Tail:	3

Small Angles Assumed:

Yes

Vortex Sheet Geometry:

Flat, Aligned with Free Stream

The aerodynamic coefficients obtained from the vortex lattice analysis of this case are presented in Table 3. These data served as input to the nominal open loop stability analysis, the results of which are presented in Tables 4 and 5. The following conclusions were drawn from Table 4:

- Longitudinal instability of our nominal configuration occurs around $b_w/H = 1.2$, and is due to the heave mode becoming unstable. The instability of the heave mode can be ascribed to the value of C_{Lz} exceeding the critical value obtained from the Routh criterium that demands positive “heave stiffness”. (By this we mean that the “heave stiffness” coefficient, which is the coefficient of s^0 in the approximated 3rd order characteristic polynomial of phugoid-and-heave motion presented in Chapter 4, must be positive.)
- Neglecting the $\dot{\alpha}$ - and $\dot{q}$ -derivatives in Table 2 has an insignificant influence on the characteristic roots of phugoid-and-heave motion. This is not unexpected, because these derivatives account for unsteady aerodynamic effects, which are negligible when considering states of motion, such as the phugoid and heave modes of motion, of which the characteristic time constants are much longer than the time it takes the free stream to traverse $\frac{1}{2}$ of the wing chord.
- It must be noted here that the θ -derivatives are identically zero far from the ground, because a change in pitch angle *at constant angle of attack*, as required by the *definition of the state derivatives as partial derivatives*, does not produce any change in the aerodynamic forces or moments far from the ground. (*It does in ground proximity, however*, because it changes the combined geometry of the vortex sheets and their images.) The fact that our vortex lattice algorithm does not produce zero θ -derivatives far from the ground was reason for concern, initially. We did, however, construct an exact symbolic algorithm that treats only a single pair of horseshoe vortices, one on the wing and one on the tail, which did turn out to yield exactly zero θ -derivatives far from the ground. A subsequent comparison of the other aerodynamic derivatives produced for the same case by our vortex lattice algorithm (in which small

angle assumptions are made) was found to give good agreement indeed. We then made a number of additional checks and cross checks, which involved the removal of some small angle assumptions in the computation of the vortex lattice co-ordinates and the influence coefficients. It was concluded that our non-zero θ -derivatives are due partly to these small angle assumptions, and partly to the fact that position co-ordinates were resolved in a geodetic frame. (Co-ordinate transformations that use small angle assumptions do not always yield identical cancellation of terms obtained when exact transformations are used.) Fair agreement was obtained between the stability results obtained using the roots of the *full 5th order characteristic equation, including the θ -derivatives*, and results obtained using those obtained from the approximate *reduced 3rd order characteristic equation that excludes them*, indicating that the θ -derivatives have an insignificant influence *on the essential nature of longitudinal stability* in ground proximity. (This conclusion was finally drawn only after we considered the influence of including the θ - and q -derivatives in the expressions for the critical values of C_{Lz} and C_{Dz} , which revealed that they do influence the critical height, but not the essential conclusions where it comes to identifying either C_{Lz} or C_{Dz} as the cause of the instability. We decided for these reasons that it was not worth the trouble to reformulate our geometric description of the lattice geometry in terms of large angles.

- the q -derivatives of C_L and C_D affect the magnitude of the phugoid roots by less than 2 % in the height range down to the critical height, and the heave root by less than 0.2 %. This indicates that C_{Lq} and C_{Dq} may be safely omitted from the formulation of both the stability criteria. The derivative C_{Mq} , *on the other hand, does affect the magnitude of the phugoid roots by 5 to 10 %, and the heave root by less than 0.5 % in the same height range*. Exclusion of C_{Mq} from the formulation of our stability criteria was found to affect the predicted value of the critical height somewhat, but the *essential conclusions regarding the cause of the instability remained unaffected by this simplification*. And it is a major simplification in the case of the expression for C_{Dzerit} , an expression that was found by using Matlab's Symbolics Toolbox to be 33 times longer (1325 characters vs 40) when including all the q -derivatives than when excluding them, and 93 times longer when including all q - and θ -derivatives (3741

characters vs 40). This, incidentally, indicates furthermore that it is probably futile to brutally formulate (by symbolic software) exact expressions for the various stability criteria from the full fifth order characteristic equation, because they will be impossible to interpret as the stability criteria of the simplified 3rd order characteristic equation could.

- The analytical approximation of the phugoid-and-heave characteristic roots, an approximation that is based primarily on the assumptions a) that the pitch moment of inertia is negligible in the slow modes of motion, and b) that at least one of the phugoid-and-heave roots are much smaller in magnitude than the other(s) is quite accurate up to the onset of instability. (The discrepancy of up to 8 % between this approximation and the roots of the unsimplified fifth order characteristic equation is due primarily to the neglect of C_{Mq} , not to the fact that it is an approximation, and the neglect of C_{Mq} has been argued above not to affect the *nature* of the longitudinal instability in ground proximity significantly.)
- The accuracy of the approximation of the short period roots computed using a reduced second order characteristic equation is on the order of 2 %, compared to the short period roots of the full fifth order system, at least in the a height range down to $\frac{1}{4}$ of the wing span. The reduced second order system mentioned here was obtained on the assumption that the lift and pitch moment perturbations due to the small speed and height changes expected in the short period mode of motion are negligible, as shown in Chapter 4. This assumption is thus seen to be quite reasonable.
- The instability that occurs at about $b/H = 1.2$ (for a CM placed at 20.63% of the wing chord) is shown by the signs of the listed stability margins of C_{Dz} and C_{Lz} in Table 3 to be *due to C_{Lz} exceeding the critical limit C_{Lzcrit} , not due to C_{Dz} exceeding the critical limit C_{Dzcrit}* . The limit C_{Lzcrit} is obtained from the requirement of “positive heave stiffness”, also known as static height stability. The instability occurs because this limit, which depends strongly on C_{Mz} , decreases more rapidly with decreasing height than does C_{Lz} itself, so that C_{Lz} eventually exceeds C_{Lzcrit} . We will show in the sensitivity analysis presented later, however, that the *longitudinal instability encountered by some other configurations is caused by C_{Dz} exceeding the critical limit C_{Dzcrit}* , a limit obtained from the less obvious dynamic stability requirement (by

Routh's criteria) that $BC - AD > 0$ if $A > 0$ in the characteristic equation $As^3 + Bs^2 + Cs + D = 0$ of phugoid-and-heave motion. One finds that the limit C_{Dzcrit} depends on C_{Mz} too, so that C_{Dz} eventually exceeds C_{Dzcrit} in the same way that C_{Lz} exceeds its limit in the nominal configuration presented above.

Inspection of the eigenvectors listed in Table 5 shows that the phugoid mode of motion is indeed dominated by large changes in air speed, pitch angle and height when far from the ground, and that the heave mode is neutrally stable, as expected. This is still the case in the height range around $b/H = 1.2$, where the longitudinal instability sets in, but the heave mode, which is the one that becomes unstable in this example, now displays a quite significant change in the air speed. The heave mode in ground proximity is therefore also a mode of motion that primarily involves the translational motion of the aircraft, just like the phugoid mode far from the ground in this respect.

5.3 Practical Significance of Instability in Ground Proximity

The longitudinal instability encountered in close ground proximity is not insignificant at all. Root locus plots, such as those in Figure R3 b), have shown us that the divergent instability can easily have a dimensionless damping (exponential growth rate) of $\sigma = 0.005$ radians *per air second* at a height around $\frac{1}{2}$ the wing span. Recalling that an air second is the time it would take the free stream to traverse half the wing chord it follows that, at 50 m/s, a Cherokee air second is approximately 16 real milliseconds, so that the exponential growth rate in real time is $\sigma/0.016 = 0.3125$ rad/sec. This means that the exponential *real time term* represented by the typical unstable real pole at 0.005 in the dimensionless s-plane (at a typical height of half the span) has a time to double amplitude, i.e. the time it would take a disturbance to grow to twice its initial value, of $\ln(2)/(\sigma/0.016) = \ln(2)/(0.005/0.016) = 2.22$ *real seconds*. This is clearly not an insignificant growth rate, in the sense that, if an aircraft *does* exhibit such an instability, then the pilot, consciously or not, will have to keep providing a control input that counters the unstable tendency, because it may arguably take longer than 2 seconds to touch down from a height of half the wing span. An experienced pilot could well be unaware of this stabilising control input, but it might add up to quite a work load when

flying in close ground proximity for an extended period. A remotely piloted vehicle, on the other hand, will definitely have to be stabilised artificially when required to operate inside the height range where the longitudinal instability occurs.

5.4 Parameter Sensitivity

The first response of a design engineer to an instability that may be objectionable would be to investigate the possibility of alleviating the instability by prudent design changes. The parameters that can be expected to have an influence on the longitudinal stability of a fixed wing aircraft are, amongst others:

- Position of the CM, customarily expressed as a percentage of wing chord aft of the leading edge
- Tail length l_t
- Tail height h_t
- Trimmed lift coefficient C_{L0}

Concerning the convergence checks we made before performing the sensitivity analysis, we mention in passing that, during numerous numerical experiments, a set of horseshoe vortices as small as $(N_c; N_s) = (2; 4)$ on the wing and $(2; 3)$ on the tail was found to be sufficient for capturing the essential nature of the longitudinal instability. Smaller values for the critical height is obtained, however, as the chordwise (N_c) and spanwise (N_s) numbers of vortices are increased, but $(N_c; N_s) = (3; 6)$ on the wing and $(2; 4)$ on the tail was found to give not much improvement at all over $(N_c; N_s) = (2; 6)$ and $(2; 3)$ for the wing and tail, respectively, least of all any tendency towards essentially different stability characteristics.

Table 6 contains the results of a sensitivity analysis with respect to CM position, results that are summarised in Figures R4 a) to R4 f). Sensitivity analyses with respect to other design parameters were performed too, but the results of these are presented even more concisely, using one pair of graphs per parameter, in Figures R5 a) and R5 b) to R8 a) and R8 b), rather than producing a similar set of graphs to Figures R4 a) to f) for each parameter.

5.4.1 Sensitivity to CM Position

Figure R4 a), which summarises Figures R4 c) to f), shows clearly that a rearward shift of the centre of mass is destabilising in ground proximity, in that it increases the critical height at which the longitudinal instability sets in. The *essential nature of the longitudinal instability*, however, which is a monotonically divergent heave mode caused by C_{Lz} exceeding C_{Lzcrit} in the nominal case, is shown by Figure R4 b) to be *practically unaffected by a shift in CM position* – it merely *changes the critical height*, as indicated by the displacement of the characteristic roots along the basic shape of the root loci.

The small values of the maximum real part of the characteristic roots in Figures R4 c) and of the stability margin $\Delta C_{Lz} = C_{Lzcrit} - C_{Lz}$ shown in Figure R4 d) do not show the exact height at which the instability occurs clearly. They do indicate, however that the instability occurs at a height well above 2/3 of the wing span in the whole of this CM range, whereas Figure R4 d) shows the stability margin $\Delta C_{Dz} = C_{Dzcrit} - C_{Dz}$ to become negative well below 2/3 of the wing span. The *instability is therefore due to $C_{Lz} > C_{Lzcrit}$, not due to $C_{Dz} > C_{Dzcrit}$, irrespective of CM position in the range considered.*

The quantity $h_{Nz} = -C_{Mz}/C_{Lz}$ can be regarded as a static longitudinal (pitch moment) stability margin (with respect to height perturbation), similar to the well-known static longitudinal stability margin $h_N = -C_{M\alpha}/C_{L\alpha}$ (with respect to angle of attack perturbation). The stability requirement $\Delta C_{Lz} = C_{Lzcrit} - C_{Lz} > 0$, arising from the requirement of positive “heave stiffness” (i.e. $D > 0$ in $A s^3 + B s^2 + C s + D = 0$, if $A > 0$), can easily be shown to yield an entirely equivalent form $\Delta h_{Nz} = h_{Nzcrit} - h_{Nz} > 0$. The latter formulation can be expected to provide a more accurate estimation of the critical height when interpolating linearly using data such as those in Figure R4 f) than the critical height estimation obtained by linear interpolation in Figure R4 d). This point should be borne in mind when interpreting similar results in Figures R5 a) to R8 a).

5.4.2 Sensitivity to Other Design Parameters

Figure R5 a) shows that the critical height is less sensitive to a change in wing aspect ratio than to a change in CM position, and that a configuration with a wing of large aspect ratio will cause instability sooner than one with small aspect ratio. The shapes of the root loci, however, are seen in Figure R5 b) to be affected slightly by a change in aspect ratio. At the smaller aspect ratios the root loci tend to shift closer to the imaginary axis, but it is still the heave mode that eventually becomes unstable. This shift in the root loci is a bit surprising. One would, after all, expect the root loci to approach the imaginary axis at high aspect ratio, because the phugoid roots are on the imaginary axis when the wing and tail aspect ratios are both infinite, and the heave root must remain at the origin. (This is because C_{Di} and all its derivatives will then be zero, which causes B and D to go to zero in the 3rd order characteristic equation of phugoid-and-heave motion.) The observed root locus shift in Figure R5 a) may, however, be due to the fact that the parasite drag still remains, so the phugoid roots far from the ground will still be to the left of the imaginary axis, and a prediction of the root locus trend based on 2D *inviscid* theory is may not be reliable.

Figure R6 a) shows the influence of tail length on the critical height to be somewhat more pronounced than that of CM position or wing aspect ratio. The tendency that a larger wing to tail distance l_t destabilises an aircraft in ground proximity is perhaps surprising at first. However, this brings the tail closer to the ground at a given CM height, because the height of the tail above the ground is influenced more strongly by an increase in tail distance than by the slight decrease in angle of attack expected from the smaller tail force required for moment equilibrium of the “long tail” configuration. This reduction in tail height (above the *ground*) gives a higher C_{Mz} , which lowers the stability boundary of C_{Lz} . A change in tail length is seen in Figure R6 b) to affect the shape of the root loci, and therefore the essential longitudinal stability characteristics, just as little as does a change in CM position.

Figure R7 a) shows the tail height above the wing chord line, on the other hand, to have a very strong influence on the critical height of the CM. The critical CM height increases

sharply from about $H = 1.2 b$ to $H = 2 b$ with an increase in tail height from 0 to 1.5 wing chords above the wing chord line, which implies that an aircraft with a high tail is much more stable at a given CM height than one with a low tail. This is because the value of C_{Mz} is much lower for a high tail configuration than for a low tail, as seen in Table 8 compared to Table 3. This is mainly because a high tail is much less affected by the ground, thus yielding a much smaller change in tail force with closer ground proximity, resulting in a much smaller value of C_{Mz} than in the case of a low tail. The decrease in the value of C_{Mz} has a strong stabilising effect, because it lifts the stability boundaries C_{Lzcrit} and C_{Dzcrit} of both C_{Lz} and C_{Dz} . An increase in tail height, however, must increase the stability margin ΔC_{Lzcrit} more sharply than it does ΔC_{Dzcrit} , because Figure R7 a) shows clearly that, contrary to the case of a low tail, the *instability of our high tail configuration is due to C_{Dz} exceeding the critical limit C_{Dzcrit} , not due to C_{Lz} exceeding the critical limit C_{Lzcrit}* . Figure R7 b), furthermore, read in conjunction with the numerical values of the phugoid-and-heave roots in Table 8, shows that the *instability of our high tail configuration is due to the phugoid mode becoming unstable*. Initially it is an *oscillating instability, but it becomes a (second order) monotonically divergent instability below a height of about 1/3 of the wing span, while the heave mode of motion remains stable, and, furthermore, becomes increasingly more stable as the CM height decreases*.

We conclude our sensitivity analysis by noting that an increase in the load factor C_w , i.e., the equilibrium lift coefficient C_{L0} , has a similar influence on the longitudinal stability characteristics as an increase in tail height, but not as pronounced, as shown in Figures R8 a) and b). It follows from these sensitivity considerations that an aircraft with a high tail flying in ground proximity at a high C_{L0} (i.e. flying at low air speed and/or with a heavy load), will become unstable at a much lower height than one with a low tail flying at low C_{L0} (i.e. flying at high air speed and/or with a light load). We were initially quite surprised, if not seriously concerned, at finding the instability of our nominal configuration to be a heave instability, as opposed to the phugoid instability reported by Staufienbiel¹⁹. The results of the sensitivity analysis presented above, however, indicate that Staufienbiel's phugoid instability could be due to the high $C_{L0} = 1.8$, compared to our $C_{L0} = 0.543$, which decreases the phugoid damping. Another source of initial concern

was the positive C_{Lz} listed in Staufienbiel's Table A1, but we concluded that this must have been a typing error, because he cites negative C_{Lz} elsewhere, as we obtained throughout the whole range of CM height, down to 1/8 of the wing span, of all the configurations we considered.

5.5 Alleviation of Instability by Feedback Control

The expressions for $C_{Lz_{crit}}$ and $C_{Dz_{crit}}$ in Chapter 4, along with typical numerical results obtained for the relevant aerodynamic coefficients and derivatives, revealed that an increase in C_{Mz} causes a deterioration of the longitudinal stability in ground proximity, in that it increases the critical height at which instability sets in. This result indicates that any design change that decreases C_{Mz} can be expected to alleviate the instability, i.e., to decrease the critical height, of which the most striking example is the influence of increased tail height discussed above. It is therefore reasonable to expect that, if the tail angle δ_t (positive nose down) is decreased significantly when a positive (upward) height disturbance z occurs, then the resulting nose down pitch moment would yield a negative contribution to C_{Mz} . This can be realised easily by measuring and feeding back the height perturbation to the elevator control input. We argued that a substantial reduction of C_{Mz} should yield a substantial improvement in longitudinal stability, because the aircraft is inherently stable far from the ground, where $C_{Mz} = 0$. A control strategy was therefore formulated according to which the feedback gain is scheduled such, as a function of height, that C_{Mz} is reduced by a constant ratio compared to its open loop value, irrespective of the height, as described in Chapter 4.

Table 7 contains the results of a feedback control analysis of the nominal case, where the gain with which the height perturbation was fed back to the elevator control input was scheduled such that C_{Mz} is reduced by 60%, regardless of height. Figure R9 a) shows the influence of such a feedback control strategy on the shape of the root loci, with the reduction ratio $K_{CMz} = (C_{Mz})_{CL}/(C_{Mz})_{OL}$ of the open loop and closed loop values of C_{Mz} varied from $K_{CMz} = 1$ (the open loop case) to $K_{CMz} = 0.4$. It is concluded that:

- the sensitivity of the phugoid roots and the heave root to changes in CM height is reduced drastically by this feedback, as shown by the closer spacing of the roots with decreasing K_{CMz} in Figure R9 a)
- a reduction ratio of $K_{CMz} = 0.4$ stabilises the aircraft completely down to a CM height of $\frac{1}{4}$ of the wing span, as shown in Table 7 c), whereas the open loop system became unstable at a height of around 85 % of the wing span.
- the long term response to an impulsive elevator input, shown in Figure R9 b) over the time interval of 0 to 5000 Cherokee air seconds (approximately 80 real seconds), confirms that the aircraft is stabilised successfully by this feedback control strategy. The frequency of the closed loop phugoid mode is a bit higher at $\frac{1}{4}$ span than far from the ground, as expected from Figure R9 b). The short term response is left practically unaffected, as shown in Figure R9 c). Note that the long term response in Figure R9 b) was computed for heights down to $\frac{1}{4}$ of the wing span, whereas the open loop case of Figure R3 g) was only computed down to $1/2.4$ times the span due to its instability. Note furthermore that the short term response in Figure R9 c) is shown over the time interval of 0 to 200 air seconds = 3.2 real seconds, whereas an interval of 0 to 100 air seconds was used in Figure R3 f).
- the closed loop values of C_{Lz} and C_{Dz} are seen at the end of Table 7 to differ by a much smaller percentage from their open loop values than that of C_{Mz} , which one should expect, because a change in the tail angle affects the pitch moment much more than it affects the lift and drag. This accounts for the success of the strategy, because the large change in C_{Mz} cause a large upward shift of the stability boundaries C_{Lzcrit} and C_{Dzcrit} , while C_{Lz} and C_{Dz} themselves are not changed as much.

We conclude finally that the control strategy suggested here is highly effective in stabilising fixed wing aircraft in ground proximity, and does not affect other aspects of handling qualities significantly, especially not the frequency of the α -mode, neither its damping ratio. The control strategy suggested by Staufenberg¹⁹ (a strategy that relies on perfect air speed regulation, and feeds back both height and rate of descent to the elevator), on the other hand, yields a sharp decrease in the damping ratio of the short period mode, which is quite undesirable from a handling qualities point of view.

REFERENCES

1. Lanchester, W. B., *Aerodynamics*, Constable, London, 1907.
2. Kutta, W. M., "Auftriebskräfte in strömenden Flüssigkeiten", *Illustr. Aeronaut. Mitteilungen*, pp. 133, 1902.
3. Prandtl, L., "Tragflügeltheorie: I. Mitteilung", *Nachrichten der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse*, pp. 151-177, 1918. Obtained from *Ludwig Prandtl Gesammelte Abhandlungen zur angewandten Mechanik, Hydro- und Aerodynamik*, Ed. Riegels, F. W., Vol. 1, pp. 322-345, Springer-Verlag, Berlin, 1961.
4. Betz, A., "Auftrieb und Widerstand einer Tragfläche in der Nähe einer horizontale Ebene", *Z. Flugtechn. Motorl. Jg. 2*, pp. 217, 1912.
5. Lagally, M., "Zur Theorie der Wirbelschichten", *Sitzungsber. d. Bayr. Akad. Wiss. Math.-phys., Klasse*, pp. 79, 1915.
6. Munck, M., "Isoperimetrische Aufgaben aus der Theorie des Fluges", Göttingen, 1918.
7. Wieselsberger, C., "Beitrag zur Erklärung des Winkelfluges einiger Zugvögel", *Z. Flugtechn. Motorl. Jg. 5*, pp. 225, 1914.
8. Wieselsberger, C., "Wing Resistance Near the Ground", NACA TM77, April 1922.
9. Weissinger, J., "The Lift Distribution of Swept-Back Wings", NACA TM 1120, Trans. Grumman, H. R. from "Über die Auftriebverteilung von Pfeilflugeln", ZWB Berlin-Adlershof FB 1553, 1942.

10. Von Mises, R., *Theory of Flight*, 1st Edition, McGraw-Hill, New York, 1945.
11. Milne-Thomson, L. M., *Theoretical Aerodynamics*, 4th Edition, St Martin's Press, New York, 1966.
12. Karamcheti, K., *Principles of Ideal-Fluid Aerodynamics*, John Wiley, New York, 1966.
13. Wagner, S., "On the Singularity Method of Subsonic Lifting-Surface Theory", *Journal of Aircraft*, Vol. 6, No. 6, pp. 549-558, Nov.-Dec. 1969.
14. Lan, C.E., "A Quasi-Vortex-Lattice Method in Thin Wing Theory", *Journal of Aircraft*, Vol. 11, No. 9, pp. 518-527, Sept. 1974.
15. Miranda, L. R., Elliot, D. and Baker, W. M., "A Generalised Vortex Lattice Method for Subsonic and Supersonic Flow Applications", NASA Contractor Report 2865, 1977.
16. Konstadinopoulos, P., Thrasher, D. F., Mook, D. T., Nayfeh, A. H. and Watson, L., "A Vortex-Lattice Method for General, Unsteady Aerodynamics", *Journal of Aircraft*, Vol. 22, No. 1, pp. 43-49, Jan. 1985.
17. Du Plessis, J-L., "Die Analise van Grondeffekte op Vlerke en die Stabiliteit van Grondeffekvliegtuie", M. Sc. Thesis, Dept. Mech. Eng., Univ. of Stellenbosch, RSA, 1985.
18. Chang, R. C. and Muirhead, V. U., "Effect of Sink Rate on Ground Effect of Low-Aspect-Ratio Wings", *Journal of Aircraft*, Vol. 24, No. 3, pp. 43-49, Jan. 1985.

19. Staufenbiel, R. W. and Schlichting, U.-J., "Stability of Airplanes in Ground Effect", *Journal of Aircraft*, Vol. 25, No. 4, pp. 289-294, April 1988.
20. Suh, Y. B. and Ostowari, C., "Drag Reduction Due to Ground Effect", *Journal of Aircraft*, Vol. 25, No. 11, pp. 1071-1072, Nov. 1988.
21. Nuhait, A. O. and Mook, D. T., "Numerical Simulation of Wings in Steady and Unsteady Ground Effects", *Journal of Aircraft*, Vol. 26, No. 12, pp. 1081-1089, Dec. 1989.
22. Nuhait, A. O., and Zedan, M. F., "Numerical Simulation of Unsteady Flow Induced by a Flat Plate Moving Near Ground", *Journal of Aircraft*, Vol. 30, No. 5, pp. 611-617, Sept.-Oct. 1988.
23. Elzebda, J. M., Mook, D. T. and Nayfeh, A. H., "Numerical Simulation of Steady and Unsteady Vorticity Dominated Aerodynamic Interference", *Journal of Aircraft*, Vol. 31, No. 5, pp. 1031-1036, Nov. 1994.
24. Coulliette, C. and Plotkin, A., "Aerofoil Ground Effect Revisited", *Aeronautical Journal*, pp. 65-74, February 1996.
25. Devinant, Ph., "An Approach for Unsteady Lifting-Line Time-Marching Numerical Computation", *International Journal for Numerical Methods in Fluids*, Vol. 26, pp. 177-197, 1998.
26. Prandtl, L., "Applications of Modern Hydrodynamics to Aeronautics", NACA Technical Report No. 116, pp. 7-61, 1921.

27. Bertin, J. J., and Smith, M. L., *Aerodynamics for Engineers*, Prentice-Hall, New Jersey, 1989.
28. McCormick, B. W., *Aerodynamics, Aeronautics and Flight Mechanics*, John Wiley and Sons, New York, 1979.
29. Hassenpflug, W. C., "Momentum and Vortex Theory of Rotor Blades and Wakes", *AIAA Journal*, Vol. 34, No. 8, pp. 1741-1742, 1995.
30. Ashley, H., *Engineering Analysis of Flight Vehicles*, pp. 233-238, Addison-Wesley, Massachusetts, 1974.
31. Hancock, G. J., "Comment on 'Spanwise Distribution of Induced Drag'", *Journal of Aircraft*, Vol. 8, No. 8, Aug. 1971, pp. 681-682.
32. Greenberg, M. D., *Advanced Engineering Mathematics*, Prentice-Hall, New Jersey, 1998.
33. Saunders, G. H., "Aerodynamics of Wings in Ground Effect", M. Sc. Thesis, Massachusetts Institute of Technology, June 1963.
34. Spiegel, M., *Vector Analysis*, pp. 51, McGraw-Hill, 1974.

ILLUSTRATIONS

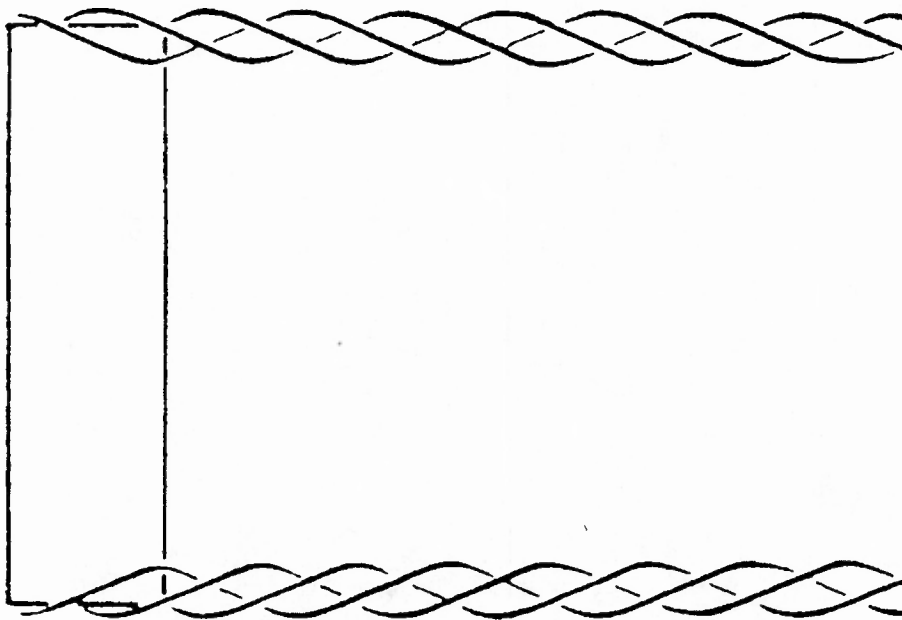


Figure 1.1: Approximate Model of Wing Tip Vortices (from Prandtl³)

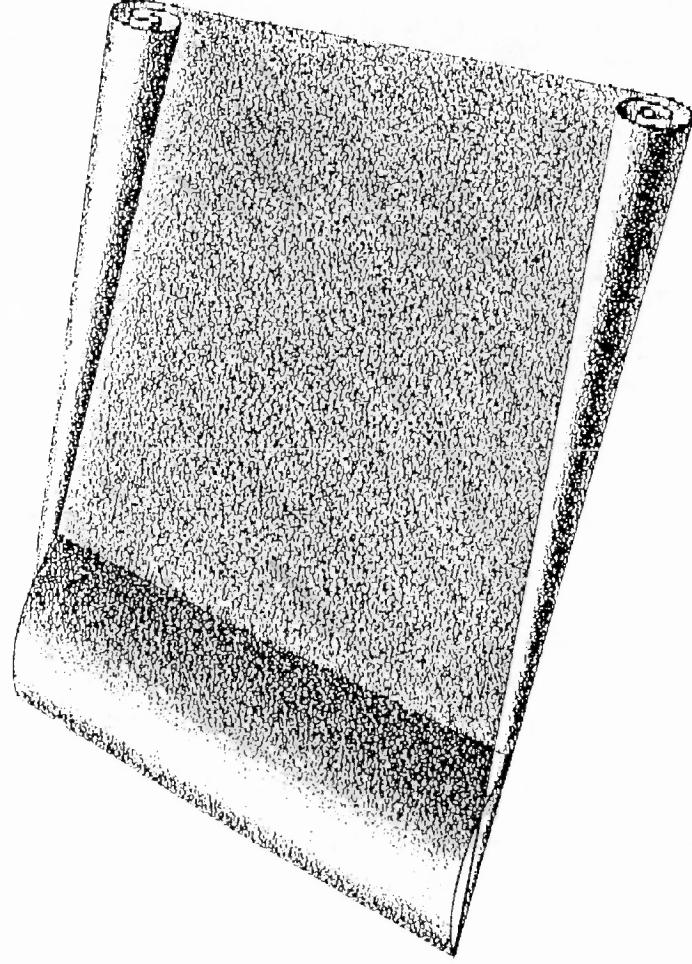


Figure 1.2: Trailing Vortex Sheet (from Karamcheti^{1,2})

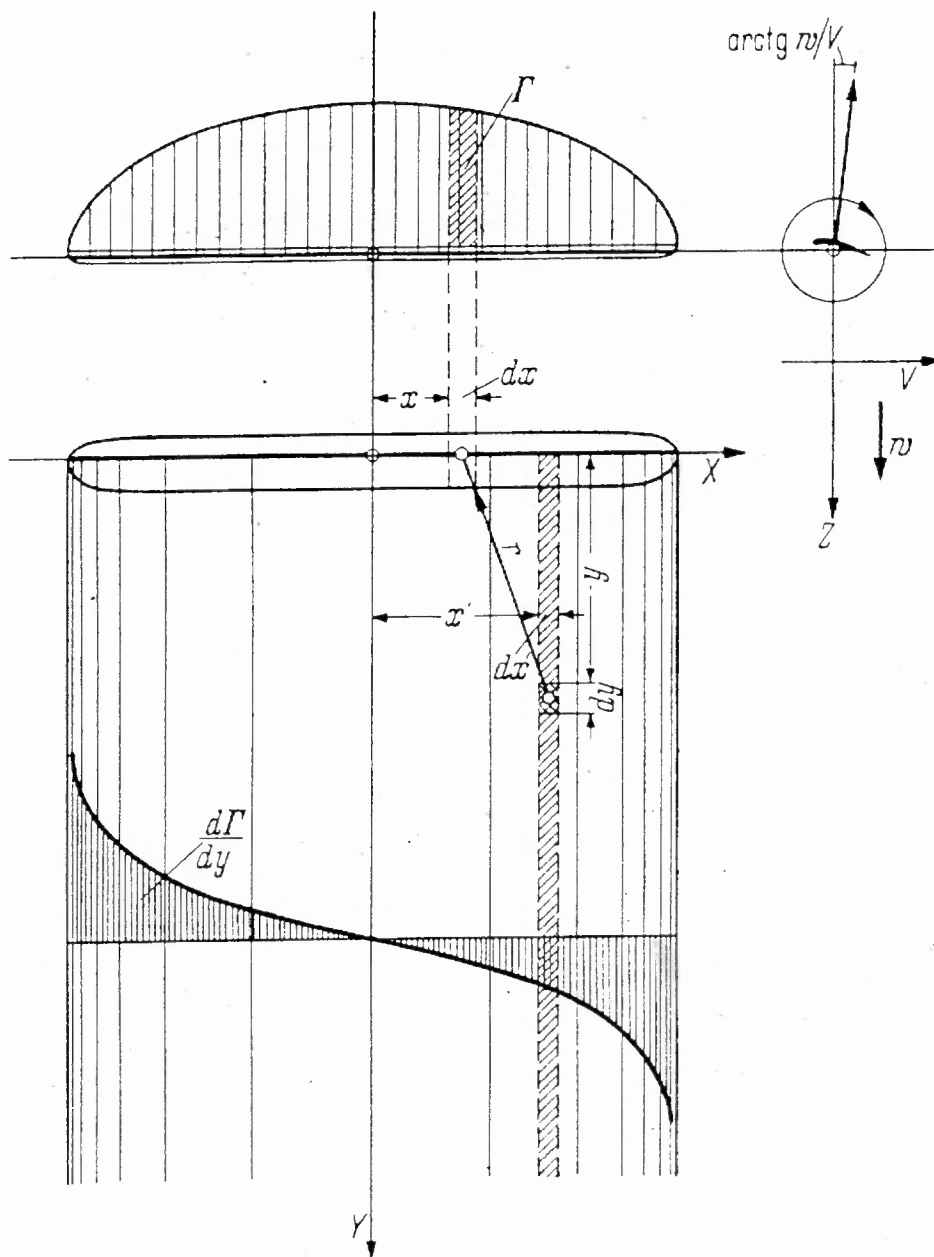


Figure 1.3 a): Prandtl's Lifting Line Model (taken from Prandtl³, where it is stated that $d\Gamma/dx$ should be read instead of $d\Gamma/dy$). Note that the lifting line is not placed at the trailing edge!

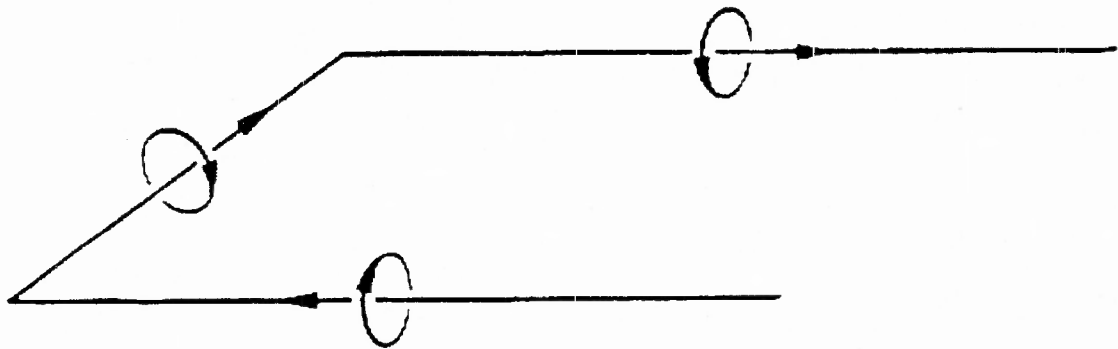


Figure 1.3 b): Simplified Horseshoe Model of Wing Vortices (from Prandtl³)

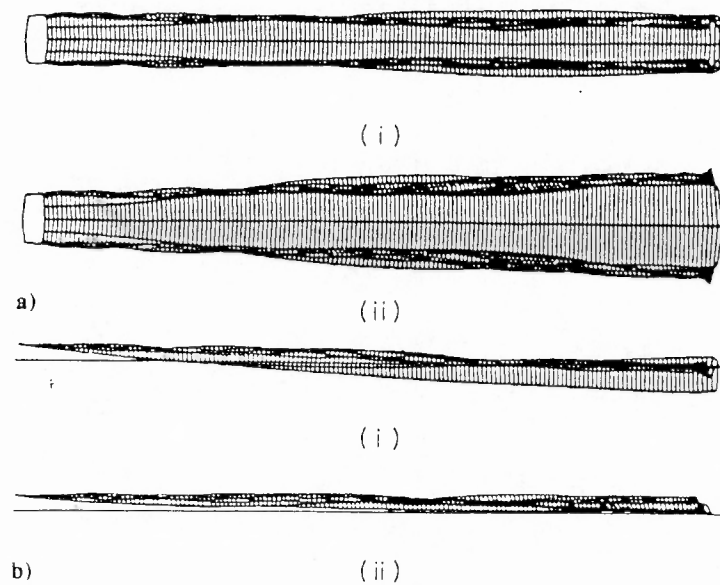


Figure 1.4: Shape of Trailing Vortex Sheet (from Nuhait and Mook²¹). a) Top View i) out of Ground Effect, and ii) in Ground Effect; b) Side View i) out of Ground Effect and ii) in Ground Effect.

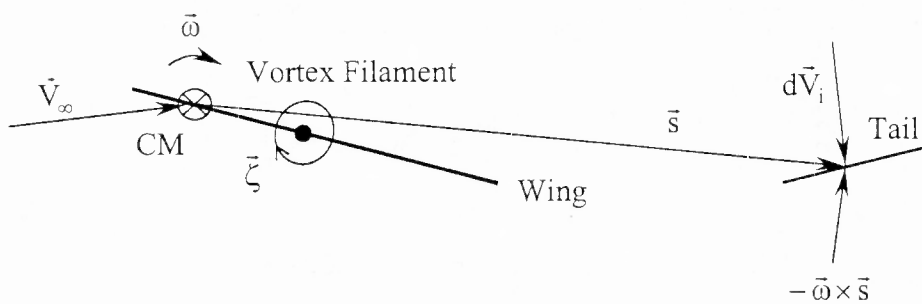


Figure 2.1: Side view of wing and tail showing velocity components $\vec{V}_\infty$ and $-\vec{\omega} \times \vec{s}$ due to translation and rotation, respectively, and infinitesimal velocity component $d\vec{V}_i$ induced at tail by a vortex filament of strength $\vec{\zeta}$ on the wing.

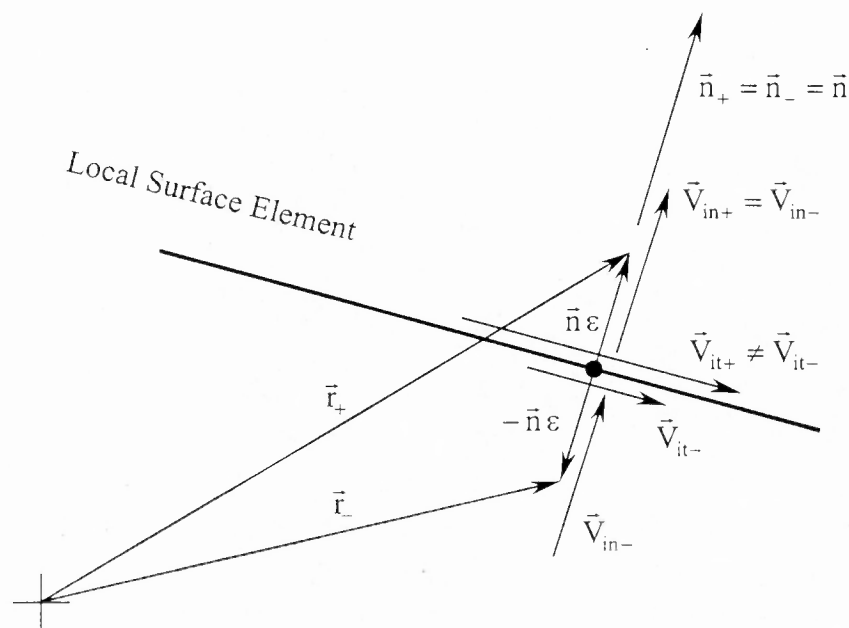


Figure 2.2: Side view of local surface element illustrating the discontinuity of the tangential component $\vec{V}_{it}$ of induced velocity and the continuity of the normal component $\vec{V}_{in}$ of induced velocity.

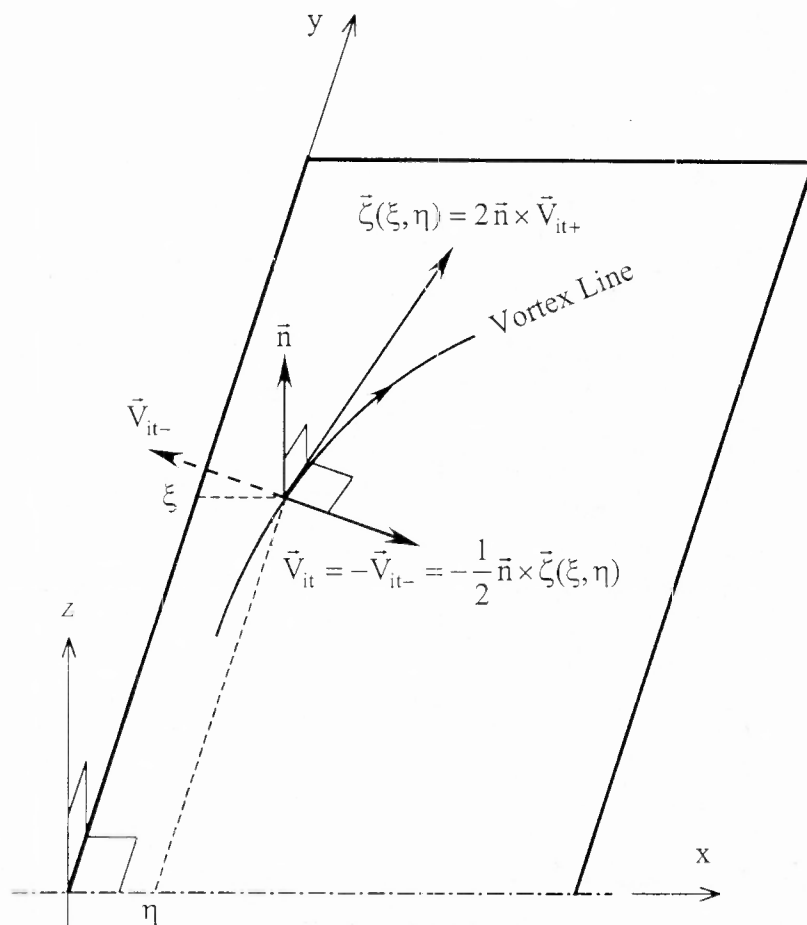


Figure 2.3: Schematic representation of the mutual relationship between the local vortex strength and the tangential discontinuity of velocity on the starboard half of a rectangular wing.

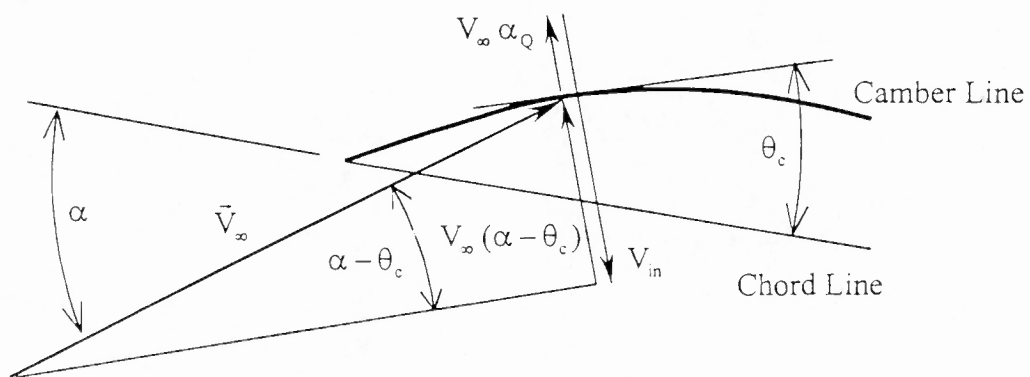


Figure 2.4: Side view of cambered wing illustrating flow tangency condition.

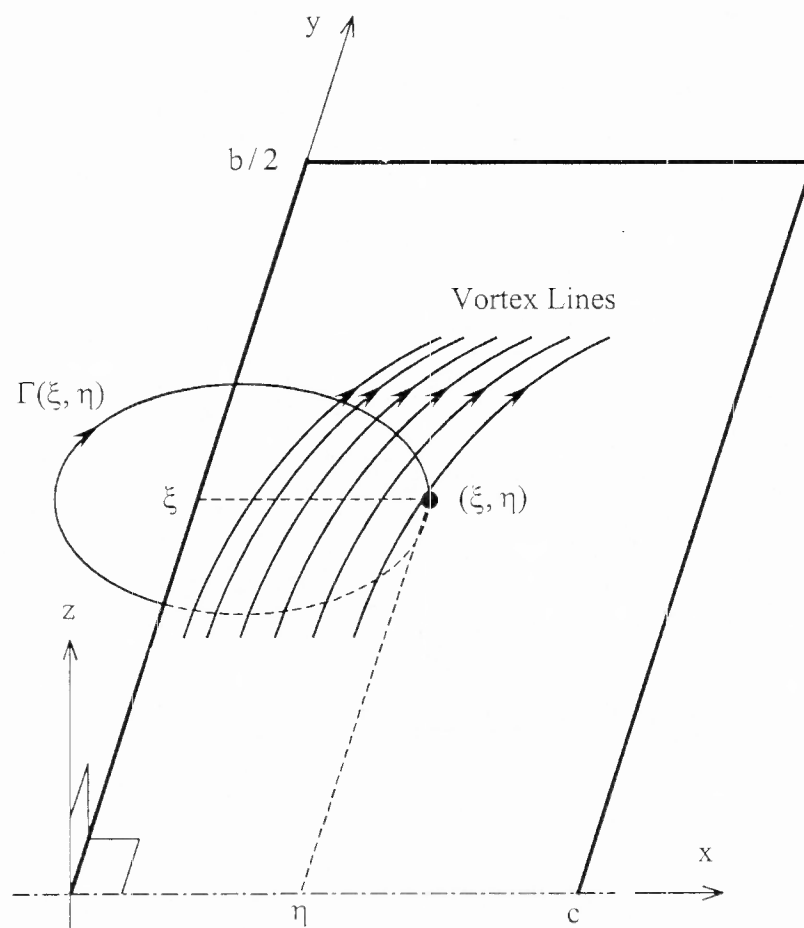


Figure 2.5: Definition of circulation function $\Gamma(\xi, \eta)$ illustrated for a rectangular wing.

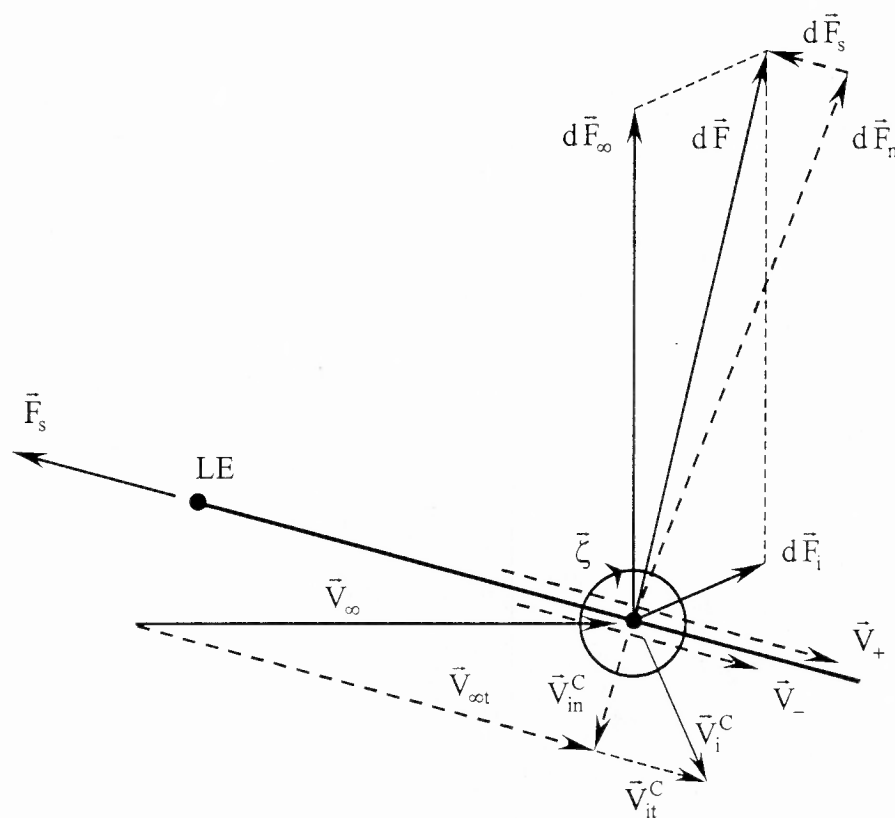


Figure 2.6: Comparison between the Kutta-Joukowski and Blasius-Bernoulli approaches. The Blasius-Bernoulli approach yields a local normal force increment $d\vec{F}_n$ alone by making explicit use of the flow tangency condition, and thus working with the local (tangential) velocity resultant. The leading edge suction force resultant $\vec{F}_s$ must be found afterwards by application of the Blasius integral there. The Kutta-Joukowski approach, on the other hand, yields two separate local force increments $d\vec{F}_\infty$ and $d\vec{F}_i$ (if the velocity components $\vec{V}_\infty$ and $\vec{V}_i$ are considered separately, rather than eliminating the parts that cancel). The local Kutta-Joukowski force resultant $d\vec{F}$ thus contains a fictitious local tangential component $d\vec{F}_s$, but one that adds up to the same leading edge force resultant (when integrated over the chord) as the Blasius Bernoulli approach, whereas the remaining part of $d\vec{F}$ is merely equal to $d\vec{F}_n$. Note that, for the sake of brevity, the force contribution due to angular velocity is not considered here, but it would merely yield additional normal and tangential velocity components to accounted for similarly.

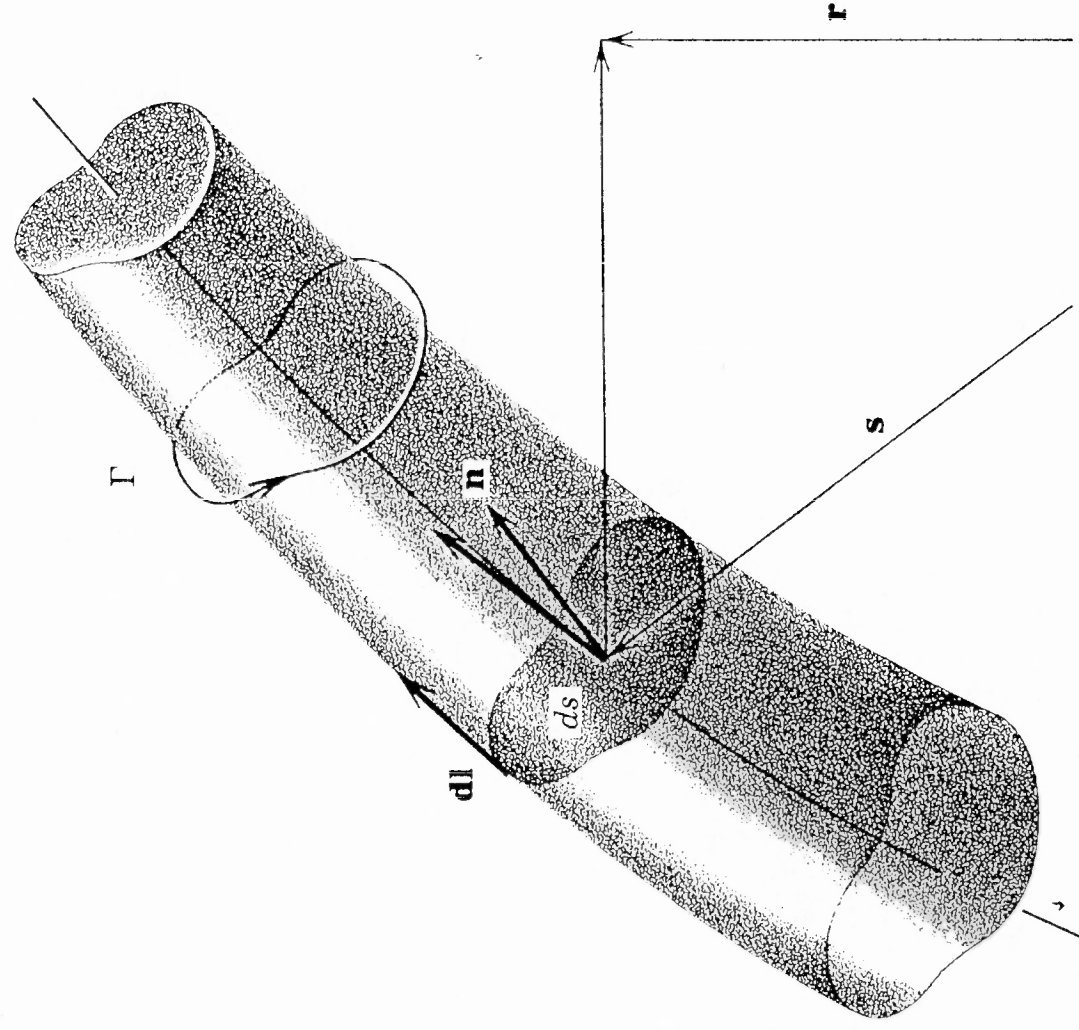


Figure 2.7: Schematic representation of a vortex tube (from Karamcheti¹²)

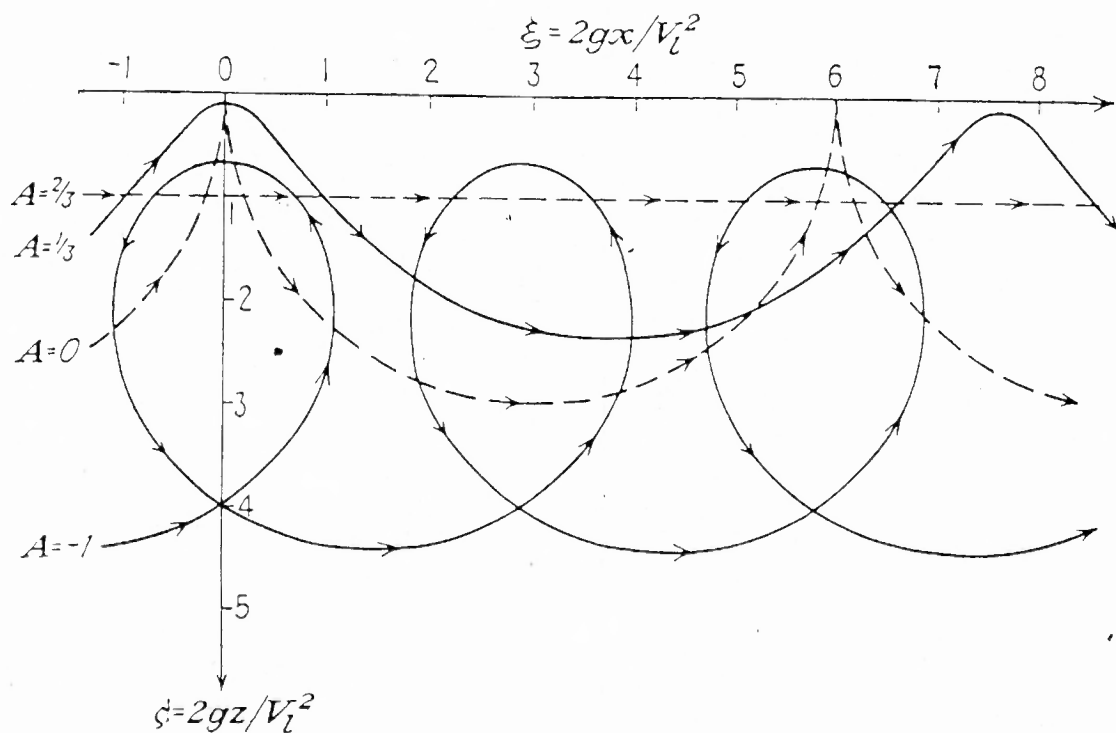


Figure 4.1: Illustration of the Lanchester's phugoids, shown for various initial conditions, as obtained from Von Mises¹⁰. Note that ξ and ζ represent the normalised horizontal and vertical position co-ordinates, respectively, of the centre of mass. The value of A is defined in terms of the initial dimensionless vertical distance below $\zeta = 0$ and the initial angle of climb γ (Von Mises used θ for angle of climb) by $A = \sqrt{\zeta} (\cos \gamma - \zeta/3)$. If the initial state is taken as $\zeta = 1$ in all of these cases, then $A = 2/3$ represents straight and level flight at a speed V_l for which the lift equals the weight. The cases of $A < 2/3$ represent increasing initial angles of climb.

TABLES OF NUMERICAL RESULTS

TABLE 1: Verification of Present VLM Results

Table 1 a): Vortex Lattice Results for Rectangular Wing with AR = 2 at H = ∞						
N _c	N _s	C _{Lα}	C _{Mα}	C _{Dii} /C _L ²	C _{Dii} /C _{Di}	Ref.
8	15	2.4707	-0.5173	0.1595	0.9972	Lan ¹⁴
8	15	2.4742	-0.5182	0.1593	1.0003	Present
3	6	2.4712	-0.5204	0.1590	0.9985	Present

Table 1 b): Vortex Lattice Results for Wing with Warren 12 Planform at H = ∞						
N _c	N _s	C _{Lα}	C _{Mα}	C _{Dii} /C _L ²	C _{Dii} /C _{Di}	Ref.
5	20	2.7382	-3.0844	0.1143	1.0054	Lan ¹⁴
5	20	2.7490	-3.0851	0.1093	0.9648	Present

Table 1 c): Values of C _L , C _M and C _D for flat wing with AR = 10 ⁶ , H = ∞, α = 5°, q = 0.5°, N _c = 3 and N _s = 20, as compared to 2D results c _ℓ , c _m and c _d from thin wing theory. Moments taken about 1/4-chord line.						
C _L	1000 C _M	1000 C _D	C _L / c _ℓ	C _M / c _m	C _D /c _d	C _D /(2 q C _M)
0.6292	-6.7749	-2.8636	0.9979	0.9885	0.9974	0.9999
Note: The surprising result of negative C _D is discussed in par. 5.1.						

TABLE 2: Data of Nominal Aircraft (Piper Cherokee data from McCormick ²⁸)	
Wing Data:	
Span:	9.144 m
Chord:	1.6002 m
Aspect ratio:	5.7143
Planform area (span x chord):	14.6322 m ²
Distance of CM behind wing LE:	0.3302 m (20.64 % of chord)
Tail Data:	
Span:	3.048 m
Chord:	0.762 m
Aspect ratio:	4
Height of tail AC above wing chord:	0 m
Distance of tail LE behind wing LE:	4.2545 m
Aerodynamic Data:	
Airfoils:	Assumed thin and symmetric
Nominal air speed :	50 m/s
Nominal C _{Lo} :	0.543
Dimensionless mass μ :	86.6
Dimensionless pitch inertia i_{yy} :	210
Equivalent flat plate area f:	0.5
Drag factor k _D :	3 (for propeller aircraft)
	: 0.0342
	: -1.29
	: -3.32
	: 0
Nominal short period roots: -0.0388+0.0567i; -0.0388-0.0567i	
Nominal long period roots: -0.000442+0.00397i; -0.000442-0.00397i	
Units of root locations are radians per nominal air second.	
An air second is the time it takes the free stream to traverse half the wing chord.	
1 nominal air second = 0.016 real seconds (for Piper Cherokee at 50 m/s).	

TABLE 3: Results of Vortex Lattice Analysis of Nominal Aircraft Configuration

TRIM STATES AND AERODYNAMIC DERIVATIVES BY VORTEX LATTICE METHOD

Results are given as function of ground height, and include:

- trim states
- stability derivatives
- control derivatives

EXAMPLE: Piper Cherokee PA-28-180

Table 3a): Some Input Data

Aspect Ratio of Wing = 5.7143
 Aspect Ratio of Tail = 4
 Ratio of Planform Areas (tail/wing) = 0.15873
 Wingspan = 9.144
 Tailspan = 3.048
 Distance of CM behind wing LE $dx_{CM}/cw = 0.20635$
 Tail distance $(x(le_t) - x(le_w))/cw = 2.6587$
 Tail height $ht/cw = 0$

 Number of chordwise segments on wing $N_c = 2$
 Number of chordwise segments on tail $N_{c_} = 2$
 Number of spanwise segments on wing $N_s = 6$
 Number of spanwise segments on tail $N_{s_} = 3$

 Angle of Descent (degrees) $\alpha_{do} = 0$
 Trim Value of Lift Coefficient $CL_o = C_w = 0.543$

Table 3b)

Influence of Ground Proximity on Trim State						
bw/H	θ_o	δ_t	CDi	CDi/CL ²	CDii/CDi	CDii_v/CDi
0	7.6330	5.3031	0.0168	0.0571	0.9973	1.0002
0.2000	7.6314	5.3034	0.0168	0.0570	0.9975	1.0002
0.4000	7.6264	5.3054	0.0168	0.0568	0.9980	1.0002
0.6000	7.6182	5.3104	0.0167	0.0565	0.9988	1.0002
0.8000	7.6069	5.3197	0.0165	0.0561	0.9999	1.0002
1.0000	7.5929	5.3341	0.0164	0.0556	1.0013	1.0002
1.2000	7.5763	5.3544	0.0162	0.0550	1.0030	1.0002
1.4000	7.5578	5.3808	0.0160	0.0544	1.0050	1.0002
1.6000	7.5375	5.4134	0.0158	0.0537	1.0072	1.0002
1.8000	7.5158	5.4517	0.0156	0.0530	1.0097	1.0002
2.0000	7.4931	5.4951	0.0154	0.0523	1.0124	1.0002
2.2000	7.4695	5.5429	0.0152	0.0515	1.0153	1.0002
2.4000	7.4453	5.5941	0.0150	0.0508	1.0185	1.0002
2.6000	7.4206	5.6481	0.0148	0.0501	1.0218	1.0002
2.8000	7.3954	5.7038	0.0146	0.0494	1.0253	1.0002
3.0000	7.3700	5.7606	0.0143	0.0487	1.0290	1.0002
3.2000	7.3444	5.8176	0.0141	0.0480	1.0328	1.0002

3.4000	7.3186	5.8744	0.0139	0.0473	1.0368	1.0002
3.6000	7.2927	5.9304	0.0137	0.0466	1.0409	1.0002
3.8000	7.2668	5.9851	0.0136	0.0460	1.0452	1.0002
4.0000	7.2407	6.0383	0.0134	0.0453	1.0495	1.0002

Table 3c)

Influence of Ground Proximity on CDi/CDi_inf

bw/H	theta_o	delta_t	CDi/CDi_inf
0	7.6330	5.3031	1.0000
0.2000	7.6314	5.3034	0.9989
0.4000	7.6264	5.3054	0.9955
0.6000	7.6182	5.3104	0.9901
0.8000	7.6069	5.3197	0.9828
1.0000	7.5929	5.3341	0.9739
1.2000	7.5763	5.3544	0.9638
1.4000	7.5578	5.3808	0.9527
1.6000	7.5375	5.4134	0.9408
1.8000	7.5158	5.4517	0.9285
2.0000	7.4931	5.4951	0.9159
2.2000	7.4695	5.5429	0.9030
2.4000	7.4453	5.5941	0.8902
2.6000	7.4206	5.6481	0.8774
2.8000	7.3954	5.7038	0.8648
3.0000	7.3700	5.7606	0.8524
3.2000	7.3444	5.8176	0.8402
3.4000	7.3186	5.8744	0.8283
3.6000	7.2927	5.9304	0.8166
3.8000	7.2668	5.9851	0.8052
4.0000	7.2407	6.0383	0.7942

Table 3d)

INFLUENCE OF GROUND PROXIMITY ON AERODYNAMIC DERIVATIVES

alpha-derivatives

bw/H	CL_a	CDi_a	CM_a
0	4.5965	0.2757	-1.0644
0.2000	4.5973	0.2754	-1.0646
0.4000	4.5998	0.2746	-1.0655
0.6000	4.6041	0.2733	-1.0673
0.8000	4.6101	0.2716	-1.0701
1.0000	4.6177	0.2695	-1.0742
1.2000	4.6268	0.2671	-1.0797
1.4000	4.6372	0.2644	-1.0866
1.6000	4.6489	0.2615	-1.0948
1.8000	4.6615	0.2585	-1.1044
2.0000	4.6750	0.2554	-1.1152
2.2000	4.6891	0.2522	-1.1271
2.4000	4.7039	0.2489	-1.1400
2.6000	4.7190	0.2456	-1.1537
2.8000	4.7346	0.2424	-1.1681
3.0000	4.7504	0.2391	-1.1832
3.2000	4.7663	0.2359	-1.1987
3.4000	4.7825	0.2327	-1.2146

3.6000	4.7987	0.2295	-1.2308
3.8000	4.8151	0.2265	-1.2473
4.0000	4.8315	0.2234	-1.2639

Table 3e)

bw/H	q-derivatives		
	CL _q	CDi _q	CM _q
0	7.8884	0.1948	-8.5659
0.2000	7.8899	0.1944	-8.5661
0.4000	7.8945	0.1930	-8.5668
0.6000	7.9019	0.1907	-8.5684
0.8000	7.9118	0.1877	-8.5713
1.0000	7.9238	0.1838	-8.5759
1.2000	7.9375	0.1793	-8.5825
1.4000	7.9525	0.1742	-8.5913
1.6000	7.9685	0.1686	-8.6025
1.8000	7.9852	0.1627	-8.6162
2.0000	8.0024	0.1564	-8.6324
2.2000	8.0199	0.1499	-8.6511
2.4000	8.0376	0.1433	-8.6723
2.6000	8.0554	0.1366	-8.6958
2.8000	8.0733	0.1298	-8.7216
3.0000	8.0914	0.1231	-8.7496
3.2000	8.1095	0.1163	-8.7796
3.4000	8.1277	0.1097	-8.8116
3.6000	8.1461	0.1032	-8.8455
3.8000	8.1647	0.0968	-8.8812
4.0000	8.1836	0.0905	-8.9186

Table 3f)

bw/H	z-derivatives		
	CL _z	CDi _z	CM _z
0	0	0	0
0.2000	0.0000	0.0000	0.0000
0.4000	-0.0001	0.0000	0.0000
0.6000	-0.0003	0.0000	0.0001
0.8000	-0.0006	0.0000	0.0003
1.0000	-0.0012	0.0001	0.0006
1.2000	-0.0021	0.0001	0.0012
1.4000	-0.0032	0.0002	0.0019
1.6000	-0.0046	0.0002	0.0029
1.8000	-0.0063	0.0003	0.0042
2.0000	-0.0083	0.0003	0.0057
2.2000	-0.0105	0.0004	0.0074
2.4000	-0.0129	0.0005	0.0094
2.6000	-0.0156	0.0005	0.0114
2.8000	-0.0184	0.0006	0.0136
3.0000	-0.0214	0.0007	0.0159
3.2000	-0.0245	0.0007	0.0183
3.4000	-0.0279	0.0008	0.0206
3.6000	-0.0313	0.0008	0.0230
3.8000	-0.0349	0.0009	0.0253

4.0000	-0.0386	0.0010	0.0277
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Table 3g)

bw/H	theta-derivatives		
	CL_the	CDi_the	CM_the
0	-0.0720	-0.0045	-0.0011
0.2000	-0.0720	-0.0045	-0.0011
0.4000	-0.0722	-0.0044	-0.0011
0.6000	-0.0723	-0.0044	-0.0013
0.8000	-0.0724	-0.0044	-0.0018
1.0000	-0.0725	-0.0043	-0.0026
1.2000	-0.0724	-0.0042	-0.0039
1.4000	-0.0721	-0.0041	-0.0057
1.6000	-0.0718	-0.0039	-0.0082
1.8000	-0.0712	-0.0037	-0.0112
2.0000	-0.0705	-0.0035	-0.0147
2.2000	-0.0697	-0.0033	-0.0188
2.4000	-0.0688	-0.0030	-0.0233
2.6000	-0.0679	-0.0027	-0.0282
2.8000	-0.0668	-0.0024	-0.0333
3.0000	-0.0658	-0.0021	-0.0387
3.2000	-0.0648	-0.0018	-0.0441
3.4000	-0.0638	-0.0015	-0.0496
3.6000	-0.0629	-0.0012	-0.0551
3.8000	-0.0621	-0.0008	-0.0605
4.0000	-0.0613	-0.0005	-0.0658

Table 3h)

bw/H	delta_t-derivatives		
	CL_del_t	CDi_del_t	CM_del_t
0	-0.6166	-0.0252	1.4655
0.2000	-0.6167	-0.0251	1.4654
0.4000	-0.6171	-0.0250	1.4654
0.6000	-0.6176	-0.0248	1.4654
0.8000	-0.6182	-0.0246	1.4654
1.0000	-0.6189	-0.0242	1.4654
1.2000	-0.6197	-0.0238	1.4656
1.4000	-0.6204	-0.0233	1.4660
1.6000	-0.6210	-0.0228	1.4666
1.8000	-0.6216	-0.0222	1.4674
2.0000	-0.6222	-0.0216	1.4684
2.2000	-0.6227	-0.0209	1.4697
2.4000	-0.6231	-0.0202	1.4714
2.6000	-0.6234	-0.0195	1.4733
2.8000	-0.6238	-0.0188	1.4756
3.0000	-0.6241	-0.0181	1.4782
3.2000	-0.6244	-0.0174	1.4811
3.4000	-0.6248	-0.0167	1.4844
3.6000	-0.6252	-0.0160	1.4880
3.8000	-0.6256	-0.0153	1.4919
4.0000	-0.6261	-0.0146	1.4962

TABLE 4: Nominal Open Loop Stability Analysis

NOMINAL LONGITUDINAL STABILITY AND CONTROL ANALYSIS OF CHEROKEE

Equations of longitudinal motion were resolved in wind axes.

Trim solution and aerodynamic derivatives were obtained by aerodynamic analysis using the Vortex Lattice Method.

Description of control strategy:

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Open loop control is used

Table 4a)

Largest real part of dominant three roots,
stability margin $CD_{z_max} - CD_z$ of CD_z
and stability margin $CL_{z_max} - CL_z$
of CL_z as function of ground height
(positive Rel implies instability;
negative margin of any z-derivative
implies instability)

b/H	1e3*Rel	CD_{z_marg}	CL_{z_marg}
0	0	0.9592	0
0.4000	-0.0063	0.9444	0.0477
0.8000	-0.0257	0.8366	0.1692
1.2000	0.0033	0.5432	-0.0729
1.6000	1.2693	0.0171	-1.2900
2.0000	3.8073	-0.7212	-3.7927
2.4000	5.6653	-1.6006	-7.4273
2.8000	7.1442	-2.5337	-11.7192
3.2000	8.3128	-3.4453	-16.1047
3.6000	9.2101	-4.2836	-20.0960
4.0000	9.8701	-5.0205	-23.3469

Table 4b)

Static pitch moment stability wrt. angle of attack and height

b/H	hN_{α}	hN_z	hN_{zcrit}
0	0.2292	NaN	NaN
0.4000	0.2292	0.3689	0.6133
0.8000	0.2297	0.4708	0.5763
1.2000	0.2309	0.5611	0.5472
1.6000	0.2331	0.6348	0.5251
2.0000	0.2361	0.6890	0.5089
2.4000	0.2400	0.7238	0.4974
2.8000	0.2443	0.7410	0.4894
3.2000	0.2491	0.7438	0.4838
3.6000	0.2541	0.7355	0.4798
4.0000	0.2593	0.7190	0.4766

Table 4c)

Dominant three root values as function of ground height
(1, 2, 3 in order of decreasing real value, were applicable)
Computed using 5th order characteristic equation)

b/H	1e3*root_1	1e3*root_2	1e3*root_3
0	0	-0.5793- 4.0896i	-0.5793+ 4.0896i
0.4000	-0.0063	-0.5782- 4.0756i	-0.5782+ 4.0756i
0.8000	-0.0257	-0.5901- 3.8881i	-0.5901+ 3.8881i
1.2000	0.0033	-0.6741- 3.0954i	-0.6741+ 3.0954i
1.6000	1.2693	-1.0065	-1.8824
2.0000	3.8073	-0.3459	-5.4880
2.4000	5.6653	-0.3139	-7.8752
2.8000	7.1442	-0.3060	-9.8933
3.2000	8.3128	-0.3001	-11.5848
3.6000	9.2101	-0.2921	-12.9612
4.0000	9.8701	-0.2811	-14.0427

Table 4d)

Analytical estimation of dominant three root values as function of
ground height by approximate factorisation of reduced 3rd order
system

b/H	1e3*root_1	1e3*root_2	1e3*root_3
0	0	-0.4416+ 4.4117i	-0.4416- 4.4117i
0.4000	-0.0062	-0.4379+ 4.3965i	-0.4379- 4.3965i
0.8000	-0.0242	-0.4270+ 4.1890i	-0.4270- 4.1890i
1.2000	0.0166	-0.4447+ 3.2953i	-0.4447- 3.2953i
1.6000	1.8813	-0.9581	-1.7892
2.0000	4.8703	-0.3595	-5.3695
2.4000	7.1917	-0.3290	-7.7139
2.8000	9.1341	-0.3237	-9.6542
3.2000	10.7468	-0.3214	-11.2621
3.6000	12.0535	-0.3178	-12.5655
4.0000	13.0806	-0.3121	-13.5917

Table 4e)

Influence of neglecting α - and $\dot{q}$ -derivatives on values
of long period roots (roots no. 1, 2 and 3)

b/H	(r_a/r)1	(r_a/r)2	(r_a/r)3
0	NaN	1.0002	1.0002
0.4000	1.0000	1.0001	1.0001
0.8000	1.0003	0.9999	0.9999
1.2000	1.0016	0.9989	0.9989
1.6000	1.0200	1.0251	0.9548
2.0000	1.0151	0.9990	0.9828
2.4000	1.0150	0.9991	0.9810
2.8000	1.0156	0.9991	0.9785
3.2000	1.0163	0.9992	0.9762
3.6000	1.0170	0.9992	0.9742
4.0000	1.0177	0.9991	0.9725

- 1, 2, 3 indicates 1st, 2nd and 3rd dominant root
- a) r INCLUDES effects of CM_alphadot AND CZ_alphadot, BUT r_a excludes them.
- b) both r and r_a INCLUDE effects of all theta- and q-derivatives
- c) r and r_a were computed using 5th order system.

Table 4f)

Influence neglecting theta-derivatives on values of long period roots (roots no. 1, 2 and 3)

b/H	(r_a/r)1	(r_a/r)2	(r_a/r)3
0	NaN	0.9921	0.9921
0.4000	0.9869	0.9919	0.9919
0.8000	0.9433	0.9896	0.9896
1.2000	4.9534	0.9740	0.9740
1.6000	1.2836	0.9516	0.9398
2.0000	1.1180	1.0349	0.9639
2.4000	1.0991	1.0473	0.9715
2.8000	1.0973	1.0583	0.9762
3.2000	1.1010	1.0718	0.9801
3.6000	1.1072	1.0891	0.9837
4.0000	1.1149	1.1115	0.9873

- 1, 2, 3 indicates 1st, 2nd and 3rd dominant root
- a) r INCLUDES effects of all theta-derivatives, BUT r_a excludes them.
- b) both r and r_a INCLUDE effects of ALL q-derivatives, and both EXCLUDE effects of CM_alphadot AND CZ_alphadot
- c) r and r_a were computed using 5th order system.

Table 4g)

Influence of neglecting CL_q and CD_q on values of long period roots (roots no. 1, 2 and 3)

b/H	(r_a/r)1	(r_a/r)2	(r_a/r)3
0	NaN	0.9811	0.9811
0.4000	1.0000	0.9811	0.9811
0.8000	0.9998	0.9812	0.9812
1.2000	0.9983	0.9819	0.9819
1.6000	0.9696	0.9980	0.9947
2.0000	0.9715	1.0015	0.9897
2.4000	0.9713	1.0013	0.9905
2.8000	0.9706	1.0012	0.9917
3.2000	0.9696	1.0012	0.9931
3.6000	0.9687	1.0012	0.9945
4.0000	0.9677	1.0013	0.9959

- 1, 2, 3 indicates 1st, 2nd and 3rd dominant root
- a) r INCLUDES effects of CL_q and CD_q, BUT r_a EXCLUDES them both
- b) both r and r_a INCLUDE effect of CM_q, BUT BOTH EXCLUDE effects of theta-derivatives AND effects of CM_alphadot AND CZ_alphadot
- c) r and r_a were computed using 5th order system.

Table 4h)

Influence of neglecting CM_q on values
of long period roots (roots no. 1, 2 and 3)

b/H	(r _a /r)1	(r _a /r)2	(r _a /r)3
0	NaN	1.1034	1.1034
0.4000	1.0000	1.1034	1.1034
0.8000	1.0007	1.1026	1.1026
1.2000	1.0057	1.0987	1.0987
1.6000	1.1494	1.0175	1.0340
2.0000	1.1432	0.9972	1.0551
2.4000	1.1422	0.9973	1.0490
2.8000	1.1430	0.9974	1.0410
3.2000	1.1444	0.9975	1.0330
3.6000	1.1460	0.9974	1.0255
4.0000	1.1478	0.9974	5.8840

1, 2, 3 indicates 1st, 2nd and 3rd dominant root

a) r INCLUDES effects of CM_q, BUT r_a does not

b) both r and r_a EXCLUDE effects CL_q and CD_q, of theta-derivatives,

AND effects of CM_{alphadot} AND CZ_{alphadot}

c) r and r_a were computed using 5th order system.

Table 4i)

Accuracy of analytical approximation of phugoid and heave roots measured by ratio of magnitudes of analytical approximation to values obtained from 5th order characteristic equation. All derivatives were taken into account in the latter. The former were computed by approximate factorisation of the reduced 3rd order system, computing

approximating two roots by assuming the third to be much smaller in magnitude, and then approximating the third one.

b/H	(r _a /r)1	(r _a /r)2	(r _a /r)3
0	NaN	1.0734	1.0734
0.4000	0.9867	1.0733	1.0733
0.8000	0.9428	1.0707	1.0707
1.2000	4.9881	1.0496	1.0496
1.6000	1.4822	0.9519	0.9505
2.0000	1.2792	1.0395	0.9784
2.4000	1.2694	1.0481	0.9795
2.8000	1.2785	1.0580	0.9758
3.2000	1.2928	1.0710	0.9721
3.6000	1.3087	1.0880	0.9695
4.0000	1.3253	1.1100	0.9679

Table 4j)

Accuracy of approximation of alpha roots			
b/H	sig2/sig5	wn2/wn5	zet2/zet5
0	1.0159	1.0052	1.0107
0.4000	1.0160	1.0051	1.0108
0.8000	1.0166	1.0049	1.0116
1.2000	1.0184	1.0042	1.0141
1.6000	1.0219	1.0028	1.0190
2.0000	1.0270	1.0006	1.0264
2.4000	1.0333	0.9977	1.0357
2.8000	1.0401	0.9943	1.0460
3.2000	1.0466	0.9906	1.0565
3.6000	1.0525	0.9869	1.0665
4.0000	1.0576	0.9834	1.0755

2 indicates that 2th order determinant were used, neglecting CZ_ad

5 indicates that 5th order determinant were used, neglecting nothing

TABLE 5: Results of Eigenvector Analysis of Nominal Configuration

EIGENVECTORS OF LONGITUDINAL MOTION OF CHEROKEE IN GROUND PROXIMITY

Equations of longitudinal motion were resolved in wind axes.

Trim solution and aerodynamic derivatives were obtained by aerodynamic analysis using the Vortex Lattice Method.

Magnitudes and phase angles(deg) of first three eigenvectors. Last row contains 1000 x real parts of 1st three characteristic roots below Magnitude columns, and repeated below phase columns. Magnitudes are normalised w.r.t fractional height change, and phase referred to that of H-element. Units of elements of eigenvectors are:

[u in % of U₀, alpha in deg, q in deg-airsec/sec, theta in deg, z in frac of c]

All are normalised w.r.t. H-element (i.e., expressed per unit change of H/c)

Value of Height Parameter			b/H = 0		
M(P1)	M(P2)	M(P3)	phi(P1)	phi(P2)	phi(P3)
0	0.6280	0.6280	0	-168.5456	168.5456
0	0.0157	0.0157	0	-342.8595	342.8595
0	0.0020	0.0020	0	-165.7440	165.7440
0	0.4760	0.4760	0	-263.8058	263.8058
1.0000	1.0000	1.0000	0	0	0
0	-0.3899	-0.3899	0	-0.3899	-0.3899

Value of Height Parameter			b/H = 0.4		
M(P1)	M(P2)	M(P3)	phi(P1)	phi(P2)	phi(P3)
0.0043	0.6277	0.6277	180.0000	191.3152	-191.3152
0.0016	0.0171	0.0171	0	15.2907	-15.2907
0.0000	0.0020	0.0020	180.0000	194.1018	-194.1018
0.0008	0.4742	0.4742	0	96.0275	-96.0275
1.0000	1.0000	1.0000	0	0	0
-0.0062	-0.3865	-0.3865	-0.0062	-0.3865	-0.3865

Value of Height Parameter			b/H = 0.8		
M(P1)	M(P2)	M(P3)	phi(P1)	phi(P2)	phi(P3)
0.0598	0.6262	0.6262	180.0000	190.5110	-190.5110
0.0162	0.0300	0.0300	0	6.4656	-6.4656
0.0000	0.0018	0.0018	180.0000	193.4478	-193.4478
0.0132	0.4505	0.4505	0	94.8185	-94.8185
1.0000	1.0000	1.0000	0	0	0
-0.0242	-0.3805	-0.3805	-0.0242	-0.3805	-0.3805

Value of Height Parameter			b/H = 1.2		
M(P1)	M(P2)	M(P3)	phi(P1)	phi(P2)	phi(P3)
0.2652	0.6236	0.6236	180.0000	188.6388	-188.6388
0.0624	0.0713	0.0713	0	0.5000	-0.5000
0.0000	0.0011	0.0011	0	193.2365	-193.2365
0.0628	0.3554	0.3554	0	90.9511	-90.9511

1.0000	1.0000	1.0000	0	0	0
0.0166	-0.4158	-0.4158	0.0166	-0.4158	-0.4158

Value of Height Parameter			b/H = 1.6		
M(P1)	M(P2)	M(P3)	phi(P1)	phi(P2)	phi(P3)
0.6099	0.7002	0.6329	180.0000	-180.0000	-180.0000
0.1493	0.1557	0.1558	0	0	0
0.0004	0.0000	0.0001	0	-180.0000	0
0.2948	0.0404	0.0599	0	0	-180.0000
1.0000	1.0000	1.0000	0	0	0
1.8521	-0.9971	-1.7375	1.8521	-0.9971	-1.7375

Value of Height Parameter			b/H = 2		
M(P1)	M(P2)	M(P3)	phi(P1)	phi(P2)	phi(P3)
0.6925	1.4089	0.5356	-180.0000	-180.0000	180.0000
0.2607	0.2933	0.2927	0	0	0
0.0027	0.0001	0.0018	0	-180.0000	0
0.6970	0.2537	0.3362	0	0	180.0000
1.0000	1.0000	1.0000	0	0	0
4.7987	-0.3571	-5.4285	4.7987	-0.3571	-5.4285

Value of Height Parameter			b/H = 2.4		
M(P1)	M(P2)	M(P3)	phi(P1)	phi(P2)	phi(P3)
0.7345	2.2871	0.4933	180.0000	180.0000	180.0000
0.3942	0.4665	0.4729	0	0	0
0.0059	0.0001	0.0034	0	180.0000	0
1.0434	0.4306	0.4295	0	0	180.0000
1.0000	1.0000	1.0000	0	0	0
7.0116	-0.3280	-7.7982	7.0116	-0.3280	-7.7982

Value of Height Parameter			b/H = 2.8		
M(P1)	M(P2)	M(P3)	phi(P1)	phi(P2)	phi(P3)
0.7660	3.2452	0.4617	180.0000	180.0000	180.0000
0.5384	0.6581	0.6809	0	0	0
0.0097	0.0002	0.0045	0	180.0000	0
1.3570	0.6231	0.4528	0	0	180.0000
1.0000	1.0000	1.0000	0	0	0
8.8324	-0.3231	-9.7566	8.8324	-0.3231	-9.7566

Value of Height Parameter			b/H = 3.2		
M(P1)	M(P2)	M(P3)	phi(P1)	phi(P2)	phi(P3)
0.7916	4.1820	0.4366	180.0000	180.0000	180.0000
0.6834	0.8525	0.8990	0	0	0
0.0136	0.0002	0.0050	0	180.0000	0
1.6359	0.8181	0.4285	0	0	180.0000
1.0000	1.0000	1.0000	0	0	0
10.3215	-0.3209	-11.3704	10.3215	-0.3209	-11.3704

Value of Height Parameter			b/H = 3.6		
M(P1)	M(P2)	M(P3)	phi(P1)	phi(P2)	phi(P3)
0.8134	5.0221	0.4163	180.0000	180.0000	180.0000
0.8228	1.0385	1.1127	0	0	0
0.0173	0.0003	0.0048	0	180.0000	0
1.8782	1.0050	0.3725	0	0	180.0000
1.0000	1.0000	1.0000	0	0	0
11.5127	-0.3175	-12.6681	11.5127	-0.3175	-12.6681

Value of Height Parameter				b/H = 4	
M(P1)	M(P2)	M(P3)	phi(P1)	phi(P2)	phi(P3)
0.8327	5.7204	0.3998	180.0000	180.0000	180.0000
0.9535	1.2095	1.3125	0	0	0
0.0206	0.0003	0.0042	0	180.0000	0
2.0846	1.1773	0.2967	0	0	180.0000
1.0000	1.0000	1.0000	0	0	0
12.4388	-0.3118	-13.5899	12.4388	-0.3118	-13.5899

TABLE 6: Results of Parameter Sensitivity with respect to CM Position

RESULTS OF SENSITIVITY ANALYSIS

Maximum real part (x1000) of roots for varying CM position

bw/H	0	0	0	0	0
0.2000	-0.0010	-0.0010	-0.0010	-0.0010	-0.0010
0.4000	-0.0066	-0.0063	-0.0061	-0.0058	-0.0058
0.6000	-0.0176	-0.0161	-0.0150	-0.0133	-0.0133
0.8000	-0.0309	-0.0261	-0.0226	-0.0167	-0.0167
1.0000	-0.0403	-0.0278	-0.0181	-0.0015	-0.0015
1.2000	-0.0337	-0.0004	0.0271	0.0784	0.0784
1.4000	0.0364	0.1587	0.2776	0.5302	0.5302
1.6000	0.6948	1.2204	1.5341	1.9558	1.9558
1.8000	2.2027	2.6290	2.8923	3.2623	3.2623
2.0000	3.3754	3.7676	4.0182	4.3778	4.3778
2.2000	4.3595	4.7504	5.0030	5.3682	5.3682
2.4000	5.2197	5.6214	5.8819	6.2593	6.2593
2.6000	5.9808	6.3990	6.6701	7.0626	7.0626
2.8000	6.6548	7.0924	7.3756	7.7847	7.7847
3.0000	7.2490	7.7079	8.0040	8.4306	8.4306
3.2000	7.7687	8.2504	8.5601	9.0047	9.0047
3.4000	8.2185	8.7244	9.0482	9.5114	9.5114
3.6000	8.6025	9.1340	9.4727	9.9550	9.9550
3.8000	8.9246	9.4834	9.8376	10.3398	10.3398
4.0000	9.1883	9.7762	10.1468	10.6700	10.6700

CM position :

0.1750 0.2063 0.2250 0.2500

Values of dCLz (x1000) for varying CM position

bw/H	0	0	0	0	0
0.2000	0.0077	0.0075	0.0075	0.0073	0.0073
0.4000	0.0501	0.0479	0.0463	0.0436	0.0436
0.6000	0.1288	0.1174	0.1089	0.0947	0.0947
0.8000	0.2094	0.1731	0.1460	0.1014	0.1014
1.0000	0.2302	0.1434	0.0791	-0.0271	-0.0271
1.2000	0.1182	-0.0546	-0.1823	-0.3927	-0.3927
1.4000	-0.1949	-0.4971	-0.7200	-1.0865	-1.0865
1.6000	-0.7584	-1.2385	-1.5921	-2.1724	-2.1724
1.8000	-1.5952	-2.3035	-2.8241	-3.6770	-3.6770
2.0000	-2.6999	-3.6851	-4.4078	-5.5893	-5.5893
2.2000	-4.0420	-5.3489	-6.3053	-7.8658	-7.8658
2.4000	-5.5726	-7.2399	-8.4572	-10.4388	-10.4388
2.6000	-7.2310	-9.2909	-10.7910	-13.2272	-13.2272
2.8000	-8.9514	-11.4294	-13.2291	-16.1445	-16.1445
3.0000	-10.6692	-13.5843	-15.6957	-19.1069	-19.1069
3.2000	-12.3239	-15.6900	-18.1210	-22.0378	-22.0378
3.4000	-13.8622	-17.6888	-20.4439	-24.8706	-24.8706
3.6000	-15.2388	-19.5320	-22.6135	-27.5502	-27.5502
3.8000	-16.4163	-21.1802	-24.5887	-30.0330	-30.0330
4.0000	-17.3653	-22.6028	-26.3378	-32.2857	-32.2857

Values of dCDz (x1000) for varying CM position

bw/H				
0	0.9626	0.9593	0.9578	0.9561
0.2000	0.9607	0.9574	0.9558	0.9542
0.4000	0.9481	0.9446	0.9428	0.9409
0.6000	0.9131	0.9087	0.9062	0.9031
0.8000	0.8442	0.8371	0.8327	0.8265
1.0000	0.7309	0.7187	0.7106	0.6980
1.2000	0.5661	0.5454	0.5309	0.5079
1.4000	0.3465	0.3132	0.2894	0.2512
1.6000	0.0729	0.0228	-0.0134	-0.0721
1.8000	-0.2499	-0.3210	-0.3726	-0.4566
2.0000	-0.6141	-0.7100	-0.7798	-0.8932
2.2000	-1.0104	-1.1341	-1.2241	-1.3705
2.4000	-1.4282	-1.5820	-1.6939	-1.8757
2.6000	-1.8574	-2.0429	-2.1775	-2.3960
2.8000	-2.2887	-2.5065	-2.6644	-2.9201
3.0000	-2.7136	-2.9640	-3.1451	-3.4378
3.2000	-3.1256	-3.4082	-3.6122	-3.9410
3.4000	-3.5194	-3.8335	-4.0597	-4.4234
3.6000	-3.8909	-4.2357	-4.4833	-4.8803
3.8000	-4.2375	-4.6119	-4.8800	-5.3087
4.0000	-4.5575	-4.9605	-5.2482	-5.7070

Values of dhNz for varying CM position

bw/H				
0	NaN	NaN	NaN	NaN
0.2000	0.3719	0.3243	0.2954	0.2565
0.4000	0.2925	0.2482	0.2213	0.1848
0.6000	0.2179	0.1764	0.1511	0.1168
0.8000	0.1482	0.1090	0.0851	0.0526
1.0000	0.0839	0.0466	0.0238	-0.0073
1.2000	0.0253	-0.0104	-0.0323	-0.0622
1.4000	-0.0270	-0.0615	-0.0827	-0.1116
1.6000	-0.0727	-0.1063	-0.1270	-0.1552
1.8000	-0.1116	-0.1447	-0.1650	-0.1927
2.0000	-0.1440	-0.1767	-0.1968	-0.2242
2.2000	-0.1700	-0.2026	-0.2225	-0.2498
2.4000	-0.1901	-0.2227	-0.2426	-0.2699
2.6000	-0.2047	-0.2375	-0.2576	-0.2849
2.8000	-0.2145	-0.2476	-0.2678	-0.2953
3.0000	-0.2199	-0.2534	-0.2738	-0.3016
3.2000	-0.2215	-0.2555	-0.2762	-0.3043
3.4000	-0.2199	-0.2545	-0.2754	-0.3039
3.6000	-0.2155	-0.2507	-0.2720	-0.3008
3.8000	-0.2087	-0.2446	-0.2663	-0.2956
4.0000	-0.2000	-0.2366	-0.2587	-0.2884

LINEARLY INTERPOLATED CRITICAL HEIGHTS FROM PRECEDING TABLES

Critical Height Ratios Predicted by Various Criteria

		(b/H)crit		
x_CM LEw	rlocus	CLzcrit	CDzcrit	hNzcrit
0.1750	1.2961	1.2755	1.6452	1.2968
0.2063	1.2005	1.1449	1.6132	1.1634
0.2250	1.0803	1.0605	1.5911	1.0847
0.2500	1.0037	0.9578	1.5554	0.9757

Criteria by which columns 1 to 4 were obtained are:

- Criterion 1: root locus crosses over to right half of s-plane
- Criterion 2: CLz exceeds CLz_crit
- Criterion 3: CDz exceeds CDz_crit
- Criterion 4: hNz exceeds hNz_crit

The NaN at $b/H = 0$ signifies neutral height stability

If ALL elements = NaN, stability over whole parameter range

is implied, PROVIDED that ALL maximum real parts of roots are ≤ 0

TABLE 7: Closed Loop Stability Analysis with Height Feedback to Elevator

CLOSED LOOP LONGITUDINAL STABILITY AND CONTROL ANALYSIS OF CHEROKEE

Equations of longitudinal motion were resolved in wind axes.

Trim solution and aerodynamic derivatives were obtained by aerodynamic analysis using the Vortex Lattice Method.

Description of control strategy:

=====

Closed loop control is used (feedback of height perturbation to elevator)

Reduction ratio of CM_z is $K_{\{CMz\}} = 0.4$ at all b_w/H

Proportional error gain therefore varies with b_w/H

Table 7 a):

Largest real part of dominant three roots,
stability margin $CD_z_{max} - CD_z$ of CD_z
and stability margin $CL_z_{max} - CL_z$
of CL_z as function of ground height
(positive Rel implies instability;
negative margin of any z-derivative
implies instability)

b/H	1e3*Rel	CD_z_marg	CL_z_marg
0	0	0.9599	0
0.2000	-0.0016	0.9584	0.0125
0.4000	-0.0123	0.9498	0.0961
0.6000	-0.0390	0.9275	0.3067
0.8000	-0.0851	0.8860	0.6776
1.0000	-0.1512	0.8208	1.2192
1.2000	-0.2362	0.7295	1.9240
1.4000	-0.3392	0.6113	2.7743
1.6000	-0.4599	0.4675	3.7500
1.8000	-0.4483	0.3007	4.8343
2.0000	-0.4279	0.1147	6.0169
2.2000	-0.4060	-0.0863	7.2943
2.4000	-0.3835	-0.2975	8.6692
2.6000	-0.3635	-0.5146	10.1485
2.8000	-0.3506	-0.7335	11.7418
3.0000	-0.3497	-0.9505	13.4591
3.2000	-0.3643	-1.1627	15.3102
3.4000	-0.3955	-1.3676	17.3038
3.6000	-0.4425	-1.5635	19.4469
3.8000	-0.5022	-1.7488	21.7450
4.0000	-0.5710	-1.9225	24.2022

Table 7 b):

Static pitch moment stability wrt. angle of attack and height

b/H	hN_alpha	hN_z	hN_zcrit
0	0.2328	NaN	NaN
0.2000	0.2328	0.1195	0.6184
0.4000	0.2329	0.1378	0.5940
0.6000	0.2331	0.1552	0.5724
0.8000	0.2334	0.1714	0.5534
1.0000	0.2339	0.1865	0.5368
1.2000	0.2347	0.2001	0.5224
1.4000	0.2356	0.2122	0.5101
1.6000	0.2369	0.2227	0.4996
1.8000	0.2383	0.2316	0.4910
2.0000	0.2399	0.2389	0.4839
2.2000	0.2418	0.2447	0.4782
2.4000	0.2438	0.2490	0.4736
2.6000	0.2460	0.2521	0.4701
2.8000	0.2483	0.2541	0.4675
3.0000	0.2507	0.2549	0.4655
3.2000	0.2531	0.2549	0.4641
3.4000	0.2556	0.2540	0.4631
3.6000	0.2582	0.2524	0.4624
3.8000	0.2608	0.2503	0.4620
4.0000	0.2634	0.2475	0.4617

Table 7 c):

Dominant three root values as function of ground height

(1, 2, 3 in order of decreasing real value, were applicable)

Computed using 5th order characteristic equation)

b/H	1e3*root_1	1e3*root_2	1e3*root_3
0	0	-0.5775- 4.0942i	-0.5775+ 4.0942i
0.2000	-0.0016	-0.5768- 4.0957i	-0.5768+ 4.0957i
0.4000	-0.0123	-0.5723- 4.1044i	-0.5723+ 4.1044i
0.6000	-0.0390	-0.5622- 4.1231i	-0.5622+ 4.1231i
0.8000	-0.0851	-0.5468- 4.1497i	-0.5468+ 4.1497i
1.0000	-0.1512	-0.5277- 4.1783i	-0.5277+ 4.1783i
1.2000	-0.2362	-0.5073- 4.2015i	-0.5073+ 4.2015i
1.4000	-0.3392	-0.4871- 4.2134i	-0.4871+ 4.2134i
1.6000	-0.4599	-0.4676- 4.2108i	-0.4676+ 4.2108i
1.8000	-0.4483- 4.1952i	-0.4483+ 4.1952i	-0.5990
2.0000	-0.4279- 4.1722i	-0.4279+ 4.1722i	-0.7563
2.2000	-0.4060- 4.1513i	-0.4060+ 4.1513i	-0.9299
2.4000	-0.3835- 4.1439i	-0.3835+ 4.1439i	-1.1140
2.6000	-0.3635- 4.1618i	-0.3635+ 4.1618i	-1.2989
2.8000	-0.3506- 4.2148i	-0.3506+ 4.2148i	-1.4714
3.0000	-0.3497- 4.3095i	-0.3497+ 4.3095i	-1.6189
3.2000	-0.3643- 4.4495i	-0.3643+ 4.4495i	-1.7319
3.4000	-0.3955- 4.6359i	-0.3955+ 4.6359i	-1.8056
3.6000	-0.4425- 4.8682i	-0.4425+ 4.8682i	-1.8407
3.8000	-0.5022- 5.1445i	-0.5022+ 5.1445i	-1.8417
4.0000	-0.5710- 5.4617i	-0.5710+ 5.4617i	-1.8156

Table 7 d):

Analytical estimation of dominant three root values as function of ground height by approximate factorisation of reduced 3rd order system

b/H	le3*root_1	le3*root_2	le3*root_3
0	0	-0.4419+ 4.4116i	-0.4419- 4.4116i
0.2000	-0.0016	-0.4409+ 4.4133i	-0.4409- 4.4133i
0.4000	-0.0124	-0.4351+ 4.4237i	-0.4351- 4.4237i
0.6000	-0.0391	-0.4209+ 4.4459i	-0.4209- 4.4459i
0.8000	-0.0851	-0.3969+ 4.4774i	-0.3969- 4.4774i
1.0000	-0.1506	-0.3628+ 4.5104i	-0.3628- 4.5104i
1.2000	-0.2346	-0.3193+ 4.5357i	-0.3193- 4.5357i
1.4000	-0.2669+ 4.5449i	-0.2669- 4.5449i	-0.3362
1.6000	-0.2055+ 4.5331i	-0.2055- 4.5331i	-0.4556
1.8000	-0.1343+ 4.4995i	-0.1343- 4.4995i	-0.5944
2.0000	-0.0524+ 4.4477i	-0.0524- 4.4477i	-0.7544
2.2000	0.0406+ 4.3852i	0.0406- 4.3852i	-0.9368
2.4000	0.1440+ 4.3225i	0.1440- 4.3225i	-1.1398
2.6000	0.2543+ 4.2720i	0.2543- 4.2720i	-1.3568
2.8000	0.3652+ 4.2476i	0.3652- 4.2476i	-1.5749
3.0000	0.4677+ 4.2632i	0.4677- 4.2632i	-1.7762
3.2000	0.5520+ 4.3312i	0.5520- 4.3312i	-1.9413
3.4000	0.6108+ 4.4601i	0.6108- 4.4601i	-2.0554
3.6000	0.6414+ 4.6531i	0.6414- 4.6531i	-2.1132
3.8000	0.6459+ 4.9080i	0.6459- 4.9080i	-2.1189
4.0000	0.6297+ 5.2187i	0.6297- 5.2187i	-2.0834

Table 7 e):

Influence of neglecting alphadot- and qdot-derivatives on values of long period roots (roots no. 1, 2 and 3)

b/H	(r_a/r)1	(r_a/r)2	(r_a/r)3
0	NaN	1.0002	1.0002
0.2000	1.0000	1.0002	1.0002
0.4000	1.0000	1.0001	1.0001
0.6000	1.0000	1.0001	1.0001
0.8000	1.0001	1.0001	1.0001
1.0000	1.0001	1.0000	1.0000
1.2000	1.0002	0.9999	0.9999
1.4000	1.0002	0.9997	0.9997
1.6000	9.2081	0.9996	0.1086
1.8000	0.9996	0.9996	0.9997
2.0000	0.9997	0.9997	0.9990
2.2000	1.0000	1.0000	0.9979
2.4000	1.0005	1.0005	0.9965
2.6000	1.0012	1.0012	0.9947
2.8000	1.0018	1.0018	0.9929
3.0000	1.0024	1.0024	0.9913
3.2000	1.0028	1.0028	0.9900
3.4000	1.0030	1.0030	0.9893
3.6000	1.0029	1.0029	0.9892
3.8000	1.0025	1.0025	0.9895
4.0000	1.0020	1.0020	0.9902

- 1, 2, 3 indicates 1st, 2nd and 3rd dominant root
a) r INCLUDES effects of CM_alphadot AND CZ_alphadot, BUT r_a excludes them.
b) both r and r_a INCLUDE effects of all theta- and q-derivatives
c) r and r_a were computed using 5th order system.

Table 7 f):

Influence neglecting theta-derivatives on values
of long period roots (roots no. 1, 2 and 3)

b/H	(r_a/r)1	(r_a/r)2	(r_a/r)3
0	NaN	0.9923	0.9923
0.2000	1.0072	0.9923	0.9923
0.4000	1.0060	0.9923	0.9923
0.6000	1.0047	0.9923	0.9923
0.8000	1.0032	0.9923	0.9923
1.0000	1.0017	0.9919	0.9919
1.2000	1.0005	0.9912	0.9912
1.4000	12.3723	0.9898	0.0800
1.6000	0.9876	0.9876	0.9997
1.8000	0.9845	0.9845	1.0007
2.0000	0.9803	0.9803	1.0032
2.2000	0.9750	0.9750	1.0074
2.4000	0.9687	0.9687	1.0133
2.6000	0.9616	0.9616	1.0208
2.8000	0.9540	0.9540	1.0297
3.0000	0.9462	0.9462	1.0398
3.2000	0.9383	0.9383	1.0507
3.4000	0.9307	0.9307	1.0621
3.6000	0.9236	0.9236	1.0733
3.8000	0.9174	0.9174	1.0834
4.0000	0.9125	0.9125	1.0916

- 1, 2, 3 indicates 1st, 2nd and 3rd dominant root
a) r INCLUDES effects of all theta-derivatives, BUT r_a excludes them.
b) both r and r_a INCLUDE effects of ALL q-derivatives, and both EXCLUDE effects of CM_alphadot AND CZ_alphadot
c) r and r_a were computed using 5th order system.

Table 7 g):

Influence of neglecting CL_q and CD_q on values
of long period roots (roots no. 1, 2 and 3)

b/H	(r_a/r)1	(r_a/r)2	(r_a/r)3
0	NaN	0.9811	0.9811
0.2000	1.0000	0.9811	0.9811
0.4000	1.0000	0.9810	0.9810
0.6000	1.0000	0.9810	0.9810
0.8000	1.0000	0.9810	0.9810
1.0000	1.0000	0.9810	0.9810
1.2000	1.0000	0.9810	0.9810
1.4000	0.9810	0.9810	1.0000
1.6000	0.9810	0.9810	0.9999
1.8000	0.9810	0.9810	0.9999

2.0000	0.9809	0.9809	1.0000
2.2000	0.9809	0.9809	1.0002
2.4000	0.9807	0.9807	1.0005
2.6000	0.9805	0.9805	1.0009
2.8000	0.9803	0.9803	1.0014
3.0000	0.9800	0.9800	1.0020
3.2000	0.9798	0.9798	1.0025
3.4000	0.9795	0.9795	1.0031
3.6000	0.9793	0.9793	1.0036
3.8000	0.9791	0.9791	1.0039
4.0000	0.9790	0.9790	1.0041

1, 2, 3 indicates 1st, 2nd and 3rd dominant root

a) r INCLUDES effects of CL_q and CD_q, BUT r_a EXCLUDES them both

b) both r and r_a INCLUDE effect of CM_q, BUT BOTH EXCLUDE effects of theta-derivatives AND effects of CM_{alphadot} AND CZ_{alphadot}

c) r and r_a were computed using 5th order system.

Table 7 h):

Influence of neglecting CM_q on values
of long period roots (roots no. 1, 2 and 3)

b/H	(r _a /r)1	(r _a /r)2	(r _a /r)3
0	NaN	1.1021	1.1021
0.2000	1.0000	1.1021	1.1021
0.4000	1.0000	1.1021	1.1021
0.6000	0.9999	1.1021	1.1021
0.8000	0.9999	1.1019	1.1019
1.0000	0.9999	1.1017	1.1017
1.2000	1.0000	1.1012	1.1012
1.4000	1.1007	1.1007	1.0000
1.6000	1.1000	1.1000	1.0000
1.8000	1.0993	1.0993	0.9998
2.0000	1.0988	1.0988	0.9993
2.2000	1.0984	1.0984	0.9982
2.4000	1.0982	1.0982	0.9967
2.6000	1.0983	1.0983	0.9946
2.8000	1.0986	1.0986	0.9921
3.0000	1.0991	1.0991	0.9895
3.2000	1.0995	1.0995	0.9869
3.4000	1.0999	1.0999	0.9844
3.6000	1.1001	1.1001	0.9824
3.8000	1.1000	1.1000	0.9810
4.0000	1.0996	1.0996	0.9802

1, 2, 3 indicates 1st, 2nd and 3rd dominant root

a) r INCLUDES effects of CM_q, BUT r_a does not

b) both r and r_a EXCLUDE effects CL_q and CD_q, of theta-derivatives,

AND effects of CM_{alphadot} AND CZ_{alphadot}

c) r and r_a were computed using 5th order system.

Table 7 i):

Accuracy of analytical approximation of phugoid and heave roots measured by ratio of magnitudes of analytical approximation to values obtained from 5th order characteristic equation. All derivatives were taken into account in the latter. The former were computed by approximate factorisation of the reduced 3rd order system, computing

approximating two roots by assuming the third to be much smaller in magnitude, and then approximating the third one.

b/H	(r_a/r)1	(r_a/r)2	(r_a/r)3
0	NaN	1.0723	1.0723
0.2000	1.0071	1.0723	1.0723
0.4000	1.0055	1.0726	1.0726
0.6000	1.0030	1.0732	1.0732
0.8000	0.9998	1.0739	1.0739
1.0000	0.9964	1.0744	1.0744
1.2000	0.9933	1.0744	1.0744
1.4000	13.4231	1.0734	0.0793
1.6000	9.8659	1.0711	0.1075
1.8000	1.0669	1.0669	0.9924
2.0000	1.0605	1.0605	0.9976
2.2000	1.0514	1.0514	1.0075
2.4000	1.0392	1.0392	1.0231
2.6000	1.0244	1.0244	1.0446
2.8000	1.0080	1.0080	1.0704
3.0000	0.9919	0.9919	1.0971
3.2000	0.9780	0.9780	1.1209
3.4000	0.9675	0.9675	1.1383
3.6000	0.9609	0.9609	1.1480
3.8000	0.9577	0.9577	1.1505
4.0000	0.9572	0.9572	1.1475

Table 7 j):

Accuracy of approximation of alpha roots

b/H	sig2/sig5	wn2/wn5	zet2/zet5
0	1.0158	1.0052	1.0106
0.2000	1.0159	1.0052	1.0106
0.4000	1.0159	1.0052	1.0106
0.6000	1.0160	1.0052	1.0108
0.8000	1.0162	1.0050	1.0112
1.0000	1.0166	1.0047	1.0118
1.2000	1.0172	1.0043	1.0128
1.4000	1.0180	1.0037	1.0143
1.6000	1.0191	1.0029	1.0161
1.8000	1.0203	1.0019	1.0184
2.0000	1.0218	1.0008	1.0210
2.2000	1.0235	0.9995	1.0239
2.4000	1.0252	0.9981	1.0271
2.6000	1.0270	0.9966	1.0305
2.8000	1.0289	0.9951	1.0340
3.0000	1.0307	0.9935	1.0375
3.2000	1.0324	0.9919	1.0409
3.4000	1.0341	0.9903	1.0443

3.6000	1.0357	0.9887	1.0475
3.8000	1.0372	0.9872	1.0506
4.0000	1.0385	0.9857	1.0536

2 indicates that 2th order determinant were used, neglecting CZ_ad
5 indicates that 5th order determinant were used, neglecting nothing

FEEDBACK CONTROL STRATEGY AND ITS INFLUENCE ON HEIGHT DERIVATIVES

Feedback Control Law used is: $\Delta t = \Delta t_{ref} - K_z \cdot z$

in which: Δt_{ref} = the part of elevator angle that is due to pilot input

z = measured height perturbation
 K_z = feedback gain

Table 7 k):

Table of Gain Values K_z with which Height Perturbation is Fed Back
Units of K_z are degrees elevator angle per unit height perturbation

bw/H	K_z
0	0
0.2000	0.0001
0.4000	0.0007
0.6000	0.0027
0.8000	0.0072
1.0000	0.0153
1.2000	0.0280
1.4000	0.0463
1.6000	0.0705
1.8000	0.1007
2.0000	0.1368
2.2000	0.1781
2.4000	0.2239
2.6000	0.2734
2.8000	0.3256
3.0000	0.3796
3.2000	0.4346
3.4000	0.4899
3.6000	0.5449
3.8000	0.5989
4.0000	0.6517

Table 7 l):

Open Loop (OL) and Closed Loop (CL) values of CM_z , CD_z and CL_z
(all x1000)

bw/H	CM_z_{OL}	CM_z_{CL}	CDi_z_{OL}	CDi_z_{CL}	CL_z_{OL}	CL_z_{CL}
0	0	0	0	0	0	0
0.2000	0.0030	0.0012	0.0008	0.0008	-0.0094	-0.0101
0.4000	0.0293	0.0117	0.0059	0.0056	-0.0777	-0.0851

0.6000	0.1150	0.0460	0.0186	0.0175	-0.2672	-0.2963
0.8000	0.3058	0.1223	0.0406	0.0376	-0.6360	-0.7134
1.0000	0.6508	0.2603	0.0721	0.0659	-1.2313	-1.3962
1.2000	1.1952	0.4781	0.1127	0.1015	-2.0863	-2.3894
1.4000	1.9737	0.7895	0.1609	0.1428	-3.2199	-3.7210
1.6000	3.0071	1.2028	0.2152	0.1884	-4.6379	-5.4017
1.8000	4.3002	1.7201	0.2742	0.2370	-6.3354	-7.4282
2.0000	5.8435	2.3374	0.3366	0.2875	-8.3001	-9.7853
2.2000	7.6151	3.0460	0.4012	0.3394	-10.5146	-12.4499
2.4000	9.5847	3.8339	0.4673	0.3924	-12.9592	-15.3940
2.6000	11.7170	4.6868	0.5343	0.4462	-15.6136	-18.5879
2.8000	13.9750	5.5900	0.6014	0.5007	-18.4582	-22.0022
3.0000	16.3225	6.5290	0.6684	0.5559	-21.4755	-25.6098
3.2000	18.7260	7.4904	0.7347	0.6114	-24.6505	-29.3866
3.4000	21.1555	8.4622	0.7998	0.6672	-27.9706	-33.3125
3.6000	23.5855	9.4342	0.8635	0.7228	-31.4262	-37.3709
3.8000	25.9949	10.3980	0.9252	0.7780	-35.0100	-41.5494
4.0000	28.3666	11.3466	0.9845	0.8322	-38.7171	-45.8386

TABLE 8: Longitudinal Stability of High Tail Configuration

NOMINAL CASE OF TABLE 2, BUT WITH TAIL HEIGHT = 1.5 TIMES WING CHORD,
INSTEAD OF ZERO TAIL HEIGHT
(COMPARE RESULTS BELOW TO THOSE OF TABLES 3 AND 4)

bw/H	z-derivatives		
	CL _z	CDi _z	CM _z
0	0	0	0
0.2000	0.0000	0.0000	0.0000
0.4000	-0.0001	0.0000	0.0000
0.6000	-0.0003	0.0000	0.0001
0.8000	-0.0006	0.0000	0.0002
1.0000	-0.0011	0.0001	0.0005
1.2000	-0.0019	0.0001	0.0008
1.4000	-0.0029	0.0002	0.0013
1.6000	-0.0041	0.0002	0.0018
1.8000	-0.0055	0.0003	0.0025
2.0000	-0.0071	0.0004	0.0033
2.2000	-0.0089	0.0005	0.0041
2.4000	-0.0108	0.0006	0.0050
2.6000	-0.0130	0.0007	0.0060
2.8000	-0.0153	0.0007	0.0071
3.0000	-0.0177	0.0008	0.0082
3.2000	-0.0203	0.0009	0.0093
3.4000	-0.0230	0.0010	0.0104
3.6000	-0.0259	0.0011	0.0116
3.8000	-0.0290	0.0011	0.0128
4.0000	-0.0321	0.0012	0.0140

Dominant three root values as function of ground height
(1, 2, 3 in order of decreasing real value, were applicable)
Computed using 5th order characteristic equation)

b/H	1e3*root_1	1e3*root_2	1e3*root_3
0	0	-0.6096- 4.0883i	-0.6096+ 4.0883i
0.2000	-0.0010	-0.6093- 4.0874i	-0.6093+ 4.0874i
0.4000	-0.0073	-0.6078- 4.0784i	-0.6078+ 4.0784i
0.6000	-0.0224	-0.6052- 4.0489i	-0.6052+ 4.0489i
0.8000	-0.0487	-0.6022- 3.9844i	-0.6022+ 3.9844i
1.0000	-0.0888	-0.5986- 3.8709i	-0.5986+ 3.8709i
1.2000	-0.1474	-0.5929- 3.6948i	-0.5929+ 3.6948i
1.4000	-0.2350	-0.5799- 3.4422i	-0.5799+ 3.4422i
1.6000	-0.3776	-0.5465- 3.0959i	-0.5465+ 3.0959i
1.8000	-0.4541- 2.6333i	-0.4541+ 2.6333i	-0.6508
2.0000	-0.1856- 2.0770i	-0.1856+ 2.0770i	-1.2872
2.2000	0.2640- 1.6724i	0.2640+ 1.6724i	-2.2949
2.4000	0.6450- 1.4066i	0.6450+ 1.4066i	-3.1727
2.6000	0.9462- 1.1525i	0.9462+ 1.1525i	-3.8963
2.8000	1.1940- 0.8614i	1.1940+ 0.8614i	-4.5169
3.0000	1.4029- 0.4519i	1.4029+ 0.4519i	-5.0620
3.2000	2.1121	1.0496	-5.5463
3.4000	2.5750	0.8901	-5.9783
3.6000	2.8958	0.8267	-6.3636
3.8000	3.1339	0.8042	-6.7058
4.0000	3.3077	0.8075	-7.0077

GRAPHICAL RESULTS

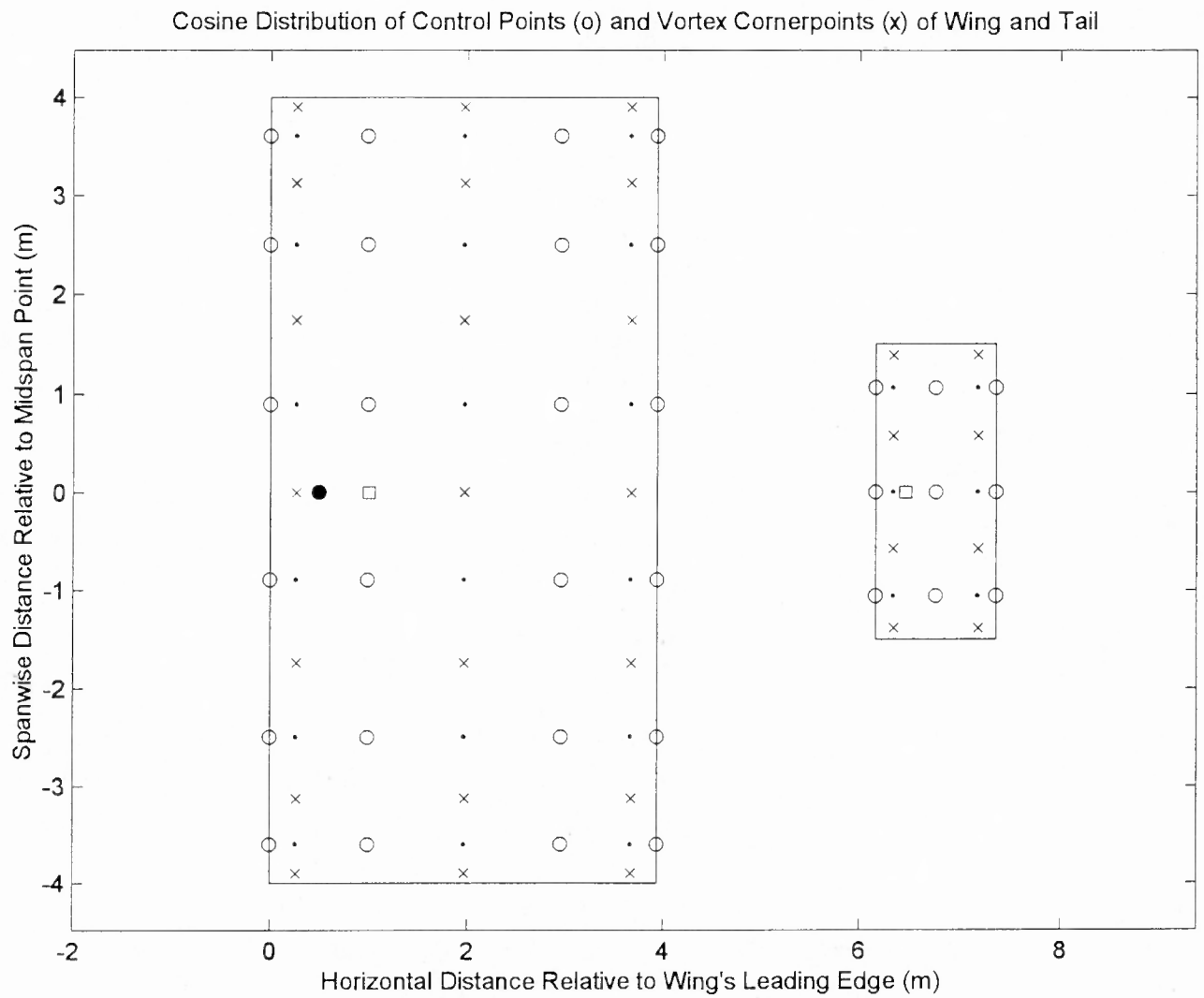


FIGURE R1 a)

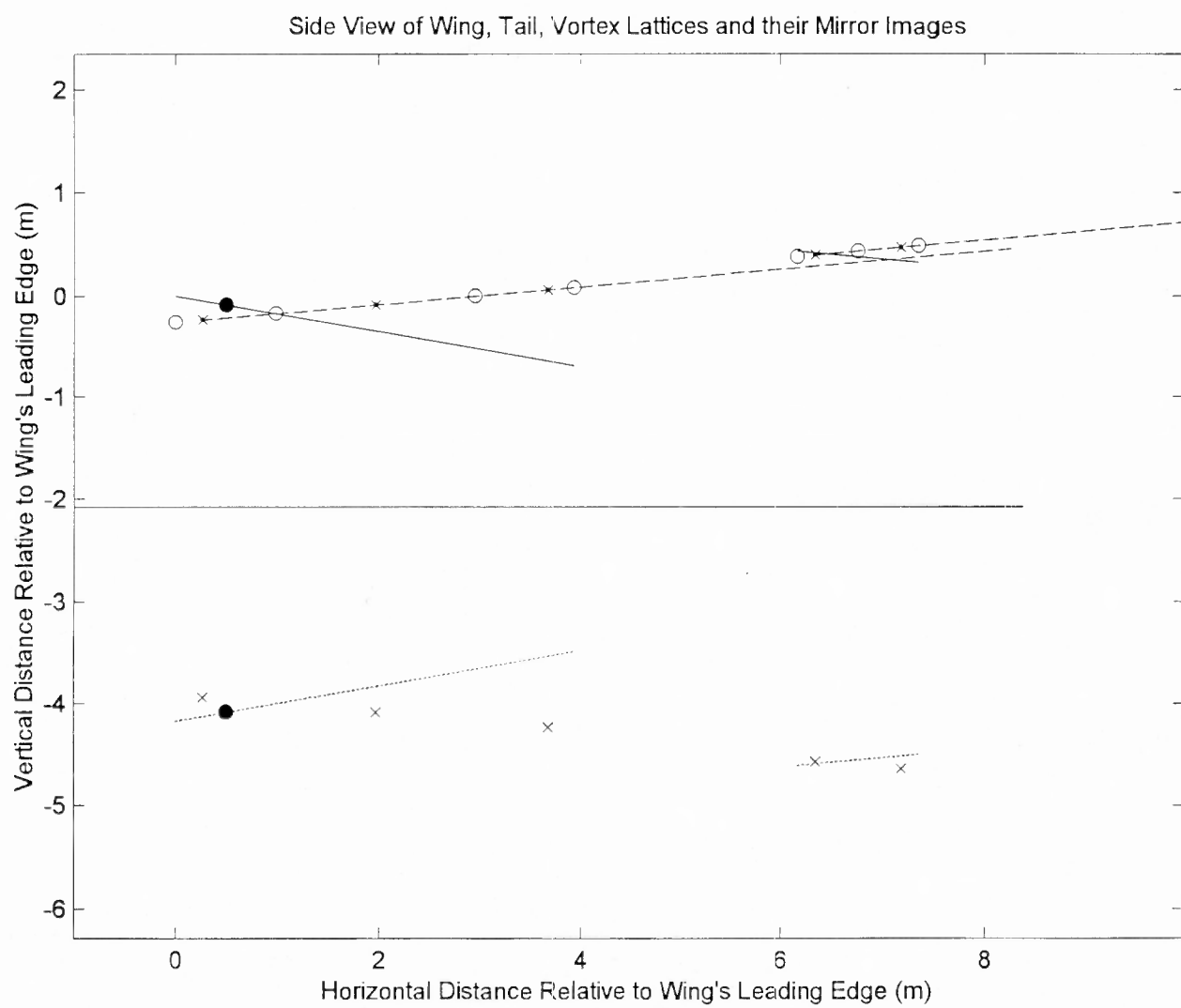


FIGURE R1 b)

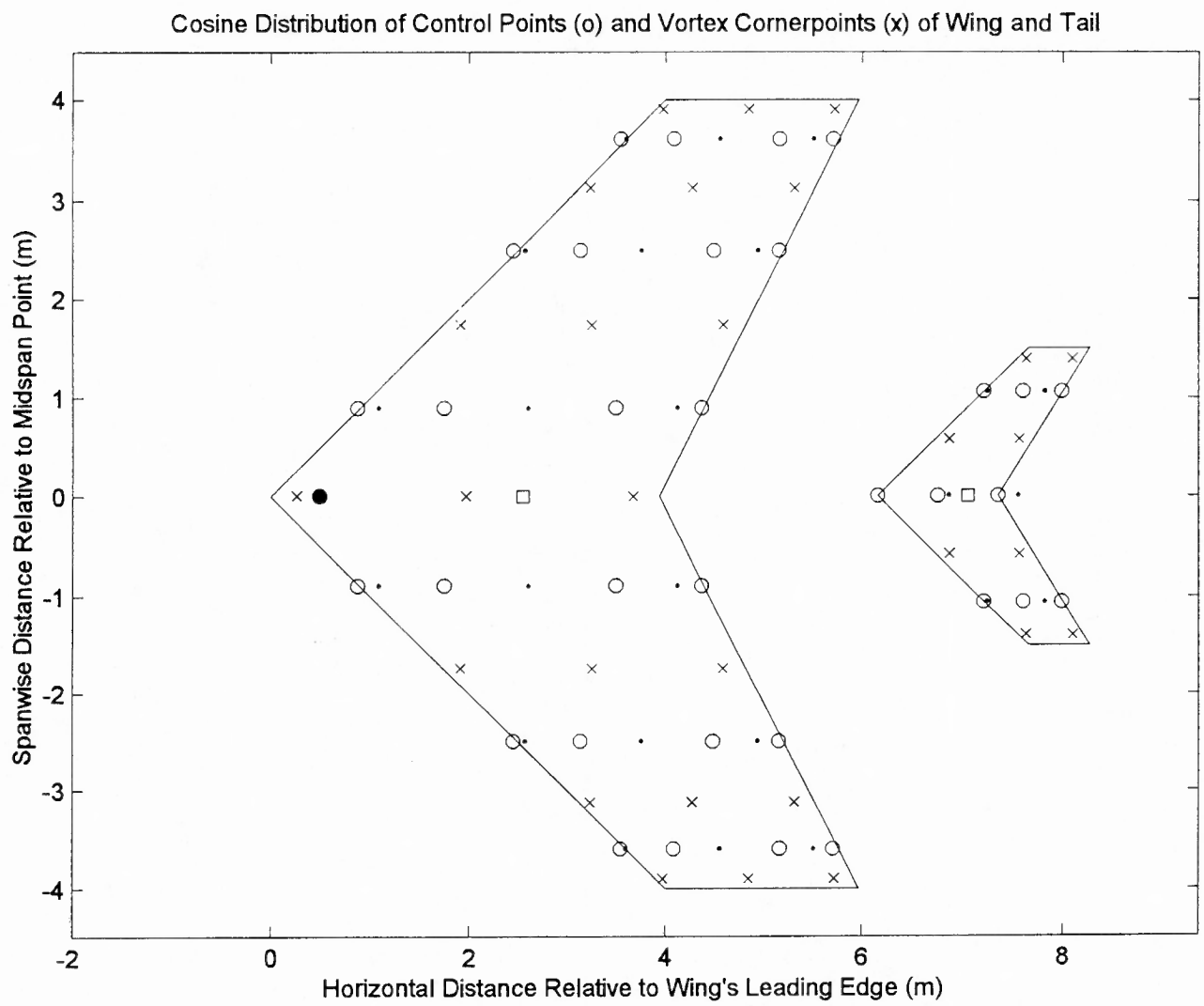


FIGURE R1 c)

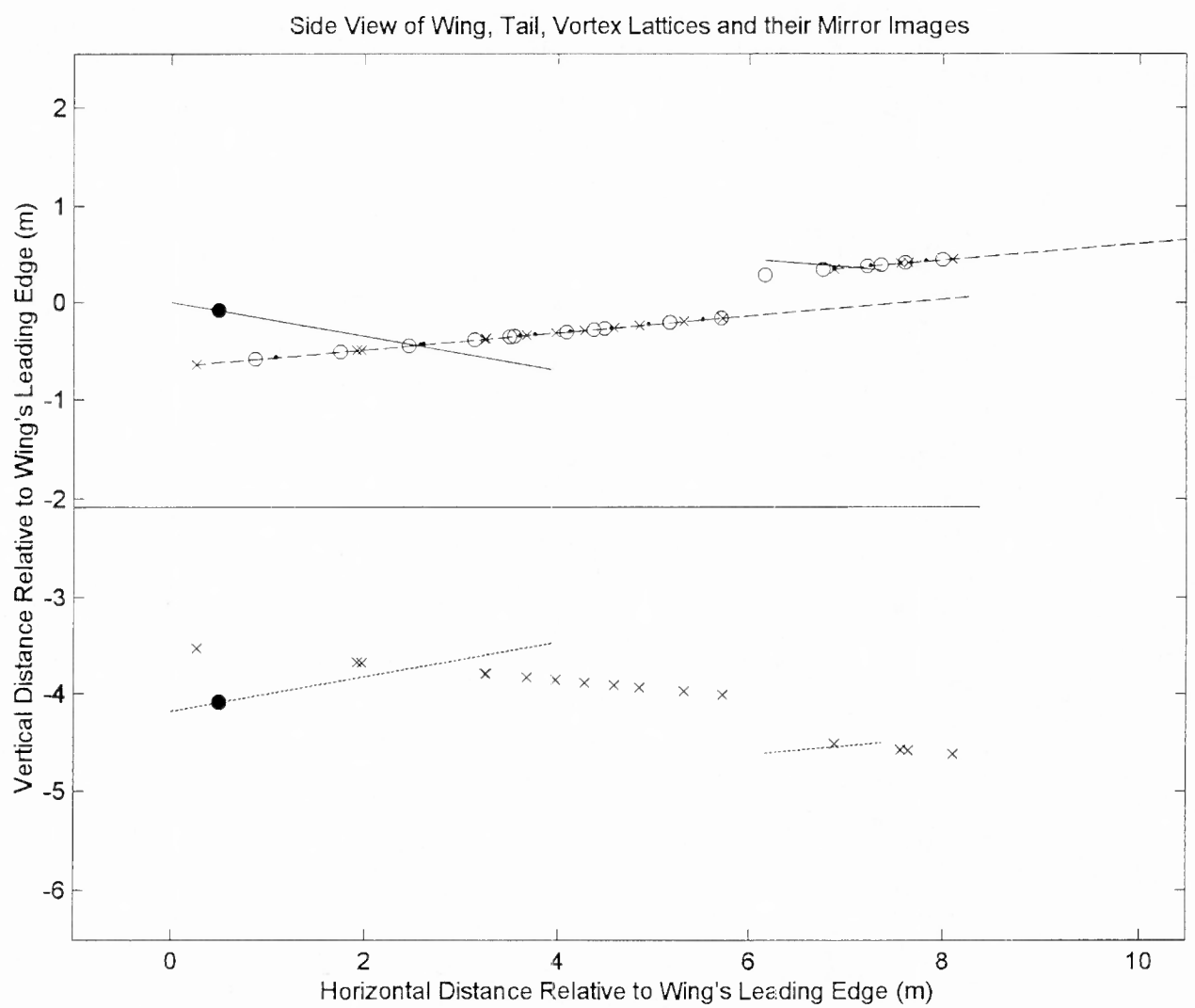


FIGURE R1 d)

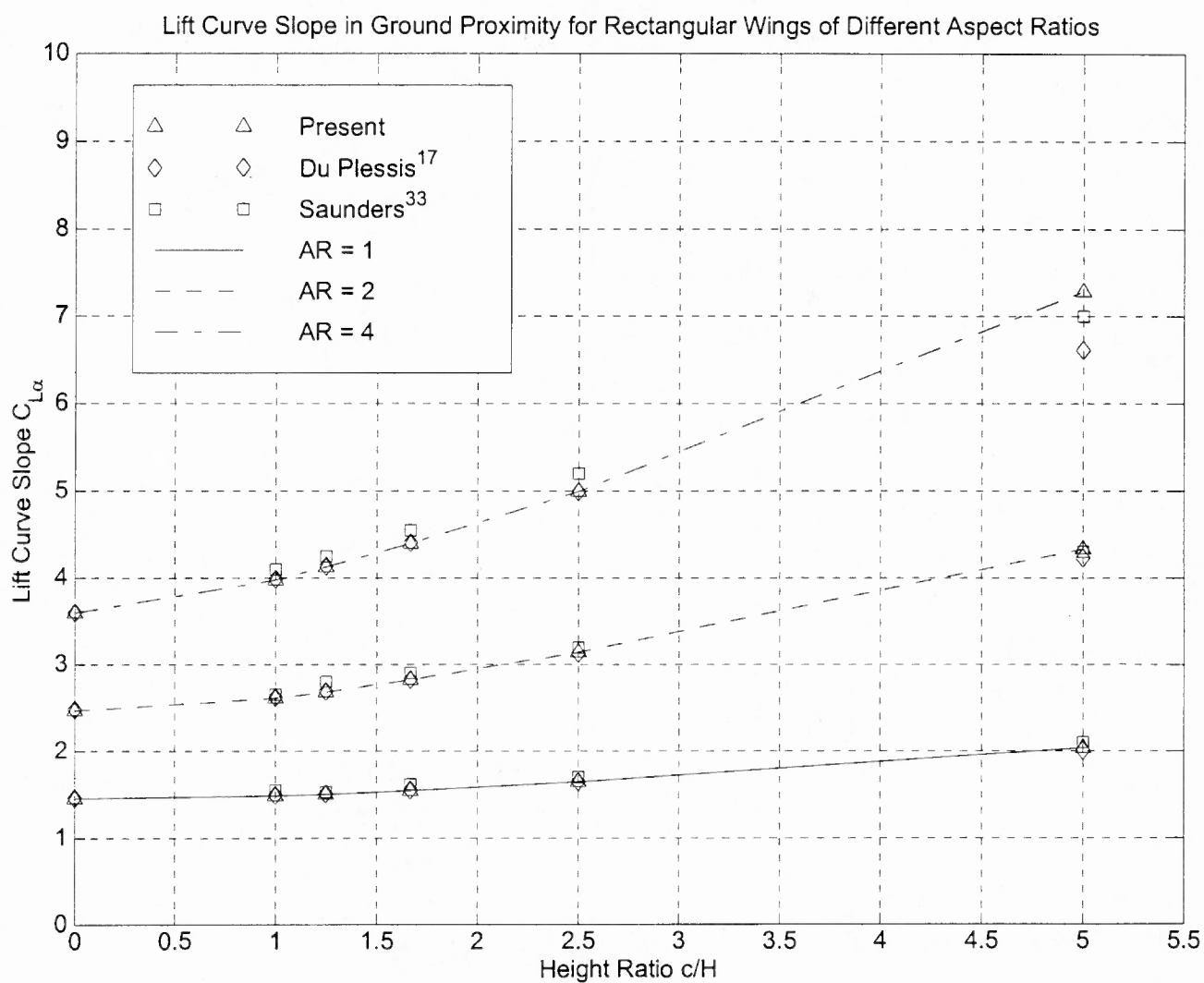


FIGURE R2 a)

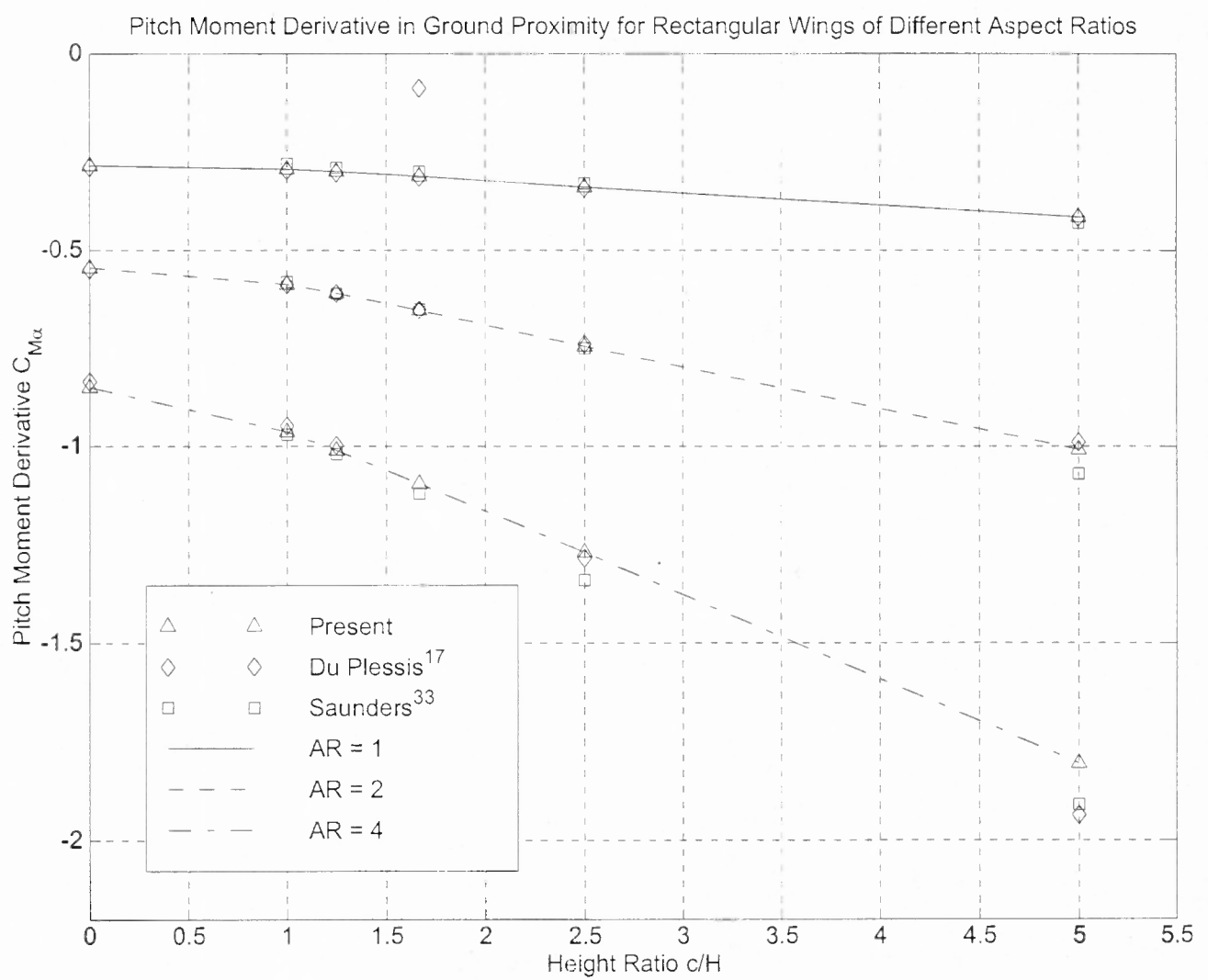


FIGURE R2 b)

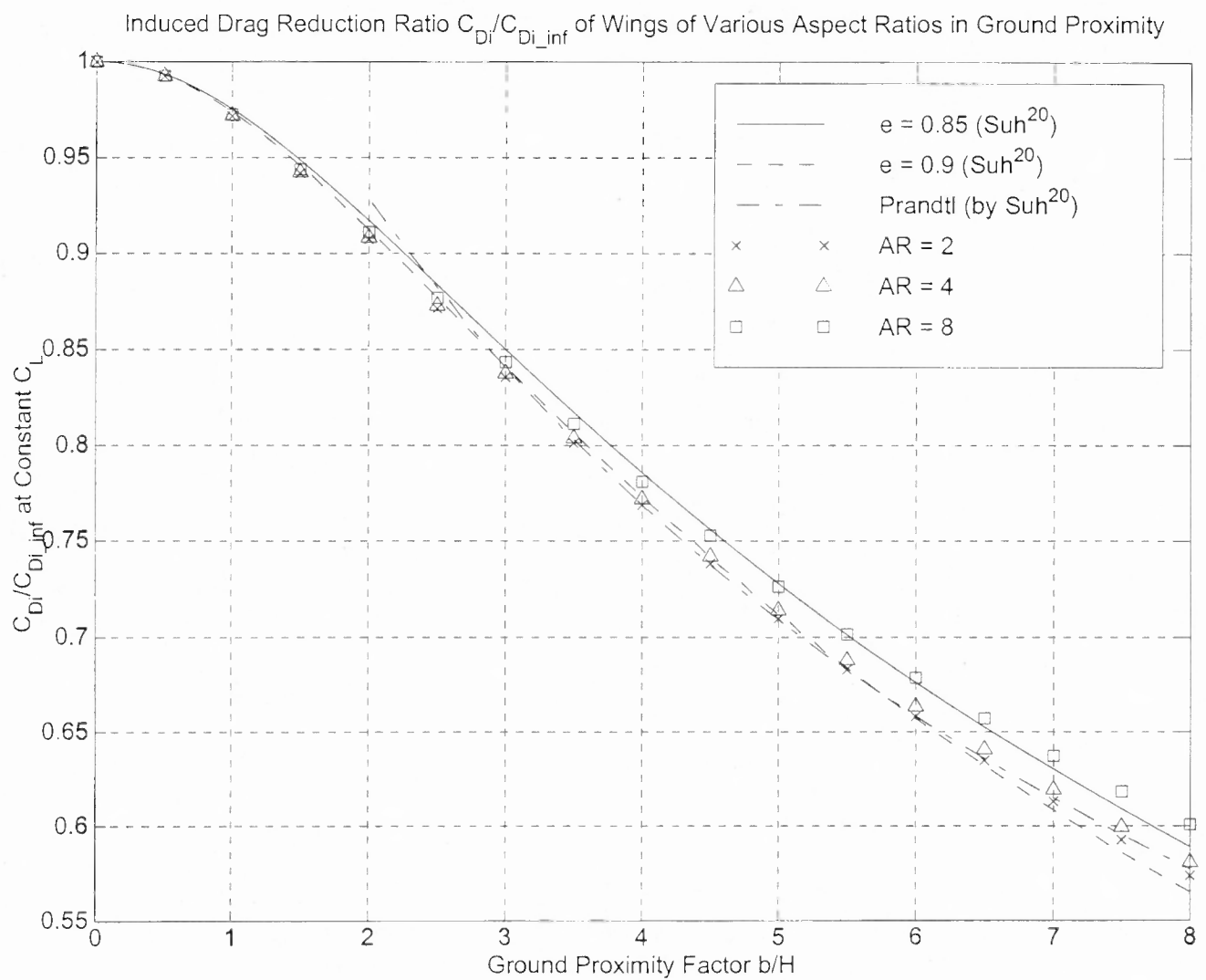


FIGURE R2 c)

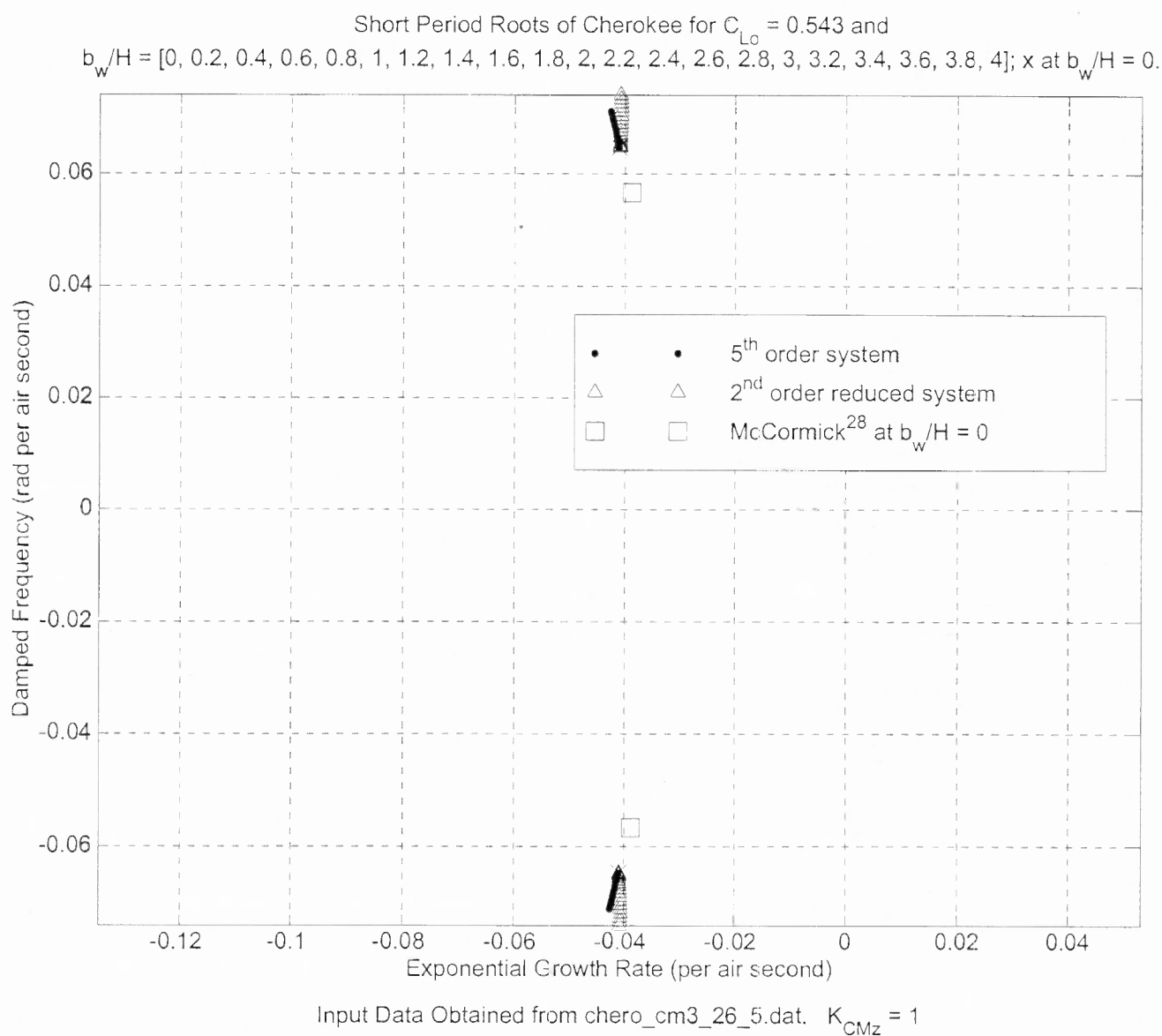
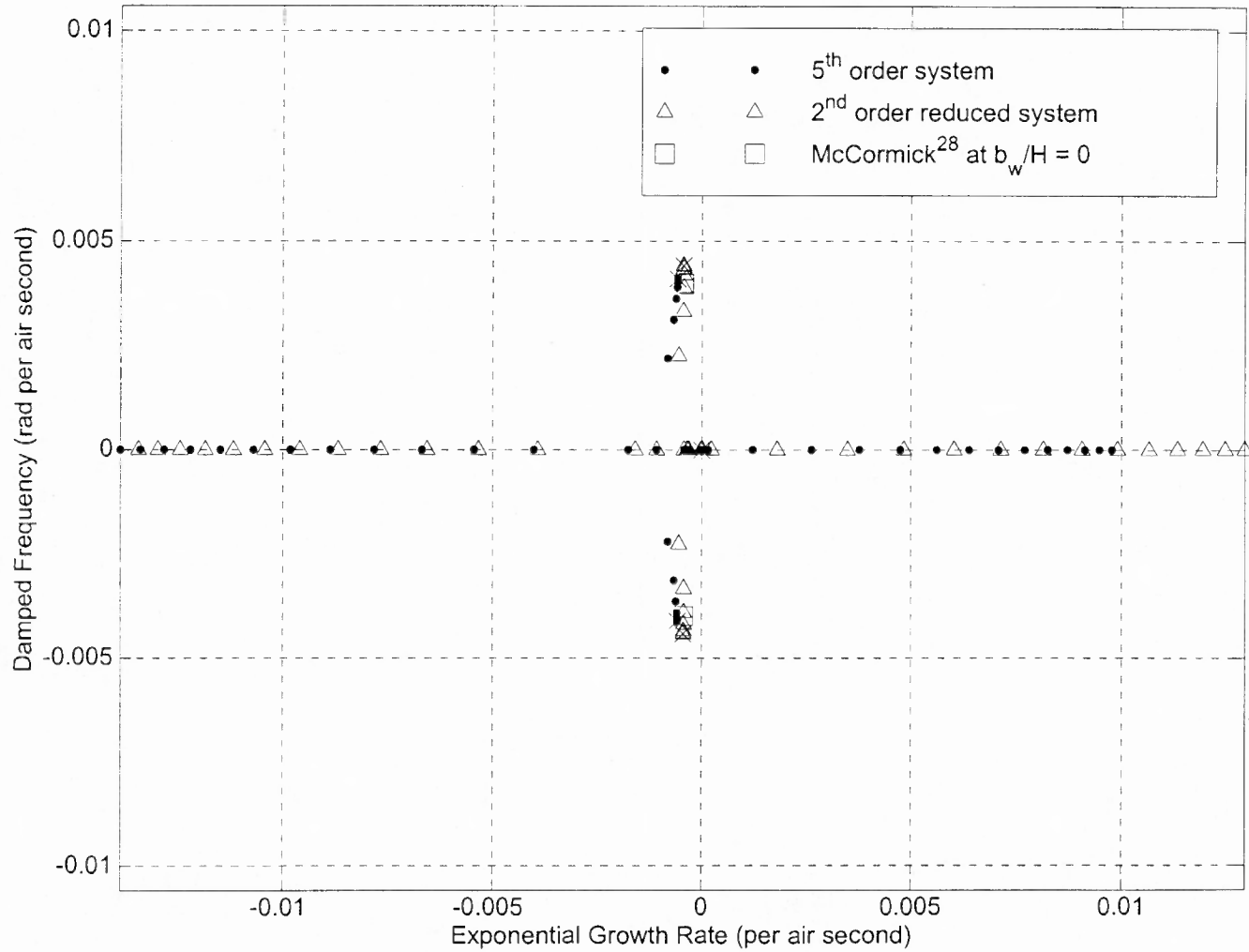


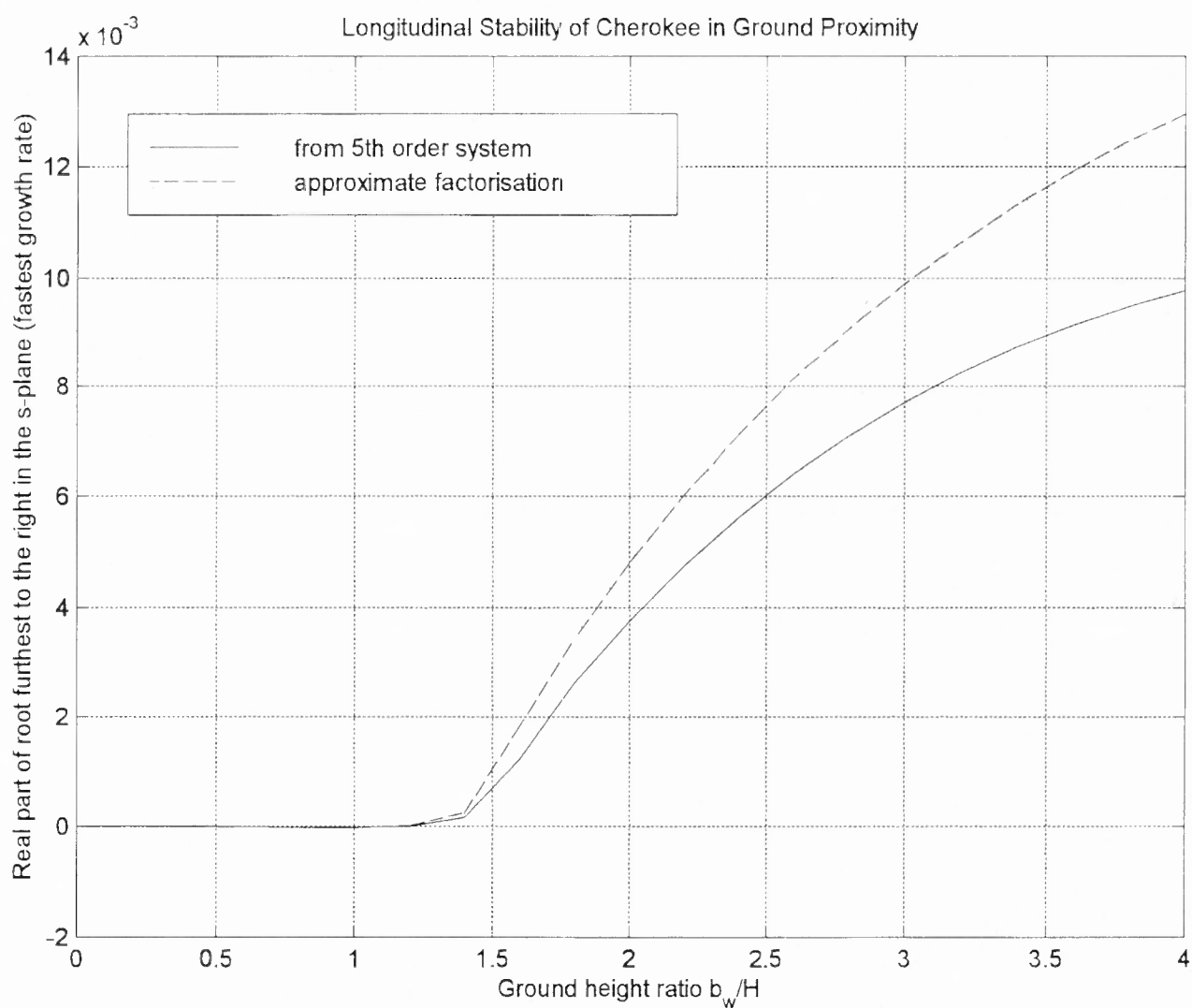
FIGURE R3 a)

Long Period Roots of Piper Cherokee for $C_{Lo} = 0.543$ and
 $b_w/H = [0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2, 2.2, 2.4, 2.6, 2.8, 3, 3.2, 3.4, 3.6, 3.8, 4]$; x at $b_w/H = 0$.



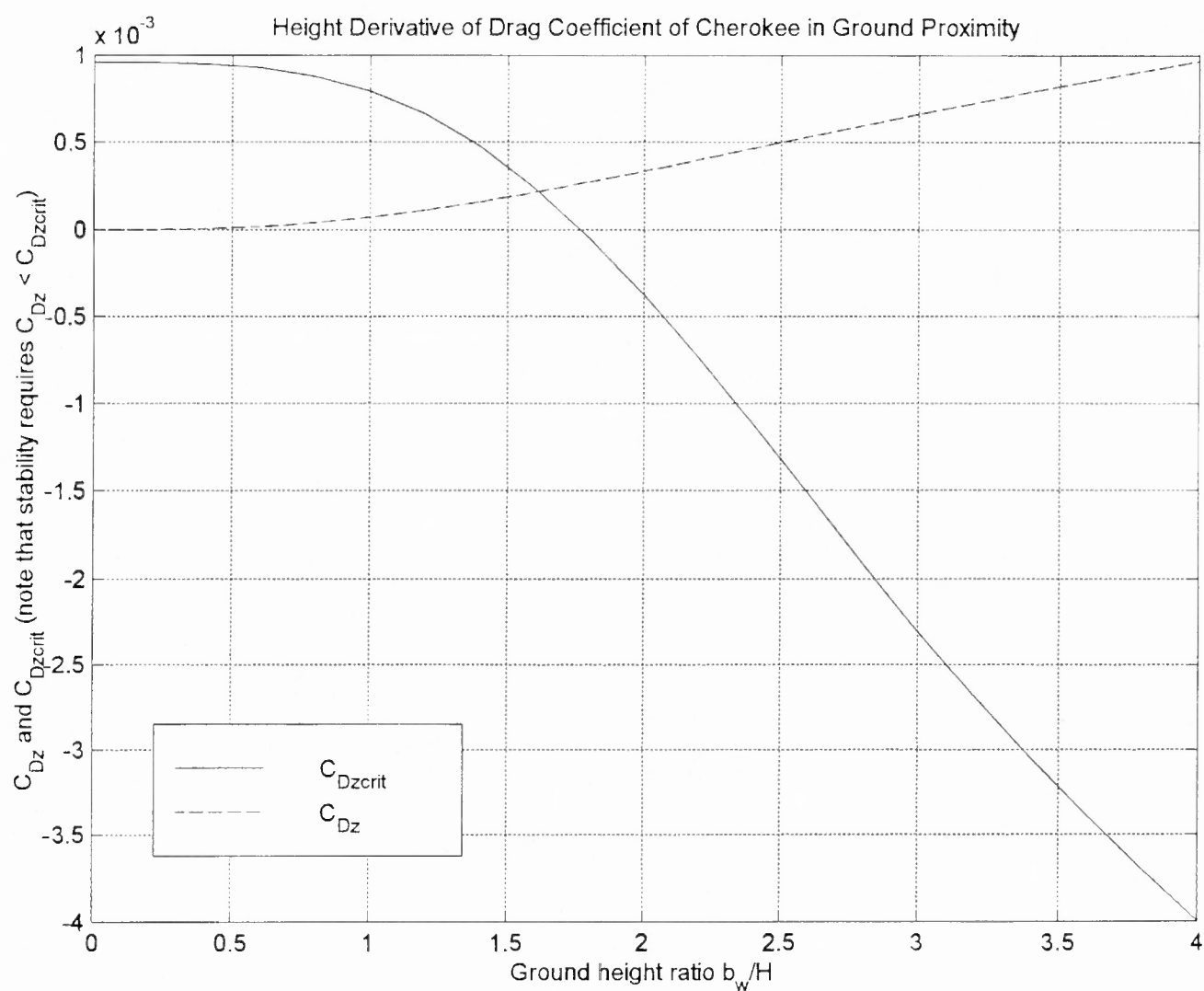
Input Data Obtained from chero_cm3_26_5.dat. $K_{CMz} = 1$

FIGURE R3 b)



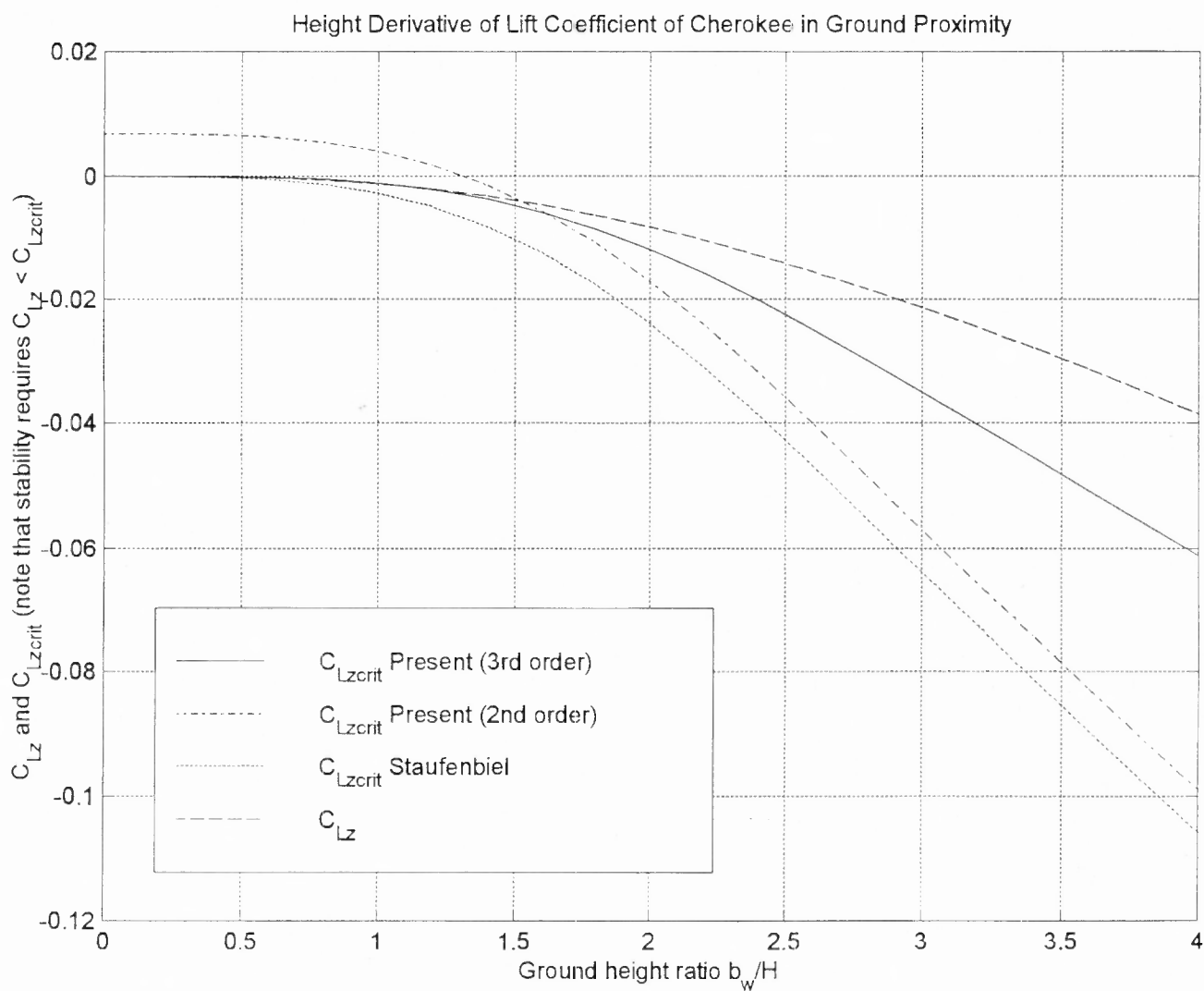
Input Data Obtained from chero_cm3_26_5.dat. $K_{CMz} = 1$

FIGURE R3 c)



Input Data Obtained from chero_cm3_26_5.dat. $K_{CMz} = 1$
 C_{Dzcrit} obtained from $B C - A D = 0$ in $A s^3 + B s^2 + C s + D = 0$ (with $A > 0$)

FIGURE R3 d)



Input Data Obtained from chero_cm3_26_5.dat. $K_{CMz} = 1$

C_{Lzcrit} (3rd order) obtained from $D = 0$ in $A s^3 + B s^2 + C s + D = 0$ (with $A > 0$)

C_{Lzcrit} (2nd order) obtained from $C = 0$ in $(A s^2 + B s + C) s = 0$ (with $A > 0$)

FIGURE R3 e)

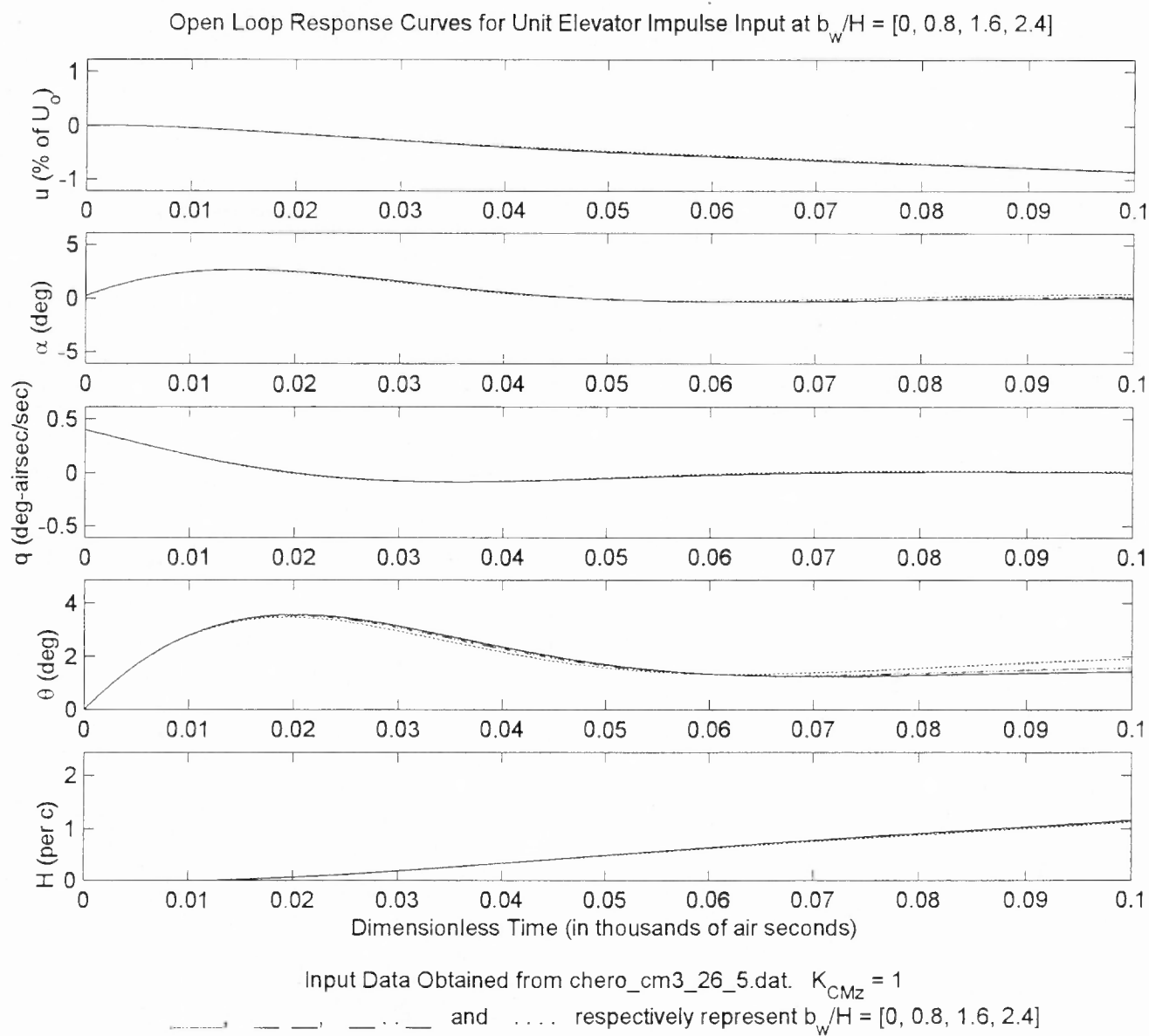


FIGURE R3 f)

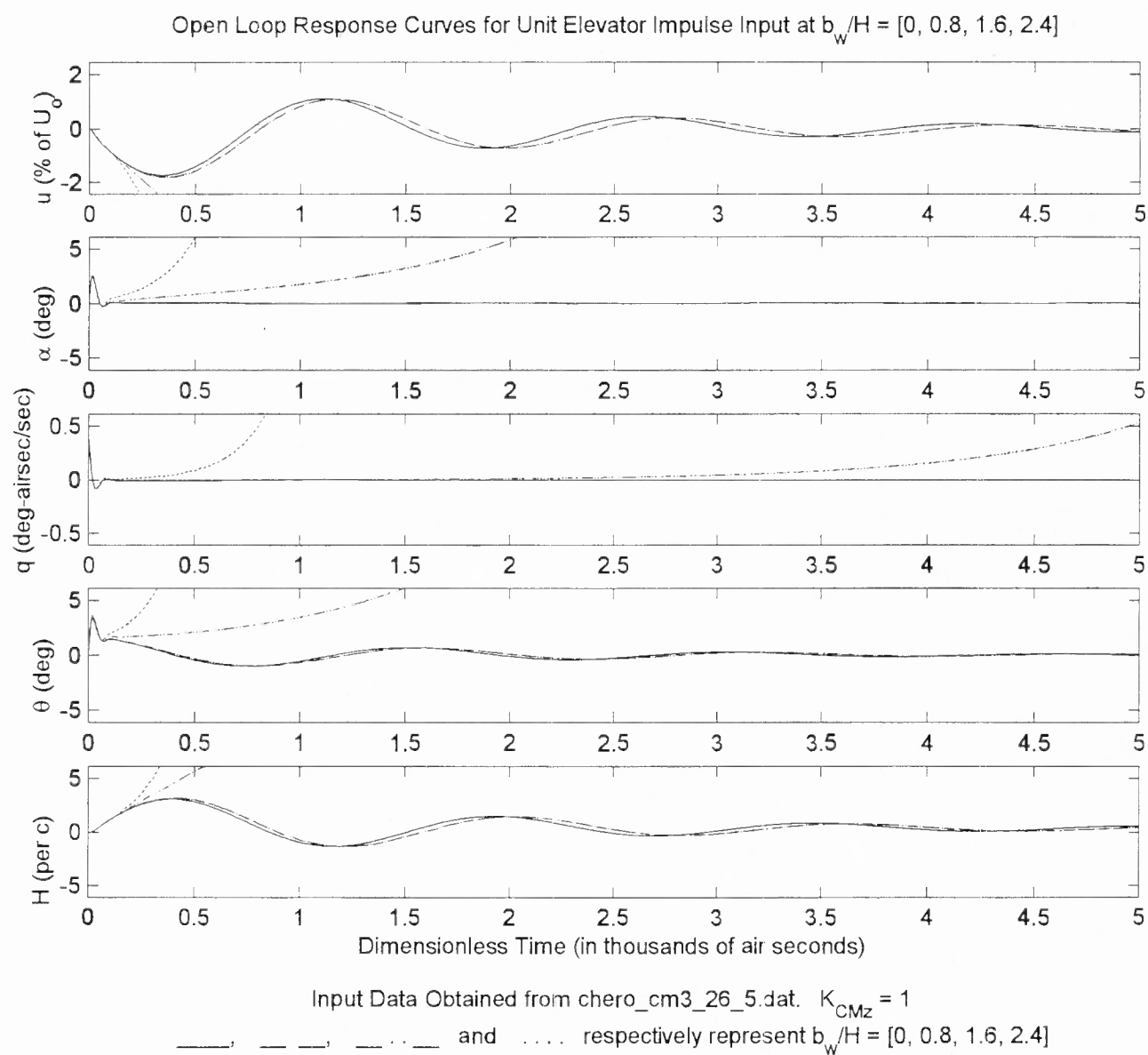


FIGURE R3 g)

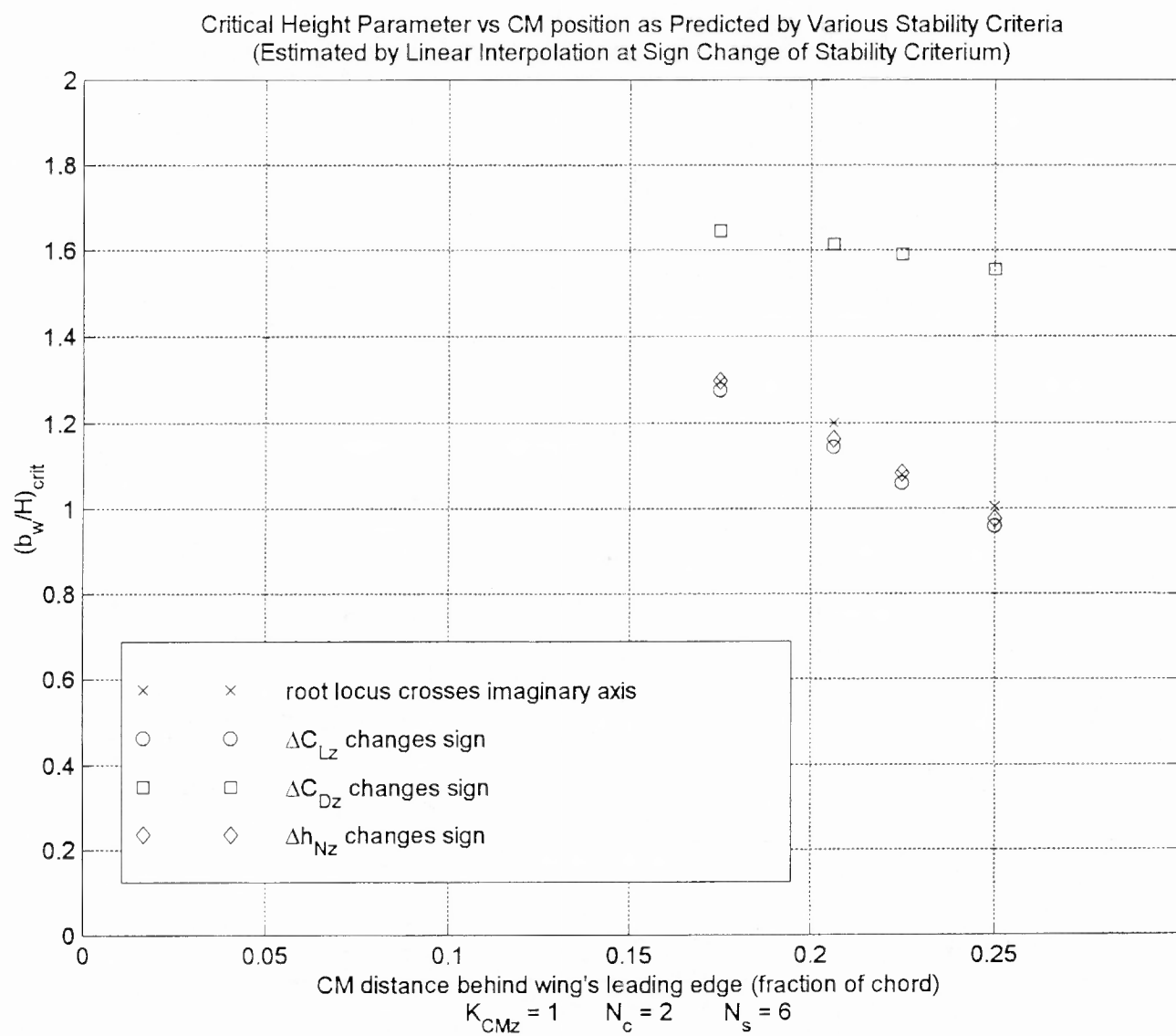


FIGURE R4 a)

Influence of CM Position on Longitudinal Stability of Cherokee

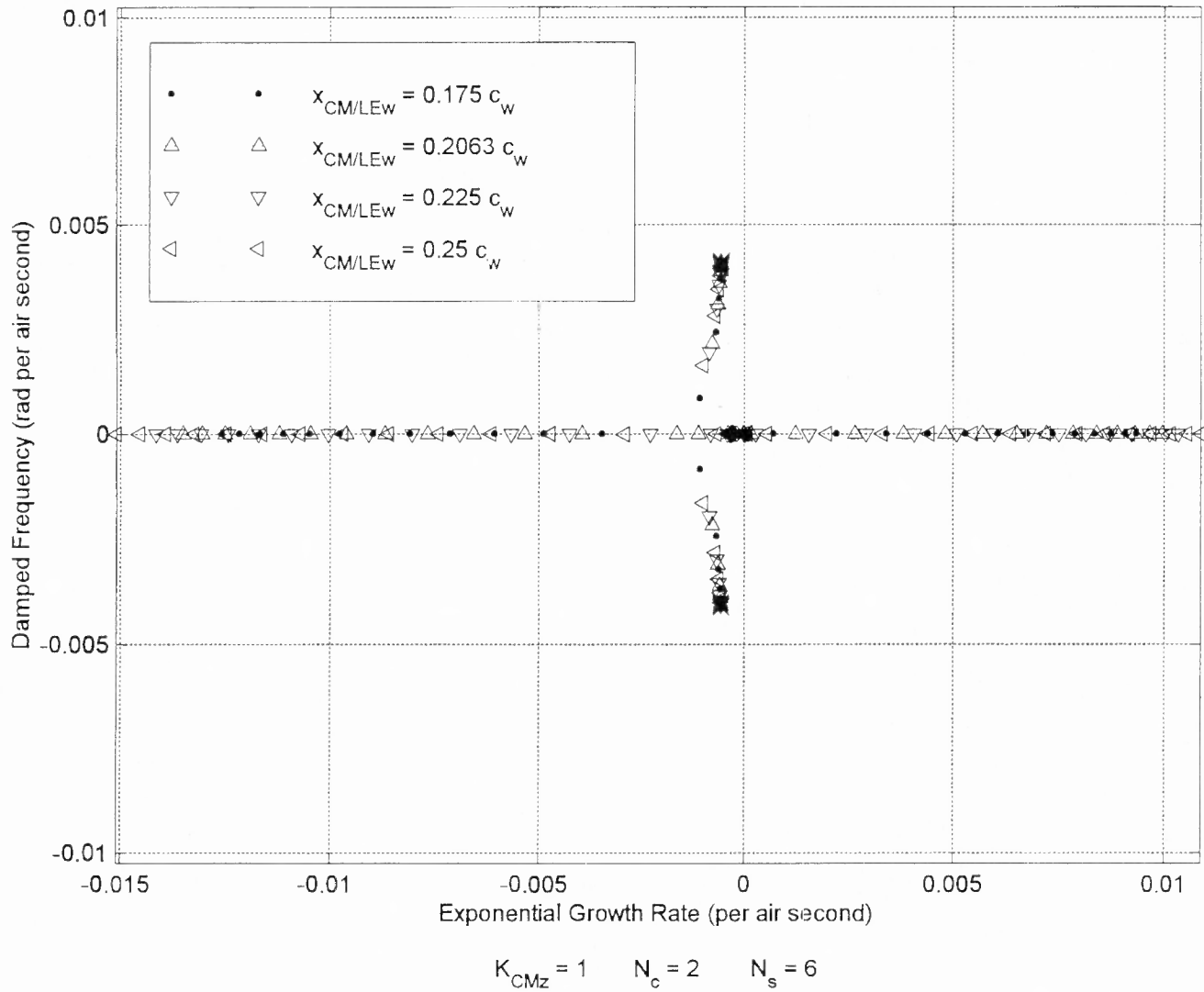


FIGURE R4 b)

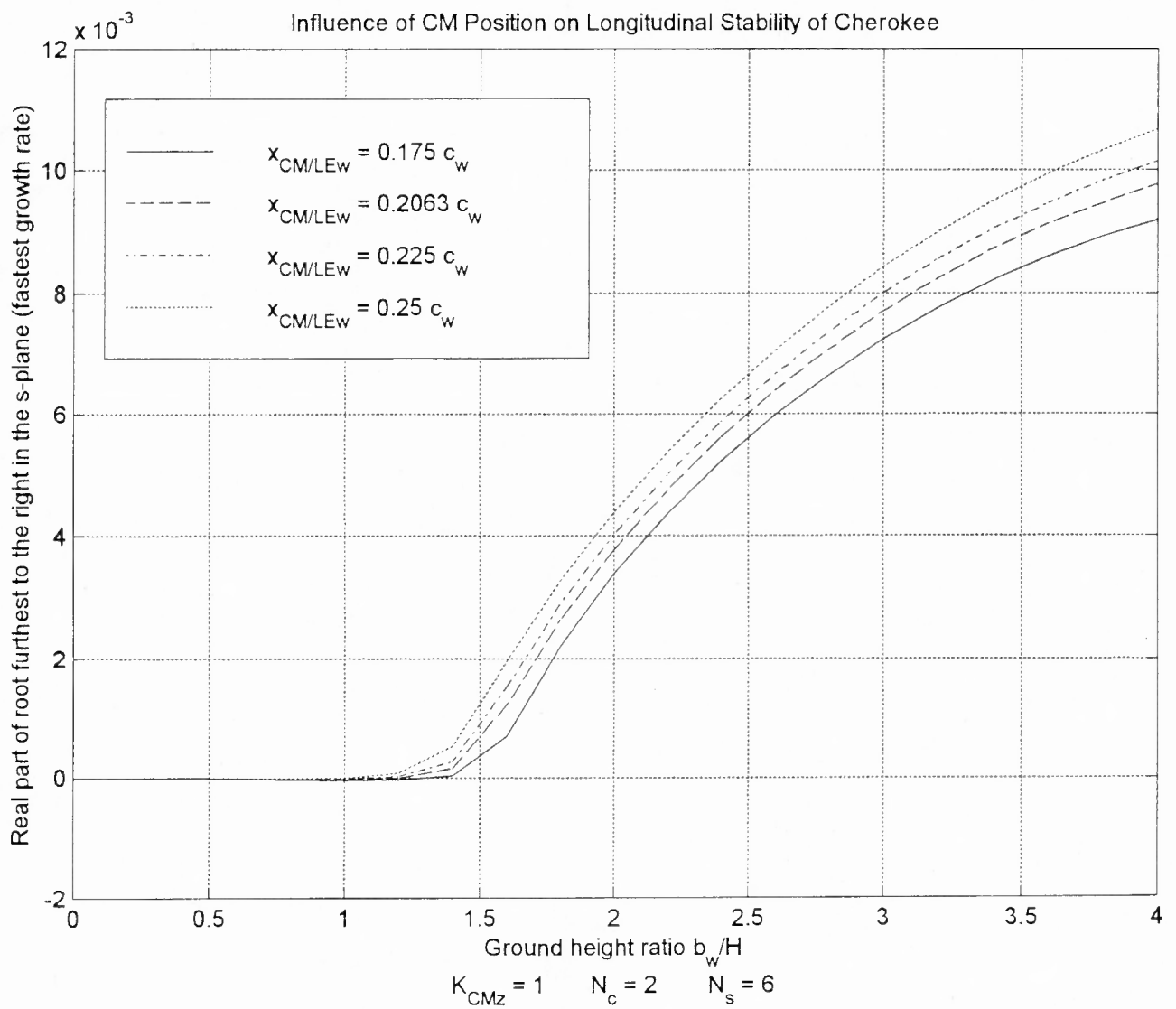


FIGURE R4 c)

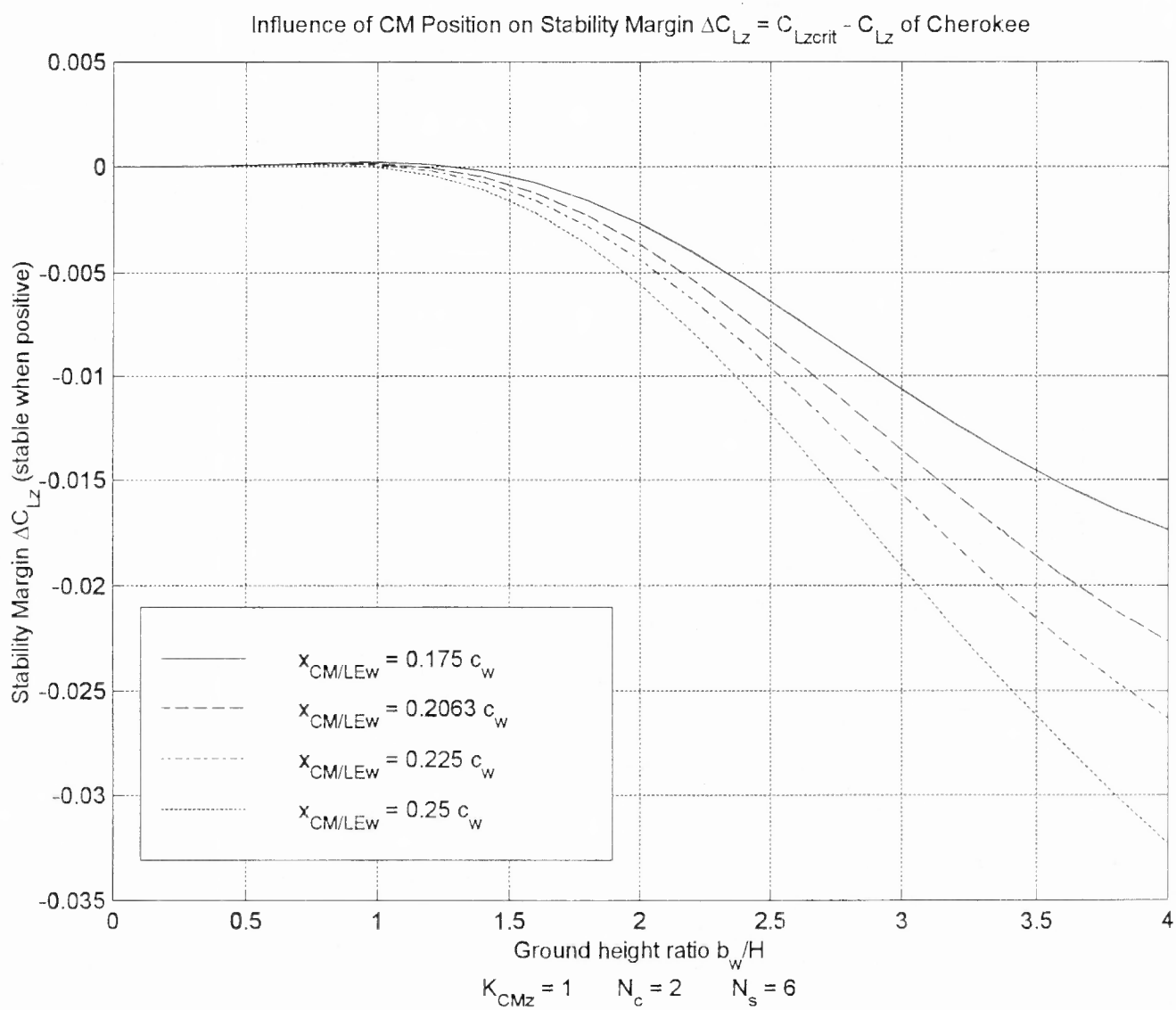


FIGURE R4 d)

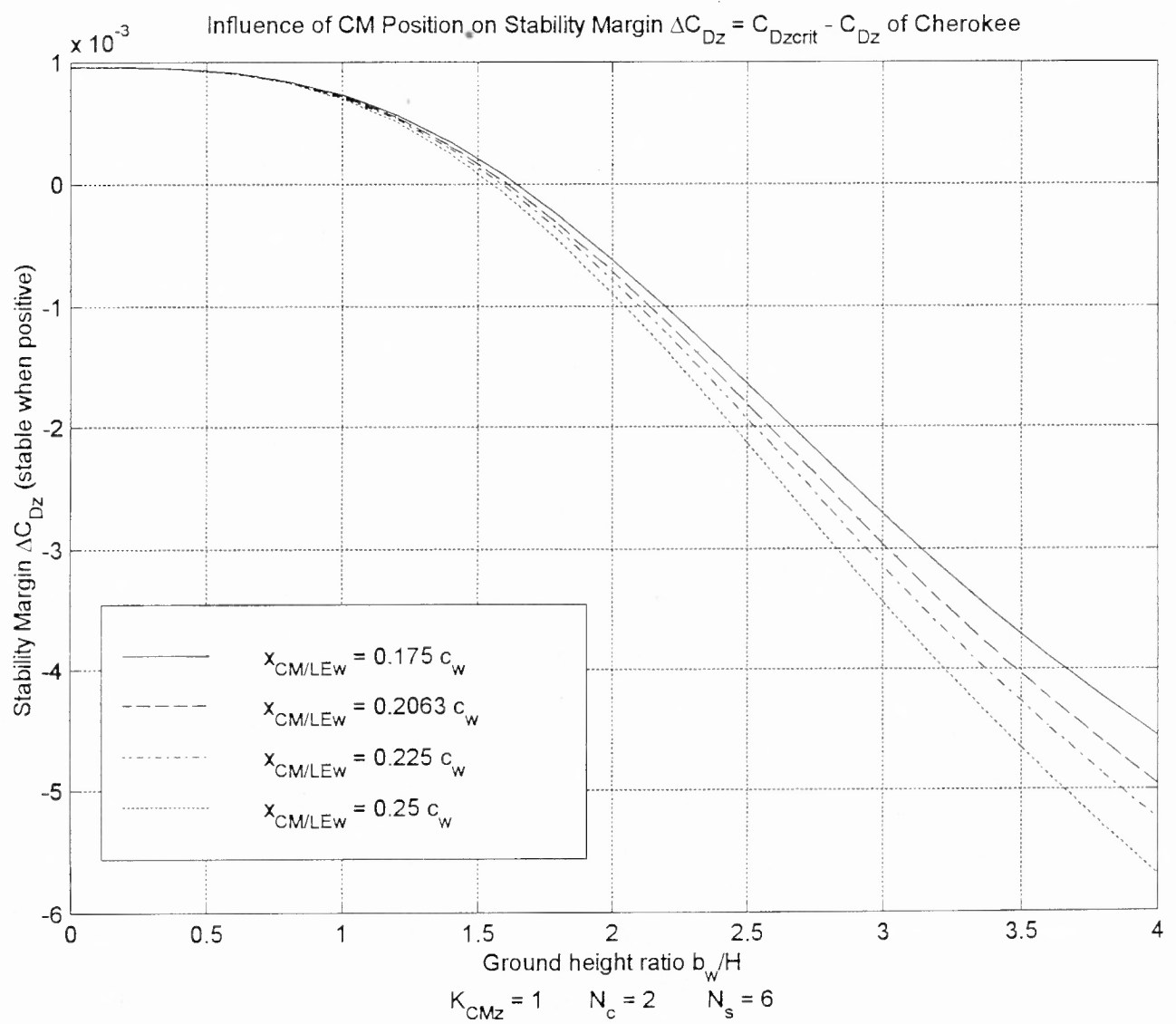


FIGURE R4 e)

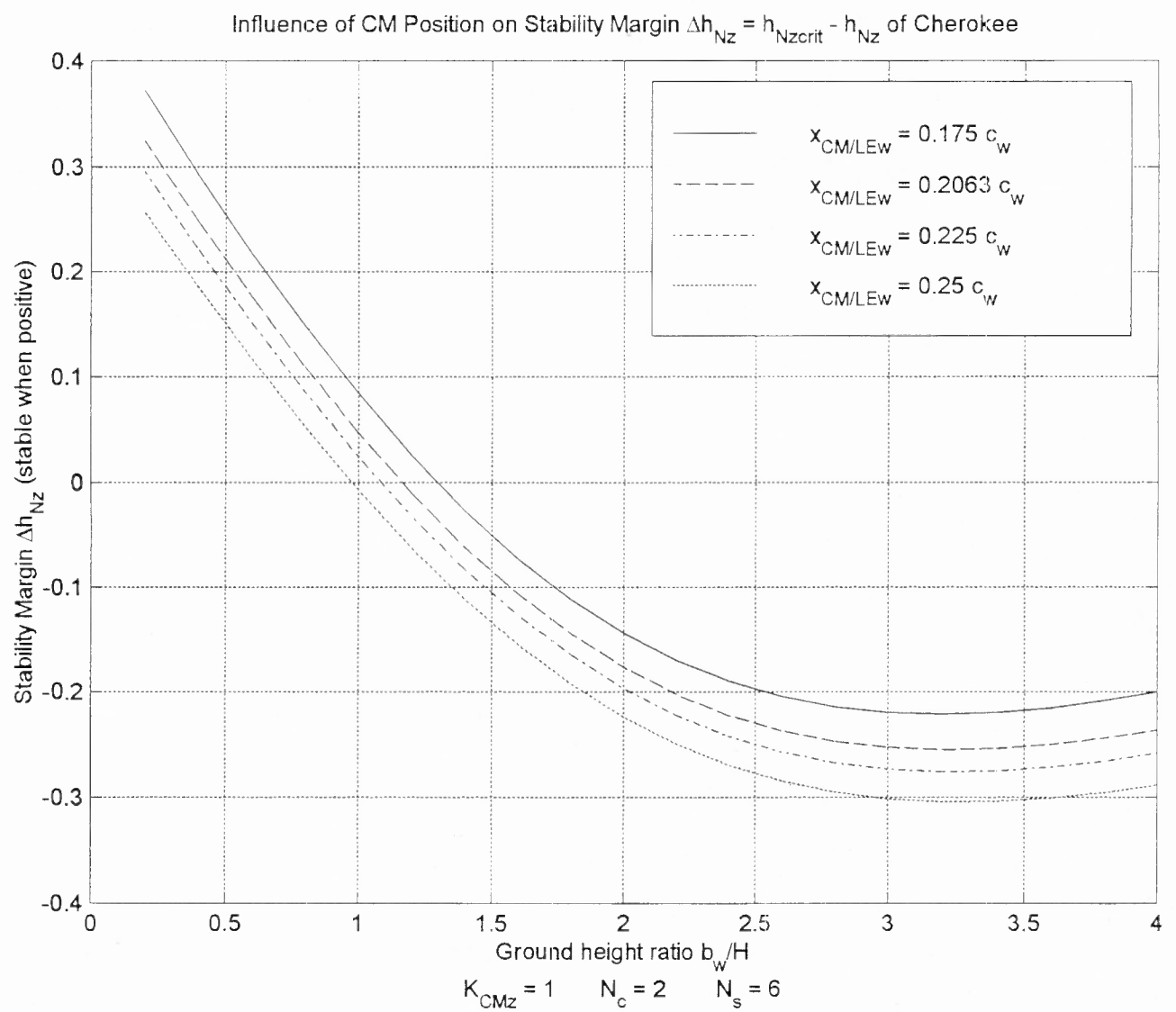


FIGURE R4 f)

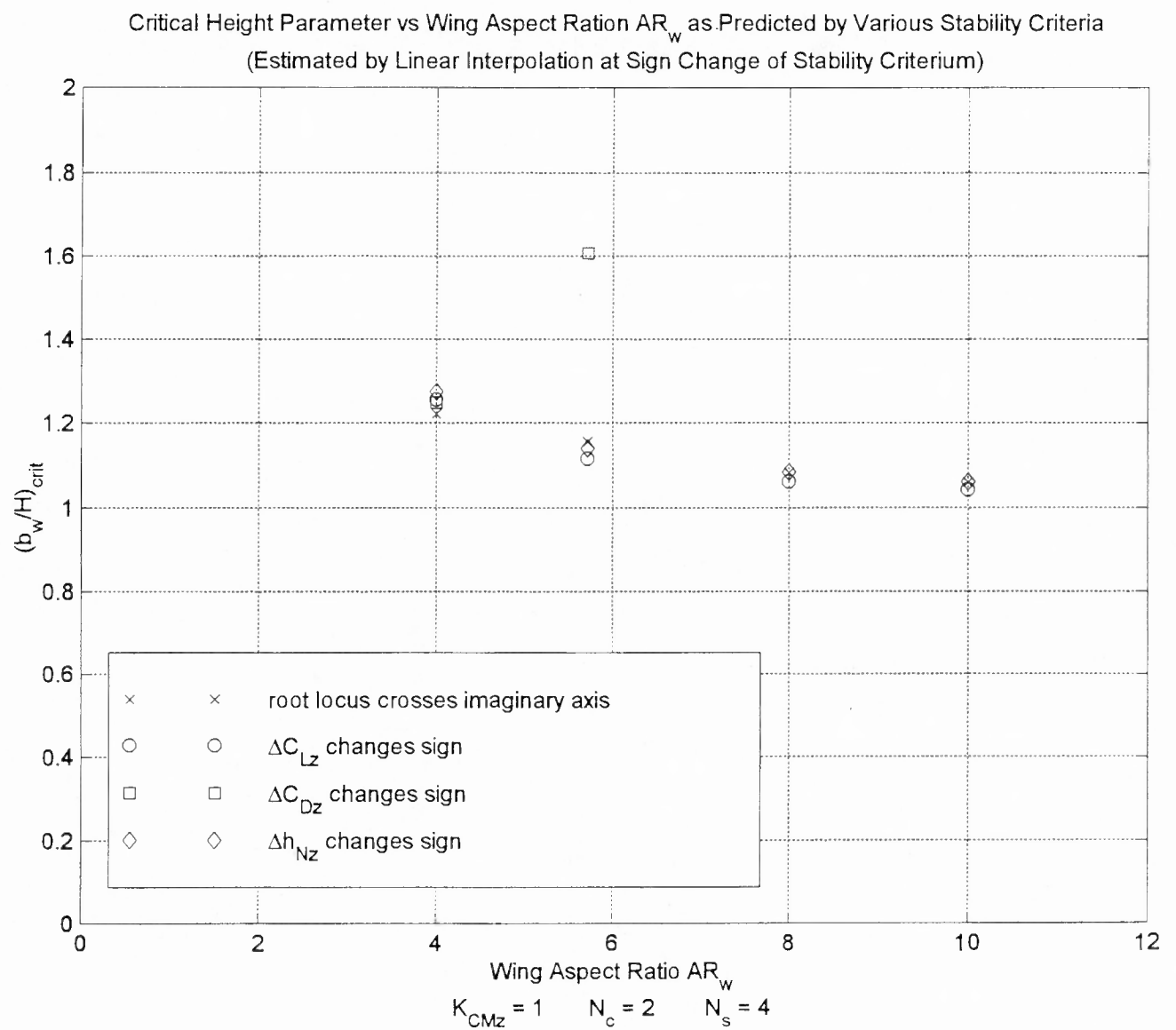


FIGURE R5 a)

Influence of Wing Aspect ratio AR_w on Longitudinal Stability of Cherokee

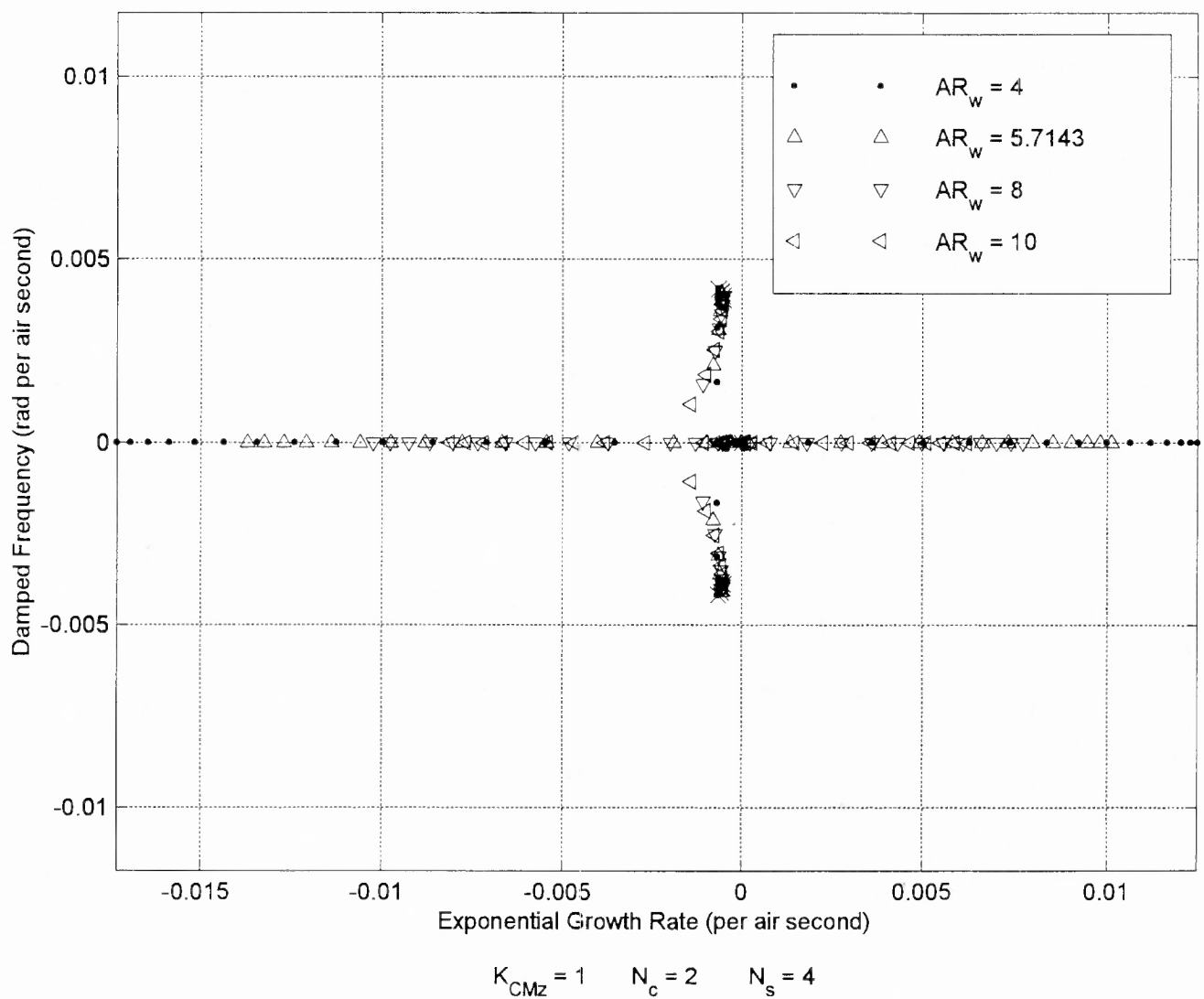


FIGURE R5 b)

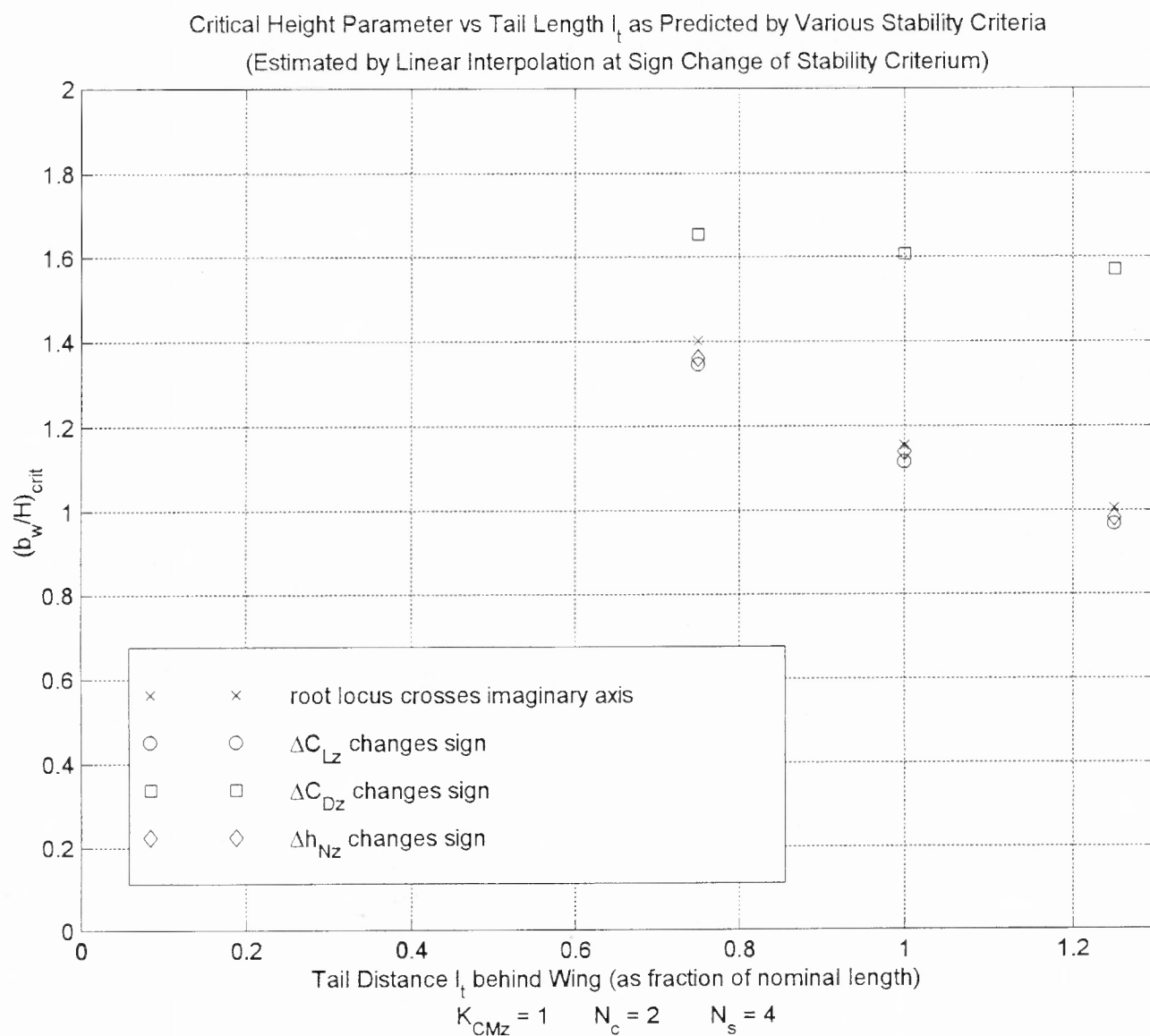


FIGURE R6 a)

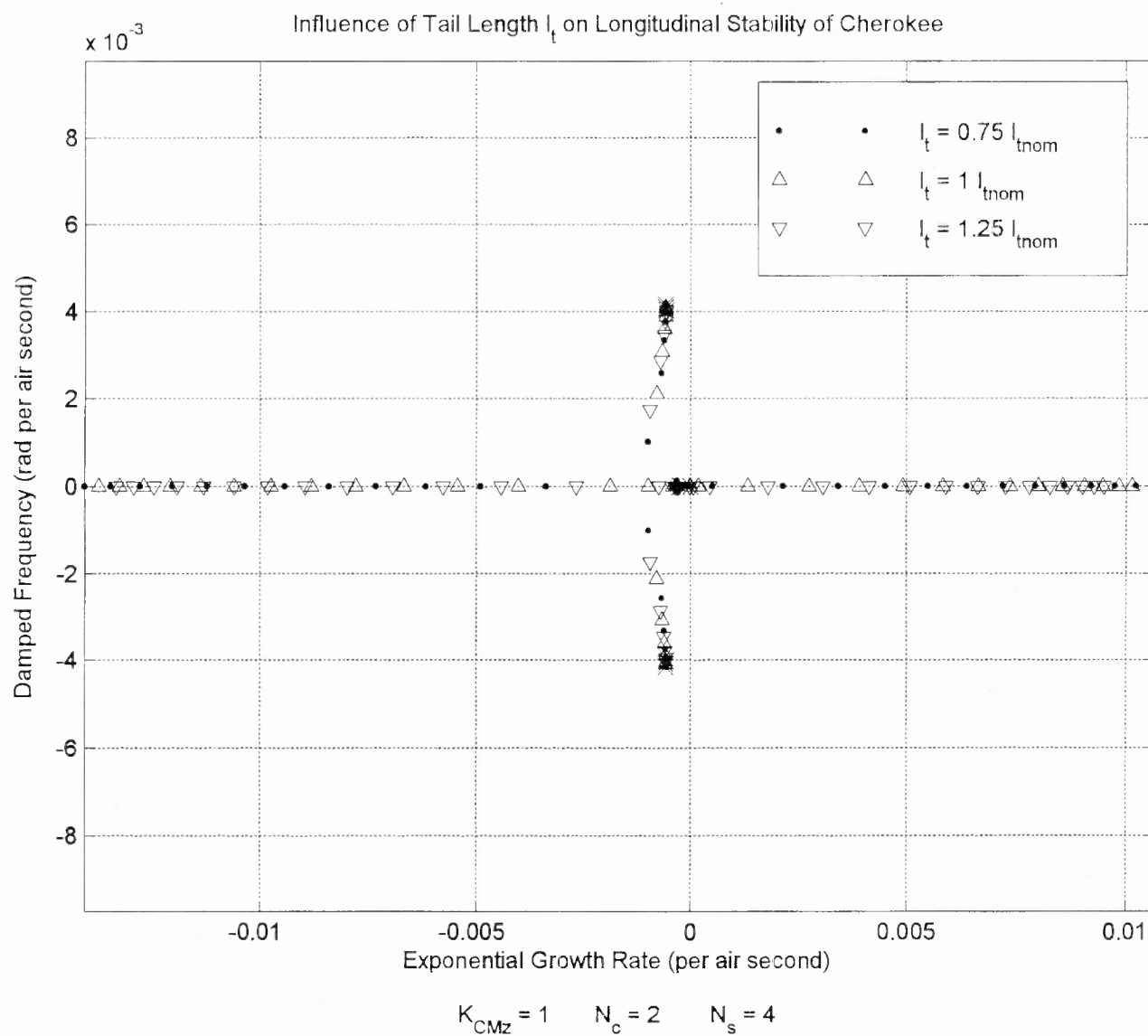


FIGURE R6 b)

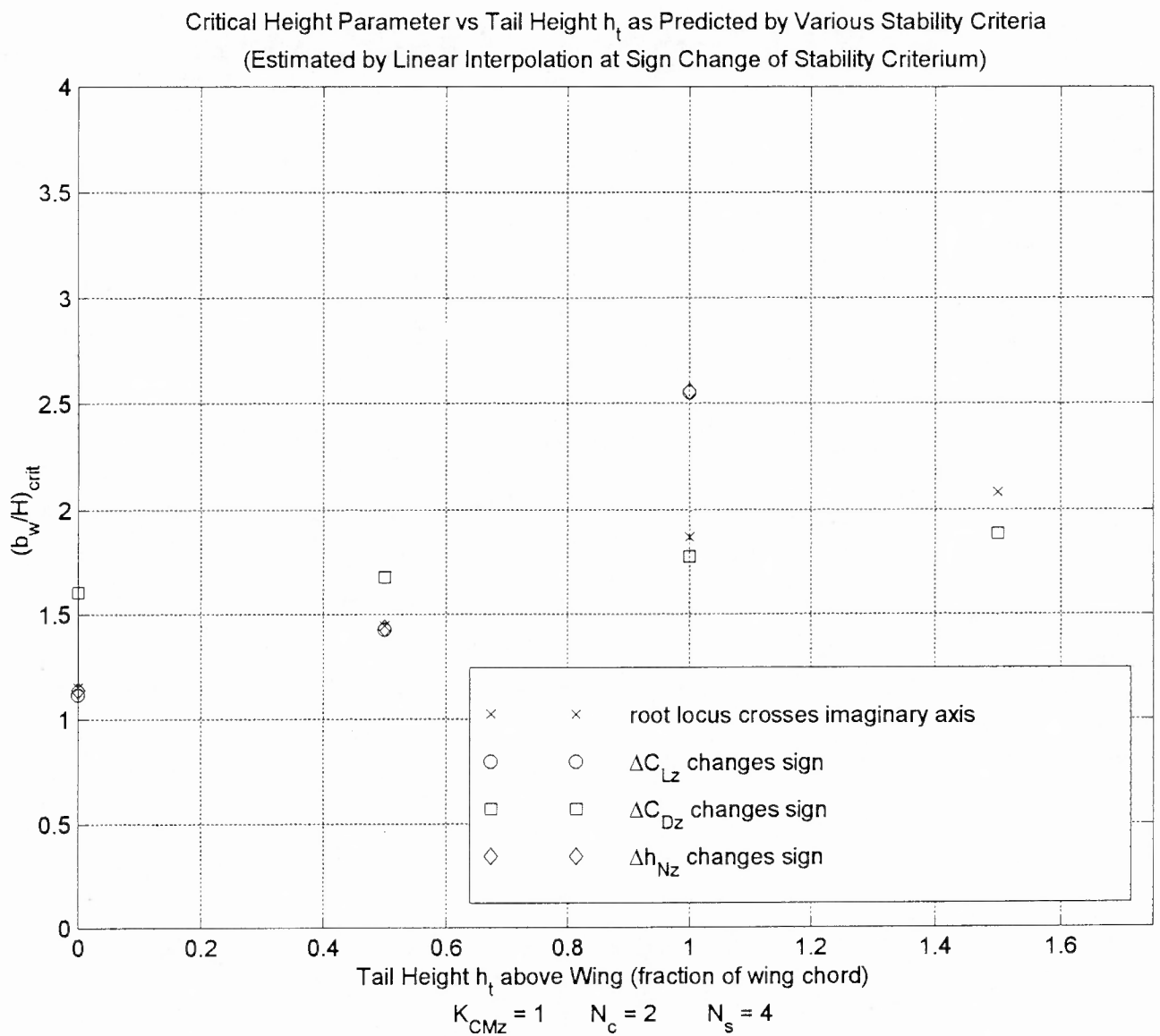


FIGURE R7 a)

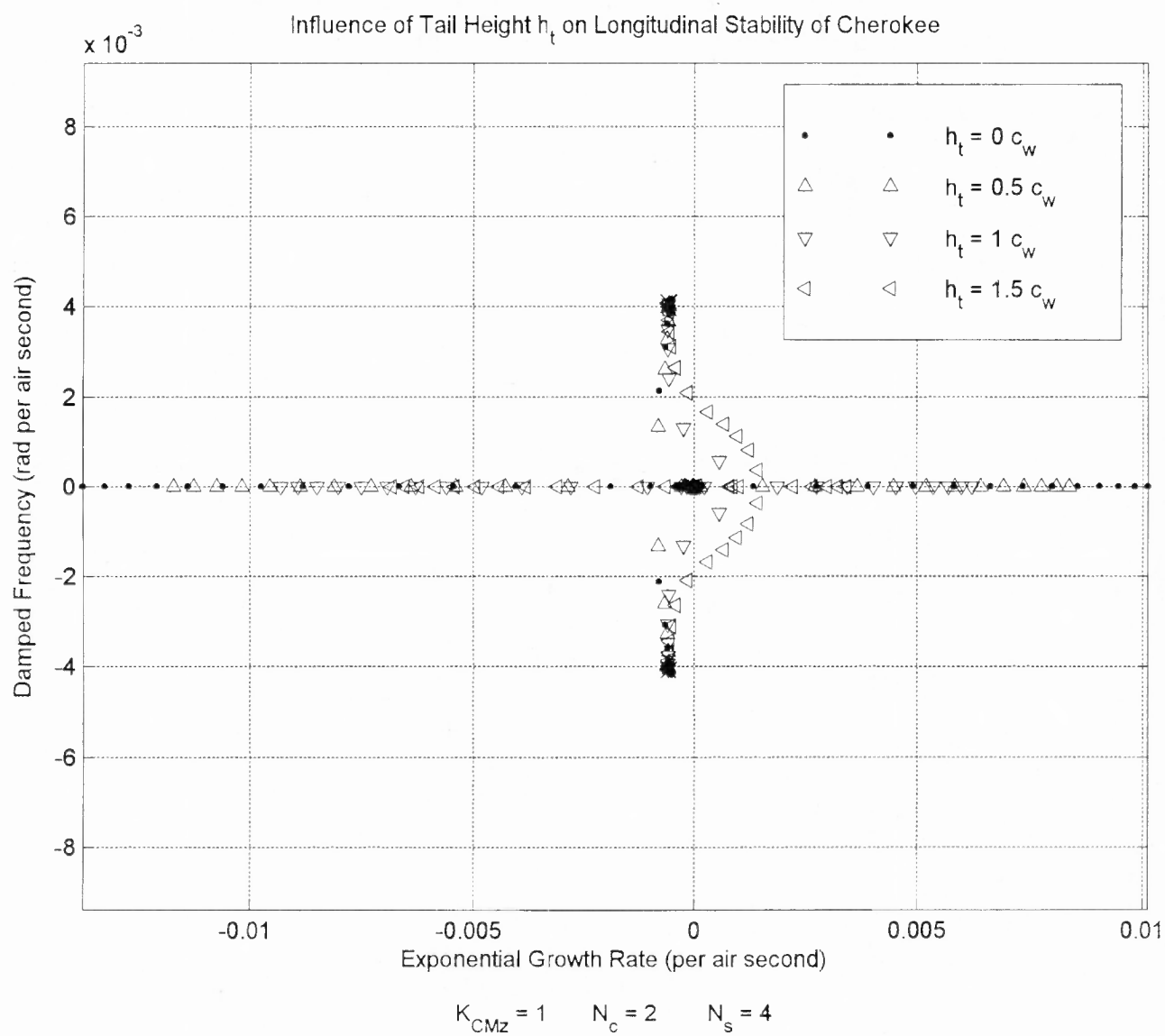


FIGURE R7 b)

Critical Height Parameter vs Load Factor C_w as Predicted by Various Stability Criteria
 (Estimated by Linear Interpolation at Sign Change of Stability Criterion)

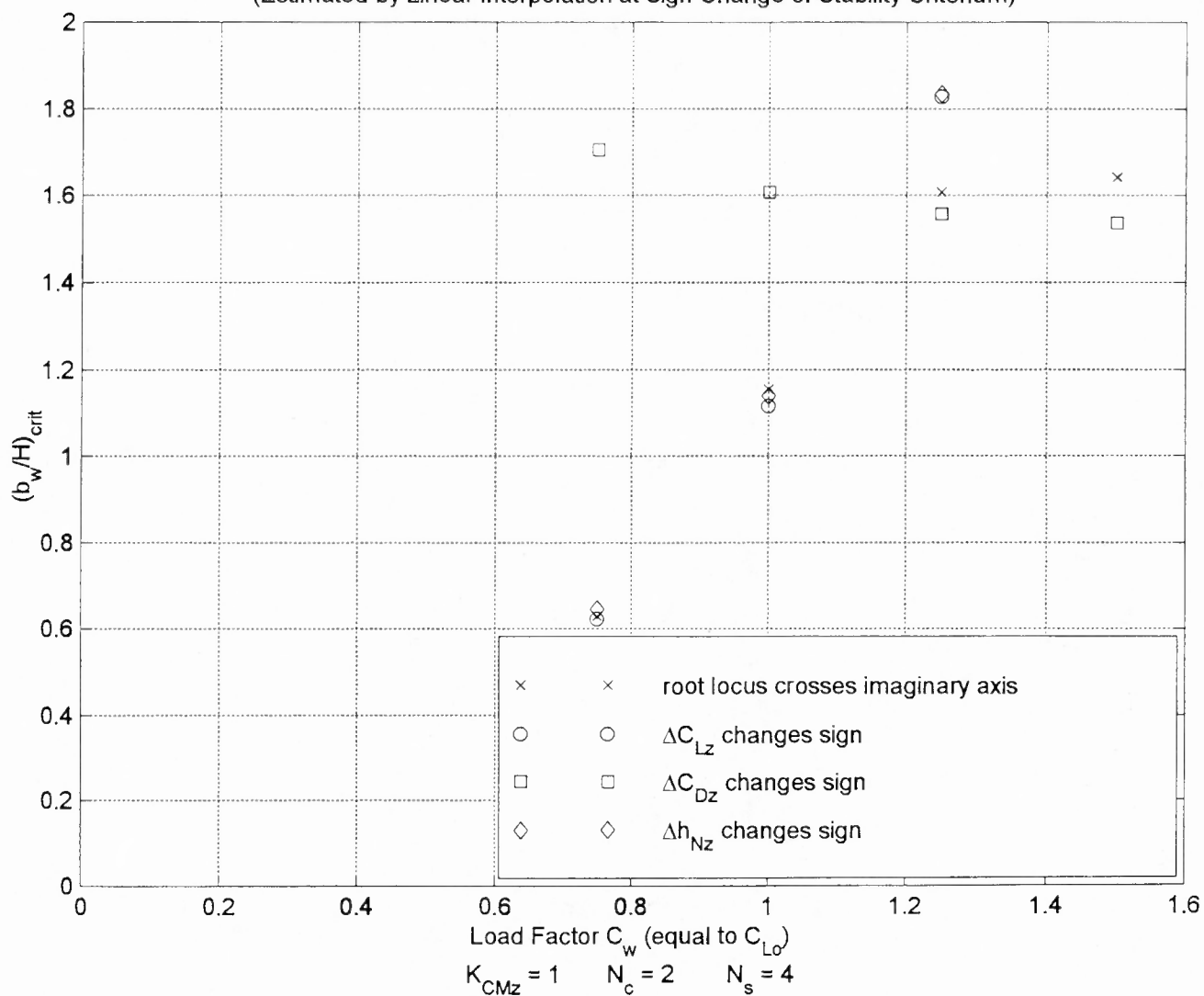


FIGURE R8 a)

Influence of Load Factor C_w on Longitudinal Stability of Cherokee

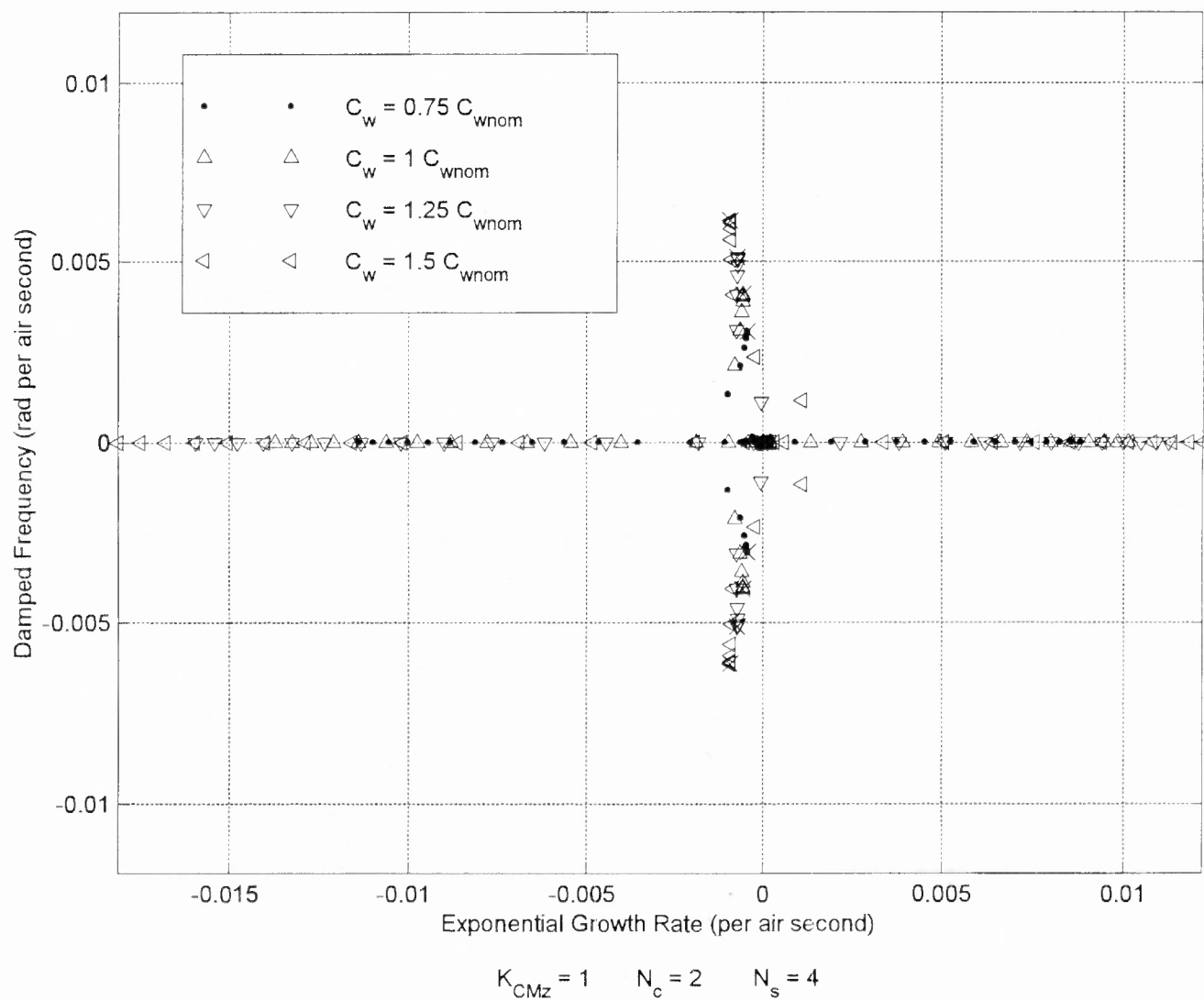


FIGURE R8 b)

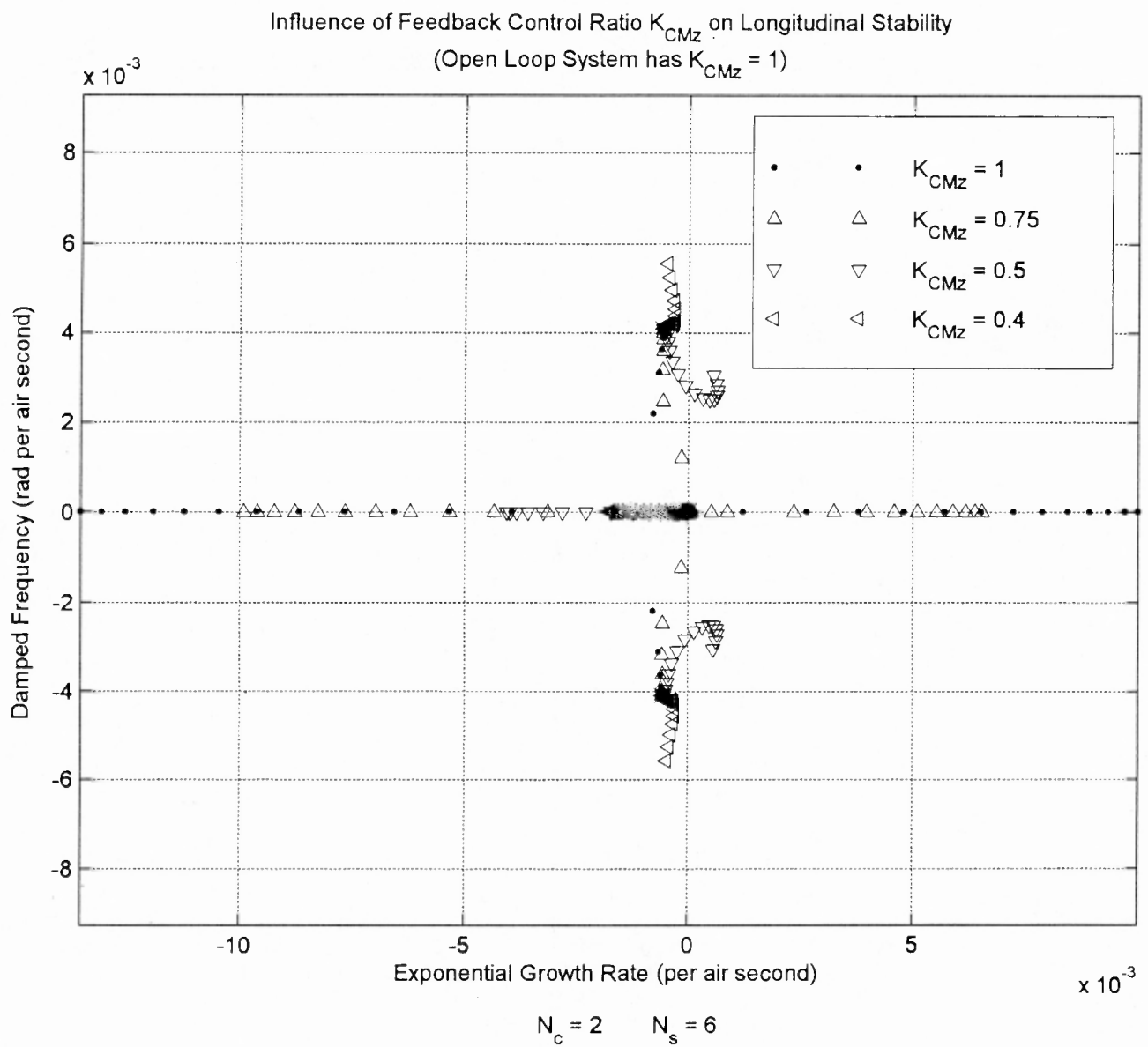
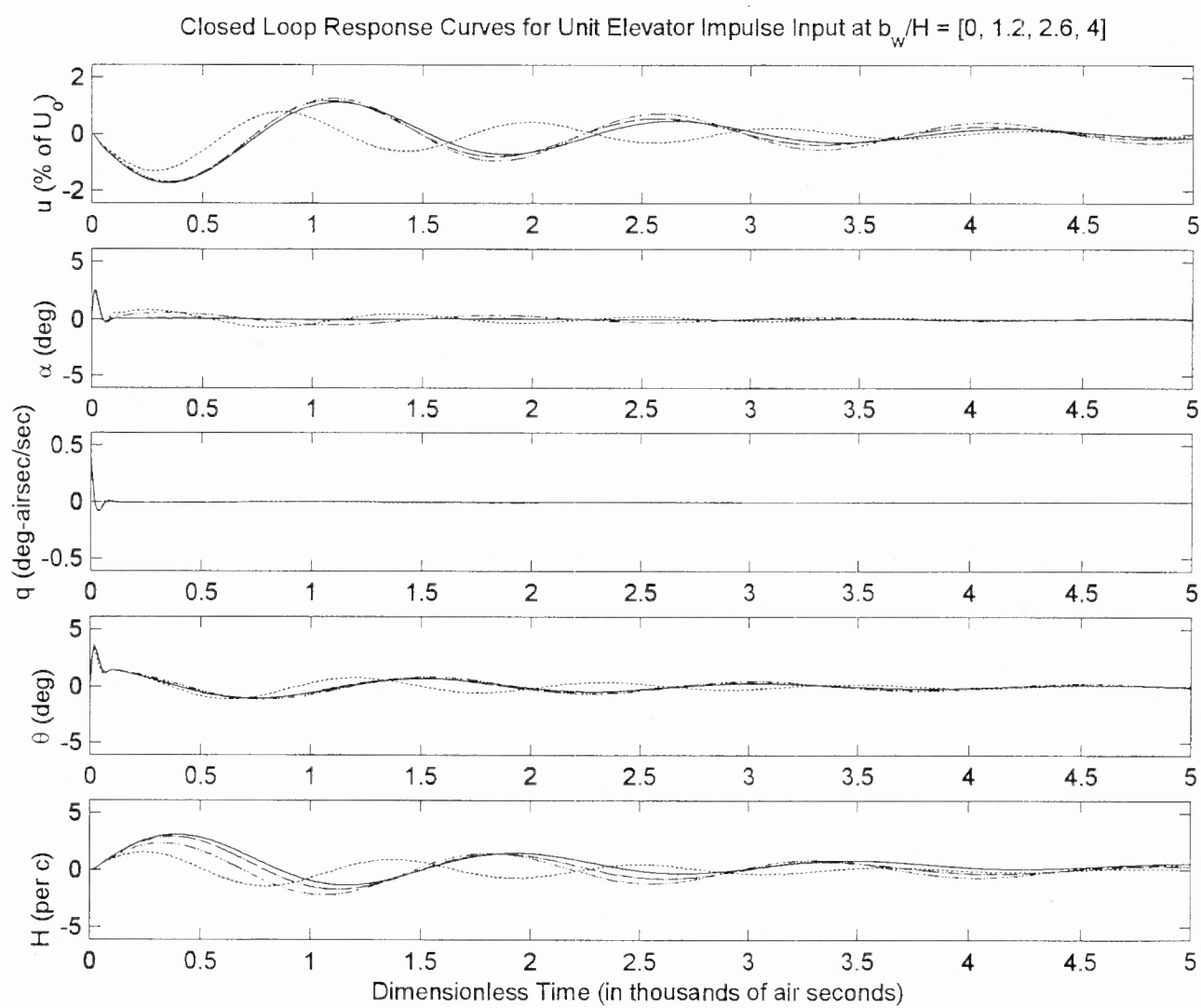


FIGURE R9 a)



Input Data Obtained from chero_cm3_26_5.dat. $K_{CMz} = 0.4$
 —, — —, — · · — and · · · · · respectively represent $b_w/H = [0, 1.2, 2.6, 4]$

FIGURE R9 b)

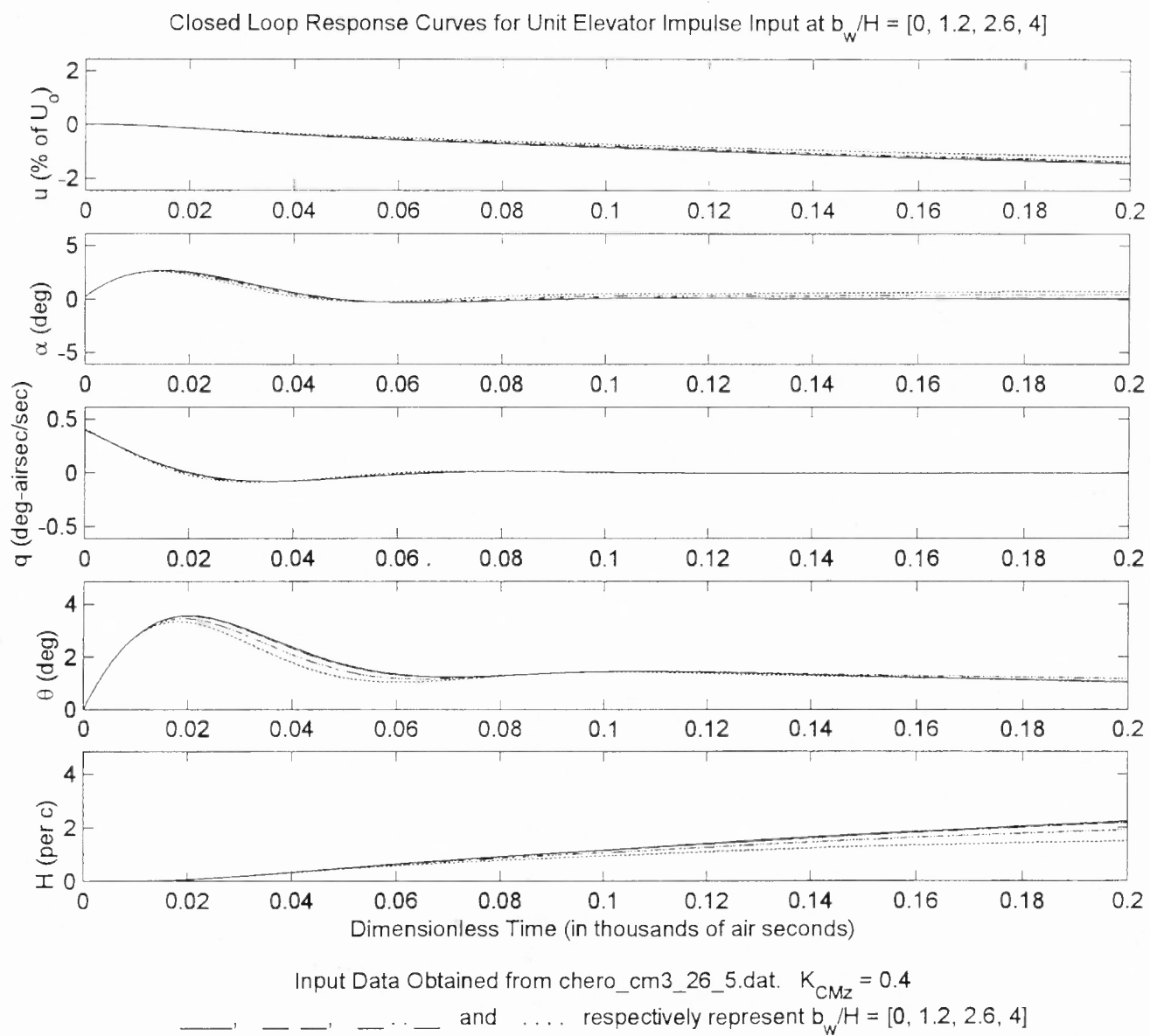


FIGURE R9 c)

APPENDIX A

INDUCED VELOCITY

Induced velocity is that component of the resultant air velocity, at a given point in a flow field, that is due to the resultant normal force acting on all lifting surfaces, and not due to the motion of the aircraft relative to the distant, undisturbed air. The resultant normal force per unit span acting at a given spanwise position on a lifting surface can be regarded as being due to the local circulation around the airfoil. The Biot-Savart Law relates induced velocity to the circulation of a vortex, and is eminently suitable in the analysis of the induced velocity of the vortices used to model the lifting surfaces of aircraft. The main reason for our interest in the induced velocity is the fact that it causes induced drag, which is to be added to the parasite drag of an aircraft when computing the total drag.

A1 Biot-Savart Law

The Biot-Savart Law according to Karamcheti¹², pp. 527-528, states that the infinitesimal velocity increment $d\vec{V}_i$ induced at position $\vec{r}$ by a vortex filament at position $\vec{s}$ with tangential position increment $d\vec{s}$ and of circulation strength Γ , as illustrated in Figure A1, is given by the relation

$$d\vec{V}_i = \frac{\Gamma}{4\pi} \frac{d\vec{s} \times (\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|^3} \quad (A1)$$

The circulation Γ is positive when taken in the right hand screw direction of $d\vec{s}$. The lifting line and vortex lattice models of an aircraft's circulation distribution, on which the computation of the force and moment coefficients depends, make extensive use of results obtained by application of the Biot-Savart Law to straight vortex segments of finite, semi-infinite and infinite length.

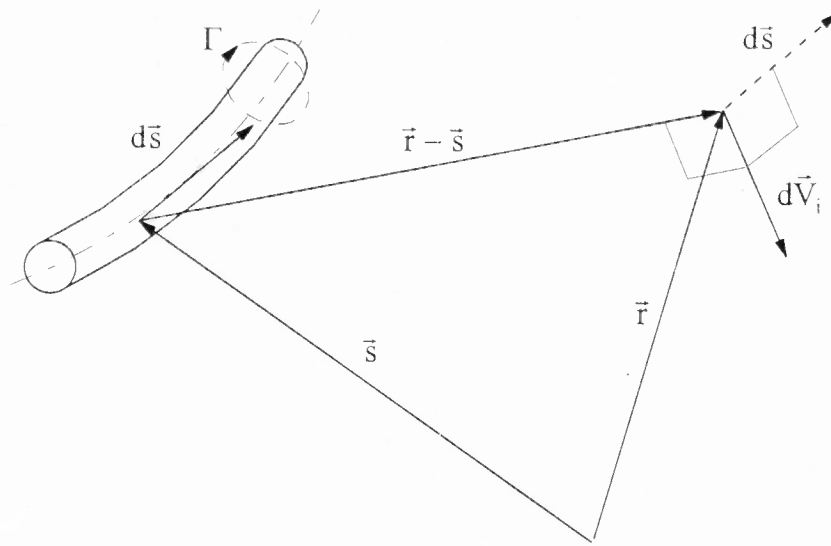


Figure A1: Infinitesimal vortex filament and its induced velocity increment

A2 Straight Vortex Segment of Finite Length

Straight vortex segments of finite circulation strength are used to model the vortex sheets bound to the lifting surfaces of an aircraft in order to compute the aerodynamic lift, drag and moment coefficients.

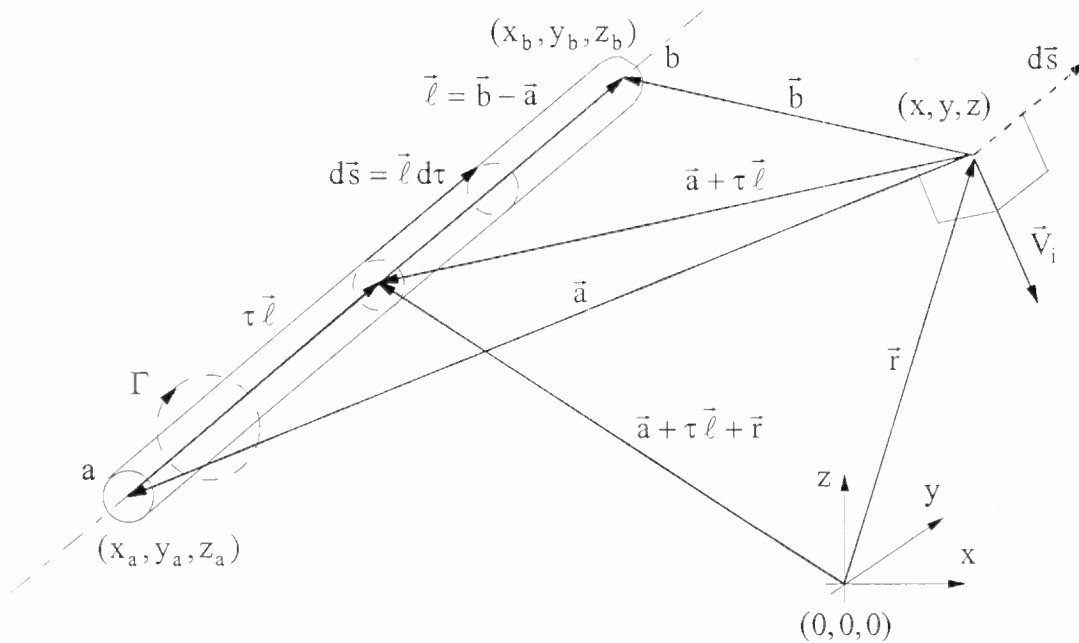


Figure A2: Straight vortex of finite length and its induced velocity

The total induced velocity of such a vortex segment is found by integrating the Biot-Savart Law over the length of the spanwise segment. We follow the approach reported by Lan¹⁴ to solve this integral, except that we restrict the formulation to incompressible flow here. The basic technique is to replace $\vec{r} - \vec{s}$ with

$$\vec{r} - \vec{s} = -(\vec{a} + \tau \vec{\ell}) \quad (\text{A2})$$

so that $d\vec{s} = \vec{\ell} d\tau$, and

$$\vec{s} - \vec{r} = \vec{a} + \tau \vec{\ell} = \begin{cases} \vec{a} & \text{at } \tau = 0 \\ \vec{a} + \vec{\ell} = \vec{b} & \text{at } \tau = 1 \end{cases} \quad (\text{A3})$$

as illustrated in Figure A2.

The numerator term in the Biot-Savart Law is thus seen to be

$$d\vec{s} \times (\vec{r} - \vec{s}) = (\vec{s} - \vec{r}) \times d\vec{s} = (\vec{a} + \tau \vec{\ell}) \times \vec{\ell} d\tau = \vec{a} \times \vec{\ell} d\tau \quad (\text{A4})$$

Noting now that $\vec{\ell} = \vec{b} - \vec{a}$ is independent under integration over the length of a straight vortex segment, that its cross product with itself is the zero vector, and that the circulation strength Γ is uniform over the length of the vortex filament the total induced velocity is given in integral form by

$$\vec{V}_i = \int d\vec{V}_i = \frac{\Gamma \vec{a} \times \vec{\ell}}{4\pi} \int_0^1 \frac{d\tau}{|\vec{r} - \vec{s}|^3} \quad (\text{A5})$$

It follows from $\vec{s} - \vec{r} = \vec{a} + \tau \vec{\ell}$ that

$$(\vec{s} - \vec{r}) \cdot (\vec{s} - \vec{r}) = \vec{a} \cdot \vec{a} + 2\tau \vec{a} \cdot \vec{\ell} + \tau^2 \vec{\ell} \cdot \vec{\ell} = A\tau^2 + B\tau + C \quad (\text{A6})$$

so that $A = \vec{\ell} \cdot \vec{\ell}$, $B = 2\vec{a} \cdot \vec{\ell}$ and $C = \vec{a} \cdot \vec{a}$, with the result that the denominator term of the Biot-Savart Law is of the form

$$|\vec{r} - \vec{s}|^3 = |(\vec{s} - \vec{r}) \cdot (\vec{s} - \vec{r})|^{3/2} = (A\tau^2 + B\tau + C)^{3/2} \quad (\text{A7})$$

Using DERIVE™ the required indefinite integral is found to be

$$\int \frac{d\tau}{(A\tau^2 + B\tau + C)^{3/2}} = -\frac{2(2A\tau + B)}{(B^2 - 4AC)\sqrt{A\tau^2 + B\tau + C}} + c \quad (\text{A8})$$

so that the total induced velocity is

$$\vec{V}_i = \frac{\Gamma \vec{a} \times \vec{\ell}}{4\pi} \left[\frac{2B}{(B^2 - 4AC)\sqrt{C}} - \frac{2(2A + B)}{(B^2 - 4AC)\sqrt{A + B + C}} \right] \quad (\text{A9})$$

Expanding the indicated vector operations here and those defining A, B and C above one can verify readily that

$$B^2 - 4AC = -4|\vec{a} \times \vec{\ell}|^2, \quad 2A + B = 2\vec{b} \cdot \vec{\ell} \quad \text{and} \quad A + B + C = |\vec{b}|^2 \quad (\text{A10})$$

Substitution of these results finally shows that the induced velocity of a straight vortex filament of uniform circulation strength is

$$\vec{V}_i = \frac{\Gamma}{4\pi} \frac{\vec{a} \times \vec{\ell}}{|\vec{a} \times \vec{\ell}|^2} \left[\left(\frac{\vec{a} + \vec{\ell}}{|\vec{a} + \vec{\ell}|} - \frac{\vec{a}}{|\vec{a}|} \right) \cdot \vec{\ell} \right] \quad (\text{A11})$$

Elimination of $\vec{\ell} = \vec{b} - \vec{a}$ yields, after some rearrangement, the alternative formulation

$$\vec{V}_i = \frac{\Gamma}{4\pi} \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|^2} (|\vec{a}| + |\vec{b}|) \left(1 - \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right) \quad (\text{A12})$$

which is useful in the treatment of vortex filaments of semi-infinite length. Both finite and semi-infinite vortices feature in the analysis of the horseshoe vortices of which the vortex lattice is composed, where they are used in the formulation of the flow tangency condition that determines the circulation distribution of the aircraft.

A3 Semi-Infinite Vortex Segment

A straight semi-infinite vortex segment with its left end bounded at point a is illustrated in Figure A3 below. Its other end b tends to infinity along the right hand screw direction of positive circulation indicated by the unit vector $\vec{e}_\Gamma$, so that $\vec{\ell} \rightarrow \vec{e}_\Gamma \infty$.

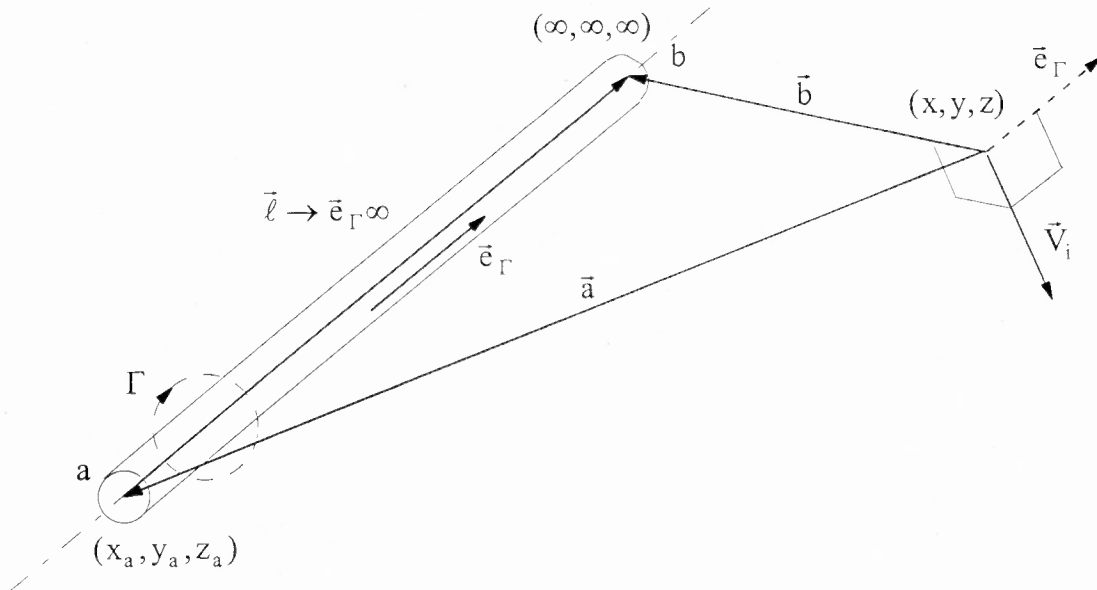


Figure A3: Semi-infinite vortex of given axial direction

The magnitude of $\vec{a}$ vanishes compared to that of $\vec{b}$ as $\vec{\ell}$ approaches the infinity vector $\vec{e}_r \infty$, and furthermore, $\vec{b} = \vec{e}_r |\vec{b}|$ in the limit of $\vec{\ell} = \vec{e}_r \infty$. Inserting this result into the latter formulation of the induced velocity above and taking the limit of infinite $|\vec{b}|$ yields

$$\vec{V}_i = \frac{\Gamma}{4\pi} \lim_{|\vec{b}| \rightarrow \infty} \frac{\vec{a} \times \vec{e}_r |\vec{b}|}{|\vec{a} \times \vec{e}_r |\vec{b}||^2} (|\vec{a}| + |\vec{b}|) \left(1 - \frac{\vec{a} \cdot \vec{e}_r |\vec{b}|}{|\vec{a}| |\vec{b}|} \right) \quad (\text{A13})$$

$$\text{or} \quad \vec{V}_i = \frac{\Gamma}{4\pi} \frac{\vec{a} \times \vec{e}_r}{|\vec{a} \times \vec{e}_r|^2} \left(\lim_{|\vec{b}| \rightarrow \infty} \frac{|\vec{a}|}{|\vec{b}|} + 1 \right) \left(1 - \frac{\vec{a} \cdot \vec{e}_r}{|\vec{a}|} \right) \quad (\text{A14})$$

which finally yields

$$\vec{V}_i = \frac{\Gamma}{4\pi} \frac{\vec{a} \times \vec{e}_r}{|\vec{a} \times \vec{e}_r|^2} \left(1 - \frac{\vec{a} \cdot \vec{e}_r}{|\vec{a}|} \right) \quad (\text{A15})$$

A4 Infinite Vortex Segment

The induced velocity of a straight infinite vortex segment is found by first defining the position of the point at which the induced velocity is required relative to the vortex by a vector $\vec{h} = \vec{e}_n h$ as shown in Figure A4 below.

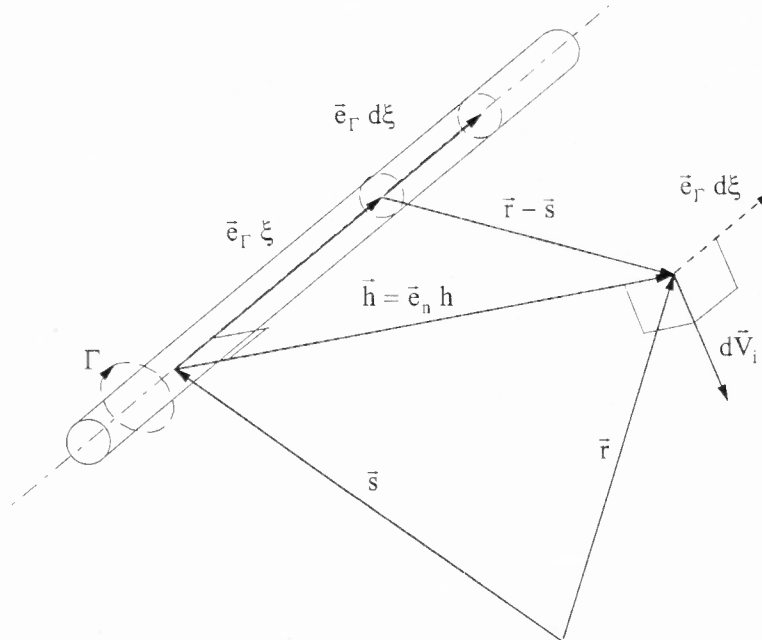


Figure A4: Nomenclature for analysis of infinite vortex section

The position vector of an arbitrary infinitesimal vortex increment ahead of this point is defined by $\vec{e}_\Gamma \xi$. We see that

$$\vec{s} - \vec{r} = \vec{e}_\Gamma \xi - \vec{h} \quad (\text{A15})$$

from which it follows, noting that $\vec{r}$ and $\vec{h}$ are independent of ξ , that

$$d\vec{s} = \vec{e}_\Gamma d\xi \quad (\text{A16})$$

This yields the forms

$$d\vec{s} \times (\vec{r} - \vec{s}) = (\vec{s} - \vec{r}) \times d\vec{s} = -\vec{h} \times \vec{e}_\Gamma d\xi \quad (\text{A17})$$

and

$$(\vec{s} - \vec{r}) \cdot (\vec{s} - \vec{r}) = \xi^2 - 2\xi \vec{e}_\Gamma \cdot \vec{h} + \vec{h} \cdot \vec{h} = \xi^2 + h^2 \quad (\text{A18})$$

in the numerator and denominator terms of the Biot-Savart Law, which leads to an induced velocity increment of

$$d\vec{V}_i = -\frac{\Gamma}{4\pi} \frac{\vec{h} \times \vec{e}_\Gamma d\xi}{(\xi^2 + h^2)^{3/2}} = \frac{\Gamma}{4\pi} \frac{\vec{e}_\Gamma \times \vec{h} d\xi}{(\xi^2 + h^2)^{3/2}} \quad (\text{A19})$$

The required indefinite integral is

$$f(\xi) = \int \frac{d\xi}{(\xi^2 + h^2)^{3/2}} = \frac{\xi}{h^2 \sqrt{\xi^2 + h^2}} + c \quad (\text{A20})$$

from which it follows that the required definite integral is

$$\int_{-\infty}^{\infty} \frac{d\xi}{(\xi^2 + h^2)^{3/2}} = \lim_{\xi \rightarrow \infty} f(\xi) - \lim_{\xi \rightarrow -\infty} f(\xi) = \frac{2}{h^2} \quad (\text{A21})$$

Recalling that $\vec{h} = \vec{e}_n h$ the total induced velocity due to a straight infinite vortex filament is found by letting

$$\vec{V}_i = \int d\vec{V}_i = \frac{\Gamma \vec{e}_\Gamma \times \vec{e}_n h}{4\pi} \int_{-\infty}^{\infty} \frac{d\xi}{(\xi^2 + h^2)^{3/2}} \quad (\text{A22})$$

which finally yields

$$\vec{V}_i = \frac{\Gamma \vec{e}_\Gamma \times \vec{e}_n}{2\pi h} \quad (\text{A23})$$

A5 Horseshoe Vortex

The results obtained above for vortex segments of finite and semi-infinite length find direct application in the computation of the velocity distribution induced by a horseshoe vortex. The vortex lattice method models the vortex system of a lifting surface as being composed of a finite number of horseshoe vortices, of which each comprises a segment of finite length bound to the surface and two semi-infinite segments trailing back towards infinity along the direction of the free stream velocity.

The purpose of this analysis is to provide a computational algorithm by which the induced velocity distribution of a lifting surface can be found. The general results obtained above in physical vector form is recast, to this end, in component vector form.

A5.1 Bound Vortex Segment

Interpreting Figure A2 as an illustration of the bound part alone of a horseshoe vortex with left and right corner points a and b the induced velocity of the bound vortex segment expressed in terms of $\vec{a}$ and $\vec{b}$ is found from equation (A11), on substitution of $\vec{\ell} = \vec{b} - \vec{a}$, to be

$$\vec{V}_i = \frac{\Gamma}{4\pi} \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|^2} \left[\left(\frac{\vec{b}}{|\vec{b}|} - \frac{\vec{a}}{|\vec{a}|} \right) \cdot (\vec{b} - \vec{a}) \right] \quad (A24)$$

The position vectors of a and b are not shown in Figure A2, but it is clear that, relative to the origin of the reference frame, they are given by

$$\vec{r}_a = \vec{i} x_a + \vec{j} y_a + \vec{k} z_a \quad \text{and} \quad \vec{r}_b = \vec{i} x_b + \vec{j} y_b + \vec{k} z_b \quad (A25)$$

Our eventual description of the geometry of the vortex lattice will supply, besides the components of $\vec{r}_a$ and $\vec{r}_b$ along some suitable reference axes, also those of

$$\vec{r} = \vec{i} x + \vec{j} y + \vec{k} z \quad (A26)$$

It follows then that

$$\vec{a} = \vec{r}_a - \vec{r} = \vec{i} (x_a - x) + \vec{j} (y_a - y) + \vec{k} (z_a - z) \quad (A27)$$

and

$$\vec{b} = \vec{r}_b - \vec{r} = \vec{i} (x_b - x) + \vec{j} (y_b - y) + \vec{k} (z_b - z) \quad (A28)$$

whereas the position of b relative to a is

$$\vec{\ell} = \vec{b} - \vec{a} = \vec{i}(x_b - x_a) + \vec{j}(y_b - y_a) + \vec{k}(z_b - z_a) \quad (\text{A29})$$

Defining the co-ordinate differences

$$X_a = x_a - x, \quad Y_a = y_a - y, \quad Z_a = z_a - z \quad (\text{A30})$$

$$X_b = x_b - x, \quad Y_b = y_b - y, \quad Z_b = z_b - z \quad (\text{A31})$$

$$\Delta X = X_b - X_a, \quad \Delta Y = Y_b - Y_a, \quad \text{and} \quad \Delta Z = Z_b - Z_a \quad (\text{A32})$$

we reformulate

$$\vec{a} = \vec{i}X_a + \vec{j}Y_a + \vec{k}Z_a, \quad \vec{b} = \vec{i}X_b + \vec{j}Y_b + \vec{k}Z_b \quad (\text{A33})$$

$$\text{and} \quad \vec{b} - \vec{a} = \vec{i}\Delta X + \vec{j}\Delta Y + \vec{k}\Delta Z \quad (\text{A34})$$

In terms of the co-ordinate differences X , Y and Z we define furthermore

$$R_a = |\vec{a}| = \sqrt{X_a^2 + Y_a^2 + Z_a^2} \quad \text{and} \quad R_b = |\vec{b}| = \sqrt{X_b^2 + Y_b^2 + Z_b^2} \quad (\text{A35})$$

The indicated cross product in the expression for the induced velocity above, expressed in terms of these co-ordinate differences, is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ X_a & Y_a & Z_a \\ X_b & Y_b & Z_b \end{vmatrix} = \vec{i}(Y_a Z_b - Z_a Y_b) + \vec{j}(Z_a X_b - X_a Z_b) + \vec{k}(X_a Y_b - Y_a X_b) \quad (\text{A36})$$

of which the components are signified by

$$C_X = Y_a Z_b - Z_a Y_b, \quad C_Y = Z_a X_b - X_a Z_b \quad \text{and} \quad C_Z = X_a Y_b - Y_a X_b \quad (\text{A37})$$

In addition we signify the square of the magnitude of $\bar{a} \times \bar{b}$ by

$$C = |\bar{a} \times \bar{b}|^2 = C_X^2 + C_Y^2 + C_Z^2 \quad (\text{A38})$$

so that

$$\frac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|^2} = \frac{1}{C} (\bar{i} C_X + \bar{j} C_Y + \bar{k} C_Z) \quad (\text{A39})$$

Defining

$$D = \left(\frac{\bar{b}}{|\bar{b}|} - \frac{\bar{a}}{|\bar{a}|} \right) \cdot (\bar{b} - \bar{a}) = \frac{\bar{b} \cdot (\bar{b} - \bar{a})}{R_b} + \frac{\bar{a} \cdot (\bar{b} - \bar{a})}{R_a} \quad (\text{A40})$$

and recalling that

$$\bar{a} = \bar{i} X_a + \bar{j} Y_a + \bar{k} Z_a, \quad \bar{b} = \bar{i} X_b + \bar{j} Y_b + \bar{k} Z_b \quad (\text{A41})$$

and $\bar{b} - \bar{a} = \bar{i} \Delta X + \bar{j} \Delta Y + \bar{k} \Delta Z$

it follows that

$$D = \frac{1}{R_b} (X_b \Delta X + Y_b \Delta Y + Z_b \Delta Z) - \frac{1}{R_a} (X_a \Delta X + Y_a \Delta Y + Z_a \Delta Z) \quad (\text{A42})$$

Making use of the preceding definitions we can now define the quantities

$$G_X = \frac{C_X D}{C}, \quad G_Y = \frac{C_Y D}{C} \quad \text{and} \quad G_Z = \frac{C_Z D}{C} \quad (\text{A43})$$

in terms of which the induced velocity of the bound segment of a horseshoe vortex is given by

$$\vec{V}_i = \frac{\Gamma}{4\pi} (\vec{i} G_X + \vec{j} G_Y + \vec{k} G_Z) \quad (\text{A44})$$

The formulation above yields a finite induced velocity vector at finite distances from the centre line of the bound vortex segment, zero induced velocity on the centre line outside the interval between a and b, but it yields an infinite induced velocity on the centre line between a and b. The latter result, however, must be disregarded in the computational algorithm, because the core of a vortex filament is modelled as a solidly rotating cylinder of air, at the centre of which the self-induced velocity is zero.

A5.2 Semi-Infinite Trailing Vortex Segments

The induced velocity of a trailing vortex segment emanating from point a in Figure A2 has been shown to be

$$\vec{V}_i = \frac{\Gamma}{4\pi} \frac{\vec{a} \times \vec{e}_r}{|\vec{a} \times \vec{e}_r|^2} \left(1 - \frac{\vec{a} \cdot \vec{e}_r}{|\vec{a}|} \right) \quad (\text{A45})$$

where $\vec{a}$ is the position vector, measured relative to the point at which the induced velocity is required, of the point from which the trailing vortex is shed, $\vec{e}_r$ is the unit vector along the right hand screw sense of its circulation and Γ is its circulation strength. However, we adopt the convention that semi-infinite vortex filaments with clockwise circulation directions looking rearward along the free stream direction are shed from all points signified by b (the right corner points), and that semi-infinite vortex filaments vortices with clockwise circulation directions looking forward toward

the oncoming free stream join up with all bound vortex segments at points signified by a (the left corner points). Noting that the direction of the circulation of the latter semi-infinite vortex filament is opposite to that of the case for which we have derived the result above we replace Γ by $-\Gamma$ in our original formulation, so that

$$\bar{V}_{ia} = -\frac{\Gamma}{4\pi} \frac{\bar{\mathbf{a}} \times \bar{\mathbf{e}}_\Gamma}{|\bar{\mathbf{a}} \times \bar{\mathbf{e}}_\Gamma|^2} \left(1 - \frac{\bar{\mathbf{a}} \cdot \bar{\mathbf{e}}_\Gamma}{|\bar{\mathbf{a}}|} \right) \quad (\text{A46})$$

In the case of a vortex filament emanating from point b, however, we use our original formulation as is, except that we replace $\bar{\mathbf{a}}$ with $\bar{\mathbf{b}}$, so that

$$\bar{V}_{ib} = \frac{\Gamma}{4\pi} \frac{\bar{\mathbf{b}} \times \bar{\mathbf{e}}_\Gamma}{|\bar{\mathbf{b}} \times \bar{\mathbf{e}}_\Gamma|^2} \left(1 - \frac{\bar{\mathbf{b}} \cdot \bar{\mathbf{e}}_\Gamma}{|\bar{\mathbf{b}}|} \right) \quad (\text{A47})$$

If we select a reference frame of which the x-axis points rearward along the free stream direction, then $\bar{\mathbf{e}}_\Gamma = \bar{\mathbf{i}}$. Recalling that

$$\bar{\mathbf{a}} = \bar{\mathbf{i}} X_a + \bar{\mathbf{j}} Y_a + \bar{\mathbf{k}} Z_a \quad \text{and} \quad \bar{\mathbf{b}} = \bar{\mathbf{i}} X_b + \bar{\mathbf{j}} Y_b + \bar{\mathbf{k}} Z_b \quad (\text{A48})$$

it follows from $\bar{\mathbf{i}} \times \bar{\mathbf{i}} = \bar{\mathbf{0}}$, $\bar{\mathbf{j}} \times \bar{\mathbf{i}} = -\bar{\mathbf{k}}$ and $\bar{\mathbf{k}} \times \bar{\mathbf{i}} = \bar{\mathbf{j}}$ that

$$\bar{\mathbf{a}} \times \bar{\mathbf{e}}_\Gamma = \bar{\mathbf{j}} Z_a - \bar{\mathbf{k}} Y_a \quad \text{and} \quad \bar{\mathbf{b}} \times \bar{\mathbf{e}}_\Gamma = \bar{\mathbf{j}} Z_b - \bar{\mathbf{k}} Y_b \quad (\text{A49})$$

Recalling furthermore that

$$|\bar{\mathbf{a}}| = \sqrt{X_a^2 + Y_a^2 + Z_a^2}, \quad |\bar{\mathbf{b}}| = \sqrt{X_b^2 + Y_b^2 + Z_b^2} \quad (\text{A50})$$

and noting that $\bar{\mathbf{a}} \cdot \bar{\mathbf{e}}_\Gamma = X_a$ and $\bar{\mathbf{b}} \cdot \bar{\mathbf{e}}_\Gamma = X_b$ it follows that

$$\bar{V}_{ia} = -\frac{\Gamma}{4\pi} \frac{\bar{j} Z_a - \bar{k} Y_a}{Y_a^2 + Z_a^2} \left(1 - \frac{X_a}{\sqrt{X_a^2 + Y_a^2 + Z_a^2}} \right) \quad (\text{A51})$$

and

$$\bar{V}_{ib} = \frac{\Gamma}{4\pi} \frac{\bar{j} Z_b - \bar{k} Y_b}{Y_b^2 + Z_b^2} \left(1 - \frac{X_b}{\sqrt{X_b^2 + Y_b^2 + Z_b^2}} \right) \quad (\text{A52})$$

If, on the other hand, the zero lift axes are selected as reference axes, then the free stream direction is defined by the small angle of attack α upwards relative to the x-y-plane, and the small sideslip angle β towards the port wing tip relative to the x-z-plane.

The small angle approximation of the unit vector along the right trailing vortex for this case is

$$\bar{e}_r = \bar{i} - \bar{j}\beta + \bar{k}\alpha \quad (\text{A53})$$

The dot product terms are therefore

$$\bar{a} \cdot \bar{e}_r = X_a - \beta Y_a + \alpha Z_a \quad \text{and} \quad \bar{b} \cdot \bar{e}_r = X_b - \beta Y_b + \alpha Z_b \quad (\text{A54})$$

whereas the cross product terms are seen to be

$$\bar{a} \times \bar{e}_r = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ X_a & Y_a & Z_a \\ 1 & -\beta & \alpha \end{vmatrix} = \bar{i}(\alpha Y_a + \beta Z_a) + \bar{j}(Z_a - \alpha X_a) - \bar{k}(\beta X_a + Y_a) \quad (\text{A55})$$

$$\text{and } \vec{b} \times \vec{e}_\Gamma = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ X_b & Y_b & Z_b \\ 1 & -\beta & \alpha \end{vmatrix} = \vec{i}(\alpha Y_b + \beta Z_b) + \vec{j}(Z_b - \alpha X_b) - \vec{k}(\beta X_b + Y_b) \quad (\text{A56})$$

In conclusion we define

$$K_{a1} = |\vec{a} \times \vec{e}_\Gamma|^2 = (\alpha Y_a + \beta Z_a)^2 + (Z_a - \alpha X_a)^2 + (\beta X_a + Y_a)^2 \quad (\text{A57})$$

$$K_{b1} = |\vec{b} \times \vec{e}_\Gamma|^2 = (\alpha Y_b + \beta Z_b)^2 + (Z_b - \alpha X_b)^2 + (\beta X_b + Y_b)^2$$

$$K_{a2} = 1 - \frac{X_a - \beta Y_a + \alpha Z_a}{\sqrt{X_a^2 + Y_a^2 + Z_a^2}} \quad \text{and} \quad K_{b2} = 1 - \frac{X_b - \beta Y_b + \alpha Z_b}{\sqrt{X_b^2 + Y_b^2 + Z_b^2}} \quad (\text{A58})$$

so that the required expressions for the induced velocity contributions of the left and right trailing vortex segments are finally given *in zero lift axes* by

$$\vec{V}_{ia} = -\frac{\Gamma K_{a2}}{4\pi K_{a1}} \left[\vec{i}(\alpha Y_a + \beta Z_a) + \vec{j}(Z_a - \alpha X_a) - \vec{k}(\beta X_a + Y_a) \right] \quad (\text{A59})$$

and

$$\vec{V}_{ib} = \frac{\Gamma K_{b2}}{4\pi K_{b1}} \left[\vec{i}(\alpha Y_b + \beta Z_b) + \vec{j}(Z_b - \alpha X_b) - \vec{k}(\beta X_b + Y_b) \right] \quad (\text{A60})$$

A5.3 Semi-Infinite and Infinite Vortices of Infinitesimal Circulation Strength

All the vortex segments treated thus far were of finite circulation strength. However, vortex filaments of semi-infinite or infinite length but infinitesimal circulation strength are of considerable interest in the analysis of lifting surfaces of finite aspect ratio. The total circulation Γ around a particular section of such a surface varies along the span because of the continuous shedding of infinitesimally weak free vortices

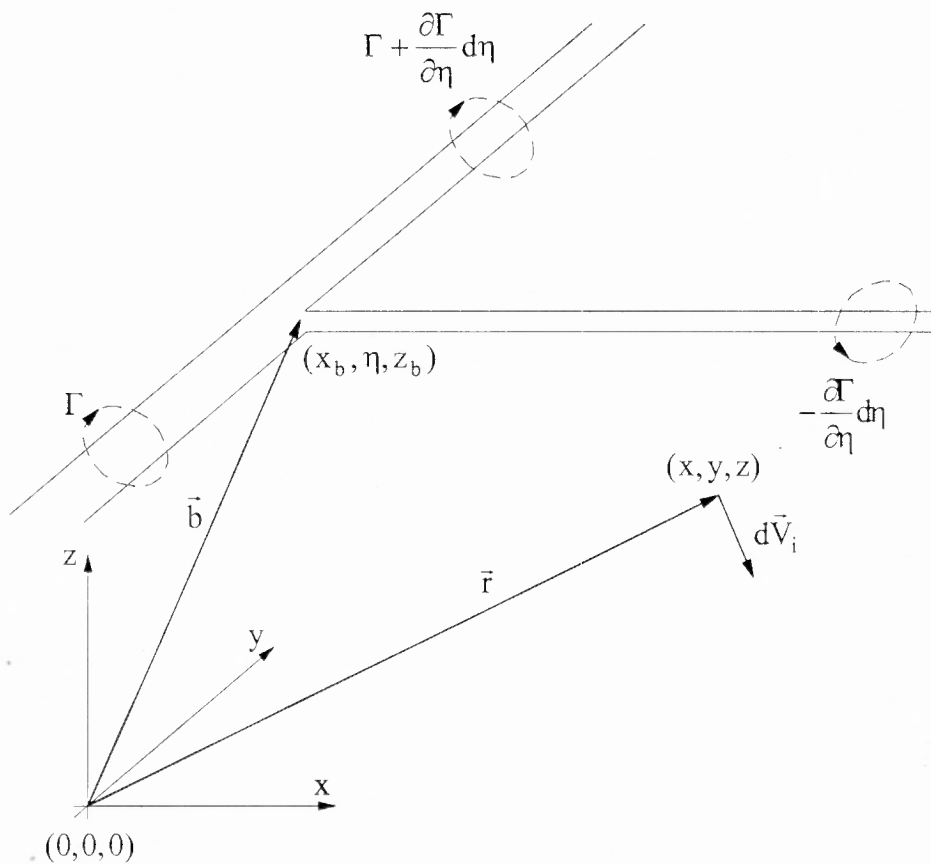


Figure A5: Induced velocity of semi-infinite vortex of infinitesimal circulation strength

Thus, if $\frac{\partial \Gamma}{\partial \eta}$ is negative, as is normally the case on the starboard side, then the trailing vortex filament has positive circulation in the direction shown in Figure A5, and $d\Gamma$ will be negative. Using the results of the preceding section the induced velocity increment of the semi-infinite trailing vortex filament at an arbitrary point is seen to be

$$d\bar{V}_i = -\frac{1}{4\pi} \frac{\partial \Gamma}{\partial \eta} d\eta \frac{\bar{j}Z_b - \bar{k}Y_b}{Y_b^2 + Z_b^2} \left(1 - \frac{X_b}{\sqrt{X_b^2 + Y_b^2 + Z_b^2}} \right) \quad (A61)$$

A simple but nevertheless instructive application of this result is the case of a bound vortex that coincides with the y-axis. The induced velocity increment at $(0, y, 0)$ due

$$d\bar{V}_i = -\frac{1}{4\pi} \frac{\partial \Gamma}{\partial \eta} d\eta \frac{\bar{j} Z_b - \bar{k} Y_b}{Y_b^2 + Z_b^2} \left(1 - \frac{X_b}{\sqrt{X_b^2 + Y_b^2 + Z_b^2}} \right) \quad (\text{A61})$$

A simple but nevertheless instructive application of this result is the case of a bound vortex that coincides with the y-axis. The induced velocity increment at $(0, y, 0)$ due to a trailing vortex shed from $(0, \eta, 0)$ is obtained by noting that $X_b = Z_b = 0$, and $Y_b = \eta - y$ so that

$$d\bar{V}_i = \frac{\bar{k}}{4\pi Y_b} \frac{\partial \Gamma}{\partial \eta} d\eta = \frac{\bar{k}}{4\pi} \frac{\partial \Gamma}{\partial \eta} \frac{1}{\eta - y} d\eta \quad (\text{A62})$$

The total induced velocity of all the trailing vortex filaments shed between the port and starboard wing tips $y = -b/2$ and $y = b/2$ is therefore

$$\bar{V}_i = \frac{\bar{k}}{4\pi} \int_{-b/2}^{b/2} \frac{\partial \Gamma}{\partial \eta} \frac{1}{\eta - y} d\eta \quad (\text{A63})$$

A well-known example to which this result applies is that of a wing with an elliptic circulation distribution, that is

$$\Gamma = \Gamma_0 \sqrt{1 - \left(\frac{2\eta}{b} \right)^2} \quad (\text{A64})$$

In this case

$$\frac{\partial \Gamma}{\partial \eta} = -\frac{4\Gamma_0 \eta}{b^2} \left[1 - \left(\frac{2\eta}{b} \right)^2 \right]^{-1/2} \quad (\text{A65})$$

and

$$\bar{V}_i = \frac{\bar{k}}{4\pi} \int_{-b/2}^{b/2} \frac{\partial \Gamma}{\partial \eta} \frac{1}{\eta - y} d\eta = -\frac{\bar{k}}{4\pi} \frac{2\pi \Gamma_0}{b} = -\frac{\bar{k} \Gamma_0}{2b} \quad (\text{A66})$$

The striking feature of a straight vortex with an elliptic circulation distribution is that the velocity induced at the bound vortex by the trailing vortex sheet is uniformly distributed along the span. Recalling that the sectional lift coefficient is expressed in terms of the sectional circulation by

$$c_\ell c = \frac{2\Gamma}{V_\infty} \quad (\text{A67})$$

and that the total lift coefficient is defined in terms of the sectional lift coefficient by

$$C_L = \frac{1}{S} \int_{-b/2}^{b/2} c_\ell c \, dy \quad (\text{A68})$$

it follows that the lift coefficient of a bound vortex with an elliptic circulation distribution is

$$C_L = \frac{1}{S} \int_{-b/2}^{b/2} \frac{2\Gamma(\eta)}{V_\infty} d\eta = \frac{2\Gamma_0}{SV_\infty} \int_{-b/2}^{b/2} \sqrt{1 - \left(\frac{2\eta}{b}\right)^2} d\eta = \frac{\pi \Gamma_0 b}{2SV_\infty} \quad (\text{A69})$$

Defining the induced angle of attack α_i as the reduction in angle of attack due to the downward velocity component $-w_i$ induced by the trailing vortex sheet it follows from the expression obtained for the induced velocity above that

$$\alpha_i = \frac{\Gamma_0}{2V_\infty b} \quad (\text{A70})$$

Finally, because the induced angle of attack of the elliptic circulation distribution is uniform the induced drag defined by the sectional induced drag coefficient

$$c_{di} = \frac{2\Gamma\alpha_i}{V_\infty} \quad (A71)$$

and the total induced drag coefficient

$$C_{Di} = \frac{1}{S} \int_{-b/2}^{b/2} c_{di} c \, dy \quad (A72)$$

is found to be given by

$$C_{Di} = \frac{\alpha_i}{S} \int_{-b/2}^{b/2} c \, dy = C_L \alpha_i = \frac{\pi\Gamma_0 b}{2SV_\infty} \frac{\Gamma_0}{2V_\infty b} = \frac{\pi\Gamma_0^2}{4SV_\infty^2} \quad (A73)$$

APPENDIX B

THEORY OF A THIN INFINITE WING

A thin wing with small camber and small angle of attack can be modelled to first order approximation by a flat vortex sheet in the plane of the undisturbed free stream. The analysis of the *infinite* thin wing presented subsequently is based on the exposition of Bertin and Smith²⁷, pp. 207-217, the main difference being that we make allowance for non-zero pitch rate here. The analysis requires, firstly, the solution of the flow tangency problem, and secondly, the formulation of expressions for the aerodynamic coefficients.

B1 Flow Tangency Problem

The flow tangency problem of a wing with small camber and small pitch rate is the problem of finding the vortex distribution that satisfies the flow tangency requirement expressed by equation (2.45). Small camber allows the assumption of a flat vortex sheet in the x-y-plane, everywhere on which $Z = 0$. And if the wing is of infinite span width, then its flow field is two-dimensional. This means that it has no spanwise velocity component, which implies a zero chordwise component of vortex strength $\zeta_x = 0$. The spanwise component of vortex strength $\zeta_y = \gamma(\xi)$ will then be uniformly distributed along the span, but not along the chord, so that the flow tangency condition of equation (2.45), applied to a 2D wing, reduces to

$$w_i^c(x, y) = \frac{1}{4\pi} \int_0^c \gamma(\xi) \int_{-\infty}^{\infty} \frac{X}{(X^2 + Y^2)^{3/2}} d\eta d\xi = -V_{\infty}(\alpha - \theta_c + \alpha_Q) \quad (B1)$$

Recalling from equations (2.35) and (2.36) the definitions $X = \xi - x$ and $Y = \eta - y$, it is not very difficult to verify that the induced velocity in equation (B1) becomes

$$w_i^c(x) = \frac{1}{4\pi} \int_0^c \frac{\gamma(\xi) Y}{X(X^2 + Y^2)^{1/2}} \bigg|_{-\infty}^{\infty} d\xi = \frac{1}{2\pi} \int_0^c \frac{\gamma(\xi)}{\xi - x} d\xi \quad (B2)$$

Converting to angular co-ordinates, using the transformations

$$\xi = \frac{c}{2}(1 - \cos \phi) \quad \text{and} \quad x = \frac{c}{2}(1 - \cos \theta) \quad (\text{B3})$$

equation (B2) yields

$$w_i^c(\theta) = -\frac{1}{2\pi} \int_0^\pi \frac{\gamma(\phi) \sin \phi}{\cos \phi - \cos \theta} d\phi \quad (\text{B4})$$

A vortex distribution given according to Bertin and Smith²⁷ by

$$\gamma(\phi) = 2V_\infty \left(A_0 \frac{1 + \cos \phi}{\sin \phi} + \sum_{n=1}^{\infty} A_n \sin n\phi \right) \quad (\text{B5})$$

will identically satisfy the *Kutta condition of zero spanwise component of vortex strength at the trailing edge*, and will produce a normal component of induced velocity distributed according to

$$w_i^c(\theta) = -\frac{V_\infty A_0}{\pi} \int_0^\pi \frac{1 + \cos \phi}{\cos \phi - \cos \theta} d\phi - \frac{V_\infty}{\pi} \sum_{n=1}^{\infty} \int_0^\pi \frac{A_n \sin n\phi \sin \phi}{\cos \phi - \cos \theta} d\phi \quad (\text{B6})$$

Both of the indicated integrals in this distribution are improper, in that their integrands are singular at $\phi = \theta$. These integrals must therefore be evaluated using their Cauchy principal values, to be found by making use of identities that, according to Karamcheti¹², are given by

$$\int_0^\pi \frac{\sin n\phi \sin \phi}{\cos \phi - \cos \theta} d\phi = -\pi \cos n\theta \quad (\text{B7})$$

$$\text{and} \quad \int_0^\pi \frac{\cos n\phi}{\cos \phi - \cos \theta} d\phi = \begin{cases} 0 & \text{if } n = 0 \\ \frac{\pi \sin n\theta}{\sin \theta} & \text{if } n \geq 1 \end{cases} \quad (\text{B8})$$

Substitution from equations (B7) and (B8) into equation (B6) yields an induced upwash distribution of

$$w_i^c(\theta) = -V_\infty A_0 + V_\infty \sum_{n=1}^{\infty} A_n \cos n\theta \quad (\text{B9})$$

Subsequent substitution from equation (B9) into equation (B1) shows that the vortex distribution of equation (B5) will satisfy the flow tangency condition if

$$A_0 - \sum_{n=1}^{\infty} A_n \cos n\theta = \alpha - \theta_c(\theta) + \alpha_Q \quad (\text{B10})$$

The local angle of attack increment due to pitch rate, as expressed in terms of the local angular co-ordinate, can easily be shown to be

$$\alpha_Q = \alpha_{QMC} - \Delta\alpha_Q \cos \phi \quad (\text{B11})$$

in which the midchord increment α_{QMC} and the increment $\Delta\alpha_Q$ per half chord are given respectively by

$$\alpha_{QMC} = \frac{Qx_{MC}}{V_\infty} \quad \text{and} \quad \Delta\alpha_Q = \frac{Qc}{2V_\infty} \quad (\text{B12})$$

where x_{MC} is the distance of the midchord point behind the centre of pitch. For example, if $x_{MC} = c/2$, then $\alpha_Q = q(1 - \cos \phi)$. The slope of the camber line of an arbitrary airfoil, if its shape is available in analytical form, can be expressed directly as a cosine series

$$\theta_c(\theta) = \sum_{m=1}^{\infty} \theta_{cLEm} \cos m\theta \quad (B13)$$

If the slope of the camber line is available in the form of normal co-ordinates z_c versus chordwise position x , on the other hand, then a polynomial curve fit can be made, from which the values of θ_{cLEm} can be solved after substituting $x = c(1 - \cos\theta)/2$ and then converting from powers of $\cos\theta$ to cosines of multiples of θ .

Substituting from equation (B11) into (B10), and then integrating over the chord interval $0 \leq \theta \leq \pi$, but retaining the slope of the camber line as is, for the moment, the flow tangency requirement becomes

$$\pi A_0 - \sum_{m=1}^{\infty} \int_0^{\pi} A_m \cos m\theta d\theta = \pi \alpha - \int_0^{\pi} \theta_c(\theta) d\theta + \pi \alpha_{QMC} \quad (B14)$$

This yields, by making use of the identity

$$\int_0^{\pi} \cos k\theta d\theta = 0 \quad \text{for all } k \quad (B15)$$

the result that the coefficient A_0 in the vortex distribution of equation (B5) must be related to the angle of attack α , the slope of the camber line $\theta_c(\theta)$ and the midchord contribution of pitch rate α_{QMC} by

$$A_0 = \alpha - \frac{1}{\pi} \int_0^{\pi} \theta_c(\theta) d\theta + \alpha_{QMC} \quad (B16)$$

Noting furthermore that

$$\int_0^{\pi} \cos m\theta \cos n\theta d\theta = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n \end{cases} \quad (\text{B17})$$

it follows by integration of $\cos n\theta$ times equation (B10) over the chord interval $0 \leq \theta \leq \pi$, making use of equations (B11) and (B15) again, and then solving for A_n that

$$A_n = \begin{cases} \frac{2}{\pi} \int_0^{\pi} \theta_c(\theta) \cos \theta d\theta + \Delta\alpha_Q & \text{if } n = 1 \\ \frac{2}{\pi} \int_0^{\pi} \theta_c(\theta) \cos n\theta d\theta & \text{if } n > 1 \end{cases} \quad (\text{B18})$$

We note in passing that a parabolic airfoil that is symmetric about the $\frac{1}{2}$ -chord point, as a simple example, has a camber line slope described by small leading and trailing edge values of the camber line slope given by

$$\theta_{cLEI} = -\theta_{cTE} \quad (\text{B19})$$

and that the slope of its camber line elsewhere is

$$\theta_c(\theta) = \theta_{cLEI} \cos \theta \quad (\text{B20})$$

while all the other coefficients θ_{cLEm} in equation (B13) are zero. One thus finds, *for a parabolic arc*, that

$$A_0 = \alpha + \alpha_{QMC} \quad \text{and} \quad A_1 = \theta_{cLEI} + \Delta\alpha_Q \quad (\text{B21})$$

and that all other A_n in equation (B5) are zero.

B2 Sectional Force and Moment Coefficients

The lift per unit span of an infinite wing out of ground effect, as represented by that of a flat bound vortex sheet aligned with the wind, can be deduced easily from equation (2.82). Its chordwise component of vortex strength ζ_x is identically zero by virtue of its two-dimensional flow field. The chordwise induced velocity u_i is identically zero if there are no other lifting surfaces in the vicinity, nor any image vortices below the ground plane to be considered. And the pitch rate Q produces zero chordwise velocity perturbation $Q v$ by virtue of the assumption of a flat vortex sheet. These observations show that the lift per unit span for an infinite wing represented by a vortex sheet with a spanwise component of vortex strength $\zeta_y = \gamma(\xi)$ follows from equation (2.82) to be given by

$$L' = \rho V_\infty \int_0^c \gamma(\xi, \eta) d\xi \quad (B22)$$

Normalising with respect to $\rho V_\infty^2 c / 2$ and converting to the angular co-ordinate one finds the sectional lift coefficient to be given in terms of the vortex distribution by

$$c_\ell = \frac{1}{V_\infty} \int_0^\pi \gamma(\phi) \sin \phi d\phi \quad (B23)$$

which yields, after substitution from equation (B5), and by making use of the appropriate trigonometric identities, the result that

$$c_\ell = 2\pi \left(A_0 + \frac{1}{2} A_1 \right) = 2\pi \left[\alpha + \alpha_{QMC} + \frac{1}{2} (\theta_{cLEI} + \Delta\alpha_Q) \right] \quad (B24)$$

Note that $\Delta\alpha_Q$, as defined in equation (B12), is precisely the same as the dimensionless perturbation of pitch rate q , as used in the equations of longitudinal motion derived in Appendix D.

The pitch moment per unit span, taken about the $1/4$ -chord point, is similarly found, on consideration of equation (2.83), to be given by

$$M'_{c/4} = -\rho V_{\infty} \int_0^c (\xi - c/4) \gamma(\xi) d\xi \quad (B25)$$

Normalising with respect to $\rho V_{\infty}^2 c^2 / 2$ and converting to the angular co-ordinate one finds the sectional pitch moment coefficient about the $1/4$ -chord point to be given in terms of the vortex distribution by

$$c_{mc/4} = -\frac{1}{2V_{\infty}} \int_0^{\pi} \gamma(\phi) \left(\frac{1}{2} - \cos \phi \right) \sin \phi d\phi \quad (B26)$$

which yields, after substitution from equation (B5) and by making use of the appropriate trigonometric identities, the result that

$$c_{mc/4} = -\frac{\pi}{4} (A_1 - A_2) \quad (B27)$$

In the case of the parabolic airfoil mentioned above this result becomes

$$c_{mc/4} = -\frac{\pi}{4} (\theta_{cLEI} + \Delta\alpha_Q) \quad (B28)$$

In addition to the lift force one finds, surprisingly, that an infinite wing experiences some drag if the pitch rate is not zero! This drag is easily seen, on comparison of equation (2.81) with (2.83), to be related to the pitch moment about the $1/4$ -chord point merely by

$$D' = M' Q / V_{\infty} = 2q M' / c \quad (B29)$$

Normalising with respect to $\rho V_\infty^2 c / 2$, as in the case of the lift coefficient, and recalling that the pitch moment coefficient above was defined by normalising with respect to $\rho V_\infty^2 c^2 / 2$, one obtains the result that

$$c_d = 2q c_{mc/4} = -\frac{\pi q}{2} (\theta_{cLE1} + q) \quad (B30)$$

It was surprising enough to find that the drag is not zero when the pitch rate is not zero. But it is much more surprising to find negative drag (that is thrust!) when the pitch rate is positive. One finds, however, that the (nose-up) moment exerted by this pitch induced drag, when taken about the centre of pitch, as shown in Figure B1, is exactly equal to the nose-down moment produced by the distortion of the vortex distribution induced by the pitch rate. One can therefore conclude that the infinite wing, even at a steady pitch rate, *does* fly with zero power, as expected, even though the negative drag coefficient creates the illusion of a positive power output. (See Chapter 5 for more comments on this point.)

The total circulation of the infinite wing, incidentally, is found after conversion to the angular co-ordinate to be given by

$$\Gamma = \frac{c}{2} \int_0^\pi \gamma(\phi) \sin \phi \, d\phi \quad (B31)$$

which, on substitution from equation (B5), yields the general result

$$\Gamma = \pi V_\infty c \left(A_0 + \frac{1}{2} A_1 \right) \quad (B32)$$

It follows therefore, on comparison with equation (B24), that the 2D lift coefficient is related to the circulation around the wing by

$$c_l = \frac{2\Gamma}{V_\infty c} \quad (\text{B33})$$

as expected from consideration of the Kutta-Joukowski Theorem.

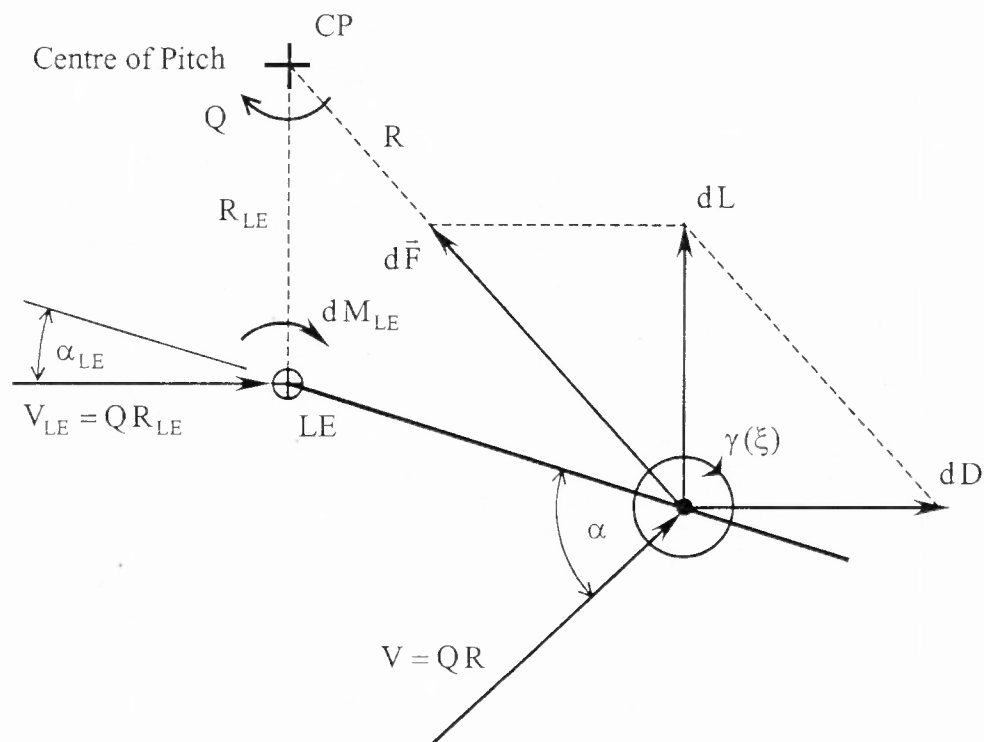


Figure B1: Side view of pitching 2D wing showing increments of lift, pitch moment and drag. The resultant moment about the centre of pitch must be identically zero, because no force increment $d\vec{F}$ exerts a moment about CP. However, every force increment has a *positive forward component* (i.e. a *negative drag component* dD). This requires that *the resultant drag D times R_{LE} , which represents a moment about CP in the nose-up direction, must be cancelled identically by a negative pitch moment resultant M_{LE} .*

APPENDIX C

VORTEX LATTICE GEOMETRY

The vortex lattice model (as discussed Chapter 3) of an aerodynamic surface is based on the assumption of a vortex distribution that is impulsively distributed in the chordwise direction and piecewise continuous in the spanwise direction. The simplest example of such a vortex distribution is a set of horseshoe vortices, each with a finite bound segment on the surface and two semi-infinite trailing segments directed along the free stream velocity vector. If one goes one step further and model not only the trailing vortex sheet, but also the bound vortex sheet as aligned with the free stream velocity, then the set of horseshoe vortices that represent a given surface are all co-planar, but not necessarily co-planar with the vortices of other aerodynamic surfaces. The geometry of such a set of horseshoe vortices is uniquely defined if the distribution of its corner points and the direction of the free stream are given. Following the approach of Lan¹⁴ we use a cosine type distribution of the corner points and control points of the horseshoe vortices, as illustrated in Figures R1 a) to R1 d), which are also discussed in Chapter 5.

Each aerodynamic surface is divided into a matrix of N_s panels spanwise and N_c panels chordwise. On each panel a horseshoe vortex is placed, with its corners on the chord lines that flank the panel. Each horseshoe vortex has a control point that is placed spanwise at the half angle point between its two trailing segments, and chordwise at the half angle point between adjacent bound vortex segments.

C1 Angular Co-ordinates of Horseshoe Vortex Reference Points

The position of the corner points and midpoints of the horseshoe vortices and their control points are defined in terms of chordwise and spanwise angular co-ordinates of a cosine type distribution. The chordwise positions of the *control points* are defined by the set of $N_c + 1$ angular co-ordinates

$$\theta_{ci} = i \Delta \theta \quad \text{in which} \quad i = 0, 1, 2, \dots, N_c \quad (C1)$$

The angular increment between not only two (chordwise) adjacent control points, but also between two adjacent *vortex corner points*, is

$$\Delta\theta = \frac{\pi}{N_c} \quad (C2)$$

The control points at $\theta_c = 0$ on the leading edge are used to find the distribution of the leading edge singularity strength. The other N_c control points are used in the formulation of the flow tangency problem, which requires one control point on each of the N_c panels into which each of the N_s chordwise strips on the surface is divided.

The chordwise positions of the bound vortex corner points are defined by the set of N_c angular co-ordinates

$$\theta_{vai} = \theta_{vbi} = \theta_{ci} - \frac{1}{2} \Delta\theta \quad \text{in which} \quad i = 1, 2, \dots, N_c \quad (C3)$$

Each vortex corner point is thus seen to be situated midway (at half angle increment) between the two adjacent control points in the chordwise direction.

The spanwise position of the control points is defined by the set of N_s angular co-ordinates

$$\phi_{cj} = j\Delta\phi \quad \text{in which} \quad j = 1, 2, \dots, N_s \quad (C4)$$

The angular increment here is

$$\Delta\phi = \frac{\pi}{(N_s + 1)} \quad (C5)$$

The spanwise position of the corner points of the horseshoe vortices are defined in terms of this angular increment and the spanwise control point positions by

$$\phi_{vaj} = \phi_{cj} - \frac{1}{2} \Delta \theta \quad \text{and} \quad \phi_{vbj} = \phi_{cj} + \frac{1}{2} \Delta \theta \quad \text{with } j = 1, 2, \dots, N_s \quad (C6)$$

Each control point is therefore placed midway (at half angle increment) between the flanking pair of vortex corner points in the spanwise direction, with no control points at the wing tips.

C2 Co-ordinates of Horseshoe Vortex Reference Points in Local Chord Plane

The chord lengths c_{aj} , c_{bj} and c_{cj} evaluated at the spanwise angular positions ϕ_{aj} , ϕ_{bj} and ϕ_{cj} , respectively, are given in terms of the taper ratio $\lambda = c_t / c_o$ and the midspan chord length c_o by

$$c_{aj} = c_o \left[1 - (1 - \lambda) |\cos \phi_{aj}| \right] \quad (C7a)$$

$$c_{bj} = c_o \left[1 - (1 - \lambda) |\cos \phi_{bj}| \right] \quad (C7b)$$

$$\text{and} \quad c_{cj} = c_o \left[1 - (1 - \lambda) |\cos \phi_{cj}| \right] \quad (C7c)$$

as illustrated in Figures R1c) and R1 d). The linear position co-ordinates of points on the leading edge (indicated by subscript ℓe) ahead of the inboard vortex corner points, and of the leading edge control points, *as measured relative to the leading tip of the midspan chord* (this measurement is indicated by superscript LE), are given by

$$x_{\ell eaj}^{LE} = \frac{b}{2} \tan \Lambda_{\ell e} |\cos \phi_{aj}| \quad (C8a)$$

$$x_{\ell e b j}^{LE} = \frac{b}{2} \tan \Lambda_{\ell e} |\cos \phi_{bj}| \quad (C8b)$$

$$x_{\ell e c j}^{LE} = \frac{b}{2} \tan \Lambda_{\ell e} |\cos \phi_{cj}| \quad (C8c)$$

$$\text{and } y_{\ell e c j}^{LE} = -\frac{b}{2} \cos \phi_{cj} \quad (C9)$$

The x-co-ordinates of equations (C8) above are used in formulating expressions for the x-co-ordinates of the inboard vortex corner points below. No vortices are placed on the leading edge, and therefore no co-ordinates y_a or y_b are required on the leading edge.

The linear position co-ordinates of the vortex corner points and the inboard control points, also superscripted with LE to indicate measurement with respect to the midspan leading tip, and of which some are illustrated in Figure C1 below, are given by

$$x_{vai,j}^{LE} = x_{\ell e a j}^{LE} + \frac{1}{2} c_{aj} (1 - \cos \theta_{vai}) \quad (C10a)$$

$$x_{vbi,j}^{LE} = x_{\ell e b j}^{LE} + \frac{1}{2} c_{bj} (1 - \cos \theta_{vbi}) \quad (C10b)$$

$$x_{ci,j}^{LE} = x_{\ell e c j}^{LE} + \frac{1}{2} c_{cj} (1 - \cos \theta_{ci}) \quad (C10c)$$

$$y_{vai,j} = -\frac{b}{2} \cos \phi_{vaj} \quad (C11a)$$

$$y_{vbi,j} = -\frac{b}{2} \cos \phi_{vbj} \quad (C11b)$$

$$\text{and} \quad y_{ci,j} = -\frac{b}{2} \cos \phi_{cj} \quad (\text{C11c})$$

Note that the superscript LE is omitted from the expressions for the y-co-ordinates, because these are the same in the chord plane as in the geodetic reference frame to which the x-co-ordinates are presently to be referred. All z-co-ordinates are by definition zero in the chord plane. We define a geodetic reference frame as one with an x-axis pointing horizontally rearward, a z-axis pointing vertically upward and a y-axis pointing starboard. We prefer a geodetic reference frame, because of the simple formulations of co-ordinate expressions it offers, mainly due to the fact that the images of the wing and tail and their vortices are symmetric about the ground plane in a longitudinal state of motion. Any suitable reference point at which the reference frame is centred may be selected, but we prefer to centre our geodetic reference frame at the leading tip of the wing's midspan chord.

C3 Co-ordinates of Vortex Reference Points in Geodetic Frame

One requires not only the co-ordinates of the vortex corner points and control points, but also the co-ordinates of the images of the vortex corner points below the ground, to be expressed in this frame. To this end the position co-ordinates of the tail relative to the wing are needed. Denoting the distance of the leading edge LE_t of the tail behind the leading edge LE_w of the wing by l_{t0} and h_{t0} , respectively, *when measured along and normal to the wing chord*, the corresponding distances in our geodetic reference frame of reference are shown in Figure C2 to be

$$\ell_t = \ell_{t0} \cos \Theta + h_{t0} \sin \Theta \approx \ell_{t0} + h_{t0} \Theta \quad (\text{C12})$$

$$\text{and} \quad h_t = -\ell_{t0} \sin \Theta + h_{t0} \cos \Theta \approx -\ell_{t0} \Theta + h_{t0} \quad (\text{C13})$$

If the assumption is made that the pitch angle and the tail angle are small, all x-co-ordinates of the wing are the same in the geodetic reference frame as those in the wing

chord plane. The x-co-ordinates of the tail, however, are referred to the leading edge of the wing by letting

$$x = \ell_t + x^{LE} \quad (C14)$$

for all vortex corner points and control points on the tail. In order to place the horseshoe vortices in planes passing thorough the $\frac{1}{4}$ -chord points of the mean aerodynamic chords, planes that are furthermore aligned with the relative wind, one also needs the distances of these $\frac{1}{4}$ -chord points (we call them AC_w and AC_t), of the wing and tail relative to their respective midspan chord tips. The chordwise distance of the point AC behind the $\frac{1}{4}$ -chord point of a tapered and swept back wing is shown by McCormick²⁸ to be given in terms of the sweep angle Λ_{qc} of the $\frac{1}{4}$ -chord line by

$$X_{AC} = \frac{1}{6} \frac{1+2\lambda}{1+\lambda} b \tan \Lambda_{qc} \quad (C15)$$

The $\frac{1}{4}$ -chord sweep angle is easily shown to be related to the sweep angle Λ_{te} of the leading edge by

$$\tan \Lambda_{qc} = \tan \Lambda_{te} - \frac{c_o}{2b} (1-\lambda) \quad (C16)$$

in which c_o is the midspan chord length. The vertical distance of the point AC_w of the wing above its leading edge is given to first approximation in terms of the pitch angle Θ of the wing chord relative to the horizontal by

$$\Delta z \approx -(c_w / 4 + X_{ACw}) \Theta \quad (C17)$$

The vertical distance of the mean aerodynamic $\frac{1}{4}$ -chord point of the tail, on the other hand, also measured above the wing's leading edge, requires the distance of its leading

edge behind that of the wing, and the nose down angle δ_t of the tail relative to the wing chord too, and is thus given to first order approximation by

$$\Delta z \approx h_t - (c_t / 4 + X_{ACt})(\Theta - \delta_t) \quad (C18)$$

Allowance for the fact that the trailing vortex sheets (assumed to be flat) are inclined upward at an angle equal to the angle of descent α_d is finally made, for vortex corner points and control points on both the wing and tail, by adding the extra vertical distance due to the upward inclination to yield

$$z \approx \Delta z + (x - c / 4 - X_{AC})\alpha_d \quad (C19)$$

Equations (C17) to (C19) supply the means of computing the z-co-ordinates, resolved in the geodetic frame, of all vortex corner points and control points of the wing and tail. All that remains is to find the z-co-ordinates of the images (below the ground plane) of the vortex corner points, because these are required in both the flow tangency problem and the induced drag computation when accounting for ground proximity. The x- and y-co-ordinates of the image vortices are the same as that of the real vortices. (This is the main reason for our selection of the geodetic base as reference frame!). The z-co-ordinates z_i of the image vortices, in contrast, are related in terms of the height H_{LEw} of the wing's leading edge above the ground to the co-ordinates z of their parent real vortices by

$$z_i \approx -z - 2H_{LEw} \quad (C20)$$

If it is the height H of the CM that is specified, the H_{LEw} is related to CM in terms of the CM distance behind the leading tip LE_w of the wing by

$$H_{LEw} = H + \Theta \ell_{CM/LEw} \quad (C21)$$

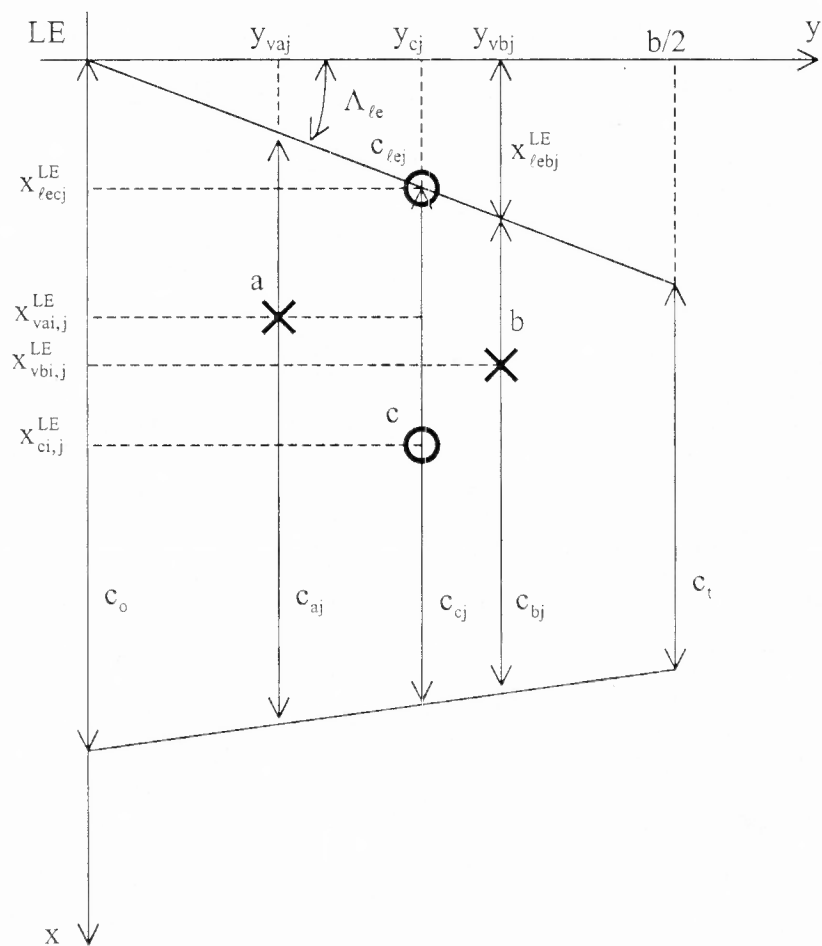


Figure C1: Definitions of some co-ordinates that define vortex lattice geometry

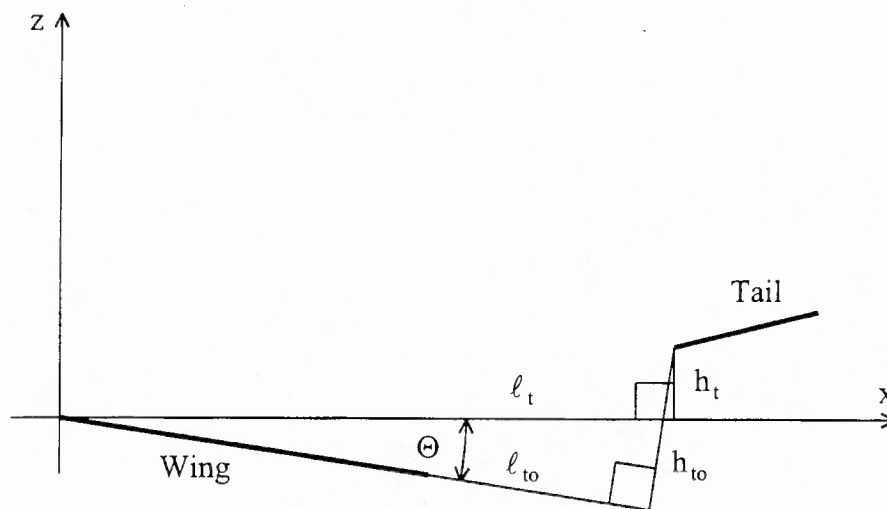


Figure C2: Side view of wing and tail showing distances ℓ_{to} and h_{to} between their leading edges, as measured along and normal to wing chord, respectively, and their counterparts ℓ_t and h_t as measured along the x- and z-axes of the geodetic reference frame.



APPENDIX D

EQUATIONS OF MOTION

The derivation of the required equations of motions, reasoning from basic principles, is presented here. The formulations are quite general at the outset, but are restricted later to the case of longitudinal motion, and later still, by the assumption of small perturbations. We do, however, present non-dimensional equations of longitudinal motion for large perturbations before making the latter assumption, rather than making small perturbational assumptions from the outset.

D1 Translational Motion

The equation of translational motion of an arbitrary system of particles (not necessarily rigidly connected to one another) is expressed by Newton's Second Law in the vector form

$$mD_I \vec{V}_{CM} = \vec{F}_e \quad (D1)$$

in which $\vec{V}_{CM}$ is the velocity of the centre of mass of the system, and $\vec{F}_e$ is the resultant of all externally applied forces. Equation (D1) is directly applicable to a rigid body too.

The symbolic operator

$$D_I() \equiv \frac{d()}{dt} \quad (D2)$$

employed here, in accordance with the notation of Spiegel³⁴, is a derivative operator that denotes time derivation with respect to an inertial reference frame. It is related to the derivative operator D_r with respect to a rotating reference frame by

$$D_I() \equiv D_r() + \vec{\omega}_r \times () \quad (D3)$$

in which $\vec{\omega}_r$ is the angular velocity of the rotating reference frame relative to an arbitrary inertial reference frame. An inertial reference frame, also known as a Galilelean reference frame, is defined as one that does not accelerate or rotate relative to the distant stars. The angular velocity of the earth and its centripetal acceleration towards the sun, however, is so small that *a reference frame with constant velocity and zero angular velocity relative to an earth-fixed reference frame is a quite adequate approximation of an inertial reference frame for the purposes of terrestrial flight mechanics*. The acceleration of the centre of mass (CM) is thus expressed alternatively by

$$D_1 \vec{V}_{CM} = D_r \vec{V}_{CM} + \vec{\omega}_r \times \vec{V}_{CM} \quad (D4)$$

The motion of a rigid body is considered here with reference to a unit triad of which the x-member points forward along the trajectory of motion, the y-element points to the right perpendicular to the plane that contains the curved local trajectory segment (or horizontally to the right, if the trajectory is straight), whereas the z-element completes the right-handed set. Resolved in a Cartesian frame r of which this is the unit triad the translational velocity vector is expressed with reference to Figure D1 by

$$\vec{V}_{CM} = \vec{e}_{xr} V_\infty \quad (D5)$$

whereas the angular velocity of the reference frame is

$$\vec{\omega}_r = \vec{e}_{yr} \frac{d\gamma}{dt} \quad (D6)$$

The definition of D_r as a derivative in which the absolute time rates of change of the unit vectors of the rotating reference frame do not feature implies that

$$D_r \vec{V}_{CM} = \vec{e}_{xr} \frac{dV_\infty}{dt} \quad (D7)$$

Noting that $\bar{e}_{yr} \times \bar{e}_{xr} = -\bar{e}_{zr}$ it follows furthermore from equations (D5) and (D6) that

$$\bar{\omega}_r \times \bar{V}_{CM} = -\bar{e}_{zr} V_\infty \frac{d\gamma}{dt} \quad (D8)$$

with the result that the translational inertia term in equation (D1) is given by

$$D_1 \bar{V}_{CM} = \bar{e}_{xr} \frac{dV_\infty}{dt} - \bar{e}_{zr} V_\infty \frac{d\gamma}{dt} \quad (D9)$$

The resultant external force on an aircraft in longitudinal motion, as illustrated in Figure D1, is usually expressed in terms of components due to its lift L , drag D , thrust T and weight mg . Resolved along the unit vectors mentioned above the resultant force is

$$\bar{F}_e = \bar{e}_{xr} (T - D - mg \sin \gamma) + \bar{e}_{zr} (mg \cos \gamma - L) \quad (D10)$$

It follows by substitution from equations (D9) and (D10) into equation (D1) that

$$m \left(\bar{e}_{xr} \frac{dV_\infty}{dt} - \bar{e}_{zr} V_\infty \frac{d\gamma}{dt} \right) = \bar{e}_{xr} (T - D - mg \sin \gamma) + \bar{e}_{zr} (mg \cos \gamma - L) \quad (D11)$$

so that the x- and z-components of the equations of longitudinal motion are respectively given by

$$m \frac{dV_\infty}{dt} = T - D - mg \sin \gamma \quad (D12)$$

$$\text{and} \quad m V_\infty \frac{d\gamma}{dt} = L - mg \cos \gamma \quad (D13)$$

The small component of the actual thrust force that may contribute (or deprive the aircraft of) some lift is also assumed to be negligible. These are the only simplifying assumptions made thus far.

D2 Rotational Motion

The equations of rotational rigid body motion are expressed by Euler's Law of angular momentum in vector form by

$$D_1 \vec{H}_{CM} = \vec{M}_{CMe} \quad (D14)$$

in which $\vec{M}_{CMe}$ is the resultant moment of all externally applied forces, taken about the CM, and $\vec{H}_{CM}$, the angular momentum about the CM, is the dot product of the inertia tensor and the angular velocity vector. In the case of rotation at angular velocity $\vec{\omega} = \vec{e}_y Q$ about a y-axis of fixed orientation $\vec{H}_{CM}$ takes the simple form

$$\vec{H}_{CM} = \vec{e}_y I_{yy} Q \quad (D15)$$

The direction of the unit vector $\vec{e}_y$ remains fixed, because our reference frame r has no angular velocity components about its other axes, and the pitch moment of inertia I_{yy} of the body is constant. If the resultant moment has no components about the x- and z- axes, and if the pitch moment is denoted by M , then

$$\vec{M}_{CMe} = \vec{e}_y M \quad (D16)$$

so that equations (D14) to (D16) yield the equation of pitch motion in the scalar form

$$I_{yy} \frac{dQ}{dt} = M \quad (D17)$$

It is to be noted that the description of rotational motion with non-zero angular velocity components about the x- and z- axes requires consideration of the full inertia tensor of the body. This becomes necessary when the lateral-directional modes of motion are of interest. These modes, however, can be treated independently in the majority of fixed wing aircraft, mainly due to symmetry about the x-z-plane, not only of the inertia properties, but also of the aerodynamic characteristics.

D3 Kinematic Equations

In order to complete the description of longitudinal motion the equations of motion must be augmented by three kinematic equations. Two of these are differential equations: one that relates pitch angle to pitch rate, and another that relates climb rate to angle of climb. But the third is an algebraic equation, one that relates the angle of climb to the pitch angle and the angle of attack. These kinematic equations are

$$\frac{d\Theta}{dt} = Q \quad (D18)$$

$$\frac{dH}{dt} = V_{\infty} \sin \gamma \quad (D19)$$

$$\text{and} \quad \gamma = \Theta - \alpha \quad (D20)$$

D4 State Equations

The two equations of translational motion, the equation of pitch motion, and the kinematic equations of pitch rate and climb rate form a set of five first order differential equations. These, the *non-linear state equations of longitudinal motion in dimensional form*, have thus been shown to be

$$m \frac{dV_{\infty}}{dt} = T - D - mg \sin(\Theta - \alpha) \quad (D21)$$

$$mV_{\infty} \frac{d\alpha}{dt} = mV_{\infty} Q - L + mg \cos(\Theta - \alpha) \quad (D22)$$

$$I_{yy} \frac{dQ}{dt} = M \quad (D23)$$

$$\frac{d\Theta}{dt} = Q \quad (D24)$$

$$\text{and} \quad \frac{dH}{dt} = V_{\infty} \sin(\Theta - \alpha) \quad (D25)$$

Note that equation (D21) is obtained by substitution from equation (D20) into equation (D12), whereas equation (D22) is obtained by substitution from both equations (D18) and (D20) into equation (D13).

D5 Reference State of Flight

The eventual purpose of our analysis is to investigate the stability of perturbed longitudinal motion. We define a dimensionless perturbation u of airspeed in terms of the airspeed $V_{\infty 0}$ in the reference state of motion such that

$$V_{\infty} = V_{\infty 0} (1 + u) \quad (D26)$$

The perturbation u is not necessarily small, and furthermore, $V_{\infty 0}$ is not necessarily constant, because the reference state of motion may be any state of flight, even accelerated flight. However, we select steady, straight and level flight at zero pitch rate as reference state for our present investigation, because we are interested in the influence of ground proximity on longitudinal stability during a typical landing approach. It is especially the height range of around one wingspan that is of interest, and we assume that the longitudinal and normal acceleration, the pitch rate and the rate of descent, *specifically of the reference state of flight*, will often be negligible in this height range.

Our selection of steady, straight and level flight at zero pitch rate as reference state of flight implies that $\alpha_0 = \Theta_0$, because the angle of climb is zero. This reference state implies furthermore that $V_{\infty 0}$, α_0 , Θ_0 and H_0 are constant, and that $Q_0 = 0$. Considering equations (D21) to (D25) it follows therefore that *the equations of longitudinal equilibrium in steady, straight and level flight at zero pitch rate are*

$$T_0 - D_0 = 0 \quad (D27)$$

$$-L_0 + mg = 0 \quad (D28)$$

$$M_0 = 0 \quad (D29)$$

$$\frac{d\Theta_0}{dt} = 0 \quad (D30)$$

$$\text{and} \quad \frac{dH_0}{dt} = 0 \quad (D31)$$

The aerodynamic coefficients of longitudinal motion, i.e., the lift coefficient C_L , the drag coefficient C_D , the thrust coefficient C_T and the pitch moment coefficient C_M , are considered next.

D6 Lift, Drag and Pitch Moment Coefficients

The lift, drag and pitch moment coefficients are customarily defined such that

$$L = \frac{1}{2} \rho V_{\infty}^2 S_w C_L \quad (D32)$$

$$D = \frac{1}{2} \rho V_{\infty}^2 S_w C_D \quad (D33)$$

$$M = \frac{1}{2} \rho V_{\infty}^2 S_w c_w C_M \quad (D34)$$

in which S_w is the planform area of the wing and c_w is the mean aerodynamic wing chord. (Note that some authors prefer to define the pitch moment coefficient with reference to the mean geometric wing chord, or even the midspan wing chord!) Expressed in terms of the reference airspeed and its perturbation the values of L , D and M of equations (D32) to (D34) are alternatively related to their respective coefficients C_L , C_D and C_M by

$$L = \frac{1}{2} \rho V_{\infty 0}^2 (1+u)^2 S_w C_L \quad (D35)$$

$$D = \frac{1}{2} \rho V_{\infty 0}^2 (1+u)^2 S_w C_D \quad (D36)$$

$$\text{and} \quad M = \frac{1}{2} \rho V_{\infty 0}^2 (1+u)^2 S_w c_w C_M \quad (D37)$$

These expressions are to be substituted into equations (D21) to (D23) when converting the state equations to their dimensionless form.

D7 Propulsion Analysis

Recalling that the equilibrium thrust T_0 is equal to the equilibrium drag D_0 , we *define a thrust controlled propulsion unit* as one designed such that a control input δ_T produces a thrust T of

$$T = D_0 (1 + \delta_T) \quad (D38)$$

This expression for the thrust force can be used directly to eliminate T from the x -equation of longitudinal motion. It follows from equation (D36) that

$$D_0 = \frac{1}{2} \rho V_{\infty 0}^2 S_w C_{D0} \quad (D39)$$

This implies that the thrust produced by the thrust controlled propulsion unit defined above can be expressed alternatively in terms of the equilibrium drag coefficient C_{D0} and the thrust control input δ_T , so that

$$T = \frac{1}{2} \rho V_{\infty 0}^2 S_w C_{D0} (1 + \delta_T) \quad (D40)$$

A *power controlled propulsion unit*, on the other hand, is defined here as one that yields a power output of

$$P = P_0 (1 + \delta_p) \quad (D41)$$

An expression for the thrust T of such a propulsion unit is found by noting that the mechanical power output, i.e. the mechanical power delivered by the propulsion unit to the aircraft, is

$$P = TV_{\infty} \quad (D42)$$

Recalling that $T_0 = D_0$ it follows that the power output in the reference flight condition is

$$P_0 = D_0 V_{\infty 0} \quad (D43)$$

It follows from equations (D41) and (D42), after substitution of P_0 from equation (D43) and solving for T that the thrust produced by a power controlled propulsion unit is

$$T = D_0 V_{\infty 0} V_{\infty}^{-1} (1 + \delta_p) \quad (D44)$$

Substitution of D_0 from equation (D39) and V_∞ from equation (D26) shows that the thrust delivered by the power controlled propulsion unit defined here, expressed in terms of the equilibrium drag coefficient C_{D0} and the power control input δ_P , is given by

$$T = \frac{1}{2} \rho V_{\infty 0}^2 S_w C_{D0} (1+u)^{-1} (1+\delta_P) \quad (D45)$$

We note in passing that a useful approximation of equation (D45) is obtained from the first order truncation of its Taylor series expansion. This approximation, which is quite accurate for small u and δ_P , is

$$T = \frac{1}{2} \rho V_{\infty 0}^2 S_w C_{D0} (1-u+\delta_P) \quad (D46)$$

D8 Dimensionless Form of State Equations

Following the customary approach adopted in flight mechanics the state equations (D21) to (D25) are converted to dimensionless form next. To this end we define a weight coefficient C_W such that

$$mg = \frac{1}{2} \rho V_{\infty 0}^2 (1+u)^2 S_w C_W \quad (D47)$$

The time derivatives and the angular rates, of which the units are radians per second, are converted to dimensionless form too. The time it takes an aircraft to cover a distance of half its mean aerodynamic wing chord when travelling at its reference airspeed is $c_w / (2V_{\infty 0})$, a quantity that is sometimes referred to as an air second, according to McCormick²⁸. In terms of this quantity a dimensionless time parameter τ is defined such that it is related to real time t by

$$\tau = \frac{2V_{\infty 0} t}{c_w} \quad (D48)$$

The time derivatives with respect to real time in the state equations can thus be eliminated in favour of time derivatives with respect to this dimensionless time unit by using the conversion

$$\frac{d(\quad)}{dt} = \frac{2V_{\infty 0}}{c_w} \frac{d(\quad)}{d\tau} \quad (D49)$$

In addition we replace the height H of the CM above the ground with

$$H = H_0 + c_w z \quad (D50)$$

where z is a dimensionless height perturbation, and not necessarily a small one either.

The mass m and the pitch inertia I_{yy} are then eliminated in favour of their respective dimensionless counterparts μ and i_{yy} , which are defined by McCormick²⁸ such that

$$m = \frac{1}{2} \mu \rho S_w c_w \quad (D51)$$

$$\text{and } I_{yy} = \frac{1}{8} i_{yy} \rho S_w c_w^3 \quad (D52)$$

The conversion of the state equation to dimensionless form is accomplished by first substituting D , L and M from equations (D35) to (D37), substituting T from either equation (D40) or equation (D45), depending on the design of the propulsion system, substituting mg from equation (D47), and by using equation (D49) to replace all $d()/dt$ in equations (D21) to (D25) with their counterparts in terms of τ . This is followed, secondly, by division of equations (D21) and (D22) by $\rho V_{\infty 0}^2 S_w / 2$, equation (D23) by $\rho V_{\infty 0}^2 S_w c_w / 2$, equation (D24) by $2V_{\infty 0} / c_w$ and equation (D25) by $V_{\infty 0}$. We note finally

that $\Theta = \Theta_0 + \theta$, in which Θ_0 is constant and θ is a perturbation of pitch angle, which brings us to the *dimensionless form of the non-linear state equations*

$$2\mu \frac{du}{d\tau} = C_{D0}(1 + \delta_T) - (1 + u)^2 C_D - C_W \sin(\Theta - \alpha) \quad (D53a)$$

or
$$2\mu \frac{du}{d\tau} = C_{D0}(1 + u)^{-1}(1 + \delta_P) - (1 + u)^2 C_D - C_W \sin(\Theta - \alpha) \quad (D53b)$$

$$2\mu \frac{d\alpha}{d\tau} = 2\mu q - (1 + u)^2 C_L + C_W \cos(\Theta - \alpha) \quad (D54)$$

$$i_{yy} \frac{dq}{d\tau} = (1 + u)^2 C_M \quad (D55)$$

$$\frac{d\theta}{d\tau} = q \quad (D56)$$

and
$$2 \frac{dz}{d\tau} = \sin(\Theta - \alpha) \quad (D57)$$

Equation (D53a) applies when the propulsion unit controls the thrust directly in response to an input δ_T , and equation (D53b) applies when the propulsion unit controls power output directly in response to an input δ_P . The former is approximately true for jet propulsion, and the latter is approximately true for propeller engines.

We emphasise that the perturbations of the state variables in equations (D53) to (D57) have not been assumed to be small yet, so that these equations are restricted only by the assumptions of

- negligible inertia coupling and aerodynamic coupling between the longitudinal and lateral-directional modes of motion,
- negligible normal component of the thrust vector, and
- zero pitch rate in the reference flight condition.

The assumption of zero steady climb rate in the reference flight condition has not been used thus far. The restriction of zero steady pitch rate, in addition, can be relaxed easily (e.g. if a reference state of steady pitching is of interest) by replacing q in equation (D54) with $Q_0 + q$, in which the steady initial pitch rate Q_0 is normalised with respect to $2V_{\infty 0}/c_w$ too.

D9 Small Perturbational Form of State Equations

The small perturbational form of the state equations is obtained by performing a first order Taylor series expansion of each non-linear term in equations (D53) to (D57) and subtracting from the resulting equations the following dimensionless version of the equilibrium equations

$$0 = C_{T0} - C_{D0} \quad (D58)$$

$$0 = -C_{L0} + C_{0w} \quad (D59)$$

$$0 = C_{M0} \quad (D60)$$

$$\frac{d\Theta_0}{d\tau} = 0 \quad (D61)$$

$$\text{and} \quad \frac{dH_0}{d\tau} = 0 \quad (D62)$$

This yields the penultimate formulation of the linearised equations of longitudinal motion, in that expressions for small perturbations of C_L , C_D and C_M are still to be substituted. The results are

$$2\mu \frac{du}{d\tau} = C_{D0}\delta_T - \Delta C_D - 2C_{D0}u - C_w(\theta - \alpha) \quad (D63a)$$

$$\text{or} \quad 2\mu \frac{du}{d\tau} = C_{D0}\delta_P - \Delta C_D - 3C_{D0}u - C_W(\theta - \alpha) \quad (\text{D63b})$$

$$2\mu \frac{d\alpha}{d\tau} = 2\mu q - 2C_{L0}u - \Delta C_L \quad (\text{D64})$$

$$i_{yy} \frac{dq}{d\tau} = \Delta C_M \quad (\text{D65})$$

$$\frac{d\theta}{d\tau} = q \quad (\text{D66})$$

$$\text{and} \quad 2 \frac{dz}{d\tau} = \theta - \alpha \quad (\text{D67})$$

We emphasise that α in equations (D63) to (D67) is the *small perturbation of the angle of attack now and from here on, not the total angle of attack that it signified before*. (Without this change of definition one would have had to use $\Delta\alpha$, or some other symbol further on.) The perturbations of all the other state variables have also been assumed to be small now. The final step is to formulate expressions for the perturbations of the lift, drag and pitch moment coefficients ΔC_L , ΔC_D , and ΔC_M in terms of their partial derivatives with respect to the relevant state variables and the control variables (and, in addition, the partial derivatives with respect to the rates of change of the state variables, whenever the influence of unsteady aerodynamics is expected to be significant).

The derivatives of C_L , C_D and C_M with respect to u are all ignored, because they reflect Reynolds number effects, which we assume to be negligible in a first order analysis of subsonic flight. The perturbations of C_L , C_D and C_M are thus approximated to first order by

$$\Delta C_L = C_{L\alpha}\alpha + C_{Lq}q + C_{L\theta}\theta + C_{Lz}z + C_{L\dot{\alpha}}\dot{\alpha} + C_{L\dot{q}}\dot{q} + C_{L\delta t}\delta_t \quad (\text{D68})$$

$$\Delta C_D = C_{D\alpha}\alpha + C_{Dq}q + C_{D\theta}\theta + C_{Dz}z + C_{D\dot{\alpha}}\dot{\alpha} + C_{D\dot{q}}\dot{q} + C_{D\delta t}\delta_t \quad (D69)$$

$$\Delta C_M = C_{M\alpha}\alpha + C_{Mq}q + C_{M\theta}\theta + C_{Mz}z + C_{M\dot{\alpha}}\dot{\alpha} + C_{M\dot{q}}\dot{q} + C_{M\delta t}\delta_t \quad (D70)$$

We note that the derivatives of C_L , C_D and C_M with respect to θ only come into play in close ground proximity, and their effect is quite small even then. The derivatives with respect to $\dot{\alpha}$ and $\dot{q}$, on the other hand, are found to affect the damping ratio of the short period mode of motion significantly. They have, however, only a small influence on the phugoid mode, a mode of motion in which even the q -derivatives play but a small role, albeit not entirely negligible. (Note that the influence of the derivative with respect to $\dot{u}$ has been neglected here, but they too have to be considered when “added mass” effects are significant in stability analysis, as in the case of airships, or other craft that depend on buoyancy.)

Equations (D63) to (D67), along with equations (D68) to (D70), are the *linearised equations of longitudinal motion*. We do not substitute from equations (D68) to (D70) here, because no better perspective is gained by the fairly lengthy expressions one then obtains. This is done effectively, however, in the discussions of longitudinal stability in Chapter 4, where the formulations of equations (D68) to (D70) are accounted for, along with the other contributions from equations (D63) to (D67), in the A- and B-matrices of the generalised eigenvalue problem of longitudinal motion.

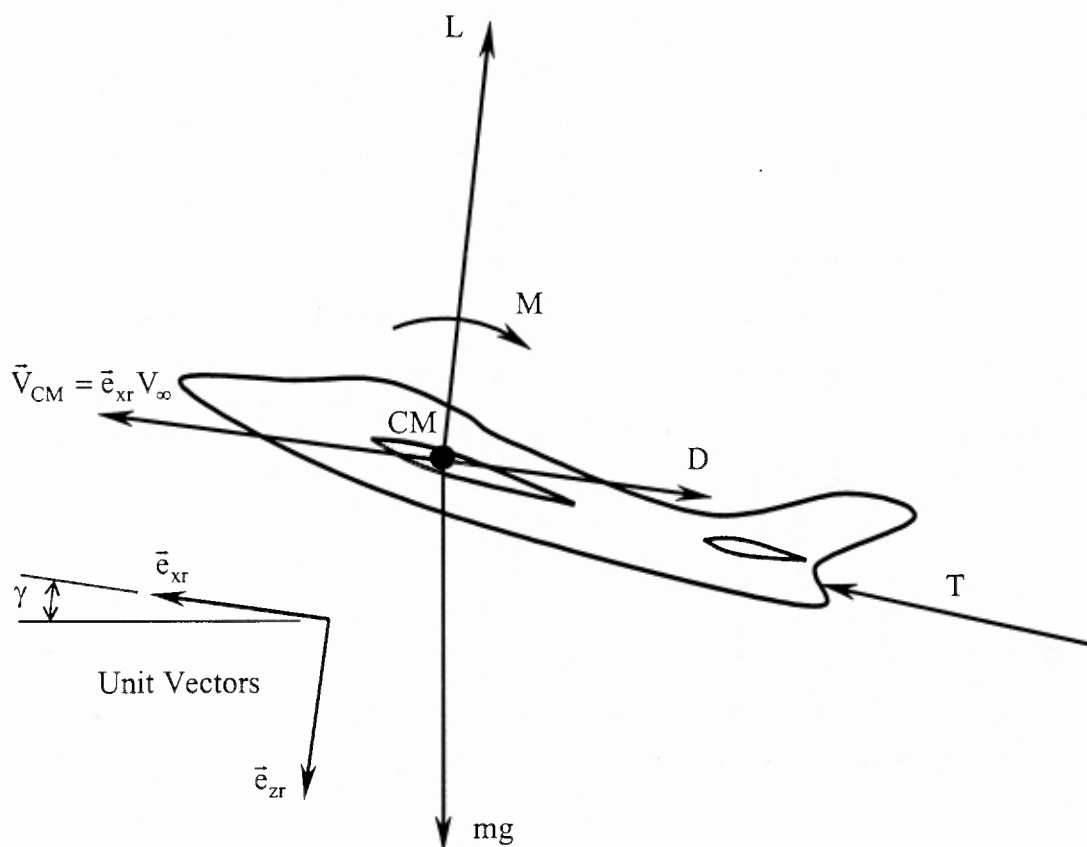


Figure D1: Schematic side view of aircraft in longitudinal motion illustrating the various force components and the pitch moment.

APPENDIX E

TRIM ANALYSIS

The trim problem of longitudinal motion of an aircraft amounts essentially to the solution of the angle of attack and the tail angle (i.e. the setting angle of the all-movable horizontal stabilizer, or the elevator angle, depending on which tail design is used) that would provide the required lift and ensure equilibrium of pitch moments at given air speed and angle of climb. It may also involve the simultaneous solution of the required throttle setting, if the thrust contributes significantly to the pitch moment, but should be acceptable to ignore this contribution in a first order analysis. Even though the lift depends primarily on the angle of attack, whereas the pitch moment depends primarily on the tail angle, the overall lift at fixed angle of attack depends to some extent on the tail angle, with the result that the equilibrium equations of lift and pitch moment must usually be solved simultaneously, rather than one after the other.

E1 Trim Conditions

We assume that the contributions of the thrust force, both to the overall lift and to the pitch moment, is negligible compared to that of the lift forces of the wing and tail (the force on the latter is usually downward). The trimmed lift condition for steady rectilinear flight at zero pitch rate under these assumptions, and at small angle of climb γ , is

$$C_L - C_W \approx 0 \quad (\text{E1})$$

where C_W is the weight coefficient, which is merely the weight $W = mg$ normalised with respect to the product of dynamic pressure and wing planform area. The trimmed moment condition, on the other hand, is merely

$$C_M = 0 \quad (\text{E2})$$

If the equilibrium thrust is required, it can be computed afterwards using

$$C_T = C_D + C_w \gamma \quad (\text{E3})$$

We note that C_L and C_M depend primarily on the circulation distributions of the wing and tail, which in turn depend primarily on the angles of attack α_w and α_t at the wing and tail. (These exclude the downwash contributions.) And we note furthermore that, in equilibrium, these angles of attack are $\alpha_w = \alpha_o$ and $\alpha_t = \alpha_o - \delta_t$, where α_o is the equilibrium angle of attack of the wing, and δ_t is the setting angle of the all-movable tail (positive when the leading edge goes down). The dependence of C_L and C_M on α is non-linear, even though the non-linearity is probably quite weak at small angles of attack. The solution of the values of α and δ_t that satisfy the trim conditions in equations (E1) and (E2) therefore requires, in general, the iterative solution of two non-linear equations, even though a direct solution based on the assumption of linear dependence of C_L and C_M on α and δ_t may yield a very good first approximation.

E2 Mathematical Formulation of Trim Problem

We formulate the trim problem mathematically by defining

$$f_1(\alpha, \delta_t) = C_L - C_w \quad (\text{E4})$$

$$\text{and } f_2(\alpha, \delta_t) = C_M \quad (\text{E5})$$

from which the trim state variables $\bar{x}$ are collected in the vector form

$$\bar{x} = [\alpha \quad \delta_t]^T \quad (\text{E6})$$

The two functions, i.e. the trim conditions, of which the zeros are required are then collected in the vector form

$$\bar{f}(\bar{x}) = [f_1(\bar{x}) \quad f_2(\bar{x})]^T \quad (\text{E7})$$

in which the bar on f indicates that two different functions f_1 and f_2 are considered, whereas the bar on x indicates that each of f_1 and f_2 may be a function of the two independent variables $x_1 = \alpha$ and $x_2 = \delta_t$.

The mathematical statement of the trim problem, which we intend to solve using a multivariable variation of the Newton-Raphson algorithm, is completed by defining the Jacobian matrix of this particular problem to be given by

$$\bar{J}(\bar{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial \alpha} & \frac{\partial f_1}{\partial \delta_t} \\ \frac{\partial f_2}{\partial \alpha} & \frac{\partial f_2}{\partial \delta_t} \end{bmatrix} \quad (E8)$$

We note in passing that a more general definition of the Jacobian matrix is

$$\bar{J}(\bar{x}) = \frac{\partial \bar{f}}{\partial \bar{x}} = \bar{f}_{\bar{x}} \quad (E9)$$

in which $\bar{f}_{\bar{x}}$ is defined for an n -dimensional problem by

$$\bar{f}_{\bar{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (E10)$$

The mathematical statement of the trim problem is therefore to find the values of the elements of $\bar{x}$ for which $\bar{f}(\bar{x}) = \bar{0}$.

One method of solving the problem is obtained by noting that the first order truncation of the Taylor series yields, in matrix-vector form, the result that

$$\bar{f}(\bar{x}_2) \approx \bar{f}(\bar{x}_1) + \bar{J}(\bar{x}_1)(\bar{x}_2 - \bar{x}_1) \quad (\text{E12})$$

This approximation is fairly accurate if no element of $\bar{x}_2$ differs much from those of $\bar{x}_1$. If, in addition, $\bar{x}_1$ happens to be close to the exact solution of $\bar{f}(\bar{x}) = \bar{0}$, then one may assume that $\bar{f}(\bar{x}_2) \approx \bar{0}$, which yields the result, on substitution into equation (E11), that

$$\bar{0} \approx \bar{f}(\bar{x}_1) + \bar{J}(\bar{x}_1)(\bar{x}_2 - \bar{x}_1) \quad (\text{E13})$$

Solving for $\bar{x}_2$, which obviously requires that $\bar{J}(\bar{x}_1)$ be non-singular, it follows that

$$\bar{x}_2 \approx \bar{x}_1 - [\bar{J}(\bar{x}_1)]^{-1} \bar{f}(\bar{x}_1) \quad (\text{E14})$$

This is the multivariable version of the Newton-Raphson algorithm for finding the value(s) of $\bar{x}$ at which all of the multivariable functions in $\bar{f}(\bar{x})$ are zero. In the case of a single function $f(x)$ of one variable x it reduces to the well-known form

$$x_2 \approx x_1 - f(x_1)/f'(x_1) \quad (\text{E15})$$

E3 Numerical Implementation

E3.1 Case 1

If the elements $\frac{\partial f_i}{\partial x_j}$ of the Jacobian matrix are all available as explicit functions of $\bar{x}$, then the numerical procedure is as follows:

1. Obtain an initial estimate $\bar{x} = \bar{x}_1$, and specify a set of convergence tolerances $\bar{\epsilon}$
2. Compute $\bar{f} = \bar{f}(\bar{x})$
3. Compute $\bar{J} = \bar{J}(\bar{x})$
4. Compute $\Delta \bar{x} = -\bar{J}^{-1} \bar{f}$
5. Check if the elements of $\Delta \bar{x}$ are all smaller than the corresponding elements of $\bar{\epsilon}$. If they are, then $\bar{x}$ is the required solution of $\bar{f}(\bar{x}) = \bar{0}$ and the procedure is terminated. If not, then substitute $\bar{x} \leftarrow \bar{x} + \Delta \bar{x}$ and repeat steps 2 to 5.

E3.2 Case 2

If the elements $\frac{\partial f_i}{\partial x_j}$ of $\bar{J}(\bar{x})$ are not available explicitly, however, or are too cumbersome to obtain, then the numerical procedure is as follows:

1. Same as above
2. Select a set of increments $[\Delta x_1 \quad \Delta x_2 \quad \cdots \quad \Delta x_n]$ of which no elements are zero.

Then define

$$\Delta \bar{x}_1 = [\Delta x_1 \quad 0 \quad 0 \quad \cdots \quad 0]^T$$

$$\Delta \bar{x}_2 = [0 \quad \Delta x_2 \quad 0 \quad \cdots \quad 0]^T$$

$\vdots$

$$\Delta \bar{x}_n = [0 \quad 0 \quad 0 \quad \cdots \quad \Delta x_n]^T$$

and compute $\bar{x}_2 = \bar{x}_1 + \Delta \bar{x}_1$

$$\bar{x}_3 = \bar{x}_2 + \Delta \bar{x}_2$$

$\vdots$

$$\bar{x}_n = \bar{x}_{n-1} + \Delta \bar{x}_{n-1}$$

and, finally, let $\bar{x} = \bar{x}_n + \Delta \bar{x}_n$

3. Compute $\bar{f}_1 = \bar{f}(\bar{x}_1)$

$$\bar{f}_2 = \bar{f}(\bar{x}_2)$$

$\vdots$

- $$\bar{f}_n = \bar{f}(\bar{x}_n)$$
- and $\bar{f} = \bar{f}(\bar{x})$
4. Define $\bar{X}_1 = [\bar{x}_1 \quad \bar{x}_2 \quad \cdots \quad \bar{x}_{n-1} \quad \bar{x}_n]$
 $\bar{X}_2 = [\bar{x}_2 \quad \bar{x}_3 \quad \cdots \quad \bar{x}_n \quad \bar{x}]$
 $\bar{F}_1 = [\bar{f}_1 \quad \bar{f}_2 \quad \cdots \quad \bar{f}_{n-1} \quad \bar{f}_n]$
 $\bar{F}_2 = [\bar{f}_2 \quad \bar{f}_3 \quad \cdots \quad \bar{f}_n \quad \bar{f}]$
and compute $\Delta \bar{X} = \bar{X}_2 - \bar{X}_1$
and $\Delta \bar{F} = \bar{F}_2 - \bar{F}_1$
 5. Compute $\bar{J} = \Delta \bar{F} (\Delta \bar{X})^{-1}$
 6. Compute $\Delta \bar{x} = -\bar{J}^{-1} \bar{f}$
 7. Terminate the procedure if the convergence criteria are all satisfied. If not, then let
 $\bar{x}_{old} \leftarrow \bar{x}$ and $\bar{f}_{old} \leftarrow \bar{f}$, substitute $\bar{x} \leftarrow \bar{x} + \Delta \bar{x}$ and compute $\bar{f} = \bar{f}(\bar{x})$
Substitute $\bar{x}_1 \leftarrow \bar{x}_2$ and $\bar{f}_1 \leftarrow \bar{f}_2$
 $\bar{x}_2 \leftarrow \bar{x}_3$ and $\bar{f}_2 \leftarrow \bar{f}_3$
 $\vdots$
 $\bar{x}_{n-1} \leftarrow \bar{x}_n$ and $\bar{f}_{n-1} \leftarrow \bar{f}_n$ and
 $\bar{x}_n \leftarrow \bar{x}_{old}$ and $\bar{f}_n \leftarrow \bar{f}_{old}$.

Then repeat from step 4.

E4 Computation of Aerodynamic Derivatives

After solving the trim problem for each of the selected aircraft configurations, and at each of the selected heights in the height range of interest, the aerodynamic coefficients were computed for the following small perturbations of state around the trimmed states:

- the angle of attack and the pitch angle were both perturbed (in turn, of course) by 0.1 degree above and below the trim values trimmed value $\alpha = \alpha_o$ and $\theta = \theta_o$, in the numerical computation of the α -derivatives and the θ -derivatives

- perturbations of $-q$ and q were applied when the q -derivatives were computed, where q was selected such that it produces a difference in angle of attack of 0.1 degree between the leading edges of the wing and tail. Note that q , as defined in the nomenclature, is expressed in terms of the pitch rate Q by $q = \Delta Q c / (2 V_{\infty 0})$. This implies that q is approximately equal to the amount (in rad) by which the *local* angle of attack increases per $\frac{1}{2}$ -chord in the downwind direction of an aircraft that simultaneously moves at a speed of $V_{\infty 0}$ m/s and pitches at a *small* rate ΔQ rad/s. Only when q is selected on this basis can ΔQ be regarded as a sufficiently small perturbation for the numerical computation of the q -derivatives to be reasonably accurate.
- the height was perturbed by 10% of the wing chord above and below the trimmed height H_0 when the z -derivatives were computed

These perturbations were found to be small enough to give sufficiently accurate numerical estimations of the aerodynamic derivatives of C_L , C_D and C_M with respect to α , q , θ and z . The aerodynamic derivatives were then estimated, for example in the case of $C_{L\alpha}$, by letting

$$C_{L\alpha} = \frac{\partial C_L}{\partial \alpha} \approx \frac{C_{L2} - C_{L1}}{\alpha_2 - \alpha_1} = \frac{1}{2} \frac{\Delta C_L}{\Delta \alpha} \quad (\text{E16})$$

$$\text{if } \alpha_1 = \alpha_0 - \Delta \alpha \quad \text{and} \quad \alpha_2 = \alpha_0 + \Delta \alpha \quad (\text{E17})$$

were used. In order that the aerodynamic derivatives required in our linearised equations of motions be obtained, *each state variable had to be perturbed while keeping the other state variables constant at their trimmed values*. This required, when α was perturbed, that the *perturbation of α be realised by prescribing perturbed angles of descent of $\alpha_{d1} = \alpha_{d0} - \Delta \alpha$ and $\alpha_{d2} = \alpha_{d0} + \Delta \alpha$ at constant $\Theta = \Theta_0$ in the computation of C_{L1} and C_{L2} , as required in equation (E16) above. (The vortex lattice algorithm returns C_{D1} and C_{D2} , and C_{M1} and C_{M2} as well when the C_L 's are computed, so that the α -derivatives of these coefficients were obtained in the same manner in the same run.) Note that an*

increase in θ affects the aerodynamic coefficients in a different way than does an increase in the rate of descent α_d . *But this is true only when in ground proximity, where an increase in θ brings the tail significantly closer to its image below the ground, which an increase in α_d does not do.*

The θ -derivatives, on the other hand, were obtained by perturbing not only θ , but the angle of climb too, and by exactly the same amount, in order that the angle of attack remains $\alpha = \alpha_0$, as required by the definition of the θ -derivatives. (The θ -derivatives were found to be of negligible consequence in our stability analyses.)

The computation of the q -derivatives, the z -derivatives and the δ_t -derivatives, finally, were performed directly (i.e., without the subtle approach required in the computation of the α - and θ -derivatives), because perturbations of q , z and δ_t do not affect one another, nor the values of α and θ .

E5 Concluding Remarks

The second of the Newton-Raphson procedures outlined above was used to obtain the trim solutions of α_0 and δ_t as function of the height of the centre of mass above the ground. (The influence of non-zero equilibrium angle of descent was also investigated briefly.) The procedure was found to converge very rapidly indeed, typically to better than 10^{-6} degrees in less than 6 iterations, including the initial three computations of $\bar{f}$. This perhaps indicates that the trim problem is only mildly non-linear, because if it were linear, then exact convergence would have been obtained on the fourth iteration.

We note in conclusion that the procedure may become less effective with an increasing number of variables if the Jacobian is not available in explicit form, because in the case of an n -variable problem one finds that $n+1$ computations of $\bar{f}$ must be performed before even starting the iteration loop.

APPENDIX F

COMPUTATIONAL STRATEGY AND MATLAB PROCEDURES

A number of Matlab programs were developed to perform the vortex lattice analysis of the aerodynamics of a wing and tail, and a stability and control analysis of its small perturbational longitudinal motion relative to its trim state. The computational strategy followed in these programs is discussed briefly here. The Matlab code of the programs themselves are listed at the end of this Appendix.

F1 Functional Description of Programs

The brunt of our computations were done by programs contained in the following m-files:

- **AeroCoefVLM.m**, a function file that accepts data describing the aircraft geometry and a given state of flight as input. It computes and returns, firstly, the discrepancy between the actual lift coefficient and the required lift coefficient, and secondly, the pitch moment coefficient about the centre of mass. These are used to solve the trim problem. It also returns the lift coefficient and the drag coefficients computed by the leading edge singularity method, by the conventional vortex lattice method, and by the far-field method. In addition the separate contributions of the wing and tail to the lift and drag coefficients are returned, should these be required.
- **TestAeroCoefVLM.m**, a function file that, after judicious commenting and uncommenting of program instructions, calls **AeroCoefVLM** to analyse a rectangular flat wing alone with $AR = 2$ out of ground effect, a wing alone with a Warren 12 planform out of ground effect, and a rectangular flat wing alone of very high aspect ratio at non-zero angle of attack and non-zero pitch rate. The results of the former two cases were found to compare very well with the results of Lan¹⁴, and the results of the latter case was found to be very close indeed to the exact results obtained using the two-dimensional thin wing theory as described in Appendix B.
- **TrimDerivsVLM.m**, a function file that, based on the multivariable Newton-Raphson iteration procedure described in Appendix E, and using the values of C_L and C_M obtained from **AeroCoefVLM**, computes the trim states of a given aircraft

at given lift coefficient, angle of descent and pitch rate over a specified range of heights. It then proceeds to compute all the aerodynamic derivatives, those with respect to the five longitudinal state variables, and those with respect to the elevator control input. These results are displayed on screen, if required, and written to data files that are later used in the stability and control analysis.

- **TrimDerivsMain.m**, a function file that defines a range of heights above the ground, *varies selected design parameters through suitable ranges*, and then calls **TrimDerivsVLM** to compute and store sets of trim states and aerodynamic derivatives, *one set of results (one that covers the specified the height range) for each selected aircraft configuration*.
- **StabContrAnalysis.m**, a function file that reads the stability and control derivatives produced beforehand by **TrimDerivsVLM** and **TrimDerivsMain**, and then performs an analysis of longitudinal stability as function of height for a given set of design parameters. The stability derivatives (with respect to the state variables) are used in assessing the stability of perturbed longitudinal motion. In addition to these the control derivatives (with respect to the two control inputs) are used to compute the elevator impulse response by simultaneous state space solution of the five linearised longitudinal equations of motion that were formulated in Chapter 4 and Appendix D. The results of the stability analysis are displayed in root locus plots of the short period and long period modes of motion with height as parameter. Plots are produced of the largest real part of all stability roots against height, and of C_{Lz} and C_{Dz} and their critical values C_{Lzcrit} and C_{Dzcrit} against height, as obtained by application of Routh's stability criteria, *plots that serve to explain the causes of longitudinal instability*. Finally response plots of the five state variables to an impulsive elevator input are produced for selected heights above the ground, *plots that serve to illustrate the nature of the instability of the long period modes of motion of a stick fixed, fixed wing aircraft in perturbed longitudinal motion in close ground proximity*. The control analysis involves the feedback of height to the elevator, which was found to stabilise the long period modes of motion without significantly affecting the short period mode of motion.

- **StabContrPostProc.m**, a function file that loops through ranges of values of various design parameters. It calls **StabContrAnalysis** each time to do a stability analysis (recall that **StabContrAnalysis** uses previously computed vortex lattice results), and then produces screen plots that illustrate the sensitivity of longitudinal stability in ground proximity. The sensitivity with respect to design parameters such as CM position, load, tail length, tail height, et cetera, is thus illustrated.
- **SummaryResults.m**, a script file called by **StabContrPostProc** to summarise the results of the stability analysis. It produces tables and plots of the critical height as function of CM position, height feedback gain constant, load factor, tail height, tail length or wing aspect ratio. The critical height is estimated by linear interpolation at the points where sign changes of the various stability criteria occur. The purpose is, on the one hand, to illustrate the influence of these design parameters on longitudinal stability in ground proximity, and on the other hand, to give an indication of the cause of the instability.

Examples of the results produced by these programs are listed in Tables 3 to 8, and illustrated in Figures R2 to R9.

F2 Structure of Programs

The vortex lattice program **AeroCoefVLM** discretises the wing and tail using a cosine type distribution, as illustrated in Figures R1 a) to d). This involves the computation of the x-, y- and z-co-ordinates of all vortex corner points, vortex control points and vortex midpoints on the wing and tail. Using these co-ordinates the influence matrices required in the solution of the flow tangency problem are computed, along with those required for the computation of the induced drag by the three methods mentioned above. The flow tangency problem is then solved to find the required circulation strength of the horseshoe vortices by which the vortex sheets of the wing and tail are modelled. These circulation values are then used to compute the induced velocities at the bound vortex midpoints for computation of the induced drag by the conventional vortex lattice method. They are also used for the direct computation of the lift and pitch moment coefficients, and, in a more indirect manner, for the computation of the induced drag coefficient by the leading edge singularity approach and the far-field approach. Screen

plots that illustrate the aircraft configuration and the lattice geometry are produced, if required, and a summary of the aerodynamic coefficients is displayed on the text screen (Matlab Command Window).

In addition to the main function (program) contained in **AeroCoefVLM**, the file also contains four separate nested functions. These are:

- **discret**, for performing the cosine discretisation, including provision for non-zero sweep angle and non-zero taper ratio
- two functions, **inf_matz** and **inf_mavi**, that accept the relevant vortex co-ordinates as input, and then compute and return all the required influence matrices.
- a function **InflueCoeffs** called on by **inf_matz** and **inf_mavi** to compute the influence coefficients of a single horseshoe vortex (with given corner points) at another given point where the induced velocity is required.

The trim and derivative program **TrimDerivsVLM** is a function file without any nested functions. It accepts the description of the aircraft and its required trim condition as input, and then computes the angle of attack and the tail angle that simultaneously provides the required lift coefficient and zero pitch moment coefficient, using the computational procedure outlined in Appendix E. The results of these computations are displayed on the text screen, and printed to a number of text files. One of these contains the same mixture of text and numbers as the screen display, but the others contain only numerical results, to be read when performing the stability analysis.

As mentioned above, **StabContrAnalysis** reads the stability and control derivatives produced by **TrimDerivsVLM** and performs a longitudinal stability analysis. It contains three nested functions, one that constructs the coefficient matrices of the state space equations for a given set of aerodynamic derivatives, another one that computes and saves the eigenvectors corresponding to the characteristic roots obtained for these aerodynamic derivatives, and a third that merely constructs a character string that is used in the titles of the root locus plots to indicate the range of heights to which the loci apply. The main actions taken by **StabContrAnalysis** are:

- extracting the aerodynamic coefficients in the required form and order from the data files produced by **TrimDerivsVLM**
- computing coefficients matrices for various combinations of options in order to assess the importance of the separate influences of the various aerodynamic derivatives.
- computing eigenvalues using various simplifying assumptions. The influence of these assumptions are assessed by tabling the eigenvalues that they yield and comparing them by visual inspection with eigenvalues that were obtained *without* making that particular assumption
- computing eigenvectors, scaling them suitably in order that their physical meaning can be interpreted, and saving them for later inspection
- application of Routh's criteria to produce stability margins of the height derivatives of the lift and drag coefficients. These are plotted later against height in order to reveal which of them are responsible for the longitudinal instability encountered in ground proximity
- solving the state equations for an impulsive elevator input, with and without feedback of ground height to the elevator
- producing screen tables, text files and graphs of results

The post-processing program **StabContrPostProc** calls firstly on **StabContrAnalysis** to perform the actions outlined above, looping through assigned ranges of the values of a number of design parameters, such as CM position, tail height, et cetera. It produces the following:

- one set of graphs for each design parameter considered. Each member of this set depicts the influence of height on a particular stability margin, showing a separate curve for each value of the selected design parameter.
- one summarising graph that displays the critical height as function of the value of the selected design parameter, as judged by four different stability criteria. The critical height in this graph is estimated by linear interpolation at the points where sign changes of the various stability criteria occur. The purpose of the graph is, on the one hand, to illustrate the influence of these design parameters on longitudinal

stability in ground proximity, and on the other hand, to give an indication of the cause of the instability.

F3 Listings of Matlab Code

F3.1 To Get Input Data, Call **CheroInput.m**

```
function [bw, cw, Sw, ARw, bt, ct, St, ARt, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...  
    Lle, Lle_, lam, lam_, Cw] = CheroInput  
% function [bw, cw, Sw, ARw, bt, ct, St, ARt, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...  
%     Lle, Lle_, lam, lam_, Cw] = CheroInput  
  
% RAW DATA OF CHEROKEE WING AND TAIL OBTAINED FROM McCORMICK pp. 148  
bw = 360*0.0254;           % Wing span in m  
cw = 63*0.0254;           % Wing chord in m  
bt = 120*0.0254;          % Tail span in m  
ct = 30*0.0254;           % Tail chord in m  
l_AcT_CM = 162*0.0254;    % Distance of tail AC behind nominal CM in m  
l_CM_LEw = 13*0.0254;     % Distance of nominal CM behind wing LE in m  
h_AcT_ACw = 0;            % Height of tail AC above wing AC in m  
h_LEt_LEw = 0;            % Height of tail LE above wing LE in m  
  
% WING AND TAIL PLANFORM AREAS, ASPECT RATIOS AND HORIZONTAL TAIL VOLUME  
Sw = bw*cw;               % Wing planform area in m^2  
St = bt*ct;               % Tail planform area in m^2  
ARw = bw/cw;              % Aspect ratio of wing  
ARt = bt/ct;              % Aspect ratio of tail  
Vt = St/Sw*l_AcT_CM/cw;   % Horizontal tail volume  
  
% WING AND TAIL SWEEP ANGLES AND TAPER RATIOS  
Lle = 0; Lle_ = 0; lam = 1; lam_ = 1;  
  
% DISTANCE OF TAIL'S LEADING EDGE BEHIND WING'S LEADING EDGE  
l_ACw_LEw = cw/4;         % Distance of 2D wing AC behind wing LE in m  
l_AcT_LEt = ct/4;         % Distance of 2D tail AC behind tail LE in m  
l_ACw_CM = l_ACw_LEw - l_CM_LEw; % Distance of wing AC behind CM in m  
l_AcT_ACw = l_AcT_CM - l_ACw_CM; % Distance of tail AC behind wing AC in m  
l_LEt_LEw = l_AcT_ACw + l_ACw_LEw - l_AcT_LEt; % Distance of tail LE behind wing LE in m  
  
% Nominal weight coefficient  
Cw = 0.543;               % Used in trim analysis by setting CLo = Cw
```

F3.2 To Get More Input Data, Call **MoreCheroInput.m**

```
function [bw, cw, Sw, mu, iyy, f, CDmin, CM_ad, CZ_ad, CM_qd, Qo, kd, ...  
    E_Short_0, E_Long_0] = MoreCheroInput  
% function [bw, cw, Sw, mu, iyy, f, CDmin, CM_ad, CZ_ad, CM_qd, Qo, kd, ...  
%     E_Short_0, E_Long_0] = MoreCheroInput  
  
% Call CheroInput (just to get bw, cw and Sw)  
OldDir = cd;  
InDir = 'c:\my_documents\Navorsing\Phd00\Matlab5\ThesisJunJul2000\InputData';  
feval('cd', InDir)  
eval('['bw, cw, Sw] = CheroInput;')  
feval('cd', OldDir)  
  
% Additional aircraft data (required in stability analysis) from McCormick  
mu = 86.6; % Dimensionless mass (pp. 562)  
iyy = 210; % Dimensionless pitch moment of inertia (pp. 562)  
f = 0.5; % Equivalent flat plate area in m^2 McCormick pp. 562  
CDmin = f/Sw; % Parasite drag coefficient  
CM_ad = -3.32; % From McCormick pp. 564  
CZ_ad = -1.29; % From McCormick pp. 563  
CM_qd = 0; % Assumed by McCormick (negligibly small compared to iyy)  
Qo = 0; % For straight and level flight  
kd = 3; % For propeller aircraft  
  
% Short period and long period roots of Cherokee obtained from McCormick (pp. 571)  
% at 1500 m altitude and 50 m/s (pp. 562)  
E_Short_0 = [[-0.0388+0.0567i];[-0.0388-0.0567i]];  
E_Long_0 = [[-0.000442+0.00397i];[-0.000442-0.00397i]];
```


F3.3 To Perform VLM Analysis, Call **AeroCoefVLM.m**

```
function [f, Coeffs, yva, yvb, Gam_mat, Gam_mat_] = ...

    AeroCoefVLM(theta, alpha_d, delta_t, Cw, H_g_b, ...
    Nc, Nc_, Ns, Ns_, b, c, S, b_, c_, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
    Lle, Lle_, lam, lam_, q, SheetOption, LargeAngles, option)

%
%
% #####
% VORTEX LATTICE ANALYSIS OF WING AND TAIL WITH PROVISION FOR GROUND
% PROXIMITY, SWEEP ANGLE AND PITCH RATE. WING ALONE CAN BE ANALISED BY
% SPECIFYING VERY SMALL SPAN AND CHORD FOR TAIL. LIFTING LINE THEORY CAN
% BE APPROXIMATED BY SPECIFYING SINGLE CHORDWISE SEGEMENT, BUT MORE SPAN-
% WISE SEGMENTS. LANCASTER MODEL (SINGLE HORSESHOE VORTEX PER SURFACE) IS
% ALSO ALLOWED FOR.
% #####
%
% function [f, Coeffs] = ...
% AeroCoefVLM(theta, alpha_d, delta_t, Cw, H_g_b, ...
% Nc, Nc_, Ns, Ns_, b, c, S, b_, c_, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
% Lle, Lle_, lam, lam_, q, SheetOption, LargeAngles, option)
%
%
% INPUT DATA
% =====
%
%   theta = nose up pitch angle of wing chord
%   alpha_d = angle of descent
%   delta_t = nose down tail angle relative to wing chord
%   Cw = weight coefficient (= REQUIRED lift coefficient)
%   H_g_b = height above ground per wing span
%   Nc = no. of chord panels on wing
%   Nc_ = no. of chord panels on tail
%   Ns = no. of span panels on wing
%   Ns_ = no. of span panels on tail
%   b = wing span
%   c = wing chord
%   S = wing planform area
%   b_ = tail span
%   c_ = tail chord
%   l_LEt_LEw = distance of leading edge of tail behind leading edge of wing
%   h_LEt_LEw = height of leading edge of tail above leading edge of wing
%               (the latter two are along and normal to wing chord, resp.)
%   l_CM_LEw = distance of centre of mass behind leading edge of wing
%   Lle = leading edge sweep angle of wing
%   Lle_ = leading edge sweep angle of tail
%   lam = taper ratio c_tip/c_midspan of wing
%   lam_ = taper ratio of tail
%   q = normalised pitch rate  $q = Q \cdot c / (2 \cdot V_{inf})$ 
%   SheetOption = 0 for horisontal flat vortex sheets passing through 1/4-chords
%                 = 1 for flat vortex sheets along free stream velocity (including
%                 relative wind due to descent rate, and again passing through
%                 1/4-chords)
%   LargeAngles = 0 to make small angle assumptions everywhere
%                 = 1 to make no small angle assumptions in vortex lattice geometry,
```

```

%      except that small angle of trailing vortex upsweep is assumed,
%      and also statement of flow tangency requirements
%      option = 3-vector of display options, e.g. [1, 1, 0] to give screen display
%      of heading and notes, and of aerodynamic coefficients, but no plots
%      option(1) = 1 to display heading and notes, 0 not to
%      option(2) = 1 to display aerodynamic coefficients, 0 not to
%      option(3) = 1 to produce screen plots, 0 not to
%      If these options are omitted from input argument list, then all screen display is
%      suppressed.
%
%
% OUTPUT DATA
% =====
%
%      f = [CL - Cw; CM] = object function of trim problem
%      Coeffs = [CL, CD_v, CD_v/CL^2, CM, CLw, CLt, ...
%      CDw_v, CDt_v, CD/CD_LL, CD_v/CD_LL];
%      CL = lift coefficient of wing and tail, based on wing area
%      CD_v = drag coefficient by CVLM method
%      CD_v/CL^2 = drag polar coefficient
%      CM = pitch moment coefficient (wing and tail) about CM base on average
%      chord c_aver = S_wing/b_wing and on wing area
%      CLw = lift coefficient of wing based on wing area
%      CLt = lift coefficient of tail based on tail area
%      CDw_v = drag coefficient of wing (by CVLM) based on wing area
%      CDt_v = drag coefficient of tail (by CVLM) based on tail area
%      CD/CD_LL = ratio of leading edge singularity estimation of drag to estimation
%      CD_LL based on the lifting lines obtained by VLM
%      CD_v/CD_LL = ration of CLVM drag estimation to CD_LL
%
%
S = 1/2*b*c*(1+lam);
S_ = 1/2*b_ *c_*(1+lam_);
c_AC = 2/3*c*(1+lam+lam^2)/(1+lam);
c_AC_ = 2/3*c_*(1+lam_+lam_^2)/(1+lam_);
c_aver = S/b; c_aver_ = S_/b_;

% Angle of attack
alpha = theta + alpha_d;

% Upsweep and downsweep angles of vortices and their images
alpha_vR = alpha_d*SheetOption;
alpha_vI = -alpha_d*SheetOption;

% Vortex counts
Nw = Ns*Nc; Nv = Nw + Ns_*Nc_; % Nw = number of vortices on wing
% Nv = total number of vortices on wing and tail
Nt = Ns_*Nc_;

% Sines and cosines of relevant angles
if LargeAngles == 1 % Don't make small angle assumptions in lattice geometry
    sin_theta = sin(theta); cos_theta = cos(theta);
    sin_theta_ = sin(theta-delta_t); cos_theta_ = cos(theta-delta_t);
    s_the_mat = ones(Nv,1)*[ones(1,Nw)*sin(theta)];
    c_the_mat = ones(Nv,1)*[ones(1,Nw)*cos(theta)];
    s_the_le = ones(Ns+Ns_,1)*[ones(1,Nw)*sin(theta)];

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```

c_the_le = ones(Ns+Ns_,1)*[ones(1,Nw)*cos(theta)];
s_the_LL = ones(Ns+Ns_,1)*[ones(1,Ns)*sin(theta)];
c_the_LL = ones(Ns+Ns_,1)*[ones(1,Ns)*cos(theta)];
s_the_mat_ = ones(Nv,1)*[ones(1,Nt)*sin(theta-delta_t)];
c_the_mat_ = ones(Nv,1)*[ones(1,Nt)*cos(theta-delta_t)];
s_the_le_ = ones(Ns+Ns_,1)*[ones(1,Nt)*sin(theta-delta_t)];
c_the_le_ = ones(Ns+Ns_,1)*[ones(1,Nt)*cos(theta-delta_t)];
s_the_LL_ = ones(Ns+Ns_,1)*[ones(1,Ns_)*sin(theta-delta_t)];
c_the_LL_ = ones(Ns+Ns_,1)*[ones(1,Ns_)*cos(theta-delta_t)];
sin_alpha = sin(alpha); cos_alpha = cos(alpha);
sin_alpha_d = sin(alpha_d); cos_alpha_d = cos(alpha_d);
sin_alpha_w = sin(theta+alpha_d); cos_alpha_w = cos(theta+alpha_d);
sin_alpha_w_ = sin(theta-delta_t+alpha_d);
cos_alpha_w_ = cos(theta-delta_t+alpha_d);
elseif LargeAngles == 0 % Do make small angle assumptions in lattice geometry
sin_theta = theta; cos_theta = 1;
sin_theta_ = theta-delta_t; cos_theta_ = 1;
s_the_mat = ones(Nv,1)*[ones(1,Nw)*sin_theta];
c_the_mat = ones(Nv,1)*[ones(1,Nw)*cos_theta];
s_the_le = ones(Ns+Ns_,1)*[ones(1,Nw)*sin_theta];
c_the_le = ones(Ns+Ns_,1)*[ones(1,Nw)*cos_theta];
s_the_LL = ones(Ns+Ns_,1)*[ones(1,Ns)*sin_theta];
c_the_LL = ones(Ns+Ns_,1)*[ones(1,Ns)*cos_theta];
s_the_mat_ = ones(Nv,1)*[ones(1,Nt)*sin_theta_];
c_the_mat_ = ones(Nv,1)*[ones(1,Nt)*cos_theta_];
s_the_le_ = ones(Ns+Ns_,1)*[ones(1,Nt)*sin_theta_];
c_the_le_ = ones(Ns+Ns_,1)*[ones(1,Nt)*cos_theta_];
s_the_LL_ = ones(Ns+Ns_,1)*[ones(1,Ns_)*sin_theta_];
c_the_LL_ = ones(Ns+Ns_,1)*[ones(1,Ns_)*cos_theta_];
sin_alpha = alpha; cos_alpha = 1;
sin_alpha_d = alpha_d; cos_alpha_d = 1;
sin_alpha_w = theta+alpha_d; cos_alpha_w = 1;
sin_alpha_w_ = theta-delta_t+alpha_d; cos_alpha_w_ = 1;
end

% Discretise wing and tail surfaces (x and x_ are w.r.t. respective leading edges,
% and all z's are zero, because co-ordinates are returned here w.r.t. chord planes!)
[xc, x_le_a, x_le_b, x_le_c, xv, xva, xvb, yc, y_le_c, yva, yvb, ca, cb, cc] = ...
    discret(Ns, Nc, c, b, Lle, lam);
[xc_, x_le_a_, x_le_b_, x_le_c_, xv_, xva_, xvb_, yc_, y_le_c_, yva_, yvb_, ...
    ca_, cb_, cc_] = discret(Ns_, Nc_, c_, b_, Lle_, lam_);

% *****
% CO-ORDINATES ARE REFERRED TO MIDSPAN LEADING EDGE OF WING BELOW
% =====

% Height of CM above ground plane
H_g = H_g_b*b; b_H_g = 1/H_g_b;

% Height of LE
H_LEw = H_g + l_CM_LEw*theta;

% Horiz. and vert. distance between leading edges of wing and tail
lt0 = l_LEt_LEw; ht0 = h_LEt_LEw;
lt = lt0*cos_theta + ht0*sin_theta;
ht = -lt0*sin_theta + ht0*cos_theta;

```

```

% Positions of leading and trailing edges
xLE = 0; zLE = 0; xLE_ = lt; zLE_ = ht;
xTE = c*cos_theta; zTE = -c*sin_theta; xTE_ = lt+c*cos_theta_;
zTE_ = ht-c*sin_theta_;

% Distances of 2D estimations of AC's above and behind wing's leading edge
XA = (1+2*lam)/(1+lam)*b/6*(tan(Lle)-c/2/b*(1-lam));
XA_ = (1+2*lam_)/(1+lam_)*b_/6*(tan(Lle_)-c_/2/b_*(1-lam_));
dz = -(c/4+XA)*sin_theta;
dz_ = ht-(c_/4+XA_)*sin_theta_;
dx = (c/4+XA)*cos_theta; dx_ = lt+(c_/4+XA_)*cos_theta_;
dzI = -dz - 2*H_LEw; dzI_ = -dz_ - 2*H_LEw;

% All vortices are in planes that pass through respective 1/4-chords, and
% z-co-ordinates are defined accordingly below. We also allow for the vortices
% to be aligned with a free stream velocity vector that INCLUDES the relative
% wind due to the descent rate, if specified such by SheetOption = 1, as
% opposed to SheetOption <> 1, which specifies a horizontal vortex plane.
% If no small angle assumptions are made (ang == 1), then the positions
% of all vortex control points and corner points (they are placed in the wind
% plane) are on the vertical lines that pass through the physical surface
% points that they represent.
if SheetOption ~= 1
    zc = dz*ones(size(xc)); zc_ = dz_*ones(size(xc_));
    z_le_c = dz*ones(size(x_le_c)); z_le_c_ = dz_*ones(size(x_le_c_));
    zva = dz*ones(size(xva)); zva_ = dz_*ones(size(xva_));
    zvb = dz*ones(size(xvb)); zvb_ = dz_*ones(size(xvb_));
    zv = dz*ones(size(xv)); zv_ = dz_*ones(size(xv_));
else
    zc = dz+((xc-c/4-XA))*sin_alpha_d; z_le_c = dz+(x_le_c-c/4-XA)*sin_alpha_d;
    zva = dz+(xva-c/4-XA)*sin_alpha_d; zvb = dz+(xvb-c/4-XA)*sin_alpha_d;
    zv = dz+(xv-c/4-XA)*sin_alpha_d; zc_ = dz+(xc_-c_/4-XA_)*sin_alpha_d;
    z_le_c_ = dz+((x_le_c_-c_/4-XA_))*sin_alpha_d; zva_ = dz+(xva_-c_/4-XA_)*sin_alpha_d;
    zvb_ = dz+(xvb_-c_/4-XA_)*sin_alpha_d; zv_ = dz+(xv_-c_/4-XA_)*sin_alpha_d;
end

% Add tail distance to tail's x-coordinates. Then concatenate co-ordinates
xc = [xc*cos_alpha_d, lt+xc*cos_alpha_d];
x_le_c = [x_le_c*cos_alpha_d, lt+x_le_c*cos_alpha_d];
xva = [xva*cos_alpha_d, lt+xva*cos_alpha_d];
xvb = [xvb*cos_alpha_d, lt+xvb*cos_alpha_d];
xv = [xv*cos_alpha_d, lt+xv*cos_alpha_d];
yc = [yc, yc_]; y_le_c = [y_le_c, y_le_c_];
yva = [yva, yva_]; yvb = [yvb, yvb_];
zc = [zc, zc_]; z_le_c = [z_le_c, z_le_c_];
zva = [zva, zva_]; zvb = [zvb, zvb_]; zv = [zv, zv_];

% Define vortex midpoints halfway LINEARLY between flanking corner points
xv = (xva+xvb)/2; yv = (yva+yvb)/2; zv = (zva+zvb)/2;

% Define image vortex co-ordinates
xIva = xva; xIvb = xvb;
yIva = yva; yIvb = yvb;
zIva = -zva - 2*H_LEw; zIvb = -zvb - 2*H_LEw;
% REFERENCE OF CO-ORDINATES TO MIDSPAN LEADING EDGE OF WING COMPLETED

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% =====

% *****
% INFLUENCE MATRICES ARE COMPUTED HERE
% =====

% Compute required influence matrices of all vortices
[HxR, Hx_leR, Hx_LL_R, HzR, Hz_leR, Hz_LL_R] = ...
    inf_matz(Ns, Ns_, Nc, Nc_, xc, x_le_c, xv, xva, xvb, yc, y_le_c, yva, yvb, ...
        zc, z_le_c, zv, zva, zvb, ...
        dx, dx_, dz, dz_, dzI, dzI_, alpha_vR, 0);

[Hx_vR, Hz_vR] = inf_mavi(Ns, Ns_, Nc, Nc_, xv, xva, xvb, yc, yva, yvb, ...
    zv, zva, zvb, alpha_vR, 0);

% Compute required influence matrices of all vortice images
if H_g~inf
    [HxI, Hx_leI, Hx_LL_I, HzI, Hz_leI, Hz_LL_I] = ...
        inf_matz(Ns, Ns_, Nc, Nc_, xc, x_le_c, xv, xIva, xIvb, yc, y_le_c, ...
            yIva, yIvb, zc, z_le_c, zv, zIva, zIvb, ...
            dx, dx_, dz, dz_, dzI, dzI_, alpha_vI, 1);

    Hx = HxR - HxI; Hx_le = Hx_leR - Hx_leI; Hx_LL = Hx_LL_R - Hx_LL_I;
    Hz = HzR - HzI; Hz_le = Hz_leR - Hz_leI; Hz_LL = Hz_LL_R - Hz_LL_I;
    H = Hx.*[s_the_mat, s_the_mat_] + Hz.*[c_the_mat, c_the_mat_];
    H_le = Hx_le.*[s_the_le, s_the_le_] + Hz_le.*[c_the_le, c_the_le_];
    H_LL = Hx_LL.*[s_the_LL, s_the_LL_] + Hz_LL.*[c_the_LL, c_the_LL_];

    [Hx_vI, Hz_vI] = inf_mavi(Ns, Ns_, Nc, Nc_, xv, xIva, xIvb, yc, yIva, yIvb, ...
        zv, zIva, zIvb, alpha_vI, 1);
    Hx_v = Hx_vR - Hx_vI;
    Hz_v = Hz_vR - Hz_vI;
else
    Hx_v = Hx_vR; Hz_v = Hz_vR;
    H = HxR.*[s_the_mat, s_the_mat_] + HzR.*[c_the_mat, c_the_mat_];
    H_le = Hx_leR.*[s_the_le, s_the_le_] + Hz_leR.*[c_the_le, c_the_le_];
    H_LL = Hx_LL_R.*[s_the_LL, s_the_LL_] + Hz_LL_R.*[c_the_LL, c_the_LL_];
end
% COMPUTATION OF INFLUENCE MATRICES COMPLETED
% =====

% *****
% COMPUTATION OF VORTEX DISTRIBUTION
% =====

% Find circulation increments by solving flow tangency problem
Nw = Ns*Nc; Nv = Nw + Ns_*Nc_;
alphas = ones(Nw,1)*sin_alpha + 2*q*(xc(1:Nw) - l_CM_LEw)/c*cos_theta ...
    - 2*q*zc(1:Nw)/c*sin_theta;
alphas_ = ones(Ns_*Nc_,1)*sin_alpha_w_ + ...
    2*q*(xc(Nw+1:Nv) - l_CM_LEw)/c*cos_theta_ ...
    - 2*q*zc(Nw+1:Nv)/c*sin_theta_;
Gamma_ = H^(-1)*([-alphas; -alphas_]);
Gam_mat = reshape(Gamma_(1:Nw,1), Ns, Nc)';
Gam_mat_ = reshape(Gamma_(Nw+1:Nv,1), Ns_, Nc_)';
% COMPUTATION OF VORTEX DISTRIBUTION COMPLETED
% =====

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% *****
% COMPUTATION OF AERODYNAMIC COEFFICIENTS
% =====
% Lift distribution cl_c (required in leading edge singularity analysis)
if Nc>1; cl_c = 2*sum(Gam_mat); else; cl_c = 2*Gam_mat; end
if Nc_>1; cl_c_ = 2*sum(Gam_mat_); else; cl_c_ = 2*Gam_mat_; end
% Spanwise dimensions dy of bound vortex segments
dy = (yvb(1:Ns)-yva(1:Ns)); dy_ = (yvb(Nw+1:Nw+Ns)-yva(Nw+1:Nw+Ns_));

% Upwash at leading edge, leading edge singularity parameter and induced drag
% coefficient by modified version of Lan's Quasi Vortex Lattice Method (QVLM)
wi_le = H_le*Gamma_; % Induced upwash at wing and tail LE's
aQLE = 2*q*(x_le_c(1:Ns) - l_CM_LEw)/c*cos_theta ...
-2*q*z_le_c(1:Ns)/c*sin_theta; % Upwash at wing LE due to pitch rate
aQLE_ = 2*q*(x_le_c(Ns+1:Ns+Ns_) - l_CM_LEw)/c*cos_theta ...
-2*q*z_le_c(Ns+1:Ns+Ns_)/c*sin_theta; % Upwash at tail LE due to pitch rate
alphas_LE = sin_alpha + aQLE; % Required downwash at wing LE
alphas_LE_ = sin_alpha_w_ + aQLE_; % Required downwash at tail LE
C = cos(Lle)/Nc * (wi_le(1:Ns)' + alphas_LE);
C_ = cos(Lle_)/Nc_ * (wi_le(Ns+1:Ns+Ns_)' + alphas_LE_);
ct = pi*C./cos(Lle); ct_ = pi*C_./cos(Lle_);
CD = 1/S * (cl_c * dy' * sin_alpha - (ct.*cc) * dy' * cos_alpha ...
+ cl_c_ * dy_ * sin_alpha_w_ - (ct_.*cc_) * dy_ * cos_alpha_w_);

% Far-field induced downwash and far-field induced drag coefficient
% by analysis of far wake. This is the same as a lifting line drag
% approximation based on the circulations obtained by VLM method. It
% cannot predict influence of pitch rate accurately, because the
% influence of Q depends strongly on chord length.
% Influence of pitch rate is approximated based on angle of attack increment
% at respective 1/4-chords, BUT WITHOUT ANY THEORETICAL FOUNDATION!
if Nc>1
    Gam_mat_w = sum(Gam_mat);
else
    Gam_mat_w = Gam_mat;
end
if Nc_>1
    Gam_mat_t = sum(Gam_mat_);
else
    Gam_mat_t = Gam_mat_;
end
Gamma_LL = [Gam_mat_w, Gam_mat_t]'; % Distribution of circulation strength
% of lifting lines of wing and tail
alpha_i = -H_LL*Gamma_LL; % Distr. of downwash angle on these
CDi = 1/S * ((cl_c.*alpha_i(1:Ns)')*dy' + (cl_c_.*alpha_i(Ns+1:Ns+Ns_))'*dy_');
CDiw = ((cl_c.*alpha_i(1:Ns)')*dy')/b/c;
CDit = ((cl_c_.*alpha_i(Ns+1:Ns+Ns_))'*dy_)/b_/c_;
% Influence of pitch rate (note that q is angle of attack increment due to pitch
% rate over distance of 1/2 wing chord, thus q/2 is increment over 1/4 wing chord)
CDQ = -1/S * ((cl_c.*(aQLE+q/2))*dy' + (cl_c_.*(aQLE_+q/2*c_/c))*dy_');
CD_LL = CDi + CDQ;

% CONVENTIONAL VORTEX LATTICE COMPUTATION OF AERODYNAMIC COEFFICIENTS (BY A
DIRECT

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% KUTTA-JOUKOWSKI APPROACH)
uig_v = Hx_v*Gamma_; % x-component (geodetic) of induced V at vortex 1/2 angle points
wig_v = Hz_v*Gamma_; % z-component (geodetic) of induced V at vortex 1/2 angle points
wi_v = -uig_v*sin_alpha_d + wig_v*cos_alpha_d; % Induced w normal to vortex plane
VxQ = -2*q*zv/c; VzQ = 2*q*xv/c;
Vx = cos_alpha_d + VxQ + uig_v'; Vz = sin_alpha_d + VzQ + wig_v';
if Nc>1
    cFx_c = -2*sum(Gam_mat.*reshape(Vz(1:Nw),Ns,Nc));
    cFz_c = 2*sum(Gam_mat.*reshape(Vx(1:Nw),Ns,Nc));
    cm_c21 = - 2*sum(Gam_mat.*reshape(Vx(1:Nw),Ns,Nc)'.*reshape(xv(1:Nw), Ns, Nc));
    cm_c22 = - 2*sum(Gam_mat.*reshape(Vz(1:Nw),Ns,Nc)'.*reshape(zv(1:Nw), Ns, Nc));
else
    cFx_c = -2*Gam_mat.*reshape(Vz(1:Nw),Ns,Nc);
    cFz_c = 2*Gam_mat.*reshape(Vx(1:Nw),Ns,Nc);
    cm_c21 = - 2*Gam_mat.*reshape(Vx(1:Nw),Ns,Nc)'.*reshape(xv(1:Nw), Ns, Nc);
    cm_c22 = - 2*Gam_mat.*reshape(Vz(1:Nw),Ns,Nc)'.*reshape(zv(1:Nw), Ns, Nc);
end
cm_c2 = cm_c21 + cm_c22;
if Nc >1
    cFx_c_ = -2*sum(Gam_mat_.*reshape(Vz(Nw+1:Nv),Ns_,Nc_));
    cFz_c_ = 2*sum(Gam_mat_.*reshape(Vx(Nw+1:Nv),Ns_,Nc_));
    cm_c21_ = - 2*sum(Gam_mat_.*reshape(Vx(Nw+1:Nv),Ns_,Nc_)'.*reshape(xv(Nw+1:Nv),Ns_,Nc_));
    cm_c22_ = - 2*sum(Gam_mat_.*reshape(Vz(Nw+1:Nv),Ns_,Nc_)'.*reshape(zv(Nw+1:Nv),Ns_,Nc_));
else
    cFx_c_ = -2*Gam_mat_.*reshape(Vz(Nw+1:Nv),Ns_,Nc_);
    cFz_c_ = 2*Gam_mat_.*reshape(Vx(Nw+1:Nv),Ns_,Nc_);
    cm_c21_ = - 2*Gam_mat_.*reshape(Vx(Nw+1:Nv),Ns_,Nc_)'.*reshape(xv(Nw+1:Nv),Ns_,Nc_);
    cm_c22_ = - 2*Gam_mat_.*reshape(Vz(Nw+1:Nv),Ns_,Nc_)'.*reshape(zv(Nw+1:Nv),Ns_,Nc_);
end
cm_c2_ = cm_c21_ + cm_c22_;
CFx = 1/S * (cFx_c * dy' + cFx_c_ * dy_');
CFz = 1/S * (cFz_c * dy' + cFz_c_ * dy_');
CL_VLM = CFz*cos_alpha_d - CFx*sin_alpha_d;
CD_VLM = CFz*sin_alpha_d + CFx*cos_alpha_d;

CM_LEw_VLM = 1/S/c_aver * (cm_c2 * dy' + cm_c2_ * dy_'); % About LE of wing
CM_VLM = CM_LEw_VLM + CFz*1_CM_LEw*cos_theta/c_aver + ... % About C of M
    CFx*1_CM_LEw*sin_theta/c_aver;
CL = CL_VLM;
CD_v = CD_VLM;
CM = CM_VLM;
% COMPUTATION OF AERODYNAMIC COEFFICIENTS COMPLETED
% =====

% Output data
f = [CL - Cw; CM]; % Object function of trim problem
Coeffs = [CL, CD_v, CD_v/CL^2, CM, NaN, NaN, NaN, NaN, CD/CD_LL, CD_v/CD_LL];
%
% Separate contributions of wing and tail to aerodynamic coefficients not extracted
% here, but they can easily be made available, if required.
%Coeffs = [CL, CD_v, CD_v/CL^2, CM, CLw, CLt, CDw_v, CDt_v, CD/CD_LL, CD_v/CD_LL];
% Aerodynamic coefficients (note
% that CD of CVLM is returned!)

% *****
% SCREEN DISPLAY

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% =====

if nargin == 25
if option(1) == 1
    clc
    disp('          VORTEX LATTICE ANALYSIS OF WING AND TAIL')
    disp('          *****')
    disp(' ')
    disp(' ')
    disp(' NOTES:')
    disp(' =====')
    disp(' ')
    disp(' ')
    if LargeAngles == 1
        disp(' NO SMALL ANGLE ASSUMPTION WAS MADE IN VORTEX LATTICE GEOMETRY !')
    elseif LargeAngles == 0
        disp(' SMALL ANGLE ASSUMPTION WAS MADE IN VORTEX LATTICE GEOMETRY !')
    end
    disp(' ')
    disp(' ')
    disp('1 HORSESHOE VORTICES ARE IN HORIZONTAL PLANE IF SheetOption = 0, BUT ARE IN')
    disp(' WIND PLANE (SWEPT UP AT NON-ZERO ANGLE OF DESCENT) WHEN SheetOption = 1.')
    disp(' IN BOTH CASES THE VORTEX PLANES PASS THROUGH THE 2D ESTIMATIONS OF THE')
    disp(' AERODYNAMIC CENTRES OF THE WING AND TAIL, RESPECTIVELY.')
    disp(' ')
    disp('2 ALL POSITION COORDINATES ARE RESOLVED IN GEODETIC REFERENCE FRAME')
    disp(' (I.E. WITH HORIZONTAL X-Y-PLANE)')
    disp(' ')
    disp('3 DRAG COEFFICIENT OF QVLM (BY LE SINGULARITY APPROACH) SEEMS SENSITIVE TO')
    disp(' ANGLE OF DESCENT. DRAG RESULT OF QVLM CHANGES MUCH MORE THAN THAT OF CVLM. ')
    disp(' PERHAPS BECAUSE OF CHANGE IN z-CO-ORDINATE OF LEADING EDGE?')
    disp(' ')
    disp('4 DRAG COEFFICIENT BY VLM IS COMPUTED USING LINEAR MIDPOINTS OF BOUND')
    disp(' VORTEX SEGMENTS, RATHER THAN ANGULAR MIDPOINTS, EVEN THOUGH A COSINE')
    disp(' DISTRIBUTION IS USED SPANWISE. THIS WORKS BETTER FOR A SWEPT WING, BUT')
    disp(' MAKES NEGLIGIBLE DIFFERENCE IN VLM DRAG RESULT OF A RECTANGULAR WING.')
    disp(' ')
    disp(' ')
    if q ~= 0
        disp('5 NOTE THAT THE FAR-FIELD DRAG ANALYSIS IS PROBABLY NOT VERY ACCURATE')
        disp(' AT NON-ZERO PITCH RATE, ESPECIALLY IF Q IS NOT SMALL. THIS IS BECAUSE')
        disp(' THE FAR-FIELD DRAG ANALYSIS IS ESSENTIALLY A LIFTING LINE APPROACH,')
        disp(' WHICH FAILS TO CAPTURE THE "PITCH DRAG", BECAUSE THIS DEPENDS')
        disp(' STRONGLY ON THE CHORD LENGTH, AND c IS ASSUMED TO BE NEGLIGIBLE')
        disp(' IN A LIFTING LINE APPROACH.')
        disp(' ')
        disp(' ')
    end
end

% Numerical screen display of output data
if option(2) == 1
    disp(...)
    ' b/H_g    CL    CM    CD/CL^2 CD/CD_LL CD_v/CD_LL CD_v/CD')
    disp([b_H_g, CL, CM, CD/CL^2, CD/CD_LL, CD_v/CD_LL, CD_v/CD])
end

```



```

% Some plots
if option(3) == 1
    plot(xva,yva, 'x'), hold on, plot(xvb,yvb, 'rx'), plot(xc,yc, 'o'); axis('equal')
    plot(xv,yc, '.')
    plot(x_le_c,y_le_c, 'o');
    plot(xIva,yIva, 'cx'), hold on, plot(xIvb,yIvb, 'bx'); axis('equal')
    plot(l_CM_LEw, 0, '.', 'MarkerSize', 20)
    plot(c/4+XA, 0, 's', 'MarkerSize', 6)
    plot(lt+(c_/4+XA_)*cos_theta_, 0, 's', 'MarkerSize', 6)

% Wing outline
plot([b/2*tan(Lle), 0], [-b/2, 0]), plot([0, b/2*tan(Lle)], [0, b/2])
plot([b/2*tan(Lle), b/2*tan(Lle)+c*cos_theta*lam], [b/2, b/2]),
plot([b/2*tan(Lle)+c*cos_theta*lam, c*cos_theta], [b/2, 0])
plot([c*cos_theta, b/2*tan(Lle)+c*cos_theta*lam], [0, -b/2])
plot([b/2*tan(Lle)+c*cos_theta*lam, b/2*tan(Lle)], [-b/2, -b/2]),

% Tail outline
plot([lt+b_/2*tan(Lle_), lt], [-b_/2, 0])
plot([lt, lt+b_/2*tan(Lle_)], [0, b_/2])
plot([lt+b_/2*tan(Lle_), lt+b_/2*tan(Lle_)+c_*lam_], [b_/2, b_/2]),
plot([lt+b_/2*tan(Lle_)+c_*lam_, lt+c_], [b_/2, 0])
plot([lt+c_, lt+b_/2*tan(Lle_)+c_*lam_], [0, -b_/2])
plot([lt+b_/2*tan(Lle_)+c_*lam_, lt+b_/2*tan(Lle_)], [-b_/2, -b_/2]),

dx = 2.2*c/4; axis([-dx, lt+c_+dx], 1.1*[-b/2, b/2]), axis('equal')
dx = 3*c/4; axis([-dx, lt+c_+dx], 1.1*[-b/2, b/2]), axis('equal')
title(['Cosine Distribution of Control Points (o) and ', ...
    'Vortex Cornerpoints (x) of Wing and Tail'])
xlabel('Horizontal Distance Relative to Wing"s Leading Edge (m)')
ylabel('Spanwise Distance Relative to Midspan Point (m)')

figure
if SheetOption == 1; alpha_v = alpha_d; else; alpha_v = 0; end
plot(xva,zva, 'x'), hold on, plot(xvb,zvb, 'x'), plot(xc,zc, 'o');
plot(xv,zv, '.')
plot(x_le_c,z_le_c, 'o');
plot([xLE, xTE], [zLE, zTE])
[xvL, II] = min(xva(1:Ns)); zvL = zva(II);
plot([xvL, xvL+2*c_], [zvL, zvL+2*c_*alpha_v], '--') % Vortex plane
plot([xLE_, xTE_], [zLE_, zTE_])
[xvL_, II] = min(xva(Ns*Nc+1:Ns*Nc+Nc_)); zvL_ = zva_(II);

plot([xvL_, xvL_+3*c_], [zvL_, zvL_+3*c_*alpha_v], '--') % Vortex plane image
plot(l_CM_LEw, -l_CM_LEw*theta, '.', 'MarkerSize', 20)
axis('equal')
title('Side View of Wing and Tail')
xlabel('Horizontal Distance Relative to Wing"s Leading Edge (m)')
ylabel('Vertical Distance Relative to Wing"s Leading Edge (m)')

if H_g<10*b
    plot(xIva,zIva, 'rx'), plot(xIvb,zIvb, 'rx')
    plot([xLE, xTE], [-zLE-2*H_LEw, -zTE-2*H_LEw], ':')
    plot([xLE_, xTE_], [-zLE_-2*H_LEw, -zTE_-2*H_LEw], ':')
    plot(l_CM_LEw, l_CM_LEw*theta-2*H_LEw, '.', 'MarkerSize', 20)

```

```

    plot([xLE-c/4, xTE_+c/4], [-H_LEw, -H_LEw], 'LineWidth', 2)
    title('Side View of Wing, Tail, Vortex Lattices and their Mirror Images')
end
end
end
% SCREEN DISPLAY ENDS
% =====

return

% =====
% END OF MAIN PROGRAM
% *****

% #####
% SUBROUTINES
% =====

% *****
% COSINE DISCRETISATION
% =====

function [xc, x_le_a, x_le_b, x_le_c, xv, xva, xvb, yc, y_le_c, yva, yvb, ...
    ca, cb, cc] = discret(Ns, Nc, c, b, Lle, lam)
%
% function [xc, x_le_a, x_le_b, x_le_c, xv, xva, xvb, yc, y_le_c, yva, yvb, ...
%     ca, cb, cc] = discret(Ns, Nc, c, b, Lle, lam)
%
% Discretises a given rectangular surface
%
% INPUT:
%
%   Ns = required number of spanwise increments
%   Nc = required number of chordwise increments
%   c = chord length
%   b = span length
%   Lle = sweep angle of leading edge
%   lam = taper ratio
%
% OUTPUT:
%
%   xc = 1 x Ns*Nc vector of x-co-ordinates of all inner control points
%   x_le_a = 1 x Ns vector of x-co-ordinates of left leading edge corner points
%   x_le_b = 1 x Ns vector of x-co-ordinates of right leading edge corner points
%   x_le_c = 1 x Ns vector of x-co-ordinates of all leading edge control points
%   xv = 1 x Ns*Nc vector of x-co-ordinates of bound vortex midpoints
%   xva = 1 x Ns*Nc vector of x-co-ordinates of left corner points of bound vortices
%   xvb = 1 x Ns*Nc vector of x-co-ordinates of right corner points of bound vortices
%   yc = 1 x Ns*Nc vector of y-co-ordinates of all inner control points
%   y_le_c = 1 x Ns vector of y-co-ordinates of all leading edge control points
%   yva = 1 x Ns*Nc vector of y-co-ordinates of port vortex cornerpoints
%   yvb = 1 x Ns*Nc vector of y-co-ordinates of starboard vortex cornerpoints
%   ca = 1 x Ns vector of chord lengths evaluated at left corner points
%   cb = 1 x Ns vector of chord lengths evaluated at right corner points

```

```

% cc = 1 x Ns vector of chord lengths evaluated at control points
%
% CO-ORDINATES ARE REFERRED TO MIDSPAN LEADING TIP. z-CO-ORDINATES ARE ASSUMED ZERO.
%

% Angular increments dthe and dphi in chordwise and spanwise directions
dthe = pi/Nc; dphi = pi/(Ns+1);

% Cosines of chordwise angular positions of ALL vortex control points
% and cornerpoints
for ii = 1:Nc
    the_c_inboard_ii = ii*dthe;
    the_v_ii = the_c_inboard_ii - dthe/2;
    cos_the_v(ii) = cos(the_v_ii);
    cos_the_c(ii) = cos(the_c_inboard_ii);
end

% Leading edge co-ordinates, with allowance for sweepback
for jj = 1:Ns
    % Spanwise angular co-ordinates of ALL vortex control points and cornerpoints
    phi_c_jj = jj*dphi;
    phi_va_jj = phi_c_jj - dphi/2;
    phi_vb_jj = phi_va_jj + dphi;
    % Cosines of same
    cos_phi_a(jj) = cos(phi_va_jj);
    cos_phi_b(jj) = cos(phi_vb_jj);
    cos_phi_c(jj) = cos(phi_c_jj);

    % Chord distribution
    ca(jj) = c*(1-(1-lam)*abs(cos_phi_a(jj)));
    cb(jj) = c*(1-(1-lam)*abs(cos_phi_b(jj)));
    cc(jj) = c*(1-(1-lam)*abs(cos_phi_c(jj)));
    % Linear co-ordinates of leading edge control points and corner points
    x_le_a(jj) = b/2*tan(Lle)*abs(cos_phi_a(jj));
    x_le_b(jj) = b/2*tan(Lle)*abs(cos_phi_b(jj));
    x_le_c(jj) = b/2*tan(Lle)*abs(cos_phi_c(jj));
    y_le_c(jj) = -b/2 * cos_phi_c(jj);
end

% Inboard vortex lattice co-ordinates, with allowance for sweepback
for ii = 1:Nc
    for jj = 1:Ns
        % Linear co-ordinates of inboard control points and corner points
        xa(ii,jj) = x_le_a(jj) + 1/2*ca(jj)*(1-cos_the_v(ii));
        xb(ii,jj) = x_le_b(jj) + 1/2*cb(jj)*(1-cos_the_v(ii));
        xc(ii,jj) = x_le_c(jj) + 1/2*cc(jj)*(1-cos_the_c(ii));
        xv(ii,jj) = x_le_c(jj) + 1/2*cc(jj)*(1-cos_the_v(ii));
        yva(ii,jj) = -b/2 * cos_phi_a(jj);
        yvb(ii,jj) = -b/2 * cos_phi_b(jj);
        yvc(ii,jj) = -b/2 * cos_phi_c(jj);
    end
end

% Convert inboard co-ordinates from matrix form to row vector form
xva = reshape(xa', 1,Nc*Ns); xvb = reshape(xb', 1,Nc*Ns);
xc = reshape(xc', 1,Nc*Ns); xv = reshape(xv', 1,Nc*Ns);

```

```

yva = reshape(yva', 1, Nc*Ns); yvb = reshape(yvb', 1, Nc*Ns);
yc = reshape(yc', 1, Nc*Ns);

% Place vortices at c/4 when Nc = 1
if Nc == 1
    xva = xva - c/4;
    xvb = xvb - c/4;
    xc = xc - c/4;
    xv = xv - c/4;
    x_le_a = x_le_a - c/4;
    x_le_b = x_le_a - c/4;
    x_le_c = x_le_a - c/4;
end

% Set vortex width to pi/4*b if Ns = 1
if Ns == 1
    f = (pi/4)/(2*1/2/sqrt(2));
    yva = yva*f; yvb = yvb*f; yc = yc*f;
    yc = reshape(yc', 1, Nc*Ns);
    y_le_c = y_le_c*f;
end

return
% COSINE DISCRETISATION ENDS
% =====

% *****
% COMPUTATION OF INFLUENCE MATRICES
% =====

function [Hx, Hx_le, Hx_LL, Hz, Hz_le, Hz_LL] = ...
    inf_matz(Ns, Ns_, Nc, Nc_, xc, x_le_c, xv, xva, xvb, yc, y_le_c, yva, yvb, ...
        zc, z_le_c, zv, zva, zvb, ...
        dx, dx_, dz, dz_, dzl, dzl_, alpha_v, ImVort)

%
% [Hx, Hx_le, Hx_LL, Hz, Hz_le, Hz_LL] = ...
%     inf_matz(Ns, Ns_, Nc, Nc_, xc, x_le_c, xv, xva, xvb, yc, y_le_c, yva, yvb, ...
%         zc, z_le_c, zv, zva, zvb, ...
%         dx, dx_, dz, dz_, dzl, dzl_, alpha_v, ImVort)
%
% Compute influence matrices for given discretisation

a_v = alpha_v; % Upsweep angle of free vortices

% Influence matrix at control points
Nv = Nc*Ns + Nc_*Ns_;
for j = 1:Nv
    for k = 1:Nv
        [H_Fx, H_Fy, H_Fz, H_Bx, H_Bz] = ...
            InflueCoeffs(xc(j), xva(k), xvb(k), yc(j), yva(k), yvb(k), ...
                zc(j), zva(k), zvb(k), a_v, 3);
        Hx(j,k) = H_Bx + H_Fx; Hz(j,k) = H_Bz + H_Fz;
    end
end
end

```

```

% Influence matrix at leading edge
for j = 1:length(y_le_c)
    for k = 1:Nv
        [H_Fx, H_Fy, H_Fz, H_Bx, H_Bz] = ...
            InflueCoeffs(x_le_c(j), xva(k), xvb(k), y_le_c(j), yva(k), yvb(k), ...
                z_le_c(j), zva(k), zvb(k), a_v, 3);
        Hx_le(j,k) = H_Bx + H_Fx; Hz_le(j,k) = H_Bz + H_Fz;
    end
end

% Influence coefficients of semi-infinite trailing vortices at the lifting lines. It
% is used for far-field drag computation, which is approximated here by a lifting
% line analysis. Lifting lines are straight and run through 2D estimations of the
% respective aerodynamic centres of wing and tail. Lifting line co-ordinates are:
x_LL_va = [dx*ones(1,Ns), dx_*ones(1,Ns_)]; x_LL_vb = x_LL_va;
x_LL_v = [dx*ones(1,Ns), dx_*ones(1,Ns_)];
y_LL_va = [yva(1:Ns), yva(Nc*Ns+1:Nc*Ns+Ns_)];
y_LL_vb = [yvb(1:Ns), yvb(Nc*Ns+1:Nc*Ns+Ns_)];
y_LL_v = y_le_c;
z_LL_va = [dz*ones(1,Ns), dz_*ones(1,Ns_)]; z_LL_vb = z_LL_va;
z_LL_v = [dz*ones(1,Ns), dz_*ones(1,Ns_)];

% Allow for height when influence of image of lifting lines is considered
if ImVort == 1
    z_LL_va = [dzI*ones(1,Ns), dzI_*ones(1,Ns_)]; z_LL_vb = z_LL_va;
end

% Influence coefficients are:
for j = 1:length(y_le_c)
    for k = 1:length(y_LL_va)
        [H_Fx, H_Fy, H_Fz] = ...
            InflueCoeffs(x_LL_v(j), x_LL_va(k), x_LL_vb(k), ...
                y_LL_v(j), y_LL_va(k), y_LL_vb(k), ...
                z_LL_v(j), z_LL_va(k), z_LL_vb(k), a_v, 2);
        Hx_LL(j,k) = H_Fx; Hz_LL(j,k) = H_Fz;
    end
end

return
% *****

function [Hx_vI, Hz_vI] = ...
    inf_mavi(Ns, Ns_, Nc, Nc_, xv, xIva, xIvb, yc, yIva, yIvb, ...
        zv, zIva, zIvb, alpha_vI, ImVort);

%
% function [Hx_vI, Hz_vI] = ...
%     inf_mavi(Ns, Ns_, Nc, Nc_, xv, xIva, xIvb, yc, yIva, yIvb, ...
%         zv, zIva, zIvb, alpha_vI, ImVort);
%
% Compute influence matrix of image vortices at real vortex midpoints

a_vI = alpha_vI; % Downward angle of image of free vortices

Nv = Nc*Ns + Nc_*Ns_;
for j = 1:Nv
    for k = 1:Nv

```

```

[H_Fx, H_Fy, H_Fz, H_Bx, H_Bz] = ...
    InflueCoeffs(xv(j), xIva(k), xIvb(k), yc(j), yIva(k), yIvb(k), ...
        zv(j), zIva(k), zIvb(k), a_vI, 3);
if j == k & ImVort == 0 % Avoid singularity or ill-conditioned coeffs.
    Hx_vI(j,k) = H_Fx;
    Hz_vI(j,k) = H_Fz;
else
    Hx_vI(j,k) = H_Bx + H_Fx;
    Hz_vI(j,k) = H_Bz + H_Fz;
end
end
end

return

% *****

function [H_Fx, H_Fy, H_Fz, H_Bx, H_Bz] = ...
    InflueCoeffs(x, xva, xvb, y, yva, yvb, z, zva, zvb, a_v, option)
%
% function [H_Fx, H_Fy, H_Fz, H_Bx, H_Bz] = ...
%     InflueCoeffs(x, xva, xvb, y, yva, yvb, z, zva, zvb, a_v, option)
%
% Subroutine for computation of influence coefficients of a horseshoe vortex
% (utility for use in computation of influence matrices)

% Required relative co-ordinates X, Y, Z and R
Xa = xva - x; Xb = xvb - x;
Ya = yva - y; Yb = yvb - y;
Za = zva - z; Zb = zvb - z;
dX = xvb - xva; dY = yvb - yva; dZ = zvb - zva;
Ra = sqrt(Xa^2 + Ya^2 + Za^2); Rb = sqrt(Xb^2 + Yb^2 + Zb^2);
CX = Ya*Zb - Za*Yb; CY = Za*Xb - Xa*Zb; CZ = Xa*Yb - Ya*Xb;
C = CX^2 + CY^2 + CZ^2;
D = (Xb*dX + Yb*dY + Zb*dZ)/Rb - (Xa*dX + Ya*dY + Za*dZ)/Ra;

% Influence coefficients due to bound leg
if option == 1 | option == 3
    if C ~= 0 % Avoid singularity of self-induced velocity
        H_Bx = 1/4/pi*CX*D/C; % of bound vortex legs - the numerical equivalent
        H_Bz = 1/4/pi*CZ*D/C; % of the Cauchy integral is found by summing later
    else
        H_Bx = 0; H_Bz = 0;
    end
end

% Influence coefficients due to trailing legs
if option == 2 | option == 3
    Ka = (Ra - Xa - a_v*Za)/Ra/((1+a_v^2)*Ya^2 + (Za-a_v*Xa)^2);
    Kb = (Rb - Xb - a_v*Zb)/Rb/((1+a_v^2)*Yb^2 + (Zb-a_v*Xb)^2);
    H_Fz = -1/4/pi*(Yb*Kb - Ya*Ka);
    H_Fy = 1/4/pi*((Zb-a_v*Xb)*Kb - (Za-a_v*Xa)*Ka);
    H_Fx = -a_v*H_Fz;
end

```

return

% =====

% END OF SUBROUTINES

% #####

F3.4 To Perform VLM Test Calculations, Call **TestAeroCoefVLM.m**

```
function TestAeroCoefVLM

% Test program for aerodynamic analysis of wing and tail by Vortex Lattice Method
%
% Tests results of AeroCoefVLM.m against:
%
% - Lan's results for rectangular wing with AR = 2
% - Lan's results for wing with Warren 12 planform shape
% - exact 2D-results for airfoil with both alpha and q non-zero

% REMOVE THESE?
% Test commands for interactive application, Do not uncomment here!!!!
% TestCoeffs = Coeffs; save VLMTestData TestCoeffs
% close all, testvlm, load VLMTestData; Test = max(abs(Coeffs)-abs(TestCoeffs))

clear all, close all

%Examp = 'Rectang2';
%Examp = 'Warren12';
%Examp = 'Exact_2D';

% Geometric data for rectangular wing alone of AR = 2
alpha = pi*5/180; % Angle of attack
alpha_d = pi*0/180; % Angle of descent
alpha_q = pi*1/180; % Required incidence increment at tail's LE due to Q (N/A)
delta_t = pi*0/180; % Tail setting angle (nose down wrt wing chord line) (N/A)
theta = alpha - alpha_d; % Pitch angle of wing chord
l_LEt_LEw = 6; % Distance of tail LE behind wing LE in m
h_LEt_LEw = 1.5; % Height of tail LE above wing LE
l_CM_LEw = 0; % Distance of C of M behind wing LE (take CM about this point)

b = 8; c = 4; % Wing span and chord in m
b = 1e6*c;
%b = 2*sqrt(2); c = 1.5;
b_ = 1e-10; c_ = 1e-10; % Tail span and chord in m (N/A)
S = b*c; % Wing planform area in m^2

Lle = 0*pi/180; % Sweep angle of wing LE
%Lle = 53.543173*pi/180; % Sweep angle of wing LE
Lle_ = 0*pi/180; % Sweep angle of tail LE
lam = 1; % Taper ratio of wing: lam = 1 is rectangular
%lam = 1/3; % Taper ratio of wing: lam = 1 is rectangular
lam_ = 1; % Taper ratio of tail
H_g_b = .25; % Ratio of C of M height (above ground level) to wing span
H_g_b = inf; % Ratio of C of M height to wing span
Cw = 0.5; % Not used in comparison with Lan

% Computational options
SheetOption = 1; % Flat vortex sheet with upsweep angle = alpha_d wrt horizontal
LargeAngles = 0; % All angles assumed to be small
%Nc = 5; Ns = 20; % Chordwise (Nc) and spanwise (Ns) no. of wing vortices
%Nc = 8; Ns = 15; % Chordwise (Nc) and spanwise (Ns) no. of wing vortices
Nc = 3; Ns = 20; % Chordwise (Nc) and spanwise (Ns) no. of wing vortices
%Nc = 3; Ns = 12; % Chordwise (Nc) and spanwise (Ns) no. of wing vortices
```



```

Nc = 3; Ns = 6;          % Chordwise (Nc) and spanwise (Ns) no. of wing vortices
Nc_ = 1; Ns_ = 1;       % Chordwise (Nc_) and spanwise (Ns_) no. of tail vortices

% Normalised pitch rate that would yield selected upwash angle at tail
% (note that  $q = Q \cdot c / (2 \cdot V_{\infty}) = daQ$  !!!!)
q = 1/2*c/(l_LEt_LEw - l_CM_LEw)*alpha_q;
q = alpha_q/2;

%OutFile1 = 'CompareLan1.res';      % Rectangular, Nc = 8; Ns = 15
%OutFile1 = 'CompareLan2.res';      % Rectangular, Nc = 3; Ns = 6
%OutFile1 = 'CompareLan3.res';      % Warren 12, Nc = 5; Ns = 20
%OutFile1 = 'CompareLan4.res';      % Warren 12, Nc = 3; Ns = 6
OutFile1 = 'NewCompare_2D_1.res';    % Comparison with exact 2D results
OutFile1 = 'NewCompare_2D_3.res';    % Comparison with exact 2D results

if exist(OutFile1) == 2; delete(OutFile1); end

diary(OutFile1), diary on

tic
option = [1, 1, 1];
[f, Coeffs] = AeroCoefVLM(theta, alpha_d, delta_t, Cw, H_g_b, Nc, Nc_, Ns, Ns_, ...
    b, c, S, b_c, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
    Lle, Lle_, lam, lam_, q, SheetOption, LargeAngles, option);
% Coeffs = [CL, CD_v, CD_v/CL^2, CM, CLw, CLt, CDw_v, CDt_v, CD/CD_LL, CD_v/CD_LL];

disp([' SheetOption = ', num2str(SheetOption)])
disp([' LargeAngles = ', num2str(LargeAngles)])
disp([' Nc = ', num2str(Nc), ' (no. of chordwise segments)'])
disp([' Ns = ', num2str(Ns), ' (no. of spanwise segments)'])
disp([' alpha = ', num2str(alpha*180/pi), ' degrees'])
disp(' ')

disp(' Note that CD/CD_LL above means the same as CDii/CDi of Lan, i.e. the ratio')
disp(' of induced drag CDii by LE singularity approach to CDi of far-field method.')
disp(' '), disp(' ')

if strcmp(Examp, 'Warren12')|strcmp(Examp, 'Rectang2')
    disp(' Comparison with Published Results')
    disp(' Warren 12 Planform')
    %disp(' Rectangular Wing with AR = 2')
    disp(' ')
    disp([' Present Analysis: Nc = ', num2str(Nc), ' and Ns = ', num2str(Ns)])
    disp(' CL/alpha CM/alpha CDii/CL^2 CDii/CDi')
    disp([Coeffs(1)/alpha, Coeffs(4)/alpha, Coeffs(3)*Coeffs(9)/Coeffs(10), Coeffs(9)])
    disp(' ')
    %disp([' Lan, also with: Nc = ', num2str(8), ' and Ns = ', num2str(15)])
    disp([' Lan, also with: Nc = ', num2str(5), ' and Ns = ', num2str(20)])
    disp(' CL/alpha CM/alpha CDii/CL^2 CDii/CDi')
    %disp([2.4707, -0.5173, 0.1595, 0.9972])
    disp([2.7382, -3.0844, 0.11428, 1.0054])
    disp(' ')
end

if strcmp(Examp, 'Exact_2D')
    CL = Coeffs(1);

```

```

CD = Coeffs(2);
CM = Coeffs(4);

CM_c4 = CM + (c/4-l_CM_LEw)/c*CL; % Pitch moment coeff about 1/4 chord

% Difference daQ in pitch incidence between trailing edge and midchord point
daQ = q;

% Pitch incidence aQMC at midchord
aQMC = daQ - daQ * 2*l_CM_LEw/c;

% Exact 2D results in terms of aQMC, daQ and alpha
cl_2D = 2*pi*(alpha + aQMC + 1/2*daQ);
cm_c4_2D = -pi/4*daQ;
cd_2D = -2*pi*((alpha+aQMC)*aQMC - 1/2*alpha*daQ);

% Alternative formulation in terms of xCM_, q and alpha
xCM_ = l_CM_LEw/c;
cl_2D = 2*pi*(alpha + 2*q*(3/4 - xCM_));
cl_2D = 2*pi*(alpha + 3*q/2); % VALID ONLY IF xCM_ = 0 !!!
cm_c4_2D = -pi/4*q;
cd_2D = -2*pi*q*(2*alpha*(1/4 - xCM_) + q*(1 - 2*xCM_)^2);
cd_2D = -2*pi*q*(alpha/2 + q); % VALID ONLY IF xCM_ = 0 !!!
% max(abs([cl_2D, cm_c4_2D, cd_2D] - [cl_2D_, cm_c4_2D_, cd_2D_])) shows that
% the latter formulation agrees within machine accuracy with the former

disp('          Various angles (in degrees)')
disp(' alpha alpha_d theta delta_t alpha_q q')
disp([alpha, alpha_d, theta, delta_t, alpha_q, q]/pi*180)

disp(' Compare QVLM results (very high AR, H = inf) with exact 2D results')
disp([' Aspect ratio = ', num2str(b/c)])
disp([' Height H_g/b = ', num2str(H_g_b)])

% CM IS MOMENT ABOUT LEADING EDGE !!!!
% CENTRE OF PITCH IS ON PERPENDICLAR OF WIND PLANE ABOVE CENTRE OF MASS !!!!
% (THIS YIELDS PURE ROTATION, EVEN THOUGH ANGLE OF ATTACK IS NON-ZERO !!!!)
disp(' CL/cl_2D 1e3*CM_c4 1e3*cm_c4 1e3*CD 1e3*cd 1e3*(CM-CD/2/q)')
disp([CL/cl_2D, CM_c4*1e3, cm_c4_2D*1e3, CD*1e3, cd_2D*1e3, 1e3*(CM-CD/2/q)])

disp(' ')
disp(' NOTE THAT CD < 0 ! (O.K., BECAUSE POWER COEFFICIENT IS ZERO!)')
disp(' ')

disp(' Exact results are:')
disp(' ')
disp(' CL = cl_2D = 2*pi*(alpha + 2*q*(3/4 - xCM_))')
disp(' in which xCM_ = l_CM_LEw/c')
disp(' cd_2D = -2*pi*q*(2*alpha*(1/4 - xCM_) + q*(1 - 2*xCM_)^2)')
disp(' CM_c4 = cm_c4_2D = -pi/4*q')
disp(' -CD/2/q+CM_c4 = 0 (power coefficient must be zero!)')
disp(' ')
disp(' If xCM_ = 0, then:')
disp(' ')
disp(' cl_2D = 2*pi*(alpha + 3*q/2)')
disp(' and: cd_2D = -2*pi*q*(alpha/2 + q)')

```

```

disp(' ')
disp([' QVLM results for large b/c = ', num2str(b/c), ' and infinite H/b '])
disp([' exact 2D results very closely if Nc = ', num2str(Nc), ...
      ' and Ns = ', num2str(Ns)])
disp(' ')
end

disp([' Elapsed time = ', num2str(toc), ' sec'])

time = clock; t = [num2str(time(4)),':',num2str(time(5)),':',num2str(time(6))];
disp(' ')
disp([' Date: ', date])
disp([' Time: ', t])

diary off

```

F3.5 To Find Trim Solution and Aerodynamic Derivatives, Call **TrimDerivs.m**

```
function TrimDerivsVLM(Nc, Ns, Nc_, Ns_, SheetOption, LargeAngles, EpsNewt, ...
    alpha_do, Qo, All_H_bw, Sens_option, ParamFact, kk, ...
    ParNam, Examp, Example, InDir, OutDir)
% function TrimDerivsVLM(Nc, Ns, Nc_, Ns_, SheetOption, LargeAngles, EpsNewt, ...
%     alpha_do, Qo, All_H_bw, Sens_option, ParamFact, kk, ...
%     ParNam, Examp, Example, InDir, OutDir)
%
% =====
% Vortex Lattice Computation of trim state, stability derivatives
% and control derivatives as function of ground height.
% =====
% INPUT DATA:  Obtained by eval(InputFile), which calls function file InputFile.m,
% =====
%     in which InputFile is the character string:  [Examp, 'Input']
%
%     (see for example CheroInput.m)
%
% OUTPUT DATA (THEY ALL WRITTEN TO DATA FILES):
% =====
%
% The data computed here are, firstly:
%
%     AllTrimCoeffs = [AllTrimCoeffs; Coeffs]
%
% with Coeffs = [CL, CDi, CDi/CL^2, CM, CLw, CLt, CDiw, CDit, CDii/CDi, CDii_v/CDi];
%
% where Coeffs are obtained using function file AeroDeriv.m. Other data, furthermore,
%
% are:     AllTrimAngles(kk,:) = [theta, delta_t]*180/pi
%          = [pitch angle, tail angle] in degrees
%          AllTrimCoeffs(kk,2:3) = [CDi_v, CDi_v/CL^2];
%          AllTrimCoeffs(kk,9:10) = [CDii/CDi, CDii_v/CDi];
%          AllTrimCoeffs(kk,2)/AllTrimCoeffs(1,2) = CDi_v/CDi_v_inf
%          ControlDerivs(ii,:) = [CL_del_t, CDi_del_t, CM_del_t];
%
% and finally:
%
% AllDerivs(ii,:) = [CL_a, CDi_a, CM_a, CL_q, CDi_q, CM_q, CL_z, CDi_z, CM_z, ...
%     CL_the, CDi_the, CM_the];
%
% These data define:
%
% TrimRes1 = [1./All_H_bw', AllTrimAngles, AllTrimCoeffs(:,2:3), AllTrimCoeffs(:,9:10)];
% TrimRes2 = [1./All_H_bw', AllTrimAngles, AllTrimCoeffs(:,2)/AllTrimCoeffs(1,2)];
% TrimRes3 = [1./All_H_bw', AllTrimAngles, AllTrimCoeffs];
% TrimRes4 = [1./All_H_bw', AllTrimAngles, AllTrimCoeffs, AllDerivs];
% TrimRes5 = [1./All_H_bw', ControlDerivs];
%
% of which TrimRes1 is displayed in the TEXT file OutFile1, defined by:
%
% OutFile1 = [Examp, ParNam, num2str(Nc), num2str(Ns), '_1.res'];
%
```

```
% in which Examp is an abbreviated name of the Case Study, and ParNam is an
% abbreviation of the name of the parameter with respect to which the sensitivity
% is required. The results TrimRes2, TrimRes3, TrimRes4 and TrimRes5 are respectively
% stored in the DATA files:
```

```
%
% OutFile2 = [Examp, ParNam, num2str(Nc), num2str(Ns), '_2.dat'];
% OutFile3 = [Examp, ParNam, num2str(Nc), num2str(Ns), '_3.dat'];
% OutFile4 = [Examp, ParNam, num2str(Nc), num2str(Ns), '_4.dat'];
% OutFile5 = [Examp, ParNam, num2str(Nc), num2str(Ns), '_5.dat'];
%
%
```

```
=====
```

```
options = [0, 0, 0]; % Suppress all screen output from AeroCoefVLM
```

```
InputFile = [Examp, 'Input'];
OutputRoot = [Examp, ParNam];
OutFile1 = [OutputRoot, num2str(Nc), num2str(Ns), '_1.res'];
OutFile2 = [OutputRoot, num2str(Nc), num2str(Ns), '_2.dat'];
OutFile3 = [OutputRoot, num2str(Nc), num2str(Ns), '_3.dat'];
OutFile4 = [OutputRoot, num2str(Nc), num2str(Ns), '_4.dat'];
OutFile5 = [OutputRoot, num2str(Nc), num2str(Ns), '_5.dat'];
```

```
OldDir = cd;
feval('cd', InDir)
eval(['[bw, cw, Sw, ARw, bt, ct, St, ARt, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ', ...
'Le, Lle_, lam, lam_, Cw] = ', InputFile, ';'])
feval('cd', OldDir)
```

```
% #####
% CALCULATION OF TRIM STATES
% =====
```

```
Cw_nom = Cw;
l_CM_LEw_nom = l_CM_LEw;
h_LEt_LEw_nom = h_LEt_LEw;
l_LEt_LEw_nom = l_LEt_LEw;
bw_nom = bw; cw_nom = cw;
bt_nom = bt; ct_nom = ct;
ARw_nom = bw_nom/cw_nom;
ARt_nom = bt_nom/ct_nom;
St_nom = bt_nom*ct_nom;
Nc_nom = Nc;
%alpha_do = 0; % Equilibrium angle of descent
%Qo = 0; % Equilibrium pitch rate
```

```
epsilon = EpsNewt; % Convergence criterium of Newton-Raphson iteration
```

```
tic % Start stop watch
```

```
% Select parameter value according to specified parameter sensitivity option
% (i.e. assign parameter value other than its nominal value)
ParVal = ParamFact(kk);
if Sens_option == 1
    Cw = ParamFact(kk)*Cw_nom; ParNam = 'Cw/Cwnom';
end
```

```

if Sens_option == 2
    l_CM_LEw = ParamFact(kk)*l_CM_LEw_nom; ParNam = 'l_CM/cw';
end
if Sens_option == 3
    h_LEt_LEw = ParamFact(kk)+h_LEt_LEw_nom; ParNam = 'ht/htnom';
end
if Sens_option == 4
    l_LEt_LEw = ParamFact(kk)*l_LEt_LEw_nom; ParNam = 'lt/ltnom';
end
if Sens_option == 5
    ARw = ParamFact(kk)*ARw_nom; ParNam = 'ARw'; ParVal = ARw;
    f_b = sqrt(ParamFact(kk)); f_c = 1/f_b; % Ensures that Sw =bw*cw remains unchanged!
    bw = f_b*bw_nom; cw = f_c*cw_nom;
end
if Sens_option == 6
    ARt = ParamFact(kk)*ARt_nom; ParNam = 'ARt'; ParVal = ARt;
    f_b = sqrt(ParamFact(kk)); f_c = 1/f_b; % Ensures that St =bt*ct remains unchanged!
    bt = f_b*bt_nom; ct = f_c*cw_nom;
end
if Sens_option == 7
    St = ParamFact(kk)*St_nom; ParNam = 'St/Stnom';
    f_b = sqrt(ParamFact(kk)); f_c = f_b; % Ensures that ARt =bt/ct remains unchanged!
    bt = f_b*bt_nom; ct = f_c*cw_nom;
end
if Sens_option == 8
    Nc = ParamFact(kk); Nc_ = Nc; ParNam = 'Nc'; ParVal = Nc;
end
if Sens_option == 9
    EpsNewt = ParamFact(kk);
end
%disp([' Computing trim states for parameter # ', num2str(Sens_option), ...
% ' ... please wait'])
disp([' Computing trim states for ', ParNam, ' # ', num2str(kk), ...
' ... please wait'])

% Loop through range of ground height ratios H/bw specified in All_H_bw
AllTrimAngles = []; AllTrimCoeffs = [];
count_res = 0;
for jj = 1:length(All_H_bw)
    H_bw = All_H_bw(jj);

    % Initialisation of Newton-Raphson algorithm for trim analysis
    theta_o1 = 5*pi/180; % First estimate of pitch angle
    delta_t1 = theta_o1; % First estimate of tail angle
    dtheta_o1 = pi/180; % First increment of pitch angle
    ddelta_t1 = dtheta_o1; % First increment of tail angle
    dx1 = [dtheta_o1; 0];
    dx2 = [0; ddelta_t1];

    % First three values of state vector
    x1 = [theta_o1; delta_t1];
    x2 = x1+dx1;
    x = x2+dx2;

    % First three values of object function (excess of lift and pitch moment)
    f1 = AeroCoefVLM(x1(1), alpha_do, x1(2), Cw, H_bw, Nc, Nc_, Ns, Ns_, ...

```

```

        bw, cw, Sw, bt, ct, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
        Lle, Lle_, lam, lam_, Qo, SheetOption, LargeAngles, options);
f2 = AeroCoefVLM(x2(1), alpha_do, x2(2), Cw, H_bw, Nc, Nc_, Ns, Ns_, ...
        bw, cw, Sw, bt, ct, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
        Lle, Lle_, lam, lam_, Qo, SheetOption, LargeAngles, options);
f = AeroCoefVLM(x(1), alpha_do, x(2), Cw, H_bw, Nc, Nc_, Ns, Ns_, ...
        bw, cw, Sw, bt, ct, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
        Lle, Lle_, lam, lam_, Qo, SheetOption, LargeAngles, options);

% Newton-Raphson iteration
dx = dx2; count = 0;
while max(abs(dx)) > epsilon & count < 20
    count = count + 1;
    X1 = [x1, x2]; X2 = [x2, x]; dX = X2 - X1;
    F1 = [f1, f2]; F2 = [f2, f]; dF = F2 - F1;

    J = dF*dX^(-1);
    dx = -Jf;
    x_old = x;
    x = x + dx;
    f_old = f;
    [f, Coeffs] = AeroCoefVLM(x(1), alpha_do, x(2), Cw, H_bw, Nc, Nc_, Ns, Ns_, ...
        bw, cw, Sw, bt, ct, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
        Lle, Lle_, lam, lam_, Qo, SheetOption, LargeAngles, options);

    x1 = x2; x2 = x_old;
    f1 = f2; f2 = f_old;

end

% Coeffs = [CL, CDi, CDi/CL^2, CM, CLw, CLt, CDiw, CDit, CDii/CDi, CDii_v/CDi];
AllTrimCoeffs = [AllTrimCoeffs; Coeffs];
% Note that AllTrimAngles(kk,:) = [theta, delta_t]*180/pi
%           = [pitch angle, tail angle] in degrees
AllTrimAngles = [AllTrimAngles; x'*180/pi];
CDi = Coeffs(2); CDi_CL2 = Coeffs(3); CDii_CD_i = Coeffs(9); CDiiv_CD_i = Coeffs(10);
count_res = count_res + 1;
if count_res == 1
    disp(' 1/H_bw theta delta_t CDi CDi_CL2 CDii_CD_i CDiiv_CD_i')
end
disp([1/H_bw, x'*180/pi, CDi, CDi_CL2, CDii_CD_i, CDiiv_CD_i])
% 1/H_bw theta delta_t CDi CDi_CL2 CDii_CD_i CDiiv_CD_i
% 0.0000 7.6946 5.1377 0.0168 0.0569 1.0036 1.0001
end
TrimRes1 = [1./All_H_bw', AllTrimAngles, AllTrimCoeffs(:,2:3), AllTrimCoeffs(:,9:10)];
TrimRes2 = [1./All_H_bw', AllTrimAngles, AllTrimCoeffs(:,2)/AllTrimCoeffs(1,2)];
TrimRes3 = [1./All_H_bw', AllTrimAngles, AllTrimCoeffs];
% CALCULATION OF TRIM STATES COMPLETED
% #####

% COMPUTATION OF AERODYNAMIC DERIVATIVES
% =====
% Increments of state variables used in computation of aerodynamic derivatives
dang = 0.1*pi/180;
dalp_h_d = dang;

```

```

dthe = dang;
dq = 1/2*cw/l_LEt_LEw*dang;    % Selected pitch rate perturbation per unit air speed
dH_bw = 0.1*cw/bw;
dz = dH_bw*bw/cw;

% Loop through all trim states computed above and find aerodynamic derivatives
for ii = 1:size(AllTrimAngles,1)
    theta_o = AllTrimAngles(ii,1)*pi/180;
    delt_to = AllTrimAngles(ii,2)*pi/180;
    H_bw = All_H_bw(ii);

    % Compute alpha-derivatives (perturb alpha at constant theta by perturbing
    % alpha_d !!!!)
    % Coeffs = [CL, CDi, CDi/CL^2, CM, CLw, CLt, CDiw, CDit, CDii/CDi, CDii_v/CDi];
    alpha_d1 = alpha_do - dalpha_d;
    alpha_d2 = alpha_do + dalpha_d;
    [f, Coeffs1] = AeroCoefVLM(theta_o, alpha_d1, delt_to, Cw, H_bw, Nc, Nc_, Ns, Ns_, ...
        bw, cw, Sw, bt, ct, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
        Lle, Lle_, lam, lam_, Qo, SheetOption, LargeAngles, options);
    [f, Coeffs2] = AeroCoefVLM(theta_o, alpha_d2, delt_to, Cw, H_bw, Nc, Nc_, Ns, Ns_, ...
        bw, cw, Sw, bt, ct, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
        Lle, Lle_, lam, lam_, Qo, SheetOption, LargeAngles, options);
    CL1 = Coeffs1(1); CL2 = Coeffs2(1);
    CDi1 = Coeffs1(2); CDi2 = Coeffs2(2);
    CM1 = Coeffs1(4); CM2 = Coeffs2(4);
    CL_a = (CL2 - CL1)/(2*dalpha_d);
    CDi_a = (CDi2 - CDi1)/(2*dalpha_d);
    CM_a = (CM2 - CM1)/(2*dalpha_d);

    % Compute q-derivatives
    % Coeffs = [CL, CDi, CDi/CL^2, CM, CLw, CLt, CDiw, CDit, CDii/CDi, CDii_v/CDi];
    [f, Coeffs1] = AeroCoefVLM(theta_o, alpha_do, delt_to, Cw, H_bw, Nc, Nc_, Ns, Ns_, ...
        bw, cw, Sw, bt, ct, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
        Lle, Lle_, lam, lam_, Qo-dq, SheetOption, LargeAngles, options);
    [f, Coeffs2] = AeroCoefVLM(theta_o, alpha_do, delt_to, Cw, H_bw, Nc, Nc_, Ns, Ns_, ...
        bw, cw, Sw, bt, ct, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
        Lle, Lle_, lam, lam_, Qo+dq, SheetOption, LargeAngles, options);
    CL1 = Coeffs1(1); CL2 = Coeffs2(1);
    CDi1 = Coeffs1(2); CDi2 = Coeffs2(2);
    CM1 = Coeffs1(4); CM2 = Coeffs2(4);
    CL_q = (CL2 - CL1)/(2*dq);
    CDi_q = (CDi2 - CDi1)/(2*dq);
    CM_q = (CM2 - CM1)/(2*dq);

    % Compute z-derivatives
    % Coeffs = [CL, CDi, CDi/CL^2, CM, CLw, CLt, CDiw, CDit, CDii/CDi, CDii_v/CDi];
    [f, Coeffs1] = AeroCoefVLM(theta_o, alpha_do, delt_to, Cw, ...
        H_bw-dH_bw, Nc, Nc_, Ns, Ns_, ...
        bw, cw, Sw, bt, ct, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
        Lle, Lle_, lam, lam_, Qo, SheetOption, LargeAngles, options);
    [f, Coeffs2] = AeroCoefVLM(theta_o, alpha_do, delt_to, Cw, ...
        H_bw+dH_bw, Nc, Nc_, Ns, Ns_, ...
        bw, cw, Sw, bt, ct, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
        Lle, Lle_, lam, lam_, Qo, SheetOption, LargeAngles, options);
    CL1 = Coeffs1(1); CL2 = Coeffs2(1);
    CDi1 = Coeffs1(2); CDi2 = Coeffs2(2);

```



```

CM1 = Coeffs1(4); CM2 = Coeffs2(4);
CL_z = (CL2 - CL1)/(2*dz);
CDi_z = (CDi2 - CDi1)/(2*dz);
CM_z = (CM2 - CM1)/(2*dz);

% Compute theta-derivatives
% Coeffs = [CL, CDi, CDi/CL^2, CM, CLw, CLt, CDiw, CDit, CDii/CDi, CDii_v/CDi];
theta_o1 = theta_o - dthe; theta_o2 = theta_o + dthe;
alpha_d1 = alpha_do + dthe; alpha_d2 = alpha_do - dthe;
% Constant alpha is maintained by increasing angle of descent by same amount
% by which pitch angle is decreased!
[f, Coeffs1] = AeroCoefVLM(theta_o1, alpha_d1, delt_to, Cw, H_bw, Nc, Nc_, Ns, Ns_, ...
    bw, cw, Sw, bt, ct, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
    Lle, Lle_, lam, lam_, Qo, SheetOption, LargeAngles, options);
[f, Coeffs2] = AeroCoefVLM(theta_o2, alpha_d2, delt_to, Cw, H_bw, Nc, Nc_, Ns, Ns_, ...
    bw, cw, Sw, bt, ct, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
    Lle, Lle_, lam, lam_, Qo, SheetOption, LargeAngles, options);
CL1 = Coeffs1(1); CL2 = Coeffs2(1);
CDi1 = Coeffs1(2); CDi2 = Coeffs2(2);
CM1 = Coeffs1(4); CM2 = Coeffs2(4);
CL_the = (CL2 - CL1)/(2*dthe);
CDi_the = (CDi2 - CDi1)/(2*dthe);
CM_the = (CM2 - CM1)/(2*dthe);

AllDerivs(ii,:) = [CL_a, CDi_a, CM_a, CL_q, CDi_q, CM_q, CL_z, CDi_z, CM_z, ...
    CL_the, CDi_the, CM_the];

% Compute delta_t-derivatives
% Coeffs = [CL, CDi, CDi/CL^2, CM, CLw, CLt, CDiw, CDit, CDii/CDi, CDii_v/CDi];
delt_t1 = delt_to - dang; delt_t2 = delt_to + dang;
[f, Coeffs1] = AeroCoefVLM(theta_o, alpha_do, delt_t1, Cw, H_bw, Nc, Nc_, Ns, Ns_, ...
    bw, cw, Sw, bt, ct, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
    Lle, Lle_, lam, lam_, Qo, SheetOption, LargeAngles, options);
[f, Coeffs2] = AeroCoefVLM(theta_o, alpha_do, delt_t2, Cw, H_bw, Nc, Nc_, Ns, Ns_, ...
    bw, cw, Sw, bt, ct, l_LEt_LEw, h_LEt_LEw, l_CM_LEw, ...
    Lle, Lle_, lam, lam_, Qo, SheetOption, LargeAngles, options);
CL1 = Coeffs1(1); CL2 = Coeffs2(1);
CDi1 = Coeffs1(2); CDi2 = Coeffs2(2);
CM1 = Coeffs1(4); CM2 = Coeffs2(4);
CL_del_t = (CL2 - CL1)/(2*dang);
CDi_del_t = (CDi2 - CDi1)/(2*dang);
CM_del_t = (CM2 - CM1)/(2*dang);

ControlDerivs(ii,:) = [CL_del_t, CDi_del_t, CM_del_t];
end

TrimRes4 = [1./All_H_bw', AllTrimAngles, AllTrimCoeffs, AllDerivs];
TrimRes5 = [1./All_H_bw', ControlDerivs];

% COMPUTATION OF AERODYNAMIC DERIVATIVES COMPLETED
% #####

% SCREEN DISPLAY OF RESULTS, AND OUTPUT OF TABLES OF RESULTS TO TEXT FILE
% =====
clc

```

```

feval('cd', OutDir)
if exist(OutFile1) == 2; delete(OutFile1); end; feval('diary', OutFile1)
disp(' TRIM STATES AND AERODYNAMIC DERIVATIVES COMPUTED BY VORTEX LATTICE METHOD')
disp(' =====')
disp(' ')
disp(' Results are given as function of ground height, and include:')
disp(' - trim states')
disp(' - stability derivatives')
disp(' - control derivatives')
disp(' ')
disp([' EXAMPLE: ', Example])
disp(' ')
disp([' Aspect Ratio of Wing = ', num2str(ARw)])
disp([' Aspect Ratio of Tail = ', num2str(ARt)])
disp([' Ratio of Planform Areas (tail/wing) = ', num2str(St/Sw)])
disp([' Wingspan = ', num2str(bw)])
disp([' Tailspan = ', num2str(bt)])
disp([' Distance of CM behind wing LE dx_CM/cw = ', num2str(l_CM_LEw/cw)])
disp([' Tail distance (x(le_t)-x(le_w))/cw = ', num2str(l_LEt_LEw/cw)])
disp([' Tail height ht/cw = ', num2str(h_LEt_LEw/cw)])
disp(' ')
disp([' Number of chordwise strips on wing Nc = ', num2str(Nc)])
disp([' Number of chordwise strips on tail Nc_ = ', num2str(Nc_)])
disp([' Number of spanwise strips on wing Ns = ', num2str(Ns)])
disp([' Number of spanwise strips on tail Ns_ = ', num2str(Ns_)])
disp(' ')
disp([' Angle of Descent (degrees) alpha_do = ', num2str(alpha_do*180/pi)])
disp([' Trim Value of Lift Coefficient CLo = Cw = ', num2str(Cw)])
disp(' ')
disp(' Influence of Ground Proximity on Trim State')
disp(...)
' bw/H theta_o delta_t CDi CDi/CL^2 CDii/CDi CDii_v/CDi')
disp(TrimRes1)

disp(' ')
disp(' Influence of Ground Proximity on CDi/CDi_inf')
disp(' bw/H theta_o delta_t CDi/CDi_inf')
disp(TrimRes2)

disp(' ')
disp(' INFLUENCE OF GROUND PROXIMITY ON AERODYNAMIC DERIVATIVES')
disp(' alpha-derivatives')
disp(' bw/H CL_a CDi_a CM_a')
disp([1./All_H_bw', AllDerivs(:,1:3)])
disp(' q-derivatives')
disp(' bw/H CL_q CDi_q CM_q')
disp([1./All_H_bw', AllDerivs(:,4:6)])
disp(' z-derivatives')
disp(' bw/H CL_z CDi_z CM_z')
disp([1./All_H_bw', AllDerivs(:,7:9)])
disp(' theta-derivatives')
disp(' bw/H CL_the CDi_the CM_the')
disp([1./All_H_bw', AllDerivs(:,10:12)])
disp(' delta_t-derivatives')
disp(' bw/H CL_del_t CDi_del_t CM_del_t')
disp([1./All_H_bw', ControlDerivs])

```

```

disp([' Elapsed time = ', num2str(toc), ' sec'])

time = clock; t = [num2str(time(4)),':',num2str(time(5)),':',num2str(time(6))];
disp(' ')
disp([' Date: ', date])
disp([' Time: ', t])
diary off

% SAVE RESULTS IN OUTPUT DATA FILES
% =====
feval('save', OutFile2, 'TrimRes2', '-ascii', '-double')
feval('save', OutFile3, 'TrimRes3', '-ascii', '-double')
feval('save', OutFile4, 'TrimRes4', '-ascii', '-double')
feval('save', OutFile5, 'TrimRes5', '-ascii', '-double')
feval('cd', OldDir)

return

```

F3.6 To Find Sets of Trim Solutions and Aerodynamic Derivatives, Call **TrimDerivsMain.m**

```
% MAIN PROGRAM FOR COMPUTATION OF LONGITUDINAL TRIM STATE AND STABILITY
% AND CONTROL DERIVATIVES EVALUATED IN THE TRIM STATE
%
% TrimDerivsMain

clear all

% Names of input and output directories
InDir = 'c:\my_documents\Navorsing\Phd00\Matlab5\ThesisJunJul2000\InputData';
OutDir = 'c:\my_documents\Navorsing\Phd00\Matlab5\ThesisJunJul2000\VLM_Results';

% Required numbers of chordwise and spanwise segments on wing and tail
%Nc = 5; Ns = 16;
%Nc = 3; Ns = 6;
%Nc = 1; Ns = 1;
%Nc = 1; Ns = 4;
%Nc = 1; Ns = 6;
Nc = 2; Ns = 4;
%Nc = 2; Ns = 6;
%Nc = 3; Ns = 6;
Nc_ = 2; Ns_ = 3;
%Nc_ = 2; Ns_ = 2;

if Nc == 1; Nc_ = 1; end % Approximation of Prandtl's Monoplane Theory
if Ns == 1; Ns_ = 1; end % Lanchester approximation (one vortex per surface)

% Some computational options
SheetOption = 1; % Vortex sheet aligned along descent path
%SheetOption = 0; % Vortex sheet in horizontal plane
LargeAngles = 0; % Small angles assumed everywhere
%LargeAngles = 1; % Small angles assumed only in vortex sheet angle
EpsNewt = 1e-4; % Nominal convergence criterium of trim problem

% Angle of descent and pitch rate in required trim state
alpha_do = 0*pi/180;
Qo = 0;

% Abbreviated title of example to be considered (Cherokee was analysed extensively,
% but the Astra not yet)
%Examp = 'Astra';
Examp = 'Chero';
% Long title of example
if strcmp(Examp, 'Astra'); Example = 'Pilatus PC-7 Astra'; end
if strcmp(Examp, 'Chero'); Example = 'Piper Cherokee PA-28-180'; end

% Required range of height ratios b/H
All_H_bw = [[inf 1/0.2 1/0.4 1/0.6 1/0.8 1/1], ...
            [1/1.2 1/1.4 1/1.6 1/1.8 1/2.0 1/2.2 1/2.4 1/2.6], ...
            [1/2.8 1/3.0 1/3.2 1/3.4 1/3.6 1/3.8 1/4.0]];
%All_H_bw = [[inf 1/0.4 1/0.8 1/1.2 1/1.6 1/2.0 1/2.4 1/2.8 1/3.2 1/3.6 1/4.0]];

% Sensitivity analysis of Cherokee
if strcmp(Examp, 'Chero')
% for Sens_option = 2:2
```

```

% for Sens_option = [1 4 5]
for Sens_option = 5
    if Sens_option == 1;
        ParNam = 'Cw'; ParamFact = [0.75 1.0 1.25 1.5];
    end
    if Sens_option == 2;
        % Note that wing chord cw = 63*0.0254, and
        % hN_nom = 13*0.0254/(63*0.0254) = 0.2063; (2D estimation!)
        % for Cherokee
        ParNam = 'CM'; ParamFact = [0.15 0.175 0.2063 0.225 0.25]/0.2063;
    end
    if Sens_option == 3;
        % Note that ht_nom = 63*0.0254 for Cherokee;
        ParNam = 'ht'; ParamFact = [0 0.5 1.0 1.5]*63*0.0254;
    end
    if Sens_option == 4;
        ParNam = 'lt'; ParamFact = [0.75 1.0 1.25];
    end
    if Sens_option == 5;
        ARw_nom = 360*0.0254/(63*0.0254);
        % Note that ARw_nom = 360*0.0254/(63*0.0254) = 5.7143 for Cherokee;
        ParNam = 'ARw'; ParamFact = [4 ARw_nom 8 10 1e6]/ARw_nom;
    end
    if Sens_option == 6;
        ARt_nom = 120*0.0254/(30*0.0254);
        % Note that ARt_nom = 120*0.0254/(30*0.0254) = 4;
        ParNam = 'ARt'; ParamFact = [2 ARt_nom 4 6]/ARt_nom;
    end
    if Sens_option == 7;
        St_nom = 120*0.0254*(30*0.0254);
        % Note that St_nom = 120*0.0254*(30*0.0254) = 2.3226;
        ParNam = 'St'; ParamFact = [0.75 1 1.25];
    end
    if Sens_option == 8;
        ParNam = 'Nc'; ParamFact = [1 2 3];
    end
    if Sens_option == 9;
        ParNam = 'Eps'; ParamFact = [0.1 1.0 10 100]*1e4;
    % ParNam = 'Eps'; ParamFact = [100]*1e4;
    end

% for kk = 1:1 % Just first one
% for kk = 3:3 % Some other one
% for kk = length(ParamFact):length(ParamFact) % Just last one
for kk = 1:length(ParamFact) % All of them
    ParamFactVal = ParamFact(kk);
    ParNam2 = ['_', ParNam, num2str(kk), '_'];
    TrimDerivsVLM(Nc, Ns, Nc_, Ns_, SheetOption, LargeAngles, EpsNewt, ...
        alpha_do, Qo, All_H_bw, Sens_option, ParamFact, kk, ...
        ParNam2, Examp, Example, InDir, OutDir)
end
end
end

% Sensitivity analysis of Astra
if strcmp(Examp, 'Astra')

```

```

return                                % Option not built in yet !!!!
Sens_option = 2;
ParNam = 'CM'; ParamFact = [0.19 0.21 0.23 0.25 0.27]/0.23;
for kk = 1:length(ParamFact)
    ParamFactVal = ParamFact(kk);
    ParNam2 = ['_', ParNam, num2str(kk), '_'];
    TrimDerivsVLM(Nc, Ns, Nc_, Ns_, SheetOption, LargeAngles, EpsNewt, ...
        alpha_do, Qo, All_H_bw, Sens_option, ParamFact, kk, ...
        ParNam2, Examp, Example, InDir, OutDir)
end
end

```

F3.7 To Perform Stability and Control Analysis, Call **StabContrAnalysis.m**

```
function [bw_H, max_real, CL_z_, CL_z_max3, CDi_z_, CD_z_max, h_z_, h_z_crit, E_q_the] = ...
    StabContrAnalysis(Examp, ParamName, ParamNum, Nc, Ns, K_CMz, ii_H, t_fin_)
% function [bw_H, max_real, CL_z_, CL_z_max3, CDi_z_, CD_z_max] = ...
%     StabContrAnalysis(Examp, ParamName, ParamNum, Nc, Ns, K_CMz, ii_H, t_fin_)
% STABILITY ANALYSIS OF PIPER CHEROKEE BY VORTEX LATTICE METHOD
%
% Equations of X- and Z- motion are resolved in wind axes, i.e., with x-axis
% into the wind, and z-axis pointing downward.
%
% Stability criteria of longitudinal stability in ground proximity are computed for
% Piper Cherokee with decreasing height at constant CLo = 0.543

% Specify line types to be used in time traces of impulse response
Lyn = str2mat(' - ', '- ', '- ', '- ', '- ', '- ', '- ', '- ', '- ');
Lyn = str2mat(' - ', '- ', '- ', '- ', '- ', '- ', '- ', '- ', '- ');
LegStr = ['\_\_\_\_', ' \_\_ \_\_', ' \_\_ . . \_\_', ' and . . . ', ...
    ' respectively represent '];

if strcmp(Examp, 'Astra'); Example = 'Pilatus PC-7 Astra'; end
if strcmp(Examp, 'Chero'); Example = 'Piper Cherokee PA-28-180'; end

% Input and output file directories
OldDir = cd;
InDir = 'c:\my_documents\Navorsing\Phd00\Matlab5\ThesisJunJul2000\Saved_VLM_Results';
InDir2 = 'c:\my_documents\Navorsing\Phd00\Matlab5\ThesisJunJul2000\InputData';
OutDir = ['c:\my_documents\Navorsing\Phd00\Matlab5\ThesisJunJul2000\', ...
    'FinalResults\StabContrOutData'];

% Input and output file names
InFile1 = [Examp, '_', ParamName, num2str(ParamNum), '_', ...
    num2str(Nc), num2str(Ns), '_4.dat'];
InFile2 = [Examp, '_', ParamName, num2str(ParamNum), '_', ...
    num2str(Nc), num2str(Ns), '_5.dat'];
InFile2_ = [Examp, '\_', ParamName, num2str(ParamNum), '\_', ...
    num2str(Nc), num2str(Ns), '\_5.dat'];
OutFile1 = [Examp, '_', ParamName, num2str(ParamNum), ...
    '_K', num2str(floor(100*K_CMz)), '_', ...
    num2str(Nc), num2str(Ns), '_1.txt'];
OutFile2 = [Examp, '_', ParamName, num2str(ParamNum), ...
    '_K', num2str(floor(100*K_CMz)), '_', ...
    num2str(Nc), num2str(Ns), '_2.txt'];

% Read required data from output files of VLM analysis and from input files
% Read files containg trim data and aerodynamic derivatives from VLM analysis
feval('cd', InDir)
feval('load', InFile1, '-ascii'); feval('load', InFile2, '-ascii')
feval('cd', InDir2)
% Read additional aircraft data (required in stability analysis) obtained from McCormick
% (see MoreCheroInput.m)
[bw, cw, Sw, mu, iyy, f, CDmin, CM_ad, CZ_ad, CM_qd, Qo, kd, ...
    E_Short_0, E_Long_0] = MoreCheroInput;
feval('cd', OldDir)

% Extract values of all bw/H from InFile1
```

```

eval(['bw_H = ', InFile1(1:length(InFile1)-4), '(:,1)";'])

% Construct a character string that contains these values of bw/H
% (to be used in captions of screen plots of results of stability analysis)
AppendStr = 'x at b_w/H = 0.';
bw_H_str = MakeCapStr(bw_H, AppendStr);

% Construct character string that contains values of bw/H for which time traces
% are produced (to be used in captions of screen plots of time traces of impulse response)
Some_bw_H = bw_H(ii_H); AppendStr = [];
bw_H_str_p = MakeCapStr(Some_bw_H, AppendStr);

% Extract trim states, aerodynamic coefficients, and aerodynamic derivatives
eval(['AllTrimAngles = ', InFile1(1:length(InFile1)-4), '(:,2:3);'])
eval(['AllTrimCoeffs = ', InFile1(1:length(InFile1)-4), '(:,4:13);'])
eval(['AllTrimDerivs = ', InFile1(1:length(InFile1)-4), '(:,14:25);'])
eval(['AllControlDerivs = ', InFile2(1:length(InFile2)-4), '(:,2:4);'])

% Make provision for change in density with altitude (NOT USED, BUT RETAIN!)
d_rho_dH = -1.0494e-4; % Density derivative (kg/m^4) at H = 1500 m from Zucrow pp. 708
Cr_z = d_rho_dH*cw; % Dimensionless derivative of density ratio
Cr_z = 0; % COMMENT OUT IF DENSITY DERIVATIVE IS NOT TO BE IGNORED !!!

% Extract CLo and CDio (subscript _ indicates column vector)
CLo_ = AllTrimCoeffs(:,1);
CDio_ = AllTrimCoeffs(:,2);

% Extract derivatives of CL, CDi and CM (subscript _ indicates column vector)
% alpha- and q-derivatives
CL_a_ = AllTrimDerivs(:,1); CL_q_ = AllTrimDerivs(:,4);
CDi_a_ = AllTrimDerivs(:,2); CDi_q_ = AllTrimDerivs(:,5);
CM_a_ = AllTrimDerivs(:,3); CM_q_ = AllTrimDerivs(:,6);

% theta- and z-derivatives
CL_th_ = AllTrimDerivs(:,10); CL_z_ = AllTrimDerivs(:,7);
CDi_th_ = AllTrimDerivs(:,11); CDi_z_ = AllTrimDerivs(:,8);
CM_th_ = AllTrimDerivs(:,12); CM_z_ = AllTrimDerivs(:,9);

% Extract control derivatives
CL_del_t_ = AllControlDerivs(:,1);
CDi_del_t_ = AllControlDerivs(:,2);
CM_del_t_ = AllControlDerivs(:,3);

% Select/compute vector of proportional feedback gain factors for the various heights
% for selected ratio K_CMz of Closed Loop to Open Loop values of CMz
Kz_ = (1 - K_CMz)*CM_z_/CM_del_t_; % Gain as function of height

% Apply feedback control of height perturbation to elevator input
CDi_z_OL = CDi_z_; CL_z_OL = CL_z_; CM_z_OL = CM_z_; % Open Loop z-derivatives
% Closed Loop z-derivatives
CDi_z_ = CDi_z_ + Kz_*CDi_del_t_;
CL_z_ = CL_z_ + Kz_*CL_del_t_;
CM_z_ = CM_z_ - Kz_*CM_del_t_;

% Compute height stability margin h_z and its critical value
CDo_ = CDmin+CDio_;

```



```

RH = (CL_a_/CM_a_).*(1-2/kd*CLo_/CDo_.*(CDi_a_/CL_a_));
nRH = length(RH);
if CL_z_(1) ~= 0
    LH = (1-(2/kd*CLo_/CDo_).*(CDi_z_/CL_z_));
else
    LH(1,1) = NaN;
    LH(2:nRH,1) = (1-(2/kd*CLo_(2:nRH,1)/CDo_(2:nRH,1)).* ...
        (CDi_z_(2:nRH,1)/CL_z_(2:nRH,1)));
end
if RH(1) ~= 0
    CMz_CLz_crit = LH./RH;
else
    CMz_CLz_crit(1,1) = NaN;
    CMz_CLz_crit(2:nRH,1) = LH(2:nRH,1)./RH(2:nRH,1);
end
h_z_crit = -CMz_CLz_crit;
if CL_z_(1) ~= 0
    h_z_ = -CM_z_/CL_z_;
else
    h_z_(1,1) = NaN;
    h_z_(2:nRH,1) = -CM_z_(2:nRH,1)/CL_z_(2:nRH,1);
end

% Open output file to which eigenvectors are to be saved
feval('cd', OutDir)
if exist(OutFile2) == 2; delete(OutFile2); end
diary(OutFile2), diary on
feval('cd', OldDir)

% Display caption of output file for eigenvectors
disp(' EIGENVECTORS OF LONGITUDINAL MOTION OF CHEROKEE IN GROUND PROXIMITY')
disp(' '), disp(' ')
disp(' Equations of longitudinal motion were resolved in wind axes.')
disp(' '), disp(' ')
disp(' Trim solution and aerodynamic derivatives were obtained by')
disp(' aerodynamic analysis using the Vortex Lattice Method.')
disp(' '), disp(' ')
disp(' ')
disp(' '), disp(' ')

% Loop through selected ranges of CM heights
ii_Lyn = 0;
for ii = 1:length(bw_H)

    % Equilibrium lift and drag coefficients
    CLo = CLo_(ii);
    CDo = CDmin + CDio_(ii);

    % Aerodynamic stability derivatives
    CD_a = CDi_a_(ii);    CD_z = CDi_z_(ii);    CD_q = CDi_q_(ii);
    CL_a = CL_a_(ii);    CL_z = CL_z_(ii);    CL_q = CL_q_(ii);
    CM_a = CM_a_(ii);    CM_z = CM_z_(ii);    CM_q = CM_q_(ii);
    CD_the = CDi_th_(ii); CL_the = CL_th_(ii); CM_the = CM_th_(ii);

    % Control derivatives
    CDi_del_t = CDi_del_t_(ii);

```

```

CL_del_t = CL_del_t(ii);
CM_del_t = CM_del_t(ii);

% Allow for density change with altitude (ineffective when Cr_z = 0 is uncommented above!)
CL_z = CLo*Cr_z + CL_z;      % Add contribution of density derivative
CD_z = CDo*Cr_z + CD_z;      % Add contribution of density derivative

% Compute coefficient matrices using function A_B_Matrices defined below
CL_q = 0; CD_q = 0;
[A_0_0, A_q_0, A_q_the, B_0, B_add, C_c] = ...
    A_B_Matrices(kd, CDo, CLo, mu, iyy, CD_a, CL_a, CM_a, CD_q, CL_q, CM_q, ...
        CD_the, CL_the, CM_the, CD_z, CL_z, CM_z, CZ_ad, CM_ad, CM_qd, ...
        CDi_del_t, CL_del_t, CM_del_t);
A_q0_0 = A_q_0; % A-matrix with CM_q accounted for, but zero CL_q and CD_q

% Restore CL_q and CD_q for A-matrix that includes non-zero CL_q, CD_q and CM_q
CL_q = CL_q(ii); CD_q = CD_q(ii); %A_B_Matrices;
[A_0_0, A_q_0, A_q_the, B_0, B_add, C_c] = ...
    A_B_Matrices(kd, CDo, CLo, mu, iyy, CD_a, CL_a, CM_a, CD_q, CL_q, CM_q, ...
        CD_the, CL_the, CM_the, CD_z, CL_z, CM_z, CZ_ad, CM_ad, CM_qd, ...
        CDi_del_t, CL_del_t, CM_del_t);

% Compute AA- and BB-matrices of 5-th order state equations
AA = B_add\A_q_the;
BB = B_add\C_c;

AAA = B_add(2:5,2:5)\A_q_the(2:5,2:5);

% Determine whether results of this H-loop have been selected to plot
ii_p = 0;
for kkk = 1:length(ii_H)
    if ii == ii_H(kkk)
        ii_p = 1;
        ii_Lyn = ii_Lyn + 1;
    end
end

% Compute and plot response to unit impulsive elevator input
% Scroll through various time ranges (short, medium, long, where applicable)
for jj_t = 1:length(t_fin_)
    t_fin = t_fin_(jj_t);
    tt = linspace(0,t_fin, 501);
    sys = ss(AA,BB,eye(size(AA)),0);
    Y = impulse(sys, tt);
    Scales = [100, 180/pi, 180/pi, 180/pi, 1];
    Y_Labels = str2mat('u (% of U_o)', 'alpha (deg)', 'q (deg-airsec/sec)', ...
        'theta (deg)', 'H (per c)');

% Plot time traces if height is one that has been pre-selected for a plot
if ii_p == 1
    figure(8+jj_t-1)
    for iii = 1:5
        subplot(5,1,iii), plot(tt/1e3, Scales(iii)*Y(:,iii,2), ...
            deblank(Lyn(ii_Lyn,:))), hold on
    end
    if ii_p == 1
        Asse(iii,:) = axis;
    end
end

```

```

    Asse(iiii,:) = [Asse(iiii,1:2), 1.05*Asse(iiii,3:4)];
end
axis(Asse(iiii,:));
if iiii == 1
    if K_CMz == 1
        title(['Open Loop Response Curves for Unit Elevator Impulse Input ', ...
            'at b_w/H = ', bw_H_str_p])
    else
        title(['Closed Loop Response Curves for Unit Elevator Impulse Input ', ...
            'at b_w/H = ', bw_H_str_p])
    end
end
ylabel(Y_Labels(iiii,:))
end
end
Kz = Kz(ii);
xlabel(['Dimensionless Time (in thousands of air seconds)', ' ', ...
    ['Input Data Obtained from ', InFile2_, ' ', ...
    ' K_{CMz} = ', num2str(K_CMz)], [LegStr, 'b_w/H = ', bw_H_str_p]})
end

% Eigenvalues for zero q- and theta-derivatives; no added mass
[P, E1] = eig(B_0\A_0_0);
E_temp = diag(E1);
[dum, I_E] = sort(real(-E_temp));
E(:,ii) = E_temp(I_E); % EXCLUDES effects of CL_q, CD_q AND CM_q

% Eigenvalues for zero theta-derivatives, but non-zero q-derivatives; no added mass
[P, E2] = eig(B_0\A_q_0); % _0 on A_q indicates zero theta-derivatives
E_temp = diag(E2);
[dum, I_E] = sort(real(-E_temp));
E_q(:,ii) = E_temp(I_E); % INCLUDES effects of CL_q, CD_q AND CM_q

% Eigenvalues for zero theta-derivatives, with CL_q = CD_q = 0, but non-zero CM_q,
% and no added mass
[P, E_dum] = eig(B_0\A_q0_0); % 0_0 on A_q indicates no theta-derivs, CL_q=CD_q=0
E_temp = diag(E_dum);
[dum, I_E] = sort(real(-E_temp));
E_q0(:,ii) = E_temp(I_E); % INCLUDES effect of CM_q, BUT NOT CL_q nor CD_q

% Eigenvalues for non-zero q-derivatives, and non-zero theta-derivatives; no added mass
[P, E3] = eig(B_0\A_q_the);
E_temp = diag(E3);
[dum, I_E] = sort(real(-E_temp));
E_q_the(:,ii) = E_temp(I_E);

% Eigenvalues for non-zero q-derivatives, and non-zero theta-derivatives; with added mass
[P, E4] = eig(B_add\A_q_the);
E_temp = diag(E4);
[dum, I_E] = sort(real(-E_temp));
E_q_the_add(:,ii) = E_temp(I_E);
P0 = P(:, I_E);

% [P, E6] = eig(B_add(2:5,2:5)\A_q_the(2:5,2:5));
% [P, E6] = eig(B_add(2:4,2:4)\A_q_the(2:4,2:4));
% E_temp = diag(E6); [dum, I_E] = sort(real(-E_temp)); E6(:,ii) = E_temp(I_E);

```

```

% junk = [junk, E6(:,ii)];
% close all, disp(ii), disp(junk'), pause

EigVecs(E, ii, P0, cw, bw_H) % Computes, displays and saves eigenvectors in OutFile2

% Eigenvalues from reduced 2nd order system, with non-zero q- and theta-derivatives,
% and with added mass
[P, E5] = eig(B_add(2:3, 2:3)\A_q_the(2:3, 2:3));
E_temp = diag(E5);
[dum, I_E] = sort(real(-E_temp));
E_q_the_add_alpha(:,ii) = E_temp(I_E);

% Rate of decay, damping ratio, damped and undamped natural frequencies
% obtained for alpha mode from 5th order determinant
sigl(ii) = -real(E_q_the_add(4,ii));
wnl(ii) = abs(E_q_the_add(4,ii));
zetl(ii) = sigl(ii)/wnl(ii);
wdl(ii) = abs(imag(E_q_the_add(2,ii)));
% Approximations of same from 2nd order determinant and with zero CL_ad
sigl_appr(ii) = (iyy*CL_a+CL_q*CM_ad-2*mu*(CM_ad+CM_q))/(4*iyy*mu);
sigl_appr(ii) = (iyy*CL_a-2*mu*(CM_ad+CM_q))/(4*iyy*mu);
wnl_appr(ii) = sqrt((CL_q*CM_a-(2*mu*CM_a+CL_a*CM_q))/(2*iyy*mu));
zetl_appr(ii) = sigl_appr(ii)/wnl_appr(ii);
wdl_appr(ii) = wnl(ii)*sqrt(1-zetl_appr(ii)^2);

% =====
% THIS IS THE ESSENTIAL PART OF THE STABILITY ANALYSIS
% Eigenvalues from reduced 3rd order system, with zero q- and theta-derivatives,
% and no added mass. Characteristic equation is:
%
%  $A s^5 + B s^4 + C s^3 + D s^2 + E s + F = 0$ 
%
% Assumptions are:      - negligible "added mass" derivatives
%                        - negligible q- and theta-derivatives
%
% Coefficients of characteristic equation of reduced 3rd order system
A_appr2 = 0;
B_appr2 = 0;
C_appr2 = -4*mu^2*CM_a;
D_appr2 = -2*kd*CD0*CM_a*mu;
E_appr2 = -mu*CL_a*CM_z+mu*CM_a*CL_z-2*CM_a*CLo^2;
F_appr2 = -1/2*kd*CD0*CL_a*CM_z+1/2*kd*CD0*CM_a*CL_z+CLo*CD_a*CM_z-CLo*CM_a*CD_z;
E_appr(:,ii) = roots([C_appr2, D_appr2, E_appr2, F_appr2]);
% Maximum of real parts of all characteristic roots, critical CL_z and CD_z
max_real(ii) = max(real(E_q_the_add(:,ii)));
max_real_appr(ii) = max(real(E_appr(:,ii)));
% Critical values of CD_z
CD_z_max(ii) = 1/mu*kd*CD0*CLo+1/CM_a*CD_a*CM_z; % Obtained from D*E-C*F=0 !
CD_z_max1(ii) = 1/mu*kd*CD0*CLo; % Setting CM_z = 0
CD_zcrit_q_the(ii) = ...
    CD_zcrit_new(CD_a, CD_q, CD_the, CD0, CL_a, CL_q, CL_the, CL_z, CLo, ...
    CM_a, CM_q, CM_the, CM_z, kd, mu);
CD_zcrit_q(ii) = ...
    CD_zcrit_new(CD_a, CD_q, 0, CD0, CL_a, CL_q, 0, CL_z, CLo, ...
    CM_a, CM_q, 0, CM_z, kd, mu);

```

```

% CD_z_max(ii) = CD_zcrit_q(ii);
% CD_z_max(ii) = CD_zcrit_q_the(ii);
% Critical values of CL_z
CL_z_max3(ii) = 1/CM_a*CL_a*CM_z-2/kd/CDo/CM_a*CLo*CD_a*CM_z+2/kd/CDo*CLo*CD_z;
% Obtained from F=0 !
CL_z_max2(ii) = 2*CLo^2/mu + 1/CM_a*CL_a*CM_z; % Uncoupled 2nd order appr.
CL_z_max_Stauf(ii) = 1/CM_a*CL_a*CM_z; % From Staufenbiel's static
% height stability criterium

CL_zcrit_q_the(ii) = ...
    CL_zcrit_new(CD_a, CD_the, CD_z, CDo, CL_a, CL_the, CLo, CM_a, CM_the, CM_z, kd);
CL_zcrit_q(ii) = ...
    CL_zcrit_new(CD_a, 0, CD_z, CDo, CL_a, 0, CLo, CM_a, 0, CM_z, kd);
hN_zcrit_q_the(ii) = ...
    hN_zcrit_new(CD_a, CD_the, CD_z, CDo, CL_a, CL_the, CL_z, CLo, CM_a, CM_the, kd);
hN_zcrit_q(ii) = ...
    hN_zcrit_new(CD_a, 0, CD_z, CDo, CL_a, 0, CL_z, CLo, CM_a, 0, kd);
% disp(CL_zcrit_q_the(ii) - CL_zcrit_q(ii)), disp(hN_zcrit_q_the(ii) - hN_zcrit_q(ii)), pause
% END OF ESSENTIAL PART OF THE STABILITY ANALYSIS
% =====

% Estimation of phugoid and heave roots by approximate factorisation
% of reduced 3rd order system
p1(ii) = F_appr2/E_appr2;
prod_p2_p3(ii) = E_appr2/C_appr2;
sum_p2_p3(ii) = D_appr2/C_appr2 - p1(ii);
aa(ii) = E_appr2/C_appr2;
bb(ii) = D_appr2/C_appr2 - p1(ii);
p_2_3(1:2,ii) = roots([1, -bb(ii), aa(ii)]);
PP(:,ii) = [-p1(ii); -p_2_3(1:2,ii)];
[dum, I_pp] = sort(real(-PP(:,ii)));
E_appr3(:, ii) = PP(I_pp,ii);
wn3_appr(ii) = E_appr2/C_appr2;
sig3_appr(ii) = (F_appr2/E_appr2-D_appr2/C_appr2)/2;
% Second approach to same estimation
a2 = D_appr2/C_appr2; a1 = E_appr2/C_appr2; a0 = F_appr2/C_appr2;
p1_(ii) = 1/2*(a2+sqrt(a2^2-4*a1));
p2_(ii) = 1/2*(a2-sqrt(a2^2-4*a1));
p3_(ii) = F_appr2/C_appr2/p1_(ii)/p2_(ii);
PP(:,ii) = [-p1_(ii); -p2_(ii); -p3_(ii)];
[dum, I_PP_] = sort(real(-PP(:,ii)));
E_appr4(:, ii) = PP_(I_PP_,ii);

p_crit1(ii) = -F_appr2/E_appr2;
p_crit2(ii) = -1/2*(D_appr2/C_appr2-sqrt((D_appr2/C_appr2)^2-4*E_appr2/C_appr2));

end
E_appr4 = [real(E_appr4(1,:)); E_appr4(2:3,:)];
diary off

% Assess accuracy of various approximations of eigenvalues
Acc_Slow = abs(E_appr3)./abs(E_q_the_add(1:3,:)); % Accuracy of phug. appr. 1
Acc_Slow_ = abs(E_appr4)./abs(E_q_the_add(1:3,:)); % Accuracy of phug. appr. 2
Acc_add = abs(E_q_the(1:3,:))./abs(E_q_the_add(1:3,:)); % Influence of ad and qd
Acc_the = abs(E_q(1:3,:))./abs(E_q_the(1:3,:)); % Influence of theta
Acc_q0 = abs(E_q0(1:3,:))./abs(E_q(1:3,:)); % Influence of CL_q and CD_q
Acc_q_LD = abs(E(1:3,:))./abs(E_q0(1:3,:)); % Influence of CM_q

```

```

clc
% Open output file to which results of stability analysis are to be saved
feval('cd', OutDir)
if exist(OutFile1) == 2; delete(OutFile1); end
diary(OutFile1), diary on
feval('cd', OldDir)

% Caption of output file for stability roots
disp(' RESULTS OF LONGITUDINAL STABILITY AND CONTROL ANALYSIS OF CHEROKEE')
disp(' '), disp(' ')
disp(' Equations of longitudinal motion were resolved in wind axes.')
disp(' '), disp(' ')
disp(' Trim solution and aerodynamic derivatives were obtained by')
disp(' aerodynamic analysis using the Vortex Lattice Method.')
disp(' '), disp(' ')
disp(' Description of control strategy:')
disp(' =====')
disp(' ')
if K_CMz == 1
    disp(' Open loop control is used')
else
    disp(' Closed loop control is used (feedback of height perturbation to elevator)')
    disp([' Reduction ratio of CM_z is ', ...
        ' K_{CMz} = ', num2str(K_CMz), ' at all b_w/H'])
    disp(' Proportional error gain therefore varies with b_w/H')
end
disp(' '), disp(' ')

disp(' Largest real part of dominant three roots,')
disp(' stability margin CD_z_max - CD_z of CD_z')
disp(' and stability margin CL_z_max - CL_z')
disp(' of CL_z as function of ground height')
disp(' (positive Re1 implies instability;')
disp(' negative margin of any z-derivative')
disp(' implies instability)')
disp(' ...')
disp(' b/H 1e3*Re1 CD_z_marg CL_z_marg')
disp(['bw_H', 1e3*[max_real, CD_z_max-CDi_z_, CL_z_max3-CL_z_]])

disp(' Static pitch moment stability wrt. angle of attack and height')
disp(' ...')
disp(' b/H hN_alpha hN_z hN_zcrit')
disp(['bw_H', -CM_a_/CL_a_, h_z_, h_z_crit])

disp(' '), disp(' ')
disp(' Dominant three root values as function of ground height')
disp(' (1, 2, 3 in order of decreasing real value, were applicable)')
disp(' Computed using 5th order characteristic equation')
disp(' ...')
disp(' b/H 1e3*root_1 1e3*root_2 1e3*root_3')
disp(['bw_H', 1e3*[E_q_the_add(1:3,:)]])

disp(' '), disp(' ')
disp(' Analytical estimation of dominant three root values as function of')

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```

disp(' ground height by approximate factorisation of reduced 3rd order system')
disp( ...
' b/H      1e3*root_1      1e3*root_2      1e3*root_3')
disp([bw_H', 1e3*E_appr3'])

disp(' '), disp(' ')
disp(' Influence of neglecting alphadot- and qdot-derivatives on values')
disp(' of long period roots (roots no. 1, 2 and 3)')
disp( ...
' b/H (r_a/r)1 (r_a/r)2 (r_a/r)3')
disp([bw_H', Acc_add'])
disp(' 1, 2, 3 indicates 1st, 2nd and 3rd dominant root')
disp(' a) r INCLUDES effects of CM_alphadot AND CZ_alphadot, BUT r_a excludes them.')
disp(' b) both r and r_a INCLUDE effects of all theta- and q-derivatives')
disp(' c) r and r_a were computed using 5th order system.')

disp(' '), disp(' ')
disp(' Influence neglecting theta-derivatives on values')
disp(' of long period roots (roots no. 1, 2 and 3)')
disp( ...
' b/H (r_a/r)1 (r_a/r)2 (r_a/r)3')
disp([bw_H', Acc_the'])
disp(' 1, 2, 3 indicates 1st, 2nd and 3rd dominant root')
disp(' a) r INCLUDES effects of all theta-derivatives, BUT r_a excludes them.')
disp(' b) both r and r_a INCLUDE effects of ALL q-derivatives, and both EXCLUDE')
disp(' effects of CM_alphadot AND CZ_alphadot')
disp(' c) r and r_a were computed using 5th order system.')

disp(' '), disp(' ')
disp(' Influence of neglecting CL_q and CD_q on values')
disp(' of long period roots (roots no. 1, 2 and 3)')
disp( ...
' b/H (r_a/r)1 (r_a/r)2 (r_a/r)3')
disp([bw_H', Acc_q0'])
disp(' 1, 2, 3 indicates 1st, 2nd and 3rd dominant root')
disp(' a) r INCLUDES effects of CL_q and CD_q, BUT r_a EXCLUDES them both')
disp(' b) both r and r_a INCLUDE effect of CM_q, BUT BOTH EXCLUDE effects of')
disp(' theta-derivatives AND effects of CM_alphadot AND CZ_alphadot')
disp(' c) r and r_a were computed using 5th order system.')

disp(' '), disp(' ')
disp(' Influence of neglecting CM_q on values')
disp(' of long period roots (roots no. 1, 2 and 3)')
disp( ...
' b/H (r_a/r)1 (r_a/r)2 (r_a/r)3')
disp([bw_H', Acc_q_LD'])
disp(' 1, 2, 3 indicates 1st, 2nd and 3rd dominant root')
disp(' a) r INCLUDES effects of CM_q, BUT r_a does not')
disp(' b) both r and r_a EXCLUDE effects CL_q and CD_q, of theta-derivatives,')
disp(' AND effects of CM_alphadot AND CZ_alphadot')
disp(' c) r and r_a were computed using 5th order system.')

disp(' '), disp(' ')
disp(' Accuracy of analytical approximation of phugoid and heave roots')
disp(' measured by ratio of magnitudes of analytical approximation to values')
disp(' obtained from 5th order characteristic equation. All derivatives')

```

```

disp(' were taken into account in the latter. The former were computed')
disp(' by approximate factorisation of the reduced 3rd order system, computing')
disp(' approximating two roots by assuming the third to be much smaller in')
disp(' magnitude, and then approximating the third one.')
disp( ...
'   b/H   (r_a/r)1 (r_a/r)2 (r_a/r)3')
disp([bw_H', Acc_Slow'])

disp(' '), disp(' ')
disp(' Accuracy of approximation of alpha roots')
disp( ...
'   b/H   sig2/sig5 wn2/wn5 zet2/zet5')
disp([bw_H', sig1_appr'./sig1', wn1_appr'./wn1', zet1_appr'./zet1'])
disp(' 2 indicates that 2th order determinant were used, neglecting CZ_ad')
disp(' 5 indicates that 5th order determinant were used, neglecting nothing')

if K_CMz ~= 1
    disp(' '), disp(' ')
    disp(' FEEDBACK CONTROL STRATEGY AND ITS INFLUENCE ON HEIGHT DERIVATIVES')
    disp(' '), disp(' ')
    disp(' Feedback Control Law used is: delta_t = delta_t_ref - Kz*z')
    disp(' ')
    disp([' in which: delta_t_ref = the part of elevator angle that is due to ', ...
        'pilot input'])
    disp('          z = measured height perturbation')
    disp('          Kz = feedback gain')
    disp(' '), disp(' ')
    disp(' Table of Gain Values Kz with which Height Perturbation is Fed Back')
    disp(' Units of Kz are degrees elevator angle per unit height perturbation')
    disp('   bw/H   Kz   ')
    disp([bw_H', Kz_*180/pi])
    disp(' '), disp(' ')
    disp(' Open Loop (OL) and Closed Loop (OL) values of CM_z, CD_z and CL_z (all x1000)')
    disp('   bw/H   CM_z_OL   CM_z_CL   CDi_z_OL   CDi_z_CL   CL_z_OL   CL_z_CL')
    disp([bw_H', [CM_z_OL, CM_z_, CDi_z_OL, CDi_z_, CL_z_OL, CL_z_] * 1e3])
end

disp(' '), disp(' ')
disp([' ', date])
diary off

disp(' Influence of q-and theta-derivatives on stability criteria')
disp( ...
'   b/H   hN_alpha   hN_z   hN_zcrit')
%size(bw_H'), size(CL_z_max3), size(CL_zcrit_q_the)
%size(h_z_crit'), size(hN_zcrit_q_the), size(CD_z_max), size(CD_zcrit_q)
disp([bw_H', CL_z_max3'./CL_zcrit_q_the', h_z_crit./hN_zcrit_q_the', ...
    CD_z_max'./CD_zcrit_q_the', CD_zcrit_q'./CD_zcrit_q_the'])

% Plot short period results
n_H = length(bw_H);
figure(1)
S = 10; ms = 10;
E = E_q_the_add_alpha;

```



```

plot(real(E(1,2:n_H)), imag(E(1,2:n_H)), 'r.', 'MarkerSize', ms)
hold on
E = E_q_the_add(4:5,:);
plot(real(E(1,2:n_H)), imag(E(1,2:n_H)), 'r^', 'MarkerSize', 6)

E = E_Short_0;
plot(real(E(1)), imag(E(1)), 'rs', 'MarkerSize', S)
plot(real(E(2)), imag(E(2)), 'bs', 'MarkerSize', S)

E = E_q_the_add_alpha;
plot(real(E(2,1)), imag(E(2,1)), 'bx', 'MarkerSize', S)
plot(real(E(2,2:n_H)), imag(E(2,2:n_H)), 'b.', 'MarkerSize', ms)

E = E_q_the_add(4:5,:);
plot(real(E(1,1)), imag(E(1,1)), 'rx', 'MarkerSize', S)
plot(real(E(2,2:n_H)), imag(E(2,2:n_H)), 'b^', 'MarkerSize', 6)

E = E_Short_0;
plot(real(E(1)), imag(E(1)), 'rs', 'MarkerSize', S)
plot(real(E(2)), imag(E(2)), 'bs', 'MarkerSize', S)

axis('equal'), grid on
title(['Short Period Roots of Cherokee for C_{Lo} = ', num2str(CLo), ' and'], ...
      ['b_w/H = ', bw_H_str])
xlabel(['Exponential Growth Rate (per air second)', [' '], ...
      ['Input Data Obtained from ', InFile2_, '. ', ...
      ' K_{CMz} = ', num2str(K_CMz)])
ylabel('Damped Frequency (rad per air second)')
legend('5th order system', '2nd order reduced system', ...
      'McCormick, at b_w/H = 0')

% Plot long period results
figure(2)
S = 10;
E = E_q_the_add; % Eigenvalues of unsimplified 5th order system
plot(real(E(1,2:n_H)), imag(E(1,2:n_H)), 'r.', 'MarkerSize', ms)
hold on
E = E_appr; % Eigenvalues of partly decoupled 3rd order system
plot(real(E(1,2:n_H)), imag(E(1,2:n_H)), 'r^', 'MarkerSize', 6)
E = E_q_the_add;

E = E_Long_0; % Eigenvalues from McCormick
plot(real(E(1)), imag(E(1)), 'rs', 'MarkerSize', S)
plot(real(E(2)), imag(E(2)), 'bs', 'MarkerSize', S)

E = E_q_the_add;
plot(real(E(1,1)), imag(E(1,1)), 'rx', 'MarkerSize', S)
plot(real(E(2,1)), imag(E(2,1)), 'bx', 'MarkerSize', S)
plot(real(E(3,1)), imag(E(3,1)), 'kx', 'MarkerSize', S)

plot(real(E(2,2:n_H)), imag(E(2,2:n_H)), 'b.', 'MarkerSize', ms)
plot(real(E(3,2:n_H)), imag(E(3,2:n_H)), 'k.', 'MarkerSize', ms)
axis('equal'), grid on
title(['Long Period Roots of Piper Cherokee for C_{Lo} = ', num2str(CLo), ' and'], ...
      ['b_w/H = ', bw_H_str])
xlabel(['Exponential Growth Rate (per air second)', [' '], ...

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```

    ['Input Data Obtained from ', InFile2_, '. ', ...
    ' K_{CMz} = ', num2str(K_CMz)]]}
ylabel('Damped Frequency (rad per air second)')

E = E_appr;
plot(real(E(1,1)), imag(E(1,1)), 'rx', 'MarkerSize', S)
plot(real(E(2,1)), imag(E(2,1)), 'bx', 'MarkerSize', S)
plot(real(E(3,1)), imag(E(3,1)), 'kx', 'MarkerSize', S)
plot(real(E(1,2:n_H)), imag(E(1,2:n_H)), 'r^', 'MarkerSize', 6)
plot(real(E(2,2:n_H)), imag(E(2,2:n_H)), 'b^', 'MarkerSize', 6)
plot(real(E(3,2:n_H)), imag(E(3,2:n_H)), 'k^', 'MarkerSize', 6)
legend('5th order system', '3rd order reduced system', ...
'McCormick, at b_w/H = 0')

%E6 = junk; figure
%plot(real(E6(1,1)), imag(E6(1,1)), 'rx', 'MarkerSize', S), hold on
%plot(real(E6(2,1)), imag(E6(2,1)), 'bx', 'MarkerSize', S)
%plot(real(E6(1,2:n_H)), imag(E6(1,2:n_H)), 'b.', 'MarkerSize', ms)
%plot(real(E6(2,2:n_H)), imag(E6(2,2:n_H)), 'k.', 'MarkerSize', ms)

% Plot influence of aerodynamic lift and drag derivatives on
% longitudinal stability in ground proximity
figure(3)
plot(bw_H, CD_z_max, 'r'), hold on
plot(bw_H, CDi_z, 'b--')
title('Height Derivative of Drag Coefficient of Cherokee in Ground Proximity')
xlabel(['Ground height ratio b_w/H', ' ', ...
    ['Input Data Obtained from ', InFile2_, '. ', ...
    ' K_{CMz} = ', num2str(K_CMz)], ...
    ['C_{Dzcrit} obtained from B C - A D = 0 in A s^3 + B s^2 + C s + D = 0', ...
    ' (with A > 0)']})
ylabel('C_{Dz} and C_{Dzcrit} (note that stability requires C_{Dz} < C_{Dzcrit})')
legend('C_{Dz}_{crit}', 'C_{Dz}')
grid on

figure(4)
plot(bw_H, CD_z_max - CDi_z, 'r'), hold on
plot(bw_H, CDi_z, 'b--')
title(['Stability Margin of Height Derivative of Drag Coefficient ', ...
    '\Delta C_{Dz} = C_{Dz}_{crit} - C_{Dz} of Cherokee'])
xlabel(['Ground height ratio b_w/H', ' ', ...
    ['Input Data Obtained from ', InFile2_, '. ', ...
    ' K_{CMz} = ', num2str(K_CMz)], ...
    ['C_{Dzcrit} obtained from B C - A D = 0 in A s^3 + B s^2 + C s + D = 0', ...
    ' (with A > 0)']})
ylabel('C_{Dz} and \Delta C_{Dz} (note that stability requires \Delta C_{Dz} > 0)')
legend('\Delta C_{Dz}', 'C_{Dz}_{crit}')
grid on

figure(5)
plot(bw_H, CL_z_max3 - CL_z, 'r'), hold on
plot(bw_H, CL_z_max2 - CL_z, 'r-'), plot(bw_H, CL_z_max_Stauf - CL_z, 'r:')
plot(bw_H, CL_z, 'b--')
title(['Stability Margin of Height Derivative of Lift Coefficient ', ...
    '\Delta C_{Lz} = C_{Lz}_{crit} - C_{Lz} of Cherokee'])
xlabel(['Ground height ratio b_w/H', ' ', ...

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```

    ['Input Data Obtained from ', InFile2_, '. ', ...
     ' K_{CMz} = ', num2str(K_CMz)]]))
ylabel('C_{Lz} and \Delta C_{Lz} (note that stability requires \Delta C_{Lz} > 0)')
legend('\Delta C_{Lz} Present (3rd order)', ...
       '\Delta C_{Lz} Present (2nd order)', ...
       '\Delta C_{Lz} Staufenbiel', 'C_{Lz}')
grid on

figure(6)
plot(bw_H, CL_z_max3', 'r'), hold on
plot(bw_H, CL_z_max2', 'r-'), plot(bw_H, CL_z_max_Stauf', 'r')
plot(bw_H, CL_z_, 'b--')
title('Height Derivative of Lift Coefficient of Cherokee in Ground Proximity')
xlabel(['Ground height ratio b_w/H', [' '], ...
       ['Input Data Obtained from ', InFile2_, '. ', ...
        ' K_{CMz} = ', num2str(K_CMz)], ...
       ['C_{Lzcrit} (3rd order) obtained from D = 0 in A s^3 + B s^2 + C s + D = 0 ', ...
        '(with A > 0)', ...
       ['C_{Lzcrit} (2nd order) obtained from C = 0 in (A s^2 + B s + C) s = 0 ', ...
        '(with A > 0)']]))
ylabel('C_{Lz} and C_{Lzcrit} (note that stability requires C_{Lz} < C_{Lzcrit})')
legend('C_{Lzcrit} Present (3rd order)', ...
       'C_{Lzcrit} Present (2nd order)', ...
       'C_{Lzcrit} Staufenbiel', 'C_{Lz}')
grid on

figure(7)
plot(bw_H, max_real')
title('Longitudinal Stability of Cherokee in Ground Proximity')
xlabel(['Ground height ratio b_w/H', [' '], ...
       ['Input Data Obtained from ', InFile2_, '. ', ...
        ' K_{CMz} = ', num2str(K_CMz)]]))
ylabel('Real part of root furthest to the right in the s-plane (fastest growth rate)')
grid on, hold on
plot(bw_H, real(E_appr3(1,:)), 'r--')
legend('from 5th order system', 'approximate factorisation')

return

% #####
function EigVecs(E, ii, P0, cw, bw_H)

E_P = E;          % All eigenvalues

% Indices that sort eigenvalues: heave mode first, then phugoid mode
iii1 = 1; iii2 = 2; iii3 = 3; comp_f = 0;
if abs(imag(E_P(1,ii))) > 1e-6 & abs(imag(E_P(2,ii))) > 1e-6
    iii1 = 3; iii2 = 1; iii3 = 2; comp_f = 1;
end

% Scaling matrix
S1 = [[100 0 0 0 0]; [0 180/pi 0 0 0]; [0 0 180/pi 0 0]; [0 0 0 180/pi 0]; [0 0 0 0 1]];
% Scaled eigenvectors
PS1 = S1*P0;

```

```

% Extract 1st three eigenvectors: heave mode first, then phugoid mode
PS1_p_a = PS1(:,[iii1, iii2, iii3]);

% Scaling matrix for normalisation w.r.t. H-component
S2 = abs([1/PS1_p_a(5, 1), 0, 0]; [0, 1/PS1_p_a(5, 2), 0]; [0, 0, 1/PS1_p_a(5, 3)]);

% Extract 1st three eigenvectors: heave mode first, then phugoid mode
Mag = [abs(PS1_p_a)*S2]; % Normalise with respect to H-element
phase_p_a = [180/pi*(angle(PS1_p_a) - ... % Subtract phase of H-element
[ones(5,1)*angle(PS1_p_a(5, 1)), ones(5,1)*angle(PS1_p_a(5, 2)), ...
ones(5,1)*angle(PS1_p_a(5, 3))]);
% Phase of H-element is subtracted from others above

if ii == 1
disp(' Magnitudes and phase angles(deg) of first three eigenvectors. Last row')
disp(' contains 1000 x real parts of 1st three characteristic roots below Magnitude')
disp(' columns, and repeated below phase columns. Magnitudes are normalised w.r.t')
disp(' fractional height change, and phase referred to that of H-element. Units of')
disp(' elements of eigenvectors are:')
disp(' [u in % of Uo, alpha in deg, q in deg-airsec/sec, theta in deg, z in frac of c]')
disp(' All are normalised w.r.t. H-element (i.e., expressed per unit change of H/c)')
disp(' ')
end
disp([' Value of Height Parameter b/H = ', num2str(bw_H(ii))])
disp( ...
' M(P1) M(P2) M(P3) phi(P1) phi(P2) phi(P3)')
E_p_a = 1e3*E_P([iii1, iii2, iii3], ii);
disp([Mag, phase_p_a; real([E_p_a, E_p_a])])

%tt = 1/(2*50/cw);
%disp([bw_H(ii)*ones(5,1), Mag, phase_p_a; [bw_H(ii), tt./real([E_p_a, E_p_a])]])
%disp([bw_H(ii), E_p_a])

return

% #####
function bw_H_str = MakeCapStr(bw_H, AppendStr)
bw_H_str = '';
for ii = 1:length(bw_H)
if ii == 1
bw_H_str = [bw_H_str, num2str(bw_H(ii))];
else
bw_H_str = [bw_H_str, ', ', num2str(bw_H(ii))];
end
end
bw_H_str = [bw_H_str, ']', AppendStr];
return

% #####
function [A_0_0, A_q_0, A_q_the, B_0, B_add, C_c] = ...
A_B_Matrices(kd, CDo, CLo, mu, iyy, CD_a, CL_a, CM_a, CD_q, CL_q, CM_q, ...
CD_the, CL_the, CM_the, CD_z, CL_z, CM_z, CZ_ad, CM_ad, CM_qd, ...
CDi_del_t, CL_del_t, CM_del_t);
% CD_the, CL_the, CM_the, CD_z, CL_z, CM_z, CZ_ad, CM_ad, CM_qd);
% COEFFICIENT MATRICES OF LONGITUDINAL DYNAMICS RESOLVED IN WIND AXES

```

% A-matrix with no q- or theta-derivatives:

```
A_0_0 = ...
[[ -kd*CD0 -CD_a + CL0 0 -CL0 -CD_z]
[-2*CL0 -CL_a 2*mu 0 -CL_z]
[ 0 CM_a 0 0 CM_z]
[ 0 0 1 0 0]
[ 0 -1/2 0 1/2 0 ]];
```

% A-matrix with q-derivatives, but no theta-derivatives:

```
A_q_0 = ...
[[ -kd*CD0 -CD_a + CL0 -CD_q -CL0 -CD_z]
[-2*CL0 -CL_a 2*mu -CL_q 0 -CL_z]
[ 0 CM_a CM_q 0 CM_z]
[ 0 0 1 0 0]
[ 0 -1/2 0 1/2 0 ]];
```

% A-matrix with both q-derivatives and theta-derivatives:

```
A_q_the = ...
[[ -kd*CD0 -CD_a + CL0 -CD_q -CL0 -CD_the -CD_z]
[-2*CL0 -CL_a 2*mu -CL_q -CL_the -CL_z]
[ 0 CM_a CM_q CM_the CM_z]
[ 0 0 1 0 0]
[ 0 -1/2 0 1/2 0 ]];
```

% B-matrix with no added mass terms:

```
B_0 = ...
[[ 2*mu 0 0 0 0]
[ 0 2*mu 0 0 0]
[ 0 0 iyy 0 0]
[ 0 0 0 1 0]
[ 0 0 0 0 1]];
```

% B-matrix with added mass terms CZ_ad, CM_ad and CM_qd:

```
B_add = ...
[[ 2*mu 0 0 0 0]
[ 0 2*mu - CZ_ad 0 0 0]
[ 0 -CM_ad iyy - CM_qd 0 0]
[ 0 0 0 1 0]
[ 0 0 0 0 1]];
```

% Control coefficients (C_c-matrix):

```
C_c = ...
[[ CD0 -CDi_del_t]
[ 0 -CL_del_t]
[ 0 CM_del_t]
[ 0 0]
[ 0 0 ]];
```

return

F3.8 To Perform Post-processing of Stability and Control Results, Call **StabContrPostProc.m**

```
function [bw_H, all_max_real, all_dCLz, all_dCDz, all_dhz] = ...

    StabContrPostProc(ParamName, K_CMz, Nc, Ns)
% e.g. StabContrPostProc('cm', 1, 2, 2)
%function [bw_H, max_real, CL_z_, CL_z_crit_, CD_z_, CD_z_crit_, h_z_, h_z_crit_] = ...
% StabContrPostProc(ParamName, K_CMz, Nc, Ns)
%
% e.g. StabContrPostProc('cm', 0.75) displays previously computed results that
% illustrates influence of CM position on stability characteristics when K_CMz = 0.75
%
% Let:
%
% ParamName = 'cm'; for influence of CM position
% ParamName = 'cw'; for influence of load  $C_w = W/(\frac{1}{2} \rho V^2 S)$ 
% ParamName = 'ht'; for influence of tail height
% ParamName = 'lt'; for influence of tail length
% ParamName = 'arw'; for influence of wing aspect ratio
%
% Influence of factor K_CMz by which feedback system is designed to change CM_z
% K_CMz = 1; % Selection of K_CM_z = 1 implies no feedback (CM_z unchanged)
% Selection of K_CM_z = 0.75 implies CM_z = 0.75*CM_z_Open_Loop, etc.
% Note that K_CM_z = Closed Loop to Open Loop ratio of CMz required
% (this is realised by feedback of height perturbation to elevator)
% K_CMz = 0.75;
% K_CMz = 0.5;
% K_CMz = 0.45;
% K_CMz = 0.25;
% K_CMz = 0;
%
% RESULTS OF PREVIOUSLY EXECUTED VORTEX LATTICE ANALYSIS ARE USED IN STABILITY
% ANALYSIS HERE !
%
% Post-processing of trim data and aerodynamic derivatives for assessing sensitivity
% of stability and control characteristics to selected design parameters

Examp = 'chero';
LineTypes = str2mat('-', '--', '-.', '.');

% VLM RESULTS FOR SPECIFIED CHORD- AND SPANWISE NUMBERS OF VORTICES MUST BE
% AVAILABLE !!

%All_H_bw = [[inf 1/0.2 1/0.4 1/0.6 1/0.8 1/1], ...
% [1/1.2 1/1.4 1/1.6 1/1.8 1/2.0 1/2.2 1/2.4 1/2.6], ...
% [1/2.8 1/3.0 1/3.2 1/3.4 1/3.6 1/3.8 1/4.0]];
%All_H_bw = [[inf 1/0.4 1/0.8 1/1.2 1/1.6 1/2.0 1/2.4 1/2.8 1/3.2 1/3.6 1/4.0]];
ii_H = [1 6 11 14]; % Range of height indices for which response is to be computed
ii_H = [1 8 12 14]; % Range of height indices for which response is to be computed
ii_H = [1 6 14 24];
ii_H = [1 5 9 13];
%ii_H = [1 3 5 7];
t_fin_ = ...
[100, 2000, 5000]; % Range of time spans of required impulse response

OutDir = ['c:\my_documents\Navorsing\Phd00\Matlab5\ThesisJunJul2000\', ...
```

```

'FinalResults\SummarySensTables'];
OutFile3 = [Examp, '_', ParamName, ...
            '_', num2str(floor(100*K_CMZ)), '_', ...
            num2str(Nc), num2str(Ns), '_3.txt'];

MainCap_CM1 = 'Influence of CM Position on Longitudinal Stability of Cherokee';
MainCap_CM2 = ['Influence of CM Position on Stability Margin ', ...
               '\Delta C_{Lz} = C_{Lzcrit} - C_{Lz} of Cherokee'];
MainCap_CM3 = ['Influence of CM Position on Stability Margin ', ...
               '\Delta C_{Dz} = C_{Dzcrit} - C_{Dz} of Cherokee'];
MainCap_CM4 = ['Influence of CM Position on Stability Margin ', ...
               '\Delta h_{Nz} = h_{Nzcrit} - h_{Nz} of Cherokee'];
X_Label = 'Ground height ratio b_w/H';
Y_Label1 = 'Real part of root furthest to the right in the s-plane (fastest growth rate)';
Y_Label2 = 'Stability Margin \Delta C_{Lz} (stable when positive)';
Y_Label3 = 'Stability Margin \Delta C_{Dz} (stable when positive)';
Y_Label4 = 'Stability Margin \Delta h_{Nz} (stable when positive)';

MainCap_Cw1 = 'Influence of Load Factor C_w on Longitudinal Stability of Cherokee';
MainCap_Cw2 = ['Influence of Load Factor C_w on Stability Margin ', ...
               '\Delta C_{Lz} = C_{Lzcrit} - C_{Lz} of Cherokee'];
MainCap_Cw3 = ['Influence of Load Factor C_w on Stability Margin ', ...
               '\Delta C_{Dz} = C_{Dzcrit} - C_{Dz} of Cherokee'];
MainCap_Cw4 = ['Influence of Load Factor C_w on Stability Margin ', ...
               '\Delta h_{Nz} = h_{Nzcrit} - h_{Nz} of Cherokee'];
MainCap_ht1 = 'Influence of Tail Height h_t on Longitudinal Stability of Cherokee';
MainCap_ht2 = ['Influence of Tail Height h_t on Stability Margin ', ...
               '\Delta C_{Lz} = C_{Lzcrit} - C_{Lz} of Cherokee'];
MainCap_ht3 = ['Influence of Tail Height h_t on Stability Margin ', ...
               '\Delta C_{Dz} = C_{Dzcrit} - C_{Dz} of Cherokee'];
MainCap_ht4 = ['Influence of Tail Height h_t on Stability Margin ', ...
               '\Delta h_{Nz} = h_{Nzcrit} - h_{Nz} of Cherokee'];
MainCap_lt = 'Influence of Tail Length l_t on Longitudinal Stability of Cherokee';
MainCap_lt1 = 'Influence of Tail Length l_t on Longitudinal Stability of Cherokee';
MainCap_lt2 = ['Influence of Tail Length l_t on Stability Margin ', ...
               '\Delta C_{Lz} = C_{Lzcrit} - C_{Lz} of Cherokee'];
MainCap_lt3 = ['Influence of Tail Length l_t on Stability Margin ', ...
               '\Delta C_{Dz} = C_{Dzcrit} - C_{Dz} of Cherokee'];
MainCap_lt4 = ['Influence of Tail Length h_t on Stability Margin ', ...
               '\Delta h_{Nz} = h_{Nzcrit} - h_{Nz} of Cherokee'];
MainCap_ARw = 'Influence of Wing Aspect ratio AR_w on Longitudinal Stability of Cherokee';
MainCap_ARw1 = 'Influence of Wing Aspect ratio AR_w on Longitudinal Stability of Cherokee';
MainCap_ARw2 = ['Influence of Wing Aspect ratio AR_w on Stability Margin ', ...
               '\Delta C_{Lz} = C_{Lzcrit} - C_{Lz} of Cherokee'];
MainCap_ARw3 = ['Influence of Wing Aspect ratio AR_w on Stability Margin ', ...
               '\Delta C_{Dz} = C_{Dzcrit} - C_{Dz} of Cherokee'];
MainCap_ARw4 = ['Influence of Wing Aspect ratio AR_w on Stability Margin ', ...
               '\Delta h_{Nz} = h_{Nzcrit} - h_{Nz} of Cherokee'];

% Construct legend string for plot of influence of CM position l_CM_LEw/cw
ParamVals = [0.15 0.175 0.2063 0.225 0.25]; % These are all values of l_CM_LEw/cw
ii_PV = [2 3 4 5]; % These are indices of some l_CM_LEw/cw
%ii_PV = [2]; % These are indices of some l_CM_LEw/cw
%ii_PV = [3]; % These are indices of some l_CM_LEw/cw
ParamStr1 = 'x_{CM/LEw} = '; ParamStr2 = 'c_w';
LegStr = MakeLegStr(ParamVals, ii_PV, ParamStr1, ParamStr2);

```

```

ii_F = 1;
if strcmp(ParamName, 'cm')
    close all
    [bw_H, all_max_real, all_dCLz, all_dCDz, all_dhz] = ...
    MakeSensPlots(ii_F, ii_PV, LineTypes, LegStr, ...
        Examp, ParamName, Nc, Ns, K_CMz, ii_H, t_fin, ...
        MainCap_CM1, MainCap_CM2, MainCap_CM3, MainCap_CM4, X_Label, Y_Label1, ...
        Y_Label2, Y_Label3, Y_Label4);
    nnF = 5; AxLims = [0 0.3 0 2];
    ShortParName = ' x_CM_LEw'; LongParName = 'CM position';
    X_Label = ['CM distance behind wing's leading edge (fraction of chord)'];
    SummaryResults
    disp(MainCap_CM1)
    return
end

% Construct legend string for plot of influence of load factor Cw
% ParNam = 'Cw'; ParamFact = [0.75 1.0 1.25 0.5];
Cw_nom = 0.543;
ParamVals = [0.75 1.0 1.25 1.5];
% These are values of Cw for which sensitivity plots are required
ii_PV = [1 2 3 4]; % These are the corresponding indices of Cw
ParamStr1 = 'C_w = '; ParamStr2 = ' C_{wnom}';
LegStr = MakeLegStr(ParamVals, ii_PV, ParamStr1, ParamStr2);
ii_F = 1;
if strcmp(ParamName, 'cw')
    close all
    [bw_H, all_max_real, all_dCLz, all_dCDz, all_dhz] = ...
    MakeSensPlots(ii_F, ii_PV, LineTypes, LegStr, ...
        Examp, ParamName, Nc, Ns, K_CMz, ii_H, t_fin, ...
        MainCap_Cw1, MainCap_Cw2, MainCap_Cw3, MainCap_Cw4, X_Label, Y_Label1, ...
        Y_Label2, Y_Label3, Y_Label4);
    nnF = 5; AxLims = [0 1.6 0 2];
    ShortParName = ' Cw'; LongParName = 'Load Factor C_w';
    X_Label = ['Load Factor C_w (equal to C_{Lo})'];
    SummaryResults
    disp(MainCap_Cw1)
    return
end

% Construct legend string for plot of influence of tail height ht
% ParNam = 'ht'; ParamFact = [0 0.5 1.0 1.5]*63*0.0254;
ii_PV = [1 2 3 4]; % These are indices of ht/ht_nom for which plots are required
ht_nom = 63*0.0254;
ParamVals = [0 0.5 1.0 1.5];
ParamStr1 = 'h_t = '; ParamStr2 = ' c_w';
LegStr = MakeLegStr(ParamVals, ii_PV, ParamStr1, ParamStr2);
ii_F = 1;
if strcmp(ParamName, 'ht')
    close all
    [bw_H, all_max_real, all_dCLz, all_dCDz, all_dhz] = ...
    MakeSensPlots(ii_F, ii_PV, LineTypes, LegStr, ...
        Examp, ParamName, Nc, Ns, K_CMz, ii_H, t_fin, ...
        MainCap_ht1, MainCap_ht2, MainCap_ht3, MainCap_ht4, X_Label, Y_Label1, ...
        Y_Label2, Y_Label3, Y_Label4);
    nnF = 5; AxLims = [0 1.1 0 2];

```



```

nnF = 5; AxLims = [0 1.75 0 4];
ShortParName = 'ht'; LongParName = 'Tail Height h_t';
X_Label = ['Tail Height h_t above Wing (fraction of wing chord)'];
SummaryResults
disp(MainCap_ht1)
return
end

% Construct legend string for plot of influence of tail length lt
% ParNam = 'lt'; ParamFact = [0.75 1.0 1.25];
%ParName = 'lt';
lt_nom = 63*0.0254;
ParamVals = [0.75 1.0 1.25];
ii_PV = [1 2 3]; % These are indices of lt/lt_nom for which plots are required
ParamStr1 = 'l_t = '; ParamStr2 = 'l_{tnom}';
LegStr = MakeLegStr(ParamVals, ii_PV, ParamStr1, ParamStr2);
ii_F = 1;
if strcmp(ParamName, 'lt')
close all
[bw_H, all_max_real, all_dCLz, all_dCDz, all_dhz] = ...
    MakeSensPlots(ii_F, ii_PV, LineTypes, LegStr, ...
        Examp, ParamName, Nc, Ns, K_CMz, ii_H, t_fin, ...
        MainCap_lt1, MainCap_lt2, MainCap_lt3, MainCap_lt4, X_Label, Y_Label1, ...
        Y_Label2, Y_Label3, Y_Label4)
nnF = 5; AxLims = [0 1.3 0 2];
ShortParName = 'lt'; LongParName = 'Tail Length l_t';
X_Label = ['Tail Distance l_t behind Wing (as fraction of nominal length)'];
SummaryResults
disp(MainCap_lt1)
return
end

% Construct legend string for plot of influence of wing aspect ratio ARw
% ParNam = 'ARw'; ParamFact = [4 ARw_nom 8 10 12]/ARw_nom;
ParName = 'ARw'; ARw_nom = 360*0.0254/(63*0.0254);
% Note that ARw_nom = 360*0.0254/(63*0.0254) = 5.7143;
ParamVals = [4 ARw_nom 8 10 12]/ARw_nom;
ParamVals = [4 ARw_nom 8 10 12];
ii_PV = [2 3 4 5]; % These are indices of ARw_nom for which plots are required
ii_PV = [1 2 3 4]; % These are indices of ARw_nom for which plots are required
ParamStr1 = 'AR_w = '; ParamStr2 = '';
LegStr = MakeLegStr(ParamVals, ii_PV, ParamStr1, ParamStr2);
ii_F = 1;
if strcmp(ParamName, 'arw')
close all
[bw_H, all_max_real, all_dCLz, all_dCDz, all_dhz] = ...
    MakeSensPlots(ii_F, ii_PV, LineTypes, LegStr, ...
        Examp, ParamName, Nc, Ns, K_CMz, ii_H, t_fin, ...
        MainCap_ARw1, MainCap_ARw2, MainCap_ARw3, MainCap_ARw4, X_Label, Y_Label1, ...
        Y_Label2, Y_Label3, Y_Label4)
nnF = 5; AxLims = [0 12 0 2];
ShortParName = 'ARw'; LongParName = 'Wing Aspect Ration AR_w';
X_Label = 'Wing Aspect Ratio AR_w';
SummaryResults
disp(MainCap_ARw1)
return

```

```
end
return
```

```
% #####
```

```
function LegStr_cm = MakeLegStr(ParamVals, ii_PV, ParamStr1, ParamStr2)
```

```
% Matrix of legend captions
```

```
LegStrMat = ['', ParamStr1, num2str(ParamVals(ii_PV(1))), ParamStr2, ''];
```

```
for ii = ii_PV(2:length(ii_PV))
```

```
    LegStrMat = str2mat(LegStrMat, ['', ParamStr1, num2str(ParamVals(ii)), ParamStr2, '']);
```

```
end
```

```
% String of legend captions
```

```
LegStr = [legend('deblank(LegStrMat(1,:))];
```

```
for ii = 2:length(ii_PV)
```

```
    LegStr = [LegStr, ', ', deblank(LegStrMat(ii,:))];
```

```
end
```

```
LegStr_cm = [LegStr, ''];
```

```
LegStrMat_cm = LegStrMat;
```

```
return
```

```
% #####
```

```
function [bw_H, all_max_real, all_dCLz, all_dCDz, all_dhz] = ...
```

```
    MakeSensPlots(ii_F, ii_PV, LineTypes, LegStr, ...
```

```
    Examp, ParamName, Nc, Ns, K_CMz, ii_H, t_fin, ...
```

```
    MainCap_CM1, MainCap_CM2, MainCap_CM3, MainCap_CM4, X_Label, Y_Label1, ...
```

```
    Y_Label2, Y_Label3, Y_Label4)
```

```
All_E = [];
```

```
for ResNum = 1:length(ii_PV)
```

```
    [bw_H, max_real, CL_z, CL_z_crit, CD_z, CD_z_crit, h_z, h_z_crit, E] = ...
```

```
    StabContrAnalysis(Examp, ParamName, ii_PV(ResNum), Nc, Ns, K_CMz, ii_H, t_fin);
```

```
All_E = [All_E; E];
```

```
all_max_real(ResNum,:) = max_real;
```

```
all_dCLz(ResNum,:) = CL_z_crit - CL_z;
```

```
all_dCDz(ResNum,:) = CD_z_crit - CD_z;
```

```
all_dhz(ResNum,:) = (h_z_crit - h_z);
```

```
% if strcmp(ParamName, 'cm') & ResNum == 1, pause, end
```

```
% if strcmp(ParamName, 'cm') & ResNum == 2, pause, end
```

```
% if strcmp(ParamName, 'cm') & ResNum == 3, pause, end
```

```
% if strcmp(ParamName, 'cm') & ResNum == 4, pause, end
```

```
% if strcmp(ParamName, 'ht') & ResNum == 1, pause, end
```

```
% if strcmp(ParamName, 'ht') & ResNum == 2, pause, end
```

```
% if strcmp(ParamName, 'ht') & ResNum == 3, pause, end
```

```
% if strcmp(ParamName, 'ht') & ResNum == 4, pause, end
```

```
% if strcmp(ParamName, 'cw') & ResNum == 3, pause, end
```

```
% if strcmp(ParamName, 'cw') & ResNum == 4, pause, end
```

```
% if strcmp(ParamName, 'arw') & ResNum == 1, pause, end
```

```
    close all
```

```
end
```

```
for ResNum = 1:length(ii_PV)
```

```
    figure(ii_F), plot(bw_H, all_max_real(ResNum,:), deblank(LineTypes(ResNum,:))), hold on
```

```
    figure(ii_F+1), plot(bw_H, all_dCLz(ResNum,:), deblank(LineTypes(ResNum,:))), hold on
```

```
    figure(ii_F+2), plot(bw_H, all_dCDz(ResNum,:), deblank(LineTypes(ResNum,:))), hold on
```

```
    figure(ii_F+3), plot(bw_H, all_dhz(ResNum,:), deblank(LineTypes(ResNum,:))), hold on
```

```

end
X_Label = {X_Label, ['K_{CMz} = ', num2str(K_CMz), '    N_c = ', num2str(Nc), ...
    '    N_s = ', num2str(Ns)]};
figure(ii_F), title(MainCap_CM1), xlabel(X_Label), ylabel(Y_Label1)
grid on, eval(LegStr)
figure(ii_F+1), title(MainCap_CM2), xlabel(X_Label), ylabel(Y_Label2)
grid on, eval(LegStr)
figure(ii_F+2), title(MainCap_CM3), xlabel(X_Label), ylabel(Y_Label3)
grid on, eval(LegStr)
figure(ii_F+3), title(MainCap_CM4), xlabel(X_Label), ylabel(Y_Label4)
grid on, eval(LegStr)

figure(6)
S = 10; n_H = size(All_E,2);
Symbols = ['.^v<']; ms_ = [10, 6, 6, 6];

for ResNum = 1:length(ii_PV)
    M1 = ['r',Symbols(ResNum)];
    ms = ms_(ResNum);
    ii1 = 1+5*(ResNum-1); ii2 = ii1 + 4;
    E = All_E(ii1:ii2,:);
    plot(real(E(1,2:)), imag(E(1,2:)), M1, 'MarkerSize', ms), hold on
end

for ResNum = 1:length(ii_PV)
    M1 = ['r',Symbols(ResNum)]; M2 = ['b',Symbols(ResNum)]; M3 = ['k',Symbols(ResNum)];
    ms = ms_(ResNum);
    ii1 = 1+5*(ResNum-1); ii2 = ii1 + 4;
    E = All_E(ii1:ii2,:);
    plot(real(E(1,2:n_H)), imag(E(1,2:n_H)), M1, 'MarkerSize', ms), hold on
    plot(real(E(1,1)), imag(E(1,1)), 'rx', 'MarkerSize', S)
    plot(real(E(2,1)), imag(E(2,1)), 'bx', 'MarkerSize', S)
    plot(real(E(3,1)), imag(E(3,1)), 'kx', 'MarkerSize', S)
    plot(real(E(2,2:n_H)), imag(E(2,2:n_H)), M2, 'MarkerSize', ms)
    plot(real(E(3,2:n_H)), imag(E(3,2:n_H)), M3, 'MarkerSize', ms)
    title(MainCap_CM1)
    xlabel({'Exponential Growth Rate (per air second)', [' '], ...
        ['K_{CMz} = ', num2str(K_CMz), '    N_c = ', num2str(Nc), ...
        '    N_s = ', num2str(Ns)]});
    ylabel('Damped Frequency (rad per air second)')
end
axis('equal')
grid on, eval(LegStr)

%return

% Produce plots of root loci for CM position #2 and for various K_CMz
if ~strcmp(ParamName, 'cm'); return; end
%if ~strcmp(ParamName, 'cw'); return; end

All_E = [];
ParamVals = [1 0.8 0.6 0.4];
ParamVals = [1 0.75 0.5 0.4];
ParamStr1 = 'K_{CMz} = '; ParamStr2 = ' ';
LegStr = MakeLegStr(ParamVals, [1 2 3 4], ParamStr1, ParamStr2);

```

```

for K_CMz = ParamVals
    [bw_H, max_real, CL_z_, CL_z_crit_, CD_z_, CD_z_crit_, h_z_, h_z_crit_, E] = ...
        StabContrAnalysis(Examp, ParamName, ii_PV(2), Nc, Ns, K_CMz, ii_H, t_fin_);
%    StabContrAnalysis(Examp, ParamName, ii_PV(2), Nc, Ns, K_CMz, ii_H, t_fin_);
    All_E = [All_E; E];
end
close all

figure(7)
for ResNum = 1:length(ii_PV)
    M1 = ['r', Symbols(ResNum)];
    ms = ms_(ResNum);
    ii1 = 1+5*(ResNum-1); ii2 = ii1 + 4;
    E = All_E(ii1:ii2,:);
    plot(real(E(1,2)), imag(E(1,2)), M1, 'MarkerSize', ms), hold on
end
for ResNum = 1:length(ii_PV)
    M1 = ['r', Symbols(ResNum)]; M2 = ['b', Symbols(ResNum)]; M3 = ['k', Symbols(ResNum)];
    ms = ms_(ResNum);
    ii1 = 1+5*(ResNum-1); ii2 = ii1 + 4;
    E = All_E(ii1:ii2,:);
    plot(real(E(1,2:n_H)), imag(E(1,2:n_H)), M1, 'MarkerSize', ms), hold on
    plot(real(E(1,1)), imag(E(1,1)), 'rx', 'MarkerSize', S)
    plot(real(E(2,1)), imag(E(2,1)), 'bx', 'MarkerSize', S)
    plot(real(E(3,1)), imag(E(3,1)), 'kx', 'MarkerSize', S)
    plot(real(E(2,2:n_H)), imag(E(2,2:n_H)), M2, 'MarkerSize', ms)
    plot(real(E(3,2:n_H)), imag(E(3,2:n_H)), M3, 'MarkerSize', ms)
end
title({'Influence of Feedback Control Ratio  $K_{CMz}$  on Longitudinal Stability', ...
    '(Open Loop System has  $K_{CMz} = 1$ )'})
xlabel({'Exponential Growth Rate (per air second)', [' '], ...
    ['     $N_c =$ ', num2str(Nc), '     $N_s =$ ', num2str(Ns)]});
ylabel('Damped Frequency (rad per air second)')
axis('equal')
grid on, eval(LegStr)

return

```

F3.9 To Summarise Post-processed Stability and Control Results, Call **SummaryResults.m**

```
% SummaryResults
%
% Program that produces tables and a graph that summarise results of sensitivity analysis
% (called by StabContrPostProc)

% Estimate critical heights by interpolation at sign changes of stability criteria
ParamSelect = ParamVals(ii_PV);
bw_Hcrit = ones(4, size(all_max_real, 1))*NaN;
for ii = 1:size(all_max_real, 1)
    for jj = 2:length(bw_H)
        if sign(all_max_real(ii, jj)) ~= sign(all_max_real(ii, jj-1)) & ...
            all_max_real(ii, jj)*sign(all_max_real(ii, jj-1)) ~= 0
            bw_Hcrit(1, ii) = bw_H(1, jj-1) - (bw_H(1, jj) ...
                - bw_H(1, jj-1))*all_max_real(ii, jj-1)/ ...
                (all_max_real(ii, jj) - all_max_real(ii, jj-1));
        end
        if sign(all_dCLz(ii, jj)) ~= sign(all_dCLz(ii, jj-1)) & ...
            all_dCLz(ii, jj)*sign(all_dCLz(ii, jj-1)) ~= 0
            bw_Hcrit(2, ii) = bw_H(1, jj-1) - (bw_H(1, jj) ...
                - bw_H(1, jj-1))*all_dCLz(ii, jj-1)/ ...
                (all_dCLz(ii, jj) - all_dCLz(ii, jj-1));
        end
        if sign(all_dCDz(ii, jj)) ~= sign(all_dCDz(ii, jj-1)) & ...
            all_dCDz(ii, jj)*sign(all_dCDz(ii, jj-1)) ~= 0
            bw_Hcrit(3, ii) = bw_H(1, jj-1) - (bw_H(1, jj) ...
                - bw_H(1, jj-1))*all_dCDz(ii, jj-1)/ ...
                (all_dCDz(ii, jj) - all_dCDz(ii, jj-1));
        end
        if sign(all_dhz(ii, jj)) ~= sign(all_dhz(ii, jj-1)) & ...
            all_dhz(ii, jj)*sign(all_dhz(ii, jj-1)) ~= 0
            bw_Hcrit(4, ii) = bw_H(1, jj-1) - (bw_H(1, jj) ...
                - bw_H(1, jj-1))*all_dhz(ii, jj-1)/ ...
                (all_dhz(ii, jj) - all_dhz(ii, jj-1));
        end
    end
end
end

% Define captions of figures, figure axes and tables according to specified parameter
FigCap1 = ['Critical Height Parameter vs ', LongParName, ...
    ' as Predicted by Various Stability Criteria'];
FigCap2 = '(Estimated by Linear Interpolation at Sign Change of Stability Criterium)';
TabCap_r = [' Maximum real part (x1000) of roots for varying ', LongParName];
TabCap_L = [' Values of dCLz (x1000) for varying ', LongParName];
TabCap_D = [' Values of dCDz (x1000) for varying ', LongParName];
TabCap_hN = [' Values of dhNz for varying ', LongParName];
TabCap1 = ' bw/H';
TabFoot1 = [' ', LongParName, ' :'];
TabCap2 = str2mat(' LINEARLY INTERPOLATED CRITICAL HEIGHTS FROM PRECEDING TABLES', ...
    ' Critical Height Ratios Predicted by Various Criteria', ...
    ' (b/H)crit', ...
    [ShortParName, ' rlocus CLzcrit CDzcrit hNzcrit']);
TabFoot2 = str2mat('Criteria by which columns 1 to 4 were obtained are:', ...
    ' Criterion 1: root locus crosses over to right half of s-plane', ...
    ' Criterion 2: CLz exceeds CLz_crit', ...
```

```

' Criterium 3: CDz exceeds CDz_crit', ...
' Criterium 4: hNz exceeds hNz_crit', ' ', ...
' The NaN at b/H = 0 signifies neutral height stability', ...
' If ALL elements = NaN, stability over whole parameter range', ...
' is implied, PROVIDED that ALL maximum real parts of roots are <= 0');

% Write summary of results of sensitivity analysis to file
OldDir = cd;
feval('cd', OutDir)
if exist(OutFile3) == 2; delete(OutFile3); end
diary(OutFile3), diary on
feval('cd', OldDir)
disp(' RESULTS OF SENSITIVITY ANALYSIS')
disp(' ')
disp(' ')
disp(TabCap_r)
disp(TabCap1)
disp([bw_H', 1e3*all_max_real'])
disp(TabFoot1)
disp(ParamSelect)

disp(TabCap_L)
disp(TabCap1)
disp([bw_H', 1e3*all_dCLz'])

disp(TabCap_D)
disp(TabCap1)
disp([bw_H', 1e3*all_dCDz'])

disp(TabCap_hN)
disp(TabCap1)
disp([bw_H', all_dhz'])

disp(TabCap2)
disp([ParamSelect', bw_Hcrit'])
disp(TabFoot2)

disp(' '), disp(' ')
disp([' ', date])
diary off

% Produce plots that illustrate influence of selected parameter on critical height,
% Intended also to provide indication of the cause of the instability!
figure(nnF), plot(ParamSelect, bw_Hcrit(1,:), 'x'), hold on
if ~isempty(AxLims); axis(AxLims); end
plot(ParamSelect, bw_Hcrit(2,:), 'o')
plot(ParamSelect, bw_Hcrit(3,:), 's')
plot(ParamSelect, bw_Hcrit(4,:), 'd')
title({FigCap1, FigCap2})
X_Label = {X_Label, ['K_{CMz} = ', num2str(K_CMz), '    N_c = ', num2str(Nc), ...
    '    N_s = ', num2str(Ns)]};
xlabel(X_Label)
ylabel('(b_w/H)_{crit}')
grid on
legend('root locus crosses imaginary axis', '\Delta C_{Lz} changes sign', ...
    '\Delta C_{Dz} changes sign', '\Delta h_{Nz} changes sign')

```