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Master Thesis

The KLT Relations In Unimodular Gravity

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I, Daniel Johannes Burger, declare that this thesis titled,

The KLT Relations In Unimodular Gravity and the work presented in it are my own. I confirm that:

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Abstract

Here we initiate a systematic study of some of the symmetry properties of unimodular gravity, building on much of the known structure of general relativity, and utilizing the powerful technology developed in that context, such as the spinor helicity formalism. In particular, we show, up to five-points and tree-level, that the KLT relations of perturbative gravity hold for trace free or unimodular gravity. This work is in conjunction with a paper written with A. Welman, J. Murugan and G.F.R. Ellis (ARXIV:1511.08517)

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Contents

Chapter 1

Introduction

Of all the remarkable developments in physics and mathematics during the last hundred years, Einstein's Theory of General Relativity (GR) and Quantum Theory have altered the way we understand physical phenomena the most. In particle physics, quantum field theory has become the norm for studying a vast array of problems, from Quantum Electrodynamics to non-Abelian gauge theories such as Quantum Chromodynamics. Apart from particle physics this has been extended to general systems such as condensed matter physics and inflationary cosmology. In this framework the effect of forces is described by the exchange of virtual particles. Successful in describing the electro-weak as well as strong forces, the next question is one of how one formulates gravity in a field theoretical framework.

Classically GR has been particularly successful in modelling gravitational phenomena, for example planetary orbits, black holes and so forth. The insight offered here by GR has come mostly from studying the classical solutions of the Einstein field equations.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \tag{1.1}$$

But at the largest scales when considering cosmologies describing an accelerating universe as a result of an effective cosmological constant, the theory runs into a contradiction with quantum field theory. When taking the cosmological constant as arising from the vacuum energy, simple estimates in quantum field theory predict a value between 60 and 120 orders of magnitude greater than that of the valued fixed by astronomical observation. This large discrepancy is a serious conflict in quantum field theory and general relativity. Among the suggested ways to solve this problem is that of modified theories of gravity. There is considerable variation in these modifications of GR, and the one we are interested in is Unimodular or Trace free Gravity (UG) that goes all the way back to Einstein in 1919. Since then it was mostly considered ignored in favour of GR until recently brought to light in [1], in the context of the cosmological constant problem.

While it does not completely resolve the issue of the cosmological constant, UG does relegate it to an integration constant to be fixed by empirical data. It does so by decoupling fluctuations of the quantum vacuum from gravitational physics rendering an entirely viable classical theory of gravity [2].

The trace free field equations,

$$R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = \kappa^2(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T), \tag{1.2}$$

can be derived in a number of ways, the simplest of which involve a rescaling of the metric that fixes the determinant of the metric to one. We start of with the action for unimodular gravity, with a matter term,

$$S_{UG} = \frac{1}{2\kappa^2}\int_M d^4x\,(\hat{R}[\hat{g}_{\mu\nu}] + \mathcal{L}_m), \tag{1.3}$$

where $\hat{R}[\hat{g}_{\mu\nu}]$ is the Ricci scalar as a functional of the rescaled metric $\hat{g}_{\mu\nu} \to g^{-\frac{1}{4}} g_{\mu\nu}$. It is important to note that the Christoffel symbols transform under this rescaling as

$$\hat{\Gamma}\left[\hat{g}\right]^{\alpha}_{\mu\nu} = \Gamma\left[g\right]^{\alpha}_{\mu\nu} + P\left[g\right]^{\alpha}_{\mu\nu} = \Gamma\left[g\right]^{\alpha}_{\mu\nu} - \frac{1}{4}\left(\Gamma\left[g\right]^{\beta}_{\mu\beta}\,\delta^{\alpha}_{\nu} + \Gamma\left[g\right]^{\beta}_{\nu\beta}\,\delta^{\alpha}_{\mu} - \Gamma\left[g\right]^{\beta}_{\gamma\beta}\,g^{\alpha\gamma}g_{\mu\nu}\right)$$

Here we define $\hat{\Gamma}\left[\hat{g}\right]^{\alpha}_{\mu\nu}$ as the the Christoffel symbol as relating to the rescaled metric $\hat{g}_{\mu\nu}$, and the tensor $P\left[g\right]^{\alpha}_{\mu\nu}$ as the difference between the Christoffel symbols of the rescaled and standard metrics. From this we can deduce that the Riemann tensor will transform as

$$\hat{R}^{\alpha}_{\mu\rho\nu} = R^{\alpha}_{\mu\rho\nu} + f[g]^{\alpha}_{\mu\rho\nu},$$

where, the explicit form of the function $f[g]^{\alpha}_{\mu\rho\nu}$ is given in chapter 3. We can also determine that $\det(\hat{g}) = 1$. Now applying the rescaling and varying with respect to the inverse rescaled metric $g^{\mu\nu}$, and in much the same way we would in the Einstein Hilbert action, we have

$$\delta S_{UG} = \frac{1}{2\kappa^2}\int_M (\delta g^{\frac{1}{4}}R + g^{\frac{1}{4}}\delta g^{\mu\nu}R_{\mu\nu} + g^{\frac{1}{4}}g^{\mu\nu}\delta R_{\mu\nu} + \delta(g^{\frac{1}{4}}g^{\mu\nu}\delta^{\alpha}_{\rho}f[\Gamma[g]]^{\alpha}_{\mu\rho\nu}) + \delta\mathcal{L}_m)$$

We can show that the third and fourth terms in the integrand offers no contribution to the integral by rewriting them as total derivatives, see [3] for an example of reducing the fourth term. Discarding the non-contributing terms we have,

$$\delta S_{UG} \;=\; \frac{1}{2\kappa^2}\int_M (-\frac{1}{4}g^{\frac{1}{4}}g_{\mu\nu}\delta g^{\mu\nu}R + g^{\frac{1}{4}}\delta g^{\mu\nu}R_{\mu\nu} + \delta\mathcal{L}_m).$$

We can then write the Euler-Lagrange equations:

$$\frac{1}{g^{\frac{1}{4}}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} = \frac{1}{2\kappa^2}(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) + \frac{1}{g^{\frac{1}{4}}}\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}}.$$

Defining the variation of the matter Lagrangian $\delta\mathcal{L}_m$ in terms of the energy momentum tensor as $\delta\mathcal{L}_m = \frac{1}{2}g^{\frac{1}{4}}\hat{T}_{\mu\nu}\delta g^{\mu\nu}$, where $\hat{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T$ is the traceless part of the energy momentum tensor. We then have the trace free field equations given by

$$R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = \kappa^2(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T).$$

Note that unlike Einstein's 1919 theory we here require the right hand side of the equations to be trace free as well. When taking the second contracted Bianchi identity, $\bigtriangledown^{\mu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 0$, and imposing conservation of energy $\bigtriangledown^{\mu}T_{\mu\nu} = 0$ we recover the Einstein field equations with a cosmological constant in the following way:

$$\begin{aligned}
\bigtriangledown^{\mu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{1}{4}g_{\mu\nu}R) &= \kappa^2 \bigtriangledown^{\mu}(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T) \\
\bigtriangledown^{\mu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + \frac{1}{4}g_{\mu\nu}\bigtriangledown^{\mu}R &= \kappa^2 \bigtriangledown^{\mu}T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}8\kappa^2 \bigtriangledown^{\mu}T \\
\frac{1}{4}g_{\mu\nu}\bigtriangledown^{\mu}(R + \kappa^2 T) &= 0.
\end{aligned}$$

Now we set $R + \kappa^2 T = 4\Lambda$, and substituting back into the field equations we have

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \kappa^2 T_{\mu\nu}.$$

We have now recovered Einstein's field equations with a cosmological constant Λ. Although at first glance this doesn't seem all that great since we can consider this as a classical gauge fixing, i.e. the physical results should not differ. The significance, however, is that we now have a cosmological constant that arises as an integration constant as opposed to a fundamental quantity in the action, as mentioned above.

This offers a different perspective from which to view the cosmological constant. On a more practical note, this means that the value for Λ can be fixed by observation. Although this does not shed light on how the observed value of the cosmological constant arises, it does give us a means of bypassing the large discrepancy between the measured and observed values of the cosmological constant. With the equivalence to the classical field equations of general relativity this makes the theory very attractive for use in cosmological problems since the classical dynamics of unimodular gravity are equivalent to general relativity. This also provides a problem in its own right; that at the classical level UG is expected to be completely indistinguishable from GR [4, 5] (see also the extended discussion in [6]) even though the former only preserves a Weyl transverse subgroup, *WTDiff(M)* of the full *Diff(M)* symmetry group of GR. This difference will, however, manifest at the quantum level [4]. With an ultimate goal of exploring the quantum differences between unimodular gravity and GR in mind, it is certainly important to understand the extent of their similarities.

This leads us naturally to the purpose of this treatise: to better analyse the similarities and differences that GR and UG have at the quantum level. As stated previously the vast majority of our understanding of the universe stemming from GR is as a result of solutions to the Einstein field equations. Here we consider the development independent of the classical solutions and return to the question of how we can express the dynamics of GR in terms of a quantum field theoretic framework. One of the most significant developments in this sense is that GR as an effective field theory encodes the low energy dynamics of a massless spin -2 particle, the graviton. In this framework GR arises from a hard-graviton, $h_{\mu\nu}$, propagating on a flat background, $\eta_{\mu\nu}$. From the semi-classical perspective the graviton can be seen as a small perturbation of spacetime. At the linear level the dynamics of this spin-2 two particle is given by the Pauli-Fierz Lagrangian. In the light of self-consistency of the graviton coupling to matter, it is required to couple to its own energy momentum tensor. This then sources the next order in the graviton interaction Lagrangian. This process when repeated facilitates the bootstrap to the full non-linear Einstein-Hilbert action [7].

A more recent development is the realization that GR can be found in the low energy limit of the critical superstring. In this context, the structure of the Einstein-Hilbert action is determined from the leading order contribution to the vanishing of the beta functions that guarantee the one-loop cancellation of the conformal anomaly of the worldsheet sigma model [9]. This remarkable discovery that the closed string spectrum contains a massless, spin-2 field with all the characteristics of the graviton, soon precipitated the observation that the closed string may be decomposed into a product of two open strings.

$$|\text{closed string state}\rangle = |\text{open string state}\rangle \otimes |\text{open string state}\rangle$$

This result has incited a silent revolution in our understanding of the nature of gravity that can be summarised schematically as

$$\text{gravity} \sim (\text{gauge theory})^2,$$

and whose precise statement is embodied in the KLT (Kawai, Lewellen and Tye) relations that connect gauge theory and gravity at the level of scattering amplitudes (see, for example, [11] for an outstanding review of the state of the art).

KLT Relations

The KLT relations [10] offer a simple yet powerful way to study the scattering amplitudes in a perturbative gravity regime. They were first derived in the context of string theory where it was noticed that one could "sew" two open string states together to form a closed string state. More formally that the vertex operator for the emission of a closed string state can be decomposed into the product of two open string vertex operators, as can be found in [12].

$$V^{closed} = V^{open}_{left} \otimes \bar{V}^{open}_{right}.$$

This decomposition holds at the integrand level before world sheet integration, which can be clearly seen in the Koba-Nielsen form of string amplitudes [13]. But Kawai, Lewellen and Tye were able to demonstrate that the decomposition holds even after world sheet integration and the factorisation structure is reflected in the complete string amplitudes. A simple diagrammatic example to demonstrate this property of the string amplitudes is the factorization of the four-point partial amplitude of the massless heterotic closed superstring into a product of the partial four point amplitude of open superstring theory propagating massless modes.

$$A_4^{open} = -\frac{1}{2} g^2 \frac{\Gamma(\alpha' s)\Gamma(\alpha' t)}{\Gamma(1+\alpha' s+\alpha' t)} \xi^A \xi^B \xi^C \xi^D K_{ABCD}(k_1, k_2, k_3, k_4), \tag{1.4}$$

where α' is the inverse tension of the open string and K is the gauge invariant kinematic coefficient dependant on the momenta k_i. Depending on the particle type the ξ^A can be vector polarisations, spinors or group theory matrices. The four-point partial amplitude of the massless heterotic closed superstring is given by:

$$M_4^{closed} = \kappa^2 \sin \frac{\alpha' \pi t}{4} \frac{\Gamma(\alpha' s/4)\Gamma(\alpha' t/4)}{\Gamma(1+\alpha' s/4+\alpha' t/4)} \frac{\Gamma(\alpha' t/4)\Gamma(\alpha' u/4)}{\Gamma(1+\alpha' t/4+\alpha' u/4)} \xi^{AA'} \xi^{BB'} \xi^{CC'} \xi^{DD'} \times K_{ABCD}(k_1/2, k_2/2, k_3/2, k_4/2) K_{A'B'C'D'}(k_1/2, k_2/2, k_3/2, k_4/2) \tag{1.5}$$

where α' is the closed string inverse tension, or twice the open string Regge slope. Up to some coefficients the replacements $\xi^A \xi^{A'} \to \xi^{AA'}$ and substituting $k_i \to k_i/2$, the closed string amplitude (1.5) is a product of open string partial amplitudes (1.4). Hence the amplitude of any closed string tree amplitude can be written as a product of two open string states. What we are interested in, is that in the low energy limit of the strings, this corresponds to the factorisation of a gravity theory state into a product of two gauge theory states.

Consider this schematic factorisation in four dimensions, we have that each of the two physical degrees of freedom of the graviton are given by the product of two vector boson states of the same helicity. One can then also factorise other spin states, for example a spin $\frac{3}{2}$ gravitino state is factorised into a product of a spin 1 vector and a spin $\frac{1}{2}$ fermion. This does not seem to be that significant since in the free theory it is simply the decomposition of higher spin states into direct products of lower spin states, but these KLT relations hold for the interacting theory as well. Since these relations hold in the broad framework of string theory as well as the numerous compactifications thereof this implies that it should also hold in the low energy limit in which one recovers gravity field theories. The explicit relations for three, four, five and six point amplitudes are given by, [14]

$$\begin{aligned} M_3(123) &= A_3[123]^2, \\ M_4(1234) &= -s_{12} A_4[1234] A_4[1243], \\ M_5(12345) &= s_{23} s_{45} A_5[12345] A_5[13254] + (3 \leftrightarrow 4), \\ M_6(123456) &= -s_{12} s_{45} A_6[123456] \left(s_{35} A_6[153462] + (s_{34}+s_{35}) A_6[154362]\right) + \mathcal{P}(2,3,4), \end{aligned} \tag{1.6}$$

where $M_i(1...i)$ and $A_i(1...i)$ are the gravity and color stripped gauge theory amplitudes respectively, and $\mathcal{P}$ denotes a permutation of the legs in the scattering diagram. Since their discovery, the KLT relations have revealed similar connections between 4-dimensional GR and Yang-Mills theory; 4-dimensional axion-dilaton gravity and Yang-Mills theory and even $\mathcal{N} = 8$ supergravity and $\mathcal{N} = 4$ super Yang-Mills theory [14], where it has proven particularly useful in probing the UV finiteness of the supergravity theory.

This opens up a very powerful avenue to calculate gravity scattering amplitudes since the relations (1.6) allow us to write the gravity amplitudes in terms of the much more easily calculated gauge theory amplitudes. When using standard Feynman techniques to see the difference in level of complexity one need only look up DeWitt's review on quantum gravity from the 60's [15] where he calculates the three point amplitude for perturbative GR, and compare to the gauge theory calculations which can be found in nearly any quantum field theory text, for example [16].

Suffice it to say, the KLT relations and their generalisations have led to a completely novel way of looking at gravity at both the quantum and classical levels that call into question our understanding of such foundational ideas such as locality, causality and perhaps even spacetime itself [18]. The goal here, however, is far less lofty; we look only to answer the question:

Do the KLT relations still hold for modifications of GR?

Due to the large number of "modified gravity" theories: $f(R), f(G), f(T)$, massive gravity, new massive gravity, Lovelock gravity, pure Lovelock gravity and braneworld gravity, to name but a few [17]. We will focus on the particular modification of UG and refine our question:

To what extent do the KLT relations hold in unimodular gravity?

Before continuing, it is worth noting that there are two important developments needed in the calculation of the required scattering amplitudes; the *spinor helicity formalism* and *twistor calculus* that provide essential mathematical tools to make the computation of scattering amplitudes on both sides of the KLT map tractable [19]. Tiwstor theory as we use it here was first proposed by Penrose in the 60's; it was meant to be a path to a consistent theory of quantum gravity but it was only recently that the spinor helicity formalism was developed in the framework of twistor theory as an application to the calculation of scattering amplitudes. This development has allowed for the computation of large numbers of scattering amplitudes by effectively breaking down the amplitudes to a set of recursion relations, most notably the BCFW recursion relations [20]. These relations allow for the calculation of all tree level amplitudes in pure gauge theory by constructing the amplitudes from three point amplitudes and propagators.

With our goal in mind we can now delve into the construction of the theory. We start with a review of the spinor helicity formalism in chapter 2 which will form the basis in which all the amplitude calculations are done. In chapter 3 we derive the perturbative expansions of both GR and UG. We then calculate the propagators of GR and UG and review the method for extracting the Feynman rules necessary for the amplitude calculations from the perturbative Lagrangians. Finally, we calculate the scattering amplitudes up to five-point and tree level in both GR and UG.

Chapter 2

Twistors

As stated in the introduction, the mathematical machinery used to compute the scattering amplitudes considered in this theory is the spinor helicity formalism which is itself a manifestation of twistor calculus. Here we review the spinor helicity formalism to establish the conventions and notation. This is a working formalism and as such we do not go into the formal group theoretical structure of twistor theory but rather set up the formalism using an applicable framework.

2.1 Formalism

Our review of the spinor helicity formalism in this section will follow quite closely the excellent treatment given in [14]. Throughout the paper we use a mostly plus flat metric $\eta_{\mu\nu} = diag(-1,+1,+1,+1)$. Lower case Greek letters designate spacetime indices and run $\mu = \{0,1,2,3\}$ and $a, b, \dot{a}, \dot{b} = \{1,2\}$ are 2-spinor indices. We use μ_i and ν_i to respectively label the left- and right-handed spacetime indices of external states of the various amplitudes, where i runs over the number of particles in the interaction. To construct the spinor helicity formalism consider first the Dirac Lagrangian

$$\mathcal{L}_D = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi, \tag{2.1.1}$$

where Ψ is a four-spinor and $\bar{\Psi}$ is the Dirac conjugate of Ψ defined by

$$\bar{\Psi} = \Psi^\dagger\beta, \quad \beta = \begin{pmatrix} 0 & \delta^{\dot{a}}_{\dot{b}} \\ \delta^b_a & 0 \end{pmatrix}. \tag{2.1.2}$$

The Euler-Lagrange equations are, of course, the Dirac equations for Ψ and $\bar{\Psi}$,

$$\begin{aligned} (-i\gamma^\mu\partial_\mu + m)\,\Psi &= 0 \\ (i\gamma^\mu\partial_\mu + m)\,\bar{\Psi} &= 0. \end{aligned} \tag{2.1.3}$$

These equations admit plane wave solutions which, for Ψ take the form

$$\Psi \approx u(p)e^{ipx} + v(p)e^{-ipx}, \tag{2.1.4}$$

with $p^2 = p_\mu p^\mu = -m^2$ (and similarly for $\bar{\Psi}$). In momentum space the Dirac equations (2.1.3) reduce to

$$\begin{aligned} (\gamma^\mu p_\mu + m)\,u(p) &= 0 \\ (-\gamma^\mu p_\mu + m)\,v(p) &= 0. \end{aligned} \tag{2.1.5}$$

Then (2.1.5) has two independent solutions, one for each value of $s = \pm$ where, for massless fermions '$\pm$' denotes the particle helicity, i.e.

$$\Psi(x) = \sum_{s=\pm}\int \frac{d^3p}{(2\pi)^3 2E_p}\Big(b_s(p)u_s(p)e^{ipx} + d^\dagger_s(p)v_s(p)e^{-ipx}\Big), \tag{2.1.6}$$

with a similar expression for $\bar{\Psi}$ involving instead $d_s(p)$ and $b_s^\dagger(p)$. The $d_s^{(\dagger)}$ and $b_d^{(\dagger)}$ are as usual fermionic annihilation and creation operators and $u_s(p)$ and $v_s(p)$ are four component commuting spinors that encode the Grassmann nature of the particles. The vacuum of the theory is defined such that

$$\begin{aligned} b_\pm(p)|0\rangle &= 0, \\ b_\pm^\dagger(p)|0\rangle &= |p,\pm\rangle. \end{aligned}$$

For consistency we take all the particles to be outgoing, so that $v_\pm(p)$ represents an anti-fermion and $\bar{u}_\pm(p)$ a fermion, obtained from the expansion of $\bar{\Psi}$.

In the high-energy limit, in which the rest mass energy of the fermions is negligible relative to their kinetic energy, we can consider the particles as massless. The corresponding massless equations of motion

$$\begin{aligned} \gamma^\mu p_\mu v_\pm(p) &= 0, \\ \bar{u}_\pm(p)\gamma^\mu p_\mu &= 0, \end{aligned} \tag{2.1.7}$$

each have two solutions which can be written in terms of 2-component spinors as

$$\begin{aligned} v_+(p) &= \begin{pmatrix} |p]_a \\ 0 \end{pmatrix} \\ v_-(p) &= \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix} \\ \bar{u}_+(p) &= \begin{pmatrix} [p|^a, 0 \end{pmatrix} \\ \bar{u}_-(p) &= \begin{pmatrix} 0, \langle p|_{\dot{a}} \end{pmatrix}. \end{aligned} \tag{2.1.8}$$

These angle- and square-brakets, central to the spinor-helicity notation are nothing but the commuting 2-component spinors. The kets here are outgoing anti-fermions and the bras outgoing fermions. Starting with a momentum 4-vector $p^\mu = \left(p^0, p^i\right) = \left(E, p^i\right)$ with $p^\mu p_\mu = -m^2$ and the Dirac gamma-matrices defined as usual,

$$\gamma^\mu p_\mu = \begin{pmatrix} 0 & p_{a\dot{b}} \\ p^{\dot{a}b} & 0 \end{pmatrix}, \tag{2.1.9}$$

the momentum bi-spinors are defined as

$$p_{a\dot{b}} = p_\mu \left(\sigma^\mu\right)_{a\dot{b}}, \quad p^{\dot{a}b} = p_\mu \left(\bar{\sigma}^\mu\right)^{\dot{a}b}. \tag{2.1.10}$$

The two component spinors that we defined in (2.1.8) then solve the massless Weyl equations

$$p^{\dot{a}b}\, |p]_b = 0, \quad [p|^b\, p_{b\dot{a}} = 0, \quad p_{a\dot{b}}\, |p\rangle^{\dot{b}} = 0, \quad \langle p|_{\dot{b}}\, p^{\dot{b}a} = 0. \tag{2.1.11}$$

2.2 Spinor-helicity properties and identities

Here we take some time to set out some of the properties and conventions of the spinors defined in (2.1.8). Firstly, we note that the 2-spinor indices are raised and lowered using the antisymmetric Levi-Civita tensor,

$$\epsilon_{ab}[p|^b = |p]_a, \quad \epsilon_{\dot{a}\dot{b}}|p\rangle^{\dot{b}} = \langle p|_{\dot{a}}, \quad \epsilon^{ab}|p]_b = [p|^a, \quad \epsilon^{\dot{a}\dot{b}}|p\rangle_{\dot{b}} = \langle p|^{\dot{a}}. \tag{2.2.12}$$

Next we consider the reality conditions of the square- and angle-spinors. The spinor field $\bar{\Psi}$ is the Dirac conjugate of Ψ. Applying this conjugation to the momentum space Dirac equations (2.1.5) necessitates that $\bar{v}_\pm = \bar{u}_\mp$ and $u_\pm = v_\mp$, if the momentum p^μ is to be real valued. This usually goes by the name of *crossing symmetry.* For real momenta then, we have that

$$[p|^a = \left(|p\rangle^{\dot{a}}\right)^*, \quad \langle p|_{\dot{a}} = \left(|p]_a\right)^*. \tag{2.2.13}$$

The Dirac 4-spinors satisfy a spin completeness relation that for $m = 0$ reads

$$\sum_{s=\pm} u_s(p)\bar{u}_s(p) = \sum_{s=\pm} v_s(p)\bar{v}_s(p) = -\gamma^\mu p_\mu. \tag{2.2.14}$$

Using the crossing symmetry this can be rewritten in spinor-helicity notation as

$$|p\rangle[p| + |p]\langle p| \quad = \quad -\gamma^\mu p_\mu, \tag{2.2.15}$$

or, in terms of the momentum bi-spinors,

$$\begin{aligned} p_{a\dot{b}} &= -|p]_a\langle p|_{\dot{b}}, \\ p^{\dot{a}b} &= -|p\rangle^{\dot{a}}[p|^b. \end{aligned} \tag{2.2.16}$$

We now introduce the notation that is the basis for writing amplitudes in the spinor helicity formalism, the angle spinor bracket $\langle pq\rangle$ and the square spinor bracket $[pq]$. For two lightlike vectors, p^μ and q^μ these are defined as

$$\langle pq\rangle = \langle p|_{\dot{a}}|q\rangle^{\dot{a}}, \quad [pq] = [p|^a|q]_a, \tag{2.2.17}$$

with all other combinations vanishing. Since the raising and lowering of the spinor indices are done with the completely antisymmetric tensor these brackets are anti-symmetric,

$$\langle pq\rangle = -\langle qp\rangle, \quad [pq] = -[qp]. \tag{2.2.18}$$

Reality of the momenta translates into spinor-helicity language as

$$([pq])^* = \langle qp\rangle. \tag{2.2.19}$$

When working with higher spin states and interactions expressions like $\bar{u}_-(p)\gamma^\mu v_+(q)$ are frequently encountered. Using (2.1.8) and the definition of the gamma-matrices as in (2.1.9), we define the angle-square bracket,

$$\langle p|\gamma^\mu|q] = \bar{u}_-(p)\gamma^\mu v_+(q) = \begin{pmatrix} 0, \langle p|_{\dot{a}} \end{pmatrix} \begin{pmatrix} 0 & p_{a\dot{b}} \\ p^{\dot{a}b} & 0 \end{pmatrix} \begin{pmatrix} |p]_a \\ 0 \end{pmatrix}, \tag{2.2.20}$$

with a similar expression defining $[p|\gamma^\mu|q\rangle$ while for *same helicity* fermions the product vanishes. These angle-square brackets satisfy

$$\begin{aligned} \langle p|\gamma^\mu|q] &= [q|\gamma^\mu|p\rangle, \\ (\langle p|\gamma^\mu|q])^* &= \langle q|\gamma^\mu|p], \\ \langle p|P|q] &= P_\mu\langle p|\gamma^\mu|q] = \langle p_{\dot{a}}|P^{\dot{a}b}|q_b] = \langle p_{\dot{a}}|\left(-|P^{\dot{a}}\rangle[P^b|\right)|q_b] = -\langle pP\rangle[Pq], \end{aligned} \tag{2.2.21}$$

where, in the last line we take P^μ to be a lightlike vector. The *Fierz identity* is given by

$$\langle p_1|\gamma^\mu|p_2]\langle p_3|\gamma_\mu|p_4] = 2\langle p_1p_3\rangle[p_2p_4]. \tag{2.2.22}$$

From this it follows quite easily that $k^\mu = \frac{1}{2}\langle k|\gamma^\mu|k]$ while two lightlike vectors p^μ and q^μ will satisfy

$$(p+q)^2 = 2p\cdot q = \langle pq\rangle[pq]. \tag{2.2.23}$$

The next important identity encodes the conservation of momentum, which in spinor notation becomes

$$\sum_{i=1}^n p_i^\mu = 0 \to \sum_{i=1}^n \langle pi\rangle[ik] = 0. \tag{2.2.24}$$

Lastly we have the so-called *Schouten identity*. This identity encompasses the rather simple fact that three 2-dimensional vectors, say $|p\rangle, |q\rangle, |k\rangle$, cannot all be linearly independent. Any one of them must be a linear combination of the other two, $|p\rangle = a|q\rangle + b|k\rangle$. We can then 'dot in' $\langle p|, \langle q|, \langle k|$ as appropriate to determine the constant coefficients a and b giving

$$|p\rangle\langle qk\rangle + |q\rangle\langle kp\rangle + |k\rangle\langle pq\rangle = 0. \tag{2.2.25}$$

A similar statement also holds for square-spinors.

2.3 Gauge theory

Now let's put the formalism to use to (eventually) compute the tree-level scattering amplitudes in pure Yang-Mills gauge theory. We start with the general Yang Mills Lagrangian,

$$\mathcal{L}_{YM} = -\frac{1}{4}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right), \tag{2.3.26}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}}\left[A_\mu, A_\nu\right]$, $A_\mu = T_a A^a_\mu$. Here the lower case Latin indices are color indices and run over $a, ... = 1, 2, ..., N^2-1$ where we have N colors with the gluon fields in the adjoint representation. The gauge group is then $G = SU(N)$. The generators T^a are normalized such that $\mathrm{Tr}T^aT^b = \delta^{ab}$ and $\left[T^a, T^b\right] = -\bar{f}^{abc}T^c$. Where $\bar{f}^{abc}$ are the color structure constants. Next we fix the gauge redundancy in the Lagrangian by choosing an amplitude friendly gauge, the *Gervais-Neveu* gauge. For which the gauge fixing term is $\mathcal{L}_{gf} = -\frac{1}{2}\mathrm{Tr}(H^\mu_\nu)^2$ with $H_{\mu\nu} = \partial_\mu A_\nu - \frac{ig}{\sqrt{2}}\left[A_\mu, A_\nu\right]$. Once gauge fixed the Lagrangian reads;

$$\mathcal{L}_{YM} = \mathrm{Tr}\left(-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - ig\sqrt{2}\partial^\mu A^\nu A_\nu A_\mu + \frac{g^2}{4}A^\mu A^\nu A_\nu A_\mu\right), \tag{2.3.27}$$

From the above Lagrangian we can extract the Feynman rules and see that the three and four vertex rules have color factors on the forms $\bar{f}^{abc}$ and $\bar{f}^{abe}\bar{f}^{ecd}+$ permutations respectively. These colored amplitudes can be sorted into separate group theory structures each dressed with kinematic factors. Take for example the four point amplitude, the s-, t-, and u-channel contributions each carry the color factors

$$c_s = \bar{f}^{a_1a_2b}\bar{f}^{ba_3a_4} \quad, c_t = \bar{f}^{a_1a_3b}\bar{f}^{ba_4a_2} \quad, c_u = \bar{f}^{a_1a_4b}\bar{f}^{ba_2a_3}. \tag{2.3.28}$$

Now using the $SU(N)$ identity $(T^a)_i^j(T^a)_l^k = \delta_i^k\delta_l^j - \frac{1}{N}\delta_i^j\delta_l^k$, we can rewrite the color factor dependence of the contributions to the amplitude in terms of single-trace group theory factors of the generators. We can therefore write the full gluon tree amplitude of n external states as

$$A_n^{full,tree} = g^{n-2}\sum_{\text{perm }\sigma} A_n[1\sigma(2...n)]\mathrm{Tr}(T^{a_1}T^{\sigma(a_2}\cdots T^{a_n)}). \tag{2.3.29}$$

Where A_n is the color-ordered amplitude. These amplitudes are calculated from the diagrams in which the external lines appear in fixed order and there are no lines crossed. These amplitudes have various properties that significantly reduce the number of independent diagrams, these properties will be discussed in detail in section 2.6. Now that we have effectively color stripped the Lagrangian we can continue with building our formalism.

To construct the spinor-helicity representation of the spin-1 particles we 'dot-in' the photon polarization vectors. These are constructed from the spinor-helicity variables as

$$\begin{aligned}\epsilon^\mu_-(p) &= -\frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[pq]}\\ \epsilon^\mu_+(p) &= -\frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle pq\rangle},\end{aligned} \tag{2.3.30}$$

with the massless Weyl equation ensuring that $p_\mu\epsilon^\mu_\pm(p) = 0$. In what follows, we have now put aside the kinematic factors of the color structure in order to better analyse the vertex structure and extract the Feynman rules for the *color-ordered* amplitudes. The 3-vertex expression for example is then given by

$$V^{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) = -\sqrt{2}\left(\eta^{\mu_1\mu_2}p_1^{\mu_3} + \eta^{\mu_2\mu_3}p_2^{\mu_1} + \eta^{\mu_3\mu_1}p_3^{\mu_2}\right), \tag{2.3.31}$$

where each η consists of two spin-1 polarisation vectors. The rules for extracting the amplitude from the vertex are as follows:

- To any stand alone momentum p_i, say, we associate a square-angle bracket $p_i^\mu \to \frac{1}{2}\langle i|\gamma^\mu|i]$.
- For contracted momenta, for example $p_1^{\mu_2} \to \epsilon_\mu(p_2)p_1^\mu$ and
- For each η factor, say, $\eta^{\mu_1\mu_2} \to \epsilon^\mu(p_1)\epsilon_\mu(p_2)$

Given this, the amplitude from the vertex expression can be written down as

$$V^{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = -\sqrt{2}\Big((\epsilon^{\mu_1}\epsilon^{\mu_2})(\epsilon^{\mu_3}p_1) + (\epsilon^{\mu_2}\epsilon^{\mu_3})(\epsilon^{\mu_1}p_2) + (\epsilon^{\mu_3}\epsilon^{\mu_1})(\epsilon^{\mu_2}p_3)\Big).$$

With the notation $p_1 \to 1$ etc, the associated *color-ordered* amplitude is then

$$A_3[1,2,3] = -\sqrt{2}\Big((\epsilon^1\epsilon^2)(\epsilon^3 p_1) + (\epsilon^2\epsilon^3)(\epsilon^1 p_2) + (\epsilon^3\epsilon^1)(\epsilon^2 p_3)\Big). \tag{2.3.32}$$

To take the calculation further we need to first assign helicities to the individual particles. For this example we will choose particles 1 and 2 to have helicity -1 and particle 3 to have +1 helicity. We can then substitute from equations (2.3.30) to get,

$$\begin{aligned} A_3\left[1^-,2^-,3^+\right] &= \frac{1}{2[q_1 1][q_2 2]\langle q_3 3\rangle}\Big(\langle 1|\gamma^\mu|q_1]\langle 2|\gamma_\mu|q_2]\langle q_3|\gamma^\nu|3]\langle 1|\gamma_\nu|1] \\ &+ \langle 2|\gamma^\mu|q_2]\langle q_3|\gamma_\mu|3]\langle 1|\gamma^\nu|q_1]\langle 2|\gamma_\nu|2] + \langle q_3|\gamma^\mu|3]\langle 1|\gamma_\mu|q_1]\langle 2|\gamma^\nu|q_2]\langle 3|\gamma_\nu|3]\Big) \\ &= \frac{1}{[q_1 1][q_2 2]\langle q_3 3\rangle}\Big(\langle 12\rangle[q_1q_2]\langle q_3 1\rangle[31] \\ &+ \langle 2q_3\rangle[q_2 3]\langle 12\rangle[q_1 2] + \langle q_3 1\rangle[3q_1]\langle 23\rangle[q_2 3]\Big). \end{aligned} \tag{2.3.33}$$

We now have to consider 3-particle special kinematics. For now, it will be sufficient to consider the expression

$$\langle 12\rangle[12] = (p_1+p_2)^2 = p_3^2 = 0. \tag{2.3.34}$$

For this to be true, either the angle spinor bracket or square spinor bracket must vanish. We must either choose $|1] \propto |2] \propto |3]$ or $|1\rangle \propto |2\rangle \propto |3\rangle$. The choice is made by considering the dimension of the expression. This is also a result of the little group scaling which will be introduced in section 2.5. In this case, we set $|1] \propto |2] \propto |3]$, killing off the first term so that

$$A_3\left[1^-,2^-,3^+\right] = \frac{-1}{[q_1 1][q_2 2]\langle q_3 3\rangle}\Big(\langle q_3 2\rangle[3q_2]\langle 12\rangle[2q_1] + \langle q_3 1\rangle[3q_1]\langle 23\rangle[3q_2]\Big) \tag{2.3.35}$$

This result is still dependant on the arbitrary reference spinors q_i. This can be eliminated by multiplying each term by the appropriate representation of 1. This allows the use of conservation of momentum to change the structure of the brackets, (2.2.24) so that, for example,

$$\langle 13\rangle[3q_2] = -\langle 12\rangle[2q_2] - \langle 11\rangle[1q_2] = -\langle 12\rangle[2q_2]. \tag{2.3.36}$$

Substituting back into the amplitude,

$$\begin{aligned} A_3\left[1^-,2^-,3^+\right] &= \frac{-1}{[q_1 1][q_2 2]\langle q_3 3\rangle}\left(\frac{\langle q_3 2\rangle\langle 12\rangle[2q_2]\langle 12\rangle\langle 31\rangle[1q_1]}{\langle 13\rangle\langle 32\rangle} + \frac{\langle q_3 1\rangle\langle 21\rangle[1q_1]\langle 23\rangle\langle 12\rangle[2q_2]}{\langle 23\rangle\langle 13\rangle}\right) \\ &= \frac{-\langle 12\rangle^2}{\langle q_3 3\rangle}\left(\frac{\langle q_3 2\rangle\langle 13\rangle - \langle q_3 1\rangle\langle 23\rangle}{\langle 13\rangle\langle 23\rangle}\right). \end{aligned} \tag{2.3.37}$$

Finally we apply the Schouten identity (2.2.25) and simplify to get

$$\begin{aligned} A_3\left[1^-,2^-,3^+\right] &= \frac{-\langle 12\rangle^2}{\langle q_3 3\rangle}\Big(\frac{\langle q_3 2\rangle\langle 13\rangle + \langle q_3 1\rangle\langle 32\rangle}{\langle 13\rangle\langle 23\rangle}\Big) \\ &= \frac{-\langle 12\rangle^2}{\langle q_3 3\rangle}\left(\frac{\langle q_3 3\rangle\langle 12\rangle}{\langle 13\rangle\langle 23\rangle}\right) \end{aligned}$$

$$= \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}. \tag{2.3.38}$$

This then is the color-ordered 3-point amplitude for non-abelian Yang-Mills gauge theory. This can now be dressed with color factors and interaction strength to obtain the full three-gluon amplitude.

2.4 Gravity

Now that we have constructed a working spinor helicity formalism for fermions (spin-$\frac{1}{2}$) and bosons (spin-1) we still need to define the spinor helicity representation for gravitons (spin-2). In graviton scattering the external legs on the diagrams each carry two Lorentz indices, this encodes the two degrees of freedom of the massless spin two particle in four dimensions. Like in the case of the spin 1 particle the way to obtain the spinor helicty representation of the particle one has to "dot-in" the polarization vector. For a massless spin 2 particle this is constructed simply as a product of two spin 1 states of the same helicity

$$\eta^{\mu\nu}_{\pm}(p_i) = \epsilon^{\mu}_{\pm}(p_i)\epsilon^{\nu}_{\pm}(p_i) \tag{2.4.39}$$

2.5 Little Group Scaling

Now that we have introduced the various spinor helicity representations of the spin-$\frac{1}{2}$, -1 and -2 particles it is of interest to introduce Little Group Scaling. With the introduction of square and angle spinors as solutions to the massless Weyl equations and their relationship to the momentum of a given particle

$$\begin{aligned} p_{a\dot{b}} &= -|p]_a\langle p|_{\dot{b}}, \\ p^{\dot{a}b} &= -|p\rangle^{\dot{a}}[p|^b, \end{aligned} \tag{2.5.40}$$

we note that these quantities are invariant under the following rescaling

$$|p\rangle \to t|p\rangle \quad , \quad |p] \to t^{-1}|p]. \tag{2.5.41}$$

This is called the Little group scaling, which is the set of transformations that leave the momenta of on-shell particles invariant. To evaluate this in a more common framework; for massless particles we can go to a frame in which $p_\mu = (E, 0, 0, E)$, in which the momentum is invariant under rotations in the xy-plane. Therefore the little group is $SO(2) = U(1)$, in the spinor representation these little group transformations are realised as the scaling (2.5.41). Considering the reality conditions of the angle and square spinors (2.2.13) we can let t be any non-zero complex number.

We now need to ask how this scaling relates to the calculation of amplitudes. To do this we need to consider the separate parts that an amplitude is made of: propagators, vertices and external lines. Clearly the propagators and internal vertices do not scale under this transformation. Now knowing that only external lines scale under the little group transformations, and that any massless amplitude can be written in terms of angle and square spinors only, we can set up the following overall scalings of the external lines.

For Weyl fermions of helicity $h_i = \pm\frac{1}{2}$ we have the general scaling determined by (2.5.41), t^{-2h_i}. For bosons, which have helicity $h_i = \pm 1$, we can find the general scaling by applying the scaling to the polarisation vector

$$\epsilon^{\mu}_{-}(p) = \frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[pq]} \to \frac{t\langle p|\gamma^\mu|q]}{\sqrt{2}t^{-1}[pq]} = t^2\frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[pq]} = t^{-2h_i}\epsilon^{\mu}_{-}(p) \tag{2.5.42}$$

Finally we consider gravitons, which carry helicity $h_i = \pm 2$, and following a similar process as the above for vector bosons we end up with the general rescaling of t^{-2h_i}. We therefore have that the external leg of any particle subject to the scaling (2.5.41) scales in the same way dependent on the helicity of the particle. We then have the powerful result that under little group scaling the amplitude as a function of helicity, angle- and square spinors scales as

$$\begin{aligned} &A_n(\{|1\rangle, |1], h_1\}, ..., \{t|i\rangle, t^{-1}|i], h_i\}, ..., \{|n\rangle, |n], h_n\}) \\ &= t^{-2h_i} A_n(\{|1\rangle, |1], h_1\}, ..., \{|i\rangle, |i], h_i\}, ..., \{|n\rangle, |n], h_n\}) \end{aligned} \tag{2.5.43}$$

The amazing thing we can take away from this is that the little group scaling along with locality uniquely determines the amplitude of a massless tree level scattering process. To demonstrate, consider for example the three gluon tree amplitude $A[1^-, 2^-, 3^+]$. From section 2.3 we know that the amplitude depends only on angle spinor brackets, this also gives the correct little group scaling. We now write the general Anzats where the amplitude depends solely on square spinor brackets.

$$A_3\left[1^-, 2^-, 3^+\right] \quad = \quad c[12]^{x_{12}}[23]^{x_{23}}[13]^{x_{13}}. \tag{2.5.44}$$

From the scaling relation established in (2.5.43) through the general rescaling of the amplitude, and rescaling all external states by $|i] \to t_i^{-1}|i]$ we find that

$$\begin{aligned} x_{12} + x_{13} = 2h_1 \quad &, x_{12} + x_{23} = 2h_2 \quad &, x_{13} + x_{23} = 2h_3 \\ x_{12} = h_1 + h_2 - h_3 \quad &, x_{13} = h_1 - h_2 + h_3 \quad &, x_{23} = -h_1 + h_2 + h_3. \end{aligned}$$

Substituting this back into (2.5.44) along with the helicities we have

$$A_3\left[1^-, 2^-, 3^+\right] \quad = \quad c\frac{[23][13]}{[12]^3}. \tag{2.5.45}$$

Remembering that the n-particle amplitude in four dimensions must have mass dimension $4-n$, but both angle and square spinors have mass dimension 1. From this we see that the coupling constant c has mass dimension 2. But if the coupling has this dimension then the interaction can only come from a term in the Lagrangian that looks like $cAA\frac{\partial}{\Box}A$ which is not found in the local Lagrangian. We can therefore conclude that the amplitude for $A[1^-, 2^-, 3^+]$ depending only on square spinor brackets is not allowed. This method is applicable to any theory that can be expressed in spinor helicity variables. We can therefore determine the three point amplitude in a gravity theory in the same manner. Firstly we choose the helicities to be $h_1 = h_2 = -h_3 = -2$ and then write down the general formula

$$M_3\left[1^-, 2^-, 3^+\right] \quad = \quad \kappa\langle 12\rangle^{x_{12}}\langle 23\rangle^{x_{23}}\langle 13\rangle^{x_{13}}. \tag{2.5.46}$$

From the invariance of the amplitude under little group scaling we can solve for x_{12}, x_{13} and x_{23} as above to produce,

$$M_3(1^{h_1}2^{h_2}3^{h_3}) = \kappa\langle 12\rangle^{h_3-h_1-h_2}\langle 13\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3}, \tag{2.5.47}$$

substituting the helicities of the particles we than have

$$M_3\left[1^-, 2^-, 3^+\right] \quad = \quad \kappa\frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 13\rangle^2}. \tag{2.5.48}$$

By quickly checking this with dimenssional analysis, this requires the coupling constant κ to be of mass dimension -1. Which, as we will see in later sections, corresponds to general relativity. This is then the only viable three point amplitude that a massless theory of gravity can produce. We will also see in later sections how this amplitude is calculated explicitly from the vertex expressions derived from the perturbative gravity Lagrangian.

2.6 Color Ordered Amplitude Properties

In this section we briefly review some of the properties of the color ordered amplitudes that will greatly simplify the calculation of all the distinct amplitudes. Firstly we need to introduce the celebrated Parke-Taylor amplitude [21],

$$A_n[1^+,2^+,...,i^-,j^-,...,n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle...\langle ij\rangle...\langle (n-1)n\rangle\langle n1\rangle}, \qquad (2.6.49)$$

this expression holds for all gauge theory amplitudes in which two adjacent particles have negative helicity and the rest carry positive helicity. The anti-Parke-Taylor amplitude holds for the change of all the helicity states; all except two states have negative helicity, this is simply the Park Taylor amplitude but with all square spinor brackets. With this amplitude it is now simple to calculate at least one form of all MHV n-tree color ordered amplitudes in gauge theory, e.g.

$$A_4[1^-2^-3^+4^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \qquad (2.6.50)$$

Now we need to consider some of the other properties that color ordered gauge theory amplitudes offer. Firstly the amplitudes are cyclic, which can be seen from the trace-structure in (2.3.29); $A_n[1,2,...,(n-1),n] = A_n[2,...,(n-1),n,1] = ...A_n[n,1,2,...,(n-1)]$. The next is the reflection property $A_n[1,2,...,n] = (-1)^n A_n[n,...,2,1]$. We then also have the $U(1)$ decoupling identity, or the photon decoupling identity, which follows from taking one of the generators of T^a to be proportional to the identity matrix, $\sum_{\sigma\in\text{Cyclic}} A_n[1,\sigma(2,3,...,n)] = 0$ or alternately $A_n[1,2,...,n]+A_n[2,1,...,n]+ ...+A_n[2,...,1,n] = 0$. Next we also know that the trace basis is over-complete which imply that there has to be further linear relations. These are called the *Kleiss-Kuijf* relations [22] and can most generally be written as

$$A_n[1,\{\alpha\},n,\{\beta\}] = (-1)^{|\beta|} \sum_{\sigma\in\text{OP}(\{\alpha\},\{\beta^{\text{T}}\})} A_n[1,\sigma,n]. \qquad (2.6.51)$$

In this expression $\{\beta^{\text{T}}\}$ denotes the reverse order of the set $\{\beta\}$. The sum is over all the ordered permutations of the joined set $\{\alpha\}\cup\{\beta^{\text{T}}\}$ such that the ordering in the individual sets $\{\alpha\}$ and $\{\beta^{\text{T}}\}$ is preserved. As a specific example consider the five point amplitude $A_5[1,\{2\},5,\{3,4\}]$. we then have the set $\{\alpha\}\cup\{\beta^{\text{T}}\} = \{2\}\cup\{4,3\}$, and the sum runs over the ordered permutations $\sigma = \{2,4,3\},\{4,3,2\},\{4,2,3\}$. Giving the Kleiss-Kuijf relation to be

$$A_n[1,2,5,3,4] = A_n[1,2,4,3,5] + A_n[1,4,3,2,5] + A_n[1,4,2,3,5]. \qquad (2.6.52)$$

This along with the other properties discussed in this section breaks down the number of independent n-gluon amplitudes to $(n-2)!$. There is yet one more set of linear relations that further reduces the number of independent amplitudes to $(n-3)!$, these are the BCJ relations, named for Bern, Carrasco and Johansson [23]. These relations derive from the structure of the kinematic factors of the full amplitudes and the other properties and relations stated above. We state here the relations that are of interest in this handling of the amplitudes. These are

$$\begin{aligned} s_{14}A_4[1,2,3,4] - s_{13}A_4[1,2,4,3] &= 0 \\ s_{12}A_5[2,1,3,4,5] + s_{23}A_5[1,3,2,4,5] + (s_{23}+s_{24})A_5[1,3,4,2,5] &= 0 \end{aligned} \qquad (2.6.53)$$

With this in hand it is now a simple matter to determine the gauge theory side of the KLT relations (1.6).

Consider the the right hand side of the four point KLT relation. For the helicity choice $h_1 = h_2 = -1 = -h_3 = -h_4$ determining both of these is a simple matter since they fulfil the requirements of the Parke-Taylor amplitude. We therefore have

$$s_{12}A_4[1,2,3,4]A_4[1,2,4,3] = \langle 12\rangle[12]\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle}$$

$$= \frac{\langle 12\rangle^7 [12]}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2} \quad (2.6.54)$$

For the five point KLT relation it is a bit more involved. The gauge theory side is

$$s_{23}s_{45}A_5[1,2,3,4,5]A_5[1,3,2,5,4] + s_{24}s_{35}A_5[1,2,4,3,5]A_5[1,4,2,5,3] \quad (2.6.55)$$

Now for the helicity choice $h_1 = h_2 = -1 = -h_3 = -h_4 = -h_5$ the first amplitude in each term can be found with the Parke-Taylor amplitude. But the other two require some more work. Firstly one can use the Kleiss-Kuijf relation to rewrite the amplitude as

$$A_5[1,3,2,5,4] = A_5[1,3,4,5,2] + A_5[1,4,3,5,2] + A_5[1,4,5,3,2], \quad (2.6.56)$$

one then uses the cyclic property on the right hand side terms to get expressions that can be calculated using the simple Parke-Taylor amplitude,

$$A_5[1,3,2,5,4] = A_5[3,4,5,2,1] + A_5[4,3,5,2,1] + A_5[4,5,3,2,1], \quad (2.6.57)$$

A similar process is followed for the remaining amplitude. Substituting and simplifying one then gets the amplitude

$$\frac{\langle 12\rangle^7(\langle 24\rangle\langle 35\rangle[23][45] - \langle 23\rangle\langle 45\rangle[24][35])}{\langle 13\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 24\rangle\langle 25\rangle\langle 34\rangle\langle 35\rangle\langle 45\rangle}. \quad (2.6.58)$$

2.7 BCFW Recursion relations

Although we have the power of the Parke-Taylor amplitude to construct all the necessary tree level n-point gauge theory amplitudes, the Parke-Taylor amplitude can be proven to hold by means of the BCFW recursion relations. Due to the large impact that these relations have had on the computation of amplitudes and the extension thereof to general relativity we now give a basic review of these relations. Firstly note than on-shell amplitudes are characterised by the momenta of external particles and their helicities. For the theories we are considering we assume massless particles, $p_i^2 = 0$ for $i = 1, ..., n$ and imposing conservation of momentum we have that $\sum_{i=1}^n p_i^\mu = 0$. We now introduce n complex valued vectors r_i^μ with the following restrictions:

$$\sum_{i=1}^{n} r_i^\mu = 0 \quad (2.7.59)$$

$$r_i \cdot r_j = 0 \quad , \quad \text{for all } i, j = 1, 2, ..., 0 \quad (2.7.60)$$

$$p_i \cdot r_i = 0 \quad , \quad \text{for each } i. \quad (2.7.61)$$

We use these vectors are used to define a complex shifted momentum

$$\tilde{p} = p_i + z r_i \quad , \quad z \in \mathbb{C}. \quad (2.7.62)$$

Next we note the following (a) that by (2.7.59) shifted momenta is conserved, (b) by (2.7.60) and (2.7.61) the shifted momenta are lightlike and (c) for a non-trivial subset of momenta $\{p_i\}_{i\in I}$ define $P_I^\mu = \sum_{i\in I} p_i^\mu$. Then the shifted momenta $\tilde{P}_i^2$ is linear in z,

$$\tilde{P}_i^2 = \left(\sum_{i\in I} \tilde{p}_i\right)^2 = P_I^2 + z 2 P_I R_I \quad (2.7.63)$$

where $R_I = \sum_{i\in I} r_i$. Next we define $z_I = -\frac{P_I^2}{2P_I R_I}$ and can then write

$$\tilde{P}_i^2 = -\frac{P_I^2}{z_I}(z - z_I). \quad (2.7.64)$$

Next we consider the amplitude $\tilde{A}_n(z)$ which is simply the amplitude in terms of the shifted momenta. We can do this due to the results (a) and (b) above, ensuring that conservation of momentum and the on-shell condition for external states hold. We study this amplitude as a function of z, from which we can recover the amplitude with unshifted momenta, A_n by setting $z = 0$. Next we fix the amplitude to be tree level, this ensures a very simply analytic structure that only depends on the poles, which are only produced by the shifted propagators $1/\tilde{P}_I^2$, where $\tilde{P}_I$ is a sum of non-trivial subset of shifted momenta. We can see this dependence clearly when considering Feynman diagrams. By the result (c) above, $1/\tilde{P}_I^2$ gives a simple pole at $z_I = 0$. Also for generic momenta no diagram can have two or more of the same propagator, therefore all the poles of the propagators in a given diagram are in different positions in the z-plane. $\tilde{A}_n(z)$ therefore only has simple poles for generic momenta. Note the implicit assumption of locality. When we consider the expression $\frac{\tilde{A}_n(z)}{z}$ in the z-plane and take a contour that surrounds the simple pole at the origin, the residue at this pole is nothing but the unshifted amplitude, $\text{Res}\left(\frac{\tilde{A}_n(z)}{z}, 0\right) = A_n(0)$. By expanding this contour to enclose all the poles we can see that by Cauchy's theorem,

$$A_n = -\sum_{z_I} \text{Res}\left(\frac{\tilde{A}_n(z)}{z}, z_I\right) + B_n, \tag{2.7.65}$$

where B_n is the residue of the pole at $z = \infty$. The interesting result of this lies in the fact that at a z_I pole the propagator $1/\tilde{P}_I^2$ goes on-shell and shifted amplitude can be factorized into two on-shell sub amplitudes, $\tilde{A}_L$ and $\tilde{A}_R$, allowing us to write,

$$\text{Res}\left(\frac{\tilde{A}_n(z)}{z}, z_I\right) = -\tilde{A}_L(z_I)\frac{1}{\tilde{P}_I^2}\tilde{A}_R(z_I). \tag{2.7.66}$$

It is useful to note that unlike standard Feynman diagrams the internal line (the propagator) in (2.7.66) is on-shell, $\tilde{P}_I^2 = 0$ and the subamplitudes are shifted on-shell amplitudes evaluated at $z = z_I$. By necessity the subamplitudes are constructed of fewer than n external states, this is the basis on which the recursion relations are built. By summing over all z_I in (2.7.66) we have the following

$$A_n = -\sum_{z_I} \text{Res}\left(\frac{\tilde{A}_n(z)}{z}, z_I\right) + B_n = \sum_{z_I} \tilde{A}_L(z_I)\frac{1}{\tilde{P}_I^2}\tilde{A}_R(z_I) + B_n. \tag{2.7.67}$$

From this we can produce the amplitude A_n from lower point amplitudes and propagators. One should note that the sum runs over all possible factorization channels as well as over all the possible on-shell particle states that can be exchanged via the propagator. There is one problem that remains, the B_n term. As stated earlier this is the residue of the pole at $z = \infty$, and as of yet there is no general structure for dealing with it. The most common way of getting rid of this term is to demonstrate that $\tilde{A}_n(z) \to 0$ as $z \to \infty$.

We now get to the BCFW recursion relations. This is a special case of the general statements above in which one only shifts two of the external momenta, say i and j, and the rest are trivial, $r_k^\mu = 0$ for all $k \neq i, j$. The shift is implemented as follows

$$|\tilde{i}] = |i] + z|j], \qquad |\tilde{j}] = |j], \qquad |\tilde{i}\rangle = |i\rangle, \qquad |\tilde{j}\rangle = |j\rangle - z|i\rangle, \tag{2.7.68}$$

and no other spinors are shifted. We call the shift above an $[i, j\rangle$-shift. The validity of this recursion relation relies on the boundary term B_n vanishing. The typical approach is once again to show that $\tilde{A}_n(z) \to 0$ as $z \to \infty$. In pure Yang-Mills theory for adjacent i, j of given helicity the large z behaviour of the shifted amplitude is given by

$[i, j\rangle$	$[-, -\rangle$	$[-, +\rangle$	$[+, +\rangle$	$[+, -\rangle$
$\tilde{A}_n(z) \approx$	$\frac{1}{z}$	$\frac{1}{z}$	$\frac{1}{z}$	z^3

(2.7.69)

This means that the first three are valid shifts for the recursion relations of the gluon tree amplitudes. These recursion relations can now be used to calculate higher orders of tree level amplitudes once the propagators and 3-amplitudes are known, It can also be used for other purposes such as constructing an inductive proof of the Parke-Taylor amplitude or deriving the KLT relations for an arbitrary number of external states.

Chapter 3

Perturbative Structure of GR and UG

3.1 GR Lagrangian

Since the weak field expansion of general relativity is covered extensively in the literature we only include this section as a means to establish our notation. While the treatment given here can certainly be extended to generally curved backgrounds, we will restrict our attention to a graviton propagating on a flat Minkowski geometry whose metric tensor $\eta_{\mu\nu} = \text{diag}(-1,+1,+1,+1)$. Consequently, we take $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ and expand the Einstein-Hilbert action

$$S_{EH} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{|g|}\,R, \tag{3.1.1}$$

in powers of $\kappa h_{\mu\nu}$. Here, as is standard in the literature, we define the gravitational coupling as $\kappa^2 \equiv 8\pi G_N$. Unlike, for example, Maxwell electrodynamics, this expansion generates an infinite series in $h_{\mu\nu}$, due essentially to the presence of the inverse metric in the Ricci scalar

$$R = g^{\mu\nu}\left(\partial_\nu\Gamma^\lambda_{\mu\lambda} - \partial_\lambda\Gamma^\lambda_{\mu\nu} + \Gamma^\tau_{\mu\lambda}\Gamma^\lambda_{\tau\nu} - \Gamma^\tau_{\mu\nu}\Gamma^\lambda_{\tau\lambda}\right), \tag{3.1.2}$$

and the square root of the determinant of the metric in the volume form. Expanded in $h_{\mu\nu}$ up to cubic order, these factors contribute

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\lambda}h^\nu_\lambda - \kappa^3 h^{\mu\lambda}h_{\lambda\rho}h^{\rho\nu} + O(h^4), \tag{3.1.3}$$

and, respectively,

$$\begin{aligned}\sqrt{|g|} &= \prod_{n=1}^{\infty}\left[\sum_{m=0}^{\infty}\left(\frac{1}{m!}\left[\frac{(-1)^{n-1}}{2n}\,(h^n)^\mu_\mu\right]\right)\right] \qquad (3.1.4)\\ &= 1 + \frac{\kappa}{2}h + \frac{\kappa^2}{8}\left(h^2 - 2h^{\mu\nu}h_{\mu\nu}\right) + \frac{\kappa^3}{48}\left(h^3 - 6hh^{\mu\nu}h_{\mu\nu} + 8h^{\mu\nu}h_{\nu\lambda}h^{\lambda\mu}\right) + O(h^4),\end{aligned}$$

Substituting this into the Einstein-Hilbert Lagrangian and organizing the resulting expansion in powers of $h_{\mu\nu}$ gives the formal series,

$$\mathcal{L} = \mathcal{L}_2 + \kappa\mathcal{L}_3 + \kappa^2\mathcal{L}_4 + \mathcal{L}_5 + ... \tag{3.1.5}$$

This as one can imagine is an example of extreme indicidal manipulation. But with the use of Mathematica packages like xAct and specifically xTensor, one can very simply expand the given Lagrangian in perturbations around a flat metric. This package also allowed the automatic application of various rules to better manipulate the various stages of the expansion to forms more applicable to the calculation. In order to extract the Feynman rules from this Lagrangian, we still need to fix a gauge. A common choice in the perturbative gravity literature is the *de Donder gauge* in which $\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu h^\mu_\mu$ bringing the quadratic terms in the Lagrangian into the form $-\frac{1}{2}h_{\mu\nu}\,\Box h^{\mu\nu} + \frac{1}{4}h^\mu_\mu\,\Box h_\nu^{\ \nu}$. This will facilitate reading off of the de Donder gauge propagator a little later but at this point, up to quintic order (good for 5-point graviton scattering), the perturbative Lagrangian of GR in all its indicial glory, reads

$$
\begin{aligned}
\mathcal{L}_2 \;=\;& -\frac{1}{8}\partial_\alpha h\,\partial^\alpha h + \frac{1}{4}\partial_\gamma h^{\alpha\beta}\,\partial^\gamma h_{\alpha\beta} \\
\mathcal{L}_3 \;=\;& +\frac{1}{4}h^{\alpha\gamma}\,\partial_\beta h\,\partial^\beta h_{\alpha\gamma} - \frac{1}{16}h\,\partial_\beta h\,\partial^\beta h - \frac{1}{4}h^{\alpha\gamma}\,\partial_\alpha h_{\beta\delta}\,\partial_\gamma h^{\beta\delta} \\
& +\frac{1}{2}h^{\alpha\gamma}\,\partial_\gamma h_{\beta\delta}\,\partial^\delta h_\alpha^{\ \beta} - \frac{1}{2}h^{\alpha\gamma}\,\partial_\delta h_{\gamma\beta}\,\partial^\delta h_\alpha^{\ \beta} + \frac{1}{8}h\,\partial^\delta h_{\alpha\beta}\,\partial_\delta h^{\alpha\beta} \\
\mathcal{L}_4 \;=\;& \frac{1}{4}h^{\alpha\delta}h^{\beta\mu}\,\partial_\nu h_{\delta\mu}\,\partial^\nu h_{\alpha\beta} - \frac{1}{4}h^{\alpha\delta}h^{\beta\mu}\,\partial_\nu h_{\beta\mu}\,\partial^\nu h_{\alpha\delta} - \frac{1}{16}h^{\alpha\delta}h^{\alpha\delta}\,\partial_\nu h_{\beta\mu}\,\partial^\nu h_{\beta\mu} + \frac{1}{32}hh\,\partial_\nu h_{\alpha\beta}\,\partial^\nu h^{\alpha\beta} \\
& -\frac{1}{2}h^\beta_\alpha h^{\alpha\delta}\,\partial_\beta h_{\mu\nu}\,\partial^\nu h^\mu_\delta + \frac{1}{2}hh^{\delta\beta}\,\partial_\beta h_{\mu\nu}\,\partial^\nu h^\mu_\delta + \frac{1}{2}h^\beta_\alpha h^{\alpha\delta}\,\partial_\nu h_{\beta\mu}\,\partial^\nu h^\mu_\delta - \frac{1}{4}hh^{\delta\beta}\,\partial_\nu h_{\beta\mu}\,\partial^\nu h^\mu_\delta \\
& -\frac{1}{2}h^{\alpha\delta}h^{\beta\mu}\,\partial_\beta h^\nu_\alpha\,\partial_\delta h_{\mu\nu} + \frac{1}{4}h^\beta_\alpha h^{\alpha\delta}\,\partial_\beta h_{\mu\nu}\,\partial_\delta h^{\mu\nu} - \frac{1}{8}hh^{\delta\beta}\,\partial_\beta h_{\mu\nu}\,\partial_\delta h^{\mu\nu} - \frac{1}{8}h^\beta_\alpha h^{\alpha\delta}\,\partial_\beta h\,\partial_\delta h \\
& +\frac{1}{2}h^{\alpha\delta}h^{\beta\mu}\,\partial^\nu h_{\alpha\delta}\,\partial_\mu h_{\beta\nu} + \frac{1}{8}h_{\alpha\delta}h^{\alpha\delta}\,\partial^\nu h^{\beta\mu}\,\partial_\mu h_{\beta\nu} - \frac{1}{16}hh\,\partial^\nu h^{\beta\mu}\,\partial_\mu h_{\beta\nu} + \frac{1}{4}hh^{\delta\beta}\partial^\nu h^\mu_\delta\partial_\mu h_{\beta\nu} \\
& -\frac{1}{4}h^{\alpha\delta}h^{\beta\mu}\,\partial_\beta h_{\alpha\delta}\,\partial_\mu h + \frac{1}{4}h^\beta_\alpha h^{\alpha\delta}\,\partial_\beta h^\mu_\delta\,\partial_\mu h - \frac{1}{4}hh^{\delta\beta}\,\partial_\beta h^\mu_\delta\,\partial_\mu h + \frac{1}{4}h^{\alpha\delta}h^{\beta\mu}\,\partial_\delta h_{\alpha\beta}\,\partial_\mu h \\
& +\frac{1}{2}h^{\alpha\delta}h^{\beta\mu}\,\partial_\beta h^\nu_\alpha\,\partial_\mu h_{\delta\nu} - \frac{1}{2}h^{\alpha\delta}h^{\beta\mu}\,\partial^\nu h_{\alpha\beta}\,\partial_\mu h_{\delta\nu} - \frac{1}{4}h^\beta_\alpha h^{\alpha\delta}\,\partial_\mu h\,\partial^\mu h_{\delta\beta} + \frac{1}{8}hh^{\delta\beta}\,\partial_\mu h\,\partial^\mu h_{\delta\beta} \\
\mathcal{L}_5 \;=\;& -\frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\beta h^\nu_\mu\partial_\gamma h_{\sigma\nu} + h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\mu h^\nu_\beta\partial_\gamma h_{\sigma\nu} - \frac{1}{4}h^\alpha_\alpha h^{\beta\gamma}h^{\mu\sigma}\partial_\mu h^\nu_\beta\partial_\gamma h_{\sigma\nu} \\
& +\frac{1}{4}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\beta h_{\mu\sigma}\partial_\gamma h^\nu_\nu + \frac{1}{4}h^{\alpha\beta}h^{\gamma\mu}h^{\sigma\nu}\partial_\gamma h_{\alpha\beta}\partial_\mu h_{\sigma\nu} - \frac{1}{4}h^\gamma_\alpha h^{\alpha\beta}h^\mu_\beta\partial_\gamma h^{\sigma\nu}\partial_\mu h_{\sigma\nu} \\
& +\frac{1}{8}h^\alpha_\alpha h^\mu_\beta h^{\beta\gamma}\partial_\gamma h^{\sigma\nu}\partial_\mu h_{\sigma\nu} + \frac{1}{16}h_{\alpha\beta}h^{\alpha\beta}h^{\gamma\mu}\partial_\gamma h^{\sigma\nu}\partial_\mu h_{\sigma\nu} - \frac{1}{32}h^\alpha_\alpha h^\beta_\beta h^{\gamma\mu}\partial_\gamma h^{\sigma\nu}\partial_\mu h_{\sigma\nu} \\
& +\frac{1}{4}h^\gamma_\alpha h^{\alpha\beta}h^\mu_\beta\partial_\gamma h^\sigma_\sigma\partial_\mu h^\nu_\nu + \frac{1}{2}h^{\alpha\beta}h^{\gamma\mu}h^{\sigma\nu}\partial_\mu h_{\beta\nu}\partial_\sigma h_{\alpha\gamma} - \frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\mu h^\nu_\beta\partial_\sigma h_{\gamma\nu} \\
& +\frac{1}{4}h^\alpha_\alpha h^{\beta\gamma}h^{\mu\sigma}\partial_\mu h^\nu_\beta\partial_\sigma h_{\gamma\nu} + \frac{1}{4}h^\alpha_\alpha h^{\beta\gamma}h^{\mu\sigma}\partial_\gamma h_{\beta\mu}\partial_\sigma h^\nu_\nu + \frac{1}{4}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\mu h_{\beta\gamma}\partial_\sigma h^\nu_\nu \\
& -\frac{1}{8}h^\alpha_\alpha h^{\beta\gamma}h^{\mu\sigma}\partial_\mu h_{\beta\gamma}\partial_\sigma h^\nu_\nu + \frac{1}{4}h^\alpha_\alpha h^\mu_\beta h^{\beta\gamma}\partial_\mu h^\sigma_\gamma\partial_\sigma h^\nu_\nu + \frac{1}{8}h_{\alpha\beta}h^{\alpha\beta}h^{\gamma\mu}\partial_\mu h^\sigma_\gamma\partial_\sigma h^\nu_\nu \\
& -\frac{1}{16}h^\alpha_\alpha h^\beta_\beta h^{\gamma\mu}\partial_\mu h^\sigma_\gamma\partial_\sigma h^\nu_\nu + \frac{1}{4}h^\gamma_\alpha h^{\alpha\beta}h^\mu_\beta\partial_\sigma h^\nu_\nu\partial^\sigma h_{\gamma\mu} - \frac{3}{4}h^\alpha_\alpha h^\mu_\beta h^{\beta\gamma}\partial_\sigma h^\nu_\nu\partial^\sigma h_{\gamma\mu} \\
& -\frac{1}{16}h_{\alpha\beta}h^{\alpha\beta}h^{\gamma\mu}\partial_\sigma h^\nu_\nu\partial^\sigma h_{\gamma\mu} + \frac{1}{32}h^\alpha_\alpha h^\beta_\beta h^{\gamma\mu}\partial_\sigma h^\nu_\nu\partial^\sigma h_{\gamma\mu} - \frac{5}{24}h^\gamma_\alpha h^{\alpha\beta}h_{\beta\gamma}\partial_\sigma h^\nu_\nu\partial^\sigma h^\mu_\mu \\
& +\frac{1}{96}h^\alpha_\alpha h^\beta_\beta h^\gamma_\gamma\partial_\sigma h^\nu_\nu\partial^\sigma h^\mu_\mu - \frac{1}{4}h^{\alpha\beta}h^{\gamma\mu}h^{\sigma\nu}\partial_\sigma h_{\alpha\gamma}\partial_\nu h_{\beta\mu} - \frac{1}{2}h^{\alpha\beta}h^{\gamma\mu}h^{\sigma\nu}\partial_\gamma h_{\alpha\beta}\partial_\nu h_{\mu\sigma} \\
& -\frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\gamma h^\nu_\beta\partial_\nu h_{\mu\sigma} - \frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\sigma h_{\mu\nu}\partial^\nu h_{\beta\gamma} + \frac{1}{4}h^\alpha_\alpha h^{\beta\gamma}h^{\mu\sigma}\partial_\sigma h_{\mu\nu}\partial^\nu h_{\beta\gamma} \\
& +\frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\nu h_{\mu\sigma}\partial^\nu h_{\beta\gamma} - \frac{1}{8}h^\alpha_\alpha h^{\beta\gamma}h^{\mu\sigma}\partial_\nu h_{\mu\sigma}\partial^\nu h_{\beta\gamma} - \frac{1}{2}h^\alpha_\alpha h^{\beta\gamma}h^{\mu\sigma}\partial_\sigma h_{\gamma\nu}\partial^\nu h_{\beta\mu} \\
& +\frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\nu h_{\gamma\sigma}\partial^\nu h_{\beta\mu} + \frac{1}{8}h^\alpha_\alpha h^{\beta\gamma}h^{\mu\sigma}\partial_\nu h_{\gamma\sigma}\partial^\nu h_{\beta\mu} - \frac{1}{2}h^\alpha_\alpha h^\mu_\beta h^{\beta\gamma}\partial_\mu h_{\sigma\nu}\partial^\nu h^\sigma_\gamma \\
& -\frac{1}{4}h_{\alpha\beta}h^{\alpha\beta}h^{\gamma\mu}\partial_\mu h_{\sigma\nu}\partial^\nu h^\sigma_\gamma + \frac{1}{8}h^\alpha_\alpha h^\beta_\beta h^{\gamma\mu}\partial_\mu h_{\sigma\nu}\partial^\nu h^\sigma_\gamma - \frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^\mu_\beta\partial_\sigma h_{\mu\nu}\partial^\nu h^\sigma_\gamma \\
& -\frac{1}{4}h^\alpha_\alpha h^\mu_\beta h^{\beta\gamma}\partial_\sigma h_{\mu\nu}\partial^\nu h^\sigma_\gamma - \frac{1}{8}h_{\alpha\beta}h^{\alpha\beta}h^{\gamma\mu}\partial_\sigma h_{\mu\nu}\partial^\nu h^\sigma_\gamma + \frac{1}{16}h^\alpha_\alpha h^\beta_\beta h^{\gamma\mu}\partial_\sigma h_{\mu\nu}\partial^\nu h^\sigma_\gamma \\
& +\frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^\mu_\beta\partial_\nu h_{\mu\sigma}\partial^\nu h^\sigma_\gamma + \frac{1}{4}h^\alpha_\alpha h^\mu_\beta h^{\beta\gamma}\partial_\nu h_{\mu\sigma}\partial^\nu h^\sigma_\gamma + \frac{1}{8}h_{\alpha\beta}h^{\alpha\beta}h^{\gamma\mu}\partial_\nu h_{\mu\sigma}\partial^\nu h^\sigma_\gamma \\
& -\frac{1}{16}h^\alpha_\alpha h^\beta_\beta h^{\gamma\mu}\partial_\nu h_{\mu\sigma}\partial^\nu h^\sigma_\gamma - \frac{1}{12}h^\gamma_\alpha h^{\alpha\beta}h_{\beta\gamma}\partial_\sigma h_{\mu\nu}\partial^\nu h^{\mu\sigma} + \frac{1}{16}h^\alpha_\alpha h_{\beta\gamma}h^{\beta\gamma}\partial_\sigma h_{\mu\nu}\partial^\nu h^{\mu\sigma} \\
& -\frac{1}{96}h^\alpha_\alpha h^\beta_\beta h^\gamma_\gamma\partial_\sigma h_{\mu\nu}\partial^\nu h^{\mu\sigma} + \frac{1}{24}h^\gamma_\alpha h^{\alpha\beta}h_{\beta\gamma}\partial_\nu h_{\mu\sigma}\partial^\nu h^{\mu\sigma}
\end{aligned}
$$

$$-\frac{1}{32}h^{\alpha}_{\alpha}h_{\beta\gamma}h^{\beta\gamma}\partial_\nu h_{\mu\sigma}\partial^\nu h^{\mu\sigma}+\frac{1}{192}h^{\alpha}_{\alpha}h^{\beta}_{\beta}h^{\gamma}_{\gamma}\partial_\nu h_{\mu\sigma}\partial^\nu h^{\mu\sigma}, \tag{3.1.6}$$

and will form the basis for comparison to unimodular gravity below.

3.2 Unimodular Lagrangian

Before writing down the equivalent perturbative expansion for unimodular gravity, it will be useful to recall the symmetries of the theory. The Einstein-Hilbert action of GR is famously invariant under the full group of diffeomorphisms on the spacetime manifold, Diff(M), under which $g_{\mu\nu}\to g_{\mu\nu}+\nabla_\mu\xi_\nu+\nabla_\nu\xi_\mu$ or, in infinitesimal form, $h_{\mu\nu}\to h_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu$. The defining characteristic of Einstein's 1919 trace free theory is that the metric determinant is held fixed, to unity in the special case of unimodular gravity. This unimodularity condition breaks Diff(M) to the proper subgroup of *transverse* diffeomorphisms, TDiff(M) under which $h_{\mu\nu}\to h_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu$, with $\partial_\mu\xi^\mu=0$. This is, of course, just a classical gauge fixing of the GR action and the reason why the two theories are classically indistinguishable[1] [2, 6] (modulo the important issue of the interpretation of the cosmological constant). In fact the theory enjoys an additional Weyl symmetry under which $g_{\mu\nu}\to e^{2\sigma(x)}g_{\mu\nu}$, enhancing its symmetry to WTDiff(M) with a corresponding four generators per spacetime point.

There are, in fact, many ways of implementing the WTDiff(M) symmetry into an action functional that range from the most obvious enforcing of the fixed metric determinant through a Lagrange multiplier [24] so that

$$S_{EH}\to S_{UG}=\int_M d^4x\ \left[\sqrt{|g|}R+\lambda\left(\sqrt{|g|}-1\right)\right], \tag{3.2.7}$$

to Henneaux and Teitelboim's [25] more sophisticated formulation in which the trace free equations are derived from the fully covariant action

$$S_{HT}=\frac{1}{2\kappa^2}\left(\int_M d^4x\ \sqrt{|g|}(R+2\lambda)+\int_M A_3\wedge d\lambda\right), \tag{3.2.8}$$

where A_3 and λ are a spacetime 3-form and scalar respectively. All these formalisms have been treated extensively in the literature, and all produce Einstein's 1919 equations.

$$R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R=8\pi G_N\hat{T}_{\mu\nu}, \tag{3.2.9}$$

with $\mathrm{tr}\left(\hat{T}_{\mu\nu}\right)=0$. Since we are interested in making *on-shell* statements about the theory, any of these various action principles will suffice for our purposes. However, for convenience, we will use the one that begins with a rescaling of $g_{\mu\nu}\to\hat{g}_{\mu\nu}\equiv g^{-1/4}g_{\mu\nu}$, as done for the classical field equations in the introduction. The resulting action, formulated in terms of $\hat{g}_{\mu\nu}$ is unimodular since $\hat{g}=\det(\hat{g}_{\mu\nu})=1$ and reads quite simply,

$$S_{UG}=\frac{1}{2\kappa^2}\int_M d^4x\ \hat{R}(\hat{g}_{\mu\nu}). \tag{3.2.10}$$

The perturbative expansion for unimodular gravity then proceeds in much the same way from (3.2.10) as that for GR follows from the Einstein-Hilbert action. Again, we

[1] The reader will no doubt have noticed that we have also not been distinguishing between "trace free" and "unimodular" since, from our perspective, the only difference between the two is the value of the constant that the determinant of the metric is fixed at. This will have no effect on any scattering amplitudes.

write $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. Since the Christoffel symbols transform under this rescaling as

$$\hat{\Gamma}\left[\hat{g}\right]^{\alpha}_{\mu\nu} = \Gamma\left[g\right]^{\alpha}_{\mu\nu} + P\left[g\right]^{\alpha}_{\mu\nu} = \Gamma\left[g\right]^{\alpha}_{\mu\nu} - \frac{1}{4}\left(\Gamma\left[g\right]^{\beta}_{\mu\beta}\delta^{\alpha}_{\nu} + \Gamma\left[g\right]^{\beta}_{\nu\beta}\delta^{\alpha}_{\mu} - \Gamma\left[g\right]^{\beta}_{\gamma\beta}g^{\alpha\gamma}g_{\mu\nu}\right),$$

we can then write the rescaled Ricci scalar $\hat{R}(\hat{g}_{\mu\nu})$ in terms of the unscaled metric $g_{\mu\nu}$. This can easily be shown to be,

$$\hat{R} = \hat{g}^{\mu\nu}\hat{R}_{\mu\nu} = g^{\frac{1}{4}}g^{\mu\nu}(R_{\mu\nu} + \nabla_{\nu}P^{\alpha}_{\mu\alpha} - \nabla_{\alpha}P^{\alpha}_{\mu\nu} + P^{\alpha}_{\mu\rho}P^{\rho}_{\mu\alpha} - P^{\alpha}_{\alpha\rho}P^{\rho}_{\mu\nu}). \quad (3.2.11)$$

From this we can expand $g^{\frac{1}{4}}$ and $g_{\mu\nu}$ as in GR to produce the perturbative Lagrangian for unimodular gravity. Alternately we can write $\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \kappa\hat{h}_{\mu\nu}$, where $\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu}h$ and expand $\hat{R}(\hat{g}_{\mu\nu})$ directly in $h_{\mu\nu}$[2]. This cannot be interpreted as a field redefinition since the trace, h, of $h_{\mu\nu}$ cannot be recovered from $\hat{h}$. This substitution then yields the perturbative Lagrangian (by order in h) for unimodular gravity. Formally, $\hat{\mathcal{L}} = \hat{\mathcal{L}}_2 + \kappa\hat{\mathcal{L}}_3 + \ldots$, where for example,

$$\begin{aligned}
\hat{\mathcal{L}}_2 &= -\frac{3}{32}\partial_\alpha h\partial^\alpha h + \frac{1}{4}\partial_\gamma h^{\alpha\beta}\partial^\gamma h_{\alpha\beta} + \frac{1}{4}\partial^\alpha h\partial^\beta h_{\alpha\beta} - \frac{1}{2}\partial_\alpha h^{\alpha\beta}\partial^\gamma h_{\gamma\beta}\\
\hat{\mathcal{L}}_3 &= +\frac{3}{8}h^{\alpha\gamma}\partial_\beta h\partial^\beta h_{\alpha\gamma} - \frac{17}{128}h\partial_\beta h\partial^\beta h - \frac{1}{4}h^{\alpha\gamma}\partial^\beta h\partial^\gamma h_{\alpha\beta} - \frac{1}{4}h^{\alpha\gamma}\partial_\alpha h^{\beta\delta}\partial_\gamma h_{\beta\delta} + \frac{5}{32}h^{\alpha\gamma}\partial_\alpha h\partial_\gamma h\\
&\quad -\frac{1}{2}h^{\alpha\gamma}\partial^\beta h_{\alpha\beta}\partial_\gamma h - \frac{1}{2}h^{\alpha\gamma}\partial^\beta h_{\alpha\gamma}\partial^\delta h_{\beta\delta} + \frac{5}{16}h\partial^\beta h\partial^\alpha h_{\alpha\beta} + \frac{1}{2}h^{\alpha\gamma}\partial_\gamma h_{\alpha\beta}\partial_\delta h^{\beta\delta} - \frac{1}{4}h\partial_\gamma h^{\gamma\beta}\partial^\delta h_{\beta\delta}\\
&\quad +\frac{1}{2}h^{\alpha\gamma}\partial^\beta h_{\alpha\beta}\partial^\delta h_{\gamma\delta} + \frac{1}{2}h^{\alpha\gamma}\partial_\gamma h_{\beta\delta}\partial^\delta h^\beta_\alpha - \frac{1}{2}h^{\alpha\gamma}\partial_\delta h_{\gamma\beta}\partial^\delta h^\beta_\alpha - \frac{1}{8}h\partial_\beta h_{\gamma\delta}\partial^\delta h^{\gamma\beta} + \frac{3}{16}h\partial_\delta h_{\gamma\beta}\partial^\delta h^{\gamma\beta}.
\end{aligned}$$

As a check that we do indeed have the correct invariance required of the theory, let's consider the quadratic piece $\hat{\mathcal{L}}_2$, from which we will derive the propagator. Under a general field redefinition $h_{\mu\nu} \to h_{\mu\nu} + \delta h_{\mu\nu}$, and up to total derivatives,

$$\delta\hat{\mathcal{L}}_2 = +\frac{3}{16}\delta h\partial_\alpha\partial^\alpha h - \frac{1}{2}\delta h^{\alpha\beta}\partial_\gamma\partial^\gamma h_{\alpha\beta} - \frac{1}{4}\left(\delta h\partial^\alpha\partial^\beta h_{\alpha\beta} + \delta h_{\alpha\beta}\partial^\alpha\partial^\beta h\right) + \frac{1}{2}\delta h^{\alpha\beta}\partial^\gamma\partial_{(\alpha}h_{\beta)\gamma}.$$

Evidently, under the restricted set of gauge transformations $\delta h_{\alpha\beta} \to 2\partial_{(\alpha}\xi_{\beta)} + \frac{1}{2}\eta_{\mu\nu}\phi$ with the parameters obeying the transversality condition $\partial_\alpha\xi^\alpha = 0$, the first and third terms as well as a combination of the second and fourth terms are all invariant. As promised, the traceless perturbative Lagrangian is WTDiff-invariant. We have checked that $\hat{\mathcal{L}}_3$ (and higher order in h terms) also exhibit the same invariance under WTDiff(M), using again the xTensor package in Mathematica. It remains only to fix the additional gauge redundancies by applying de Donder gauge again, yielding

$$\begin{aligned}
\hat{\mathcal{L}}_2 &= -\frac{3}{32}\partial_\alpha h\partial^\alpha h + \frac{1}{4}\partial_\gamma h^{\alpha\beta}\partial^\gamma h_{\alpha\beta}\\
\hat{\mathcal{L}}_3 &= +\frac{1}{8}h^{\alpha\gamma}\partial_\beta h\partial^\beta h_{\alpha\gamma} - \frac{5}{128}h\partial_\beta h\partial^\beta h - \frac{1}{4}h^{\alpha\gamma}\partial_\alpha h_{\beta\delta}\partial_\gamma h^{\beta\delta} + \frac{1}{2}h^{\alpha\gamma}\partial_\gamma h_{\beta\delta}\partial^\delta h^\beta_\alpha\\
&\quad -\frac{1}{2}h^{\alpha\gamma}\partial_\delta h_{\gamma\beta}\partial^\delta h^\beta_\alpha + \frac{3}{16}h\partial^\delta h_{\alpha\beta}\partial_\delta h^{\alpha\beta} - \frac{1}{8}h\partial_\beta h^{\alpha\gamma}\partial_\gamma h^\beta_\alpha + \frac{1}{32}h^{\alpha\gamma}\partial_\alpha h\partial_\gamma h\\
\hat{\mathcal{L}}_4 &= \frac{1}{4}h^{\alpha\delta}h^{\beta\mu}\partial_\nu h_{\delta\mu}\partial^\nu h_{\alpha\beta} - \frac{1}{4}h^{\alpha\delta}h^{\beta\mu}\partial_\nu h_{\beta\mu}\partial^\nu h_{\alpha\delta} - \frac{1}{16}h^{\alpha\delta}h^{\alpha\delta}\partial_\nu h_{\beta\mu}\partial^\nu h_{\beta\mu} + \frac{7}{64}hh\partial_\nu h_{\alpha\beta}\partial^\nu h^{\alpha\beta}\\
&\quad -\frac{1}{2}h^\beta_\alpha h^{\alpha\delta}\partial_\beta h_{\mu\nu}\partial^\nu h^\mu_\delta + \frac{5}{8}hh^{\delta\beta}\partial_\beta h_{\mu\nu}\partial^\nu h^\mu_\delta + \frac{1}{2}h^\beta_\alpha h^{\alpha\delta}\partial_\nu h_{\beta\mu}\partial^\nu h^\mu_\delta - \frac{1}{2}hh^{\delta\beta}\partial_\nu h_{\beta\mu}\partial^\nu h^\mu_\delta\\
&\quad -\frac{1}{2}h^{\alpha\delta}h^{\beta\mu}\partial_\beta h^\nu_\alpha\partial_\delta h_{\mu\nu} + \frac{1}{4}h^\beta_\alpha h^{\alpha\delta}\partial_\beta h_{\mu\nu}\partial_\delta h^{\mu\nu} - \frac{1}{4}hh^{\delta\beta}\partial_\beta h_{\mu\nu}\partial_\delta h^{\mu\nu} - \frac{1}{16}h^\beta_\alpha h^{\alpha\delta}\partial_\beta h\partial_\delta h\\
&\quad +\frac{1}{2}h^{\alpha\delta}h^{\beta\mu}\partial^\nu h_{\alpha\delta}\partial_\mu h_{\beta\nu} + \frac{1}{8}h_{\alpha\delta}h^{\alpha\delta}\partial^\nu h^{\beta\mu}\partial_\mu h_{\beta\nu} - \frac{1}{8}hh\partial^\nu h^{\beta\mu}\partial_\mu h_{\beta\nu} + \frac{1}{8}hh^{\delta\beta}\partial^\nu h^\mu_\delta\partial_\mu h_{\beta\nu}\\
&\quad -\frac{1}{4}h^{\alpha\delta}h^{\beta\mu}\partial_\beta h_{\alpha\delta}\partial_\mu h + \frac{1}{8}h^\beta_\alpha h^{\alpha\delta}\partial_\beta h^\mu_\delta\partial_\mu h - \frac{1}{4}hh^{\delta\beta}\partial_\beta h^\mu_\delta\partial_\mu h + \frac{1}{4}h^{\alpha\delta}h^{\beta\mu}\partial_\delta h_{\alpha\beta}\partial_\mu h
\end{aligned}$$

[2]Another way of producing the WTDiff(M) symmetry in the perturbative Lagrangian is simply to make the substitution $\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu}h$ in the perturbative GR Lagrangian. Although this ensures the correct symmetry it is not an acceptable transformation.

$$
\begin{aligned}
& +\frac{1}{2}h^{\alpha\delta}h^{\beta\mu}\partial_\beta h^\nu_\alpha\partial_\mu h_{\delta\nu}-\frac{1}{2}h^{\alpha\delta}h^{\beta\mu}\partial^\nu h_{\alpha\beta}\partial_\mu h_{\delta\nu}-\frac{1}{4}h^\beta_\alpha h^{\alpha\delta}\partial_\mu h\partial^\mu h_{\delta\beta}+\frac{1}{4}hh^{\delta\beta}\partial_\mu h\partial^\mu h_{\delta\beta} \\
& +\frac{1}{16}hh^{\delta\beta}\partial_\beta h\partial_\delta h+\frac{1}{128}h^{\alpha\delta}h_{\alpha\delta}\partial^\mu h\partial_\mu h-\frac{13}{512}hh\partial_\mu h\partial^\mu h \\
\hat{\mathcal{L}}_5 \;=\; & -\frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\beta h^\nu_\mu\partial_\gamma h_{\sigma\nu}+h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\mu h^\nu_\beta\partial_\gamma h_{\sigma\nu}-\frac{5}{8}h^\alpha_\alpha h^{\beta\gamma}h^{\mu\sigma}\partial_\mu h^\nu_\beta\partial_\gamma h_{\sigma\nu} \\
& +\frac{1}{4}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\beta h_{\mu\sigma}\partial_\gamma h^\nu_\nu-\frac{1}{4}h^{\alpha\beta}h^{\gamma\mu}h^{\sigma\nu}\partial_\gamma h_{\alpha\beta}\partial_\mu h_{\sigma\nu}-\frac{1}{4}h^\gamma_\alpha h^{\alpha\beta}h^\mu_\beta\partial_\gamma h^{\sigma\nu}\partial_\mu h_{\sigma\nu} \\
& +\frac{5}{16}h^\alpha_\alpha h^\mu_\beta h^{\beta\gamma}\partial_\gamma h^{\sigma\nu}\partial_\mu h_{\sigma\nu}-\frac{5}{32}h^\alpha_\alpha h^\beta_\beta h^{\gamma\mu}\partial_\gamma h^{\sigma\nu}\partial_\mu h_{\sigma\nu}+\frac{1}{8}h^\gamma_\alpha h^{\alpha\beta}h^\mu_\beta\partial_\gamma h^\sigma_\sigma\partial_\mu h^\nu_\nu \\
& -\frac{1}{8}h^\alpha_\alpha h^\mu_\beta h^{\beta\gamma}\partial_\gamma h^\sigma_\sigma\partial_\mu h^\nu_\nu-\frac{3}{64}h_{\alpha\beta}h^{\alpha\beta}h^{\gamma\mu}\partial_\gamma h^\sigma_\sigma\partial_\mu h^\nu_\nu+\frac{11}{256}h^\alpha_\alpha h^\beta_\beta h^{\gamma\mu}\partial_\gamma h^\sigma_\sigma\partial_\mu h^\nu_\nu \\
& +\frac{1}{2}h^{\alpha\beta}h^{\gamma\mu}h^{\sigma\nu}\partial_\mu h_{\beta\nu}\partial_\sigma h_{\alpha\gamma}-\frac{1}{4}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\gamma h^\nu_\nu\partial_\sigma h_{\beta\mu}-\frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\mu h^\nu_\beta\partial_\sigma h_{\gamma\nu} \\
& +\frac{5}{8}h^\alpha_\alpha h^{\beta\gamma}h^{\mu\sigma}\partial_\mu h^\nu_\beta\partial_\sigma h_{\gamma\nu}-\frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\gamma h_{\beta\mu}\partial_\sigma h^\nu_\nu+\frac{3}{8}h^\alpha_\alpha h^{\beta\gamma}h^{\mu\sigma}\partial_\gamma h_{\beta\mu}\partial_\sigma h^\nu_\nu \\
& +\frac{3}{8}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\mu h_{\beta\gamma}\partial_\sigma h^\nu_\nu-\frac{3}{16}h^\alpha_\alpha h^{\beta\gamma}h^{\mu\sigma}\partial_\mu h_{\beta\gamma}\partial_\sigma h^\nu_\nu+\frac{1}{4}h^\gamma_\alpha h^{\alpha\beta}h^\mu_\beta\partial_\mu h^\sigma_\gamma\partial_\sigma h^\nu_\nu \\
& -\frac{1}{16}h^\alpha_\alpha h^\mu_\beta h^{\beta\gamma}\partial_\mu h^\sigma_\gamma\partial_\sigma h^\nu_\nu-\frac{1}{32}h^\alpha_\alpha h^\beta_\beta h^{\gamma\mu}\partial_\mu h^\sigma_\gamma\partial_\sigma h^\nu_\nu-\frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^\mu_\beta\partial_\sigma h^\nu_\nu\partial^\sigma h_{\gamma\mu} \\
& +\frac{13}{32}h^\alpha_\alpha h^\mu_\beta h^{\beta\gamma}\partial_\sigma h^\nu_\nu\partial^\sigma h_{\gamma\mu}+\frac{1}{8}h_{\alpha\beta}h^{\alpha\beta}h^{\gamma\mu}\partial_\sigma h^\nu_\nu\partial^\sigma h_{\gamma\mu}-\frac{3}{16}h^\alpha_\alpha h^\beta_\beta h^{\gamma\mu}\partial_\sigma h^\nu_\nu\partial^\sigma h_{\gamma\mu} \\
& +\frac{1}{16}h^\gamma_\alpha h^{\alpha\beta}h_{\beta\gamma}\partial_\sigma h^\nu_\nu\partial^\sigma h^\mu_\mu-\frac{17}{256}h^\alpha_\alpha h_{\beta\gamma}h^{\beta\gamma}\partial_\sigma h^\nu_\nu\partial^\sigma h^\mu_\mu+\frac{25h^\alpha_\alpha h^\beta_\beta h^\gamma_\gamma\partial_\sigma h^\nu_\nu\partial^\sigma h^\mu_\mu}{1024} \\
& -\frac{1}{4}h^{\alpha\beta}h^{\gamma\mu}h^{\sigma\nu}\partial_\sigma h_{\alpha\gamma}\partial_\nu h_{\beta\mu}-\frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\nu h_{\mu\sigma}\partial^\nu h_{\beta\gamma}+\frac{5}{16}h^\alpha_\alpha h^{\beta\gamma}h^{\mu\sigma}\partial_\nu h_{\mu\sigma}\partial^\nu h_{\beta\gamma} \\
& -\frac{1}{4}h^\alpha_\alpha h^{\beta\gamma}h^{\mu\sigma}\partial_\sigma h_{\gamma\nu}\partial^\nu h_{\beta\mu}+\frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^{\mu\sigma}\partial_\nu h_{\gamma\sigma}\partial^\nu h_{\beta\mu}-\frac{3}{16}h^\alpha_\alpha h^{\beta\gamma}h^{\mu\sigma}\partial_\nu h_{\gamma\sigma}\partial^\nu h_{\beta\mu} \\
& -\frac{1}{4}h^\alpha_\alpha h^\mu_\beta h^{\beta\gamma}\partial_\mu h_{\sigma\nu}\partial^\nu h^\sigma_\gamma+\frac{1}{4}h^\alpha_\alpha h^\beta_\beta h^{\gamma\mu}\partial_\mu h_{\sigma\nu}\partial^\nu h^\sigma_\gamma-\frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^\mu_\beta\partial_\sigma h_{\mu\nu}\partial^\nu h^\sigma_\gamma \\
& +\frac{3}{8}h^\alpha_\alpha h^\mu_\beta h^{\beta\gamma}\partial_\sigma h_{\mu\nu}\partial^\nu h^\sigma_\gamma-\frac{1}{16}h^\alpha_\alpha h^\beta_\beta h^{\gamma\mu}\partial_\sigma h_{\mu\nu}\partial^\nu h^\sigma_\gamma+\frac{1}{2}h^\gamma_\alpha h^{\alpha\beta}h^\mu_\beta\partial_\nu h_{\mu\sigma}\partial^\nu h^\sigma_\gamma \\
& -\frac{3}{8}h^\alpha_\alpha h^\mu_\beta h^{\beta\gamma}\partial_\nu h_{\mu\sigma}\partial^\nu h^\sigma_\gamma+\frac{1}{16}h^\alpha_\alpha h^\beta_\beta h^{\gamma\mu}\partial_\nu h_{\mu\sigma}\partial^\nu h^\sigma_\gamma-\frac{1}{64}h^\alpha_\alpha h^\beta_\beta h^\gamma_\gamma\partial_\sigma h_{\mu\nu}\partial^\nu h^{\mu\sigma} \\
& +\frac{1}{128}h^\alpha_\alpha h^\beta_\beta h^\gamma_\gamma\partial_\nu h_{\mu\sigma}\partial^\nu h^{\mu\sigma} \qquad (3.2.12)
\end{aligned}
$$

At this point it is worth noticing that, on comparison with the corresponding expression for the gauge-fixed form for GR, the Lagrangians differ only by numerical coefficients in terms involving h; the index structure of the terms in the overall expression remain unchanged. This has important bearing on what follows.

Chapter 4

Propagators and Vertex Rules

4.1 Propagators

Before deriving expressions for the vertices for graviton scattering central to the computation of amplitudes in the theory, we first need to find the appropriate expressions for the graviton propagator which itself derives from the quadratic contribution to the perturbative Lagrangian. The quadratic terms in GR and UG differ only in the coefficient of the term containing factors of the trace $h = h^{\mu}_{\mu}$, so we expect that the computation of the propagator itself will be nearly identical. We will content ourselves with deriving this in GR, and then deducing the corresponding expression in UG. To this end, consider the gauge fixed expression for $\mathcal{L}_2$ from (3.1.6) which is of the form,

$$\mathcal{L}_2 = \frac{1}{2}\partial_\gamma h_{\alpha\beta} V^{\alpha\beta\mu\nu} \partial^\gamma h_{\mu\nu}, \tag{4.1.1}$$

with $V^{\alpha\beta\mu\nu} \equiv \frac{1}{4}\eta^{\alpha\beta}\eta^{\mu\nu} - \frac{1}{2}\eta^{\alpha\mu}\eta^{\beta\nu}$. Recognising that the the right hand side of this expression is symmetric with respect to $\alpha \leftrightarrow \beta$, $\mu \leftrightarrow \nu$ and $(\alpha\beta) \leftrightarrow (\mu\nu)$ allows us to trade the rank-2 tensor $h_{\mu\nu}$ for a vector Ψ_i where, since there are only ten independent combinations of $\alpha\beta$ and $\mu\nu$ we use the following translation between tensor and vector indices

$\alpha\beta, \mu\nu$	00	11	22	33	01	02	03	12	13	23
i,j	1	2	3	4	5	6	7	8	9	10

(4.1.2)

With this dictionary in place, the quadratic Lagrangian reads

$$\begin{aligned}\mathcal{L}_2 &= \frac{1}{2}\sum_{i=5}^{10} \partial_\mu \Psi^i \partial^\mu \Psi^i + \frac{1}{4}\sum_{i=1}^{4} \partial_\mu \Psi^i \partial^\mu \Psi^i - \frac{1}{8}\sum_{i=1}^{4} \partial_\mu \Psi^i \sum_{j=1}^{4} \partial^\mu \Psi^j \\ &\equiv \frac{1}{2}\sum_{i,j} \partial_\mu \Psi^i V_{ij} \partial^\mu \Psi^j,\end{aligned} \tag{4.1.3}$$

where the symmetric matrix

$$V_{ij} = \begin{cases} \delta_{ij} & \text{for} \quad i,j \geq 5 \\ \begin{pmatrix} 1/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 1/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 1/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 1/4 \end{pmatrix} & \text{for} \quad 1 \leq i,j \leq 4 \end{cases} \tag{4.1.4}$$

The propagator itself is computed by Fourier transforming to momentum space, which as usual transforms (4.1.3) into an algebraic equation in the momentum k^μ. The propagator then solves the (formal) matrix equation $k^2\mathbf{VP} = \mathbf{I}$ where the identity matrix is now symmetrised as above *i.e.* $\mathbf{P} = \frac{1}{k^2}\mathbf{V}^{-1}$. Inverting $\mathbf{V}$ is simple enough and gives,

$$(V^{-1})_{ij} = \begin{cases} \delta_{ij} & if \quad i,j \geq 5 \\ \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} & if \quad 1 \leq i,j \leq 4 \end{cases} \tag{4.1.5}$$

Then translating back to tensor indices with the same key, (4.1.2), we find that $(V^{-1})_{ij}$ corresponds to the matrix

$$\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}, \tag{4.1.6}$$

which, in turn gives the celebrated graviton propagator in GR,

$$P^{\mu_1\nu_1,\mu_2\nu_2}(k) = \frac{\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2} + \eta^{\mu_1\nu_2}\eta^{\nu_1\mu_2} - \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}}{k^2}. \tag{4.1.7}$$

Using the same translation between tensor and vector indices as in the propagator section of the GR case, the propagator for unimodular gravity can be derived in a similar way. The gauge fixed expression for the quadratic Lagrangian in (3.2.12) reads

$$\hat{\mathcal{L}}_2 \quad = \quad \frac{1}{2}\partial_\gamma h_{\alpha\beta}\, \hat{V}^{\alpha\beta\mu\nu}\, \partial^\gamma h_{\mu\nu}, \tag{4.1.8}$$

with $\hat{V}^{\alpha\beta\mu\nu} \equiv \dfrac{3}{16}\eta^{\alpha\beta}\eta^{\mu\nu} - \dfrac{1}{2}\eta^{\alpha\mu}\eta^{\beta\nu}$. Again, this can be put into the form,

$$\hat{\mathcal{L}}_2 = \frac{1}{2}\sum_{i,j} \partial_\mu \Psi^i \hat{V}_{ij} \partial^\mu \Psi^j, \tag{4.1.9}$$

where the symmetric matrix

$$\hat{V}_{ij} = \begin{cases} \delta_{ij} & \text{for} \quad i,j \geq 5 \\ \begin{pmatrix} 5/16 & -3/16 & -3/16 & -3/16 \\ -3/16 & 5/16 & -3/16 & -3/16 \\ -3/16 & -3/16 & 5/16 & -3/16 \\ -3/16 & -3/16 & -3/16 & 5/16 \end{pmatrix} & \text{for} \quad 1 \leq i,j \leq 4 \end{cases} \tag{4.1.10}$$

Inverting $\hat{\mathbf{V}}$ gives,

$$(\hat{V}^{-1})_{ij} = \begin{cases} \delta_{ij} & if \quad i,j \geq 5 \\ \begin{pmatrix} 1/2 & -3/2 & -3/2 & -3/2 \\ -3/2 & 1/2 & -3/2 & -3/2 \\ -3/2 & -3/2 & 1/2 & -3/2 \\ -3/2 & -3/2 & -3/2 & 1/2 \end{pmatrix} & if \quad 1 \leq i,j \leq 4 \end{cases} \tag{4.1.11}$$

and, translating back to rank-2 indices with (4.1.2), we find that $(\hat{V}^{-1})_{ij}$ corresponds to the matrix

$$\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \frac{3}{2}\eta^{\mu\nu}\eta^{\alpha\beta}, \tag{4.1.12}$$

which leads to the unimodular gravity propagator

$$\hat{P}^{\mu_1\nu_1,\mu_2\nu_2}(k) = \frac{\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2} + \eta^{\mu_1\nu_2}\eta^{\nu_1\mu_2} - \frac{3}{2}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}}{k^2}. \tag{4.1.13}$$

As alluded to earlier, the differences between the quadratic actions of GR and UG are such that the index structure of the propagators are the same with the only change coming in one of the coefficients in the numerator of (4.1.7).

4.2 Vertex Rules

Having now derived the perturbative Lagrangians for both GR and UG, found the corresponding propagators and set up the formalism to calculate the amplitudes it remains only to extract the Feynman rules for graviton scattering. We begin by assigning a particle number to each of the gravitons in a given expression *from left to right*. We also designate the left index of a particular graviton by μ_i and its right index by ν_i. Then, contracting (left) indices on two particles, say i and j, produces a factor $\eta^{\mu_i\mu_j}$ with similar factors for left-right, right-left and right-right contractions. Similarly, contracting a derivative of graviton i with another graviton, j, produces a factor $k_i^{\mu_j}$ while contracting indices on two derivatives gives a product of the momenta of the corresponding gravitons, $k_i \cdot k_j$.

As an example, applying the Feynman rules above to the following term encountered at cubic order in the perturbative GR Lagrangian,

$$h^{\alpha\gamma}\partial_\gamma h_{\beta\delta}\partial^\delta h^\beta_\alpha, \tag{4.2.14}$$

results in a cubic vertex factor

$$k_2^{\mu_1}k_3^{\nu_2}\eta^{\mu_2\nu_3}\eta^{\mu_1\mu_3}. \tag{4.2.15}$$

But we then permute the vertex rule through all the permutations of the external legs of the diagram, *i.e.* permute (k_i, μ_i, ν_i) through $i = 1, 2, 3$, keeping in mind the symmetry in $(\mu_i\nu_i)$. This particular term has six distinct permutations. To account for this, we introduce the notation P_k, where P permutes the particle labels among the external legs and k designates the number of distinct permutations. The complete rule for this example then reads

$$P_6\left(k_2^{\mu_1}k_3^{\nu_2}\eta^{\mu_2\nu_3}\eta^{\mu_1\mu_3}\right). \tag{4.2.16}$$

With this we can now extract the vertex rules of both GR and UG from the respective Lagrangians. Although this is not a complicated process with the length of the gauge fixed Lagrangians it does add up to a number of manipulations that are impractical to do by hand. The explicit vertex expressions in both GR and UG were extracted in Mathematica to be later used in the calculation of the amplitudes.

Chapter 5

GR Amplitudes

As we have mentioned before there are well established methods of calculating scattering amplitudes in general relativity. Most commonly we could have used either the KLT relations or the BCFW recursion relations. But as a precursor to calculating the scattering amplitudes in unimodular gravity we first review the techniques used in the framework of general relativity. This is work that has been done previously but we include it here as a means to establish the methods and conventions we use. Since we have already determined the propagator and the method by which the vertex rules can be extracted, we move on to calculate the three point, four point and five point amplitudes. This also serves as the basis on which the calculations in unimodular gravity are done with the bulk of the calculation shown in this section.[1]

5.1 Three Point Amplitude

We start with the three point amplitudes for GR, beginning with the graviton 3-vertex given by the Feynman diagram in Figure 1.

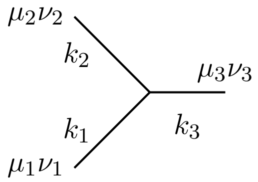

Figure1: 3-graviton vertex

Given the gauge fixed cubic Lagrangian $\mathcal{L}_3$ in GR, (3.1.6), we extract the following 3-vertex rule

$$\begin{aligned} V^{\mu_1\nu_1;\mu_2\nu_2;\mu_3\nu_3}(k_1,k_2,k_3) &= \bigg(\frac{1}{4}P_6\left(k_1\cdot k_2\eta^{\mu_1\nu_1}\eta^{\mu_2\mu_3}\eta^{\nu_2\nu_3}\right) - P_3\left(k_1\cdot k_2\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}\eta^{\nu_3\mu_1}\right) \\ &+ \frac{1}{4}P_3\left(k_1\cdot k_2\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\eta^{\mu_3\nu_3}\right) - \frac{1}{8}P_3\left(k_1\cdot k_2\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}\right) \\ &- \frac{1}{2}P_3\left(k_1^{\mu_3}k_2^{\nu_3}\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\right) + \frac{1}{2}P_6\left(k_1^{\mu_3}k_2^{\mu_1}\eta^{\nu_1\mu_2}\eta^{\nu_2\nu_3}\right)\bigg). \quad (5.1.1) \end{aligned}$$

With this in place, we can now calculate the amplitude by applying the same method as for the gluon 3-amplitude in section 2.3. Before deciding on the helicities of the particles it will be useful to first consider the special kinematics of the particles. Depending on the helicity structure, we will either choose $|i\rangle \propto |j\rangle$, which implies $\langle ij\rangle = 0$, or $|i] \propto |j]$, which implies $[ij] = 0$, for all particles i and j. Irrespective of our choice though, terms containing a dot product of momenta $k_i \cdot k_j$ will vanish

[1] The majority of calculations and manipulations done in this chapter, as well as the next, were done in Mathematica using the xAct and xTensor packages, which can be found at http://www.xact.es

since $k_i \cdot k_j = \langle i|\gamma^\mu|i]\langle j|\gamma_\mu|j] = \frac{1}{2}\langle ij\rangle[ij]$. This allows us to simplify the 3-vertex rule to

$$V^{\mu_1\nu_1;\mu_2\nu_2;\mu_3\nu_3}(k_1,k_2,k_3) = -\frac{1}{2}P_3\left(k_1^{\mu_3}k_2^{\nu_3}\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\right) + \frac{1}{2}P_6\left(k_1^{\mu_3}k_2^{\mu_1}\eta^{\nu_1\mu_2}\eta^{\nu_2\nu_3}\right).$$

To illustrate how this computation works in some detail, let's choose a specific case, say $M_3(k_1^-,k_2^-,k_3^+)$. For this amplitude, we choose the special kinematics $|i] \propto |j]$ for all particles i and j so that,

$$k_i^{\mu_3} = \epsilon^{\mu_3}k_i = -\frac{\langle q_3|\gamma^\mu|3]\langle i|\gamma_\mu|i]}{2\sqrt{2}\langle q_3 3\rangle} = -\frac{\langle q_3 i\rangle[3i]}{\sqrt{2}\langle q_3 3\rangle}, \tag{5.1.2}$$

and any term containing $k_i^{\mu_3}$ or $k_i^{\nu_3}$ for $i=1,2$ will necessarily vanish, further reducing the 3-vertex rule to

$$\begin{aligned} V^{\mu_1\nu_1;\mu_2\nu_2;\mu_3\nu_3}(k_1^-,k_2^-,k_3^+) &= -\frac{1}{2}\left(k_2^{\mu_1}k_3^{\nu_1}\eta^{\mu_2\mu_3}\eta^{\nu_2\nu_3} + k_3^{\mu_2}k_1^{\nu_2}\eta^{\mu_3\mu_1}\eta^{\nu_3\nu_1}\right) \\ &+ \frac{1}{2}\left(k_2^{\mu_1}k_3^{\mu_2}\eta^{\nu_2\mu_3}\eta^{\nu_3\nu_1} + k_1^{\mu_2}k_3^{\mu_1}\eta^{\nu_1\mu_3}\eta^{\nu_3\nu_2}\right). \end{aligned} \tag{5.1.3}$$

Now we rewrite the vertex rule in terms of spinor brackets using the conventions set out in section 2.3, starting with the decomposition into polarisation vectors so that, for example, $p_1^{\mu_2} = \epsilon_\mu(p_2)p_1^\mu$ and $\eta^{\mu_1\mu_2} = \epsilon^\mu(p_1)\epsilon_\mu(p_2)$, followed by the translation into spinor-helicity variables through,

$$\begin{aligned} p_1^\mu &= \frac{1}{2}\langle 1|\gamma_\mu|1], \\ \epsilon_\mu^+(p_1) &= -\frac{\langle q_1|\gamma_\mu|1]}{\sqrt{2}\langle q_1 1\rangle}, \\ \epsilon_\mu^-(p_1) &= -\frac{\langle 1|\gamma_\mu|q_1]}{\sqrt{2}[q_1 1]}, \end{aligned} \tag{5.1.4}$$

where here, and in what follows below, the q_i are arbitrary reference spinors which will not feature in the final expression for the amplitude. After this initial substitution we can then contract the associated angle and square brackets to give the amplitude in square- and angle-spinor brackets, using the relation

$$\langle i|\gamma_\mu|j]\langle k|\gamma^\mu|l] = 2\langle ik\rangle[jl].$$

This leads to the following expression for the three-point amplitude,

$$\begin{aligned} M_3(k_1^-,k_2^-,k_3^+) &= \frac{-1}{[q_1 1]^2[q_2 2]^2\langle q_3 3\rangle^2}\Big(\langle 21\rangle[q_2 1]\langle 23\rangle[q_2 3]\langle 1q_3\rangle^2[q_1 3]^2 \\ &+ \langle 12\rangle[q_1 2]\langle 13\rangle[q_1 3]\langle 2q_3\rangle^2[q_2 3]^2 - \langle 21\rangle[q_2 1]\langle 13\rangle[q_1 3]\langle 1q_3\rangle[q_1 3]\langle 2q_3\rangle[q_2 3] \\ &- \langle 12\rangle[q_1 2]\langle 23\rangle[q_2 3]\langle 1q_3\rangle[q_1 3]\langle 2q_3\rangle[q_2 3]\Big). \end{aligned} \tag{5.1.5}$$

This expression can then be simplified using the antisymmetry property of the spinor brackets along with conservation of momentum which, in spinor-helicity language reads $\sum_j \langle ij\rangle[jk] = 0$. This gives,

$$M_3(k_1^-,k_2^-,k_3^+) = \frac{\langle 12\rangle^4}{\langle q_3 3\rangle^2}\left(\left(\frac{\langle q_3 1\rangle}{\langle 13\rangle}\right)^2 - 2\frac{\langle q_3 1\rangle\langle q_3 2\rangle}{\langle 13\rangle\langle 23\rangle} + \left(\frac{\langle q_3 2\rangle}{\langle 23\rangle}\right)^2\right) \tag{5.1.6}$$

This can then be factorized and the Schouten identity, $\langle ij\rangle\langle kl\rangle + \langle ik\rangle\langle lj\rangle + \langle il\rangle\langle jk\rangle = 0$, can be used to simplify as follows

$$M_3(k_1^-,k_2^-,k_3^+) = \frac{\langle 12\rangle^4}{\langle q_3 3\rangle^2}\left(\frac{\langle q_3 1\rangle}{\langle 13\rangle} - \frac{\langle q_3 2\rangle}{\langle 23\rangle}\right)^2$$

$$
\begin{aligned}
&= \frac{\langle 12\rangle^4}{\langle q_3 3\rangle^2}\left(\frac{\langle q_3 1\rangle\langle 23\rangle + \langle q_3 2\rangle\langle 31\rangle}{\langle 13\rangle\langle 23\rangle}\right)^2 \\
&= \frac{\langle 12\rangle^4}{\langle q_3 3\rangle^2}\left(\frac{-\langle q_3 3\rangle\langle 12\rangle}{\langle 13\rangle\langle 23\rangle}\right)^2 \\
&= \frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 23\rangle^2}. \qquad (5.1.7)
\end{aligned}
$$

At this point, we note that, first, the q_i's have dropped out as promised and, second, the final result depends only on *angle* brackets without a square bracket in sight. This is, of course, a consequence of the 3-particle special kinematics. Finally, and more to the point, comparing this to the corresponding one in section 2.3 for the color-ordered 3-point gluon scattering amplitude in Yang-Mills theory,

$$
A_3\left[1^-, 2^-, 3^+\right] = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}, \qquad (5.1.8)
$$

indeed shows that

$$
M_3(k_1, k_2, k_3) = \left(A_3[k_1, k_2, k_3]\right)^2, \qquad (5.1.9)
$$

which is nothing but the celebrated KLT relation between GR and Yang-Mills theory at 3-points.

5.2 Four point Amplitude

Now let's consider four graviton scattering. As in the previous computation, we will focus on the detailed calculation of the 4-point amplitude in GR, identify the differences with UG and then deduce the associated amplitude in umimodular gravity. We will focus on the maximal helicity violating (MHV) amplitude where all but two of the gravitons have one helicity. At tree level, the complete amplitude receives contributions from four distinct diagrams that can be constructed for the choice of particles. These are the basic four graviton vertex, and the s, t and u channel respectively. As usual, we will take all momenta to be outgoing.

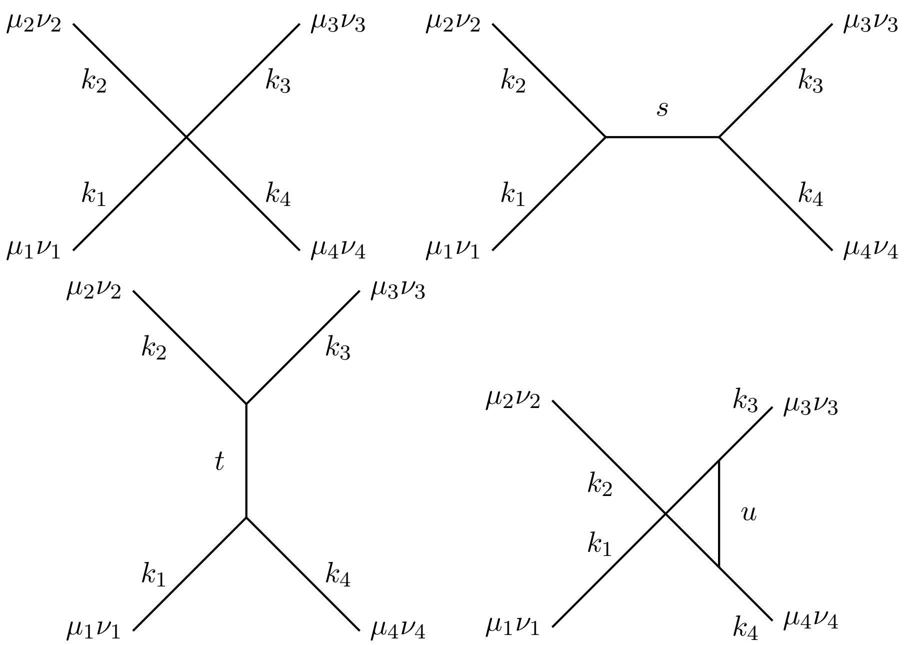

Figure 2: 4-graviton scattering diagrams

To compute the 4-point amplitude, we start with the gauge fixed quartic Lagrangian in the perturbation series(3.1.6). Following the same reasoning as we used in the 3-point amplitude computation, we can extract the four-vertex expression for GR.

$$
\begin{aligned}
& V^{\mu_1\nu_1;\mu_2\nu_2;\mu_3\nu_3;\mu_4\nu_4}(p_1,p_2,p_3,p_4) = \frac{1}{4}P_{24}(p_3\cdot p_4\eta^{\mu_1\mu_4}\eta^{\nu_1\mu_3}\eta^{\mu_2\nu_4}\eta^{\nu_2\nu_3}) \\
- & \frac{1}{4}P_{24}(p_3\cdot p_4\eta^{\mu_1\mu_4}\eta^{\nu_1\nu_4}\eta^{\mu_2\mu_3}\eta^{\nu_2\nu_3}) - \frac{1}{16}P_{24}(p_3\cdot p_4\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\eta^{\mu_3\mu_4}\eta^{\nu_3\nu_4}) \\
- & \frac{1}{32}P_{24}(p_3\cdot p_4\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\mu_4}\eta^{\nu_3\nu_4}) - \frac{1}{2}P_{24}(p_3^{\mu_1}p_4^{\mu_3}\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_4}\eta^{\nu_3\nu_4}) \\
+ & \frac{1}{2}P_{24}(p_3^{\mu_2}p_4^{\mu_3}\eta^{\mu_1\nu_1}\eta^{\nu_2\mu_4}\eta^{\nu_3\nu_4}) + \frac{1}{2}P_{24}(p_3\cdot p_4\eta^{\mu_1\mu_2}\eta^{\nu_1\mu_3}\eta^{\nu_2\mu_4}\eta^{\nu_3\nu_4}) \\
- & \frac{1}{4}P_{24}(p_3\cdot p_4\eta^{\mu_1\nu_1}\eta^{\mu_2\mu_4}\eta^{\nu_2\mu_3}\eta^{\mu_3\nu_4}) - \frac{1}{2}P_{24}(p_3^{\mu_2}p_4^{\mu_1}\eta^{\nu_1\mu_3}\eta^{\nu_2\mu_4}\eta^{\nu_3\nu_4}) \\
+ & \frac{1}{4}P_{24}(p_3^{\mu_1}p_4^{\mu_2}\eta^{\nu_1\nu_2}\eta^{\mu_3\mu_4}\eta^{\nu_3\nu_4}) - \frac{1}{8}P_{24}(p_3^{\mu_2}p_4^{\nu_2}\eta^{\mu_1\nu_1}\eta^{\mu_3\mu_4}\eta^{\nu_3\nu_4}) \\
- & \frac{1}{8}P_{24}(p_3^{\mu_1}p_4^{\mu_2}\eta^{\nu_1\nu_2}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_4}) + \frac{1}{2}P_{24}(p_3^{\mu_4}p_4^{\mu_2}\eta^{\mu_1\mu_3}\eta^{\nu_1\nu_3}\eta^{\nu_2\nu_4}) \\
+ & \frac{1}{8}P_{24}(p_3^{\mu_4}p_4^{\mu_3}\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\eta^{\nu_3\nu_4}) - \frac{1}{16}P_{24}(p_3^{\mu_4}p_4^{\mu_3}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\nu_3\nu_4}) \\
+ & \frac{1}{4}P_{24}(p_3^{\mu_4}p_4^{\mu_3}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\nu_2\nu_4}) - \frac{1}{4}P_{24}(p_3^{\mu_2}p_4^{\nu_2}\eta^{\mu_1\mu_3}\eta^{\nu_1\nu_3}\eta^{\mu_4\nu_4}) \\
+ & \frac{1}{4}P_{24}(p_3^{\mu_1}p_4^{\mu_3}\eta^{\nu_1\mu_2}\eta^{\nu_2\nu_3}\eta^{\mu_4\nu_4}) - \frac{1}{4}P_{24}(p_3^{\mu_2}p_4^{\mu_3}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_4\nu_4}) \\
+ & \frac{1}{4}P_{24}(p_3^{\mu_1}p_4^{\mu_2}\eta^{\nu_1\mu_3}\eta^{\nu_2\nu_3}\eta^{\mu_4\nu_4}) + \frac{1}{2}P_{24}(p_3^{\mu_2}p_4^{\nu_2}\eta^{\mu_1\mu_3}\eta^{\nu_1\mu_4}\eta^{\nu_3\nu_4}) \\
- & \frac{1}{2}P_{24}(p_3^{\mu_4}p_4^{\mu_2}\eta^{\mu_1\mu_3}\eta^{\nu_1\nu_4}\eta^{\nu_2\nu_3}) - \frac{1}{4}P_{24}(p_3\cdot p_4\eta^{\mu_1\mu_2}\eta^{\nu_1\mu_4}\eta^{\nu_2\nu_4}\eta^{\mu_3\nu_3}) \\
+ & \frac{1}{8}P_{24}(p_3\cdot p_4\eta^{\mu_1\nu_1}\eta^{\mu_2\mu_4}\eta^{\nu_2\nu_4}\eta^{\mu_3\nu_3}). \qquad (5.2.10)
\end{aligned}
$$

Unlike with the 3-point computation, this is not sufficient since there are also the s-, t- and u-channel diagrams that need to be evaluated. This is, however, easily taken care of with the contraction of two appropriate three point vertex rules. For example, for the s-channel diagram the appropriate vertex factor is given by

$$V^{\mu_1\nu_1;\mu_2\nu_2;\mu_s\nu_s}(p_1,p_2,p_s)V^{\mu_s\nu_s;\mu_3\nu_3;\mu_4\nu_4}(p_s,p_3,p_4), \qquad (5.2.11)$$

where the contraction between the two three-vertices is taken over the "particle" label s. Momentum conservation relates its momentum to the external particle momenta through $p_s = -p_1 - p_2 = p_3 + p_4$. The propagators of the two 3-vertices containing the internal graviton line act together as a place holder for the particle propagator of the theory ultimately sewing together the correct factors of the two three-point vertices. Take, for example, the term

$$(p_1^{\mu_s}p_2^{\nu_s}\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2})\,(p_s^{\mu_4}p_3^{\nu_4}\eta^{\mu_s\mu_3}\eta^{\nu_s\nu_3})\,. \qquad (5.2.12)$$

We first expand this explicitly (including the index structure) as,

$$\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}p_s^{\mu_4}p_3^{\nu_4}\,(p_1)_\mu\,(p_2)_\nu\,(\epsilon^{\mu_3})_\alpha\,(\epsilon^{\nu_3})_\beta\,(\epsilon^{\mu_s})^\mu\,(\epsilon^{\nu_s})^\nu\,(\epsilon^{\mu_s})^\alpha\,(\epsilon^{\nu_s})^\beta\,. \qquad (5.2.13)$$

Then replacing the internal momentum p_s with the appropriate representation in the external momenta p_1, p_2, p_3, p_4, and the factor $(\epsilon^{\mu_s})^\mu\,(\epsilon^{\nu_s})^\nu\,(\epsilon^{\mu_s})^\alpha\,(\epsilon^{\nu_s})^\beta$ with the particle propagator of the theory (4.1.7), in this case,

$$(\epsilon^{\mu_s})^\mu\,(\epsilon^{\nu_s})^\nu\,(\epsilon^{\mu_s})^\alpha\,(\epsilon^{\nu_s})^\beta = P^{\mu\nu\alpha\beta} = \frac{\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\eta^{\mu\nu}\eta^{\alpha\beta}}{s_{12}}, \qquad (5.2.14)$$

where $s_{ij} \equiv -(p_i+p_j)^2$. This process is repeated for all other terms in the s-channel amplitude as well as the t- and u-channels, with appropriate choice of momentum in the denominator [2].

[2] This result is of specific importance when considering even higher numbers of graviton scattering

Now on to the amplitude calculation. We choose the helicities to be $h_1 = h_2 = -2$ and $h_3 = h_4 = +2$ ensuring that we have a non-vanishing amplitude since it is MHV. By necessity we assign values to the arbitrary reference momenta for each of the external legs. One such choice, $q_1 = q_2 = p_4$ and $q_3 = q_4 = p_1$, will allow us to simplify the expressions we need to calculate to a more manageable size. It is needed to calculate the the factors that will survive this choice of reference momenta. Of all the possible contractions between external particles, the only non-vanishing factors are

$$\begin{aligned}
\eta^{\mu_2\mu_3} &= \frac{\langle 21\rangle[43]}{[42]\langle 13\rangle}, & p_2^{\mu_1} &= -\frac{\langle 12\rangle[42]}{\sqrt{2}[41]}, & p_3^{\mu_1} &= -\frac{\langle 13\rangle[43]}{\sqrt{2}[41]}, \\
p_1^{\mu_2} &= -\frac{\langle 21\rangle[41]}{\sqrt{2}[42]}, & p_3^{\mu_2} &= -\frac{\langle 23\rangle[43]}{\sqrt{2}[42]}, & p_2^{\mu_3} &= -\frac{\langle 13\rangle[32]}{\sqrt{2}\langle 13\rangle}, \\
p_4^{\mu_3} &= -\frac{\langle 14\rangle[34]}{\sqrt{2}\langle 13\rangle}, & p_2^{\mu_4} &= -\frac{\langle 12\rangle[42]}{\sqrt{2}\langle 14\rangle}, & p_3^{\mu_4} &= -\frac{\langle 13\rangle[43]}{\sqrt{2}\langle 14\rangle}.
\end{aligned} \tag{5.2.15}$$

Evidently with these choices, the explicit four-point vertex as well as the t-channel diagram both give no contribution to the 4-point amplitude while the remaining two diagrams are greatly simplified giving a final result of

$$\begin{aligned}
M_4(p_1^-, p_2^-, p_3^+, p_4^+) &= \frac{\langle 12\rangle^4\langle 13\rangle^2[24]^2[34]^4\left(\langle 12\rangle[12] + \langle 13\rangle[13]\right)}{\langle 14\rangle^2\langle 24\rangle^2[13]^2[14]^2\langle 12\rangle[12]\langle 13\rangle[13]} \\
&= \frac{\langle 12\rangle^7[12]}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2}.
\end{aligned} \tag{5.2.16}$$

On the other hand, the corresponding color-ordered tree-level 4-point MHV gluon scattering amplitude as determined in section 2.6(alternately see for example [14]), with the same helicity choice as above, is given by

$$A_4[1^-, 2^-, 3^+, 4^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}, \tag{5.2.17}$$

so that

$$A_4[1^-, 2^-, 3^+, 4^+]A_4[1^-, 2^-, 4^+, 3^+] = \frac{\langle 12\rangle^7[12]}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2}\ \frac{1}{\langle 12\rangle[12]}. \tag{5.2.18}$$

In other words, dropping the helicity labels on the scattering particles,

$$M_4^{tree}(1234) = -s_{12}A_4^{tree}[1234]\, A_4^{tree}[1243], \tag{5.2.19}$$

precisely as expected for the KLT relations at 4-points.

5.3 Five point amplitude

Lastly for this work we consider the five point graviton scattering amplitude. This as one can well imagine is much more tedious exercise in index manipulation than that of the previous amplitudes. For brevity a condensed version of the calculation will be shown.

As with the previous amplitudes we choose the helicities to be MHV, $h_1 = h_2 = -2$ and $h_3 = h_4 = h_5 = +2$, ensuring a non-vanishing amplitude. By necessity we assign acceptable values to the reference spinors in each of the polarisation vectors, $q_1 = q_2 = p_5$ and $q_3 = q_4 = q_5 = p_1$, this greatly reduces the number of non-vanishing contractions of polarisation vectors. The non-vanishing contributions are

$$\eta^{\mu_2\mu_3} = \frac{\langle 21\rangle[53]}{[52]\langle 13\rangle}, \qquad \eta^{\mu_2\mu_4} = \frac{\langle 21\rangle[54]}{[52]\langle 14\rangle}, \qquad p_2^{\mu_1} = -\frac{\langle 12\rangle[52]}{\sqrt{2}[51]},$$

$$
\begin{aligned}
p_3^{\mu_1} &= -\frac{\langle 13\rangle[53]}{\sqrt{2}[51]}, & p_1^{\mu_2} &= -\frac{\langle 21\rangle[51]}{\sqrt{2}[52]}, & p_3^{\mu_2} &= -\frac{\langle 23\rangle[53]}{\sqrt{2}[42]},\\
p_2^{\mu_3} &= -\frac{\langle 13\rangle[32]}{\sqrt{2}\langle 13\rangle}, & p_4^{\mu_3} &= -\frac{\langle 14\rangle[34]}{\sqrt{2}\langle 13\rangle}, & p_2^{\mu_4} &= -\frac{\langle 12\rangle[42]}{\sqrt{2}\langle 14\rangle},\\
p_3^{\mu_4} &= -\frac{\langle 13\rangle[43]}{\sqrt{2}\langle 14\rangle}. & p_2^{\mu_5} &= -\frac{\langle 12\rangle[52]}{\sqrt{2}\langle 15\rangle}, & p_3^{\mu_5} &= -\frac{\langle 13\rangle[53]}{\sqrt{2}\langle 15\rangle},\\
p_4^{\mu_1} &= -\frac{\langle 14\rangle[54]}{\sqrt{2}[51]}, & p_4^{\mu_2} &= -\frac{\langle 24\rangle[54]}{\sqrt{2}[52]}, & p_4^{\mu_5} &= -\frac{\langle 14\rangle[54]}{\sqrt{2}\langle 15\rangle},\\
p_5^{\mu_3} &= -\frac{\langle 15\rangle[35]}{\sqrt{2}\langle 13\rangle}, & p_5^{\mu_4} &= -\frac{\langle 14\rangle[54]}{\sqrt{2}\langle 15\rangle}.
\end{aligned}
\tag{5.3.20}
$$

Now that we know which factor in the vertex expressions survive, and letting all other contraction vanish in our program, we can analyse the graphs needed to calculate the amplitude. There are three distinct sets of graphs needed to in a five point amplitude, given in Figure 3 below.

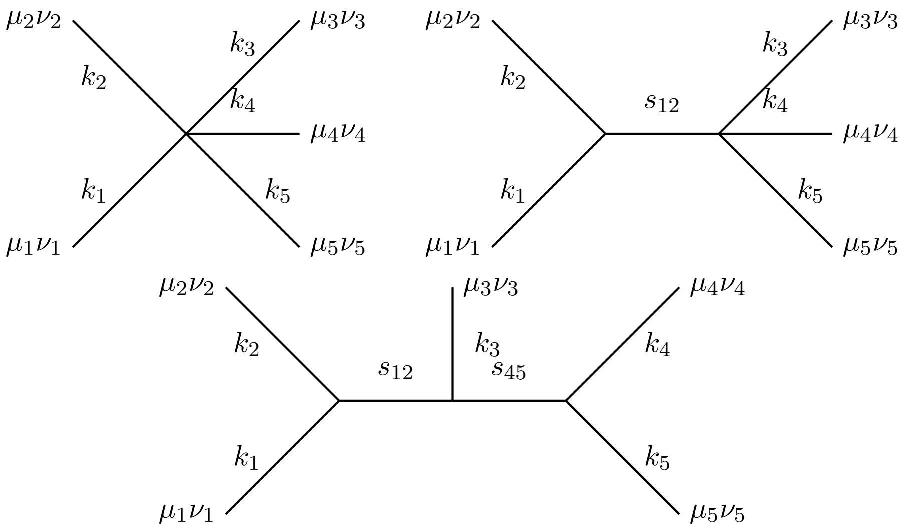

Figure 3: 5-graviton scattering diagrams

One should also take into consideration all unique permutations of the external legs. This comes to a total of 26 distinct diagrams.

We follow the same contraction procedure as in the four point amplitude to connect the various vertices. As a quick example we now consider one of each of the different types of diagrams. Firstly the five vertex expression is

$$
\begin{aligned}
&V^{\mu_1\nu_1;\mu_2\nu_2;\mu_3\nu_3;\mu_4\nu_4;\mu_5\nu_5}(p_1,p_2,p_3,p_4,p_5) =\\
&\frac{1}{4}(\eta^{\mu_3\mu_5})^2 p_2^{\mu_4} p_3^{\nu_4}(\eta^{\mu_1\mu_2})^2 - \frac{1}{2}\eta^{\mu_3\mu_4}\eta^{\mu_3\mu_5} p_2^{\mu_4} p_3^{\nu_5}(\eta^{\mu_1\mu_2})^2\\
&+\frac{1}{2}p_2p_3\eta^{\mu_1\mu_4}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_4}\eta^{\mu_3\mu_5}\eta^{\mu_1\mu_2} + \frac{1}{2}\eta^{\mu_1\mu_3}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_4}p_2^{\mu_4}p_3^{\nu_5}\eta^{\mu_1\mu_2}\\
&+\frac{1}{4}\eta^{\mu_1\mu_4}\eta^{\mu_2\nu_4}\eta^{\mu_3\nu_3}p_2^{\mu_5}p_3^{\nu_5}\eta^{\mu_1\mu_2} - \frac{1}{4}\eta^{\mu_1\mu_3}\eta^{\mu_2\nu_4}\eta^{\mu_3\mu_4}p_2^{\mu_5}p_3^{\nu_5}\eta^{\mu_1\mu_2}\\
&-\frac{1}{8}p_2p_3\eta^{\mu_1\nu_1}(\eta^{\mu_2\nu_5})^2(\eta^{\mu_3\mu_4})^2 - \frac{1}{16}p_2p_3(\eta^{\mu_1\mu_4})^2\eta^{\mu_2\nu_2}(\eta^{\mu_3\mu_5})^2\\
&-\frac{1}{32}p_2p_3\eta^{\mu_1\nu_1}(\eta^{\mu_2\nu_3})^2(\eta^{\mu_4\nu_5})^2 + \frac{1}{2}p_2p_3\eta^{\mu_1\mu_3}\eta^{\mu_1\mu_4}(\eta^{\mu_2\nu_5})^2\eta^{\mu_3\mu_4}\\
&+\frac{1}{8}p_2p_3(\eta^{\mu_1\mu_4})^2\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_5} + \frac{1}{8}p_2p_3\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_4}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_4}\eta^{\mu_3\mu_5}\\
&+\frac{1}{32}p_2p_3\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}(\eta^{\mu_3\mu_5})^2\eta^{\mu_4\nu_4} - \frac{1}{16}p_2p_3\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_4}\\
&+\frac{1}{24}p_2p_3\eta^{\mu_1\mu_4}\eta^{\mu_1\mu_5}(\eta^{\mu_2\nu_3})^2\eta^{\mu_4\nu_5} + \frac{1}{2}p_2p_3\eta^{\mu_1\mu_3}\eta^{\mu_1\mu_4}\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_5}\eta^{\mu_4\nu_5}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{5}{24}p_2p_3\eta^{\mu_1\mu_4}\eta^{\mu_1\mu_5}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_5}+\frac{1}{4}p_2p_3\eta^{\mu_1\mu_3}\eta^{\mu_1\mu_4}\eta^{\mu_2\nu_2}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_5}\\
&+\frac{1}{4}p_2p_3\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_4}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_5}-\frac{3}{4}p_2p_3\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\mu_4}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_5}\\
&+\frac{1}{192}p_2p_3\eta^{\mu_1\nu_1}(\eta^{\mu_2\nu_3})^2\eta^{\mu_4\nu_4}\eta^{\mu_5\nu_5}+\frac{1}{96}p_2p_3\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_4}\eta^{\mu_5\nu_5}\\
&+\frac{1}{4}\eta^{\mu_1\mu_4}(\eta^{\mu_2\nu_5})^2\eta^{\mu_3\nu_3}p_2^{\mu_4}p_3^{\nu_1}-\frac{1}{2}\eta^{\mu_1\mu_4}\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_5}p_2^{\mu_4}p_3^{\nu_1}\\
&-\frac{1}{2}\eta^{\mu_1\mu_4}\eta^{\mu_2\nu_4}(\eta^{\mu_3\mu_5})^2p_2^{\mu_1}p_3^{\nu_2}+\frac{1}{16}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}(\eta^{\mu_4\nu_5})^2p_2^{\mu_3}p_3^{\nu_2}\\
&-\frac{1}{8}(\eta^{\mu_1\mu_4})^2\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_5}p_2^{\mu_3}p_3^{\nu_2}+\frac{1}{16}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_4}p_2^{\mu_3}p_3^{\nu_2}\\
&-\frac{1}{12}\eta^{\mu_1\mu_4}\eta^{\mu_1\mu_5}\eta^{\mu_2\nu_3}\eta^{\mu_4\nu_5}p_2^{\mu_3}p_3^{\nu_2}-\frac{1}{2}\eta^{\mu_1\mu_3}\eta^{\mu_1\mu_4}\eta^{\mu_2\nu_5}\eta^{\mu_4\nu_5}p_2^{\mu_3}p_3^{\nu_2}\\
&-\frac{1}{4}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_4}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_5}p_2^{\mu_3}p_3^{\nu_2}-\frac{1}{96}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_4\nu_4}\eta^{\mu_5\nu_5}p_2^{\mu_3}p_3^{\nu_2}\\
&+\frac{1}{4}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_5}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_5}p_2^{\mu_4}p_3^{\nu_2}-\frac{1}{2}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_5}p_2^{\mu_4}p_3^{\nu_2}\\
&+\frac{1}{4}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_5}(\eta^{\mu_3\mu_4})^2p_2^{\mu_5}p_3^{\nu_2}+\frac{1}{8}(\eta^{\mu_1\mu_4})^2\eta^{\mu_2\nu_5}\eta^{\mu_3\nu_3}p_2^{\mu_5}p_3^{\nu_2}\\
&-\frac{1}{2}\eta^{\mu_1\mu_3}\eta^{\mu_1\mu_4}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_4}p_2^{\mu_5}p_3^{\nu_2}-\frac{1}{4}(\eta^{\mu_1\mu_4})^2\eta^{\mu_2\nu_3}\eta^{\mu_3\mu_5}p_2^{\mu_5}p_3^{\nu_2}\\
&-\frac{1}{2}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_4}\eta^{\mu_3\mu_4}\eta^{\mu_3\mu_5}p_2^{\mu_5}p_3^{\nu_2}-\frac{1}{16}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_5}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_4}p_2^{\mu_5}p_3^{\nu_2}\\
&+\frac{1}{8}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_4}p_2^{\mu_5}p_3^{\nu_2}+\frac{1}{8}\eta^{\mu_1\nu_1}(\eta^{\mu_2\nu_3})^2\eta^{\mu_4\nu_5}p_2^{\mu_5}p_3^{\nu_4}\\
&+\eta^{\mu_1\mu_4}\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_4}p_2^{\mu_1}p_3^{\nu_5}-\frac{1}{4}\eta^{\mu_1\mu_4}(\eta^{\mu_2\nu_3})^2\eta^{\mu_4\nu_5}p_2^{\mu_1}p_3^{\nu_5}\\
&+\frac{1}{4}\eta^{\mu_1\mu_4}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_5}p_2^{\mu_1}p_3^{\nu_5}+\frac{1}{4}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_4}\eta^{\mu_2\nu_5}\eta^{\mu_3\nu_3}p_2^{\mu_4}p_3^{\nu_5}\\
&-\frac{1}{4}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_4}p_2^{\mu_4}p_3^{\nu_5}+\frac{1}{16}(\eta^{\mu_1\mu_4})^2(\eta^{\mu_2\nu_3})^2p_2^{\mu_5}p_3^{\nu_5}\\
&-\frac{1}{2}\eta^{\mu_1\mu_3}\eta^{\mu_1\mu_4}\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_4}p_2^{\mu_5}p_3^{\nu_5}-\frac{1}{8}\eta^{\mu_1\nu_1}(\eta^{\mu_2\nu_4})^2\eta^{\mu_3\nu_3}p_2^{\mu_5}p_3^{\nu_5}\\
&+\frac{1}{4}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_4}\eta^{\mu_3\mu_4}p_2^{\mu_5}p_3^{\nu_5}-\frac{1}{32}\eta^{\mu_1\nu_1}(\eta^{\mu_2\nu_3})^2\eta^{\mu_4\nu_4}p_2^{\mu_5}p_3^{\nu_5}.
\end{aligned}
$$

The structure of these terms ensure that all the terms vanish when the vanishing factors, those not mentioned in (5.3.20), are taken into account. Next we consider the structure of the diagram with one propagator, i.e. the contraction of a four-vertex with a three-vertex. The structure of this diagram is set up as

$$
V^{\mu_1\nu_1;\mu_2\nu_2;\mu_3\nu_3;\mu_s\nu_s}(p_1,p_2,p_3,p_s)V^{\mu_s\nu_s;\mu_4\nu_4;\mu_5\nu_5}(p_s,p_4,p_5), \tag{5.3.21}
$$

where the sum runs over the s index. Taking a term this produces at random we have the following structure

$$
\left(p_1^{\mu_s}p_2^{\nu_s}\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\eta^{\mu_3\nu_3}\right)\left(p_s^{\mu_5}p_4^{\nu_5}\eta^{\mu_s\mu_4}\eta^{\nu_s\nu_5}\right). \tag{5.3.22}
$$

We first expand this explicitly (including the index structure) as,

$$
\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\eta^{\mu_3\nu_3}p_s^{\mu_5}p_4^{\nu_5}\,(p_1)_\mu\,(p_2)_\nu\,(\epsilon^{\mu_4})_\alpha\,(\epsilon^{\nu_4})_\beta\,(\epsilon^{\mu_s})^\mu\,(\epsilon^{\nu_s})^\nu\,(\epsilon^{\mu_s})^\alpha\,(\epsilon^{\nu_s})^\beta\,. \tag{5.3.23}
$$

Then replacing the internal momentum p_s with the appropriate representation in the external momenta p_1,p_2,p_3,p_4, in this case $p_s=-p_1-p_2-p_3$ and the factor $(\epsilon^{\mu_s})^\mu\,(\epsilon^{\nu_s})^\nu\,(\epsilon^{\mu_s})^\alpha\,(\epsilon^{\nu_s})^\beta$ with the particle propagator of the theory (4.1.7), in this case,

$$
(\epsilon^{\mu_s})^\mu\,(\epsilon^{\nu_s})^\nu\,(\epsilon^{\mu_s})^\alpha\,(\epsilon^{\nu_s})^\beta=P^{\mu\nu\alpha\beta}=\frac{\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\eta^{\mu\nu}\eta^{\alpha\beta}}{s_{123}}, \tag{5.3.24}
$$

where $s_{ijk} \equiv -(p_i+p_j+p_k)^2$. As mentioned in the four point case this is crucial when considering higher point scattering. The painful part of this is that this contraction of lower order vertices produces a total of 13608 terms per graph, before general simplification. Luckily it is a simple matter to automate the simplification process, and due to the choices of the reference momenta we find that none of these terms offer a contribution to the amplitude in question.

This now only leaves the 15 diagrams with two propagators, for which the contraction will be

$$V^{\mu_1\nu_1;\mu_2\nu_2;\mu_{s_{12}}\nu_{s_{12}}}(p_1,p_2,p_{s_{12}})V^{\mu_{s_{12}}\nu_{s_{12}};\mu_3\nu_3;\mu_{s_{45}}\nu_{s_{45}}}(p_{s_{12}},p_3,p_{s_{45}})V^{\mu_{s_{45}}\nu_{s_{45}};\mu_4\nu_4;\mu_5\nu_5}(p_{s_{45}},p_3,p_4), \tag{5.3.25}$$

where we contract over the s_{ij} indices. Once again taking a term this produces at random we have

$$\left(p_1^{\mu_{s_{12}}}p_2^{\nu_{s_{12}}}\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\right)\left(p_{s_{12}}^{\mu_{s_{45}}}p_3^{\nu_{s_{45}}}\eta^{\mu_{s_{12}}\mu_3}\eta^{\nu_{s_{12}}\nu_3}\right)\left(p_{s_{45}}^{\mu_5}p_4^{\nu_5}\eta^{\mu_{s_{45}}\mu_4}\eta^{\nu_{s_{45}}\nu_4}\right). \tag{5.3.26}$$

Expanding this in the same manner as above yields

$$\begin{aligned}\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}p_4^{\nu_5}p_{s_{45}}^{\mu_5}(p_1)_\mu(p_2)_\nu(p_{s_{12}})^\alpha(p_3)^\beta(\epsilon_3)_\gamma(\epsilon_3)_\sigma(\epsilon_4)^\rho(\epsilon_4)^\lambda \\ \times(\epsilon_{s_{12}})^\mu(\epsilon_{s_{12}})^\nu(\epsilon_{s_{12}})^\gamma(\epsilon_{s_{12}})^\sigma(\epsilon_{s_{45}})_\alpha(\epsilon_{s_{45}})_\beta(\epsilon_{s_{45}})_\lambda(\epsilon_{s_{45}})_\rho.\end{aligned} \tag{5.3.27}$$

We then substitute the appropriate propagators for $(\epsilon_{s_{12}})^\mu(\epsilon_{s_{12}})^\nu(\epsilon_{s_{12}})^\gamma(\epsilon_{s_{12}})^\sigma$ and $(\epsilon_{s_{45}})_\alpha(\epsilon_{s_{45}})_\beta(\epsilon_{s_{45}})_\lambda(\epsilon_{s_{45}})_\rho$, as in the previous case, and the appropriate substitutions for internal momenta, i.e. $p_{s_{12}} = -p_1 - p_2$ and $p_{s_{45}} = -p_4 - p_5$. In general this multiplication of vertices produces a total of 78732 terms for each diagram. But once converting to spinor helicity variables and simplifying the amplitude reduces considerably to a mere 140 terms. After simplifying this using the various identities established in the spinor helicity formalism up to this point and conservation of momentum this produces the amplitude

$$M_5[1^-2^-3^+4^+5^+] = \frac{\langle 12\rangle^7(\langle 24\rangle\langle 35\rangle[23][45] - \langle 23\rangle\langle 45\rangle[24][35])}{\langle 13\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 24\rangle\langle 25\rangle\langle 34\rangle\langle 35\rangle\langle 45\rangle}, \tag{5.3.28}$$

which, when compared to the gauge theory side of the KLT relation as calculated in section 2.6, confirms the 5-point KLT relation.

Chapter 6

UG Amplitudes

Now that the techniques used in calculating the amplitudes are established we move on to determine the amplitudes in unimodular gravity. The brute force calculation is also a simple matter of adapting the programs we wrote for the GR case to that of UG, which we did as a check on the calculation. For this section we use nearly identical methods to the GR case. The main method though depends on the similarities of the GR (3.1.6) and UG (3.2.12) Lagrangians. Since we have already done the bulk of the necessary calculations in the GR section, the focus here is to determine the differences that UG show with regard to GR and from there to calculate the amplitudes in UG.

6.1 Three Point Amplitude

Now we can apply the methods established in the GR section to the perturbative unimodular Lagrangian, (3.2.12). Extracting the 3-vertex rule from the Lagrangian gives in this case,

$$\begin{aligned}\hat{V}^{\mu_1\nu_1;\mu_2\nu_2;\mu_3\nu_3}(p_1,p_2,p_3) &= \frac{1}{8}P_6\left(p_1\cdot p_2\eta^{\mu_1\nu_1}\eta^{\mu_2\mu_3}\eta^{\nu_2\nu_3}\right) - P_3\left(p_1\cdot p_2\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}\eta^{\nu_3\mu_1}\right)\\ &+ \frac{3}{8}P_3\left(p_1\cdot p_2\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\eta^{\mu_3\nu_3}\right) - \frac{5}{64}P_3\left(p_1\cdot p_2\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}\right)\\ &- \frac{1}{2}P_3\left(p_1^{\mu_3}p_2^{\nu_3}\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\right) + \frac{1}{2}P_6\left(p_1^{\mu_3}p_2^{\mu_1}\eta^{\nu_1\mu_2}\eta^{\nu_2\nu_3}\right)\\ &+ \frac{1}{16}P_3\left(p_1^{\mu_3}p_2^{\nu_3}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\right) - \frac{1}{8}P_6\left(p_1^{\mu_2}p_2^{\mu_3}\eta^{\mu_1\nu_1}\eta^{\nu_2\nu_3}\right). \qquad (6.1.1)\end{aligned}$$

As in the case of GR, the 3-particle special kinematics kills off any term with a momentum dot product, leaving us with

$$\begin{aligned}\hat{V}^{\mu_1\nu_1;\mu_2\nu_2;\mu_3\nu_3}(p_1,p_2,p_3) &= -\frac{1}{2}P_3\left(p_1^{\mu_3}p_2^{\nu_3}\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\right) + \frac{1}{2}P_6\left(p_1^{\mu_3}p_2^{\mu_1}\eta^{\nu_1\mu_2}\eta^{\nu_2\nu_3}\right)\\ &+\frac{1}{16}P_3\left(p_1^{\mu_3}p_2^{\nu_3}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\right) - \frac{1}{8}P_6\left(p_1^{\mu_2}p_2^{\mu_3}\eta^{\mu_1\nu_1}\eta^{\nu_2\nu_3}\right).\end{aligned}$$

We can now choose the same helicities as in the GR calculation, i.e. $M_3(p_1^-,p_2^-,p_3^+)$, which forces the 3-particle special kinematics to be $|i] \propto |j]$ for all particles i and j, eliminating all terms containing $p_i^{\mu_3}$ or $p_i^{\nu_3}$ for $i=1,2$ due to the antisymmetry of the square- and angle-spinor brackets. At this point, we deviate from the GR computation, noticing that any trace of the positive helicity particle will also vanish, getting rid of any terms containing $\eta^{\mu_3\nu_3}$, thereby reducing the 3-vertex rule to

$$\begin{aligned}\hat{V}^{\mu_1\nu_1;\mu_2\nu_2;\mu_3\nu_3}(p_1^-,p_2^-,p_3^+) &= -\tfrac{1}{2}\left(p_2^{\mu_1}p_3^{\nu_1}\eta^{\mu_2\mu_3}\eta^{\nu_2\nu_3} + p_3^{\mu_2}p_1^{\nu_2}\eta^{\mu_3\mu_1}\eta^{\nu_3\nu_1}\right)\\ &+\tfrac{1}{2}\left(p_2^{\mu_1}p_3^{\mu_2}\eta^{\nu_2\mu_3}\eta^{\nu_3\nu_1} + p_1^{\mu_2}p_3^{\mu_1}\eta^{\nu_1\mu_3}\eta^{\nu_3\nu_2}\right), \quad (6.1.2)\end{aligned}$$

which is, of course, equivalent to the rule for the 3-vertex in GR. Since we are considering the same external states as in the GR case, we can follow the same substitution rules when converting to spinor variables. This, along with the fact that the vertex expressions are equivalent, yields,

$$\hat{M}_3(p_1^-, p_2^-, p_3^+) = \frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 23\rangle^2}, \tag{6.1.3}$$

and confirms the KLT relations to 3-points in Unimodular Gravity.

6.2 Four Point Amplitude

To compute the four-point amplitude in unimodular gravity we *could* follow the same procedure the GR case, noticing that the 4-vertex given by,

$$\begin{aligned}
& V^{\mu_1\nu_1;\mu_2\nu_2;\mu_3\nu_3;\mu_4\nu_4}(p_1,p_2,p_3,p_4) = \\
& \frac{1}{4}P_{24}(p_3\cdot p_4\eta^{\mu_1\mu_4}\eta^{\nu_1\mu_3}\eta^{\mu_2\nu_4}\eta^{\nu_2\nu_3}) - \frac{1}{4}P_{24}(p_3\cdot p_4\eta^{\mu_1\mu_4}\eta^{\nu_1\nu_4}\eta^{\mu_2\mu_3}\eta^{\nu_2\nu_3}) \\
- & \frac{1}{16}P_{24}(p_3\cdot p_4\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\eta^{\mu_3\mu_4}\eta^{\nu_3\nu_4}) + \frac{7}{64}P_{24}(p_3\cdot p_4\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\mu_4}\eta^{\nu_3\nu_4}) \\
- & \frac{1}{2}P_{24}(p_3^{\mu_1}p_4^{\mu_3}\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_4}\eta^{\nu_3\nu_4}) + \frac{5}{8}P_{24}(p_3^{\mu_2}p_4^{\mu_3}\eta^{\mu_1\nu_1}\eta^{\nu_2\mu_4}\eta^{\nu_3\nu_4}) \\
+ & \frac{1}{2}P_{24}(p_3\cdot p_4\eta^{\mu_1\mu_2}\eta^{\nu_1\mu_3}\eta^{\nu_2\mu_4}\eta^{\nu_3\nu_4}) - \frac{1}{2}P_{24}(p_3\cdot p_4\eta^{\mu_1\nu_1}\eta^{\mu_2\mu_4}\eta^{\nu_2\mu_3}\eta^{\mu_3\nu_4}) \\
- & \frac{1}{2}P_{24}(p_3^{\mu_2}p_4^{\mu_1}\eta^{\nu_1\mu_3}\eta^{\nu_2\mu_4}\eta^{\nu_3\nu_4}) + \frac{1}{4}P_{24}(p_3^{\mu_1}p_4^{\mu_2}\eta^{\nu_1\nu_2}\eta^{\mu_3\mu_4}\eta^{\nu_3\nu_4}) - \frac{1}{4}P_{24}(p_3^{\mu_2}p_4^{\nu_2}\eta^{\mu_1\nu_1}\eta^{\mu_3\mu_4}\eta^{\nu_3\nu_4}) \\
- & \frac{1}{16}P_{24}(p_3^{\mu_1}p_4^{\mu_2}\eta^{\nu_1\nu_2}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_4}) + \frac{1}{2}P_{24}(p_3^{\mu_4}p_4^{\mu_2}\eta^{\mu_1\mu_3}\eta^{\nu_1\nu_3}\eta^{\nu_2\nu_4}) \\
+ & \frac{1}{8}P_{24}(p_3^{\mu_4}p_4^{\mu_3}\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\eta^{\nu_3\nu_4}) - \frac{1}{8}P_{24}(p_3^{\mu_4}p_4^{\mu_3}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\nu_3\nu_4}) \\
+ & \frac{1}{8}P_{24}(p_3^{\mu_4}p_4^{\mu_3}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\nu_2\nu_4}) - \frac{1}{4}P_{24}(p_3^{\mu_2}p_4^{\nu_2}\eta^{\mu_1\mu_3}\eta^{\nu_1\nu_3}\eta^{\mu_4\nu_4}) \\
+ & \frac{1}{8}P_{24}(p_3^{\mu_1}p_4^{\mu_3}\eta^{\nu_1\mu_2}\eta^{\nu_2\nu_3}\eta^{\mu_4\nu_4}) - \frac{1}{4}P_{24}(p_3^{\mu_2}p_4^{\mu_3}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_4\nu_4}) \\
+ & \frac{1}{4}P_{24}(p_3^{\mu_1}p_4^{\mu_2}\eta^{\nu_1\mu_3}\eta^{\nu_2\nu_3}\eta^{\mu_4\nu_4}) + \frac{1}{2}P_{24}(p_3^{\mu_2}p_4^{\nu_2}\eta^{\mu_1\mu_3}\eta^{\nu_1\mu_4}\eta^{\nu_3\nu_4}) \\
- & \frac{1}{2}P_{24}(p_3^{\mu_4}p_4^{\mu_2}\eta^{\mu_1\mu_3}\eta^{\nu_1\nu_4}\eta^{\nu_2\nu_3}) - \frac{1}{4}P_{24}(p_3\cdot p_4\eta^{\mu_1\mu_2}\eta^{\nu_1\mu_4}\eta^{\nu_2\nu_4}\eta^{\mu_3\nu_3}) \\
+ & \frac{1}{4}P_{24}(p_3\cdot p_4\eta^{\mu_1\nu_1}\eta^{\mu_2\mu_4}\eta^{\nu_2\nu_4}\eta^{\mu_3\nu_3}) + \frac{1}{16}P_{24}(p_3^{\mu_2}p_4^{\nu_2}\eta^{\mu_1\nu_1}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_4}) \\
+ & \frac{1}{128}P_{24}(p_3\cdot p_4\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_4}) - \frac{13}{512}P_{24}(p_3\cdot p_4\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_4}),
\end{aligned} \tag{6.2.4}$$

is similar to the GR expression but with different constant coefficients *etc.* However, recalling that, for the MHV 4-graviton scattering with our choice of reference momenta, only the s- and u-channel diagrams contributed and these in turn were constructed by sewing together 3-point amplitudes which we've already determined to be the same in UG and GR, we deduce that the 4-point tree-level MHV amplitude in unimodular gravity must be

$$\begin{aligned}
\hat{M}_4^{tree}(p_1^-, p_2^-, p_3^+, p_4^+) &= \frac{\langle 12\rangle^7[12]}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2} \\
&= -s_{12}A_4^{tree}[1234]\,A_4^{tree}[1243],
\end{aligned} \tag{6.2.5}$$

and the KLT relations hold. It is interesting to note that in the case of gauge theory amplitudes, when considering the color-stripped 4-point amplitude, the diagrams are restricted to those that have no crossing legs, i.e. the u-channel diagram is not included. Also, when the amplitude is calculated explicitly one finds that the t-channel diagram offers no contribution.

6.3 Five point amplitude

Finally for this work we calculate the five point amplitude in unimodular gravity. We proceed in a similar way as before. The five point vertex expression is,

$$
\begin{aligned}
&V^{\mu_1\nu_1;\mu_2\nu_2;\mu_3\nu_3;\mu_4\nu_4,\mu_5\nu_5}(p_1,p_2,p_3,p_4,p_5) = \\
&\frac{1}{8}p_2p_3\eta^{\mu_2\nu_2}(\eta^{\mu_3\mu_5})^2(\eta^{\mu_1\mu_4})^2 - \frac{3}{64}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}p_2^{\mu_5}p_3^{\nu_5}(\eta^{\mu_1\mu_4})^2 \\
&-\frac{1}{2}p_2p_3\eta^{\mu_1\mu_3}(\eta^{\mu_2\nu_5})^2\eta^{\mu_3\mu_4}\eta^{\mu_1\mu_4} + \frac{1}{2}p_2p_3\eta^{\mu_1\mu_2}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_4}\eta^{\mu_3\mu_5}\eta^{\mu_1\mu_4} \\
&+\frac{1}{2}p_2p_3\eta^{\mu_1\mu_3}\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_5}\eta^{\mu_4\nu_5}\eta^{\mu_1\mu_4} + \frac{1}{16}p_2p_3\eta^{\mu_1\mu_5}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_5}\eta^{\mu_1\mu_4} \\
&-\frac{1}{2}p_2p_3\eta^{\mu_1\mu_3}\eta^{\mu_2\nu_2}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_5}\eta^{\mu_1\mu_4} + \frac{1}{4}(\eta^{\mu_2\nu_5})^2\eta^{\mu_3\nu_3}p_2^{\mu_4}p_3^{\nu_1}\eta^{\mu_1\mu_4} \\
&-\frac{1}{2}\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_5}p_2^{\mu_4}p_3^{\nu_1}\eta^{\mu_1\mu_4} - \frac{1}{2}\eta^{\mu_1\mu_3}\eta^{\mu_2\nu_5}\eta^{\mu_4\nu_5}p_2^{\mu_3}p_3^{\nu_2}\eta^{\mu_1\mu_4} \\
&+\frac{1}{4}\eta^{\mu_1\mu_2}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_5}p_2^{\mu_5}p_3^{\nu_2}\eta^{\mu_1\mu_4} - \frac{1}{2}\eta^{\mu_2\nu_4}\eta^{\mu_2\nu_5}\eta^{\mu_3\nu_3}p_2^{\mu_1}p_3^{\nu_5}\eta^{\mu_1\mu_4} \\
&+\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_4}p_2^{\mu_1}p_3^{\nu_5}\eta^{\mu_1\mu_4} - \frac{1}{4}\eta^{\mu_2\nu_2}\eta^{\mu_3\mu_4}\eta^{\mu_3\mu_5}p_2^{\mu_1}p_3^{\nu_5}\eta^{\mu_1\mu_4} \\
&-\frac{1}{4}(\eta^{\mu_2\nu_3})^2\eta^{\mu_4\nu_5}p_2^{\mu_1}p_3^{\nu_5}\eta^{\mu_1\mu_4} + \frac{1}{8}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_5}p_2^{\mu_1}p_3^{\nu_5}\eta^{\mu_1\mu_4} \\
&-\frac{1}{2}\eta^{\mu_1\mu_3}\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_4}p_2^{\mu_5}p_3^{\nu_5}\eta^{\mu_1\mu_4} + \frac{3}{8}\eta^{\mu_1\mu_2}\eta^{\mu_2\nu_4}\eta^{\mu_3\nu_3}p_2^{\mu_5}p_3^{\nu_5}\eta^{\mu_1\mu_4} \\
&+\frac{5}{16}p_2p_3\eta^{\mu_1\nu_1}(\eta^{\mu_2\nu_5})^2(\eta^{\mu_3\mu_4})^2 - \frac{17}{256}p_2p_3\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}(\eta^{\mu_4\nu_5})^2 \\
&-\frac{3}{16}p_2p_3\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_4}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_4}\eta^{\mu_3\mu_5} - \frac{3}{16}p_2p_3\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}(\eta^{\mu_3\mu_5})^2\eta^{\mu_4\nu_4} \\
&+\frac{1}{16}p_2p_3\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_4} - \frac{3}{8}p_2p_3\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_4}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_5} \\
&+\frac{13}{32}p_2p_3\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\mu_4}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_5} + \frac{1}{128}p_2p_3\eta^{\mu_1\nu_1}(\eta^{\mu_2\nu_3})^2\eta^{\mu_4\nu_4}\eta^{\mu_5\nu_5} \\
&+\frac{25p_2p_3\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_4}\eta^{\mu_5\nu_5}}{1024} - \frac{1}{16}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_4}p_2^{\mu_3}p_3^{\nu_2} \\
&+\frac{3}{8}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_4}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_5}p_2^{\mu_3}p_3^{\nu_2} - \frac{1}{64}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_4\nu_4}\eta^{\mu_5\nu_5}p_2^{\mu_3}p_3^{\nu_2} \\
&-\frac{1}{16}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_5}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_5}p_2^{\mu_4}p_3^{\nu_2} - \frac{1}{4}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_5}p_2^{\mu_4}p_3^{\nu_2} \\
&-\frac{1}{4}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_4}\eta^{\mu_3\mu_4}\eta^{\mu_3\mu_5}p_2^{\mu_5}p_3^{\nu_2} - \frac{1}{32}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_5}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_4}p_2^{\mu_5}p_3^{\nu_2} \\
&+\frac{1}{4}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_3\mu_5}\eta^{\mu_4\nu_4}p_2^{\mu_5}p_3^{\nu_2} - \frac{1}{4}(\eta^{\mu_1\mu_2})^2(\eta^{\mu_3\mu_5})^2p_2^{\mu_4}p_3^{\nu_4} \\
&+\frac{5}{16}\eta^{\mu_1\nu_1}(\eta^{\mu_2\nu_3})^2\eta^{\mu_4\nu_5}p_2^{\mu_5}p_3^{\nu_4} - \frac{1}{8}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_5}p_2^{\mu_5}p_3^{\nu_4} \\
&+\frac{3}{8}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_4}\eta^{\mu_2\nu_5}\eta^{\mu_3\nu_3}p_2^{\mu_4}p_3^{\nu_5} + \frac{1}{2}\eta^{\mu_1\mu_2}\eta^{\mu_1\mu_3}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_4}p_2^{\mu_4}p_3^{\nu_5} \\
&-\frac{5}{8}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_5}\eta^{\mu_3\mu_4}p_2^{\mu_4}p_3^{\nu_5} - \frac{3}{16}\eta^{\mu_1\nu_1}(\eta^{\mu_2\nu_4})^2\eta^{\mu_3\nu_3}p_2^{\mu_5}p_3^{\nu_5} \\
&-\frac{1}{4}\eta^{\mu_1\mu_2}\eta^{\mu_1\mu_3}\eta^{\mu_2\nu_4}\eta^{\mu_3\mu_4}p_2^{\mu_5}p_3^{\nu_5} + \frac{5}{8}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_2\nu_4}\eta^{\mu_3\mu_4}p_2^{\mu_5}p_3^{\nu_5} \\
&-\frac{5}{32}\eta^{\mu_1\nu_1}(\eta^{\mu_2\nu_3})^2\eta^{\mu_4\nu_4}p_2^{\mu_5}p_3^{\nu_5} + \frac{11}{256}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_4}p_2^{\mu_5}p_3^{\nu_5}
\end{aligned}
\tag{6.3.6}
$$

We employ the same choices for helicity of the external legs along with the choices for reference momenta as in the GR case. When considering the different types of diagrams we find that the pure five-vertex vanishes along with the contracted four- and three vertex vertex, as in the GR case. We are therefore left with the two propagator graphs, i.e. the amplitude is built up out of 3-point amplitudes tagged together by the propagators (4.1.13) with appropriate Mandelstam variables. After translation to spinor helicity variables and simplification we find the amplitude to be

$$\hat{M}_5[1^-2^-3^+4^+5^+] = \frac{\langle 12\rangle^7(\langle 24\rangle\langle 35\rangle[23][45] - \langle 23\rangle\langle 45\rangle[24][35])}{\langle 13\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 24\rangle\langle 25\rangle\langle 34\rangle\langle 35\rangle\langle 45\rangle}. \tag{6.3.7}$$

This is in agreement with the five point KLT relation as calculated in section 2.6. With this we have now established that the KLT relations hold for unimodular gravity up to five points. With the similarities that we have now uncovered between unimodular gravity and general relativity we expect this to hold up for an arbitrarily high number of external states at tree-level.

Chapter 7

Discussion

Historically general relativity has proven to be a remarkable theory. In addition to the myriad of classical solutions it offers, at both the levels of the solar system and the whole universe, the application of field theoretic methods to it have exposes a completely different aspect, allowing one to use it as an effective field theory. This development has given rise to the analysis of such structures as the KLT relations (1.6).

Here, we asked the question: To what extent do these KLT relations extend to 'deformations' of GR? Focussing on the specific subset of unimodular gravity. At the level of the Lagrangians, once perturbed in the hard graviton around flat space, we find the extraordinary result that the structure of the individual terms in the expansion of unimodular gravity is similar to that of general relativity. This is only voided by the fact that the terms containing the trace of $h_{\mu\nu}$ differs by a numerical coefficient. This has significant implications on the vertex expressions of the theories. Take for example the propagators of both GR and UG

$$P^{\mu_1\nu_1,\mu_2\nu_2}(k) = \frac{\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2} + \eta^{\mu_1\nu_2}\eta^{\nu_1\mu_2} - \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}}{k^2}.$$

$$\hat{P}^{\mu_1\nu_1,\mu_2\nu_2}(k) = \frac{\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2} + \eta^{\mu_1\nu_2}\eta^{\nu_1\mu_2} - \frac{3}{2}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}}{k^2}.$$

Here we see the direct result of the difference in numerical coefficients has on the structure of the scattering rules. This is even more pronounced in the higher order vertices. This difference in numerical coefficients is the result in the differences in symmetry between UG and GR. As we have stated GR is invariant under the full set of diffeomorphisms on the spacetime manifold, Diff(M), which is parametrised in the perturbative limit by $h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$. We can then view the subset of transverse diffeomorphisms, TDiff(M), as a classical gauge fixing on the full set of Diff(M). Both GR and UG is invariant under this subset of diffeomorphisms parametrized by $h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$, with the additional restriction $\partial^{\mu}\xi_{\mu} = 0$. In d-dimensions this gauge symmetry only has $(d-1)$ independent arbitrary functions, which is not enough to be gauge fixed by De Donder gauge, which has d independent conditions due to the free index (see [27] for a good review on the subject). But luckily unimodular gravity carries an additional Weyl symmetry, $h_{\mu\nu} \to h_{\mu\nu} + e^{2\sigma(x)}\eta_{\mu\nu}$, producing a Weyl transverse diffeomorphism invariant theory, WTDiff(M). This additional symmetry allows for the use of gauge fixing that is linear in the derivatives, i.e. De Donder gauge, to bring the unimodular perturbative Lagrangian into a form that can be be used to extract the vertex rules necessary for amplitude calculations.

Now that we have explicitly calculated the amplitudes in both GR and UG, we have compared them to the gauge theory amplitudes that arise as the one side to the KLT relations. We have now found that the KLT relations hold for both GR (as is well known) and UG up to five point tree-level scattering in pure gravity with the same corresponding gauge theory. With this in mind it is interesting to note that the KLT relations which were first derived in the framework of string theory and holds in GR also hold for a deformation of GR, namely UG. Although, as we have stated that UG and GR share many of the same structure aspects even at the level of the Lagrangian, this raises two important questions. One is the extent to which the KLT relations hold for other deformations of gravity? And the other, what is needed to break this degeneracy?

As an introductory look at the first of these questions we can construct a simple case in which the KLT relations do not hold. Let's consider a 3-graviton scattering process in an $f(R) = R^2$ gravity theory and compute the MHV amplitude $M_3(1^-2^-3^+)$. Fortunately, little group scaling and locality completely fix the massless 3-particle amplitudes [14] as

$$M_3(1^{h_1}2^{h_2}3^{h_3}) = \tilde{\kappa}\langle 12\rangle^{h_3-h_1-h_2}\langle 13\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3}, \tag{7.1}$$

where $\tilde{\kappa}$ is the coupling associated with the R^2 operator and $h_i = \pm 2$, the helicities of the gravitons. With $h_1 = h_2 = -2$ and $h_3 = +2$, this gives

$$M_3(1^-2^-3^+) = \tilde{\kappa}\frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 23\rangle^2}, \tag{7.2}$$

which looks promising until one realizes that the mass-dimension 2 of the kinematic part requires that the coupling $\tilde{\kappa}$ have mass-dimension -1 in order to ensure that the whole amplitude have the correct mass-dimension of 1. However a quick dimensional analysis check of the Lagrangian reveals that in this case $[\tilde{\kappa}] = 0$. *A priori* then, we would not expect generic $f(R)$ gravity theories to exhibit the KLT structure.

As a modification of GR, UG is different from the above example. Since the determinant of $g_{\mu\nu}$, $|g|$ does not contribute to the dimensional analysis, the gravitational coupling in GR and UG have the same mass dimension. We would therefore expect tree-level results like the KLT relations to hold, as we have shown to be the case up to 5-points. Another facet of this is that the tree-level scattering processes only encode the semi-classical interactions and as such we would expect them to be the same as in GR since the theories are classically equivalent. So, even though UG and GR exhibit significant differences in the structure of the vertex rules, once physical assignations have been made to the particles the amplitudes reduce to the same expressions.

General relativity and unimodular gravity are however expected to differ at the quantum level [4]. So in answer to the second of our new questions, the study of graviton scattering necessary to break the degeneracy of the classical UG and GR. To this end, what is needed are the 1-loop and higher scattering amplitudes. This would normally be a formidable task but due to the development of machinery such as unitarity methods (see [11] and references therein), one can obtain loop amplitudes directly from trees. It will be interesting to extend the calculations done here to the loop level.

Another point of interest is the BCFW recursion relations [20]. These allow for the construction of all higher point tree amplitudes from only the 3-vertex and propagators, a property that is known to extend also to GR and that was used to give an explicit proof of the n-point KLT relations [26]. This can also be seen explicitly in the amplitudes calculated in chapter 5, where we found that the diagrams containing any vertex of order 4 or higher does not contribute to the amplitude. Although we did not explicitly establish the recursion relations in these sections. In the case of the UG amplitudes, chapter 6, we see that the same general premise holds in the case of UG up to five points. With such similarity between the basic structure in GR and UG, we anticipate that a version of the BCFW recursion relations will also be applicable in unimodular gravity.

Then there is the issue of coupling to matter. One of the key phenomenological motivations for UG is the fact that, unlike in GR, gravity no longer couples to matter potentials [28]. This necessarily means that graviton-matter scattering should differ in the two theories. As a general example of this expected difference we can consider the scattering of two particles, with energy momentum tensors $T^{\mu\nu}_{(1)}$ and $T^{\mu\nu}_{(2)}$, as mediated by the graviton [1]. The scattering process in GR can than be expressed as $GT^{\mu\nu}_{(1)}P_{\mu\nu\alpha\beta}(k)T^{\alpha\beta}_{(2)}$ [7], where $P_{\mu\nu\alpha\beta}(k)$ is the GR graviton propagator (4.1.7). After contractions we have

$$GT^{\mu\nu}_{(1)}P_{\mu\nu\alpha\beta}(k)T^{\alpha\beta}_{(2)} = \frac{G}{k^2}\left(2T^{\mu\nu}_{(1)}T_{(2)\mu\nu} - T_{(1)}T_{(2)}\right). \tag{7.3}$$

When considering the same scattering problem in UG we have,

$$\begin{aligned} G\hat{T}^{\mu\nu}_{(1)}\hat{P}_{\mu\nu\alpha\beta}(k)\hat{T}^{\alpha\beta}_{(2)} &= \frac{G}{k^2}\left(2\hat{T}^{\mu\nu}_{(1)}\hat{T}_{(2)\mu\nu} - \frac{3}{2}\hat{T}_{(1)}\hat{T}_{(2)}\right) \\ &= \frac{G}{k^2}\left(2T^{\mu\nu}_{(1)}T_{(2)\mu\nu} - \frac{1}{2}T_{(1)}T_{(2)}\right) \\ &= GT^{\mu\nu}_{(1)}P_{\mu\nu\alpha\beta}(k)T^{\alpha\beta}_{(2)} + \frac{G}{2k^2}T^{\mu\nu}_{(1)}\eta_{\mu\nu}\eta_{\alpha\beta}T^{\alpha\beta}_{(2)}. \end{aligned} \tag{7.4}$$

Where the propagator, $\hat{P}_{\mu\nu\alpha\beta}(k)$ is the UG graviton propagator (4.1.13) and, $\hat{T}^{\mu\nu} = T^{\mu\nu} - \frac{1}{4}\eta^{\mu\nu}T$, as stipulated that UG couples to the traceless part of the energy momentum tensor.

The theory coupling to the trace free part of the energy momentum tensor is the result of the restriction placed by the spacetime part of the Euler Lagrange equations, as can be seen in the classical case in the Introduction. This can be extended to the perturbative case where we find the same result. Expanding the action for UG (1.3) in terms of $h_{\mu\nu}$, and varying the action with $\delta h^{\mu\nu}$ we find at the Euler-Lagrange equations at linear level:

$$\frac{\delta\mathcal{L}}{\delta h^{\mu\nu}} = -\frac{1}{2}\partial^2 h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial^\gamma h_{\gamma\nu} - \frac{1}{2}\partial_\nu\partial^\gamma h_{\gamma\mu} + \frac{1}{4}\partial_\mu\partial_\nu h + \frac{1}{4}\eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} - \frac{3}{16}\eta_{\mu\nu}\partial^2 h - \frac{\kappa}{2}\hat{T}_{\mu\nu} \tag{7.5}$$

Which is manifestly trace free if we let gravity couple to the trace-free energy momentum tensor, $\hat{T} = 0$. As of yet we do not clearly understand how this will influence the consistency of UG. It is our hope that this difference will be clarified when the amplitudes of graviton matter scattering are studied in more detail in the framework of UG. In this regard we once again turn to the KLT relations, which in particular allow for such scattering amplitudes to be computed (at least in some restricted cases) [29]. We would be curious to see how these amplitudes change in unimodular gravity.

In summary we have now shown that as far as the tree-level scattering amplitudes of both general relativity and unimodular gravity are concerned the two theories are equivalent. In other words this computation does not yet offer a means to distinguish between GR and UG, but what is does show is that the KLT relations for GR and UG both hold with the corresponding gauge theory being Yang-Mills. To break this degeneracy it is needed to calculate the loop level amplitudes in the theories. There is also the difference in the matter coupling of GR and UG that merits further investigation.

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Bibliography

[1] S. Weinberg, "The Cosmological Constant Problem," Rev. Mod. Phys. **61**, 1 (1989).

[2] G. F. R. Ellis, H. van Elst, J. Murugan and J. P. Uzan, "On the Trace-Free Einstein Equations as a Viable Alternative to General Relativity," Class. Quant. Grav. **28**, 225007 (2011) [arXiv:1008.1196 [gr-qc]].

[3] E. Alvarez and R. Vidal, Phys. Rev. D **81**, 084057 (2010) doi:10.1103/PhysRevD.81.084057 [arXiv:1001.4458 [hep-th]].

[4] E. Alvarez, "The Weight of matter," JCAP **1207**, 002 (2012) [arXiv:1204.6162 [hep-th]].

[5] E. Alvarez, S. Gonzalez-Martin, M. Herrero-Valea and C. P. Martin, "Unimodular Gravity Redux," Phys. Rev. D **92**, no. 6, 061502 (2015) [arXiv:1505.00022 [hep-th]].

[6] A. Padilla and I. D. Saltas, "A note on classical and quantum unimodular gravity," arXiv:1409.3573 [gr-qc].

[7] A. Zee, "Quantum Field Theory in a Nutshell," Princeton University Press (2010).

[8] S. Deser, "Gravity from self-interaction redux," Gen. Rel. Grav. **42**, 641 (2010) [arXiv:0910.2975 [gr-qc]].

[9] C. G. Callan, Jr., E. J. Martinec, M. J. Perry and D. Friedan, "Strings in Background Fields," Nucl. Phys. B **262**, 593 (1985).

[10] H. Kawai, D. C. Lewellen and S. H. H. Tye, "A Relation Between Tree Amplitudes of Closed and Open Strings," Nucl. Phys. B **269**, 1 (1986).

[11] Z. Bern, "Perturbative quantum gravity and its relation to gauge theory," Living Rev. Rel. **5**, 5 (2002) [gr-qc/0206071].

[12] M.B. Green, J.H. Schwarz and E. Witten, "Superstring theory, Cambridge Monographs On Mathematical Physics," Cambridge University Press (1987).

[13] Z. Koba and H.B. Nielsen, "Manifestly crossing invariant parametrization of N meson amplitude," Nucl. Phys. B, **12**, 517 (1969)

[14] H. Elvang and Y. t. Huang, "Scattering Amplitudes," arXiv:1308.1697 [hep-th].

[15] B.S. DeWitt, "Quantum Theory of Gravity. III. Applications of the Covariant Theory," Phys. Rev. **162**, 5, 1239–1256, (1967)

[16] M.E. Peskin and D.V. Schroeder, "An Introduction to Quantum Field Theory," Perseus Books Publishing (1995)

[17] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, Phys. Rept. **513**, 1 (2012) doi:10.1016/j.physrep.2012.01.001 [arXiv:1106.2476 [astro-ph.CO]].

[18] N. Arkani-Hamed and J. Trnka, "The Amplituhedron," JHEP **1410**, 030 (2014) [arXiv:1312.2007 [hep-th]].

[19] E. Witten, "Perturbative gauge theory as a string theory in twistor space," Commun. Math. Phys. **252**, 189 (2004) [hep-th/0312171].

[20] R. Britto, F. Cachazo, B. Feng and E. Witten, "Direct proof of tree-level recursion relation in Yang-Mills theory," Phys. Rev. Lett. **94**, 181602 (2005) [hep-th/0501052].

[21] S.J. Parke and T.R. Taylor, "Amplitude for n-Gluon Scattering," Phys. Rev. Lett. **56**, 23, 2459 (1986)

[22] R. Kleiss and H. Kuijf, Nucl. Phys. B **312**, 616 (1989). doi:10.1016/0550-3213(89)90574-9

[23] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. D **78**, 085011 (2008) doi:10.1103/PhysRevD.78.085011 [arXiv:0805.3993 [hep-ph]].

[24] J. J. Lopez-Villarejo, "TransverseDiff gravity is to scalar-tensor as unimodular gravity is to General Relativity," JCAP **1111**, 002 (2011) [arXiv:1009.1023 [hep-th]].

[25] M. Henneaux and C. Teitelboim, "The Cosmological Constant and General Covariance," Phys. Lett. B **222**, 195 (1989).

[26] N. E. J. Bjerrum-Bohr, P. H. Damgaard, B. Feng and T. Sondergaard, "Proof of Gravity and Yang-Mills Amplitude Relations," JHEP **1009**, 067 (2010) [arXiv:1007.3111 [hep-th]].

[27] E. Alvarez, D. Blas, J. Garriga and E. Verdaguer, "Transverse Fierz-Pauli symmetry," Nucl. Phys. B **756**, 148 (2006) doi:10.1016/j.nuclphysb.2006.08.003 [hep-th/0606019].

[28] G. F. R. Ellis, "The Trace-Free Einstein Equations and inflation," Gen. Rel. Grav. **46**, 1619 (2014) [arXiv:1306.3021 [gr-qc]].

[29] Z. Bern, A. De Freitas and H. L. Wong, "On the coupling of gravitons to matter," Phys. Rev. Lett. **84**, 3531 (2000) [hep-th/9912033].