A measure for the number of commuting subgroups in compact groups

by

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Abstract

The present thesis is devoted to the construction of a probability measure which counts the pairs of closed commuting subgroups in infinite groups. This measure turns out to be an extension of what was known in the finite case as subgroup commutativity degree and opens a new approach of study for the class of near abelian groups, recently introduced in [24, 27].

The extremal case of probability one characterises the topologically quasihamiltonian groups, studied originally by K. Iwasawa [30, 31] in the abstract case and then by F. Kümmich [35, 36, 37], C. Scheiderer [45, 46], P. Diaconis [11] and S. Strunkov [48] in the topological case. Our probability measure turns out to be a useful tool in describing the distance of a profinite group from being topologically quasihamiltonian.

We have been inspired by an idea of H. Heyer in the present context of investigation and in fact we generalise some of his techniques, in order to construct a probability measure on the space of closed subgroups of a profinite group. This has been possible because the space of closed subgroups of a profinite group may be approximated by finite spaces and the consequence is that our probability measure may be approximated by finite probability measures.

While we have a satisfactory description for profinite groups and compact groups, the case of locally compact groups remains open in its generality.

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Declaration

I declare that **A measure for the number of commuting subgroups in compact groups** is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

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Funmilayo Eniola Kazeem June 26, 2019

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Dedication

To my family, for their continuous support and for always making me laugh.

Contents

Introduction

The *commutativity degree* of a finite group G is defined by

$$\mathrm{d}(G) = \frac{|\{(x,y) \in G \times G : [x,y] = 1\}|}{|G|^2}$$

and was introduced by Erdős and Turán [12, 13, 14, 15, 16, 17, 18] almost fifty years ago in a series of fundamental contributions, which are at the origin of probabilistic group theory. It gives a measure of how far G is from being abelian, because $\mathrm{d}(G) = 1$ if and only if G is abelian.

There are more recent contributions on the notion of commutativity degree for compact groups in [28, 29] and the importance of the subject becomes more clear when we look at [1, 2, 3, 4, 5, 6] which investigate the role of commuting elements in classifying spaces, simplicial spaces and similar structures in algebraic topology.

Studying the problem from the perspective of lattice theory, some work has been done in [40, 50, 51] and the corresponding notion of *subgroup commutativity degree*

$$\mathrm{sd}(G) = \frac{|\{(H,K) \in \mathcal{L}(G) \times \mathcal{L}(G) \mid HK = KH\}|}{|\mathcal{L}(G) \times \mathcal{L}(G)|}$$

has been introduced for a finite group G, where $\mathcal{L}(G)$ denotes the subgroup lattice of G. The number $\mathrm{sd}(G)$ also gives important information on how far G is from being abelian, in fact this time we are not describing the probability of commuting elements, but that of commuting subgroups and there are several circumstances in which the two probabilities may differ significantly.

Finite groups with $\mathrm{sd}(G) = 1$ are the groups in which subgroups permute with each other (i.e. *quasihamiltonian finite groups*). These groups were classified by K. Iwasawa (see [47, Chapter 2], or the original works [30, 31]) and among them we find the abelian groups and the well-known quaternion group Q_8 of order 8, in which all the subgroups are normal.

The way of looking at the abelianess of groups via the number of their commuting subgroups is the subject of this thesis and we will focus on classifications which generalise the ideal case of quasihamiltonian groups, focusing on profinite groups. The motivation for our work is due to the recent contributions [24, 27], which are devoted to classifying topological groups with many closed commuting subgroups. In particular, we answer [44, Remark 2.11] reported below:

...Since $sd(G)$ is defined in terms of counting measures over finite G, it is reasonable to expect that the context of infinite compact groups could be treated in the same way, once $sd(G)$ is replaced by appropriate Haar measures. No results are known on this point, to the best of our knowledge, because the topologies on subgroup lattices of compact groups are quite delicate to investigate in this perspective. One of the main problems is that we require some effort in order to find a good measure, with the same properties of the Haar measure, on the subgroup lattice of a compact group. Already, the case in which the probability collapses to one, presents difficulties, since there are few results of structure on the so-called "non abelian topologically quasihamiltonian groups"...

We will in fact, define a new measure in Chapter 7, answering the above question via a variation of [25, Theorem 1.2.17]. This will provide a good framework for generalisation within the finite case. In particular, profinite groups in which each pair of closed subgroups commute are called *topologically quasihamiltonian* and have been classified in [27]. We will define an infinite version of the subgroup commutativity degree, looking at the extremal case of probability *one* as the case characterising all profinite topologically quasihamitonian groups.

In our contribution (see Theorem 7.1), we take a profinite group and consider the space of closed subgroups on it, we then show that the subgroup commutativity degree can be lifted to a probability measure in an infinite case. This allows us to generalise the subgroup commutativity degree (see Corollary 7.3) and gives a new notion in Definition 7.2 which has a lot of interesting consequences and applications (see Examples 7.6, 7.7 and 7.8). Further ideas, which are related to Theorem 7.1, are contained in [33] but not reported in the present thesis.

The moment we move to consider compact groups, analogies are possible, so we have a satisfactory notion of subgroup commutativity degree for compact groups, but the space of closed subgroups becomes very complicated when the group is locally compact (see [10] and several examples in [24]), so different techniques must be involved for an appropriate notion of subgroup commutativity degree when we deal with locally compact groups. This is one of the main problems, which prevent us from adapting the argument of Lemma 6.1. There are also other difficulties for a more general approach, due to the nature of the space of closed subgroups (see Remark 6.3 and Question 6.4).

There are several mathematical fields, which must be involved for the proof of our main results, so the chapters are organised in order to give preliminary notions for the arguments contained in the proofs of Chapter 7. For instance, Chapter 1 describes some elementary group theory, while classical topological notions are placed in Chapter 2. We then get to Chapter 3, where we find some elementary notions on topological groups. Basically, Chapter 4 gives two examples of the presence of topological groups, one in differential geometry and the other in algebraic topology. This is to help the reader with concrete situations, where topological groups can be encountered. In Chapter 5, we describe the space of closed subgroups of profinite groups and we give some notions of abstract harmonic analysis in Chapter 6 which are useful for the main result and its applications in Chapter 7. Notations and terminologies follow the references [25, 26, 42, 47].

Finally, we point out that the present thesis is based on the manuscripts [32] and [33].

1. Preliminaries of abstract group theory

In the present chapter we recall the axioms of an abstract group, that is, the classical notion of a group without the additional presence of a topological structure on it. Then we examine subgroups and quotients of an abstract group, offering classical examples. Some large abelian subgroups and quotients are also considered in this chapter, involving the center and the commutator subgroup.

Given a set G and the product set $G \times G = \{(a, b) \mid a, b \in G\}$, the map

$$* : (a, b) : G \times G \longmapsto *(a, b) = a * b \in G$$

is called a binary operation on G and the pair $(G, *)$ is a *group* if $*$ satisfies some prescribed properties.

Definition 1.1. *A pair $(G, *)$ is an abstract group, or briefly a group, if*

(i) for every $a, b, c \in G$, one has $a * (b * c) = (a * b) * c$;

(ii) $\exists\ 1 \in G$ such that $\forall\ a \in G, a * 1 = 1 * a = a$;

(iii) $\forall\ a \in G, \exists\ a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = 1$.

It is well known that Definition 1.1 (i) describes the associativity of $*$ and this axiom corresponds to a condition of symmetry for the binary operation, if we look at it in the equivalent way $*(a, *(b, c)) = *(*(a, b), c)$. Also Definition 1.1 (ii), that is, the existence of an element $*(a, 1) = *(1, a) = a$, may be interpreted as another condition of symmetry for $*$. This axiom describes the existence of a left and right neutral element 1 for $*$. The same happens for the final axiom in Definition 1.1 (iii), that is, the existence of an element

a^{-1} such that $*(a, a^{-1}) = *(a^{-1}, a) = 1$ for all $a \in G$ may be interpreted again as a condition of symmetry for $*$. This final axiom describes the existence of a left and right inverse, or briefly of the inverse, for any element of G.

As usual, we will replace $*$ with $\cdot$ and will briefly refer to G, instead of $(G, \cdot)$ each time. Moreover, we will use the adjective *abelian* if, in addition to (i), (ii) and (iii) of Definition 1.1, the condition $ab = ba$ is satisfied $\forall\ a, b \in G$. Apparently we have another condition of symmetry between the axioms of an abstract group.

Some examples of groups and non-groups are seen in the following example.

Example 1.2.

(1). The pairs $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$ are abelian groups with respect to the usual addition. The same happens for the pairs $(\mathbb{R}, +)$, $(\mathbb{C}, +)$, where $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ the set of all complex numbers.

(2). The sets $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ are abelian groups with respect to the usual multiplication. The identity element in each case is 1 and the inverse of any element a is $1/a$. On the other hand, $\mathbb{Z}^*$ is not a group with respect to the usual multiplication.

(3). Given the set of all rational numbers $\mathbb{Q}$, the pair $(\mathbb{Q}, \cdot)$ is *not* a group, since it is not possible to define the inverse of $0 \in \mathbb{Q}$ with respect to the usual multiplication between two rational numbers.

(4). Given the set of all integers $\mathbb{Z}$ and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, the pair $(\mathbb{Z}^*, \cdot)$ is *not* a group, since the inverse $a^{-1} = 1/a$ of every $a \in \mathbb{Z}^*$ does not belong to $\mathbb{Z}^*$ (unless $a = \pm 1$).

The consideration of $\mathbb{Z}^*$ shows that it is possible to have a subset H of G and consider a restriction of the binary operation from $H \times H$ to H, but in general one does not know whether H is stable under the binary operation or not. Consequently, one does not know in general whether a restriction of a binary operation on G satisfies the axioms in Definition 1.1 or not.

For instance,

$$\cdot : (a, b) \in \mathbb{Z}^* \times \mathbb{Z}^* \to \cdot(a, b) = ab \in \mathbb{Z}^*$$

does not define a group on $\mathbb{Z}^*$, but if we restrict to

$$\cdot : (a, b) \in \{1, -1\} \times \{1, -1\} \to \cdot(a, b) = ab \in \{1, -1\},$$

then we have the structure of a group on $\{1, -1\}$.

Definition 1.3. *A subset H of a group G is an abstract subgroup of G, or briefly a subgroup of G, if the axioms of Definition 1.1 are satisfied replacing G with H.*

A classical example of nonabelian group is given by the symmetric group.

Example 1.4. Let S_n be the set of all permutations on the set $\{1, 2, \ldots, n\}$. Here the binary operation is the composition $\circ$ between two permutations of elements of $\{1, 2, \ldots, n\}$ and $(S_n, \circ)$ is a group with respect to $\circ$. The identity element 1 is the permutation that fixes every element. Denoting by (12) the permutation fixing 3 and sending 1 to 2 and 2 to 1, and similarly (23), (13), and denoting by (123) the cycle sending 1 to 2, 2 to 3 and 3 to 1, the group $S_3 = \{1, (12), (23), (13), (123), (132)\}$ is finite of order six and is nonabelian because $(12)(13) \neq (13)(12)$. It is easy to check that S_n is abelian if and only if $n \leq 2$.

Another important source of nonabelian groups, but this time infinite, is given by the linear groups. In order to introduce these well-known groups, we briefly recall some elementary notions of linear algebra.

Given the abelian group $(\mathbb{R}, +)$ with neutral element 0 and the abelian group $(\mathbb{R}^*, \cdot)$ with neutral element 1, it is possible to connect the operations $\cdot$ and $+$ by adding three extra axioms. In fact $(\mathbb{R}, +, \cdot)$ is called a *vector space* on $\mathbb{R}$ of *dimension 1*, if, in addition to the structure of abelian groups on $(\mathbb{R}, +)$ and $(\mathbb{R}^*, \cdot)$, the distributive property $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, the scalar multiplication $(x \cdot y)\ z = x\ (y \cdot z)$ and the compatibility condition $1 \cdot x = x$ are satisfied for all $x, y, z \in \mathbb{R}$. A *vector space* on $\mathbb{R}$ of *dimension 2* may be defined in analogy, considering the additive abelian group on $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ via the pointwise addition $u+v = (x_1+x_2, y_1+y_2)$, where $u = (x_1, y_1)$ and $v = (x_2, y_2)$ are elements of $\mathbb{R}^2$ and $(0, 0)$ is the neutral element, and we define multiplication by scalars by $r \cdot (x, y) := (r \cdot x, r \cdot y)$. It is clear now what we mean for a *vector space* on $\mathbb{R}$ of *dimension n*, when $n \geq 2$.

The set of all $(n \times n)$-matrices with real coefficients may be endowed with the structure of the vector space $\mathbb{R}^{n^2}$ of dimension n^2 on $\mathbb{R}$ (see for instance [26, Examples 1.6]), so we may list some important examples.

Example 1.5. The set $\mathrm{GL}_n(\mathbb{R}) = \{T \in \mathbb{R}^{n^2} : \det T \neq 0\}$ is a group with respect to the usual product row by columns between two matrices, called *general linear group of dimension n on $\mathbb{R}$*. Here the neutral element is given by the identity matrix and the inverse of each matrix can be determined by the classical computation of the inverse of a square matrix with the methods of linear algebra. One can check that for $n \geq 2$, $\mathrm{GL}_n(\mathbb{R})$ is an infinite nonabelian group. Moreover, the set $\mathrm{SL}_n(\mathbb{R}) = \{T \in \mathbb{R}^{n^2} : \det T = 1\}$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$ and is called *special linear group of dimension n on $\mathbb{R}$*. Important properties are described in [26, Examples 1.6., Exercise E1.1, Exercise E1.2].

We move on to describe a dual notion of a subgroup. The idea is "to make a cut" from a given group, preserving the axioms of groups in the new structure.

Definition 1.6. *Let G be a group, $g \in G$ and H a subgroup of G. The sets*

$$gH = \{gh \mid h \in H\} \text{ and } Hg = \{hg \mid h \in H\}$$

are called the left and right cosets of H in G, respectively. The subgroup H is normal in G, if $gH = Hg$ for all $g \in G$.

The reader may refer to [42, Chapter 1] for the classical properties of normal subgroups and of left and right cosets. Since the maps $h \mapsto gh$ and $h \mapsto hg$ are bijective, gH and Hg have the same cardinality, so in any group G and for any subgroup H we always have $|gH| = |Hg|$ but this does not imply $gH = Hg$.

Definition 1.7. *Let G be a group and H a subgroup of G. The cardinality of the set G/H of the left (or right) cosets is denoted by $|G : H|$ and is called the index of H in G.*

One could ask how far H is from having the same properties as G and the notion of index serves this purpose. In fact, if H is normal in G. then the set $G/H = \{gH \mid g \in G\}$ satisfies Definition 1.1, where the binary operation is

$$\cdot : (xH, yH) \in G/H \times G/H \to \cdot(xH, yH) = (xH)(yH) = (xy)H \in G/H$$

and it is easy to check that it is well-defined. In particular, Definition 1.7 may be specialised.

Definition 1.8. *Let H be a normal subgroup of a group G. The index of H in G is the cardinality of the quotient group G/H.*

Note that for a finite group G, the cardinality $|G|$ is called the *order* of G, so Definition 1.8 may be rephrased saying that a normal subgroup H of G has finite index when the order of G/H is finite. Of course, any subgroup of an abelian group is in this situation, because all subgroups of abelian groups are normal.

Example 1.9. Given a prime p, the set $p\mathbb{Z} = \{px \mid x \in \mathbb{Z}\}$ is a subgroup of the additive group $\mathbb{Z}$ and the quotient group $\mathbb{Z}/p\mathbb{Z} = \mathbb{Z}(p)$ is an abelian group called the additive group of the integers modulo p. One can define the direct sum $\mathbb{Z}(p)^{(2)} = \mathbb{Z}(p) \oplus \mathbb{Z}(p) = \{(x, y) \mid x, y \in \mathbb{Z}(p)\}$ and this forms a group with respect to the pointwise operation $(x\mathbb{Z}(p)) + (y\mathbb{Z}(p)) = (x + y)\mathbb{Z}(p)$. On the other hand, one can define the multiplication pointwise and check without problems that $(\mathbb{Z}(p)^{(2)}, +, \cdot)$ satisfies the axioms of a vector space on $\mathbb{Z}(p)$. This is a classical example of vector space of finite dimension. In analogy with Example 1.5, one can introduce $\mathrm{GL}_2(\mathbb{Z}(p))$, the general linear group of degree 2 on $\mathbb{Z}(p)$, and the corresponding special linear group $\mathrm{SL}_2(\mathbb{Z}(p))$.

Due to the importance of groups like $\mathbb{Z}(p)$ and $\mathbb{Z}(p) \oplus \mathbb{Z}(p)$, there is a special terminology for them. In fact $\mathbb{Z}(p)$ is called a *p-elementary abelian group of rank 1*, $\mathbb{Z}(p)^{(2)} = \mathbb{Z}(p) \oplus \mathbb{Z}(p)$ is a *p-elementary abelian group of rank 2* and so on, and $\mathbb{Z}(p)^{(k)}$ is called a *p-elementary abelian group of rank k* if we have exactly k factors isomorphic to $\mathbb{Z}(p)$.

When H is a normal subgroup of G, the map

$$\pi : g \in G \to gH \in G/H$$

is called a *canonical projection* and has several interesting properties. One of them is that π preserves both the group structure in the domain and in the codomain, that is, $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$ for all $g_1, g_2 \in G$. This is a classical example of homomorphism of abstract groups.

Definition 1.10. *Given two groups A and B, a map $f : A \to B$ is an homomorphism between A and B, if $f(a_1 a_2) = f(a_1)f(a_2)$ for all $a_1, a_2 \in A$.*

There are some classical results, known as theorems of isomorphisms of groups [42, 1.4.3, 1.4.4, 1.4.5], which have their own counterpart in the context of topological groups (for instance, compare [42, 1.4.3] and [26, Proposition 1.10 (iv)]). We recall one of these theorems for abstract groups.

Lemma 1.11 (Correspondence Theorem)**.** *Let G be a group and N a normal subgroup of G. Then the assignment $K \mapsto K/N$ determines a bijection from the set of subgroups of G which contain N to the set of subgroups of G/N. Furthermore, K/N is normal in G/N if and only if K is normal in G.*

Proof. See 1.4.6 in [42]. □

Among abstract abelian groups, it is possible to describe very well the homomorphisms. For instance, the map $f : x \in A \mapsto x^{-1} \in A$ is homomorphism of groups if and only if A is abelian. Moreover, literature has plenty of detailed results of structure for the automorphism group of abelian groups (see [42]).

On the other hand, it is more difficult when we deal with abelian groups which admit a topological structure (we will discuss later on the notion of "topological group"). The description of their automorphism group is indicated in [26, Proposition 9.85] and technical difficulties originate from the presence of an additional structure. Their automorphism group requires much more attention (see [26, Theorem 9.86, Corollary 9.87]) and may involve homological methods (see [26, Part 5, Chapter 8]). We do not go into the details of the problems but just show a few classical examples of homomorphisms of abelian groups.

Example 1.12. Consider $\mathbb{R}$ with respect to addition and fix $n \geq 0$. Then the function $\phi : x \in \mathbb{R} \mapsto nx \in \mathbb{R}$ which sends x to the multiple nx is a homomorphism of abelian groups. When $n \neq 0$, it is clear that ϕ is an isomorphism. In particular, when $n = 2\pi$, $\phi : t \in \mathbb{R} \mapsto \phi(t) = 2\pi t \in \mathbb{R}$ is an isomorphism.

Another specialisation, always in the abelian case, is the following.

Example 1.13. The function $\phi : (\mathbb{R}, +) \to (\mathbb{C}^*, \cdot)$ defined by $\phi(t) = e^{2\pi i t}$ is a homomorphism because if we take any $t_1, t_2 \in \mathbb{R}$, then

$$\phi(t_1 + t_2) = e^{2\pi i(t_1+t_2)} = e^{2\pi i t_1} \cdot e^{2\pi i t_2} = \phi(t_1) \cdot \phi(t_2).$$

A reason why Example 1.9 is important is mentioned below.

Example 1.14. According to [42, Example 1.5 (11), Page 30], the set of all automorphisms $\mathrm{Aut}(\mathbb{Z}(p)^{(k)})$ of $\mathbb{Z}(p)^{(k)}$ forms a group and $\mathrm{Aut}(\mathbb{Z}(p)^{(k)})$ is isomorphic to the general linear group $\mathrm{GL}_k(\mathbb{Z}(p))$ of $k \times k$ matrices with coefficients in $\mathbb{Z}(p)$.

In general, Example 1.4 shows that S_n is not abelian (when $n \geq 3$) and so one could ask for the largest abelian subgroup of such a group, or for the largest abelian quotient. This motivates the following notion.

Definition 1.15. *The centre $Z(G)$ of a group G is defined as the set*

$$\{g \in G : gx = xg \ \forall \ x \in G\}.$$

It is easy to check that $Z(G)$ is a normal subgroup of G. The importance of the notion of center will be clear later on. In fact this is not only an abstract subgroup but, in any topological group G, it is also closed. Details will be given later on. About abelian quotients, we need to introduce the following notion.

Definition 1.16 (See [26], Definition 5.56)**.** *If A and B are subsets of a group G, then*

$$[A, B] = \langle [a, b] \mid a \in A, b \in B \rangle$$

denotes the subgroup generated by all the commutators $[a, b] = aba^{-1}b^{-1}$*. In particular,* $[G, G]$ *(also denoted* G'*) is a subgroup called the commutator subgroup of* G*.*

Once we have the notion of commutator subgroup, we may iterate and get the *derived series*

$$G = G^{[0]} \supseteq G^{[1]} = [G, G] \supseteq G^{[2]} = [[G, G], [G, G]] \supseteq \cdots$$

of an abstract group G consists of the subgroups defined by the recursion $G^{[n+1]} := [G^{[n]}, G^{[n]}]$ for $n \in \mathbb{N}$.

A group G is *solvable* if $G^{[n]} = 1$ for some $n \in \mathbb{N}$, that is, if its derived series reaches the trivial subgroups after finitely many steps. Some elementary properties of solvable groups are recalled from [42].

Proposition 1.17. *Given a group* G*,*

(i). the quotient group G/G' *is abelian;*

(ii). if $G' \subseteq H \subseteq G$*, then* G/H *is abelian;*

(iii). for all $n \in \mathbb{N}$ *the factor* $G^{[n]}/G^{[n+1]}$ *is abelian.*

Proof. (i.) For any $x, y \in G$, take $xG', yG' \in G/G'$. Then

$$xG' \cdot yG' = xyG' = yxx^{-1}y^{-1}xyG' = yxG'.$$

(ii). Suppose H is normal in G and $G' \subseteq H$. For any $x, y \in G$, we have

$$\begin{aligned} xH \cdot yH &= xyH \\ &= yxx^{-1}y^{-1}xyH \\ &= yx[x, y]H \quad \text{since } [x, y] \in H \\ &= yxH = yH \cdot xH. \end{aligned}$$

(iii). Of course, $G^{[n+1]}$ is normal in $G^{[n]}$ and it is easy to check that $[G^{[n]}/G^{[n+1]}, G^{[n]}/G^{[n+1]}] = 1$.

□

In particular solvable groups have a finite series with abelian factors.

The *upper central series* $1 = Z_0(G) \subseteq Z_1(G) \subseteq \ldots$ of a group G consists of the sets defined by

$$Z_1(G) = Z(G), \quad Z\left(\frac{G}{Z(G)}\right) = \frac{Z_2(G)}{Z(G)}, \quad Z\left(\frac{G}{Z_2(G)}\right) = \frac{Z_3(G)}{Z_2(G)}, \quad \ldots$$

and it is not difficult to check that each $Z_i(G)$ is a normal subgroup of G. The group G is called *nilpotent* if $G = Z_n(G)$ for some $n \in \mathbb{N}$, that is, if the central series reaches G after finitely many steps.

By duality we may define the *lower central series*, $G = \gamma_1(G) \supseteq \gamma_2(G) \subseteq \ldots$ of a group G as a series consisting of the sets defined by $\gamma_2(G) = [\gamma_1(G,),G] = G'$, $\gamma_3(G) = [\gamma_2(G),G] = [G',G]$, and so on. Alternatively we say that G is nilpotent if its lower central series reaches the trivial subgroup after finitely many steps.

It is not difficult to check that all the factors $Z_{n+1}(G)/Z_n(G)$ are abelian and all the factors $\gamma_n(G)/\gamma_{n+1}(G)$, so both the notion of central series and of derived series give a generalisation of non abelian groups in terms of series with abelian factors. It is in fact clear that abelian groups are groups in which both the central series and the derived series stop after just one step, but one has to be careful since nilpotent groups are solvable but there are non nilpotent solvable groups.

Example 1.18. The alternating group A_4 of order 4 is a classical example of a finite group which is solvable but not nilpotent, so the notion of nilpotence is stronger than the notion of solvability.

We may produce an example of infinite solvable non nilpotent group by direct products and $A_4 \times \mathbb{Z}$ is one of them.

2. Some classical notions of topology

In the present chapter we formulate classical notions of topology which will be useful when we combine the algebraic structure with the topological one in a group.

Definition 2.1 (See Definition 3.1, [52])**.** *Let X be a set and $\mathcal{T}$ a collection of subsets of X satisfying:*

(i) $\emptyset, X \in \mathcal{T}$*;*

(ii) if $A_1, A_2, \ldots, A_n \in \mathcal{T}$*, then* $\bigcap_{i=1}^{n} A_i \in \mathcal{T} \in \mathcal{T}$*;*

(iii) if I is an arbitrary index set and $A_i \in \mathcal{T}$ with $i \in I$, then $\bigcup_{i \in I} A_i \in \mathcal{T}$*.*

$\mathcal{T}$ is called a topology for X and the pair $(X, \mathcal{T})$, a topological space. The elements of $\mathcal{T}$ are usually called *open* sets of X and their complements in X are called *closed*. Moreover, if $x \in X$ and U_x is an open set in X containing x, we refer to U_x as an *open neighbourhood* of x, or briefly a *neighbourhood* of x.

Of course, $\mathcal{T} = \{\emptyset, X\}$ satisfies Definition 2.1 and is called *trivial topology*, while the opposite situation is described by $\mathcal{T} = \mathcal{P}(X)$, that is, $\mathcal{T}$ equal to the set of all subsets $\mathcal{P}(X)$ of X; this turns out to be a topology on X, called the *discrete topology* on X. The reader may refer to [52] for notions of general topology. A classical example of a topological space is $\mathbb{R}$ with the *Euclidean topology* (see [52, Example 3.2b]).

The pair $(X, \mathcal{T})$ is said to be a T_2 space, or a *Hausdorff space*, if for any two distinct points $x, y \in X$ there exists open sets U and V such that for

$x \in U, y \in V, U \cap V = \emptyset$. Again $\mathbb{R}$ (with the Euclidean topology) is the classical example.

Note that a topological space X is *connected*, if it cannot be split into two non trivial open sets, that is, if $X = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$ for two open sets A_1, A_2 of X, then either $(A_1, A_2) = (\emptyset, X)$ or $(A_1, A_2) = (X, \emptyset)$. Another important property of connected spaces is that they are invariant under unions, provided that there is a common point.

Proposition 2.2 (See Theorem 9.6, [34]). *Suppose that $\{Y_j : j \in J\}$ is a collection of connected subsets of a space X. If $\bigcap_{j \in J} Y_j \neq \emptyset$ then $Y = \bigcup_{j \in J} Y_j$ is connected.*

Proposition 2.2 gives the idea to split a non connected topological space into smaller pieces which are connected. In fact the maximal connected subsets of a topological space are called *connected components* and one can see that they form a partition and their union is the whole space (which cannot be necessarily connected). When these connected components collapse to points, the space is very far from being connected. A topological space is *totally disconnected* if all its connected components are singletons (see [26, 52]).

Now we make an analogy with the axioms in Definition 2.1 and a classical notion of functional analysis, which can be found in [43, Chapter 1].

Definition 2.3 (See Definition 1.3, [43]).

A family $\mathcal{A}$ of subsets of a set X is a σ-algebra on X, if:

(i) $\emptyset, X \in \mathcal{A}$;

(ii) *if* $A \in \mathcal{A}$, *then* $X \setminus A \in \mathcal{A}$;

(iii) *if* I *is an index set of* $|I| \leq |\mathbb{N}|$ *and* $A_i \in \mathcal{A}$ *with* $i \in I$, *then* $\bigcup_{i \in I} A_i \in \mathcal{A}$.

Note that elements of $\mathcal{A}$ are called *measurable sets* of X and the pair $(X, \mathcal{A})$, described in Definition 2.3, may be indicated simply with X, referring to X as a *measurable space*. Of course, a trivial case is given by $\mathcal{A} = \{\emptyset, X\}$ and another by $\mathcal{A} = \mathcal{P}(X)$. The analogy between Definitions 2.1 and 2.3

appears immediately once the axiom (ii) is modified, requiring the "complement property" instead of the stability with respect to finitely many intersections, and the axiom (iii) is modified, restricting the cardinality of I to be countable.

An example of a topology which is not a σ-algebra is given below:

Example 2.4. Consider $X = \{1, 2, 3\}$ and $\mathcal{T} = \{\emptyset, \{1, 2\}, \{2\}, \{2, 3\}, X\} \subseteq \mathcal{P}(X)$. Clearly, $\mathcal{T}$ is a topology: it is enough to check (i), (ii) and (iii) in Definition 2.1, but $X - \{2, 3\} = \{1\} \notin \mathcal{T}$, so $\mathcal{T}$ is not a σ-algebra, because (ii) of Definition 2.3 is not satisfied.

A more general example of a σ-algebra which is not a topology is shown below:

Example 2.5. Let X be any uncountable set and $\mathcal{A}$ the collection of all subsets of X which are either countable or have countable complements. Clearly, $\mathcal{A}$ is closed under complements, so both (i) and (ii) of Definition 2.3 are satisfied. Now if we have a countable union of elements of $\mathcal{A}$, all of which are countable, then the union is countable. Else, at least one element is a set whose complement in X is countable, hence so is the union and (iii) of Definition 2.3 is satisfied. Therefore $\mathcal{A}$ is a σ-algebra, but it is not a topology. If $\mathcal{A}$ were a topology, every singleton set would have to be open. Hence the topology is discrete. It follows that every subset of X will have to belong to $\mathcal{A}$. However, there is a subset Y not countable and whose complement is not countable either. If we let $\chi : X \to X \times \{0, 1\}$ be any bijection, then $Y := \chi^{-1}(X \times \{0\})$ is a suitable set which serves the purpose.

One could ask whether the assumption of uncountability on X in Example 2.5 may be avoided or not. The answer is negative because it is not difficult to see that every σ-algebra on a set X of $|X| \leq |\mathbb{N}|$ is necessarily a topology on X.

On the other hand, we can connect the two structures on X and consider the smallest σ-algebra containing a topology on a set. In fact it is possible to show that given a topological space $(X, \mathcal{T})$, there exists a σ-algebra in $\mathcal{P}(X)$ containing $\mathcal{T}$ and among those containing $\mathcal{T}$ we can choose the smallest σ-algebra containing $\mathcal{T}$ (see [43, Theorem 1.10]). This result is classical in functional analysis.

Definition 2.6 (See 1.11, [43]). *Given a topological space $(X, \mathcal{T})$, the smallest σ-algebra containing $\mathcal{T}$ is called the Borel σ-algebra on X. An element of the Borel σ-algebra on X is called a Borel set.*

Later on, the notions recalled in Definitions 2.1, 2.3 and 2.6, will be used in a classical theorem of functional analysis (see Theorem 5.7 below). This result will describe the properties of the measure which we want to construct on the space of closed subgroups of a compact group.

We now come back to the description of the peculiar maps which are related to topological spaces and σ-algebras. If we have a map f between two topological spaces X and Y, the notion of continuity plays the role of the notion of homomorphism in the context of abstract groups, that is, continuous maps between two topological spaces preserve the topological structure. The reader may refer to [52] for a complete review on topological spaces and continuous maps. Here we report just few facts, which will be useful later on.

Following [43, Definition 1.2 (c)], a function f between two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is *continuous*, or *globally continuous*, if $f^{-1}(V) \in \mathcal{T}_Y$ for all $V \in \mathcal{T}_X$. On the other hand, we say that f is *locally continuous*, or *continuous at* $x_0 \in X$ if for all neighbourhoods $U_{f(x_0)}$ of $f(x_0)$ in Y there corresponds a neighbourhood U_{x_0} of x_0 in X such that $f(U_{x_0}) \subseteq U_{f(x_0)}$.

Proposition 2.7 (See Proposition 1.5, [43]). *Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, the map $f : X \to Y$ is continuous if and only if it is continuous for all $x \in X$.*

At the level of the σ-algebras, the functions preserving the structure are the measurable functions. Recall from [43, Definition 1.3 (b)] that a function f from a measurable space X to a topological space Y is a *measurable function* if $f^{-1}(V)$ is measurable in X for all V open in Y. Some classical properties of measurable functions are listed below:

Proposition 2.8 (See Theorem 1.12, [43]). *Given a measurable space $(X, \mathcal{A})$, a topological space $(Y, \mathcal{T})$ and a map $f : X \to Y$:*

(i) the set $\{E \subseteq Y \mid f^{-1}(E) \in \mathcal{A}\}$ is a σ-algebra on Y;

(ii) if f is measurable and E is a Borel set in Y, then $f^{-1}(E) \in \mathcal{A}$;

(iii) if $Y = \mathbb{R}$ and $f^{-1}((a, +\infty]) \in \mathcal{A}$ for all $a \in \mathbb{R}$, then f is measurable.

The above properties are well-known in the theory of Lebesgue measure (see [43, Chapter 2]). Again we report this fact, in order to be familiar with the formulation of the main results which will appear in the successive chapters.

After we show in parallel the axioms of topological spaces and measurable spaces, discussing the peculiar functions of these structures, we examine substructures of topological spaces and of measurable spaces and their quotients. The same approach has been used from an abstract point of view in the context of groups in the previous chapter.

Definition 2.9 (See Definition 6.1, [52]). *Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$. The set $\{Y \cap U : U \in \mathcal{T}\}$ defines a topology on Y inherited from X and is called the subspace topology of Y.*

It is easy to check that the set defined in Definition 2.9 satisfies Definition 2.1.

Definition 2.10 (See Definition 9.1, [52]). *Suppose that $f : X \to Y$ is a surjective map from a topological space X onto a set Y. The quotient topology on Y with respect to f is the set $\{U \subseteq Y : f^{-1}(U)$ is open in $X\}$.*

Again it is easy to check that the set in Definition 2.10 satisfies Definition 2.1. In analogy, if $Y \subseteq X$ and X is a measurable space, one can define the induced σ-algebra on Y, checking that the set $\{Y \cap E : E \in \mathcal{A}\}$ satisfies Definition 2.3.

We now introduce a well-known notion of approximation, which may be reformulated in different ways and in different contexts.

Definition 2.11 (See [34], Definitions 7.1, 7.2, 7.3, 7.4). *A cover (or covering) of a subset Y of a set X is a collection of subsets $\{U_j : j \in J\}$ of X such that $Y \subseteq \bigcup_{j \in J} U_j$. If in addition the indexing set J is finite then $\{U_j : j \in J\}$ is said to be a finite cover. Suppose that $\{U_j : j \in J\}$ and $\{V_k : k \in K\}$ are covers of the subset Y of X. If for all $j \in J$ there is a $k \in K$ such that $U_j = V_k$ then we say that $\{U_j : j \in J\}$ is a subcover of $\{V_k : k \in K\}$. The* *cover* $\{U_j : j \in J\}$ *is an open cover of* Y *if each* U_j*, is an open subset of* X*.* Y *is said to be compact if every open cover of* S *has a finite subcover.*

Definition 2.11 can be formulated locally, saying that a space X is *locally compact* if each point x of X possesses a compact neighbourhood U_x.

We end with the notion of product both for topological spaces and σ-algebras.

Definition 2.12. *Given two topological spaces* $(X, \mathcal{T}_X)$ *and* $(Y, \mathcal{T}_Y)$ *along with* $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ *and the projections* $p : X \times Y \to X$ *and* $q : X \times Y \to Y$*, the product topology* $\mathcal{T}_{X\times Y}$ *on* $X \times Y$ *is defined as the coarsest topology (that is, the topology with the smallest collection of open sets) for which both* p *and* q *are continuous.*

In the case of a product of infinitely many topological spaces, the idea is again, to define the product topology as the coarsest for which all the projections are continuous simultaneously.

The reader may refer to [52, Pages 52-58] for the details. In particular, we mention a classical result of general topology in connection with the topological products:

Theorem 2.13 (Tychonoff Theorem)**.** *If* $(X_\alpha, \mathcal{T}_\alpha)$ *are compact topological spaces for each* $\alpha \in A$*, then* $X = \prod_{\alpha\in A} X_\alpha$*, endowed with the product topology is also compact.*

A proof of this important result is known with different methods and we will offer one with the use of the filters in the following chapters. Finally, for σ-algebras we have the following notion of product:

Definition 2.14. *Let* $(X, \mathcal{A}_X)$ *and* $(Y, \mathcal{A}_Y)$ *be two measurable spaces. The product* $X\times Y$ *of measure spaces is the set* $X\times Y$ *with the* σ*-algebra* $\mathcal{A}_{X\times Y} = \{A \times B \mid A \in \mathcal{A}_X, B \in \mathcal{A}_Y\}$*.*

It is easy to check that $\mathcal{A}_{X\times Y}$ satisfies Definition 2.3, but there are a series of interesting analytical properties in the product of spaces of measures which are described in [43, Chapter 7]. Again the case of more than two measure spaces requires formalisation and details can be found in [43].

3. Elementary notions on topological groups

We can define a topology $\mathcal{T}$, as well as a σ-algebra, on a group G.

Definition 3.1 (See Definition 1.1(i), [26])**.** *A topological group G is a group together with a topology such that multiplication $(x, y) \mapsto xy : G \times G \to G$ and inversion $x \mapsto x^{-1} : G \to G$ are continuous functions.*

A topological group is connected, compact, Hausdorff, if its topology is connected, compact, Hausdorff, respectively. With Definition 3.1 we are requiring that Definition 1.1 is satisfied and that the binary operation (that is, multiplication) in G preserves the topological structure on $G \times G$ and G. Up to here, one has the notion of *paratopological groups* (see [9, 52]). In fact the axiom that the inversion is continuous cannot be omitted, since there are examples of paratopological groups which are not topological groups, e.g. the Sorgenfrey line (see [52]).

On the other hand, if the topology of a group G is compact and the multiplication is continuous, then one can see that the axiom of continuity of the inversion is automatically satisfied.

We will work with Hausdorff topological groups so the adjective "Hausdorff" will be assumed from now until the end, anytime we refer to a topological group. A topological group is *locally compact* if the identity has a compact neighbourhood.

The corresponding notion which we must use, in analogy with Definitions 1.3 and 2.9, is that of closed subgroups for topological groups.

Definition 3.2. *A closed subgroup H of a topological group G is an abstract subgroup of G which is closed with respect to the topology of G.*

Of course, we ask that H is closed because we want to preserve both the algebraic and topological structures when we look at sub-structures of topological groups.

Definition 3.3 (See Definition 1.9(ii), [26])**.** *Given a closed normal subgroup H of a topological group G, the quotient group G/H is a topological group with the quotient topology induced by the canonical projection $\pi : G \to G/H$.*

Details and comments on Definition 3.3 are made in [26, Proposition 1.10 (i), (ii) and (iii)]. The notion of a product of topological groups is exactly what one expects: it is defined algebraically as the product group of the given factors and topologically it has the product topology in Definition 2.12.

Example 3.4. The center is a subgroup which is always closed. In a compact group, we see this as an application of [26, Proposition 1.10 (i)]. The commutator subgroup however is not always closed. An example is seen in [26, Exercise E6.6] of a compact, totally disconnected group. Of course, in the finite case the commutator subgroup is always closed.

To define the notion of a filter basis on a compact group and to give a proof of Proposition 3.13, we will use the notion of filters on a topological space. There are in fact some important filters which allow us to approximate compact groups very well.

Definition 3.5 (See Chapter 1, Section 6 of [9])**.** *Given a topological space X, a set $\mathcal{F} \subseteq \mathcal{P}(X)$ is a filter for X, if*

(i) $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$;

(ii) for all $n \in \mathbb{N}$ and $F_1, \ldots, F_n \in \mathcal{F}$, $\bigcap_{i=1}^{n} F_i \in \mathcal{F}$;

(iii) $B \subseteq A \subseteq X$ with $B \in \mathcal{F}$, then $A \in \mathcal{F}$.

The condition (ii) shows that $\mathcal{F}$ is stable under finite intersections, while (iii) shows that $\mathcal{F}$ is closed upwards. It may be useful to note that a *filter* *basis* of $\mathcal{F}$ is a set $\mathcal{B} \subseteq \mathcal{F}$ such that for all $S \in \mathcal{F}$, there exists $T \in \mathcal{B}$ such that $T \subseteq S$.

If G is a topological group, the *identity component* is the connected component of the identity element in G. The identity component, usually denoted G_0, is a closed normal subgroup of G such that the quotient G/G_0 is totally disconnected (see Exercise E1.12(iii) of [26]). Moreover, a subgroup H containing an element $1 \in G$ is called a neighbourhood of 1 if there exists U open with $1 \in U \subseteq H$. Thus an open neighbourhood of 1 is simply an open subset containing 1. We will see that these notions allow us to formulate approximations of topological groups via finite groups. Details will be shown later on. The table below gives some examples of topological groups and their properties. Let I_n denote the $n \times n$-identity matrix.

Group	Identity	Inverse	Topology	Compact	Connected
$(\mathbb{R}, +)$	0	$\forall\ a \in \mathbb{R}, -a \in \mathbb{R}$	Natural	No	Yes
$(\mathbb{R}^*, \cdot)$	1	$\forall a \in \mathbb{R}^*, \frac{1}{a} \in \mathbb{R}^*$	Induced by $\mathbb{R}$	No	Yes
$(\mathbb{Q}, +)$	0	$\forall\ a \in \mathbb{Q}, -a \in \mathbb{Q}$	Induced by $\mathbb{R}$	No	No
$\mathrm{GL}_n(\mathbb{R})$	I_n	Inverse matrices	Induced by $\mathbb{R}^{n^2}$	No	No
$\mathrm{O}(n)$	I_n	Inverse matrices	Induced by $\mathbb{R}^{n^2}$	Yes	No
$\mathrm{SO}(n)$	I_n	Inverse matrices	Induced by $\mathbb{R}^{n^2}$	Yes	Yes

Table 3.1: Some classical examples of topological groups.

For topological groups, there are some nice circumstances in which Definition 3.5 shows a series of important additional properties.

For instance, if G is a compact group, then

$$\mathcal{N}(G) = \{H \text{ closed normal subgroup of } G \mid G/H \text{ is finite}\}$$

is a filter basis for G and its properties can be found in [26, Proposition 1.33].

We will now give the notion of projective limits below since large classes of topological groups can be constructed as projective limits of finite groups and are usually studied alongside their topologies. We report some notions below which can be found in [26].

Definition 3.6 (Projective system, see [26], Page 17)**.** *Let J be a directed set, that is, a set with a reflexive, transitive and antisymmetric relation $\leq$ such that every finite non empty subset has an upper bound. A* ***projective system*** *of topological groups over J is a family of morphisms $\{f_{jk} : G_k \to G_j \mid (j,k) \in J \times J, j \leq k\}$, where G_j are topological groups for all $j \in J$ satisfying the following conditions:*

(i) $f_{jj} = id_{G_j}$ for all $j \in J$,

(ii) $f_{jk} \circ f_{kl} = f_{jl}$ for all $j,k,l \in J$ with $j \leq k \leq l$.

If $P = \{f_{jk} : G_k \to G_j \mid (j,k) \in J \times J, j \leq k\}$ is a projective system of topological groups, then

$$G = \{(g_j)_{j\in J} \in P \mid j \leq k \ (\forall j,k \in J) \Rightarrow f_{jk}(g_k) = g_j\}$$

is called the ***projective limit*** *of the groups G_j and is written*

$$G = \lim_{j\in J} G_j.$$

The mappings (morphisms) $f_j : G \to G_j$ are called ***limit maps*** *and the morphisms $f_{jk} : G_k \to G_j$ are called* ***bonding maps****.*

In particular, given a projective system of finite groups, a *profinite group* is a topological group which is obtained as the projective limit of finite groups, each endowed with the discrete topology. We recall [26, Page 17].

Lemma 3.7. *Assume* $G = \lim_{j\in J} G_j$ *with* J *directed set.*

(i). If $\mathrm{inc} : G \to \prod_{j\in J} G_j$ *denotes the inclusion and* $\mathrm{pr}_j : \prod_{j\in J} G_j \to G_j$ *the projection, then the function* $f_j = \mathrm{pr}_j \circ \mathrm{inc} : G \to G_j$ *is a morphism of topological groups for all* $j \in J$*, and for* $j \leq k$ *in* J *the relation* $f_j = f_{jk} \circ f_k$ *is satisfied.*

(ii). If all groups G_j *in the projective system are compact, then* $\prod_{j\in J} G_j$ *and* G *are compact groups.*

The proof is reported from [26, page 17] for the convenience of the reader.

Proof.

(i). Assume that $j \leq k$ in J and define

$$G_{jk} = \Big\{ (g_l)_{l\in J} \in \prod_{j\in J} G_j \mid f_{jk}(g_k) = g_j \Big\}.$$

Of course, f_{jk} is a homomorphism of topological groups, since it is the composition of two continuous homomorphism of groups, and so G_{jk} is a subgroup of $\prod_{j\in J} G_j$. Again the continuity of f_{jk} implies that G_{jk} is a closed subgroup. But the intersection of closed subgroups $G = \bigcap_{\substack{(j,k)\in J\times J \\ j\leq k}} G_{jk}$ shows that G must be closed, by one of the axioms of a topological space. Finally, if $j \leq k$, then $f_{jk} \circ f_k = f_j$ which follows from the definition of G_{jk}.

(ii). If all G_j are compact, then $\prod_{j\in J} G_j$ is compact by Tychonoff's theorem (see Theorem 2.13), and thus G as a closed subgroup of $\prod_{j\in J} G_j$ is compact, too. □

Now we illustrate an important example of infinite compact groups. Let p be a prime number. Given an integer $n > 0$, we can write n in base p:

$$n = a_0 + a_1 p + a_2 p^2 + \ldots + a_k p^k$$

with $0 \leq a_i < p$.

Example 3.8 (Additive group of p-adic integers). We report some classical notions, which can be found in [26, Chapter 1] for the material in the present example. A p-adic integer is a series

$$a_0 + a_1 p + a_2 p^2 + \ldots$$

with $0 \leq a_i < p$. The set of p-adic integers is denoted by $\mathbb{Z}_p$. If we cut out an element $\alpha \in \mathbb{Z}_p$ at its kth term

$$n = a_0 + a_1 p + a_2 p^2 + \ldots + a_{k-1} p^{k-1},$$

we get a well-defined element of $\mathbb{Z}(p^k)$. This yields mappings $\mathbb{Z}_p \to \mathbb{Z}(p^k)$.

A sequence of $\alpha_k, k > 0$, such that $\alpha_k \bmod p^{k'} \equiv \alpha_{k'}$ for all $k' < k$ defines a unique p-adic integer $\alpha \in \mathbb{Z}_p$ (when $k = 1, \alpha_1 = a_0$, when $k = 2, \alpha_1 = a_0 + a_1 p$). We then have the bijection

$$\mathbb{Z}_p = \lim_{\leftarrow} \mathbb{Z}/p^k\mathbb{Z}.$$

Here, $f_{jk} : \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p^j\mathbb{Z}$ is the natural projection for $j \leq k$ and the projective limit defined by

$$\lim_{\leftarrow} \mathbb{Z}/p^j\mathbb{Z} = \{(x_j)_j \in \prod_j \mathbb{Z}/p^j\mathbb{Z} \mid f_{jk}(x_k) = x_j, j \leq k\}$$

is the group $\mathbb{Z}_p$ of p-adic integers.

The following theorem gives a characterisation of profinite groups.

Theorem 3.9 (See Theorem 1.34, [26]).

Given a compact group G, the following are equivalent:

(i) G is totally disconnected;

(ii) the filter of neighbourhoods of the identity has a basis of open normal subgroups;

(iii) G is a projective limit of finite groups.

Dealing with profinite groups (and more generally one can note a similar behaviour for compact groups), we work in a topology which is a subspace of the product topology. This shows the importance of thinking of projective limits as suitable subspaces of the topological product. For such a reason, we conclude the present chapter with the proof of Theorem 2.13. To prove this theorem, we will use an equivalent notion of compactness using the finite intersection property and the existence of maximal filters. The idea of the proof uses arguments from [9].

Definition 3.10. *A collection $\mathcal{F}$ of subsets of a set X has the finite intersection property (FIP) if for every finite subcollection $\{F_1, F_2, \ldots, F_n\}$ of $\mathcal{F}$, we have $\bigcap_{i=1}^{n} F_i \neq \emptyset$.*

By Definition 3.5, any filter satisfies the FIP.

Theorem 3.11. *A topological space X is compact if and only if for every collection $\mathcal{F}$ of closed subsets of X with the FIP, we have $\bigcap \mathcal{F} \neq \emptyset$.*

We use the notion of a maximal collection of sets with the FIP. These collections turn out to satisfy the properties of being maximal filters.

Definition 3.12. *Let $\mathcal{C}$ be a collection of subsets of a topological space X with the FIP. $\mathcal{C}$ is maximal if whenever $\mathcal{A}$ is a collection of subsets with the FIP such that $\mathcal{C} \subseteq \mathcal{A}$, then $\mathcal{C} = \mathcal{A}$.*

Now the point is to show the existence of sets like $\mathcal{C}$ in Definition 3.12. The answer is positive and mentioned in the following result.

Proposition 3.13. *For every collection $\mathcal{C}$ of closed subsets with the FIP, there exists a maximal $\mathcal{A}$ such that $\mathcal{C} \subseteq \mathcal{A}$.*

The proof of this proposition uses Zorn's Lemma which we state below:

Lemma 3.14 (Zorn). *Let $(P, \leq)$ be a non empty partially ordered set such that every linearly ordered subset has an upper bound. Then P contains a maximal element.*

We will use the partial order of the family $\mathcal{U}$ of all collections of subsets of a space X with the FIP that contain $\mathcal{C}$ with the order $\subseteq$. So $\mathcal{U} \neq \emptyset$ since $\mathcal{C} \in \mathcal{U}$.

Proof of Proposition 3.13. Let $\mathcal{C}$ be a collection of subsets of X with the FIP. Let A be a linearly ordered subset of $\mathcal{U}$. We need to show that A has an upper bound.

We claim that $\bigcup_{\mathcal{F} \in A} \mathcal{F}$ is an upper bound for A.

Clearly, $\mathcal{F} \subseteq \bigcup_{\mathcal{F}\in A}\mathcal{F}$ for each $\mathcal{F} \in A$. Then what is left is to show that $\bigcup_{\mathcal{F}\in A}\mathcal{F}$ has the FIP. Let $F_1, F_2, \ldots, F_n \in \bigcup_{\mathcal{F}\in A}\mathcal{F}$, then there are $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ such that $F_i \in \mathcal{F}_i$, $1 \leq i \leq n$. Since A is linearly ordered there is a $1 \leq N \leq n$ such that $\mathcal{F}_i \subseteq \mathcal{F}_N$ for all $1 \leq i \leq n$. Then $\mathcal{F}_i \in \mathcal{F}_N$ for all $1 \leq i \leq n$ and because $\mathcal{F}_N$ has the FIP,

$$\bigcap_{i=1}^{n} F_i \in \mathcal{F}_N \subseteq \bigcup_{\mathcal{F}\in A}\mathcal{F}.$$

So Lemma 3.14 applies and $\mathcal{U}$ has a maximal element as desired. □

Because of Proposition 3.13, we may introduce the following notion.

Definition 3.15. *A filter $\mathcal{F}$ is maximal, if it is maximal with respect to the FIP.*

The next result shows that any subset of the power set $\mathcal{P}(X)$ which is maximal with respect to the FIP must necessarily be a filter.

Proposition 3.16. *A collection of subsets $\mathcal{F}$ of a set X that is maximal with respect to the FIP is a filter.*

Proof. Let $F_1, F_2, \ldots, F_n \in \mathcal{F}$. Then $\bigcap_{i=1}^{n} F_i \neq \emptyset$ by the FIP.

Clearly, $\{\bigcap_{i=1}^{n} F_i\} \bigcup \mathcal{F}$ has the FIP and since $\mathcal{F}$ is maximal, we have

$$\bigcap_{i=1}^{n} F_i \in \mathcal{F} = \Big\{\bigcap_{i=1}^{n} F_i\Big\} \bigcup \mathcal{F}.$$

Since $\mathcal{F}$ has the FIP, $\emptyset \notin \mathcal{F}$. Thus $X \cap F = F \neq \emptyset$ for each $F \in \mathcal{F}$. So $\mathcal{F} \bigcup \{X\}$ has the FIP. By the maximality of $\mathcal{F}$, $X \in \mathcal{F} = \mathcal{F} \bigcup \{X\}$. The missing axiom can be proved in analogy, so Definition 3.5 is satisfied and the result follows. □

The following result will let us know how to identify whether a set belongs to the maximal filter or not.

Proposition 3.17. *Suppose that $\mathcal{F}$ is a maximal filter on a set X and $A \subseteq X$. The following are equivalent:*

(i) $A \in \mathcal{F}$;

(ii) $\forall F \in \mathcal{F},\ A \cap F \neq \emptyset$.

Proof. We begin by showing that (i) implies (ii). Suppose that $A \in \mathcal{F}$. Since $\mathcal{F}$ has the FIP, (ii) is clear.

Vice versa, we want to show that (ii) implies (i). Suppose that $\forall F \in \mathcal{F},\ A \cap F \neq \emptyset$. We show that $\{A\} \cup \mathcal{F}$ has the FIP, in which case the result follows by the maximality of $\mathcal{F}$. In fact, if $F_1, \dots, F_n \in \mathcal{F}$, we know that $\mathcal{F}$ is a filter and so $\emptyset \neq \bigcap_{i=1}^n F_i \in \mathcal{F}$. This means that $(\bigcap_{i=1}^n F_i) \bigcap A \neq \emptyset$ and the result follows. □

We say that $\mathcal{B}$ is a *basis* for a topological space $(X, \mathcal{T})$, if $\mathcal{B} \subseteq \mathcal{T}$ and each open set of $\mathcal{T}$ can be written as a union of elements of $\mathcal{B}$. A weaker notion can be formulated: $\mathcal{S}$ is a *subbasis* of a topological space $(X, \mathcal{T})$ if $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{S}$ with the set of finite intersections of elements of $\mathcal{S}$ is a basis for $\mathcal{T}$. Now we have all that we need to prove Theorem 2.13.

Proof of Theorem 2.13. Let $\mathcal{C}$ be a collection of closed subsets of $\prod X_\alpha$ with the FIP. We claim that $\bigcap \mathcal{C} \neq \emptyset$.

Extend $\mathcal{C}$ to a maximal filter $\mathcal{F}$ (by Propositions 3.13 and 3.16). Since $\mathcal{C} \subseteq \mathcal{F}$, we get that $\bigcap \mathcal{F} \subseteq \bigcap \mathcal{C}$. In fact $\bigcap_{F \in \mathcal{F}} \overline{F} \subseteq \bigcap \mathcal{C}$ and so it is enough to show that $\emptyset \neq \bigcap_{F \in \mathcal{F}} \overline{F}$.

We now transfer the filter down, using the usual projection maps $\pi_\alpha : X \to X_\alpha$.

Define $\mathcal{F}_\alpha = \{\pi_\alpha(F) : F \in \mathcal{F}\}$. This collection has the FIP because $\emptyset \neq \pi_\alpha(F_1 \bigcap F_2) \subseteq \pi_\alpha(F_1) \bigcap \pi_\alpha(F_2)$. Thus the collection $\{\overline{\pi_\alpha(F)} : F \in \mathcal{F}\}$ of closed sets has the FIP.

Now since X_α is compact, we can choose $x_\alpha \in \bigcap_{F \in \mathcal{F}} \overline{\pi_\alpha(F)}$ for each $\alpha \in A$. Thus $\underline{x} = (x_\alpha) \in \prod_{\alpha \in A} X_\alpha$.

Our claim will follow once $\underline{x} \in \bigcap_{F \in \mathcal{F}} \overline{F}$ has been shown.

The idea is to show that every open set B (of a basis of the topology) containing $\underline{x}$ has the property that $B \in \mathcal{F}$. This will tell us that for all such open neighbourhoods B of $\underline{x}$ the property $\forall F \in \mathcal{F},\ B \bigcap F \neq \emptyset$ is satisfied, or equivalently, $\forall F \in \mathcal{F},\ \underline{x} \in \overline{F}$. Thus we have our claim.

In order to show that every open set of a basis containing $\underline{x}$ is in $\mathcal{F}$, it is sufficient to show that every open set of a subbasis containing $\underline{x}$ is in $\mathcal{F}$, because $\mathcal{F}$ is closed under finite intersections, and every open set of a basis is a finite intersection of open sets of a subbasis. Therefore we may restrict to claim that all open sets of a subbasis containing $\underline{x}$ are in $\mathcal{F}$. This is easier to show, because if $S = \pi_\alpha^{-1}(U_\alpha)$ is an open set of a subbasis containing $\underline{x}$, where U_α is open in $X_\alpha = \pi_\alpha^{-1}(X)$, then $x_\alpha \in U_\alpha$. By our choice of x_α, we have $x_\alpha \in \bigcap_{F \in \mathcal{F}} \overline{\pi_\alpha(F)}$. Thus $x_\alpha \in U_\alpha \bigcap \overline{\pi_\alpha(F)}$ for each $F \in \mathcal{F}$. So $\emptyset \neq \pi_\alpha^{-1}(U_\alpha) \bigcap F = S \bigcap F$ and by Proposition 3.17 we have $S \in \mathcal{F}$ so the original claim is proved. $\square$

4. Two classical examples of topological groups

Here we show some relations between compact groups and some classical notions in differential geometry. This is done, in order to show the importance of compact groups in various branches of pure and applied mathematics. Another relation is discussed in connection with algebraic topology.

Definition 4.1 (See Definition 2.37, [26]). *Let G be a topological group. G has no small subgroups (resp. no small normal subgroups), if there is a neighbourhood U of the identity such that for every subgroup (resp. normal subgroup) H of G, the relation $H \subseteq U$ implies $H = \{1\}$.*

Small subgroups may be used to define compact Lie groups in a topological perspective.

Definition 4.2 (See Definition 2.41, [26]). *A compact group G is called a compact Lie group, if it has no small subgroups.*

In particular, one can replace the absence of small subgroups with equivalent conditions:

Proposition 4.3 (See Corollary 2.40, [26]). *For a compact group G, the following statements are equivalent:*

(i) G is isomorphic as a topological group to a compact group of orthogonal (or unitary) matrices;

(ii) G has no small subgroups;

(iii) G has no small normal subgroups.

There are additional conditions, involving Banach algebras and finite dimensional representations in [26, Corollary 2.40], but we report only the conditions above, because it is what we need for our purposes. In fact we focus on the equivalent conditions between the property of being a compact Lie group, the absence of small normal subgroups, and the concrete model of orthogonal groups. Somehow Proposition 4.3 describes compact Lie groups in terms of structure.

On the other hand, one can describe any compact group via a theorem of approximation and, in doing this, the notion of projective limit must be used.

Theorem 4.4 (See Corollary 2.43, [26])**.** *Every compact group is a projective limit of compact Lie groups.*

The relevance of Theorem 4.4 has been discussed by several authors and brings about important results of structure for compact groups (see Levi–Mal'cev Structure Theorem in [26, Theorems 9.23 and 9.24]) via approximations of Lie groups. The reader may refer to the main results in [26, Chapters 8 and 9] for the structure of compact groups.

On the other hand, there is an important connection between Definition 4.3 and differential geometry, because the approximation via compact Lie groups may transfer a differential structure on a compact group. This idea is quite technical to explain here, but we invite the reader to refer to [26, Chapter 5] for a formal approach. Roughly speaking, first of all one needs to formulate the notion of *analytic group* [26, Page 139] and see how an analytic manifold (modelled on $\mathbb{R}^n$) of dimension n can be associated to a Lie group. Then one discovers that:

Theorem 4.5 (See Corollary 5.73, [26])**.** *Every compact Lie group is a finite dimensional analytic group.*

Theorem 4.5 shows that a compact Lie group may be endowed with a differential structure and this makes compact groups very interesting in theoretical physics and mathematical physics. Details are quite sophisticated and are not reported here.

A second classical example can be found in the area of algebraic topology. Here the reader may refer to [34] for details and constructions, which we sketch briefly in the following lines.

Definition 4.6. *Given a topological space X, the map $f : t \in [0,1] \to X$ is a path in X, if it is continuous. The points $f(0)$ and $f(1)$ are called the initial and final points of f, respectively.*

If $f : t \in [0,1] \to X$ is a path in X, the *inverse path* of f is the path $f^{-1} : t \in [0,1] \to X$ given by $f^{-1}(t) = f(1-t)$. According to [34, Definition 12.3];

Definition 4.7. *A topological space X is said to be path connected if given any two points $x_0, x_1 \in X$, there is a path in X from x_0 to x_1.*

Note that the notion of connectedness is more general than path connectedness but they are equivalent in $\mathbb{R}^n$. Details can be found in [34, 52]. Next we define the *star product* in order to construct a new path from old ones.

If f and g are two paths in X such that $f(1) = g(0)$, then the map $f * g : [0,1] \to X$, defined by

$$(f * g)(t) = \begin{cases} f(2t), & 0 \le t \le 1/2 \\ g(2t-1), & 1/2 \le t \le 1 \end{cases}$$

turns out to be a path in X. This is possible via the glueing lemma [34, Lemma 12.2], which we report below.

Lemma 4.8 (Glueing Lemma)**.** *Given topological spaces W and X, suppose that $W = A \cup B$ where A and B are closed subsets of W. If $f : A \to X$ and $g : B \to X$ are continuous such that $f(w) = g(w)$ for all $w \in A \cap B$, then $h : W \to X$ defined by*

$$h(w) = \begin{cases} f(w), & \text{if } w \in A \\ g(w), & \text{if } w \in B \end{cases}$$

is continuous.

Next, we consider the notion of *homotopy* of one path into another. Given two continuous maps $f, g : X \to Y$, where X and Y are prescribed topological spaces, we say that f is *homotopic* to g (denoted by $f \sim g$), if there is a continuous map (called *homotopy*) $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ (see [34, Definition 13.1]). The following definition is a more general notion of homotopy and is reported by [34, Definition 13.2].

Definition 4.9. *Suppose $A \subseteq X$, we say that f and g are homotopic relative to A (and we write $f \sim_A g$) if there is a homotopy $F : X \times [0,1] \to Y$ between f and g such that $F(a,t) = f(a)$ for all $a \in A$ and $t \in [0,1]$.*

Note that the relation $\sim_A$ is an equivalence relation on the set of paths in X (see [34, Lemma 13.3]). Recalling [34, Definition 15.1], a path f in X is a *loop* based at x, if $f(0) = f(1) = x$. Now we may specialise the notion of homotopy for loops.

Definition 4.10. *Two loops f and g are said to be homotopic if there is a continuous map $F : (t,s) \in [0,1] \times [0,1] \to F(t,s) \in X$ such that $F(t,0) = f(t), F(t,1) = g(t)$ for all $t \in [0,1]$ and $F(0,s) = f(0), F(1,s) = f(1)$, $g(1) = g(0)$ for all $s \in [0,1]$.*

The equivalence class of a loop based at $x \in X$ is usually denoted by $[f]$ and

$$\pi(X,x) = \{[f] \mid f \text{ is a loop of basis } f(0) = f(1) = x\},$$

where

$$[f] = \{g \text{ loops on } X \mid f \sim_{\{0,1\}} g\}.$$

Now one can check that the set $\pi(X,x)$ is a group under the operation $[f] \cdot [g] = [f * g]$, for all $[f], [g] \in \pi(X,x)$ (see [34, Theorem 15.2]) since

1. the operation $\cdot$ on f and g is well defined;

2. it is associative, i.e.: for all $[f], [g], [h] \in \pi(X,x)$ one can check that the loops $f * (g * h)$ and $(f * g) * h$ are homotopic;

3. there is $\varepsilon = [x] \in \pi(X,x)$ such that $[f] \cdot \varepsilon = \varepsilon \cdot [f] = [f]$;

4. for all $[f] \in \pi(X, x)$, there is $[f]^{-1}$ such that $[f] \cdot [f]^{-1} = [f]^{-1} \cdot [f] = \varepsilon$.

By the axioms above, $\pi(X, x)$ is a group, called the *fundamental group* of X based at x.

The following result is based on [34, Exercise 15.18(d)].

Proposition 4.11. *If G is a topological group, then $\pi(G, 1)$ is an abelian group.*

Proof. Take two loops f and g in G based at 1 and consider

$$(f * g)(t) = \begin{cases} f(2t), & 0 \le t \le 1/2 \\ g(2t-1), & 1/2 \le t \le 1 \end{cases}$$

We want to show that $(f * g)(t)$ is homotopic to the loop given by the pointwise product of $f(t)$ and $g(t)$, $f(t) \cdot g(t)$. By the definition of a loop, $f(1) = g(0) = 1$. We first show that $\cdot$ and the usual multiplication $*$ on $\pi(G, 1)$ are actually the same binary operation. Consider

$$f'(t) = \begin{cases} f(2t), & 0 \le t \le 1/2 \\ 1, & 1/2 \le t \le 1 \end{cases} \qquad g'(t) = \begin{cases} 1, & 0 \le t \le 1/2 \\ g(2t-1), & 1/2 \le t \le 1. \end{cases}$$

Since $f(1) = 1 = g(0)$, $1 * g'(t) = g'(t)$, $f'(t) * 1 = f'(t)$, we have that

$$(f * g)(t) = f'(t) * g'(t).$$

We next show that $f * g$ is homotopic to $f \cdot g$. Define two functions $h_1, h_2 : [0, 1] \times [0, 1] \to G$ by

$$h_1(s, t) = \begin{cases} f\left(\dfrac{2t}{1+s}\right), & 0 \le t \le \dfrac{1+s}{2} \\ 1, & \dfrac{1+s}{2} \le t \le 1 \end{cases}$$

$$h_2(s,t) = \begin{cases} 1, & 0 \leq t \leq \dfrac{1-s}{2} \\ g\left(\dfrac{2t-1+s}{1+s}\right), & \dfrac{1-s}{2} \leq t \leq 1. \end{cases}$$

At $s = 0$, we have the first half of the loop $f * g$ since h_1 does the whole loop f as t goes from 0 to $1/2$ and then rests at 1. As s increases, h_1 does the whole loop f as t goes from 0 to $(1+s)/2$ and at $s = 1$, the whole loop is done at normal speed. Similarly for the function h_2, at $s = 0$ we have the second half of the loop $f * g$ while at $s = 1$, this is exactly the loop g.

Now, we will define an homotopy $H(s,t)$ which is defined when $s = 0$, $s = 1$ and when $t = 0$, $t = 1$. We will see that $H(s,t)$ is continuous since it is the product of two continuous functions (see [34, Lemma 14.2]). At $s = 0$,

$$H(0,t) = h_1(0,t) * h_2(0,t) = f(2t) * g(2t-1) = (f * g)(t)$$

and at $s = 1$,

$$H(1,t) = h_1(1,t) * h_2(1,t) = f(t) * g(t) = f(t) \cdot g(t).$$

Also,

$$\begin{aligned} H(s,0) &= h_1(s,0) * h_2(s,0) = f(0) * 1 = f(0) \\ H(s,1) &= h_1(s,1) * h_2(s,1) = 1 * g(1) = g(1) \end{aligned}$$

Clearly, $H(s,0) = H(s,1) = 1$ for all s, so H is a homotopy of loops between $f * g$ and $f \cdot g$. Repeating the same procedure for $g * f$, we have that $f \cdot g$ is homotopic to $g * f$ and we then conclude that $\pi(G,1)$ is abelian. □

The following corollary is straightforward.

Corollary 4.12. *A compact group induces an abelian fundamental group on itself, based at the identity element.*

5. The space of closed subgroups of a profinite group

Every topological (Hausdorff) group G supports a variety of uniformities compatible with the topology of G. The notion of uniformity and the properties of uniformities can be found in [52], but we are going to recall the necessary information in the following lines, where we deal with a specific topology.

Denoting by $\mathcal{C}(G)$ the *set of all non empty compact subsets* of G, if $\mathcal{U}$ is a compatible uniformity on G, then $\mathcal{C}(G)$ has a natural induced uniformity, compatible with the Vietoris topology on $\mathcal{C}(G)$. A formal definition of the Vietoris topology can be found in [38, Definition 1.7] and usually one can define this topology via a local basis. In our case, if G is a compact group, then G has a unique compatible uniformity (that we may suppose to be $\mathcal{U}$) and we therefore describe the topology on the space $\mathcal{S}(G)$ *of all closed subgroups of* G via the local basis for $\mathcal{U}$ at the point H in $\mathcal{S}(G)$

$$B(H,U) = \{K \in \mathcal{S}(G) : H \subseteq K \cdot U \text{ and } K \subseteq H \cdot U\}, \tag{5.1}$$

where U is an open neighbourhood of the identity.

Giving the Vietoris topology on $\mathcal{S}(G)$ turns out to be equivalent to giving the topology induced by the local basis (5.1). Note that $\mathcal{S}(G)$ turns out to be compact in this situation and we will actually see this fact with more details in Proposition 5.5 below.

There is a more general notion of projective limit, which may be formulated (see [26, Appendix 3, Pages 739–743]) via the notion of pullback in arbitrary categories (not necessarily for topological groups) with corresponding morphisms, but we focus mainly on profinite groups since we do not have examples which are not too different from approximations of finite spaces when we want to generalise the notion of subgroup commutativity degree.

Definition 5.1 (Pullback, see Definition A3.43 (iii) in [26])**.** *Given the objects A, B, C in a category $\mathcal{C}$ and morphisms f and g in $\mathcal{C}$, the object $P = A \times_{\mathcal{C}} B$ is called a pullback of f and g, if there are morphisms $p : P \to A$ and $q : P \to B$ such that $f \circ p = g \circ q$ and that for each pair of morphisms $\alpha : X \to A$ and $\beta : X \to B$ (where X is a given object in $\mathcal{C}$) with $f \circ \alpha = g \circ \beta$, there is a unique morphism $\xi : X \to P$ such that the following diagram*

$$\begin{tikzcd}
X \arrow[dr, dashed, "\xi"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& P \arrow[r, "p"] \arrow[d, "q"'] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

is commutative, that is, $\alpha = p \circ \xi$ and $\beta = q \circ \xi$.

A classical result of the theory of categories shows that a category has pullbacks, if it has finite products and intersections of retracts (see [26, Corollary A3.46] for details and terminology). One can also show that a category has arbitrary limits (in particular, projective limits in the sense of Definition 3.6) if it has arbitrary products and intersections of retracts (see [26, Theorem A3.48]). This is just to say that Definition 3.6 and Lemma 3.7 may be formulated categorically, but we focus mainly on the category of profinite groups.

Example 5.2 (See Exercise EA3.30 in [26]). Classical examples of categories with pullbacks and projective limits are given by the category of compact (Hausdorff) groups, by the category of profinite groups, by the category of topological spaces and by the category of rings.

Subsequently, we work only with $I = \mathbb{N}$, that is, with projective limits of countably many finite groups. This assumption is not very restrictive because of the following theorem which is reported from [26, Theorem 9.91].

Theorem 5.3 (Countable Layer Theorem)**.** *Any compact group G has a canonical countable descending sequence*

$$G = \Omega_0(G) \supseteq \Omega_1(G) \ldots \supseteq \Omega_n(G) \supseteq \ldots$$

of closed characteristic subgroups $\Omega_n(G)$ of G such that

$$\bigcap_{n\in\mathbb{N}} \Omega_n(G) = Z_0(G_0),$$

where $Z_0(G_0)$ is the identity component of $Z(G_0)$.

Moreover $\Omega_n(G)/\Omega_{n+1}(G)$ is a strictly reductive group for each $n \in \mathbb{N}$, that is, a product of centerfree simple compact connected Lie groups, or discrete cyclic groups of prime order, or discrete simple (non abelian) finite groups.

There is another important way to describe compact groups via filters (see [26, Corollary 2.43]) and it is the "Theorem of Approximation" for compact groups. We report the corollary below, noting that a compact totally disconnected group is profinite.

Theorem 5.4. *Suppose G is profinite and $\mathcal{N}(G)$ is the set of all closed normal subgroups H of G such that G/H is finite. Then*

$$\lim_{H\in\mathcal{N}(G)} G/H = G.$$

After these classical notions on profinite groups, we go back to our description of $\mathcal{S}(G)$. In [20], there is a usual description of $\mathcal{S}(G)$ in terms of the filter basis introduced in Theorem 5.4. We report its proof from [21], specialised to our case.

Proposition 5.5. *If G is a profinite group, then $\mathcal{S}(G) = \lim_{H\in\mathcal{N}(G)} \mathcal{S}(G/H)$.*

Proof. Let $\mathcal{N}(G)$ denote the filter basis of all $H = \overline{H}$ such that G/H is a finite group. From Theorem 5.4, $\lim_{i\in I} G_i = G$ where $G_i = G/H_i$ with $H_i \in \mathcal{N}(G)$. Here $i \leq j$ iff $H_i \supseteq H_j$. If $i \leq j$, there is a canonical homomorphism $f_{ij} : G_j \to G_i$ given by $gH_j \mapsto gH_i$. Thus, (G_i, f_{ij}) forms a projective system of finite groups in the sense of Definition 3.6 and we may write $G = \lim_{i\in I} G_i$ as shown in the diagram below:

$$
\begin{tikzcd}
G_0 & G_1 \arrow[l, "f_{01}"'] & G_2 \arrow[l, "f_{12}"'] & \cdots \arrow[l] & G_i \arrow[l, "f_{i-1i}"'] & G_{i+1} \arrow[l, "f_{ii+1}"'] \cdots \\
 & & & & G \arrow[llll, "f_0"] \arrow[lllu, dashed] \arrow[u, "f_i"] \arrow[ru, "f_{i+1}"'] &
\end{tikzcd}
\tag{5.2}
$$

The f_i are recalled in Lemma 3.7. By analogy we can take canonical maps $\phi_i : \mathcal{S}(G) \to \mathcal{S}(G_i)$ where ϕ_i are the limit maps from $\mathcal{S}(G)$ to $\mathcal{S}(G_i)$, namely, $\phi_i(L) = LH_i/H_i$. Then we have

$$
\begin{tikzcd}
 & & & \mathcal{S}(G) \arrow[llld, "\phi_0"'] \arrow[lld] \arrow[ld] \arrow[d, dashed] \arrow[rd, "\phi_i"] \arrow[rrrd, dashed] & & & \\
\mathcal{S}(G_0) & \mathcal{S}(G_1) \arrow[l] & \mathcal{S}(G_2) \arrow[l, "\pi_1"'] & \cdots \arrow[l, "\pi_2"'] & \mathcal{S}(G_i) \arrow[l, "\pi_{i-1}"'] & \mathcal{S}(G_{i+1}) \arrow[l, "\pi_i"'] & \cdots \arrow[l]
\end{tikzcd}
\tag{5.3}
$$

One can note that the (continuous) epimorphisms of finite groups $f_{ii+1} : G_{i+1} \to G_i$ induces the isomorphism $G_i \simeq G_{i+1}/N$ of finite groups, where $N = \ker f_{ii+1}$ is consequently finite. Since G_i is a quotient of G_{i+1}, the inclusion $\mathcal{L}(G_i) \subseteq \mathcal{L}(G_{i+1})$ is possible because of the correspondence theorem (see Lemma 1.11). This means that $\mathcal{S}(G_i) \subseteq \mathcal{S}(G_{i+1})$ so the Diagram (5.2) induces the projective system in Diagram (5.3) and we can lift the $\mathcal{S}(G_i)$ getting $\lim_{i\in I} \mathcal{S}(G_i) = \mathcal{S}(G)$. The result follows. $\square$

Fisher and Gartside [20, 21] gave a series of contributions on the hyperspace $\mathcal{C}(G)$ of all non empty compact subsets of G and they presented a similar version of Proposition 5.5. This proposition may be more general but we focus only on profinite groups.

Now we recall some notions of functional analysis, which are fundamental for our main result.

Take a locally compact topological space E and K, a compact subset of E. Define

$$
\mathcal{K}_{\mathbb{C}}(E, K) = \{f : E \to \mathbb{C} \mid f \text{ is continuous and } \operatorname{supp}(f) \subseteq K\}. \tag{5.4}
$$

This is the family of complex-valued continuous functions with compact support on the locally compact space E. Note that the support of the function is given by

$$\operatorname{supp}(f) = \overline{E - f^{-1}(0)} = \overline{\{x \in E \mid f(x) \neq 0\}}. \tag{5.5}$$

In particular, we take the family

$$\mathcal{K}_{\mathbb{R}}(E, K) = \{f : E \to \mathbb{R} \mid f \text{ is continuous and } \operatorname{supp}(f) \subseteq K\} \tag{5.6}$$

and write

$$\mathcal{K}_{\mathbb{R}}(E) = \bigcup_{\substack{K \subseteq E \\ K \text{ compact}}} \mathcal{K}_{\mathbb{R}}(E, K). \tag{5.7}$$

The meaning of $\mathcal{K}_{[0,+\infty)}(E)$ is exactly that of $\mathcal{K}_{\mathbb{R}}(E)$, replacing the role of $\mathbb{R}$ with that of $[0, +\infty)$. A *Radon measure* $\mu : \mathcal{K}_{\mathbb{R}}(E) \to \mathbb{R}$ is a measure satisfying [43, Theorem 6.10], that is, a functional $f \mapsto \mu(f) = \int_E f \mathrm{d}\mu$ such that $|\mu(f)| \leq M_K \|f\|$, where $\|f\| = \sup_{x \in E} |f|$ and M_K a constant. Note that a measure $\mu \in \mathcal{K}_{\mathbb{R}}(E)$ is *positive*, that is, $\mu \geq 0$ if $\mu(f) \geq 0$ for all $f \in \mathcal{K}_{[0,+\infty)}(E)$ (see [25, page 18]).

The set

$$\mathcal{M}(E) = \{\mu : \mathcal{K}_{\mathbb{R}}(E) \to \mathbb{R} : \mu \text{ is a Radon measure}\} \tag{5.8}$$

is very important for what we will later on see. In particular,

$$\mathcal{M}_+(E) = \{\mu : \mathcal{K}_{[0,+\infty)}(E) \to [0, +\infty) \mid \mu \text{ is a Radon measure}\} \tag{5.9}$$

is the set of all positive Radon measures and we have

$$\mathcal{K}_{[0,+\infty)}(E) = \bigcup_{\substack{K \subseteq E \\ K \text{compact}}} \mathcal{K}_{[0,+\infty)}(E, K), \tag{5.10}$$

where

$$\mathcal{K}_+(E, K) = \mathcal{K}_{[0,+\infty)}(E, K) \tag{5.11}$$
$$= \{f : E \to [0, +\infty) \mid f \text{ is continuous and } \operatorname{supp}(f) \subseteq K\}.$$

The following notions of measure theory are classical and can be found in [25, 43].

Definition 5.6 (Regularity, see Chapters 1 and 2 of [43]). *Let $(X, \mathcal{T})$ be a Hausdorff topological space and let $\mathcal{A}$ be the Borel σ-algebra on X that contains the topology $\mathcal{T}$. Then a measure μ on the measurable space $(X, \mathcal{A})$ is called inner regular if, for every set A in $\mathcal{A}$,*

$$\mu(A) = \sup\{\mu(K) \mid K \subseteq A, K \text{ is compact}\},$$

and outer regular if, for every set A in $\mathcal{A}$,

$$\mu(A) = \inf\{\mu(V) : A \subseteq V, V \text{ is open}\}.$$

Among regular measures, Radon measures are well-known and so we focus on them. Now we are going to adapt [25, Theorem 1.2.17 and Lemma 1.2.18] to our context of study and we will recall a result of functional analysis from [43].

Theorem 5.7 (Riesz Representation Theorem). *Let X be a locally compact Hausdorff space, and let Λ be a positive linear functional on*

$$C_c(X) = \{f : X \to \mathbb{C} \text{ continuous} \mid \operatorname{supp}(f) \text{ is compact}\}.$$

Then there exists a σ-algebra $\mathcal{A}$ in X which contains all Borel sets in X, and there exists a unique positive measure μ on $\mathcal{A}$ which represents Λ in the sense that

(i). $\Lambda f = \int_X f d\mu$ *for every* $f \in C_c(X)$,

(ii). $\mu(K) < \infty$ *for every compact set* $K \subseteq X$,

(iii). For every $E \in \mathcal{A}$, *we have*

$$\mu(E) = \inf\{\mu(V) : E \subseteq V, V \text{ is open}\},$$

(iv). The relation

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ is compact}\}$$

holds for every open set E, and for every $E \in \mathcal{A}$ with $\mu(E) < \infty$,

(v). If $E \in \mathcal{A}, A \subseteq E$, and $\mu(E) = 0$, then $A \in \mathcal{A}$.

The conditions (iii) and (iv) are recalled by Definition 5.6, while the condition (v) describes a standard behaviour of μ with respect to the sets of zero measure. For the sake of clarity, let us be more explicit about the meaning of the word "positive". Λ is assumed to be a linear functional on $C_c(X)$, with the additional property that Λf is a non negative real number for every f whose range consists of non negative real numbers.

Remark 5.8. We mentioned Theorem 5.7 because we do not know in general that the properties (i), (ii), (iii), (iv) and (v) are invariant under projective limits. This is the motivation for explicitly recalling the regularity in Definition 5.6. In fact one of our main theorems, Theorem 7.1, will show conditions in which we may do projective limits of measures as in Theorem 5.7 without losing regularity, finiteness on the compact sets and the other classical properties of measures satisfying the Riesz Representation Theorem.

There are more technical notions of measure theory, which are reported in [7, 39] and they involve the methods of the geometric measure theory. We are going to construct a projective limit of measures in the next chapter via a direct approach without using category theory and geometric measure theory.

In fact we will construct an approximation of the subgroup commutativity degree of finite topological spaces.

6. Notions of abstract harmonic analysis

The first result of this chapter shows that for a profinite group G, any positive function $f \in \mathcal{K}_+(\mathcal{S}(G))$ admits an approximating function $g \in \mathcal{K}_+(\mathcal{S}(G))$, which is constant on a large enough set and vanishes out of it. Note that $\mathcal{K}_{\mathbb{R}}(\mathcal{S}(G))$ must be endowed with the uniform norm, so $f \in \mathcal{K}_{\mathbb{R}}(\mathcal{S}(G))$ has $\|f\| = \sup_{H \in \mathcal{S}(G)} |f(H)|$, but $\mathcal{S}(G)$ is compact, so the maximum is always reached (by the Weierstrass Theorem) and we write briefly $\|f\| = |f|$.

Lemma 6.1 below has been adapted to what we have in [25, Lemma 1.2.18].

Lemma 6.1. *Let G be a metrisable profinite group and f a nonnegative real valued continuous function on $\mathcal{S}(G)$ whose support is contained in a given compact subset S of $\mathcal{S}(G)$. Given $\varepsilon > 0$ and an open set U of $\mathcal{S}(G)$ containing S one can find a partition of $\mathcal{S}(G)$ into open and closed sets and a nonnegative real valued function g on $\mathcal{S}(G)$, constant on each member of the partition and zero outside of U, such that*

$$|f(L) - g(L)| < \varepsilon, \quad \forall L \in \mathcal{S}(G).$$

Proof. Since f is continuous and $\mathcal{S}(G)$ is compact, the function f is uniformly continuous on $\mathcal{S}(G)$. It follows that there is a covering $\mathcal{V}$ of $\mathcal{S}(G)$ such that for each $V \in \mathcal{V}$ and any subgroups L and L' in V the estimate

$$|f(L) - f(L')| < \varepsilon$$

holds. Moreover, one can achieve that $V \cap S \neq \emptyset$ always implies $V \subseteq U$. Since G is metrisable there is a countable filter base $\mathcal{N}(G) = (H_i)_{i\geq 1}$ of open normal subgroups of G (see e.g. Proposition 5.5.). Since $\mathcal{S}(G)$ is equipped with the Vietoris topology, the open covering $\mathcal{V}$ can be replaced by a refinement to become a partition by sets of the form

$$V_i(K) := \{L \leq_c G : LH_i = KH_i\}.$$

Further refinement allows us to assume for some finite set J,

$$\mathcal{V} = \{V_i(K_j) : j \in J\}, \tag{$*$}$$

that is, to assume that $i \geq 1$ is fixed and $(K_j)_{j \in J}$ is a finite set of closed subgroups of G. Next we define our candidate for the approximating function $g : \mathcal{S}(G) \to [0, +\infty)$ as follows:

$$g(L) := \begin{cases} f(K_j) & \text{if } L \in V_i(K_j) \text{ and } V_i(K_j) \cap S \neq \emptyset, \\ 0 & \text{else.} \end{cases}$$

Observe that g is continuous since it is constant on the open and closed sets of the partition. Pick L arbitrarily in $S(G)$. We still need to prove $|f(L) - g(L)| < \varepsilon$. Since $\mathcal{V}$ is a partition there exists a unique $j \in J$ with $L \in V_i(K_j)$. If $V_i(K_j) \cap S \neq \emptyset$ then

$$|f(L) - g(L)| = |f(L) - f(K_j)| < \varepsilon,$$

by the continuity property of f on $V_i(K_j)$. Finally, if $V_i(K_j) \cap S = \emptyset$ then certainly $L \cap S = \emptyset$ and therefore

$$|f(L) - g(L)| = |0 - 0| < \varepsilon.$$

Since, by assumption, $V_i(K_j) \cap S \neq \emptyset$ implies $V_i(K_j) \subseteq U$ it follows that g vanishes outside of U. □

Remark 6.2. One may, alternatively, construct from the set of locally constant functions arising in the proof of Lemma 6.1, a point-separating Banach algebra $\mathcal{A}$ which contains the constant functions and apply the Stone-Weierstrass Theorem to $\mathcal{A}$. Little extra work is needed for achieving that g vanishes outside U.

One could wonder whether the above proof works for projective limits of topological spaces which are not necessarily related to topological groups.

The following remark explains why generalisations must be done very carefully.

Remark 6.3. We have fundamental restrictions which prevent us from a general treatment for locally compact groups.

(1) $\mathcal{S}(G)$ is a hyperspace on G which preserves the projective limit structure on G, that is, $\mathcal{S}(\lim_{i\in I} G_i) = \lim_{i\in I} \mathcal{S}(G_i)$. This property is not always satisfied when we replace G_i with an arbitrary topological space X_i and it is a first obstruction to generalising Proposition 5.5.

(2) We begin with a compact space X and know that $\mathcal{S}(X)$ is very special and has the Vietoris topology, which makes $\mathcal{S}(X)$ compact. In general, if we begin with a generic topological space X and take an arbitrary hyperspace on X, then the behaviour of the Vietoris topology should be properly investigated.

(3) Finally [20, Example 5.2] shows that $\mathcal{S}(\mathbb{T})$ is homeomorphic to the space $\{0\} \cup \{1/n \mid n \in \mathbb{N}\}$. In other words, even if we begin with a connected non profinite compact group G, then $\mathcal{S}(G)$ is not very different from an approximation of finite spaces. Therefore, even if we stay among topological groups, and look at connected examples, there are not too many differences from the profinite case. This means that the assumptions of Lemma 6.1 are general enough.

Remark 6.3 motivates the following problem:

Question 6.4. *Assume G is a locally compact group, $\mathfrak{X}$ a class of groups containing the class $\mathfrak{L}$ of all compact Lie groups and $\mathcal{S}(G)$ approximable (as projective limit) by groups in $\mathfrak{X}$. Which properties should $\mathfrak{X}$ have, in order to be the maximal class such that the argument of Lemma 6.1 applies?*

The consideration of [10, 24] shows that the space of closed subgroups of locally compact groups is very rich and difficult to describe.

7. The main result and its applications

Now we are ready to prove one of the main results of this thesis.

We may consider the bonding maps π_i defined in Equation (5.3) and the following functions:

$$\pi^*_{i-1} : \varphi_i \in \mathcal{K}_+(\mathcal{S}(G_i)) \longmapsto \pi^*_{i-1}(\varphi_i) = \varphi_i \circ \pi^{-1}_{i-1} \in \mathcal{K}_+(\mathcal{S}(G_{i-1})),$$

$$\phi^*_i : \varphi \in \mathcal{K}_+(\mathcal{S}(G)) \longmapsto \phi^*_i(\varphi) = \varphi_i \in \mathcal{K}_+(\mathcal{S}(G_i))$$

where $\varphi|\mathcal{S}(G_i) = \varphi_i$.

Theorem 7.1. *Let $G = \lim_{i\in I} G_i$ be a profinite group. If $(\mathcal{S}(G_i), \pi_i, I)$ is the projective system realising $\mathcal{S}(G)$ and $\mu_i \in \mathcal{M}_+(\mathcal{S}(G_i))$ such that $\phi^*_{i-1} = \pi^*_{i-1} \circ \phi^*_i$ and $\mu_i = \pi^*_{i-1} \circ \mu_{i-1}$, then there exists exactly one $\mu \in \mathcal{M}_+(\mathcal{S}(G))$ such that $\mu|\mathcal{S}(G_i) = \mu_i$ and the following diagram*

$$\begin{tikzcd}
 & & & \mathcal{K}_+(\mathcal{S}(G)) \arrow[dlll, dashed] \arrow[dl, "\phi^*_{i-1}"'] \arrow[dr, "\phi^*_i"] \arrow[drr, dashed] & & \\
\mathcal{K}_+(\mathcal{S}(G_0)) \arrow[drrr, dashed] & \cdots \arrow[l] & \mathcal{K}_+(\mathcal{S}(G_{i-1})) \arrow[l, dashed, no head] \arrow[dr, "\mu_{i-1}"'] & & \mathcal{K}_+(\mathcal{S}(G_i)) \arrow[ll, "\pi^*_{i-1}"'] \arrow[dl, "\mu_i"] & \cdots \arrow[l, "\pi^*_i"'] \arrow[dll, dashed] \\
 & & & (0, +\infty) & &
\end{tikzcd} \tag{7.1}$$

is commutative.

The proof of Theorem 7.1 is split into two parts. In the first part we prove the existence of μ and in the other, we prove the uniqueness of this μ.

Proof. We want to apply an argument of approximation as in Lemma 6.1 or Remark 6.2, replacing the role of $\mathcal{A}$ with

$$\mathcal{V} = \{\, f : \mathcal{S}(G) \to \mathbb{R} \mid f = 1 \ \forall\ X \in \mathcal{S}(G) \text{ with } XK_i = HK_i \ (H \in \mathcal{S}(G)) \\ \text{and } f = 0 \text{ otherwise}, \ i = 1, 2, 3, \ldots \}$$

where $\mathcal{S}(G) = \lim_{i \in I} \mathcal{S}(G/K_i)$ and $K_i \in \mathcal{N}(G)$. Again we may omit the evident condition that $\operatorname{supp}(f)$ is compact. It is clear that $\mathcal{V}$ coincides with $\mathcal{A}$ in Remark 6.2, when we consider the case $S = \mathcal{S}(G)$ in $\mathcal{A}$.

If $\varphi \in \mathcal{K}_+(\mathcal{S}(G))$, we can easily check that $\mu(\varphi) = \mu_i(\varphi_i)$ where $\mu = \phi_i^* \circ \mu_i$, $\phi_{i-1}^* = \pi_{i-1}^* \circ \phi_i^*$ and $\mu_i = \pi_{i-1}^* \circ \mu_{i-1}$.

The value $\mu(\varphi) := \mu_i(\varphi_i)$ is independent of the choice of i because for each i, the same argument works. In fact, one can note from the following diagram that $\varphi_i \circ (\pi_l \circ \pi_{l-1} \circ \cdots \circ \pi_i) = \varphi_l$

$$\begin{tikzcd}
\ldots\mathcal{S}(G_{i+1}) \arrow[drrrr, "\varphi_i"'] & \mathcal{S}(G_{i+2}) \arrow[l, "\pi_{i+1}"'] & \cdots \arrow[l, "\pi_{i+2}"'] \arrow[drr, dashed] & \mathcal{S}(G_{l-1}) \arrow[l] & \mathcal{S}(G_l) \arrow[l, "\pi_{l-1}"'] \arrow[d, "\varphi_{l-1}"] & \mathcal{S}(G_{l+1})\ldots \arrow[l, "\pi_l"'] \arrow[dl, "\varphi_l"] \\
& & & & \mathbb{R} &
\end{tikzcd} \tag{7.2}$$

and hence

$$\mu_l(\varphi_l) = \mu_i(\varphi_i).$$

Similarly one obtains $\mu_l(\varphi_l) = \mu_k(\varphi_k)$ for different k and l in I and the independence of $\mu(\varphi)$ is established.

Therefore $\mu \in \mathcal{M}_+(\mathcal{S}(G))$ exists and is an extension of $\mu_i \in \mathcal{M}_+(\mathcal{S}(G_i))$ as indicated in Diagram (7.1).

Now, to show the uniqueness of μ we show that whenever μ and μ' are measures in $\mathcal{M}_+(\mathcal{S}(G))$ constructed as extensions of $\mu_i \in \mathcal{M}_+(\mathcal{S}(G_i))$ as indicated in Diagram (7.1), then $\mu = \mu'$. In fact, if $v \in \mathcal{K}_+(\mathcal{S}(G))$, then we have for all $i \in I$ that

$$\mu(v) = (\phi_i^* \circ \mu_i)(v) = (\phi_i^* \circ \mu_i')(v) = \mu'(v).$$

So the result follows. $\square$

From Theorem 7.1, the moment we assign a profinite group $G = \lim_{i\in I} G_i$ and a family of measures $\{\mu_i \in \mathcal{M}_+(\mathcal{S}(G_i)) \mid i \in I\}$ on each $\mathcal{S}(G_i)$, there is a unique measure μ on $\mathcal{S}(G) = \lim_{i\in I} \mathcal{S}(G_i)$, obtained as the projective limit of μ_i.

Definition 7.2. *Assume $G = \lim_{i\in I} G_i$ is profinite and $\{\mu_i \in \mathcal{M}_+(\mathcal{S}(G_i)) \mid i \in I\}$ are given measures on $\mathcal{S}(G_i)$. The unique measure on G, ensured by Theorem 7.1, will be denoted by $\mu = \lim_{i\in I} \mu_i$. We define the subgroup commutativity degree* $\mathrm{sd}(G)$ *of G as the measure*

$$\mathrm{sd}(G) = \lim_{i\in I} \mathrm{sd}(G_i) \tag{7.3}$$

where $\mathrm{sd}(G_i)$ *is the subgroup commutativity degree of a finite group G_i.*

In the previous definition, one has to note that $\mathrm{sd}(G_i)$ counts the commuting subgroups of G_i in $\mathcal{S}(G_i) \times \mathcal{S}(G_i) = \mathcal{L}(G_i) \times \mathcal{L}(G_i)$ and this number is then normalised by the size $|\mathcal{L}(G_i)|^2$ of the space $\mathcal{L}(G_i) \times \mathcal{L}(G_i)$. When we look at $\mathrm{sd}(G)$, we should count closed commuting subgroups of G in $\mathcal{S}(G) \times \mathcal{S}(G)$ and normalise by the size of $\mathcal{S}(G) \times \mathcal{S}(G)$, which is of course bounded and compact. In both cases we have a probability measure on a product space, so if G is profinite and μ_i are given measures on $\mathcal{S}(G_i)$, one can apply Theorem 7.1, find a unique measure $\mu = \lim_{i\in I}$ on $\mathcal{S}(G)$, and can look at

$$(\mu \times \mu)(\mathcal{S}(G) \times \mathcal{S}(G)) = (\lim_{i\in I} \mu_i \times \lim_{i\in I} \mu_i)(\mathcal{S}(G_i) \times \mathcal{S}(G_i)).$$

This is an alternative and more formal version of (7.3).

Following Heyer [25, see pages 27 and 28], let us suppose that we have two locally compact spaces E and F (where $E = F = \mathcal{S}(G)$), and measures $\mu \in \mathcal{M}_+(E)$ and $\mu \in \mathcal{M}_+(F)$. It is known that there exists exactly one measure on the product space $(\mu \times \mu) \in \mathcal{M}_+(\mathcal{S}(G) \times \mathcal{S}(G))$ such that for all $f, g \in \mathcal{K}_+(\mathcal{S}(G))$, we can define the integral

$$\int_{\mathcal{S}(G)\times\mathcal{S}(G)} f(x)g(y)\, d(\mu\times\mu) = \left(\int_{\mathcal{S}(G)} f(x)\, d\mu\right)\left(\int_{\mathcal{S}(G)} g(y)\, d\mu\right). \tag{7.4}$$

We may apply for all $h \in \mathcal{K}_+(\mathcal{S}(G) \times \mathcal{S}(G))$, the theorem of Fubini, that is,

$$\int_{\mathcal{S}(G)\times\mathcal{S}(G)} h(A,B)\ d(\mu \times \mu) = \int_{\mathcal{S}(G)} d\mu \int_{\mathcal{S}(G)} h(A,B)\ d\mu. \tag{7.5}$$

The measure $(\mu \times \mu)$ so defined is called the product measure of two copies of μ. The mapping $(\mu, \mu) \to \mu \times \mu$ from $\mathcal{M}_+(\mathcal{S}(G)) \times \mathcal{M}_+(\mathcal{S}(G))$ into $\mathcal{M}_+(\mathcal{S}(G) \times \mathcal{S}(G))$ is bilinear and the extension to $(\mu \times \mu)-$integrable functions on $\mathcal{S}(G) \times \mathcal{S}(G)$ of the product formula is given by the theorem of Fubini. Clearly, we have

$$\mathcal{M}_+(\mathcal{S}(G)) \times \mathcal{M}_+(\mathcal{S}(G)) \subseteq \mathcal{M}_+(\mathcal{S}(G) \times \mathcal{S}(G)).$$

Corollary 7.3. *If G is finite, then Equation (7.3) becomes the subgroup commutativity degree of a finite group.*

Proof. Looking at the previous definitions and at Theorem 7.1, here $\mathcal{S}(G) = \mathcal{L}(G)$ and $\mu = \mu_i$ is the counting measure on $G = G_i$. The result then follows. □

The profinite case is enough to describe the compact case, because any compact group G has the quotient group G/G_0 which is profinite. Therefore we get;

Corollary 7.4. *Assume G is a compact group. If $G_0 = G$, then $\mathrm{sd}(G/G_0) = 0$. Otherwise, $\mathrm{sd}(G/G_0)$ is the subgroup commutativity degree of the profinite group G/G_0.*

We now focus on some applications. We want to describe the behaviour of (7.3) under direct products. It is important to recall a result of Suzuki [47, Theorem 1.6.5] that shows that two finite groups G and H satisfy $\mathcal{L}(G\times H) \simeq \mathcal{L}(G) \times \mathcal{L}(H)$ if and only if the elements of G have coprime order with those of H. Actually Suzuki's theorem shows the same for more than two factors, that is,

$$\mathcal{L}(G_1 \times G_2 \times \ldots \times G_i \times G_{i+1} \times \ldots) \simeq \mathcal{L}(G_1) \times \mathcal{L}(G_2) \times \ldots \mathcal{L}(G_i) \times \mathcal{L}(G_i + 1) \times \ldots$$

if and only if G_i is coprime with G_j for all $i, j \in I$ with $i \neq j$. Given two profinite groups G and H one can find variations of Suzuki's theorem.

in order to ensure that $\mathcal{S}(G \times H) \simeq \mathcal{S}(G) \times \mathcal{S}(H)$. A first case, in which this happens, is when both G and H are profinite torsion groups (see [26, Appendix I] in the abelian case), but the elements of G have co-prime order with those of H. A second case is when G is profinite torsion-free and H is profinite torsion. We do not want to give more details since we will directly use the condition $\mathcal{S}(G \times H) \simeq \mathcal{S}(G) \times \mathcal{S}(H)$ in the next result.

Proposition 7.5. *Assume that $G = \lim_{i \in I} G_i$ and $H = \lim_{j \in J} H_j$ are profinite groups, $\mu_i \in \mathcal{M}_+(\mathcal{S}(G_i))$ and $\lambda_j \in \mathcal{M}_+(\mathcal{S}(H_j))$. If $\mathcal{S}(G \times H) \simeq \mathcal{S}(G) \times \mathcal{S}(H)$, then*

$$\begin{aligned}((\mu \times \mu) \times (\lambda \times \lambda))&\Big((\mathcal{S}(G) \times \mathcal{S}(G)) \times (\mathcal{S}(H) \times \mathcal{S}(H))\Big) \\ &= (\mu \times \mu)\Big(\mathcal{S}(G) \times \mathcal{S}(G)\Big) \cdot (\lambda \times \lambda)\Big(\mathcal{S}(H) \times \mathcal{S}(H)\Big).\end{aligned}$$

Proof. There is a $h \in \mathcal{K}_+(\mathcal{S}(G) \times \mathcal{S}(G) \times \mathcal{S}(H) \times \mathcal{S}(H))$ such that

$$h(A, B, C, D) = \psi(\{1\}, \{1\}, C, D) \cdot \varphi(A, B, \{1\}, \{1\})$$

with $\psi \in \mathcal{K}_+\big(\{1\}, \mathcal{S}(G) \times \mathcal{S}(G)\big) \simeq \mathcal{K}_+(\mathcal{S}(G) \times \mathcal{S}(G))$ and $\varphi \in \mathcal{K}_+\big(\mathcal{S}(H) \times \mathcal{S}(H), \{1\}\big) \simeq \mathcal{K}_+(\mathcal{S}(H) \times \mathcal{S}(H))$.

We have the diagram below:

$$\begin{tikzcd}
\mathcal{S}(\{1\}) \times \mathcal{S}(\{1\}) \times \mathcal{S}(H) \times \mathcal{S}(H) \arrow[dr, "\psi"] & \\
\mathcal{S}(G) \times \mathcal{S}(G) \times \mathcal{S}(H) \times \mathcal{S}(H) \arrow[u, "p_{3,4}"] \arrow[d, "p_{1,2}"'] \arrow[r, "h"] & (0, +\infty) \\
\mathcal{S}(G) \times \mathcal{S}(G) \times \mathcal{S}(\{1\}) \times \mathcal{S}(\{1\}) \arrow[ur, "\varphi"'] &
\end{tikzcd} \tag{7.6}$$

where $p_{1,2}(A, B, C, D) = (\{1\}, \{1\}, C, D)$ is the projection fixing the third and fourth components and sending the first and second to the trivial subgroup and $p_{3,4}(A, B, C, D) = (A, B, \{1\}, \{1\})$ analogously. We then look at

$$\int_{\mathcal{S}(G) \times \mathcal{S}(G)} h \, \mathrm{d}(\mu \times \mu) = \int_{\mathcal{S}(G) \times \mathcal{S}(G)} \psi\varphi \, \mathrm{d}(\mu \times \mu)$$

and we have by Fubini's Theorem

$$\int_{\mathcal{S}(G)\times\mathcal{S}(G)} \psi\varphi \,\mathrm{d}(\mu\times\mu) = \left(\int_{\mathcal{S}(G)\times\mathcal{S}(G)} \psi \,\mathrm{d}(\mu\times\mu)\right)\cdot\left(\int_{\mathcal{S}(H)\times\mathcal{S}(H)} \varphi \,\mathrm{d}(\lambda\times\lambda)\right) \tag{7.7}$$

since $\mathcal{S}(G\times H) \simeq \mathcal{S}(G)\times\mathcal{S}(H)$.

Therefore from Equation (7.7) we get

$$(\mu\times\mu)\Big(\mathcal{S}(G)\times\mathcal{S}(G)\Big)\cdot(\lambda\times\lambda)\Big(\mathcal{S}(H)\times\mathcal{S}(H)\Big). \tag{7.8}$$

□

Some consequences of this result are seen below.

Example 7.6. Take $A = \mathbb{Z}_p$, the group of $p-$adic integers with p odd and $B = Q_8$. Since A is abelian and B is Hamiltonian (see [42, Theorem 5.3.7]), Proposition 7.5 implies,

$$\mathrm{sd}(A\times B) = \mathrm{sd}(A)\times\mathrm{sd}(B) = 1.$$

The above example shows a first number of cases in which the subgroup commutativity degree of an infinite profinite group is realised just as a split of the subgroup commutativity degree of an abelian factor by the subgroup commutativity degree of a finite factor (and in the finite case we have literature, see [50]). Proposition 7.5 is then very important and applies to these cases. One can generalise Example 7.6, looking at the finite factor in a more general way, but keeping the abelian factor and the splitting of the subgroup commutativity degree (that is, Proposition 7.5).

Example 7.7. Take $A = \mathbb{Z}_p$ with p odd and $B = Q_{2^m}$ (the generalised quaternion group) with

$$\mathrm{sd}(Q_{2^m}) = \frac{(m-3)2^{m+1} + m2^m + (m-1)^2 + 8}{(m-1+2^{m-1})^2}, \qquad m > 3 \tag{7.9}$$

as seen in [50, Theorem 3.2.1].

Note here that Q_{2^m} $(m \geq 3)$ is a finite $2-$group with a presentation of the form $\langle x, y \mid x^{2^{m-1}} = 1, y^2 = x^{2^{m-2}}, y^{-1}xy = x^{-1}\rangle$ (see [42, Pages 136–141]).

By Proposition 7.5, we have

$$\mathrm{sd}(\mathbb{Z}_p\times Q_{2^m}) = \mathrm{sd}(\mathbb{Z}_p)\times\mathrm{sd}(Q_{2^m}) = \mathrm{sd}(Q_{2^m}) \neq 1.$$

We now consider an example not arising from direct products. Here we cannot apply Proposition 7.5 but we should really construct the bonding maps and the projective system, in order to evaluate the subgroup commutativity degree.

Example 7.8. For a profinite group G, consider $\mu_i = \mathrm{sd}(G_i)$ and $\mu = \mathrm{sd}(G)$. We have $\mu_{|G_i} = \mu_i$ and $\lim_{i\in\mathbb{N}} \mu_i = \mu$ by Theorem 7.1. If μ_i is strictly decreasing and $i \in \mathbb{N}$, then $\lim_{i\in\mathbb{N}} \mu_i = \mu_1$ because each μ_i is an extension of μ_{i+1}. We have the same behaviour on the product space when μ_i is decreasing. In fact, $\lim_{i\in\mathbb{N}}(\mu_i \times \mu_i) = \mu_1 \times \mu_1$.

Consider $G_i = \mathbb{Z}(2) \ltimes \mathbb{Z}(p^i)$ for p odd prime and $i \in \mathbb{N}$. Then

$$\{1\} \xleftarrow{f_0} \mathbb{Z}(2)\ltimes\mathbb{Z}(p) \xleftarrow{f_1} \mathbb{Z}(2)\ltimes\mathbb{Z}(p^2) \xleftarrow{f_2} \ldots \mathbb{Z}(2)\ltimes\mathbb{Z}(p^i) \xleftarrow{f_i} \mathbb{Z}(2)\ltimes\mathbb{Z}(p^{i+1}) \xleftarrow{f_{i+1}} \ldots$$

with

$$\begin{aligned} \mathbb{Z}(2) \ltimes \mathbb{Z}(p^i) &= \langle x \rangle \ltimes \langle y \rangle \\ &= \langle x, y \mid x^2 = 1, y^{p^i} = 1, x^{-1}y^{p^i}x = (y^{p^i})^{-1} \rangle \end{aligned}$$

and

$$f_i : (x, y) \in \mathbb{Z}(2) \ltimes \mathbb{Z}(p^i) \to (x, g_i(y)) \in \mathbb{Z}(2) \ltimes \mathbb{Z}(p^{i-1}),$$

where g_i is the bonding map realising the definition of $\mathbb{Z}_p$ (see [26, Exercise E1.10]). Noting that $D_{2\cdot p^i} = G_i$ where G_i is the dihedral group of order $2p^i$ described in [42, Page 51], we have by [50, Theorem 3.1.1], that

$$\mathrm{sd}(D_{2p^i}) = \frac{\tau(p^i)^2 + 2\tau(p^i)\sigma(p^i) + g(p^i)}{(\tau(p^i) + \sigma(p^i))^2},$$

where

$$\tau(p^i) = \text{number of all divisors of } p^i, \quad \sigma(p^i) = \text{ sum of all divisors of } p^i$$

and

$$|\mathcal{L}(G_i)| = |\mathcal{S}(G_i)| = \tau(p^i) + \sigma(p^i)$$

so that

$$g(p^i) = \frac{(2i + 1)p^{i+2} - (2i + 3)p^{i+1} + p + 1}{(p - 1)^2}$$

for any odd p and $i \in \mathbb{N}$.

We compute $\mathrm{sd}(G)$ for $G = \lim_{i\in\mathbb{N}} G_i$ and get

$$\mathrm{sd}(G) = \lim_{i\in\mathbb{N}} \mu_i(\mathcal{S}(G_i) \times \mathcal{S}(G_i)) = (\mu_1 \times \mu_1)(\mathcal{S}(G_1) \times \mathcal{S}(G_1)) = \mathrm{sd}(G_1).$$

Now $\tau(p) = 2$, $\sigma(p) = p + 1$ and

$$g(p) = \frac{3p^3 - 5p^2 + p + 1}{(p-1)^2}.$$

So,

$$\begin{aligned}
\mathrm{sd}(G_1) &= \frac{\tau(p)^2 + 2\tau(p)\sigma(p) + g(p)}{(\tau(p) + \sigma(p))^2} \\
&= \frac{4 + 2 \cdot 2 \cdot (p+1)}{(p+3)^2} + \frac{3p^3 - 5p^2 + p + 1}{(p-1)^2(p+3)^2} \\
&= \frac{4p+8}{(p+3)^2} + \frac{(p-1)^2(3p+1)}{(p-1)^2(p+3)^2} \\
&= \frac{7p+9}{(p+3)^2} < 1.
\end{aligned}$$

With Examples 7.6 and 7.7 in mind, we may replace the $p-$adic integer $\mathbb{Z}_p$ by a topologically quasihamiltonian profinite group in [27]. So, if A is a topologically quasihamiltonian group, then $\mathrm{sd}(A) = 1$ and Example 7.6 may be generalised by replacing $\mathbb{Z}_p$ with A of such a form.

8. Bibliographical notes and conclusion

The literature, which has been mentioned in the present thesis, may be organised in four portions.

The first portion collects information about the commutativity degree in profinite groups, compact groups and finite groups. The subgroup commutativity degree has been reported from the context of finite groups, where originally it was born. The theorems of structure, which characterise the cases of probability equal to one, belong to this portion.

A second one deals with measure theory and refers to classical theorems in abstract harmonic analysis on topological groups. A more careful analysis of the references can induce the reader to find connections between the theory of the random walks and that of the dynamical systems, but there is no explicit discussion (in the whole thesis) about connections of this nature. On the other hand, some of the references use techniques and methods in these two areas of "applied mathematics".

A third portion deals with commuting pairs in other topological structures, which are more close to the area of algebraic topology. We have not discussed a series of important analogies which could be found with the theory of classifying spaces and corresponding generalisations. This portion shows that there are a series of possible directions and interactions with different branches of topology.

Finally, there are some recent contributions in probabilistic group theory, where it is possible to find methods of combinatorics and of representation theory. Again these areas have interactions and connections with the topics of the present thesis, but the perspective may be different, so further investigations should be made.

In conclusion, to find an appropriate measure for spaces of closed subgroups of non profinite groups, we should reformulate the notion of projective limits for measure spaces with different methods and new ideas. This has been done very recently (in [33]) via the notion of presheaves in [53] and of pullbacks (see Definition 5.1). The existence of some pathological examples shows some technical obstacles and so the notion of the subgroup commutativity degree needs a more sophisticated framework of functional analysis and topology when we want to extend it to wider contexts.

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